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Superprocesses: Construction and Characterization

by

Guillaume Leduc, M.Math.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

October 1995

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CONSTRUCTION AND CHARACTERIZATION**

submitted by

Guillaume Leduc, B.Sc., M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

for Dr. Vlastimil Dlab, Chair
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Abstract

In this thesis we investigate the problem of construction and characterization of branching measure valued processes. We first exhibit a sort of canonical way to construct a large class of branching particle systems. This is then used to construct multitype superprocesses and a class of superprocesses with unbounded branching rate. We then use it to fully characterize a large class of branching measure valued processes, subject to mild conditions. In the last chapter we obtain a characterization of superprocesses in terms of a martingale problem.

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Chapter 0

Introduction

0.1 Basic concepts

A superprocess is a measure valued process arising as the limit of systems of indistinguishable particles which move according to the law of a Markov process ξ and which die at random times leaving a random number of offspring. The limit is taken as the mass of the particles tend to zero but the rate at which they branch tends to infinity.

A branching measure valued process is a Markov process $X = (X_t, \mathfrak{F}, P_{r,\mu})$ such that:

$$P_{r,\mu+\nu} = P_{r,\mu} * P_{r,\nu} \quad (0.1.1)$$

where $*$ denotes the convolution of measures. This *branching property* means that different parts of the random measure evolve independently: if $\mu = \mu_1 + \mu_2$ and if (X_t^1, P_{r,μ_1}) and (X_t^2, P_{r,μ_2}) are independent, then $(X_t, P_{r,\mu})$ has the same law as $X_t^1 + X_t^2$.

In fact, processes satisfying (0.1.1) are sometimes called weak branching pro-

cesses; however, all superprocesses (in so far as this concept is well defined) satisfy also a stronger property which is essentially a consequence of the passage to the limit:

$$\log P_{r,\mu} \left(e^{-\langle f, X_t \rangle} \right) = \int \mu(dx) \log P_{r,\delta_x} \left(e^{-\langle f, X_t \rangle} \right) \quad (0.1.2)$$

A process satisfying (0.1.1) and (0.1.2) is called a *strong branching process*. Since our work is devoted to superprocesses, the word branching process will be used in here to designate the strong branching processes.

0.2 Historical overview

Branching processes were the subject of an important study for the first time in the work of Galton and Watson [WaG] whose work was published in 1874 and was motivated by the problem of family name extinction. At that time, probability theory itself was not developed enough to naturally lead to the study of non-discrete branching processes. The first non-discrete model of branching appeared only in 1951 in a paper due to Feller [Fe]. It was constructed as a limit of Galton-Watson processes. This paper initiated the systematic application of diffusion processes to population biology. The result, the method and the concepts involved in that paper led to the introduction by Jirina [Jr1] in 1958, of the concept of “stochastic branching process with continuous state space”. In fact, the state space referred to is the space $M_f(E)$ of all finite positive measures on a finite set E . The research on branching measure valued processes was thus initiated and Lamperti [La] characterized in 1967 the class of all the Markov processes on $[0, \infty)$ that can be obtained as a limit of Galton-Watson branching processes. In fact he characterized all the time homogeneous right continuous BMV processes with one-point base space E . Silverstein [Si1] proved in 1968 independently using a different method that all these processes belong to a certain class which he described.

A generalization to more general state space E was a natural development. Results on continuous state branching processes in general space E , were first published in 1968 by Watanabe [W1] and in 1969 by Silverstein [Si2]. Watanabe obtained a class of branching measure valued processes as a limit of branching particle systems (superprocesses) and he characterized all the time homogeneous branching measure valued processes. Silverstein characterized a slightly more general class of processes.

In addition to its application to population biology [Fe], the theories of branching particle systems, branching measure valued processes, random measures and related topics, found, in the late sixties and early seventies, applications in physics such as the study of the fluctuations of macroscopic physical variables due to thermal noise [Sz1] and [Sz2]. They have also found applications in the study of stochastic population models (see for example [Se], [PrNiB], [Dav1], [Dav2] and [IkNaW]). Also birth, death and migration models were studied for spatially distributed populations [Bai]. Spacetime epidemic models [Bar] and stochastic population genetics models for geographically structured populations [Fle] and [Ma] were developed.

On the other hand, it was the case that no really satisfactory models were available for the study of the above mentioned fields of research. Dawson in 1975 constructed in [Da1] a large class of measure valued processes which provided suitable models for several applications. Those random measures were obtained as the limit of particle systems. This paper establishes an important relation between the study of measure valued processes and the study of stochastic evolution equations. In another paper [Da2] published in 1977, the author introduced "the critical measure diffusion processes". Together with Watanabe's paper [W1], these papers contributed greatly to establish the foundations and the richness of the theory of superprocesses. In fact, the "critical measure valued diffusion processes" are now known as the Dawson-Watanabe superprocesses and they are typical examples of superprocesses. Their study, for which the authors are major contributors, has been

of considerable importance in the evolution of the superprocesses theory.

Hence the theory had reached in the second half of the seventies a point where it was shown to be rich from the strict point of view of mathematics, linked to other important mathematical theories and fully justified from the point of view of applications. The eighties and the early nineties have seen the appearance of several new kinds of superprocesses and an explosion of the number of publications on the topic. Among others we list: the catalytic superprocesses, the multitype superprocesses, the superprocesses with general additive functional, the multilevel superprocesses, the historical superprocesses and the superprocesses with interactions.

The multitype processes have been first considered by a note of Kolmogorov and Dmitriev [KoD] in 1947. Several authors have given a partial treatment to the subject, when their work easily generalized to the multitype case, but it is only recently that substantial progress has been made in their understanding. Particularly interesting are the papers of Joffe and Métivier [JoM], Ivanoff [Iv1, Iv2], Gorostiza and Lopez-Mimbela [GoLM], Gorostiza and Roelly-Coppoletta [GoRC], Gorostiza, Roelly and Wakobinger [GoRoW] and Li [Li].

A common feature to all the branching measure valued processes mentioned here and to all the branching measure valued processes that were known in the early 90's was that they could always be characterized by an integral equation, called an evolution equation. This is an integral equation solved by the log-Laplace functional

$$v_{r,t}(f)(x) := P_{r,\delta_x}(e^{-\langle X_t, f \rangle}), f \geq 0$$

of a random measure. The discovery of new kinds of superprocesses thus always led to a larger and larger class of evolution equations. In 1993, Dynkin, Kuznetsov and Skorohod proved in an important paper, [DyKuS], that under mild regularity conditions, every branching measure valued process can be characterized by an evolution equation, and they precisely describe the class of possible evolution equations. That was a major breakthrough for the characterization of all the superprocesses.

In fact, that problem had been studied until [DyKuS], only in the case of finite base space E : for the one point base space E , all time-homogeneous right continuous branching measure valued processes were described in Lamperti [La] and Silverstein [Si1], and for the finite base space E , it was proved (by different method) in [W2] and [RyS] that every "regular" CBS belongs to the class described by [DyKuS].

The result in [DyKuS] obviously opens the door to a complete characterization of the superprocesses: it remains only to show that every evolution equation in the class described in [DyKuS] has a log-Laplace functional that solves it. The problem is considered in [Dy7]. A large class of superprocesses is constructed, but it is smaller than the targeted class. A gap remained to be filled, and the problem of the characterization of the superprocesses was still open.

A martingale approach, related to Itô's stochastic differential equations method and based originally on works of Stroock and Varadhan [StV], can be used to construct processes. If π_x is the distribution (on the space of càdlàg trajectories) of a homogeneous E -valued Feller process with associated semigroup T and generator A , it is well known (see for instance [EtK]) that for any ϕ in the domain, $\mathcal{D}(A)$, of A

$$\phi(\xi_t) - \phi(\xi_0) - \int_0^t A\phi(\xi_s)ds \quad (0.2.3)$$

is a π_x -martingale. Moreover, for every $x \in E$, π_x is the only probability measure such that $\pi_x(\xi_0 = x) = 1$ and (0.2.3) is a martingale. If $(E, \mathcal{E})$ is a measurable space, $b\mathcal{E}$ is the set of bounded $\mathcal{E}$ -measurable functions and $A(s, \phi)(x)$ is a mapping from $R_+ \times b\mathcal{E}$ to $b\mathcal{E}$, we say there exists a solution to the martingale problem

$$\phi(\xi_t) - \phi(\xi_r) - \int_r^t A(s, \phi)(\xi_s)ds \quad (0.2.4)$$

if for every $r > 0$ and every $x \in E$ there exists a distribution $\pi_{r,x}$ on the space of càdlàg trajectories such that (0.2.4) is a martingale. If such a solution is unique we say the martingale problem is well posed.

The method can be adapted to measure valued processes, which is done in [Da6, Chapter 5]. The martingale problem characterization has some advantages (although it also has some disadvantages) to it which are going to be discussed below. For some classes of measure valued processes, a characterization via an evolution equation does not exist. In fact the characterization via evolution equation is restricted to the branching measure valued processes. In this historical overview, we will concentrate our attention only on the martingale problems related to the branching measure valued processes. In fact, branching measure valued processes can be linked to a large and important class of measure valued processes via Dawson-Girsanov's transformation [Da6, Th. 7.2.2], providing the martingale problem for the branching measure valued processes is first solved. This is one of the motivations to study the martingale problem characterization for superprocesses.

A measure valued process whose log-Laplace function $v_{r,t}(f)(x)$ solves the integral equation

$$v_{r,t}(x) = \pi_{r,x}f(\xi_t) - \pi_{r,x} \int_r^t \psi(s, \xi_s, v_{s,t})k(ds)$$

where $\xi = (\xi_t, \mathfrak{F}, \pi_{r,x})$ is a Markov process, $k(ds)$ is an additive functional of ξ and ψ is an operator is called a (ξ, ψ, k) -superprocesses. In view of the advantages and the intrinsic interest in characterizing superprocesses by a martingale problem, Roelly-Coppoletta [RoC] posed and solved in 1986 the martingale problem for the $(\xi, (\cdot)^2, ds)$ -superprocesses where ξ is a Feller process. In 1987, El-Karoui and Roelly-Coppoletta extended the result to a large class of (ξ, ψ, ds) -superprocesses where ξ is a Feller process. In 1988, Fitzsimmons obtained some results on the martingale problem for the (ξ, ψ, ds) -superprocesses (ξ a right process¹) and in particular he showed that important properties can be derived from a solved martingale problem. Multitype superprocesses were characterized by martingale problems in

¹A càdlàg Markov process $\xi = (\xi_t, \mathfrak{F}, \pi_{r,x})$ is a right process if, $\pi_{r,\mu}$ -almost surely, the processes $s \mapsto \pi_{s,\xi_s}f(\xi_t)$ are right continuous for every f which belongs to $b\mathcal{E}$, the set of bounded $\mathcal{E}$ -measurable functions.

1990 by Gorostiza and Lopez-Mimbela [GoLM]. In 1992, Fitzsimmons solved the martingale problem for the (ξ, ψ, ds) -superprocesses for ξ a right process and

$$\psi(x, \lambda) = b(x)\lambda^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) n(x, du)$$

where the measure $n(x, du)$ satisfies some properties. No results until recently were available when the branching additive functional rate $k(ds)$ is not the Lebesgue measure. Difficulties are inherent even in the statement of a martingale problem for superprocesses with branching rates $k(ds)$ which are not absolutely continuous with respect to the Lebesgue measure. In 1994, Dawson and Fleischmann showed in [DaF4] that the one point catalytic super Brownian motion, that is the $(\xi, (\cdot)^2, L_t^c)$ -superprocesses (where L_t^c is the local time of the Brownian motion ξ at time t), solves a martingale problem related to the density of the occupation time process.

0.2.1 A comment on the history of superprocesses

While mathematics is a science, this is not the case of history, even when applied to mathematics. But it is the expression of a profound lack of culture not to recognize the intrinsic interest of historical works. As it is the case for all the humanities, historical works are tainted by the interests, the culture and the -never complete- amount of knowledge their authors have about the subjects that they treat. We are obviously not an exception to that rule. Our hope is that this brief historical overview will be of interest for the reader and that it will place the problems treated in this thesis in an historical context that in turn may serve as a motivation for their study.

It seems that a general pattern can be abstracted from the development of mathematical concepts and that most of the important fields in mathematics have had a similar evolution. In the early stages of the theory, a concept, useful for a problem of interest, is invented. It allows mathematicians to solve a larger class of problems than the ones originally considered. Through the study of its applications, ap-

pears the necessity to know more about its properties and the concept itself evolves through the various applications in which it is involved. The notion becomes a subject of research on its own, as a useful mathematically interesting tool. It is often only after a concept has already found important applications that it is brought to maturity and defined in its full generality.

As an illustration of that idea, we relate the development of the notion of group. It is the result of research on the problem of finding all solutions to a polynomial equation. It is in fact because of his work on that problem, that Al-Khawarizmi (who died before 850) got the title of "father of algebra"². In 1826, Abel (1802-1829), motivated by the same problem, implicitly introduced the algebraic structure of *field*. Galois (1811-1832), in his spectacular and genius solution to the (same) general problem, studied only the structure of "groups of substitutions". And it is only years later, in 1854, that Cayley (1821-1895) defined the modern idea of finite *abstract group*.

What has happened to the theory of superprocesses seems also to fit into the general pattern we referred to a moment ago. Mainly driven by problems from genetics, the notion of branching processes has been invented (Galton Watson [WaG]), and has led by a passage to the limit to a primitive idea of superprocesses (Feller [Fe]). The same idea was used in various other applications. The notions of "continuous state branching" processes (Jirina [Jr1]) and "Branching measure valued"³ process (Watanabe [W1]) have been given. In the second half of the seventies, Feller's ideas were abstracted and generalized (Dawson [Da1] and [Da2]) revealing the potential

²Indeed, the word algebra, derived from the arabic al-jabr, comes from the title of Al-Khawarizmi's book "*al-jabr w'al muqábala*". The "word by word" translation of this is "the restoration (of the balance) and the reduction": it refers to the operations of subtracting a quantity on both sides of an equation (restoring the balance of an equation) and the cancellation of like terms on opposite sides of the equation. Al-Khawarizmi studied methods to find solutions to what is now known as "Diophantian equations". See [Boyer, p.252] for more details on the subject.

³Note that this concept has evolved and Dynkin's definition differs from Watanabe's one.

of applications and the mathematical richness of the concept, as well as linking it to other important fields of mathematics. We have now reached the stage where the theory has established behind any doubt its richness and usefulness, and it has also found important applications in differential equation theory (Dynkin [Dy6]). The notion has given birth to several other kinds of related processes such as multitype superprocesses, superprocesses with interactions, etc. While numerous examples of superprocesses are given, the actual concept under which they fall, the concept of superprocess, is still *vague*. We have reached that stage where a suitable and definitive characterization of that concept can be given. Indeed, put together, [Dy7], [DyKuS] and this thesis, constitute a characterization of a large class of branching measure valued processes. But cases not yet covered remain to be found and this is still a lively and most challenging open problem.

0.3 General description of the problems considered in this work

In fact, the main subject of this thesis is the characterization and the construction of superprocesses. The problem of characterization is important in itself, and as we just saw it, has been treated by some of the best probabilists.

Different aspects of the characterization problem can be considered, and different characterizations are possible. The problem can consist of characterizing a whole class of superprocesses as did Dynkin, Kuznetsov and Skorohod, or it can consist of giving another characterization to an already constructed process (which can also serve to characterize classes of superprocesses). For example, while the characterization via an evolution equation is practical and relatively easy to handle, it is totally useless for processes with interactions. On the other hand, a characterization of the same process via a martingale problem will have the advantage of directly leading,

via Dawson-Girsanov's transformations, to the characterization of processes with interactions.

Superprocesses can be characterized in three ways. They can be described via evolution equations, via martingale problems or via stochastic evolution equations. The first two approaches will be treated in this thesis.

In this thesis, we have also considered the problem of the construction of multitype superprocesses. This subject is in itself independent of the characterization problems. The topic is now the subject of a rapidly growing literature, and is crucial for certain applications. In a certain manner it is also useful for the problem of characterization: the multitype superprocesses provide a rich source for the construction of new kinds of superprocesses. We support that affirmation by giving an example of a new class of superprocesses tightly linked to their multitype property. Several interesting properties and special features are described in recent works on multitype superprocess. Those which have catalytic branching rate provide suitable models for chemical reactions and metabolism problems. Those processes are contained in our construction of multitype superprocesses, and they have been constructed for the first time here. Their study will be the subject of future work by the author.

0.4 General description of chapter 1

In this chapter we consider the construction of Particle Systems, which are the building blocks of the whole Superprocesses Theory.

Rigorous construction of particle systems is a problem which was the subject of many publications since the early stages of the measure valued processes theory. While the problem was of obvious interest at the beginning of the development of that theory, it has surprisingly continued to be the subject of new results until the present. That may suggest that no canonical manner to construct them has been

found⁴. As a few examples of different constructions of different particle systems, one can see Harris 1963, Ikeda, Nagasawa and Watanabe (1968-69), Chauvin 1986, Neveu 1986, Gorostiza and Lopez-Mimbela (1990), Dynkin 1991, Dawson 1993⁵, Dynkin 1994... And several other sources treat different constructions. This is in sharp contrast with "ordinary" classes of random variables or Markov processes: there is no need to prove their existence since it is based on Kolmogorov's or Ionescu-Tulcea's Theorems. Particle systems are also, in general, Markov processes. In fact, what complicates the picture, in their case, is that they are often defined not by the behavior of the whole random measure, but rather by the individual behavior of the particles composing the random cloud.

Our construction of particle systems is not only very general and -at least do we believe it- a sort of canonical construction, but moreover it exhibits branching particle systems which had never been constructed before. In fact, we show in chapter three, that simply by carefully choosing one of our branching particle systems, the 14 pages proof of [Dy7, Th. 4.4.1] which are needed to construct the (ξ, Φ, k) -superprocesses with "admissible" k , could be reduced to only a few line proof. Indeed, this proof even solves an open problem the author raises in his book [Dy7, p.113].

Closely related to the problem of construction of branching particle systems (abbreviated BPS), is the problem of their characterization. A powerful tool to characterize BPS's is to express their log-Laplace functional as the (unique) solution to some evolution equations. These are integral equations, related to the BPS's. This topic may not be properly covered without the study of the Feynman-Kac formulae, which establish existence and uniqueness to (large) classes of integral equations.

⁴It seems to the author that while all families of random variables are almost always constructed via Kolmogorov's Theorem or Ionescu-Tucea's Theorem, in the case of branching particle systems, everybody has his own *ad hoc* method to construct them.

⁵Several different constructions are presented in Dawson's reference book. The author's construction of Moran particle systems is particularly interesting.

In fact, these important formulae, are four theorems established in [Dy7, Ch. 4] which generalize the classical Feynman-Kac formula. This family of theorems was incomplete. One of them didn't have the uniqueness part established. This is done in chapter one. This formula is crucial for the proof of existence of some important BPS's considered in this thesis.

In a "supplementary chapter" we give a unified manner to construct extremely general multitype particle systems that would be an analogue to Kolmogorov's and Ionescu-Tulcea's canonical constructions. It includes particle systems with interactions which are not even Markovian. The topic is a little on the margin of the main subject of this thesis, and we have put it in a "supplementary chapter", to be read by the interested reader only.

The particle systems are needed for the construction of superprocesses.

0.5 General description of Chapter 2

The first part of the second chapter of this thesis is devoted to the construction of a of multitype superprocesses. Those superprocesses are natural extensions of the single type processes. If we see the random measures as random clouds constituted of particles moving independently, the multitype superprocesses are random clouds where each particle is assigned a particular type. The multitype superprocesses are pictured as colored random clouds.

Obviously, the multitype superprocesses model a larger class of phenomenon than the single type ones. Their study becomes even more interesting with the additional property of being catalytic superprocesses. Catalytic superprocesses are processes for which branching occurs only when a particle has spent enough time in the presence of a (possibly random) catalyst. In particular, the branching can consist simply in a change of type. In some cases, a chemical reaction may occur only in the presence of catalysts. A chemical reaction might be seen as the transformation of a

substance of a given type into another substance of a different type, via intermediate types. Intuitively, multitype catalytic superprocesses seem to model the simple chemical reactions. Moreover, when the branching occurs after a particle has spent enough time in presence of a catalyst given by a (possibly random) measure, the corresponding “branching rate additive functional”, which is called the collision local time, does not satisfy the “admissibility condition” stated in [Dy7] (Dawson and Fleischmann [a paper to appear]). This is yet another motivation to enlarge the class of superprocesses constructed in [Dy7] as we do in chapter three. Catalytic multitype superprocesses with collision local time catalyst will be the subject of future work by the author.

The multitype extension of the class of superprocesses constructed in [Dy2] is a problem which has also been considered by Dynkin. Multitype branching particle systems and the multitype superprocesses were independently constructed by Dynkin and published in [Dy7], about at the same time we completed our own construction. We want to stress that we obtained them in a *completely* independent manner. Our approach is different and we hope that the reader will find it suggestive and interesting. Since it is the result of original work, and several parts of it are distinct and not contained in [Dy7], we believe that it deserves a place here.

A common feature to all the BPS's and superprocesses constructed until now, is that the rate at which the particles may possibly branch in a finite interval, is always finite. This condition is imposed to insure that after finite length of time after the death of an individual, the population constituted of his descendents is *finite*. We give an example in this thesis where even the expected value of the branching rate of the siblings of a particle, is *infinite*. In particular, those superprocesses cannot be constructed by methods given in [Dy7]. In our case, the problem of explosion is solved by guarantying that such a thing, when it happens, is the result of a population which has expended by being spread on the entire space, until “infinity”,

in such a way that only finitely many siblings have been produced at a finite distance from the death location of the original ancestor.

Condition to prevent explosion in BMV processes had always taken the form of requiring that the evolution equation

$$v_{r,t}(f)(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \psi(s, \xi_s, v_{s,t}) k(ds)$$

related to the superprocess, could be given in terms of a branching mechanism ψ such that at least for every $r < t$

$$\sup_{x \in E} \pi_{r,x} \int_r^t \psi(s, \xi_s, 1) k(ds) < \infty.$$

In fact they typically satisfy a stronger condition of the form

$$\begin{aligned} \psi(s, x, 0) &= 0 \\ 0 \leq \psi(s, x, u) \leq \psi(s, x, v) &\quad \text{for every } 0 \leq u \leq v \\ |\psi(s, x, u) - \psi(s, x, v)| &\leq C(s, x) \|u - v\|_\infty \\ \sup_{x \in E} \pi_{r,x} \int_r^t C(s, \xi_s) k(ds) &< \infty. \end{aligned}$$

Typically, this property is crucial for their construction.

We have constructed an example for which

$$\sup_{x \in E} \pi_{r,x} \int_r^t \psi(s, \xi_s, 1) k(ds) = \infty.$$

In addition to the intrinsic interest of this superprocess, this may suggest that there are advantages to treat multitype superprocesses in a distinct manner from the single type ones, specially in the problem of characterizing -via log-Laplace equation for instance- the class of branching measure valued processes in which they belong. Single and multitype superprocesses can formally be separated by defining the single type processes as the processes in which the birth position of the siblings is the same position as the death position of their parents.

0.6 General description of Chapter 3

In this chapter we solve one open problem mentioned in [Dy7]: we fully characterize a class of superprocesses satisfying some mild regularity conditions. We derive, from methods developed in this chapter, a theorem suggesting the existence of superprocesses with discontinuous branching. This is another open problem stated in [Dy7], we hope our result can throw some light on it. We also derive from these methods a criterion for the convergence of superprocesses, which is of independent interest. In particular, it is used in chapter 4 to show that the martingale problem for a large class of BMV processes is well posed.

Let $(E, \mathcal{B})$ be a metrizable Luzin space. Consider a critical⁶ branching $M_f(\mathcal{B})$ -valued right process $X = (X_t, \mathcal{F}, P_{r,\mu})$. Assume that for all $t \in R_+, \mu \in M_f$

$$\sup_{r < t} P_{r,\mu} < X_t, 1 >^2 < \infty \quad (0.6.5)$$

It is proven in [DyKuS] that the function

$$p(r, t, x, dy) := P_{r,\delta_x}(X_t(dy))$$

is the transition probability of some right process $\xi = (\xi_t, \mathfrak{F}, \pi_{r,x})$. That process is called the projection of X , or equivalently, that X is a superprocess over ξ .

Suppose that X and its projection ξ are Hunt processes⁷. Then the quadratic variation of the martingale $\langle X_t, 1 \rangle$ is a continuous additive functional, and it belongs to the special class of additive functionals $A(ds)$, called the linear additive functionals, for which

$$P_{r,\mu}A(r, t) = \int \mu(dx) P_{r,\delta_x}A(r, t).$$

A one to one correspondence is established in [DyKuS] between the linear additive functional of X and the additive functional of its projection ξ . For a linear additive

⁶that is $\langle X_t, 1 \rangle$ is a martingale

⁷A right process $\xi = (\xi_t, \mathfrak{F}, \pi_{r,x})$ is a Hunt process if it is quasi left continuous, that is, $\pi_{r,\mu}$ -almost surely, $\xi_{\tau_n} \rightarrow \xi_\tau$, for every sequence of stopping time $\tau_n \rightarrow \tau$, with $\tau_n < \tau$.

functional $A(ds)$ of X the corresponding additive functional $a(ds)$ of ξ is called the projection of $A(ds)$. Let $A(ds)$ be the quadratic variation of $\langle X_t, 1 \rangle$ and let $a(ds)$ be its projection. Assume that $a(ds)$ is continuous.

In [DyKuS] (see also [Dy7] for more details) it is established that the log-Laplace functional $v_{r,t}(f)(x)$ of X satisfies a typically non-linear equation of the form

$$v_{r,t}(f)(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \psi(s, \xi_s v_{s,t}(f)) k(ds) \quad (0.6.6)$$

where some conditions are imposed on the mapping ψ and on the additive functional k . Hence all the Hunt critical branching measure valued process with Hunt projection for which the projection of the quadratic variation of $\langle X_t, 1 \rangle$ is continuous would be fully characterized if for every integral equation of the form (0.6.6) satisfying the conditions imposed in [DyKuS] we could construct a branching measure valued process having its log-Laplace functional satisfying that equation.

The problem is considered in [Dy7]. But the superprocesses were not constructed without assuming other conditions. A gap was remaining to be filled. We solve the problem in this thesis.

In fact all the processes constructed by Dynkin satisfy a condition called the "admissibility condition", and even adding that condition, the above mentioned "gap" was still open. In this thesis, the admissibility condition is not assumed, and hence, superprocesses with substantially more general parameters are constructed. As shown by recent work of Dawson and Fleischmann, [to be published soon], the admissibility condition is not natural, and it would have been necessary to remove it, in particular for the study of general catalytic superprocesses and for applications.

Another open problem stated in [Dy7] is: are there some processes for which the projection $a(ds)$ of the quadratic variation of the martingale $\langle X_t, 1 \rangle$ is not a continuous additive functional? We give here a criteria for the existence of such processes.

In chapter 3, we also give a criteria for the convergence of superprocesses. In

addition to the intrinsic interest of the result, it is essential in chapter 4 where it is used to solve the martingale problem of a large class of superprocesses.

0.7 General description of Chapter 4

We construct approximating sequences of smooth additive functionals for arbitrary admissible additive functionals, and we use the convergence criteria of chapter 3 to construct an approximating sequence of superprocesses to a given superprocess. We solve and pose the martingale problem for a class of superprocesses with singular additive functionals, using the approximating superprocesses.

In fact, to state the martingale problem, we first need to extend the notion of lifting and projection introduced in [DyKuS]. Let $(E, \mathcal{E})$ be a measurable space. Given an $M_f(\mathcal{E})$ -valued Hunt process $X = (X_t, \mathfrak{F}, P_{r,\mu})$ and a E -valued Hunt process ξ , Dynkin, Kuznetsov and Skorohod defined the lifting $A(ds)$ of an additive functional $a(ds)$ of ξ as an additive functional $A(ds)$ of X such that for every $r, t \in R_+$, for every $\mu \in M_f(\mathcal{E})$, for every $x \in E$ and for every bounded measurable $\phi(\cdot)$

$$P_{r,\mu} A(r, t] = \int_E \mu(dx) P_{r,\delta_x} A(r, t] \quad (0.7.7)$$

and

$$P_{r,\delta_x} \int_r^t \langle X_s, \phi \rangle A(ds) = \pi_{r,x} \int_r^t \phi(\xi_s) a(ds). \quad (0.7.8)$$

If $A(ds)$ is a lifting of $a(ds)$, then $a(ds)$ is said to be the projection of $A(ds)$. And in fact, given a *linear* additive functional $A(ds)$ of X , that is an additive functional such that (0.7.7) is verified, one can find an additive functional $a(ds)$ of ξ which is the projection of A . The authors proved that the lifting-projection relation establishes a one to one correspondence between the additive functionals of ξ and the linear additive functionals of X . Their proof makes use of the Markov property of X^6 . For our purposes, it was necessary to reduce that condition to the assumption that

⁶In fact even the Hunt property is invoked.

a certain *partial-martingale problem*⁹ is verified. We thus slightly extended the concept of lifting developed in [DyKuS]. In fact we also obtained a criteria for the convergence of the liftings of a convergent sequence of additive functionals which may also be of independent interest. The lifting's existence allowed us to define a process, M_t , playing the role of the martingale problem statement that "for every $r > 0$ and every $\mu \in M_f(\mathcal{E})$ there exists only one distribution, $P_{r,\mu}$, on the space of càdlàg trajectories in $M_f(\mathcal{E})$, such that $P_{r,\mu}(X_r = \mu) = 1$ and M_t is a $P_{r,\mu}$ -martingale for $t \geq r$ ".

The proof relies on a sequence of superprocesses that we can construct to approximate our given superprocess. The approximating superprocesses have the property that their branching additive functional rates $k(ds)$ are absolutely continuous with respect to the Lebesgue measure: $k(ds) = \phi(s, \xi_s)ds$. But then, El Karoui and Roelly-Coppoletta's proof can be decomposed into different steps making sense in terms of the approximating superprocesses, and finally passing to the limit, their proof finds an expression in our more general context.

This raises another problem of independent interest: the differentiability of functions satisfying

$$v_{r,t}(x) = \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^t \psi(s, \xi_s, v_{s,t}(\xi_s))ds.$$

These functions are central in superprocess theory and the question of whether or not they are differentiable is perhaps the most natural and elementary question one may ask about them. For some cases, a positive answer to that question was rigorously established. For instance it was well know that

$$v_t(x) = S_t f(x) - \int_0^t S_s (\psi(\cdot, v_{t-s}(\cdot))) (x) ds \quad (0.7.9)$$

is continuously differentiable, where S is the semigroup associated with the Brownian motion and where $\psi(x, \lambda) = \lambda^2$ (See for instance [Iscoe, Ph.D. Thesis]). El-Karoui

⁹This is just a martingale problem that the solution to the true martingale problem has to verify, but which is not a well posed martingale problem.

and Roelly-Coppoletta claim that for the large class of mechanisms ψ they considered in [EKaRC], the solution $v_t(x)$ to (0.7.9) is also continuously differentiable. Perkins pointed out that this is false for the most general cases. Fitzsimmons, in a first paper [F1], published a solution to the martingale problem for a large class of (ξ, Φ, ds) -superprocesses assuming also their log-Laplace functional is continuously differentiable. In 1992, he published a completely new proof making no use of the differentiability property of $v_t(x)$. Under what conditions (0.7.9) is continuously differentiable was an open problem. We improve the known results on the question, substantially extend the cases where $v_t(x)$ can be said to be continuously differentiable and give a rigorous and original proof to the question. The result is essential to our solution to the martingale problem.

0.8 On the structure of this thesis

Due to the nature of a Ph.D. thesis, it has seemed to us appropriate to separate our results from the ones taken from the literature. We implemented this rule by systematically putting in appendix the results which are not ours.

Chapter 1

Canonical Branching Particle Systems

1.1 Introduction

1.2 Discrete time homogeneous Markov chains on “special” spaces

1.2.1 Basic ingredients

In the following, the word kernel will always mean a *subprobability* kernel. A kernel from the measure space (S_1, σ_{S_1}) to the measure space (S_2, σ_{S_2}) is, a mapping $m(., .)$ such that for every $x \in S_1$, $m(x, .)$ is a subprobability measure on σ_{S_2} and for every $A \in \sigma_{S_2}$, the mapping $x \mapsto m(x, A)$ is σ_{S_1} -measurable.

Let $(D, \mathcal{D})$ be a measurable space. Let $(E, \mathcal{E})$ be another measurable space which we call the **space of types**. In fact the *types* will be interpreted in the sequel as representing the *birth position* of a particle. Note that in simple situations, $(D, \mathcal{D}) = (E, \mathcal{E})$. Let $\mathbf{N} = \{1, 2, \dots\}$. For $n \geq 1$, we interpret the vector

$$(d, e, e_1, \dots, e_n, n) \in D \times E^{n+1} \times \mathbf{N}$$

as describing an element d whose birth position (type) is e , which has had n offspring whose birth positions are respectively $e_1, \dots, e_n$. the vector

$$(d, e, 0)$$

is interpreted as an element d whose birth position is e and which has had no offspring. For $n \geq 1$, The vector

$$(d, e, -n)$$

is interpreted as an already *dead* element who *died* n generations ago. This is introduced to facilitate the calculations, and more precisely that will allow us to define some *infinite* Markov chains that would otherwise not exist.

Consider the space

$$B := \left(\bigcup_{n=1}^{\infty} D \times E \times E^n \times \{n\} \right) \cup \left(\bigcup_{n=0}^{\infty} D \times E \times \{-n\} \right)$$

furnished with the σ -algebra

$$\mathcal{B} := \sigma \left(\left(\bigcup_{n=1}^{\infty} \mathcal{D} \times \mathcal{E} \times \mathcal{E}^n \times \{n\} \right) \cup \left(\bigcup_{n=0}^{\infty} \mathcal{D} \times \mathcal{E} \times \{-n\} \right) \right).$$

The space $(B, \mathcal{B})$ describes the *life* of a single particle and we call it the **single particle-branch space**. We introduce now the space $(G, \mathcal{G})$ defined by

$$\begin{aligned} G &:= \bigcup_{p=1}^{\infty} \prod_{k=1}^p B \\ \mathcal{G} &:= \sigma \left(\bigcup_{p=1}^{\infty} \prod_{k=1}^p \mathcal{B} \right) \end{aligned}$$

will be called the **generation space**. It describes the life of all the individuals belonging to the same generation. Finally, we introduce the space $(T, \mathcal{T})$

$$\begin{aligned} T &:= \prod_{k=1}^{\infty} G \\ \mathcal{T} &:= \prod_{k=1}^{\infty} \mathcal{G} \end{aligned}$$

which is called the **space of trees**. We interpret any element of T as describing a genealogical tree.

Assume that there is given a kernel $\pi_{d,e}(dx)$ from $D \times E$ to $\mathcal{B}$. That kernel induces another kernel

$$P_{(d,e,e_1,\dots,e_N,N)}(dx) := \begin{cases} \pi_{d,e_1}(dx) \times \dots \times \pi_{d,e_N}(dx) & \text{if } N \geq 1 \\ \delta_{(d,e,N-1)}(dx) & \text{if } N \leq 0 \end{cases}$$

which is now a kernel form B to $\mathcal{G}$. This kernel finally induces another kernel

$$P_{(b_1, \dots, b_N)}(dx) := P_{b_1}(dx) \times \dots \times P_{b_N}(dx)$$

which is now a kernel form G to $\mathcal{G}$. We will call it the $\pi_{d,e}$ -induced kernel.

With the $\pi_{d,e}$ -induced kernel, $P(\cdot)$, and Ionescu-Tulcea's Theorem, we construct a discrete Markov chain X_N on the state space G , having $P(\cdot)$ as its (time homogeneous) transition probability. As is almost always the case in the literature, we will keep the symbol $P(\cdot)$ to denote the conditional probability of the Markov process $\{X_n\}_{n \in \mathbb{N}}$.

We will now examine some elementary properties of that Markov chain.

1.2.2 Elementary properties

Assume that a point $x \in G$ is given. Since our Markov chain is time homogeneous we obviously have

$$P_{X_n=x}(f(X_{n+1}, X_{n+2}, \dots)) = P_{X_{n+1}=x}(f(X_{n+2}, X_{n+3}, \dots)). \quad (1.2.1)$$

If now $x = (d, e, -n)$ for $n \geq 0$, then the Markov chain starting at $X_0 = x$ is absolutely deterministic and we have

$$X_k = (d, e, -n + k),$$

for every $k \geq 0$, with $P_{X_0=x}$ -probability one.

Let us now say that a real valued function f on G is **multiplicative** if it has the property that $f((b_1, \dots, b_n)) = f(b_1) \dots f(b_n)$. We will also say that f is **additive** if it has the property that $f((b_1, \dots, b_n)) = f(b_1) + \dots + f(b_n)$. Obviously a product of multiplicative functions is a multiplicative function.

Proposition 1.2.0 *Suppose that the functions $0 \leq f_n \leq 1$ are multiplicative. Then the function*

$$x \mapsto P_x(f_1(X_1)f_2(X_2)\dots)$$

is also multiplicative.

Proof: it is immediate from the definition of P that the function

$$P_{X_n=x}(f_{n+1}(X_{n+1}))$$

is multiplicative. Therefore, since, by definition of transition probability,

$$P_x(f_1(X_1) \dots f_{n+1}(X_{n+1})) = P_x(f_1(X_1) \dots f_n(X_n) P_{X_n}(f_{n+1}(X_{n+1}))), \quad (1.2.2)$$

an induction argument gives that (1.2.2) is multiplicative for every $n \geq 1$. Passing to the limit we obtain the desired result. $\square$

1.2.3 Supplementary ingredients and recurrence formula

Assume that $\hat{\Xi}$ is a measurable non negative additive function from G to R_+ and consider the random variables $\hat{\Xi}(X_n)$ on our space T . Define the random variable Ξ by

$$\Xi := \sum_{n=0}^{\infty} \hat{\Xi}(X_n).$$

Note that from (1.2.1)

$$P_{X_1=x} \left(e^{-\sum_{n=2}^{\infty} \hat{\Xi}(X_n)} \right) = P_{X_0=x} \left(e^{-\sum_{n=1}^{\infty} \hat{\Xi}(X_n)} \right)$$

and moreover, if

$$\hat{\Xi}(x) = 0$$

then

$$P_{X_1=x} \left(e^{-\sum_{n=1}^{\infty} \hat{\Xi}(X_n)} \right) = P_{X_0=x} (e^{-\Xi}). \quad (1.2.3)$$

A Markov chain is uniquely determined by its transition probability and an initial distribution. We take $\{\pi_{d,e}(\cdot)\}_{d \in D, e \in E}$ as a collection of initial distributions for the Markov chain $X_n, n \geq 0$.

Define the mapping $\hat{b}_i$, from the subset $\cup_{k \geq i} D \times E \times E^k \times \{k\}$ of G to the single particle-branch space B by

$$\hat{b}_i(d, e, e_1, \dots, e_N) = (d, e_i)$$

for $N \geq 1$, and put

$$v(d, e) := \pi_{d,e} \left(P_{X_0} \left(e^{-\Xi} \right) \right)$$

Lemma 1.2.0 (recurrence formula) Assume that for every $N \geq 1$,

$$1_{D \times E \times E^N \times \{N\}}(X_0) \hat{\Xi}(X_0) = 0 \quad (1.2.4)$$

with $\pi_{d,e}$ -probability one. Then

$$\begin{aligned} v(d, e) &= \pi_{d,e} \left(1_{D \times E \times \{N \leq 0\}}(X_0) P_{X_0} \left(e^{-\Xi} \right) \right) \\ &\quad + \pi_{d,e} \left(\sum_{N=1}^{\infty} 1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N v(\hat{b}_i(X_0)) \right). \end{aligned} \quad (1.2.5)$$

Proof: The assumption (1.2.4) made on $\pi_{d,e}()$ implies that for every $N \geq 1$,

$$\pi_{d,e} \left(1_{D \times E \times E^N \times \{N\}}(X_0) P_{X_0} \left(e^{-\Xi} \right) \right) = \pi_{d,e} \left(1_{D \times E \times E^N \times \{N\}}(X_0) P_{X_0} \left(e^{-\sum_{n=1}^{\infty} \hat{\Xi}(X_n)} \right) \right)$$

Taking the conditional expectation and invoking Proposition 1.2.0, we can continue with

$$= \pi_{d,e} \left(1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N \pi_{\hat{b}_i(X_0)} \left(P_{X_1} \left(e^{-\sum_{n=1}^{\infty} \hat{\Xi}(X_n)} \right) \right) \right).$$

Obviously, we have

$$\begin{aligned} \int_B \left(P_{X_1=x} \left(e^{-\sum_{n=1}^{\infty} \hat{\Xi}(X_n)} \right) \right) \pi_{d,e}(dx) &= \int_B \left(P_{X_0=x} \left(e^{-\sum_{n=0}^{\infty} \hat{\Xi}(X_n)} \right) \right) \pi_{d,e}(dx) \\ &= \int_B \left(P_{X_0=x} \left(e^{-\Xi} \right) \right) \pi_{d,e}(dx) \\ &= v(d, e). \end{aligned}$$

Thus

$$\pi_{d,e} \left(1_{D \times E \times E^N \times \{N\}}(X_0) P_{X_0} \left(e^{-\Xi} \right) \right) = \pi_{d,e} \left(1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N v(\hat{b}_i(X_0)) \right).$$

Therefore, the equation

$$\begin{aligned} v(d, e) &= \pi_{d,e} (P_{X_0} (e^{-\Xi})) \\ &= \pi_{d,e} (1_{D \times E \times \{N \leq 0\}}(X_0) P_{X_0} (e^{-\Xi})) \\ &\quad + \pi_{d,e} \left(\sum_{N=1}^{\infty} 1_{D \times E \times E^N \times \{N\}}(X_0) P_{X_0} (e^{-\Xi}) \right) \end{aligned}$$

yields (1.2.5). $\square$

1.3 Weak Branching Particle Systems

1.3.1 Basic ingredients

We will define the weak branching particle systems, as being special cases of the Markov chains we considered in the previous section. In fact, to obtain them, we only restrict ourself to the particular case where $(E, \mathcal{E})$ is any metrizable Luzin space and where the space and $(D, \mathcal{D})$ is the space of trajectories, that is, the space defined in the following way:

Definition 1.3.0 *Let $D_E(R)$ be the set of càdlàg trajectories in E . We pose*

$$D := \{(\alpha, \xi, \beta) : -\infty \leq \alpha < \beta \leq \infty, \xi \in D_E(R)\}.$$

For any $\xi \in D_E(R)$, let ξ_t denote the t^{th} projection on E . We define a sigma algebra $\mathcal{D}$ on D by taking the sigma algebra generated by the functions

$$F(\alpha, \xi, \beta) = f^\alpha(\xi_\alpha) 1_{\alpha < t_1 < t_2 < \beta} Y g^\beta(\xi_\beta)$$

where f, g are progressively measurable, and where Y is measurable with respect to the universal completion of $\mathcal{F}(t_1, t_2)^$ of $\mathcal{F}(t_1, t_2) := \sigma(\xi_t : t \in (t_1, t_2))$.*

Remark 1.3.0 *More generally, the space $D_E(R)$ can be replaced by $\prod_{t \in [0, \infty]} E$. In the first case we talk about the càdlàg canonical process and in the second case, we talk about the canonical process.*

Let $\mathcal{B}(R)$ denotes the Borel sets of the real line. Recall that given a Markov process $\xi = (\xi_t, \mathfrak{F}, \pi_{r,x})$, a function $k : \Omega \times \mathcal{B}(R) \mapsto [0, \infty]$ is called an **additive functional** if

- For every $\omega \in \Omega$, $k(\omega, \cdot)$ is a σ -finite measure on $\mathcal{B}(R)$,
- For every open interval I , $k(\cdot, I)$ is $\mathfrak{F}(I)^*$ -measurable (where $\mathfrak{F}(I)^*$ denotes the universal completion of $\mathfrak{F}(I)$).

If the condition that $k(\omega, \cdot)$ is σ -finite is replaced by the condition that

$$k(\omega, \sum_n I_n) = \prod_{n=1}^{\infty} k(\omega, I_n)$$

for any disjoint union of Borel sets of R , then we say that k is a **multiplicative functional**.

In order to construct the weak branching particle systems we need, in addition to our measurable space $(E, \mathcal{E})$:

1. a E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$.
2. a non negative continuous additive functional $k(ds)$ of ξ
3. an **offspring generating functional** ϕ , that is, a functional having the form having the form

$$z \mapsto \phi(s, x, z) = \sum_{n=0}^{\infty} \int_{E^n} z(y_1) \dots z(y_n) q(s, x, n, dy_1, \dots, dy_n)$$

where z is a measurable function from E to $[0, 1]$, $q_0(s, x)$ is an $R_+ \times E$ measurable function such that $0 \leq q_0(s, x) \leq 1$ and where $q_n(s, x, \cdot) := q(s, x, n, \cdot)$, $n = 1, 2, \dots$ is a collection of kernels from $R_+ \times E$ to E^n , such that for every s, x

$$\sum_{n=0}^{\infty} q(s, x, n, E^n) = 1.$$

Let d^* denotes an element $(\alpha^*, \xi^*, \beta^*) \in D$, and $e^* \in E$. Let F be a measurable function from B to R_+ . For $\beta^* < \infty$, we define the $\tilde{\pi}^t(\cdot)$ by

$$\begin{aligned} \tilde{\pi}_{d^*, e^*}^t(F) &:= \pi_{\beta^*, e^*} \left(e^{-k(\beta^*, t)} F(\beta^*, \xi, \infty, e^*, 0) \right) \\ &+ \sum_{N=0}^{\infty} \pi_{\beta^*, e^*} \left(\int_{\beta^*}^t e^{-k(\beta^*, s)} q_N(s, \xi, F(\beta^*, \xi, s, e^*, \cdot, N)) k(ds) \right). \end{aligned} \quad (1.3.6)$$

where $q_N(s, x, F(\cdot))$ denotes the integral of $F(\vec{z})$ with respect to $q_N(s, x, d\vec{z})$. For $\beta^* = \infty$, we simply put

$$\tilde{\pi}_{d^*, e^*}^t(F) := F(d^*, e^*, -1).$$

Note that $\tilde{\pi}_{d^*, e^*}^t$ is a kernel from $D \times E$ to B (which doesn't charge the subset $D \times E \times \{\dots, -2, -1\}$ of B). Note also that $\tilde{\pi}_{d^*, e^*}^t(1) = 1$ since for $\beta^* < \infty$

$$\begin{aligned} \tilde{\pi}_{d^*, e^*}^t(1) &= \pi_{d^*, e^*} \left(e^{-k(\beta^*, t)} \right) \\ &+ \pi_{d^*, e^*} \left(\int_{\beta^*}^t e^{-k(\beta^*, s)} k(ds) \right) \\ &= 1 \end{aligned}$$

With this kernel we get the $\tilde{\pi}^t(\cdot)$ -induced kernel on G , from which we define a Markov chain $X_n, n \geq 0$ on the product space $(T, \mathcal{T})$. As we did in the previous section we take $\{\tilde{\pi}_{d, e}^t(\cdot)\}_{d \in D, e \in E}$ as a collection of initial distribution for our Markov chain. Thus we denote

$$P_{d, e}(\cdot) := \int_S P_{X_0=x}(\cdot) \tilde{\pi}_{d, e}^t(dx).$$

We will now define some random measures on the space $(T, \mathcal{T}, P_{d, e})$.

1.3.2 Random measures and Laplace equation

Let $x \in B$ have the form

$$x = (\alpha, \xi, \beta, e, e_1, \dots, e_{n_1}, n) \text{ or } x = (\alpha, \xi, \beta, e, 0),$$

and define the measure $\hat{\Xi}_t(x)$ by

$$\hat{\Xi}_t(x)(f) = 1_{\{\alpha, \beta\}}(t) f(\xi_t).$$

If now $x \in B$ has the form

$$x = (\alpha, \xi, \beta, e, -n) \text{ with } n \geq 1,$$

define $\hat{\Xi}_t(x)$ by

$$\hat{\Xi}_t(x)(f) = 0.$$

Thus $\hat{\Xi}_t(x)$ is a measure for every $x \in B$. If now $x \in G$ and $x = (b_1, \dots, b_n)$, define

$$\hat{\Xi}_t(x)(f) = \sum_{i=1}^n \hat{\Xi}_t(b_i)(f).$$

That gives a collection $\hat{\Xi}_t(X_n)$, $n \geq 0$ of random measures from which we define another random measure

$$\Xi_t := \sum_{n=0}^{\infty} \hat{\Xi}_t(X_n).$$

Let $f \in pb\mathcal{E}$, the set of non negative bounded $\mathcal{E}$ -measurable functions. As we did in the previous section we define

$$v(d, e) = P_{d,e} \left(P_{X_0}(e^{-\Xi_t(f)}) \right) := \tilde{\pi}_{d,e}^t \left(P_{X_0}(e^{-\Xi_t(f)}) \right).$$

Recall that Lemma 1.2.0 states that

$$\begin{aligned} v(d, e) &= \tilde{\pi}_{d,e}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) P_{X_0} \left(e^{-\Xi_t(f)} \right) \right) \\ &\quad + \tilde{\pi}_{d,e}^t \left(\sum_{N=1}^{\infty} 1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N v(\hat{b}_i(X_0)) \right) \end{aligned} \quad (1.3.7)$$

providing that for every $N \geq 1$

$$\tilde{\pi}_{d,e}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) \hat{\Xi}_t(X_0)(f) \right) = 0.$$

One notes that if $d = (\alpha, \xi, r) \in D$ and $x \in E$, the initial distributions $\pi_{r,x}^t(\cdot)$ really depends only on r and x whenever $r < \infty$. Let us thus write $v_{r,t}(x)$ instead of $v(d, x)$. The functional $v_{r,t}(x)(f)$ is known as the **Laplace functional** of Ξ . In this context of branching particle systems, the recurrence formula, Lemma 1.2.0, has the following expression:

Lemma 1.3.0 (Laplace equation) *The Laplace functional of the random measure Ξ_t satisfies the equation*

$$\begin{aligned} v_{r,t}(x) &= \pi_{r,x} \left(e^{-k(r,t)} e^{-f(\xi_t)} \right. \\ &\quad \left. + \pi_{r,x} \left(\int_r^t e^{-k(r,s)} \phi(s, \xi_s, v_{s,t}) k(ds) \right) \right) \end{aligned} \quad (1.3.8)$$

where $q_N(s, x, f) := \int f(z) q_N(s, x, dz)$.

Proof: Note first that the hypothesis of Lemma 1.2.0 are satisfied since

$$\begin{aligned} \hat{\pi}_{r,x}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) \hat{\Xi}_t(X_0)(f) \right) \\ = \pi_{r,x} \left(\int_r^t e^{-k(r,s)} \hat{\Xi}_t(r, \xi, s, N)(f) q_N(s, \xi_s, E^N) k(ds) \right) \end{aligned}$$

and

$$1_{[r,t]}(s) \hat{\Xi}_t(r, \bar{e}, s, N)(f) = 0$$

by the definition of $\hat{\Xi}_t$.

We will now find another expression for

$$\hat{\pi}_{r,x}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) P_{X_0} \left(e^{-\Xi(f)} \right) \right).$$

Indeed, we have

$$\hat{\pi}_{r,x}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) P_{X_0} \left(e^{-\Xi(f)} \right) \right) = \pi_{r,x} \left(e^{-k(r,t)} P_{(r,\xi,\infty,0)} \left(e^{-\Xi(f)} \right) \right)$$

but starting from $X_0 = (r, \xi, \infty, 0)$, the Markov chain is completely deterministic:

$$X_n = (r, \xi, \infty, -n), \text{ for } n \geq 0.$$

Thus, by definition of Ξ ,

$$P_{(r,\xi,\infty,0)} \left(e^{-\Xi(f)} \right) = P_{(r,\xi,\infty,0)} \left(e^{-\hat{\Xi}(X_0)(f)} \right) = e^{-f(\xi_t)},$$

and therefore,

$$\hat{\pi}_{r,x}^t \left(1_{D \times E \times \{N \leq 0\}}(X_0) \hat{\Xi}_t(X_0)(f) \right) = \pi_{r,x} \left(e^{-k(r,t)} e^{-f(\xi_t)} \right). \quad (1.3.9)$$

We will now find another expression for

$$\tilde{\pi}_{r,x}^t \left(\sum_{N=1}^{\infty} 1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N v(\hat{b}_i(X_0)) \right).$$

In fact, by definition of $\tilde{\pi}_{r,x}^t$,

$$\begin{aligned} & \tilde{\pi}_{r,x}^t \left(1_{D \times E \times E^N \times \{N\}}(X_0) \prod_{i=1}^N v(\hat{b}_i(X_0)) \right) \\ &= \pi_{r,x} \left(\int_r^t e^{-k(r,s)} q_N \left(s, \xi_s, \prod_{i=1}^N v_{s,i} \right) k(ds) \right). \end{aligned} \quad (1.3.10)$$

Thus, putting (1.3.7), (1.3.9) and (1.3.10) together, we get

$$\begin{aligned} v_{r,t}(x) &= \pi_{r,x} \left(e^{-k(r,t)} e^{-f(\xi_t)} \right. \\ &\quad \left. + \sum_{N=1}^{\infty} \pi_{r,x} \left(\int_r^t e^{-k(r,s)} q_N \left(s, \xi_s, \prod_{i=1}^N v_{s,i} \right) k(ds) \right) \right) \end{aligned}$$

which is precisely (1.3.8). $\square$

Remark 1.3.0 We have previously introduced the kernels $\tilde{\pi}_{d,e}^t(\cdot)$ on B by (1.3.6).

Let $b = (\alpha, \xi, r)$ and $e = x \in E$. Note that $\tilde{\pi}_{d,e}(\cdot)$ defined by

$$\begin{aligned} \tilde{\pi}_{d,e}(F) &:= \pi_{r,x} \left(e^{-k(r,t)} F(r, \xi, \infty, e, 0) \right) \\ &\quad + \sum_{N=0}^{\infty} \pi_{r,x} \left(\int_r^t e^{-k(r,s)} q_N(s, \xi_s, F(r, \xi, s, e, N)) k(ds) \right) \end{aligned} \quad (1.3.11)$$

is also a kernel from $D \times E$ to $\mathcal{G}$, where $q_N(s, x, f) := \int_{E^N} f(z) q_N(s, x, dz)$. Taking $\tilde{\pi}_{d,e}(\cdot)$ instead of $\tilde{\pi}_{d,e}^t(\cdot)$, one would obtain exactly the same Laplace equation as (1.3.8) for the Laplace functional of Ξ_t . Hence the random measures $\{\Xi_t\}_{t \geq 0}$ can be constructed on a same space. In particular, it is clear that the process $t \mapsto \Xi_t$ is càdlàg.

Definition 1.3.0 1. A family of probability measures $\{P_{r,\mu}(\cdot)\}$ will be said to have the weak branching property if

$$P_{r,\mu+\nu} = P_{r,\mu} * P_{r,\nu} \quad (1.3.12)$$

for every μ, ν .

2. A family of probability measures $\{P_{r,t,\mu}(\cdot)\}$ on $M_f(\mathcal{E})$ such that

$$P_{r,t,\mu}(e^{-\langle \cdot, f \rangle}) = - \int_E \mu(dx) \log (P_{r,t,\delta_x}(e^{-\langle \cdot, f \rangle}))$$

will be said to have the (strong) branching property.

Let $M_f^N(E)$ denote the set of random measures on $\mathcal{E}$ with values in $\{0, 1, 2, 3, \dots\}$, and let $\mathcal{B}(M_f^N(\mathcal{E}))$ denotes the Borel sets of $M_f^N(\mathcal{E})$.

Definition 1.3.0 (Weak branching particle system) A 4-tuple $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ will be called a weak branching particle system, abbreviated weak BPS, if

- $(T, \mathcal{T})$ is a measurable space,
- $\{P_{r,\mu}(\cdot) : r \geq 0, \mu \in M_f^N(\mathcal{E})\}$ is a family of probability measures on $(T, \mathcal{T})$,
- $\{\Xi_t\}_{t \geq 0}$ is a càdlàg $M_f^N(\mathcal{E})$ -valued process defined on $(T, \mathcal{T})$,
- the family of probabilities $\{P_{r,\mu}\}$ has the weak branching property.

Theorem 1.3.1 (Existence of weak BPS) Assume that are given

1. a measurable space $(E, \mathcal{E})$,
2. a E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$,
3. a non negative continuous additive functional $k(ds)$ of ξ ,
4. an offspring generating functional ϕ (given by a collection of kernels $\{q_n(s, x)\}_{n \geq 0}$).

Then there exists a weak BPS $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ such that if

$$v_{r,t}(f)(x) := P_{r,\delta_x}(e^{-\langle \Xi_t, f \rangle}) \tag{1.3.13}$$

then $v_{r,t}(f)(x)$ satisfies (1.3.8).

Proof: With $\pi_{r,x}$ we define the kernel $\hat{\pi}_{d,e}(\cdot)$ from $D \times E$ to $\mathcal{B}$ from which we get the $\hat{\pi}_{d,e}(\cdot)$ -induced kernel from G to $\mathcal{G}$. That gives a G -valued Markov chain X_n defined on the space $(T, \mathcal{T}, P_{r,x})$ from which we define the càdlàg $M_f^N(\mathcal{E})$ -valued process $\{\Xi_t\}_{t \geq 0}$.

Let us denote $P_{r,\delta_x}(\cdot) := P_{r,\cdot}(\cdot)$. If, for $f \in pb\mathcal{E}$, we define $v_{r,t}(f)(x)$ by (1.3.13), the $v_{r,t}(f)(x)$ satisfies (1.3.8) according to Lemma 1.3.0.

We are allowed to define $P_{r,\delta_{x_1} + \dots + \delta_{x_n}}$ by recurrence

$$P_{r,\delta_{x_1} + \dots + \delta_{x_n}}(\cdot) := P_{r,\delta_{x_1}}(\cdot) * P_{r,\delta_{x_2} + \dots + \delta_{x_n}}(\cdot)$$

which yields (1.3.12). $\square$

Definition 1.3.1 *The weak branching particle system constructed in Theorem 1.3.1 will be called the (ξ, ϕ, k) -weak BPS.*

Remark 1.3.1 *A weak BPS $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ induces another collection of probability measures on $M_f(\mathcal{E})$, $\{P_{r,\mu}(\cdot) : r \geq 0, \mu \in M_f^N(\mathcal{E})\}$, which are defined by*

$$P_{r,t,\mu}(A) := P_{r,\mu}(\Xi_t^{-1}(A))$$

for every Borel measurable subset A of $M_f(\mathcal{E})$.

1.4 Weak BPS with killing and Mass creation

In this section we introduce weak BPS where random killing and random mass creation occur *at the level of the individual particles*. As it will clearly appear, killing and mass creation are tightly related to the important general Feynman-Kac formula which can be found in appendix. Three of the four General Feynman-Kac formulae had their converses established. We show in this chapter that this is also true for the fourth one. This will later be essential.

Motion with killing

The following is essentially taken from [Dy7, p. 14]. Suppose that α_t^r is a multiplicative functional of a right process ξ , such that $0 \leq \alpha_t^r \leq 1$ for all $r < t$. Then there exists a Markov process $\tilde{\xi}$ with transition function

$$\tilde{p}(r, x; t, B) = \pi_{r,x} 1_{\xi_t \in B} \alpha_t^r.$$

Moreover, ξ and $\tilde{\xi}$ can be defined on the same space $\tilde{\Omega}$ in such a way that the life of $\tilde{\xi}$ does not exceed the life of ξ and the transition probabilities of $\tilde{\xi}$ and ξ are related by the formula

$$\tilde{\pi}_{r,x} \int_{(r,t]} g(s, \tilde{\xi}_s) K(ds) = \pi_{r,x} \int_{(r,t]} \alpha_s^r g(s, \xi_s) K(ds)$$

for any additive functional K of ξ and every non negative Borel measurable function g . We say that $\tilde{\xi}$ is obtained from ξ by killing corresponding to α . In particular, for every additive function $A(ds)$ of ξ there correspond the additive functional $\tilde{A}(ds)$ of $\tilde{\xi}$ which is equal to $A(ds)$ restricted to the life interval of $\tilde{\xi}$, and the formula

$$\tilde{\pi}_{r,x} \int_{(r,t]} \tilde{A}(ds) = \pi_{r,x} \int_{(r,t]} \alpha_s^r A(ds)$$

is verified (cf. [Dy7, p.71-72]).

One particular case will play an important role in this thesis, and we will call it the *jump-killing functional*. Let us first introduce the following definitions:

Definition 1.4.1 (nearly Borel functions) *Given a Markov process ξ with state space $(E, \mathcal{E})$, a function f on $R \times E$ is said to be nearly Borel if, for every r, μ , there exists Borel functions $f_1 \leq f \leq f_2$ such that $f_1(t, \xi_t)$ and $f_2(t, \xi_t)$ are $\pi_{r,\mu}$ -indistinguishable on the set $[r, \infty)$. The nearly Borel σ -algebra is the σ -algebra generated by all the nearly Borel functions. If $(Z, \mathcal{Z})$ is another measurable space, a function f defined on the space $R \times E \times Z$ will be called nearly Borel if f is measurable with respect to the product of the nearly Borel σ -algebra and $\mathcal{Z}$.*

A nearly Borel function $\rho^s(x, y)$ defined on $R \times E \times E$ will be called a **jump-killing functional** if $0 \leq \rho \leq 1$ and $\rho^s(x, x) = 1$ for every $x \in E$.

Put $\alpha_t^r := \prod_{s \in (r, t]} \rho^s(\xi_{s-}, \xi_s)$. The process $\tilde{\xi}$ obtained from ξ by killing corresponding to α_t^r will be called the ξ -process with jump-killing functional ρ , and we denote it by ξ^ρ .

The additive functional $\tilde{A}$ obtained from additive functionals A of ξ by restricting A to the life interval of ξ^ρ will be denoted by A^ρ .

Remark 1.4.1 Clearly, for any additive functional $k(ds)$ of a right process ξ , any offspring generating functional ϕ and any jump-killing functional ρ , there exists the (ξ^ρ, ϕ, k^ρ) -BPS.

Weak BPS with mass creation

Remark 1.4.1 Let α_t^r be a non negative multiplicative functional α_t^r of the Markov process ξ and let $k(ds)$ be a non negative additive functional of ξ . Assume also that for any real numbers β^* and any $t \geq \beta^*$, we have

$$\pi_{\beta^*, e^*} \left(\alpha_t^{\beta^*} + \int_{\beta^*}^t \alpha_s^{\beta^*} k(ds) \right) \leq 1. \quad (1.4.14)$$

Then, if, in the kernel $\tilde{\pi}_{\beta^*, e^*}^t(F)$ given by (1.3.6), we replace the multiplicative functional $e^{-k(r, t)}$ by α_t^r , that is, if we put

$$\begin{aligned} \tilde{\pi}_{\beta^*, e^*}^t(F) &:= \pi_{\beta^*, e^*} \left(\alpha_t^{\beta^*} F(\beta^*, \xi, \infty, e^*, 0) \right) \\ &+ \sum_{N=0}^{\infty} \pi_{\beta^*, e^*} \left(\int_{\beta^*}^t \alpha_s^{\beta^*} q_N(s, \xi_s, F(\beta^*, \xi, s, e^*, \cdot, N)) k(ds) \right) \end{aligned}$$

for $\beta^* < \infty$, then strictly no change in the argument should be made to construct a weak BPS $(\{\Xi\}_t, T, \mathcal{T}, \{P_{r, \mu}\})$ such that if

$$v_{r, t}(f)(x) := P_{r, \delta_x}(e^{-\langle \Xi_t, f \rangle}) \quad (1.4.15)$$

then

$$\begin{aligned} v_{r, t}(x) &= \pi_{r, x} \left(\alpha_t^r e^{-f(\xi_t)} \right) \\ &+ \pi_{r, x} \left(\int_r^t \alpha_s^r \phi(s, \xi_s, v_{s, t}) k(ds) \right). \end{aligned} \quad (1.4.16)$$

Let $C(ds)$ be a continuous non negative additive functional of ξ . We will call such an additive functional a **mass creation additive functional**. The reason is that, given a multiplicative functional m_t^r , $C(ds)$ induces a multiplicative functional α_t^r defined by $\alpha_t^r := e^{C(r,t)} m_t^r$.

Remark 1.4.1 *The expression mass creation additive functional is appropriate only when $C(ds)$, plays the role of mass creation in a particle system. It will be used only in that context and only to emphasize which role is played by $C(ds)$.*

Definition 1.4.1 *The weak BPS such that (1.4.15) implies (1.4.16) will be called the (ξ, ϕ, k, α) -BPS. If $C(ds)$ is any non negative additive functional of ξ , and if α is given by $\alpha_t^r := e^{C(r,t)} e^{-k(r,t)}$, the (ξ, ϕ, k, α) -BPS (which doesn't exist if (1.4.14) is not satisfied) will be called the (ξ, ϕ, k) -BPS with mass creation C . If ρ is a jump-killing functional, the (ξ^ρ, ϕ, k^ρ) -BPS with mass creation C^ρ will be called the (ξ, ϕ, k) -BPS with killing ρ and mass creation C .*

Remark 1.4.1 *For the (ξ, ϕ, k) -BPS with killing ρ and mass creation C to exist, it is sufficient that*

$$\pi_{r,x}^\rho \left(e^{C^\rho(r,t)} \right) \leq 1$$

for every r, t and x , since in that case (1.4.14) is clearly verified.

The missing converse to the general Feynman-Kac formulae

The fourth chapter of Dynkin's book "An introduction to branching measure-valued processes" is entitled "General Feynman-Kac formula". It is constituted of four theorems (and the secondary results needed to establish them) from which the author derives the classical Feynman-Kac formula. Each of these theorems gives an integral equation satisfied by $\pi_{r,x}(m(r,t))$ where $m(ds)$ is a certain random measure. Three of them have their converses established: if the solution to the integral equation makes sense (i.e. we reject any "solution" equivalent to $\infty = \infty$) then the solution

to the integral equation is unique (and is therefore equal to $\pi_{r,x}(m(r, t))$). We prove here that the fourth theorem also has a valid converse. This will later be essential. Note that this concerns the third Feynman-Kac formula Dynkin established, which is [Dy7, Th. 4.3.1].

Using the General Feynman-Kac formulae, we characterize the evolution equation of weak BPS with killing and mass creation in the next subsection.

Theorem 1.4.2 (General Feynman-Kac formulae) formula 3 *Let C_0 a continuous positive functional of right process ξ . Suppose that $\rho^s(x, y)$ is a nearly Borel function such that $0 \leq \rho \leq 1$ and $\rho^s(x, x) = 1$ for all s, x . Put*

$$\hat{H}(I) := e^{-C_0(I)} \prod_{s \in I} \rho^s(\xi_{s-}, \xi_s).$$

Let B be a signed additive functional of ξ . Assume that

$$\pi_{r,x} |B|(r, t) < \infty. \quad (1.4.17)$$

Then the function

$$g(r, x) = \pi_{r,x} \int_{(r,t]} \hat{H}(r, s) B(ds) \quad (1.4.18)$$

satisfies the equation

$$\begin{aligned} g(r, x) &+ \pi_{r,x} \int_r^t g(s, \xi_s) C_0(ds) \\ &+ \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho^s(\xi_{s-}, y)] g(s, y) N_\xi(ds, dy) = \pi_{r,x} B(r, t], \end{aligned} \quad (1.4.19)$$

where $N_\xi(ds, dy)$ is the Lévy measure of ξ . Conversely, if g is finite and if the right side of (1.4.19) is well defined then (1.4.19) implies (1.4.18).

Proof that (1.4.18) implies (1.4.19): see [Dy7, Th. 4.3.1] $\square$

Proof that (1.4.19) implies (1.4.18): Let g be a solution to (1.4.19). Let Ψ be a nearly Borel function such that $0 < \Psi \leq 1$ and

$$\pi_{r,x} \int_{(r,t]} \int_E \Psi^s(\xi_{s-}, y) N_\xi(ds, dy) \leq 1$$

for all r, x . The existence of Ψ is guarantied by Theorem 1.7.7. Define ψ by

$$\psi^s(x, y) := 1_{\{|g(s, y)| \leq 1\}} \Psi^s(x, y) + 1_{\{|g(s, y)| > 1\}} \frac{\Psi^s(x, y)}{|g(s, y)|}.$$

Hence ψ is nearly Borel, satisfies $0 < \psi \leq 1$ and verifies the inequality

$$\pi_{r,x} \int_{(r,t]} \int_E \psi^s(\xi_{s-}, y) |g(s, y)| N_\xi(ds, dy) \leq 1 \quad (1.4.20)$$

for all r, x . Put $\rho_\epsilon = \rho + \epsilon 1_{\{\rho=0\}} \psi$. Note that

$$\begin{aligned} \pi_{r,x} B^\epsilon(r, t] = & g(r, x) + \pi_{r,x} \int_{(r,t]} g(s, \xi_s) C_0(ds) \\ & + \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho_\epsilon^s(\xi_{s-}, y)] g(s, y) N_\xi(ds, dy) \end{aligned} \quad (1.4.21)$$

where

$$B^\epsilon(I) := B(I) - \epsilon \int_I \int_E \psi^s(\xi_{s-}, y) g(s, y) N_\xi(ds, dy). \quad (1.4.22)$$

Put

$$C^\epsilon(I) := C_0(I) - \sum_{s \in I} \log \rho_\epsilon^s(\xi_{s-}, \xi_s)$$

and for every additive functional $A(ds)$ define the expressions $\tilde{\pi}_{r,x} A(r, t]$ and $\tilde{\pi}_{r,x}^\epsilon A(r, t]$ by

$$\tilde{\pi}_{r,x} A(r, t] := \pi_{r,x} \int_{(r,t]} \bar{H}(r, s) A(ds)$$

and

$$\tilde{\pi}_{r,x}^\epsilon A(r, t] := \pi_{r,x} \int_{(r,t]} \bar{H}^\epsilon(r, s) A(ds)$$

where $\bar{H}^\epsilon(I) := e^{-C_0(I)} \prod_{s \in I} \rho_\epsilon^s(\xi_{s-}, \xi_s)$.

We have, from the definition of the Levy measure (1.7.51),

$$\pi_{r,x} \int_{(0,t]} \int_E [1 - \rho_\epsilon^s(\xi_{s-}, y)] g(s, y) N_\xi(ds, dy) = \pi_{r,x} \sum_{s \in (r,t]} [1 - \rho_\epsilon^s(\xi_{s-}, y)] g(s, \xi_s).$$

Returning to (1.4.21), that gives

$$\begin{aligned} \pi_{r,x} B^\epsilon(r, t] = & g(r, x) + \pi_{r,x} \int_{(r,t]} g(s, \xi_s) C_0(ds) \\ & + \pi_{r,x} \sum_{s \in (r,t]} (1 - \exp(-C^\epsilon\{s\})) g(s, \xi_s). \end{aligned}$$

where $\rho_t^s(\xi_{s-}, \xi_s)$ has been replaced by $\exp(-C^s\{s\})$. Using now Theorem 1.7.5 (Formula 1)¹ applied to the above equation, we derive that

$$g(r, x) = \tilde{\pi}_{r,x}^\epsilon B^\epsilon(r, t) = \tilde{\pi}_{r,x}^\epsilon \left[B(r, t) - \epsilon \int_{(r,t]} \int_E \psi^s(\xi_{s-}, y) g(s, y) N_\xi(ds, dy) \right]$$

But because $\tilde{\pi}_{r,x}^\epsilon A(r, t) \leq \pi_{r,x} A(r, t)$ for every positive additive functional A we can write

$$\tilde{\pi}_{r,x}^\epsilon B(r, t) - \epsilon \pi_{r,x} \int_{(r,t]} \int_E \psi^s(\xi_{s-}, y) g(s, y) N_\xi(ds, dy) \leq g(r, x) \leq \tilde{\pi}_{r,x}^\epsilon B(r, t)$$

and therefore $g(r, x) = \tilde{\pi}_{r,x} B(r, t)$, since from Lebesgue's Theorem, $\tilde{\pi}_{r,x}^\epsilon B(r, t) \searrow \tilde{\pi}_{r,x} B(r, t)$. $\square$

The evolution equation for weak BPS's with killing and mass creation

Lemma 1.4.2 *Assume that are given*

Theorem 1.4.3 1. *a measurable space $(E, \mathcal{E})$,*

2. *a E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$,*

3. *a non negative continuous additive functional $k(ds)$ of ξ ,*

4. *an offspring generating functional ϕ (given by a collection of kernels $\{q_n(s, x)\}_{n \geq 0}$),*

5. *a jump-killing functional ρ and a mass creation additive functional $C(ds)$*

And assume also that

- $\pi_{r,x}(k(r, t) + C(r, t)) < \infty$ for every r, t and x ,
- $\pi_{r,x}^\rho \left(e^{C^\rho(r,t)} + \int_r^t e^{C^\rho(r,s)} k^\rho(ds) \right) \leq 1$ for every r, t and x .

¹the finiteness condition in Theorem 1.7.5 follows from the assumption that all the terms involved are finite and from (1.4.20)

Then there exists a weak BPS $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ such that the function v defined by

$$v_{r,t}(f)(x) := P_{r,\delta_x}(e^{-\langle \Xi_t, f \rangle}) \quad (1.4.23)$$

satisfies

$$v_{r,t}(x) = \pi_{r,x}^{\rho,C}(e^{-f(\xi_t)}) + \pi_{r,x}^{\rho,C} \int_r^t (\phi(s, \xi_s, v_{s,t}) - v_{s,t}(\xi_s)) k(ds) \quad (1.4.24)$$

Proof: According to remark 1.4.1, there exists a weak BPS $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ such that if v is defined by (1.4.23) then v satisfies

$$\begin{aligned} v_{r,t}(x) &= \pi_{r,x}^{\rho} \left(e^{C^{\rho}(r,t)} e^{-k^{\rho}(r,t)} e^{-f(\xi_t^{\rho})} \right) \\ &\quad + \pi_{r,x}^{\rho} \int_r^t e^{C^{\rho}(r,s)} e^{-k^{\rho}(r,s)} \phi(s, \xi_s^{\rho}, v_{s,t}) k^{\rho}(ds). \end{aligned}$$

using Remark 1.7.5, we get that

$$\begin{aligned} v_{r,t}(x) &= \pi_{r,x}^{\rho}(e^{-f(\xi_t^{\rho})}) + \pi_{r,x}^{\rho} \int_r^t \phi(s, \xi_s^{\rho}, v_{s,t}) k^{\rho}(ds) \\ &\quad + \pi_{r,x}^{\rho} \int_r^t v_{s,t}(\xi_s^{\rho}) (C^{\rho}(ds) - k^{\rho}(ds)). \end{aligned}$$

Invoking now Theorem 1.7.5 Formula 2 we obtain (1.4.24). $\square$

1.5 An important example of weak BPS with killing and mass creation

This section is devoted to the study of a special example of a weak BPS with killing and mass creation. Such weak BPS had never been constructed before, and they will play an important role in the problem of characterizing the whole class of critical Hunt BMV processes with Hunt projection and continuous branching characteristics. Before we need to establish the following Combined Feynman-Kac formula.

A Combined Feynman-Kac Formula

The Feynman-Kac formulae given in Theorem 1.4.2 and Theorem 1.7.5 deal with situations where killing or mass creation are introduced alone. It is easy to derive the following corollary where killing and mass creation are introduced together.

Given an additive functional A of ξ , we define the expression $\pi_{r,x}^{\rho,C}(A(r,t])$ by

$$\pi_{r,x}^{\rho,C}(A(r,t]) := \pi_{r,x}^{\rho} \left(\int_{(r,t]} e^{C^{\rho}(r,s)} A^{\rho}(ds) \right).$$

Corollary 1.5.3 (Combined Feynman-Kac Formula) *Let $N_{\xi}(ds, dy)$, the Lévy measure of ξ , ρ , a jump-killing functional and $C(ds)$, a mass creation additive functional. Let also $A(ds)$ be an additive functional of ξ . The function $g(r, x)$ defined by*

$$g(r, x) := \pi_{r,x}^{\rho,C}(A(r,t]) \quad (1.5.25)$$

satisfies the equation

$$\begin{aligned} g(r, x) = & \pi_{r,x} A(r, t] + \pi_{r,x} \int_{(r,t]} g(s, \xi_s) C(ds) \\ & - \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho^s(\xi_{s-}, y)] g(s, y) N_{\xi}(ds, dy), \end{aligned} \quad (1.5.26)$$

providing that all the terms involved are finite. Conversely, if all the terms involved are finite, then the only solution g to (1.5.26) is given by (1.5.25).

Proof:

If $g(r, x)$ is given by (1.5.25) then from Theorem 1.7.5 (Formula 2)

$$g(r, x) = \pi_{r,x}^{\rho} \int_{(r,t]} e^{C^{\rho}(r,s)} A^{\rho}(ds) = \pi_{r,x}^{\rho}(A^{\rho}(r,t]) + \pi_{r,x}^{\rho} \int_{(r,t]} g(s, \xi_s^{\rho}) C^{\rho}(ds). \quad (1.5.27)$$

Putting $B(ds) := A(ds) + g(s, \xi_s) C(ds)$, the above equation is identical to

$$g(r, x) = \pi_{r,x} \int_{(r,t]} \prod_{u \in (r,s]} \rho^u(\xi_{u-}, \xi_u) B(ds). \quad (1.5.28)$$

An appeal to Theorem 1.4.2 (Formula 3) gives

$$g(r, x) = \pi_{r,x} B(r, t] - \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho^s(\xi_{s-}, y)] g(s, y) N_{\xi}(ds, dy)$$

which is precisely (1.5.26).

The converse is proven the same way, invoking the same Theorems (the converse parts), following the reversed order. $\square$

A special example of killing ρ and mass creation C

Let us recall that $(E, \mathcal{E})$ is a metrizable Luzin space, ξ , a E -valued right process, $N_\xi(ds, dy)$, the Lévy measure of ξ , ϕ and offspring generating functional, $k(ds)$, a continuous non negative additive functional of ξ , ρ , a jump-killing functional and $C(ds)$, a mass creation additive functional. This notation remains in force *for the rest of this chapter*.

For the rest of this chapter, we will also restrict ourself to the special case where

$$C(ds) = \int_E (1 - \rho^s(\xi_{s-}, y)) N_\xi(ds, dy). \quad (1.5.29)$$

Note that, if ξ is not a Hunt process, it is not always the case that the additive functional $\int_E (1 - \rho^s(\xi_{s-}, y)) N_\xi(ds, dy)$ is continuous. (In our case, it is assumed to be continuous since C is a mass creation additive functional.)

For this special example of mass creation, the integral equation (1.5.26) of the Combined Feynman-Kac Formula has another expression:

Lemma 1.5.3 *For any measurable non negative function $g(r, x)$, the equation*

$$\begin{aligned} g(r, x) = & \pi_{r,x} A(r, t) + \pi_{r,x} \sum_{s \in (r, t]} g(s, \xi_{s-}) [1 - \rho^s(\xi_{s-}, \xi_s)] \\ & - \pi_{r,x} \sum_{s \in (r, t]} g(s, \xi_s) [1 - \rho^s(\xi_{s-}, \xi_s)] \end{aligned} \quad (1.5.30)$$

is equivalent to (1.5.26).

Proof: First, use the definition of the Lévy measure to transform (1.5.30) into

$$\begin{aligned} g(r, x) = & \pi_{r,x} A(r, t) + \pi_{r,x} \int_{(r, t)} \int_E g(s, \xi_{s-}) [1 - \rho^s(\xi_{s-}, y)] N_\xi(ds, dy) \\ & - \pi_{r,x} \int_{(r, t)} \int_E g(s, y) [1 - \rho^s(\xi_{s-}, y)] N_\xi(ds, dy) \end{aligned}$$

Obviously,

$$\begin{aligned} \int_E g(s, \xi_{s-}) [1 - \rho^s(\xi_{s-}, y)] N_\xi(ds, dy) &= g(s, \xi_{s-}) \int_E [1 - \rho^s(\xi_{s-}, y)] N_\xi(ds, dy) \\ &= g(s, \xi_{s-}) C(ds) \end{aligned}$$

In order to complete the proof, it remains only to show that $g(s, \xi_{s-}) C(ds) = g(s, \xi_s) C(ds)$. But $C(ds)$ is continuous, and therefore, doesn't charge countable sets. $\square$

Existence of weak BPS for our special example of killing and mass creation

We will now prove that (ξ, ϕ, k) -BPS with killing ρ and mass creation C exists. In order to do this, we need the following “reversed” version of the generalized Gronwall inequality.

Lemma 1.5.3 (Reversed Gronwall inequality) *Let $h(s, x)$ be progressively measurable, and assume that*

$$h(r, x) \geq a + b \int_r^t h(s, \xi_s) k(ds) \quad (1.5.31)$$

where $k(ds)$ is a continuous (positive) additive functional of ξ and a and b are positive constants. Then

$$h(r, x) \geq a \pi_{r,x} e^{bk(r,t)}. \quad (1.5.32)$$

Proof: Using (1.5.31) and induction it is easy to verify that for each $n \geq 1$

$$\begin{aligned} h(r, x) &\geq a \sum_{k=0}^n b^k \pi_{r,x} \int \dots \int 1_{\{r < s_1 < \dots < s_k < t\}} k(ds_1) \dots k(ds_k) \\ &= a \sum_{k=0}^n b^k \pi_{r,x} k(r, t)^k / k!. \end{aligned}$$

Letting n tends to infinity gives (1.5.32). $\square$

Notation: Let $t > 0$ be given. Given a function f , the expression

$$A_{f,t}(I) := f(\xi_t) 1_I(t)$$

defines an additive functional of ξ . $A_{f,t}(r, t]$ is simply denoted by $f(\xi_t)$. The expression $f(\xi_t^\rho)$ is interpreted as $A_{f,t}^\rho(r, t]$. In the special case where $f \equiv 1$, $f(\xi_t)$ is naturally denoted by 1 while $f(\xi_t^\rho)$ is denoted by 1^ρ .

The proof of the existence of a (ξ, ϕ, k) -BPS with killing ρ and mass creation C is based on the following Lemma.

Lemma 1.5.3 *Assume that*

$$\pi_{r,x} (k(r, t) + C(r, t)) < \infty \text{ for every } r, t \text{ and } x. \quad (1.5.33)$$

is verified. Then we have that

$$\pi_{r,x}^{\rho,C}(1) = 1.$$

Proof: Trivially, $g(r, x) \equiv 1$ is a solution to the equation:

$$\begin{aligned} g(r, x) = & \pi_{r,x}(1) + \pi_{r,x} \sum_{s \in (r,t]} g(s, \xi_{s-})(1 - \rho^s(\xi_{s-}, \xi_s)) \\ & - \pi_{r,x} \sum_{s \in (r,t]} g(s, \xi_s)(1 - \rho^s(\xi_{s-}, \xi_s)). \end{aligned}$$

By Lemma 1.5.3, this equation is equivalent to

$$\begin{aligned} g(r, x) = & \pi_{r,x}(1) + \pi_{r,x} \int_{(r,t]} g(s, \xi_s) C(ds) \\ & - \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho^s(\xi_{s-}, y)] g(s, y) N_\xi(ds, dy). \end{aligned}$$

Invoking the converse part of Feynman-Kac Formula 3, Theorem 1.4.2, we obtain that

$$g(r, x) = \pi_{r,x}^\rho(1^\rho) + \pi_{r,x}^\rho \int_{(r,t]} g(s, \xi_s^\rho) C^\rho(ds).$$

Since $1 \equiv g(r, x)$ we can use the reversed Gronwall inequality and write

$$1 \equiv g(r, x) \geq \pi_{r,x}^\rho(e^{C^\rho(r,t)}) \equiv \pi_{r,x}^{\rho,C}(1).$$

Indeed, we are allowed to use the converse part in Feynman-Kac Formula 1, Theorem 1.7.5, and this yields

$$1 \equiv g(r, x) = \pi_{r,x}^{\rho,C}(1),$$

which is precisely what we wanted. $\square$

Corollary 1.5.3 *Assume that (1.5.39) is verified. Then there exists the (ξ, ϕ, k) -weak BPS with killing ρ and mass creation C .*

Proof: This is immediate from Lemma 1.5.3 which guaranties that the hypothesis of Theorem 1.3.1 are verified. $\square$

1.6 (Strong) branching particle systems

We introduce now the notion of (strong) branching particle systems. As we will see it, weak BPS and BPS are tightly related one to the other. Assume that a weak BPS $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}\})$ is given. By giving a small mass ϵ to each particle and by randomly choosing the number of initial particles according to a Poisson point measure, we construct random measures having the (strong) branching property. More generally this procedure can be done if for each ϵ we have a different weak BPS. The following definition is more rigorous:

Definition 1.6.3 (Branching Particle System) Let $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}^\epsilon\}), \epsilon > 0$, be a collection of weak BPS's.

We define the expression $Q_{r,t,\mu}^\epsilon(\cdot)$ by

$$Q_{r,t,\mu}^\epsilon(A) := P_{r,q_\epsilon}^\epsilon(\epsilon \Xi_t^{-1}(A)),$$

where q_ϵ is the Poisson point measure with intensity $\frac{\mu}{\epsilon}$.

The family

$$\{Q_{r,t,\mu}^\epsilon(\cdot)\} = \{Q_{r,t,\mu}^\epsilon(\cdot) : 0 < \epsilon; 0 \leq r \leq t; \mu \in M_f(\mathcal{E})\}$$

will be called the **associated branching particle system**.

The BPS associated to the a collection of $(\xi^\epsilon, \phi^\epsilon, k^\epsilon)$ -weak BPS's will be called the $\{(\xi^\epsilon, \phi^\epsilon, k^\epsilon)\}$ -BPS

The weak BPS's $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}^\epsilon\})$ will be called the **underlying weak BPS's**.

A family of $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ will be called a **BPS** if $Q_{r,t,\mu}^\epsilon(\cdot)$ is the associated BPS of some collection of weak BPS.

Remark 1.6.3 Let $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}^\epsilon\}), \epsilon > 0$, be a collection of weak BPS's. Let $\overline{P_{r,\mu}^\epsilon}$ be the law of $\epsilon \Xi_t$. Then we have

$$\overline{P_{r,\mu}^\epsilon}(\epsilon^{-\langle \Xi_t, f \rangle}) = P_{r,\mu}^\epsilon(\epsilon^{-\langle \Xi_t, f \rangle}).$$

Let $\overline{\overline{P_{r,t,\mu}^\epsilon}}(\cdot)$ be defined as in remark 1.3.1. Define finally

$$\overline{\overline{P_{r,t,\mu}^\epsilon}}(\cdot) := \overline{\overline{P_{r,t,q_\mu^\epsilon}^\epsilon}}(\cdot).$$

Then

$$\overline{\overline{P_{r,t,\mu}^\epsilon}}(\cdot) \equiv Q_{r,t,\mu}^\epsilon(\cdot)$$

where $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ is the associated BPS.

Lemma 1.6.3 Let $(\{\Xi_t\}, T, \mathcal{T}, \{P_{r,\mu}^\epsilon\}), \epsilon > 0$, be a collection of weak BPS's, and let $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ be the associated BPS. Put

$$v_{r,t}^\epsilon(f)(x) := P_{r,\delta_x}^\epsilon(e^{-\langle \Xi_t, f \rangle}).$$

Then

$$Q_{r,t,\mu}^\epsilon(e^{-\langle \Xi_t, f \rangle}) = e^{-\langle \mu, \frac{1-v_{r,t}^\epsilon(f)}{\epsilon} \rangle}. \quad (1.6.34)$$

Moreover $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ posses the (strong) branching property.

Proof: The equality (1.6.34) is a simple (but a little tricky) calculation which is performed in [Dy2, p. 1159-60]. The fact that $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ posses the branching property follows immediately from equation (1.6.34). $\square$

Theorem 1.6.4 Assume that the following are given:

1. a measurable space $(E, \mathcal{E})$,
2. a E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$,
3. a collection of non negative continuous additive functional $k^\epsilon(ds)$ of ξ , for $\epsilon > 0$,
4. an offspring generating functional ϕ^ϵ (given by a collection of kernels $\{q_n^\epsilon(s, x)\}_{n \geq 0}$), for $\epsilon > 0$,

5. A jump-killing functional ρ and a mass creation C .

Assume that the conditions (1.5.29) and (1.5.33) are satisfied.

Then there exists a branching particle system $\{Q_{r,t,\mu}^\epsilon\}$ such that

$$Q_{r,t,\mu}^\epsilon(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, \frac{1-v_{r,t}^\epsilon(\cdot f)}{\epsilon} \rangle}$$

where $v_{r,t}^\epsilon(f)$ satisfies

$$\pi_{r,x}^{\rho,C}(e^{-f(\xi_t)}) + \pi_{r,x}^{\rho,C} \int_r^t (\phi^\epsilon(s, \xi_s, v_{s,t}^\epsilon(f)) - v_{s,t}^\epsilon(f)(\xi_s)) k^\epsilon(ds)$$

Proof: This is an immediate consequence of Theorem 1.3.1, Corollary 1.5.3 and Lemma 1.6.3. $\square$

1.7 Appendix

1.7.1 General Feynman-Kac formulae

For sake of completeness, we include here the proof of the important Feynman-Kac formulae.

Their proofs rely on the following “calculus type” lemma:

Lemma 1.7.4 *Let $a(t)$ be a right continuous function of a bounded variation and F be a right function of class C^1 . For every $r < t$,*

$$F(a(t)) - F(a(r)) = \int_r^t F'(a(s)) da_c(s) + \sum_{s \in (r,t]} \Delta F(a(s)) \quad (1.7.35)$$

where

$$\Delta F(a(s)) = F(a(s)) - F(a(s-))$$

and

$$a_c(t) = a(t) - \sum_{s \in (r,t]} \Delta a(s)$$

is the continuous part of a .

Proof: See [Dy7, Lemma 4.1.1]. $\square$

Remark: Let $h(s) = e^{a(s)}$. By applying formula (1.7.35) to $F(u) = e^u$, we get

$$\int_{(r,t]} f(s) dh(s) = \int_r^t f(s) h(s) da_c(s) + \sum_{s \in (r,t]} f(s) \Delta h(s) \quad (1.7.36)$$

for every bounded Borel function f . The last term can be rewritten by using the formula

$$\Delta h(s) = h(s-) [\exp -\Delta a(s) - 1] = h(s) [1 - \exp -\Delta a(s)].$$

Theorem 1.7.5 (General Feynman-Kac's formulae) Formula1 *Let C be a positive functional of ξ . Let C_c be the continuous part of C . Put $H(I) := \exp C(I)$. Let B be a continuous signed additive functional of ξ . Assume that*

$$\pi_{r,x} |B|(r, t] < \infty. \quad (1.7.37)$$

Then the function

$$g(r, x) = \pi_{r,x} B(r, t] \quad (1.7.38)$$

satisfies the equation

$$\begin{aligned} g(r, x) = & \pi_{r,x} \int_{(r,t]} H(r, s)^{-1} B(ds) + \pi_{r,x} \int_r^t H(r, s)^{-1} g(s, \xi_s) C_c(ds) \\ & + \pi_{r,x} \sum_{s \in (r,t]} H(r, s)^{-1} (1 - \exp(-C\{s\})) g(s, \xi_s). \end{aligned} \quad (1.7.39)$$

If g is finite and if the right side of (1.7.39) is well defined² then (1.7.39) implies (1.7.38).

The function

$$g(r, x) = \pi_{r,x} \int_{(r,t]} H(r, s)^{-1} B(ds) \quad (1.7.40)$$

satisfies the equation

$$\begin{aligned} g(r, x) + \pi_{r,x} \sum_{s \in (r,t]} (1 - \exp(-C\{s\})) g(s, \xi_s) \\ + \pi_{r,x} \int_r^t g(s, \xi_s) C_c(ds) = \pi_{r,x} B(r, t]. \end{aligned} \quad (1.7.41)$$

²That is all terms are finite

If g is finite and if the left side of (1.7.41) is well-defined then (1.7.41) implies (1.7.40).

Formula 2 Let C a positive functional of ξ . Let C_c be the continuous part of C . Put $H(I) := \exp C(I)$. Let B be a continuous signed additive functional of ξ . Assume that

$$\pi_{r,x} \int_{(r,t]} H(r,s) |B|(ds) < \infty. \quad (1.7.42)$$

Then the function (1.7.38) satisfies the equation

$$\begin{aligned} g(r,x) + \pi_{r,x} \int_r^t H(r,s)^{-1} g(s, \xi_s) C_c(ds) \\ + \pi_{r,x} \sum_{s \in (r,t)} H(r,s) (\exp C\{s\} - 1) g(s, \xi_s) = \pi_{r,x} \int_{(r,t]} H(r,s) B(ds). \end{aligned} \quad (1.7.43)$$

The function

$$g(r,x) = \pi_{r,x} \int_{(r,t]} H(r,s) B(ds) \quad (1.7.44)$$

satisfies the equation

$$\begin{aligned} g(r,x) = \pi_{r,x} B(r,t] + \pi_{r,x} \int_r^t g(s, \xi_s) C_c(ds) \\ + \pi_{r,x} \sum_{s \in (r,t)} H(r,s) (\exp C\{s\} - 1) g(s, \xi_s). \end{aligned} \quad (1.7.45)$$

If g is finite and if the right side of (1.7.45) is well defined, then (1.7.45) implies (1.7.44).

Proof of Formula 1: 1°) First we assume that B is positive. Let g be given by (1.7.38). By applying (1.7.36) to $f(s) = -g(s, \xi_s)$ and $a(s) = -C(r,s]$, we get

$$\begin{aligned} \int_r^t H(r,s)^{-1} g(s, \xi_s) C_c(ds) + \sum_{s \in (r,t)} H(r,s) (1 - \exp(-C\{s\})) g(s, \xi_s) \\ = - \int_{(r,t]} g(s, \xi_s) d_s \{H(r,s)^{-1}\}. \end{aligned}$$

since $g(t,y) = 0$ we can replace the sum over $s \in (r,t]$ by a sum over $s \in (r,t)$. On the other hand, by the strong Markov property of ξ , we have for every stopping time σ

$$\begin{aligned} \pi_{r,x} B(\sigma, t] &= \pi_{r,x} B(\sigma \wedge t, t] \\ &= \pi_{r,x} \pi_{\sigma \wedge t, \xi_{\sigma \wedge t}} B(\sigma \wedge t, t] = \pi_{r,x} \pi_{\sigma, \xi_\sigma} B(\sigma, t] \end{aligned}$$

Thus, $\pi_{r,x}$ -a.s. surely, the right continuous process³ $\pi_{s,\xi} B(s, t]$ is the optional projection of $B(s, t]$ and according to [DeM, VI.57] we have that

$$\pi_{r,x} \int_{(r,t]} g(s, \xi_s) d_s \{H(r, s)^{-1}\} = \pi_{r,x} \int_{(r,t]} B(s, t] d_s \{H(r, s)^{-1}\}. \quad (1.7.46)$$

By Fubini's theorem, the right side of (1.7.46) is equal to

$$\pi_{r,x} \int_{r < s < u < t} B(du) d_s \{H(r, s)^{-1}\} = \pi_{r,x} \int_{(r,t]} [H(r, u)^{-1} - 1] B(du).$$

Hence the right side of (1.7.39) is equal to

$$\pi_{r,x} \int_{(r,t]} [H(r, u)^{-1} + (1 - H(r, u)^{-1})] B(du) = \pi_{r,x} B(r, t].$$

If g is given by (1.7.40), then, applying (1.7.36) to $f(s) = H(r, s)^{-1} g(s, \xi_s)$ and $a(s) = C(r, s]$, we get

$$\begin{aligned} \int_r^t g(s, \xi_s) C_c(ds) + \sum_{s \in (r,t]} (1 - \exp(-C\{s\})) g(s, \xi_s) \\ = \int_{(r,t]} H(r, s)^{-1} g(s, \xi_s) d_s \{H(r, s)^{-1}\}. \end{aligned} \quad (1.7.47)$$

We have (cf. (1.7.46))

$$\begin{aligned} \int_{(r,t]} H(r, s)^{-1} g(s, \xi_s) d_s \{H(r, s)^{-1}\} &= \int_{(r,t]} H(r, s)^{-1} \int_{(s,t]} H(s, u)^{-1} B(du) d_s \{H(r, s)^{-1}\} \\ &= \int_{r < s < u \leq t} H(r, u)^{-1} B(du) d_s \{H(r, s)^{-1}\} \\ &= \int_{(r,t]} H(r, u)^{-1} (1 - H(r, u)^{-1}) B(du). \end{aligned}$$

In conjunction with (1.7.47), this formula implies (1.7.41).

2°) Now let $B = B_+ - B_-$. By (1.7.37), the functions $g_+(r, x) = \pi_{r,x} B_+(r, t]$ and $g_-(r, x) = \pi_{r,x} B_-(r, t]$ are finite. We proved that g_+ and g_- satisfy (1.7.39). Each term in the right side of (1.7.39) is finite. Hence (1.7.39) holds for $g = g_+ - g_-$. Similar arguments are applicable to (1.7.41).

3°) Suppose that g satisfies (1.7.39). Then (1.7.37) holds for

$$\hat{B}(I) = B(I) + \int_I g(s, \xi_s) C_c(ds) + \sum_{s \in I} (1 - \exp(-C\{s\})) g(s, \xi_s).$$

³according to [Dy, p.30]

Formula (1.7.39) can be rewritten in the form (1.7.40) with B replaced by $\dot{B}$ and, since (1.7.40) implies (1.7.41) we have

$$\begin{aligned} g(r, x) &= \pi_{r,x} \int_r^t g(s, \xi_s) C_c(ds) + \pi_{r,x} \sum_{s \in (r,t]} (1 - \exp(-C\{s\})) g(s, \xi_s) \\ &= \pi_{r,x} \dot{B}(r, t] \\ &= \pi_{r,x} \left(B(r, t] + \int_{(r,t]} g(s, \xi_s) C_c(ds) + \sum_{s \in (r,t]} (1 - \exp(-C\{s\})) g(s, \xi_s) \right). \end{aligned}$$

this implies (1.7.38) because all the terms in both sides are finite.

4°) Implication (1.7.41) $\Rightarrow$ (1.7.41) can be proved analogously. $\square$

Proof of Formula 2: Similar. $\square$

Remark 1.7.5 *Formula 1 and 2 can be extended to the case of a signed additive functional C with bounded variation $|C|[0, t]$. (The crucial point is that the relation (1.7.46) remains true for such functionals.) For that extension to be true, condition (1.7.37) should be replaced by the condition*

$$\pi_{r,x} |B|(r, t] + \pi_{r,x} \int_{(r,t]} H^{-1}(r, s) |B(ds)| < \infty$$

in formula 1, while condition (1.7.42) should be replaced by

$$\pi_{r,x} |B|(r, t] + \pi_{r,x} \int_{(r,t]} H(r, s) |B(ds)| < \infty.$$

The reader will note that the extended form of formula 1 and 2 coincide and that it contains as a particular case both formula 1 and 2.

If the discontinuous part $C_d = C - C_c$ of the functional C is concentrated on the jumps of the process ξ , then formulae (1.7.39), (1.7.41), (1.7.43) and (1.7.45) can be rewritten in terms of the Levy measure of ξ . The third Feynman-Kac's formula has been included in the corpus of the thesis. For sake of completeness we include also here the fourth one, although we will not use it.

Theorem 1.7.6 formula 4 *Let C_0 be a continuous positive functional of ξ . Suppose that $\rho^s(x, y)$ is a nearly Borel function such that $\rho \geq 1$ and $\rho^s(x, x) = 1$ for all s, x .*

$$H(I) := e^{C_0(I)} \prod_{s \in I} \rho^s(\xi_{s-}, \xi_s) \quad (1.7.48)$$

Let B be a signed additive functional of ξ . Assume that

$$\pi_{r,x} \int_{(r,t]} |B|(ds) < \infty.$$

Then the function

$$g(r, x) = \pi_{r,x} B(r, t] \quad (1.7.49)$$

satisfies the equation

$$\begin{aligned} g(r, x) + \pi_{r,x} \int_r^t g(s, \xi_s) C_0(ds) \\ + \pi_{r,x} \int_{(r,t]} \int_E [1 - \rho^s(\xi_{s-}, y)] g(s, y) N(ds, dy) = \pi_{r,x} B(r, t]. \end{aligned} \quad (1.7.50)$$

where $N(ds, dy)$ is the Lévy measure of ξ . Conversely, if g is finite and if the left side of (1.7.50) is well defined then (1.7.50) implies (1.7.49).

Proof of Formula 4: See [Dy7, Th. 4.3.2]. $\square$

1.7.2 Review of some classical notions and theorems used in this chapter

Definition 1.7.6 *Let ξ be a right Markov process and let $(Z, \mathcal{Z})$ be a Luzin space. A measure valued additive functional (MVAF) of ξ is a random measure $\Gamma(\omega; dt, dz)$ on $R_+ \times Z$ such that for every open interval I and every $B \in \mathcal{Z}$, $\Gamma(I \times B)$ is $F(I)^*$ -measurable, where $F(I)^*$ is the universal completion of $F(I)$.*

Definition 1.7.6 *Denote by $\mathcal{P}_r$ the σ -algebra in $[r, \infty) \times \Omega$ generated by all left continuous processes adapted to the filtration $\mathcal{F}[r, t]$. ($\mathcal{P}_r$ is the predictable σ -algebra for the filtration $\mathcal{F}[r, t]$.)*

Definition 1.7.6 A natural MVAF $N(ds, dy)$ in the space $(Z, \mathcal{Z}) = (E, \mathcal{B})$ of a Markov process ξ is called the Lévy measure for ξ if N is concentrated on the set $\{(s, y) : \xi_{s-} \neq y\}$ and, for every non-negative $\mathcal{P}_r \times \mathcal{B}$ -measurable function

$$\pi_{r,x} \sum_{s>r} F(s, \xi_s) 1_{\xi_{s-} \neq \xi_s} = \pi_{r,x} \int_{(r,\infty) \times E} F(s, y) N(ds, dy) \quad (1.7.51)$$

Remark: For every r -predictable non negative Y and every Borel non-negative f such that $f^s(x, x) = 0$ for all s, x we have

$$\pi_{r,x} \sum_{s>r} Y_s f^s(\xi_{s-}, \xi_s) 1_{\xi_{s-} \neq \xi_s} = \pi_{r,x} \int_{(r,\infty) \times E} Y_s f^s(\xi_{s-}, y) N(ds, dy) \quad (1.7.52)$$

Proposition 1.7.6 If $K(ds)$ is a natural additive functional of ξ then

$$\int_{(r,t]} f(s, \xi_{s-}) K(ds) = \int_{(r,t]} f(s, \xi_s) K(ds) \quad \pi_{r,\mu} - a.s.$$

Proof:[Dy7, eq. (2.4.2)]□

Theorem 1.7.7 For every right process ξ there exists the unique Lévy measure N . And there is a strictly positive nearly Borel function ψ such that

$$\pi_{r,x} \int_{(r,\infty) \times E} \psi^s(\xi_{s-}, y) N(ds, dy) \leq 1 \quad (1.7.53)$$

for all r, x .

Proof:[Dy7, Th. 2.5.6]□

Theorem 1.7.8 Let $A \ll B$ be continuous MVAF's of a right process ξ . Suppose that there exists a strictly positive nearly Borel function ψ such that

$$\pi_{r,x} \int_{(r,\infty) \times E} \psi^s(\xi_{s-}, y) B(ds, dy) \leq 1$$

for all r, x . Then there exists a nearly Borel function ρ in $S \times \mathcal{Z}$ such that $0 \leq \rho \leq 1$ and

$$A(ds, dz) = \rho^s(\xi_{s-}, z) B(ds, dz).$$

Proof: [DeM, XV.13] see also [Dy7, Th. 2.4.4] $\square$

Theorem 1.7.9 *Suppose that a continuous functional B of a right process ξ and a MVAF A satisfy the condition*

$$A(I, Z) \leq B(I) \text{ for all } I$$

Then there exists a Borel kernel $\gamma^(x, dz)$ such that*

$$A(ds, dz) = \gamma^*(\xi_{s-}, dz)B(ds).$$

Proof:[DeM, XV.34] see also [Dy7, Th. 2.4.5] $\square$

Chapter 2

Multitype superprocesses

2.1 Introduction

By passing to the limit of a class of previously constructed multitype particle systems, we construct a class of multitype superprocesses. This class is the natural multitype extension of the superprocesses constructed in [Dy2], and in fact was originally an attempt to generalize that class. The publication in 1994 of [Dy7] has considerably reduced the novelty of our class, but nevertheless, this contains some multitype superprocesses which cannot be constructed by Dynkin's method. Until now, every superprocess (with no interaction) had a log-Laplace functional

$$v_{r,t}(f)(x) := P_{r,\delta_x} \left(e^{-\langle X_t, f \rangle} \right)$$

which solves an evolution equation of the form

$$v_{r,t}(f)(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \psi(s, \xi_s, v_{s,t}(f)) k(ds)$$

where

$$\sup_x \pi_{r,x} \int_r^t \psi(s, \xi_s, 1) k(ds) < \infty.$$

This property is essential for this type of construction. We include an example of multitype superprocess where in fact this is not the case. A formal definition of single type superprocess could be a *superprocess in which the offspring birth location*

is identical to the parent's death location. A multitype superprocess would then be a superprocess which is not a single type superprocess. Our new superprocesses can be seen as another example suggesting that multitype superprocesses have in their nature features that are not shared by the single type superprocesses and that for the problem of characterizing all the superprocesses, it might be advantageous to give them a distinct treatment.

2.2 A class of Multitype Superprocesses

2.2.1 The Multitype Branching Particle System

Let the following “ingredients” be given:

1. a measure space $(E, \mathcal{E})$ and a countable set of types Δ
2. an E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$
3. a continuous additive functional $k^\tau(ds)$ of ξ for each $\tau \in \Delta$. We assume that there exists another continuous additive functional $k(ds)$ such that $k^\tau(I) \leq k(I)$ for every interval I which is such that for every $q > 0$

$$\sup_{r,x} \pi_{r,x} e^{qk(r,t)} < \infty$$

4. a collection of non negative measurable functions $\{p_{r,\zeta}(s, x)\}_{r,\zeta \in \Delta}$ such that

$$\sum_{\zeta \in \Delta} p_{r,\zeta}(s, x) = 1$$

5. a branching mechanism (see [Da6, p.62]; for simplicity we assume that it is the critical binary branching, but general mechanisms can be used...).

Notation: We will refer the above elements as the “*basic assumptions*”.

In order to construct the appropriate particle systems needed to derive our multitype superprocesses, we will need the following result which is a primitive form of Dynkin's general Feynman-Kac formulae.

Theorem 2.2.10 *Let $h(.,.) \geq 0$ be a progressively measurable function. If*

$$g(r, x) = \pi_{r,x} \left(e^{-k(r,t)} f(\xi_t) \right) + \pi_{r,x} \left(\int_r^t e^{-k(r,s)} h(s, \xi_s) k(ds) \right)$$

then

$$g(r, x) = \pi_{r,x} (f(\xi_t)) + \pi_{r,x} \left(\int_r^t (h(s, \xi_s) - g(s, \xi_s)) k(ds) \right)$$

Proof: [Dy2, Lemma 2.3]□

Theorem 2.2.11 *Under the basic assumptions, for every $\epsilon > 0$, there exists a probability space $(\mathcal{T}, \mathfrak{F}, P_{r,\mu}^\epsilon, \mu \in M_f(E \times \Delta))$ such that if*

$$v_{r,t}^\epsilon(x, \tau) := P_{r,\delta(x,\tau)}^\epsilon \left(e^{-\langle \Xi_t, f \rangle} \right) \quad (2.2.1)$$

then

$$\begin{aligned} v_{r,t}^\epsilon(x, \tau) = & \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} \right) - \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{r,\zeta}(s, \xi_s) \left(v_{s,t}^\epsilon(\xi_s, \zeta) - v_{s,t}^\epsilon(\xi_s, \tau) \right) k^\tau(ds) \\ & - \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{r,\tau}(s, \xi_s))) \frac{1}{2} \left(v_{s,t}^\epsilon(\xi_s, \tau) \right)^2 k^\tau(ds) \end{aligned} \quad (2.2.2)$$

and such that moreover,

$$P_{r,\mu}^\epsilon(e^{-\langle \Xi_t, f \rangle}) = e^{-\langle \mu, v_{r,t}^\epsilon \rangle}.$$

Proof: Consider the space $E \times \Delta$ furnished with the σ -algebra $\sigma(A \times \{\tau\} : A \in E, \tau \in \Delta)$. On this space we define the Markov process (ξ, τ) with transition probability given by

$$\pi_{r,(x,\tau)} f(\xi_t, \tau_t) = \pi_{r,x} f(\xi_t, \tau).$$

That is, this new process simply describes the process obtained by first assigning to ξ a type τ which the process is going to keep during its whole life. To the process (ξ, τ) we associate the additive functional $K(ds)$ with

$$K(ds)(\omega) = k^\tau(ds)(\omega)$$

for any trajectory ω on the subspace $E \times \{\tau\}$. As guaranteed by Theorem 1.3.1, for any offspring distribution with generating function ϕ , we can construct a space

$(\mathcal{T}, \mathfrak{I}, P_{r,\mu}, \mu \in M_f(E \times \Delta))$ and a random $M_f(E \times \Delta)$ -valued variable, Ξ_t , on that space such that if

$$v_{r,t}(x, \tau) := P_{r,\delta_{(x,\tau)}}(e^{-\langle \Xi_t, f \rangle}) \quad (2.2.3)$$

then

$$v_{r,t}(x, \tau) = \pi_{r,x}(e^{-k^\tau(r,t)}e^{-f(\xi_t)}) + \pi_{r,x}\left(\int_r^t e^{-k^\tau(r,s)}\phi(s, \xi_s, v_{s,t})k^\tau(ds)\right). \quad (2.2.4)$$

Moreover, Theorem 1.3.1 asserts that

$$P_{r,\mu}(e^{-\langle \Xi_t, f \rangle}) = e^{-\langle \mu, v_{r,t} \rangle} \quad (2.2.5)$$

for any measure $\mu \in M_f(E \times \Delta)$. Using Theorem 2.2.10 we conclude that for any offspring distribution with generating function ϕ , there exists a space $(\mathcal{T}, \mathfrak{I}, P_{r,\mu}, \mu \in M_f(E \times \Delta))$ and a random $M_f(E \times \Delta)$ -valued variable, Ξ_t , on that space such that the function $v_{r,t}^\tau(x)$ given by (2.2.3) satisfies

$$v_{r,t}(x, \tau) = \pi_{r,x}(e^{-f(\xi_t)}) + \pi_{r,x}\left(\int_r^t (\phi(s, \xi_s, v_{s,t}) - v_{s,t}(\xi_s, \tau))k^\tau(ds)\right), \quad (2.2.6)$$

and such that (2.2.5) is verified.

Therefore, if for every $\epsilon > 0$ we define

$$\begin{aligned} \psi_\epsilon(s, x, \tau, \phi) &:= \sum_{\zeta \neq \tau} \epsilon p_{r,\zeta}(s, x) \phi(x, \zeta) \\ &\quad + (1 - \epsilon(1 - p_{r,\tau}(s, x))) \epsilon \left[\frac{\epsilon}{2} \left(\frac{1 - \phi(x, \tau)}{\epsilon} \right)^2 + \frac{1}{\epsilon} \phi(x, \tau) \right] \\ &= \sum_{\zeta \neq \tau} \epsilon p_{r,\zeta}(s, x) \phi(x, \zeta) + (1 - \epsilon(1 - p_{r,\tau}(s, x))) \phi(x, \tau) \\ &\quad + (1 - \epsilon(1 - p_{r,\tau}(s, x))) \frac{\epsilon^2}{2} \left(\frac{1 - \phi(x, \tau)}{\epsilon} \right)^2 \\ k_\epsilon^\tau(ds) &:= \frac{1}{\epsilon} k^\tau(ds) \end{aligned}$$

Then ψ_ϵ is the generating function of an offspring distribution, k_ϵ^τ is a continuous additive functional and ξ a Markov process. Hence there exists $(\mathcal{T}, \mathfrak{I}, (\mathcal{T}, \mathfrak{I}, \hat{P}_{r,\mu}^\epsilon, \mu \in M_f(E \times \Delta)))$ and the random measure Ξ_t on that space such that if

$$\tilde{v}_{r,t}^\epsilon(x, \tau) := \hat{P}_{r,\delta_{(x,\tau)}}^\epsilon(e^{-\langle \Xi_t, f \rangle})$$

then

$$\tilde{v}_{r,t}^{\epsilon}(x, \tau) = \pi_{r,x}(\epsilon^{-\frac{1}{\epsilon}k(r,t)} \epsilon^{-f(\xi_t, \tau)}) + \int_r^t \epsilon^{-\frac{1}{\epsilon}k^{\tau}(r,s)} \psi_{\epsilon}(s, \xi_s, \tilde{v}_{s,t}^{\epsilon}) \frac{1}{\epsilon} k^{\tau}(ds)$$

and

$$P_{r,\mu}(\epsilon^{-\langle \Xi_t, f \rangle}) = \epsilon^{-\langle \mu, \tilde{v}_{r,t}^{\epsilon} \rangle}.$$

Let $\hat{P}_{r,\delta(x,\tau)}^{\epsilon}(\cdot)$ be the Law of the random measure $\epsilon \Xi_t$ (with Ξ_t having distribution $\hat{P}_{r,\mu}^{\epsilon}(\cdot)$). Thus we have

$$\hat{P}_{r,\mu}^{\epsilon}(\epsilon^{-\langle \Xi_t, f \rangle}) := \hat{P}_{r,\mu}^{\epsilon}(\epsilon^{-\langle \Xi_t, f \rangle}),$$

and if

$$\tilde{v}_{r,t}^{\epsilon}(x, \tau) := \hat{P}_{r,\delta(x,\tau)}^{\epsilon}(\epsilon^{-\langle \Xi_t, f \rangle})$$

then

$$\tilde{v}_{r,t}^{\epsilon}(x, \tau) = \pi_{r,x}(\epsilon^{-\frac{1}{\epsilon}k(r,t)} \epsilon^{-f(\xi_t, \tau)}) + \int_r^t \epsilon^{-\frac{1}{\epsilon}k^{\tau}(r,s)} \psi_{\epsilon}(s, \xi_s, \tilde{v}_{s,t}^{\epsilon}) \frac{1}{\epsilon} k^{\tau}(ds),$$

and

$$\hat{P}_{r,\mu}^{\epsilon}(\epsilon^{-\langle \Xi_t, f \rangle}) = \epsilon^{-\langle \mu, \tilde{v}_{r,t}^{\epsilon} \rangle}. \quad (2.2.7)$$

In fact, using Theorem 2.2.10, we get that $\tilde{v}_{r,t}^{\epsilon}(x, \tau)$ solves also

$$\tilde{v}_{r,t}^{\epsilon}(x, \tau) = \pi_{r,x}(e^{-\epsilon f(\xi_t, \tau)}) + \pi_{r,x} \int_r^t \left(\psi_{\epsilon}(s, \xi_s, \tilde{v}_{s,t}^{\epsilon}) - \tilde{v}_{s,t}^{\epsilon}(\xi_s, \tau) \right) \frac{1}{\epsilon} k^{\tau}(ds). \quad (2.2.8)$$

Note that

$$\begin{aligned} (\psi_{\epsilon}(s, x, \tau, \phi) - \phi(x, \tau)) &= \sum_{\zeta \neq \tau} \epsilon p_{\tau, \zeta}(s, x) (\phi(x, \zeta) - \phi(x, \tau)) \\ &\quad + (1 - \epsilon(1 - p_{\tau, \tau}(s, x))) \frac{\epsilon^2}{2} \left(\frac{1 - \phi(x, \tau)}{\epsilon} \right)^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{v}_{s,t}^{\epsilon}(x, \tau) &= \pi_{r,x}(e^{-\epsilon f(\xi_t, \tau)}) + \pi_{r,x} \int_r^t \left(\psi_{\epsilon}(s, \xi_s, \tilde{v}_{s,t}^{\epsilon}) - \tilde{v}_{s,t}^{\epsilon}(\xi_s, \tau) \right) \frac{1}{\epsilon} k^{\tau}(ds) \\ &= \pi_{r,x}(e^{-\epsilon f(\xi_t, \tau)}) \\ &\quad + \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{\tau, \zeta}(s, \xi_s) \left(\tilde{v}_{s,t}^{\epsilon}(\xi_s, \zeta) - \tilde{v}_{s,t}^{\epsilon}(\xi_s, \tau) \right) k^{\tau}(ds) \\ &\quad + \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{\tau, \tau}(s, \xi_s))) \frac{\epsilon^2}{2} \left(\frac{1 - \tilde{v}_{s,t}^{\epsilon}(\xi_s, \tau)}{\epsilon} \right)^2 k^{\tau}(ds) \end{aligned}$$

Putting

$$v_{r,t}^{\epsilon}(x, \tau) = \frac{1 - \tilde{v}_{r,t}^{\epsilon}(x, \tau)}{\epsilon}. \quad (2.2.9)$$

that gives

$$\begin{aligned} \tilde{v}_{s,t}^{\epsilon}(x, \tau) &= \pi_{r,x}(e^{-\epsilon f(\xi_t, \tau)}) \\ &+ \epsilon \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{\tau, \zeta}(s, \xi_s) \left(v_{s,t}^{\epsilon}(\xi_s, \tau) - v_{s,t}^{\epsilon}(\xi_s, \zeta) \right) k^{\tau}(ds) \\ &+ \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{\tau, \tau}(s, \xi_s))) \frac{\epsilon}{2} \left(v_{s,t}^{\epsilon}(\xi_s, \tau) \right)^2 k^{\tau}(ds) \end{aligned}$$

And thus we have

$$\begin{aligned} v_{s,t}^{\epsilon}(x, \tau) &= \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} \right) \\ &- \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{\tau, \zeta}(s, \xi_s) \left(v_{s,t}^{\epsilon}(\xi_s, \zeta) - v_{s,t}^{\epsilon}(\xi_s, \tau) \right) k^{\tau}(ds) \\ &- \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{\tau, \tau}(s, \xi_s))) \frac{1}{2} \left(v_{s,t}^{\epsilon}(\xi_s, \tau) \right)^2 k^{\tau}(ds), \end{aligned}$$

and (cf. (2.2.7))

$$\tilde{P}_{r, \frac{\epsilon}{2}}^{\epsilon}(e^{-\langle \Xi_t, f \rangle}) = e^{-\langle \mu, \frac{1 - \tilde{v}_{r,t}^{\epsilon}(x, \tau)}{\epsilon} \rangle}.$$

Hence, if we put

$$P_{r, \mu}^{\epsilon}(\cdot) := \tilde{P}_{r, \frac{\epsilon}{2}}^{\epsilon}(\cdot),$$

we get the desired result. $\square$

2.2.2 Uniqueness of the solution to an integral equation

Let us first recall the generalized Gronwall inequality:

Lemma 2.2.11 (general Gronwall inequality) *If $h^r(x)$ is a bounded progressively measurable function then the inequality*

$$h^r(x) \leq a + q \pi_{r,x} \int_r^t h^s(\xi_s) k(ds)$$

implies

$$h^r(x) \leq a \pi_{r,x} e^{qk(r,t)}$$

where $k(ds)$ is a continuous additive functional, and a, q are non negative constants.

Proof: [Dy2, Lemma 3.2] $\square$

Theorem 2.2.12 Suppose Ψ is a mapping such that for all bounded measurable functions z^1 and z^2 we have

$$\|\Psi(s, x, z^1) - \Psi(s, x, z^2)\|_{\infty} \leq C \left(\|z^1\|_{\infty} \vee \|z^2\|_{\infty} \right) |z^1(x, \zeta) - z^2(x, \zeta)|$$

where $C(\cdot)$ is any non negative real valued function. Then, under the "basic assumptions", there cannot be two distinct bounded solutions to an equation of the form

$$v_{r,t}f(x, \tau) = \pi_{r,x}f(\xi_t, \tau) + \pi_{r,x} \int_r^t \Psi(s, \xi_s, v_{r,t}f) k^{\tau}(ds)$$

Proof: Let's denote

$$\Delta v_{r,t}(x) := \sup_{\tau \in \Delta} |v_{r,t}^1(x, \tau) - v_{r,t}^2(x, \tau)|$$

If v^1 and v^2 are two such solutions, then

$$\begin{aligned} |v_{r,t}^1(x, \tau) - v_{r,t}^2(x, \tau)| &\leq \pi_{r,x} \int_r^t |\Psi(s, \xi_s, v_{r,t}^1f) - \Psi(s, \xi_s, v_{r,t}^2f)| k^{\tau}(ds) \\ &\leq C(\|v^1\|_{\infty} \vee \|v^2\|_{\infty}) \pi_{r,x} \int_r^t \Delta v_{r,t}(\xi_s) k^{\tau}(ds) \end{aligned}$$

Hence, since $k^{\tau}(I) \leq k(I)$ for every interval I we get

$$\Delta v_{r,t}(x) \leq C(\|v^1\|_{\infty} \vee \|v^2\|_{\infty}) \pi_{r,x} \int_r^t \Delta v_{r,t}(\xi_s) k(ds).$$

The generalized Gronwall inequality gives that

$$\begin{aligned} \Delta v_{r,t}(x) &\leq 0 \left\| \pi_{r,x} e^{C(\|v^1\|_{\infty} \vee \|v^2\|_{\infty})k(r,t)} \right\|_{\infty} \\ &= 0 \end{aligned}$$

which completes the proof. $\square$

2.2.3 From particle systems to superprocesses

Let (E, E) be a metrizable Luzin space. We will say that a mapping $(f, x) \mapsto v_{r,t}(f)(x)$, $x \in E$, $f \in pbE$ is a *log-Laplace functional* if there exists a probability

space $(T, T, P_{r,\mu}, \mu \in M_f(E))$ and a random measures $\{\Xi_t\}_{t \geq 0}$ on that space such that

$$P_{r,\mu}(e^{-\langle \Xi_t, f \rangle}) = e^{-\langle \mu, v_{r,t}(f) \rangle}$$

for every $f \in pbE, 0 \leq r \leq t, \mu \in M_f(E)$. A log-Laplace functional is said to be the log-Laplace functional of a Markov process if Ξ is a Markov process.

Theorem 2.2.13 *There exists a Markov measure valued process having a bounded log-Laplace functional solving the integral equation*

$$\begin{aligned} v_{r,t}(x, \tau) = & \pi_{r,x}(f(\xi_t, \tau) - \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{r,\zeta}(s, \xi_s) (v_{s,t}(\xi_s, \zeta) - v_{s,t}(\xi_s, \tau)) k^\tau(ds) \\ & - \pi_{r,x} \int_r^t \frac{1}{2} (v_{s,t}(\xi_s, \tau))^2 k^\tau(ds). \end{aligned} \quad (2.2.10)$$

for every $f \in pbE$.

Proof: We have previously shown in Theorem 2.2.11 that there is a log-Laplace function, $v_{s,t}^\epsilon(x, \tau)$, which is a solution to the integral equation

$$\begin{aligned} v_{r,t}^\epsilon(x, \tau) = & \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} \right) \\ & - \pi_{r,x} \int_r^t \sum_{\zeta \neq \tau} p_{r,\zeta}(s, \xi_s) (v_{s,t}^\epsilon(\xi_s, \zeta) - v_{s,t}^\epsilon(\xi_s, \tau)) k^\tau(ds) \\ & - \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{r,\tau}(s, \xi_s))) \frac{1}{2} (v_{s,t}^\epsilon(\xi_s, \tau))^2 k^\tau(ds). \end{aligned} \quad (2.2.11)$$

Note that, by construction of v^ϵ , (cf. 2.2.9),

$$0 \leq v_{r,t}^\epsilon f(x, \tau) = \frac{1 - \tilde{v}_{r,t}^\epsilon f(x, \tau)}{\epsilon} = \tilde{P}_{r,x,\tau}^\epsilon \left(\frac{1 - e^{-\langle \Xi_t, f \rangle}}{\epsilon} \right) \leq \frac{1}{\epsilon} < \infty.$$

From (2.2.11), and from the facts that

$$\begin{aligned} 0 \leq & \pi_{r,x} \int_r^t \left(\sum_{\zeta} p_{r,\zeta}(s, \xi_s) v_{s,t}^\epsilon(\xi_s, \zeta) \right) k^\tau(ds) \\ & + \pi_{r,x} \int_r^t (1 - \epsilon(1 - p_{r,\tau}(s, \xi_s))) \frac{1}{2} (v_{s,t}^\epsilon(\xi_s, \tau))^2 k^\tau(ds), \end{aligned}$$

and

$$1 = \sum_{\zeta} p_{r,\zeta}(s, \xi_s),$$

we can conclude that

$$v_{r,t}^\epsilon(x, \tau) - \pi_{r,x} \int_\tau^t v_{s,t}^\epsilon(\xi_s, \tau) k^\tau(ds) \leq \left\| \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} \right) \right\|_\infty \leq \|f\|_\infty.$$

Consequently,

$$\begin{aligned} v_{r,t}^\epsilon(x, \tau) &\leq \|f\|_\infty + \pi_{r,x} \int_\tau^t v_{s,t}^\epsilon(\xi_s, \tau) k^\tau(ds) \\ &\leq \|f\|_\infty + \pi_{r,x} \int_\tau^t v_{s,t}^\epsilon(\xi_s, \tau) k(ds). \end{aligned}$$

Using the general Gronwall inequality, we get

$$\|v_{\cdot,t}^\epsilon(\cdot, \cdot)\|_\infty \leq \|f\|_\infty \sup_{r,x} \pi_{r,x} e^{k(r,t)} =: M$$

where the supremum is taken over all r such that $0 \leq r \leq t$ and all $x \in E$.

Note that the log-Laplace functional, v , that we will construct, is obtained as the limit as ϵ goes to zero of v^ϵ . Therefore, v is also bounded by M . Let us define

$$\Delta v_{r,t}(x) := \sup_{\tau \in \Delta} |v_{r,t}^\epsilon(x, \tau) - v_{r,t}^\delta(x, \tau)|.$$

We have:

$$\begin{aligned} |v_{r,t}^\epsilon(x, \tau) - v_{r,t}^\delta(x, \tau)| &\leq \left\| \frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} - \frac{1 - e^{-\delta f(\xi_t, \tau)}}{\delta} \right\|_\infty + \\ &\quad + 2\pi_{r,x} \int_\tau^t \sum_{\zeta \neq \tau} p_{\tau,\zeta}(s, \xi_s) \Delta v_{s,t}(\xi_s) k(ds) \\ &\quad + \pi_{r,x} \int_\tau^t \frac{1}{2} \left| \left(v_{s,t}^\epsilon(\xi_s, \tau) \right)^2 - \left(v_{s,t}^\delta(\xi_s, \tau) \right)^2 \right| k(ds) \\ &\quad + \pi_{r,x} \int_\tau^t \epsilon (1 - p_{\tau,\tau}(s, \xi_s)) \left(v_{s,t}^\epsilon(\xi_s, \tau) \right)^2 k(ds) \\ &\quad + \pi_{r,x} \int_\tau^t \delta (1 - p_{\tau,\tau}(s, \xi_s)) \left(v_{s,t}^\delta(\xi_s, \tau) \right)^2 k(ds) \\ &\leq \left\| \frac{1 - e^{-\epsilon f(\xi_t, \tau)}}{\epsilon} - \frac{1 - e^{-\delta f(\xi_t, \tau)}}{\delta} \right\|_\infty \\ &\quad + (\epsilon + \delta) M^2 \|\pi_{\cdot,\cdot} k(\cdot, t)\|_\infty \\ &\quad + (2 + M) \pi_{r,x} \int_\tau^t \Delta v_{s,t}(\xi_s) k(ds). \end{aligned}$$

Thus, by general Gronwall inequality,

$$\Delta v_{r,t}(x) \leq \left(C + \left\| \frac{1 - e^{-\epsilon f(\xi_t, \cdot)}}{\epsilon} - \frac{1 - e^{-\delta f(\xi_t, \cdot)}}{\delta} \right\|_\infty \right) \|\pi_{r,x} e^{Mk(r,t)}\|_\infty$$

where

$$C := \epsilon \vee \delta M^2 \|\pi_{\cdot,\cdot} k(\cdot, t)\|_\infty.$$

That shows that

$$\|v_{r,t}^\epsilon - v_{r,t}^\delta\|_\infty \rightarrow 0$$

as ϵ, δ goes to zero, and that moreover, the convergence is uniform on the set of functions f such that $0 \leq f \leq c$. Consequently, there exists a log-Laplace functional $v_{r,t}(x, \tau)$ solving (2.2.10).

To prove that it is the log-Laplace functional of a Markov process it remains only to show that v is a semigroup. As shown in [Da6, Th. 4.4.1 step 3] this is the case if the solution is unique. But this is guaranteed by Theorem 2.2.12 and the proof is complete. $\square$

2.3 A new class of (Multitype) Superprocesses

Assume that are given:

1. a measured space $(E, \mathcal{E})$ and a (countable) set of types $\Delta = \{\tau_1, \tau_2, \dots\}$; For convenience, let's assume that Δ is the set of all natural numbers.
2. an E -valued Markov process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$
3. a continuous additive functional $k(ds)$ such that for every $q > 0$

$$\sup_{r,x} \pi_{r,x} e^{qk(r,t)} < \infty$$

4. a branching mechanism $\Phi(s, x, \phi(x))$ (see [Da6, p.62]; for simplicity, we assume that Φ is the critical binary branching mechanism).
5. a sequence of non negative numbers such that $\sum_{\zeta \in \Delta} p_{\tau,\zeta}(s, x) = 1$.

It is not our purpose to construct the most general possible class of superprocesses here. For that reason we will only consider the case where $p_{\tau,\xi}(s, x) = p_{\tau,\xi}$ and $\Phi(s, x, \lambda) = \lambda^2$. Indeed we want to construct superprocesses with "unbounded" branching rate additive functional.

Consider the following multitype superprocesses which are given here via their log-Laplace functionals as solutions of the following evolution equations:

$$\begin{aligned} v_{r,t}^N(f)(x, \tau) = & \pi_{r,x}(f(\xi_t, \tau)) \\ & - \pi_{r,x} \int_r^t \sum_{n=1}^{\infty} p_{\tau, \tau_n} \left(v_{s,t}^N(f)(\xi_s, \tau_n) - v_{s,t}^N(f)(\xi_s, \tau) \right) k_N^{\tau}(ds) \\ & - \pi_{r,x} \int_r^t \left(v_{s,t}^N(f)(\xi_s, \tau) \right)^2 k_N^{\tau}(ds) \end{aligned} \quad (2.3.12)$$

where

$$k_N^{\tau}(ds) = 1_{\{m \leq N\}} m k(ds)$$

Trivially, for $N = \infty$, these additive functionals are not bounded. Note that the processes with log-Laplace functional solving (2.3.12) are special cases of the ones considered in the previous section. And therefore in particular, uniqueness of the solution to the log-Laplace equation is guaranteed.

Note that

$$\lim_{n, N \rightarrow \infty} k_N^{\tau}(I) = \infty$$

for any interval I such that $k(I) > 0$. Moreover, for every non null function $0 \leq \phi(s)$ such that $\int_0^t \phi(s) k(ds) > 0$

$$\sup_{\tau} \pi_{r,x}^{\tau} \int_0^t \phi(s) k^{\tau}(ds) = \infty$$

where $k^{\tau}(ds) := k_{\infty}^{\tau}(ds)$. We will construct superprocesses with additive functional $k^{\tau}(ds)$. Remark that this is a novelty, since, even at the particle system's level, the fact that we have possibly unbounded branching rate does not *a priori* guarantee that no explosion will occur.

Remark also that we have for $i \leq N$:

$$\begin{aligned} v_{r,t}^N(f)(x, \tau_i) & \leq \pi_{r,x}(f(\xi_t, \tau_i)) + \pi_{r,x} \int_r^t v_{s,t}^N(f)(\xi_s, \tau_i) k_N^{\tau}(ds) \\ & = \pi_{r,x}(f(\xi_t, \tau_i)) + \pi_{r,x} \int_r^t v_{s,t}^N(f)(\xi_s, \tau_i) i k(ds) \end{aligned}$$

and therefore, by Gronwall's inequality:

$$v_{r,t}^N(x, \tau_i) \leq \|f(\cdot, \tau_i)\|_{\infty} \left\| \pi_{r,x} e^{ik(\cdot, t)} \right\|_{\infty} \quad \text{for every } i = 1, 2, \dots \quad (2.3.13)$$

Let $\mathcal{C}$ denote the set of functions charging only finitely many types:

$$\mathcal{C} := \{f(., .) : \|f(., .)\|_\infty < \infty \text{ and } 0 < \|f(., \tau)\|_\infty \text{ for only finitely many type } \tau\}$$

Assume that $f(., .) \in \mathcal{C}$. We will adopt the following notation:

Notation: For every $f \in \mathcal{C}$ we will denote

$$K(f) := \min\{N : f(., \tau_n) = 0 \text{ for every } n \geq N\},$$

and the expression

$$\text{"for every } \tau \leq K(f)\text{"}$$

will be interpreted as

$$\text{"for every } \tau \in \{\tau_1, \tau_2, \dots, \tau_{K(f)}\}."$$

We adopt a similar convention for the expression "for every $\tau \geq K(f)$ ".

Indeed, for every $\tau_i \geq K(f)$ we have

$$0 \leq v_{r,t}^N(x, \tau_i) \leq \|f(., \tau_i)\|_\infty \left\| \pi_{r,x} e^{ik(r,t)} \right\|_\infty = 0. \quad (2.3.14)$$

Note that the set of continuous functions with compact support is contained in $\mathcal{C}$. (The topology put on $E \times \Delta$ is the topology generated by the sets of the form $\mathcal{O} \times \{\tau\}$ where $\tau \in \Delta$ and $\mathcal{O}$ is open in E .)

Theorem 2.3.14 *For every $f \in \mathcal{C}$, there exists a unique non negative bounded semigroup $v_{r,t}(f)(x, \tau)$ solving the integral equation*

$$\begin{aligned} v_{r,t}(f)(x, \tau) &= \pi_{r,x}(f(\xi_t, \tau)) \\ &\quad - \pi_{r,x} \int_r^t \sum_{n=1}^\infty p_{r,\xi}(v_{s,t}(f)(\xi_s, \zeta) - v_{s,t}(f)(\xi_s, \tau)) k^\tau(ds) \\ &\quad - \pi_{r,x} \int_r^t (v_{s,t}(f)(\xi_s, \tau))^2 k^\tau(ds). \end{aligned} \quad (2.3.15)$$

Moreover,

$$v_{r,t}(f)(x, \tau) = \lim_{N \rightarrow \infty} v_{r,t}^N(f)(x, \tau)$$

where $v_{r,t}^N(f)$ is the log-Laplace functional solving (2.3.12).

Proof: Take $f \in \mathcal{C}$. Let $N \geq K(f)$ and let $v_{r,t}^N(f)$ be given by (2.3.12). For every $\tau \geq K(f)$ we have

$$\|v_{r,t}^N(f)(\cdot, \tau)\|_{\infty} = 0$$

and therefore, for $\tau \leq K(f)$

$$\begin{aligned} v_{r,t}^N(f)(x, \tau) &= \pi_{r,x}(f(\xi_t, \tau)) \\ &\quad - \pi_{r,x} \int_r^t \sum_{n=1}^{K(f)} p_{r,\tau_n} v_{s,t}^N(f)(\xi_s, \tau_n) k_N^{\tau}(ds) \\ &\quad - \pi_{r,x} \int_r^t \left[\left(v_{s,t}^N(f)(\xi_s, \tau) \right)^2 - v_{s,t}^N(f)(\xi_s, \tau) \right] k_N^{\tau}(ds) \end{aligned}$$

and because $k_N^{\tau}(ds) = 1_{\{m \leq \tau\}} m k(ds)$ we have that for every $\tau_i \leq K(f)$ and every $N \geq K(f)$

$$\begin{aligned} v_{r,t}^N(f)(x, \tau_i) &= \pi_{r,x}(f(\xi_t, \tau_i)) \\ &\quad - \pi_{r,x} \int_r^t \sum_{n=1}^{K(f)} p_{r,\tau_n} v_{s,t}^N(f)(\xi_s, \tau_n) i k(ds) \\ &\quad - \pi_{r,x} \int_r^t \left[\left(v_{s,t}^N(f)(\xi_s, \tau_i) \right)^2 - v_{s,t}^N(f)(\xi_s, \tau_i) \right] i k(ds). \end{aligned} \quad (2.3.16)$$

But we have already proven that such integral equations have only one bounded non-negative solution (cf. Theorem 2.2.12). Consequently, for every $N, M \geq K(f)$ and every $\tau_i \leq K(f)$

$$v_{r,t}^N(f)(x, \tau_i) \equiv v_{r,t}^M(f)(x, \tau_i)$$

and since, for $i \geq K(f)$

$$v_{r,t}^N(f)(x, \tau_i) \equiv 0$$

we obtain that

$$v_{r,t}^N(f) \equiv v_{r,t}^M(f) \text{ for every } N, M \geq K(f), \quad (2.3.17)$$

and therefore $v_{r,t}(f) = \lim_{N \rightarrow \infty} v_{r,t}^N(f) = v_{r,t}^{K(f)}(f)$ is a solution to (2.3.15).

Let us now show that (2.3.15) has a unique bounded solution. We derive from (2.3.15) that any bounded solution $u_{r,t}(f)$ to (2.3.15) satisfies the inequality

$$\begin{aligned} u_{r,t}(f)(x, \tau) &\leq \pi_{r,x}(f(\xi_t, \tau)) + \pi_{r,x} \int_r^t \sum_{n=1}^{K(f)} p_{r,\tau_n} u_{s,t}(f)(\xi_s, \tau) k^{\tau}(ds) \\ &\leq \pi_{r,x}(f(\xi_t, \tau)) + \pi_{r,x} \int_r^t u_{s,t}(f)(\xi_s, \tau) k^{\tau}(ds) \end{aligned}$$

which implies, by general Gronwall inequality, that

$$0 \leq u_{r,t}(x, \tau) \leq \|f(\cdot, \tau)\|_{\infty} \|\pi_{r,x} e^{k^{\tau}(r,t)}\|_{\infty} = 0.$$

Consequently, $u_{r,t}(x, \tau) \equiv 0$ for every $\tau \geq K(f)$, and thus $u_{r,t}$ solves the integral equation (2.3.16) which implies that $u_{r,t}(f)(x, \tau) = v_{r,t}^{K(f)}(f)(x, \tau)$ since for every $f \in C$, (2.3.16) has a unique bounded solution (according to Theorem 2.2.12). Thus the solution to (2.3.15) is unique.

Let us show now that it is a semigroup. But this is immediate from the fact that $v_{r,t}^{K(f)}(f)(x, \tau) = v_{r,t}(f)(x, \tau)$ and that $v_{r,t}^{K(f)}(f)(x, \tau)$ is a semigroup. $\square$

2.3.1 The solution to the evolution equation is a log-Laplace functional

Theorem 2.3.15 *For every $f \in C$, the solution $v_{r,t}(f)(x, \tau)$ to (2.3.15) is the log-Laplace functional of a Markov process.*

Proof: Let $\overline{E \times \Delta}$ be the compactification of $E \times \Delta$. Let $v_{r,t}^N(f)$ be given by (2.3.12). All the random measures X_t^N with log-Laplace functionals $v_{r,t}^N(f)(x, \tau)$ can be seen as random measures on $\overline{E \times \Delta}$. But in that case, they are tight and there exists a random measure $\overline{X}_t$ on $\overline{E \times \Delta}$ and a subsequence $X_t^{N_k}$ such that $X_t^{N_k}$ converges weakly to $\overline{X}_t$. But then, for all continuous functions with compact support we have that $L_{N_k}(f) \rightarrow \overline{L}(f)$, where $L_{N_k}(f)$ denotes the Laplace functional of $X_t^{N_k}$ and $\overline{L}(f)$ denotes the Laplace functional of $\overline{X}_t$. But if we consider the restriction X_t to $E \times \Delta$ of $\overline{X}_t$, that is, if we consider the random measure $X_t(A) := \overline{X}_t(A \cap E \times \Delta)$, then by definition of Laplace functional, it is obvious that $L(f) = \overline{L}(f)$ for every function f with compact support, where $L(f)$ denotes the Laplace functional of X_t . Thus for every continuous function f with compact support we have that

$$L(f) = e^{-v_{r,t}(f)(x, \tau)},$$

where

$$v_{r,t}(f)(x, \tau) := \lim_{k \rightarrow \infty} v_{r,t}^{N_k}(f)(x, \tau) = \lim_{n \rightarrow \infty} v_{r,t}^n(f)(x, \tau).$$

Now let f be an arbitrary bounded non negative function with $K(f) < \infty$. Suppose that the functions $\{f_n\}$ are converges bounded pointwise to and $K(f_n) = K(f)$. Assume that

$$L(f_n) = e^{-v_{r,t}(f_n)(x, \tau)} = e^{-v_{r,t}^{K(f_n)}(f_n)(x, \tau)}.$$

Let's denote $K(f)$ simply by K . Since $L(\cdot)$ is a Laplace functional, it must be continuous for the bounded pointwise convergence; Thus

$$L(f) = e^{-v_{r,t}(f)(x, \tau)} = e^{-v_{r,t}^{K(f)}(f)(x, \tau)} \quad (2.3.18)$$

since $v_{r,t}^{K(\cdot)}(\cdot)$ is also continuous for the bounded pointwise convergence (being a log-Laplace functional). But the set of functions for which (2.3.18) holds contains the continuous functions g with compact supports such that $K(g) < \infty$. Hence, since it is also a bp-closed set, it contains all the bounded non-negative functions g with $K(g) < \infty$. $\square$

Chapter 3

A complete characterization of a class of BMV processes

3.1 Introduction

Let $(E, \mathcal{E})$ be a measure space. Let $M_f(\mathcal{E})$ denote the set of finite measures on E . We say that an $M_f(\mathcal{E})$ -valued Markov process X is a branching measure-valued process (that we abbreviate as BMV process) if its transition probability $P_{r,\mu}$ satisfies the following formulae:

$$\begin{aligned}P_{r,\mu+\nu} &= P_{r,\mu} * P_{r,\nu} \\v_{r,t}f(\mu) &= \int_E \mu(dx) v_{r,t}f(\delta_x)\end{aligned}$$

where $*$ means the convolution of measures and

$$v_{r,t}f'(\mu) = -\log P_{r,\mu} e^{-\langle X_t, f \rangle}, \quad f \in bp\mathcal{E}, \quad \mu \in M_f(\mathcal{E}).$$

$v_{r,t}f$ is called the log-Laplace functional and is a semigroup, $v_{r,s}(v_{s,t}(f))(\mu) = v_{r,t}f(\mu)$, for $r < s < t$. $v_{r,t}$ uniquely characterizes the process X : if for every μ the operator $\{v_{r,t}(\cdot)\}_{0 \leq r \leq t < \infty}$ is a semigroup and the log-Laplace functional of some measure-valued process, and if for every $f \in pb\mathcal{E}$ the mapping $\mu \mapsto v_{r,t}(f)(\mu)$ is measurable, then there exists one and only one Markov measure valued process with log-Laplace functional $v_{r,t}$. A natural question arises: what is the most general form of this semigroup?

In all known cases, the log-Laplace functional satisfies an equation of the form

$$v_{r,t}f(\mu) + \psi_{r,t}(v(f))(\mu) = T_{r,t}f(\mu)$$

where $\psi_{r,t}$ is a (typically) nonlinear operator, and T is the semigroup generated by some E -valued Markov process.

A special class of BMV processes called superprocesses consists of the ones arising as limits of systems of indistinguishable particles which move according to the law of a Markov process ξ and which die at random times leaving a random number of offspring. The limit is taken as the mass of each individual particle tends to zero but the expected total mass of particles remains the same. A natural question is: what is the place of the superprocesses among the BMV processes? In fact it is conjectured that every BMV is a superprocess.

Since all the actually known superprocesses have their log-Laplace functionals characterized as the unique solution to some integral equation, a natural strategy to solve the above mentioned question would be: to first find a class of integral equations such that any BMV process has a log-Laplace functional which is a solution to one of those integral equations; second, show that for each of these integral equations, there exists a superprocess having its log-Laplace functional solving the equation.

This would solve at the same time all the questions we previously posed: it would fully characterize the semigroups of the BMV processes and show that every BMV is a superprocess. In an important paper, [DyKuS], Dynkin, Kuznetsov and Skorohod proved the following:

Theorem 3.1.16 *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a Hunt BMV process and $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$ be a Hunt process with state space $(E, \mathcal{E})$. Assume X and ξ are related by the formula¹*

$$P_{r,\mu} \langle X_t, f \rangle = \int_E \mu(dx) \pi_{r,x} f(\xi_t) \text{ for every } f \in b\mathcal{E}. \quad (3.1.1)$$

¹we say that ξ is the projection of X if the relation (3.1.1) is verified. If X is critical, that is if

and that

$$\sup_{r < t} P_{r,\mu} < 1, X_t >^2 < \infty, \quad (3.1.2)$$

then the log-Laplace equation for X has the form

$$v_t + (C_t + D_t)v_t = T_t f \quad (3.1.3)$$

where

$$\begin{aligned} v_t(r, x) &= v_{r,t}f(x) \\ T_t f(r, x) &= \pi_{r,x}f(\xi_t) \\ C_t v(r, x) &= \pi_{r,x} \int_r^t v_t(s, \xi_s)^2 Q(ds), \\ D_t v(r, x) &= \pi_{r,x} \int_r^t \int_{M_f} \mathcal{E} < v_t(s, \cdot), \mu > \ell(ds, d\mu) \end{aligned}$$

where

$$\mathcal{E}(u) := e^{-u} + u - 1$$

and where the branching characteristics Q and ℓ satisfy the (necessary) conditions:

N1 The additive functional Q is natural and $\sup_{x \in E} \pi_{r,x} Q(r, t] < \infty$ for all $r < t \in R_+$.²

N2 The measure-valued additive functional ℓ is natural and it satisfies the condition

$$\sup_{x \in E} \pi_{r,x} \int_r^t \int_{M_f} |\mu|^2 \ell(ds, d\mu) < \infty$$

where $|\mu| = \mu(E)$.

N3 The measure

$$\tilde{\ell}(A, B) = \int_{M_f} \mu(B) \ell(A, d\mu)$$

is dominated by the Lévy measure $N_\xi(ds, dy)$ of the process ξ in the following sense: for every positive measurable function $f^*(x, y)$ on $S \times E$,

$$\int 1_{\xi_{s-} \neq y} f^*(\xi_{s-}, y) \tilde{\ell}(ds, dy) \leq \int f^*(\xi_{s-}, y) N_\xi(ds, dy) \text{ a.s.}$$

$< X_t, 1 >$ is a martingale, the formula $p(r, x, t, B) := P_{r,x} X_t(B)$ defines the transition probability of a Markov process ξ and that process ξ is the projection of X .

²This is obviously the same thing as saying that $\pi_{r,\mu} Q(r, t] < \infty$ for all $r < t \in R_+$, and all $\mu \in M_f$ because if one could find a sequence $\{x_n\}$ such that $\pi_{r,x_n} Q(r, t] > 2^n$, then the measure $\mu(dx) := \sum 2^{-n} \delta_{x_n}(\cdot)$ is finite but $\pi_{r,\mu} Q(r, t] = \sum 2^{-n} \pi_{r,x_n} Q(r, t] = \infty$.

where $S := [0, \infty) \times E$ is furnished with the product σ -algebra.

Let Q and ℓ be continuous branching characteristics satisfying the conditions N1, N2 and N3 for the right process ξ . As shown in [Dy7, Chapter 7.3.4], if we put

$$K(I) := \int_I \int_{M_I} |\mu|^2 \ell(ds, d\mu) + Q(I)$$

then with the Radon-Nikodym Theorems (Theorem 1.7.8 and Theorem 1.7.9) we can find a measurable function $b^*(x)$ and a kernel $\ell_x^*(d\mu)$ satisfying

$$0 \leq b \leq 1 \text{ and } 0 \leq \int_{M_I} |\mu|^2 \ell_x^*(d\mu) \leq 1$$

and such that

$$\begin{aligned} Q(ds) &= b^*(\xi_s) K(ds) \\ \ell(ds, d\mu) &= \ell_{\xi_s}^*(d\mu) K(ds). \end{aligned}$$

Put

$$\begin{aligned} \Phi^c(s, x, z) &:= b^*(x) (z(x))^2 \\ \Phi^d(s, x, z) &:= \int_{M_I} \mathcal{E} \langle z, \mu \rangle \ell_x^*(d\mu). \\ \Phi(s, x, z) &:= \Phi^c(s, x, z) + \Phi^d(s, x, z) \end{aligned} \tag{3.1.4}$$

The equation (3.1.3) can be rewritten as³

$$v_{r,t}(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, v_{s,t}) K(ds). \tag{3.1.5}$$

When (3.1.5) has a log-Laplace functional solution, the corresponding superprocess will be called the (ξ, Φ, K) -superprocess. On the other hand, it will sometimes be more practical to refer to the (ξ, Φ, K) -superprocess as the (ξ, b, ℓ, K) -superprocess, which we will occasionally do.

Let $\mathcal{H}$ be the class of all the Hunt BMV processes X with Hunt projection ξ and continuous characteristics Q and ℓ for which the second moment condition (3.1.2) is verified. Necessary conditions are stated in Theorem 3.1.16 concerning the existence

³Note that for any measurable function g , since $k(ds)$ is continuous and therefore doesn't charge countable sets, we have that $g(s, \xi_{s-}) K(ds) = g(s, \xi_s) K(ds)$. This will be used from now on without special mention.

of the processes that belong to $\mathcal{H}$. For those conditions to be sufficient and therefore to fully characterize all the processes in $\mathcal{H}$, it is required to prove that a log-Laplace solution to (3.1.5) always exists. This will be established in this chapter. We prove:

Theorem 3.1.17 *Let $\xi = (\xi_s, \mathcal{F}, \pi_{r,x})$ be a right process. Let Φ be a branching mechanism given by (3.1.4) and $K(ds)$ a continuous additive functional of ξ . Suppose that the following conditions are satisfied:*

NS1 *the mapping b is measurable, $0 \leq b \leq 1$ and*

$$\sup_{x \in E} \pi_{r,x} \left(\int_r^t b^s(\xi_s) K(ds) \right) < \infty$$

for all $r < t \in \mathbb{R}_+$.

NS2 *the kernel $\ell_x^s(d\mu)$ satisfies $0 \leq \int_{M_f} |\mu|^2 \ell_x^s(d\mu) \leq 1$ and*

$$\sup_{x \in E} \pi_{r,x} \left(\int_r^t \int_{M_f} |\mu|^2 \ell_{\xi_s}^s(d\mu) K(ds) \right) < \infty.$$

NS3 *the measure*

$$\bar{\ell}(A, B) = \int_A \int_{M_f} \mu_{\xi_s}(B) \ell_{\xi_s}^s(d\mu) K(ds)$$

is dominated by the Levy measure of ξ in the sense that for every positive measurable function $f^s(x, y)$ on $S \times E$,

$$\int f^s(\xi_{s-}, y) \bar{\ell}(ds, dy) \leq \int f^s(\xi_{s-}, y) N_\xi(ds, dy) \text{ a.s.,}$$

where $\mu_x(A) := \mu(A - \{x\})$.

Under those conditions there exists a superprocess with log-Laplace functional $v_{r,t}(f)(x)$ solving (3.1.5).

The same problem has been treated in [Dy7]. Conditions NS1-NS3 together with the additional conditions Add1-Add4 below were proven to be sufficient for the existence of a log-Laplace solution to (3.1.5). A gap remained to be filled, and this is what we do here.

Add1 $\|\pi_{r,x}K(r,t)\|_\infty \rightarrow 0$ as $t - r \rightarrow s$ for every s

Add2 The functions

$$\int_{M_f} |\mu_x| \ell^s(x, d\mu), \quad \int_{0 \leq |\mu| \leq 1} \mu \{x\}^2 \ell^s(x, d\mu) \text{ and } \ell^s(x, \{\mu : |\mu| > 1\})$$

are bounded on every set $[0, T] \times E$.

Add3 The function

$$\int_{0 \leq |\mu| \leq \epsilon} [|\mu_x| + \mu \{x\}^2] \ell^s(x, d\mu)$$

tends to 0 uniformly on every set $[0, T] \times E$ as $\epsilon \rightarrow 0$.

Add4 The function

$$\ell^s(x, \{\mu : |\mu| > n\})$$

tends to 0 uniformly on every set $[0, T] \times E$ as $n \rightarrow \infty$.

In this chapter, we also study the question of convergence between superprocesses, a topic of obvious interest. We derive a criterion for the convergence of superprocesses. This criterion is then used in the next chapter of this thesis to solve the martingale problem for a large class of superprocesses.

Theorem 3.1.18 *Suppose that $\{(\xi, \Phi, k^n)\}_{n=1, \dots, \infty}$ is a sequence of superprocesses for which the conditions of Theorem 3.1.17 are satisfied. Denote by $P_{r,\mu}^n$ the transition probability of the superprocess with parameters (ξ, Φ, k^n) . If for every $r \leq s \leq t$ and every $x \in E$ we have that*

$$\pi_{r,x}k^n(s, t] \rightarrow \pi_{r,x}k^\infty(s, t] \quad (3.1.6)$$

then the superprocess (ξ, Φ, k^n) converges f.d.d. to (ξ, Φ, k^∞) in the sense that

$$P_{r,\mu}^n(f_1(X_{t_1}) \dots f_n(X_{t_n})) \rightarrow P_{r,\mu}(f_1(X_{t_1}) \dots f_n(X_{t_n})). \quad (3.1.7)$$

for all functions $f_1, \dots, f_n$ such that $\lim_{|\mu| \rightarrow \infty} f(\mu) = 0$, where $|\mu| = \mu(E)$.

An open problem mentioned in [Dy7] is the question of the existence of superprocess with discontinuous branching characteristics. The following theorem gives a criterion for their existence.

Theorem 3.1.19 *Let ξ be a time homogeneous right process. Let Φ be a branching mechanism of the form*

$$\Phi(s, x, z) := \int_{M_f} \mathcal{E} < \mu, z > \ell(d\mu)$$

Lets $k^n(ds), n = 1, 2, \dots$ be a sequence of continuous additive functionals $k^n(ds), n = 1, 2, \dots$ of ξ . Assume that the additive functionals are uniformly admissible in the sense that

$$\sup_{n, x} \pi_x k^n(r, t) \rightarrow 0 \text{ as } t - r \rightarrow 0$$

and that $k(ds)$ is also admissible but that it isn't continuous. If for every x , $k^n(r, t)$ converges weakly to k , π_x -almost surely, or if $\pi_x |k^n[0, t] - k[0, t]| \rightarrow 0$ for every x , then there exists a superprocess with parameters (ξ, Φ, k) .

3.2 Proof of our Main Results

Proof of Theorem 3.1.17:

Note: throughout all the paper we will assume that all the conditions of Theorem 3.1.17 are satisfied. Moreover, by putting

$$\begin{aligned} b^s(x) &:= \frac{b^s(x)}{b^s(x) + \int_{M_f} |\mu|^2 \ell_x^s(d\mu)} \\ \ell_x^s(d\mu) &:= \frac{\ell_x^s(d\mu)}{b^s(x) + \int_{M_f} |\mu|^2 \ell_x^s(d\mu)} \\ k(ds) &:= \left(b^s(\xi_s) + \int_{M_f} |\mu|^2 \ell_{\xi_s}^s(d\mu) \right) k(ds) \end{aligned}$$

it is always possible to assume that

$$\pi_{r, x} k(r, t) < \infty.$$

For now on, we assume that $k(ds)$ satisfies that property.

We first prove Theorem 3.1.17 under the following additional conditions:

C1 For every $x \in E$ and $s \geq 0$, $\int_{M_f} (|\mu| \vee 1) \ell_x^s(d\mu) < \infty$.

C2 For every $r, t \geq 0$, $\sup_{x \in E} \pi_{r,x} \int_r^t \int_{M_f} (|\mu| \vee 1) \ell_{\xi_s}^s(d\mu) K(ds) < \infty$

Part A: when ℓ and K satisfy the additional conditions C1 and C2

Step 1. The approximating Branching Particle System

The transition probabilities $P_{r,t,\mu}(\cdot)$ of the desired superprocess, will be obtained as limits of certain probability measures $Q_{r,t,\mu}^\epsilon(\cdot) \in M_1(B(M_f(E)))$.⁴ These probability measures are described in the following Lemma:

Lemma 3.2.19 *There exists probability measures $\{Q_{r,t,\mu}^\epsilon(\cdot) : \epsilon > 0; 0 \leq r \leq t < \infty; \mu \in M_f(E)\}$, having the (strong) branching property, such that if v^ϵ is defined by*

$$v_{r,t}^\epsilon(x) := Q_{r,t,\delta_x}^\epsilon(e^{-\langle \cdot, f \rangle})$$

for $f \in bpE$, then

$$v_{r,t}^\epsilon(x) = \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t)}}{\epsilon} \right) - \pi_{r,x} \int_r^t \Phi_\epsilon(s, \xi_s, v_{s,t}^\epsilon) K(ds) \quad (3.2.8)$$

where

$$\Phi_\epsilon(s, x, z) = b^s(x) (z(x))^2 + \int_{M_f} \mathcal{E}(\langle \mu, z \rangle) \ell_x^s(d\mu) + \epsilon z(x) \ell_x^s(M_f).$$

Proof: The measures $Q_{r,t,\mu}^\epsilon$ are tightly related to some BPS with killing and compensating mass creation. The underlying weak BPS will be obtained by an appropriate choice of a $(\xi, \phi^\epsilon, k^\epsilon)$ -weak BPS.

⁴ $M_f(E)$ is furnished with the topology of weak convergence. $B(M_f(E))$ denotes the Borel σ -algebra on $M_f(E)$. $M_1(B(M_f(E)))$ is the set of probability measures on $B(M_f(E))$.

Choice of the offsprings generating functional ϕ_ϵ and choice of K_ϵ

Let $\mathcal{E}(u) := e^{-u} - 1 + u$ and $\tilde{\mathcal{E}}_x(<\mu, z>) = \mathcal{E}(<\mu, z>) - <\mu_x, z>$. Define

$$\begin{aligned}\phi_\epsilon^c(s, x, z) &:= \frac{b^s(x)}{\epsilon} \left((1 - z(x))^2 + 2z(x) \right) = \frac{b^s(x)}{\epsilon} (1 + z^2(x)) \\ \phi_\epsilon^d(s, x, z) &:= \int_{M_f} \left(<\mu, z> + \epsilon + \epsilon \mathcal{E} <\mu, \frac{1-z}{\epsilon}> \right) \ell_x^s(d\mu) \\ &= \int_{M_f} \left(<\mu, 1> + \epsilon e^{-|\frac{z}{\epsilon}|} \sum_{n=0}^{\infty} \frac{\epsilon^{-n}}{n!} <\mu, z>^n \right) \ell_x^s(d\mu) \\ &= \int_{M_f} \left(\mu\{x\}z(x) + |\mu_x| + \epsilon + \epsilon \tilde{\mathcal{E}}_x <\mu, \frac{1-z}{\epsilon}> \right) \ell_x^s(d\mu)\end{aligned}\quad (3.2.9)$$

Note that

$$\phi_\epsilon^c(s, x, 1) + \phi_\epsilon^d(s, x, 1) = 2 \frac{b^s(x)}{\epsilon} + \int_{M_f} (|\mu| + \epsilon) \ell_x^s(d\mu) < \infty.$$

Hence if we define $\phi_\epsilon(s, x, z)$ by

$$\phi_\epsilon(s, x, z) := \left(\phi_\epsilon^c(s, x, 1) + \phi_\epsilon^d(s, x, 1) \right)^{-1} \left(\phi_\epsilon^c(s, x, z) + \phi_\epsilon^d(s, x, z) \right)$$

then $\phi_\epsilon(s, x, 1) = 1$, so ϕ_ϵ is an offsprings generating functional. Put

$$K_\epsilon(ds) := \left(\phi_\epsilon^c(s, \xi_s, 1) + \phi_\epsilon^d(s, \xi_s, 1) \right) K(ds).$$

Choice of an appropriate process with killing and mass creation

Let $\hat{\ell}^s(x, dy)$ be defined by

$$\hat{\ell}^s(x, dy) = \int_{M_f} \mu_x(dy) \ell^s(x, d\mu), \quad (3.2.10)$$

Choose⁵ a nearly Borel ρ such that $0 \leq \rho \leq 1$, $\rho^s(x, x) = 1$ and

$$\hat{\ell}^s(\xi_{s-}, dy) K(ds) = (1 - \rho^s(\xi_{s-}, y)) N_\xi(ds, dy). \quad (3.2.11)$$

This is possible because NS3 allows us to appeal to the Radon-Nikodym Theorem 1.7.8 and to find a Radon-Nikodym derivative $\gamma^s(\xi_{s-}, y)$ that we then rewrite in the form $\gamma^s(\xi_{s-}, y) = (1 - \rho^s(\xi_{s-}, y))$. The fact that N_ξ doesn't charge $\{(s, y) : \xi_{s-} = y\}$ allows us to assume $\rho^s(x, x) = 1$.

⁵The killing function ρ has been introduced by Dynkin in [Dy7.Th. 5.2.1].

Define the positive continuous additive functional C by

$$C(ds) := \int_{M_f} |\mu_{\xi_s}| \ell^s(\xi_s, d\mu) K(ds). \quad (3.2.12)$$

Note that

$$\begin{aligned} C(ds) &= \int_E \hat{\ell}(\xi_{s-}, dy) K(ds) \\ &= \int_E (1 - \rho^s(\xi_{s-}, y)) N_\xi(ds, dy). \end{aligned}$$

Therefore, according to Theorem 1.6.4, the $(\xi, \phi^\epsilon, k^\epsilon)$ -BPS with killing ρ and mass creation C exists.

The evolution equation of the $(\xi, \phi^\epsilon, k^\epsilon)$ -BPS with killing ρ and mass creation C

Let $\{Q_{r,t,\mu}^\epsilon(\cdot)\}$ be the $(\xi, \phi^\epsilon, k^\epsilon)$ -BPS with killing ρ and mass creation C . According to Theorem 1.6.4,

$$Q_{r,t,\mu}^\epsilon(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, \frac{1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)}{\epsilon} \rangle}$$

where

$$\tilde{v}_{r,t}^\epsilon(f)(x) = \pi_{r,x}^{\rho,C}(e^{-f(\xi_t)}) + \pi_{r,x}^{\rho,C} \int_r^t (\phi^\epsilon(s, \xi_s, \tilde{v}_{s,t}^\epsilon(f)) - \tilde{v}_{s,t}^\epsilon(f)(\xi_s)) k^\epsilon(ds).$$

We will show that

$$v_{r,t}^\epsilon(x) := \frac{1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)}{\epsilon}$$

is a solution to (3.2.8).

By definition of ϕ^ϵ and k^ϵ we have

$$\begin{aligned} \tilde{v}_{r,t}^\epsilon(\epsilon f)(x) &= \pi_{r,x}^{\rho,C}(e^{-\epsilon f(\xi_t)}) + \pi_{r,x}^{\rho,C} \int_r^t \frac{k^\epsilon(\xi_s)}{\epsilon} \left(1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)(\xi_s)\right)^2 K(ds) \\ &\quad + \pi_{r,x}^{\rho,C} \int_r^t \int_{M_f} |\mu_{\xi_s}| \left(1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)(\xi_s)\right) \ell_{\xi_s}^s(d\mu) K(ds) \\ &\quad + \pi_{r,x}^{\rho,C} \int_r^t \int_{M_f} \epsilon \left(1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)(\xi_s)\right) \ell_{\xi_s}^s(d\mu) K(ds) \\ &\quad + \pi_{r,x}^{\rho,C} \int_r^t \int_{M_f} \epsilon \tilde{\mathcal{E}}_{\xi_s} < \mu, \frac{1 - \tilde{v}_{s,t}^\epsilon(\epsilon f)}{\epsilon} > \ell_{\xi_s}^s(d\mu) K(ds). \end{aligned} \quad (3.2.13)$$

Using the equality $E_x < \mu, z \rangle = E < \mu, z \rangle - < \mu_x, z \rangle$, we get

$$\begin{aligned} v_{r,t}^\epsilon(x) &= \pi_{r,x}^{\rho,C}\left(\frac{1 - e^{-\epsilon f(\xi_t)}}{\epsilon}\right) - \pi_{r,x}^{\rho,C} \int_r^t \Phi_\epsilon(s, \xi_s, v_{s,t}^\epsilon) K(ds) \\ &\quad - \pi_{r,x}^{\rho,C} \int_r^t \int_{M_f} |\mu_{\xi_s}| v_{s,t}^\epsilon(\xi_s) \ell_{\xi_s}^s(d\mu) K(ds) \\ &\quad + \pi_{r,x}^{\rho,C} \int_r^t \int_{M_f} < \mu_{\xi_s}, v_{s,t}^\epsilon > \ell_{\xi_s}^s(d\mu) K(ds). \end{aligned} \quad (3.2.14)$$

Invoking the Combined Feynman-Kac formula, Corollary 1.5.3, we obtain

$$\begin{aligned}
v_{r,t}^\epsilon(x) = & \pi_{r,x}\left(\frac{1-e^{-\epsilon f(\xi_t)}}{\epsilon}\right) - \pi_{r,x} \int_r^t \Phi_\epsilon(s, \xi_s, v_{s,t}^\epsilon) K(ds) \\
& - \pi_{r,x} \int_r^t \int_{M_f} |\mu_{\xi_s}| v_{s,t}^\epsilon(\xi_s) \ell_{\xi_s}^\epsilon(d\mu) K(ds) \\
& + \pi_{r,x} \int_r^t \int_{M_f} \langle \mu_{\xi_s}, v_{s,t}^\epsilon \rangle \ell_{\xi_s}^\epsilon(d\mu) K(ds) \quad (3.2.15) \\
& + \pi_{r,x} \int_r^t v_{s,t}^\epsilon(\xi_s) C(ds) \\
& - \pi_{r,x} \int_r^t \int_E v_{s,t}^\epsilon(y) (1 - \rho^\epsilon(\xi_{s-}, y)) N_\epsilon(ds, dy).
\end{aligned}$$

By definition of ρ and C , this is exactly (3.2.8). $\square$

Notation: In the sequel, $v_{r,t}^\epsilon f(x)$ will always denote a solution to (3.2.8). Similarly, $v_{r,t} f(x)$ will always denote a solution to (3.1.5).

Step 5. Convergence of $v_{r,t}^\epsilon f(x)$ for small f

Lemma 3.2.19 *There exists a $\delta > 0$ such that $\|f\|_\infty \leq \delta$ implies that $v_{r,t}(x) := \lim_{\epsilon \rightarrow 0} v_{r,t}^\epsilon f(x)$ exists and is a solution to the evolution equation (3.1.5).*

Proof: Note first that

$$v_{r,t}^\epsilon f(x) \leq \pi_{r,x} \left(\frac{1 - e^{-\epsilon f(\xi_t)}}{\epsilon} \right) \leq \|f\|_\infty.$$

Let $\bar{v}_{r,t}(x) := \limsup_{\epsilon \rightarrow 0} v_{r,t}^\epsilon f(x)$ and $\underline{v}_{r,t}(x) := \liminf_{\epsilon \rightarrow 0} v_{r,t}^\epsilon f(x)$. We have

$$\begin{aligned}
\bar{v}_{r,t}(x) & \leq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, \bar{v}_{s,t}) K(ds) \\
\underline{v}_{r,t}(x) & \geq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, \underline{v}_{s,t}) K(ds).
\end{aligned} \quad (3.2.16)$$

Let us check now that Φ is locally uniformly Lipschitz. For any z_1, z_2 such that $0 \leq z_i \leq c$ we have

1.

$$|\mathcal{E}(\langle \mu, z_1 \rangle) - \mathcal{E}(\langle \mu, z_2 \rangle)| = |\langle \mu, z_1 - z_2 \rangle| (1 - e^{-\theta})$$

for some θ in the interval $[\langle \mu, z_1 \wedge z_2 \rangle, \langle \mu, z_1 \vee z_2 \rangle]$ and consequently, since $1 - e^{-\theta} \leq \theta$,

$$|\mathcal{E}(\langle \mu, z_1 \rangle) - \mathcal{E}(\langle \mu, z_2 \rangle)| \leq c \|\mu\|^2 \|z_1 - z_2\|_\infty.$$

2.

$$|z_1^2(x) - z_2^2(x)| = 2c \|z_1 - z_2\|_\infty$$

Therefore Φ is uniformly Lipschitz and we even have

$$\|\bar{v} - \underline{v}\|_\infty \leq \|\bar{v} - \underline{v}\|_\infty \|f\|_\infty \left\| \pi_{r,x} \int_r^t \left(2b^s(\xi_s) + \int_{M_f} |\mu|^2 \ell_{\xi_s}^*(d\mu) \right) K(ds) \right\|_\infty \quad (3.2.17)$$

and hence for f small enough, that is f such that

$$\|f\|_\infty \left\| \pi_{r,x} \int_r^t \left(2b^s(\xi_s) + \int_{M_f} |\mu|^2 \ell_{\xi_s}^*(d\mu) \right) K(ds) \right\|_\infty \leq \frac{1}{2}, \quad (3.2.18)$$

we can conclude from the relations (3.2.16) and (3.2.17) that $v_{r,t}(x) := \lim_{\epsilon \rightarrow 0} v'_{r,t} f(x)$ exists and is a solution to the evolution equation (3.1.5) according to Lebesgue theorem. $\square$

Remark 3.2.19 *The solution is unique (for small f) since if u and v are two solutions to (3.1.5) then they satisfy the relations (3.2.16) and therefore (3.2.17), from which the two solutions must coincide.*

Remark 3.2.19 *In a more abstract way we showed that there exists a function $c(s, x, \lambda)$ such that*

$$|\Phi(s, x, z_1) - \Phi(s, x, z_2)| \leq c(s, x, \|z_1\|_\infty \vee \|z_2\|_\infty) \|z_1 - z_2\|_\infty.$$

and

$$\sup_{x \in E} \pi_{r,x} \int_{(r,t]} c(s, \xi_s, \lambda) K(ds) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+$$

from which we concluded, using non negativity and monotonicity of Φ , that $\lim_{\epsilon \rightarrow 0} v'_{r,t} f(x)$ exists and solves (3.1.5).

Step 6. Continuity in f of $v_{r,t} f(x)$ for small f

Lemma 3.2.19 *There exists $\delta > 0$ such that over the set $\{f : 0 \leq f \leq \delta\}$, the mapping $f \mapsto v_{r,t}(f)(x)$ (is well defined) and continuous for the bounded pointwise convergence.*

Proof: Let f and g be small (as specified by (3.2.18)). If $f \leq g$ then $v_{r,t}f \leq v_{r,t}g$ because v is limit of objects having this property. Now without loss of generality assume $v_{r,t}f(x) \leq v_{r,t}g(x)$. Then $v_{r,t}g(x) \leq v_{r,t}(f + |g - f|)(x)$ for all r, t and x . So if $\Delta\Phi_{s,t}^{s,x}(f, g) := \Phi(s, x, v_{s,t}(f + |g - f|)) - \Phi(s, x, v_{s,t}(f))$ then $\Delta\Phi_{s,t}^{s,x}(f, g)$ is always non negative, and hence

$$\begin{aligned} 0 \leq v_{r,t}g(x) - v_{r,t}f(x) &\leq v_{r,t}(f + |g - f|)(x) - v_{r,t}f(x) \\ &= \pi_{r,x} |g(\xi_t) - f(\xi_t)| - \pi_{r,x} \int_r^t \Delta\Phi_{s,t}^{s,x}(f, g) K(ds) \\ &\leq \pi_{r,x} |g(\xi_t) - f(\xi_t)|. \end{aligned} \tag{3.2.19}$$

Therefore $v_{r,t}(\cdot)(x)$ is continuous under bounded pointwise convergence, and more generally, if $f_n(\xi_t)$ converges to $f(\xi_t)$ in $L^1(\pi_{r,x})$ then $v_{r,t}(f_n)(x) \rightarrow v_{r,t}(f)(x)$. $\square$

Step 7. Convergence of $v_{r,t}^c f(x)$ for every f

Up to now, given a measure space $(E, \mathcal{E})$ we have sometimes denoted by $M_f(E)$ the set of finite measures on $\mathcal{E}$. Obviously, $M_f(E)$ depends on the sigma algebra we put on E . The notation is therefore imprecise. In the next Proposition we want to avoid any ambiguity. We will therefore write $M_f(\mathcal{E})$ instead of $M_f(E)$. And if d is a metric on E , we will write $\mathcal{B}_d$ to design the Borel sets of E generated by the metric d . In the cases where ambiguity is still possible, we will write $\mathcal{B}_d(E)$ instead of $\mathcal{B}_d$. Moreover, we denote by $H_0^c(\mathcal{E})$ the set of all $\mathcal{E}$ -measurable functions such that $0 \leq f \leq c$.

By definition, for any metrizable Luzin space $(E, \mathcal{E})$, there exists $(\hat{E}, d)$, a compact metric space, such that $E \subseteq \hat{E}$ and $E \cap \mathcal{B}_d(\hat{E}) = \mathcal{E}$. The proof of Lemma (3.2.19) below is based on the following results:

(A) Let $(E, \mathcal{E})$ be a Radon space. The set $\mathfrak{K} := \{e^{-\langle \cdot, f \rangle} : f \in H_0^c(\mathcal{E})\}$ is separating.
(i.e. if $P^* e^{-\langle \cdot, f \rangle} = P e^{-\langle \cdot, f \rangle}$ for every $f \in H_0^c(\mathcal{E})$ then $P^* = P$.)

Proof: [Dy7, 3.3.4A] $\square$

(B) Assume L_{P_n} is the Laplace functional of some $M_f(\mathcal{E})$ -valued random variable where $(E, \mathcal{E})$ is a Radon space. Assume there exists c such that $L_{P_n}(f) \rightarrow L(f)$ for every f such that $0 \leq f \leq c$ and that L is continuous on that set. Then there exists a probability measure P on E such that $L = L_P$.

Proof: [Dy7, 3.3.4C] $\square$

(C) Suppose that a set $\mathfrak{R}$ is separating, and that a collection of distributions $\{P^n\}$ is tight. Then $\mathfrak{R}$ is convergence determining. (i.e. $P^n(f) \rightarrow P(f)$ for every $f \in \mathfrak{R}$ implies $P^n \Rightarrow P$).

Proof: [EtK, Lemma 3.4.3] $\square$

(D) Suppose that $(\hat{E}, d)$ is a compact metric space. Let $f \in b\mathcal{B}_d(\hat{E})$. Define d_f by

$$d_f(x, y) := d(x, y) + |f(x) - f(y)|. \quad (3.2.20)$$

Then $(\hat{E}, d_f)$ is also compact and $\mathcal{B}_d(\hat{E}) = \mathcal{B}_{d_f}(\hat{E})$.

Proof: See appendix. $\square$

Lemma 3.2.19 Assume L_{P_n} is the Laplace functional of some $M_f(\mathcal{E})$ -valued random variable where $(E, \mathcal{E})$ is a Radon space. Assume there exists c such that $L_{P_n}(f) \rightarrow L(f)$ for every measurable f such that $0 \leq f \leq c$ and that L is continuous on that set. Then there exists a probability measure P_∞ on $\mathcal{E}$ such that $L = L_{P_\infty}$ and $L_{P_n}(f) \rightarrow L_{P_\infty}(f)$ for every f in $pb\mathcal{E}$.

Proof: First let P_∞ be the distribution (given by (B)) such that $L = L_{P_\infty}$. Let $(\hat{E}, d)$ be a compact metric space, such that $E \subseteq \hat{E}$ and $E \cap \mathcal{B}_d(\hat{E}) = \mathcal{E}$. Choose an arbitrary $f_0 \in pb\mathcal{E}$, extend it to $\hat{E} - E$ by putting $f_0(x) = 0$ for every x in $\hat{E} - E$. Define d_{f_0} as in (3.2.20). The distributions P_n , $n = 1, \dots, \infty$, can obviously

be seen as distributions of $M_f(\mathcal{B}_{d_{f_0}}(\hat{E}))$ -valued random variables since from (D) $\mathcal{B}_{d_{f_0}}(\hat{E}) = \mathcal{B}_d(\hat{E})$. Moreover, $(\hat{E}, d_{f_0})$ is compact by (D) again. Hence $\{P_n\}$ is tight in the weak topology inherited from the metric d_{f_0} . Moreover the set $\mathfrak{R} := \{e^{-\langle \cdot, f \rangle} : f \in H_0^c(\mathcal{B}_d(\hat{E}))\}$ is separating by (A). Consequently by (C) $\mathfrak{R}$ is also convergence determining. And hence $P_n \Rightarrow P_\infty$ where " $\Rightarrow$ " refers to the weak convergence inherited from the metric d_{f_0} . But in that metric, f_0 is uniformly continuous. Hence $L_{P_n}(f_0) \rightarrow L_{P_\infty}(f_0)$ as desired. $\square$

Corollary 3.2.19 *For every $f \in pbE$, $v_{r,t}f(x) := \lim_{\epsilon \rightarrow 0} v_{r,t}^\epsilon f(x)$ exists and solves (3.1.5). Moreover, there exists $P_{r,t,\mu}(\cdot) \in M_1(B(M_f(E)))$ such that*

$$P_{r,t,\mu}(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, v_{r,t}f \rangle}$$

for every $f \in pbE$.

Proof: Immediate from Lemma 3.2.19, Lemma 3.2.19 and Lemma 3.2.19. $\square$

Step 8. θ -analytic uniqueness of v .

We will say that a function $v_{r,t}f(x)$ is θ -analytic if for fixed r, t and x , $v_{r,t}(\theta f)(x)$ is an analytic function of $\theta \in (0, \infty)$. Uniqueness of v for small f has already been shown in the remark of step 5.

Remark 3.2.19 *Let u and v be two θ -analytic solutions to the evolution equation (3.1.5). Because $u_{s,t}(\theta f)(x) = v_{s,t}(\theta f)(x)$ when θ is small enough, the equality must also hold for every $\theta \geq 0$ and in particular for $\theta = 1$.*

Step 9. Semigroup property of v .

Lemma 3.2.19 *The unique θ -analytic solution v to (3.1.5) is a semigroup.*

Proof: Calculating $v_{r,s}(v_{r,t}f)(x)$ from the definition of v we obtain

$$\begin{aligned} v_{r,s}(v_{r,t}f)(x) &= \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^s \Phi(u, \xi_u, v_{u,s}(v_{r,t}))K(du) \\ &\quad - \pi_{r,x} \int_s^t \Phi(u, \xi_u, v_{u,t})K(du). \end{aligned}$$

For small f , uniqueness of the solution implies that $v_{r,t}f(x) = v_{r,s}(v_{s,t}f)(x)$. For an arbitrary f , it is enough to show that $v_{r,t}(\theta f)(x)$ and $v_{r,s}(v_{s,t}(\theta f))$ are analytic in $\theta \in (0, \infty)$ to get the semigroup property of v , since for small θ we have $v_{r,t}(\theta f)(x) = v_{r,s}(v_{s,t}(\theta f))(x)$.

But it is well known that if P is a probability on (M_f, B_{M_f}) then the mapping $\theta \mapsto P(e^{-\theta \langle \cdot, f \rangle})$ is analytic in $\theta \in (0, \infty)$ for every $f \in pb(E)$. Therefore $\theta \mapsto e^{-v_{r,t}(\theta f)(x)}$ is analytic in $\theta \in (0, \infty)$.

At that point of the proof, all we have constructed is probability measures $P_{r,t,\mu}(\cdot)$ (arising as the weak limits of the laws $P_{r,t,\mu}^x(\cdot)$ of the particle system) and operators $v_{r,t}(f)(x)$ such that $P_{r,t,\mu}(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, v_{r,t}(f) \rangle}$. We conclude from the facts that the functions $e^{-\langle \cdot, f \rangle}$ form an algebra containing the constants and from the fact that $e^{-\langle \cdot, v_{r,t}(f) \rangle}$ is a measurable mapping, that the mapping $\mu \mapsto P_{r,t,\mu}(F)$ is measurable for every positive B_{M_f} -measurable F . That allows us to define the probability $P_{r,s,t,\delta_x}(F) := \int_{M_f} P_{s,t,\mu}(F) P_{r,s,\delta_x}(d\mu)$.

So we have

$$\begin{aligned} P_{r,s,t,\delta_x}(e^{-\theta \langle \cdot, f \rangle}) &= \int_{M_f} P_{s,t,\mu}(e^{-\theta \langle \cdot, f \rangle}) P_{r,s,\delta_x}(d\mu) \\ &= \int_{M_f} e^{-\langle \mu, v_{s,t}(\theta f) \rangle} P_{r,s,\delta_x}(d\mu) \\ &= e^{-v_{r,s}(v_{s,t}(\theta f))(x)} \end{aligned}$$

from which $v_{r,s}(v_{s,t}(\theta f))(x)$ has also to be analytic in $\theta \in (0, \infty)$. Consequently $v_{r,t}f(x) = v_{r,s}(v_{s,t}f)(x)$. $\square$

Step 10. Conclusion: We have constructed a family of probabilities $P_{r,t,\mu}(\cdot)$ on (M_f, B_{M_f}) such that $P_{r,t,\mu}(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, v_{r,t}(f) \rangle}$ and such that $v_{r,t}f(x) = v_{r,s}(v_{s,t}f)(x)$, where v is the unique θ -analytic solution to (3.1.5). Therefore this family is the transition probability of some Markov measure valued process and Part A of the Theorem is proven.

Interlude

At that point it is worthwhile to summarize the following part of the previous reasoning:

Theorem 3.2.20 Let $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$ be a Markov process with metrizable Luzin state space $(E, \mathcal{E})$. Assume the operators $v_{r,t}^\epsilon f(x)$ are the log-Laplace functionals of a family of laws $P_{r,t,\mu}^\epsilon$ (that is $P_{r,t,\mu}^\epsilon(e^{-\langle \cdot, f \rangle}) = e^{-\langle \mu, v_{r,t}^\epsilon f \rangle}$ for every $f \in b\mathcal{E}$). Let $\psi(s, x, z)$ be measurable, non negative and non decreasing in z such that

$$|\psi(s, x, z_1) - \psi(s, x, z_2)| \leq c(s, x, \|z_1\|_\infty \vee \|z_2\|_\infty) \|z_1 - z_2\|_\infty.$$

Let $K(ds)$ be a natural additive functional of ξ . Suppose that

$$\sup_{x \in E} \pi_{r,x} \int_{(r,t]} c(s, \xi_s, \lambda) K(ds) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+$$

Define

$$\begin{aligned} \bar{v}_{r,t} f(x) &= \lim_{\epsilon \rightarrow 0} \sup v_{r,t}^\epsilon f(x) \\ \underline{v}_{s,t} f(x) &= \lim_{\epsilon \rightarrow 0} \inf v_{r,t}^\epsilon f(x). \end{aligned}$$

If

$$\begin{aligned} \bar{v}_{r,t} f(x) &\leq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_{(r,t]} \psi(s, \xi_s, \underline{v}_{s,t}) K(ds) \\ \underline{v}_{s,t} f(x) &\geq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_{(r,t]} \psi(s, \xi_s, \bar{v}_{s,t}) K(ds) \end{aligned}$$

for every f in a set $H_0^c = \{f : 0 \leq f \leq c\}$, then there exists a Markov process

$X = (X_t, \mathfrak{F}, P_{r,\mu})$ with log-Laplace functional satisfying

$$v_{r,t} f(x) = \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_{(r,t]} \psi(s, \xi_s, v_{s,t}) K(ds)$$

and $v_{r,t} f(x) = \lim_{\epsilon \rightarrow 0} v_{r,t}^\epsilon f(x)$ for every $f \in pb\mathcal{E}$.

Proof: Follows from step 5 through 10. $\square$

Part B: When b, ℓ and K satisfy only NS1-NS3

Define $\ell^n(s, x, d\mu) = 1_{\{\frac{1}{n} \leq |\mu| \leq n\}} \ell(s, x, d\mu)$. Then

$$1 \geq \int_{M_f} |\mu|^2 \ell^n(s, x, d\mu) \geq \frac{1}{n} \int_{M_f} |\mu| \ell^n(s, x, d\mu) \geq \left(\frac{1}{n}\right)^2 \int_{M_f} \ell^n(s, x, d\mu) \geq 0$$

and therefore ℓ^n and K satisfy the additional condition C1 and C2 and there exists a superprocess with parameters (ξ, b, ℓ^n, K) . Let $v_{r,t}^n f(x)$ be the log-Laplace functional

of the superprocess (ξ, b, c, K) . From Fatou's Lemma and monotonicity of $\mathcal{E}$ we obtain for $c_{\sup}(x) := \limsup c'_j f(x)$ and $c_{\inf}(x) := \liminf c'_j f(x)$ that the relations

$$\begin{aligned} c_{\sup}(x) &\leq \pi_{\tau_1}(f(\xi_1)) = \pi_{\tau_1} \int \Phi(s, \xi_s, c'_j) K(ds) \\ c_{\inf}(x) &\geq \pi_{\tau_1}(f(\xi_1)) = \pi_{\tau_1} \int \Phi(s, \xi_s, c'_j) K(ds) \end{aligned}$$

are still verified and with Theorem 3.2.20, we obtain a unique “ θ -analytic” solution $c_{\sup} f(x) = c_{\inf}(x) = c_{\theta}(x)$ to the evolution equation (3.1.5) and that this solution is the log-Laplace functional of the (ξ, b, c, K) superprocess. $\square$

3.2.1 Convergence of superprocesses

We give here a criteria for the convergence of a sequence of superprocesses to another superprocess. In the next chapter this is used to solve the martingale problem for non-Feller superprocesses. We need first some Lemmas

Lemma 3.2.20 *Let $(\Omega, \mathbf{R}, P)$ be a probability space and $a_n(ds)$ be a sequence of random measures on $\mathbf{R}_+$ almost surely weakly converging to the continuous random measure $a(ds)$ in such a way that for every $t > 0$ $P\{|a_n([0, t]) - a([0, t])| > 0\} \rightarrow 0$. Assume also $P\{a([0, t])\} < \infty$ for every $t > 0$. Let $\phi_n(s)$ be a sequence of (possibly random) measurable non-negative uniformly bounded functions of s converging almost surely to $\phi(s)$. Then:*

(i) $P \int_0^t \phi_n(s) a_n(ds)$ converges to $P \int_0^t \phi(s) a(ds)$.

(ii) If in addition $a_n(ds)$ converges weakly to $a(ds)$ almost surely then $\int_0^t \phi_n(s) a_n(ds)$ converges almost surely to $\int_0^t \phi(s) a(ds)$ as well as in L^1 .

The proof of the above lemma is based on another lemma:

Lemma 3.2.20 *Let $(\Omega, \mathbf{R}, P)$ be a probability space and $a_n(ds)$ be a sequence of random measures on $\mathbf{R}_+$. If $a(ds)$ is such that $P\{a > 0, t\} = a([0, t]) < \infty$ for every $t \geq 0$, then there exists a subsequence a_{n_k} such that P -a.s. $a_{n_k} \rightarrow a$.*

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Proof: With the use of Cantor's diagonalization method one finds a subsequence a_{n_k} such that $P \{ |a_{n_k}[0, q] - a[0, q]| \rightarrow 0 \text{ for every rational } q \geq 0 \} = 1$. But then, because the mappings $t \mapsto a_n[0, t]$ are increasing, that implies that P -a.s. $a_{n_k} \Rightarrow a$. $\square$

Proof (of Lemma 3.2.20): we first prove (ii): Using elementary inequalities one gets

$$\begin{aligned} \overline{\lim} \left(\int_0^t \phi_n(s) a^n(ds) \right) &\leq \int_0^t \left(\overline{\lim} \phi_n(s) \right) a(ds) \\ \underline{\lim} \left(\int_0^t \phi_n(s) a^n(ds) \right) &\geq \int_0^t \left(\underline{\lim} \phi_n(s) \right) a(ds) \\ \overline{\lim} \left(P \int_0^t \phi_n(s) a^n(ds) \right) &\leq P \int_0^t \left(\overline{\lim} \phi_n(s) \right) a(ds) \\ \underline{\lim} \left(P \int_0^t \phi_n(s) a^n(ds) \right) &\geq P \int_0^t \left(\underline{\lim} \phi_n(s) \right) a(ds) \end{aligned} \quad (3.2.21)$$

therefore, if $\phi_n(s) \rightarrow \phi(s)$ we get

$$\begin{aligned} \overline{\lim} \left(\int_0^t \phi_n(s) a^n(ds) \right) &\leq \int_0^t \phi(s) a(ds) \leq \underline{\lim} \left(\int_0^t \phi_n(s) a^n(ds) \right) \\ \overline{\lim} \left(P \int_0^t \phi_n(s) a^n(ds) \right) &\leq P \int_0^t \phi(s) a(ds) \leq \underline{\lim} \left(P \int_0^t \phi_n(s) a^n(ds) \right). \end{aligned}$$

And the L^1 convergence follows from Lebesgue's Theorem. $\square$

Proof of (i): Assume that there exists a subsequence such that

$$P \left| \int_0^t \phi_n(s) a^n(ds) - \int_0^t \phi(s) a(ds) \right| \geq \epsilon \quad (3.2.22)$$

for some $\epsilon > 0$. Then with Lemma 3.2.20, one finds a subsequence of that subsequence for which (3.2.22) is impossible. $\square$

Proof of Theorem 3.1.18: (3.1.7) can be derived from the Stone-Weierstrass Theorem and the above convergence

$$P_{r,\mu}^n \left(e^{-\langle X_{t_1}, f_1 \rangle} \dots e^{-\langle X_{t_n}, f_n \rangle} \right) \rightarrow P_{r,\mu} \left(e^{-\langle X_{t_1}, f_1 \rangle} \dots e^{-\langle X_{t_n}, f_n \rangle} \right)$$

for every non negative measurable functions $f_1, \dots, f_n$. That last convergence can also be derived, by taking conditional expectation, from the convergence

$$P_{r,\mu}^n \left(e^{-\langle X_t, f \rangle} \right) \rightarrow P_{r,\mu} \left(e^{-\langle X_t, f \rangle} \right)$$

for every non negative measurable f . That last convergence is verified if and only if $v_{r,t}^n(f)(x) \rightarrow v_{r,t}(f)(x)$ for every $f \in b\mathcal{E}$. From (3.2.21) one gets

$$\begin{aligned}\bar{v}_{r,t}(x) &\leq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, \underline{v}_{s,t}) K(ds) \\ \underline{v}_{s,t}(x) &\geq \pi_{r,x}(f(\xi_t)) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, \bar{v}_{s,t}) K(ds)\end{aligned}$$

where $\bar{v}_{r,t}(x) := \limsup v_{r,t}^n f(x)$ and $\underline{v}_{r,t}(x) := \liminf v_{r,t}^n f(x)$. Consequently, a limit, $v_{r,t} f(x)$, exists for small f . By Lemma 3.2.20 $v_{r,t} f(x)$ solves (3.1.5) for small f . Using Lemma 3.2.19 $v_{r,t}^n f(x)$ converges to $v_{r,t} f(x)$ for every measurable non negative f . And another use of Lemma 3.2.20 shows that $v_{r,t} f(x)$ solves (3.1.5). $\square$

3.2.2 A criterion for the existence of a class of superprocesses with discontinuous characteristics

Proof of Theorem 3.1.19.

Let us now consider a branching mechanism of the form

$$\Phi(s, x, z) := \int_{M_f} \mathcal{E} < \mu, z > \ell(d\mu)$$

for some measure $\ell(d\mu)$ for which

$$\int_{M_f} |\mu|^2 \ell(d\mu) \leq 1.$$

Note that Φ is time and space homogeneous and that the log-Laplace functional of (ξ, Φ, k^n) can be written

$$v_{r,t}^n(x) = \pi_x f(\xi_{t-r}) - \pi_x \int_r^t \Phi(v_{s,t}^n) k^n(ds). \quad (3.2.23)$$

We now consider a sequence of continuous additive functional $k^n(ds), n = 1, 2, \dots$ of a time homogeneous right process ξ . Assume that the additive functionals are uniformly admissible in the sense that

$$\sup_{n,x} \pi_x k^n(r, t) \rightarrow 0 \text{ as } t - r \rightarrow 0.$$

Assume also that $k(ds)$ is admissible but that it isn't continuous. Suppose also that for every $x \in E$ and for every $0 \leq t$ (3.1.6) is verified.⁶ Pick $f \in pb\mathcal{E}$. Choose $x \in E$ and $0 < t$. Note that the log-Laplace functionals $v_{r,t}^n(x)$ of the superprocesses (ξ, Φ, k^n) are uniformly continuous uniformly in n : by this we mean that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|s - r| < \delta$ implies $\sup_n \|v_{s,t}^n - v_{r,t}^n\|_\infty < \epsilon$. This can be verified as follows:

$$\begin{aligned} \|v_{s,t}^n - v_{r,t}^n\|_\infty &\leq (f_{M_f} \|f\|_\infty |\mu|^2 \ell(d\mu)) \|\pi_x k^n(r, t)\|_\infty \\ &\leq \|f\|_\infty \|\pi_x k^n(r, t)\|_\infty. \end{aligned}$$

Therefore our claim is proved. But consequently, given $\epsilon > 0$, there exists a $\delta(\epsilon) =: \delta > 0$ such that $|s - r| < \delta$ implies

$$\sup_n |\Phi(v_{s,t}^n) - \Phi(v_{r,t}^n)| < \epsilon$$

and

$$\sup_{n,x} \pi_x k^n(r, s) \leq 1.$$

Choose $\epsilon > 0$. Take a partition $0 = t_1 < \dots < t_n = t$ of $[0, t]$ with mesh smaller than $\delta(\epsilon)$. Then we have that

$$v_{r,t}^n(x) = \pi_x f(\xi_{t-r}) - \sum_{r \leq t_i \leq t} \Phi(v_{t_i,t}^n) \pi_x \int_{t_i}^{t_{i+1}} k^n(ds) + R_n \quad (3.2.24)$$

where

$$\begin{aligned} R_n &= \pi_x \int_{t_i}^{t_{i+1}} \sup_n |\Phi(v_{t_i,t}^n) - \Phi(v_{s,t}^n)| k^n(ds) \\ &\leq \epsilon \pi_x \int_{t_i}^{t_{i+1}} k^n(ds) \\ &\leq \epsilon. \end{aligned} \quad (3.2.25)$$

Put $\bar{v}_{r,t}(x) := \limsup v_{r,t}^n f(x)$ and $\underline{v}_{r,t}(x) := \liminf v_{r,t}^n f(x)$. From (3.2.24) and (3.2.25) we get

$$\begin{aligned} \|\bar{v}_{r,t}(\cdot) - \underline{v}_{r,t}(\cdot)\|_\infty &\leq (\|f\|_\infty f_{M_f} |\mu|^2 \ell(d\mu) \|\pi_x k(r, t)\|_\infty) \|\bar{v}_{r,t}(\cdot) - \underline{v}_{r,t}(\cdot)\|_\infty + \epsilon \\ &\leq (\|f\|_\infty \|\pi_x k(r, t)\|_\infty) \|\bar{v}_{r,t}(\cdot) - \underline{v}_{r,t}(\cdot)\|_\infty + \epsilon \end{aligned}$$

⁶One can also suppose instead that $k^n \Rightarrow k$ π_x -almost surely for every x and then put $k_N^n(0, t) := k^n(0, t) \wedge N$ from which (3.1.6) is verified. The modifications are easy to do.

and for f small enough that gives

$$\|\bar{v}_{.,t}(\cdot) - \underline{v}_{.,t}(\cdot)\|_{\infty} \leq 2\epsilon.$$

Hence $v_{r,t}^n(f)(x)$ converges for f small enough, and Lebesgue's Theorem proves that its limit $v_{r,t}(f)(x)$ is a solution to

$$v_{r,t}(x) = \pi_x f(\xi_{t-r}) - \pi_x \int_r^t \Phi(v_{s,t}) k(ds). \quad (3.2.26)$$

Therefore we can appeal to Theorem 3.2.20 and get that $v_{r,t}^n(f)(x)$ converges for every f to the log-Laplace functional of some superprocess with log-Laplace functional solving (3.2.26). $\square$

3.3 Appendix

We prove here statement (D) made in step 4 of the proof of Theorem 3.1.17. For the convenience of the reader, we rewrite the statement here.

Proposition 3.3.20 *Let (E, d) be a compact metric space, let $\mathcal{B}_d$ be its Borel sigma algebra, and let $f \in b\mathcal{B}_d$. Define a new metric d_f by $d_f(x, y) := d(x, y) + |f(x) - f(y)|$. Then (E, d_f) is still a compact space and the Borel sigma algebra related to d_f , $\mathcal{B}_{d_f}$, is the same as $\mathcal{B}_d$.*

Suppose that $(\hat{E}, d)$ is a compact metric space. Let $f \in b\mathcal{B}_d(\hat{E})$. Define d_f by

$$d_f(s, y) := d(x, y) + |f(x) - f(y)|.$$

Then $(\hat{E}, d_f)$ is also a compact space and $\mathcal{B}_{d_f}(\hat{E}) = \mathcal{B}_d(\hat{E})$.

Proof: Recall first that for metric spaces the concept of compactness and sequential compactness are equivalent [Munkres, Th. 3.7.4], and that the Borel sigma algebra is the sigma algebra generated by the open balls.

To prove that (E, d_f) is compact we take an arbitrary sequence $\{x_n\}$ of points in E . Since (E, d) is compact, we can extract a convergent subsequence (still denoted

$\{x_n\}$ out of that sequence. Moreover the sequence $\{f(x_n)\}$ being bounded we can again extract another convergent subsequence out of $\{x_n\}$. That subsequence is convergent with respect to d_f , and that completes the proof.

To prove $\mathcal{B}_{d_f} = \mathcal{B}_d$, remark first that the mappings

$$x \mapsto d(x, y) + |f(x) - f(y)|$$

are measurable, and hence, so are the sets

$$\{x : d(x, y) + |f(x) - f(y)| \leq r\}.$$

Thus the d_f -balls belong to $\mathcal{B}_d$ and $\mathcal{B}_{d_f} \subseteq \mathcal{B}_d$. Since $d_f \geq d$, the d_f -topology is finer than the d -topology, and therefore $\mathcal{B}_{d_f} \supseteq \mathcal{B}_d$. $\square$

3.4 Martingale problem: statement

Let us define once and for all the following objects: let $T > 0$ be a constant, we consider only our processes during the time interval $[0, T]$. Let $\xi = (\xi_t, \mathcal{F}, \pi_x)$ be a (time homogeneous) Hunt process on a metrizable Luzin space $(E, \mathcal{E})$. It is always the case that π_x induces a positive contraction semigroup $S_t(f)(x)$ on $b\mathcal{E}$ defined by:

$$S_t(f)(x) := \pi_x f(\xi_t).$$

Let L be the set of bounded measurable functions f such that $S_t(f)(x)$ is *strongly* continuous, that is $\|S_t(f)(\cdot) - S_{t+h}(f)(\cdot)\|_\infty \rightarrow 0$ as $h \rightarrow 0$. For example, the function $f(x) \equiv 1$ always belongs to L . Let $\{U^\alpha\}_{\alpha>0}$ be the resolvent of ξ , that is,

$$U^\alpha(f)(x) := \int_0^\infty e^{-\alpha t} S_t(f)(x) dt.$$

If A is the (strong infinitesimal) generator of S_t and ${}^w A$ its weak generator, then obviously $\mathcal{D}(A) \subseteq \mathcal{D}({}^w A)$. Moreover, $\mathcal{D}({}^w A) \subseteq L$ since for every $f \in \mathcal{D}({}^w A)$

$$f(\xi_t) - f(\xi_0) = \int_0^t {}^w A f(\xi_s) ds$$

is a π_x -martingale, and hence

$$\begin{aligned} \|S_t(f)(\cdot) - S_{t+h}(f)(\cdot)\|_\infty &= \|\pi.f(\xi_t) - \pi.f(\xi_{t+h})\|_\infty \\ &= \left\| \pi. \int_t^{t+h} {}^W A f(\xi_s) ds \right\|_\infty \\ &\leq \int_t^{t+h} \left\| {}^W A f \right\|_\infty ds \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

From the Hille-Yosida Theorem, we get that $\mathcal{D}(A)$ is dense in $(L, \|\cdot\|_\infty)$. Also according to [F1, p.355] $\mathcal{D}({}^W A)$ is dense in $b\mathcal{E}$, when $b\mathcal{E}$ is furnished with the topology of bounded pointwise convergence. The same must be true for $\mathcal{D}(A)$ since $\mathcal{D}(A)$ is dense in L which contains $\mathcal{D}({}^W A)$. We assume that L is an algebra. The Hunt property of ξ is preserved by a passage to the topology generated by metric $\Delta(x, y) = d(x, y) + \rho(x, y)$ (cf. appendix). Throughout this chapter we will always assume that E is furnished with the metric Δ . Note that Δ generates the smallest topology containing the topologies generated by d and ρ .

Let $k(ds)$ be a continuous non negative additive functional of ξ satisfying the condition

$$h_t^r(x) := \pi_x k(r, t) \rightarrow 0 \text{ uniformly in } x \text{ as } t - r \rightarrow 0. \quad (3.4.27)$$

(Note that this is equivalent to the so called “admissibility condition” in [DyKuS] according to [Dy7, Lemma 3.3.1]) We assume that $h_t^r(\cdot) \in L$ for every r, t . Let also $b(x)$ and $\ell(x, d\mu)$ be respectively a measurable function and a kernel satisfying the following conditions:

$$0 \leq b(x) \leq 1, \quad 0 \leq \int_0^\infty u^2 \ell(x, du) \leq 1. \quad (3.4.28)$$

Throughout this paper we pose

$$\Phi(x, f) = b(x)f^2(x) + \int_0^\infty \mathcal{E}(uf) \ell(x, du) \quad (3.4.29)$$

where $\mathcal{E}(z) = e^{-z} + z - 1$. In the same spirit as [EKaRC], we assume that for every $\phi \in \mathcal{D}(A)$, $\Phi(x, \phi) \in L$. We will state and prove the martingale problem for the superprocess with parameters (ξ, Φ, k) .

Notation: All measure valued processes considered in this chapter will be canonical càdlàg processes and the triple $X = (X_t, \mathfrak{F}, P_{r,\mu})$ will always denote such process. Indeed:

- X_t denotes the mapping from any of the spaces $D_{[r,\infty)}(M_f)$, $r \geq t$, to the space M_f , which is defined by

$$X_t(\omega) = \omega_t, \text{ for } \omega \in D_{[r,\infty)}(M_f)$$

- $\mathfrak{F}$ denotes the collection of filtrations $\{\mathfrak{F}_t^r\}_{t \in [r,\infty)}$ defined by

$$\mathfrak{F}_t^r = \bigcap_{\epsilon > 0} \sigma(X_s : r \leq s \leq t + \epsilon)^*$$

where the symbol $*$ denotes the universal completion of σ -algebra. i.e. the completion with respect to all finite measures.

- $P_{r,\mu}$ denotes a distribution on $D_{[r,\infty)}(M_f)$ and we *always* assume that $P_{r,\mu}(X_r = \mu) = 1$.

The reader will note that in our proofs, the processes could have simply been chosen to be right continuous processes in the (weak) original topology with left limits in the (weak) Ray topology instead of being canonical càdlàg measure valued processes. But this last assumption appeared more natural to us.

We will later need the following definition.

Definition 3.4.20 *Let $(\Omega, \mathfrak{R}, P)$ be a probability space and $\{a^n(ds)\}_{n=1,\dots,\infty}$ a random measures defined on a subset I of the real line. Then we will say that $a^n(ds)$ **converges in $L^1(P)$** to $a^\infty(ds)$ if for every $t \geq 0$, $a^n(I \cap [0, t])$ converges to $a^\infty(I \cap [0, t])$ in $L^1(P)$. The convergence is said to be strong if t can be replaced by any finite stopping time τ .*

In general, a martingale problem can be formulated in the following way: first, to any (canonical càdlàg) process X , a real valued process $t \mapsto (M_G^r)_t$, $t \geq r$, is defined up to $P_{r,\mu}$ -indistinguishability, for every function G in a set S . A càdlàg process $X = (X_t, \mathfrak{F}, P_{r,\mu})$ is said to be a **solution to the martingale problem** if the processes $(M_G^r)_t$ are $P_{r,\mu}$ -martingales for every G in S . The martingale problem $((M_G^r), S)$ is said to be **well posed** if there exists one and only one solution to the martingale problem.

We see a martingale problem as a “test” which characterizes a process. Pick a process $X = (X_t, \mathfrak{F}, P_{r,\mu})$. The test is:

- For every $G \in S$, check if the process $(M_G^r)_t$ is a $P_{r,\mu}$ -martingale.

If the test is a succes, X is a solution to the $((M_G^r), S)$ martingale problem.

In the above test, the order in which the processes $t \mapsto (M_G^r)_t$ are tested has no importance. We introduce here a slight modification to this procedure. Let $S = S_1 \cup S_2$ where S_1 and S_2 are two disjoint sets. Our new test goes like this:

1. Test whether or not X is a solution to the $((M_G^r), S_1)$ -martingale problem
2. If the test has not failed continue, otherwise, stop.
3. Test whether or not X is a solution to the $((M_G^r), S_2)$ -martingale problem

The difference between the two “tests” is the following: it is possible that given an arbitrary $X = (X_t, \mathfrak{F}, P_{r,\mu})$, the processes $(M_G^r)_t$ are not well defined up to $P_{r,\mu}$ -indistinguishability. But on the other hand, all the processes $(M_G^r)_t$ are well defined for every $G \in S_1$, and given that X is a solution to the $((M_G^r), S_1)$ -martingale problem, then, for every $G \in S_2$ all the processes $(M_G^r)_t$ are well defined as well up to $P_{r,\mu}$ -indistinguishability.

In that case, the (non well posed) martingale problem $((M_G^r), S_1)$ will be called the **partial martingale problem**. A solution to the partial martingale problem will be

called a solution to the full martingale problem if it is a solution to the $((M_G^r), S_2)$ -martingale problem.

Let us now rigorously introduce those concepts.

Definition 3.4.20 (partial-martingale problem for A) *The process $X = (X_t, \mathfrak{F}, P_{r,\mu})$ will be said to be a solution to the partial martingale problem for A if for every $\phi \in \mathcal{D}(A)$*

$$\langle X_t, \phi \rangle - \langle X_r, \phi \rangle - \int_r^t \langle X_s, A\phi \rangle ds \quad (3.4.30)$$

is a $P_{r,\mu}$ -martingale for $s \in [r, t]$.

The full martingale problem requires for its statement the notion of a lifting of an additive functional:

Definition 3.4.20 (Lifting) *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a (canonical càdlàg M_f -valued) process and let $a(ds)$ be an additive functional of ξ . A natural⁷ right continuous additive functional $A(ds)$ of X will be called a **lifting** of $a(ds)$ if for every $t \geq r$*

$$A(r, s] + \langle X_s, \pi, a(s, t] \rangle$$

is a $P_{r,\mu}$ -martingale for $s \in [r, t]$.

The following theorem (which will be proved in a further section) guaranties the existence and uniqueness of liftings for every solution $X = (X_t, \mathfrak{F}, P_{r,\mu})$ to the partial martingale problem.

Theorem 3.4.21 (liftings existence and uniqueness) *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem for A. Then for every additive functional $a(ds)$ of ξ satisfying (3.4.27), there exists a unique lifting $A(ds)$ of X .*

⁷An additive functional $A(ds)$ is said to be natural if the processes $t \mapsto A(r, t]$ are predictable.

Notation: Let f be a progressively measurable bounded function. Then the additive functional $\Phi(\xi_s, f(s, \xi_s))k(ds)$ satisfies (3.4.27), and we will denote by $K^{\Phi(f)dk}(ds)$ the lifting of $\Phi(\xi_s, f(s, \xi_s))k(ds)$.

The next Theorem characterizes the (ξ, Φ, k) -superprocess in terms of a martingale problem. Theorem 3.4.21, is needed for the statement of this martingale problem to be well defined. This theorem is the central result of this chapter.

Theorem 3.4.22 *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem, and let $P_{r,\mu}^{(\xi, \Phi, k)}$ be the distribution of the (ξ, Φ, k) -superprocess. If*

$$\begin{aligned} \exp(-\langle X_t, \phi \rangle) + \int_r^t \exp(-\langle X_s, \phi \rangle) \langle X_s, A\phi \rangle ds \\ - \int_r^t \exp(-\langle X_s, \phi \rangle) K^{\Phi(\phi)dk}(ds) \end{aligned} \quad (3.4.31)$$

is a $P_{r,\mu}$ -martingale for every $\phi \in \mathcal{D}(A)$, then $P_{r,\mu} = P_{r,\mu}^{(\xi, \Phi, k)}$. Conversely, the (ξ, Φ, k) -superprocess is solution to the partial martingale problem and (3.4.31) is a $P_{r,\mu}^{(\xi, \Phi, k)}$ -martingale.

Definition 3.4.22 *A solution $X = (X_t, \mathfrak{F}, P_{r,\mu})$ to the partial martingale for A which is such that (3.4.31) is a $P_{r,\mu}$ -martingale will be called a solution to the full martingale problem for (A, Φ, k) .*

Remark: In this work, we will consider only the partial martingale problem for A , and only full martingale problem for (A, Φ, k) . Thus we may sometimes refer to them simply as the *partial martingale problem* and the *full martingale problem*.

We will thus prove in this chapter that the martingale problem for the (ξ, Φ, k) -superprocess is well posed in the sense that there exists one and only one solution to the partial martingale problem which is also a solution to the full martingale problem.

3.5 The partial martingale problem

Several consequences can be derived from the knowledge that a process $X = (X_t, \mathfrak{F}, P_{r,\mu})$ is a solution to the partial martingale problem. In this section we investigate some of the properties shared by all the solutions to the partial martingale problem. One of those consequences is that for such processes, liftings exist, and therefore, the full martingale problem can be stated. Moreover, as we will see, the liftings share also some important regularity properties.

3.5.1 Path regularity and Fitzsimmons Projection Theorem

We first establish some path regularity shared by all the solutions to the partial martingale problem. We then state Fitzsimmons projection theorem which will be used at several places in this chapter.

Theorem 3.5.23 (path regularity of the solutions to the partial martingale problem) *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem. Then*

- (i) *the processes $\langle X_t, f \rangle$ are optional and belong to class (D) for every $f \in b\mathcal{E}$.*
- (ii) *the processes $M_t(\phi)$ given by (3.4.30) are indistinguishable from càdlàg processes for every $\phi \in \mathcal{D}(A)$*

Proof of (i): Since $f \equiv 1$ is a uniformly continuous function we have that $\langle X_t, 1 \rangle$ is càdlàg. Since $f \equiv 1 \in \mathcal{D}(A)$, it is also a martingale (from (3.4.30)). We get from Conseq6 that it belongs to class (D). The fact that this is therefore true for every $f \in bE$ is immediate. Because X_t is right continuous, it is itself optional, and $F(X_t)$ is optional for every real valued measurable mapping F and in particular, $\langle X_t, f \rangle$ is optional. $\square$

Proof of (ii) Note⁸ that for $g \in \mathcal{R}$, and $f = U^1 g \in \mathcal{D}({}^W A) \cap \mathcal{R}$, we have that ${}^W A f = f - g$. Indeed, $U^1(b\mathcal{E}) \supseteq \mathcal{D}({}^W A)$, and for every $f \in \mathcal{D}({}^W A)$, there exists $g \in b\mathcal{E}$ such that $f = U^1 g$ and

$${}^W A f = f - g.$$

Let the process $M'_t(g)$ be defined by

$$M'_t(g) := \langle X_t, U^1(g) \rangle - \langle X_r, U^1(g) \rangle - \int_r^t \langle X_s, U^1(g) - g \rangle ds.$$

When $g \in \mathcal{R}$, the process $M'_t(g)$ is a càdlàg (martingale), since $t \mapsto \langle X_t, U^1(g) \rangle$ is a càdlàg process ($U^1(g)$ being in $\mathcal{R}$). Therefore, for every bounded r -stopping⁹ time τ , we have that

$$P_{r,\mu} M'_\tau(g) = 0.$$

A monotone class argument shows that this is true for every $g \in b\mathcal{E}$. In particular, given $f \in \mathcal{D}({}^W A)$, this is true for functions $g \in b\mathcal{E}$ such that $f = U^1 g$ and $Af = U^1(g) - g$, and we obtain that

$$P_{r,\mu} M_\tau(f) = P_{r,\mu} M'_\tau(g) = 0.$$

But then, for every sequence of bounded decreasing r -stopping times $\tau_n \searrow \tau$, we have that

$$\lim_{n \rightarrow \infty} P_{r,\mu} M_{\tau_n}(f) = 0 = P_{r,\mu} M_\tau(f).$$

Because the process is optional, we conclude, using Conseq4, that it is almost surely right continuous (and therefore also càdlàg by a similar reasoning invoking Conseq1).

□

The following result is an important theorem due to Fitzsimmons [F2, Corollary 2.8 and remark to the Corollary]. It establishes -via the partial martingale problem-

⁸See [F1, p.355] for more details.

⁹Let's recall that this is simply a stopping time with respect to the filtration $\{\mathfrak{F}_t\}_{t \in [r, \infty)}$.

a link between the BMV-processes and their projections, the particles. Note that [F2, Remark to Corollary 2.8] is more general than Theorem 3.5.24 (iii); indeed, we preferred to put the emphasis on the case treated here, since it is the one we will need to use (in the proof of the existence of a solution to the full martingale problem).

Theorem 3.5.24 (Fitzsimmons Projection Theorem) *Let $X = (X_t, \mathfrak{Z}, P_{r,\mu})$ be a solution to the partial martingale problem and let*

$$S_{t-\tau}(f)(x) =: S_t^r(f)(x) = \pi_{r,x}f(\xi_t) := \pi_x f(\xi_{t-\tau})$$

be the semigroup of the time homogenous Hunt process ξ . We have:

(i) if τ is a bounded r -stopping time then for all $f \in b\mathcal{E}$

$$P_{r,\mu}^{\mathfrak{Z}_\tau} \langle X_{\tau+t}, f \rangle = \langle X_\tau, S_t f \rangle, \text{ for every } t \geq 0$$

where $P_{r,\mu}^{\mathfrak{Z}_\tau}$ denotes the conditional expectation with respect to $\mathfrak{Z}_\tau$.

(ii) if τ is a predictable bounded r -stopping time strictly bigger than zero then

$$P_{r,\mu}^{\mathfrak{Z}_{\tau-}} \langle X_{\tau+t}, f \rangle = \langle X_{\tau-}, S_t f \rangle, \text{ for every } t \geq 0$$

where $P_{r,\mu}^{\mathfrak{Z}_{\tau-}}$ denotes the conditional expectation with respect to $\mathfrak{Z}_{\tau-}$.

(iii) for every $\phi \in \mathcal{D}(A)$

$$P_{r,\mu} \left\{ \lim_{s \nearrow t} \langle X_s, \phi \rangle = \langle X_{t-}, \phi \rangle \text{ for every } t > r \right\} = 1$$

Proof: For (i) and (ii) we refer to [F2, Corollary 2.8] for the proof. For (iii), first choose $f \in \mathcal{D}(A)$. Then $M_t(f)$ is a càdlàg martingale according to part Theorem 3.5.23. Therefore, from the optional sampling theorem, we have that for every bounded r -stopping time τ ,

$$P_{r,\mu} M_\tau(f) = 0.$$

Also, the process

$$t \mapsto \int_r^t \langle X_s, Af \rangle ds$$

is left continuous and belongs to class (D). From this we can derive that for any increasing sequence of r -stopping times $\tau_n \nearrow \tau$, with $\tau_n < \tau$ we have that

$$\lim_{n \rightarrow \infty} P_{r,\mu} \langle X_{\tau_n}, f \rangle = P_{r,\mu} \langle X_\tau, f \rangle. \quad (3.5.32)$$

But

$$P_{r,\mu} \langle X_\tau, f \rangle = P_{r,\mu} \left(P_{r,\mu}^{\mathfrak{A}_{\tau-}} (\langle X_\tau, f \rangle) \right) = P_{r,\mu} \langle X_{\tau-}, f \rangle.$$

Thus, if $L(t)$ denotes the process

$$L(t) := \lim_{s \nearrow t} \langle X_s, f \rangle$$

then $L(t)$ is predictable and from (3.5.32) and Lebesgue's theorem we get that for every predictable r -stopping time τ

$$P_{r,\mu} L(\tau) = P_{r,\mu} \lim_{s \nearrow \tau} \langle X_s, f \rangle = P_{r,\mu} \lim_{n \rightarrow \infty} \langle X_{\tau_n}, f \rangle = P_{r,\mu} \langle X_{\tau-}, f \rangle.$$

Thus, by Conseq1, we can conclude that the process $L(t)$ is indistinguishable from the process $\langle X_{t-}, f \rangle$. $\square$

3.5.2 Existence of liftings

The following Lemma will be used in the proof of existence and uniqueness for liftings.

Lemma 3.5.24 *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem for A . Then for every $f \in b\mathcal{E}$, and every bounded r -stopping time τ we have that*

$$P_{r,\mu} \langle X_\tau, S_{T-\tau} f \rangle = \langle \mu, S_{T-\tau} f \rangle$$

Proof: If $f \in \mathcal{D}(A)$, then $S_t f \in \mathcal{D}(A)$ for every $t \geq 0$ and therefore $\langle X_t, f \rangle$ is a càdlàg optional process. Form Theorem 3.5.24 (i), we have that

$$P_{r,\mu}^{\mathfrak{A}_t} \langle X_{t+s}, S_{T-t-s}(f) \rangle = \langle X_t, S_{T-t}(f) \rangle$$

and hence the process $\langle X_t, S_{T-t}f \rangle$ is a martingale. By Conseq7 it admits a right continuous modification, $M_t^+(f)$. Note that $\langle X_t, S_{T-t}f \rangle$ and $M_t^+(f)$ belong to (D) since they are bounded by a constant times $\langle X_t, 1 \rangle$ which belongs to class (D) according to part (i) of Theorem 3.5.23. Let τ be a bounded r -stopping time and $\{\tau_n\}$ be a sequence of r -stopping times taking only *finitely* many different values and decreasing uniformly to τ . From the optional sampling theorem we get that

$$P_{r,\mu} \langle X_{\tau_n}, S_{T-\tau_n}f \rangle = \langle \mu, S_{T-\tau}f \rangle$$

and

$$P_{r,\mu} M_\tau^+ = P_{r,\mu} M_\tau^+ = P_{r,\mu} \langle X_\tau, S_{T-\tau}f \rangle = \langle \mu, S_{T-\tau}f \rangle.$$

Since $S_{\tau_n}f \in \mathcal{D}(A)$, the processes

$$t \mapsto \langle X_t, S_{T-\tau_n}f \rangle$$

are right continuous and belong to class (D) (According to Theorem 3.5.23). Thus we have for a fixed N ,

$$\lim_{n \rightarrow \infty} P_{r,\mu} \langle X_{\tau_n}, S_{T-\tau_n}f \rangle = P_{r,\mu} \langle X_\tau, S_{T-\tau_N}f \rangle.$$

On the other hand, since τ_n converges uniformly to τ , we conclude that

$$\lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \langle X_{\tau_n}, |S_{T-\tau_N}f - S_{T-\tau_n}f| \rangle \leq \lim_{N \rightarrow \infty} \sup_{n \geq N} \|S_{T-\tau_N}f - S_{T-\tau_n}f\|_\infty \langle \dot{X}_\tau, 1 \rangle = 0$$

and moreover

$$\lim_{N \rightarrow \infty} \langle X_\tau, S_{T-\tau_N}f \rangle = \langle X_\tau, S_{T-\tau}f \rangle.$$

Consequently, since $\langle X_t, 1 \rangle$ belongs to class (D) , we obtain that

$$\begin{aligned} P_{r,\mu} M_\tau^+ &= \langle \mu, S_{T-\tau}f \rangle \\ &= \lim_{n \rightarrow \infty} P_{r,\mu} \langle X_{\tau_n}, S_{T-\tau_n}f \rangle \\ &= \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} P_{r,\mu} \langle X_{\tau_n}, S_{T-\tau_N}f \rangle \\ &= \lim_{N \rightarrow \infty} P_{r,\mu} \langle X_\tau, S_{T-\tau_N}f \rangle \\ &= P_{r,\mu} \langle X_\tau, S_{T-\tau}f \rangle. \end{aligned}$$

That proves, according to Conseq5, that the optional martingale $\langle X_t, S_{T-t}f \rangle$ is indistinguishable from $M_t^+(f)$ and is therefore almost surely càdlàg. The extension to arbitrary $f \in b\mathcal{E}$ follows from the fact that $D(A)$ is dense, for the bounded pointwise convergence, in $b\mathcal{E}$. $\square$

Consider now the function $h_T^r(x) := \pi_x a(r, T]$ which is called the **characteristic** of the additive functional $a(ds)$. Assume that h is bounded. Note that by Markov property, for every $0 \leq s \leq r$ we have

$$S_s(h_T^r)(x) = \pi_x(\pi_{\xi_s} a(r, T]) = h_T^r(x).$$

If now $r \leq s \leq 2r$, then we get

$$S_s(h_T^r)(x) = S_r(S_{s-r}(h_T^r))(x) = S_r(h_T^r)(x) = h_T^r(x).$$

Thus a simple induction argument gives that

$$S_t(h_T^r)(x) = h_T^r(x) \text{ for every } t \geq 0.$$

We use this in the following proof of the existence and uniqueness of liftings for the solutions to the partial martingale problem.

Proof of Theorem 3.4.21: According to Theorem 3.5.24 (i) we have

$$P_{r,\mu}^{\mathfrak{A}_t} \langle X_{t+s}, f \rangle = \langle X_t, S_s f \rangle, \text{ } P_{r,\mu}\text{-almost surely for every } f \in b\mathcal{E}$$

Consequently,

$$P_{r,\mu}^{\mathfrak{A}_t} \langle X_{t+s}, h_T^{t+s} \rangle = \langle X_t, S_s(h_T^{t+s}) \rangle = \langle X_t, h_T^{t+s'} \rangle \leq \langle X_t, h_T^t \rangle$$

and therefore the process $x_t := \langle X_t, h_T^t \rangle$ is a supermartingale.

Choose t' in $[r, T]$. Then the process $\langle X_t, h_T^{t'} \rangle$ is optional and belongs to class (D) according to Theorem 3.5.23 (i). Moreover, according to Lemma 3.5.24, we have that for any bounded r -stopping time τ

$$\begin{aligned} P_{r,\mu} \langle X_\tau, h_T^{\tau'} \rangle &= P_{r,\mu} \langle X_\tau, S_{T-\tau}(h_T^{t'}) \rangle \\ &= \langle \mu, S_{T-\tau}(h_T^{t'}) \rangle \\ &= \langle \mu, h_T^{t'} \rangle \end{aligned}$$

which proves, according to Conseq4, that the process $\langle X_t, h_T^{t'} \rangle$ is right continuous. Thus, for a sequence of nested partitions $\Lambda_n := r = t_0^n < \dots < t_n^n = T$ with $\text{mesh}\{\Lambda_n\} \rightarrow 0$, we have that the processes

$$x_t^n := \sum_{i=0}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(t) \langle X_t, h_T^{t_{i+1}^n} \rangle$$

are right continuous. Moreover, due to the fact that $\langle X_t, 1 \rangle$ is càdlàg,

$$\sup_{s \in [r, T]} \langle X_s, 1 \rangle < \infty$$

and since (3.4.27) this implies that

$$1_{[t_i^n, t_{i+1}^n)}(t) \left\| h_T^{t_{i+1}^n} - h_T^t \right\|_{\infty} \rightarrow 0.$$

We can thus conclude that

$$\lim_{n \rightarrow \infty} \sup_{t \in [r, T]} |x_t^n - x_t| = 0.$$

The uniform limit of a sequence of right continuous functions being also a right continuous function, we obtain that x_t is right continuous (and therefore, being a supermartingale, also càdlàg).

Thus, by Doob-Meyer decomposition theorem, x_t has a unique compensator $A(ds)$ (which is, by definition of lifting, the unique lifting of $a(ds)$) and in fact:

$$\begin{aligned} A(r, t] &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r, \mu}^{\mathfrak{A}_{t_i}^-} \left\{ \langle X_{t_i}, h_T^{t_i} \rangle - \langle X_{t_{i+1}}, h_T^{t_{i+1}} \rangle \right\} \\ &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r, \mu}^{\mathfrak{A}_{t_i}^-} \left\{ \langle X_{t_i}, h_T^{t_i} \rangle - \langle X_{t_i}, h_T^{t_{i+1}} \rangle \right\} \\ &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r, \mu}^{\mathfrak{A}_{t_i}^-} \left\{ \langle X_{t_i}, h_{t_{i+1}}^{t_i} \rangle \right\} \end{aligned} \quad (3.5.33)$$

weakly in $L^1(P_\mu)$ as Λ runs over a standard sequence of partitions $\Lambda = \{r = t_0 < t_1 < \dots < t_n = t\}$ of the interval $[r, t]$. Moreover, the convergence in (3.5.33) is strong when A is continuous. And in fact, from the criteria for continuity in Doob-Meyer decomposition theorem, A is continuous if and only if for every sequence of r -stopping times $\tau_n \nearrow \tau$, with $\tau_n < \tau$, we have $E_{r, \mu} x_{\tau_n} \searrow E_{r, \mu} x_\tau$. $\square$

3.5.3 Continuity of liftings

We now establish that the liftings are in fact continuous additive functionals. Thus will be used in the proof that the full-martingale problem is well posed.

Theorem 3.5.25 *Every lifting $A(ds)$ of an additive functional $a(ds)$ satisfying (3.4.27) is a continuous additive functional.*

Proof: Let the r -stopping times τ_n increase to τ . Choose ϵ and pick δ such that $|s - s'| \leq \delta$ implies $\|\pi_{r,\cdot} k(s, s')\|_\infty \leq \epsilon$. We have

$$E_{r,\mu} x_{\tau_n} \geq E_{r,\mu} x_\tau \geq E_{r,\mu} x_{\tau \vee \tau_n + \delta} = E_{r,\mu} 1_{\{\tau_n + \delta < \tau\}} x_\tau + E_{r,\mu} 1_{\{\tau_n + \delta \geq \tau\}} x_{\tau_n + \delta}$$

But because x belongs to class (D) , for n big enough we have that

$$E_{r,\mu} x_{\tau_n + \delta} \leq E_{r,\mu} x_\tau + \epsilon.$$

On the other hand we get from Theorem 3.5.24 (i) and the fact that $k(ds)$ satisfies (3.4.27)

$$\begin{aligned} E_{r,\mu} \langle X_{\tau+\delta}, h_T^{\tau+\delta} \rangle &\leftarrow \sum_{i=0}^N E_{r,\mu} 1_{\{\frac{i}{N}T \leq \tau_n < \frac{i+1}{N}T\}} \langle X_{\tau+\delta}, h_T^{\frac{i}{N}T+\delta} \rangle \\ &= \sum_{i=0}^N E_{r,\mu} 1_{\{\frac{i}{N}T \leq \tau_n < \frac{i+1}{N}T\}} \langle X_\tau, h_T^{\frac{i}{N}T+\delta} \rangle \\ &\rightarrow E_{r,\mu} \langle X_\tau, h_T^{\tau+\delta} \rangle. \end{aligned}$$

Thus for n big enough,

$$\begin{aligned} 0 \leq E_{r,\mu} x_{\tau_n} - E_{r,\mu} x_\tau &\leq \epsilon + E_{r,\mu} (x_{\tau_n} - x_{\tau_n + \delta}) \\ &= \epsilon + E_{r,\mu} (\langle X_{\tau_n}, h_T^{\tau_n} \rangle - \langle X_{\tau_n}, h_T^{\tau_n + \delta} \rangle) \\ &= \epsilon + E_{r,\mu} (\langle X_{\tau_n}, h_T^{\tau_n + \delta} \rangle) \\ &\leq \epsilon(1 + |\mu|) \end{aligned}$$

which shows that $E_{r,\mu} x_{\tau_n} \searrow E_{r,\mu} x_\tau$ and therefore that the compensator A of x is continuous according to the criteria given in Doob-Meyer decomposition theorem. $\square$

3.5.4 Induced convergence for liftings

We now establish a criteria from which the L^1 convergence of additive functionals of ξ implies the L^1 convergence of their liftings. This will be needed for the proof of uniqueness to the solution to the full martingale problem. The result is in fact a corollary to the following theorem taken from [DeM, VII.19 and 20].

Theorem 3.5.26 *Let x^n , $n = 1, \dots, \infty$ be right continuous supermartingales and A^n , $n = 1, \dots, \infty$ their compensators. Assume the x^n belong uniformly to the class (D). Assume also that for every stopping time τ , x_τ^n converges weakly in L^1 to x_τ^∞ and that $\sup_{0 \leq s \leq T} |x_s^n - x_s|$ converges to zero in probability. Then for every stopping time τ , A_τ^n converges to A_τ^∞ in L^1 . $\square$*

Corollary 3.5.26 *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem for A , and let $\{a^n(ds)\}_{n=1,\dots,\infty}$ be a collection of additive functional of ξ satisfying (3.4.27). Suppose that for every $t > 0$, $a^n(ds)$ converge to $a^\infty(ds)$ in $L^1(\pi_\mu)$. If $\sup_n \|\pi_x a^n([0, T])\|_\infty < \infty$, then the liftings A^n of the a^n converge to the lifting A^∞ of a^∞ in $L^1(P_{r,\mu})$ for every $t > 0$.*

Proof: Let $x_t^n := \langle X_t, {}^n h_T^t \rangle$ where ${}^n h_T^t(x) := \pi_x a^n(t, T]$. The lifting A^n of a^n is (by definition of lifting) the compensator of x_t^n . Since the characteristics ${}^n h_T^t(\cdot)$ are uniformly bounded and since $\langle X_t, 1 \rangle$ belongs to class (D), then the x^n uniformly belong to class (D).

Since ${}^n h_T^t(x) \rightarrow h_T^t(x)$ for every t and x , then by Lebesgue's Theorem, $P_{r,\mu} \{M < X_\tau, {}^n h_T^\tau >\} \rightarrow P_{r,\mu} \{M < X_\tau, h_T^\tau >\}$ for every bounded random variable M and every r -stopping time τ . Thus x^n converges weakly to x .

Now let $\tau_\epsilon := \inf \{r \leq s \leq T : |x_s^n - x_s| \geq \epsilon\}$. Then

$$\begin{aligned} P_{r,\mu} \left\{ \sup_{0 \leq s \leq T} |x_s^n - x_s| > \epsilon \right\} &\leq P_{r,\mu} \left\{ |x_{\tau_\epsilon}^n - x_{\tau_\epsilon}| \geq \epsilon \right\} \\ &\leq \frac{1}{\epsilon} P_{r,\mu} \{ \langle X_{\tau_\epsilon}, |{}^n h_T^{\tau_\epsilon} - h_T^{\tau_\epsilon}| \rangle > \epsilon \} \end{aligned}$$

which again converges to zero by Lebesgue's Theorem. Thus $\sup_{r \leq s \leq T} |x_s^n - x_s|$ converges to zero in probability.

Therefore Theorem 3.5.26 applies and A_τ^n converges to A_τ^∞ in $L^1(P_\mu)$ for every r -stopping time τ . $\square$

Corollary 3.5.26 *Let $X = (X_t, \mathfrak{F}, P_{r,\mu})$ be a solution to the partial martingale problem for A , and let $\psi_s^n(x)$ be a sequence of measurable functions such that $\sup_{s,x,n} |\psi_s^n(x)| < \infty$ and $\psi_s^n(x) \rightarrow \psi_s(x)$ for every $x \in E$ and every $s \geq r$. Suppose that $\lambda^n(s, \xi_s)ds$ converges to $k(ds)$ in $L^1(\pi_x)$ for every $x \in E$, and that $\sup_{x \in E} \left\| \pi_x \int_r^t \lambda^n(s, \xi_s)ds \right\|_\infty < \infty$ for every $r, t \geq 0$. Let $K_i^n(ds), i = 1, 2$, be given by*

$$\begin{aligned} K_1^n(ds) &= \langle X_s, \Phi(., \psi_s^n(.))\lambda^n(s, .) \rangle ds \\ K_2^n(ds) &= K^{\Phi(\psi^n)dk}(ds). \end{aligned}$$

Then $K_i^n(ds)$ converges in $L^1(P_{r,\mu})$ to the lifting $K^{\Phi(\psi)dk}(ds)$ of $\Phi(\xi_s, \psi_s(\xi_s))k(ds)$.

Proof: According to Lemma 3.2.20, $k_i^n(ds)$ converges in $L^1(\pi_\mu)$ to $\Phi(\xi_s, v_{s,T}(\xi_s))k(ds)$, where

$$\begin{aligned} k_1^n(ds) &= \Phi(\xi_s, \psi_s^n(\xi_s))\lambda^n(s, \xi_s)ds \\ k_2^n(ds) &= \Phi(\xi_s, \psi_s^n(\xi_s))k(ds). \end{aligned}$$

Since $\|\Phi(., \psi^n(.))\|_\infty < \infty$, we can apply Corollary 3.5.26 to get that their liftings, $K_i^n(ds)$, converge to $K^{\Phi(\psi)dk}(ds)$ in $L^1(P_{r,\mu})$ for $i = 1, 2$. $\square$

3.6 Martingale problem: existence

In this section we prove the converse to Theorem 3.4.22, that is, we show that the distribution $P_{r,\mu}^{(\xi, \Phi, k)}$ of the (ξ, Φ, k) -superprocess is a solution to the full martingale problem. The proof is based on the following well known results.

3.6.1 Some well known results

Theorem 3.6.27 *Let X be the superprocess with parameters (ξ, Φ, k) . Then X is a Hunt process¹⁰, the lifting of every natural additive functional with finite characteristic exists and the lifting of $\ell(\xi_s, du)k(ds)$ is the modified Levy measure $L(ds, d\mu)$ of X . In particular for every bounded measurable real valued function f we have*

$$P_{r,\mu} \int_r^t \int_{M_f} f(<\mu, \phi>) L(ds, d\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty f(u\phi(\xi_s)) \ell(\xi_s, du) k(ds). \quad (3.6.34)$$

The following moment formulae are satisfied:

- (i) $P_{r,\mu} < X_t, f > = \pi_\mu f(\xi_{t-r})$
- (ii) $P_{r,\mu} < X_t, f >^2 = (\pi_\mu f(\xi_{t-r}))^2$
 $+ \pi_\mu \int_r^t (\pi_{\xi_s} f(\xi_{t-s}))^2 (b(\xi_s) + \int_0^\infty u^2 \ell(\xi_s, du)) k(ds)$

Proof: The E -valued process ξ is a Hunt process when E is furnished with its original topology and its Ray topology [Sh, Th. 47.6]. Therefore the superprocess X is also a Hunt in the weak convergence induced by the original topology and the Ray topology (see [F1] or [Ku] or [Dy7, Th.3.3.2, and Th.6.2.2]). The existence and uniqueness of liftings is established in [DyKuS]¹¹ (see [Dy7, Ch. 6.2.5]). The fact that the modified Levy measure $L(ds, d\mu)$ is the lifting of $\ell(\xi_s, du)k(ds)$ is established [DyKuS] (See [Dy7, Ch. 6.1.2, 6.3.2 and 6.8.1]). The formula (3.6.34) follows from Dynkin's definition of lifting, see [Dy7, equation 6.2.13a]. The moment formulae were established in [Dy1].

3.6.2 Proof of the existence of a solution to the martingale problem

Let $P_{r,\mu}^{(\xi, \Phi, k)}$ be the distribution of the (ξ, Φ, k) -superprocess. Clearly, (3.4.30) is a $P_{r,\mu}^{(\xi, \Phi, k)}$ -martingale for every $\phi \in \mathcal{D}(A)$. Existence and uniqueness of liftings is one

¹⁰in the weak convergence induced by the original topology and the Ray topology on E

¹¹We will systematically refer to [Dy7] for the proofs of the results in [DyKuS]. The reason is that the exposition of [DyKuS] made in [Dy7] is slightly more systematic and self contained, and that it covers a larger scale of problems.

of the (well known) results stated in Theorem 3.6.27. From Theorem 3.5.24 (iii) we have

$$\sum_{r < s \leq t} \Delta \langle X_s, \phi \rangle = \sum_{r < s \leq t} \langle \Delta X_s, \phi \rangle$$

for every $\phi \in \mathcal{D}(A)$. Let $C_t(\phi)$ be the quadratic variation of the continuous martingale part of the semimartingale $\langle X_t, \phi \rangle$. Then Itô's formula implies that

$$\begin{aligned} & (\langle X_t, \phi \rangle)^2 + \frac{1}{2} \int_r^t \langle X_s, \phi \rangle \langle X_s, A\phi \rangle ds + C_t(\phi) \\ & + \sum_{r < s \leq t} ((\langle X_{s-} + \Delta X_s, \phi \rangle)^2 - (\langle X_{s-}, \phi \rangle)^2 - 2 \langle X_{s-}, \phi \rangle \langle \Delta X_s, \phi \rangle) \end{aligned}$$

is a $P_{r,\mu}$ -martingale for every $\phi \in \mathcal{D}(A)$. Simplifying we obtain that

$$\begin{aligned} & \frac{1}{2} (\langle X_t, \phi \rangle)^2 + \frac{1}{2} \int_r^t \langle X_s, \phi \rangle \langle X_s, A\phi \rangle ds \\ & + C_t(\phi) + \sum_{r < s \leq t} (\langle \Delta X_s, \phi \rangle)^2 \end{aligned}$$

is a $P_{r,\mu}$ -martingale for every $\phi \in \mathcal{D}(A)$. By definition of the modified Levy measure, this is the same thing as saying that

$$\begin{aligned} & \frac{1}{2} (\langle X_t, \phi \rangle)^2 + \frac{1}{2} \int_r^t \langle X_s, \phi \rangle \langle X_s, A\phi \rangle ds \\ & + C_t(\phi) + \int_r^t \int_{M_f} \langle \mu, \phi \rangle^2 L(ds, d\mu) \end{aligned}$$

is a $P_{r,\mu}$ -martingale for every $\phi \in \mathcal{D}(A)$, where $\int_{M_f} \langle \mu, \phi \rangle^2 L(ds, d\mu)$ is the lifting of $\int_0^\infty (u\phi)^2(\xi_s) \ell(\xi_s, du) k(ds)$. Note that (by definition of lifting)

$$P_{r,\mu} \int_r^t \int_{M_f} \langle \mu, \phi \rangle^2 L(ds, d\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty (u\phi)^2(\xi_s) \ell(\xi_s, du) k(ds)$$

and in fact, we can calculate that

$$P_{r,\mu} \left(\frac{1}{2} C_t(\phi) \right) = \pi_{r,\mu} \int_0^t \frac{1}{2} b(\xi_s) \phi^2(\xi_s) k(ds)$$

and thus

$$P_{r,\mu} \left(\frac{1}{2} C_t(\phi) - \int_0^t \hat{Q}(\phi)(ds) \right) = 0$$

where $\hat{Q}(\phi)(ds)$ is the lifting of the additive functional $\frac{1}{2} b(\xi_s) \phi^2(\xi_s) k(ds)$. Therefore, since X_t is a Markov process, this implies that

$$\frac{1}{2} C_t(\phi) - \int_0^t \hat{Q}(\phi)(ds)$$

is a martingale. But because $\frac{1}{2}C_t(\phi) - \int_0^t \hat{Q}(\phi)(ds)$ is also a right continuous predictable process of integrable variation, we obtain that $\frac{1}{2}C_t(\phi) = \int_0^t \hat{Q}(\phi)(ds)$. We can apply Itô's formula which gives that

$$\begin{aligned} & \exp(-\langle X_t, \phi \rangle) + \int_r^t \exp(-\langle X_s, \phi \rangle) \langle X_s, A\phi \rangle ds \\ & - \int_r^t \exp(-\langle X_s, \phi \rangle) \hat{Q}(\phi)(ds) + \int_r^t \exp(-\langle X_s, \phi \rangle) \int_{M_f} \mathcal{E}(-\langle \mu, \phi \rangle) L(ds, d\mu) \end{aligned}$$

is a $P_{r,\mu}^{(\xi, \Phi, k)}$ -martingale. But since $\hat{Q}(\phi)(ds)$ is the lifting of $\frac{1}{2}b(\xi_s)\phi^2(\xi_s)k(ds)$ and $\int_{M_f} \exp(-\langle \mu, \phi \rangle) L(ds, d\mu)$ is the lifting of $\int_0^\infty \mathcal{E}(u\phi(\xi_s)) \ell(\xi_s, du)k(ds)$ this can be rewritten to give that

$$\begin{aligned} & \exp(-\langle X_t, \phi \rangle) + \int_r^t \exp(-\langle X_s, \phi \rangle) \langle X_s, A\phi \rangle ds \\ & - \int_r^t \exp(-\langle X_s, \phi \rangle) K^{\Phi(\phi)dk}(ds) \end{aligned}$$

is a $P_{r,\mu}^{(\xi, \Phi, k)}$ -martingale. $\square$

3.7 Martingale problem: uniqueness

We now prove uniqueness to the full martingale problem. Assume for now on that $(X_t, \mathfrak{F}, P_{r,\mu})$ is a solution to the full martingale problem. Our goal, in the next section, is to establish that $(X_t, \mathfrak{F}, P_{r,\mu})$ is a solution to an "extended" form of the full martingale problem.

3.7.1 Extension of the martingale problem to time dependent functions

We need the following definition.

Definition 3.7.27 *We say that a mapping $\psi(s, x)$ is smooth for the strong generator $(A, \mathcal{D}(A))$ of ξ , or simply that $\psi(s, x)$ is smooth for A , if*

- 1) $\psi(s, \cdot)$ belongs to $\mathcal{D}(A)$ for every s
- 2) $\frac{\partial}{\partial s} \psi(s, x)$ exists and $\left\| \frac{\psi(s+h, \cdot) - \psi(s, \cdot)}{h} - \frac{\partial}{\partial s} \psi(s, \cdot) \right\|_\infty \rightarrow 0$
- 3) $\psi, \frac{\partial}{\partial s} \psi$ and $A\psi$ are bounded and strongly continuous

The "extended" form of the full martingale problem for (A, Φ, k) is given in the following theorem:

Theorem 3.7.28 *Let X_t be a solution to the full martingale problem for (A, Φ, k) and let ψ be smooth for A . Then*

$$\begin{aligned} \exp(-\langle X_t, \psi_t \rangle) + \int_r^t \exp(-\langle X_s, \psi_s \rangle) \langle X_s, A\psi_s + \frac{\partial}{\partial s} \psi_s \rangle ds \\ - \int_r^t \exp(-\langle X_s, \psi_s \rangle) K^{\Phi(\psi)dk}(ds) \end{aligned}$$

is a $P_{r,\mu}$ -martingale, where $K^{\Phi(\psi)dk}(ds)$ is the lifting of $\Phi(\xi_s, \psi_s)k(ds)$.

Proof: The proof is a generalization of Lemma 4.3.4 in [EtK]. First let

$$\begin{aligned} u(s, X_t) &:= \exp(-\langle X_t, \psi_s \rangle) \\ v(s, X_t) &:= \exp(-\langle X_t, \psi_s \rangle) \langle X_t, \frac{\partial}{\partial s} \psi_s \rangle \\ w(s, X_t) &:= \exp(-\langle X_t, \psi_s \rangle) \langle X_t, A\psi_s \rangle \end{aligned}$$

Then we have

$$\begin{aligned} u(t_2, X_{t_2}) - u(t_1, X_{t_1}) &= - \int_{t_1}^{t_2} v(s, X_{t_2}) ds \\ E^{\mathfrak{G}_{t_1}} [u(t_2, X_{t_2}) - u(t_1, X_{t_1})] &= -E^{\mathfrak{G}_{t_1}} \left[\int_{t_1}^{t_2} w(t_1, X_s) ds \right] \\ &\quad - E^{\mathfrak{G}_{t_1}} \left[\int_{t_1}^{t_2} u(t_1, X_s) K^{\Phi(\psi_{t_1})dk}(ds) \right]. \end{aligned}$$

Therefore, if Λ^n is a partition of $[t_1, t_2]$ with $\text{mesh}\{\Lambda^n\} \rightarrow 0$ and ψ^n and X^n are defined by

$$\begin{aligned} \psi^n(s, x) &:= \sum_{i=1}^n \psi(t_i^n, x) 1_{[t_i^n, t_{i+1}^n)}(s) \\ X^n(s) &:= \sum_{i=1}^n X_{t_{i+1}^n} 1_{[t_i^n, t_{i+1}^n)}(s) \end{aligned}$$

then, since from (3.5.33),

$$\begin{aligned} \sum_{i=1}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(s) K^{\Phi(\xi_s, \psi_{t_i^n}(\xi_s))dk}(ds) &= K^{\sum_{i=1}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(s) \Phi(\xi_s, \psi_{t_i^n}(\xi_s))dk}(ds) \\ &= K^{\Phi(\xi_s, \psi_s^n(\xi_s))dk}(ds) \end{aligned}$$

(where $K^{\eta(s, \xi_s)dk}(ds)$ denotes here the lifting of $\eta(s, \xi_s)k(ds)$) we get that

$$\begin{aligned} E^{\mathfrak{G}_{t_1}} [u(t_2, X_{t_2}) - u(t_1, X_{t_1})] &= -E^{\mathfrak{G}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s^n, \psi_s \rangle) \langle X_s^n, \frac{\partial}{\partial s} \psi_s \rangle ds \right] \\ &\quad - E^{\mathfrak{G}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s, \psi_s^n \rangle) \langle X_s, A\psi_s^n \rangle ds \right] \\ &\quad - E^{\mathfrak{G}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s, \psi_s^n \rangle) K^{\Phi(\psi^n)dk}(ds) \right]. \end{aligned} \tag{3.7.35}$$

We want to show that (3.7.35) converges to

$$\begin{aligned} E^{\mathfrak{A}_{t_1}} [u(t_1, X_{t_2}) - u(t_1, X_{t_1})] &= -E^{\mathfrak{A}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s, \psi_s \rangle) \langle X_s, \frac{\partial}{\partial s} \psi_s \rangle ds \right] \\ &\quad -E^{\mathfrak{A}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s, \psi_s \rangle) \langle X_s, A\psi_s \rangle ds \right] \\ &\quad -E^{\mathfrak{A}_{t_1}} \left[\int_{t_1}^{t_2} \exp(-\langle X_s, \psi_s \rangle) K^{\Phi(\psi)dk}(ds) \right] \end{aligned} \quad (3.7.36)$$

which would complete the proof of the Theorem.

1°) Let us start by showing that the first two members of the right hand side of (3.7.35) converge to the first two members of the right hand side of (3.7.36).

Note that by Fubini's theorem, we can interchange the order of the conditional expectation and the integration with respect to the Lebesgue measure. From Lebesgue's theorem¹², it suffices to prove that for a fixed $s \geq t_1$, we have that $P_{r,\mu}$ -almost surely

$$\begin{aligned} \langle X_s^n, \psi_s \rangle &\rightarrow \langle X_s, \psi_s \rangle, \\ \langle X_s^n, A\psi_s \rangle &\rightarrow \langle X_s, A\psi_s \rangle, \\ \langle X_s^n, \frac{\partial}{\partial s} \psi_s \rangle &\rightarrow \langle X_s, \frac{\partial}{\partial s} \psi_s \rangle \end{aligned}$$

in order to get the desired result.

Only the last convergence is not straightforward. Indeed, $P_{r,\mu}$ -almost surely, all the processes

$$t \mapsto \langle X_t, \frac{\psi_{s+h} - \psi_s}{h} \rangle, \quad h \text{ rational}$$

are right continuous since $\frac{\psi_{s+h} - \psi_s}{h} \in \mathcal{D}(A)$.

But since $\langle X_t, 1 \rangle$ is a càdlàg process, we have that

$$\sup_{r \leq t \leq T} \langle X_t, 1 \rangle < \infty,$$

and using the fact that $\frac{\psi_{s+h} - \psi_s}{h} \rightarrow \frac{\partial}{\partial s} \psi_s$ uniformly in r , we obtain that $P_{r,\mu}$ -almost surely, the process $t \mapsto \langle X_t, \frac{\partial}{\partial s} \psi_s \rangle$ is the uniform limit of right continuous pro-

¹²Lebesgue's theorem can be invoked since $\psi, A\psi, \frac{\partial}{\partial s} \psi$ are bounded, since the family $\{\langle X_s^n, 1 \rangle\}_{n \in \mathbb{N}, r \leq s \leq T}$ is uniformly integrable (with respect to $P_{r,\mu}$), and since $E^{\mathfrak{A}_{t_1}} \langle X_s^n, 1 \rangle = \langle X_{t_1}, 1 \rangle$ which is Lebesgue integrable on the interval $[t_1, t_2]$.

cesses, and is therefore also a right continuous process. Hence $\langle X_s^n, \frac{\partial}{\partial s} \psi_s \rangle \rightarrow \langle X_s, \frac{\partial}{\partial s} \psi_s \rangle$ as wanted.

2°) We now have to show that the third member of the right hand side of (3.7.35) converges to the third member of the right hand side of (3.7.36).

Note that according to Corollary 3.5.26, $K^{\Phi(\psi^n)dk}(ds)$ converges to $K^{\Phi(\psi)dk}(ds)$ in $L^1(P_{r,\mu})$. Moreover, from Lemma 3.5.25, the additive functional $K^{\Phi(\psi)dk}(ds)$ is continuous. We can thus invoke Lemma 3.2.20 to obtain the result we wanted. $\square$

3.7.2 Proof of uniqueness for the full martingale problem

We are now ready to show that the solution to the full martingale problem for (A, Φ, k) is unique, as stated in Theorem 3.4.22. We need the following definition:

Definition 3.7.28 *We will say that a sequence of superprocesses X^n with parameters (ξ, Φ, k^n) approximates the superprocess X with parameters (X, Φ, k) , if:*

- $k^n(ds)$ converges to $k(ds)$ in $L^1(\pi_x)$ for every $x \in E$,
- $\sup_{x \in E} \|\pi_x k^n(r, t)\|_\infty < \infty$ for every $r, t \geq 0$,
- the log-Laplace functional v^n of X^n converges to the log-Laplace functional v of X .

Similarly, we may talk about an **approximating sequence** X^n for the superprocess with parameters (X, Φ, k) , if X^n approximates the (ξ, Φ, k) -superprocess.

If in addition:

- v^n is smooth for A ,
- $Av_{s,T}^n + \frac{\partial}{\partial s} v_{s,T}^n = \Phi(., v_{s,T}^n) \lambda^n(s, .)$,

*we will say that the (ξ, Φ, k^n) -superprocesses form an **A-smooth approximating sequence** for the (ξ, Φ, k) -superprocess.*

Proof of the uniqueness part in Theorem 3.4.22:

- We prove the theorem under the assumption that there exists an A -smooth approximating sequence for the (ξ, Φ, k) -superprocess, which has the form $(\xi, \Phi, \lambda^n(s, \xi_s)ds)$. The existence of such a sequence is guaranteed by Theorem 3.7.29 and Theorem 3.7.30 which will be stated and proved in further sections.

Let v^n be the log-Laplace functional of the $(\xi, \Phi, \lambda^n(s, \xi_s)ds)$ -superprocess. From Theorem (3.7.28) we get that

$$\exp(-\langle X_t, v_{t,T}^n \rangle) + \int_r^t \exp(-\langle X_s, v_{s,T}^n \rangle) \langle X_s, Av_{s,T}^n + \frac{\partial}{\partial s} v_{s,T}^n \rangle ds - \int_r^t \exp(-\langle X_s, v_{s,T}^n \rangle) K^{\Phi(\xi_s, v_{s,T}^n)dk}(ds)$$

is a martingale. Putting $x_t^n = \langle X_t, v_{t,T}^n \rangle$, the equality

$$Av_{s,T}^n(\cdot) + \frac{\partial}{\partial s} v_{s,T}^n(\cdot) = \Phi(\cdot, v_{s,T}^n) \lambda^n(s, \cdot)$$

gives

$$e^{-x_t^n} = M_t^n(\phi) + \int_r^t e^{-x_t^n} K_1^n(ds) - \int_r^t e^{-x_t^n} K_2^n(ds) \quad (3.7.37)$$

where

$$\begin{aligned} K_1^n(ds) &:= \langle X_s, \Phi(\cdot, v_{s,T}^n) \lambda^n(s, \cdot) \rangle ds \\ K_2^n(ds) &:= K^{\Phi(v_{s,T}^n)dk}(ds). \end{aligned}$$

Clearly $e^{-x_t^n} \rightarrow e^{-x_t}$ pointwise and in $L^1(P_{r,\mu})$ where $x_t := \langle X_t, v_{t,T} \rangle$ and v is the log-Laplace functional of the superprocess (ξ, Φ, k) .

According to Corollary 3.5.26, $K_i^n(ds)$ in $L^1(P_{r,\mu})$ to $K(ds) := K^{\Phi(v, \cdot)dk}(ds)$ for $i = 1, 2$. Therefore, from Lemma (3.2.20), we get that

$$\int_0^t e^{-x_t^n} K_1^n(ds) - \int_0^t e^{-x_t^n} K_2^n(ds) \rightarrow 0 \quad (3.7.38)$$

where the convergence holds in $L^1(P_{r,\mu})$.

That forces $M_t^n(\phi)$ to converge in $L^1(P_\mu)$ to a limit $M_t(\phi)$ which has to be a martingale, and we get

$$P_{r,\mu}(e^{-x_T}) = P_{r,\mu}(e^{-x_r}),$$

Which is precisely

$$P_{r,\mu}(\exp(-\langle X_T, \phi \rangle)) = \exp(-\langle \mu, v_{r,T} \rangle).$$

In other words, X_t is the superprocess with parameters (ξ, Φ, k) . $\square$

3.7.3 Construction of approximating superprocesses

In the proof of the uniqueness part of Theorem 3.4.22, we assumed that there exists an A -smooth approximating sequence of the form $(\xi, \Phi, \lambda^n(s, \xi_s)ds)$ for the superprocess with parameters (ξ, Φ, k) . This is what we prove here. In fact, in Theorem 3.7.29, we construct an approximating sequence of the form $(\xi, \Phi, \lambda^n(s, \xi_s)ds)$ for the superprocess with parameters (ξ, Φ, k) , and in Theorem 3.7.30 we prove that it is an A -smooth approximating sequence.

Notation: in the sequel, the support of a real valued function f will be denoted by $\text{supp}\{f\}$; recall that $\text{supp}\{f\}$ is the closure of the set of points where $f \neq 0$. We will also denote the set of infinitely differentiable functions with compact support by $C_c^\infty(\mathbf{R}_+)$, and the non negative members of that class by $pC_c^\infty(R_+)$.

Theorem 3.7.29 *There exists an A -smooth approximating sequence of the form $(\xi, \Phi, \lambda^n(s, \xi_s)ds)$ for the superprocess with parameters (ξ, Φ, k) . Moreover, we can find $\lambda^n(s, \xi_s)ds$ such that*

$$\lambda^n(s, x) = \sum_{i=1}^n h_i^n(x) f_i^n(s)$$

where the $h_i^n(x)$ are uniformly bounded, for $i = 1, \dots, n$, and where the $f_i^n(s)$ belong to $pC_c^\infty(R_+)$ and have disjoint supports, for $i = 1, \dots, n$.

Proof: The Doob-Meyer decomposition Theorem provides us with a sequence of purely atomic additive functionals $k_n^\infty(ds)$, approximating a given additive functional, $k^\infty(ds)$, namely,

$$k_n^\infty(ds) := \sum_{i=1}^{n-1} h_{t_{i+1}}^{t_i^n}(\xi_s) \delta_{t_i^n}(ds),$$

where ${}^\infty h_t^r(x) := \pi_x(k^\infty(r, t])$ is the characteristic of k^∞ . Let $\Lambda_n \equiv \{0 = t_0^n < t_1^n < \dots < t_n^n := T\}$ be a sequence of partitions of the interval $[0, T]$ such that $\text{mesh}\{\Lambda_n\} > \frac{1}{2^n}$, but that nevertheless $\lim_{n \rightarrow \infty} \text{mesh}\{\Lambda_n\} = 0$.

We will approximate each $k_n^\infty(ds)$ by a “smooth” additive functional $\lambda^n(ds) =: \lambda^n(s, \xi_s)ds$ in such a way that $k_n^n(ds)$ approximates also $k(ds)$.

More precisely, let f_i^n be a function in $pC_c^\infty(\mathbf{R}_+)$ such that

- 1) $\text{supp}\{f_i^n\} \subseteq [t_i^n, t_i^n + \frac{1}{2^n}]$,
- 2) $\int f_i^n(s)ds = 1$,
- 3) assume also (for simplicity) that $f_i^n(s)$ is a translation of $f_j^n(s)$.

Put

$$k_n^n(ds) := \sum_{i=0}^{n-1} {}^\infty h_{t_{i+1}^n}^{t_i^n}(\xi_s) f_i^n(s) ds.$$

Note that $k_n^n(ds)$ clearly has the form

$$k_n^n(ds) =: \lambda^n(s, \xi_s)ds =: \sum_{i=1}^n h_i^n(x) f_i^n(s).$$

Moreover,

$${}^n h_T^0(x) := \pi_x k_n^n(0, T] = \sum_{i=0}^{n-1} {}^\infty h_{t_{i+1}^n}^{t_i^n}(x) \int_{t_i^n}^{t_{i+1}^n} f_i^n(s) ds = {}^\infty h_T^0(x),$$

and in fact, more generally,

$$\begin{aligned} {}^n h_T^{t_i^n}(x) &= {}^\infty h_T^{t_i^n}(x) \\ |{}^n h_T^s(x) - {}^\infty h_T^s(x)| &\leq \max_i \| {}^\infty h_{t_{i+1}^n}^{t_i^n}(x) \|_\infty \rightarrow 0 \end{aligned} \tag{3.7.39}$$

It is easy to check that the processes ${}^n h_T^s(\xi_s)$ are supermartingales, which are right continuous according to [Dy7, p.31]¹³. With no difficulty, one verifies that $k^n(ds)$ is the compensator of the semimartingale ${}^n h_T^s(\xi_s)$. Let us check that we meet the conditions of Theorem 3.5.26: our supermartingales are uniformly bounded, and therefore they must belong uniformly to class (D). Moreover (3.7.39) shows that

¹³In fact, $h_T^r(x)$ is an exit rule and for every exit rule $f(r, x)$ the process $f(s, \xi_s)$ is almost surely right continuous. See [Dy7, p. 29] for more details.

${}^n h_T(\cdot)$ converges uniformly to ${}^\infty h_T(\cdot)$. Therefore Theorem 3.5.26 applies and we conclude that

$$k^n[0, t] \rightarrow k^\infty[0, t] \text{ in } L^1 \text{ for every } t \in [0, T].$$

According to Theorem 3.1.18, $v_{t,T}^n(x)$ converges to $v_{t,T}(x)$. $\square$

3.7.4 Smoothness property of a class of log-Laplace functionals

In this section we state and prove Theorem 3.7.30 which was used above in the proof of Theorem 3.7.29.

Let $\lambda(s, x)$ be bounded, non negative measurable mapping having strongly continuous derivatives $\frac{\partial^i}{\partial s^i} \lambda(s, x)$ uniformly bounded in x for $i = 1, 2$. We show in this section that the log-Laplace functionals of superprocesses with parameters $(\xi, \Phi(x, z), \lambda(s, \xi_s) ds)$ have log-Laplace functionals smooth for the generator A of ξ . Note that this is also the superprocess with parameters $(\xi, \Phi(x, z) \lambda(s, x), ds)$.

We will investigate the differentiability (and the continuity of that derivative) of solutions to integral equations of the form

$$v_t(f)(x) = \pi_x f(\xi_t) - \pi_x \int_0^t \Phi(s, \xi_s, v_{t-s}(\xi_s)) ds.$$

A result analogous to ours but substantially weaker had been obtained in [EKaRC]: they showed that the function $e^{-v_t(x)}$ can be differentiated and they obtained

$$\frac{\partial}{\partial t} e^{-v_t(x)} = P_x \left(e^{-\langle X_t, \phi \rangle} \langle X_t, A\phi + \Phi(\cdot, \phi) \rangle \right). \quad (3.7.40)$$

Then, since $A\phi(x)$ and $\Phi(x, \phi)$ were uniformly continuous in x and X has no fixed time discontinuities, this shows also that $t \mapsto \frac{\partial}{\partial t} e^{-v_t(x)}$ is continuous. Moreover, the fact that $\frac{\partial}{\partial t} e^{-v_t(x)}$ exists can be used to show that v_t belongs to the weak generator of the process ξ , and that moreover

$$\frac{\partial}{\partial t} v_t(x) = Av_t(x) + \Phi(x, v_t). \quad (3.7.41)$$

But since the functions $\phi(x)$, $A\phi(x)$, and $\Phi(x, \phi)$, as functions of x , are uniformly continuous for every ϕ in the domain of A , then $x \mapsto \frac{\partial}{\partial t} v_t(x)$ must also be uniformly continuous.

They could then use Lemma 4.3.4 in [EtK] to prove uniqueness to the martingale problem. In the above argument, the fact that the semigroup generated by π_x acts on $C_0(E)$ is absolutely crucial.

In our case, L is arbitrary and can not be assumed to be $C_0(E)$. This is one of the difficulties we will have to solve. Moreover, we are not looking for differentiation and continuity but rather to strong differentiation and strong continuity (where the word strong refers to uniformity in x). These facts make our proof much more involved. We first give only the main steps of our proof and leave the computations to an appendix for the interested reader. They are tedious, sometimes a little tricky, but never really difficult.

Remark that in contrast to [EKaRC], our proof is completely analytical, and remains true even when v_t is not the log-Laplace functional of some superprocess, which will be the case in general when $\Phi(s, x, \lambda)$ is not a branching mechanism. In particular, our method gives a way to construct solutions to those equations even in cases not covered by branching mechanisms.

Definition 3.7.29 *We will say that a mapping $S(t, x)$ is **strongly continuous** if*

$$\lim_{t \rightarrow s} \sup_{x \in E} |S(t, x) - S(s, x)| = 0$$

Definition 3.7.29 *We will say that a mapping $\Gamma(s, x, \lambda)$ is **locally in λ strongly continuous** if for every $\Lambda \geq 0$*

$$\lim_{t \rightarrow s} \sup_{x \in E, 0 \leq \lambda \leq \Lambda} |\Gamma(s, x, \lambda) - \Gamma(t, x, \lambda)| = 0$$

We will use the following elementary proposition.

Proposition 3.7.29 (i) If $v(t, x)$ is a bounded strongly continuous mapping and if $\Gamma(s, x, \lambda)$ is a locally in λ strongly continuous mapping such that $\frac{\partial}{\partial \lambda} \Gamma$ exists and

$$\sup_{0 \leq s < \infty, x \in E, 0 \leq \lambda \leq \Lambda} \left| \frac{\partial}{\partial \lambda} \Gamma(s, x, \lambda) \right| < \infty$$

then the mapping $\Gamma(t, x, v(t, x))$ is strongly continuous

(ii) the product of two strongly continuous functions is still a strongly continuous function.

Proof:(i) Let $\|v\|_\infty =: \Lambda$, let $\sup_{0 \leq s < \infty, x \in E, 0 \leq \lambda \leq \Lambda} \left| \frac{\partial}{\partial \lambda} \Gamma(s, x, \lambda) \right| =: M$ and let $|\Gamma(t, x, v(t, x)) - \Gamma(s, x, v(s, x))| =: o(t - s, x)$. Because v is bounded and Γ is locally strongly continuous, we have that $\|o(t - s, \cdot)\|_\infty \rightarrow 0$. We also have:

$$|\Gamma(s, x, v(t, x)) - \Gamma(s, x, v(s, x))| \leq M \|v(t, \cdot) - v(s, \cdot)\|_\infty.$$

Thus

$$|\Gamma(t, x, v(t, x)) - \Gamma(s, x, v(s, x))| \leq M \|v(t, \cdot) - v(s, \cdot)\|_\infty + \|o(t - s, \cdot)\|_\infty \rightarrow 0$$

and the statement is proven. $\square$

Proof:(ii) is immediate. $\square$

Theorem 3.7.30 Let $(\xi, \mathcal{F}, \pi_x)$ be a right process with value in the metrizable Luzin space $(E, \mathcal{E})$. Let S be the strongly continuous semigroup generated by ξ on the set $L \subseteq b\mathcal{E}$. Let $(A, \mathcal{D}(A))$ be the (strong) generator of S . Let $\Phi(s, x, \lambda)$ be a non negative mapping such that $\Phi(s, x, \phi(x)) \in L$ for every $\phi \in \mathcal{D}(A)$ and such that for each $\Lambda, T \in R_+$,

$$\|\Phi'_s\|_\infty \vee \|\Phi''_s\|_\infty \vee \|\Phi'_\lambda\|_\infty \vee \|\Phi''_\lambda\|_\infty =: M(\Lambda, T) =: M < \infty \quad (3.7.42)$$

where the supremum is taken over the triples (s, x, λ) such that $0 \leq s \leq T, x \in E, 0 \leq \lambda \leq \Lambda$. Assume that Φ and its derivatives are locally in λ strongly continuous. Then for each $\phi \in \mathcal{D}(A)$, there exists a unique solution v to the equation

$$v_{t,T}(\phi)(x) = S_{T-t}\phi(x) - \int_t^T S_{r-t} [\Phi(r, v_r)](x) dr, \quad (3.7.43)$$

$v_{.,T}(\cdot)$ is smooth for A and

$$\frac{\partial}{\partial t} v_{t,T}(x) + A v_t(x) = \Phi(t, x, v_t(x)).$$

Proof: Note first that from general Gronwall inequality, [Dy2, Lemma 3.2], there cannot be two solutions to equations of the form (3.7.43). Once and for all the remaining of the proof, choose a certain Φ satisfying the hypothesis of the Theorem, choose $\phi \in \mathcal{D}(A)$, let $\Lambda := \|\phi\|_\infty$, pick $T \geq 0$ and find T^* small enough such that $T^* M(\Lambda, T) \leq \frac{1}{2}$, where $M := M(\Lambda, T)$ is determined by (3.7.42). Define

$$c(\phi) := c(T^*, \Lambda, \phi) := 2(\|A\phi\|_\infty + M)$$

and for any mapping u from $[0, T^*] \times E$ define the following conditions:

- $c_1(u) : \|u\|_\infty \leq \|\varphi\|_\infty \leq \Lambda$
- $c_2(u) : u$ is strongly continuous
- $c_3(u) : u$ is uniformly Lipschitz with constant $c(T^*, \Lambda, \phi)$
- $c_4(u) : u^{(1)}(t, x) := \frac{\partial}{\partial t} u(t, x)$ exists and $\left\| \frac{u(t+h, \cdot) - u(t, \cdot)}{h} - u^{(1)}(t, \cdot) \right\|_\infty \rightarrow 0$
- $c_5(u) : u(t, \cdot) \in \mathcal{D}(A)$ for every $t \in [0, T^*]$

Define also

$$G(u)(t, x) := S_t \varphi(x) - \int_0^t S_r [\Phi(t-r, u(t-r))] (x) dr.$$

Then we will verify that:

- $c_1(u)$ implies $c_1(G(u))$
- $c_1(u)$ and $c_2(u)$ implies $c_2(G(u))$
- $c_1(u)$ and $c_3(u)$ implies $c_3(G(u))$
- $c_1(u) - c_4(u)$ implies $c_5(G(u))$ and

$$\begin{aligned} AG(u(t)) = & S_t A \phi - \int_0^t S_r [\Phi'_s(t-r, u(t-r))] dr \\ & - \int_0^t S_r [\Phi'_\lambda(t-r, u(t-r)) u'_s(t-r)] dr \\ & + S_t [\Phi(0, u(0))] - \Phi(t, u(t)). \end{aligned}$$

- $c_1(u) - c_3(u)$ and $c_5(u)$ implies $c_4(G(u))$ and

$$\frac{\partial}{\partial t}G(u)(t) = AG(u)(t) - \Phi(t, u(t)).$$

- $c_1(u) - c_5(u)$ and $c_2(\frac{\partial}{\partial t}u)$ implies $c_1(u) - c_5(u)$, $c_2(\frac{\partial}{\partial t}u)$ and $c_2(Au)$
- $c_1(u)$ and $c_1(w)$ implies

$$\|G(u) - G(w)\|_{\infty} \leq \frac{1}{2} \|u - w\|_{\infty}$$

- $c_1(u) - c_5(u)$ and $c_1(w) - c_5(w)$ implies there exists some $K := K(T^*, \Lambda, \phi)$ such that

$$\begin{aligned} \|AGG(u) - AGG(w)\|_{\infty} &\leq K \|Gu - Gw\|_{\infty} + K \|u - w\|_{\infty} \\ &\quad + \frac{1}{2} \|AG(u) - AG(w)\|_{\infty} \end{aligned}$$

- $c_1(u) - c_5(u)$ implies $c_2(AG(G(u)))$.

Let $G^0(t, x) := \varphi(x)$ and $G^{n+1}(t, x) := G(G^n)(t, x)$ Clearly $G^0(., .)$ satisfies the conditions $c_1 - c_5$. Therefore, from the previous lemmas, $G^n(t, x)$ satisfies as well all those conditions for every $n \geq 0$. Since for $n \geq 1$ we have

$$\|G^{n+1} - G^n\|_{\infty} \leq \frac{1}{2} \|G^n - G^{n-1}\|_{\infty}$$

then G^n converges uniformly to a solution u to $u = G(u)$. It is straightforward that u satisfies $c_1 - c_3$ since it is the uniform limit of objects satisfying those conditions and that the space of strongly continuous mappings from $[0, T^*] \times E$ to R is complete while furnished with the sup norm.

We now show u satisfies c_5 from which, since $c_1(u) - c_3(u)$ and $c_5(u)$ implies $c_4(G(u))$, we get $c_4(u)$. But the fact that there exists a constant K such that

$$\begin{aligned} \|AG^{n+3} - AG^{n+2}\|_{\infty} &\leq K \|G^{n+2} - G^{n+1}\|_{\infty} \\ &\quad + K \|G^{n+1} - G^n\|_{\infty} \\ &\quad + \frac{1}{2} \|AG^{n+2} - AG^{n+1}\|_{\infty}. \end{aligned}$$

implies that AG^n is Cauchy in the sup norm. Thus, for every $0 \leq s \leq T^*$, $(G^n(s), AG^n(s))$ converges to a point in the graph of A since that graph is closed. In other words, u belongs to the domain of A .

Let us now prove that Au and $\frac{\partial}{\partial t}u$ are strongly continuous. Because

$$\frac{\partial}{\partial t}u_t(x) = Au_t(x) - \Phi(t, x, u_t(x))$$

then, Au is strongly continuous if and only if $\frac{\partial}{\partial t}u$ is strongly continuous¹⁴. But Au is the uniform limit of strongly continuous functions, and it therefore has to be strongly continuous as well.

Put now

$$\Phi^*(h, x, \lambda) = \Phi(T - h, x, \lambda),$$

and let

$$u_t^*(x) = S_t f(x) - \int_0^t S_r \left(\Phi^*(t - r, u_{t-r}^*) \right) (x) dr \text{ for } 0 \leq t \leq T^*.$$

Define

$$v_{t,T}(x) := u_{T-t}^*(x), \text{ for } t \in [(T - T^*) \vee 0, T].$$

Then, for $t \in [(T - T^*) \vee 0, T]$ we have:

$$\begin{aligned} v_{t,T}(x) &= S_{T-t} f(x) - \int_0^{T-t} S_r \left(\Phi^*(T - (r + t), u_{T-(r+t)}^*) \right) (x) dr \\ &= S_{T-t} f(x) - \int_0^{T-t} S_r \left(\Phi(r + t, u_{T-(r+t)}^*) \right) (x) dr \\ &= S_{T-t} f(x) - \int_0^{T-t} S_r \left(\Phi(r + t, v_{r+t,T}) \right) (x) dr \\ &= S_{T-t} f(x) - \int_t^T S_{r-t} \left(\Phi(r, v_{r,T}) \right) (x) dr \end{aligned}$$

And thus, the theorem is proven for t in the interval $[(T - T^*) \vee 0, T]$. To extend the result to the whole interval $[0, T]$, first replace $f(x)$ by $v_{T-T^*,T}(x)$, and replace T by $T - T^*$. Then get our "nice" solution $v_{t,T-T^*}(v_{T-T^*,T})(x)$ to

$$v_{t,T-T^*}(v_{T-T^*,T})(x) = S_{T-T^*-t}(v_{T-T^*,T})(x) - \int_t^{T-T^*} S_{r-t} \left(\Phi(r, v_{r,T-T^*}(v_{T-T^*,T})) \right) (x) dr$$

¹⁴According to Proposition 3.7.29, because u is strongly continuous and because $\Phi(s, x, \lambda)$ is locally in λ strongly continuous then $\Phi(x, u_t)$ is strongly continuous.

and define

$$v_{t,T}(x) := \begin{cases} v_{t,T-T^*}(v_{T-T^*,T})(x) & \text{for } t \in [(T - 2T^*) \vee 0, (T - T^*) \vee 0] \\ v_{t,T}(x) & \text{for } t \in [(T - T^*) \vee 0, T] \end{cases}$$

Then, as shown in [Da6, step 3, p.66], $v_{t,T}(x)$ is the *unique* solution to

$$v_{t,T}(x) = S_{T-t}f(x) - \int_t^T S_{r-t}(\Phi(r, v_{r,T}))(x)dr \text{ for } (T - 2T^*) \vee 0 \leq t \leq T.$$

And thus the Theorem is proven for the interval $[(T - 2T^*) \vee 0, T]$. By induction, the Theorem is verified on the whole interval $[0, T]$. $\square$

3.8 Appendix

3.8.1 Smoothness property of a class of log-Laplace functionals: details of the calculations...

We complete here the details of the proof of Theorem 3.7.30. Let us first introduce some new notation:

$$\begin{aligned} u_h &:= u_h(.) =: u(h) \\ \Phi(h, u) &:= \Phi(h, ., u_h) \\ \Phi(h, u(r)) &:= \Phi(h, ., u(r)) \\ \Phi'_s(h, x, \lambda) &:= \frac{\partial}{\partial s} \Phi(s, x, \lambda) \text{ evaluated at } s = h \\ \Phi'_x(h, \lambda) &:= \frac{\partial}{\partial x} \Phi(s, ., \lambda) \\ \Phi'_x(h, x, \lambda) &:= \frac{\partial}{\partial x} \Phi(s, y, \lambda) \text{ evaluated at } x = y \\ u'(h, y) &:= \frac{\partial}{\partial t} u(t, y) \text{ evaluated at } t = h \\ u'(h) &:= u'(h, .) \\ \Delta_h \Phi(r, u) &:= \Phi(r + h, u) - \Phi(r, u) \\ \Delta_h u(r) &:= u(r + h) - u(r) \end{aligned}$$

For further notation, refer to Theorem 3.7.30.

Lemma 3.8.30 $c_1(u)$ implies $c_1(G(u))$

Proof: By definition of $G(u)$ we have

$$\|G(u)\|_\infty \leq \max(\|\varphi\|_\infty, T^*M\|u\|_\infty)$$

but because $T^*M \leq \frac{1}{2}$, $c_1(u)$ implies $c_1(G(u))$. $\square$

Lemma 3.8.30 $c_1(u)$ and $c_2(u)$ implies $c_2(G(u))$

Proof:

$$\begin{aligned} \|G(u)(t) - G(u)(s)\|_\infty &\leq \|S_t\varphi - S_s\varphi\|_\infty + \left\| \int_s^t S_r[\Phi(t-r, u)]dr \right\|_\infty \\ &\quad + \left\| \int_0^s \{S_r[\Phi(t-r, u)] - S_r[\Phi(s-r, u)]\} dr \right\|_\infty \\ &\leq \|S_t\varphi - S_s\varphi\|_\infty + \left\| \int_s^t S_r[\Phi(t-r, u)]dr \right\|_\infty \\ &\quad + \left\| \int_0^s \{S_r[\Phi(s-r, u(t-r)) - \Phi(s-r, u)]\} dr \right\|_\infty \\ &\quad + \left\| \int_0^s \{S_r[\Phi(t-r, u(t-r)) - \Phi(s-r, u(t-r))]\} dr \right\|_\infty \\ &\leq \|A\varphi\|_\infty (t-s) + M(t-s) \\ &\quad + T^*M \sup_{r \in [0, s]} \|u(t-r) - u(s-r)\|_\infty \\ &\quad + T^*M(t-s) \end{aligned}$$

but because u is (uniformly) continuous so must be $G(u)(t)$ $\square$

Lemma 3.8.30 $c_1(u)$ and $c_3(u)$ implies $c_3(G(u))$

Proof: Let $0 \leq s \leq t \leq T^*$. We have:

$$\begin{aligned} |G(u)(s, x) - G(u)(t, x)| &\leq \left| \int_s^t S_r A\varphi(x) dr \right| + \left| \int_s^t dr S_r[\Phi(t-r, u)](x) \right| \\ &\quad + \left| \int_0^s dr S_r[\Phi(s-r, u) - \Phi(t-r, u)](x) \right| \\ &\leq \|A\varphi\|_\infty (t-s) + M(t-s) + T^*Mc(T^*, \Lambda, \varphi)(t-s) \end{aligned}$$

which is bounded by $c(T^*, \lambda, \varphi)(t-s)$ since $T^*M \leq \frac{1}{2}$. $\square$

Lemma 3.8.30 $c_1(u) - c_4(u)$ implies $c_5(G(u))$ and

$$\begin{aligned} AG(u(t)) = & S_t A\phi - \int_0^t S_r [\Phi'_s(t-r, u) + \Phi'_x(t-r, u)u'_s(t-r)]dr \\ & + S_t [\Phi(0, u(0))] - \Phi(t, u(t)) \end{aligned} \quad (3.8.44)$$

Proof: By definition of $G(u)$:

$$\begin{aligned}
\left(\frac{S(h)-I}{h}\right) G(u)(t) &= \left(\frac{S(h)-I}{h}\right) S_t \varphi - \int_0^t \frac{1}{h} S_{r+h}[\Phi(t-r, u)] - S_r[\Phi(t-r, u)] dr \\
&= \left(\frac{S(h)-I}{h}\right) S_t \varphi \\
&\quad - \frac{1}{h} \left(\int_0^{t+h} S_r[\Phi(t-r+h, u)] dr - \int_0^t S_r[\Phi(t-r, u)] dr \right) \\
&= \left(\frac{S(h)-I}{h}\right) S_t \varphi - \frac{1}{h} \left(\int_0^t S_r[\Phi(t-r+h, u)] - \Phi(t-r, u) dr \right) \\
&\quad - \frac{1}{h} \int_t^{t+h} S_r[\Phi(t-r+h, u)] dr + \frac{1}{h} \int_0^h S_r[\Phi(t-r, u)] dr
\end{aligned}$$

Then we have:

$$1. \left\| \left(\frac{S(h)-I}{h}\right) S_t \varphi - S_t A \varphi \right\|_\infty \rightarrow 0$$

2. If $I(h)$ is defined by

$$I(h) := \left\| \int_0^t S_r \left[\frac{\Delta_h \Phi(t-r, u)}{h} - \{ \Phi'_s(t-r, u) + \Phi'_x(t-r, u) u'_s(t-r) \} \right] dr \right\|_\infty$$

then $I(h) \rightarrow 0$.

Proof: The above integral is clearly bounded by

$$\int_0^t \left\| \frac{\Delta_h \Phi(t-r, u)}{h} - \Phi'_s(t-r, u) + \Phi'_x(t-r, u) u'_s(t-r) \right\|_\infty dr \quad (3.8.45)$$

but by Taylor's formula

$$\begin{aligned}
\frac{\Phi(t-r+h, u) - \Phi(t-r, u)}{h} &= \Phi'_s(t-r, u) + \Phi'_x(t-r, u) \frac{u(t-r+h) - u(t-r)}{h} \\
&\quad + \Phi''_s(\xi, \eta) h + \Phi''_{sx}(\xi, \eta) h \left(\frac{u(t-r+h) - u(t-r)}{h} \right) \\
&\quad + \Phi''_{xx}(\xi, \eta) h \left(\frac{u(t-r+h) - u(t-r)}{h} \right)^2
\end{aligned}$$

where $t-r \leq \xi \leq t-r+h$ and $u(t-r) \leq \eta \leq u(t-r+h)$.

The second derivatives of Φ (and therefore $\frac{u(t-r+h) - u(t-r)}{h}$) being bounded by M we have¹⁵

$$\left\| \frac{\Delta_h \Phi(t-r, u)}{h} - \Phi'_s(t-r, u) - \Phi'_x(t-r, u) \frac{\Delta_h u(t-r)}{h} \right\|_\infty \leq 3M^3 h.$$

¹⁵See the beginning of this subsection for notation.

Consequently, (3.8.45) can be bounded by

$$3T^* M^3 h + M \int_0^t \left\| \frac{u(t-r+h) - u(t-r)}{h} - u'_s(t-r) \right\|_\infty dr. \quad (3.8.46)$$

And since

$$\left\| \frac{u(t-r+h) - u(t-r)}{h} - u'_s(t-r) \right\|_\infty \leq 2M$$

we conclude from Lebesgue's Theorem, (3.8.46) and $c_4(u)$ that $I(h) \rightarrow 0$. $\square$

3. If $I(h)$ is defined by

$$I(h) := \left\| \frac{1}{h} \int_t^{t+h} S_r[\Phi(t-r+h, u(t-r+h))] dr - S_t[\Phi(0, u(0))] \right\|_\infty$$

then $I(h) \rightarrow 0$.

Proof:

$$\begin{aligned} I(h) &\leq \frac{1}{h} \int_t^{t+h} \|S_r[\Phi(t-r+h, u(t-r+h))] - S_t[\Phi(0, u(0))]\|_\infty dr \\ &\leq \frac{1}{h} \int_t^{t+h} \|S_r[\Phi(t-r+h, u(t-r+h))] - S_r[\Phi(0, u(0))]\|_\infty dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|S_t[\Phi(0, u(0))] - S_r[\Phi(0, u(0))]\|_\infty dr \\ &\leq \frac{1}{h} \int_t^{t+h} \|\Phi(t-r+h, u(0)) - \Phi(0, u(0))\|_\infty dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|\Phi(t-r+h, u(t-r+h)) - \Phi(t-r+h, u(0))\|_\infty dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|S_t[\Phi(0, u(0))] - S_r[\Phi(0, u(0))]\|_\infty dr \\ &\leq \frac{1}{h} \int_t^{t+h} \|\Phi'_s\|_\infty h dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|\Phi'_s\|_\infty \|u(t-r+h) - u(0)\|_\infty dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|S_t[\Phi(0, u(0))] - S_r[\Phi(0, u(0))]\|_\infty dr \\ &\leq h \|\Phi'_s\|_\infty + \frac{1}{h} \|\Phi'_r\|_\infty c(T^*, \lambda, \phi) \frac{1}{h} \int_t^{t+h} (t-r+h) dr \\ &\quad + \frac{1}{h} \int_t^{t+h} \|S_t[\Phi(0, u(0))] - S_r[\Phi(0, u(0))]\|_\infty dr \\ &\leq h \|\Phi'_s\|_\infty + \|\Phi'_r\|_\infty c(T^*, \lambda, \phi) h \\ &\quad + \frac{1}{h} \int_t^{t+h} \|S_t[\Phi(0, u(0))] - S_r[\Phi(0, u(0))]\|_\infty dr \end{aligned}$$

which converges to zero by the fundamental theorem of calculus.

4. The same reasoning shows that

$$\left\| \frac{1}{h} \int_0^h S_r[\Phi(t-r, u(t-r))] dr - \Phi(t, u(t)) \right\|_\infty \rightarrow 0$$

And we conclude that $G(u(t))$ is in the domain of A and that $AG(u)$ satisfies (3.8.44). $\square$

Lemma 3.8.30 $c_1(u) - c_3(u)$ and $c_5(u)$ implies $c_4(G(u))$ and

$$\frac{\partial}{\partial t}G(u)(t) = AG(u)(t) - \Phi(t, u(t)) \quad (3.8.47)$$

Proof:

$$\begin{aligned} \frac{u_{t+h} - u_t}{h} &= \left(\frac{S_h - I}{h}\right)(S_t f) - \frac{1}{h} \left(\int_0^{t+h} S_r [\Phi(t+h-r, u(t+h-r))] dr \right) \\ &\quad - \frac{1}{h} \left(\int_0^t S_r [\Phi(t-r, u(t-r))] dr \right) \\ &= \left(\frac{S_h - I}{h}\right)(S_t f) - \frac{1}{h} \left(\int_0^t \{S_{r+h}[\Phi(t-r, u(t-r))] - S_r[\Phi(t-r, u(t-r))]\} dr \right) \\ &\quad - \frac{1}{h} \int_0^h S_r [\Phi(t+h-r, u(t+h-r))] dr \\ &= \left(\frac{S_h - I}{h}\right)(S_t f - \int_0^t S_r [\lambda u^2(t-r)] dr) - \frac{1}{h} \int_0^h S_r [\Phi(t+h-r, u(t+h-r))] dr \\ &= \left(\frac{S_h - I}{h}\right)G(u)(t) - \frac{1}{h} \int_0^h S_r [\Phi(t+h-r, u(t+h-r))] dr \end{aligned}$$

which converges uniformly to $AG(u)(t) - \Phi(t, u(t))$ as shown in Lemma 3.8.30 and yields (3.8.47). $\square$

Lemma 3.8.30 $c_1(u) - c_5(u)$ and $c_2(\frac{\partial u}{\partial t})$ implies $c_1(G(u)) - c_5(G(u))$, $c_2(\frac{\partial G(u)}{\partial t})$ and $c_2(AG(u))$.

Proof: From the previous Lemmas, we already know that $c_1(u) - c_5(u)$ implies $c_1(G(u)) - c_5(G(u))$. It remains therefore only to show that if we add the condition $c_2(\frac{\partial u}{\partial t})$ then both $c_2(\frac{\partial G(u)}{\partial t})$ and $c_2(AG(u))$ will follow. But in fact, because of (3.8.47), it is enough to show that either $c_2(\frac{\partial G(u)}{\partial t})$ or $c_2(AG(u))$ is satisfied. We will show that $c_2(AG(u))$ is satisfied. Using (3.8.44) we get

$$\begin{aligned} AG(u(t)) &= S_t A\phi - \int_0^t S_r [\Phi'_s(t-r, u) + \Phi'_r(t-r, u)u'_s(t-r)] dr \\ &\quad + S_t [\Phi(0, u(0))] - \Phi(t, u(t)). \end{aligned}$$

Clearly, $S_t A\phi$ and $S_t [\Phi(0, u(0))]$ are strongly continuous, since $A\phi$ and $\Phi(0, u(0))$ belongs to L , the set of functions for which the semigroup S is strongly continuous.

From Proposition 3.7.29, the function $\sigma(t, x) := \Phi'_s(t - r, u) + \Phi'_x(t - r, u)u'_s(t - r)$ is strongly continuous. Consequently, given and $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\sigma(t, \cdot) - \sigma(s, \cdot)\|_\infty < \epsilon$. For $|t - s| < \delta$ we therefore have

$$\begin{aligned} \sup_{x \in E} \left| \int_0^t S_r[\sigma(t, \cdot)](x) dr - \int_0^s S_r[\sigma(s, \cdot)](x) dr \right| &\leq \int_s^t \|S_r[\sigma(t, \cdot)]\|_\infty \\ &\quad + \int_0^s S_r[\|\sigma(t, \cdot) - \sigma(s, \cdot)\|_\infty] dr \\ &\leq (t - s) \|S_r[\sigma(t, \cdot)]\|_\infty \\ &\quad + \epsilon s \\ &\leq (t - s) \|S_r[\sigma(t, \cdot)]\|_\infty \\ &\quad + \epsilon T^*. \end{aligned}$$

Therefore $\int_0^t S_r[\sigma(t, \cdot)](x) dr$ is strongly continuous, and the proof is complete. $\square$

Lemma 3.8.30 $c_1(u)$ and $c_1(w)$ implies

$$\|G(u)(s, x) - G(w)(s, x)\|_\infty \leq \frac{1}{2} \|u - w\|_\infty \quad (3.8.48)$$

Proof: By definition of $G(u)$ we have:

$$\begin{aligned} |G(u)(t, x) - G(w)(t, x)| &\leq \left| \int_0^t S_r[\Phi(t - r, u(t - r)) - \Phi(t - r, w(t - r))](x) dr \right| \\ &\leq T^* M \|u - w\|_\infty \end{aligned}$$

from what (3.8.48) follows since $T^* M \leq \frac{1}{2}$. $\square$

Lemma 3.8.30 Assume $c_1(u) = c_5(u)$ and $c_1(w) = c_5(w)$. Then there exists a constant K (depending on T^*, Λ and ϕ) such that

$$\begin{aligned} \|AG(G(u)) - AG(G(w))\|_\infty &\leq K \|G(u) - G(w)\|_\infty \\ &\quad + K \|u - w\|_\infty \\ &\quad + \frac{1}{2} \|AG(u) - AG(w)\|_\infty \end{aligned} \quad (3.8.49)$$

Proof: Combining the (3.8.44) and (3.8.47) one obtains the formula

$$\begin{aligned} AG(G(u))(t) &= S_t A\varphi - \int_0^t S_r [\Phi'_s(t - r, G(u)(t - r))] \\ &\quad - \int_0^t S_r [\Phi'_x(t - r, G(u)(t - r)) (AG(u)(t - r) - \Phi(t - r, u(t - r)))] \\ &\quad + S_t \Phi(0, G(u)(0)) - \Phi(t, G(u)(t)). \end{aligned}$$

Note that from (3.8.47) and Lemma 3.8.30

$$\|AG(u)\|_{\infty} \leq M + c(T^*, \Lambda, \phi)$$

from which :

$$\begin{aligned} \|AG(G(u)) - AG(G(w))\|_{\infty} &\leq \int_0^t S_r [\|\Delta\Phi'_s(t-r, G(\cdot))(t-r)\|_{\infty}] dr \\ &\quad + \int_0^t S_r [\|\Phi'_x(t-r, Gu)\|_{\infty} \|\Delta AG(\cdot)(t-r)\|_{\infty}] dr \\ &\quad + \int_0^t S_r [\|\Phi'_x(t-r, Gu)\|_{\infty} \|\Delta\Phi(t-r, \cdot)\|_{\infty}] dr \\ &\quad + \int_0^t S_r [\|\Delta\Phi'_x(t-r, G)\|_{\infty} \|AG(w)(t-r) - \Phi(t-r, w)\|_{\infty}] dr \\ &\quad + \|\Delta\Phi(0, G)\|_{\infty} + \|\Delta\Phi(t, G)\|_{\infty}. \end{aligned}$$

Therefore

$$\begin{aligned} \|AG(G(u)) - AG(G(w))\|_{\infty} &\leq MT^* \|G(u) - G(w)\|_{\infty} \\ &\quad + M^2 T^* \|AG(u) - AG(w)\|_{\infty} \\ &\quad + M^2 T^* \|u - w\|_{\infty} \\ &\quad + MT^* (M + c(T^*, \lambda, \phi)) \|G(u) - G(w)\|_{\infty} \\ &\quad + 2M \|G(u) - G(w)\|_{\infty} \end{aligned}$$

from which (3.8.49) follows by the choice of $T^* M \leq \frac{1}{2}$. $\square$

3.8.2 Review of classical results on martingale theory and related topics

Let $(\Omega, \{\mathfrak{F}\}_{t \geq 0}, P)$ be a filtered probability space. A substantial part of the results in the general theory of stochastic processes depends on the following conditions which are called the *usual hypotheses*:

1. P is complete;
2. $\mathfrak{F}_t$ contains all sets of P -measure 0;
3. $\mathfrak{F}_t$ is right continuous, i.e., $\mathfrak{F}_t$ coincide with the intersection of $\mathfrak{F}_u$ over all $u > t$.

The following consequences of those hypotheses are fundamental results of the martingale theory.

Conseq1 A predictable process X_t of class (D) has a.s., left limits X_{t-} for all t if and only if the limit of $\lim EX_{\tau_n}$ exists for every increasing sequence of stopping times τ_n . It is left continuous if and only if, in addition, $\lim EX_{\tau_n} = E[\lim X_{\tau_n}]$. (see [DeM, VI.49 and 50(i)] for the proof).

Conseq2 Two predictable processes X and Y of class (D) are indistinguishable if and only if $EX_\tau = EY_\tau$ for every predictable stopping time, i.e. for every stopping time τ which is the limit of a sequence of increasing stopping times strictly smaller than τ . (see [DeM, IV.86 and 87(b)] for the proof)

Conseq3 All predictable processes are optional and all optional processes are progressively measurable. (see [DeM, IV.67 and 64] for the proof).

Conseq4 An optional process X_t of class (D) has a.s., right limits X_{t+} for all t if and only if the limit of $\lim EX_{\tau_n}$ exists for every decreasing sequence of stopping times τ_n . It is right continuous if and only if, in addition, $\lim EX_{\tau_n} = E[\lim X_{\tau_n}]$. (see [DeM, VI.48] for the proof).

Conseq5 Two optional processes X and Y of class (D) are indistinguishable if and only if $EX_\tau = EY_\tau$ for all stopping times τ . (see [DeM, IV.86 and 87(b)] for the proof).

Conseq6 Every positive right continuous submartingale X belongs to class (D) . (See [Dy7, Lemma A.1.1] for the proof).

Conseq7 Let X_t be a supermartingale. If the mapping $t \mapsto E(X_t)$ is right continuous then X_t has a right continuous modification.

Theorem (Optional Sampling Theorem) Let X_t be a supermartingale. Then:

- (i) if τ and σ are bounded stopping times taking only finitely many values then we have $E^{\mathfrak{F}_\sigma}(X_\tau) \leq X_{\tau \wedge \sigma}$ with equality if X is a martingale. (See [EtK, Lemma 2.2.2] for proof)
- (ii) if X_t is a right continuous supermartingale then for every bounded stopping time τ and σ we have that $E^{\mathfrak{F}_\sigma}(X_\tau) \leq X_{\tau \wedge \sigma}$ with equality if X is a martingale. (See [EtK, Th.2.2.13] for proof)

Theorem (Doob-Meyer Decomposition) Let X_t be a right continuous supermartingale of class (D) for $t \in [a, b]$. Then there exists a unique predictable right continuous increasing process A_t with $A_a = 0$ and $E A_b < \infty$ such that $M = X + A$ is a martingale. Moreover, for all $r < t$ in $[a, b]$,

$$A_t^r := A_t - A_r = \lim_{\Lambda} \sum_{i=0}^{n-1} E\{(X_{t_i} - X_{t_{i+1}}) | \mathfrak{F}_{t_i-}\}$$

weakly in $L^1(P)$ as $\Lambda = \{r = t_0 < t_1 < \dots < t_n = t\}$ runs over a sequence of partitions $\Lambda_n \subseteq \Lambda_{n+1}$ with $\text{mesh}\{\Lambda_n\} \rightarrow 0$. Moreover, the convergence is strong if A is continuous.

The following condition is necessary and sufficient for continuity of $A : EX_{\tau_n} \rightarrow EX_\tau$ for every sequence of stopping times $\tau_n \uparrow \tau$, such that $\tau_n < \tau$. (See [Dy7, Th. A.1.1] for the proof).

Semimartingale Decomposition A process X is a semimartingale if it admits a decomposition

$$X = M + A$$

where M is a local martingale and A is an adapted right continuous process of finite variation. The decomposition is not unique but the continuous martingale part of M is determined uniquely by X . It is called the continuous martingale part of X and it is denoted by X^c . (See [Dy7, Appendix Section 2.2]).

Ito's formula Suppose that $X^1, \dots, X^n$ are semimartingales in the same filtered probability space. Let $X = (X^1, \dots, X^n)$ and let f be a function of class C^2 in R^n . Then

$$f(X_t) = f(X_0) + \int_0^t \sum_i D_i f(X_{s-}) dX_s^i + \frac{1}{2} \sum_{i,j} D_i D_j f(X_{s-}) d \langle X^{ic}, X^{jc} \rangle_s - \sum_{s \leq t} [f(X_s) - f(X_{s-}) - \sum_i D_i f(X_{s-}) \Delta X_s^i]$$

where X^{ci} is the continuous martingale part of X^i and $D_i f$ means the partial derivative of f with respect to x_i . (See [DeM, VIII.25, 56(c) and 27] for the proof).

3.8.3 Review of Ray-Knight compactification, Ray topology and related topics

The following is essentially taken from [Sh, II.17; V.47 and A2] and the reader may consult that source for more details.

Definition 3.8.30 *A metric space (E, d) is said to be a metric Luzin space if it is homeomorphic to a compact metric space.*

Proposition 3.8.30 *Let (E, d) be a metric Luzin space. Then there exists a countable set of bounded uniformly continuous functions $\{f_n\}$ which is uniformly dense in the set of uniformly continuous functions on (E, d) . The topology generated by d is the same as the topology generated by the uniformly continuous functions. And the metric d' defined by*

$$d'(x, y) := \sum \frac{1}{2^n \|f_n\|_\infty} |f_n(x) - f_n(y)| \quad (3.8.50)$$

is equivalent to d in the sense that it generates the same topology.

Proof: Left to the reader. $\square$

For now on, in this section, let (E, d) be a metric Luzin space and $\mathcal{E}$ the universal completion of its d -Borel σ -algebra. Let $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$ be a E -valued Hunt process

with Markov resolvent $\{U^\alpha\}_{\alpha>0}$. That is, the U^α are kernels on $(E, \mathcal{E})$ satisfying

$$U^\alpha = U^\beta + (\beta - \alpha)U^\alpha U^\beta \text{ for every } \beta > 0$$

$$\alpha U^\alpha 1_E = 1_E$$

Let Q denote the set of rational numbers, Q^+ the set of non-negative members of Q and Q^{++} the set of strictly positive members of Q . The Q -cone generated by a family Y of positive bounded functions is defined to be the set of Q linear combinations of members of Y . Given a Q -cone Y set

$$\Lambda(Y) := \{k_1 \wedge \dots \wedge k_n : n \geq 1, k_i \in Y\}$$

$$\mathcal{U}(Y) := \{U^{\alpha_1} k_1 \wedge \dots \wedge U^{\alpha_n} k_n : n \geq 1, \alpha_i \in Q^{++}, k_i \in Y\}.$$

Let C' be a family of functions satisfying

1. C' is a countable set of non negative d -uniformly continuous functions
2. 1_E belongs to C'
3. the linear span of C' is uniformly dense in $C_d(E) :=$ the set of d -uniformly continuous functions of E .

Let $\mathcal{H}$ the Q -cone generated by C' . Let

$$\mathcal{R}_0 := \bigwedge(\mathcal{H}).$$

Let

$$\mathcal{R}_{n+1} := \bigwedge \{\mathcal{R}_n + \mathcal{U}(\mathcal{R}_n)\}$$

where $\{\mathcal{R}_n + \mathcal{U}(\mathcal{R}_n)\} := \{f + g : f \in \mathcal{R}_n, g \in \mathcal{U}(\mathcal{R}_n)\}$. Set

$$\mathcal{R} := \bigcup_{n \geq 1} \mathcal{R}_n.$$

Then $\mathcal{R}$ is called the rational ray cone generated by C' and $\{U^\alpha\}$. $\mathcal{R}$ is countable,

$\mathcal{R} := \{g_1, g_2, \dots\}$ and the metric

$$\rho(x, y) := \sum \frac{1}{2^n \|g_n\|_\infty} |g_n(x) - g_n(y)|$$

defines a metric on $(E, \mathcal{E})$, which generates $\mathcal{E}$ as its Borel σ -algebra. Moreover, the completion of (E, ρ) is a compact space, and the Ray topology, the topology generated by ρ , is independent of the choice of C' . Remark that (E, ρ) is obviously a metric Luzin space.

Recall that we assume (E, d) is a metric Luzin space and $\mathcal{E}$ is the d -Borel σ -algebra. Recall also that $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$ is a Hunt process. Let ρ be the metric of the Ray topology; then we have:

Theorem 3.8.31 *The Hunt property of ξ is preserved by a passage to the Ray topology.*

Proof: See [Sh, Th. V.47.5]. $\square$

Corollary 3.8.31 *The Hunt property of ξ is preserved by a passage to smallest topology generated by d and ρ .*

Proof: Let $\{f_n\}$ be a countable set of bounded uniformly continuous functions which is uniformly dense in the set of uniformly continuous functions on (E, d) . Let $\mathcal{R} := \{g_1, g_2, \dots\}$. Put

$$H := \{f_n\} \cup \{g_n\} := \{h_n\}.$$

Then ξ is obviously a Hunt process in the topology generated the metric $\Delta(x, y)$ defined by

$$\Delta(x, y) := \sum \frac{1}{2^n \|h_n\|_\infty} |h_n(x) - h_n(y)|$$

which is precisely the topology generated by d and ρ . $\square$

Chapter 4

Appendix: general particle systems

4.1 Colored trees

Let Δ be a countable set of colors, and $(D, \mathcal{D})$ a measured space. We want to construct a measured space describing the set of trees. We think of the trees as being genealogical trees: starting from one individual, is issued a set of offsprings which constitute the *first generation*. The set of all descendants of the individuals of the first generation constitute the *second generation*, and so on. To keep the image of a tree, we also think of a generation as being constituted of a set of branches. Hence we may talk indistinguishably of the *individuals in the k^{th} generation* or of the *branches in the k^{th} generation*. Similarly, we may talk of the *individuals of a tree* or of the *branches of a tree*. Each branch of the tree is characterized by a color τ , a point $d \in D$ and a vector $\tilde{n} \in N_f^\Delta := \{\tilde{n} \in \mathbf{N}^\Delta, \text{ with } \tilde{n} \text{ having only finitely many components different than zero}\}$ describing the number of offspring of different colors that the branch has had.

It may occur that a branch of a tree has no offspring. Such a branch is called a *terminating branch*. It will turn out later on that those terminating branches complicate the construction of random colored trees. For that reason we will artificially prolong the terminating branches: remark first that a terminating branch always has the form $(d, \tau, \vec{0})$; we prolong it by gluing on it another branch $(d, \tau, -1)$ and on

this branch gluing again another branch $(d, \tau, -2)$ and so on...

With this in mind, we define the "Branch Space". $(B, \mathcal{B})$ by

$$B := \{(d, \tau, \tilde{n}) : d \in D, \tau \in \Delta, \tilde{n} \in \bar{N}_f^\Delta\}$$

where $\bar{N}_f^\Delta := N_f^\Delta \cup \{-1, -2, -3, \dots\}$. We put on B the smallest sigma algebra $\mathcal{B}$, containing the subsets of B having the form

$$\{(d, \tau, \tilde{n}) : d \in A \in \mathcal{D}, \tau \in \Delta, \tilde{n} \in \bar{N}^\Delta\}.$$

We also define the "Generation Space", $(G, \mathcal{G})$ by

$$\begin{aligned} G &:= \bigcup_{1 \leq p < \infty} \bigotimes_{i=1}^p B \\ \mathcal{G} &:= \sigma \left(\bigcup_{1 \leq p < \infty} \bigotimes_{i=1}^p \mathcal{B} \right). \end{aligned}$$

Finally we define the set of trees and its sigma algebra, the "Tree Space" $(\mathcal{T}, \mathfrak{Z})$, by

$$\mathcal{T} = \prod_{k=0}^{\infty} G_k, \quad \mathfrak{Z} = \prod_{k=0}^{\infty} \mathcal{G}_k$$

where

$$(G_k, \mathcal{G}_k) := (G, \mathcal{G}).$$

For each tree T in $\mathcal{T}$, we have the representation:

$$T = (g^k)_{k \geq 0} \text{ where } g^k = \left(b_i^k \right)_{i=1}^{p_k} \text{ where } b_i^k = (d_i^k, \tau_i^k, \tilde{n}_i^k) \text{ and } \tilde{n}_i^k = \left(n_{\tau_i}^k \right)_{\tau \in \Delta}.$$

We think T as being the tree. The tree can be decomposed into generations, $T = (g^k)_{k \geq 0}$. In fact, g^0 is the initial generation, g^1 is the set of branches issued from the branches in the initial generation g^0 , and so on. Each generation g^k is constituted of a certain number p_k of branches $b_i^k : g^k = \left(b_i^k \right)_{i=1}^{p_k}$. Now each branch describes the life of one individual: it says the state d_i^k in which the individual has lived, it says the color τ_i^k the individual has¹, and finally $\tilde{n}_i^k = \left(n_{\tau_i}^k \right)_{\tau \in \Delta}$ says how many offspring of each color the individual has had.

¹ may be we should talk about the color the individual was associated to

Notation: We will write $\vec{n} \leq 0$ when $\vec{n} = \vec{0}$ or $\vec{n} \notin \mathbf{N}^\Delta$. We define similarly $\vec{n} < 0$, $\vec{n} > 0$, $\vec{n} \geq 0$, etc. If $\vec{n} \leq 0$ and k is a integer, then $\vec{n} - k$ is defined in a trivial manner².

Notation:

$$\begin{array}{ll}
B_{\vec{n}} := \{(d, \tau, \vec{k}) \in B : \vec{k} = \vec{n}\} & B_{\vec{n}} := B \cap B_{\vec{n}} \\
B_{\vec{n}} := \{(d, \zeta, \vec{n}) \in B : \zeta = \tau\} & B_{\vec{n}} := B \cap B_{\vec{n}} \\
B_{\tau, \vec{n}} := \{(d, \zeta, \vec{k}) \in B : \zeta = \tau \text{ and } \vec{k} = \vec{n}\} & B_{\tau, \vec{n}} := B \cap B_{\tau, \vec{n}} \\
B_{\tau, \geq 0} := \{(d, \zeta, \vec{k}) \in B : \zeta = \tau \text{ and } \vec{k} \geq 0\} & B_{\tau, \geq 0} := B \cap B_{\tau, \geq 0} \\
B_{\geq 0} := \{(d, \tau, \vec{k}) \in B : \vec{k} \geq 0\} & B_{\geq 0} := B \cap B_{\geq 0}
\end{array}$$

4.2 Random colored rooted trees (RCRT) and discrete Markov chains

We think of a Random Colored Rooted Tree³ as a triplet $(\mathcal{T}, \mathfrak{Z}, P)$ where $(\mathcal{T}, \mathfrak{Z})$ is the Tree Space describe in the previous section and P is a probability on that (measured) space. Not all the members T in $\mathcal{T}$ describe *admissible trees*: for instance if the initial generation g^0 in T is constituted of the element $(b, \tau, \vec{0})$, then we want the first generation g^1 to be equal to $(b, \tau, -1)$. But the Tree Space contains elements for which this is not the case. Therefore we will define a Random Colored Rooted Tree as a triplet $(\mathcal{T}, \mathfrak{Z}, P)$ where P is concentrated on the set of admissible trees.

A particular class of random colored rooted trees is the one where the random variables g^k form a (discrete) Markov chain with transition probability $P_{k+1}^k(\cdot, \cdot)$ from $(G_k, \mathcal{F}_k)$ to $(G_{k+1}, \mathcal{F}_{k+1})$. As it is always the case for Markov chains, we construct it via Ionescu-Tulcea's Theorem⁴. What Ionescu-Tulcea's Theorem says is that there

²that is if for instance $\vec{n} = -8$ and $k = 7$, then $\vec{n} - k = -15$.

³that we abbreviate by RCRT

⁴Proposition V.2.1 page 168, Neveu

exists probabilities $P_{k,g^k}(\cdot)$ on $(\mathcal{T}, \mathfrak{F})$ such that

$$P_{k,g^k}(Y) = \int_{G_{k+1}} P_{k+1}^k(g^k, dg^{k+1}) \dots \int_{G_N} P_N^{N-1}(g^{N-1}, dg^N) Y(g^k, \dots, g^N) \quad (4.2.1)$$

for all measurable Y . Moreover the mapping $g^k \mapsto P_{k,g^k}(Y)$ is G_k measurable and if $G_k = G_{k+1}$ and $P_{k+1}^k = P_{k+2}^{k+1}$ then $P_{k,g^k}(\cdot) = P_{k+1,g^{k+1}}(\cdot)$.⁵

Therefore, for any probability P^0 on $(G_0, \mathcal{F}_0)$ we can associate a probability P on $(\mathcal{T}, \mathfrak{F})$ given by

$$P := P^0(P_{g^0}(\cdot)) \quad (4.2.2)$$

where $P_{g^0}(\cdot) = P_{0,g^0}(\cdot)$ is given by (4.2.1).

Definition 4.2.31 (Admissibility for RCRT) A tree $T = (g^k)_{k \geq 0} \in \mathcal{T}$ is called a rooted tree if

$$g^0 \in B_{\geq 0} = \{(d, \tau, \vec{k}) \in B : \vec{k} \geq 0\} \subseteq G_0^6.$$

A tree $T = (g^k)_{k \geq 0} \in \mathcal{T}$ is called an admissible tree if for every $k \geq 0$ we have that $g^{k+1} = (b_j^k)_{j=1}^{p_k} \in G(g^k)$ where $\bigotimes_{j=1}^{p_k} G(b_j^k)$ is defined by

$$G(b, \zeta, \vec{n}) := \begin{cases} \bigotimes_{\substack{\tau \in \Delta \\ n_\tau \neq 0}} \bigotimes_{i=1}^{n_\tau} B_{\tau, \geq 0} & : \vec{n} > 0 \\ \{(b, \zeta, \vec{n} - 1)\} & : \vec{n} \leq 0 \end{cases} \quad (4.2.3)$$

and where $B_{\tau, \geq 0} = \{(b, \xi, \vec{k}) \in B : \xi = \tau, \vec{k} \geq 0\}$ and $\vec{n} = (n_\tau)_{\tau \in \Delta}$. (Note that $G(d, \tau, \vec{n})$ can be thought as the set of admissible offspring issued from $(d, \tau, \vec{n})$)

Let's now formally define a RCRT:

Definition 4.2.31 Assume that a probability P^0 on $(G_0, \mathcal{F}_0)$ is given together with transition probabilities P_{k+1}^k from $(G_k, \mathcal{F}_k)$ to $(G_{k+1}, \mathcal{F}_{k+1})$. Assume also that P is given by (4.2.2). Then the triplet $(\mathcal{T}, \mathfrak{F}, P)$ is called a **Random Colored Rooted Tree with colors Δ , state space $(D, \mathcal{D})$, transition probability P_{k+1}^k and initial distribution P^0** if:

⁵By definition, $P_{k,g^k}(Y)$ is defined on G_k but $G_k = G_0$, and most of the time no confusion is possible, so we will simply write $P_{g^k}(Y)$ instead of $P_{k,g^k}(Y)$.

RCRT1) P -almost surely $T = (g^k)_{k \geq 0}$ has a single root, or in other words: P^0 is concentrated on the subset $B_{\geq 0} = \{(d, \tau, \vec{k}) \in B : \vec{k} \geq 0\}$ of G_0 .⁷

RCRT2) P is concentrated on the set of $\mathcal{T}$ describing admissible colored trees, or in other words for any $g = (d_i, \tau_i, \vec{n}_i)_{i=1}^p \in G$,

$$P_{k+1}^k \left(g, \bigotimes_{i=1}^p G(d_i, \tau_i, \vec{n}_i) \right) = 1$$

where $G(d, \tau, \vec{n})$ is defined by (4.2.3)

The above definition can obviously be generalized to the case where $(\mathcal{T}, \mathfrak{I}, P)$ is not constructed as a Markov chain:

Definition 4.2.31 We say that a triplet $(\mathcal{T}, \mathfrak{I}, P)$ is a **Random Colored Rooted Tree** $(\mathcal{T}, \mathfrak{I}, P)$, with colors Δ and state space $(D, \mathcal{D})$ if the following conditions are satisfied:

- P -almost surely $T = (g^k)_{k \geq 0}$ has a single root
- P is concentrated on the set of $\mathcal{T}$ describing admissible trees

In the next Section, the Multitype Particle Systems are defined as particular cases of the above Random Colored Rooted Trees.

4.3 Multitype Particle System

We will define a Multitype Particle System as being a special case of the RCRT that we obtained in the previous section as discrete Markov chains. In fact, to obtain them, we only consider the particular case where the state space $(\mathcal{D}, D)$ of the first section is equal to $(\mathcal{W}, \mathcal{L})$, where $(\mathcal{W}, \mathcal{L})$ is called the space of trajectories and is defined by:

⁷Recall that $G_0 = \bigcup_{1 \leq p < \infty} \bigotimes_{i=1}^p B$.

Definition 4.3.31 Let $(E, \mathcal{E})$ be a measured space. Let

$$W := \prod_{t \in R} E$$

We pose

$$\mathcal{W} := \{(\alpha, w, \beta) : \alpha < \beta \leq \infty, w \in W\}.$$

For any $w \in W$, let w_t denote the t^{th} projection on E . We define a sigma algebra $\mathcal{L}$ on $\mathcal{W}$ by taking the sigma algebra generated by the functions

$$F(\alpha, w, \beta) = f^\alpha(w_\alpha) 1_{\alpha < t_1 < t_2 < \beta} Y g^\beta(w_\beta)$$

where f, g are progressively measurable, and where Y is measurable with respect to the universal completion of $\mathcal{A}(t_1, t_2)$, where $\mathcal{A}(t_1, t_2) := \sigma(w_t : t \in (t_1, t_2))$.

(We keep the same definition as in the previous sections for the Branch Space $(\mathcal{B}, B)$, the Generation Space $(\mathcal{G}, G)$ and the Tree Space $(\mathcal{T}, \mathfrak{T})$). Remark that each branch of a tree $T \in \mathcal{T}$ contains the complete information about a particle; it contains:

- its birth time α , its death time β and its trajectory w between α and β .
- its type τ .
- its number of offspring of each type, $\vec{n}$.

It may happen that a particle never die. We want that those particle produce no offspring. Hence a tree $T = (g^k)_{k \geq 0}$ for which g^0 is, for example, equal to $(\alpha, w, \infty, \tau, \vec{n})$ should not have offsprings, and therefore we want that $(\alpha, w, \infty, \tau, \vec{n}) = (\alpha, w, \infty, \tau, 0)$. We call those branches the *admissible branches* and we denote them by:

$$\hat{B} := \{(\alpha, w, \beta, \tau, \vec{n}) \in B : \beta < \infty \text{ or } \vec{n} \leq 0\}.$$

Definition 4.3.31 (Admissibility for MPS) A branch $b = (\alpha, w, \beta, \tau, \vec{n})$ is called admissible if $b \in \hat{B}$. A generation $g = (b_j)_{j=1}^{p_k} \in G$ is called admissible if b_j is admissible for $j = 1, \dots, p_k$. A tree $T = (g^k)_{k \geq 0} \in \mathcal{T}$ is called a rooted tree (with admissible root) if $g^0 \in \hat{B}_{\geq 0} = \{(d, \tau, \vec{k}) \in \hat{B} : \vec{k} \geq 0\} \subseteq G_0$. A tree $T = (g^k)_{k \geq 0} = ((b_j^k)_{j=1}^{p_k})_{k \geq 0} \in \mathcal{T}$ is called an admissible tree if for every $k \geq 0$ we have that $g^{k+1} \in \hat{G}(g^k)$ where

$$\hat{G}(g^k) = \bigotimes_{j=1}^{p_k} \hat{G}(b_j^k)$$

where $\hat{G}(\alpha, w, \beta, \zeta, \vec{n})$ is defined by

$$\hat{G}(\alpha, w, \beta, \zeta, \vec{n}) := \begin{cases} \bigotimes_{\substack{\tau \in \Delta \\ \vec{n}_\tau \neq 0}} \bigotimes_{v=1}^{\vec{n}_\tau} \hat{B}_{\tau, \geq 0} & \vec{n} > 0 \\ \{(\alpha, w, \beta, \zeta, \vec{n} - 1)\} & \vec{n} \leq 0 \end{cases} \quad (4.3.4)$$

and where $\hat{B}_{\tau, \geq 0} = \{(\alpha, w, \beta, \xi, \vec{k}) \in \hat{B} : \xi = \tau, \vec{k} \geq 0\}$ and $\vec{n} = (n_\tau)_{\tau \in \Delta}$.

Let us now define the Multitype Particle Systems:

Definition 4.3.31 A Colored Random Rooted Tree $(T, \mathcal{Z}, P)$ with colors Δ , state space $(\mathcal{W}, \mathcal{L})$, transition probabilities $P_{k+1}^k(\dots)$ and initial probability $\hat{P}$ is said to be a Multitype Particle System with types Δ , state space $(E, \mathcal{E})$, transition probabilities $P_{k+1}^k(\dots)$ and initial distribution $\hat{P}$ if P is given by (4.2.2) and:

MPS1) P -almost surely ω has a possible root: P is concentrated on

$$\bigcup_{\vec{n} \geq 0} \hat{B}_{\vec{n}} = \hat{B}_{\geq 0}$$

MPS2) P is concentrated on the subset of $\mathcal{T}$ describing possible trees: that is, if $(\alpha_i^k, w_i^k, \beta_i^k, \tau_i^k, \vec{n}_i^k)$ is an admissible branch for $i = 1, 2, \dots, p_k$ then, $P_{k+1}^k \left((\alpha_i^k, w_i^k, \beta_i^k, \tau_i^k, \vec{n}_i^k)_{i=1}^{p_k}, \cdot \right)$ is concentrated on $\bigotimes_{i=1}^{p_k} \hat{G}(\alpha_i^k, w_i^k, \beta_i^k, \tau_i^k, \vec{n}_i^k)$ where

$$\hat{G}(\alpha, w, \beta, \zeta, \vec{n}) := \begin{cases} \bigotimes_{\substack{\tau \in \Delta \\ \vec{n}_\tau \neq 0}} \bigotimes_{v=1}^{\vec{n}_\tau} \hat{B}_{\tau, \geq 0} & \vec{n} > 0 \\ \{(\alpha, w, \beta, \zeta, \vec{n} - 1)\} & \vec{n} \leq 0 \end{cases}$$

and where again $\hat{B}_{\tau, \geq 0} = \{(\alpha, w, \beta, \xi, \vec{k}) \in \hat{B} : \xi = \tau, \vec{k} \geq 0\}$ and $\vec{n} = (n_\tau)_{\tau \in \Delta}$. Note that $G(\alpha, w, \beta, \tau, \vec{k})$ can be thought as the set of admissible offspring issued from $(\alpha, w, \beta, \tau, \vec{k})$.

4.4 Multitype Branching Particle System

In this section we will define the Multitype Branching Particle System as a special case of the Multitype Particle Systems above. We assume now that a family of kernels $\{\hat{\pi}_b^\tau(\cdot)\}_{\tau \in \Delta}$ is given such that

$$(MBPS) \begin{cases} b \in \hat{B}_{>0} & \Rightarrow \hat{\pi}_b^\tau(\hat{B}_{\geq 0}) = 1 \\ (\alpha, w, \beta, \tau, \vec{n}) \in \hat{B}_{\leq 0} & \Rightarrow \hat{\pi}_{(\alpha, w, \beta, \tau, \vec{n})}^\tau(\cdot) = \delta_{(\cdot, \alpha, w, \beta, \tau, \vec{n}-1)}(\cdot) \\ b \in B - \hat{B} & \Rightarrow \hat{\pi}_b^\tau(\cdot) = \delta_b(\cdot) \\ \hat{\pi}_{(\alpha, w, \beta, \tau, \vec{n})}^\tau(\{(\alpha^0, w^0, \beta^0, \tau^0, \vec{n}^0) : \alpha^0 = \beta\}) = 1. \end{cases}$$

We will now construct the $P_{k+1}^k(g^k, \cdot)$. Assume first that $g^k = (\alpha^k, w^k, \beta^k, \tau^k, \vec{n}^k)$, that is, that the k^{th} generation has only one member. Define

$$P_{k+1}^k(g^k, \cdot) := \begin{cases} = \bigotimes_{\substack{\zeta \in \Delta \\ n_\zeta^k \neq 0}} \bigotimes_{u=1}^{n_\zeta^k} \hat{\pi}_{g^k}^\zeta(\cdot) & : \beta^k < \infty, \vec{n}^k > 0 \\ = \delta_{(\alpha^k, w^k, \beta^k, \tau^k, \vec{n}^k-1)}(\cdot) = \hat{\pi}_{g^k}^{\tau^k}(\cdot) & : \vec{n}^k \leq 0 \\ = \delta_{(\alpha^k, w^k, \beta^k, \tau^k, \vec{n}^k)}(\cdot) = \hat{\pi}_{g^k}^{\tau^k}(\cdot) & : \beta^k = \infty, \vec{n}^k > 0 \end{cases} \quad (4.4.5)$$

Recall that

$$g^k = (b_j^k)_{j=1}^{p_k}$$

where

$$b_i^k = (\alpha_i^k, w_i^k, \beta_i^k, \tau_i^k, \vec{n}_i^k).$$

Define

$$P_{k+1}^k(g^k, \cdot) := \bigotimes_{j=1}^{p_k} P_{k+1}^k(b_j^k, \cdot)$$

The $P_{k+1}^k(.,.)$ are probability transitions $P_{k+1}^k(g^k, dg^{k+1})$ from $(G_k, \mathcal{G}_k)$ to $(G_{k+1}, \mathcal{G}_{k+1})$. Exactly as above, we can therefore define probabilities P_{k,g^k} on $(\mathcal{T}, \mathfrak{Z})$ by:

$$P_{k,g^k}(Y) = \int_{G_{k+1}} P_{k+1}^k(g^k, dg^{k+1}) \dots \int_{G_N} P_N^{N-1}(g^{N-1}, dg^N) Y(g^0, \dots, g^N) \quad (4.4.6)$$

for all measurable Y . And for any probability $\hat{P}$ on $(G_0, \mathcal{F}_0)$ we can associate a probability P on $(\mathcal{T}, \mathfrak{Z})$ given by

$$P := \hat{P}(P_{g^0}(.)) \quad (4.4.7)$$

where $P_{g^0}(.):= P_{0,g^0}(.)$ is given by (4.4.6) with $k = 0$. The probability $\hat{\pi}_b^r(.)$ on B can be seen as a probability on $G_0 = \bigcup_{p \geq 1} \bigotimes_{j=1}^p B$ concentrated on the subset B of G_0 . Hence, defining

$$P_b^r(.):= \hat{\pi}_b^r(P_{g^0}(.)) \quad (4.4.8)$$

it is easy to check that $\hat{\pi}_b^r$ and $P_{k+1}^k(.,.)$ satisfy the requirements MPS1 and MPS2. Hence $(\mathcal{T}, \mathfrak{Z}, P_b^r(.))$ is a Multitype Particle System.

Definition 4.4.31 *Let $(E, \mathcal{E})$ be a measured space and Δ a countable set of types. Let $\{\hat{\pi}_b^r(.)\}_{r \in \Delta}$ be a family of kernels on the Branch Space $(B, \mathcal{B})$ which satisfy the condition (MBPS). Let $P_{k+1}^k(.)$ be defined by (4.4.5). Then the Multitype Particle System with types Δ , state space $(E, \mathcal{E})$, transition probabilities $P_{k+1}^k(.)$ and initial distribution $P_b^r(.)$ given by (4.4.8) is called the Multitype Particle System with types Δ , state space $(E, \mathcal{E})$ and particle's law $\{\hat{\pi}_b^r(.)\}_{r \in \Delta}$.*

4.5 Multitype Empirical Measure

Let $(\mathcal{T}, \mathfrak{Z}, P_b^r(.))$ be a MBPS with Δ the countable set of types, $(E, \mathcal{E})$ the state space, and $\{\hat{\pi}_b^r(.)\}_{r \in \Delta}$ the law of the particles. Let's recall our notation: $T = (g^k)_{k \geq 0}$ where $g^k = (b_i^k)_{i=1}^{p_k}$ where $b_i^k = (\alpha_i^k, w_i^k, \beta_i^k, \tau_i^k, \tilde{n}_i^k)$ and $\tilde{n}_i^k = (n_{\tau_i}^k)_{\tau_i \in \Delta}$. The object

w_i^k is a trajectory with state space E and $w_i^k(t)$ represents the position of that trajectory at time t . Note that the mapping

$$T = (g^k)_{k \geq 0} \mapsto g^k$$

is obviously a measurable mapping for every $k \geq 0$. The same is true for the mappings

$$\begin{aligned} g = (b_i)_{i=1}^p \in G & \mapsto b_i \\ b = (\alpha, w, \beta, \tau, \vec{n}) \in B & \mapsto w \\ w \in W & \mapsto w_t \end{aligned}$$

The object $w_j^k(t)(T)$ that we denote simply by $w_j^k(t)$ should be understood as the composition of all the above mappings.

The empirical measure of type ζ at the k^{th} generation, $\Xi_t^{k,\zeta}$, is defined by:

$$\begin{aligned} \Xi_t^{k,\zeta} : \mathcal{T} & \longrightarrow M_F(E \times \Delta) \\ T & \longmapsto \sum_{\substack{j, \tau_j^k = \zeta \\ \alpha_j^k \leq t < \beta_j^k \\ \pi_j^k \geq 0}} \delta_{(w_j^k(t), \zeta)}(\cdot) \end{aligned}$$

The empirical measure at the k^{th} generation, Ξ_t^k , is defined by:

$$\Xi_t^k(\cdot) = \sum_{\zeta \in \Delta} \Xi_t^{k,\zeta}(\cdot)$$

Remark: Although $\Xi_t^k(\cdot)$ is a function of $\mathcal{T}$ it really depends only on the k^{th} generation G_k , and we will often consider it as a function of G_k . If $g^k = (b_j^k)_{j=1}^{p_k}$ then $\Xi_t^k(g^k) = \sum_{j=1}^{p_k} \Xi_t^k(b_j^k)$. We have that for a fixed $t \geq 0$:

$$\Xi_t^0(g^0) = 0 \Rightarrow \forall k > 0, \Xi_t^k(g^k) = 0;$$

this is true because the condition (MBPS) implies that either the descendants of g^0 born at a time greater than or equal to the death time of g^0 , or, g^0 has no descendants. In both cases $\Xi_t^k(g^k) = 0$ for every $k > 0$.

4.6 Laplace equation for the Multitype Branching Particle Systems

Let Δ be a countable set of types, $(E, \mathcal{E})$ a state space and $\{\tilde{\pi}_\tau^r(\cdot)\}_{\tau \in \Delta}$ be a collection of kernels from $(B, \mathcal{B})$ to $(B, \mathcal{B})$ satisfying (MBPS). Let $(\mathcal{T}, \mathfrak{I}, P_b^r)$ be the corresponding Multitype Branching Particle System. Let

$$V_t f(b, \tau) := P_b^r(e^{-\langle \Xi_t, f \rangle}) \quad (4.6.9)$$

for $f \in pb(E \times \Delta)$.⁸ Then we have:

Theorem 4.6.32 *Let $(\mathcal{T}, \mathfrak{I}, P)$ be a MBPS with type Δ , state space (E, E) and particle's law $\{\pi_b^r(\cdot)\}_{\tau \in \Delta}$. Let $V_t(b, \tau)$ be given by (4.6.9). Then (keeping our notation $b^0 = (\alpha^0, w^0, \beta^0, w^0, \vec{n}^0)$) we have:*

$$\begin{aligned} V_t(b, \tau) = & \int \tilde{\pi}_b^r(db^0) \mathbf{1}_{\{\beta > t\}}(b^0) \mathbf{1}_{\{\beta^0 > t\}}(b^0) e^{-f(w_t^0, \tau)} \\ & + \sum_{\vec{n} \in N_f^\Delta} \int \tilde{\pi}_b^r(db^0) \mathbf{1}_{\{\beta^0 \leq t, \vec{n}^0 = \vec{n}\}}(b^0) \prod_{\zeta \in \Delta} [V_t(b^0, \zeta)]^{n_\zeta} \end{aligned} \quad (4.6.10)$$

The proof of Theorem 4.6.32 is based on intermediate steps:

Proposition 4.6.32 *Under the assumption of Theorem 4.6.32, let $b = (\alpha, w, \beta, \tau, \vec{n})$, $\vec{n} = (n_\zeta)_{\zeta \in \Delta}$ and assume that $t > \alpha$. Then we have:*

$$\mathbf{1}_{\{\beta > t\}}(b) P_b(e^{-\langle \Xi_t, f \rangle}) = \mathbf{1}_{\{\beta > t\}}(b) e^{-f(w_t, \tau)}$$

Proof: we have $\Xi_t(T) = \sum_{l=0}^{\infty} \Xi_t^l(g^l) = \sum_{l=0}^{\infty} \sum_{j=0}^{P_l} \Xi_t^l(b_j^l)$. The condition (MBPS) implies that $b_j^l = (\alpha_j^l, w_j^l, \beta_j^l, \tau_j^l, \vec{n}_j^l)$ is such that $\alpha_j^l \geq \beta$ for every $l \geq 1$. And by definition of Ξ_t , $\Xi_t(b_j^l) = 0$ if $t \leq \alpha_j^l$. Because $\Xi_t(b) = \delta_{(w_t, \tau)}$ we can conclude. $\square$

⁸ $V_t(b, \tau)$ will often be abbreviated by $V_t f(b, \tau)$.

Proposition 4.6.32 *Under the assumptions of Theorem 4.6.32, let $b = (\alpha, w, \beta, \tau, \vec{n})$, $\vec{n} = (n_\zeta)_{\zeta \in \Delta}$ and assume that $t > \alpha$. Then we have:*

$$\mathbf{1}_{\{\beta^0 \leq t, \vec{n}^0 = \vec{0}\}}(b^0) P_{b^0}(e^{-\langle \Xi_t, f \rangle}) = \mathbf{1}_{\{\beta^0 \leq t, \vec{n}^0 = \vec{0}\}}(b^0)$$

Proof: we have $\Xi_t(T) = \sum_{l=0}^{\infty} \Xi_t^l(g^l) = \sum_{l=0}^{\infty} \sum_{j=0}^{p_l} \Xi_t^l(b_j^l)$. But whenever $\vec{n}_j^l \leq 0$, then, by definition of Ξ_t , $\Xi_t(b_j^l) = 0$. Now because T is admissible (with probability one) we have that $\vec{n}^0 = 0$ implies that $\vec{n}_j^l < 0$ for every $l \geq 1$. Hence $\Xi_t(T) = 0$. $\square$

Proposition 4.6.32 *Under the assumption of Theorem 4.6.32, let $g^k = (b_j^k)_{j=1}^{p_k}$ where, $b_j^k = (\alpha_j^k, w_j^k, \beta_j^k, \tau_j^k, \vec{n}_j^k)$ is admissible for $i = 1, \dots, p_k$. Then we have:*

$$P_{k,g^k}(\prod_{l>k} e^{-\langle \Xi_l, f \rangle}) = \prod_{j=1}^{p_k} P_{k,b_j^k}(\prod_{l>k+1} e^{-\langle \Xi_l, f \rangle}) \quad (4.6.11)$$

Proof: Let's write $P_{g^l}(\cdot)$ instead of $P_{l,g^l}(\cdot)$. Clearly to prove (4.6.11), it is enough to show that,

$$P_{g^k}(\prod_{l=k+1}^N e^{-\langle \Xi_l, f \rangle}) = \prod_{j=1}^{p_k} P_{b_j^k}(\prod_{l=k+1}^N e^{-\langle \Xi_l, f \rangle})$$

The proof goes by recurrence. **Basis of the induction:** We first prove

$$P_{g^{N-1}}(e^{-\langle \Xi_t^N, f \rangle}) = \prod_{j=1}^{p_{N-1}} P_{b_j^{N-1}}(e^{-\langle \Xi_t^N, f \rangle}).$$

Remember that by definition

$$P_N^{N-1}((b_j^{N-1})_{j=1}^{p_{N-1}}, \cdot) = \bigotimes_{j=1}^{p_{N-1}} P_N^{N-1}(b_j^{N-1}, \cdot)$$

and that $P_N^{N-1}(b_j^{N-1}, \cdot)$ is concentrated on a subset on a certain subset $G(\vec{n}_j^{N-1})$ of G . Moreover for any $g = ((b_j)_{j=1}^{p_n})_{n=1}^N \in G$ we have that

$$e^{-\langle \Xi_t^N(g), f \rangle} = e^{-\sum_{n=1}^N \langle \Xi_t^N((b_j)_{j=1}^{p_n}), f \rangle} = \prod_{n=1}^N e^{-\langle \Xi_t^N((b_j)_{j=1}^{p_n}), f \rangle}.$$

But trivially, if a measure v as the form $v = \bigotimes_{n=1}^N \mu(dx_n)$, then

$$\int \prod_{n=1}^N f(x_n) \bigotimes_{n=1}^N \mu(dx_n) = \prod_{n=1}^N \int f(x) \mu(dx)$$

Hence we have:

$$\int_{\Omega_N} P_N^{N-1}(g^{N-1}, dg^N) e^{-\langle \Xi_i^N(g^N), f \rangle} = \prod_{j=1}^{p_{N-1}} \int_{G(\vec{n}_j^{N-1})} P_N^{N-1}(b_j^{N-1}, dg^N) e^{-\langle \Xi_i^N(g^N), f \rangle}$$

But because $P_N^{N-1}(b_j^{N-1}, dg^N)$ is concentrated on $G(\vec{n}_j^{N-1})$ we can continue with

$$\begin{aligned} &= \prod_{j=1}^{p_{N-1}} \int_{\Omega_N} P_N^{N-1}(b_j^{N-1}, dg^N) e^{-\langle \Xi_i^N(g^N), f \rangle} \\ &= \prod_{j=1}^{p_{N-1}} P_{b_j^{N-1}} \left(e^{-\langle \Xi_i^N(g^N), f \rangle} \right) \end{aligned}$$

Induction hypothesis: Now assume that

$$P_{g^{k+1}} \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) = \prod_{j=1}^{p_{k+1}} P_{b_j^{k+1}} \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right)$$

is true, where as always $g^{k+1} = (b_j^{k+1})_{j=1}^{p_{k+1}}$ and $b_j^{k+1} = (\alpha_j^{k+1}, w_j^{k+1}, \beta_j^{k+1}, \tau_j^{k+1}, \vec{n}_j^{k+1})$.

Then

$$P_{g^k} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) = \int_{G_{k+1}} P_{k+1}^k(g^k, dg^{k+1}) e^{-\langle \Xi_i^{k+1}(g^{k+1}), f \rangle} P_{g^{k+1}} \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) \quad (4.6.12)$$

But again

$$P_{k+1}^k(g^k, dg^{k+1}) = P_{k+1}^k((b_j^{k+1})_{j=1}^{p_{k+1}}, \cdot) = \bigotimes_{j=1}^{p_k} P_{k+1}^k(b_j^k, \cdot)$$

and again $P_{k+1}^k(b_j^k, \cdot)$ is concentrated on $G(\vec{n}_j^k)$ and by the induction hypothesis, if $g = ((b_j)_{j=1}^{p_n})_{n=1}^N \in G$ then

$$\begin{aligned} P_g \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) &= \prod_{n=1}^N \prod_{j=1}^{p_n} P_{b_j} \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) \\ &= \prod_{n=1}^N P_{(b_j)_{j=1}^{p_n}} \left(\prod_{l=k+2}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) \end{aligned}$$

and also, by definition of Ξ_t^l :

$$e^{-\langle \Xi_t^{k+1}((b_j)_{j=1}^{p_n})_{n=1}^N, f \rangle} = \prod_{n=1}^N e^{-\langle \Xi_t^{k+1}((b_j)_{j=1}^{p_n}), f \rangle}.$$

Therefore with the same reasoning as in the basis of the induction we can continue (4.6.12) by writing

$$P_{g^k} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) = \prod_{j=1}^{p_k} \int_{G(\vec{n}_j^k)} P_{k+1}^k(b_j^k, dg^{k+1}) P_{g^{k+1}} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right).$$

Again, because $P_{k+1}^k(b_j^k, dg^{k+1})$ is concentrated on $G(\vec{n}_j^k)$ we can continue with

$$\begin{aligned} P_{g^k} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) &= \prod_{j=1}^{p_k} \int_{G_{k+1}} P_{k+1}^k(b_j^k, dg^{k+1}) P_{g^{k+1}} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right) \\ &= \prod_{j=1}^{p_k} P_{b_j^k} \left(\prod_{l=k+1}^N e^{-\langle \Xi_l^l(g^l), f \rangle} \right). \end{aligned}$$

The proof is complete. $\square$

Proposition 4.6.32 *Under the assumptions of Theorem 4.6.32, let $b = (\alpha, w, \beta, \tau, \vec{n})$, $\vec{n} = (n_\zeta)_{\zeta \in \Delta}$ and assume that $t > \beta$. Then we have*

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \prod_{\zeta \in \Delta} [V_t(b, \zeta)]^{n_\zeta} \quad (4.6.13)$$

Proof: by definition of $P_{0,b}(\cdot)$ we have that

$$\begin{aligned} P_{0,b}(e^{-\langle \Xi_t, f \rangle}) &= e^{-\langle \Xi_t^0(b), f \rangle} \int_{\Omega_1} P_1^0(b, dg^1) P_{1,g^1} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^l(\omega^l), f \rangle} \right) \\ &= \int_{\Omega_1} P_1^0(b, dg^1) P_{1,g^1} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^l(g^l), f \rangle} \right). \end{aligned}$$

But by definition of Ξ_t , $\Xi_t(\alpha, w, \beta, \tau, \vec{n}) = 0$ whenever $t > \beta$. Hence we obtain

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \int_{\Omega_1} P_1^0(b, dg^1) P_{1,g^1} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^l(g^l), f \rangle} \right) \quad (4.6.14)$$

But because $b = (\alpha, w, \beta, \tau, \vec{n})$, then by definition of P_1^0 , we have that $P_1^0(b, \cdot)$

is concentrated on $\bigotimes_{\substack{\zeta \in \Delta \\ n_\zeta \neq 0}} \bigotimes_{u=1}^{n_\zeta} B_\zeta$ and is given by

$$P_1^0(b, \cdot) = \bigotimes_{\substack{\zeta \in \Delta \\ n_\zeta \neq 0}} \bigotimes_{u=1}^{n_\zeta} \hat{\pi}_b^\zeta(\cdot).$$

Hence, replacing this in (4.6.14) gives

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \prod_{\zeta \in \Delta n_\zeta \neq 0} \prod_{u=1}^{n_\zeta} \int_{\hat{B}_\zeta} \hat{\pi}_b^\zeta(dg^1) P_{1,g^1} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^t(g^l), f \rangle} \right)$$

Rewritting the equation above by putting $\hat{g}^0 = g^1$ we obtain

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \prod_{\zeta \in \Delta n_\zeta \neq 0} \prod_{u=1}^{n_\zeta} \int_{\hat{B}_\zeta} \hat{\pi}_b^\zeta(d\hat{g}^0) P_{1,\hat{g}^0} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^t(g^l), f \rangle} \right). \quad (4.6.15)$$

But for any $g \in G$ we have that

$$P_{1,\hat{g}^0} \left(\prod_{l \geq 2} e^{-\langle \Xi_l^t(g^l), f \rangle} \right) = \lim_{N \rightarrow \infty} P_{1,\hat{g}^0} \left(\prod_{l=2}^N e^{-\langle \Xi_l^t(g^l), f \rangle} \right)$$

and by definition of $P_{1,\hat{g}^0}(\cdot)$,

$$P_{1,\hat{g}^0} \left(\prod_{l=1}^N e^{-\langle \Xi_l^t(g^l), f \rangle} \right) = \int_{G_2} P_2^1(\hat{g}^0, dg^2) \dots \int_{G_N} P_N^{N-1}(g^{N-1}, dg^N) \prod_{l=2}^N e^{-\langle \Xi_l^t(g^l), f \rangle}.$$

But because $G_k = G_{k+1}$, $P_{k+1}^k(\cdot, \cdot) = P_{k+2}^{k+1}(\cdot, \cdot)$ and $\Xi_t^l = \Xi_t^{l+1}$, putting $\hat{g}^k = g^{k+1}$ and rewriting the above equation we obtain:

$$P_{1,\hat{g}^0} \left(\prod_{l=1}^N e^{-\langle \Xi_l^t(g^l), f \rangle} \right) = \int_{G_1} P_1^0(\hat{g}^0, d\hat{g}^1) \dots \int_{G_{N-1}} P_{N-1}^{N-2}(\hat{g}^{N-2}, d\hat{g}^{N-1}) \prod_{l=1}^{N-1} e^{-\langle \Xi_l^t(\hat{g}^l), f \rangle}$$

and consequently

$$P_{1,\hat{g}^0} \left(\prod_{l=1}^N e^{-\langle \Xi_l^t(g^l), f \rangle} \right) = P_{0,\hat{g}^0} \left(\prod_{l=1}^{N-1} e^{-\langle \Xi_l^t(\hat{g}^l), f \rangle} \right). \quad (4.6.16)$$

Therefore, passing to the limit in (4.6.16) and using the fact that $\Xi_t^0 = \Xi_t^1$, we get:

$$\begin{aligned} e^{-\langle \Xi_t^1(\hat{g}^0), f \rangle} P_{1,\hat{g}^0} \left(\prod_{l \geq 2} e^{-\langle \Xi_l^t(g^l), f \rangle} \right) &= e^{-\langle \Xi_t^0(\hat{g}^0), f \rangle} P_{0,\hat{g}^0} \left(\prod_{l \geq 1} e^{-\langle \Xi_l^t(\hat{g}^l), f \rangle} \right) \\ &= P_{0,\hat{g}^0} \left(\prod_{l \geq 0} e^{-\langle \Xi_l^t(\hat{g}^l), f \rangle} \right) \\ &= P_{0,\hat{g}^0} (e^{-\langle \Xi_t, f \rangle}) \\ &= P_{0,g^1} (e^{-\langle \Xi_t, f \rangle}). \end{aligned}$$

Consequently, continuing (4.6.15) we can write

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \prod_{\zeta \in \Delta_{n_\zeta} \neq 0} \prod_{u=1}^{n_\zeta} \int_{\dot{B}_\zeta} \hat{\pi}_b^\zeta(dg^0) P_{0,g^0}(e^{-\langle \Xi_t, f \rangle}).$$

The last equation is precisely the same as

$$P_{0,b}(e^{-\langle \Xi_t, f \rangle}) = \prod_{\zeta \in \Delta_{n_\zeta} \neq 0} \prod_{u=1}^{n_\zeta} [P_b^\zeta(e^{-\langle \Xi_t, f \rangle})]$$

and we derive (4.6.13) from it. $\square$

Conclusion: proof of Theorem 4.6.32: Simply using the definitions of $V_t(b, \tau)$ and $P_b^\tau(\cdot)$ we write

$$\begin{aligned} V_t(b, \tau) &= P_b^\tau(e^{-\langle \Xi_t, f \rangle}) \\ &= \int \hat{\pi}_b^\tau(db^0) \mathbf{1}_{\{\beta > t\}}(b^0) P_{0,b^0}(e^{-\langle \Xi_t, f \rangle}) \\ &\quad + \sum_{\tilde{\pi} \in N_f^\Delta} \int \hat{\pi}_b^\tau(db^0) \mathbf{1}_{\{\beta \leq t, \tilde{\pi}^0 = \tilde{\pi}\}}(b^0) P_{0,b^0}(e^{-\langle \Xi_t, f \rangle}) \end{aligned}$$

(4.6.10) follows from Step1, Step2 and Step4. $\square$

4.7 Ordinary Multitype Branching Particle System

In this section we will construct a canonical version of common Multitype Branching Particule System. Intuitively, they are described as follows. A particle of a certain type τ evolves in a state space $(E, \mathcal{E})$. After a random time controlled by an additive functional $k(ds)$, the particle dies and produces a random number of offspring of different types, independent one of the others and following the same pattern as its parent. We consider only the case where the set of types is countable.

In order to describe such a system one needs:

1. a measure space $(E, \mathcal{E})$ and a (countable) set of types Δ

2. a E -valued Markov process $w = (w_t, \mathcal{F}, \pi_{t,x}^\tau)$ for each $\tau \in \Delta$
3. a continuous additive functional $k^\tau(ds)$ of ξ for each $\tau \in \Delta$
4. a distribution for the number of offspring produced at time t , position x type τ : that is we need a (measurable) function $0 \leq p(t, x, \tau, \vec{n}) \leq 1$ such that $\sum_{\vec{n} \geq 0} p(t, x, \tau, \vec{n}) = 1$.

We will define the ordinary Multitype Branching Particle System as a particular case of the Multitype Branching Particle Systems of the preceding Section. For $b^* = (\alpha^*, w^*, s^*, \tau^*, \vec{n}^*) \in B$, and for a measurable function $F(\alpha, w, \beta, \tau, \vec{n})$ we define

$$\begin{aligned} \hat{\pi}_{b^*}^{\hat{\tau}}(F) &:= \pi_{\beta^*, w_{\beta^*}}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(\beta^*, \infty)} F(\beta^*, w, \infty, \hat{\tau}, 0) \right) \\ &+ \sum_{\vec{n} \geq 0} \pi_{\beta^*, w_{\beta^*}}^{\hat{\tau}} \left(\int_{\beta^*}^{\infty} e^{-k^{\hat{\tau}}(r, s)} p(s, w_s, \hat{\tau}, \vec{n}) F(\beta^*, w, s, \hat{\tau}, \vec{n}) k^{\hat{\tau}}(ds) \right). \end{aligned} \quad (4.7.17)$$

Remark that if $F \equiv 1$, we have

$$\begin{aligned} \hat{\pi}_{b^*}^{\hat{\tau}}(1) &= \pi_{\beta^*, w_{\beta^*}}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(\beta^*, \infty)} \right) \\ &+ \pi_{\beta^*, w_{\beta^*}}^{\hat{\tau}} \left(\int_{\beta^*}^{\infty} e^{-k^{\hat{\tau}}(r, s)} k^{\hat{\tau}}(ds) \right) \\ &= 1. \end{aligned}$$

Therefore $\hat{\pi}^{\hat{\tau}}(\cdot)$ is a kernel from $(B, \mathcal{B})$ to $(B, \mathcal{B})$. With this kernel we define a Multitype Branching Particle Sytem corresponding to the ordinary Multitype Branching Particle System.

4.8 Laplace equation of the ordinary Branching Particle System

Define again as a previous Section $V_t(b, \tau) := P_b^\tau(e^{-\langle \Xi_t, f \rangle})$. Let $b = (\alpha, w, \beta, \tau, \vec{n})$.

Recall that

$$P_b^\tau(\cdot) = \int \hat{\pi}_b^\tau(db^0) P_{0, b^0}(\cdot).$$

By definition, $\hat{\pi}_b^\tau(\cdot)$ depends only on the type τ , the death time β and the position at death time w_β . Choosing the special trajectory $w_t \equiv x$, and $\beta = r$ we see that $V_t(b, \tau) = V_t(r, x, \tau)$ is really only a function of the three arguments x, β and τ . Hence, we will simply write $v_{r,t}^\tau(x)$ instead of $V_t(b, \tau)$. We have:

Theorem 4.8.33 *The Laplace equation of the ordinary Branching Particle System is*

$$\begin{aligned} v_{r,t}^\tau(x) &= \pi_{r,x} \left(e^{-k^\tau(r,t)} e^{-f(w_t)} \right) \\ &+ \sum_{\tilde{n} \geq 0} \pi_{r,x}^\tau \left(\int_r^t e^{-k^\tau(r,s)} p(s, w_s, \tau, \tilde{n}) \prod_{\zeta \in \Delta} [v_{s,t}^\zeta(w_s)]^{n_\zeta} k^\tau(ds) \right) \end{aligned} \quad (4.8.18)$$

Proof: Let $b^* = (r^*, w^*, \beta^*, \tau^*, \tilde{n}^*) \in B$ be such that $w_t^* \equiv x$ and $\beta^* = r$. Take $t > r$. With the help of Theorem 4.6.32 we write (recall that with our notation $db = d(\alpha, w, \beta, \tau, \tilde{n})$)

$$\begin{aligned} V_t(b^*, \hat{\tau}) &= \int \hat{\pi}_{b^*}^{\hat{\tau}}(db) \mathbf{1}_{\{\beta > t\}}(b) e^{-f(w_t, \tau)} \\ &+ \sum_{\tilde{k} \geq 0} \int \hat{\pi}_{b^*}^{\hat{\tau}}(db) \mathbf{1}_{\{\beta \leq t, \tilde{n} = \tilde{k}\}}(b) \prod_{\zeta \in \Delta} [V_t(b, \zeta)]^{n_\zeta}. \end{aligned} \quad (4.8.19)$$

Next we replace $\hat{\pi}_{b^*}^{\hat{\tau}}$ by its definition (4.7.17) and we get

$$\begin{aligned} \hat{\pi}_{b^*}^{\hat{\tau}} \left(\mathbf{1}_{\{\beta > t\}} e^{-f(w_t, \tau)} \right) &= \hat{\pi}_{b^*}^{\hat{\tau}} \left(\mathbf{1}_{\{\beta > t\}} e^{-f(w_t, \hat{\tau})} \right) \\ &= \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(\beta^*, \infty)} \mathbf{1}_{\{\infty > t\}} e^{-f(w_t, \hat{\tau})} \right) \\ &+ \sum_{\tilde{n} \geq 0} \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(\int_{\beta^*}^\infty e^{-k^{\hat{\tau}}(r,s)} p(s, w_s, \hat{\tau}, \tilde{n}) \mathbf{1}_{\{\beta > t\}} e^{-f(w_t, \hat{\tau})} k^{\hat{\tau}}(d\beta) \right) \\ &= \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(\beta^*, \infty)} e^{-f(w_t, \hat{\tau})} \right) \\ &+ \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-f(w_t, \hat{\tau})} \int_t^\infty e^{-k^{\hat{\tau}}(r,s)} \sum_{\tilde{n} \geq 0} p(s, w_s, \hat{\tau}, \tilde{n}) k^{\hat{\tau}}(d\beta) \right) \\ &= \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(\beta^*, \infty)} e^{-f(w_t, \hat{\tau})} \right) \\ &+ \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-f(w_t, \hat{\tau})} e^{-k^{\hat{\tau}}(\beta^*, \infty)} (1 - e^{-k^{\hat{\tau}}(r,t)}) \right). \end{aligned}$$

Hence we can finally conclude that

$$\hat{\pi}_{b^*}^{\hat{\tau}} \left(\mathbf{1}_{\{\beta > t\}} e^{-f(w_t, \tau)} \right) = \pi_{\beta^*, w_{\beta^*}^*}^{\hat{\tau}} \left(e^{-k^{\hat{\tau}}(r,t)} e^{-f(w_t, \hat{\tau})} \right). \quad (4.8.20)$$

Again with the help of (4.7.17) one calculates that

$$\hat{\pi}_{b^*}^{\hat{\tau}} \left(\mathbf{1}_{\{\beta \leq t, \tilde{n} = \tilde{k}\}}(b) \prod_{\zeta \in \Delta} [V_t(b, \zeta)]^{n_\zeta} \right)$$

is equal to

$$\sum_{\vec{n} \geq 0} \pi_{\beta^*, w_{\beta^*}}^{\vec{n}} \left(\int_{\beta^*}^t e^{-k^{\vec{n}}(r, \beta)} p(\beta, w_{\beta}, \vec{n}, \vec{n}) \left[v_{\beta, t}^{\vec{n}}(w_{\beta}) \right]^{\vec{n}} k^{\vec{n}}(d\beta) \right)$$

and finally, replacing this and (4.8.20) in (4.8.19), precisely gives (4.8.18). $\square$

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