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STIFFNESS BEHAVIOUR IN TWO DEGREE OF FREEDOM MECHANISMS

by

CHAD ENGLISH, BEng, MEng

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of

Doctor of Philosophy
Mechanical Engineering

**Department of Mechanical and Aerospace Engineering
Ottawa-Carleton Institute
for Mechanical and Aerospace Engineering**

Carleton University
Ottawa, Ontario
December 29th, 1999
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the Faculty of Graduate Studies and Research
acceptance of the thesis**

**STIFFNESS BEHAVIOUR IN TWO DEGREE
OF FREEDOM MECHANISMS**

submitted by

**Chad English, B.Eng., M.Eng.
in partial fulfilment of the requirements for
the degree of Doctor of Philosophy**

Thesis Supervisor

External Examiner

Chair, Dept. of Mechanical and Aerospace Engineering

**Carleton University
May 2000**

Abstract

The ultimate goal of upper limb prosthesis development is to duplicate the behaviour of the natural arm. To duplicate the natural behaviour we must first understand it. The current work investigates the behaviour of stiffness in a two-link manipulator based on a musculo-skeletal model of the human arm. The focus of the work is on the development of an above-elbow upper limb prosthesis but the results are relevant to robotic manipulators, haptic devices, human movement studies, and any linkage mechanism that interacts with external forces.

Derivation of the limb stiffness as seen at the hand shows two distinct types of stiffness: the endpoint spring stiffness that is dependent on actuator stiffness (such as muscles) and the geometric stiffness that is solely dependent on the changing geometry and is often neglected in stiffness studies. A key accomplishment of the current work is the decomposition of the endpoint spring stiffness into a sum of projections of antagonistic actuator stiffnesses, each contributing to the endpoint stiffness in a unique way. Limitations, techniques, and desired properties for the control of endpoint stiffness are investigated. One consequence of these limitations is that natural limb behaviour cannot be reproduced in an artificial manipulator without replacing the stiffness effects of bi-articular muscles in the upper arm.

The geometric stiffness is investigated and shown to be significant at relatively low contact forces, potentially causing instability with forces on the order of 50N (10 lb) for typical joint stiffness ranges seen in human limb studies. The geometric stiffness has predictable behaviour based on the direction of endpoint force. The current work characterizes this behaviour in terms of stiffness size, shape, and orientation with respect to the direction of applied force. External forces toward the shoulder tend to decrease the endpoint stiffness and potentially destabilize the arm. Forces away from the shoulder increase the stiffness and further stabilize it. Methods for using and compensating for the geometric stiffness are discussed.

We develop several analytical tools to understand stiffness. Mohr's circle is adapted to represent a general stiffness, including stiffnesses that are non-conservative and unstable. Traditional stiffness representations do not allow such cases. We also develop a figure of merit for determining when the geometric stiffness is negligible.

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I had not originally intended to pursue a PhD, or even a Master's for that matter. It has been a long and sometimes difficult road, but I do not regret it. I could not have gone this far without the support of so many people.

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Nomenclature

Symbol	Description
A_1, A_2, A_3	parameters defined in (8.2.3) for simplification
B_{12}, B_{13}, B_{23}	parameters defined in (8.2.3) for simplification
B_e	rotational damping of shoulder joint
B_s	rotational damping of elbow joint
C	center stiffness in Mohr's circle, arithmetic mean of $k_{\max}$ and $k_{\min}$
C_d	$\cos(\theta_d)$
C_{dJ}	$\cos(\theta_d - {}^J\theta_k)$
C_e	$\cos(\theta_e)$
C_{es}	$\cos(\theta_{es})$
C_{esJ}	$\cos(\theta_{es} - {}^J\theta_k)$
C_{kG}	$\cos(\theta_k - {}^G\theta_k)$
C_{kJ}	$\cos(\theta_k - {}^J\theta_k)$
C_s	$\cos(\theta_s)$
C_t	$\cos(\theta_t)$
C_{tJ}	$\cos(\theta_t - {}^J\theta_k)$
$d\delta$	vector containing $d\delta_s$, $d\delta_e$, and $d\delta_d$
$d\delta_d$	$= d\delta_{dx} = -d\delta_{df}$, a small change in the double-joint spring stretches
$d\delta_e$	$= d\delta_{ex} = -d\delta_{ef}$, a small change in the single-joint elbow spring stretches
$d\delta_{ln}$	$= \phi_{\Delta ln}$, left-nullspace of $\mathbf{R}$ and $\mathbf{P}$
$d\delta_s$	$= d\delta_{sx} = -d\delta_{sf}$, a small change in the single-joint shoulder spring stretches
D_J	determinant of $\mathbf{J}^{-1}$
D_k	$k_{xy} - k_{yx}$, anti-symmetric stiffness coefficient
E	energy
E_{df}	energy stored in double-joint flexor spring
E_{dx}	energy stored in double-joint extensor spring
E_{ef}	energy stored in single-joint elbow flexor spring
E_{ex}	energy stored in single-joint elbow extensor spring
E_H	energy stored in linear spring with high stiffness
E_L	energy stored in linear spring with low stiffness
E_{sf}	energy stored in single-joint shoulder flexor spring
E_{spring}	nominal potential energy function for springs
E_{sx}	energy stored in single-joint shoulder extensor spring
E_p	potential energy
${}^J E_p$	potential energy associated with ${}^J \mathbf{K}$
${}^G E_p$	potential energy associated with ${}^G \mathbf{K}$
$\mathbf{F}$	general force, endpoint force
$\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2$	reference endpoint forces in Figure 8.3.1

Nomenclature (continued)

Symbol	Description
F_a	vector of applied (external) forces
F_{ax}	applied force in the x-direction
F_{ay}	applied force in the y-direction
F_{ext}	externally applied force
F_e	internal force at elbow
F_{e-ext}	external force at elbow
F_{fmax}	maximum friction force on endpoint
F_H	force in linear spring with high stiffness
F_L	force in linear spring with low stiffness
F_s	internal force at shoulder; alternately, general spring force
F_{s-ext}	external force at shoulder; alternately, general spring force
F_{uv}	endpoint force vector with respect to δu - δv coordinate system
G	$G_s M_s + G_e M_e$
G_e	$dJ_s^{-T}/d\theta$
G_F	$(G_x \cos(\theta_F) + G_y \sin(\theta_F))J^{-1}$
G_{Fv}	vector form of G_F
G_s	$dJ_e^{-T}/d\theta$
G_x	matrix defined in (7.4.1)
G_y	matrix defined in (7.4.1)
H_{ed}, H_{sd}, H_{se}	parameters defined in (9.1.4)
J	Jacobian matrix relating endpoint space to joint space
J_e^{-T}	second column of J^{-T}
${}^J J^{-1}_k$	$J^{-1} {}^J R_k$
J_s^{-T}	first column of J^{-T}
k	general linear spring stiffness
$k_{12(max)}$	maximum coupled stiffness (orthogonal force and displacement)
k_{avg}	size of stiffness, geometric mean of k_{max} and k_{min}
k_H	high linear spring stiffness
k_L	low linear spring stiffness
k_{max}	maximum endpoint stiffness
${}^J k_{max}$	maximum endpoint spring stiffness (${}^J K$)
${}^G k_{max}$	maximum geometric stiffness (${}^G K$)
k_{min}	minimum endpoint stiffness
k_{xx}	stiffness relating F_x to a displacement in x, element (1,1) of K
k_{xy}	stiffness relating F_x to a displacement in y, element (1,2) of K
k_{yx}	stiffness relating F_y to a displacement in x, element (2,1) of K
k_{yy}	stiffness relating F_y to a displacement in y, element (2,2) of K

Nomenclature (continued)

Symbol	Description
${}^G k_{xx}$	geometric stiffness relating F_x to x displacement, element (1,1) of ${}^G \mathbf{K}$
${}^G k_{xy}$	geometric stiffness relating F_x to y displacement, element (1,2) of ${}^G \mathbf{K}$
${}^G k_{yx}$	geometric stiffness relating F_y to x displacement, element (2,1) of ${}^G \mathbf{K}$
${}^G k_{yy}$	geometric stiffness relating F_y to y displacement, element (2,2) of ${}^G \mathbf{K}$
${}^J k_{xx}$	endpoint spring stiffness relating F_x to x displacement, element (1,1) of ${}^J \mathbf{K}$
${}^J k_{xy}$	endpoint spring stiffness relating F_x to y displacement, element (1,2) of ${}^J \mathbf{K}$
${}^J k_{yx}$	endpoint spring stiffness relating F_y to x displacement, element (2,1) of ${}^J \mathbf{K}$
${}^J k_{yy}$	endpoint spring stiffness relating F_y to y displacement, element (2,2) of ${}^J \mathbf{K}$
$\mathbf{K}$	endpoint stiffness matrix
${}^G \mathbf{K}$	geometric stiffness matrix, an effect of changing geometry
${}^J \mathbf{K}$	endpoint spring stiffness matrix, transformation of spring stiffnesses
$\mathbf{K}_a$	anti-symmetric component of $\mathbf{K}$
${}^J \mathbf{K}_k$	${}^J \mathbf{K}$ matrix relative to the axes of the ${}^J \mathbf{K}$ isopotential ellipse
${}^G \mathbf{K}_k$	${}^G \mathbf{K}$ matrix relative to the axes of the ${}^G \mathbf{K}$ isopotential ellipse
$\mathbf{K}_r$	requested endpoint stiffness matrix
$\mathbf{K}_v$	vector form of $\mathbf{K}_r$ containing the three unique stiffness elements
$\mathbf{K}_s$	symmetric component of $\mathbf{K}$
$\mathbf{K}_{uv}$	endpoint stiffness matrix with respect to δu - δv coordinate system
${}^J \mathbf{K}_v$	stiffness vector containing the three unique elements of ${}^J \mathbf{K}$
${}^G \mathbf{K}_v$	stiffness vector containing the three unique elements of ${}^G \mathbf{K}$
l_0	rest length of spring
L_d	virtual link length associated with double-joint stiffness projection
L_e	length from elbow to endpoint (forearm)
L_s	length from shoulder to elbow (upper arm)
L_t	length to tip from shoulder
m_e	mass of link from elbow to endpoint (forearm)
m_s	mass of link from shoulder to elbow (upper arm)
M	general joint moment
$\mathbf{M}$	vector of joint moments
M_e	internal elbow moment
M_{e-ext}	externally applied elbow moment
${}^d M_e$	elbow moment component from double-joint springs
${}^s M_e$	elbow moment component from single-joint elbow springs
M_s	internal shoulder moment
M_{s-ext}	externally applied shoulder moment
${}^d M_s$	shoulder moment component from double-joint springs
${}^d \mathbf{M}$	vector of double-joint moments
${}^s \mathbf{M}$	vector of single-joint moments
${}^s M_s$	shoulder moment component from single-joint shoulder springs

Nomenclature (continued)

Symbol	Description
$\mathbf{M}_v$	vector of single- and double-joint moment components
$\mathbf{p}$	endpoint position vector
$\mathbf{P}$	$\mathbf{R}\mathbf{J}^{-1}$, endpoint projection matrix
$\mathbf{P}_k$	$\mathbf{P}$ matrix rotated by θ_k
${}^J\mathbf{P}_{kv}$	projection matrix to transform ${}^J\mathbf{K}$
$\mathbf{P}_v$	projection matrix to transform vector notation of stiffness by (6.2.30)
$r_{\max}$	maximum radius (semi-major axis) of $\mathbf{K}$ isopotential ellipse
${}^Jr_{\max}$	maximum radius (semi-major axis) of ${}^J\mathbf{K}$ isopotential ellipse
$r_{\min}$	minimum radius (semi-minor axis) of $\mathbf{K}$ isopotential ellipse
${}^Jr_{\min}$	minimum radius (semi-minor axis) of ${}^J\mathbf{K}$ isopotential ellipse
$\mathbf{Q}$	$\det(\mathbf{R}^T\mathbf{R})\mathbf{P}$
Q_F	θ_F in some plots
${}^J\mathbf{Q}_v$	${}^J\mathbf{P}_{kv}^{-1}$
R	radius of Mohr's circle for stiffness; alternately, general joint radius
$\mathbf{R}$	matrix or joint radii defined by (6.2.2)
R_d	virtual joint radius associated with double-joint stiffness projection
dR_e	joint radius for double-joint springs attachment about elbow
sR_e	joint radius for single-joint elbow springs attachment
$\mathbf{R}_k$	rotation matrix through θ_k
${}^J\mathbf{R}_k$	rotation matrix through ${}^J\theta_k$
${}^G\mathbf{R}_k$	rotation matrix through ${}^G\theta_k$
$\mathbf{R}_{kG}$	${}^G\mathbf{R}_k^{-1}\mathbf{R}_k$
$\mathbf{R}_{kJ}$	${}^J\mathbf{R}_k^{-1}\mathbf{R}_k$
dR_s	joint radius for double-joint springs attachment about shoulder
sR_s	joint radius for single-joint elbow springs attachment
$\mathbf{R}_v$	matrix of radii to solve stiffness as vectors, defined in (6.2.28)
S_d	$\sin(\theta_d)$
S_d	stiffness matrix due to changing double-joint spring displacements
S_{dJ}	$\sin(\theta_d - {}^J\theta_k)$
S_e	$\sin(\theta_e)$
S_e	stiffness matrix due to changing single-joint elbow spring displacements
S_{es}	$\sin(\theta_{es})$
S_{esJ}	$\sin(\theta_{es} - {}^J\theta_k)$
S_k	stiffness shape factor
S_{kG}	$\sin(\theta_k - {}^G\theta_k)$
S_{kJ}	$\sin(\theta_k - {}^J\theta_k)$
$S_{\max}$	stiffness projection vector of cosines corresponding to ${}^Jk_{\max}$
$S_{\min}$	stiffness projection vector of sines corresponding to ${}^Jk_{\min}$

Nomenclature (continued)

Symbol	Description
S_s	$\sin(\theta_s)$
S_s	stiffness matrix due to changing single-joint shoulder spring displacements
S_t	$\sin(\theta_t)$
S_{ij}	$\sin(\theta_i - {}^j\theta_k)$
T_1, T_2, T_{J1}, T_{J2}	arctan parameters defined by (6.3.12)
T_{ext}	external torque applied to torsion spring
T_s	torsion spring torque
W	work
x	horizontal coordinate of endpoint; alternately, general spring end position
x^*	general spring end position (opposite end to x)
y	vertical coordinate of endpoint
α	unitless parameter defined in (6.3.4); limited range of stiffness orientations
β	unitless parameter defined by (6.1.40); max. range of stiffness orientations
γ	unitless parameter defined by (6.1.46); max. force angle in Figure 8.3.1
γ_e, γ_s	parameters defined in (9.1.4)
δ	general spring displacement (stretch)
δ	vector of spring displacements (stretches)
δ_0	general spring displacement initial value; alternately, parameter in (7.1.2)
δ_H	displacement of linear spring with high stiffness
δ_L	displacement of linear spring with low stiffness
δ_{df}	displacement of double-joint flexor spring
δ_{dx}	displacement of double-joint extensor spring
δ_{ef}	displacement of single-joint elbow flexor spring
δ_{ex}	displacement of single-joint elbow extensor spring
δ_{sf}	displacement of single-joint shoulder flexor spring
δ_{sx}	displacement of single-joint shoulder extensor spring
δ_{pos}	nominal maximum stretch of spring
δp_{uv}	displacement of endpoint from virtual point with respect to δu - δv system
δu	displacement in direction of maximum stiffness from virtual point
δv	displacement in direction of minimum stiffness from virtual point
δx	displacement in x direction from virtual point
δy	displacement in y direction from virtual point
Δ	general angular displacement of joint or torsion spring
Δ	vector of joint displacements
Δ_0	reference joint displacement for spring design

Nomenclature (continued)

Symbol	Description
Δ_e	angular displacement of elbow joint
${}^s\Delta_{ev}$	$\theta_{ev} - {}^s\theta_{ev}$, single-joint elbow virtual displacement
${}^d\Delta_{ev}$	$\theta_{ev} - {}^d\theta_{ev}$, double-joint elbow virtual displacement
Δ_s	angular displacement of shoulder joint
${}^s\Delta_{sv}$	$\theta_{sv} - {}^s\theta_{sv}$, single-joint shoulder virtual displacement
${}^d\Delta_{sv}$	$\theta_{sv} - {}^d\theta_{sv}$, double-joint shoulder virtual displacement
Δ_v	vector of joint virtual displacements
ε	dummy variable for integration
ϕ	general spring force function
Φ	vector of spring forces
ϕ_{df}	spring force in double-joint flexor spring
ϕ_{dx}	spring force in double-joint extensor spring
ϕ_{ef}	spring force in single-joint elbow flexor spring
ϕ_{ex}	spring force in single-joint elbow extensor spring
ϕ_{sf}	spring force in single-joint shoulder flexor spring
ϕ_{sx}	spring force in single-joint shoulder extensor spring
Φ_{Δ}	vector containing $\phi_{\Delta s}$, $\phi_{\Delta e}$, and $\phi_{\Delta d}$
$\phi_{\Delta d}$	$\phi_{dx} - \phi_{df}$
$\phi_{\Delta e}$	$\phi_{ex} - \phi_{ef}$
$\phi_{\Delta ln}$	$= d\delta_{ln}$, left-nullspace of $\mathbf{R}$ and $\mathbf{P}$
$\phi_{\Delta s}$	$\phi_{sx} - \phi_{sf}$
Φ	general torsion spring torque function
λ	dummy variable for integration
λ_1, λ_2	eigenvalues of $\mathbf{K}$, equal to k_{max} and k_{min}
μ	parameters in (7.1.2) and (7.1.3) for spring function design
μ_s	static friction coefficient
θ	general joint angle; alternately, angle of one end of a torsion spring
θ^*	angle of one end of a torsion spring (opposite end to θ)
Θ	vector of joint angles
θ_1, θ_2	principle angles of intersection points of ${}^J\mathbf{K}$ and $\mathbf{K}$ isopotential ellipses
θ_{c1}, θ_{c2}	orientations in which force and displacement are co-linear
θ_d	projection direction associated with double-joint stiffness
θ_e	angle of forearm with respect to extension of upper arm
θ_{es}	angle of forearm with respect to x-axis
θ_{ev}	virtual (unloaded equilibrium) elbow angle with respect to upper arm

Nomenclature (continued)

Symbol	Description
${}^d\theta_{ev}$	virtual elbow angle due to double-joint springs
${}^s\theta_{ev}$	virtual elbow angle due to single-joint elbow springs
θ_{esv}	virtual (unloaded equilibrium) elbow angle with respect to x-axis
θ_F	direction of externally applied endpoint force with respect to x-axis
θ_k	orientation of endpoint stiffness, direction of k_{max} with respect to x-axis
${}^J\theta_k$	orientation of ${}^J\mathbf{K}$, direction of ${}^Jk_{max}$ with respect to x-axis
${}^G\theta_k$	orientation of ${}^G\mathbf{K}$, direction of ${}^Gk_{max}$ with respect to x-axis
θ_s	angle of upper arm with respect to x-axis
θ_{sv}	virtual (unloaded equilibrium) shoulder joint angle
${}^s\theta_{sv}$	virtual shoulder angle due to single-joint shoulder springs
${}^d\theta_{sv}$	virtual shoulder angle due to double-joint springs
θ_t	angle of tip direction (shoulder to endpoint) from x-axis
θ_v	general virtual (unloaded equilibrium) joint angle
${}^s\theta_v$	general virtual (unloaded equilibrium) single-joint angle
σ	vector of spring inputs
${}^k\sigma$	vector of spring inputs to change stiffness
σ_{df}	input position of double-joint flexor spring
${}^e\sigma_{df}$	elbow side position of double-joint flexor spring
${}^k\sigma_{df}$	input of double-joint flexor spring to change stiffness
${}^m\sigma_{df}$	input of double-joint flexor spring to move virtual position
${}^s\sigma_{df}$	shoulder side position of double-joint flexor spring
σ_{dx}	input position of double-joint extensor spring
${}^e\sigma_{dx}$	elbow side position of double-joint extensor spring
${}^k\sigma_{dx}$	input of double-joint extensor spring to change stiffness
${}^m\sigma_{dx}$	input of double-joint extensor spring to move virtual position
${}^s\sigma_{dx}$	shoulder side position of double-joint extensor spring
σ_{ef}	input position of single-joint elbow flexor spring
${}^e\sigma_{ef}$	elbow side position of single-joint elbow flexor spring
${}^k\sigma_{ef}$	input of single-joint elbow flexor spring to change stiffness
${}^m\sigma_{ef}$	input of single-joint elbow spring to move virtual position
σ_{ex}	input position of single-joint elbow extensor spring
${}^e\sigma_{ex}$	elbow side position of single-joint elbow extensor spring
${}^k\sigma_{ex}$	input of single-joint elbow extensor spring to change stiffness
${}^m\sigma_{ex}$	input of single-joint elbow extensor spring to move virtual position
σ_f	general flexor spring input

Nomenclature (continued)

Symbol	Description
$^k\sigma_f$	general flexor spring input to change stiffness
$^m\sigma_f$	general flexor spring input to move joint virtual angle
σ_{sf}	input position of single-joint shoulder flexor spring
$^k\sigma_{sf}$	input of single-joint shoulder flexor spring to change stiffness
$^m\sigma_{sf}$	input of single-joint shoulder flexor spring to move virtual position
$^s\sigma_{sf}$	shoulder side position of single-joint shoulder flexor spring
σ_{sx}	input position of single-joint shoulder extensor spring
$^k\sigma_{sx}$	input of single-joint shoulder extensor spring to change stiffness
$^m\sigma_{sx}$	input of single-joint shoulder extensor spring to move virtual position
$^s\sigma_{sx}$	shoulder side position of single-joint shoulder extensor spring
σ_x	general extensor spring input
$^k\sigma_x$	general extensor spring input to change stiffness
$^m\sigma_x$	general extensor spring input to move joint virtual angle
τ	dummy variable for integration of time
ψ	general spring stiffness function
ψ_{df}	spring stiffness in double-joint flexor spring
ψ_{dx}	spring stiffness in double-joint extensor spring
ψ_{ef}	spring stiffness in single-joint elbow flexor spring
ψ_{ex}	spring stiffness in single-joint elbow extensor spring
ψ_{sf}	spring stiffness in single-joint shoulder flexor spring
ψ_{sx}	spring stiffness in single-joint shoulder extensor spring
ψ_{Σ}	actuator stiffness matrix with $\psi_{\Sigma s}$, $\psi_{\Sigma e}$, and $\psi_{\Sigma d}$ on the diagonal
$\psi_{\Sigma c}$	actuator stiffness matrix required to compensate for ${}^G\mathbf{K}$
$\psi_{\Sigma cv}$	vector containing the critical values $\psi_{\Sigma sc}$, $\psi_{\Sigma ec}$, and $\psi_{\Sigma dc}$
$\psi_{\Sigma d}$	$\psi_{df} + \psi_{dx}$
$\psi_{\Sigma dc}$	critical value of $\psi_{\Sigma d}$ at which endpoint stiffness becomes unstable
$\psi_{\Sigma e}$	$\psi_{ef} + \psi_{ex}$
$\psi_{\Sigma ec}$	critical value of $\psi_{\Sigma e}$ at which endpoint stiffness becomes unstable
$\psi_{\Sigma r}$	actuator stiffness matrix to achieve requested $\mathbf{K}$ after ${}^G\mathbf{K}$ compensation
$\psi_{\Sigma rv}$	vector form of $\psi_{\Sigma r}$
$\psi_{\Sigma s}$	$\psi_{sf} + \psi_{sx}$
$\psi_{\Sigma sc}$	critical value of $\psi_{\Sigma s}$ at which endpoint stiffness becomes unstable
$\psi_{\Sigma v}$	vector containing $\psi_{\Sigma s}$, $\psi_{\Sigma e}$, and $\psi_{\Sigma d}$
Ψ	general joint stiffness
Ψ	joint stiffness matrix

Nomenclature (continued)

Symbol	Description
${}^d\Psi$	joint stiffness matrix containing only double-joint stiffnesses
${}^s\Psi$	joint stiffness matrix containing only single-joint stiffnesses
Ψ_d	general double-joint stiffness
Ψ_d	matrix containing only double-joint stiffness components
Ψ_{de}	coupled stiffness of shoulder torque relative to elbow displacement
Ψ_{ds}	coupled stiffness of elbow torque relative to shoulder displacement
Ψ_e	elbow joint stiffness
Ψ_e	matrix containing Ψ_e in position (2,2) and zeros elsewhere
${}^s\Psi_e$	elbow stiffness component due to single-joint elbow springs
${}^d\Psi_e$	elbow stiffness component due to double-joint springs
$\Psi_{\max}$	nominal maximum joint stiffness
$\Psi_{\min}$	nominal minimum joint stiffness
Ψ_s	shoulder joint stiffness
Ψ_s	matrix containing Ψ_s in position (1,1) and zeros elsewhere
${}^s\Psi_s$	shoulder stiffness component due to single-joint shoulder springs
${}^d\Psi_s$	shoulder stiffness component due to double-joint springs
Ψ_v	vector containing the three unique elements of Ψ
ξ_0	reference joint stiffness error for spring function design
ξ_I	Ellipse Comparison Index

Chapter 1

Introduction

1.1 Purpose

The history of powered upper limb prostheses has seen many developments in controller design, signal processing, and mechanical transmissions. However, none of the approaches to date have been based on the natural mechanics of the human arm. There have been advancements in the development of artificial muscle-like actuators, but these remain experimental and have not been implemented in a commercial prosthesis at present. The current work is part of an ongoing project to develop the mechanics of a new above-elbow powered prosthesis that uses efficient antagonistic actuation. The prosthesis will operate and appear more natural in its motion and can efficiently interact with the environment.

The current work is based on the behaviour of a model of the natural limb. In particular, the properties of interest are the ability to control stiffness in the natural limb and the behaviour of the limb in free motion and contact. Figure 1.1 shows a simplified model of the muscles in the human upper arm. Although there are many muscles in the

upper arm, they fall into three general categories: those that cross only the shoulder joint, those that cross only the elbow joint, and those that cross both joints. Within these three categories there are two sub-categories: flexors and extensors. Flexors and extensors act antagonistically. That is, one set pulls the joint one way while the other set pulls in the opposite direction. The obvious reason for this type of actuation is that muscles cannot push. However, when these muscles are co-activated (activated at the same time), there is no net torque at the joint and hence the joint does not move, but the joint stiffness increases. This occurs because the muscle stiffnesses are non-linear. The joint stiffnesses largely govern the dynamic behaviour and the forces of interaction with the environment (pushing, pulling, lifting, shaking hands, etc.). It is therefore important to understand this ability to modulate stiffness when attempting to implement a device that is based on the behaviour of the natural limb. The results are applicable to prosthesis development, robotic manipulators, haptic (human-machine) interfaces, and human studies of motion and muscle control. Since the direct goal is the development of above-elbow prostheses, the research will generally be interpreted with respect to this application.

Figure 1.1: Muscle groups of the upper arm. Mono-articular muscles cross only one joint. Bi-articular muscles cross two joints. These classifications provide three categories of muscles: mono-articular shoulder, mono-articular elbow, and bi-articular. The sub-categories of flexors (bend the arm) and extensors (straighten the arm) provide the six groups as shown. There are more than six muscles in the upper arm, but each of them falls into one of the six groups shown.

1.2 Motivation

The development of powered prostheses over the last 30 years has largely consisted of improvements to feedback controller design, signal acquisition from the amputees, and signal processing, mostly based on practical engineering decisions. In general, neuromuscular signals above some threshold activate some aspect of the prosthesis movement such as position or velocity. Neither the activation nor the mechanics of the prosthesis operate as in the natural limb. The prosthesis mechanics typically consist of the motor, transmission system, and possibly an unpowered free swing mode that is tuned to match the natural frequency of a swaying arm when walking. These prostheses typically do not perform well during contact tasks either. Similarly, industrial robots have typically performed poorly for contact tasks. Even during unconstrained motion they typically do not move as fluidly as a natural limb.

There are several theories as to why these devices do not perform as well as a natural limb. In particular, they tend to assume an ideal actuator such as an ideal torque motor. On the other hand, the actuators of the natural limb, i.e., muscles, have inherent stiffness and damping properties. These properties can largely govern the dynamics and contact behaviour of the limb. There is much evidence that the Central Nervous System (CNS) in humans uses these properties to control limb motion and interaction. These theories will be discussed in much greater detail in Chapter 3.

Regardless of the natural control strategies, the fact remains that the natural limb does contain stiffness and damping properties that affect the dynamic behaviour. Such properties are not included in the design of current above-elbow prostheses. Furthermore, previous research, which will also be discussed in Chapter 3, has shown that actuating a prosthesis (or robot manipulator) antagonistically provides greater energy efficiency. In addition, control of stiffness in this manner allows for the management of contact forces as well as unconstrained motion. It is known that there are efficiency advantages to using contact forces when performing constrained motions (see Appendix A, for example). Therefore mimicking the actuation system of the natural limb also offers practical engineering and efficiency improvements. The current research is carried out with respect to these principles:

- (1) Using natural limb mechanics in a prosthesis can only make the motion appear more natural.
- (2) Using natural limb mechanics allows for stable interaction with the environment.
- (3) Natural limb mechanics appear to offer improved efficiency of operation.
- (4) The CNS appears to use natural limb mechanics to control motion and contact forces in a unified control strategy.

The above principles describe the motivation for duplicating the natural limb mechanics in a prosthesis. However, we do not yet fully understand the mechanics of the

natural limb, particularly which aspects of the natural actuation system are important and how they affect the behaviour of the limb. These details are the underlying basis for the current work.

1.3 Objectives

The principal objective of this research is to investigate the properties of interaction of a multi-joint arm with the ability to modulate joint stiffnesses. This investigation will include the development of the relationship between the individual muscle stiffnesses (modeled as non-linear springs) and the stiffness as seen at the point of interaction (i.e., the hand). We will also investigate the effect of contact forces on stiffness.

The goal of this investigation will be to better understand the control stiffness in a two link manipulator system modelled after the human arm, particularly with respect to the development of an above-elbow powered prosthesis. It is hoped that this research will help to optimize such a design to best duplicate the natural ability to modulate stiffness as well as realize the limitations and practical design restrictions of such an approach.

The specific objectives will be discussed in greater detail in Chapter 4 once the issues and shortcomings of previous research has been fully investigated.

1.4 Outline

The research presented in this thesis is divided into three main sections. Chapters 1-4 introduced the concepts and issues that will be investigated in later chapters. Chapter 2 will explain the basic background concepts necessary to understand the issues related to stiffness in a two-link manipulator. We will investigate previous research in the area of muscle and stiffness control in the natural limb in Chapter 3. We present the problem statement, summarizing the shortcomings of the previous research with respect to our application of an above-elbow prosthesis, in Chapter 4.

The second section of this thesis develops the understanding of stiffness in a two-link manipulator necessary to answer the problem statement presented in Chapter 4. This section constitutes the majority of the research work. In Chapter 5, we define the model, parameters, and variables that will be used to investigate stiffness. We use these definitions in Chapter 6 to develop relationships between stiffnesses as seen from different coordinate systems (e.g., muscles, joints, hand), different categories of stiffness, and tools for representing and measuring stiffness. We study the stiffness relationships developed in Chapter 6 in much greater detail numerically in Chapter 7.

The final section of this thesis applies the knowledge and understanding obtained in Chapters 5-7 to practical issues. In Chapter 8 we apply this knowledge to strategies for controlling stiffness in a desirable way. We adapt our understanding of stiffness to

various applications in Chapter 9, including our main focus on prosthesis design, robotics, haptic interfaces, and studies of human posture and movement. Finally, Chapter 10 summarizes the conclusions and recommendations from this research.

Chapter 2

Background Concepts

There are a variety of concepts in the current work that may not be entirely clear from standard usage. In order to understand the issues relating to previous research and properties that we investigate here, this chapter provides a brief outline of the concepts and terminology that we use in later chapters.

2.1 Stiffness

The term *stiffness*, although in frequent use, has a somewhat ambiguous meaning. In this research we use the concepts of non-linear spring stiffness, joint stiffness, endpoint stiffness, and linearized stiffness matrices. It is therefore important to explain what we mean by these terms as well as stiffness in general.

2.1.1 Stiffness in Various Fields of Study

The term *stiffness* itself is used in a variety of fields, either conceptually or as a specific parameter name. In magnetic studies, especially with respect to levitation using semiconductors, stiffness refers to the rate of change of magnetic force with respect to

distance from the magnet (e.g., [Lehndorff et al., 1995]). (Incidentally, an alternate term used for the same purpose, *variable spring constant*, is even more ambiguous since it is neither constant nor part of a spring.)

In the study of fluid dispersion and surface tension, *stiffness* is a position dependent matrix that is the second derivative of potential energy describing the rate of force change between molecules in three dimensions (e.g., [Parray and Boulter, 1996]). In other areas of molecular and atomic studies, *stiffness* describes a resistance to twist; that is, it describes the relationship between changes of molecular or atomic orientation and the resulting changes in force (e.g., [Van Leeumen et al., 1996]). In the field of electronics, it is even used sometimes to refer to the relationship between charge and voltage in a voltage source [Horowitz, 1989].

It is clear that in many of these applications, *stiffness* is used in analogy to the more familiar use in a mechanical spring. In general, mechanical stiffness refers to a relationship between force and deformation of a solid material. However, even in mechanics the term has a variety of uses. In Finite Element Method (FEM) analysis, stiffness is a non-linear matrix in three dimensions representing both the rate of material stress change with deformation and rate of stress change due to motion of the node positions within the material [Cesar de Sa and Owen, 1987; Doltsinis, 1989].

For linearized analysis of thin plate deformation, *stiffness* is used interchangeably with *modulus of flexural rigidity*, a parameter defined by the thickness of the plate and

material properties. It is a scalar parameter that is indicative of the rate of stress increase with deformation of the plate material [Duvant and Lions, 1976].

The definition of *stiffness* is not even clear with respect to springs. Some authors only apply the term *stiffness* to the proportionality parameter between force and displacement of linear springs [Cheng, 1959; Fung, 1965; Lindsay, 1961; Lunn, 1991; Smith and Smith, 1990], $k = F/x$, while others apply it to the derivative of the force function in a (generally) non-linear spring, $k(x) = dF/dx$ [Cabannes, 1968]. In this context, *stiffness* is a function of spring stretch except in the case of linear springs where it is a constant as described above.

2.1.2 Use of Stiffness in the Current Work

The system proposed in this research is modeled after the musculo-skeletal structure of the human upper arm, shown in Figure 2.1.1. A two link arm is operated by a mechanical stiffness control system consisting of a set of two non-linear antagonistic springs acting about each joint, and one set of two non-linear springs acting about both joints combined. In this system, *stiffness* is used in several contexts discussed below. We review and discuss the details and justifications for the stiffness definitions of this model in Chapter 5.

Figure 2.1.1: Simplified model of human upper arm. The two joint radii (single- and double-joint springs) at each joint are fixed together. In the human arm, motion is performed by activating muscles internally. The actuation system for this model is not shown. The springs represent only the reaction effects from displacement of the arm from its set equilibrium position.

Antagonistic (Muscle) Stiffness

Each of the six springs shown in Figure 2.1.1 have their own force-displacement characteristics which are generally non-linear. We denote these force functions as $\phi\{\delta\}$, where δ is the displacement of the spring from its defined zero position ($\delta = 0$). The spring stiffnesses are then defined as $\psi\{\delta\} = d\phi/d\delta$. In this definition, stiffness is a function of displacement. This is consistent with the definition of non-linear spring stiffness given by Cabannes [1968].

Joint Stiffness

The springs are connected to the links such that they apply moments about the joints as they are stretched. Therefore there is the relationship between joint displacements and the resulting joint moments. These relationships depend on the spring force functions and the moment arms through which the springs act. For the model in Figure 2.1.1 the moment arms are the joint radii at which the springs attached, and are constant here. We define joint stiffness, Ψ , as the rate of change of the joint moment with respect to joint displacement, $\Psi = dM/d\theta$. Since there are two joint moments and two joint displacements, it is possible to define four joint stiffnesses. We write these joint stiffnesses into the *joint stiffness matrix* given by

$$\Psi = \frac{dM}{d\theta} = \begin{bmatrix} \Psi_s & \Psi_{ds} \\ \Psi_{de} & \Psi_e \end{bmatrix} \quad (2.1.1)$$

where $d\mathbf{M}$ is a vector whose elements are the differential changes in moment at each joint, $d\theta$ is a vector whose elements are the differential changes in angular position at each joint, and Ψ is the joint stiffness matrix. In Chapter 5 we decompose the elements of the joint stiffness matrix into those caused by each set of springs.

Equation (2.1.1) defines the instantaneous joint stiffness, the rate at which the moments change for an infinitesimal change in joint position. For small displacements from the zero position (where there are no joint moments), the net moment can be approximated by this linear relationship such that

$$\mathbf{M} = \Psi \delta \theta \quad (2.1.2)$$

where $\mathbf{M}$ is the net moment and $\delta \theta$ is the angular displacement from the zero moment position.

Latash and Zatsiorsky objected to this definition of joint stiffness because there is no actual deformation of a material at the joint [1993]. They suggested the use of the term *quasi-stiffness* should be used in this case since it is merely a transformation of stiffness from the muscle coordinates to the joint coordinates. We maintain the use of the term *joint stiffness* here because it is indistinguishable from the deformation of material at the joint (with joint coupling as described in Chapter 5) and because the terminology has been used by the majority of those in related fields. Furthermore, the use of *stiffness* as

an analogy in other fields, as seen in Section 2.1.1, supports the use of this term. In short, it looks and acts like a stiffness and so we call it a stiffness.

Endpoint Stiffness

Displacement of the endpoint of the arm in Figure 2.1.1 results in forces that attempt to return the arm to its equilibrium position. These forces come from the spring forces that are translated by the moment arms at the joints and the link lengths to the endpoint. It is important to note that these forces do not generally act in the opposite direction to the displacement. Endpoint displacement can occur in two dimensions, and so the forces and displacements are typically decomposed into horizontal and vertical directions (x and y). Since there are two force directions and two displacement directions there are four possible endpoint stiffnesses. These are typically given in the endpoint stiffness matrix. We define this matrix through the force displacement relationship

$$\mathbf{K} = \frac{d\mathbf{F}}{d\mathbf{p}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \quad (2.1.3)$$

where $d\mathbf{F}$ is the differential endpoint force vector, $d\mathbf{p}$ is the differential endpoint displacement vector, and $\mathbf{K}$ is the endpoint stiffness matrix. As with the joint stiffness, this defines the instantaneous rate of force change with displacement. For small displacements from the zero position (no endpoint force), this linear relationship approximates the net force:

$$\mathbf{F} = \mathbf{K}\delta\mathbf{p} \quad (2.1.4)$$

where $\mathbf{F}$ is the net force and $\delta\mathbf{p}$ is the small displacement from the zero position.

Since the stiffness in (2.1.3) is also a transformation of the spring (muscle) stiffnesses, its description as a stiffness is also somewhat ambiguous. However, the spring stiffnesses as seen from hand are indistinguishable from a two-dimensional spring system acting at the endpoint. We therefore use the term *endpoint stiffness* to describe it here. The stiffness of the arm is typically represented from the perspective of the endpoint. This is the usual point of interaction with the environment. It is also simpler to graphically represent the arm stiffness at this point, as we will show in the next section. In Chapter 5, we will discuss the sign conventions, displacements, and forces in greater detail. We will also decompose the endpoint stiffness into effects from different sources.

2.1.3 Conservative Stiffness

A conservative system is defined as one that obeys the principle of conservation of energy. That is, no net energy is added to, or removed from, the system when it is returned to its original state. This means that the energy in the system is uniquely defined by its state. The model in Figure 2.1.1 contains only ideal springs and has no elements that can produce or dissipate energy, hence it is conservative. This means that the stiffness matrices, $\mathbf{K}$ and Ψ , must be symmetric (see Appendix B for details),

$$k_{xy} = k_{yx} \quad (2.1.3a)$$

$$\Psi_{ds} = \Psi_{de} \quad (2.1.3b)$$

A conservative system will automatically produce symmetrical stiffness matrices.

However, non-conservative systems can also produce symmetrical stiffness matrices. In such cases, the stiffness behaves as though the system was conservative, and hence we say such a system *appears* conservative and this requirement defines a *spring-like* system [Hogan, 1985a].

We can separate any general endpoint stiffness matrix into the sum of its symmetric, $\mathbf{K}_s$, and anti-symmetric, $\mathbf{K}_a$, components:

$$\mathbf{K}_s = \frac{1}{2}(\mathbf{K} + \mathbf{K}^T) = \begin{bmatrix} k_{xx} & \frac{1}{2}(k_{xy} + k_{yx}) \\ \frac{1}{2}(k_{xy} + k_{yx}) & k_{yy} \end{bmatrix} \quad (2.1.4a)$$

$$\mathbf{K}_a = \frac{1}{2}(\mathbf{K} - \mathbf{K}^T) = \frac{1}{2}(k_{xy} - k_{yx}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (2.1.4b)$$

$$\mathbf{K} = \mathbf{K}_s + \mathbf{K}_a \quad (2.1.4c)$$

Since $\mathbf{K}_s$ is symmetric, it represents a conservative system and we define it as a conservative stiffness. If $\mathbf{K}_a$ is non-zero, then the overall endpoint stiffness $\mathbf{K}$ is not conservative. As discussed above, the model in Figure 2.1.1 is conservative. The natural arm generally *appears* conservative. We will discuss this further when reviewing

previous research in Chapter 3.

2.2 Ellipse Representations

The endpoint stiffness, $\mathbf{K}$, from (2.1.3) provides only a mathematical representation of stiffness with respect to a chosen coordinate system. That is, the same stiffness will produce a different endpoint stiffness matrix if the coordinate system is rotated. The endpoint stiffness matrix is then not unique for a given stiffness. The elements of the matrix alone do not provide much information as to the behaviour of the arm. There are several graphical and analytical representations of endpoint stiffness that provide better understanding of the stiffness state. There are two common graphical techniques that use ellipses to represent the stiffness. However, the ellipses have very different interpretations in the two representations, so it is important not to confuse the two methods.

2.2.1 Isopotential Ellipse

The system that (2.1.3) describes is a two-dimensional linear spring system. Like normal springs, energy can be stored and returned. Since the only source of energy change in the system of Figure 2.1.1 is mechanical work, all work done on the system is stored as potential energy in the springs, and all work done by the system comes from the potential energy stored in the springs. The work done on the system during a small displacement is the product of force and the infinitesimal distance, $W = \mathbf{F}^T d\mathbf{p}$.

Integrating this over the whole (small) displacement, and using the linearized force from (2.1.4), the net energy stored in the *symmetrical* stiffness, for small displacements, is

$$E_p = \frac{1}{2} \mathbf{F}^T \delta \mathbf{p} = \frac{1}{2} \delta \mathbf{p}^T \mathbf{K} \delta \mathbf{p} = \frac{1}{2} (k_{xx} \delta x^2 + (k_{xy} + k_{yx}) \delta x \delta y + k_{yy} \delta y^2) \quad (2.2.1)$$

When plotted for positive values of E_p , this equation produces an ellipse with the origin at the endpoint of the hand, as Figure 2.2.1(a) shows. The distance from the endpoint to any point on this ellipse represents the displacement required to store the reference energy value, E_p . This means that the major axis of the ellipse represents the lowest stiffness direction because it requires the largest displacement to store the energy. Conversely, the minor axis represents the highest stiffness direction because it requires the least displacement to store the same amount of energy.

Since the isopotential ellipse is a constant potential energy curve, based on the chosen value of E_p , no work is required to move the limb endpoint around this path. Hence the restoring force is perpendicular to the ellipse curve. In general, the force does not point directly back to the origin, but rather follows a curved path back. This can be misinterpreted as not spring-like (e.g., [Haggard et al., 1994]), but does indeed result from spring-like behaviour. The exception is on the principal axes of the ellipse. In these directions the force points directly back to the equilibrium position. The ellipse radii of the major and minor axes are

Figure 2.2.1: Isopotential representation of stiffness. (a) The isopotential ellipse for the symmetric component of stiffness. The ellipse is a constant potential energy contour. Distances from the endpoint to points on the ellipse represent endpoint displacements required to achieve the reference energy level. Restoring forces are perpendicular to the ellipse, and do not generally point towards the equilibrium point. (b) Pincushion diagram for the anti-symmetrical stiffness component. Force vectors are orthogonal to the displacements. The force magnitude is proportional to the displacement magnitude.

$$r_{\max} = (2E_p/k_{\min})^{1/2} \quad (2.2.2a)$$

$$r_{\min} = (2E_p/k_{\max})^{1/2} \quad (2.2.2b)$$

where $k_{\max}$ and $k_{\min}$ are the maximum and minimum stiffnesses and $r_{\max}$ and $r_{\min}$ are the major and minor axes of the ellipse. The principal stiffnesses, $k_{\max}$ and $k_{\min}$, are the eigenvalues of the stiffness matrix.

When stiffness increases, it takes a smaller displacement to achieve the same energy storage and therefore the isopotential ellipse gets smaller. It is important to use the same value of E_p when comparing stiffnesses since different E_p values will also produce different sized ellipses. This is a disadvantage of the isopotential ellipse representation.

The anti-symmetric stiffness component is typically represented separately in a pincushion diagram as shown in Figure 2.2.1(b) [Hogan, 1985a]. The resulting force is always orthogonal to the displacement with a magnitude proportional to the displacement. In general, the overall stiffness is a combination of those shown by the isopotential ellipse and pincushion diagram.

A slight variation on the isopotential ellipse is to plot the potential energy in a third dimension [Hogan, 1985a]. The resulting plot is a bowl-shaped energy surface with the lowest point at the origin as shown in Figure 2.2.2. The cross-section of the bowl at a

particular value of E_p is the isopotential ellipse as shown in Figure 2.2.1(a). The slope of the bowl wall indicates the force direction and magnitude for a displacement of the endpoint to that point in the δx - δy plane. This avoids the disadvantage of choosing a specific value of E_p , but adds the disadvantages of requiring a three-dimensional plot and it is more difficult to plot many isopotential ellipses in a single diagram to compare them.

Figure 2.2.2: Energy surface representation of stiffness. The cross-sections at different energy levels are the isopotential ellipses. The slope of the surface wall indicates the direction and magnitude of force.

2.2.2 Stiffness Ellipse

The second ellipse representation of stiffness uses the force vectors to form the ellipse shape [Mussa-Ivaldi et al., 1985; Dolan et al., 1993]. Consider the force from a small displacement, normalized on the displacement length,

$$\mathbf{F}/|\mathbf{dp}| = \mathbf{K}_s \mathbf{dp}/|\mathbf{dp}| \quad (2.2.3)$$

When the unit displacement vector $\mathbf{dp}/|\mathbf{dp}|$ is multiplied by the stiffness matrix, the effect is to rotate and stretch it. The resulting vector has the units of stiffness (force per unit displacement) and acts in the direction of the resulting spring force. When the unit displacement vector is applied in all directions about the equilibrium position, the tails of the resulting normalized force vectors trace out an ellipse, as shown in Figure 2.2.3(a). In this case the distance from the equilibrium position to a point on the ellipse *is not* a displacement value, but rather a value of stiffness. Note that, in general, it does not correspond to a displacement in the same direction but rather some other displacement direction that is not shown by this ellipse representation. In this case, the major axis of the ellipse is $k_{\max}$ and the minor axis is $k_{\min}$.

Inclusion of the anti-symmetric component of stiffness still results in an ellipse; however, it changes the ellipse appearance. This new ellipse is indistinguishable from a symmetric stiffness with the same size, shape, and orientation. In other words, the fact

that the stiffness is anti-symmetric is not apparent. A more appropriate method for inclusion of the anti-symmetric stiffness is to create separate stiffness ellipses for K_s and K_a . The anti-symmetric stiffness ellipse is always a circle. Though not proposed in the previous works on stiffness ellipses, the anti-symmetric stiffness circle can be included directly on the stiffness ellipse plot, as shown in Figure 2.2.3(b), as an indicator of the anti-symmetric stiffness size.

Figure 2.2.3: Stiffness ellipse representation of stiffness. (a) Formulation of stiffness ellipse. Force vectors are normalized by the magnitudes of the displacements that created them, and the resulting vectors are placed with the tip at the limb endpoint. The tails of the vectors form an ellipse and the vectors have units of stiffness. Distances from the endpoint to a point on the ellipse are stiffness values and do not represent endpoint displacements as in Fig. 1. (b) Symmetrical and anti-symmetrical components of stiffness along with a calibration circle.

2.2.3 Comparison of Ellipse Representations

Both ellipse representations are characterized by three parameters. The usual parameters chosen are the size, shape, and orientation. These parameters are all functions of the eigenvectors and eigenvalues of $\mathbf{K}_s$. The eigenvalues of $\mathbf{K}_s$ are $k_{\max}$ and $k_{\min}$ as described earlier. The size of the stiffness is an indication of the overall level of stiffness, normally defined by the product of $k_{\max}$ and $k_{\min}$. The shape describes the preference of the endpoint to displace in one direction over another, indicated by the ratio of $k_{\max}/k_{\min}$. A shape near 1 indicates an ellipse that is nearly round (in both representations) and so the stiffness is roughly equal in all direction. A large ratio indicates a long, thin ellipse for both representations, meaning the endpoint is much easier to displace in one direction. The orientation is the direction of ellipse and is usually taken as the angle between the chosen δx -axis and the direction of $k_{\max}$, indicated by the principal eigenvector of $\mathbf{K}_s$.

Equation (2.2.2) relates the two ellipse representations. The isopotential ellipse has its major axis in the direction of the minimum stiffness, whereas the stiffness ellipse has its major axis in the direction of the maximum stiffness. Hence the two ellipses are orthogonal. Furthermore, the axis ratio for the isopotential ellipse is

$$\frac{r_{\max}}{r_{\min}} = \sqrt{\frac{k_{\max}}{k_{\min}}} \quad (2.2.4)$$

where $k_{\max}/k_{\min}$ is also the axis ratio for the stiffness ellipse representation. For a long, thin stiffness ellipse with a high axis ratio, the isopotential will have a smaller axis ratio

and hence is less long and thin. This means that the isopotential ellipse will be generally less elongated than the stiffness ellipse. It is also less sensitive to changes in the stiffness shape ($k_{\max}/k_{\min}$) as noted by Mussa-Ivaldi et al. [1985].

There are advantages and disadvantages to each of the two ellipse representations of stiffness. The isopotential representation is more intuitive and informative in that it demonstrates the behaviour of the system; the result of a displacement is shown in terms of the energy requirements and the direction of force. This representation also translates easily to non-linear form for larger displacements; the energy storage for large displacements can often be calculated analytically or numerically. This makes it more appropriate for theoretical studies. The magnitude of spring force is not directly shown by the ellipse; however, it can be inferred from the relative displacements: directions of high stiffness (small displacement) produce higher forces to reach the same energy level as directions of low stiffness (large displacement). Therefore, displacements along the minor axis have the highest forces while displacements on the major axis have the lowest forces, as seen in Figure 2.2.1(a). Finally, the size of the isopotential ellipse has little meaning. A reference energy level must be chosen to construct the ellipse, but there is no intuitive link between this energy level and the stiffness values; the same ellipse can be produced by generally low stiffness and a low reference energy or by higher stiffness and higher energy. Hence the isopotential ellipse is mostly useful for relative comparisons. The energy surface interpretation avoids these disadvantages, but adds the graphical

disadvantages as previously discussed.

By contrast, the stiffness ellipse representation allows no direct intuitive interpretation. The ellipse radius does not represent a displacement, and there is no connection between displacement directions and points on the ellipse with the exception of the principal axes. Directions other than the major and minor axes have little useful meaning because the stiffness values in these directions do not generally correspond to displacements in the same directions. In the isopotential ellipse, the off-axis directions do provide useful information with respect to force direction and effort (energy) required to displace the endpoint to that point. The most useful information from the stiffness ellipse comes from the axes stiffness values and orientation, and so the whole ellipse is not necessary.

However, the stiffness ellipse representation does directly indicate the stiffness values. This is particularly useful when measuring stiffness, such as in human limb studies. Once a reference stiffness value is shown, as in Fig. 2.2.3(b), the ellipse stiffness values can be readily seen. Energy storage measurements are much more difficult, making the isopotential ellipse less useful in experimental cases. Furthermore, the stiffness ellipse is more sensitive to differences or changes in stiffness than the isopotential ellipse, as noted above.

Summarizing the above comparison, the isopotential ellipse appears to be more useful for theoretical analysis and discussion whereas the stiffness ellipse appears to be more useful for measurements of stiffness where real stiffness values are necessary. In the current work we will be dealing with analytical interpretations and non-linear representation of stiffness, and therefore we will be using the isopotential ellipse representation.

2.3 Summary of Background Concepts

In this chapter we examined the definitions of stiffness in normal one-dimensional springs, both linear and non-linear, as well as linear stiffness in higher dimensional systems. We also discussed conservative systems and how this relates to the symmetry of the two-dimensional stiffness matrices. These concepts are fundamental to understanding the review of previous research in Chapter 3, and the issues that arise from it in Chapter 4 that we will be investigating in the current work.

Chapter 3

Review of Previous Research

The proper design for a prosthetic limb draws on research from a variety of areas. The obvious area for review is the history of limb prostheses design. This review will give us an indication of the state-of-the-art and shortcomings of the technology. We can also draw on technology from the robotics industry since it must also deal with manipulator motions and contact tasks, and can provide successful engineering solutions as well as unsuccessful techniques to avoid. Finally, since the goal of prosthetic limbs is to replace the natural limb, studies of natural limb mechanics are essential to understanding what characteristics are important for inclusion in a prosthesis. These are the areas of research that we'll review in the current chapter.

3.1 Upper Limb Prosthesis Development

Previous works provide a detailed history of upper limb prosthesis development [Mann, 1981; English, 1996]. We present a summary of this development here with an emphasis on the dynamics and contact abilities of the technology.

The first notable powered upper limb prosthesis was the Boston Elbow developed in the 1960s [Williams, 1986]. The elbow was controlled by the natural nerve signals used to activate muscles, known as myo-electric signals (MES), measured on the skin surface above the amputee's bicep and tricep muscles. The proper interpretation of these signals, i.e., determining what action they would have caused in the natural arm, is difficult. In the Boston Elbow, the activation level of the muscle signals was used to control the arm velocity. Consideration of environmental contact and constrained motion was limited to subtracting external forces from motor applied forces thereby requiring a constant muscular effort from the amputee to maintain a constant force [Rothchild & Mann, 1966].

The 1970s saw several advancements in prosthetic control, mainly in the areas of myoelectric control sources and signal processing. The Temple Arm project showed some success in coordinating wrist and elbow motion using myoelectric signals from the chest and back muscles to drive velocity-controlled servos [Taylor & Finley, 1974; Mann, 1981]. However, proper control of the hand was less successful and eventually delegated to signals from unrelated muscles. The Swedish Hand was successful in control of wrist motion, finger flexion, and thumb rotation using separable patterns obtained through multiple electrodes placed on the forearm stump [Lawrence & Kadefors, 1974; Mann, 1981]. Since forearm level measurements were required, the successful control could not be adapted to above elbow amputations.

During the same period, intense research in muscle modeling and upper limb prosthesis design resulted in the development of the Utah Arm [Jacobsen and Mann, 1973; Jacobsen et al., 1982]. Excellent results were achieved in predicting joint torques using rigorous computer models and electromyographic (EMG) signal measured on the chest and back. In final arm design, EMG activity in numerous shoulder muscles was used to control elbow flexion, forearm rotation, and wrist flexion [Mann, 1981].

While all of the above mentioned prosthesis developments represented clear advances in signal acquisition, processing, and interpretation, there was still little concern for the mechanics and dynamics involved in contact tasks and constrained motion. Yet today, the Utah Arm and Boston Elbow still represent the state-of-the-art in upper limb prosthesis design. In fact, there has been very little research in the area of prosthesis interaction at all. These powered prostheses have not been well accepted by amputees [Abul-Haj & Hogan, 1990a,b].

It has been proposed that there are three main problems in current powered prosthesis design [Abul-Haj, 1987]: (1) myoelectric signal acquisition, (2) myoelectric control algorithms, and (3) socket and harnessing. Clearly such problems are being dealt with readily. A large amount of research in power prostheses tends towards the first two while socket and harnessing design is making progress through advanced material technology and fitting techniques, such as stump shape scanning. A fourth problem,

proposed by Miller [1987], is dynamic behaviour, referring to synchronization and inertia behaviour of the artificial limb while performing such tasks as walking, standing up, balancing, opening drawers, and turning sharply. Interaction can be viewed as a problem of both control and dynamics, though it requires more than management of inertia, suggesting a fifth problem.

There are two general approaches to the modification of prosthesis behaviour in the limited study of prosthesis interaction. The first approach uses the addition of a passive compliance element to the device. Examples include a compliant wrist [Jacobs, 1993], and the addition of a low stiffness 'free swing' mode to the Utah Arm, allowing the user to switch the elbow behaviour between an active rigid joint and a passive compliant one [Jacobson et al., 1982]. These implementations are an attempt to restore, at least in part, some function of the intact arm.

The second approach to prosthesis interaction deals directly with the mechanics of interaction. Collaborative work at M.I.T. [Abul-Haj & Hogan, 1990a,b; Popat et al., 1993] used a prosthesis emulator to compare the control algorithms of the Boston Elbow and Utah Arm to a "natural controller" proposed by Hogan. This "natural controller" regulated the dynamic properties of the prosthesis through a feedback algorithm. Specifically, the controlled dynamic properties were the mechanical impedances -- namely stiffness, damping, and apparent mass. This control technique, known as

impedance control, was developed earlier by Hogan as an approach to manipulation both in the prosthetic and robotic fields [Hogan 1980a, 1982, 1985b,c,d]. The approach was partially derived from the natural techniques used by humans (and other animals) when performing interactive tasks, and is currently used primarily in the robotics research field. (Impedance control will be discussed in more depth in the next section on robotic interaction research.)

The emulator comparison described above consisted of a constrained motion requiring test amputees to turn a crank using the prosthesis emulator with various controller algorithms. Results from the tests provided two significant conclusions. First, arm motions performed using the Impedance Control technique appear to match more closely with those of the natural arm. Second, improvements in prosthesis control that result in more natural appearing motion seem to make power prostheses more acceptable to amputees. A later theoretical comparison between these controllers also suggested that energy expenditures in a prosthesis could be reduced up to twenty times during constrained motions by providing user control of joint stiffness [Russell, 1993].

The concept of modulating joint stiffnesses as a means of controlling dynamic and constrained properties in a prosthesis was first proposed by Hogan [1980a,b, 1982, 1984a]. This early work suggested implementation of variable stiffness through antagonistic actuation similar to muscular systems acting about human joints. Actuation

research since that time has provided several muscle-like actuators that would allow such an implementation, including polymer gels [Meghdari et al., 1993], rubber, collagen, shape memory alloys, piezoelectric polymers and ceramics, magnetorestrictive devices, electrostatic motors, and pneumatic bladders [Immega, 1987; Greenhill, 1993]. The simplest and most basic method of producing antagonistic actuation is through the series combination of nonlinear stiffness elements, such as springs, and common driving elements, such as dc-motors [Laurin-Kovitz et al., 1991; English, 1996; English & Russell, 1997, 1999a,b].

English and Russell [English, 1996; English & Russell, 1996] also showed that the use of stiffness elements in series with the actuators can also improve the efficiency of operating a prosthesis by both permitting management of contact forces and allowing the actuators to be non-backdrivable. This means the motors do not have to produce a constant energy drain to maintain joint torque as in existing prostheses or even human muscles. The stiffness elements can both maintain the force and allow reduced forces at the stump for unexpected impacts of the prosthesis.

3.2 Robotic Manipulator Development

A complete review of robot manipulator development is not necessary to understand the issues applicable to upper limb prosthesis development. There have been several reviews of the history of robot manipulator control that provide those details

[Whitney, 1987; Sasaki, 1995]. We present here a summary of the issues relevant to the development of prosthesis design.

One of the major conclusions from the development of industrial robot manipulator controllers is that kinematic properties (position, velocity, acceleration) must be controlled for unconstrained motion, whereas contact forces must be managed in constrained directions. Successful motion controllers have included:

(1) Servo (position) Control

Derived from early 3-axis numerical controls, this method simply compares the desired position with the current position and applies a torque proportional to the difference.

Highest accuracy occurs with high feedback gain (proportionality constant), but produces unstable forces in contact.

(2) Velocity Control

Similar to position control, but torques are proportional to differences of input and actual velocity.

(3) Acceleration Control [Hori, 1988; Ohnishi, 1993]

All dynamic effects are viewed as disturbances. A disturbance observer uses acceleration feedback to suppress unmodelled dynamics. Because of the shorter response time

required with acceleration feedback, this method only became practical recently with fast switching devices and high-speed digital signal processors.

(4) Adaptive and Learning Control

Robot manipulators improve performance by learning from past behaviour. Such methods work best with regularly repeated tasks such as assembly work.

Although many of the above approaches have found success in applications where contact forces are negligible, there are several associated problems. The primary problem with these controllers is the necessity to perform inverse kinematics [Sasaki, 1995]. The ultimate goal of these schemes is to move the manipulator endpoint to a desired point and orientation, sometimes with the requirement of following a set trajectory. This requires one to calculate the joint positions associated with the desired endpoint position, known as the *inverse kinematics*. There is no unique method for solving all inverse kinematic problems. To make matters worse, there are sometimes multiple solutions to the inverse kinematics, such as when joint axes line up or redundancies are present.

Contact force controllers have generally had less success than motion controllers. Some successful force control methods have included:

(1) Explicit Force Control [Nevin & Whitney, 1973]

Contact force is directly commanded and measured and the joint torques are proportional to the difference. The goal is for the contact forces to follow the commanded input force as closely as possible.

(2) Accommodation (Damping Control) [Whitney, 1977]

Manipulator velocities are regulated in contact. Measured contact forces modify the commanded velocities through an admittance matrix (velocity/force).

(3) Passive Compliance [Watson, 1976; Drake & Sumunovic, 1977]

Contact forces are reduced by placing compliant elements (springs) in the wrist or joints. This allows the workpiece to control the exact location of the end effector so the requirements of precise position control are reduced. Remote Center Compliance (RCC) [Drake & Sumunovic, 1977] has been proven to be very successful for peg-in-hole tasks.

(4) Stiffness Control [Salisbury, 1980; Salisbury & Craig, 1982]

Desired manipulator position is controlled. Contact forces are converted to position modifications through a controlled stiffness matrix. The system effectively simulates joint springs with programmable stiffness.

(5) Hybrid Force-Position Control [Mason, 1981; Railbert & Craig, 1981]

Two control algorithms are combined. An explicit force control algorithm is used in constrained directions and position control is used in unconstrained directions. Yoshikawa [1986] included manipulator dynamics to improve the performance.

(6) Impedance Control [Hogan, 1980a, 1984b, 1985b,c,d]

The apparent dynamics of the manipulator, including the apparent stiffness (as in stiffness control), damping (as in damping control), and inertial forces are controlled. The impedance control approach utilizes the perspective that the net effect of any controller is to change the natural dynamics of the manipulator to appear as a different dynamic system to the environment. Therefore controlling the apparent dynamic properties (impedances) of the manipulator through position, velocity, and force feedback can allow constrained and free motion to occur as desired.

Contact force controllers have generally been plagued with worse problems than motion controllers, particularly with respect to contact instability. General problems in force control include sensing of contact and collocation (sensors and actuators located at different positions). There are essentially two approaches that attempt to control the dynamics when unconstrained and contact forces when constrained. Hybrid control uses two control schemes to achieve this but contains some theoretical inconsistencies, particularly when extended to include orientation. It also views the world as either constrained or unconstrained, leaving little room for less rigid contact.

On the other hand, impedance control recognizes that the net effect of feedback is to change the apparent impedances (stiffness, damping, inertia) of the manipulator [Hogan, 1987; Colgate & Hogan, 1989]. It essentially combines stiffness control, damping control, and explicit force control but uses the feedback gains to set the desired apparent impedances. This allows a single control system for both constrained and unconstrained tasks and can guarantee stability if the apparent impedances are chosen such that the manipulator appears to be a stable passive system [Hogan, 1988]. Furthermore, impedance control can circumvent the necessity of inverse kinematics by creating a virtual position at which the endpoint tries to follow, and the apparent impedances control how close it follows. This approach follows from the *equilibrium point hypothesis* for human motion presented in the next section. The largest disadvantages to the impedance control approach include the large amount of computation and the necessity to select a larger number of control parameters. There is also a limitation on achievable apparent impedances. When disturbance torques become large, the manipulator's natural impedances and the back EMF damping of the motors tend to dominate [Lawrence, 1989].

In summary, contact force controllers have not yet achieved the success of unconstrained motion controllers. The most promising approach that controls unconstrained motion and contact forces is impedance control, which also has ties to theories of human motor control. Such an approach is technically feasible using existing

robot control technology and so can be applied to prostheses.

3.3 Human Movement Studies

The ultimate goal of prosthetic limbs is to exactly duplicate the control and behaviour of the natural limb. It is important to understand the mechanics and control of the natural limb. The mechanical properties are reasonably well understood at the level of the muscular action. However, how the mechanics relate to overall global behaviour is somewhat less well understood, and in some cases there is significant disagreement. Control of the limb is much more controversial. There are a variety of theories as to how the body controls limb motion and forces and how significant a role the limb mechanics play in this behaviour. These are the issues that will be addressed in this section. We will look at limb stiffness, the role of bi-articular muscles, and motion control theories as they apply to the design of upper limb prostheses.

3.3.1 Limb Stiffness

It is well known that muscles are more than simple force generators. Most joints in the human skeletal system are operated by antagonistic (opposing) muscles pairs. Activation of both sets of muscles, referred to as co-activation, alters impedance properties involved in body motion and posture, including equilibrium position, stiffness, and damping [Hogan, 1980b, 1984a, 1985a; Winters et al., 1988; Dolan et al., 1993; Lan & Crago, 1994; Lacquarti et al., 1993; Latash & Zatsiorsky, 1993; Kornecki &

Zschorlich, 1994; Smith, 1996]. This is important for application to prosthetics because the torque motors used in traditional upper limb prostheses are essentially modelled as ideal torque generators, and the control loops around these motors tend not to try to duplicate any properties of the natural limb.

The muscle itself is typically modelled as a three element system containing a contractile component (CC), parallel elastic component (PEC), and series elastic component (SEC) as shown in Figure 3.3.1(a) [Hill, 1938]. The contractile component represents the contractile proteins, actin and myosin, which are responsible for shortening of muscle upon activation and may also exhibit some elasticity. The PEC represents the compliance of the connective tissues surrounding the muscle fibres. The SEC represents the elasticity of the tendons [Nordin & Frankel, 1989]. The resulting tension-length relationship is the combination of active tension from the contractile component and passive tension from the PEC and SEC, as shown in Figure 3.3.1(b). The effect of the activation level of the muscle is to select a particular tension-length curve as shown in Figure 3.3.1(c). The stiffness of the muscle at any time is the slope of the selected curve, and so stiffness is a non-linear function of activation level and muscle stretch.

Since muscles are only capable of providing tension, they tend to act in pairs. Agonistic muscles flex the joint whereas antagonistic muscles extend the joint. When agonist and antagonist are co-activated, both muscle stiffnesses tend to increase. Since

they work in parallel at the joint, the effective joint stiffness is proportional to the addition of these two muscle stiffnesses. Therefore co-activation tends to increase the joint stiffness. This increased stiffness can stabilize the joint in the face of unstable loading and is fundamental to control of posture [Hogan, 1984a; Hogan, 1985a; Mussa-Ivaldi et al., 1985; Winters et al., 1988; Kornecki and Zhorlich, 1994].

The net joint stiffness will generally be the combination of the inherent muscle stiffnesses, as described above, and reflexive response. The reflexive response indicates an increase in muscle activation level due to displacement of the limb. The reflex force is therefore a function of muscle displacement. For small displacements, this relationship can be approximated as linear,

$$F_{\text{reflex}} = K(x-x_0) \quad (3.3.1)$$

where K is the feedback gain and $x-x_0$ is the displacement. In this manner, the feedback gain acts as the stiffness constant. Even in non-linear form, reflexive response can appear as a non-linear spring. It is difficult to distinguish between inherent stiffness and reflexive stiffness when attempting to measure the net joint stiffness. Several studies suggest that each contribute stiffness at approximately the same order of magnitude [Bennett et al., 1994; Zhang, 1997].

Figure 3.3.1: Muscle characteristics. (a) Muscle model. The contractile element (CC) represents actin and myosin filaments that shorten the muscle, the parallel elastic element (PEC) represents connective tissue around the muscle fibres, and the series elastic element (SEC) represents the tendons [Hill, 1938]. (b) The tension-length curve for muscles is a combination of active tension due to muscle activation level and the passive elastic properties (adapted from Nordin & Frankel [1989]). (c) The muscle activation level chooses a specific tension-length curve for the muscle (adapted from Nordin & Frankel [1989] and Hogan [1984a]).

There is a trade-off between the two methods. Reflex control is bounded by feedback transmission delays on the order of 50-100 milliseconds and is limited by the bandwidth characteristics of the muscles and sensors. Antagonistic co-activation is unaffected by neural transmitter delays because it operates in open-loop fashion, but at the cost of continuous metabolic energy expenditure that performs no mechanical work. Hogan [1984a] suggested that the two are combined by the central nervous system to achieve an optimal balance. Generally, the two methods of controlling stiffness (or apparent stiffness) are not separated because they have the same effects. That is, it is generally impossible to tell the difference between a passive and active stiffness control scheme from the perspective of the environment. The exception to this similarity is the appearance of conservativeness.

The inherent muscle stiffness must act conservatively (see Section 2.1.3 and Appendix B) since it supplies no means of continuous energy input. However, reflexive action can introduce non-conservative stiffness. To do this, the cross coupling terms must have different gains. That is, the gain relating displacements of one joint to torques about another is different than the gain for the opposite relationship [Hogan, 1985a]. Currently there appears to be some disagreement as to the conservative appearance of stiffness in the human arm. Mussa-Ivaldi et al. [1985] demonstrated that the combined inherent and reflexive muscle stiffness results in a strongly conservative-appearing system when considering the shoulder and elbow joint, confirmed later by Dolan et al.

[1993]. This contradicts the results of Lacquanti et al. [1993] who found significant non-conservative stiffness when considering the elbow and wrist joints. It should be noted that in the study of Lacquanti et al. the subjects were told to consciously resist any disturbances and this may have been a factor in the discrepancy.

Actual measured stiffness values also have some discrepancy. Mussa-Ivaldi et al. [1985] measured postural stiffness with direct shoulder stiffness ranging from 16.4-29.9 Nm/rad and direct elbow stiffness From 13.4- 41.3 Nm/rad. Bennett et al. [1992, 1993] measured similar, though slightly lower, stiffnesses during reaching movements: at the shoulder during single-joint movements, 22-30 Nm/rad at motion start, 20-24 Nm/rad during motion, and 30-36 Nm/rad at motion end; at elbow, 3-14 Nm/rad during single-joint movements and 5-21 Nm/rad during multi-joint movements. These were consistent with measurements of Gomi and Kawato [1996]. By contrast, Lacquanti et al. [1993] measured direct elbow stiffness ranging from 81.6-154.2 Nm/rad during tasks in which the subjects resisted perturbations, gravity loads were present, and subjects prepared to catch a ball. These results taken as a whole seemed to imply that postural stiffness lowers when performing reaching movements, with higher values during acceleration and deceleration, and greatly increases when consciously resisting disturbances, gravity loads, and preparing for changes in loading. However, measurement of stiffness is difficult and is known to be somewhat inconsistent [Smith, 1996; Latash & Latash, 1996]. The best methods to date involve pseudo-random perturbations [Bennet et al., 1992] or partially

contracted (tonic) bias forces [Bennet et al, 1993].

In several of the above mentioned studies, the endpoint stiffness was directly measured. Mussa-Ivaldi et al. [1985] concluded that geometrical factors dominate the endpoint stiffness as seen at the hand. This is consistent with Dolan et al. [1993] who found that an increase in stiffness in one direction was usually accomplished by increasing the stiffness in all directions, i.e., the size of the stiffness increased rather than a change in shape occurred. They concluded from this that muscles tend to work in unison rather than independently. Joint stiffness, however, appears to be less of a function of position. Although the moment arms through which the muscles act about the joints change with angular position, the effect on joint stiffness seems to be small except at the extremes of the workspace [Lacquaniti et al, 1993].

3.3.2 Motion Control Theories

Ideally if a prosthesis were to replace a natural limb it would operate in the same manner, including motion control and contact force management. The current state of motion control is still somewhat unresolved and is certainly not detailed enough to duplicate it exactly in a prosthesis or artificial manipulator. However, if the elements necessary for proper motion control are present in the prosthesis, it may be possible to allow the CNS to control the motion directly as in the natural limb.

There are several leading theories towards motion control in humans. Some researches suggest that the CNS performs inverse kinematic and dynamic calculations to determine the necessary changes in muscle forces. There are two problems with this approach. First, it requires that the CNS is precisely aware of limb inertias, center of mass, and muscle moment arms. Even small errors can result in significantly different motions [Bizzi et al., 1992]. Second, there are multiple solutions to many of the inverse kinematics and dynamics problems because of limb and muscle redundancies. It would be necessary to define another set of constraints to choose the appropriate solution.

Perhaps the most dominant theory of motion is the *equilibrium point hypothesis*. This approach relies on an equilibrium defined by the tension-length relationship of antagonistic muscles acting about a joint. If all loading were removed (including gravitational) and the muscle activations remained constant, the position at which the arm would come to rest is this defined equilibrium position. The terminology can be somewhat confusing since the arm can be in equilibrium with an external load at a different position. Therefore the no-load equilibrium position is usually referred to as the *virtual position* or *attractor* to avoid confusion [Bizzi et al., 1992].

The hypothesis for the creation of motion is that the virtual position is a control parameter that can be moved by the CNS. As the virtual position moves from the actual position, joint moments result as a function of the difference,

$$\mathbf{M} = \Psi(\theta - \theta_v) \quad (3.3.2)$$

where $\mathbf{M}$ is the vector of joint moments, θ is the actual position, θ_v is the virtual position, and Ψ is the stiffness matrix. The dynamic properties (damping, inertia) keep the actual position from exactly following the trajectory of the virtual position. The muscles then act as both force generators and tunable springs.

There are several variations of the hypothesis. The *equilibrium point-invariant* hypothesis, sometimes referred to as the *λ -model*, originated with Feldman [1966, 1986]. The theory essentially states that a centrally controlled parameter, λ , may set the zero stretch length of the muscle such that muscle force is a function of stretch as well as λ , $F_{\text{muscle}} = f(x - \lambda)$, where f is a threshold function such that $F_{\text{muscle}} = 0$ for $x \leq \lambda$. The virtual position and the activation level of the muscles are therefore selected by the choice of λ . The force function itself is invariant (hence the terminology), and so λ varies θ_v rather than the stiffness in (3.3.2). Figure 3.3.2(a) shows the invariant characteristic and the shifting of the zero position. Fundamental to this theory is the necessity of both inherent muscle stiffness properties and reflexive feedback.

The other leading variation is the *equilibrium point-stiffness* hypothesis, sometimes referred to as the *α -model* (though the group most associated with this

approach claims that the α -model only partly expresses their view [Bizzi et al., 1992]). In this theory, activation levels of the muscles select a particular tension-length curve and so the virtual position is defined by the relative co-activation of the antagonistic muscles. This has the effect of changing the stiffness in (3.3.2). The virtual position is shifted by changing the relative activation levels of the muscles as shown in Figure 3.3.2(b).

A motion is created by shifting the virtual position along a desired trajectory from the starting position to a desired end position. The path is known as the *virtual trajectory*. The hand will follow closely, but not exactly because of the limb's dynamic properties. When a constraint is encountered, force can be controlled by moving the virtual point inside the constraint surface and the resulting forces will be a function of the difference between actual and virtual positions, and the endpoint stiffness.

The major difference between the α - and λ -models is the role of reflexes in performance of a movement. Both agree that reflexes may modify the central commands, but the λ -model suggests that reflexive action dominates the control of the movement, which isn't supported by experimentation on monkeys [Bizzi et al, 1992]. Since the α -model does not distinguish between inherent muscular stiffness properties and reflexive stiffness, the λ -model can be considered to be a subset of the α -model [Bizzi et al., 1992].

(a)

(b)

Figure 3.3.2: Equilibrium point hypotheses. (a) Equilibrium point-invariant hypothesis. The control parameter λ sets the zero crossing, and thereby the virtual position, of an invariant force function. (b) Equilibrium point-stiffness hypothesis. The virtual position is a function of the relative activation of antagonistic muscles. Each intersection represents the virtual position for those two activation levels. The virtual point is shifted by varying the contribution from the antagonistic muscles. (Adapted from Winters et al. [1988].)

The basis for the α -model derives from experimentation with monkeys [Bizzi et al., 1984]. Monkeys that had been deprived of sensory feedback were still able to move their hand between two targets, but performed the movement more sluggishly than when the feedback was intact. This meant that the performance of point-to-point motion was possible without the need for feedback, contradicting the requirements of the λ -model. A second experiment consisted again of moving between two points, but an applied assisting torque pushed the monkey's hand ahead of the presumed virtual position. When released, the hand moved back toward the starting position, presumably towards the current location of the virtual position, before continuing on towards the target. This clearly indicated that the arm motion was not planned as a function of the current arm position, but rather a predetermined timed trajectory. Finally, a third experiment was performed in which all feedback and visual cues to the arm position were removed. The arm was displaced from the starting position to the final position, unknown to the monkey, and released when the target light was turned on. The hand would initially move back towards the starting position before returning back to the final target, even though it was initially located at the target position. These experiments suggest a virtual trajectory existed that attracted the hand.

Various other studies have supported the notion of a virtual trajectory guiding the hand. Shadmehr et al. [1993] found that postural forces were comparable to static forces generated while attempting to perform reaching movements, linking postural forces with

those induced to generate a movement. This is consistent with the hypothesis that movements are generated shifting a virtual point towards a target. Flash [1987] used stiffness and damping characteristics derived from experimentally measured values to generate a simulation model that used a shifting virtual point. The virtual point was assumed to shift between origin and target in a straight line. The simulations successfully captured the observed behaviour of limb motions. Though the virtual trajectory was straight, the limb dynamics caused the actual limb endpoint to follow a slightly curved path, consistent with observed movement patterns. These results showed that it is possible for the CNS to plan motions in endpoint coordinates and ignore the inertial and damping effects of the limb. The motion only required a planned trajectory and velocity profile in endpoint coordinates. This approach circumvents the necessity to perform inverse kinematic and dynamic calculations.

There have been a variety of objections to the equilibrium point hypotheses. Van Ingen Schenau et al. [1992] and Doorenbach & van Ingen Schenau [1995] argue that position alone cannot define static force and movement directions, both position changes and force directions must be defined separately. However, their examples are easily explainable within the framework of the equilibrium point hypothesis. They also suggest that the equilibrium point hypothesis requires that one of the agonist-antagonist pair must lengthen while the other shortens in order to produce the spring-like behaviour, but observed that during push-off in jumping both muscle sets shortened [van Ingen Schenau,

1989]. To the contrary, the equilibrium point hypothesis allows shortening of both groups when increasing joint stiffness. It has been observed that stiffness tends to increase during acceleration and deceleration of the joints [Bennett et al., 1992; Bennet, 1993; Lan & Crago, 1994], which is exactly what would be expected during a jumping push-off.

The most damaging objection to the equilibrium point hypothesis is the size of the necessary stiffnesses to properly perform the observed fast arm movements. Flash's simulations had shown that the actual limb position would follow the virtual trajectory closely for slower speeds, but faster movements required significantly higher stiffness (scaled with the square of the velocity increase). Experimental evidence actually showed a decrease in stiffness during motion [Bennett et al., 1992, 1993]. The only other way to perform fast motion under the equilibrium point hypothesis was by changing the virtual trajectory. However, the necessary trajectory would be complex and require significant additional computation by the CNS, thereby removing the simplicity of control suggested by the equilibrium point hypothesis [Kawato et al, 1992; Gomi & Kawato, 1996].

A proposed variation of the α -model may be able to circumvent the problems of low stiffness. McIntyre & Bizzi [1993] proposed that the muscular activations could be controlled by the CNS defined virtual trajectory (position & velocity) and supplemented by feedback position and derivative controllers. The virtual trajectory signal would be

used as position and velocity reference values. Then the feedforward virtual trajectory alone is sufficient to drive the limb to the target, but more sluggishly than with intact feedback controllers. It is interesting that this is exactly what was observed in the monkey experiments when they were deprived of feedback.

Another common objection is the inability to prove, or disprove, the hypothesis because the antagonistic muscles will inherently define a virtual position and trajectory, and therefore proving the existence of such an attractor is somewhat moot. The issue is whether the virtual trajectory is a control parameter itself or a consequential result of some other control scheme. So far, determining the control parameters used by the CNS has been an impossibility and so there is no way to prove or disprove the hypothesis [Hollerbach & Atkeson, 1987]. However, Won & Hogan [1995] point out that it can be disproved if the virtual trajectory is too weak to dominate the movement. Even if a virtual trajectory is inherently defined, the trajectory need not be a simple path between points and the arm does not need to follow this trajectory closely if other forces dominate. In fact, these are necessary factors to the objections raised regarding the insufficient stiffness [Bennett et al., 1992, 1993; Kawato et al., 1992; Gomi & Kawato, 1996].

Won & Hogan [1995] provide an elegant demonstration that reaching movements strongly tended to follow an attractor (virtual) trajectory. No stiffness measurements, limb models, or linearized stiffness properties were assumed. They set up a manipulator

system such that test subjects could move freely between two target positions. During some trials, unknown to the subject, the manipulator was constrained to move in an arc between the two points. In other trials this constraint was only applied through the initial part of the motion. Results from the constrained movements clearly showed strong forces attracting the hand towards the unconstrained trajectory rather than towards the target. Furthermore, they demonstrated, without assuming any specific limb model, that these forces could not be attributed to dynamic forces of the motion. This clearly demonstrated that the limb was attempting to follow a virtual trajectory. By bounding its limits, they also showed that this virtual trajectory must lie very close to the unconstrained trajectory. In the temporary constrained trials, the limb tended to return to this trajectory rather than recalculate a new trajectory based on the current limb position.

The ultimate conclusion of the above experiment was that the limb had a predetermined trajectory between the two points and the effect of the constraint was to induce spring-like forces in the limb to return it to this trajectory. Furthermore, the subjects were unaware which trials would be constrained or unconstrained. This suggests that the motions were all planned using the same control scheme. This indicates that stable contact forces, as seen in the experiment, can be produced using a purely unconstrained based trajectory and the inherent stiffness mechanics of the limb.

These results are consistent with those of McIntyre et al. [1995], who performed a

virtual peg-in-hole task in which the subjects were to slide a peg along a surface and insert it in a hole. The task was performed using a force feedback joystick and a simulated surface. Unknown to the subjects, the surface was occasionally inclined or declined by 10°. The contact force consistently increased during inclined trials, and decreased during declined trials. The conclusion was that the contact force was not being controlled, but rather the hand was following a predetermined path based on an a priori model of the surface, and the forces resulted from the compliance of the limb. Contact forces were therefore produced by following a virtual trajectory inside the constraint surface as predicted by the α -model.

Clearly the motion control scheme of human arm movement is not without controversy. This makes it difficult to implement a realistic control system within a prosthesis to duplicate the natural processes. Even if the control scheme was well understood, providing feedback to the CNS for command modifications is not practically possible at this point. One thing that all of the leading motion control theories have in common is that the impedance properties of the arm play a large role in the dynamic behaviour of motion. The equilibrium point hypothesis requires the stiffness properties of the muscles to dominate the control of movement, even without direct control by the CNS. It is therefore prudent to include such properties in a prosthesis. Unfortunately, the control of the stiffness during motion or contact is not well understood or investigated.

3.3.3 Bi-articular Muscles

In an above-elbow amputation, the mono-articular (cross one joint) shoulder muscles remain intact. The mono-articular elbow muscles are lost, and it is these muscles we hope to replace in a antagonistically actuated prosthesis. However, the bi-articular muscles (crossing both shoulder and elbow) would not be replaced in this approach because the shoulder joint remains biological while the elbow is artificial. We should then be aware of what effect these bi-articular muscles play in the natural arm.

The bi-articular muscles contribute to the joint stiffness at both shoulder and elbow. Hogan [1985a] noted that they allow greater control over the endpoint stiffness characteristics. In particular, he demonstrated that the range of positions in which the stiffness could be made isotropic (equal in all directions) was reduced to a single point with the removal of the bi-articular muscles. He also noted that without them, there was a coupling between stiffness characteristics. This occurs because there are three endpoint stiffness characteristics – size, shape, and orientation (see Section 2.1) – and normally three control inputs, namely the stiffness contributions from the two mono-articular sets and the bi-articular set. Removal of the bi-articular set reduces the stiffness inputs to two. Kumamoto et al. [1994] noted a change in stiffness orientation in simulation by removal of the bi-articular stiffness elements. Lan [1995] noted that bi-articular muscles can contribute more towards the stability of the limb than the mono-articular sets.

Bi-articular muscles are known to contribute to limb mechanics in ways other than their stiffness contribution. They can contribute to converting rotational motions of joints to smooth translations of the limb [van Ingen Shenau, 1989; Kumamoto, 1994]. This is somewhat related to the transmission of mechanical work and power from one joint to another. As one joint reaches its maximum extension, the mechanical energy must be dissipated to avoid overextension. As well, at this point the work that can be done across the joint is minimal. Bi-articular muscles can transfer this to another joint where it is more useful. Van Ingen Schenau [1989] demonstrated this property using a simple physical model of a person jumping. The model was constrained to move in the vertical direction only, and had movable hip, knee, and ankle joints. A spring was attached across the outside of the knee. The model was then compressed and released, causing it to jump. A wire was then attached across the back of the knee and ankle, simulating the bi-articular gastrocnemius muscle. When compressed by the same amount, the model with the bi-articular wire could jump almost twice as high. The wire was not compliant and was slack when the model was compressed, and therefore offered no additional energy storage. While the same amount of energy had to be dissipated in both cases, the amount transferred towards useful work was higher in the bi-articular case. This power transfer capability has been demonstrated by other researchers as well [Wells, 1988; Jacobs et al., 1996].

Some suggested benefits of bi-articular muscles are less clear. Gielen & van

Ingen Schenau [1992] argued that some tasks required joint displacements and forces to act in opposite directions, thereby doing negative work (i.e, the environment does work on the muscles). They argued that this was inefficient because it required muscles to dissipate energy, and could be avoided by proper distribution of joint moments between mono- and bi-articular muscles so as to minimize, or remove, the negative work. No experimental evidence was offered in support of this hypothesis. The problem with this argument is that it assumes that negative work is inherently inefficient. In fact, negative work can be more efficient [Alexander, 1997]. In fact, negative work falls well within the framework of the equilibrium point hypothesis for motion control. Since the environment is doing work on the muscles, they are being stretched. Rather than being dissipated, this energy may be temporarily stored in the elastic stretch of the muscles and later returned to aid in the movement. In addition, Figure 3.3.1(b) shows that a muscle stretched by external forces can supply larger forces than a muscle that is shorter than its rest length.

Other suggestions on the use of bi-articular muscles include the allowance of independent control of force and motion [van Ingen Schenau, 1989; van Ingen Shenau et al., 1992; Doorenbosch & van Ingen Schenau, 1995; van Bolhuis et al., 1998]. In particular, Bolhuis et al. [1998] found that bi-articular muscle activation tended to be a function of force magnitude and direction, but didn't vary with respect to direction of motion under given loads. Conversely, mono-articular muscle activation tended to be a

function of endpoint force magnitude and movement direction, but not force direction. They found that the mono-articular elbow muscles showed the most activation in a direction that they claimed was the direction of greatest shortening of the elbow angle. (It should be noted that we could not reproduce the same maximum shortening directions even with the aid of the article's lead author. In our calculations, the maximum direction of elbow angle shortening was always when the hand moved towards the shoulder, regardless of postural position.)

In summary, the bi-articular muscles appear to offer a variety of advantages for both control and mechanics of the limbs, including smoothing of motions and energy efficiency. However, it is unclear from the literature what quantitative limitations on limb control and stiffness the removal of bi-articular muscles introduces. The bulk of this information is in the form of specific examples and qualitative descriptions.

Chapter 4

Problem Statement

As stated in Chapter 1, the focus of the current work is the application of natural limb mechanics and control to an above-elbow prosthesis. This requires a firm understanding of both the behaviour of the limb, and the mechanics and control that produce the behaviour. In Chapter 3 we investigated current knowledge in these areas. It is apparent from the literature that existing prostheses do not adequately reproduce the behaviour of the natural limb, particularly with respect to constrained motions. From the field of robotics we have learned that natural-appearing motion and contact force management can be implemented in practical mechanisms by controlling the impedance properties of the manipulator, in this case the human arm. This approach offers several advantages: it can guarantee stable interaction, it avoids the inverse kinematics problem, the dynamics can be controlled to match natural motion, and it uses a unified approach to both constrained and unconstrained motion.

A prosthesis emulator using the impedance control approach appears to have been more acceptable to amputees. Unfortunately, the large amounts of computation and

energy inefficiency make this approach less practical for implementation into a prosthesis. This limitation is a result of the virtual nature of the impedance control approach; apparent impedance values are implemented in a feedback controller around a torque motor. The responses must then be calculated within a computer. The storage of energy is simulated; it is actually dissipated upon input and re-created from the motor power source on output. These disadvantages can be avoided by controlling the real impedances of the prosthesis. Responses would not have to be calculated, but rather would be a natural result of the mechanics. Energy would be stored in real stiffness elements rather than simulated.

From human movement studies it seems apparent that we naturally use a combination of the above approaches. There is much debate as to the actual scheme that the CNS uses to control motion and contact forces, but it appears to be universally accepted that mechanical properties of muscles play an important role in the behaviour of the limb. In fact, the 'equilibrium point-stiffness' hypothesis suggests that the muscle mechanical properties are the largest contributor to limb behaviour and the limb stiffness in particular is largely responsible for driving limb movements and managing contact forces. It is fortunate then, that the 'equilibrium point-stiffness' hypothesis of human movement control and impedance control in robotics are related. In fact, Hogan's impedance control is largely based on the α -model. Both approaches use a virtual position to guide the limb and allow the impedances (real and virtual) to guide the

manipulator's actual behaviour.

It seems clear from the above arguments that mechanical control of stiffness is a wise approach to actuation in an above-elbow prosthesis. We have previously developed a conceptual design for such a prosthesis [Russell & English, 1995; English, 1996; English & Russell, 1997, 1999a, 1999b]. However, there is insufficient information in the literature to understand what stiffness properties are required to duplicate the natural behaviour of the limb and what limitations exist. Existing research describes qualitatively that co-activation of muscles control the limb stiffness as seen at the endpoint, but it does not provide the analytical relationships necessary to understand how to control the muscles to produce the desired changes. There is also little information as to what stiffness properties are desirable. This lack of information is best summed up in the following problem statement:

There is insufficient understanding in the control of stiffness in multi-joint manipulators to properly implement mechanical stiffness control in an above-elbow prosthesis.

The problem statement covers many subjects. We can essentially break down the problem into eight major questions:

1. How do the antagonistic (muscle) stiffnesses relate to the endpoint stiffness?

We hope to answer this question by analytically relating the stiffness in actuator

coordinates to the endpoint coordinates. This will tell us the endpoint stiffness characteristics given a set of actuator stiffnesses. This relationship should be invertible such that we can know what changes in the actuator stiffnesses are necessary to produce the desired changes in endpoint stiffness.

2. What limitations are there on endpoint stiffness?

We know that control of the actuator stiffnesses affect the three characteristics of the endpoint stiffness (size, shape, orientation). But is there a limitation on these characteristics based either on limited stiffness values from the antagonistic actuators or from the geometry? This can be solved by studying the behaviour and limits of the relationships determined from Question 1 above.

3. How do the muscle stiffnesses affect stability of the arm?

Lan [1995] noted that changing muscle stiffnesses affected the stability margin of the limb, but provided a limited demonstration of changes in stability. Is stability important in the selection of actuator stiffness values and if so, what conditions and ranges should be avoided when selecting actuator stiffness values? We hope to answer this question by looking for a pattern in the stability of the limb for given changes of actuator stiffnesses, or analytically solving for some stability measure as a function of the stiffness setting.

4. What function does the geometry play in setting the stiffness?

Mussa-Ivaldi et al. [1985] found a consistency in stiffness characteristics across subjects and trials that suggested the geometry of the limb dominated the characteristics of the endpoint stiffness. We hope to determine if this is true and to what degree one has voluntary control over the endpoint stiffness characteristics. This question is important to the design of stiffness elements because it might provide some requirements as to force-displacement functions, or show that the choice of functions is essentially irrelevant. Are the tension-length relationships in the antagonistic actuators (e.g., muscles) important to the control of stiffness, or is the non-linear behaviour dominated by the geometry? We expect to solve through numerical comparisons and substitution of finite stiffness ranges into the relationships from Question 1.

5. How does loading of the limb affect the stiffness?

Does loading inherently change the characteristics of the endpoint stiffness? If so, to what degree? Are the changes good or bad? For example, rotation of the stiffness such that the minimum stiffness points in the direction of the constraint would allow the greatest resolution for controlling contact force, and maximum stiffness along the constraint allows greatest control of the position on the surface. Does this occur naturally? Can proper design of the actuator force-displacement functions allow this to occur? We hope to answer these questions by relating the endpoint stiffness characteristics to the endpoint force, if there is any such relationship.

6. Does the linear stiffness approximation adequately describe the behaviour of the limb given its non-linear nature?

Endpoint stiffness is meant to describe the state of the limb and aid in the prediction of its reactions. The linear approximation (see Chapter 2) describes the behaviour for small displacements, but it could be that the stiffness changes greatly given slight larger displacements and this could have a significant impact on the limb's behaviour. It is also not clear how large a displacement is required for the approximation to fail. We can solve this problem by calculating energy storage for a range of displacements and plotting the non-linear isopotentials. How well these match up to the ellipse shape will indicate their appropriateness. It will also provide us some insight into the global behaviour of the non-linear stiffness.

7. What importance are bi-articular muscles for controlling stiffness and can their influence be simulated in an above-elbow prosthesis?

As we discussed in Chapter 3, an above-elbow prosthesis negates the practical possibility of using bi-articular muscles or actuators. Hogan [1985a] demonstrated the usefulness of bi-articular muscles in the possible range of isometric endpoint stiffness. We would like to expand on this specific example to see what limitations exist by removal of the bi-articular stiffness. If the limitations are severe, and they cannot be compensated for, the whole approach of replacing the mono-articular elbow actuators to allow mechanical control of stiffness may be futile. We can check this from the

analytical relationship between the bi-articular actuators and the endpoint stiffness characteristics from Question 1. Substitution of realistic values may be necessary to show the limitation, and perhaps increasing stiffness ranges in the mono-articular actuators or a feedback control mechanism might be able to compensate for the loss.

8. What strategies could be used for selecting appropriate stiffness values (endpoint and antagonistic)?

Given a desired motion or contact task, what endpoint stiffness characteristics are desirable, and given the relationships from above (stability, geometry, contact force, loss of bi-articulars), what appropriate actuator stiffness values should be chosen? Are certain areas of the workspace better for performing contact tasks? Is there a preferred direction in which to apply contact forces? These questions can be answered by understanding the above relationships and setting up criteria from which to judge the choices.

It is important to keep in mind the context of these questions. The goal is to *understand* stiffness in a two-link manipulator modelled after the human upper arm. The main application of this understanding is to the development of an above-elbow prosthesis. However, the current work will not involve designing or implementing a control scheme or designing a mechanism. The understanding we develop in this work may be applicable elsewhere as well. It can have applications in robot manipulator design and control, other man-machine interfaces such as haptic devices, and implications

towards human biomechanics and movement control.

The understanding we hope to achieve will be built up slowly, starting from the basics. We will first develop the model definitions and governing relationships. We will investigate properties and representations of the actuator stiffnesses, joint stiffnesses, and endpoint stiffness. Next we will relate the stiffnesses. Finally we will investigate the relationships as per the posed questions above. We will develop necessary tools (representations, measurement indexes, criteria) as needed to simplify our understanding. Each major section or chapter will have a summary of the achievements in that section to observe the progress and avoid losing track of the goal.

Chapter 5

Model Conventions

This chapter develops the model used for study throughout this thesis. We define what we mean by *stiffness* in terms of each coordinate system and point of interaction, including the actuator space, joint space, and endpoint space. We will also show the sign conventions used in the model and develop the relationship between forces, torques, and energy in each space.

5.1 Spring Conventions

In this section we will look at the traditional spring conventions and generalize them so that we can properly define what we mean by the term *stiffness* with respect to linear and non-linear springs, sign conventions for force and displacement in both translational and rotational springs, and energy storage in springs.

5.1.1 Stiffness

The most common use of the term *stiffness* in springs refers to a constant of proportionality between force and spring stretch in a linear spring. The first mathematical

ambiguity appears when defining the zero position of the spring, especially when both ends are movable.

Figure 5.1.1(a) shows a linear translational spring with a fixed end. (We use the term *translational* to represent motions that occur in a straight line rather than by twisting.) In this case the position of the free end can be measured relative to the position it takes when there is no force produced in the spring. The stretch of the spring is then $\delta = x$, and so the relationship between the force in the spring, F_s , and the spring stretch is

$$F_s = k\delta = kx \quad (5.1.1)$$

The spring force F_s is usually assumed positive in tension (force resisting stretch) and is opposite to the external force. (More on this later.)

Figure 5.1.1(b) shows the case with one fixed end and the position of the free end is measured from ground. This situation often appears when there are other components measured from a common reference point. In such a case, the rest position is given by $x = l_0$. The spring stretch then becomes $\delta = x - l_0$, resulting in the relationship

$$F_s = k\delta = k(x - l_0) \quad (5.1.2)$$

In some systems both ends of the spring are free to move relative to some reference point. This situation is shown in Figure 5.1.1(c). This case often occurs when

each end is attached to an independent moving object. One end of the spring is designated by x while the other end is x^* . The spring stretch becomes $\delta = x - l_0 - x^*$, resulting in the relationship

$$F_s = k\delta = k(x - l_0 - x^*) \quad (5.1.3)$$

In all three cases of Figure 5.1.1, the stiffness is defined as

$$k = \frac{F_s}{\delta} \quad (5.1.4)$$

Notice that in cases (b) and (c), $k \neq F_s/x$. The reason this occurs is that stiffness is defined by the force-stretch relationship, not the relationship of force to some global position variable.

The above definition of stiffness can be expanded for a general non-linear spring by writing it in differential form. Let the spring force be a non-linear function with respect to spring deflection, $F_s = \phi\{\delta\}$. The spring stiffness is defined as

$$k = \psi\{\delta\} = \frac{d\phi\{\delta\}}{d\delta} \quad (5.1.5)$$

Figure 5.1.1: Spring parameters. (a) one end fixed, other end measured from rest position, (b) one end fixed, other end measured from ground, (c) both ends free, measured from ground.

In this case, the stiffness k is no longer constant. It varies with the spring stretch, $k = \psi\{\delta\}$. In other words, the spring stiffness is defined here as the rate of change of spring force with respect to spring stretch. This can also be interpreted as the slope of the force-stretch relationship. Note that although (5.1.5) is the strict definition for stiffness here, it is also true that

$$\begin{aligned}\frac{\partial \phi\{x, x^*\}}{\partial x} &= \frac{d\phi\{\delta\}}{d\delta} \cdot \frac{\partial \delta\{x, x^*\}}{\partial x} = \frac{d\phi\{\delta\}}{d\delta} = \psi\{\delta\} \\ \frac{\partial \phi\{x, x^*\}}{\partial x^*} &= \frac{d\phi\{\delta\}}{d\delta} \cdot \frac{\partial \delta\{x, x^*\}}{\partial x^*} = -\frac{d\phi\{\delta\}}{d\delta} = -\psi\{\delta\}\end{aligned}$$

where x and x^* are the locations of the two ends as in Figure 5.1.1. (For cases with a fixed end, $x^* = 0 = \text{constant}$ so differentiation by x would become a full derivative rather than partial.) Hence in the general case with two movable ends,

$$\psi\{\delta\} = \frac{d\phi\{\delta\}}{d\delta} = \frac{\partial \phi\{x, x^*\}}{\partial x} = -\frac{\partial \phi\{x, x^*\}}{\partial x^*} \quad (5.1.6)$$

Differentiation with respect to x works here in spite of the fact that dividing by x did not work in the linear case. The reason for this is that, mathematically speaking, partial differentiation by one variable implies that all other variables are held constant.

Therefore a change in x is identical to a change in δ . The underlying assumption here is that x is measured in a direction such that moving the end of the spring associated with x in a positive direction (with the other end fixed) results in an increase in δ . This is why

differentiation by x^* in Figure 5.1.1(c) results in the negative of the stiffness. Positive movements of the x^* end decrease the spring stretch. The full differential representation of spring force is

$$\begin{aligned} d\phi &= \frac{d\phi}{d\delta} d\delta = \psi(dx - dx^*) = \psi dx - \psi dx^* \\ &= \frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial x^*} dx^* \end{aligned} \quad (5.1.7)$$

It is only recommended to use the alternative representations of stiffness in (5.1.6), rather than the strict definition in (5.1.5), if the relationship between spring stretch and the global variables, especially the sign conventions, is well understood. The definition of stiffness in (5.1.5) works universally, and so it will be used whenever possible.

5.1.2 Sign Convention

Spring force cannot generally be given an absolute direction because it acts in different absolute directions at each end of the spring. For example, in Figure 5.1.1(c) the spring force acts to the left at the right port (where the external force is applied), but the same spring force acts to the right on the other (left) port. The sign of the force refers to whether the spring is a *tensile force* (pulls external objects toward the spring) or a *compressive force* (pushes objects away from the spring). The sign of the force does not refer to some global coordinate system (such as x in Figure 5.1.1(a)). Traditionally, a tensile spring force is defined to be positive while a compressive spring force is negative.

Spring force is typically assumed to be equal and opposite to the external force. This is strictly true in static equilibrium. A difference between external force and spring force (as a function of stretch) can exist if we consider the acceleration of the spring mass as it stretches or the inherent damping in the spring material. In cases where these terms are not negligible they will typically be modeled as separate components, and are therefore not considered in the current work.

The spring stretch δ must follow similar rules as the force. The sign on δ usually refers to whether the spring is longer (positive sign) or shorter (negative sign) than its rest length, l_0 . Since the rest length is usually defined as the spring length at which there is no spring force, a spring with positive stretch *usually* corresponds to a positive spring force, and a negative stretch *usually* corresponds to a negative spring force. The particular application of signs depends on how the rest length is defined and on the spring force-stretch relationship. This is an important point: In normal spring convention, *a positive spring force acts in the opposite direction to a positive deflection of the spring.*

Consider the case where a non-linear spring has the force-stretch relationship shown in Figure 5.1.2(a). There are multiple values of spring stretch that result in a zero spring force. The selection of the zero stretch position is somewhat arbitrary so the graph in Figure 5.1.2(a) can be shifted to the left or right. Note too that the zero stretch position can be stable or unstable.

(a)

(b)

Figure 5.1.2: Examples of non-linear force functions with ambiguous zero stretch lengths. (a) multiple zero force stretches, (b) zero stretch defined by application.

The case of Figure 5.1.2(b) comes from our previous work in the study of antagonistic control of joint stiffness [English, 1996; English and Russell, 1999a]. The springs proposed in this work are based on rollers moving over a bent metal band [English and Russell, 1999b]. The force function was derived based on desired properties at the joint. The zero stretch position for the springs was defined such that the joint stiffness was minimum, not the point at which there was zero force in the springs. This kept the mathematics simple, rather than introducing an offset term to move the zero stretch position to the zero force position. In such cases positive force and positive displacement do not always correspond. We use the definitions:

1. Positive spring force pulls the spring ends towards each other.
2. Positive spring displacement corresponds to an increase in distance between the spring ends measured relative to some defined zero displacement.

5.1.3 Torsion Springs

The analysis of torsion springs is analogous to that of translational springs in the previous sections. Spring torque T , replaces spring force F_s , angles θ and θ^* replace the positions x and x^* , and angular displacement Δ replaces translational displacement δ . These variables are shown in Figure 5.1.3.

Figure 5.1.3: Torsion spring variables.

The spring torque in a linear torsion spring is given by

$$T_s = K\Delta \quad (5.1.8)$$

In a translational spring the displacement δ often includes a rest length, as seen in equations (5.1.2) and (5.1.3). In a torsion spring there is usually no physical reason to include a “rest angle”. That is, when the spring is at rest, $\theta = \theta^* = 0$. An exception may be the case where a torsion spring is manufactured such that the ends (hooks, etc.) are not aligned when at rest, and these endpieces are used to measure the angles as in Figure 5.1.3. It is not necessary to examine this case for the current research since it only adds an offset term. The torsion spring’s angular displacement is then

$$\Delta = \theta - \theta^* \quad (5.1.9)$$

The *torsional spring stiffness* is

$$K = \frac{T_s}{\Delta} \quad (5.1.10)$$

which is the same as equation (5.1.4) for a translational spring using the analogous parameters. For a general non-linear torsion spring, the torque is a non-linear function of angular displacement,

$$T_s = \Phi\{\Delta\} \quad (5.1.11)$$

Following from the translational analogy, the strict definition of stiffness for a non-linear torsion spring is

$$\Psi\{\Delta\} = \frac{d\Phi\{\Delta\}}{d\Delta} \quad (5.1.12)$$

Again, as with the translational springs,

$$\Psi\{\Delta\} = \frac{d\Phi\{\Delta\}}{d\Delta} = \frac{\partial\Phi\{\theta, \theta^*\}}{\partial\theta} = -\frac{\partial\Phi\{\theta, \theta^*\}}{\partial\theta^*} \quad (5.1.13)$$

As before, it is important to understand the relationship between the externally measured variables θ and θ^* and the spring angular displacement Δ before applying alternative representations of stiffness in (5.1.13). In this case the relationship given in equation (5.1.9) is fairly straightforward. The definition in (5.1.12) applies universally and so it will be used whenever possible. The full derivative form for the torque is

$$\begin{aligned} d\Phi &= \frac{d\Phi}{d\Delta} d\Delta = \Psi(d\theta - d\theta^*) = \Psi d\theta - \Psi d\theta^* \\ &= \frac{\partial\Phi}{\partial\theta} d\theta - \frac{\partial\Phi}{\partial\theta^*} d\theta^* \end{aligned} \quad (5.1.14)$$

The sign convention for torsion springs is a little more difficult than for translational springs. There is no angular equivalence to the concepts of tension and compression. (In a non-symmetrical spring such a distinction may be made. For example, in the case of a wound spring as in Figure 5.1.3 a twist in one direction would

put the outer fibers in tension and cause the spring diameter to decrease, whereas a twist in the opposite direction would put the outer fibers in compression and cause the spring diameter to increase. This is not the general case, however.) The process for assigning the sign convention is as follows. First, the positive direction of θ must be defined relative to some external reference plane. One end of the spring must then be designated as the output end of the spring to which θ is measured. The other end of the spring becomes the input end of the spring to which θ^* is measured. The positive direction of Δ is the same as for θ , as shown in Figure 5.1.3.

The sign convention on torque readily falls into place once angular displacement is defined. As with the translational spring, a positive spring torque resists a positive Δ . In Figure 5.1.3, Δ is positive counterclockwise when looking from the θ^* end to the θ end. Positive spring torque, T_s , is then in the clockwise direction at the output port when looking in the same direction. It is important to note that a positive T_s at the input end (θ^*) acts in the counterclockwise direction. In other words, a positive external torque tends to increase θ , balanced by a positive spring torque acting in the opposite direction.

5.1.4 Energy Storage in Springs

Mechanical work is the dot product of force through a distance, meaning only the components of force and distance that are co-linear do work. An external force acting on a spring does a differential amount of work on the spring given by

$$dW = F_{\text{ext}}d\delta$$

The spring force is considered to be equal and opposite to the external force at all times, subject to the assumptions discussed in Section 5.1.2. Because the spring force is defined positive in the opposite direction to the external force, it can replace the external force in the work equation above, and hence

$$dW = F_s d\delta = \phi(\delta)d\delta \quad (5.1.15)$$

For a typical spring where $\delta = 0$ and $F_s = 0$ at the same point, when the spring is stretched both $d\delta$ and F_s are positive, resulting in positive work being done on the spring. When the spring is compressed both $d\delta$ and F_s are negative, again resulting in positive work being done on the spring. If the spring is initially stretched ($\delta > 0$, $F_s > 0$) and then released some (but not to $\delta = 0$), F_s is still positive but $d\delta$ becomes negative. This means that negative work is done, which is interpreted as the spring doing work on the environment. This case is also true for springs in which positive force can be achieved for $\delta < 0$ or negative force can be achieved for $\delta > 0$, as in the example of Figure 5.1.2(b).

In such cases, the force is helping to move the spring in the same direction and so the spring is doing work.

Integrating (5.1.15) between arbitrary stretches δ_0 and δ , the total work done on the spring is given by

$$W\{\delta\} = \int_{\delta_0}^{\delta} \phi\{\epsilon\}d\epsilon + W\{\delta_0\} \quad (5.1.16)$$

where ϵ is a dummy variable for integration. In modeling the spring we do not include any losses to internal friction or damping. Therefore all the work done on the spring must become stored energy. Thus (5.1.16) can be rewritten as a relation for the energy stored in the spring,

$$E\{\delta\} = \int_{\delta_0}^{\delta} \phi\{\epsilon\}d\epsilon + E\{\delta_0\} \quad (5.1.17)$$

where ϵ is a dummy variable of integration. If the spring energy is defined such that $E\{\delta_0\} = 0$ (no energy in the spring), the last term in (5.1.18) can be dropped.

The form of equation (5.1.17) implies that the force function $\phi\{\delta\}$ is known. It may be the case that it is the spring stiffness function that is either known, or to be specified in a design. Recall that by definition in equation (5.1.5),

$$\psi\{\delta\} = \frac{d\phi\{\delta\}}{d\delta}$$

Rearranging this relationship gives

$$d\phi = \psi d\delta \quad (5.1.18)$$

Integrating between arbitrary stretches δ_0 and δ ,

$$F_s = \phi\{\delta\} = \int_{\delta_0}^{\delta} \psi\{\varepsilon\} d\varepsilon + \phi\{\delta_0\} \quad (5.1.19)$$

where ε is once again a dummy variable for integration. If $\phi\{\delta_0\} = 0$, as in a typical monotonic spring (for $\delta_0 = 0$), the last term can be dropped. Note that this isn't the case for the example in Figure 5.1.2(b).

Substituting the force function from (5.1.19) into the spring energy function of (5.1.17) gives the relationship between spring stiffness and spring energy,

$$E\{\delta\} = \int_{\delta_0}^{\delta} \int_{\delta_0}^{\lambda} \psi\{\varepsilon\} d\varepsilon d\lambda + \phi\{\delta_0\}\delta + E\{\delta_0\} \quad (5.1.20)$$

where ε and λ are dummy variables for δ in the integration. We can now use either (5.1.19) or (5.1.20) to evaluate the total energy stored in a spring.

We can apply the same methods of evaluating force and energy to torsion springs by substituting the rotational terms for their translational counterparts. A spring torque can be evaluated by

$$T_s = \Phi\{\Delta\} = \int_{\Delta_0}^{\Delta} \Psi\{\epsilon\} d\epsilon + \Phi\{\Delta_0\} \quad (5.1.21)$$

and the energy stored in a torsional spring is

$$E\{\Delta\} = \int_{\Delta_0}^{\Delta} \Phi\{\epsilon\} d\epsilon + E\{\Delta_0\} \quad (5.1.22)$$

in terms of the spring torque, or in terms of the torsional spring stiffness:

$$E\{\Delta\} = \int_{\Delta_0}^{\Delta} \int_{\Delta_0}^{\lambda} \Psi\{\epsilon\} d\epsilon d\lambda + \Phi\{\Delta_0\}\Delta + E\{\Delta_0\} \quad (5.1.23)$$

In Section 5.1 we have now defined and demonstrated:

- stiffness in linear and non-linear springs, both translational and rotational,
- the sign convention for forces and displacements in translational and rotational springs,
- the difference between external forces and spring forces and how they relate,
- energy storage in springs, and,
- the relationship between stiffness, force, and energy in springs.

5.2 Upper Arm Model

In this section we will introduce the simplified model of the human upper arm that is being studied in this work. We will apply the conventions defined and derived in Section 5.1 to the model in three spaces: the actuator space representing the translational motion of the muscles (modeled here as springs), the joint space, and the endpoint space representing the hand (with a rigid wrist). The two-dimensional motion of the endpoint space will lead us to expand our definitions of Section 5.1 to include additional concepts.

5.2.1 Sign Conventions

The system investigated in this research is modeled after the musculo-skeletal system of the human upper arm as shown in Figure 1.1. The model includes two planar links connected to a fixed ground representing the body. The chosen sign convention is shown in Figure 5.2.1.

The joint angles are assigned positive in the counter-clockwise direction from the view shown. The shoulder angle, θ_s , is measured from an arbitrarily selected ground position, shown here as a horizontal line. The elbow angle, θ_e , is measured relative to the first link. The absolute elbow angle relative to the body link is then $\theta_{es} = \theta_s + \theta_e$. In the human arm analogy, this model would have a zero angle home position ($\theta_s = \theta_e = 0$) with the arm stretched straight out to the side of the body, looking down from overhead.

The *virtual position* of the arm is defined here to be the static position of the arm with all external loads, including gravity loads, removed [Bizzi et al., 1992]. The model acts in the horizontal plane for the purposes of this research, and so gravity loads are neglected anyway. Often the term *equilibrium position* is used to describe this position, as discussed in Chapter 3. However, this can lead to confusion with the position of the arm when it is in static equilibrium with external forces, hence the present terminology is adopted.

An equivalent definition of the virtual position is the steady-state position of the arm at which there are no net joint moments, $M_s = M_e = 0$. Since all external and dynamics loads are absent, the virtual position is defined only by spring forces in this model. The joint angles at the virtual position are defined to be the virtual angles, θ_{sv} and θ_{ev} (and consequently $\theta_{esv} = \theta_{sv} + \theta_{ev}$). The virtual angles are identical to the input position, θ^* , of the torsion springs defined in Section 5.1.3. In that system, when all loads are removed, $\theta = \theta^*$. Hence when external forces displace the arm from this virtual position, the resulting joint displacements are defined as

$$\begin{aligned}\Delta_s &= \theta_s - \theta_{sv} \\ \Delta_e &= \theta_e - \theta_{ev}\end{aligned}\tag{5.2.1}$$

Figure 5.2.1: Sign convention for upper arm model.

This convention matches that used for torsion springs in equation (5.1.9), and so it can be imagined that the joints contain torsion springs where one end controls the virtual angle of the joint, θ_{iv} ($i = s, e$), and the other end represents the actual angle between the links, θ_i .

In keeping with the convention for torsion spring displacements from equation (5.1.9), the resulting joint spring torque is assumed positive in the direction that resists an increasing Δ . Therefore the internal joint torques resulting from external forces are positive in the clockwise direction (from this perspective), shown as M_s and M_e in Figure 5.2.1. The symbol M is used to denote these torques to recognize that they are the net internal *moments* at the joint (equal and opposite to the externally applied moments).

5.2.2 Joint Stiffnesses

When an external force results in joint displacements, given by (5.2.1), the springs become stretched or compressed. This produces a moment acting at the joint, much like a torsion spring. The relationships between the joint displacements of (5.2.1) and the resulting joint moments are called joint stiffnesses (despite the objection of Latash and Zatsiorsky [1993] as discussed in Section 2.1.2). Continuing with the convention for torsion springs, from equation (5.1.12) the joint stiffnesses will be defined in the form

$$\Psi_i = \frac{\partial M_i}{\partial \Delta_i} \quad (i = s, e) \tag{5.2.2}$$

In order to derive the joint stiffnesses from equation (5.2.2) it is first necessary to determine the joint moments, M_j . The human arm does not actually use torsion springs, or a biological equivalent, to create or support moments at joints. Instead, muscle forces acting with moment arms are responsible for the joint moments. In the model we are studying here, these muscles will be represented as non-linear springs in series with a position actuator (e.g., a motor) that controls the input end of the spring.

Figure 5.2.2 shows the full internal model we are studying. The spring coordinates give the location of both sides of each spring. The nomenclature conventions include:

- ${}^k\sigma_{ij}$ denotes a location variable for both ends of each spring, where
 - $i = s,e,d$ signifies whether the spring acts about the shoulder, elbow, or double (both) joints
 - $j = x,f$ signifies whether the spring is on the extensor side (rear) or flexor (front) side of the arm
 - $k = s,e,\{\text{blank}\}$ signifies whether the location is on the shoulder side of the spring (for springs that act at the shoulder), elbow (for springs that act at the elbow), or is an input from the position actuator (superscript left blank).

Figure 5.2.2: Internal spring model.

The positive directions of these variables are shown in Figure 5.2.2. (The actuator inputs of $\frac{1}{2}\sigma_{df}$ and $\frac{1}{2}\sigma_{dx}$ occur because of the choice of actuator input representation in the figure. In reality, σ_{df} and σ_{dx} correspond to contraction inputs to muscles.)

Starting at the shoulder, the displacement across the extensor side single-joint (mono-articular) shoulder spring is

$$\delta_{sx} = \sigma_{sx} - {}^s\sigma_{sx} \quad (5.2.3)$$

However, the shoulder side of the spring moves as the shoulder rotates. Taking the sign conventions from Figure 5.2.2, the following relationship results:

$${}^s\sigma_{sx} = -{}^sR_s\theta_s \quad (5.2.4)$$

Substituting ${}^s\sigma_{sx}$ into equation (5.2.3), and performing similar approaches to the other springs, gives the following relationships:

$$\begin{aligned} \delta_{sx} &= \sigma_{sx} - {}^s\sigma_{sx} = \sigma_{sx} - (-{}^sR_s\theta_s) = \sigma_{sx} + {}^sR_s\theta_s \\ \delta_{sf} &= \sigma_{sf} - {}^s\sigma_{sf} = \sigma_{sf} - {}^sR_s\theta_s \\ \delta_{ex} &= \sigma_{ex} - {}^e\sigma_{ex} = \sigma_{ex} - (-{}^eR_e\theta_e) = \sigma_{ex} + {}^eR_e\theta_e \\ \delta_{ef} &= \sigma_{ef} - {}^e\sigma_{ef} = \sigma_{ef} - {}^eR_e\theta_e \\ \delta_{dx} &= {}^e\sigma_{dx} - {}^s\sigma_{dx} = (\sigma_{dx} + {}^dR_d\theta_d) - (-{}^dR_s\theta_s) = \sigma_{dx} + {}^dR_s\theta_s + {}^dR_d\theta_d \\ \delta_{df} &= {}^e\sigma_{df} - {}^s\sigma_{df} = (\sigma_{df} - {}^dR_e\theta_e) - (-{}^dR_s\theta_s) = \sigma_{df} - {}^dR_s\theta_s - {}^dR_e\theta_e \end{aligned} \quad (5.2.5)$$

Define for each spring a force function $F_{ij} = \phi_{ij}(\delta_{ij})$. The spring stiffness is then defined by (5.1.5),

$$\psi_{ij}\{\delta_{ij}\} = \frac{d\phi_{ij}}{d\delta_{ij}} \quad (5.2.6)$$

The free body diagrams for the two joints are shown in Figure 5.2.3. Here M_{s-ext} and M_{e-ext} are the externally applied moments. In static equilibrium, summing the moments results in

$$M_{s-ext} = {}^sR_s(\phi_{sx} - \phi_{sf}) + {}^dR_s(\phi_{dx} - \phi_{df})$$

$$M_{e-ext} = {}^sR_e(\phi_{ex} - \phi_{ef}) + {}^dR_e(\phi_{dx} - \phi_{df})$$

Following the convention for a torsion spring in Figure 5.1.3, the internal moments resulting from the spring forces are equal to the external moments, but are assumed positive in the opposite direction. Thus $M_s = -M_{s-ext}$ and $M_e = -M_{e-ext}$, and hence

$$\mathbf{M} = \begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} {}^sR_s(\phi_{sx}\{\delta_{sx}\} - \phi_{sf}\{\delta_{sf}\}) + {}^dR_s(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \\ {}^sR_e(\phi_{ex}\{\delta_{ex}\} - \phi_{ef}\{\delta_{ef}\}) + {}^dR_e(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \end{bmatrix} \quad (5.2.7)$$

The positive directions of spring moments M_s and M_e are shown in Figure 5.2.1. These moments can be decomposed according to the contributions of the different sets of springs,

$$\mathbf{M} = \begin{bmatrix} {}^s\mathbf{M}_s + {}^d\mathbf{M}_s \\ {}^s\mathbf{M}_e + {}^d\mathbf{M}_e \end{bmatrix} \quad (5.2.8)$$

where

$$\begin{aligned} {}^s\mathbf{M}_s &= {}^s\mathbf{R}_s(\phi_{st}\{\delta_{st}\} - \phi_{sb}\{\delta_{sb}\}) \\ {}^d\mathbf{M}_s &= {}^d\mathbf{R}_s(\phi_{dt}\{\delta_{dt}\} - \phi_{db}\{\delta_{db}\}) \\ {}^s\mathbf{M}_e &= {}^s\mathbf{R}_e(\phi_{et}\{\delta_{et}\} - \phi_{eb}\{\delta_{eb}\}) \\ {}^d\mathbf{M}_e &= {}^d\mathbf{R}_e(\phi_{dt}\{\delta_{dt}\} - \phi_{db}\{\delta_{db}\}) \end{aligned} \quad (5.2.9)$$

Figure 5.2.3: Free body diagram of shoulder and elbow joints. M_{s-ext} and M_{e-ext} are externally applied moments.

The single-joint shoulder springs contribute sM_s to the total shoulder moment, the single-joint elbow springs contribute sM_e to the total elbow moment, and the double-joint springs contribute dM_s to the shoulder and dM_e to the elbow. This decomposition is shown in Figure 5.2.4 as a set of torsion springs acting in parallel at the joints. This analogy shows why the present system being studied is more complex than simple torsion springs acting at the joints; the double-joint springs (modeling bi-articular muscles in humans) act like two coupled torsion springs. Using the analogy in Figure 5.2.4, the joint stiffnesses can be defined from Equation (5.1.12),

$$\begin{aligned}
 {}^s\Psi_s &= \frac{\partial {}^sM_s}{\partial \Delta_s} = \frac{\partial {}^sM_s}{\partial \theta_s} \\
 {}^d\Psi_s &= \frac{\partial {}^dM_s}{\partial \Delta_s} = \frac{\partial {}^dM_s}{\partial \theta_s} \\
 {}^s\Psi_e &= \frac{\partial {}^sM_e}{\partial \Delta_e} = \frac{\partial {}^sM_e}{\partial \theta_e} \\
 {}^d\Psi_e &= \frac{\partial {}^dM_e}{\partial \Delta_e} = \frac{\partial {}^dM_e}{\partial \theta_e} \\
 \Psi_{de} &= \frac{\partial {}^dM_s}{\partial \Delta_e} = \frac{\partial {}^dM_s}{\partial \theta_e} \\
 \Psi_{ds} &= \frac{\partial {}^dM_e}{\partial \Delta_s} = \frac{\partial {}^dM_e}{\partial \theta_s}
 \end{aligned} \tag{5.2.10}$$

Figure 5.2.4: Torsion spring analogy. The single-joint spring have the same effect as normal torsion springs. Double-joint springs appear as two torsion springs that are coupled.

where

${}^s\Psi_s$ is the shoulder joint stiffness due to the single-joint shoulder springs

${}^d\Psi_s$ is the shoulder joint stiffness due to the double-joint springs

${}^s\Psi_e$ is the elbow joint stiffness due to the single-joint elbow springs

${}^d\Psi_e$ is the elbow joint stiffness due to the double-joint springs

Ψ_{de} is the joint coupling stiffness due to elbow displacement

Ψ_{ds} is the joint coupling stiffness due to shoulder displacement

These stiffnesses can be readily formed by substituting the moments from (5.2.9) into (5.2.10) using the spring stretch definitions from (5.2.5),

$$\begin{aligned}
 {}^s\Psi_s &= {}^sR_s^2 (\psi_{sx} \{\delta_{sx}\} + \psi_{sf} \{\delta_{sf}\}) \\
 {}^d\Psi_s &= {}^dR_s^2 (\psi_{dx} \{\delta_{dx}\} + \psi_{df} \{\delta_{df}\}) \\
 {}^s\Psi_e &= {}^sR_e^2 (\psi_{ex} \{\delta_{ex}\} + \psi_{ef} \{\delta_{ef}\}) \\
 {}^d\Psi_e &= {}^dR_e^2 (\psi_{dx} \{\delta_{dx}\} + \psi_{df} \{\delta_{df}\}) \\
 \Psi_{de} &= {}^dR_s {}^dR_e (\psi_{dx} \{\delta_{dx}\} + \psi_{df} \{\delta_{df}\}) = \Psi_d \\
 \Psi_{ds} &= {}^dR_s {}^dR_e (\psi_{dx} \{\delta_{dx}\} + \psi_{df} \{\delta_{df}\}) = \Psi_d
 \end{aligned} \tag{5.2.11}$$

Notice that the coupling stiffness is the same whether viewed from the shoulder or the elbow, $\Psi_{ds} = \Psi_{de}$, which we define here as Ψ_d . This will be shown to be a requirement of conservative stiffness as discussed in Chapter 2 (and Appendix B).

If we hold the spring inputs constant and displace the arm by an external force, the

resulting shoulder moment is a function of both the shoulder displacement Δ_s and the elbow displacement Δ_e (because of the double-joint springs). We can rearrange the relationships in (5.2.10) to get differential moments in the form

$$d^j M_i = {}^j \Psi_i d\Delta_i$$

then substitute them into (5.2.8), giving us

$$d\mathbf{M} = \begin{bmatrix} dM_s \\ dM_e \end{bmatrix} = \begin{bmatrix} {}^s \Psi_s + {}^d \Psi_s & \Psi_d \\ \Psi_d & {}^s \Psi_e + {}^d \Psi_e \end{bmatrix} \begin{bmatrix} d\Delta_s \\ d\Delta_e \end{bmatrix}. \quad (5.2.12)$$

or $d\mathbf{M} = \Psi d\Delta$

Here $d\mathbf{M}$ refers to a small *change* in joint moments due to a small *change* in joint displacements. If we apply (5.2.12) to a small displacement from the virtual position, we can write it as

$$\mathbf{M} = \begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} {}^s \Psi_s + {}^d \Psi_s & \Psi_d \\ \Psi_d & {}^s \Psi_e + {}^d \Psi_e \end{bmatrix} \begin{bmatrix} \delta\Delta_s \\ \delta\Delta_e \end{bmatrix} \quad (5.2.13)$$

or $\mathbf{M} = \Psi \delta\Delta$

where $\delta\Delta$ is a small absolute displacement from the virtual position (rather than a small change in displacement).

We can now see that it is no coincidence that $\Psi_{de} = \Psi_{ds}$ in (5.2.11). The displacement of springs by an external force is passive. The off-diagonal terms of Ψ in (5.2.12) must be identical in order for the system to be conservative as we discussed in Section 2.1.3.

We can also relate the joint stiffnesses to the decomposed moments of each antagonistic pair as given in (5.2.8). Equation (5.2.13) then becomes

$$\mathbf{M} = {}^s\mathbf{M} + {}^d\mathbf{M} = {}^s\Psi\delta\Delta + {}^d\Psi\delta\Delta$$

where

$${}^s\mathbf{M} = \begin{bmatrix} {}^sM_s \\ {}^sM_e \end{bmatrix}; \quad {}^d\mathbf{M} = \begin{bmatrix} {}^dM_s \\ {}^dM_e \end{bmatrix}$$

$${}^s\Psi = \begin{bmatrix} {}^s\Psi_s & 0 \\ 0 & {}^s\Psi_e \end{bmatrix}; \quad {}^d\Psi = \begin{bmatrix} {}^d\Psi_s & \Psi_d \\ \Psi_d & {}^d\Psi_e \end{bmatrix}$$

This shows us that both joint moments and joint stiffnesses can be derived by analyzing antagonistic pairs of springs separately and combining the results. This will simplify some derivations later.

5.2.3 Spring Inputs

Consider only a set of single-joint antagonistic springs as shown in Figure 5.2.5. We have explored the mechanics of this system in previous work [English, 1996; English & Russell, 1999a]. Suppose that this set of springs acts alone at a joint (no other springs attached), and the joint is unloaded. If we move σ_f to the left and σ_x to the right the same amount, the springs do not change in length but the angle of the joint changes. Now suppose we pull both σ_f and σ_x to the left such that there is no net moment at the joint. Both springs stretch, but the changes in spring forces cancel each other out at the joint. However, the joint stiffness is the sum of the individual spring stiffnesses $\psi_f\{\delta_f\}$ and $\psi_x\{\delta_x\}$, so this action changes the joint stiffness. The spring inputs can then be divided into two parts: an amount that goes to moving the position of the joint, ${}^m\sigma_i$, and an amount that goes into changing the joint stiffness, ${}^k\sigma_i$ ($i = f$ or x). We can write the spring inputs then as

$$\begin{aligned}\sigma_f &= {}^k\sigma_f + {}^m\sigma_f \\ \sigma_x &= {}^k\sigma_x + {}^m\sigma_x\end{aligned}\tag{5.2.14}$$

As already implied, we must move the springs by the same amount in opposite directions so that they do not to change length when changing the angle, which results in

$${}^m\sigma_f = -{}^m\sigma_x$$

Figure 5.2.5: Single-joint spring model.

We cannot derive a similar equality for the inputs to change stiffness because they are dependent on the force functions ϕ_f and ϕ_x . Only if the springs are identical can we say that

$$^k\sigma_f = ^k\sigma_x$$

Since the arm is unloaded (in our current thought experiment), the change in joint angle is really a change in virtual angle. Moving the springs in opposite directions when the arm is loaded will most likely affect the lengths of the springs, especially if the joint is constrained to a single position. We can relate the virtual angles to the spring inputs by the simple relationship,

$$R \cdot ^s\theta_v = ^m\sigma_f = -^m\sigma_x \quad (5.2.15)$$

We give the superscript “s” to this virtual angle to signify that it only represents the virtual angle of a single-joint set of antagonistic springs. This angle is the point at which $^sM_f = 0$ or $^sM_x = 0$, not the net joint moment, because we are presently only examining the case of the single-joint springs alone.

Using the relationship in (5.2.15), we can rewrite (5.2.14) as

$$\begin{aligned} \sigma_f &= ^k\sigma_f + R \cdot ^s\theta_v \\ \sigma_x &= ^k\sigma_x - R \cdot ^s\theta_v \end{aligned} \quad (5.2.16)$$

If we apply the relationship in (5.2.16) to the single-joint springs of our arm model we get

$$\begin{aligned}
 \sigma_{sf} &= {}^k\sigma_{sf} + {}^sR_s \cdot {}^s\theta_{sv} \\
 \sigma_{sx} &= {}^k\sigma_{sx} - {}^sR_s \cdot {}^s\theta_{sv} \\
 \sigma_{ef} &= {}^k\sigma_{ef} + {}^sR_e \cdot {}^s\theta_{ev} \\
 \sigma_{ex} &= {}^k\sigma_{ex} - {}^sR_e \cdot {}^s\theta_{ev}
 \end{aligned} \tag{5.2.17}$$

The double-joint springs are a little more difficult to analyze. Figure 5.2.6 shows a model of the double-joint springs alone. As with the single-joint springs, the activation of one input exactly opposite to the activation of the other will result in a change in the joint angles (assuming the unloaded case again). For instance, if we increase σ_{df} and decrease σ_{dx} by the same amount, the two joints will both rotate in their positive directions as shown in Figure 5.2.6. Since the springs do not stretch in this case, the total length around the loop in the figure is constant, and thus acts like a belt. Activating the inputs as described here would increase the length of “belt” on top while decreasing the length on bottom. We denote these changes in length as ${}^m\sigma_{df}$ and ${}^m\sigma_{dx}$, respectively. We can then write the inputs as

$$\begin{aligned}
 \sigma_{df} &= {}^k\sigma_{df} + {}^m\sigma_{df} \\
 \sigma_{dx} &= {}^k\sigma_{dx} + {}^m\sigma_{dx}
 \end{aligned} \tag{5.2.18}$$

Figure 5.2.6: Double-joint spring model.

Some of these changes in length will go towards changing the shoulder angle and some towards changing the elbow angle. Since we are again examining the unloaded case, these angles are virtual angles. The angles relate to the changes in length by

$$\begin{aligned} {}^m\sigma_{df} &= {}^dR_s \cdot {}^d\theta_{sv} + {}^dR_e \cdot {}^d\theta_{ev} \\ {}^m\sigma_{dx} &= - {}^dR_s \cdot {}^d\theta_{sv} - {}^dR_e \cdot {}^d\theta_{ev} \end{aligned} \quad (5.2.19)$$

We add the superscript “d” to denote that these virtual angles are only specific to the double-joint spring pair, such that ${}^dM_s = {}^dM_e = 0$. It is important to realize that the double-joint springs do not control the absolute virtual angles ${}^d\theta_{sv}$ and ${}^d\theta_{ev}$. We can move the “belt” around without stretching the springs or affecting the inputs. Imagine sliding the “belt” around clockwise in Figure 5.2.6. As ${}^d\theta_{sv}$ increases, ${}^d\theta_{ev}$ decreases in proportion. The actual result of activating the double-joint springs as described above is

$${}^dR_s \cdot {}^d\theta_{sv} + {}^dR_e \cdot {}^d\theta_{ev} = \text{constant} \quad (5.2.20)$$

The absolute values of the angles are controlled by other joint moments. Therefore the actual values of ${}^d\theta_{sv}$ and ${}^d\theta_{ev}$ are not directly relevant, but rather the whole relationship in (5.2.20). The double-joint spring inputs from (5.2.18) become

$$\begin{aligned} \sigma_{df} &= {}^k\sigma_{df} + {}^dR_s \cdot {}^d\theta_{sv} + {}^dR_e \cdot {}^d\theta_{ev} \\ \sigma_{dx} &= {}^k\sigma_{dx} - {}^dR_s \cdot {}^d\theta_{sv} - {}^dR_e \cdot {}^d\theta_{ev} \end{aligned} \quad (5.2.21)$$

We next substitute all of the spring input relationships from (5.2.17) and (5.2.21)

into the spring displacements in (5.2.5),

$$\begin{aligned}
 \delta_{sx} &= {}^k\sigma_{sx} + {}^sR_s(\theta_s - {}^s\theta_{sv}) \\
 \delta_{sf} &= {}^k\sigma_{sf} - {}^sR_s(\theta_s - {}^s\theta_{sv}) \\
 \delta_{ex} &= {}^k\sigma_{ex} + {}^eR_e(\theta_e - {}^e\theta_{ev}) \\
 \delta_{ef} &= {}^k\sigma_{ef} - {}^eR_e(\theta_e - {}^e\theta_{ev}) \\
 \delta_{dx} &= {}^k\sigma_{dx} + {}^dR_s(\theta_s - {}^d\theta_{sv}) + {}^dR_e(\theta_e - {}^d\theta_{ev}) \\
 \delta_{df} &= {}^k\sigma_{df} - {}^dR_s(\theta_s - {}^d\theta_{sv}) - {}^dR_e(\theta_e - {}^d\theta_{ev})
 \end{aligned} \tag{5.2.22}$$

The problem with the relationships in (5.2.17), (5.2.21), and (5.2.22) is that they do not allow direct control of the actual virtual angles θ_{sv} and θ_{ev} . The virtual angles of the whole joints are based on the *net* joint moment being zero. From (5.2.8) this occurs when

$${}^sM_s = -{}^dM_s \quad \text{and} \quad {}^eM_e = -{}^dM_e \tag{5.2.23}$$

which can be simplified by using (5.2.9),

$$\frac{{}^sR_s}{{}^dR_s}(\phi_{sx} - \phi_{sf}) = \phi_{dx} - \phi_{df} = \frac{{}^eR_e}{{}^dR_e}(\phi_{ex} - \phi_{ef}) \tag{5.2.24}$$

Equation (5.2.23) does not require that individual moments be zero, only that they cancel. We cannot determine a direct relationship between the spring pair virtual angles and the actual virtual angles of the arm without assuming the ϕ -functions for the springs

in (5.2.24).

To see how the virtual angles may differ significantly, consider the case when the moments in (5.2.23) are all zero and the arm is unloaded. Then the virtual angles of the individual spring pairs are identical to the virtual angles of the arm,

$$\begin{aligned}\delta\theta_{sv} &= \theta_{sv} \\ \delta\theta_{ev} &= \theta_{ev} \\ {}^dR_s \cdot \delta\theta_{sv} + {}^dR_e \cdot \delta\theta_{ev} &= {}^dR_s \cdot \theta_{sv} + {}^dR_e \cdot \theta_{ev}\end{aligned}\tag{5.2.25}$$

Next, we change $\delta\theta_{sv}$ by moving the single-joint shoulder springs in opposite directions. However, we can stop this change from actually moving the shoulder joint by moving the double-joint springs in the opposite direction to the shoulder springs such that the shoulder moments cancel each other out. Activating the double-joint springs in this manner induces a moment about the elbow as well, and so we must activate the single-joint elbow springs to create a moment that cancels it. Figure 5.2.7 shows this scenario.

The end result of this thought experiment is that the arm has not moved because there are no net moments at the joint. The virtual position of the arm has not changed because the net joint moments are still zero. However, the virtual angles of the individual spring pairs have changed. In addition, we have changed the spring displacements and

have therefore likely changed the joint stiffnesses.

The scenario in Figure 5.2.7 gives us another method for controlling stiffness using θ_{sv} , θ_{ev} , $\dot{\theta}_{sv}$, and $\dot{\theta}_{ev}$. However, it is not very practical because the stiffnesses are all coupled. Changing the shoulder stiffness by the single-joint shoulder springs requires a unique response force differential in the double-joint springs to counteract the shoulder moment, which in turn requires a unique response force differential in the single-joint elbow springs to counteract the elbow moment from the double-joint springs.

How we handle this situation of two methods for affecting joint stiffness depends on the application of this analysis. If we use these results to study the control inputs to an existing system that Figure 5.2.2 models, such as the musculo-skeletal system of the human arm, then we must consider that the joint stiffness could generally be a function of both the desired method of co-contraction and the cancellation of antagonistic pair moments by another spring pair. (We should reiterate that Figure 5.2.2 is not meant as a rigorous model of the musculo-skeletal system, especially with respect to studying the commanded inputs. Rather we intend to use the model to study the relationship between individual spring (muscle) stiffnesses, joint stiffnesses, and endpoint stiffness. In this context, the inputs in this model are not directly analogous to the inputs of a muscle.)

Figure 5.2.7: Cancellation of moments of individual spring pairs.

If, on the other hand, the application of this analysis is the design or simulation of an actuation system much like that of Figure 5.2.2, we can eliminate the undesirable second method of changing joint stiffness through either the controller design or by physical reconfiguration of model. For the controller design, we would chose the stiffnesses based off of the ${}^k\sigma$ values and knowledge of the ψ -functions in (5.2.11). Next, we would chose both the single and double-joint spring virtual angles to match the desired joint virtual positions as in (5.2.25). We would then apply the spring inputs based on (5.2.17) and (5.2.21). The conditions in (5.2.25) can actually be reduced to the restrictions

$$\begin{aligned} {}^d\theta_{sv} &= {}^s\theta_{sv} \\ {}^d\theta_{ev} &= {}^s\theta_{ev} \end{aligned} \tag{5.2.26}$$

All three relationships in (5.2.23) are met identically by these restrictions.

We can also meet the conditions in (5.2.25), or rather avoid them, by reconfiguring the model as shown in Figure 5.2.8. In this case we have reduced the inputs by one. Instead of the six spring inputs we now have three spring inputs to control stiffness (${}^k\sigma_i$) and have direct control over the two virtual angles of the joints, for a total of five inputs. This reduction of inputs eliminates the need for the controller restriction given in (5.2.26).

The configuration in Figure 5.2.8 does place some design restrictions on the system, however. By definition, we have restricted the $^k\sigma$ inputs such that changes in force of opposing springs cancel so that the joint moment from that spring pair is zero (e.g., $\phi_{xx}\{^k\sigma_{xx}\} - \phi_{xf}\{^k\sigma_{xf}\} = 0$). If the antagonistic springs are identical, activation of both by the same amount results in no net moment. However, if the springs have different force functions they require different stiffness inputs. For example, consider the case where $\phi_{xx} = 2\phi_{xf}$ (identical springs except one is double the force of the other for the same displacement). In order to keep the forces equal and opposite at the joint, it is necessary to keep $^k\sigma_{xf}$ twice as large as $^k\sigma_{xx}$. We would have to design some sort of linkage system into the actuation system (not shown in Figure 5.2.8) that maintains this restriction. If the difference in required inputs is more than just a proportionality constant, the necessary linkage system may be very complicated, if not impossible, to build. Therefore the configuration in Figure 5.2.8 is only practical if the antagonistic springs are designed to be identical in terms of force-displacement (ϕ) functions.

If we restrict the controller or physical design as described above such that the conditions in (5.2.25) or (5.2.26) are met, we can use the angular displacement definitions in (5.2.1) to re-write the spring inputs in (5.2.17) and (5.2.21) as

Figure 5.2.8: Alternate configuration for stiffness control model. Non-backdrivable motors are mounted between the joints and the springs. The motors directly control the virtual angles. The springs can only be displaced in the same direction, and therefore control stiffness. The inputs are reduced by one from the configuration in Figure 5.2.2. However, for practical purposes it is necessary for antagonistic springs to be identical.

$$\begin{aligned}
\sigma_{sf} &= {}^k\sigma_{sf} + {}^sR_s\theta_{sv} \\
\sigma_{sx} &= {}^k\sigma_{sx} - {}^sR_s\theta_{sv} \\
\sigma_{ef} &= {}^k\sigma_{ef} + {}^eR_e\theta_{ev} \\
\sigma_{ex} &= {}^k\sigma_{ex} - {}^eR_e\theta_{ev} \\
\sigma_{df} &= {}^k\sigma_{df} + {}^dR_s\theta_{sv} + {}^dR_e\theta_{ev} \\
\sigma_{dx} &= {}^k\sigma_{dx} - {}^dR_s\theta_{sv} - {}^dR_e\theta_{ev}
\end{aligned} \tag{5.2.27}$$

Similarly, the displacements in (5.2.22) become,

$$\begin{aligned}
\delta_{sf} &= {}^k\sigma_{sf} - {}^sR_s(\theta_s - \theta_{sv}) = {}^k\sigma_{sf} - {}^sR_s\Delta_s \\
\delta_{sx} &= {}^k\sigma_{sx} + {}^sR_s(\theta_s - \theta_{sv}) = {}^k\sigma_{sx} + {}^sR_s\Delta_s \\
\delta_{ef} &= {}^k\sigma_{ef} - {}^eR_e(\theta_e - \theta_{ev}) = {}^k\sigma_{ef} - {}^eR_e\Delta_e \\
\delta_{ex} &= {}^k\sigma_{ex} + {}^eR_e(\theta_e - \theta_{ev}) = {}^k\sigma_{ex} + {}^eR_e\Delta_e \\
\delta_{df} &= {}^k\sigma_{df} - {}^dR_s(\theta_s - \theta_{sv}) - {}^dR_e(\theta_e - \theta_{ev}) = {}^k\sigma_{df} - {}^dR_s\Delta_s - {}^dR_e\Delta_e \\
\delta_{dx} &= {}^k\sigma_{dx} + {}^dR_s(\theta_s - \theta_{sv}) + {}^dR_e(\theta_e - \theta_{ev}) = {}^k\sigma_{dx} + {}^dR_s\Delta_s + {}^dR_e\Delta_e
\end{aligned} \tag{5.2.28}$$

We must again stress that (5.2.27) and (5.2.28) are *not* the general cases. These relationships apply *if and only if* the conditions of (5.2.25) or (5.2.26) are met. The general cases are given in (5.2.17), (5.2.21), and (5.2.22). For the current research, we are interested in studying the relationship of stiffnesses, as previously stated, so that the control strategy is essentially irrelevant. The sole purpose of the control inputs in this work is to set the desired stiffness or force properties. We are therefore free to chose the

control of the inputs as desired. For most of the current study we will use the inputs as specified in (5.2.27).

While we do have the freedom to chose input strategy, it will be important to show that the joint moments are functions of joint displacements, $\mathbf{M} = \mathbf{M}\{\Delta_s, \Delta_e\}$. This was already assumed in the torsion spring analogy that was used to derive the linearized moment relationship in (5.2.13), $\mathbf{M} = \Psi\delta\Delta$. This relationship is evident for the spring displacement described in (5.2.28) for that specific control strategy, since the moments are a function of spring displacement by (5.2.7). However, this is not true for the general spring displacements in (5.2.22). Fortunately, we can deduce that such a relationship must exist through a simple thought experiment.

We need only consider the upper link for this experiment. Imagine that the upper arm is at its virtual angle. By definition of the virtual angle there cannot be any net joint moment. If we then load the arm (even a force at the elbow will do), the upper arm will move from the virtual angle, thereby stretching the springs and creating a joint moment. The resulting moment should not be affected by the absolute value of θ_s because we chose the reference point ($\theta_s = 0$) arbitrarily. We can make the spring displacements functions of Δ_s and Δ_e simply by adding and subtracting the actual virtual angles in (5.2.22),

$$\begin{aligned}
\delta_{sx} &= {}^k\sigma_{sx} + {}^sR_s[(\theta_s - \theta_{sv}) + (\theta_{sv} - {}^s\theta_{sv})] = {}^k\sigma_{sx} + {}^sR_s(\Delta_s + {}^s\Delta_{sv}) \\
\delta_{sf} &= {}^k\sigma_{sf} - {}^sR_s[(\theta_s - \theta_{sv}) + (\theta_{sv} - {}^s\theta_{sv})] = {}^k\sigma_{sf} - {}^sR_s(\Delta_s + {}^s\Delta_{sv}) \\
\delta_{ex} &= {}^k\sigma_{ex} + {}^sR_e[(\theta_e - \theta_{ev}) + (\theta_{ev} - {}^s\theta_{ev})] = {}^k\sigma_{ex} + {}^sR_e(\Delta_e + {}^s\Delta_{ev}) \\
\delta_{ef} &= {}^k\sigma_{ef} - {}^sR_e[(\theta_e - \theta_{ev}) + (\theta_{ev} - {}^s\theta_{ev})] = {}^k\sigma_{ef} - {}^sR_e(\Delta_e + {}^s\Delta_{ev}) \quad (5.2.29) \\
\delta_{dx} &= {}^k\sigma_{dx} + {}^dR_s[(\theta_s - \theta_{sv}) + (\theta_{sv} - {}^d\theta_{sv})] + {}^dR_e[(\theta_e - \theta_{ev}) + (\theta_{ev} - {}^d\theta_{ev})] \\
&= {}^k\sigma_{dx} + {}^dR_s(\Delta_s + {}^d\Delta_{sv}) + {}^dR_e(\Delta_e + {}^d\Delta_{ev}) \\
\delta_{df} &= {}^k\sigma_{df} - {}^dR_s[(\theta_s - \theta_{sv}) + (\theta_{sv} - {}^d\theta_{sv})] - {}^dR_e[(\theta_e - \theta_{ev}) + (\theta_{ev} - {}^d\theta_{ev})] \\
&= {}^k\sigma_{df} - {}^dR_s(\Delta_s + {}^d\Delta_{sv}) - {}^dR_e(\Delta_e + {}^d\Delta_{ev})
\end{aligned}$$

where we define

$$\begin{aligned}
{}^s\Delta_{sv} &= \theta_{sv} - {}^s\theta_{sv} \\
{}^s\Delta_{ev} &= \theta_{ev} - {}^s\theta_{ev} \\
{}^d\Delta_{sv} &= \theta_{sv} - {}^d\theta_{sv} \\
{}^d\Delta_{ev} &= \theta_{ev} - {}^d\theta_{ev}
\end{aligned} \quad (5.2.30)$$

The virtual displacements in (5.2.30) are the differences between the actual arm virtual angles and the individual spring pair virtual angles. We can see that if these angles are chosen to match as in (5.2.25), the relationships in (5.2.29) reduce to those in (5.2.28). We can use these virtual displacements to affect both stiffnesses and virtual position. In fact, they are a quantification of the behavior we described in the scenario of Figure 5.2.7. With this in mind, we must be clear that the portion of inputs ${}^k\sigma$ are not necessarily the sole input for controlling stiffness, but rather the input for controlling

stiffness by co-contraction of antagonistic pairs.

Since we have not assumed any relationships between the inputs, (5.2.29) is a general form for the spring displacements. If we compare the generalized spring displacements in (5.2.29) to those in (5.2.5) we can see that the generalized spring inputs are

$$\begin{aligned}
 \sigma_{sx} &= {}^k\sigma_{sx} + {}^sR_s ({}^s\Delta_{sv} - \theta_{sv}) \\
 \sigma_{sf} &= {}^k\sigma_{sf} - {}^sR_s ({}^s\Delta_{sv} - \theta_{sv}) \\
 \sigma_{ex} &= {}^k\sigma_{ex} + {}^sR_e ({}^s\Delta_{ev} - \theta_{ev}) \\
 \sigma_{ef} &= {}^k\sigma_{ef} - {}^sR_e ({}^s\Delta_{ev} - \theta_{ev}) \\
 \sigma_{dx} &= {}^k\sigma_{dx} + {}^dR_s ({}^d\Delta_{sv} - \theta_{sv}) + {}^dR_e ({}^d\Delta_{ev} - \theta_{ev}) \\
 \sigma_{df} &= {}^k\sigma_{df} - {}^dR_s ({}^d\Delta_{sv} - \theta_{sv}) - {}^dR_e ({}^d\Delta_{ev} - \theta_{ev})
 \end{aligned} \tag{5.2.31}$$

We then must specify both the desired virtual angles, θ_{sv} and θ_{ev} , as well as the difference between these and the individual spring pair virtual angles, ${}^s\Delta_{sv}$, ${}^s\Delta_{ev}$, ${}^d\Delta_{sv}$, and ${}^d\Delta_{ev}$. For the specific control strategy specified in (5.2.25), we can directly control the virtual angles via the inputs in (5.2.27).

5.2.4 Stiffness to Moments

Recall from (5.2.7) that the joint moments are

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_s \\ \mathbf{M}_e \end{bmatrix} = \begin{bmatrix} {}^s\mathbf{R}_s(\phi_{sx}\{\delta_{sx}\} - \phi_{sf}\{\delta_{sf}\}) + {}^d\mathbf{R}_s(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \\ {}^s\mathbf{R}_e(\phi_{ex}\{\delta_{ex}\} - \phi_{ef}\{\delta_{ef}\}) + {}^d\mathbf{R}_e(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \end{bmatrix}$$

Define

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_{sf} \\ \delta_{sx} \\ \delta_{ef} \\ \delta_{ex} \\ \delta_{df} \\ \delta_{dx} \end{bmatrix}; \quad {}^k\boldsymbol{\sigma} = \begin{bmatrix} {}^k\sigma_{sf} \\ {}^k\sigma_{sx} \\ {}^k\sigma_{ef} \\ {}^k\sigma_{ex} \\ {}^k\sigma_{df} \\ {}^k\sigma_{dx} \end{bmatrix}; \quad \Delta = \begin{bmatrix} \Delta_s \\ \Delta_e \end{bmatrix}; \quad \Delta_v = \begin{bmatrix} {}^s\Delta_{sv} \\ {}^s\Delta_{ev} \\ {}^d\Delta_{sv} \\ {}^d\Delta_{ev} \end{bmatrix} \quad (5.2.32)$$

Hence we can write $\mathbf{M} = \mathbf{M}\{\boldsymbol{\delta}\}$. If we use the generalized spring displacements from (5.2.29) we can re-write the moments as $\mathbf{M} = \mathbf{M}\{{}^k\boldsymbol{\sigma}, \Delta, \Delta_v\}$. We could also use the generalized spring displacements in the form of (5.2.5), allowing us to write the moments as $\mathbf{M} = \mathbf{M}\{\boldsymbol{\sigma}, \boldsymbol{\theta}\}$, where

$$\sigma = \begin{bmatrix} \sigma_{sf} \\ \sigma_{sx} \\ \sigma_{ef} \\ \sigma_{ex} \\ \sigma_{df} \\ \sigma_{dx} \end{bmatrix}; \quad \theta = \begin{bmatrix} \theta_s \\ \theta_e \end{bmatrix} \quad (5.2.33)$$

We can then write the differential moments in terms of each of these form. For the case of $\mathbf{M} = \mathbf{M}\{\delta\}$, we can write

$$\begin{aligned} \begin{bmatrix} dM_s \\ dM_e \end{bmatrix} &= \begin{bmatrix} \frac{\partial M_s}{\partial \delta_{sf}} & \frac{\partial M_s}{\partial \delta_{sx}} \\ \frac{\partial M_e}{\partial \delta_{sf}} & \frac{\partial M_e}{\partial \delta_{sx}} \end{bmatrix} \begin{bmatrix} d\delta_{sf} \\ d\delta_{sx} \end{bmatrix} + \begin{bmatrix} \frac{\partial M_s}{\partial \delta_{ef}} & \frac{\partial M_s}{\partial \delta_{ex}} \\ \frac{\partial M_e}{\partial \delta_{ef}} & \frac{\partial M_e}{\partial \delta_{ex}} \end{bmatrix} \begin{bmatrix} d\delta_{ef} \\ d\delta_{ex} \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{\partial M_s}{\partial \delta_{df}} & \frac{\partial M_s}{\partial \delta_{dx}} \\ \frac{\partial M_e}{\partial \delta_{df}} & \frac{\partial M_e}{\partial \delta_{dx}} \end{bmatrix} \begin{bmatrix} d\delta_{df} \\ d\delta_{dx} \end{bmatrix} \end{aligned} \quad (5.2.34)$$

where δ is broken up into the antagonistic pairs. When we expand the matrix elements by using the chain rule on (5.2.7) we reduce this to

$$\begin{aligned} \begin{bmatrix} dM_s \\ dM_e \end{bmatrix} &= \begin{bmatrix} {}^sR_s\psi_{sf} & -{}^sR_s\psi_{sx} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d\delta_{sf} \\ d\delta_{sx} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ {}^sR_e\psi_{ef} & -{}^sR_e\psi_{ex} \end{bmatrix} \begin{bmatrix} d\delta_{ef} \\ d\delta_{ex} \end{bmatrix} \\ &\quad + \begin{bmatrix} {}^dR_s\psi_{df} & -{}^dR_s\psi_{dx} \\ {}^dR_e\psi_{df} & -{}^dR_e\psi_{dx} \end{bmatrix} \begin{bmatrix} d\delta_{df} \\ d\delta_{dx} \end{bmatrix} \end{aligned}$$

$$\text{or} \quad d\mathbf{M} = \mathbf{S}_s d\delta_s + \mathbf{S}_e d\delta_e + \mathbf{S}_d d\delta_d \quad (5.2.35)$$

If we write $\mathbf{M} = \mathbf{M}\{\sigma, \Delta, \Delta_v\}$ this becomes

$$\begin{aligned} \begin{bmatrix} dM_s \\ dM_e \end{bmatrix} &= \begin{bmatrix} {}^s\Psi_s + {}^d\Psi_s & \Psi_d \\ \Psi_d & {}^s\Psi_e + {}^d\Psi_e \end{bmatrix} \begin{bmatrix} d\Delta_s \\ d\Delta_e \end{bmatrix} + \begin{bmatrix} {}^s\Psi_s & 0 \\ 0 & {}^s\Psi_e \end{bmatrix} \begin{bmatrix} d^s\Delta_{sv} \\ d^s\Delta_{ev} \end{bmatrix} \\ &+ \begin{bmatrix} {}^d\Psi_s & \Psi_d \\ \Psi_d & {}^d\Psi_e \end{bmatrix} \begin{bmatrix} d^d\Delta_{sv} \\ d^d\Delta_{ev} \end{bmatrix} + \begin{bmatrix} {}^sR_s\Psi_{sf} & -{}^sR_s\Psi_{sx} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d^k\sigma_{sf} \\ d^k\sigma_{sx} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ {}^sR_e\Psi_{ef} & -{}^sR_e\Psi_{ex} \end{bmatrix} \begin{bmatrix} d^k\sigma_{ef} \\ d^k\sigma_{ex} \end{bmatrix} + \begin{bmatrix} {}^dR_s\Psi_{df} & -{}^dR_s\Psi_{dx} \\ {}^dR_e\Psi_{df} & -{}^dR_e\Psi_{dx} \end{bmatrix} \begin{bmatrix} d^k\sigma_{df} \\ d^k\sigma_{dx} \end{bmatrix} \end{aligned}$$

$$\text{or} \quad d\mathbf{M} = \Psi \cdot d\Delta + {}^s\Psi \cdot d^s\Delta_v + {}^d\Psi \cdot d^d\Delta_v + \mathbf{S}_s \cdot d^k\sigma_s + \mathbf{S}_e \cdot d^k\sigma_e + \mathbf{S}_d \cdot d^k\sigma_d \quad (5.2.36)$$

If the control strategy of (5.2.25) is used, the second and third terms in (5.2.36) disappear. Notice that the first term represents the torsion-spring-like behavior in (5.2.12). The remaining terms in (5.2.36) represent the changes in joint moments due to changes in inputs.

The third form of writing the moments, $\mathbf{M} = \mathbf{M}\{\sigma, \theta\}$, results in

$$\begin{aligned} \begin{bmatrix} dM_s \\ dM_e \end{bmatrix} &= \begin{bmatrix} {}^s\Psi_s + {}^d\Psi_s & \Psi_d \\ \Psi_d & {}^s\Psi_e + {}^d\Psi_e \end{bmatrix} \begin{bmatrix} d\theta_s \\ d\theta_e \end{bmatrix} + {}^sR_s \begin{bmatrix} \Psi_{st} & -\Psi_{sb} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d\sigma_{st} \\ d\sigma_{sb} \end{bmatrix} \\ &+ {}^sR_e \begin{bmatrix} 0 & 0 \\ \Psi_{et} & -\Psi_{eb} \end{bmatrix} \begin{bmatrix} d\sigma_{et} \\ d\sigma_{eb} \end{bmatrix} + \begin{bmatrix} {}^dR_s\Psi_{dt} & -{}^dR_s\Psi_{db} \\ {}^dR_e\Psi_{dt} & -{}^dR_e\Psi_{db} \end{bmatrix} \begin{bmatrix} d\sigma_{dt} \\ d\sigma_{db} \end{bmatrix} \end{aligned}$$

$$\text{or} \quad d\mathbf{M} = \Psi \cdot d\theta + \mathbf{S}_s \cdot d\sigma_s + \mathbf{S}_e \cdot d\sigma_e + \mathbf{S}_d \cdot d\sigma_d \quad (5.2.37)$$

We show the three differential relationships in (5.2.35), (5.2.36), and (5.2.37) to demonstrate that the joint moments are actually a function of both passive external behavior as well as active components from the internal control of the mechanism. Recall we represented the moments as $d\mathbf{M} = \Psi d\Delta$ in (5.2.12). Other researchers have represented the joint moments as $\mathbf{M} = \Psi \delta\theta$, which is essentially identical to (5.2.13) when you consider the equalities discussed when deriving (5.1.13). We can now see that the moments are actually a function of both passive terms (Ψ) and active input terms ($\mathbf{S}_i$). The previous representations were based on the assumption that all other variables were held constant. This is an important distinction to make when measuring changes in joint moments. The active components must be accounted for or removed. In human movement studies these are usually eliminated by taking reaction data only for the time frame before long-latency command responses can affect the results (as opposed to short-latency reflex action).

5.2.5 Energy

In the last few sections we examined how forces, stiffnesses, and displacements translate from the individual springs to the joints, and how each are related to one another. However, one property that must look the same from all perspectives is the stored energy. The energy is physically stored in the individual springs. We can then

write the total stored energy as the sum of energy in each spring. In Section 5.1.4 we derived the energy stored in a spring as a function of force, given by equation (5.1.17).

The total energy in the current system is then

$$\begin{aligned}
 E &= E_{sf} + E_{sx} + E_{ef} + E_{ex} + E_{df} + E_{dx} \\
 E &= \int_0^{\delta_{sf}} \phi_{sf} \{\epsilon_{sf}\} d\epsilon_{sf} + \int_0^{\delta_{sx}} \phi_{sx} \{\epsilon_{sx}\} d\epsilon_{sx} + \int_0^{\delta_{ef}} \phi_{ef} \{\epsilon_{ef}\} d\epsilon_{ef} \\
 &\quad + \int_0^{\delta_{ex}} \phi_{ex} \{\epsilon_{ex}\} d\epsilon_{ex} + \int_0^{\delta_{df}} \phi_{df} \{\epsilon_{df}\} d\epsilon_{df} + \int_0^{\delta_{dx}} \phi_{dx} \{\epsilon_{dx}\} d\epsilon_{dx}
 \end{aligned} \tag{5.2.38}$$

where each ϵ is a dummy variable of integration replacing the δ 's. We have defined here that the energy is zero in each spring at zero displacement, $E\{0\} = 0$.

We can see from (5.2.38) that the energy is a function of δ , which we showed can be written in three different ways: as δ alone, as a function of σ and θ in (5.2.5), or as a function of $^k\sigma$, Δ , and Δ_v in (5.2.29). We can then write the energy in differential form as

$$dE = \left(\frac{\partial E}{\partial \delta} \right)^T d\delta = \phi^T d\delta \tag{5.2.39}$$

where

$$\phi = \begin{bmatrix} \phi_{st} \\ \phi_{sb} \\ \phi_{et} \\ \phi_{eb} \\ \phi_{dt} \\ \phi_{db} \end{bmatrix}$$

We can also write the energy as

$$dE = \left(\frac{\partial E}{\partial \sigma} \right)^T d\sigma + \left(\frac{\partial E}{\partial \theta} \right)^T d\theta = \phi^T d\sigma + M^T d\theta \quad (5.2.40)$$

Equation (5.2.40) also shows us another way to determine the joint moments,

$$\begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} \frac{\partial E}{\partial \theta_s} \\ \frac{\partial E}{\partial \theta_e} \end{bmatrix} \quad (5.2.41)$$

It is straightforward to show the validity of (5.2.41). Using the energy from (5.2.38),

$$\begin{aligned} \frac{\partial E}{\partial \theta_s} &= \frac{dE_{st}}{d\delta_{st}} \cdot \frac{\partial \delta_{st}}{\partial \theta_s} + \frac{dE_{sb}}{d\delta_{sb}} \cdot \frac{\partial \delta_{sb}}{\partial \theta_s} + \frac{dE_{et}}{d\delta_{et}} \cdot \frac{\partial \delta_{et}}{\partial \theta_s} \\ &\quad + \frac{dE_{eb}}{d\delta_{eb}} \cdot \frac{\partial \delta_{eb}}{\partial \theta_s} + \frac{dE_{dt}}{d\delta_{dt}} \cdot \frac{\partial \delta_{dt}}{\partial \theta_s} + \frac{dE_{db}}{d\delta_{db}} \cdot \frac{\partial \delta_{db}}{\partial \theta_s} \end{aligned}$$

$$\begin{aligned}
&= \phi_{st} {}^sR_s + \phi_{sb} (-{}^sR_s) + 0 + 0 + \phi_{dt} {}^dR_s + \phi_{st} (-{}^dR_s) \\
&= {}^sR_s(\phi_{st} - \phi_{sb}) + {}^dR_s(\phi_{dt} - \phi_{db}) = M_s
\end{aligned}$$

A similar derivation can be performed for M_e .

Finally, we can also write the differential energy as

$$dE = \phi^T d^k \sigma + M^T d\Delta + M_v^T d\Delta_v \quad (5.2.42)$$

where the moment vector M_v is

$$M_v = \begin{bmatrix} {}^sM_s \\ {}^sM_e \\ {}^dM_s \\ {}^sM_e \end{bmatrix} \quad (5.2.43)$$

We can then determine the energy associated with each source by tracking the variable velocities and integrating over time. The energy can then be written as

$$\begin{aligned}
E\{t\} - E\{0\} &= \int_0^t \phi^T \dot{\delta} d\tau = \int_0^t (\phi^T \dot{\sigma} + M^T \dot{\theta}) d\tau \\
&= \int_0^t (\phi^T \dot{\sigma} + M^T \dot{\Delta} + M_v^T \dot{\Delta}_v) d\tau
\end{aligned} \quad (5.2.44)$$

where τ is a dummy variable of integration for time. For example, suppose that we wish to determine the energy associated with external forces. We can do this without the need

for measuring the external environment. If we track the velocities on both sides of the springs and know the spring forces either by monitoring them or by their known functions with displacement, we can determine the external energy by rearranging (5.2.44) to get

$$E_{\text{ext}} = \int_0^t \mathbf{M}^T \dot{\boldsymbol{\theta}} d\tau = \int_0^t \boldsymbol{\phi}^T (\dot{\boldsymbol{\delta}} - \dot{\boldsymbol{\sigma}}) d\tau$$

This external energy is associated with the stiffness ellipse representation shown in Chapter 2. It may be therefore be useful to measure this energy as a function of displacement direction to determine the isopotentials that represent the endpoint stiffness.

In Section 5.2 we have:

- introduced the model under study in the current work,
- defined the sign conventions for the model,
- defined and related the joint displacements, moments, and stiffnesses,
- discussed control inputs in terms of relevant parameters,
- derived the internal energy storage in terms of passive and active components,

5.3 Endpoint Stiffness

We showed in Section 5.2 how individual springs can act around a joint to appear similar to torsion springs. When we twist the joint, the springs respond by creating a joint torque. Similarly, if we push on the endpoint of the manipulator in Figure 5.2.2, we feel a force in response, similar to a set of translational springs. Figure 5.3.1 shows this analogy.

Recall that we defined the sign convention for springs in Section 5.1 such that a positive displacement results in a positive spring force, which acts in the opposite direction from the displacement. If we define the spring force in the x-direction and y-direction as F_x and F_y , respectively, they must act positive in the negative x and y displacement directions. We can then write the endpoint spring force as

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = - \begin{bmatrix} F_{ax} \\ F_{ay} \end{bmatrix} \quad (5.3.1)$$

That is a negative applied force F_a stretches the springs creating a positive spring force F .

Figure 5.3.2 shows a free-body diagram of the manipulator. By summing the static forces and moments we get

$$M_e = F_{ax}L_e \sin(\theta_{es}) - F_{ay}L_e \cos(\theta_{es})$$

$$M_s = M_e + F_{ax}L_s \sin(\theta_s) - F_{ay}L_s \cos(\theta_s)$$

Define the following notations:

$$\begin{aligned}
 S_s &= \sin \theta_s \\
 C_s &= \cos \theta_s \\
 S_e &= \sin \theta_e \\
 C_e &= \cos \theta_e \\
 S_{es} &= \sin \theta_{es} \\
 C_{es} &= \cos \theta_{es} \\
 L_1 S_1 &= L_1 \sin(\theta_1) = L_e S_{es} + L_s S_s \\
 L_1 C_1 &= L_1 \cos(\theta_1) = L_e C_{es} + L_s C_s
 \end{aligned} \tag{5.3.2}$$

The last two notations above follow from the geometry for the tip (endpoint) of the arm, where L_1 is the distance from the shoulder to the tip and θ_1 is the angle from the x-axis to the line connecting the shoulder to the tip.

Using this notation, and substituting the spring forces from (5.3.1), the joint moments become

$$\begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} -L_1 S_1 & L_1 C_1 \\ -L_e S_{es} & L_e C_{es} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{5.3.3}$$

or $\mathbf{M} = \mathbf{J}^T \mathbf{F}$

Figure 5.3.1: Virtual endpoint springs. F_{ix} and F_{iy} are externally applied forces.

Figure 5.3.2: Free body diagram of arm links.

where

$$\mathbf{J} = \begin{bmatrix} -L_t S_t & -L_e S_{es} \\ L_t C_t & L_e C_{es} \end{bmatrix} \quad (5.3.4)$$

is called the Jacobian. The Jacobian matrix relates endpoint forces and velocities to joint moments and angular velocities. The joint moments M_s and M_e in Figure 5.3.2 resist an increase in θ_s and θ_e , respectively, so they are consistent with the internal joint spring moments defined by (5.2.7). To get the endpoint spring forces we must invert (5.3.3),

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{1}{L_s L_e \sin \theta_e} \begin{bmatrix} L_e \cos \theta_{es} & -L_e \cos \theta_t \\ L_e \sin \theta_{es} & -L_t \sin \theta_t \end{bmatrix} \begin{bmatrix} M_s \\ M_e \end{bmatrix} \quad (5.3.5)$$

or $\mathbf{F} = \mathbf{J}^T \mathbf{M}$

We use the notation $\mathbf{J}^T = (\mathbf{J}^T)^{-1} = (\mathbf{J}^{-1})^T$. Equation (5.3.5) is the virtual spring force felt at the endpoint, related to the joint spring moments.

The position of the endpoint is

$$\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_s \cos \theta_s + L_e \cos \theta_{es} \\ L_s \sin \theta_s + L_e \sin \theta_{es} \end{bmatrix} = \begin{bmatrix} L_t \cos \theta_t \\ L_t \sin \theta_t \end{bmatrix} \quad (5.3.6)$$

In Section 5.1 we defined stiffness to mean a rate of change in force in response to a displacement. We can displace the endpoint in an infinite number of directions in two

dimensions. However, every displacement has a unique reaction force and so we must be able to write the endpoint spring force as a function of displacement from a given position $\mathbf{p}$. The endpoint stiffness is defined as

$$\mathbf{K} = \frac{d\mathbf{F}}{d\mathbf{p}} = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \quad (5.3.7)$$

Unfortunately, we cannot directly write the force as a function of position, $\mathbf{F} = \mathbf{F}(\mathbf{p})$. Equation (5.3.5) shows us that the force is a function of the joint angles and the moments. Equation (5.3.6) relates the endpoint position to the angles, but it is not invertible. Furthermore, we would have to substitute for the joint moments which requires choosing specific ϕ -functions for the springs and substituting for the joint angles again in the δ expressions. We can, however, expand (5.3.6) using the chain rule,

$$\mathbf{K} = \frac{\partial \mathbf{F}}{\partial \boldsymbol{\theta}} \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{p}} = \begin{bmatrix} \frac{\partial F_x}{\partial \theta_s} & \frac{\partial F_x}{\partial \theta_e} \\ \frac{\partial F_y}{\partial \theta_s} & \frac{\partial F_y}{\partial \theta_e} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_s}{\partial x} & \frac{\partial \theta_s}{\partial y} \\ \frac{\partial \theta_e}{\partial x} & \frac{\partial \theta_e}{\partial y} \end{bmatrix} \quad (5.3.8)$$

As above, we cannot invert (5.3.6) and so we cannot write the joint angles as a function of position directly. However,

$$\frac{d\theta}{d\mathbf{p}} = \left(\frac{d\mathbf{p}}{d\theta} \right)^{-1} \quad (5.3.9)$$

Taking the derivative of (5.3.6),

$$\frac{d\mathbf{p}}{d\theta} = \begin{bmatrix} -L_t \sin \theta_t & -L_e \sin \theta_{es} \\ L_t \cos \theta_t & L_e \cos \theta_{es} \end{bmatrix} = \mathbf{J} \quad (5.3.10)$$

This is the definition of the Jacobian matrix. We can then invert (5.3.10) to get

$$\frac{d\theta}{d\mathbf{p}} = \frac{1}{L_t L_e \sin \theta_e} \begin{bmatrix} L_e \cos \theta_{es} & L_e \sin \theta_{es} \\ -L_t \cos \theta_t & -L_t \sin \theta_t \end{bmatrix} = \mathbf{J}^{-1} \quad (5.3.11)$$

The force derivative in (5.3.8), from (5.3.5), is

$$\frac{\partial \mathbf{F}}{\partial \theta} = \frac{\partial (\mathbf{J}^{-T} \mathbf{M})}{\partial \theta} \quad (5.3.12)$$

Both $\mathbf{J}^{-T}$ and $\mathbf{M}$ are functions of the joint angles and so we must use the chain rule. To do this we first break up the Jacobian matrix into its columns. Define

$$\mathbf{J}_t^{-T} = \frac{1}{L_t L_e S_e} \begin{bmatrix} L_e C_{es} \\ L_e S_{es} \end{bmatrix}; \quad \mathbf{J}_e^{-T} = \frac{-1}{L_t L_e S_e} \begin{bmatrix} L_t C_t \\ L_t S_t \end{bmatrix} \quad (5.3.13)$$

Then the force derivative in (5.3.12) can be written as

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial \boldsymbol{\theta}} &= \mathbf{J}^{-\top} \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} + \frac{\partial \mathbf{J}_s^{-\top}}{\partial \boldsymbol{\theta}} \mathbf{M}_s + \frac{\partial \mathbf{J}_e^{-\top}}{\partial \boldsymbol{\theta}} \mathbf{M}_e \\ &= \mathbf{J}^{-\top} \boldsymbol{\Psi} + \mathbf{G}_s \mathbf{M}_s + \mathbf{G}_e \mathbf{M}_e\end{aligned}\tag{5.3.14}$$

where

$$\mathbf{G}_s = \frac{\partial \mathbf{J}_s^{-\top}}{\partial \boldsymbol{\theta}} \quad \text{and} \quad \mathbf{G}_e = \frac{\partial \mathbf{J}_e^{-\top}}{\partial \boldsymbol{\theta}}$$

The substitution of $\boldsymbol{\Psi}$ above follows from (5.2.37). We define

$$\mathbf{G} = \mathbf{G}_s \mathbf{M}_s + \mathbf{G}_e \mathbf{M}_e\tag{5.3.15}$$

If we substitute all of the above relationships back into (5.3.8) we get

$$\mathbf{K} = \mathbf{J}^{-\top} \boldsymbol{\Psi} \mathbf{J}^{-1} + \mathbf{G} \mathbf{J}^{-1} = {}^J\mathbf{K} + {}^G\mathbf{K}\tag{5.3.16}$$

where we define ${}^J\mathbf{K} = \mathbf{J}^{-\top} \boldsymbol{\Psi} \mathbf{J}^{-1}$ and ${}^G\mathbf{K} = \mathbf{G} \mathbf{J}^{-1}$.

The first term in (5.3.16) is the transformation of the joint stiffnesses to the endpoint, which we call the *endpoint spring stiffness* because it is directly due to the stretching of the springs. We denote it by ${}^J\mathbf{K}$ because it is transformed directly from the joint stiffnesses, $\boldsymbol{\Psi}$. This corresponds to the stiffness relationship normally used by other researchers (e.g., [Hogan, 1985a; Dolan et al., 1993]). The second term is the rate at

which endpoint force changes due to a change in geometry, hence we call it the *geometric stiffness* and denote it by ${}^G\mathbf{K}$.

Arguably, the geometric stiffness is not really a stiffness. It does give a rate of change of force to a displacement, so it acts similar to a stiffness. Consider a manipulator system in which the control system creates constant moments at the joints. Since the moments are constant, $\Psi = \mathbf{0}$. There must be an external force $\mathbf{F}_0$ to offset these moments in order to create a static equilibrium. Next we apply a small force $\Delta\mathbf{F}$ to the endpoint. This defines a new endpoint force $\mathbf{F}_1 = \mathbf{F}_0 + \Delta\mathbf{F}$, which takes up a different equilibrium position with the constant joint moments. If we remove $\Delta\mathbf{F}$, the arm returns to its original position. This is similar to the response of a spring.

The main difference between the geometric stiffness and the normal use of the term *stiffness* is that it is a function of the external constant force $\mathbf{F}_0$. This external force must be considered constant and any small changes in it are considered disturbances, $\mathbf{F} = \mathbf{F}_0 + d\mathbf{F}$, somewhat analogous to a pre-loaded spring. This is easier seen by the differential representation of the endpoint force. Rearranging (5.3.7), and substituting (5.3.16) for $\mathbf{K}$,

$$d\mathbf{F} = \mathbf{K} d\mathbf{p} - ({}^J\mathbf{K} + {}^G\mathbf{K})d\mathbf{p} \quad (5.3.17)$$

In other words, a small *change* in endpoint force is due to both to ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ through a

small *change* in endpoint position. Since the moments, and therefore endpoint force, are zero at the virtual position, ${}^G\mathbf{K} = \mathbf{0}$ at the virtual position. Hence for very small displacements from the virtual position, $\delta\mathbf{p}$, the total endpoint force is

$$\mathbf{F} = {}^J\mathbf{K} \delta\mathbf{p} \quad (5.3.18)$$

The difference between (5.3.17) and (5.3.18) is similar to the treatment of the moments in (5.2.12) and (5.2.13). We can therefore interpret ${}^G\mathbf{K}$ as a contact stiffness only, meaning that it is non-zero only when there is an endpoint force. ${}^J\mathbf{K}$ is both a contact stiffness and a free-motion stiffness. This distinction necessitates a clarification of what *stiffness* actually means. When in rigid, constrained contact, we can derive both ${}^G\mathbf{K}$ and ${}^J\mathbf{K}$ as above. But since the arm cannot move, these stiffnesses have little meaning. Similarly, when in free-motion, there is no endpoint force and hence no displacement from the virtual point (in equilibrium), and so stiffness has little meaning. Therefore *endpoint stiffness only has practical meaning when we consider the dynamics in free-motion or contact situations in which the endpoint is free to move in some direction.*

Most researchers in this area simply ignore the geometric stiffness term because they are only concerned with small displacements from the virtual position [Salisbury, 1980; Hogan, 1985a; Dolan et al., 1993]. Others at least acknowledge its existence, but drop the term for the same reason [Hogan, 1990; Lacquaniti et al., 1993]. Only rarely is the geometric stiffness term considered (e.g., [McIntyre, 1996]). The focus of the current

research is the behavior of stiffness in contact, including large contact forces and low stiffness, and so we cannot neglect the ${}^G\mathbf{K}$ term.

We use the following notations for the stiffness matrices in (5.3.16):

$${}^J\mathbf{K} = \begin{bmatrix} {}^Jk_{xx} & {}^Jk_{xy} \\ {}^Jk_{yx} & {}^Jk_{yy} \end{bmatrix}; \quad {}^G\mathbf{K} = \begin{bmatrix} {}^Gk_{xx} & {}^Gk_{xy} \\ {}^Gk_{yx} & {}^Gk_{yy} \end{bmatrix} \quad (5.3.19)$$

If we expand the ${}^J\mathbf{K}$ in (5.3.16) we get the elements as

$$\begin{aligned} {}^Jk_{xx} &= \left(\frac{1}{L_s L_e S_e} \right)^2 \{ (L_e C_{es})^2 \Psi_s + (L_t C_t)^2 \Psi_e - 2(L_e C_{es})(L_t C_t) \Psi_d \} \\ {}^Jk_{xy} &= \left(\frac{1}{L_s L_e S_e} \right)^2 \{ L_e^2 S_{es} C_{es} \Psi_s + L_t^2 S_t C_t \Psi_e - (L_e S_{es} L_t S_t + L_e C_{es} L_t C_t) \Psi_d \} \\ {}^Jk_{yx} &= {}^Jk_{xy} \end{aligned} \quad (5.3.20)$$

$${}^Jk_{yy} = \left(\frac{1}{L_s L_e S_e} \right)^2 \{ (L_e S_{es})^2 \Psi_s + (L_t S_t)^2 \Psi_e - 2(L_e S_{es})(L_t S_t) \Psi_d \}$$

Similarly, the $^G\mathbf{K}$ elements are

$$\begin{aligned}
 {}^G k_{xx} &= \frac{1}{L_s^2 L_e^2 S_e^3} \left\{ (L_e^2 C_{es}^2 C_e + L_s L_e C_s^2) M_s - (L_s^2 C_s^2 C_e + L_s L_e S_e^2) M_e \right\} \\
 {}^G k_{xy} &= \frac{1}{L_s^2 L_e^2 S_e^3} \left\{ (L_e^2 S_{es} C_{es} C_e + L_s L_e S_s C_s) M_s - L_s^2 C_s S_s C_e M_e \right\} \\
 {}^G k_{yx} &= {}^G k_{xy} \\
 {}^G k_{yy} &= \frac{1}{L_s^2 L_e^2 S_e^3} \left\{ (L_e^2 S_{es}^2 C_e + L_s L_e S_s^2) M_s - (L_s^2 S_s^2 C_e + L_s L_e S_e^2) M_e \right\}
 \end{aligned} \tag{5.3.21}$$

From (5.3.16) we get the total endpoint stiffness as the addition of corresponding elements in (5.3.20) and (5.3.21),

$$\begin{aligned}
 k_{xx} &= {}^J k_{xx} + {}^G k_{xx} \\
 k_{xy} &= {}^J k_{xy} + {}^G k_{xy} \\
 k_{yx} &= {}^J k_{yx} + {}^G k_{yx} \\
 k_{yy} &= {}^J k_{yy} + {}^G k_{yy}
 \end{aligned} \tag{5.3.22}$$

In Section 5.3, we have derived

- the endpoint stiffness, including endpoint spring stiffness and geometric stiffness, and
- the circumstances in which these stiffnesses apply and what they mean.

5.4 Chapter Summary

This chapter dealt with the development and definition of our model, and provided background material useful to the research goals. In Section 5.1 we defined the general description of stiffness as used here, as well as sign conventions, and how they relate to spring force and energy. Section 5.2 presented the model we are using and applied the definitions and conventions from Section 5.1. We defined the stiffnesses, virtual angles, and control inputs, in various forms. In Section 5.3 we derived the manipulator stiffness as seen at the endpoint and related it to the joint stiffnesses and moments. We also introduced the geometric stiffness component which is usually neglected by other researchers.

Chapter 6

Stiffness Transformations

In this Chapter we will be showing how the isopotential ellipse relates to other parameters, such as the stiffnesses of the actuators, endpoint force, and arm position in the workspace. In particular we will look at how the endpoint spring stiffness and geometric stiffness combine in contact situations, including the stability of the arm under contact circumstances. In Section 6.1 we will discuss the characteristics of two-dimensional stiffnesses and their representation. We will develop the relationship between actuator stiffnesses and the endpoint stiffness. A key achievement in this development will be the decomposition of endpoint stiffness into projection components from the actuator stiffnesses. In Section 6.2 we will examine and explain these projections more fully by analysing the progression of force, displacement, and stiffness through the different vector spaces. These relationships will achieve the goals of Question 1 in Chapter 4, referring to the relationship between actuator stiffnesses and endpoint stiffness. This will allow us to understand better how changes in actuator relationships affect the endpoint stiffness, and conversely, what are the actuator stiffness requirements to create a desired endpoint stiffness.

In Section 6.1 we also investigate the geometric stiffness, and in Section 6.3 we develop a tool that can be used to judge whether or not the geometric stiffness can be neglected when studying the endpoint stiffness. Since the geometric stiffness is a function of endpoint force, this helps to answer Question 5 in Chapter 4 regarding the effects of loading on stiffness.

Once the stiffness is understood, we investigate stiffness instability to answer Question 3, which asks how the muscle (actuator) stiffnesses affect stability. The physical meaning of, and conditions for, instability are discussed. An important result of the unstable stiffness investigation is the inability of common stiffness representations to depict the unstable case. To solve this problem, we develop a general stiffness tool based on Mohr's circle (commonly used for stress analysis) that can represent the stiffness under all conditions and contains all of the necessary stiffness characteristic equations in its geometry.

6.1 Isopotential Ellipse

In this section we will develop the traditional isopotential ellipse representation for two-dimensional stiffness and compare it to a non-linear representation for the endpoint stiffness. We show the relationship between actuator stiffness and the endpoint stiffness, and what limitations and strategies can be derived from this relationship. We will also show how the endpoint spring stiffness isopotential and geometric stiffness

isopotential relate to the whole isopotential ellipse.

6.1.1 Derivation

Recall from (5.3.16) that the endpoint stiffness is calculated as

$$\mathbf{K} = \mathbf{J}^T \Psi \mathbf{J}^{-1} + \mathbf{G} \mathbf{J}^{-1} = {}^J\mathbf{K} + {}^G\mathbf{K}$$

We can calculate an endpoint stiffness matrix $\mathbf{K}$ at any point that the arm make take up in its workspace using the above equation. This operating point can be at the virtual position or at some displaced position. The stiffness is a linearized local stiffness, meaning that it is a rate of change of force by a small displacement about the operating point, and is only exact at the point of calculation. The stiffness is fictitious in this sense then, because as the arm moves though the small displacement, the stiffness matrix changes for the new point. Hence it isn't an exact depiction of the forces that result from displacements. Rather, the endpoint stiffness is a description of the dynamic state of the arm and not a method for calculating endpoint forces due to displacements.

The arm model under study here is capable of movement in two dimensions. Since the endpoint stiffness matrix is also two-dimensional, it spans the workspace of the endpoint. However, it is difficult to use this matrix to understand the reaction of the endpoint to displacements in arbitrary directions. The typical way for understanding the meaning of the stiffness matrix is to examine the energy that is stored by displacement of

the endpoint. (We will examine other interpretations of the stiffness matrix in later sections.)

If we apply a small displacement $\delta\mathbf{p}$ from the virtual point, the potential energy stored in the springs is

$$\begin{aligned} E_p &= \int_0^{\delta\mathbf{p}} \mathbf{F}^T d\boldsymbol{\varepsilon} = \int_0^{\delta\mathbf{p}} (\boldsymbol{\varepsilon}^T \mathbf{K}^T) d\boldsymbol{\varepsilon} = \frac{1}{2} \delta\mathbf{p}^T \mathbf{K} \delta\mathbf{p} \\ &= \frac{1}{2} (k_{xx} \delta x^2 + (k_{xy} + k_{yx}) \delta x \delta y + k_{yy} \delta y^2) \end{aligned} \quad (6.1.1)$$

where $\boldsymbol{\varepsilon}$ is a dummy variable for integration, and we use $\mathbf{K} = \mathbf{K}^T$ since it is symmetric (see Section 2.1.3 and Appendix B). Note that $\mathbf{K}$ is assumed constant over the displacement in this integration. From (5.3.16) we know that this isn't true. Both ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ are functions of the Jacobian, which changes as the endpoint position moves. The geometric stiffness ${}^G\mathbf{K}$ is also a function of the rate at which the Jacobian changes with respect to a displacement. In addition, the joint stiffness Ψ may change in ${}^J\mathbf{K}$ as the endpoint is displaced, depending on the design of the spring stiffness functions.

However, we can still apply (6.1.1) to understand the stiffness at a point. As we described above, the stiffness matrix is meant to describe the state of the arm rather than an actual representation of what happens when the arm is displaced. So even though (6.1.1) may not provide an accurate measure of the spring energy in the presence of large

displacements, it is still useful in describing the state of the arm.

Equation (6.1.1) defines an ellipse in the $\mathcal{X}$ - $\mathcal{Y}$ plane. Figure 6.1.1 shows the parameters we will use to describe the ellipse. The ellipse itself is a constant energy contour, meaning that if we displaced the endpoint from $(\mathcal{X}, \mathcal{Y}) = (0,0)$ to any point on the ellipse, the same potential energy is stored in the springs. If we moved the endpoint around the ellipse there would be no change in the overall stored energy, though it may shift internally from one spring to another. This means that we cannot do work by moving around the potential energy contour, and so the spring force must be perpendicular to the contour. Appendix C provides further details to the decomposition of forces. We must stress again that this is a hypothetical situation based on a constant $\mathbf{K}$, and is only approximately true.

Recall from Chapter 2 that the major axis of the ellipse corresponds to a lower stiffness while the minor axis corresponds to a higher stiffness. Directions between these axes have stiffnesses that vary between the two extremes.

Figure 6.1.1: Isopotential ellipse representation.

Equation (6.1.1) allows us to define three stiffness ellipses for each endpoint position. Using the overall endpoint stiffness $\mathbf{K}$ in (6.1.1) provides the overall energy ellipse associated with the displacement. If we use ${}^J\mathbf{K}$ we get the energy ellipse associated with joint stiffness. Similarly, using ${}^G\mathbf{K}$ will provide the energy associated with the changing geometry.

We will now examine some properties of the stiffness ellipse for a general $\mathbf{K}$ matrix, which could represent the total stiffness $\mathbf{K}$, ${}^J\mathbf{K}$, or ${}^G\mathbf{K}$. We can find the major and minor axis of the ellipse by using the polar coordinates

$$\delta x = r \sin(\theta)$$

$$\delta y = r \cos(\theta)$$

where θ is a running angle around the ellipse. The substitution of this coordinate change into (6.1.1) produces an ellipse radius function given by

$$r = \sqrt{\frac{2E_p}{k_{xx} \cos^2 \theta + (k_{xy} + k_{yx}) \sin \theta \cos \theta + k_{yy} \sin^2 \theta}} \quad (6.1.2)$$

If we define θ_k as the angle between the δx - δy coordinate system and the major axis of the ellipse, as in Figure 6.1.1, the radius of the major ($r_{\max}$) and minor ($r_{\min}$) axes are

$$r_{\min} = \sqrt{\frac{2E_p}{k_{xx}\cos^2\theta_k - (k_{xy} + k_{yx})\sin\theta_k\cos\theta_k + k_{yy}\cos^2\theta_k}} \quad (6.1.3)$$

$$r_{\max} = \sqrt{\frac{2E_p}{k_{xx}\sin^2\theta_k - (k_{xy} + k_{yx})\sin\theta_k\cos\theta_k + k_{yy}\cos^2\theta_k}} \quad (6.1.4)$$

Since the forces must be perpendicular to the ellipse, the forces on the major and minor axes point directly towards the origin. As a consequence, if the ellipse axes happen to line up with the coordinate axes, the off diagonal terms of $\mathbf{K}$ must be zero. To see this, suppose that the ellipse axes were lined up with the δx - δy axes in Figure 6.1.1. If we displaced the endpoint along the δx -axis to the vertex of the ellipse the resulting force must point back toward the origin. Recall from (5.3.18),

$$\mathbf{F} = \mathbf{K}\delta\mathbf{p}$$

and hence in this example,

$$F_y = k_{yx}\delta x + k_{yy}\delta y = 0$$

We displaced the endpoint in the δx -direction and so $\delta y = 0$. Therefore we must have $k_{yx} = 0$, and consequently $k_{xy} = 0$ since the matrix is symmetrical.

We define the δu - δv coordinate system to be oriented such that δu lines up with the major axis of the ellipse and δv lines up with the minor axis, as shown in Figure 6.1.1.

We can then write the spring force as

$$\begin{bmatrix} F_u \\ F_v \end{bmatrix} = \begin{bmatrix} k_{\min} & 0 \\ 0 & k_{\max} \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$$

$$\text{or} \quad \mathbf{F}_{uv} = \mathbf{K}_{uv} \delta \mathbf{p}_{uv} \quad (6.1.5)$$

The δu - δv coordinate system is rotated from the δx - δy coordinate system by θ_k , as shown in Figure 6.1.1. That is,

$$\delta \mathbf{p} = \mathbf{R}_k \delta \mathbf{p}_{uv} \quad (6.1.6)$$

$$\text{and} \quad \mathbf{F} = \mathbf{R}_k \mathbf{F}_{uv} \quad (6.1.7)$$

$$\text{where} \quad \mathbf{R}_k = \begin{bmatrix} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{bmatrix} \quad (6.1.8)$$

(We use the subscript “uv” here to denote force and position with respect to the δu - δv coordinate system. Terms with no subscript are assumed to be relative to the δx - δy coordinate system.)

We also know that $\mathbf{F} = \mathbf{K} \delta \mathbf{p}$ for small displacements from the virtual position.

Substitution of this relationship and (6.1.6) into (6.1.7) gives us

$$\mathbf{F}_{uv} = \mathbf{R}_k^{-1} \mathbf{K} \mathbf{R}_k \delta \mathbf{p}_{uv} \quad (6.1.9)$$

A comparison of (6.1.9) with (6.1.5) yields

$$\mathbf{K}_{uv} = \mathbf{R}_k^{-1} \mathbf{K} \mathbf{R}_k \quad (6.1.10)$$

Expansion of (6.1.10) yields the relationships

$$\begin{aligned} k_{\max} &= k_{xx} \cos^2 \theta_k + (k_{xy} + k_{yx}) \sin \theta_k \cos \theta_k + k_{yy} \sin^2 \theta_k \\ k_{\min} &= k_{xx} \sin^2 \theta_k - (k_{xy} + k_{yx}) \sin \theta_k \cos \theta_k + k_{yy} \cos^2 \theta_k \end{aligned} \quad (6.1.11)$$

A comparison of (6.1.11) to (6.1.3) and (6.1.4) provides us with the major and minor axes as

$$r_{\max} = \sqrt{\frac{2E_p}{k_{\min}}}; \quad r_{\min} = \sqrt{\frac{2E_p}{k_{\max}}} \quad (6.1.12)$$

This meets the expectations as described in Chapter 2, that the stiffness on the major axis is lower than that of the minor axis. The notation in this case can become confusing due to the connotations of “max” and “min” on the axis values.

We can use the fact that $\mathbf{K} = {}^J\mathbf{K} + {}^G\mathbf{K}$ from (5.3.16) to break up (6.1.10) in the form

$$\mathbf{K}_{uv} = \mathbf{R}_k^{-1} ({}^J\mathbf{K} + {}^G\mathbf{K}) \mathbf{R}_k \quad (6.1.13)$$

This leads to the definitions

$${}^J\mathbf{K}_{uv} = \mathbf{R}_k^{-1} {}^J\mathbf{K} \mathbf{R}_k \quad (6.1.14a)$$

$${}^G\mathbf{K}_{uv} = \mathbf{R}_k^{-1} {}^G\mathbf{K} \mathbf{R}_k \quad (6.1.14b)$$

We have to be careful of our definitions at this point. The rotation matrix $\mathbf{R}_k$ rotates the endpoint stiffness matrices ${}^J\mathbf{K}_{uv}$ and ${}^G\mathbf{K}_{uv}$ in (6.1.14a,b) to the axes of the ellipse defined by the *whole* $\mathbf{K}$ matrix. That is, we can define separate ellipses for ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ by substituting them separately into (6.1.1). They will each have their own distinct shape and orientation. We denote these ellipse orientations as ${}^J\theta_k$ and ${}^G\theta_k$, respectively. Since we defined the δu - δv coordinate system based on the axes of the whole stiffness ellipse, the ${}^J\theta_k$ and ${}^G\theta_k$ orientations do not generally line up with the δu - δv coordinate system.

Hence we define

$${}^J\mathbf{K}_k = {}^J\mathbf{R}_k^{-1} {}^J\mathbf{K} {}^J\mathbf{R}_k \quad (6.1.15)$$

$${}^G\mathbf{K}_k = {}^G\mathbf{R}_k^{-1} {}^G\mathbf{K} {}^G\mathbf{R}_k \quad (6.1.16)$$

where

$${}^J\mathbf{K}_k = \begin{bmatrix} {}^Jk_{\max} & 0 \\ 0 & {}^Jk_{\min} \end{bmatrix}$$

$${}^G\mathbf{K}_k = \begin{bmatrix} {}^Gk_{\max} & 0 \\ 0 & {}^Gk_{\min} \end{bmatrix}$$

and we use the notation ${}^J\mathbf{R}_k$ and ${}^G\mathbf{R}_k$ to be the rotation matrix in (6.1.8) with ${}^J\theta_k$ and ${}^G\theta_k$ substituted for θ_k , respectively.

6.1.2 Endpoint Spring Stiffness

When the endpoint is at its virtual position there are no joint moments, by definition, and hence no endpoint force. This means that the endpoint is not in contact with a surface in the environment and the ${}^G\mathbf{K}$ term is zero. For small endpoint forces the ${}^G\mathbf{K}$ term may also be considered to be negligible as discussed in Section 5.3. In this section we will consider the stiffness ellipse in cases in which ${}^G\mathbf{K}$ can be neglected. Section 6.3 will explore in more detail the conditions under which this applies.

Equation (5.3.16) tells us that ${}^J\mathbf{K} = \mathbf{J}^T \Psi \mathbf{J}^{-1}$. Using this relationship between endpoint spring stiffness and joint stiffness in (6.1.15) yields

$${}^J\mathbf{K}_k = {}^J\mathbf{R}_k^{-1} \mathbf{J}^T \Psi \mathbf{J}^{-1} {}^J\mathbf{R}_k \quad (6.1.17)$$

Note that

$${}^J\mathbf{R}_k^{-1} = \begin{bmatrix} \cos {}^J\theta_k & \sin {}^J\theta_k \\ -\sin {}^J\theta_k & \cos {}^J\theta_k \end{bmatrix} = {}^J\mathbf{R}_k^T$$

and so

$${}^J\mathbf{R}_k^{-1} \mathbf{J}^T = {}^J\mathbf{R}_k^T \mathbf{J}^T = (\mathbf{J}^{-1} {}^J\mathbf{R}_k)^T$$

We define

$${}^J\mathbf{J}_k^{-1} = \mathbf{J}^{-1} {}^J\mathbf{R}_k \quad (6.1.18)$$

This is the Jacobian matrix with respect to the coordinate system of the ${}^J\mathbf{K}$ isopotential ellipse. That is, this Jacobian matrix relates joint coordinates to endpoint coordinates that are oriented in the direction of the major axis of the ${}^J\mathbf{K}$ ellipse. We can then rewrite (6.1.17) as

$${}^J\mathbf{K}_k = {}^J\mathbf{J}_k^{-T} \Psi {}^J\mathbf{J}_k^{-1} \quad (6.1.19)$$

Substitution of $\mathbf{J}^{-1}$ from (5.3.11) and $\mathbf{R}_k$ from (6.1.8) into the Jacobian equation in (6.1.18) yields

$${}^J\mathbf{J}_k^{-1} = \frac{1}{L_e L_e \sin \theta_e} \begin{bmatrix} L_e \cos(\theta_{es} - {}^J\theta_k) & L_e \sin(\theta_{es} - {}^J\theta_k) \\ -L_e \cos(\theta_{et} - {}^J\theta_k) & -L_e \sin(\theta_{et} - {}^J\theta_k) \end{bmatrix} \quad (6.1.20)$$

We can now see that indeed the Jacobian in (6.1.18) is the Jacobian with respect to the ${}^J\mathbf{K}$ ellipse coordinate system since the included angles are all relative to the major axis, with the exception of θ_e which is always relative to the previous link, regardless of the coordinate system.

The stiffness matrix with respect to the ${}^J\mathbf{K}$ ellipse coordinate system is given by (6.1.19). We get the matrices ${}^J\mathbf{J}_k^{-1}$ from (6.1.20) and Ψ from the expanded form in

(5.2.12). If we take the ${}^J\mathbf{K}_k$ matrix in its expanded form from (6.1.15) and equate matrix elements on each side of (6.1.19) we get the following three relationships:

$$D_{J_1}^{-2}[(L_e C_{esJ})^2 \Psi_s + (L_t C_{tJ})^2 \Psi_e - 2(L_e C_{esJ})(L_t C_{tJ}) \Psi_d] = {}^Jk_{\max} \quad (6.1.21a)$$

$$D_{J_1}^{-2}[(L_e S_{esJ})^2 \Psi_s + (L_t S_{tJ})^2 \Psi_e - 2(L_e S_{esJ})(L_t S_{tJ}) \Psi_d] = {}^Jk_{\min} \quad (6.1.21b)$$

$$D_{J_1}^{-2}[(L_e C_{esJ})(L_e S_{esJ}) \Psi_s + (L_t C_{tJ})(L_t S_{tJ}) \Psi_e - 2\{(L_e C_{esJ})(L_t S_{tJ}) + (L_e S_{esJ})(L_t C_{tJ})\} \Psi_d] = 0 \quad (6.1.21c)$$

where we use the notation

$$\begin{aligned} D_{J_1} &= \det(\mathbf{J}^{-1}) = 1/L_s L_e \sin \theta_e \\ C_{esJ} &= \cos(\theta_{es} - {}^J\theta_k) \\ S_{esJ} &= \sin(\theta_{es} - {}^J\theta_k) \\ C_{tJ} &= \cos(\theta_t - {}^J\theta_k) \\ S_{tJ} &= \sin(\theta_t - {}^J\theta_k) \\ \Psi_s &= {}^s\Psi_s + {}^d\Psi_s \\ \Psi_e &= {}^s\Psi_e + {}^d\Psi_e \end{aligned} \quad (6.1.22)$$

Because the matrices in (6.1.19) are 2×2 , we actually get four equations. However, the off-diagonal terms are both the equation given in (6.1.21c).

Equations (6.1.21a,b) provide the relationships between joint stiffnesses and the major and minor axes stiffnesses of the endpoint spring stiffness isopotential.

(Remember that we are only discussing the ${}^J\mathbf{K}$ contribution to the isopotential ellipse at this point.) From a control perspective, the joint stiffnesses as given in (6.1.21) are not the best parameters for manipulating the endpoint stiffness because they are not

independent. That is, double-joint springs contribute to Ψ_s , Ψ_e , and Ψ_d . We can expand these joint stiffnesses via the stiffness definitions in (5.2.11),

$$\begin{aligned}\Psi_s &= {}^s\Psi_s + {}^d\Psi_s = {}^sR_s^2(\psi_{sx} + \psi_{sf}) + {}^dR_s^2(\psi_{dx} + \psi_{df}) = {}^sR_s^2\psi_{\Sigma s} + {}^dR_s^2\psi_{\Sigma d} \\ \Psi_e &= {}^s\Psi_e + {}^d\Psi_e = {}^sR_e^2(\psi_{ex} + \psi_{ef}) + {}^dR_e^2(\psi_{dx} + \psi_{df}) = {}^sR_e^2\psi_{\Sigma e} + {}^dR_e^2\psi_{\Sigma d} \\ \Psi_d &= {}^dR_s {}^dR_e(\psi_{dx} + \psi_{df}) = {}^dR_s {}^dR_e \psi_{\Sigma d}\end{aligned}\quad (6.1.23)$$

where we use the notation

$$\psi_{\Sigma i} = \psi_{ix} + \psi_{if} \quad (i = s, e, d) \quad (6.1.24)$$

With these expansions in mind, we can rewrite the three equations in (6.1.21) as

$$D_{J_s}^2[(L_e {}^sR_s C_{esJ})^2\psi_{\Sigma s} + (L_t {}^sR_e C_{tJ})^2\psi_{\Sigma e} + (L_e {}^dR_s C_{esJ} - L_t {}^dR_e C_{tJ})^2\psi_{\Sigma d}] = {}^Jk_{\max} \quad (6.1.25a)$$

$$D_{J_s}^2[(L_e {}^sR_s S_{esJ})^2\psi_{\Sigma s} + (L_t {}^sR_e S_{tJ})^2\psi_{\Sigma e} + (L_e {}^dR_s S_{esJ} - L_t {}^dR_e S_{tJ})^2\psi_{\Sigma d}] = {}^Jk_{\min} \quad (6.1.25b)$$

$$\begin{aligned}D_{J_s}^2[(L_e {}^sR_s C_{esJ})(L_e {}^sR_s S_{esJ})\psi_{\Sigma s} + (L_t {}^sR_e C_{tJ})(L_t {}^sR_e S_{tJ})\psi_{\Sigma e} \\ + (L_e {}^sR_s C_{esJ} - L_t {}^sR_e C_{tJ})(L_e {}^sR_s S_{esJ} - L_t {}^sR_e S_{tJ})\psi_{\Sigma d}] = 0\end{aligned} \quad (6.1.25c)$$

The coefficients of $\psi_{\Sigma s}$ and $\psi_{\Sigma e}$ in (6.1.25) reveal an interesting pattern. In (6.1.25a), $L_e \cos(\theta_{es} - {}^J\theta_k)$ and $L_t \cos(\theta_t - {}^J\theta_k)$ are the projection of L_e and L_t , respectively, onto the direction of ${}^J\theta_k$, i.e., the minor axis of the ${}^J\mathbf{K}$ ellipse, on which the stiffness is ${}^Jk_{\max}$. Similarly, in (6.1.25b), $L_e \sin(\theta_{es} - {}^J\theta_k)$ and $L_t \sin(\theta_t - {}^J\theta_k)$ are the projection of L_e and L_t perpendicular to the direction of ${}^J\theta_k$, i.e., the major axis of the ellipse, on which the stiffness is ${}^Jk_{\min}$. In this context, we can interpret these coefficients as projections of

“weighting factors” for each of the stiffnesses. We can then say that ψ_{zs} and ψ_{ze} “act” in the directions of θ_{es} and θ_i , respectively. (These directions correspond to those that M_s and M_e are said to “act” in Appendix A.)

The above interpretation implies that ψ_{zd} may also act in a specific direction as well. For the moment we will suppose this is true and define this unknown direction as θ_d . To match the form of the coefficients for ψ_{zs} and ψ_{ze} in (6.1.25) we must also define a corresponding length L_d and joint radius R_d . We then use these values to define the ψ_{zd} coefficients in (6.1.25a) and (6.1.25b) as $(L_d \cos(\theta_d - {}^J\theta_k) R_d)^2$ and $(L_d \sin(\theta_d - {}^J\theta_k) R_d)^2$, respectively. Our new parameter definitions must then satisfy the following relationships:

$$\begin{aligned} L_d R_d \cos(\theta_d - {}^J\theta_k) &= L_e {}^dR_s \cos(\theta_{es} - {}^J\theta_k) - L_i {}^dR_e \cos(\theta_i - {}^J\theta_k) \\ L_d R_d \sin(\theta_d - {}^J\theta_k) &= L_e {}^dR_s \sin(\theta_{es} - {}^J\theta_k) - L_i {}^dR_e \sin(\theta_i - {}^J\theta_k) \end{aligned} \quad (6.1.26)$$

By using the trigonometric expansion rules for multiple angles, it is straightforward to show that ${}^J\theta_k$ can be eliminated from (6.1.26) and the two equations simplify to

$$\begin{aligned} L_d R_d \cos\theta_d &= L_e {}^dR_s \cos\theta_{es} - L_i {}^dR_e \cos\theta_i \\ L_d R_d \sin\theta_d &= L_e {}^dR_s \sin\theta_{es} - L_i {}^dR_e \sin\theta_i \end{aligned} \quad (6.1.27)$$

The sums of the squares of these two equations results in

$$(L_d R_d)^2 = (L_t {}^dR_e)^2 + (L_e {}^dR_s)^2 - 2(L_t {}^dR_e)(L_e {}^dR_s) \cos(\theta_{es} - \theta_t) \quad (6.1.28)$$

Dividing the equations in (6.1.27) yields

$$\tan \theta_d = \frac{L_e {}^dR_s \sin \theta_{es} - L_t {}^dR_e \sin \theta_t}{L_e {}^dR_s \cos \theta_{es} - L_t {}^dR_e \cos \theta_t} \quad (6.1.29)$$

Physically, equation (6.1.28) and (6.1.29) interpret as shown in Figure 6.1.2(a), where (6.1.28) is the law of cosines for the included angle.

We can combine (6.1.28) and (6.1.29) into a single equation by interpreting the link lengths as vectors,

$$\begin{aligned} R_d L_d &= {}^dR_e L_T - {}^dR_e L_e = {}^dR_e (L_s + L_e) - {}^dR_e L_e \\ &= {}^dR_e L_s + ({}^dR_e - {}^dR_e) L_e \end{aligned} \quad (6.1.30)$$

The simplification in (6.1.30) provides us with another physical interpretation as shown in Figure 6.1.2(b).

Figure 6.1.2: Interpretations of $R_d L_d$ and θ_d .

One property of the new double-joint parameters that appears from the above equations is that R_d and L_d cannot be separated, and therefore no physical interpretation can be explicitly assigned to either parameter. However, a general interpretation of these parameters can be seen in Figure 6.1.3. The double-joint springs can be interpreted as single-joint springs with radius R_d at the shoulder, and acting about a virtual joint that is located a distance L_d along the direction θ_d from the endpoint, as seen in the figure. The exact values of L_d and R_d are arbitrary, with the only restriction that $L_d R_d$ satisfies the conditions in (6.1.28).

In the special case where ${}^dR_e = {}^dR_s$, equation (6.1.30) simplifies to $R_d L_d = {}^dR_s L_s$. We could then choose $R_d = {}^dR_s$, and hence $L_d = L_s$, though this is only one of the infinite number of solutions. The virtual linkage system in Figure 6.1.3 would then become a parallelogram, and $\theta_s = \theta_d$.

We can now see that our hunch that $\psi_{\Sigma d}$ can be associated with a direction is valid. If we apply these parameters to (6.1.25) we get the stiffness relationships

$$D_{J_s}^{-2}[(L_e {}^sR_s C_{esj})^2 \psi_{\Sigma s} + (L_t {}^tR_e C_{tj})^2 \psi_{\Sigma e} + (L_d R_d C_{dj})^2 \psi_{\Sigma d}] = {}^Jk_{\max} \quad (6.1.31a)$$

$$D_{J_s}^{-2}[(L_e {}^sR_s S_{esj})^2 \psi_{\Sigma s} + (L_t {}^tR_e S_{tj})^2 \psi_{\Sigma e} + (L_d R_d S_{dj})^2 \psi_{\Sigma d}] = {}^Jk_{\min} \quad (6.1.31b)$$

$$D_{J_s}^{-2}[(L_e {}^sR_s C_{esj})(L_e {}^sR_s S_{esj})\psi_{\Sigma s} + (L_t {}^tR_e C_{tj})(L_t {}^tR_e S_{tj})\psi_{\Sigma e} + (L_d R_d C_{dj})(L_d R_d S_{dj})\psi_{\Sigma d}] = 0 \quad (6.1.31c)$$

Figure 6.1.3: Double-joint springs interpreted as single-joint springs acting about a virtual joint.

where $C_{dj} = \cos(\theta_d - {}^j\theta_k)$

$$S_{dj} = \sin(\theta_d - {}^j\theta_k)$$

We define the stiffness projection vectors

$$\begin{aligned} \mathbf{S}_{\max} &= \mathbf{D}_{j-} \begin{bmatrix} \pm L_e {}^s\mathbf{R}_s \cos(\theta_{es} - {}^j\theta_k) \\ \pm L_t {}^s\mathbf{R}_e \cos(\theta_t - {}^j\theta_k) \\ \pm L_d \mathbf{R}_d \cos(\theta_d - {}^j\theta_k) \end{bmatrix} \\ \mathbf{S}_{\min} &= \mathbf{D}_{j-} \begin{bmatrix} \pm L_e {}^s\mathbf{R}_s \sin(\theta_{es} - {}^j\theta_k) \\ \pm L_t {}^s\mathbf{R}_e \sin(\theta_t - {}^j\theta_k) \\ \pm L_d \mathbf{R}_d \sin(\theta_d - {}^j\theta_k) \end{bmatrix} \end{aligned} \quad (6.1.32)$$

and the spring stiffness matrix as

$$\boldsymbol{\Psi}_{\Sigma} = \begin{bmatrix} \boldsymbol{\Psi}_{\Sigma s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_{\Sigma e} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Psi}_{\Sigma d} \end{bmatrix} \quad (6.1.33)$$

Using these definitions, we can summarize the equations in (6.1.33) as

$$\begin{aligned} \mathbf{S}_{\max}^T \boldsymbol{\Psi}_{\Sigma} \mathbf{S}_{\max} &= k_{\max} \\ \mathbf{S}_{\min}^T \boldsymbol{\Psi}_{\Sigma} \mathbf{S}_{\min} &= k_{\min} \\ \mathbf{S}_{\max}^T \boldsymbol{\Psi}_{\Sigma} \mathbf{S}_{\min} &= \mathbf{S}_{\min}^T \boldsymbol{\Psi}_{\Sigma} \mathbf{S}_{\max} = 0 \end{aligned} \quad (6.1.34)$$

Figure 6.1.4 shows the physical interpretations of these spring stiffness weights

and their projections. The single-joint shoulder springs, ψ_{Σ_s} , carry a weight of $L_e'R_s$ and act in the θ_{es} direction. The single-joint elbow springs, ψ_{Σ_e} , carry a weight of $L_l'R_e$ and act in the θ_l direction. The double-joint springs, ψ_{Σ_d} , carry a weight of $L_d'R_d$ and act in the θ_d direction. The weighting factors project onto the major and minor axes and, once squared and scaled by $\det^2(\mathbf{J}^{-1})$, provide the contribution of each spring stiffness to the stiffnesses on the ellipse axes. Alternately, we could include $\det(\mathbf{J}^{-1})$ into all of the weighting factors before squaring. The fact that the third equation in (6.1.34) is zero is consistent with the projection of vectors onto orthogonal axes. We will examine and simplify these projection relationships further in Section 6.2 once we understand the interpretation.

To see that the spring stiffness “act” in the specified directions, let us consider the case when two of the spring stiffness sets are zero. For this example, we will use $\psi_{\Sigma_e} = \psi_{\Sigma_d} = 0$. By equation (6.1.31c), we must have

$$\sin(\theta_{es} - \theta_k) \cos(\theta_{es} - \theta_k) = 0$$

Figure 6.1.4: Projection of stiffness weights to ellipse axes. These weighting factors are projected onto the major and minor axes of the ellipse. These projections are squared and applied to the corresponding actuator stiffness marked in the figure.

If $\cos(\theta_{es} - {}^J\theta_k) = 0$, then ${}^Jk_{\max} = 0$ and ${}^Jk_{\min} = D_{j-}^2(L_e {}^sR_s)^2\psi_{\Sigma_s}$ from (6.1.31a) and (6.1.31b).

But ${}^Jk_{\min} < {}^Jk_{\max}$ by definition, otherwise their roles are reversed. Therefore we must have $\sin(\theta_{es} - {}^J\theta_k) = 0$, resulting in ${}^Jk_{\min} = 0$ and ${}^Jk_{\max} = D_{j-}^2(L_e {}^sR_s)^2\psi_{\Sigma_s}$. Recall from equation (6.1.12) that the relationship between the axes lengths and axes stiffnesses, when applied to the ${}^J\mathbf{K}$ matrix, is

$${}^Jr_{\max} = \sqrt{\frac{2E_p}{{}^Jk_{\min}}}; \quad {}^Jr_{\min} = \sqrt{\frac{2E_p}{{}^Jk_{\max}}}$$

The ellipse in this example has a finite minor axis, but the major axis is infinitely long in a direction perpendicular to θ_{es} (since the sine is zero). Since the axes are inversely proportional to the square root of the stiffnesses, ψ_{Σ_s} has the least effect in this direction. That is, the endpoint can hypothetically be displaced an infinite distance along a line perpendicular to the ${}^J\theta_k$ direction (${}^J\theta_k = \theta_{es}$ in this case) with no resulting force or energy storage. However, ψ_{Σ_s} has the most effect in the direction of the minor axis, θ_{es} in this case. Therefore we can say that ψ_{Σ_s} , representing the effect of the single-joint shoulder springs, acts in the θ_{es} direction.

Similar examples can show that ψ_{Σ_e} acts in the θ_i direction and ψ_{Σ_d} acts in the θ_d direction. Increasing a specific spring stiffness has the effect of increasing the endpoint stiffness (decreasing the ellipse radius) in the direction in which it acts. The increase in

endpoint stiffness is proportional to the increase in the spring stiffness, by (6.1.31), but the decrease in isopotential ellipse radius is proportional to its square root.

The isopotential ellipse definition also shows this separation of spring stiffness directions. Consider the basic ellipse equation given in (6.1.1),

$$\begin{aligned} E_p &= \frac{1}{2} \delta \mathbf{p}^T \mathbf{K} \delta \mathbf{p} = \frac{1}{2} \delta \mathbf{p}^T ({}^J\mathbf{K} + {}^G\mathbf{K}) \delta \mathbf{p} = \frac{1}{2} \delta \mathbf{p}^T {}^J\mathbf{K} \delta \mathbf{p} + \frac{1}{2} \delta \mathbf{p}^T {}^G\mathbf{K} \delta \mathbf{p} \\ &= {}^J E_p + {}^G E_p \end{aligned} \quad (6.1.35)$$

where we define

$$\begin{aligned} {}^J E_p &= \frac{1}{2} \delta \mathbf{p}^T {}^J\mathbf{K} \delta \mathbf{p} \\ {}^G E_p &= \frac{1}{2} \delta \mathbf{p}^T {}^G\mathbf{K} \delta \mathbf{p} \end{aligned} \quad (6.1.36)$$

We are only considering the endpoint spring stiffness ${}^J\mathbf{K}$ in this section, and so we define an endpoint spring stiffness ellipse by substituting it for $\mathbf{K}$ (as has been assumed throughout this section). Equation (5.3.16) showed us that ${}^J\mathbf{K} = \mathbf{J}^T \Psi \mathbf{J}^{-1}$ (which is true in any orthogonal coordinate system, as seen by the rotation through ${}^J\theta_k$ in (6.1.19)). If we break up Ψ into its spring pair components from (5.2.11) and (5.2.12) we can write the endpoint spring stiffness ellipse as

$$\begin{aligned} {}^J E_p &= \frac{1}{2} d\mathbf{p}^T \mathbf{J}^T \Psi \mathbf{J}^{-1} d\mathbf{p} = \frac{1}{2} d\mathbf{p}^T \mathbf{J}^T (\Psi_c + \Psi_s + \Psi_d) \mathbf{J}^{-1} d\mathbf{p} \\ &= \frac{1}{2} d\mathbf{p}^T \mathbf{J}^T \Psi_c \mathbf{J}^{-1} d\mathbf{p} + \frac{1}{2} d\mathbf{p}^T \mathbf{J}^T \Psi_s \mathbf{J}^{-1} d\mathbf{p} + \frac{1}{2} d\mathbf{p}^T \mathbf{J}^T \Psi_d \mathbf{J}^{-1} d\mathbf{p} \end{aligned} \quad (6.1.37)$$

where

$$\begin{aligned}\Psi_s &= \begin{bmatrix} {}^sR_s^2 \psi_{\Sigma s} & 0 \\ 0 & 0 \end{bmatrix} \\ \Psi_e &= \begin{bmatrix} 0 & 0 \\ 0 & {}^sR_e^2 \psi_{\Sigma e} \end{bmatrix} \\ \Psi_d &= \begin{bmatrix} {}^dR_s^2 \psi_{\Sigma d} & {}^dR_s {}^dR_e \psi_{\Sigma d} \\ {}^dR_s {}^dR_e \psi_{\Sigma d} & {}^dR_e^2 \psi_{\Sigma d} \end{bmatrix}\end{aligned}\tag{6.1.38}$$

When plotted, the three ellipses defined in (6.1.37) have the infinite major axis perpendicular to the directions in which the stiffnesses act, as described above.

The spring-to-endpoint stiffness relationships in (6.1.31) can be used for both analytical and control applications. Given known stiffness values of $\psi_{\Sigma s}$, $\psi_{\Sigma e}$, and $\psi_{\Sigma d}$, the shape of the spring stiffness ellipse can be readily found by substituting these values in (6.1.31a,b) and using the resulting k_{maj} and k_{min} in (6.1.12) to find the major and minor axes. The solution to (6.1.31c) provides the orientation of the ellipse, ${}^1\theta_k$. We can separate out ${}^1\theta_k$ by expanding the trigonometric functions using the multiple angle identities. This separation of variable simplifies to the following quadratic equation:

$$\tan^2 {}^1\theta_k - \beta \tan {}^1\theta_k - 1 = 0$$

which we solve to give

$${}^J\theta_k = \tan^{-1}\left(\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 + 4}\right) \quad (6.1.39)$$

where

$$\beta = \frac{(L_e {}^sR_s)^2(S_{es}^2 - C_{es}^2)\psi_{\Sigma s} + (L_t {}^sR_e)^2(S_t^2 - C_t^2)\psi_{\Sigma e} + (L_d R_d)^2(S_d^2 - C_d^2)\psi_{\Sigma d}}{(L_e {}^sR_s)^2S_{es}C_{es}\psi_{\Sigma s} + (L_t {}^sR_e)^2S_tC_t\psi_{\Sigma e} + (L_d R_d)^2S_dC_d\psi_{\Sigma d}} \quad (6.1.40)$$

Alternately, if one wishes to create a specific stiffness ellipse there is a unique set of $\psi_{\Sigma s}$, $\psi_{\Sigma e}$, and $\psi_{\Sigma d}$, that satisfies (6.1.31) since there are three independent equations in three unknowns. We can solve for these values by inverting the three equations to get

$$\psi_{\Sigma s} = \left(\frac{L_s L_e S_e}{L_e {}^sR_s}\right)^2 \frac{S_{dt}(S_{tJ}S_{dJ} {}^Jk_{\max} + C_{tJ}C_{dJ} {}^Jk_{\min})}{S_{dJ}S_{tJ}S_{dt} + S_{esJ}S_{dJ}S_{esd} + S_{tJ}S_{esJ}S_{tes}} \quad (6.1.41a)$$

$$\psi_{\Sigma e} = \left(\frac{L_s L_e S_e}{L_t {}^sR_e}\right)^2 \frac{S_{esd}(S_{esJ}S_{dJ} {}^Jk_{\max} + C_{esJ}C_{dJ} {}^Jk_{\min})}{S_{dJ}S_{tJ}S_{dt} + S_{esJ}S_{dJ}S_{esd} + S_{tJ}S_{esJ}S_{tes}} \quad (6.1.41b)$$

$$\psi_{\Sigma d} = \left(\frac{L_s L_e S_e}{L_d R_d}\right)^2 \frac{S_{tes}(S_{esJ}S_{tJ} {}^Jk_{\max} + C_{esJ}C_{tJ} {}^Jk_{\min})}{S_{dJ}S_{tJ}S_{dt} + S_{esJ}S_{dJ}S_{esd} + S_{tJ}S_{esJ}S_{tes}} \quad (6.1.41c)$$

where we use the notation

$$S_{ij} = \sin(\theta_i - \theta_j), \quad C_{ij} = \cos(\theta_i - \theta_j), \quad \{i, j = e, es, t, d, J\}$$

6.1.3 Geometric Stiffness

In the last section we examined the behaviour and control of the endpoint spring stiffness, $^J\mathbf{K}$. However, (5.3.16) told us that the entire endpoint stiffness includes the rate at which force changes due to a change in the geometric position of the arm, which we denoted $^G\mathbf{K}$.

Recall from (5.3.21) that the $^G\mathbf{K}$ matrix elements are

$$^Gk_{xx} = \frac{1}{L_s^2 L_e^2 S_e^3} \{ (L_e^2 C_{es}^2 C_e + L_s L_e C_s^2) M_s - (L_e^2 C_e^2 C_e + L_s L_e S_e^2) M_e \}$$

$$^Gk_{xy} = \frac{1}{L_s^2 L_e^2 S_e^3} \{ (L_e^2 S_{es} C_{es} C_e + L_s L_e S_s C_s) M_s - L_e^2 C_e S_e C_e M_e \}$$

$$^Gk_{yx} = ^Gk_{xy}$$

$$^Gk_{yy} = \frac{1}{L_s^2 L_e^2 S_e^3} \{ (L_e^2 S_{es}^2 C_e + L_s L_e S_s^2) M_s - (L_e^2 S_e^2 C_e + L_s L_e S_e^2) M_e \}$$

We can see from these elements that the geometric stiffness depends on the joint moments rather than the joint stiffnesses. The geometric stiffness is zero when the arm is at its virtual position, since the joint moments are zero at this location by definition. Presumably, the geometric stiffness is also negligible for small displacements about the virtual position because the moments are small. This assumption and the degree of displacement for which the assumption is valid are examined in Section 6.3. By contrast, the joint moments become large for large displacements from the virtual position.

Presumably at some point the endpoint spring stiffness $^J\mathbf{K}$ becomes, or can become, negligible in comparison. In this section we examine the behaviour of the geometric stiffness by itself.

Like the endpoint spring stiffness in Section 6.1.2, we can define a geometric stiffness ellipse from (6.1.36),

$$^G\mathbf{E}_p = \frac{1}{2} d\mathbf{p}^T {}^G\mathbf{K} d\mathbf{p}$$

Also like the endpoint spring stiffness, we can rotate the $^G\mathbf{K}$ matrix to the coordinate system defined by the major and minor axes of this ellipse, given in (6.1.16),

$$^G\mathbf{K}_k = {}^G\mathbf{R}_k^{-1} {}^G\mathbf{K}_{xy} {}^G\mathbf{R}_k$$

Equating the elements on both sides leads to the following three equations:

$$D_{J_1}^2/S_e [(L_e^2 C_e C_{esG}^2 + L_s L_e C_{sG}^2) M_s + (L_t^2 C_e C_{tG}^2 - L_s L_e S_e) M_e] = {}^Gk_{maj} \quad (6.1.42a)$$

$$D_{J_1}^2/S_e [(L_e^2 C_e S_{esG}^2 + L_s L_e S_{sG}^2) M_s + (L_t^2 C_e S_{tG}^2 - L_s L_e S_e) M_e] = {}^Gk_{min} \quad (6.1.42b)$$

$$D_{J_1}^2/S_e [(L_e^2 C_e S_{esG} C_{esG} + L_s L_e S_{sG} C_{sG}) M_s + (L_t^2 C_e S_{tG} C_{tG}) M_e] = 0 \quad (6.1.42c)$$

Recall from (5.2.7) that the joint moments are

$$\mathbf{M} = \begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} {}^s\mathbf{R}_s(\phi_{sx}\{\delta_{sx}\} - \phi_{sf}\{\delta_{sf}\}) + {}^d\mathbf{R}_s(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \\ {}^s\mathbf{R}_e(\phi_{ex}\{\delta_{ex}\} - \phi_{ef}\{\delta_{ef}\}) + {}^d\mathbf{R}_e(\phi_{dx}\{\delta_{dx}\} - \phi_{df}\{\delta_{df}\}) \end{bmatrix}$$

Using the shorthand notation

$$\phi_{\Delta i} = \phi_{ix} \{\delta_{ix}\} - \phi_{if} \{\delta_{if}\}, \quad (i = s, e, d) \quad (6.1.43)$$

we can separate the spring pairs in (6.1.42) by substituting for the moments to get

$$D_{J_1}^2/S_e [(L_e^2 C_e C_{esG}^2 + L_s L_e C_{sG}^2)^s R_s \phi_{\Delta s} + (L_t^2 C_e C_{tG}^2 - L_s L_e S_e)^s R_e \phi_{\Delta e} + \{(L_e^2 C_e C_{esG}^2 + L_s L_e C_{sG}^2)^d R_s + (L_t^2 C_e C_{tG}^2 - L_s L_e S_e)^d R_e\} \phi_{\Delta d}] = {}^G k_{\max} \quad (6.1.44a)$$

$$D_{J_1}^2/S_e [(L_e^2 C_e S_{esG}^2 + L_s L_e S_{sG}^2)^s R_s \phi_{\Delta s} + (L_t^2 C_e S_{tG}^2 - L_s L_e S_e)^s R_e \phi_{\Delta e} + \{(L_e^2 C_e S_{esG}^2 + L_s L_e S_{sG}^2)^d R_s + (L_t^2 C_e S_{tG}^2 - L_s L_e S_e)^d R_e\} \phi_{\Delta d}] = {}^G k_{\min} \quad (6.1.44b)$$

$$D_{J_1}^2/S_e [(L_e^2 C_e S_{esG} C_{esG} + L_s L_e S_{sG} C_{sG})^s R_s \phi_{\Delta s} + (L_t^2 C_e S_{tG} C_{tG})^s R_e \phi_{\Delta e} + \{(L_e^2 C_e S_{esG} C_{esG} + L_s L_e S_{sG} C_{sG})^d R_s + (L_t^2 C_e S_{tG} C_{tG})^d R_e\} \phi_{\Delta d}] = 0 \quad (6.1.44c)$$

Unfortunately, the coefficients of $\phi_{\Delta s}$, $\phi_{\Delta e}$, and $\phi_{\Delta d}$ do not appear in the form of projections onto the ellipse axes as they were in (6.1.25) for the ${}^J \mathbf{K}$ matrix. Some of the terms appear to have some projection context, such as $L_e C_{esG}$ and $L_e S_{esG}$ which are projections of L_e onto the ellipse axes. However, these terms are generally projected again by $\cos \theta_e$, after they are squared. Other terms don't make physical sense as projections.

We can use the relationships in (6.1.44) to analyse or control the geometric stiffness ellipse. Equations (6.1.44a,b) provide the stiffness on the major and minor axes of the ellipse. Substitution of these values into (6.1.12) yields the actual axes lengths. The ellipse orientation angle, ${}^G \theta_k$, can be found from (6.1.44c). Using the trigonometric identities for multiple angles, the ${}^G \theta_k$ terms separate out to give the quadratic

$$\tan^2 {}^G\theta_k - \gamma \tan {}^G\theta_k - 1 = 0$$

which solves to

$${}^G\theta_k = \tan^{-1}\left(\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{\gamma^2 + 4}\right) \quad (6.1.45)$$

where

$$\gamma = \frac{L_e^2 C_e (S_{es}^2 - C_{es}^2) M_s + L_s L_e (S_s^2 - C_s^2) M_s + L_t^2 C_e (S_t^2 - C_t^2) M_e}{L_e^2 C_e S_{es} C_{es} M_s + L_s L_e S_s C_s M_s + L_t^2 C_e S_t C_t M_e} \quad (6.1.46)$$

While (6.1.44) can be inverted to solve for the forces to create a desired geometric stiffness ellipse, we do not normally have the freedom to arbitrarily select these force differences because they also affect the joint moments, which in turn affect the endpoint force. However, when we are not restricted to specific joint moments, and consequently a specific endpoint force, we can choose these force differences to specify a desired geometric stiffness ellipse. This case is most likely rare, since the direction and magnitude of the endpoint force is often important. The only case where these concerns are irrelevant to the movement of the arm is an environmental contact that is rigid in all directions. However, stiffness has little meaning in such a case.

The above arguments result in the conclusion that there is little we can do to control the geometric stiffness ellipse. It is essentially a function of the endpoint force and position of the arm, both which are often specified by the task being performed. On

the other hand, humans perform many tasks that allow them to use a variety of arm orientations relative to the body. This choice does allow us some control over the geometric stiffness ellipse. We may base our orientation selection in part on the geometric stiffness, along with other factors such as the minimum effort as discussed in Appendix A. Chapter 8 discusses some strategies for choosing orientations in the workspace to optimize the effects of the geometric stiffness.

In Section 6.1 we showed:

- the traditional stiffness ellipse representation using energy isopotentials,
- the relationship between the stiffness matrix and shape and orientation of the ellipse,
- separation of the endpoint spring stiffness isopotential and geometric stiffness isopotential from the total endpoint isopotential ellipse,
- the relationship of joint stiffnesses to the endpoint spring stiffness isopotential axes,
- the projection of actuator stiffnesses onto the endpoint spring stiffness isopotential axes from both analytical and control strategy perspectives, and,
- the relationship between joint moments and actuator forces to the geometric stiffness isopotential.

6.2 Vector Spaces

Section 6.1 introduced the relationships between actuator stiffnesses and forces with the endpoint stiffness. In this section we will generalize these relationships while simplifying and understanding their derivation.

6.2.1 Joint Space to Endpoint Space

The relationship between endpoint force and joint moments was shown in (5.3.3) to be

$$\mathbf{M} = \mathbf{J}^T \mathbf{F}$$

where $\mathbf{J}$ is the Jacobian matrix from (5.3.4). Since the Jacobian is a 2×2 matrix it is full rank and is therefore invertible. The inverse of (5.3.4) is given in (5.3.5) as

$$\mathbf{F} = \mathbf{J}^{-T} \mathbf{M}$$

We define here the vector $\mathbf{F}$ as belonging to the *endpoint force space*, and the vector $\mathbf{M}$ to be the *joint moment space*. By the nature of matrix multiplication, $\mathbf{F}$ must lie in the column space of $\mathbf{J}$ (row space of $\mathbf{J}^T$) and $\mathbf{M}$ must lie in the row space of $\mathbf{J}$ (column space of $\mathbf{J}^T$). When $\mathbf{J}$ is full rank there is a direct mapping in both directions between $\mathbf{F}$ and $\mathbf{M}$, meaning that for every vector $\mathbf{F}$ there is a corresponding unique vector $\mathbf{M}$, and vice-versa, for a given position of the arm.

Figure 6.2.1. shows this relationship. Figure 6.2.1(a) shows the coordinate transformations between **F** and **M**. In (b) a matrix representation is shown where the transposed Jacobian transforms a vector in the column space to the row space of **J** and the inverse mapping reverses this process. Not much is learned from this transformation because it is a direct relationship. However, the perspective shown here will become important when examining other spaces later.

Now consider displacement of the joints and endpoint. From (5.3.10) we can see that

$$d\mathbf{p} = \mathbf{J} d\Delta$$

The inverse equation, as seen in (5.3.11), is

$$d\Delta = \mathbf{J}^{-1} d\mathbf{p}$$

We define the vector **dp** as belonging to the *endpoint displacement space* and vector **dΔ** to the *joint displacement space*. As with forces and moments, the displacements have a one-to-one relationship. For a small displacement of the joints there is a unique corresponding displacement of the endpoint, and vice versa.

Figure 6.2.2 shows the relationships. The figure is almost identical to Figure 6.1.1 except that the spaces are now displacement spaces, and the transformation takes place

directly through the Jacobian and its inverse rather than their transposes. The same Jacobian appears in both relationship (force and displacement) as a result of energy conservation. Stored energy should appear the same whether viewed in endpoint or joint coordinates,

$$dE = \mathbf{F}^T d\mathbf{p} = (\mathbf{J}^T \mathbf{M})^T (\mathbf{J} d\boldsymbol{\theta}) = \mathbf{M}^T (\mathbf{J}^{-T} \mathbf{J}) d\boldsymbol{\theta} = \mathbf{M}^T d\boldsymbol{\theta}$$

(a)

(b)

Figure 6.2.1: Projection between endpoint force space and joint moment space. (a) Coordinate representation: The two-dimensional endpoint forces map directly onto the two-dimensional joint moment space through J^T . (b) Matrix representation: Endpoint force vectors are mapped from the column space of the Jacobian to joint moments in its row space. J^{-T} does the reverse.

(a)

(b)

Figure 6.2.2: Projection between endpoint displacement space and joint displacement space. (a) Coordinate representation: The two-dimensional joint displacements map directly onto the two-dimensional endpoint displacement space through J . (b) Matrix representation: Joint displacement vectors are mapped from the row space of the Jacobian to endpoint displacements in its column space. J^{-1} does the reverse.

6.2.2 Actuator Space to Joint Space

Recall from (5.2.7) that the joint moments are

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_s \\ \mathbf{M}_e \end{bmatrix} = \begin{bmatrix} {}^s\mathbf{R}_s(\phi_{sx} - \phi_{sf}) + {}^d\mathbf{R}_s(\phi_{dx} - \phi_{df}) \\ {}^s\mathbf{R}_e(\phi_{ex} - \phi_{ef}) + {}^d\mathbf{R}_e(\phi_{dx} - \phi_{df}) \end{bmatrix}$$

We can re-write this equation as

$$\mathbf{M} = \mathbf{R}^T \phi_\Delta \quad (6.2.1)$$

where

$$\mathbf{R} = \begin{bmatrix} {}^s\mathbf{R}_s & 0 \\ 0 & {}^s\mathbf{R}_e \\ {}^d\mathbf{R}_s & {}^d\mathbf{R}_e \end{bmatrix} \quad (6.2.2)$$

$$\phi_\Delta = \begin{bmatrix} \phi_{\Delta s} \\ \phi_{\Delta e} \\ \phi_{\Delta d} \end{bmatrix} = \begin{bmatrix} \phi_{sx} - \phi_{sf} \\ \phi_{ex} - \phi_{ef} \\ \phi_{dx} - \phi_{df} \end{bmatrix} \quad (6.2.3)$$

We will define vector ϕ_Δ to be in the *actuator force space*, and $\mathbf{M}$ is in the joint moment space as before. $\mathbf{R}$ is not invertible because it is a non-square matrix. However, we can solve the inverse equation using the *pseudo-inverse* of $\mathbf{R}$, denoted $\mathbf{R}^+$, which gives us the actuator forces as

$$\phi_{\Delta c} = \mathbf{R}^{+T} \mathbf{M} \quad (6.2.4)$$

where $\phi_{\Delta c}$ denotes that ϕ_{Δ} is forced to be in the column space of $\mathbf{R}$ (row space of $\mathbf{R}^T$) in this equation. $\mathbf{R}$ has rank 2, with three rows and two columns. It may therefore have full column rank but isn't full row rank. This leaves $\mathbf{R}$ with a one-dimensional *left-nullspace* (nullspace of $\mathbf{R}^T$). Because it is full row rank, the pseudo-inverse can be written as

$$\begin{aligned}\mathbf{R}^+ &= (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T \\ &= \frac{1}{\det(\mathbf{R}^T \mathbf{R})} \begin{bmatrix} {}^s\mathbf{R}_s({}^s\mathbf{R}_e^2 + {}^d\mathbf{R}_e^2) & -{}^s\mathbf{R}_e^d {}^s\mathbf{R}_s^d \mathbf{R}_e & {}^s\mathbf{R}_e^2 {}^d\mathbf{R}_s \\ -{}^s\mathbf{R}_s^d {}^s\mathbf{R}_e^d \mathbf{R}_e & {}^s\mathbf{R}_e({}^s\mathbf{R}_s^2 + {}^d\mathbf{R}_s^2) & {}^s\mathbf{R}_s^2 {}^d\mathbf{R}_e \end{bmatrix}\end{aligned}\quad (6.2.5)$$

where $\det(\mathbf{R}^T \mathbf{R}) = {}^s\mathbf{R}_s^2 {}^s\mathbf{R}_e^2 + {}^s\mathbf{R}_s^2 {}^d\mathbf{R}_e^2 + {}^s\mathbf{R}_e^2 {}^d\mathbf{R}_s^2$

The left-nullspace of $\mathbf{R}$, denote as $\phi_{\Delta in}$, is the set of all values of ϕ_{Δ} for which

$$\mathbf{R}^T \phi_{\Delta in} = \mathbf{0} \quad (6.2.6)$$

Solving (6.2.6),

$$\phi_{\Delta in} = a \begin{bmatrix} {}^s\mathbf{R}_e^d \mathbf{R}_s \\ {}^s\mathbf{R}_s^d \mathbf{R}_e \\ -{}^s\mathbf{R}_s {}^s\mathbf{R}_e \end{bmatrix} \quad (6.2.7)$$

where a is arbitrary. We can re-write the moment equation in (6.2.1) as

$$\mathbf{M} = \mathbf{R}^T (\phi_{\Delta c} + \phi_{\Delta in}) = \mathbf{R}^T \phi_{\Delta c} \quad (6.2.8)$$

since $\mathbf{R}^T \phi_{\Delta \text{in}} = \mathbf{0}$ by definition in (6.2.6). The left-nullspace of $\mathbf{R}$ is the component of ϕ_{Δ} that does not affect the joint moments. In other words, changing ϕ_{Δ} by adding any values of actuator forces in a ratio that is a multiple of (6.2.7) *does not change the net joint moments*. These are exactly the internal moment changes discussed in Section 5.2.3, where changes in moments from opposing spring pairs cancel each other out. Equation (6.2.8) shows that these changes have no effect on the moment. The pseudo-inverse in (6.2.4) then removes any of these internal changes to give the necessary values of ϕ_{Δ} to affect the joint moments. This is equivalent to the conditions in (5.2.25) and (5.2.26) where these internal moment cancellations are removed.

Figure 6.2.3 shows the meaning of these equations. In (a), showing the coordinate transformations, the joint moments project onto a two-dimensional subspace of the actuator force space. This must be so because the moment space is only two-dimensional and cannot create extra information to develop an independent third dimension. Any values of ϕ_{Δ} that deviate from this subspace surface only contribute the moments through the orthogonal projection of ϕ_{Δ} onto this surface. The leftover portion is in the left-nullspace and does nothing to the moments. In (b) the interpretation is shown in matrix format. Any value of ϕ_{Δ} can be decomposed into portions from the column space of $\mathbf{R}$ and the left-nullspace of $\mathbf{R}$. The combination has the same projection onto the joint moment space (rowspace of $\mathbf{R}$) as just the portion from the column space. Reversing the

calculation ignores any portion that may be from the left-nullspace, resulting in only the columnspace portion, $\phi_{\Delta c}$.

Now consider the displacements of the actuators (springs) and the joints. From (5.2.29) we have the general actuator displacements as

$$\delta_{sx} = {}^k\sigma_{sx} + {}^sR_s(\Delta_s + {}^s\Delta_{sv})$$

$$\delta_{sf} = {}^k\sigma_{sf} - {}^sR_s(\Delta_s + {}^s\Delta_{sv})$$

$$\delta_{ex} = {}^k\sigma_{ex} + {}^eR_e(\Delta_e + {}^e\Delta_{ev})$$

$$\delta_{ef} = {}^k\sigma_{ef} - {}^eR_e(\Delta_e + {}^e\Delta_{ev})$$

$$\delta_{dx} = {}^k\sigma_{dx} + {}^dR_s(\Delta_s + {}^d\Delta_{sv}) + {}^dR_e(\Delta_e + {}^d\Delta_{ev})$$

$$\delta_{df} = {}^k\sigma_{df} - {}^dR_s(\Delta_s + {}^d\Delta_{sv}) - {}^dR_e(\Delta_e + {}^d\Delta_{ev})$$

For small joint displacements only (all other inputs held constant) we can write these displacements in differential form as

$$d\delta_s = d\delta_{sx} = -d\delta_{sf} = {}^sR_s d\Delta_s$$

$$d\delta_e = d\delta_{ex} = -d\delta_{ef} = {}^eR_e d\Delta_e$$

$$d\delta_d = d\delta_{dx} = -d\delta_{df} = {}^dR_s d\Delta_s + {}^dR_e d\Delta_e$$

Figure 6.2.3: Projection between joint moment space and actuator force space. (a) Coordinate representation: The joint moments project onto a two-dimensional subspace of the actuator force space. Only the orthogonal projection of ϕ_{Δ} onto this surface contributes to M . (b) Matrix representation: ϕ_{Δ} is generally a combination from the column space of $\mathbf{R}$ and the left-nullspace of $\mathbf{R}$. The combination has the same projection onto the joint moment space as just the portion from the column space. Reversing the calculation ignores any portion that may be from the left-nullspace, resulting in only the column space portion, $\phi_{\Delta c}$.

where the terms on the far left are defined here for simplification. This can be written in matrix form as

$$d\delta = \mathbf{R}d\Delta \quad (6.2.9)$$

where

$$d\delta = \begin{bmatrix} d\delta_s \\ d\delta_c \\ d\delta_d \end{bmatrix}; \quad d\Delta = \begin{bmatrix} d\Delta_s \\ d\Delta_c \end{bmatrix}$$

and $\mathbf{R}$ is given by (6.2.2). Once again we can solve the inversion problem with the pseudo-inverse to get

$$d\Delta = \mathbf{R}^+ d\delta \quad (6.2.10)$$

In this case the left-nullspace of $\mathbf{R}$ refers to the motions of the actuators that do not result in joint motions. In other words, if it was possible, the motion of the double-joint springs would cancel the motion of the single-joint springs such that no net joint motion occurs. This would require a joint position associated with each set of springs and the actual joint position was based on their average. However, this does not match with our physical model because (6.2.9) does not include the restriction that the double-joint radii are directly fixed to the single-joint radii at the joints. When a joint moves, both single- *and* double-joint springs move with it. They are not independent of each other with respect to joint motion. So we can define a left-nullspace vector $d\delta_{in}$ (which will be equal to $d\phi_{in}$), but it has no physical meaning for our model.

Figure 6.2.4 shows the meaning of these equations. It is similar to Figure 6.2.3 except it relates the *joint displacement space* to the *actuator displacement space*. Also, the left-nullspace is dotted here to recognize it exists from (6.2.9), but we cannot select any components from it. We can only pick two of the three $d\delta$ elements and the third is solved from the physical constraint,

$$d\delta_i = \frac{{}^sR_i}{{}_dR_i} d\delta_s = \frac{{}^sR_e}{{}_dR_e} d\delta_e \quad (6.2.11)$$

Figure 6.2.4: Projection between joint displacement space and actuator displacement space. (a) Coordinate representation: The joint displacements project onto a two-dimensional subspace of the actuator displacement space. Only elements of $d\delta$ can be selected from this surface because of the physical constraint in (6.2.11). (b) Matrix representation: In this case, $d\delta$ can only be selected from the column space of $\mathbf{R}$. Only interactions between the column space and row space of $\mathbf{R}$ are allowed here.

6.2.3 Actuator Space to Endpoint Space

The previous two sections derived the relationships from actuator space to joint space, and from joint space to endpoint space. We will now combine the transformations to get the relationship from actuator space to endpoint space, and vice versa.

We take the force function as given in Section 6.2.1 (from (5.3.5)) and substitute the moments from (6.2.1),

$$\mathbf{F} = \mathbf{J}^T \mathbf{M} = \mathbf{J}^T (\mathbf{R}^T \phi_\Delta) = (\mathbf{R} \mathbf{J}^{-1})^T \phi_\Delta = \mathbf{P}^T \phi_\Delta \quad (6.2.12)$$

where we define the *endpoint projection matrix* as

$$\mathbf{P} = \mathbf{R} \mathbf{J}^{-1} = \frac{1}{L_s L_e S_e} \begin{bmatrix} L_e {}^s\mathbf{R}_e C_{es} & L_e {}^s\mathbf{R}_e S_{es} \\ -L_e {}^s\mathbf{R}_e C_{et} & -L_e {}^s\mathbf{R}_e S_{et} \\ L_d R_d C_d & L_d R_d S_d \end{bmatrix} \quad (6.2.13)$$

We make use here of the $L_d R_d$ simplification as described in Section 6.1.2. The endpoint projection matrix $\mathbf{P}$ is a 3x2 matrix, as is $\mathbf{R}$. It therefore has similar properties. It is of rank 2 (at most), is generally full row rank, and has a one-dimensional left-nullspace. The reverse equation is again solved by the pseudo-inverse,

$$\phi_\Delta = \mathbf{P}^+ \mathbf{F} \quad (6.2.14)$$

Since $\mathbf{P}$ is full row rank, the pseudo-inverse is

$$\begin{aligned}
\mathbf{P}^* &= (\mathbf{P}^T \mathbf{P}) \mathbf{P}^T = (\mathbf{J}^T \mathbf{R}^T \mathbf{R} \mathbf{J})^{-1} \mathbf{R} \mathbf{J}^{-1} = \mathbf{J} (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{J}^T \mathbf{J}^{-T} \mathbf{R}^T \\
&= \mathbf{J} (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T \\
&= \mathbf{J} \mathbf{R}^* \\
&= \frac{1}{\det(\mathbf{R}^T \mathbf{R})} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \end{bmatrix}
\end{aligned} \tag{6.2.15}$$

where

$$\begin{aligned}
Q_{11} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{11} = {}^s\mathbf{R}_s ({}^d\mathbf{R}_s {}^d\mathbf{R}_e L_e S_{es} - ({}^s\mathbf{R}_e^2 + {}^d\mathbf{R}_e^2) L_1 S_1) \\
Q_{12} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{12} = {}^s\mathbf{R}_e ({}^d\mathbf{R}_s {}^d\mathbf{R}_e L_1 S_1 - ({}^s\mathbf{R}_s^2 + {}^d\mathbf{R}_s^2) L_e S_{es}) \\
Q_{13} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{13} = -{}^s\mathbf{R}_e^2 {}^d\mathbf{R}_s L_1 S_1 - {}^s\mathbf{R}_s^2 {}^d\mathbf{R}_e L_e S_{es} \\
Q_{21} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{21} = -{}^s\mathbf{R}_s ({}^d\mathbf{R}_s {}^d\mathbf{R}_e L_e C_{es} - ({}^s\mathbf{R}_e^2 + {}^d\mathbf{R}_e^2) L_1 C_1) \\
Q_{22} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{22} = -{}^s\mathbf{R}_e ({}^d\mathbf{R}_s {}^d\mathbf{R}_e L_1 C_1 - ({}^s\mathbf{R}_s^2 + {}^d\mathbf{R}_s^2) L_e C_{es}) \\
Q_{23} &= \det(\mathbf{R}^T \mathbf{R}) \mathbf{P}^*_{23} = {}^s\mathbf{R}_e^2 {}^d\mathbf{R}_s L_1 C_1 + {}^s\mathbf{R}_s^2 {}^d\mathbf{R}_e L_e C_{es}
\end{aligned}$$

and $\det(\mathbf{R}^T \mathbf{R})$ is given in (6.2.5).

The left-nullspace of $\mathbf{P}$ is identical to that in (6.2.7). We can show this by considering its definition,

$$\mathbf{P}^T \phi_{\Delta ln} = \mathbf{J}^T \mathbf{R}^T \phi_{\Delta ln} = \mathbf{J}^T(\mathbf{0}) = \mathbf{0}$$

We can also reason this result by physical interpretation. The left-nullspace in (6.2.7) is the actuator forces that produce no joint moments. If there are no joint moments, there is no endpoint force since $\mathbf{F} = \mathbf{J}^T \mathbf{M}$. We can then write the endpoint force as

$$\mathbf{F} = \mathbf{P}^T (\phi_{\Delta c} + \phi_{\Delta in}) = \mathbf{P}^T \phi_{\Delta c} \quad (6.2.16)$$

Figure 6.2.5 shows the meaning of this equation. The interpretations of (a) and (b) are similar to those in Figure 6.2.3, with the exception that the relationship is now between actuator force space and endpoint force space.

Next, we consider the endpoint motions. Recall from Section 6.2.1 that the endpoint displacement, from (5.3.10), is

$$d\mathbf{p} = \mathbf{J} d\Delta = \mathbf{J}(\mathbf{R}^* d\delta) = \mathbf{P}^* d\delta \quad (6.2.17)$$

where the $d\Delta$ substitution comes from (6.2.10), and the final substitution comes from (6.2.15). The reverse equation is then

$$d\delta = \mathbf{P} d\mathbf{p} \quad (6.2.18)$$

As with the joint displacement space, the left-nullspace of $\mathbf{P}$ here refers to displacements of the actuators that do not produce a change in endpoint position. The same restrictions apply as in Section 6.2.2. These motions are not possible because the joint radii are fixed together at the joints, resulting in the restriction of (6.2.11). Figure 6.2.6 shows the relationship.

Figure 6.2.5: Projection between endpoint force space and actuator force space. (a) Coordinate representation: The endpoint forces project onto a two-dimensional subspace of the actuator force space. Only the orthogonal projection of ϕ_{Δ} onto this surface contributes to F . (b) Matrix representation: ϕ_{Δ} is generally a combination from the column space of P and the left-nullspace of P . The combination has the same projection onto the endpoint force space as just the portion from the column space. Reversing the calculation ignores any portion that may be from the left-nullspace, resulting in only the column space portion, $\phi_{\Delta c}$.

Figure 6.2.6: Projection between endpoint displacement space and actuator displacement space. (a) Coordinate representation: The endpoint displacements project onto a two-dimensional subspace of the actuator displacement space. Only elements of $d\delta$ can be selected from this surface because of the physical constraint in (6.2.11). (b) Matrix representation: In this case, $d\delta$ can only be selected from the column space of P . Only interactions between the column space and row space of P are allowed here.

6.2.4 Projection of Stiffness

In the previous three sections we showed the transformations of position and force (including angular position and moments) between three different spaces: a two-dimensional endpoint space, a two dimensional joint space, and a three-dimensional actuator space. In this section we will show how these relationships combine to form stiffnesses in each of these spaces, and how these stiffnesses relate to one another.

First we will relate stiffnesses in joint space and endpoint space. Recall that the differential endpoint force vector, in contact with the environment, was given in (5.3.17) as

$$d\mathbf{F} = \mathbf{K} d\mathbf{p} = ({}^J\mathbf{K} + {}^G\mathbf{K})d\mathbf{p}$$

From (5.3.14) we can also write the differential force as

$$d\mathbf{F} = \mathbf{J}^T d\mathbf{M} + (d\mathbf{J}_s^T \mathbf{M}_s + d\mathbf{J}_e^T \mathbf{M}_e) \quad (6.2.19)$$

Equating these two force representations and using the definitions of ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ from (5.3.16), we can separate the forces to get

$$\mathbf{J}^T d\mathbf{M} = {}^J\mathbf{K} d\mathbf{p} \quad (6.2.20a)$$

$$d\mathbf{J}_s^T \mathbf{M}_s + d\mathbf{J}_e^T \mathbf{M}_e = {}^G\mathbf{K} d\mathbf{p} \quad (6.2.20b)$$

In this section we are only interested in the transformation of stiffness between spaces

rather than geometric effects, so we are only interested in (6.2.20a). In Section 6.2.2, relating endpoint space to joint space, the differential displacements were shown to be related from (5.3.10) as

$$d\mathbf{p} = \mathbf{J} d\Delta$$

From (5.2.37) with constant actuator inputs, we know that

$$d\mathbf{M} = \Psi d\Delta \quad (6.2.21)$$

Substituting these two equations above into (6.2.20a) and simplifying results in

$$\mathbf{J}^T \Psi = {}^J\mathbf{K} \mathbf{J} \quad (6.2.22)$$

Since $\mathbf{J} \mathbf{J}^T = \mathbf{J}^T \mathbf{J}^T = \mathbf{I}_2$ (the two-dimensional identity matrix), we can solve (6.2.22) in both directions such that

$$\Psi = \mathbf{J}^T {}^J\mathbf{K} \mathbf{J} \quad (6.2.23a)$$

$${}^J\mathbf{K} = \mathbf{J}^T \Psi \mathbf{J}^{-1} \quad (6.2.23b)$$

where (6.2.23b) is simply the definition of ${}^J\mathbf{K}$ as given (5.3.16). The relationships in (6.2.23) describe how stiffness is transformed from joint stiffness to endpoint stiffness and vice-versa. Since we used a variation of this definition above in (6.2.19), the result is no surprise. (They are re-derived here as part of the summary of the relationships between the various spaces used in studying the manipulator characteristics.)

Next, we will relate stiffness between joint space and actuator space. From Section 6.2.2, relating joint space to actuator space, (6.2.1) gives us

$$d\mathbf{M} = \mathbf{R}^T d\phi_{\Delta}$$

Substituting this relationship into (6.2.21) results in

$$\Psi d\Delta = \mathbf{R}^T d\phi_{\Delta} \quad (6.2.24)$$

We will now try to find a relationship between $d\phi_{\Delta}$ and $d\delta$. Recall from (5.2.6) that the definition of the actuator stiffness, for each spring, is

$$\psi_{ij} \{\delta_{ij}\} = \frac{d\phi_{ij}}{d\delta_{ij}} \quad (i = s, e, d \text{ and } j = x, f)$$

Also, recall from (6.2.9) that we can write the differential actuator displacements in the form

$$d\delta_i = d\delta_{ix} = -d\delta_{if} \quad (i = s, e, d)$$

We use these two above relationships to get the differential actuator forces, from the definition in (6.2.3),

$$\begin{bmatrix} d(\phi_{\Delta s}) \\ d(\phi_{\Delta e}) \\ d(\phi_{\Delta d}) \end{bmatrix} = \begin{bmatrix} d\phi_{sx} - d\phi_{sf} \\ d\phi_{ex} - d\phi_{ef} \\ d\phi_{dx} - d\phi_{df} \end{bmatrix} = \begin{bmatrix} \psi_{sx} d\delta_{sx} - \psi_{sf} d\delta_{sf} \\ \psi_{ex} d\delta_{ex} - \psi_{ef} d\delta_{ef} \\ \psi_{dx} d\delta_{dx} - \psi_{df} d\delta_{df} \end{bmatrix} = \begin{bmatrix} \psi_{\Sigma s} & 0 & 0 \\ 0 & \psi_{\Sigma e} & 0 \\ 0 & 0 & \psi_{\Sigma d} \end{bmatrix} \begin{bmatrix} d\delta_s \\ d\delta_e \\ d\delta_d \end{bmatrix}$$

$$\text{or} \quad d\phi_{\Delta} = \psi_{\Sigma} d\delta \quad (6.2.25)$$

Also, from (6.2.9) the actuator displacements are related to the joint displacement by

$$d\delta = \mathbf{R} d\Delta$$

Substituting these above two equations into (6.2.24) and simplifying,

$$\Psi = \mathbf{R}^T \psi_{\Sigma} \mathbf{R} \quad (6.2.26)$$

This equation relates the stiffnesses from actuator space to joint space. Since $\mathbf{R} \mathbf{R}^* \neq \mathbf{I}_3$, we cannot invert (6.2.26) to get the stiffness relationship from joint space to actuator space.

Finally, we relate the stiffness between actuator space and endpoint space. Substituting Ψ from (6.2.23a) into (6.2.26) and using the $\mathbf{J} \mathbf{J}^{-1} = \mathbf{I}_2$ relationship,

$${}^J\mathbf{K} = (\mathbf{R} \mathbf{J}^{-1})^T \psi_{\Sigma} (\mathbf{R} \mathbf{J}^{-1}) = \mathbf{P}^T \psi_{\Sigma} \mathbf{P} \quad (6.2.27)$$

This is the relationship we are looking for. Since $\mathbf{P} \mathbf{P}^* \neq \mathbf{I}_3$, we cannot invert (6.2.27) to get the stiffness relationship from endpoint space to actuator space. However, we can solve the inversion problems of (6.2.26) and (6.2.27) by re-writing the equations that result from the matrix multiplications. Each of Ψ and ${}^J\mathbf{K}$ contain three distinct elements since they are 2x2 symmetrical matrices,

$$\Psi = \begin{bmatrix} \Psi_s & \Psi_d \\ \Psi_d & \Psi_e \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix}$$

Multiplying out (6.2.26) results in the relationships

$$\begin{bmatrix} \Psi_s \\ \Psi_e \\ \Psi_d \end{bmatrix} = \begin{bmatrix} {}^s\mathbf{R}_s^2 & 0 & {}^d\mathbf{R}_s^2 \\ 0 & {}^s\mathbf{R}_e^2 & {}^d\mathbf{R}_e^2 \\ 0 & 0 & {}^d\mathbf{R}_s^2 {}^d\mathbf{R}_e^2 \end{bmatrix} \begin{bmatrix} \Psi_{\Sigma s} \\ \Psi_{\Sigma e} \\ \Psi_{\Sigma d} \end{bmatrix}$$

or $\Psi_v = \mathbf{R}_v \Psi_{\Sigma v}$ (6.2.28)

The subscript “v” signifies the vector notation of stiffness. The inverse relationship is then

$$\Psi_{\Sigma v} = \mathbf{R}_v^{-1} \Psi_v$$
 (6.2.29)

While such notation does not leave the stiffness in its matrix form, it does allow us to solve the inverse equations to solve for the actuator stiffnesses required to produce the desired joint stiffness elements. Similarly, for the actuator to endpoint stiffness relationship,

$$\begin{bmatrix} {}^j k_{xx} \\ {}^j k_{yy} \\ {}^j k_{xy} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11}^2 & \mathbf{P}_{21}^2 & \mathbf{P}_{31}^2 \\ \mathbf{P}_{12}^2 & \mathbf{P}_{22}^2 & \mathbf{P}_{32}^2 \\ \mathbf{P}_{11}\mathbf{P}_{12} & \mathbf{P}_{21}\mathbf{P}_{22} & \mathbf{P}_{31}\mathbf{P}_{32} \end{bmatrix} \begin{bmatrix} \Psi_{\Sigma s} \\ \Psi_{\Sigma e} \\ \Psi_{\Sigma d} \end{bmatrix}$$

or ${}^J\mathbf{K}_v = \mathbf{P}_v \psi_{\Sigma v}$ (6.2.30)

The elements of $\mathbf{P}$ can be found in (6.2.13). The endpoint to actuator space relationships are given by the inverse of (6.2.30),

$$\psi_{\Sigma v} = \mathbf{P}_v^{-1} {}^J\mathbf{K}_v \quad (6.2.31)$$

Equation (6.1.15) showed how to rotate the endpoint spring stiffness matrix to its principal axes using a rotation matrix. Applying this to (6.2.27),

$$\begin{aligned} {}^J\mathbf{K}_k &= {}^J\mathbf{R}_k^{-1} {}^J\mathbf{K} {}^J\mathbf{R}_k = {}^J\mathbf{R}_k^{-1} \mathbf{P}^T \psi_{\Sigma} \mathbf{P} {}^J\mathbf{R}_k = (\mathbf{P} {}^J\mathbf{R}_k)^T \psi_{\Sigma} (\mathbf{P} {}^J\mathbf{R}_k) \\ &= {}^J\mathbf{P}_k^T \psi_{\Sigma} {}^J\mathbf{P}_k \end{aligned} \quad (6.2.32)$$

where

$${}^J\mathbf{P}_k = \mathbf{P} \cdot {}^J\mathbf{R}_k = \frac{1}{L_e L_e S_e} \begin{bmatrix} L_e {}^J\mathbf{R}_e \cos(\theta_{e3} - {}^J\theta_k) & L_e {}^J\mathbf{R}_e \sin(\theta_{e3} - {}^J\theta_k) \\ -L_e {}^J\mathbf{R}_e \cos(\theta_{e1} - {}^J\theta_k) & -L_e {}^J\mathbf{R}_e \sin(\theta_{e1} - {}^J\theta_k) \\ L_d R_d \cos(\theta_d - {}^J\theta_k) & L_d R_d \sin(\theta_d - {}^J\theta_k) \end{bmatrix}$$

In the derivation above we make use of the fact that ${}^J\mathbf{R}_k^{-1} = {}^J\mathbf{R}_k^T$. The matrix elements in (6.2.32) are exactly the projection of stiffness equations shown in (6.1.31), with the positive or negative ambiguity solved. In fact, the columns of ${}^J\mathbf{P}_k$ are the stiffness projection vectors $\mathbf{S}_{\max}$ and $\mathbf{S}_{\min}$ from (6.1.32). (We couldn't have determined the correct signs in (6.1.32) because the elements are squared in (6.1.31).) The inverse relationships in (6.1.41) can be found by solving (6.2.31) for a rotated coordinate frame.

However, equation (6.2.27) is a more general relationship than (6.1.31) in that it shows the projection relationship exists in all coordinate frames. Likewise, the geometric stiffness equation (6.2.20b) results in the ${}^G\mathbf{K}$ stiffness elements of (6.1.44), but is also more general. We have not found a projection interpretation for ${}^G\mathbf{K}$.

In Section 6.2 we have examined:

- the vector relationship between joint space and endpoint space,
- the vector relationship between actuator space and joint space,
- the vector relationship between actuator space and endpoint space,
- the generalized projection of actuator stiffnesses on to the endpoint stiffness,
- the inversion of stiffness relationships by rewriting the stiffness matrices in vector form, and,
- a simplification of features previously determined (projections, cancelling joint moments)

6.3 Significance of Geometric Stiffness

In Section 6.1.2 we showed that the endpoint spring stiffness is a function of the arm geometry and the joint stiffnesses. In Section 6.1.3 we showed that the geometric stiffness is a function of the arm geometry and the joint moments. This is an important distinction. Joint stiffnesses represent hypothetical relationships, in this case linear approximations, between joint moments and joint displacements. They provide details on what would happen *if* there was a displacement from the present condition, but are not restricted to certain values by this condition. That is, there is no external condition that limits the allowable values of stiffness, with the exception that non-zero stiffness is required to produce endpoint forces. On the other hand, static and dynamic conditions restrict the joint moments. If the arm is not moving, static relationships require that the joint moments balance with external forces. Joint moments are very much dependent on the position of the manipulator and the external forces. This is why we concluded that we can do little to control the geometric stiffness in Section 6.1.3. In this context, we can examine the conditions in which either the ${}^G\mathbf{K}$ or ${}^J\mathbf{K}$ terms are negligible. We will look at the whole endpoint stiffness to begin, followed by the development of a method to compare the relative importance of the stiffnesses.

6.3.1 Isopotential Ellipse Relationships

Recall from (6.1.13) that the whole endpoint stiffness in the $d\mathbf{u}$ - $d\mathbf{v}$ coordinate system is

$$\mathbf{K}_{uv} = \mathbf{R}_k^{-1} ({}^J\mathbf{K} + {}^G\mathbf{K}) \mathbf{R}_k$$

Also, the endpoint spring stiffness and geometric stiffness were defined in (6.1.15) and (6.1.16) as

$${}^J\mathbf{K}_k = {}^J\mathbf{R}_k^{-1} {}^J\mathbf{K} {}^J\mathbf{R}_k$$

$${}^G\mathbf{K}_k = {}^G\mathbf{R}_k^{-1} {}^G\mathbf{K} {}^G\mathbf{R}_k$$

If we invert these two relationships and substitute for ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ in the whole stiffness matrix equation above, the relationship between the individual endpoint stiffnesses and the whole stiffness is

$$\begin{aligned} \mathbf{K}_{uv} &= \mathbf{R}_k^{-1} {}^J\mathbf{R}_k {}^J\mathbf{K}_k \mathbf{R}_k {}^J\mathbf{R}_k^{-1} + \mathbf{R}_k^{-1} {}^G\mathbf{R}_k {}^G\mathbf{K}_k \mathbf{R}_k {}^G\mathbf{R}_k^{-1} \\ &= \mathbf{R}_{kJ}^{-1} {}^J\mathbf{K}_k \mathbf{R}_{kJ} + \mathbf{R}_{kG}^{-1} {}^G\mathbf{K}_k \mathbf{R}_{kG} \end{aligned} \quad (6.3.1)$$

where

$$\mathbf{R}_{kJ} = {}^J\mathbf{R}_k^{-1} \mathbf{R}_k = \begin{bmatrix} \cos(\theta_k - {}^J\theta_k) & -\sin(\theta_k - {}^J\theta_k) \\ \sin(\theta_k - {}^J\theta_k) & \cos(\theta_k - {}^J\theta_k) \end{bmatrix} \quad (6.3.2a)$$

$$\mathbf{R}_{kG} = {}^G\mathbf{R}_k^{-1} \mathbf{R}_k = \begin{bmatrix} \cos(\theta_k - {}^G\theta_k) & -\sin(\theta_k - {}^G\theta_k) \\ \sin(\theta_k - {}^G\theta_k) & \cos(\theta_k - {}^G\theta_k) \end{bmatrix} \quad (6.3.2b)$$

Substituting the elements into each stiffness matrix yields the three equations

$$k_{\max} = C_{kJ} {}^2Jk_{\max} + S_{kJ} {}^2Jk_{\min} + C_{kG} {}^2Gk_{\max} + S_{kG} {}^2Gk_{\min} \quad (6.3.3a)$$

$$k_{\min} = S_{kJ}^2 k_{\max} + C_{kJ}^2 k_{\min} + S_{kG}^2 k_{\max} + C_{kG}^2 k_{\min} \quad (6.3.3b)$$

$$0 = S_{kJ} C_{kJ} ({}^J k_{\max} - {}^J k_{\min}) + S_{kG} C_{kG} ({}^G k_{\max} - {}^G k_{\min}) \quad (6.3.3c)$$

where $S_{kJ} = \sin(\theta_k - {}^J \theta_k)$
 $C_{kJ} = \cos(\theta_k - {}^J \theta_k)$
 $S_{kG} = \sin(\theta_k - {}^G \theta_k)$
 $C_{kG} = \cos(\theta_k - {}^G \theta_k)$

Expanding the trigonometric functions in (6.3.3c) using multiple angle identities we get the quadratic

$$\tan^2 \theta_k - \alpha \tan \theta_k - 1 = 0$$

which solves to

$$\theta_k = \tan^{-1} \left(\frac{1}{2} \alpha \pm \frac{1}{2} \sqrt{\alpha^2 + 4} \right) \quad (6.3.4)$$

where

$$\alpha = \frac{(S_J^2 - C_J^2)({}^J k_{\min} - {}^J k_{\max}) + (S_G^2 - C_G^2)({}^G k_{\min} - {}^G k_{\max})}{S_J C_J ({}^J k_{\min} - {}^J k_{\max}) + S_G C_G ({}^G k_{\min} - {}^G k_{\max})}$$

From (6.3.3) we can see that it is not only the size of the ellipse that is relevant to whether or not one of the terms is negligible, the orientation is important as well.

Consider, for instance, the case where ${}^G k_{\min} \ll {}^J k_{\min}$, ${}^G k_{\max} \ll {}^J k_{\max}$, ${}^J \theta_k = \theta_k$, and ${}^G \theta_k = \theta_k + 90^\circ$. In other words, the ${}^G K$ ellipse is much smaller than the ${}^J K$ ellipse but rotated 90° .

Equation (6.3.3c) is met under these conditions since $S_{kJ} = 0 = C_{kG}$. By (6.3.3a) the whole

stiffness matrix has a major axis stiffness of $k_{\min} = {}^Jk_{\min} + {}^Gk_{\max}$. Remember that the stiffness on the minor axis is larger than that of the major axis, and so ${}^Gk_{\max} > {}^Gk_{\min}$ and ${}^Jk_{\max} > {}^Jk_{\min}$. If ${}^Gk_{\max}$ is on the same order as ${}^Jk_{\min}$, which hasn't been ruled out here, it can be a major contributor to the overall major axis even though the geometric stiffness ellipse is much smaller than the endpoint spring stiffness ellipse.

Now consider the orientation of the stiffness ellipse. In equation (6.3.4), the α parameter shows that the *difference* between axes stiffnesses control the overall orientation. Thus even a small ${}^G\mathbf{K}$ ellipse can have as much significance as a large ${}^J\mathbf{K}$ ellipse (and vice versa) in controlling the overall $\mathbf{K}$ ellipse orientation, so long as the *difference* of the axes stiffnesses of the smaller ellipse is of similar magnitude to the larger one.

In the next section we will develop a method for comparing ellipses to determine whether or not either stiffness term has a negligible effect on the whole ellipse.

6.3.2 Ellipse Comparison Index

The typical approach to using endpoint stiffness in studies or controller design neglects the effect of the geometric stiffness, as discussed in Chapter 5. This can only apply for displacements very near the virtual position, and is therefore not a good model of stiffness in contact where forces may be higher.

The previous section showed that it is difficult to determine when ${}^J\mathbf{K}$ or ${}^G\mathbf{K}$ may be negligible. It is not clear at what distance away from the virtual point that we should take ${}^G\mathbf{K}$ into consideration. Presumably, as the arm is displaced further from the virtual point the joint moments become very large (by the nature of stiffness). At some point the ${}^J\mathbf{K}$ term may become negligible in comparison.

How can we determine the points where one term becomes negligible? It is apparent from the previous section that a direct comparison of matrix elements will not suffice. Ideally we would like to find a single parameter that can be used to compare stiffnesses. The area of each ellipse is a single parameter that is a partial description of the ellipse. However, it is insufficient in that two very different ellipses can have the same area. Furthermore, two ellipses can be the same area and shape, but may have a different orientation. Our comparison must include changes in size, shape, and orientation. One parameter that is capable of measuring all of these changes is the shifted area, shown in Figure 6.3.1. We define this *shifted area* as the area within either ellipse that is not in both.

We will first assume that ${}^G\mathbf{K}$ is negligible, suggesting that the $\mathbf{K}$ ellipse and ${}^J\mathbf{K}$ ellipse are almost identical in size, shape, and orientation. If this is true, there should be very little shifted area. Absolute shifted area has little meaning since the size of the ellipse depends on the value of E_p chosen. We can normalize the shifted area on the area

of the ${}^J\mathbf{K}$ ellipse. This results in an index that gives the fraction of the ${}^J\mathbf{K}$ ellipse area that becomes shifted when the ${}^G\mathbf{K}$ ellipse is also considered. We denote this ellipse comparison index as ξ_J .

There are three general cases for comparison between $\mathbf{K}$ and ${}^J\mathbf{K}$ isopotential ellipses: the isopotential ellipse is bigger for $\mathbf{K}$ than for ${}^J\mathbf{K}$, the isopotential ellipse is smaller for $\mathbf{K}$ than for ${}^J\mathbf{K}$, and they intersect (as in Figure 6.3.1). In the first two cases, the shifted area is simply the difference between the two ellipse areas. Normalizing this shifted area on the area of the ${}^J\mathbf{K}$ isopotential ellipse, ξ_J is (see Appendix D for details)

$$\xi_J = \left| 1 - \sqrt{\frac{\det^J \mathbf{K}}{\det \mathbf{K}}} \right| \quad (6.3.5)$$

The case where the $\mathbf{K}$ and ${}^J\mathbf{K}$ ellipses intersect is more complex. The shifted area is the area inside either ellipse that is not in both. Determining the shifted area requires integration between the intersection points defined by

$$\theta_{1,2} = \tan^{-1} \left(-\frac{{}^G k_{xy}}{{}^G k_{yy}} \pm \frac{1}{{}^G k_{yy}} \sqrt{-\det^G \mathbf{K}} \right) \quad (6.3.6)$$

where θ_1 and θ_2 are taken as the principle angle solutions of the arctan. The resulting ellipse comparison index for the intersecting ellipses is (see Appendix D for details)

Figure 6.3.1: Shifted area between ellipses.

$$\begin{aligned} \xi_J = \frac{1}{\pi} & \left| \sqrt{\frac{\det^J \mathbf{K}}{\det \mathbf{K}}} (T_2 - T_1) - (T_{J2} - T_{J1}) \right| \\ & + \frac{1}{\pi} \left| \sqrt{\frac{\det^J \mathbf{K}}{\det \mathbf{K}}} (T_2 - T_1) - (T_{J2} - T_{J1}) + \pi \left(1 - \sqrt{\frac{\det^J \mathbf{K}}{\det \mathbf{K}}} \right) \right| \end{aligned} \quad (6.3.7)$$

where

$$T_{ij} = \tan^{-1} \left(\frac{{}^i k_{yy} \tan \theta_j + {}^i k_{xy}}{\sqrt{\det^i \mathbf{K}}} \right); (i = J, blank), (j = 1, 2) \quad (6.3.12)$$

The $\mathbf{K}$ and ${}^J\mathbf{K}$ ellipses can only intersect if ${}^G\mathbf{K}$ is unstable. Appendix D provides the details. More specifically, the condition for intersection is that ${}^G\mathbf{K}$ is a *saddle*, which will be discussed in Section 6.4. If the geometric stiffness is known to be stable, or completely unstable in all directions, the determination of ξ_J in (6.3.5) can be used. The stable ${}^G\mathbf{K}$ case is shown in Figure 6.3.2.

The range of values for this index is $0 \leq \xi_J \leq \infty$. For values of $|\xi_J| \ll 1$, very little area is shifted and so it ${}^G\mathbf{K}$ can be considered negligible. For values of $\xi_J \approx 1$, most of the ellipse has shrunken down to a tiny $\mathbf{K}$ ellipse or the ellipse has grown significantly, in which case ${}^G\mathbf{K}$ is significant and should not be neglected. Values larger than 1 indicate the overall stiffness is much lower (larger isopotential ellipse) than supplied by ${}^J\mathbf{K}$ alone, hence ${}^G\mathbf{K}$ is dominant and must be considered. In such a case, ${}^G\mathbf{K}$ is completely unstable.

Figure 6.3.2: Shifted area for stable $^G\mathbf{K}$. The $\mathbf{K}$ ellipse is completely enclosed within the $^J\mathbf{K}$ ellipse. The shifted area is the difference between the areas of the ellipses.

In Section 6.3 we:

- examined the contributions of the ${}^{\text{I}}\mathbf{K}$ and ${}^{\text{G}}\mathbf{K}$ ellipses to the whole ellipse, and
- developed an index for determining when ${}^{\text{G}}\mathbf{K}$ can be neglected.

6.4 Instability

In Section 6.1 we interpreted the potential energy function in (6.1.1) as a series of ellipses based on *isopotentials* (points of equal potential energy). If we plot E_p as a third dimension we can get an alternate interpretation [Hogan, 1985a]. Figure 6.4.1 shows that this interpretation forms a energy surface. The constant energy ellipses then become cross-sections of this surface at different heights in the E_p dimension. In Figure 6.4.1(a) the surface is bowl-shaped, and we can see that the cross-sections do form ellipses. The bottom of the bowl is located at the position of the arm in equilibrium with environmental forces. With no external forces, this point is the virtual position. This three-dimensional interpretation allows us to intuitively see the physical behaviour of the endpoint by considering the energy surface to be a physical surface. Objects on at any point on the “walls” of the bowl tend to fall down the surface towards the bottom. The force is the slope of the wall at a given point. The endpoint of the arm follows this same behaviour.

Figure 6.4.1(b) and (c) show two other possibilities. There may be directions in which the energy actually decreases while moving away from the equilibrium point. In (b) this occurs in one direction while the perpendicular direction has increasing energy. This is known as a saddle point because of its shape. In (c) all directions have decreasing energy. If we interpret (b) and (c) as physical surfaces, we can see that if an object is even slightly displaced from the equilibrium point, it will continue to move further away. We call this situation unstable. From the cross-sections in (b) we can also see that the

isopotentials do not form ellipses. In (c) they do form ellipses, but always with E_p negative. The ellipse is imaginary for positive values of E_p .

There are several ways to determine if a particular $\mathbf{K}$ matrix is stable, a saddle, or unstable. A stable matrix requires that it be positive definite. The conditions for this are either:

(1) The diagonal elements k_{xx} and k_{yy} and the determinant are positive. Note that it is only necessary to show that the determinant and one of the diagonal elements are positive. Since $\det \mathbf{K} = k_{xx}k_{yy} - k_{xy}^2$, noting that $k_{yx} = k_{xy}$, it follows that the other diagonal element must also be positive.

(2) The eigenvalues of the $\mathbf{K}$ matrix are both positive.

As it happens, the eigenvalues of any $\mathbf{K}$ matrix are $k_{\max}$ and $k_{\min}$. To see this, we write out the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}(k_{xx} + k_{yy}) \pm \frac{1}{2}\sqrt{(k_{xx} - k_{yy})^2 + 4k_{xy}k_{yx}} \quad (6.4.1)$$

Figure 6.4.1: Energy surfaces. (a) stable, (b) saddle, (c) unstable.

Recall from (6.1.10) that

$$\mathbf{K}_{uv} = \mathbf{R}_k^{-1} \mathbf{K} \mathbf{R}_k$$

If we invert this equation we get

$$\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} = \begin{bmatrix} C_k^2 k_{\max} + S_k^2 k_{\min} & S_k C_k (k_{\max} - k_{\min}) \\ S_k C_k (k_{\max} - k_{\min}) & S_k^2 k_{\max} + C_k^2 k_{\min} \end{bmatrix} \quad (6.4.2)$$

Substituting these elements into (6.4.1) gives us

$$\lambda_{1,2} = \frac{1}{2} (k_{\max} + k_{\min}) \pm \frac{1}{2} (k_{\max} - k_{\min})$$

$$\text{or} \quad \lambda_1 = k_{\max}; \quad \lambda_2 = k_{\min} \quad (6.4.3)$$

Therefore any $\mathbf{K}$ matrix is stable if $k_{\max}$ and $k_{\min}$ are positive. This is also a necessity for having an ellipse with real axes, according to (6.1.12), for positive E_p values.

We can also write the determinant of the $\mathbf{K}$ matrix as

$$\det \mathbf{K} = k_{\max} k_{\min} \quad (6.4.4)$$

since changing to the δu - δv coordinate system cannot change the determinant. A saddle results when one eigenvalue is negative while the other is positive. This corresponds to a negative determinant. A completely unstable $\mathbf{K}$ matrix, as in Figure 6.4.1(c), requires both eigenvalues to be negative. According to (6.4.4), this results in a positive

determinant.

In Section 6.3 we showed that the ${}^J\mathbf{K}$ and $\mathbf{K}$ ellipses intersect only if $\det{}^G\mathbf{K} < 0$. This means that the intersection occurs only when ${}^G\mathbf{K}$ forms a saddle. When ${}^G\mathbf{K}$ is completely unstable, we can still use the simplified ellipse comparison index in (6.3.16).

We know from (6.1.33) that ${}^J\mathbf{K}$ can neither be a saddle nor completely unstable, unless we allow it by design. However, no such restrictions exist on ${}^G\mathbf{K}$. From (6.3.3) we know that

$$\begin{aligned} k_{\max} &= C_{kJ}^2 {}^Jk_{\max} + S_{kJ}^2 {}^Jk_{\min} + C_{kG}^2 {}^Gk_{\max} + S_{kG}^2 {}^Gk_{\min} \\ k_{\min} &= S_{kJ}^2 {}^Jk_{\max} + C_{kJ}^2 {}^Jk_{\min} + S_{kG}^2 {}^Gk_{\max} + C_{kG}^2 {}^Gk_{\min} \end{aligned}$$

We can conclude that if ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ are both stable, $\mathbf{K}$ will also be stable. However, whether or not $\mathbf{K}$ is stable when combining a stable ${}^J\mathbf{K}$ with an unstable ${}^G\mathbf{K}$ depends on the relative size and orientation of the ellipses. It is possible for ${}^J\mathbf{K}$ to stabilize $\mathbf{K}$ in the presence of an unstable ${}^G\mathbf{K}$ matrix. Conversely, an unstable ${}^G\mathbf{K}$ can also make the $\mathbf{K}$ matrix go unstable. This makes it paramount that the relative importance of ${}^G\mathbf{K}$ be examined by the ellipse comparison indexes of Section 6.3. Neglecting ${}^G\mathbf{K}$ completely may not only provide an incorrect stiffness ellipse, but it can also obscure the fact that the stiffness may be unstable.

The above analysis also begs the question of what physically happens when the overall stiffness is unstable. From Figure 6.4.1 it is apparent that the arm will move away from the equilibrium point. However, as the endpoint moves, both the ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ matrices change since they are functions of the position of the arm. It may be the case that the overall stiffness continues to be unstable all the way to the limit of the arm's workspace, in which case the arm moves until it reaches a physical limit. It may also be the case that as the arm moves it reaches a point where the overall stiffness is stable. However, in order for the arm to come to rest it must find a stable equilibrium position.

There is one important point that is missed in the above discussion. If an equilibrium point is unstable, how does one ever reach it? There are two answers to this question. First, the point might not have been unstable in the first place, but becomes unstable because of the increasing external force (which affects ${}^G\mathbf{K}$). For example, while pushing against a constraint, one might increase the contact force by moving the virtual point well into the constraint surface. As the virtual point moves in, the force increases, and ${}^G\mathbf{K}$ becomes more prominent in $\mathbf{K}$. If ${}^G\mathbf{K}$ is unstable, the endpoint stiffness might become unstable at that point, causing the arm to move along the constraint surface to a stable equilibrium point.

The second answer to the question is that the stability information can still be useful even if it isn't related to the actual operating point of the arm at the time. Knowing

that the arm cannot be in a particular position in the workspace, at a particular joint stiffness setting, with a particular external force applied, means that as the arm is displaced near that position, the instability will force it to go in another direction, rather than towards the unstable point. This has consequences in picking stiffness settings, choosing the area of the workspace to interact with a constraint, and orienting arm to interact in a desired way.

In Section 6.4 we examined:

- a previously developed energy surface method for understanding endpoint stiffness,
- stable, saddle, and universally unstable point descriptions,
- the criteria for each stability condition,
- the physical reaction in an unstable condition, and,
- the relevance of stability to operating condition strategies.

6.5 Representation of the Stiffness Tensor

As we have seen throughout this chapter, the isopotential ellipse is a useful visual representation of endpoint stiffness. It can be included on figures of the arm so that one can see the direction of forces that result from displacement, which are normal to the ellipse. Also, one can see in which directions it is easier or harder to displace the endpoint. There is much less resistance to displacement in the direction of the major axis, whereas it requires more effort to displace the endpoint along the minor axis.

However, there are disadvantages to the isopotential ellipse representation. No absolute properties can be determined from it; the size depends both on the stiffness values, the chosen reference energy, and any scaling done to fit the figure. It is also not useful when considering unstable stiffness; the ellipse becomes a hyperbola when one axis stiffness is negative and the whole ellipse becomes imaginary when both axis stiffnesses are negative. The stiffness (not isopotential) ellipse discussed in Chapter 2 is similarly useful but limited with respect to instability, and provides less information.

In this section we suggest another representation of stiffness for analytical purposes, especially when dealing with instabilities or stability margins, that makes use of the fact that two-dimensional stiffness is a tensor. No initial proof will be given that stiffness is a tensor, rather it will be assumed to be true and the results of analysing the stiffness as a tensor will show the assumption to be true.

A tensor is a directional quantity that is independent of the coordinate system under which it is measured. A scalar is a tensor of rank 0. A vector is a tensor of rank 1. Both a scalar and vector remain unchanged by the selection of a particular coordinate system. The representation of a vector as a 2×1 matrix does, however, depend on the chosen coordinate system. But the vector itself, and its properties (length and direction) remain constant.

A tensor of rank 2 is more difficult to visualize. Examples of rank 2 tensors include angular momentum, moments of inertia, and stress. Rank 2 tensors, for n -dimensional properties, are often represented as $n \times n$ matrices. The three stiffness matrices that we have studied, endpoint stiffness $\mathbf{K}$, joint stiffness Ψ , and actuator stiffness ψ_z , are all tensors. We can then use analytical tools that are most often applied to other tensor properties to better understand these stiffnesses. Perhaps one of the more familiar tensor tools is Mohr's Circle normally applied to stress properties. This approach can also be useful with respect to understanding the endpoint stiffness properties.

6.5.1 Symmetric Stiffness

Figure 6.5.1 shows the Mohr's Circle for a symmetric endpoint stiffness matrix. The diagonal elements k_{11} and k_{22} are drawn on the horizontal axis, and the off-diagonal terms $k_{12} = k_{21}$ are drawn on the vertical axis. The stiffness tensor is represented by the

whole circle. The selection of an endpoint coordinate system chooses the appropriate values from the circle for a matrix representation. Rotation of the coordinate system simply changes the location on the circle for which the values are taken.

The Mohr's circle in Figure 6.5.1 can be characterized by three properties: the centre distance, the radius, and the orientation with respect to the x-axis. The centre distance and radius are

$$C = \frac{1}{2} (k_{xx} + k_{yy}) \quad (6.5.1)$$

$$R = \frac{1}{2} \sqrt{(k_{xx} - k_{yy})^2 + (k_{xy} + k_{yx})^2} \quad (6.5.2)$$

There are two principle stiffness at which $k_{12} = k_{21} = 0$. These correspond to the axes stiffnesses $k_{\max}$ and $k_{\min}$. The orientation of these principle directions is easily calculated from the circle. As with stresses, we note that angles in the Mohr's Circle represent twice the angles in the physical domain. Therefore, the principle axes are at an angle given by

$$\theta_k = \frac{1}{2} \tan^{-1} \left(\frac{k_{xy} + k_{yx}}{k_{xx} - k_{yy}} \right) \quad (6.5.3)$$

The values of the principal stiffnesses, from the geometry of the circle, are

Figure 6.5.1: Mohr's Circle representation of endpoint stiffness.

$$\begin{aligned} k_{\min} &= C - R \\ k_{\max} &= C + R \end{aligned} \tag{6.5.4}$$

When C and R are substituted from (6.5.1) and (6.5.2), these are identical to the axes stiffness values determined from the eigenvalues in (6.4.1), confirming our assumption that stiffness is a tensor. Using (6.5.4) we can also write the centre distance C as

$$C = \frac{1}{2} (k_{\max} + k_{\min}) \tag{6.5.5}$$

In other words, C is the arithmetic mean of the principal stiffnesses.

We can also directly see that the coupling stiffness, k_{12} , is maximum at 45° from the principle axes, and the direct stiffnesses, k_{11} and k_{22} , are equal at that orientation. Its value is

$$k_{12(\max)} = R \tag{6.5.6}$$

The Mohr's circle representation is particularly useful with respect to stability properties. If the circle crosses the vertical axis then one of the principal stiffnesses is negative and the other is positive. As we discussed in the last section, this means that the operating point is a saddle, which is unstable. If the entire circle is to the left of the vertical axis, both principal stiffnesses are negative and the stiffness is unstable in all directions. When the circle is entirely on the right of the vertical axis, the distance to the lower principal stiffness, $k_{\min}$, can be interpreted as a stability margin. That is, it indicates

how close the stiffness is to becoming unstable. This interpretation can be useful when we consider the endpoint spring stiffness and geometric stiffness separately. Consider the endpoint stiffness matrix from (5.3.16),

$$\mathbf{K} = {}^J\mathbf{K} + {}^G\mathbf{K}$$

Figure 6.5.2 shows all three stiffness matrices as Mohr's Circles. In this illustration, ${}^G\mathbf{K}$ is a saddle but does not contribute enough to the whole stiffness to make it go unstable. However, the distance of the overall stiffness circle from the vertical axis is such that the $k_{\min}$ is closer to zero than to $k_{\max}$. This suggests that it could go unstable with only a small increase in the geometric stiffness, or decrease in the endpoint spring stiffness.

Unfortunately, the addition of the ${}^J\mathbf{K}$ and ${}^G\mathbf{K}$ circles is not as straightforward as it looks. One cannot simply superimpose the two circles to get the overall stiffness circle. For example, one cannot add the principal stiffnesses of the ${}^J\mathbf{K}$ ellipse to the principal stiffnesses of the ${}^G\mathbf{K}$ ellipse to get the overall principle stiffnesses. But we were already aware of this from (6.3.3),

$$k_{\max} = C_{kJ}^2 k_{\max} + S_{kJ}^2 k_{\min} + C_{kG}^2 k_{\max} + S_{kG}^2 k_{\min}$$

$$k_{\min} = S_{kJ}^2 k_{\max} + C_{kJ}^2 k_{\min} + S_{kG}^2 k_{\max} + C_{kG}^2 k_{\min}$$

In order to add the two circles, we must first have a common reference plane, such as the dx-dy coordinate system. It is the stiffness values in these reference planes that add. For

instance, $k_{xx} = {}^Jk_{xx} + {}^Gk_{xx}$ in Figure 6.5.2, even though these values correspond to different points on the circles. In other words, the overall stiffness circle cannot be constructed from the two other stiffness circles without knowing how they relate to a common reference plane. However, we can know that the centre of the overall stiffness circle is the sum of the centres of the other two.

$$\frac{1}{2}(k_{xx} + k_{yy}) = \frac{1}{2}(({}^Jk_{xx} + {}^Gk_{xx}) + ({}^Jk_{yy} + {}^Gk_{yy})) = \frac{1}{2}({}^Jk_{xx} + {}^Jk_{yy}) + \frac{1}{2}({}^Gk_{xx} + {}^Gk_{yy})$$

It is the radius of the overall circle that requires knowledge of the common reference plane.

Stiffness and isopotential ellipse properties are also evident from Mohr's Circle. An isometric endpoint stiffness is one in which the stiffness is equal in all directions, and hence both ellipses are circles. The Mohr's Circle representation with equal stiffnesses is a point on the horizontal axis. The rounder a stiffness ellipse becomes the smaller the Mohr's Circle *in comparison to its distance from the horizontal axis*. That is, the radius of the circle is small compared to the location of the centre, $R \ll C$, which simplifies to

$$\det \mathbf{K} = k_{\max}k_{\min} = C^2 - R^2 \gg 0 \quad (6.5.7)$$

(Note that if it is perfectly round, Mohr's circle becomes a point.) The converse state is that a large Mohr's circle, relative to the centre location, has a long, thin isopotential ellipse. We define the shape factor as

Figure 6.5.2: Mohr's Circle representation of endpoint spring stiffness ${}^I\mathbf{K}$, geometric stiffness ${}^G\mathbf{K}$, and overall endpoint stiffness $\mathbf{K}$.

$$S_k = C/R \quad (6.5.8)$$

The shape factor indicates how round or elliptical a stiffness is. The range of shape factors is $-\infty \leq S_k \leq \infty$. Large S_k values indicate an approximately round stiffness isopotential. Values near 1, but greater than 1, indicate a long, thin stiffness. At $S_k = 1$, $k_{\min} = 0$ and so the isopotential ellipse collapses. In the direction of $k_{\min}$, the stiffness is neutral and so there is no return force. Values between -1 and 1 are saddles. Values less than -1 are the same as their positive counterparts, except unstable. That is, near -1 is long and thin and large negative numbers are near round, but all are unstable.

On a Mohr's circle plot, the shape is indicated by two possible interpretations shown in Figure 6.5.3. The shape defines an angle for which a line through the origin is tangent to the circle, shown as β_1 in the figure. Alternately, the shape can define an angle β_2 for which a line through the origin passes through the highest point on the Mohr's circle, as shown in the figure. The shape factor is $S_k = \csc(\beta_1) = \cot(\beta_2)$. Stiffnesses with the same shape all have the same shape factor, and therefore the same β_1 and β_2 . In Figure 6.5.3(a) all of the circles to the right of the origin correspond to the same stable shape and all the circles to the left correspond to the same unstable shape. In (b) the same saddle shapes are shown. Note that for a saddle, one cannot define β_1 and so β_2 must be used.

Figure 6.5.3: Shape factor as seen on Mohr's circle. $S_k = \csc(\beta_1) = \cot(\beta_2)$. (a) Circles to the right of the origin all have the same stable shape. Isopotential ellipses corresponding to the stiffnesses labelled 1, 2, and 3 are shown below the Mohr's circles. (The orientation of the stiffness is not shown in the current Mohr's circle illustrations, so the isopotential ellipses can have any orientation.) Circles to the left all have the same unstable shape, which can't be shown by isopotential ellipse. (b) Saddle points of constant shape. β_1 cannot be defined here and so β_2 must be used. Isopotential ellipses cannot be defined for these stiffnesses.

The size of the stiffness, indicating an overall stiffness level, is usually defined by the stiffness ellipse area [Mussa-Ivaldi et al., 1985; Dolan et al., 1993]. We define an alternate size here as the absolute geometric mean of the principle stiffnesses,

$$k_{avg} = \sqrt{|k_{max} \cdot k_{min}|} = \sqrt{|\det \mathbf{K}|} = \sqrt{|C^2 - R^2|} \quad (6.5.9)$$

This definition of size has the advantage that it has the units of stiffness and is numerically of the same order as the principle stiffnesses. It represents a circular stiffness with the same ellipse area as the ellipse being analysed, and therefore has the same size. The same size can come from a long, thin stiffness, or from a more rounded one. It can also be stable or unstable (including saddles) since the overall level (size) of stiffness does not refer to stability. For the Mohr's circle representation, k_{avg} defines a distance from the origin to the point of tangency of a tangent line, as shown in Figure 6.5.4(a). This is the same line of tangency whose angle is β_1 in Figure 6.5.3, defining the shape factor. For saddles there is no tangency, and so k_{avg} defines a location on the vertical axis through which the saddle stiffness must pass, as shown in Figure 6.5.4(b).

Figure 6.5.4: Mohr's circles for constant stiffness size. (a) Stable and unstable stiffnesses with the same size. (b) Saddles of constant size. If k_{avg} is the same as in (a), all stiffnesses shown in the figure have the same size.

6.5.2 Anti-Symmetric Stiffness

Although the current work deals with a conservative model, and therefore has symmetric stiffness properties, the Mohr's circle representation can include anti-symmetric components. Recall from (2.1.4b) that the anti-symmetric stiffness is

$$\mathbf{K}_a = \frac{1}{2}(k_{xy} - k_{yx}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Define the anti-symmetric co-efficient

$$D_k = k_{xy} - k_{yx} \quad (6.5.10)$$

The effect of this anti-symmetric component on the Mohr's circle representation can be seen in Figure 6.5.5. It shifts the circle up by an amount D_k . It does not change the size of the circle or most of the stiffness properties (other than conservativeness). The size, shape, and orientation of the stiffness all remain the same, but now are measured from a point D_k above the origin. The maximum coupled-stiffness (k_{xy} or k_{yx} , whichever has a larger magnitude) does increase however.

There are several interesting stiffness properties that can be seen from inclusion of the anti-symmetric component. First, the stability cannot be changed by an anti-symmetric component because the circle does not move to the left or right. Second, the directions in which the force and displacement are co-linear begin to deviate from 90°

apart (180° on the Mohr's circle diagram). The orientations of these co-linear directions are labelled θ_{C1} and θ_{C2} in Figure 6.5.5. and are determined by

$$\theta_{C1,C2} = \theta_k + \pi/4 - \frac{1}{2}\cos^{-1}(D_k/R) \quad (6.5.11)$$

If D_k becomes greater than R , the anti-symmetric component has become dominant and there are no directions in which force and displacement are co-linear. The angle between directions of co-linear force and displacement (if they exist) could be used as an experimental measure of conservativeness.

Figure 6.5.5: Inclusion of the anti-symmetric component of stiffness in the Mohr's circle representation. The circle is raised in the vertical direction, but the other properties of the circle remain unchanged. If $D_k < R$, there are still two direction in which there is no coupling term to the stiffness, and so displacement and force are co-linear. However, they are no longer 90° apart (ellipse axes directions) as in the symmetric case.

In Section 6.5 we:

- developed a unified method for representing stable, unstable, and saddle stiffnesses using Mohr's circle,
- showed how the geometric stiffness and endpoint spring stiffness add in the Mohr's circle representation to get the overall stiffness.
- derived stiffness properties and their meanings with respect to the Mohr's circle diagram, including size, shape, orientation, maximum coupling stiffness, and stability, and,
- expanded the Mohr's circle plot to include anti-symmetric components of stiffness and consequently introduced a new test for conservativeness.

6.6. Chapter Summary

Keeping in mind the goals set out in Chapter 4, we will review the accomplishments in this chapter. In Section 6.1 we developed the basic relationships between the spring stiffnesses and the endpoint stiffness, and to the characteristics of the endpoint stiffness by the maximum and minimum stiffnesses and orientation. We developed similar relationships with respect to the geometric stiffness for contact cases. In Section 6.2 we simplified and generalized these relationships and provided a more straightforward method for analysing them. This achieves the goals of Question 1 in Chapter 4, which asked to find the relationship between endpoint and actuator stiffnesses. It also provides some information with respect to the double-joint (bi-articular) springs in

that they have their own unique projection direction, hinting that it may not be possible to compensate for their action. We've also shown how they (bi-articular muscles) are analogous to a parallelogram linkage system.

In Section 6.3 we investigated how to judge the relative importance of the geometric stiffness, and created a measurement tool, the ellipse comparison index, by which to make this judgement. This partially answers Question 5 in Chapter 4 regarding the effects of loading on stiffness. In Section 6.4 we found conditions on stability of the manipulator. This partially answers Question 3, given that we have some understanding of the stability, but not clearly how to deal with it yet. Finally, in Section 6.5 we developed a tool using Mohr's circle to better investigate stiffness. It is particularly useful for investigating unstable stiffness conditions, anti-symmetric stiffness (though that isn't considered in the current model), and separating the stiffness components. In addition, all of the properties of stiffness are readily derived from the diagram.

In the next chapter we will apply the knowledge gained here to numerical examples to see the patterns that they produce. This will allow us to make some qualitative conclusions about the stiffness behaviour, and by using appropriate values we can make some quantitative conclusions as well.

Chapter 7

Numerical Simulation

In this Chapter we will use a numerical model to simulate many of the stiffness features developed in Chapter 6. Section 7.1 will introduce the numerical model and parameter values. Section 7.2 will show the extent to which the linear approximation of isopotential ellipses match the non-linear potential energy contours. For some stiffness settings this approximation holds for significant displacements. This will go towards answering Question 6 in Chapter 4 regarding the ability of the linear approximation to accurately describe the stiffness state. The results will also show that the geometry plays a significant role in the shape of the non-linear isopotentials and that variations in the spring functions do not have a large effect on these shapes.

In Section 7.3 we examine the effects of the actuator stiffnesses on the endpoint stiffness and interpret them in terms of the stiffness projections found in Chapter 6. We will also characterize the endpoint spring stiffness as a function of workspace location and discover how much the geometric governs the range of endpoint stiffnesses.

In Section 7.4 we investigate the behaviour of the geometric stiffness. We will investigate under what situations it has a significant effect on the endpoint stiffness.

Dynamic simulations will show the behaviour of the system model under a variety of stiffness situations. In particular, these simulations will show what happens to the system in unstable situations. Patterns that appear in the geometric stiffness as a function of force direction will also lead to some analytical understanding of the nature of the geometric stiffness.

7.1 Numerical Model

This section shows the spring functions and model parameters chosen for numerical simulations for the remainder of the current work in order that the simulations can be reproduced. The spring force functions are based on previous research and the parameters are chosen to roughly coincide with those of a natural human arm.

7.1.1 Spring Force Functions

Up to this point there have been no criteria on which to base a desirable set of spring functions. The previous chapters are applicable to existing systems, such as the human arm, and to the design of a manipulator system that mimics the mechanics of the human arm either through physical spring systems or by simulation through feedback control. The main goal of this work, as discussed in Chapter 4, is to understand the mechanics of stiffness in a multiple degree-of-freedom manipulator so that a prosthetic limb can be properly designed *mechanically* to mimic the natural behaviours of a human arm. Specifically, we wish to include the ability to mechanically modulate the joint

stiffnesses. We are therefore free to chose spring functions to best meet these goals, and not necessarily to duplicate the force-displacement functions found in natural muscles.

Previous work has examined the design of spring functions to modulate joint stiffness [English, 1996; English & Russell, 1999a]. It was proven that only quadratic force-displacement spring functions could provide a joint stiffness that was independent of joint displacement. Such spring functions would allow open loop control of the joint stiffness such that it remained constant even under loading. Using this criteria, and the minimization of internal forces, the spring functions have the form

$$\phi(\delta) = \frac{\Psi_{\max} - \Psi_{\min}}{4R^2\delta_{\text{pos}}}\delta^2 + \frac{\Psi_{\min}}{2R^2}\delta - \frac{\delta_{\text{pos}}(\Psi_{\max}^2 - 2\Psi_{\min}^2)}{8R^2(\Psi_{\max} - \Psi_{\min})} \quad (7.1.1)$$

where the following parameters are based on *each spring pair*:

- $\Psi_{\max}, \Psi_{\min}$ are the maximum and minimum design joint stiffness,
- R is the joint radius,
- δ_{pos} is the maximum stretch of the spring, and,
- δ is the stretch of the spring.

For example, the single joint shoulder springs would be based off of the function.

$$\phi_{si}\{\delta_{si}\} = \frac{{}^s\Psi_{s(max)} - {}^s\Psi_{s(min)}}{4 {}^sR_s^2 \delta_{s(pos)}} \delta_{si}^2 + \frac{{}^s\Psi_{s(min)}}{2 {}^sR_s^2} \delta_{si} - \frac{\delta_{s(pos)} ({}^s\Psi_{s(max)}^2 - 2 {}^s\Psi_{s(min)}^2)}{8 {}^sR_s^2 ({}^s\Psi_{s(max)} - {}^s\Psi_{s(min)})}$$

$$(i = f, x)$$

The assumption on this function derivation is that the flexor and extensor springs are identical for each pair. The function was designed for a single joint antagonistic actuation system only. This makes it difficult to apply it directly to the double joint springs. They contribute the three different joint stiffness terms (shoulder, elbow, coupling) and can have different joint radii at the elbow and shoulder. To avoid this problem, we will assume that ${}^dR_s = {}^dR_e$ and select the maximum contribution to each double joint stiffness term to be the same, $\Psi_{d(max,min)} = {}^d\Psi_{s(max,min)} = {}^d\Psi_{e(max,min)}$. For simplicity, we will make all springs identical by choosing all joint radii and stiffness ranges to be identical.

Deviation from the quadratic form was also studied in the previous work on spring design. Spring functions were generated based on the changing joint stiffness such that it increased or decreased by a certain percentage at a specified reference angular displacement. Two spring force functions resulted from this analysis. The first was a joint stiffness that increases with angular displacement:

$$\phi\{\delta\} = \frac{\Psi_{max}\{\cosh(\mu\delta) - \cosh(\mu\delta_0)\} - \Psi_{min}\{\cosh(\mu(\delta_{pos} - \delta)) - \cosh(\mu(\delta_{pos} - \delta_0))\}}{2\mu R^2 \sinh(\mu\delta_{pos})} \quad (7.1.2)$$

where

$$\delta_0 = \frac{\frac{1}{\sqrt{2}} \Psi_{\max} - \Psi_{\min}}{\Psi_{\max} - \Psi_{\min}} \delta_{\text{pos}}$$

$$\mu = \frac{\cosh^{-1}(\xi_0 + 1)}{R \Delta_0}$$

ξ_0 is a reference error in joint stiffness

Δ_0 is a reference angular displacement at which the error is set

For example, if the joint stiffness is to increase (from its set value) by 50% at a displacement of 90° , we can chose $(\xi_0, \Delta_0) = (0.5, \pi)$. Since (7.1.2) only applies to increasing stiffness, we must chose $\xi_0 > 0$. For decreasing joint stiffness the spring force functions are in the form

$$\phi\{\delta\} = \frac{\Psi_{\max} \{\cos(\mu\delta) - \cos(\mu\delta_0)\} - \Psi_{\min} \{\cos(\mu(\delta_{\text{pos}} - \delta)) - \cos(\mu(\delta_{\text{pos}} - \delta_0))\}}{2\mu R^2 \sin(\mu\delta_{\text{pos}})} \quad (7.1.3)$$

where

$$\mu = \frac{\cosh^{-1}(\xi_0 + 1)}{R \Delta_0}$$

In this case, we must chose $\xi_0 < 0$. (Note that $\xi_0 = 0$ results in the quadratic form of (7.1.1)). To clarify, (7.1.3) corresponds to decreasing *joint* stiffness with *joint* displacement. The individual spring stiffness itself increases with spring displacement.

However, when two such springs are attached antagonistically about a joint, the decrease in stiffness of one as it shortens is larger than the increase in stiffness of the other as it lengthens, hence joint stiffness decreases. This is an important distinction to make because it has been suggested that increasing muscle stiffness with force (and hence length if the force increases with length as here) is sufficient to increase joint stiffness and thereby stabilize an unstable limb [McIntyre et al., 1996]. Our results show that this is only true if stiffness increases greater than linearly with length, i.e., force increases greater than quadratic with length.

We denote the force functions in (7.1.2) and (7.1.3) as error functions since they coincide with unintentional changes in joint stiffness upon displacement. The purpose of these error functions is to address the more general cases where joint stiffness is not fixed once set by the $k\sigma$ elements discussed in Chapter 5. These functions allow us to directly control the error independently from the particular set stiffness.

The error functions in (7.1.2) and (7.1.3) will only be employed when studying the effects of joint stiffness errors on the behaviour of the endpoint stiffness. Unless otherwise stated, the springs will use the force functions in (7.1.1).

7.1.2 Parameter Selection

The static parameters for the limb model so far have consisted of the link lengths and joint radii. In order to perform dynamic simulations we will also need to define the mass properties and damping in the system. We will model the links as solid cylinders with their mass centred along their length. The moments of inertia of the arms, about their respective base joints, are then in the form

$$I_i = \frac{1}{3} m_i L_i^2$$

Damping is added directly to the joints. The values of all parameters chosen for the model simulations are listed in Table 7.1 below.

Table 7.1: Parameter Selection

	PARAMETER	DESCRIPTION	VALUE
Joint Radii	sR_s	single-joint shoulder springs radius	0.4 m
	sR_e	single-joint elbow springs radius	0.4 m
	dR_s	double-joint springs, shoulder radius	0.4 m
	dR_e	double-joint springs, elbow radius	0.4 m
Link Lengths	L_s	length of upper arm	0.4 m
	L_e	length of forearm	0.4 m
Mass	m_s	mass of upper arm	1 kg
	m_e	mass of forearm	1 kg
Damping	B_s	damping of shoulder joint	1 N·m/s/rad
	B_e	damping of elbow joint	1 N·m/s/rad
Spring Function Parameters	$\Psi_{\max}$	nominal maximum joint stiffness	80 N·m/rad
	$\Psi_{\min}$	nominal minimum joint stiffness	0 N·m/rad
	R	nominal joint radius	0.4 m
	$\delta_{\max}$	nominal maximum spring stretch	0.05 m

It might seem unusually to have the joint radii selected at 0.4m since this is the same length as the links themselves. However, it is an *effective* joint radius, meaning that it is the rate at which a change in spring position translates to a change in joint angle. This can include any gear ratios between the springs and the joint that are not drawn in the models so far. The previous work in which the spring functions were designed (English, 1996) estimated that an effective radius of 0.4m was a reasonable balance

between geometric limitations and the performance parameters of the mechanism. Since we are not considering the geometric limitations in the current work, the selection of the joint radii and maximum spring stretch, δ_{pos} , are relevant only to define the force functions. The particular settings used in the simulations will be based off of their contribution to the joint stiffness in equations (7.1.1) - (7.1.3). Changing radii R therefore has no effect on the joint stiffness since the spring function automatically adjust to compensate for the change in R .

In Section 7.1 we selected:

- quadratic spring force functions that maintain constant joint stiffness,
- error spring force functions that allow us to include changes in joint stiffness due to joint displacements, and
- parameter values for the simulations.

7.2 Isopotential Ellipse

Recall from Section 6.1.1 that the isopotential ellipse representation is based on lines of constant potential energy. For small displacements the linearized potential energy function, from (6.1.1) is

$$E_p = \frac{1}{2} \delta \mathbf{p}^T \mathbf{K} \delta \mathbf{p} = \frac{1}{2} (k_{xx} \delta x^2 + (k_{xy} + k_{yx}) \delta x \delta y + k_{yy} \delta y^2)$$

The exact potential energy is given by the sum of the energy stored in all of the springs. From (5.1.17), the energy stored in one spring is

$$E_{\text{spring}} \{\delta\} = \int_{\delta_0}^{\delta} \phi(\varepsilon) d\varepsilon + E\{\delta_0\}$$

Integrating the quadratic force function in (7.1.1),

$$E_{\text{spring}} \{\delta\} = \frac{\Psi_{\max} - \Psi_{\min}}{12 R^2 \delta_{\text{pos}}} (\delta^3 - \delta_0^3) + \frac{\Psi_{\min}}{4 R^2} (\delta^2 - \delta_0^2) - \frac{\delta_{\text{pos}} (\Psi_{\max}^2 - 2 \Psi_{\min}^2)}{8 R^2 (\Psi_{\max} - \Psi_{\min})} (\delta - \delta_0) \quad (7.2.1)$$

where we have defined $E_{\text{spring}} \{\delta_0\} = 0$ so that there is zero potential energy at the operating position. The total potential energy is then given by

$$E_p = \Sigma(E_{\text{spring}})$$

Figure 7.2.1 shows both the linearized stiffness ellipses and the exact non-linear isopotential representations of the endpoint stiffness at a location of $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$. In

(a), the joint stiffness pairs are all set at 20 N·m/rad. In (b)-(d) the individual spring pair stiffnesses are doubled. The joint stiffness matrices for these figures are then

$$(a) \Psi = \begin{bmatrix} 40 & 20 \\ 20 & 40 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

$$(b) \Psi = \begin{bmatrix} 100 & 80 \\ 80 & 100 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

$$(c) \Psi = \begin{bmatrix} 100 & 20 \\ 20 & 40 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

$$(d) \Psi = \begin{bmatrix} 40 & 20 \\ 20 & 100 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

There are no endpoint forces applied. The isopotential curves are drawn for the same values of energy for both linear and non-linear versions.

The results in the figure show that the linearized stiffness ellipse representation provides a reasonably accurate depiction of the true energy curves for displacements up to approximately $1/4$ of the length of the upper arm. Larger displacements result in the true isopotentials distorting from the ellipse shape to a significant degree. While this only one example of the two representations, other positions and stiffness values give similar results. In general, the stiffness ellipse appears to be a reasonable representation for the state of the manipulator stiffness.

It is also possible that the chosen spring force functions may play a major role in the consistency between the linear and non-linear isopotentials. The quadratic spring functions in the previous simulation maintain constant joint stiffnesses despite joint displacements. Hence the agreement in potential shapes may fall apart if the joint stiffnesses were to change upon displacement. To address this possibility, we employ the error functions in 7.1.2 and 7.1.3. We select $\xi_0 = \pm 1$ and $\Delta_0 = 50^\circ$, meaning that the joint stiffness increases or decreases linearly with displacement to a 100% stiffness error at a displacement of 50° , giving a rate of $\pm 2\%$ stiffness error per degree displacement.

Figure 7.2.2 shows the comparison of non-linear isopotentials. The solid lines represent the quadratic (0% error) spring functions. The dashed lines are the increasing stiffness (+2% per degree) while the dotted lines are the decreasing stiffness (-2% per degree). As one would expect, the increasing stiffness means that the endpoint is displaced less than the constant stiffness case to obtain the same energy storage in the springs. This is indicated by the dashed lines slightly on the inside of the solid lines. Conversely, a decreasing stiffness takes more displacement to achieve the same energy level, as indicated by the dotted lines slightly external to the solid lines.

Figure 7.2.2: Comparison of non-linear isopotentials for various spring functions. Solid lines indicate 0% joint stiffness change upon displacement. Dashed lines indicate an increase of 2% in joint stiffnesses per degree of displacement. Dotted lines indicate a decrease of 2% in joint stiffnesses per degree of displacement. (a) All stiffnesses = 20 Nm/rad. (b) All double-joint stiffnesses increased to 80 Nm/rad. (c) Single-joint shoulder stiffness increased to 80 Nm/rad. (d) Single-joint elbow stiffnesses increased to 80 Nm/rad.

The results in Figure 7.2.2 indicate that the consistency between the linear and non-linear isopotentials does not largely rely on the chosen stiffness functions. The size of displacement at which they begin to deviate significantly is a function of the spring force functions. The results here suggest that the spring function must deviate significantly from a quadratic form to affect this displacement at which this deviation occurs. In other words, the joint stiffnesses must change significantly for small displacements in order for the linear isopotentials to become poor approximations.

The results in Figure 7.2.2 also suggest that the behaviour of the endpoint spring stiffness is much more dependant on the geometry than it is on the particular spring functions. Figures 7.2.3 to 7.2.6 show the linear and non-linear isopotentials, using the quadratic force functions, for $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$, $(30^\circ, 120^\circ)$, $(60^\circ, 60^\circ)$, and $(75^\circ, 30^\circ)$, respectively. These positions vary linearly straight out from the shoulder. We should note that these results can be applied across the workspace because the selection of x-y coordinate orientation, and therefore θ_s , is arbitrary. The stiffness at the endpoint is affected by the distance from the shoulder (indicating the relative elbow angle), but not the direction of the arm, i.e., the shoulder angle.

The middle position of $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$ is shown in Figure 7.2.1. As in the previous simulations, there is no external force in these cases and hence ${}^G\mathbf{K} = \mathbf{0}$. These plots show that the accuracy of the linearized isopotentials is much more dependent on

the position of the arm. The stiffness ellipse representation is more accurate around the centre zone of the workspace ($\theta_e \approx 90^\circ$). Near the outer regions or close to the shoulder the actual non-linear shapes become very distorted.

Figure 7.2.3: Linear (---) and non-linear (—) isopotentials, $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$. (a) All stiffnesses = 20 Nm/rad. (b) All double-joint stiffnesses increased to 80 Nm/rad. (c) Single-joint shoulder stiffness increased to 80 Nm/rad. (d) Single-joint elbow stiffnesses increased to 80 Nm/rad.

Figure 7.2.4: Linear (---) and non-linear (—) isopotentials, $(\theta_s, \theta_e) = (30^\circ, 120^\circ)$. (a) All stiffnesses = 20 Nm/rad. (b) All double-joint stiffnesses increased to 80 Nm/rad. (c) Single-joint shoulder stiffness increased to 80 Nm/rad. (d) Single-joint elbow stiffnesses increased to 80 Nm/rad.

Figure 7.2.5: Linear (---) and non-linear (—) isopotentials, $(\theta_s, \theta_e) = (60^\circ, 60^\circ)$. (a) All stiffnesses = 20 Nm/rad. (b) All double-joint stiffnesses increased to 80 Nm/rad. (c) Single-joint shoulder stiffness increased to 80 Nm/rad. (d) Single-joint elbow stiffnesses increased to 80 Nm/rad.

Figure 7.2.6: Linear (---) and non-linear (—) isopotentials, $(\theta_s, \theta_e) = (75^\circ, 30^\circ)$. (a) All stiffnesses = 20 Nm/rad. (b) All double-joint stiffnesses increased to 80 Nm/rad. (c) Single-joint shoulder stiffness increased to 80 Nm/rad. (d) Single-joint elbow stiffnesses increased to 80 Nm/rad.

In Section 7.2 we showed that:

- The linearized isopotential ellipse representation of endpoint stiffness is most accurate near the centre of the workspace, where it can be reasonably valid for displacements up to $\frac{1}{4}$ the length of the limb segments. It is less accurate near the shoulder or outer limits of the workspace.
- The size and shape of the non-linear isopotentials, and therefore the accuracy of the linearized versions, are more dependant on the position of the endpoint than on the chosen spring functions.

7.3 Endpoint Spring Stiffness

In Sections 6.1.2 and 6.2.4 we showed how the stiffness of the individual spring pairs contributed to the endpoint stiffness ellipse in particular directions. In this section we will investigate the relationship of these directions to the ellipse as well as the non-linear isopotentials.

Figure 7.3.1 shows the changes in the isopotential ellipse due to changes in the individual stiffness pairs. Figure 7.3.1(a) starts with stiffnesses equal, set at 20 N·m/rad as in Section 7.2. The projected directions of the actuator stiffnesses, according to (6.2.24), are shown in the figure. Figures (b)-(d) show the effect of varying the individual stiffnesses from 20 N·m/rad to their maximum values of 80 N·m/rad. These figures show that the effect of increasing these stiffnesses is the reduction of the ellipse radius in the associated directions. This means that it becomes more difficult to displace the endpoint in the direction associated with that actuator pair. In effect, the minor axis of the ellipse rotates towards the associated direction. In reality, the ellipse just gets shortened in that direction.

Figure 7.3.2 shows the same changes, but in the non-linear energy contour representation. As seen in the figure, the perpendicular to the directions associated with the actuator stiffnesses are actually curves. The representation in the figure makes the physical reasons for these direction more apparent. Consider the case where the single-

joint shoulder stiffness is increased significantly higher than the other stiffnesses. It will be difficult to displace the upper arm from its set shoulder angle. However, the elbow is much easier to move. As the elbow moves, with the shoulder held stationary, the endpoint traces out a circle. Similarly, if the elbow stiffness is high, rotation of the shoulder is much easier, and this traces out a large circle around the workspace. Finally, if the double-joint stiffness is high, the connection between the joints acts like a belt. If the joint radii for the double-joint actuators are the same at the shoulder and elbow, the effect is to keep the absolute elbow angle (θ_{es}), constant. This also traces out a circle. If the radii are not equal, the endpoint will trace out loops, depending on the relative size of the radii.

In Section 7.2 it was shown that the position of the endpoint in the workspace also affected the size, shape, and orientation of the ellipse. Figure 7.3.3 extracts the isopotential ellipses from Figures 7.2.3 to 7.2.6 and 7.2.1 and plots them together. The stiffnesses are all equal in (a), and it can be seen that moving from the shoulder towards the outer region causes the ellipse to shrink in the direction of the shoulder and stretch out in the perpendicular direction. This is similar to the effects of changing the single-joint elbow stiffness in Figure 7.3.1(d). Moving away from the body mimics increasing the single-joint elbow stiffness while moving towards the body mimics lowering it. In fact, comparing (a) with (d) we can see that the geometry has a greater effect than increasing the single-joint elbow stiffness from 20 N·m/rad to 80 N·m/rad, a fourfold increase.

Figure 7.3.1: Effects of actuator stiffnesses on isopotential ellipse, $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$. Stiffnesses are shown at 20, 40, 60, and 80 N·m/rad. The dashed lines represent the weights and directions associated with each stiffness. (a) Equal stiffnesses, all increase at the same time. (b) Double-joint stiffness increases. (c) Single-joint shoulder stiffness increases. (d) Single-joint elbow stiffness increases.

Figure 7.3.2: Effects of actuator stiffnesses on non-linear isopotentials, $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$. Stiffnesses are shown at 20, 40, 60, and 80 N·m/rad. The dashed lines represent the perpendiculars to the directions associated with each stiffness. (a) Equal stiffnesses, all increase at the same time. (b) Double-joint stiffness increases. (c) Single-joint shoulder stiffness increases. (d) Single-joint elbow stiffness increases

All Stiffnesses Increase

Increasing Double-Joint Stiffness

Increasing Shoulder Stiffness

Increasing Elbow Stiffness

Figure 7.3.3: Effect of endpoint location on the isopotential ellipse. (a) Equal stiffnesses at 20 N·m/rad. (b) Double-joint stiffness increases from 20 N·m/rad to 80 N·m/rad. (c) Single-joint shoulder stiffness increases from 20 N·m/rad to 80 N·m/rad. (d) Single-joint elbow stiffness increases from 20 N·m/rad to 80 N·m/rad.

The effect of position on the other stiffnesses can be seen in (b) and (c). The ellipse still gets stretched out as the endpoint moves towards the outer region. Also, the orientation of the ellipse gets forced towards the perpendicular to the shoulder direction. This characteristic is consistent with the projected directions discussed in Sections 6.1.2 and 6.2.4. In particular, the stiffness projection directions in Figures 6.1.2 and 6.1.4 get closer together, and point away from the shoulder, as the arm straightens.

The results in Figure 7.3.3 also suggest a limitation in the control of the stiffness orientation. Near the outer region of the workspace, the actuator stiffness directions are all nearly pointing in the same direction. As shown in Figure 7.3.1, increasing any of the actuator stiffnesses tends to rotate the stiffness towards the direction associated with that stiffness. Hence the orientation is limited to point between these directions. This limitation might be extended by adjusting the actuator stiffnesses such that the major axis shortens and minor axis lengthens until the axes switch roles, thereby changing the orientation by 90° . However, the axes switch roles at the point the stiffness becomes circular, so to avoid the limitation by such a method the stiffness must pass through circular. This is only possible for endpoint positions nearer to the shoulder, as pointed out by Hogan [1985a]. To see this, we solve the range of angles for which the stiffness can be circular. A circular stiffness, referred to as isotropic, corresponds to an endpoint spring stiffness of

$${}^J\mathbf{K} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (7.3.1)$$

Recall from (5.3.16) that ${}^J\mathbf{K} = \mathbf{J}^T \Psi \mathbf{J}^{-1}$. Inverting this relationship and substituting the isotropic stiffness from (7.3.1), the joint stiffness matrix is

$$\begin{bmatrix} {}^s\Psi_s + {}^d\Psi_s & \Psi_d \\ \Psi_d & {}^s\Psi_e + {}^d\Psi_e \end{bmatrix} = k \begin{bmatrix} L_s^2 + L_e^2 + 2L_s L_e \cos \theta_e & L_e^2 + L_s L_e \cos \theta_e \\ L_e^2 + L_s L_e \cos \theta_e & L_e^2 \end{bmatrix} \quad (7.3.2)$$

Substituting for the joint stiffness from (5.2.11), the actuator stiffnesses required to achieve isotropic endpoint spring stiffness are

$$\Psi_{\Sigma d} = \frac{kL_e}{{}^dR_s {}^dR_e} (L_e + L_s \cos \theta_e) \quad (7.3.3a)$$

$$\Psi_{\Sigma s} = \frac{k}{{}^sR_s^2} \left\{ L_s^2 + L_e^2 \left(1 - \frac{{}^dR_s}{{}^dR_e} \right) + \left(2 - \frac{{}^dR_s}{{}^dR_e} \right) L_s L_e \cos \theta_e \right\} \quad (7.3.3b)$$

$$\Psi_{\Sigma e} = \frac{kL_e}{{}^sR_e^2} \left\{ \left(1 - \frac{{}^dR_e}{{}^dR_s} \right) L_e - \frac{{}^dR_e}{{}^dR_s} L_s \cos \theta_e \right\} \quad (7.3.3c)$$

Assuming the net stiffness from the actuators at the joints can only contribute positive stiffness, such as in the human upper limb [McIntyre et al., 1996], (7.3.3) results in the following three conditions:

$$\cos\theta_e \geq -\frac{L_e}{L_s} \quad (7.3.4a)$$

$$\cos\theta_e \geq -\frac{L_s^2 {}^dR_e + L_e^2 ({}^dR_e - {}^dR_s)}{L_s L_e (2 {}^dR_e - {}^dR_s)} \quad (7.3.4b)$$

$$\cos\theta_e \leq -\frac{L_e ({}^dR_e - {}^dR_s)}{L_s {}^dR_e} \quad (7.3.4c)$$

To understand the conditions given by (7.3.4), consider the case where ${}^dR_e = {}^dR_s$.

The conditions then simplify to

$$\cos\theta_e \geq -\frac{L_e}{L_s} \quad (7.3.5a)$$

$$\cos\theta_e \geq -\frac{L_s}{L_e} \quad (7.3.5b)$$

$$\cos\theta_e \leq 0 \quad (7.3.5c)$$

Figure 7.3.4 shows these conditions. The condition in (7.3.5a) is guaranteed for all values of θ_e when $L_e \geq L_s$. The case for $L_e < L_s$ is shown in Figure 7.3.4(a). This condition limits θ_e to no more than the point where L_e and L_s are orthogonal. If the endpoint is closer than this, the stiffness cannot be made isotropic. When $L_e = L_s$, this point occurs at $\theta_e = 180^\circ$. The condition in (7.3.5b) is guaranteed if $L_s \geq L_e$. Figure 7.3.4(b) show the case for this condition if $L_s < L_e$. The position is then limited to the point where L_s and L_e become orthogonal. For endpoint positions closer than this, the

stiffness cannot be made isotropic. Again, this position occurs at $\theta_e = 180^\circ$ when $L_s = L_e$. Note that unlike the first condition, this lower limit generally depends on the relative joint radii as suggested by (7.3.3b).

The final condition in (7.3.4c) places a lower limit of $\theta_e = 90^\circ$ on the allowable positions. Endpoints further from the shoulder than this do not allow the creation of an isotropic stiffness. This limit also generally depends on the relative joint radii. Figure 7.3.4(c) shows the allowable range of endpoint positions in which we can create an isotropic endpoint spring stiffness for the case where $L_s = L_e$, and ${}^dR_s = {}^dR_e$.

Isotropic endpoint spring stiffness cannot be achieved for positions outside of the limits set by (7.3.3). We therefore cannot reverse the major and minor axes by adjusting the actuator stiffnesses in the outer region (beyond about 70% of the arm's reach for the example in Figure 7.3.4(c)) or in the region very near the shoulder if $L_s \neq L_e$ or ${}^dR_s \neq {}^dR_e$. This means that in these areas, the stiffness orientation is limited such that the minor axis must lie within the range of directions shown as β in Figure 7.3.5.

Figure 7.3.4. Limitations on isotropic endpoint spring stiffness positions, with ${}^dR_s = {}^dR_e$, (a) $L_s > L_e$, (b) $L_s < L_e$, (c) region of workspace ($0 \leq \theta_s \leq 90^\circ$, $0 \leq \theta_e \leq 180^\circ$) for which isotropic endpoint spring stiffness can be achieved when $L_s = L_e$.

It should be noted that the limitations on orientation discussed above only govern the possibility of orientation ranges. Even in cases where the orientation can be flipped 90° by reversing the major and minor axes, there will still be limitations on the size and shape of the stiffness. For example, the second closest sample endpoint in Figure 7.3.3 goes from round in (a) to a horizontal orientation in (d) by increasing the single-joint elbow stiffness. Further such increases will make the endpoint stiffness longer and thinner, but still in the horizontal direction. Reducing the single-joint elbow stiffness will have the opposite effect – increasing the vertical axis length and decreasing the horizontal axis length. This will make the stiffness longer in the vertical direction. However, the single-joint elbow stiffness cannot be reduced below 0 Nm/rad, therefore there is a limitation on how long and skinny we can make the stiffness with a vertical orientation. Such limitations occur when any of the stiffnesses reach 0 Nm/rad.

In Section 7.3 we demonstrated:

- the effect of changing the individual actuator stiffnesses on the stiffness ellipse,
- the physical nature of the projection directions based on their curved perpendiculars in the non-linear isopotentials,
- the effect of endpoint location on the stiffness ellipse and the projection directions of the actuator stiffnesses, and,
- the limitations on the orientation of the stiffness.

Figure 7.3.5: Endpoint spring stiffness orientation limitations. The minor axis of the ellipse must lie in the range shown as β . 'S', 'E', and 'D' refer to the projection directions associated with the single-joint shoulder, single-joint elbow, and double-joint actuator pairs.

7.4 Geometric Stiffness

In this section we will look at the effect of the geometric stiffness on the overall stiffness. We will look at how it can contribute to the overall stiffness ellipse, when it is negligible, and what happens when it is unstable. We will also determine an acceptable level for the ellipse comparison index for which one can neglect the geometric stiffness.

7.4.1 Contribution of Geometric Stiffness

The geometric stiffness is often neglected because it is assumed to be small when the endpoint forces are small. This leads to several questions that have been unanswered in previous work. How large does the force have to be for the geometric stiffness to be relevant? Can it be neglected simply because it is small?

We examine these questions by performing simulations under various conditions and checking the amount of change in the stiffness ellipse. For our first set of simulations, we vary the force from 0 to 10 N (2.2 lb) at various angles to the endpoint in different positions. To begin, we start with the joint stiffnesses set at

$$\Psi = \begin{bmatrix} 40 & 20 \\ 20 & 40 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

This corresponds to each actuator pair set at 25% of the 80 N·m/rad maximum, and is of consistent magnitude with measured stiffnesses of the human arm during normal tasks

[Mussa-Ivaldi et al., 1985]. In fact, it is near the high end of measured stiffnesses in normal arm motions [Gomi & Kawato, 1996], and therefore the spring stiffness is providing as large a contribution as would normally be expected. Figure 7.4.1 shows the results for $\theta_s = 45^\circ$ and $\theta_e = 90^\circ$. The force is applied at $0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, -135^\circ, -90^\circ$, and -45° consecutively from (a) to (h). The top plots show the actual change in the stiffness ellipse from 0 N, where $\mathbf{K} = {}^J\mathbf{K}$, to 10 N. The bottom plots show the ellipse comparison index, ξ_J , (referred to as ECI in the plots) from (6.3.14) and the percentage change in $k_{\max}$, $k_{\min}$, and θ_k .

The $\mathbf{K}$ isopotential ellipse is taken to be the true stiffness, and the ${}^J\mathbf{K}$ ellipse as the approximation for the overall stiffness. Any differences can then be considered errors in the approximation. The first result to note is that there are errors greater than 10% on the minimum stiffness for an applied force of less than 10 N (2.2 lb). Since the weight of the arm itself is capable of applying equivalent forces greater than this, this force is not considerably large. This suggests that the assumption of neglecting the geometric stiffness for small forces may be questionable, at least for force values that are of any significance to contact tasks. As mentioned above, the stiffnesses here are high. Lower stiffnesses produce higher errors than those observed in Figures 7.4.1.

(a) $\theta_F = 0^\circ$

Figure 7.4.1: Contribution of ${}^G\mathbf{K}$ to overall stiffness, $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$. ${}^J\mathbf{K}$ is indicated by the solid ellipse (—) and $\mathbf{K}$ is indicated by the dotted ellipse (---). Top plots show the change in ellipse by including the geometric stiffness. Bottom plots show the Ellipse Comparison index and errors in ellipse parameters by not including the geometric stiffness.

(b) $\theta_F = 45^\circ$

Figure 7.4.1 (continued)

(c) $\theta_F = 90^\circ$

Figure 7.4.1 (continued)

(d) $\theta_F = 135^\circ$

Figure 7.4.1 (continued)

(e) $\theta_F = 180^\circ$

Figure 7.4.1 (continued)

(f) $\theta_F = -135^\circ$

Figure 7.4.1 (continued)

(g) $\theta_F = -90^\circ$

Figure 7.4.1 (continued)

(h) $\theta_F = -45^\circ$

Figure 7.4.1 (continued)

The second property to note is that for some directions of force, the $\mathbf{K}$ ellipse is outside, or partially outside, the $^J\mathbf{K}$ ellipse (which is equal to the $\mathbf{K}$ ellipse at zero force). This means that some of the stiffness actually decreases with increasing force, which requires $^G\mathbf{K}$ to contribute negatively, as discussed in Section 6.4. For applied force directions of 0° , 180° , and -135° , the ellipses intersect and hence $^G\mathbf{K}$ is a saddle. For force directions of -90° and -45° , the $\mathbf{K}$ ellipse is entirely outside, and hence $^G\mathbf{K}$ is unstable in all directions. However, since the total $\mathbf{K}$ matrix is able to produce an ellipse, the $^G\mathbf{K}$ contribution is not enough to make the whole stiffness go unstable.

A third observation is that the largest errors seem to occur when the force is applied in the direction of the minor axis, i.e., the highest stiffness. The parameter with the largest resulting error is usually the smallest stiffness, $k_{\min}$. A possible explanation for this characteristic is that smaller values are more sensitive to changes. The errors also appear to be close to linear with force.

Finally, a comparison of the changes in Figure 7.4.1 shows that the percentage errors are identical for forces in opposite directions (e.g., (a) and (e)). However, they result in opposite effects on the overall ellipse. If the force in a particular direction reduces the size and orientation of the ellipse by certain percentages, then reversing the force increases the size and reverses the change in orientation by the same percentages. That is, the stiffness values increase or decrease by the same percentage, but in opposite

directions. Since only negative contributions from the geometric stiffness can increase the size of the overall ellipse, this suggests that forces in one direction create a completely stable geometric stiffness while the opposite direction creates a completely unstable geometric stiffness. In the case of a saddle, reversing the forces reverses the saddle direction (the stable direction become unstable and vice-versa).

The above directional observation can be proven analytically. Recall from (5.3.15) and (5.3.16) that

$${}^G\mathbf{K} = \mathbf{G}\mathbf{J}^{-1} = (\mathbf{G}_s\mathbf{M}_s + \mathbf{G}_e\mathbf{M}_e)\mathbf{J}^{-1}$$

From (5.3.3) the moments are

$$\begin{bmatrix} M_s \\ M_e \end{bmatrix} = \begin{bmatrix} -L_t S_t & L_t C_t \\ -L_e S_{es} & L_e C_{es} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

We can then write the geometric stiffness as

$${}^G\mathbf{K} = |\mathbf{F}| (\mathbf{G}_x \cos(\theta_F) + \mathbf{G}_y \sin(\theta_F))\mathbf{J}^{-1} \quad (7.4.1)$$

where $|\mathbf{F}|$ is the magnitude of the endpoint force

θ_F is the orientation of the force with respect to the dx-dy coordinate system

$$\mathbf{G}_x = -L_t S_t \mathbf{G}_s - L_e S_{es} \mathbf{G}_e$$

$$\mathbf{G}_y = L_t C_t \mathbf{G}_s + L_e C_{es} \mathbf{G}_e$$

Using the trigonometric identities $\cos(\theta_F + \pi) = -\cos(\theta_F)$ and $\sin(\theta_F + \pi) = -\sin(\theta_F)$, we can

see from (7.4.1) that

$${}^G\mathbf{K}\{\theta_F+\pi\} = -{}^G\mathbf{K}\{\theta_F\} \quad (7.4.2)$$

So, ${}^Gk_{\max}$ and ${}^Gk_{\min}$ both switch sign when the force is applied in the opposite direction. In other words, if ${}^G\mathbf{K}$ is stable (${}^Gk_{\max}$ and ${}^Gk_{\min}$ positive) with force applied in one direction then ${}^G\mathbf{K}$ is completely unstable (${}^Gk_{\max}$ and ${}^Gk_{\min}$ negative) with force applied in the opposite direction. Likewise, if ${}^G\mathbf{K}$ is a saddle (one of ${}^Gk_{\max}$ and ${}^Gk_{\min}$ positive, the other negative) with force applied in one direction, ${}^G\mathbf{K}$ is a saddle with the stability directions reversed (signs on ${}^Gk_{\max}$ and ${}^Gk_{\min}$ reversed).

The next set of simulations examines the effect of position in the workspace on the contribution of the geometric stiffness. Figure 7.4.2 shows the results of the simulations. The stiffness is set as in the previous simulations. The arm is varied through four positions in 15° increments, $(\theta_s, \theta_e) = (15^\circ, 150^\circ), (30^\circ, 120^\circ), (60^\circ, 60^\circ), (75^\circ, 30^\circ)$. The middle position of $(45^\circ, 90^\circ)$ can be seen in Figure 7.4.1(g). While the force direction was also varied at each position as in Figures 7.4.1, only the results for an applied force at -90° (towards shoulder), where the maximum error was seen, are shown in Figure 7.4.2. The results for the other force directions exhibited similar characteristics as those in the previous figures.

(a) $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$

Figure 7.4.2: Effect of position on contribution from $^0\mathbf{K}$. Force is applied at $\theta_F = -90^\circ$. $^1\mathbf{K}$ is indicated by the solid ellipse (—) and $\mathbf{K}$ is indicated by the dashed ellipse (---).

(b) $(\theta_s, \theta_e) = (30^\circ, 120^\circ)$

Figure 7.4.2 (continued)

(c) $(\theta_s, \theta_c) = (60^\circ, 60^\circ)$

Figure 7.4.2 (continued)

(d) $(\theta_s, \theta_e) = (75^\circ, 30^\circ)$

Figure 7.4.2 (continued)

Figure 7.4.2 shows that the maximum error tends to be much larger the further the endpoint is from the body. Errors in $k_{\min}$ of close to 20% are seen at the furthest position. Note that the force in Figure 7.4.2 is applied in the direction of the major axis for positions close to the shoulder because the ellipse is oriented towards the shoulder in this area. Since we previously noted that the largest errors seemed to appear when the force was applied along the minor axis, one could suggest that this is the reason for the apparent correlation between position and error. Figure 7.4.3 shows the force applied along the minor axis for the nearest position of $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$. The maximum error, in this case for orientation, is only about 2%, which is even smaller than the error for the same position in 7.4.2(a). We must therefore append our observation to state that errors in stiffness values, due to neglecting ${}^G\mathbf{K}$, are higher near the outer region of the workspace. Orientation errors do not seem to follow this same behaviour. Section 7.4.5 will further investigate the behaviour of ${}^G\mathbf{K}$ with respect to endpoint position.

Figure 7.4.3: Effect of ${}^G\mathbf{K}$ at $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$ with $\theta_F = 0^\circ$. ${}^I\mathbf{K}$ is indicated by the solid ellipse (—) and $\mathbf{K}$ is indicated by the dashed ellipse (---).

The characteristics exhibited in Figures 7.4.2 and 7.4.3 suggest that *overall* largest error occurs at the outer range of the workspace, with the direction of force along the minor axis (maximum stiffness), and this error is in the major axis stiffness (minimum stiffness) value. When operating close to the shoulder, the largest error is often the orientation. This error seems to be largest when the ellipse is rounder. To see this, we examine the orientation error near a point when it is round, in this case ($30^\circ, 120^\circ$) shown in Figure 7.4.2(b). Since the ellipse at 0 N is round, it has no orientation, and hence one cannot calculate an error. Figure 7.4.4 shows the ellipses at ($29^\circ, 122^\circ$), with force applied at 0° . Here the error is about 19% (of a 180° rotation) for a 10 N force. At points closer to the shoulder or further away this error is much lower. Hence the error appears to be more a function of how round the ellipse is rather than a function of the position of the arm.

Figure 7.4.4: Effect of G^K on orientation at $(\theta_s, \theta_e) = (29^\circ, 122^\circ)$, $\theta_F = 0^\circ$. $^J K$ is indicated by the solid ellipse (—) and K is indicated by the dashed ellipse (---).

Finally, we examine the effect of the joint stiffnesses on the error. We begin with the conditions as in Figure 7.4.1(a). We then increase all stiffnesses to their maximum values such that the joint stiffness matrix is given by

$$\Psi = \begin{bmatrix} 160 & 80 \\ 80 & 160 \end{bmatrix} \text{ N} \cdot \text{m} / \text{rad}$$

This result is shown in Figure 7.4.5(a). For the remaining plots in Figure 7.4.5, we start with the original stiffnesses (20 N·m/rad for each pair) and in turn increase each pair to 80 N·m/rad, resulting in the above joint stiffness matrix. In each case, the force is applied at $\theta_F = -90^\circ$. In (a), we see that increasing all of the stiffnesses has decreased the error in $k_{\min}$ to about 3.5% at 10 N. In other words, the importance of ${}^G\mathbf{K}$ depends on the relative size of ${}^I\mathbf{K}$. In (b)-(d) we can see that increasing the single-joint shoulder stiffness or double-joint stiffness significantly reduces the largest errors, but the single-joint elbow stiffness has no apparent effect on the errors. This is because the single-joint elbow stiffness only changes the isopotential ellipse in the direction perpendicular to the shoulder, which in this case is the major axis of the ellipse.

(a) Equal stiffnesses at 80 N·m/rad

Figure 7.4.5: Effect of changing stiffnesses on contribution of ${}^0\mathbf{K}$, with $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$ and the external force is applied at $\theta_F = -90^\circ$. ${}^1\mathbf{K}$ is indicated by the solid ellipse (—) and $\mathbf{K}$ is indicated by the dashed ellipse (---).

(b) Double-joint stiffness at 80 N·m/rad

Figure 7.4.5 (continued)

(c) Single-joint shoulder stiffness at 80 Nm/rad

Figure 7.4.5 (continued)

(d) Single-joint elbow stiffness at 80 N·m/rad.

Figure 7.4.5 (continued)

In reviewing all the figures of this section, it seems that significant errors, on the order of 5% to 10% tend to occur for values of the Ellipse Comparison Index, ξ_j , near 0.05. While the amount of acceptable error depends on the application, this range of error is typically considered significant and hence we suggest that the geometric stiffness not be neglected for values of $\xi_j > 0.05$. This corresponds to a 5% shift in the ellipse area.

In summary, the following characteristics can be observed:

- 1) The relevance of the geometric stiffness, ${}^G\mathbf{K}$, depends on the relative size of the endpoint spring stiffness, ${}^J\mathbf{K}$. It can have a significant effect (>20%) even for small contact forces at the levels of measured joint stiffness in human.
- 2) The geometric stiffness appears to contribute more to the overall stiffness for endpoint positions further from the body. This will be further investigated in Section 7.4.5.
- 3) The largest error usually occurs when the external force is applied towards the shoulder. The largest error in these cases is the major axis stiffness (minimum stiffness) value, and the error seems to vary linearly with applied force magnitude.
- 4) When the force is applied perpendicular to the shoulder direction, the largest error is typically the minor axis stiffness, $k_{\max}$. However, this error is much smaller than the $k_{\min}$

error when the force is applied toward the shoulder.

5) The geometric stiffness does not affect the orientation of the ellipse to as great a degree as the axes stiffnesses, except when the ellipse is near round. However, in such cases, the stiffness in all directions is nearly the same and so orientation errors are likely not a big concern.

6) Reversing the applied force direction has an opposite but equal effect on the errors. Stiffnesses that increase for force in one direction will decrease for forces in the opposite direction by the same percentage (for the same force magnitude). Orientation changes reverse when the force is reversed. The geometric ellipse changes from stable with force in one direction to unstable with force in the opposite direction. For saddles, the stable and unstable directions reverse. (Section 7.4.5 will investigate this further.)

7) While increasing any of the actuator stiffnesses reduces errors caused by neglecting the geometric stiffness, the single-joint elbow stiffness seems to have the least effect on reducing these errors.

8) The Ellipse Comparison Index, ξ_j , indicates that the geometric stiffness can become significant for values greater than 0.05.

7.4.2 Instability

In the previous section we looked at the case where the geometric stiffness contributed a sufficient amount to the overall stiffness such that it shouldn't be neglected, even for reasonably small forces. However, in none of the cases examined did the geometric stiffness dominate, or cause an instability in the overall stiffness. We now search for the force level that can lead to instability of the overall stiffness.

From the previous section we know that the geometric stiffness seems to have the most effect for endpoint positions near the outer range of the workspace, and with forces applied toward the shoulder. Figure 7.4.6 shows the results of a simulation for which the arm is at the position $(\theta_s, \theta_e) = (75^\circ, 30^\circ)$ and the force is applied at -90° . In this case the force is varied from 0 N to 200 N. In the figure we can see that the Ellipse Comparison Index goes to infinity at a force level of 51.8 N (11.6 lb) and 155.3 N (34.9 lb). This means the shifted area becomes infinite, which occurs when the major or minor axes become infinitely large ($k_{\min} = 0$ or $k_{\max} = 0$). Beyond 51.8 N, the endpoint stiffness is unstable. The fact that this instability occurs at a medium level of force (~ 50 N) and normal levels of joint stiffness suggests that the geometric stiffness may have an important role in control decisions such as stiffness settings, acceptable interactive force levels, and area of the workspace in which contact tasks are best performed. It also suggests that spring functions that cause the joint stiffness to increase upon displacement, as in (7.1.2), may be wise in light of stabilizing endpoint forces, though they may require

feedback to determine the current joint stiffness values.

Note that ξ_j shrinks for forces above 51.8 N and 155.3 N because the shifted area upon which it is based becomes negative, but finite, when the ellipse becomes a hyperbola or imaginary. The stiffness is still unstable for these forces. This brings to attention one disadvantage of the Ellipse Comparison index; it is only designed to handle overall stable cases where ellipses can be defined. However, an advantage of the index shown in this case is that as it gets large towards infinity, it is apparent that the endpoint stiffness is nearing instability, and a small increase in force might cause it to go unstable.

Figure 7.4.6: Ellipse Comparison Index for $(\theta, \theta_c) = (75^\circ, 30^\circ)$, $\theta_F = -90^\circ$.

Since no ellipse can be formed when endpoint stiffness is unstable, we use a Mohr's Circle representation. Figure 7.4.7 shows the Mohr's Circle for ${}^J\mathbf{K}$, ${}^G\mathbf{K}$, and $\mathbf{K}$ at 20 N, 60 N, and 200 N in the above simulation. The geometric stiffness has a negative ${}^Gk_{\max}$ and ${}^Gk_{\min}$, and hence it is completely unstable in all directions. The endpoint spring stiffness, ${}^J\mathbf{K}$, is naturally stable because the joint stiffnesses are all positive. The overall endpoint stiffness, $\mathbf{K}$, starts completely stable in (a) but $k_{\min}$ shifts to the right as ${}^G\mathbf{K}$ grows bigger, and it becomes negative in (b). Since $k_{\max}$ is still positive, this makes the endpoint stiffness a saddle. When the force is increased above 155.3 N, $k_{\max}$ also becomes negative, and the endpoint stiffness is unstable in all directions.

For positions closer to the shoulder, the force must be higher to produce an instability. In Figure 7.4.8 we move the position to $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$ and perform the same calculation, except with the force increased to a 250 N maximum. The endpoint stiffness becomes unstable at 70.8 N (15.9 lb) and goes unstable in all directions at 212.2 N (47.7 lb).

Figure 7.4.7: Mohr's Circles for σ_K (---), σ_{KJ} (---), and σ_K (—), $(\theta_s, \theta_e) = (75^\circ, 30^\circ)$, $\theta_F = -90^\circ$.
(a) 20 N, (b) 60 N, (c) 200 N.

Figure 7.4.8: Ellipse Comparison Index for $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$, $\theta_F = -90^\circ$.

Normal daily activities can involve forces in excess of the values discussed above, particularly those values where the stiffness becomes a saddle. Other tasks can involve forces well above 250 N (56 lb), such as weightlifting and heavy labour work. The results above therefore strongly suggest that control of joint stiffnesses, workspace position, and orientation of the forces in the workspace must take into account the effects of geometric stiffness. These issues will be addressed in Chapter 8. For now, we will address the issue of what actually happens when the endpoint is unstable.

A dynamic simulation best illustrates the consequences of an unstable endpoint. The simulation begins with the arm at the position $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$ and the joint stiffnesses each set at 20 N·m/rad as in the above simulations. A force of 90 N is applied at an angle of -45° . The starting position is in static equilibrium, but it is a saddle point and so is unstable. Figure 7.4.9 shows the results. In Figure 7.4.9(a) the motion of the arm is seen. The dotted line represents the fixed virtual position of the arm. The dashed line is the starting position as described above. The solid lines represent the finish point of the manipulator.

Figure 7.4.9: Dynamic simulation starting from an equilibrium saddle point. (a) Motion of the endpoint. The dotted line is the constant virtual point, the dashed line is the starting equilibrium position (saddle), and the dash-dotted line is the finished equilibrium position. (b) Energy surface plot. The start point is in the middle of the plot, with the finish at the low point on the right. (c) Close-up of saddle energy surface at start point. (d) Close-up of stable bowl shape at finish point.

The path of the endpoint is traced out between the start and finish point. This path illustrates the behaviour of the endpoint in the face of the instability. The arm moves away from its set position to a new point in the workspace. This point must be a stable equilibrium point in order for it to maintain static equilibrium with the applied force. This is clearly shown in (c) and (d) which are the start and finish points. The starting position in (c) is at a high point but is in equilibrium because the energy surface is horizontal at that point. This is a typical saddle point. The endpoint then slides down the energy surface to a low point at $(\theta_s, \theta_e) = (23.2^\circ, 60.9^\circ)$. This low point is a stable equilibrium point, as shown by the close-up of the energy surface in (d). Simulations of other saddle points show similar results.

The energy surface in Figure 7.4.9(b) also shows that there may be another equilibrium in the opposite direction as well. To check this situation, we perform the same simulation as above, but with the force applied at -90° . Figure 7.4.10 shows the results. In this case, the arm falls down the other side of the saddle point. It is interesting to note that the energy slope is steeper in this direction, and the arm moves faster in this direction than in the previous simulation. One could suggest then that the counterclockwise direction is “more unstable” (greater negative stability margin) than the clockwise direction, a fact that is not evident from the linearized stiffness values. By “more unstable”, we mean that the arm falls away from the starting point at a much quicker rate.

Figure 7.4.10: Dynamic simulation starting from an equilibrium saddle point. Opposite direction to Figure 7.4.9. (a) Motion of the endpoint. The dotted line is the constant virtual point, the dashed line is the starting equilibrium position (saddle), and the dash-dotted line is the finished equilibrium position. (b) Energy surface plot. The start point is in the middle of the plot, with the finish at the lowest point. (c) Close-up of saddle energy surface at start point. (d) Close-up of stable bowl shape at finish point.

A third dynamic simulation illustrates the behaviour of a completely unstable endpoint. In this case, a force of 230 N is applied at -90° . Figure 7.4.11 shows the results. Although all directions are unstable, the direction that was originally unstable in the saddle case is much steeper here and so the arm still “falls” in that direction. In other words, the fact that the starting point has become completely unstable does not affect the overall behaviour differently than when the starting point was a saddle point. The main difference is that a perturbation in the direction towards the shoulder is no longer returned to the same path as in the saddle case.

Note Figure 7.4.11(b) is shown in the y-z plane whereas Figures 7.4.9 and 7.4.10 are in the x-z plane. The orientation is changed to show the arm motion from one energy surface to another. The upper energy surface corresponds to $\theta_e \geq 0$. The lower energy surface corresponds to $\theta_e \leq 0$ (elbow bent back). The two surfaces meet at $\theta_e = 0$ (furthest reach of workspace). The endpoint path moves from one energy surface to the other and comes to rest on the lower surface ($\theta_e < 0$, as seen in (a)).

Figure 7.4.11: Dynamic simulation starting from an equilibrium point unstable in all directions. (a) Motion of the endpoint. The dotted line is the constant virtual point, the dashed line is the starting equilibrium position (saddle), and the dash-dotted line is the finished equilibrium position. (b) Energy surface plot (y-z plane). The start point is in the middle of the plot, with the finish at the lower left. The upper energy surface corresponds to $\theta_e \geq 0$. The lower energy surface corresponds to $\theta_e \leq 0$ (elbow bent back). The endpoint path moves from one energy surface to the other and comes to rest on the lower surface ($\theta_e < 0$, as seen in (a)). (c) Close-up of energy surface at start point. (d) Close-up of stable bowl shape at finish point.

7.4.3 Characteristics of the Geometric Stiffness

The previous two sections discussed how the geometric stiffness contributed to the overall stiffness in both stable, saddle, and completely unstable cases. Some of those results have hinted at the behaviour of the geometric stiffness as a function of position in the workspace and applied force direction. In this Section we will look at these behaviours specifically so that we can better characterise the geometric stiffness.

Recall from (7.4.1) that

$${}^0\mathbf{K} = |\mathbf{F}| (\mathbf{G}_x \cos(\theta_F) + \mathbf{G}_y \sin(\theta_F)) \mathbf{J}^{-1} \quad (7.4.1)$$

where $|\mathbf{F}|$ is the magnitude of the endpoint force

θ_F is the orientation of the force with respect to the dx-dy coordinate system

$$\mathbf{G}_x = -L_1 S_1 \mathbf{G}_s - L_e S_{e1} \mathbf{G}_e$$

$$\mathbf{G}_y = L_1 C_1 \mathbf{G}_s + L_e C_{e1} \mathbf{G}_e$$

So, the magnitude of the endpoint force only serves to scale the geometric stiffness. It cannot affect its orientation or behaviour as a function of workspace position or force direction.

In Section 7.4.1, simulation results suggested that the geometric stiffness contributes more near the outer range of the workspace. We examine this behaviour by plotting the Mohr's Circle representation of geometric stiffness at $(\theta_s, \theta_e) = (15^\circ, 150^\circ)$, $(30^\circ, 120^\circ)$, $(45^\circ, 90^\circ)$, $(60^\circ, 60^\circ)$, and $(75^\circ, 30^\circ)$. Figure 7.4.12 shows the results. At each

location, a force of 10 N is applied in eight different directions: 0° , 45° , 90° , 135° , 180° , -135° , -90° , and -45° . The results in previous sections also suggested that the geometric stiffness had an equal but opposite effect when the force direction is reversed. We show this by plotting the opposite force directions on the same plot. For example, Mohr's Circles for 0° and 180° are shown in the same plots. The solid circles are for 0° , 45° , 90° , and 135° , while the dotted circles are for 180° , -135° , -90° , and -45° .

The first characteristic to note is that reversing the force makes the geometric stiffness become the exact opposite, but at the same orientation. The Mohr's Circles become mirror images about the origin (mirrored about both axes). This illustrates the behaviour described by (7.4.2),

$${}^G\mathbf{K}\{\theta_F\} = -{}^G\mathbf{K}\{\theta_F + \pi\} \quad (7.4.2)$$

It is interesting that all of the forces that pull upwards on the endpoint make more stable stiffnesses (to the right in the plots) than the forces that push down on the endpoint. This makes sense from a physical perspective. Pulling on a pendulum is inherently stable while pushing on one is not.

The next characteristic of interest is the orientation of the stiffness. (Recall from Section 6.5 that orientation is indicated by half the angle between the diameter line across the Mohr's circle and the horizontal axis.) In all the plots of 7.4.12, an endpoint force

towards or away from the shoulder ($\pm 90^\circ$) creates a stiffness oriented in the same direction (either $G_{k_{\max}}$ or $G_{k_{\min}}$ towards the shoulder, depending on the position). Endpoint forces perpendicular to the shoulder direction (0° and 180°) produce stiffnesses oriented 45° to the shoulder direction (90° in the Mohr's circle). Force between these directions create orientation between these values. However, this relationship doesn't appear to be linear. The same force orientation, other than the extremes mentioned above, produces different stiffness orientations depending on the endpoint distance from the shoulder. As the position moves from close to the shoulder to further out, the stiffness orientation for these other force angles ($\pm 45^\circ$, $\pm 135^\circ$) appears to rotate, but apparently not proportionally with distance from the shoulder. (This will be looked at in more detail later in this Section.)

(a) $(\theta_s, \theta_c) = (15^\circ, 150^\circ)$

Figure 7.4.12: Mohr's Circles for geometric stiffness. Titles indicate the direction of force, θ_F . Axes are units of stiffness. Forces are applied at 0° and 180° , 45° and -135° , 90° and -90° , and 135° and -45° . Forces in the opposite direction are plotted in the same figure. Solid lines (—) are for the force directions 0° , 45° , 90° , and 135° . Dashed lines (---) are for 0° , 180° , -135° , -90° , and -45° .

(b) $(\theta_s, \theta_c) = (30^\circ, 120^\circ)$

Figure 7.4.12 (continued)

(c) $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$

Figure 7.4.12 (continued)

(d) $(\theta_s, \theta_e) = (60^\circ, 60^\circ)$

Figure 7.4.12 (continued)

(e) $(\theta_s, \theta_e) = (75^\circ, 30^\circ)$

Figure 7.4.12 (continued)

The dependence of the geometric stiffness on force direction also appears in the stiffness size. Recall from Section 6.5 that stiffness size for saddle stiffnesses is indicated by the height on the vertical axis to the intersection point with the Mohr's circle, and for other stiffnesses it is the length from the origin to the tangent point on the Mohr's circle. For positions near the body, $\theta_F = 0^\circ$ and 180° have the largest overall stiffnesses. For positions near the outer region of the workspace (beyond $\theta_c = 90^\circ$), forces in the $\pm 90^\circ$ direction produce the largest stiffnesses.

Figure 7.4.12 also shows a pattern in stiffness shape. The shape factor S_k , as defined in Section 6.5, is the ratio of the Mohr's circle center distance to radius. For $\theta_F = 0^\circ$ and 180° , the shape factor is $S_k = 0$ regardless of the arm position. This means that a force applied perpendicular to the direction from hand to shoulder will always produce a saddle stiffness whose positive and negative principle stiffnesses are equal in magnitude ($G_{k_{\max}} = G_{k_{\min}}$). Forces applied towards or away from the shoulder ($\theta_F = \pm 90^\circ$) produce long, thin stiffnesses near the shoulder (as in (a)) and far from the shoulder (as in (e)), but get rounder near the middle region of the workspace. At $\theta_c = 90^\circ$, the geometric stiffness is circular, as shown by the stiffnesses with zero radius in (c).

To understand these complex relationships between stiffness, force direction, and distance from the shoulder, we will plot these behaviours in three dimensions. First, however, we will take some cross-sections to properly understand what will be shown.

Figure 7.4.13 shows the characteristics at the position $(\theta_s, \theta_e) = (30^\circ, 120^\circ)$. The characteristics shown in the figure only apply to this arm position. The general descriptions will follow later in this section.

In Figure 7.4.13 (a) the principal stiffnesses are shown. As seen in the Mohr's Circle representations, these stiffness relationships are identical, but upside down and shifted by 180° . The orientation is given in (b), which is seen to change very rapidly (with respect to force direction) near $\theta_F = \pm 90^\circ$ (and 270° , which is equivalent to -90°). This is consistent with the behaviour seen in Figure 7.4.12. The orientation changes much slower near $\theta_F = 0^\circ$ and 180° .

Figure 7.4.13(c) shows the shape factor, S_k , versus the force direction. As in Figure 7.4.12, $S_k = 0$ at $\theta_F = 0^\circ$ and 180° , indicating a saddle with equal magnitudes for positive and negative principle stiffnesses. The shape factor is positive for force directions between 0° and 180° , indicating that it is stable in this range. A shape factor of $S_k = 1$ indicates the border between a stable and saddle stiffness. This occurs at $\theta_F = 60^\circ$ and 120° for the arm position shown in the figure. Similarly, the border between saddle and completely unstable stiffness occurs when $S_k = -1$, which occurs at $\theta_F = -60^\circ$ and 240° ($= -120^\circ$). The only stable conditions, for the example arm position in Figure 7.4.13, occur for forces applied between 60° and 120° . Large shape factor magnitudes indicate a more rounded stiffness. The roundest stable stiffness occurs at $\theta_F = 90^\circ$, while the

roudest unstable stiffness occurs at $\theta_F = -90^\circ (= 270^\circ)$.

In (d) we show the size of the stiffness as defined in Section 6.5, k_{avg} . The magnitude of the size is not important here, but the relative size with respect to force direction is of interest. We see that the size is the largest for $\theta_F = 0^\circ$ and 180° , for the example arm position in the figure. According to the shape factor in (c), these force directions correspond to saddle points. Another local maximum in size occurs at $\theta_F = \pm 90^\circ$ (and 270°). Combining this information with the results in (c), the roundest and largest stable stiffness occurs at $\theta_F = 90^\circ$ and the roundest and largest unstable stiffness occurs at $\theta_F = -90^\circ (= 270^\circ)$.

A size of $k_{avg} = 0$ indicates that one of the principal stiffnesses is zero, meaning that displacement of the endpoint in some direction will result in no force response, neither returning the endpoint back toward the virtual position nor pushing it away. In (d) this occurs at $\theta_F = \pm 60^\circ$ and $\pm 120^\circ$, corresponding to the borders between saddle and stable or completely unstable in (c) as discussed above. This result is expected however, since these borders must occur when one principal stiffness is zero (switches from positive to negative).

Figure 7.4.13: Plot of stiffness characteristics as functions of force direction for position at $(\theta_s, \theta_c) = (30^\circ, 120^\circ)$. (a) Minor axis (largest) stiffness. (b) Major axis (smallest) stiffness. (c) Stiffness orientation. Angle is with respect to the major axis stiffness. (d) Size of the stiffness.

Since the force direction repeats every 360° , it is possible to plot Figure 7.4.13 as polar plots. Unfortunately, it is difficult to read the orientation plot in this fashion. Figure 7.4.14 shows the same stiffness, shape, and size plots from Figure 7.4.13. In (a) the principal stiffnesses $^Gk_{\max}$ and $^Gk_{\min}$ are plotted as the radius for a given force angle. Both principal stiffness produce the same figure, therefore one figure suffices for both. To read stiffnesses from (a), draw a line at a given force angle θ_F as shown in the figure. It will intersect the curve twice. These are the two principal stiffnesses. (If it only intersects the curve once, then both principle stiffnesses are equal at that value.) If the line must go in the opposite direction to the angle, as in (a), the stiffness is negative. For the example θ_F shown in Figure 7.4.14(a), $^Gk_{\max}$ (thick solid line) is positive and $^Gk_{\min}$ (thick dashed line) is negative. The magnitudes of the stiffnesses are given by the lengths of the lines from the origin, and the axes provide a scaling factor for determining the size. The behaviour of the principal stiffnesses described in Figure 7.4.13 (a) can be better seen in Figure 7.4.14(a). For example, both stiffnesses are positive for only a short range between $\theta_F = 60^\circ$ and 120° , and are both negative between $\theta_F = -60^\circ$ and -120° .

Figure 7.4.14(b) shows the shape factor, S_k , as a polar plot. As in (a), the magnitude of S_k can be read as the radius of the line from the origin at a value of θ_F . There are actually two overlapping plots in (b); the first runs from $\theta_F = 0^\circ$ to 180° with positive S_k , and the second runs from $\theta_F = -180^\circ$ to 0° with negative S_k .

Finally, Figure 7.4.14(c) shows the size, k_{avg} , of the geometric stiffness as a function of force direction. Since size is an absolute value, no negative values result. However, the figure does show a double symmetry of the size. The same behaviour described above for Figure 7.4.13(d) can be seen in this figure.

Figure 7.4.14: Polar plots for the position $(\theta_s, \theta_e) = (30^\circ, 120^\circ)$, Q_F is θ_F . Axes indicate scaling factors for radial magnitudes in the figures, in units given in the figure title. (a) For each force direction, a line drawn in that direction intersects the curve twice. The radii of these two intersections are the principal stiffness values for that force direction. (b) Shape factor of the stiffness as a function of force direction. For negative θ_F , the shape is negative. (c) Stiffness size. The two horizontal petals represent saddle stiffnesses. The top petal is a stable stiffness, while the bottom petal is an unstable stiffness.

We can produce plots similar to those in Figure 7.4.14 for each position from the shoulder to fully outstretched. By piecing these cross-sections together we can get a three dimensional plot. Figures 7.4.15 to 7.4.17 show such plots for principal stiffnesses, shape, and size. The origins for the cylindrical plots are drawn by a thick black line up the middle. The axes values are not shown because their absolute values are not of concern here. The same plots are produced by any force magnitude, with only the axes values scaled. The force magnitude does not affect the shape of the three-dimensional plot since it only serves to scale the stiffness, as indicated by equation 7.4.1.

The plots in Figures 7.4.15 to 7.4.17 are generalized for endpoint location, noting that the reference direction from which to measure θ_s , and therefore its value, is arbitrary. The analysis is generalized for all directions by defining the x-y coordinate frame such that the endpoint always lies on the y-axis. This definition results in the constraint,

$$\theta_s = 90^\circ - \frac{1}{2}|\theta_e| \quad (7.4.3)$$

This definition results in the zero reference line for θ_F as being perpendicular to L_r (see Figure 5.3.1). θ_e is measured relative to the upper arm and so its value is relevant. The only limitation on the following analysis, based on the geometric stiffness, is the values chosen for L_s and L_e . We will attempt to minimize this limitation by examining the observed behaviours for different values of L_s and L_e and noting when they affect the results. The following results should therefore describe the general behaviour of the

geometric stiffness as a function of force direction and the distance of the endpoint from the shoulder (or alternately, the elbow angle θ_e).

In Figure 7.4.15(a), the cylindrical stiffness plots are shown from $\theta_e = 170^\circ$ to 30° (from closest to furthest distances from shoulder). (The stiffnesses become too large to plot at positions beyond these.) The plot in (b) shows a close-up of the middle section in (a).

From Figure 7.4.15 we can make several characterizations. First, the stiffness values are larger near to the shoulder and far away from the shoulder than they are in the middle. The cutaway in (b) shows that the principal stiffnesses become equal at one point, corresponding to the position $\theta_e = 90^\circ$ with $\theta_F = \pm 90^\circ$ direction. This indicates that the principal stiffnesses are equal at that arm position for those force directions, and so the isopotential ellipses would be round. This is consistent with the Mohr's Circle in Figure 7.4.12(c) for forces in those directions. When the principal stiffnesses are equal, the Mohr's circle shrinks to a point.

Figure 7.4.15: Cylindrical plots of principal stiffness as function of distance from shoulder. (a) General plot from $\theta_c = 170^\circ$ to 30° . The left end is closest to the shoulder. (b) Close-up and cutaway of the middle section. Note that the stiffnesses touch at one point, indicating that the principal stiffnesses are equal.

The two positive values of θ_F at the point where $G_{k_{min}}$ is zero are exactly equal in magnitude to the two negative values of θ_F at the point where $G_{k_{max}}$ is zero since they plot over the same points. These values of θ_F , where the stability state switches, changes with respect to endpoint distance from the shoulder. Figure 7.4.16 shows a 3D plot of $G_{k_{min}}$ and $G_{k_{max}}$ versus θ_F and θ_e . Only the positive values of $G_{k_{min}}$ and negative values of $G_{k_{max}}$ are shown to clearly show the intersection point with the zero plane. Remarkably, the value of θ_F at which the principal stiffness are zero appears to be proportional to θ_e . The lines of intersection correspond exactly to the thick lines drawn by the following relationships:

(1) $G_{k_{min}}$ is zero at

$$\theta_F = \frac{1}{2}\theta_e \text{ and } \theta_F = (180^\circ - \frac{1}{2}\theta_e) \quad (7.4.4a)$$

(2) $G_{k_{min}}$ is zero at

$$\theta_F = -\frac{1}{2}\theta_e \text{ and } \theta_F = -(180^\circ - \frac{1}{2}\theta_e) \quad (7.4.4b)$$

These intersection lines do not change with selection of model parameters, which are only L_s and L_e for the geometric stiffness. Variation of L_s and L_e only changes the maximum height of the $G_{k_{min}}$ surface and minimum height of the $G_{k_{max}}$ surface. This suggests an analytical solution may be possible. From (7.4.1) we know that

$$G_{\mathbf{K}} = \mathbf{F}^T (\mathbf{G}_x \cos(\theta_F) + \mathbf{G}_y \sin(\theta_F)) \mathbf{J}^{-1} \quad (7.4.1)$$

Figure 7.4.16: $G_{k_{min}}$ and $G_{k_{max}}$ as a function of θ_F and θ_e . Only the positive values of $G_{k_{min}}$ and negative values of $G_{k_{max}}$ are shown to clearly identify the intersections of the curves with zero stiffness. The zero-crossing occur as a linear function of θ_F and θ_e .

We also know from (6.5.1), (6.5.2), and (6.5.4) that

$$\begin{aligned} {}^G k_{\max} &= C + R = \frac{1}{2}({}^G k_{xx} + {}^G k_{yy}) + \frac{1}{2}\sqrt{({}^G k_{xx} - {}^G k_{yy})^2 + 4({}^G k_{xy})^2} \\ {}^G k_{\min} &= C - R = \frac{1}{2}({}^G k_{xx} + {}^G k_{yy}) - \frac{1}{2}\sqrt{({}^G k_{xx} - {}^G k_{yy})^2 + 4({}^G k_{xy})^2} \end{aligned}$$

Setting ${}^G k_{\max}$ and ${}^G k_{\min}$ to zero results in the two relationships

$$C = R \quad (7.4.5a)$$

$$C = -R \quad (7.4.5b)$$

To solve the equations for θ_F , we must square both sides to remove the square root in R.

This produces a single equation for the both above conditions,

$$C^2 = R^2 \quad (7.4.6)$$

Substituting the elements of ${}^G \mathbf{K}$, reducing the equations and solving for θ_F results in an equation of the form

$$\theta_F = \tan^{-1} \left(\frac{-E_2 \pm \sqrt{E_2^2 - 4E_1E_3}}{2E_1} \right) \quad (7.4.7)$$

where the E_i parameters ($i = 1,2,3$) are functions of L_y , L_e , and θ_e (since θ_s is set as a function of θ_e in (7.4.3)). The four solutions in (7.4.4) result from the two values of the arctan at each of the two cases for the positive and negative square root.

The principal stiffness plots in Figure 7.4.15 provide some generalizations about the behaviour of the principal stiffnesses that do not change with endpoint position relative to the shoulder. These include:

- (1) At $\theta_F = 0^\circ$, $G_{k_{\max}}$ is positive and $G_{k_{\min}}$ is equal in magnitude but is negative.
- (2) At $\theta_F = \frac{1}{2}\theta_c$, $G_{k_{\min}}$ becomes positive.
- (3) Both principal stiffnesses remain positive through $\theta_F = 90^\circ$.
- (4) The behaviour of the principal stiffnesses is symmetric about $\theta_F = 90^\circ$.
- (5) For negative θ_F (0° to -180°), similar behaviour as for positive θ_F occurs, except that it is $G_{k_{\max}}$ that changes sign at $-\frac{1}{2}\theta_c$ and both principal stiffnesses are negative in the region around $\theta_F = -90^\circ$.

There are some characteristics of the principal stiffnesses that depend on the distance of the endpoint from the shoulder, including:

- the magnitudes of the principle stiffnesses at a value of θ_F ,
- the value of θ_F at which stiffnesses are maximum or minimum, and,
- the values of θ_F at which the principal stiffnesses change sign, which is a function of θ_c as shown in (7.4.4).

Figure 7.4.17 shows the shape factor, S_k , as a function of θ_F and the distance of the endpoint from the shoulder. The plot is shown from $\theta_c = 180^\circ$ to 110° and from $\theta_c = 70^\circ$

to 0° (from closest to furthest distance from shoulder). The gap from $\theta_e = 90^\circ$ to 70° is left because the shape factor becomes $(\pm)$ infinity at $\theta_e = 90^\circ$ for $\theta_f = \pm 90^\circ$. (Recall from (6.5.8) that $S_k = C/R$ and R is zero when $q_{k_{\max}} = q_{k_{\min}}$.) The z-axis for the cylindrical coordinates (origin for polar coordinate at each θ_e value) is given as the solid black line at the bottom middle of the shape. It is clear from this figure that the form of the plot in Figure 7.4.14(b) generalizes for the range of endpoint positions. The position where this form varies the most are the point $\theta_e = 90^\circ$, where S_k becomes infinite as noted above, and near the outer reach of the arm where the bottom edge of the shape gets flatter such that at $\theta_e = 0^\circ$ the form is a semi-circle of radius $S_k = 1$.

The force directions, θ_f , at which the geometric stiffness switches from saddle to stable (or unstable) are marked by solid black lines down each side of the three-dimensional shape. These can be seen as the intersections of the shape in Figure 7.4.17 with a cylinder of radius 1. For the transitions from saddle to stable, $S_k = 1$. This occurs at the locations described in (7.4.4a) because the transition corresponds to $q_{k_{\min}}$ passing through zero. The transition from saddle to completely unstable is $S_k = -1$, and the locations are the same as in (7.4.4b). This is consistent with the plots in Figure 7.4.13(c) and 7.4.14(b) where the transitions occur at $\theta_f = \pm 60^\circ$ and $\pm 120^\circ$ for an endpoint location of $\theta_e = 60^\circ$.

Shape Factor

Figure 7.4.17: Cylindrical plot of geometric stiffness shape factor, from $\theta_c = 180^\circ$ to 110° and $\theta_c = 70^\circ$ to 0° (closest to furthest from shoulder). The gap in the middle range around $\theta_c = 45^\circ$ is left because the shape factor become infinite (or negative infinite) at this point for $\theta_F = \pm 90^\circ$.

The general behaviour of the shape factor can then be described as follows:

- (1) Force directions between $\theta_F = \pm \frac{1}{2}\theta_e$ produce saddle geometric stiffness where the positive principal stiffness magnitude is larger than the negative principal stiffness magnitude.
- (2) Force directions between $\pm(180^\circ - \frac{1}{2}\theta_e)$ produce saddle geometric stiffness where the negative principal stiffness magnitude is larger than the positive principal stiffness magnitude.
- (3) From $\theta_F = \frac{1}{2}\theta_e$ to $(180^\circ - \frac{1}{2}\theta_e)$ the geometric stiffness is stable ($S_k > 1$).
- (4) From $\theta_F = -\frac{1}{2}\theta_e$ to $-(180^\circ - \frac{1}{2}\theta_e)$, in the negative direction, the stiffness is completely unstable ($S_k < -1$).

Unique cases occur at the extremes, $\theta_e = 180^\circ$ (at shoulder) and $\theta_e = 0^\circ$ (full reach of arm). At the shoulder, S_k is between -1 and 1 and only reaches 1 (-1) at $\theta_F = 90^\circ$ (-90°). Therefore any force direction will produce a saddle, with borderline stiffnesses at $\theta_F = 90^\circ$ ($G_{k_{\min}} = 0$) and $\theta_F = -90^\circ$ ($G_{k_{\max}} = 0$). At full reach, $S_k = 1$ for all θ_F between 0° and 180° , and $S_k = -1$ for all θ_F between -180° and 0° . Therefore all force directions produce borderline stiffnesses, between saddle and stable for positive force directions and between saddle and completely unstable for negative force directions.

Figure 7.4.18 shows the three-dimensional plot in cylindrical coordinates for size

of the geometric stiffness, k_{avg} . The plot is limited to the range of θ_c from 150° to 30° because the stiffness size becomes very large near the extremes, hiding the details of the plot in the middle section. At any given distance from the shoulder, the size follows a standard shape; four sections make up a bi-symmetric shape about the origin. The sections are separated at the points described in (7.4.4a) and (7.4.4b) because k_{avg} becomes zero when one of the principal stiffnesses is zero. The four sections between these ranges have the characteristics as described for 7.4.16.

The behaviour of the geometric stiffness size is highly dependent on distance from the shoulder. It has the following pattern:

- (1) The stiffness is generally larger at the extremes: close to the shoulder and near the outer extent.
- (2) Near the shoulder the largest stiffnesses are saddles at $\theta_F = 0^\circ$ and 180° .
- (3) Near the outer extent the largest stiffnesses are either stable at $\theta_F = 90^\circ$ or completely unstable at $\theta_F = -90^\circ$.
- (4) The saddle stiffness sizes shrink as the distance from the shoulder increases.
- (5) The stable and completely unstable stiffness sizes grow as the distance from the shoulder increases.

Size

Figure 7.4.18: Cylindrical plots of stiffness size as function of distance from shoulder. (a) General plot from $\theta_c = 150^\circ$ to 30° . The left end is closest to the shoulder. (b) Close-up of the middle section. Notice that the horizontal petals (saddles) get smaller further from the shoulder and the horizontal petals (mirrored stable and unstable stiffness) get larger.

Figure 7.4.19 shows the plot of geometric stiffness orientation, θ_k , as a function of force direction and distance from the shoulder. The stiffness orientation is more difficult to understand in cylindrical coordinates and so it is shown in Cartesian coordinates. The plot shows a cyclical behaviour that differs for endpoint positions closer or further from the shoulder than $\theta_e = 90^\circ$ (70.7% of full reach). The plot appears discontinuous at some points but this is due to the cyclical properties of orientations. For instance, there appears to be a discontinuity at $\theta_F = -90^\circ$ where the stiffness orientation jumps from 90° to -90° . However, since the orientation is defined as the direction of maximum stiffness, these orientations describe the same stiffness. The plot in 7.4.18(a) repeats ad infinitum to the left, right, top, and bottom of the figure.

Consider first the orientations for the inner workspace ($\theta_e > 90^\circ$). For positions close to the shoulder, application of the force at $\theta_F = 90^\circ$ produces a stiffness with orientation $\theta_k = 0^\circ$ (or equivalently, 180°). Using information from the previous plots, we know this is a stable stiffness and so it can make an isopotential ellipse with the major axis (minimum stiffness) pointing toward the shoulder. When the force is applied at $\theta_F = -90^\circ$, the orientation is $\theta_k = \pm 90^\circ$. Since this force direction produces a completely unstable stiffness, the isopotential ellipse would be imaginary, but would be shaped and aligned exactly as the stable stiffness ellipse since the maximum stiffness has the smaller magnitude for unstable stiffnesses.

Figure 7.4.19: Three dimensional plot of stiffness orientation as a function of force direction and distance from shoulder. Note that the general behaviour reverses at the point $\theta_c = 90^\circ$ (70.7% of total reach) be. Both the stiffness orientation and force direction repeat every 360° , and so the graph repeats to the left, right, top, and bottom of the plot shown.

Force applications at $\theta_F = 0^\circ$ and 180° result in saddle stiffnesses with orientations of $\theta_k = -45^\circ$ (equivalent to $\theta_k = 135^\circ$) and $\theta_k = 45^\circ$ (equivalent to $\theta_k = -135^\circ$), respectively. These are essentially the same stiffnesses, but with the directions of the positive and negative principal stiffnesses reversed. Forces applied between the directions described above produce orientations that vary non-linearly between the above orientations.

It is not clear from the plot, but the stiffness orientations are bi-symmetric about the line from the shoulder to the endpoint and its perpendicular through the endpoint. The above examples meet this behaviour because they are mirror images about the lines of symmetry. The definition of orientation as the direction of maximum stiffness, rather than maximum stiffness magnitude, masks the symmetry across the $\theta_F = 0^\circ$ or 180° line. The orientations are shifted by 90° when comparing the results from θ_F with $-\theta_F$, but the maximum stiffness has also switched from larger in magnitude to smaller.

The behaviour described above continues to the point $\theta_e = 90^\circ$ (70.7% of full reach), with only gradual changing of the surface slope. At this point, the stiffness is round, as seen by the equal principal stiffnesses in Figure 7.4.15(b), and orientation can no longer be defined. Positions beyond this point show a sudden change in orientation for a given force direction and the slopes reverse. This behaviour is a result of the definition of orientation. In the inner workspace ($\theta_e > 90^\circ$), a force applied at $\theta_F = 90^\circ$ consistently produces a stable stiffness with the minor axis of the isopotential ellipse

pointing toward the shoulder. At $\theta_e = 90^\circ$ the stiffness is round. As the endpoint moves into the outer region ($\theta_e < 90^\circ$), the stable ellipse has the major axis pointing toward the shoulder. This means that the orientation, by the current definition, flips discontinuously at $\theta_e = 90^\circ$. The real change in stiffness is a gradual decrease of the major axis and increase in the minor axis until the axes switch roles.

If we redefine the orientation such that it refers to the same axis rather than the maximum stiffness, the discontinuity at $\theta_e = 90^\circ$ disappears. Figure 7.4.21 shows the plot from Figure 7.4.19 using this new definition of orientation for illustrative purposes. (We are not suggesting that the definition be changed, we are demonstrating the cause of the discontinuities in Figure 7.4.19.) Although the discontinuity is removed, the plot in Figure 7.4.20 shows that the orientation does still change for a given force direction near the outer workspace, which is different behaviour than in the inner workspace. At the furthest reach the stiffness takes on discrete values. Using the proper definition of orientation in Figure 7.4.19, these values are $\theta_k = 0^\circ$ or $\pm 90^\circ$. (Figure 7.4.21 shows these shifted by 90° because it doesn't use the maximum stiffness direction for the orientation.)

There are several orientations that remain consistent for all arm positions. At $\theta_F = 0^\circ, 180^\circ$, and $\pm 90^\circ$, the orientation is $\theta_k = 135^\circ$ ($= -45^\circ$), 45° ($= -135^\circ$), and $\pm 90^\circ$. These orientations are shown in Figure 7.4.20 as heavy lines, but also apply to Figure 7.4.19.

Figure 7.4.20: Three dimensional plot of stiffness orientation as a function of force direction and distance from shoulder. The orientation is redefined for illustrative purposes such that the same axis is always reference ($\sigma_{k_{\max}}$ in the inner workspace, $\sigma_{k_{\min}}$ in the outer workspace). This redefinition removes the discontinuity at $\theta_e = 90^\circ$. The thick lines indicate force directions that result in consistent stiffness orientations regardless of distance from shoulder.

In Section 7.4 we showed:

- geometric stiffness can contribute significantly to the total stiffness even for relatively small contact forces,
- generally, the geometric stiffness can have a significant effect on the overall stiffness properties for an Ellipse Comparison Index of $\xi_j > 0.05$,
- instability can result from applied forces of the magnitude seen in normal contact tasks,
- when an unstable operating point exists, the arm moves to a stable equilibrium,
- the effect of endpoint position on geometric stiffness, and,
- the effect of applied force position on geometric stiffness.

7.5 Chapter Summary

Section 7.1 introduced the parameter values and spring equations that would be used throughout the rest of the chapter. In Section 7.2 we compared the linearized isopotential ellipses to real non-linear isopotentials. The linear approximation is more accurate for larger displacements near the center of the workspace, up to about about 1/4 of the link lengths, than at the extreme positions. This answers Question 6 in Chapter 4 regarding the relative accuracy of the linear approximation. We also showed that the spring functions did not have a large effect on this accuracy, nor on the non-linear isopotential shapes. The geometry played a larger part in determining their shape. This knowledge goes towards answering Question 4 regarding the influence of the geometry.

In Section 7.3 we investigated the behaviour of the endpoint spring stiffness. Changing actuator stiffnesses increased endpoint spring stiffness in the direction associated with the corresponding actuator pair described in Section 6.1 and 6.2. Stiffness in the orthogonal direction remains unchanged. This behaviour goes towards answering Question 1 regarding the effect of actuator stiffnesses on the endpoint stiffness. We also showed how the position in the workspace could affect these changes to the endpoint stiffness and that the geometry tends to govern the endpoint stiffness. This applies to the influence of geometry in Question 4. Also in Section 7.3 we studied the limitations on orientation, showing that the possible range of orientations is very limited in the outer region of the workspace, assuming positive actuator stiffnesses. This limitation knowledge goes towards answering Question 2.

Finally Section 7.4 investigated the behaviour of the geometric stiffness. In particular, we showed that it can be a significant part of the overall endpoint stiffness even for relatively small endpoint forces. We also studied its stability, leading us to a description of the actual stiffness values as a function of force direction and position in the workspace. Forces toward the shoulder cause unstable geometric stiffness while forces away from the shoulder cause stable geometric stiffness. Force directions between these vary from stable to saddle to completely unstable, depending on the force direction and endpoint position. This understanding of the geometric stiffness applies to both the effects of geometry and the effects of loading, Questions 4 and 5 in Chapter 4.

Chapter 8

Design and Control of Stiffness

Chapters 5 through 7 have given us an understanding of how the endpoint stiffness behaves. We have not yet considered what stiffness properties we consider to be desirable nor how to control the stiffness to achieve desirable behaviour. In this chapter we will consider selecting proper orientations to minimize instability problems, controlling joint stiffness to maintain stability, compensating for undesirable geometric stiffness effects, performance of contact tasks, and limitations on control of stiffness when the double joint actuators are neglected.

8.1 Workspace Orientation

Many common tasks require interaction with the environment. In this section we will consider strategies to best make use of the known stiffness properties by choosing the position and orientation of contact forces.

One desirable condition of interactive tasks is that the manipulator remains stable. We know that the endpoint spring stiffness, ${}^I\mathbf{K}$, is guaranteed to be stable as long as the

actuator stiffnesses are positive. From Section 7.4 we also know that the geometric stiffness is also guaranteed stable when the external force is applied away from the shoulder. That is, the manipulator is pulling the external environment towards the shoulder. This suggests that, in general, it is best to perform contact tasks such that the manipulator is pulling towards the shoulder.

There are some disadvantages to the strategy above. With the manipulator pulling towards the shoulder, if the endpoint were to slip off of the contact point it would tend to accelerate towards the shoulder and may cause some damage. In addition, if the environment consists of large objects or walls, it may be difficult to orient the manipulator properly or fit the object between the shoulder and endpoint. In such cases, it appears that stability is easiest achieved by orienting the endpoint force as close as possible to pulling on the environment rather than pushing.

A second criterion may be the shape and orientation of the stiffness ellipse. High endpoint stiffness means that large forces are required to move the endpoint from its virtual position. Low stiffness means that the virtual position can be moved a larger distance to change the contact force by small amounts. This gives better resolution on controlling the force magnitude. These arguments imply that it is best to have low stiffness in the direction of the contact force, so that the force can be controlled to a finer degree, and that stiffness should be high in the direction perpendicular to the contact

force, so that the endpoint position on the constraint can be better controlled.

From Figure 7.3.2 we know that the endpoint spring stiffness tends towards long thin stiffness ellipses near the shoulder and near the outer region of the workspace. At the point near the shoulder, the low stiffness is oriented in the direction of the shoulder, with high stiffness in the perpendicular direction. At the outer region of the workspace, the directions are reversed. Our suggestion for low stiffness in the force direction and high in the other direction then indicates that for operating points close to the shoulder, the force should be directed towards or away from the shoulder (keeping in mind that pulling towards the shoulder can guarantee stability), whereas for points near the outer region the force should be applied perpendicular to the shoulder direction.

There is also a disadvantage to using low stiffness in the direction of force.

Consider a normal linear spring. Then spring force and energy stored in the spring are

$$F = k\delta$$

$$E = \frac{1}{2}k\delta^2 = \frac{1}{2}F\delta$$

where k is the stiffness in the direction of the force

δ is the displacement of the spring

For a given force magnitude, we could use either a low stiffness setting or a high stiffness setting, given by

$$F_L = F_H \Rightarrow k_L \delta_L = k_H \delta_H$$

where the subscript “L” denotes the parameters of the low stiffness spring and “H” denotes the parameters of the high stiffness spring. The ratio of low stiffness spring energy to high stiffness spring energy is then

$$\frac{E_L}{E_H} = \frac{\frac{1}{2} F_L \delta_L}{\frac{1}{2} F_H \delta_H} = \frac{\delta_L}{\delta_H} = \frac{k_H}{k_L} > 1$$

In other words, the low stiffness spring must contain more energy to supply the same spring force. For a manipulator or human limb, this energy typically must come from the actuators or muscles, and hence a higher stiffness offers better energy efficiency. In cases where the energy to displace the spring comes from the external source, such as catching a ball, *and* the actuators are non-backdrivable such that they do not have supply continuous energy to support the load, this energy consideration is irrelevant for the purposes of energy efficiency. The passive springs in the current model fall into the latter category.

If we consider the energy efficiency argument above, it may be desirably to only have the stiffness in the direction of the force low enough to supply sufficient force resolution, but no lower. That is, the stiffness is low enough that small deflection changes result in force changes that are small enough. This is what we mean by force

resolution. The level of deflection variation and acceptable force variation will depend on the task. Providing maximum stiffness, limited by required force resolution, suggests that the extreme positions near the shoulder or far from it are not ideal, but rather the operating point should be closer to the centre of the workspace. In such positions the stiffness tends to be rounder, with the orientation of ellipse much more controllable by slight variations in the actuator stiffnesses. From Figures 7.3.2 and 7.4.15 the smaller stiffness (major axis) tends to be slightly higher for both the endpoint spring stiffness and geometric stiffness in this area, as we just suggested may be desirable. However, the largest stiffness (minor axis) tends to be much lower in this area than at the extremes, and so the control of position along the constraint will also lose resolution. It may then be necessary to find a balance between these tradeoffs to pick the optimal workspace position for interaction with the environment.

The above reasoning leads to the conclusion that the ability to balance between force and position resolutions and energy efficiency can be achieved by applying the contact forces in the middle region of the workspace. This is the area of maximum control of stiffness orientation, shape, and size. At the extreme locations, the geometry largely governs the stiffness properties and limits the ability to strike a balance.

8.2 Minimum Actuator Stiffnesses

Section 7.4.2 showed cases where a manipulator can be unstable. In particular, the stiffnesses and forces were chosen to correspond to realistic values that could be obtained in the human upper arm. However, there are no known reports of human limbs going unstable in such a manner. There are several possible ways to avoid this. The central nervous system (CNS) might keep the joint stiffnesses in a range that keeps the overall stiffness stable, implying that it has some feedback measure on the stability margin. A simpler method may be to design the spring (muscle) stiffness functions such that the joint stiffness increases with joint displacement (caused by endpoint forces), such as the 'error' function of (7.1.2).

From Sections 7.3 and 7.4 we know that the geometric effects on endpoint spring stiffness and geometric stiffness are similar: higher maximum stiffness and lower minimum stiffness at the extremes, rounder in the middle, and an orientation that tends to have maximum stiffness towards the shoulder at the outer range of positions and perpendicular to the shoulder at inner positions. The geometry takes care of increasing the endpoint spring stiffness to compensate for increasing geometric stiffness for these effects. This leaves only the changes in geometric stiffness due to force magnitude and direction to compensate for. As long as the joint stiffnesses increase at a rate with respect to endpoint force (via the joint displacements it causes) that is faster than the geometric stiffness increase with force, the limb will remain stable. Human studies suggest that this

may indeed occur [McIntyre et al., 1996]. In either case, it is necessary to know the lower limit on actuator stiffness in order to maintain stability for either controller design or spring function selection. We investigate such a limit in this section.

The lowest endpoint spring stiffness required to maintain stability can be found by setting the overall endpoint stiffness to be borderline between stable and a saddle. Since the endpoint spring stiffness is a direct result of actuator stiffnesses, the limitations of the actuator stiffness can be found.

The overall endpoint stiffness becomes borderline stable when $k_{\min} = 0$, and consequently $k_{\max} \geq 0$ since it is larger or equal. From (6.4.1) we can write

$$k_{\min} = 0 = \frac{1}{2}(k_{xx} + k_{yy}) - \sqrt{\frac{1}{4}(k_{xx} - k_{yy})^2 + k_{xy}^2}$$

which reduces to

$$\det \mathbf{K} = 0 \tag{8.2.1}$$

However, in this reduction we must square the terms thereby eliminating the difference between $k_{\max}$ and $k_{\min}$. Equation (8.2.1) therefore represents the condition in which *either* $k_{\min} = 0$ (and $k_{\max} > 0$) or $k_{\max} = 0$ (and $k_{\min} < 0$). The former case is the point at which a stable stiffness becomes a saddle, whereas the latter case is the point where a saddle become completely unstable in all directions. (In the Mohr's circle representation, the

former would correspond to the circle touching the origin but located on the right and latter to touching the origin but located on the left.) We are only interested in the former case. Since (8.2.1) represents both cases, one would expect two results and we can pick the appropriate one.

From (5.3.16) we know that $\mathbf{K} = {}^J\mathbf{K} + {}^G\mathbf{K}$. Therefore we can write

$$\begin{aligned}\det \mathbf{K} = 0 &= ({}^Jk_{xx} + {}^Gk_{xx})({}^Jk_{yy} + {}^Gk_{yy}) - ({}^Jk_{xy} + {}^Gk_{xy})^2 \\ &= \det {}^J\mathbf{K} + \det {}^G\mathbf{K} + {}^Jk_{xx} {}^Gk_{yy} + {}^Jk_{yy} {}^Gk_{xx} - 2 {}^Jk_{xy} {}^Gk_{xy}\end{aligned}\quad (8.2.2)$$

But from (6.2.27) we know that ${}^J\mathbf{K} = \mathbf{P}^T \boldsymbol{\Psi} \mathbf{P}$, where $\mathbf{P}$ is the 2×3 projection matrix given in (6.2.13). From this substitution, and simplification, we get

$$\det {}^J\mathbf{K} = (P_{11}P_{22} - P_{12}P_{21})^2 \psi_{zs} \psi_{ze} + (P_{11}P_{32} - P_{12}P_{31})^2 \psi_{zs} \psi_{zd} + (P_{21}P_{32} - P_{22}P_{31})^2 \psi_{ze} \psi_{zd}$$

where P_{ij} represents the elements of the $\mathbf{P}$ matrix. Substituting $\det {}^J\mathbf{K}$ and the elements of ${}^J\mathbf{K}$ in (8.2.2) results in the relationship

$$A_1 \psi_{zs} + A_2 \psi_{ze} + A_3 \psi_{zd} + B_{12} \psi_{zs} \psi_{ze} + B_{13} \psi_{zs} \psi_{zd} + B_{23} \psi_{ze} \psi_{zd} + \det {}^G\mathbf{K} = 0 \quad (8.2.3)$$

$$\begin{aligned}\text{where } A_i &= {}^Gk_{yy} P_{i1}^2 + {}^Gk_{xx} P_{i2}^2 - 2 {}^Gk_{xy} P_{i1} P_{i2} \\ B_{ij} &= (P_{i1} P_{j2} - P_{i2} P_{j1})^2\end{aligned}$$

Given a position and force vector, we know ${}^G\mathbf{K}$ and $\mathbf{P}$. The only unknowns are the actuator stiffnesses. Equation (8.2.3) allows us to select two of the actuator

stiffnesses while the third becomes dependent. A simple inversion of (8.2.3) solves for the third actuator stiffness. However, the resulting stiffness is unique; there aren't two solutions as we expected. This leaves us with the question as to whether the solution is the stable-saddle border or the saddle-unstable border. To determine this we plot the function in (8.2.3), as shown in Figure 8.2.1. In general, the plot results in two surfaces. One surface represents the stable-saddle border and the other represents the saddle-unstable border, as seen in the figure. Actuator stiffness settings above the stable-saddle surface produce stable endpoint stiffnesses. Actuator stiffness settings below the saddle-unstable border produce completely unstable endpoint stiffnesses. Between these two surfaces, the endpoint stiffness is a saddle.

The two surfaces meet at a common point, denoted here as the *critical point*. At this actuator stiffness setting, the endpoint stiffness is neither stable nor unstable because $k_{\max} = k_{\min} = 0$. In other words, this is the point in which ${}^J\mathbf{K}$ exactly compensates for the instability of ${}^G\mathbf{K}$ such that $\mathbf{K}$ has neutral stability. We can then find this point by letting ${}^J\mathbf{K} = -{}^G\mathbf{K}$. We use the inversion method of (6.2.3) to solve for the required actuator stiffnesses,

$$\boldsymbol{\psi}_{\text{icv}} = -\mathbf{P}_v^{-1} {}^G\mathbf{K}_v \quad (8.2.4)$$

where ${}^G\mathbf{K}_v$ is the vector form of ${}^G\mathbf{K}$ (see Section 6.2) and $\boldsymbol{\psi}_{\text{icv}}$ is the vector containing the actuator stiffness values of the critical point as shown in Figure 8.2.1.

Figure 8.2.1: Plot of function in (8.2.3). The origin is located at “O”. Two surfaces are shown. The surface on the lower left is the boundary between a saddle and completely unstable endpoint stiffness. Actuator stiffness combinations below this surface result in a completely unstable endpoint stiffness. The surface on the upper right is the boundary between stable stiffness and saddle stiffness. Actuator stiffnesses above this surface are stable. Actuator stiffness combinations between the two surfaces are saddle stiffnesses. The two surfaces meet at a common point shown as the critical point. The actuator stiffnesses at this point are the critical actuator stiffnesses, and result in an neutral endpoint stiffness, hence ${}^J\mathbf{K}$ exactly compensates for ${}^G\mathbf{K}$ at this point. ${}^G\mathbf{K}$ controls the location of the critical point; when ${}^G\mathbf{K} = \mathbf{0}$, the critical point is at the origin.

The location of the critical point varies depending on the endpoint force vector, the manipulator position, and the manipulator parameters. The critical point is located at the origin when ${}^G\mathbf{K} = \mathbf{0}$ (no endpoint force). In the special case where ${}^dR_s = {}^dR_e$ the critical actuator stiffnesses simply to

$$\psi_{zsc} = -\det {}^G\mathbf{K}/A_1 \quad (8.2.5a)$$

$$\psi_{zec} = 0 \quad (8.2.5b)$$

$$\psi_{zdc} = -\det {}^G\mathbf{K}/A_3 \quad (8.2.5c)$$

where A_1 and A_3 are given in (8.2.3). When ${}^dR_s > {}^dR_e$, $\psi_{zec} > 0$. When ${}^dR_s < {}^dR_e$, $\psi_{zec} < 0$.

We know that the solution to (8.2.3) is unique. This means that a line drawn parallel to any of the axes will only intersect *one* of the two surfaces. To determine whether the solution is the stable-saddle or saddle-unstable boundary, we simply compare the actuator stiffness solutions from (8.2.3) with their critical values. If any two of the three actuator stiffnesses are lower than their critical values then the solution is the saddle-unstable border. (Note that all three actuator stiffnesses cannot be less than their critical values and still solve (8.2.3). Such a point would be *inside* the dotted box at the lower left in Figure 8.2.1, and therefore isn't on one of the surfaces.) This means that given the two chosen actuator stiffnesses, and they are less than their critical values, there is *no* third actuator stiffness value that will stabilize the endpoint stiffness.

Since we have to choose two stiffness values in order to solve (8.2.3), choosing two that are greater than their critical values would guarantee the result is on the stable-saddle border, and therefore choosing a higher actuator stiffness guarantees a stable endpoint stiffness (lower would be a saddle). This suggests that one should solve (8.2.4) for the critical values *before* choosing the stiffnesses, if they are being chosen by the controller. It is still possible to choose one of the two actuator stiffnesses to be below the critical value, but the third actuator stiffness from solving (8.2.3) must be compared to its critical value to determine the resulting endpoint stiffness. A computationally simpler method may be to solve (8.2.4) for the critical values and simply choose all three actuator stiffnesses above these values. This will guarantee stability, but may result in higher stiffness than is necessary for a task or may be desired to meet some criteria.

It is interesting that negative actuator stiffnesses do not necessarily result in an unstable (or saddle) endpoint spring stiffness. Consider the case where the geometric stiffness is zero. This places the critical point in Figure 8.2.1 at the origin. Only two of the three stiffnesses must be above zero, their critical values. The third can be negative and still produce a stable endpoint stiffness as long as it is above the stable-saddle border.

8.3 Contact Issues

8.3.1 Compensating for Geometric Stiffness

The critical actuator stiffnesses in (8.2.4) are useful in another sense. A desired endpoint stiffness can be achieved by first neutralizing the geometric stiffness and then adding on the desired stiffness. The actuator stiffness matrix can then be broken up into Ψ_{zc} that compensates for ${}^G\mathbf{K}$, and Ψ_{zr} which is the actuator stiffness matrix corresponding to the requested overall endpoint stiffness $\mathbf{K}_r$, such that

$$\Psi_z = \Psi_{zr} + \Psi_{zc} \quad (8.3.1)$$

We can solve for Ψ_{zr} by using the vector stiffness form in (6.2.31),

$$\Psi_{zrv} = \mathbf{P}_v^{-1} \mathbf{K}_{rv} \quad (8.3.2)$$

where $\mathbf{K}_{rv}$ is the vector form of $\mathbf{K}_r$. We can then compensate for ${}^G\mathbf{K}$ even if the critical values are negative, as long as the overall actuator stiffnesses in (8.3.1) are positive. If any of the overall actuator stiffnesses to achieve the desired endpoint stiffness must be negative, the closest we can come to the desired endpoint stiffness is to set those negative actuator stiffnesses to zero (assuming we cannot achieve negative actuator stiffness).

Using (8.3.1) we can then keep a constant endpoint stiffness in the face of a changing endpoint force by altering the Ψ_{zc} components to compensate, or given a

constant geometric stiffness we can alter only the ψ_{Σ} components to change a desired $\mathbf{K}$.

8.3.2 Uncompensated Geometric Stiffness

As discussed in Section 5.3, most researchers neglect the geometric stiffness effect. The endpoint stiffness setting is based solely on ${}^J\mathbf{K}$. This is sufficient for slow non-contact tasks (negligible dynamic forces) since ${}^G\mathbf{K} = \mathbf{0}$. However, during contact tasks, or when large dynamic forces are present, the geometric stiffness may become significant as shown in Chapter 7. To minimize the effect of the geometric stiffness, we must first understand what happens to the endpoint stiffness in contact.

As discussed in Chapter 6, endpoint stiffnesses are characterized by size, shape, orientation. The endpoint spring stiffness ${}^J\mathbf{K}$ has fixed characteristics for a fixed position and joint stiffnesses. Therefore increasing the endpoint force will not affect these characteristics as long as the manipulator doesn't move. Recall from (7.4.1) that the geometric stiffness can be written as

$${}^G\mathbf{K} = \mathbf{F} \mid (\mathbf{G}_x \cos(\theta_F) + \mathbf{G}_y \sin(\theta_F)) \mathbf{J}^{-1} = \mathbf{F} \mid \mathbf{G}_F$$

where $\mathbf{G}_F = (\mathbf{G}_x \cos(\theta_F) + \mathbf{G}_y \sin(\theta_F)) \mathbf{J}^{-1}$

As discussed in Section 7.4.3, this means that the shape and orientation of the geometric stiffness is independent of the endpoint force magnitude. Therefore the geometric

stiffness also has fixed shape and orientation in contact as long as the manipulator position and force direction do not change.

The effect of increasing the endpoint force is to add weight to the geometric stiffness. To see this, consider the stiffness sizes using the geometric mean stiffness defined in (6.5.9),

$$k_{avg} = \sqrt{k_{max} \cdot k_{min}} = \sqrt{\det \mathbf{K}}$$

The size of ${}^J\mathbf{K}$ is fixed as contact force increases since it is not a function of endpoint force (assuming constant actuator stiffnesses). However, the size of the geometric stiffness is

$${}^G k_{avg} = |\mathbf{F}| \sqrt{|\det \mathbf{G}_F|} \quad (8.3.3)$$

Therefore the size of the geometric stiffness is proportional to the endpoint force magnitude.

The net effect of the above size, shape, and orientation observations is that the overall stiffness ellipse starts at the size and orientation of ${}^J\mathbf{K}$ at the point of contact, where the endpoint force is zero, and approaches the shape and orientation of ${}^G\mathbf{K}$ as the contact force increases. The size of ${}^G\mathbf{K}$ increases with contact force while the ${}^J\mathbf{K}$ size

remains constant. Therefore the size of the overall stiffness does approach that of ${}^G\mathbf{K}$ as the force increases, but this is a changing size. In fact, it is this increase in size that makes ${}^G\mathbf{K}$ become significant or dominant.

Now that we know what happens to the endpoint stiffness in contact, we can minimize the effects. Ideally one should compensate for the geometric stiffness by adjusting the actuator stiffnesses as discussed in Section 8.3.1. In some cases this compensation may require negative actuator stiffnesses or values beyond their maximum. In such cases, one cannot properly compensate for the geometric stiffness. In addition, one may choose to neglect the geometric stiffness effect for calculation simplicity, lack of feedback sensors from which to obtain endpoint force or joint moment information, or because it is known that contact forces will be low and actuator stiffnesses high enough such that the geometric stiffness is known to be negligible. There are ways in which to minimize the effect of the geometric stiffness in such cases.

Upon contact, the orientation rotates from the ${}^J\mathbf{K}$ orientation towards the ${}^G\mathbf{K}$ orientation. The simplest way to minimize this problem is to make sure the orientations are the same. We know how the geometric stiffness orientation operates from Section 7.4.3. Force directions towards or away from the shoulder result in an orientation perpendicular to the shoulder direction. Forces perpendicular to the shoulder direction produce geometric stiffness orientations at 45° to the shoulder direction. Force directions

between these produce orientations that vary between these extremes, depending on the distance from the shoulder to the endpoint. Using this knowledge we can chose a force direction to best match the ${}^G\mathbf{K}$ orientation with the ${}^J\mathbf{K}$ orientation.

In a similar fashion, we can minimize the changing shape by choosing an endpoint position such that the ${}^G\mathbf{K}$ shape best matches ${}^J\mathbf{K}$. From Section 7.4.3 we know that the geometric stiffness is long and thin for endpoint near the shoulder and far from the shoulder, and much rounder near the centre of the workspace. By properly positioning the endpoint, and applying the force in the right direction, one could duplicate exactly the shape and orientation of the desired ${}^J\mathbf{K}$ stiffness. One could argue that choosing an endpoint near the centre of the workspace minimizes the size of the geometric stiffness, but from Section 7.3 we know that the endpoint spring stiffness is also of minimum size here for given actuator stiffnesses, and so such a strategy isn't ideal.

Once the shape and orientation are matched, only the size of the stiffness will change as the force increases. If the geometric stiffness is unstable (including a saddle), this increase in size will lower the overall endpoint stiffness size until it becomes unstable. If it is stable, the size increases, effectively making the endpoint more rigid. From Section 7.4.3 we know that forces towards the shoulder are most unstable and away from the shoulder are most stable with respect to the geometric stiffness. In other words, when the manipulator pushes away from the shoulder it is acting like a softening spring,

whereas it acts as a hardening spring when pulling towards the shoulder. When choosing the direction of force to match the ${}^G\mathbf{K}$ and ${}^J\mathbf{K}$ orientation, as described above, there are always two co-linear choices that produce the same orientation, but in opposite directions. It is therefore best to choose the one that results in the external force pushing away from the shoulder.

8.3.3 Friction

In the previous section we discussed strategies for minimizing the effect of the geometric stiffness when it is not compensated for and we have the choice of force direction and endpoint position. However, there may be cases where the constraint is in a fixed position and orientation relative to the shoulder, or the contact task requires the manipulator to move through a variety of positions and orientations. In these cases there is still some ability to minimize the geometric stiffness effect by taking advantage of surface friction.

Consider the case of applying a set force, F_0 , on a constraint that is a fixed distance from the shoulder as shown in Figure 8.3.1. On a frictionless surface, a net force of F_0 must be applied exactly normal to the constraint surface as indicated by the centre force vector. In the presence of friction, one can apply the same normal force by applying a net force whose normal component is F_0 and whose tangential component is less than the maximum static friction, $F_{\text{max}} = \mu_s F_0$. This defines the force cone in Figure 8.3.1 of

allowable forces, with a maximum angle $\gamma = \tan^{-1}\mu_s$. We may then choose the most appropriate force direction, within this allowable range, that best meets our criteria. Minimum moments (as in Appendix A) might be one choice, or we may select the force direction to stabilize the stiffness or reduce the effect of the geometric stiffness. For example, we might choose the direction most closely directed towards the shoulder, F_1 in Figure 8.3.1, since we know this has the most stable geometric stiffness. (Though it has the most stable stiffness, there is little range of stability in Figure 8.3.1. This choice might be more appropriate when the constraint is more aligned with the shoulder direction where the range of stability varies more.) We may also choose the force direction such that the orientation of the geometric stiffness is aligned with that of the endpoint spring stiffness. Suppose the endpoint spring stiffness is directed toward the shoulder, then selection of F_2 in Figure 8.3.1 is the only force direction that will align the geometric stiffness with this direction, albeit an unstable geometric stiffness since the contact force on the endpoint pushes toward the shoulder.

Figure 8.3.1: Use of friction to optimize constraint force. All forces within the force cone have the same normal component and are static forces. The cone angle is given by $\gamma = \tan^{-1} \mu \psi$. Force F_1 produces the most stable geometric stiffness, while force F_2 produces a geometric stiffness aligned with the shoulder direction.

8.4 Chapter Summary

In Section 8.1 we discussed some strategies for choosing the workspace location in which to apply constraint forces. This is related to Question 4 in Chapter 4 that asked how the geometry affected the stiffness. The strategies discussed in Section 8.1 apply our knowledge of how geometry affects stiffness to decide the best geometry for interaction. These strategies also answer Question 8 in Chapter 4 that asked how we might select appropriate stiffness characteristics.

Question 3 in Chapter 4 asked how muscle (actuator) stiffnesses affected stability. Section 8.2 largely answers this question by finding the minimum actuator stiffnesses to maintain stability, and showing that selection of actuator stiffnesses above these values will guarantee stability. This same analysis can be applied to Question 2 which sought the limitation of *endpoint* stiffness characteristics from geometry and *actuator* stiffness limitations. Here we found the answer to the inverse problem, showing the limitations on *actuator* stiffness based on geometry and *endpoint* stiffness.

Section 8.3 discussed contact issues, including compensation for the geometric stiffness, and strategies of interaction when there is no compensation for geometric stiffness. This does not specifically answer a posed question in Chapter 4, but is related to Question 8 that asks about strategies for choosing stiffness characteristics. It offers strategies to best meet the chosen stiffness characteristics.

Chapter 9

Applications

Chapter 4 discussed the need for this research for the development of an above-elbow prosthesis. In this chapter we will apply the knowledge gained towards that goal. However, the purpose of this work was not to deal only with direct issues of prosthesis development, but an overall understanding of the behaviour of stiffness in a two-link manipulator. This knowledge has wider applications in other manipulator systems, including robotics, haptic devices, and human studies. We will suggest applications of the knowledge to each field.

9.1 Prosthesis Design

The current work examined several behaviours relevant to our proposed prosthesis design using antagonistic actuation. The goal is to reproduce the ability to control stiffness in a manner similar to the natural limb. We will then look at the effect of double-joint stiffness, the actuator stiffnesses, and control issues.

9.1.1 Double-Joint Stiffness

The proposed method of antagonistic actuation for control of an above-elbow prosthesis replaces the single-joint elbow stiffness from the mono-articular elbow muscles. The mono-articular shoulder muscles remain intact and therefore the single-joint shoulder stiffness remains under the direction control of the CNS. However, the bi-articular muscles can no longer cross both joints. The remaining muscles are either left unattached or are connected to the humerus. We therefore lose the contribution of the double-joint stiffness, and it is important to know what effect this loss will have on the overall stiffness behaviour. Hogan [1985a] showed that loss of the double-joint muscles can limit the range of endpoint positions at which isometric endpoint stiffness (round isopotential) can be achieved, which we expanded upon in Section 7.3. In the current section, we will expand on these approaches to understand how loss of the double-joint stiffness limits the endpoint stiffness and if anything can be done to compensate for these losses. We shall examine the change in behaviour first through an example. We will then generalize the results for overall limitations.

Figure 9.1.1 shows the manipulator located at the position $(\theta_s, \theta_e) = (45^\circ, 90^\circ)$. The actuator stiffnesses are initially all set equal such that each pair contributes 20 Nm/rad at the joints. All other parameters are chosen as in Section 7.1. The joint stiffness matrix is

$$\Psi = \begin{bmatrix} 40 & 20 \\ 20 & 40 \end{bmatrix} \text{Nm / rad}$$

The resulting stiffness ellipse is shown dotted in Figure 9.1.1. The minor axis points towards the shoulder because the shoulder and double-joint stiffness contribute the same amount. When the double-joint stiffness is removed, i.e., $\psi_{2d} = 0$, the resulting ellipse is shown as the solid ellipse in Figure 9.1.1. There are several direct effects. The size of the stiffness has decreased, as indicated by the larger isopotential ellipse. This is expected because the total stiffness contributions from the actuators has decreased. The resulting effect is that the endpoint is now easier to displace except in the direction perpendicular to 'D' (double-joint stiffness direction). The shape has also become more elongated. This means that the endpoint has a more preferred direction than before the double-joint stiffness was removed; the endpoint tends to favour displacement in this direction. Finally, the orientation has become twisted. Originally the stiffness was oriented to point toward the shoulder, corresponding to the single-joint elbow stiffness direction ('E' in Figure 9.1.1). With the double-joint stiffness removed, it has rotated towards the 'S' direction.

The changes described above are the direct effects of removing the double-joint stiffness. One can compensate for them to some degree. The reduction in stiffness size can be offset by increasing the remaining stiffnesses. However, increasing the shoulder stiffness would require conscious action from the amputee. This implies increasing the elbow stiffness in the prosthesis is a more appropriate means for restoring stiffness size. Note that this action increases the stiffness in the 'E' direction, and the perpendicular

direction is unaffected. The result is an even more elongated stiffness. The shape can be restored by a decreasing both of the remaining stiffnesses. However, this lowers the stiffness size even more, contradicting our attempt to fix the size change. Also, the original shape can only be exactly duplicated if the remaining stiffnesses are reduced to same level as the double-joint stiffness, which in this case requires they be set to zero, leaving a zero endpoint stiffness. Increasing elbow stiffness and decreasing shoulder stiffness will rotate the stiffness towards the elbow direction. This also further elongates the stiffness, counter to restoring the shape. The orientation will only return exactly towards the 'E' direction if the elbow stiffness becomes infinitely large or the shoulder stiffness reduces to zero. These solutions tend to contradict each other. Size correction requires increasing both elbow and shoulder stiffnesses, shape requires reducing them, and orientation requires increasing elbow stiffness while reducing shoulder stiffness.

Since the shoulder stiffness is controlled by the CNS, it is not easy to force desired changes. Therefore the best compensation is to provide single-joint elbow stiffness in the prosthesis that is higher than normally provided by the single-joint elbow muscles. This can compensate for the size reduction and restores the orientation to some degree, but at the expense of stiffness shape. The compensated stiffness will generally have higher stiffness in the direction towards the shoulder than normal, lower stiffness in the perpendicular direction, and will be slightly rotated towards the 'S' direction.

Figure 9.1.1: Effect of removing the double joint stiffness. The dotted ellipse is the endpoint stiffness ellipse with equal contributions from each actuator pair. The solid ellipse represents the endpoint stiffness when the double-joint stiffness is removed. Directions 'S', 'E', and 'D' correspond to the directions of the single-joint shoulder, single-joint elbow, and double-joint stiffnesses.

The changes described by the example in Figure 9.1.1 can be generalized to the overall limitations due to the removal of the double-joint stiffness. First, the overall maximum stiffness size has been reduced. The maximum stiffness size with the double-joint stiffness intact corresponded to all actuator stiffnesses at their maximum values. With the double-joint stiffness removed, the remaining shoulder muscles cannot contribute any more stiffness than their maximum to compensate. The maximum size can only be restored by designing the maximum single-joint elbow stiffness in the prosthesis to be equal to both normal contributions from elbow and double-joint stiffnesses. If possible, it may be desirable to attach any remaining double-joint muscles to the humerus so that they can still provide additional stiffness to the shoulder.

Second, the possible orientation range has been reduced. In Section 7.3 we showed that increasing actuator stiffnesses tending to rotate the stiffness towards the associated projection direction. This limits the orientation to the range shown as β in Figures 9.1.1 and 7.3.4. With the double-joint stiffness removed, there is no component to rotate the stiffness towards the 'D' direction. The orientation becomes limited to the range shown as α in Figure 9.1.1.

Finally, with the double-joint stiffness removed the characteristics of the endpoint stiffness become coupled, as previously described by Hogan [1985a]. There are three endpoint stiffness characteristics: size, shape, and orientation. With intact double-joint

stiffness, there are three inputs, namely the contribution from the three actuator sets. This allows us to control the three characteristics independently, within the limits imposed by the actuators. When the double-joint stiffness is removed the inputs are reduced to two, and so we can only chose two of the three desired characteristics and the third becomes dependent.

Given the limitation we have described in this section, it is apparent that the effects of the double-joint stiffness can not be replaced using only single-joint stiffnesses. Our proposed method for restoring natural limb behaviour using antagonistic actuation in an above-elbow prosthesis can restore some features. Full restoration of natural limb characteristics would require a mechanism that can measure shoulder displacement and produce elbow torque in response as well as measure elbow displacement and produce shoulder torque. We could implement this mechanism by extending the prosthesis over the shoulder. This solution would not likely be well accepted since it would increase the size of prosthesis and appear less natural. Another solution would be to invasively attach the prosthesis to any remaining bi-articular muscles. This would require a permanently attached prosthesis and leaves the amputee open to infection. Neither solution is correctly suitable for modifying the proposed design.

9.1.2 Spring Design

Throughout the current work the stiffness relationships have been derived as

instantaneous. That is, given an instantaneous value of actuator stiffness we can give the endpoint stiffness and vice versa. How these actuators achieve the stiffness has not been an issue. As long as the spring force functions are non-linear, they allow the ability to modulate stiffness by co-contraction of antagonistic actuator sets. In this section we will use the knowledge gained in the current work to aid in choosing appropriate non-linear spring force functions.

Previous work has suggested that quadratic force functions are desirable based on the criterion of keeping constant joint stiffness in the face of joint displacements [English, 1996; English & Russell, 1999a]. This criterion has the benefit that the joint stiffness remains as set by the controller, and therefore we can control it in an open loop fashion. Joint stiffness that changes with displacement would require feedback in order for the controller to be aware of the current stiffness values. However, we have found that the geometric stiffness can tend to destabilize the manipulator, suggesting that the stiffness should increase to compensate.

McIntyre et al. [1996] examined the possibility of maintaining constant joint stiffness in humans and simulation and found that joint stiffness tends to increase to compensate for instability. They incorrectly suggested that such instability is avoidable as long as muscle stiffness increases with force. This criterion suggests that a hardening spring (stiffness increases with displacement, and force) is sufficient to guarantee

stability. From previous work [English, 1996; English & Russell, 1999a] we know that this criterion is insufficient to guarantee an increase of joint stiffness. The criterion for increasing joint stiffness is that the force function must have greater curvature than a quadratic force-displacement function. That is, for a hardening spring the stiffness-displacement curve must be concave up. A linear stiffness-displacement curve produces no net joint stiffness change, and a concave down curve produces a decrease in joint stiffness. Note that all three of these curves have stiffness that increases with force. Furthermore, even a softening spring (stiffness decreases with displacement and force) can produce a net increase in joint stiffness as long as the stiffness-displacement curve is concave up.

Figure 9.1.2 shows why the curvature of the stiffness function determines the net stiffness change. In (a) the joint is shown displaced, where one spring increases in displacement and the other decreases displacement. Figure 9.1.2(b) shows the case of a hardening spring. In the linear case, denoted by 'A', the increase in stiffness of one spring, A^+ , exactly offsets the decrease in stiffness of the other spring, A^- because $A^+ = A^-$ from the linearity as shown in the figure. Denoting the set joint stiffness, with no joint displacement, as Ψ_{A0} , and the corresponding equal muscle stiffnesses as ψ_A , the displaced joint stiffness is proportional to the sum of the muscle stiffnesses:

$$\Psi_A = R^2\{(\psi_A + A^+) + (\psi_A - A^-)\} = R^2\{2\psi_A\} = \Psi_{A0}$$

In other words, displacement of the joint causes no change in joint stiffness for a relationship in which muscle stiffness varies linearly with muscle displacement, which is equivalent to a quadratic force-displacement relationship (since force is the integral of stiffness, as discussed in Chapter 5) as shown as curve A in the ϕ - δ plot in (b).

In a concave up stiffness curve, the increase in stiffness of the stretching spring more than offsets the decrease in stiffness of the other, resulting in a net joint stiffness increase. This is labelled as curve B in the stiffness and force plots of Figure 9.1.2(b). With an undisplaced joint, the set stiffness is Ψ_{B0} . A joint displacement produces a change in muscle displacement that increases one muscle stiffness by B^+ and decreases the other by B^- . As seen in (b), $B^+ > B^-$, and hence

$$\Psi_B = R^2\{(\psi_B + B^+) + (\psi_B - B^-)\} = R^2\{2\psi_B + (B^+ - B^-)\} > \Psi_{B0}$$

This means joint stiffness Ψ increases as the joint is displaced.

In a concave down curve, the decrease in stiffness is larger and hence the joint stiffness decreases. This is shown as curve C in (b). In this case, $C^+ < C^-$, and hence

$$\Psi_C = R^2\{(\psi_C + C^+) + (\psi_C - C^-)\} = R^2\{2\psi_C + (C^+ - C^-)\} < \Psi_{C0}$$

The joint stiffness has decreased with displacement. The same reasoning for each case is shown in (c) but for softening springs. The same equations above apply.

(a)

$$\Psi_A = R^2 \{ (\psi_A + A^*) + (\psi_A - A^*) \} = R^2 \{ 2\psi_A \} = \Psi_{A0}$$

$$\Psi_B = R^2 \{ 2\psi_B + (B^* - B^*) \} > \Psi_{B0} \quad (\text{since } B^* > B^*)$$

$$\Psi_C = R^2 \{ 2\psi_C + (C^* - C^*) \} < \Psi_{C0} \quad (\text{since } C^* < C^*)$$

(b)

(c)

Figure 9.1.2: Changes in joint stiffness. (a) Joint displacement. One spring increases in length the same amount as the decrease of the other. (b) Hardening springs. Linear stiffness produces no net joint stiffness change, concave up increases stiffness, concave down decreases it. (c) Softening springs. The same conditions apply as in (b).

From the current work, we also know that an increasing joint stiffness alone is insufficient to guarantee stability. Increasing joint stiffness with joint displacement means that ${}^J\mathbf{K}$ tends to increase with endpoint displacement. However, ${}^G\mathbf{K}$ increases with force which is the cause of displacement. Stability can only be guaranteed if the rate of increase of endpoint spring stiffness with endpoint force is greater than the rate of increase in geometric stiffness. All definitions of size, including k_{av} in (6.5.9) and the stiffness ellipse area in Section 2.2.2, relate to the determinant of the stiffness matrix. We can therefore guarantee stability if

$$\frac{\partial(\det {}^J\mathbf{K})}{\partial|\mathbf{F}|} \geq \frac{\partial(\det {}^G\mathbf{K})}{\partial|\mathbf{F}|} \quad (9.1.1)$$

From (7.4.1) we can write

$$\det {}^G\mathbf{K} = |\mathbf{F}|^2 \det(\mathbf{G}_F) \quad (9.1.2)$$

where $\mathbf{G}_F = \mathbf{G}_x \cos\theta_F + \mathbf{G}_y \sin\theta_F$

The endpoint spring stiffness is not explicitly a function of endpoint force, but is related to it through joint displacements. From Section 8.2 we know

$$\det {}^J\mathbf{K} = B_{12}\psi_{\Sigma s}\psi_{\Sigma e} + B_{13}\psi_{\Sigma s}\psi_{\Sigma d} + B_{23}\psi_{\Sigma e}\psi_{\Sigma d}$$

where the B_{ij} coefficients are given in (8.2.3). Using the chain rule we can write the rate of change of each actuator stiffness in the form

$$\frac{\partial \psi_{\Sigma i}}{\partial |F|} = \frac{\partial \psi_{\Sigma i}}{\partial \Delta} \frac{\partial \Delta}{\partial |F|} = \frac{\partial \psi_{\Sigma i}}{\partial \delta_i} \frac{\partial \delta_i}{\partial \Delta} \frac{\partial \Delta}{\partial M} \frac{\partial M}{\partial |F|} \quad (i = s, e, d) \quad (9.1.3)$$

Assuming that all spring functions will be the same, we can now rewrite (9.1.1) as

$$\frac{d\psi_{\Sigma}}{d\delta} \left(\frac{\gamma_s}{{}^sR_s} H_{ed} + \frac{\gamma_e}{{}^sR_e} H_{sd} + \left(\frac{\gamma_s}{{}^dR_s} + \frac{\gamma_e}{{}^dR_e} \right) H_{se} \right) \geq 2|F| \det G_F \quad (9.1.4)$$

$$H_{ed} = (B_{12}\psi_{\Sigma e} + B_{13}\psi_{\Sigma d})$$

$$H_{sd} = (B_{12}\psi_{\Sigma s} + B_{23}\psi_{\Sigma d})$$

$$H_{se} = (B_{13}\psi_{\Sigma s} + B_{23}\psi_{\Sigma e})$$

$$\gamma_s = ({}^sR_s^2 \psi_{\Sigma s} [J_{11} \ J_{21}]^T + {}^dR_s {}^dR_e \psi_{\Sigma d} [J_{12} \ J_{22}]^T) [\cos\theta_F \ \sin\theta_F]^T$$

$$\gamma_e = ({}^sR_e^2 \psi_{\Sigma e} [J_{11} \ J_{21}]^T + {}^dR_s {}^dR_e \psi_{\Sigma d} [J_{12} \ J_{22}]^T) [\cos\theta_F \ \sin\theta_F]^T$$

Solutions for the spring stiffness functions to the partial differential equation in (9.1.4) can guarantee stability. Note that the elements of B_{ij} and J are all functions of endpoint position and G_F is a function of both endpoint position and force direction.

When solving this equation one should select the worst case for each element. As long as condition (9.1.4) is met, the manipulator will remain stable universally. Since we only have the choice of selecting spring functions for the single-joint elbow stiffness, we would require an accurate model of the existing shoulder muscle functions, and can set the double-joint stiffness to zero. This simplifies (9.1.4) because it only becomes a function of the single-joint elbow stiffness. The solution will provide the necessary elbow stiffness function to guarantee stability of the whole arm, assuming the shoulder

supplies sufficient stiffness as modelled.

Another method to ensure stability would be to make sure that each actuator stiffness contribution keeps ahead of its critical value in (8.2.4). Using ${}^G\mathbf{K}_v = |\mathbf{F}| \mathbf{G}_{F_v}$, this criterion can be written in the vector form of the stiffness elements as

$$\frac{\partial \psi_{\Sigma v}}{\partial |\mathbf{F}|} \geq -\mathbf{P}_v \mathbf{G}_{F_v} \quad (9.1.5)$$

The elements on the left side can be expanded as in (9.1.3), resulting in a set of partial differential equations. Under this condition, the actuator stiffnesses increase with displacement, and hence endpoint force, at a rate equal to, or larger than, the increase in their critical values. Thus if they are stable to begin with, they maintain stability. The problem with this method is that it sets requirements on each actuator stiffness independently. Since the double-joint stiffness is not present in the prosthesis, it cannot increase to keep ahead of its critical value. Furthermore, the intact shoulder muscles may not provide sufficient stiffness to keep ahead of their critical value. Also, if ${}^d\mathbf{R}_s = {}^d\mathbf{R}_e$, we know from (8.2.5b) that the single-joint elbow critical value is zero. Since we only have the choice of designing this stiffness in the prosthesis, it would suggest that constant elbow stiffness is sufficient. Therefore this method is better suited to the case where one is free to choose all three actuator stiffnesses, such as in a multi-joint prosthesis or robot manipulator. Equation (9.1.4) therefore supplies a more general solution.

9.1.3 Controller

Ideally the antagonistic actuation in the prosthesis would operate as the natural musculo-skeletal system. EMG signals from the CNS could be measured and applied directly to actuators. Complete duplication of the natural limb would require exact modelling of the muscles, which is beyond the scope of the proposed design. Stability could still be guaranteed in this approach to control if the spring functions are designed following the conditions in the previous section. However, manufacturing such springs may be difficult or impossible. It may be necessary to ensure stability through a feedback controller. This would require either force feedback or joint displacement feedback such that the joint stiffnesses can be modulated in the controller to keep ahead of the geometric stiffness instability following the condition in (9.1.4).

9.2 Robotics

Robot manipulators are generally modelled after the human arm. Usually the mechanics of the manipulator are driven by motors acting at the joints and interaction properties are dependent on the feedback control scheme. The current work can be applied to robot manipulators in several ways. First, it may be desirable to control the manipulator using the mechanics of the human arm similar to the model in Figure 2.1.1. This approach can be more energy efficient [English, 1996; English & Russell, 1996, 1999a], making it useful in such areas as mobile robots and space robotics where energy sources are limited. Similar approaches have been developed that allow antagonistic

control of stiffness using cable driven joints [Ma et al., 1993; Lee et al., 1994; Yokoi et al., 1995], or other antagonistic actuation [Laurin-Kovitz et al., 1991; Immege, 1987]. Antagonistically driven manipulators can also be beneficial in tight locations where size is an issue [Jacobsen et al., 1984; Jacobsen et al., 1986].

Other robot manipulators simulate the ability of muscles to modulate stiffness through a feedback controller. These include impedance control, stiffness control, and even servo (position) control. In these schemes, the commanded position is compared to the measured position and the resulting joint torques are proportional to the difference. The proportionality constant, i.e., the feedback gain, effectively acts as a stiffness.

Based on the current work, all of the above schemes should take the geometric stiffness into account when choosing an appropriate stiffness level. That is, to achieve the desired stiffness they should first compensate for the geometric stiffness effect as described in Section 8.3.1 to guarantee stability of the manipulator. Impedance control already includes force feedback to modulate apparent inertia of the manipulator, and so it is only a matter of including the geometric stiffness when calculating endpoint stiffness, using existing position and force feedback. Schemes without force feedback can either include it, or use the displacement to indirectly calculate the geometric stiffness based on the set stiffness as in (9.1.3) and (9.1.4).

Many other control schemes do not control the apparent stiffness of the manipulator. Force control schemes use force feedback to directly control the endpoint force. Since the endpoint force is maintained constant, there is no endpoint spring stiffness component. However, there is still a geometric stiffness component since it relates to the endpoint force. Without an endpoint spring stiffness to compensate, any unstable geometric stiffness would be sufficient to drive the manipulator unstable. As the unstable manipulator moves off the surface, the endpoint force is no longer present and hence it can be stable again, driving it towards the wall where it again becomes unstable. This would tend to make the manipulator chatter on the surface, a phenomenon which is known to occur in many force feedback robots. Without an endpoint spring stiffness component, one cannot compensate for the geometric stiffness in a purely force feedback manipulator. This suggests that some form of stiffness control is imperative to maintain stability when interacting with environmental forces.

9.3 Haptic Devices

A haptic device is a human-machine interface which provides a sense of realism of a virtual environment to the user. Most existing virtual reality interfaces provide visual feedback only, thus providing the look of virtual environment but not the feel. Virtual reality has uses in entertainment as well as training in areas where real interaction is expensive, dangerous, or difficult to achieve. Such areas include flight training, surgery, and astronaut training. Visual feedback is capable of training procedures, but it

does not provide the musculo-skeletal feedback necessary to learn physical skills. Haptic devices provide this physical feedback. An example of existing haptic devices is the force feedback joystick available for some computer games.

There are generally two approaches to haptic device design. The first approach attempts to define the physical properties of the virtual environment such that one can feel the interactive forces (stiffness, damping, inertia) of the virtual objects, including surface textures. One problem with this approach is determining the forces when objects begin to overlap. There have also been dynamic instability problems in this approach [Tsai & Colgate, 1995]. It has been found that performance can be improved by using real damping in the manipulator in combination with virtual damping properties [Brown & Colgate, 1997]. It may be the case that control of stiffness using real springs may also aid in the stability of such devices since they are limited to act passively, a criterion that can guarantee dynamic stability [Hogan, 1988; Colgate & Hogan, 1989]. If so, the model used in the current work may be a viable approach to such stiffness control in a haptic device.

A second approach relies completely on passive behaviour. Instead of mimicking a virtual environment, it uses a braking system to limit motion of the device to a predetermined path [Book et al., 1996; Gomes & Book, 1997]. This method cannot produce the feel of environmental objects, but it is useful when training tasks in which the

path of movement is important, such as in surgery. The current work can be applied to this type of method such that stiffness is used to define the path rather than a braking system. If we set the endpoint stiffness such that $^Jk_{\min} = 0$ and $^Jk_{\max}$ is large, there will be no endpoint force for displacements along the $^Jk_{\min}$ direction and large forces in the $^Jk_{\max}$ direction. This tends to allow a user holding the endpoint to displace it along in the $^Jk_{\min}$ direction without resistance. By orienting the stiffness in the direction of the desired path, the user can be guided in that direction. As the endpoint moves, the orientation can be rotated to define a continuous path in the workspace. The isopotential ellipse becomes two infinitely long parallel lines whose distance apart is $r_{\min}$, a function of $^Jk_{\max}$, as shown in Figure 9.3.1.

Since the goal is to have the user move along the path with no endpoint force, the geometric stiffness can be neglected. Hence $\mathbf{K} = ^J\mathbf{K}$. Recall from (6.1.31b) that $^Jk_{\min}$ is related to the actuator stiffnesses by

$$D_{J_1}^2[(L_e^T R_e S_{eJ})^2 \psi_{\Sigma e} + (L_t^T R_t S_{tJ})^2 \psi_{\Sigma t} + (L_d^T R_d S_{dJ})^2 \psi_{\Sigma d}] = ^Jk_{\min}$$

We wish the stiffness to be zero. There are several possibilities to achieve this. First, the coefficients can all be made zero. This can only occur if $\theta_{es} = \theta_t = \theta_d$, which only happens when the arm is fully extended, and so this is an unacceptable solution. Second, we can set two of the three actuator stiffnesses to zero and let the third coefficient be zero. This only allows us to use endpoint orientations that are one of the three principal directions

associated with the actuator stiffnesses. It is unacceptable as a method for general path definition. The only remaining solution to produce zero minimum stiffness is to allow negative actuator stiffnesses. This has another advantage. In Section 7.3 we showed that the stiffness orientation was limited in the outer region of the workspace, and inner region if $L_x \neq L_e$. This implies that the haptic device would have to be limited to work only within a small region of the workspace. However, this limitation was based on the assumption of only positive actuator stiffnesses in (7.3.3). With negative actuator stiffness, this limitation is removed.

The value of ${}^Jk_{\max}$ determines the forces that are felt for a deviation from the path. One can then use ${}^Jk_{\max}$ as a guide to the degree of accuracy required in the motion. Variations of this approach can control the training of a particular path. Initially one can set ${}^Jk_{\min}$ slightly negative to push the user along the path. As their performance improves, the minimum stiffness can be increased to zero, and even slightly positive values to provide resistance if the goal is to rehabilitate muscles.

The general solution for the required actuator stiffnesses can be found by combining (6.2.31) and (6.2.32). If we use the projection matrix rotated through by the desired orientation, denoted as JP_k in (6.2.32), the actuator stiffnesses required to achieve a general endpoint stiffness are

Figure 9.3.1: Concept of passive haptic device using stiffness as a trajectory guide. Setting $k_{min} = 0$ produces no resistive force in response to displacement in that direction. This is shown as a truncated isopotential ellipse with an infinite major axis. Other displacement directions produce forces back towards the direction of zero stiffness, thereby guiding a user holding the endpoint. As the endpoint moves, the orientation of the stiffness can change to produce a continuous path.

$$[\psi_{\Sigma s} \ \psi_{\Sigma c} \ \psi_{\Sigma d}]^T = {}^J\mathbf{P}_{kv}^{-1} [{}^J\mathbf{k}_{\max} \ {}^J\mathbf{k}_{\min} \ 0]^T \quad (9.3.1)$$

where ${}^J\mathbf{P}_{kv}$ is the projection matrix in the form in (6.2.30) Define ${}^J\mathbf{Q}_v = {}^J\mathbf{P}_{kv}^{-1}$. Because of the zero element, (9.3.1) simplifies to

$$\begin{bmatrix} \psi_{\Sigma s} \\ \psi_{\Sigma c} \\ \psi_{\Sigma d} \end{bmatrix} = \begin{bmatrix} {}^JQ_{11} & {}^JQ_{12} \\ {}^JQ_{21} & {}^JQ_{22} \\ {}^JQ_{31} & {}^JQ_{32} \end{bmatrix} \begin{bmatrix} {}^J\mathbf{k}_{\max} \\ {}^J\mathbf{k}_{\min} \end{bmatrix} \quad (9.3.2)$$

In the case when ${}^J\mathbf{k}_{\min} = 0$, this simplifies further to

$$[\psi_{\Sigma s} \ \psi_{\Sigma c} \ \psi_{\Sigma d}]^T = [{}^JQ_{11} \ {}^JQ_{21} \ {}^JQ_{31}]^T {}^J\mathbf{k}_{\max} \quad (9.3.3)$$

9.4 Human Movement Studies

The current work has a variety of potential applications to studies of the human musculo-skeletal system. The geometric stiffness is particularly interesting. It implies that the human arm can be made to go unstable with sufficient loading. McIntyre et al. [1996] looked at the overall stiffness (including the term denoted as geometric stiffness in the current work) and found that stiffness tended to increase in humans to compensate. This increase in stiffness appeared to occur because of the inherent muscle stiffness properties rather than a commanded stiffness increase from the CNS. However, they incorrectly concluded that any muscle tension relationship in which stiffness increased

with force was sufficient to guarantee stability. As discussed in Section 9.1.2, this is insufficient to guarantee stability or even increasing joint stiffness. We suggest a comparison of the muscular tension-length behaviour during experiments with the conditions of (9.1.4) to determine if indeed the natural muscle functions can guarantee stability, and if not, what conditions are necessary to make the human arm unstable. It may also be interesting to compare these results with those of patients with muscular disorders to determine if the geometric instability can explain some of the visible phenomena such as tremour.

The decomposition of endpoint stiffness with respect to actuator stiffness contributions may also be useful in determine the relative contribution of muscle pairs to overall stiffness. Since the double-joint stiffness direction is a function of the relative moment arms at each end of the bi-articular muscles, this decomposition may also allow one to compute this ratio by determining this direction. The decomposition might also be expanded to include more joints and a third dimension so that stiffness in the entire arm motion, including wrist and third dimension, can be understood.

Finally, a variation on the haptic devices described in the last section can be used for motor control experimentation on humans. Once a particular path is trained, random variations in stiffness and orientation can be used to determine the degree to which the arm follows a virtual trajectory. Since the drive mechanics are passive, this technique is

safer than actively controlled systems, and the defined path is not restricted as in other passive experimental devices using brakes (e.g., Won & Hogan [1995]).

9.5 Chapter Summary

In Chapter 9 we investigated some applications of the knowledge gained in the current work. Section 9.1 directly applied the knowledge to the proposed above-elbow prosthesis as discussed in Chapters 1 and 4, using antagonistic actuation of non-linear springs to control both position and stiffness. In Section 9.1.1 we investigated the possibility of compensating for the loss of the double-joint stiffness which would not be present in the proposed prosthesis. Unfortunately, there is little that can be done to compensate for the loss. The discussion answers Question 7 in Chapter 4 that had asked about the importance of bi-articular muscles and the ability to compensate for their loss. In Sections 9.1.2 and 9.1.3 we developed the conditions of the spring or controller design to ensure stability despite an unstable geometric stiffness.

Section 9.2 discussed issues regarding stiffness in robotic manipulators. The biggest consideration is the geometric stiffness which is present in all manipulators, whether the stiffness is controlled or not. A variation of the robot manipulator is a haptic device, as discussed in Section 9.3. Instead of the manipulator software completely driving the motion and forces, it is an interactive process between user and manipulator. We proposed a passive haptic device design that can train or rehabilitate individuals by

guiding them through a pre-determined path. Finally, we applied our understanding of stiffness to studies of human posture and movement control in Section 9.4. Some previous work has investigated the stability of the human arm with an unstable geometric stiffness, but there are many questions left unanswered. The decomposition of stiffness may also be beneficial for understanding contributions from different muscle sets.

Chapter 10

Conclusions and Recommendations

In this chapter we will review the major contributions of the current work and show that we met the goals of Chapter 4. To begin we will restate the Chapter 4 questions and discuss the answers individually. Additional conclusions will be discussed following this. Finally we will present recommendations for future work.

10.1 Conclusions

1. How do the actuator (muscle) stiffnesses relate to the endpoint stiffness?

We derived the relationships between the actuator stiffnesses and the endpoint spring stiffness in Sections 6.1.2 and more generally in Section 6.2.4. Specifically, (6.1.31) and (6.2.27) provide the relationships from actuator to endpoint. That is, given the actuator stiffnesses, these equations provide the corresponding endpoint stiffness. We also showed how these equations can be inverted to solve for the required actuator stiffnesses to achieve a desired endpoint stiffness. The geometric stiffness is not a function of the actuator stiffnesses, but rather the joint moments (and thus endpoint force).

This decomposition of the endpoint spring stiffness showed that the stiffnesses from each antagonistic actuator set projects onto the endpoint in a specific direction with a specific weight. The single-joint shoulder stiffness affects the endpoint stiffness in the direction of the forearm. The single-joint elbow stiffness affects the endpoint stiffness in the direction pointing toward the shoulder. The direction of the double-joint stiffness depends on the relative moment arms at the shoulder and elbow, but is parallel to the upper arm when they are equal. The decomposition also showed that the effect of the double-joint stiffness can be duplicated by using a parallelogram linkage where the double-joint stiffness becomes a single-joint stiffness on the second linkage, as in Figure 6.1.3.

Numerical examination of these stiffness directions in Section 7.3 showed that increasing an actuator stiffness increases the endpoint stiffness in its corresponding direction, but does not affect the stiffness in the perpendicular direction. This has the effect of rotating the endpoint stiffness orientation towards the direction of increasing stiffness. Shoulder and double-joint stiffnesses tend to balance each other such that one tends to rotate the endpoint stiffness one way and the other rotates it the other. The net effect of equal stiffness contributions is little change in the orientation.

2. What limitations are there on endpoint stiffness?

As one would expect, there is a limitation on the maximum size of endpoint stiffness, corresponding to all of the actuator stiffnesses simultaneously set at their maximum allowable stiffnesses. There are other limitations, however. We showed in Section 7.3 that, assuming positive actuator stiffnesses, the orientation is limited to a small range when the endpoint is in the outer region of the workspace, and achievable shapes and sizes are limited for some orientations in the inner workspace. Section 7.4 showed us the possible instability of the endpoint stiffness due to the geometric stiffness. We investigated the minimum stiffnesses to maintain stability in Section 8.2. If this stability is to be guaranteed, the spring functions or controller must be designed as suggested in Section 9.1, such that the increase of endpoint spring stiffness increases with force as much as or more than the geometric stiffness.

3. How do the muscle stiffnesses affect the stability of the arm?

The projection of actuator stiffnesses onto the endpoint spring stiffness in (6.2.27) makes it apparent that positive actuator stiffnesses produce a positive definite endpoint spring stiffness. In Section 6.4 we pointed out that a positive definite stiffness matrix is stable. Hence positive actuator stiffnesses guarantee the stability of the endpoint spring stiffness. However, in Section 7.4 we showed that the stability of the geometric stiffness is dependent on the direction of endpoint force. It is independent of actuator stiffnesses. While the actuator stiffnesses have no direct control over the stability of the geometric

stiffness, they can control the stability by compensating for the geometric stiffness as described in Section 8.2. As long as two of the three actuator stiffnesses are above their critical values and the third is above the stable-saddle border described by (8.2.3), the endpoint will be stable. If two actuator stiffnesses are below their critical values and the third is below the saddle-unstable border from (8.2.3), the manipulator is unstable in all directions. Any other cases result in a saddle point which is also unstable.

An interesting result from Section 8.2 is that while positive actuator stiffnesses guarantee a stable endpoint spring stiffness, negative actuator stiffness does not necessarily result in unstable (including saddle) endpoint spring stiffness.

4. What function does the geometry play in setting the stiffness?

It is clear from the derivation of endpoint stiffness in Section 5.3, and previous work, that both the endpoint spring stiffness and geometric stiffness are functions of position within the workspace. It is not immediately apparent what function the geometry plays. In Section 7.2 we showed that the geometry dominates the non-linear isopotentials compared to the endpoint spring functions. Changes in position produced large changes in the shapes of the isopotentials, but the spring functions essentially only produced small changes in the magnitudes of the isopotentials rather than their shape.

The endpoint position plays a large part in the size, shape, and orientation of the

isopotential ellipse, as we demonstrated in Section 7.3. The geometry tends to keep the ellipses oriented towards the shoulder, except when there is a large difference in the single-joint shoulder and double-joint stiffnesses. This orientation tendency becomes stronger as the endpoint position moves towards the outer region of the workspace. Large differences in shoulder and double-joint stiffnesses have less effect on the orientation in the outer region because the principle directions of the actuator stiffnesses are closer together. Near the shoulder, the endpoint spring stiffness tended to be long and thin, with minimum stiffness toward the shoulder. At the outer region, the endpoint spring stiffness was also long and thin, but with maximum stiffness towards the shoulder. In the middle of the workspace, the endpoint spring stiffness tended to be rounder. In Section 7.4 we showed similar behaviour of the geometric stiffness. It is long and thin at both extremes of position, and rounder in the centre. However, the orientation of the geometric stiffness is affected most by the direction of applied force.

We combined all the knowledge of geometry in the endpoint stiffness in Sections 8.1 and 8.3 to apply strategies of endpoint position and force direction that produce desirably stiffness properties.

5. How does loading affect limb stiffness.

The endpoint spring stiffness is solely a function of the geometry and the actuator stiffnesses. Loading does not affect it. However, the geometric stiffness is a function of

both geometry and the loading. In Section 6.3 we developed an index to test the significance of the geometric stiffness contribution to the overall stiffness. This indicates at what point the loading becomes significant enough that the geometric stiffness should be considered. We applied this index to numerical simulations in Section 7.4 and found that even relatively small endpoint forces can produce a significant endpoint stiffness contribution.

The direction of loading also has an influence on the geometric stiffness orientation and stability. For endpoints with $\theta_e < 90^\circ$ (outer region), external forces pulling the endpoint away from the shoulder (manipulator forces toward the shoulder) produce a stable stiffness with an orientation of *maximum* stiffness toward the shoulder ($\theta_k = 0^\circ$). External forces toward the shoulder produce an unstable stiffness with the *minimum* stiffness toward the shoulder. Endpoint forces that push sideways on the endpoint, perpendicular to the shoulder direction, produce saddle stiffnesses and $\pm 45^\circ$, depending on which way the force is pointed. Force directions between these produces geometric stiffness orientations that vary between those described. For positions with $\theta_e > 90^\circ$ (inner region), the orientation behaviour is reversed. External forces away from the shoulder still produce stable geometric stiffness, but with the *minimum* stiffness toward the shoulder, while external force toward the shoulder result in *maximum* geometric stiffness toward the shoulder. At $\theta_e = 90^\circ$ the geometric stiffness is completely round for external forces toward or away from the shoulder, and is $\pm 45^\circ$ for all other force

directions.

6. Does the linear stiffness approximation adequately describe the behaviour of the limb given its non-linear nature?

In Section 7.2 we checked the degree to which linearized isopotentials match the true non-linear shapes. The results showed that there was good agreement in the linear and non-linear shapes in the central region of the workspace. There was little deviation of the shapes for endpoint displacements up to about one quarter of the length of the forearm. As the endpoint nears the inner and outer extremes of the workspace, the non-linear isopotentials become much more distorted, and the linear approximation fails to match the shapes at displacements an order of magnitude lower than at the centre of the workspace. This means that while the linearized stiffness is adequate to describe the state of the stiffness at a given point, it is only an accurate description of the actual behaviour of the endpoint for displacements at least an order of magnitude lower than the limb dimensions for positions near the extremes. This is an important limitation when measuring the stiffness in human subjects (or other manipulators) by perturbing the hand. Displacements of the hand must be sufficiently small near the extremes to produce a reasonably accurate depiction of the stiffness at these locations. Unfortunately, smaller displacements correspond to potentially larger errors, thereby decreasing the accuracy. This implies a limit to the accuracy at which endpoint stiffness measurements can be made near the extremes of the workspace. Locations near the centre of the workspace

allow much larger displacements to maintain accuracy of the ellipse as well as accuracy of the measurement.

7. What importance are bi-articular muscles for controlling stiffness and can their influence be compensated in an above-elbow prosthesis?

We investigated the contribution and compensation for double-joint stiffness, and hence bi-articular muscles, in Section 9.1.1. The double-joint stiffness tends to act to balance the contribution of the single-joint shoulder muscles so as to orient the endpoint spring stiffness toward the shoulder. There is no way to compensate for this behaviour in an above-elbow prosthesis that does not include a bi-articular mechanism that can produce shoulder torques in response to elbow displacements and vice-versa. This implies either an external extension of the prosthesis over the shoulder, or invasive attachment of the prosthesis to the remaining bi-articular muscles. Neither of these solutions is a viable alternative at this time.

There can be minimal compensation for the loss of the double-joint stiffness by using single-joint elbow stiffness that is higher than normal. This can reduce the change in stiffness orientation due to double-joint stiffness loss, and compensate for reduction in stiffness size, but at the expense of the stiffness shape. Attaching the remaining bi-articular muscles to the humerus bone in the upper arm can also help to restore the stiffness size, as well as shoulder torques, but reduces the ability to correct for the

orientation change.

8. What strategies could be used for selecting appropriate stiffness values (endpoint and muscles or actuators)?

The criteria to select appropriate stiffness largely depend on the desired task. However, there are some general strategies that can narrow the range of desirable stiffnesses. We discussed many of these strategies in Chapter 8. In Section 8.1 we suggested that choosing force directions such that the external force is pointed away from the shoulder maximizes stability. Orienting the maximum stiffness along a constraint surface allows greatest resolution for control of position along the constraint and force into the constraint. However, producing a given force at low stiffness requires larger work input, suggesting that an appropriate contact stiffness is only low enough to achieve acceptable force resolution.

Section 8.3 suggests achieving the desired endpoint stiffness by activating the actuator stiffnesses as a combination of the desired stiffness and compensation for the geometric stiffness. It also includes strategies to minimize the effects of the geometric stiffness on the overall stiffness. By choosing appropriate endpoint position and force orientation, the geometric shape and orientation can be matched to the desired endpoint stiffness. The endpoint spring stiffness would be chosen as this same shape and orientation, with only size varying to compensate for the geometric stiffness size which is

a function of the contact force. Using this strategy, one can minimize changes in shape, size, and orientation due to the geometric stiffness contribution. We also showed appropriate use of friction can minimize the effect of the geometric stiffness.

The above answers provide the necessary understanding of stiffness in the natural arm to implement an antagonistically actuated above-elbow prosthesis with variable stiffness. In addition to the knowledge gained with respect to the above questions, we developed several new techniques and tools for measuring and understanding stiffness. In Section 6.5 we developed a unified method for representing stiffness using a modified Mohr's circle approach from stress analysis. The Mohr's circle representation contains all of the relevant stiffness characteristics. Perhaps more importantly, it contains all of the relationship equations directly in its geometry. The traditional representations of the isopoential and stiffness ellipses contain the stiffness characteristics, but the equations that relate them are not inherent in the plot. Furthermore, the Mohr's circle representation allows the inclusion of unstable stiffness and anti-symmetric (non-conservative) stiffness in the same diagram. Neither of the ellipse representations can include these.

One technique that is learned from the inclusion of the anti-symmetric component is that we can use angle between the directions of co-linear force and displacement as a measure of the conservativeness of the system. Conservative stiffness has these

directions exactly 90° apart (the ellipse axes). As the anti-symmetric component becomes larger, the angle between these directions becomes smaller until they become a single direction. Beyond this, the anti-symmetric component dominates and there is no co-linear force-displacement direction.

Finally, in Section 6.3.2 we developed the Ellipse Comparison Index as a single parameter to determine when the geometric stiffness becomes significant to the overall stiffness. It is based on the area of the isopotential ellipse that becomes shifted when the geometric stiffness is included. Numerical simulations using this index showed that errors in stiffness values can be on the order of 5-10% for index values of about 0.05 (5% of the area shifted).

10.2 Recommendations

We break down the recommendations into two categories. First, we make recommendations for antagonistic devices based on knowledge gained from the current work. Second, we recommend future work that should be undertaken to further understand control of stiffness in multi-dimensional manipulators.

The recommendations from the current work have several applications based on the discussions of Chapter 9 and the answers to the questions in this chapter.

Above-Elbow Prosthesis

Using the antagonistically controlled prosthesis as proposed, one should allow the single-joint elbow stiffness to be larger than in the natural limb. This will help to compensate for the change in size and orientation of the endpoint stiffness due to the loss of the double-joint stiffness. If possible, attachment of the remaining bi-articular muscles to the humerus can also aid in the compensation for lost size, but at the cost of reducing the orientation correction. It can also increase the available shoulder torque. Combining all these arguments, this attachment would be a wise practise where possible.

It is also advisable to design the spring functions to guarantee stability of the arm in the face of unstable geometric stiffness. The single-joint elbow springs should be designed to satisfy the condition in (9.1.4), with a double-joint stiffness of zero and a reasonably accurate model of the single-joint shoulder stiffness contribution. It appears that the natural limb is capable of avoiding instability through the tension-length relationship of the muscles, but this may not be the case with the bi-articular muscles removed.

Robotics

Robotic manipulators that use some form of stiffness control should take the geometric stiffness into account when selecting the stiffness, or designing the stiffness functions in the case of real stiffness elements such as cable-driven systems. This

compensation would ideally use force feedback, but can use positional feedback and the current set stiffness to determine the geometric stiffness. Pure force-based control has no compensation for geometric stiffness and should be redesigned to include such compensation ability, or conservatively avoid unstable geometric stiffness conditions.

Haptic Devices

Haptic devices may benefit from using real stiffness elements as opposed to virtual ones. A passive haptic device, as proposed in Section 9.3, may be beneficial in training movement skills and rehabilitation.

Human Studies

Although there has been some study of the effect of the endpoint geometric stiffness in humans, there are many questions left unanswered. For example, do the inherent muscle tension-length relationships guarantee arm stability by the condition of (9.1.4)? Do humans use some of the strategies of Chapter 8 to minimize instability, the effects of geometric stiffness, or maximize efficiency? We may also be able to apply the decomposition of endpoint spring stiffness in Section 6.2 to determine the contribution of each antagonistic set to the overall stiffness.

While the current work is comprehensive as an analysis of a two degree-of-freedom planar manipulator, much work can be further understand stiffness in manipulators. Some preliminary work has been performed on several of the following recommendations, but it is incomplete and not included in the scope of the current work. These recommendations include:

1. Investigate the behaviour of manipulators, particularly the effects of geometric stiffness, when subject to contact forces from a static constraint, such as a wall. The current work only investigated the state of the stiffness given a contact force value, and the only dynamic behaviour was shown in Section 7.4 for a constant force such as applied by a gravity load. Preliminary results suggest that unstable contact forces on a constraint cause the manipulator to move along the constraint to a stable equilibrium point on the constraint. It appears that a stable equilibrium may bifurcate to an unstable equilibrium point with two stable points in either direction along the constraint which grow further apart as contact force increases. However, this behaviour is not well understood, nor how it might apply to contact chattering or the application of friction.
2. Expand the stiffness projections to the third dimension. Adding the third dimension of motion at the shoulder joint seems relatively straightforward for expanding the stiffness analysis. However, coupling of stiffnesses at the shoulder may be more complex. The three dimensional stiffness at the endpoint will form ellipsoidal isopotentials. This may

be more easily analysed by the Mohr's Circle approach where three-dimensional stiffness can be represented in a two-dimensional plot.

3. Expand the stiffness analysis to include three or more links. In the two link case the stiffness projection directions have physical significance for force and displacement directions. For manipulators with more joints than degrees-of-freedom at the endpoint, the manipulator Jacobian will be non-square and the stiffness projections may have different interpretations. Geometric stiffness may also exhibit significantly behaviour than for the current two-link case as seen in Section 7.4. Multiple links also allow for a greater variety of coupled stiffnesses, such as three-joint stiffnesses and various combinations of two-joint and one-joint stiffnesses. There may be an optimum combination of coupled stiffnesses for controllability and other combinations in which some stiffnesses contribute little.
4. Build an experimental apparatus. The stiffness behaviour examined in the current work may be better understood by seeing the physical consequences of various stiffness states.
5. Perform human movement studies examining the decomposition of endpoint stiffness into the different muscle groups. Behaviour of muscle activation may be better understood by examining their contribution to endpoint stiffness. Also, it may be

possible to show the use of geometric stiffness effects to optimize performance of some tasks, as well as the effect of muscle activation to compensate for geometric stiffness.

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Appendix A

Minimum Effort

It is known that use of constraint forces can reduce muscular effort and improve energy efficiency. Consider the task of opening a drawer. The motion is constrained in all directions except the sliding axis of the drawer. Intuitively, one would expect that the minimum muscular effort would be to apply an endpoint force solely in the direction of free motion with no constraint force. The underlying (incorrect) assumption is that constraint forces require additional muscular force that does no work towards opening the drawer.

In a linkage system with rotational joints, the endpoint forces are the result of drive torques at the joints. In the case of the human arm, these torques are the joint moments created by muscles. Endpoint forces are then the result of these moments and the arm kinematics. The endpoint forces resulting from a pure elbow moment must act on a line that passes through the shoulder, otherwise a shoulder moment is required. Likewise, the endpoint forces resulting from a pure shoulder moment must pass through the elbow, and the forearm becomes a two-force member. Any arbitrary endpoint force is

then a superposition of these forces.

Figure A-1(a) shows a model of an arm opening a drawer and the positive direction of joint forces resulting from pure shoulder and elbow moments. The specific combination of shoulder and elbow forces, F_s and F_e , required to produce a net force tangential to the constraints, F_t , is shown in Figure A-1(b), labelled F_{st} and F_{et} , respectively. The joint moments to produce these forces are found by multiplying them by the moment arms shown in Figure A-1(a). These moments are plotted in Figure A-1(c) as M_{st} and M_{et} . If we define effort as the vector addition of these moments, the effort to produce the purely tangential force is shown as M_t in Figure 2.2(c).

Figure A-1: Forces involved in opening a drawer. (a) Directions of pure joint forces. Forces created by a pure elbow torque must point towards the shoulder, whereas forces from pure shoulder torques must pass through the elbow. (b) Combination of joint forces, F_e and F_s , that produce a net force tangential to drawer motion, F_t . The components from the elbow and shoulder are labeled as F_{et} and F_{st} . Any force with the same tangential component will be as effective in opening the drawer. The minimum effort results in the endpoint force, F_m , shown by the dashed arrow. (Joint force components F_{em} and F_{sm} are also shown.) The constraint force for this minimum effort is shown as F_c . (c) Vector plot of moments. M_t is the effort required to create F_t . M_m is the minimum effort to produce the same tangential force component.

In opening the drawer, we also have the option to apply forces in the direction of the constraints. The *operating line* in Figure A-1(b) is defined such that no matter how much constraint force is used, the net force in the direction to open the drawer is the same. Thus any of the lighter drawn forces in Figure A-1(b) would be as effective in opening the drawer. Certainly F_l is the minimum net force of these choices, but not necessarily the minimum effort. The operating line is transferred to the plot in Figure A-1(c) using combinations of F_s and F_e (and therefore M_s and M_e) that reach the operating line in Figure A-1(b). It is clear then in Figure A-1(c) that the minimum effort, in the least squares sense, is the shortest distance from the origin to this line, denoted M_m . The net force to produce this is shown in Figure A-1(b). Thus taking advantage of constraint forces can reduce the effort required at the joints.

One could question the definition of effort in this least squares sense, however a more rigorous definition by translating joint moments to muscle forces, avoided here for simplicity, produces the same result. Furthermore, both joint moments were reduced in this example. (Compare joint moments M_{sm} and M_{em} to M_{st} and M_{et} , respectively, in Figure A-1(c).) In summary, minimum joint moments are generally not associated with minimum endpoint force.

Appendix B

Conservative Stiffness

A conservative system is one in which no work is performed around any closed path. There are at least three equivalent indicators of a conservative system [Schey, 1997]. The indicators are discussed in each section below as applied to the two dimensional stiffness used throughout the current work.

B.1 Closed Path

Using the basic definition of conservativeness, the net work around any closed path is zero,

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{t}} ds = 0 \quad (\text{B-1})$$

where $\mathbf{F}$ is the force vector

$\hat{\mathbf{t}}$ is a unit vector tangent to the path

s is the distance along the path

C is the whole closed path

Equation (B-1) indicates a conservative system when it is true for any closed path C.

Figure B-1 shows a closed path which passes through the points P_1 and P_2 . The work around the path can then be broken up into two parts: the work performed moving from P_1 to P_2 along C_1 , and the work performed moving from P_2 to P_1 along C_2 . Applying the conservation rule in (B-1), the net work is

$$W = W_{1-2} + W_{2-1} = \int_{P_1}^{P_2} \mathbf{F} \cdot \hat{\mathbf{t}}_1 ds_1 + \int_{P_2}^{P_1} \mathbf{F} \cdot \hat{\mathbf{t}}_2 ds_2 = 0 \quad (\text{B-2})$$

Rearranging (B-2), we get the work between P_1 and P_2 ,

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \hat{\mathbf{t}}_1 ds_1 = -\int_{P_2}^{P_1} \mathbf{F} \cdot \hat{\mathbf{t}}_2 ds_2 = \int_{P_1}^{P_2} \mathbf{F} \cdot \hat{\mathbf{t}}_2 ds_2 \quad (\text{B-3})$$

The equality in (B-3) indicates that the work between any arbitrary points P_1 and P_2 is *path independent*.

We apply the path independence in (B-3) to some two dimensional stiffness matrix $\mathbf{K}$ given by

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad (\text{B-4})$$

and the force field is defined by

Figure B-1: Closed path passing through points P_1 and P_2 . The closed path is made up of two parts; $C_{1,2}$ travels from P_1 to P_2 and $C_{2,1}$ travels from P_2 to P_1 .

$$\mathbf{F} = [F_x \ F_y]^T = \mathbf{K} \delta \mathbf{p} \quad (\text{B-5})$$

where $\delta \mathbf{p} = [x \ y]^T$ is displacement from the equilibrium point. The work performed to reach the point (x,y) is stored in the stiffness as potential energy, given by

$$E(x, y) = \int_0^x F_x d\xi + G(y) = \frac{1}{2} k_{11} x^2 + k_{12} xy + G(y) \quad (\text{B-6})$$

where ξ is a dummy variables for integration. Taking the derivative with respect to y gives us F_y ,

$$F_y = \frac{dE(x, y)}{dy} = k_{12}x + \frac{dG(y)}{dy} \quad (\text{B-7})$$

Equating F_y in (B-7) to that in (B-5) gives us

$$\frac{dG(y)}{dy} = k_{22}y + (k_{21} - k_{12})x = k_{22}y \quad (\text{B-8})$$

The cancellation of the x term in (B-8) is necessary since $G(y)$ cannot be a function of x .

Note that this can only occur if $k_{12} = k_{21}$, i.e., $\mathbf{K}$ is symmetric. This is the condition required for the stiffness to be conservative. Integrating (B-8) and setting the integration constant to zero, since $E(0,0) = 0$, the work is

$$E(x, y) = \frac{1}{2} k_{11} x^2 + k_{12} xy + \frac{1}{2} k_{22} y^2 \quad (\text{B-9})$$

As an alternate derivation, (B-6) can be written as an integration in y ,

$$E(x, y) = \int_0^y F_y d\lambda + H(x) = k_{21}xy + \frac{1}{2}k_{22}y^2 + H(x) \quad (\text{B-10})$$

Following a similar derivation as above, we once again get the condition $k_{12} = k_{21}$, and the potential energy solves to

$$E(x, y) = \frac{1}{2}k_{11}x^2 + k_{21}xy + \frac{1}{2}k_{22}y^2 \quad (\text{B-11})$$

Clearly (B-9) and (B-11) are equivalent when $k_{12} = k_{21}$.

B.2 Force Curl

A second method for indicating a conservative appearing system is that the curl of the (force) field function is zero,

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k} = 0 \quad (\text{B-12})$$

where $\mathbf{k}$ the unit vector in the z direction. Using the force field as defined in (B-4) and (B-5), this provides a direct method for showing that a conservative field is only achieved when $k_{12} = k_{21}$.

B.3 Gradient Function

A conservative force field is also indicated when the force can be written as the gradient of a scalar function,

$$\mathbf{F} = \text{grad}(E) = \frac{\partial E}{\partial x} \mathbf{i} + \frac{\partial E}{\partial y} \mathbf{j} = \begin{bmatrix} \frac{\partial E}{\partial x} & \frac{\partial E}{\partial y} \end{bmatrix}^T \quad (\text{B-13})$$

where E is a scalar function of (x, y) . In this case, we will not assume a force field of the form of (B-5). Instead, we will *define* a scalar energy function

$$E(x, y) = \frac{1}{2} \mathbf{p}^T \mathbf{K} \mathbf{p} = \frac{1}{2} (k_{11}x^2 + (k_{12} + k_{21})xy + k_{22}y^2) \quad (\text{B-14})$$

Taking the partial derivatives of (B-14) with respect to x and y ,

$$\mathbf{F} = \begin{bmatrix} k_{11} & \frac{1}{2}(k_{12} + k_{21}) \\ \frac{1}{2}(k_{12} + k_{21}) & k_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (\text{B-15})$$

Hence the force field defined in (B-5) is conservative if $k_{12} = k_{21} = \frac{1}{2}(k_{12} + k_{21})$.

Appendix C

Decomposition of Force

In this Appendix we will discuss the decomposition of forces and stiffness into components that are conservative, non-conservative, can contribute to potential energy storage, or perform no work.

Figure C-1 shows an isopotential ellipse with two sets of orthogonal axes by which to decompose an arbitrary endpoint force. The first set, at the upper right, consists of components that are normal and tangential to the ellipse contour, F_n and F_t . The second set, shown at the lower left, consists of components that are co-linear or at right angles to the displacement from the origin, denoted F_c and F_r . The four directions corresponding to these four components are denoted $\hat{n}$, $\hat{t}$, $\hat{c}$, and $\hat{r}$, respectively.

A conservative force must act normal to the ellipse contour and therefore has no component in the $\hat{t}$ direction. If it did have a component pointing along the ellipse contour, one could do work by moving the endpoint along the isopotential and thereby violating its definition as a constant potential energy contour. In general, the normal to

the isopotential does not point directly toward the origin, except at the vertices of the ellipse. This means that a conservative force has components co-linear and perpendicular to the displacement. Only force components co-linear with displacements contribute to work, and hence potential energy storage. This leads us to the interesting conclusion that conservative forces generally have a component that produces no useful work.

Since conservative forces must be normal to the ellipse contour, one would presume that non-conservative forces must act tangential to the ellipse contour. However, such forces have a component co-linear with displacement and therefore contribute to the potential energy. Potential energy is created solely by conservative forces. Therefore non-conservative forces must be solely in the $\hat{\mathbf{r}}$ direction, perpendicular to the displacement. We can then conclude that conservative and non-conservative forces are not generally orthogonal to each other.

The above reasoning leads to three important directions: forces co-linear with displacement contribute to the potential energy, forces perpendicular to displacement do not contribute to potential energy, and forces normal to the ellipse are conservative. Non-conservative forces are a subset of those perpendicular to displacement, and the $\hat{\mathbf{t}}$ direction tangential to the ellipse curve has no qualities of interest.

Figure C-1: Decomposition of forces into components normal and tangential to the isopotential, or co-linear and perpendicular to the displacement.

We will first consider conservative forces and linear stiffness. As discussed in Appendix B, a conservative force requires that the stiffness matrix be symmetric. The symmetric and anti-symmetric parts of a general stiffness matrix $\mathbf{K}$ can be separated as follows:

$$\mathbf{K} = \mathbf{K}_s + \mathbf{K}_a \quad (\text{C-1})$$

$$\mathbf{K}_s = \frac{1}{2}(\mathbf{K} + \mathbf{K}^T) = \begin{bmatrix} k_{11} & \frac{1}{2}(k_{12} + k_{21}) \\ \frac{1}{2}(k_{12} + k_{21}) & k_{22} \end{bmatrix} \quad (\text{C-2})$$

$$\mathbf{K}_a = \frac{1}{2}(\mathbf{K} - \mathbf{K}^T) = \frac{1}{2}(k_{12} - k_{21}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (\text{C-3})$$

This provides us with the component of force, $\mathbf{F}_n$, that is conservative and normal to the ellipse contour, and the corresponding stiffness component, $\mathbf{K}_n$,

$$\mathbf{F}_n = \mathbf{K}_n \mathbf{p} \quad (\text{C-4})$$

$$\text{where } \mathbf{K}_n = \mathbf{K}_s = \frac{1}{2}(\mathbf{K} + \mathbf{K}^T) \quad (\text{C-5})$$

$\mathbf{p} = [x \ y]^T$ is the displacement of the endpoint.

To find the component of force co-linear with the displacement, we first define a unit vector along the displacement:

$$\hat{\mathbf{c}} = \frac{\mathbf{p}}{|\mathbf{p}|} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad (\text{C-6})$$

where θ is the polar angle of the displacement relative to the positive x axis. The component of force co-linear to displacement is then

$$\mathbf{F}_c = (\mathbf{F}^T \hat{\mathbf{c}}) \hat{\mathbf{c}} = \frac{\mathbf{p}^T \mathbf{K}^T \mathbf{p}}{|\mathbf{p}|} \frac{\mathbf{p}}{|\mathbf{p}|} = \frac{\mathbf{p}^T \mathbf{K} \mathbf{p}}{|\mathbf{p}|^2} \mathbf{p} = \mathbf{K}_c \mathbf{p} \quad (\text{C-7})$$

where

$$\begin{aligned} \mathbf{K}_c &= \frac{\mathbf{p}^T \mathbf{K} \mathbf{p}}{|\mathbf{p}|^2} \mathbf{I}_2 \\ &= (k_{11} \cos^2 \theta + (k_{12} + k_{21}) \sin \theta \cos \theta + k_{22} \sin^2 \theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (\text{C-8})$$

Equation (C-7) makes use of a simplification that replaces $\mathbf{K}$ with $\mathbf{K}^T$. Since $\mathbf{K}$ is not assumed to be symmetric here, this appears to be an incorrect substitution. However, the substitution is valid by using the following property of the anti-symmetric stiffness.

$$\mathbf{p} \mathbf{K}_a^T \mathbf{p} = \mathbf{p} \mathbf{K}_a \mathbf{p} = 0 \quad (\text{C-9})$$

Hence we can write the numerator of $\mathbf{K}_c$ as

$$\begin{aligned} \mathbf{p}^T \mathbf{K}^T \mathbf{p} &= \mathbf{p}^T (\mathbf{K}_s^T + \mathbf{K}_a^T) \mathbf{p} = \mathbf{p}^T \mathbf{K}_s^T \mathbf{p} + \mathbf{p}^T \mathbf{K}_a^T \mathbf{p} = \\ &= \mathbf{p}^T \mathbf{K}_s \mathbf{p} + \mathbf{p}^T \mathbf{K}_a \mathbf{p} = \mathbf{p}^T (\mathbf{K}_s + \mathbf{K}_a) \mathbf{p} = \mathbf{p}^T \mathbf{K} \mathbf{p} \end{aligned} \quad (\text{C-10})$$

Since only co-linear force and displacement contribute to the potential energy, we can write the potential energy as

$$E_p = \mathbf{p}^T \mathbf{K}_c \mathbf{p} = \mathbf{p}^T \mathbf{K}_s \mathbf{p} = \mathbf{p}^T \mathbf{K} \mathbf{p} \quad (\text{C-11})$$

The component of force at right angles to the displacement, $\mathbf{F}_r$, is found by first defining a unit vector in that direction. We rotate $\hat{\mathbf{c}}$ through 90° ,

$$\hat{\mathbf{r}} = \mathbf{R} \hat{\mathbf{c}} = \mathbf{R} \frac{\mathbf{p}}{|\mathbf{p}|} \quad (\text{C-12})$$

$$\text{where } \mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (\text{C-13})$$

$\mathbf{F}_r$ is the projection of the force along $\hat{\mathbf{r}}$,

$$\mathbf{F}_r = (\mathbf{F}^T \hat{\mathbf{r}}) \hat{\mathbf{r}} = \frac{\mathbf{p}^T \mathbf{K}^T \mathbf{R} \mathbf{p} \mathbf{R}}{|\mathbf{p}|^2} \mathbf{p} = \mathbf{K}_r \mathbf{p} \quad (\text{C-14})$$

where

$$\begin{aligned} \mathbf{K}_r &= \frac{\mathbf{p}^T \mathbf{K}^T \mathbf{R} \mathbf{p} \mathbf{R}}{|\mathbf{p}|^2} \\ &= (-k_{21} \cos^2 \theta + (k_{11} - k_{22}) \sin \theta \cos \theta + k_{12} \sin^2 \theta) \mathbf{R} \end{aligned} \quad (\text{C-15})$$

As discussed in the introduction, the component of force at right angles to the

displacement consists of a portion from the conservative (symmetric stiffness), $\mathbf{F}_s$, and a portion from the non-conservative (anti-symmetric stiffness), $\mathbf{F}_a$. We can then write $\mathbf{F}_r$ and $\mathbf{K}_r$ as the sum of these components,

$$\mathbf{F}_r = \mathbf{F}_{rs} + \mathbf{F}_{ra} = \mathbf{F}_{rs} + \mathbf{F}_a \quad (\text{C-16})$$

$$\mathbf{K}_r = \mathbf{K}_{rs} + \mathbf{K}_{ra} = \mathbf{K}_{rs} + \mathbf{K}_a \quad (\text{C-17})$$

$$\text{where } \mathbf{F}_{rs} = \mathbf{K}_{rs}\mathbf{p} \quad (\text{C-18})$$

$$\mathbf{F}_{ra} = \mathbf{F}_a = \mathbf{K}_{ra}\mathbf{p} = \mathbf{K}_a\mathbf{p} \quad (\text{C-19})$$

$$\begin{aligned} \mathbf{K}_{rs} &= \frac{\mathbf{p}^T \mathbf{K}_s \mathbf{R} \mathbf{p} \mathbf{R}}{|\mathbf{p}|^2} \\ &= \frac{1}{2}(k_{12} + k_{21})(\sin^2 \theta - \cos^2 \theta) + (k_{11} - k_{22})\sin \theta \cos \theta \end{aligned} \quad (\text{C-20})$$

$$\begin{aligned} \mathbf{K}_{ra} &= \frac{\mathbf{p}^T \mathbf{K}_a^T \mathbf{R} \mathbf{p} \mathbf{R}}{|\mathbf{p}|^2} \\ &= \left(\frac{1}{2}(k_{12} - k_{21})\cos^2 \theta + \frac{1}{2}(k_{12} - k_{21})\cos^2 \theta \right) \mathbf{R} \\ &= \frac{1}{2}(k_{12} - k_{21})\mathbf{R} \\ &= \mathbf{K}_a \end{aligned} \quad (\text{C-21})$$

Force components normal to the displacement contribute no work and therefore we expect that no potential energy is contributed by these components. This can be seen by writing out the potential functions for these stiffness components:

$$\mathbf{E}_r = \mathbf{p}^T \mathbf{K}_r \mathbf{p} = \mathbf{p}^T \mathbf{K}_{rs} \mathbf{p} = \mathbf{p}^T \mathbf{K}_a \mathbf{p} = 0 \quad (\text{C-22})$$

The fact that $\mathbf{K}_r$ produces no energy storage is similar to a nullspace for vector transformations. (See Section 6.2, for example.) Here, $\mathbf{K}_r$ represents the stiffness component that produces no potential energy through the general transformation from 'stiffness space' to 'potential energy' space.

$$E_i = \mathbf{p}^T \mathbf{K}_i \mathbf{p} \quad (\text{C-23})$$

The components $\mathbf{K}_c$ and $\mathbf{K}_a$ are then analogous to subspaces of this 'null' stiffness $\mathbf{K}_r$. We should be careful to note that this nullspace analogy does not apply to the transformation from displacement space $\mathbf{p}$ to force space $\mathbf{F}$, since these are both two-dimensional and thereby lacking a nullspace. $\mathbf{K}_r$ does produce a force. It does not produce work (or potential energy).

In summary, the force can be broken down into three main components:

$$\mathbf{F} = \mathbf{F}_c + \mathbf{F}_n + \mathbf{F}_a \quad (\text{C-24})$$

There are four categories of force components:

- $\mathbf{F}_c = \mathbf{F}_c + \mathbf{F}_n$ is the conservative force, and is symmetric about the origin. It is normal to the ellipse contour (isopotential) and has components co-linear and perpendicular to the displacement from the origin.
- $\mathbf{F}_a$ is the non-conservative force, and is anti-symmetric about the origin. It is perpendicular to the displacement from the origin.

- $\mathbf{F}_c$ is the component of force co-linear with the displacement. It is a subset of the conservative force, and it is the only component that contributes to the potential energy.
- $\mathbf{F}_r = \mathbf{F}_n + \mathbf{F}_s$ is the component of force perpendicular to the displacement. It is made up of both conservative and non-conservative (symmetric and non-symmetric) forces, and contributes nothing to the potential energy.

The stiffness can be broken up into similar components:

$$\mathbf{K} = \mathbf{K}_c + \mathbf{K}_n + \mathbf{K}_s \quad (\text{C-25})$$

$$\mathbf{K}_s = \mathbf{K}_c + \mathbf{K}_n \quad (\text{C-26})$$

$$\mathbf{K}_r = \mathbf{K}_n + \mathbf{K}_s \quad (\text{C-27})$$

As a final note, it is interesting that only $\mathbf{K}_s$ and $\mathbf{K}_a$ can be written independent of position $\mathbf{p}$. All other stiffness components must be written as a function of the direction (but not magnitude) of displacement. This occurs because $\mathbf{K}_s$ and $\mathbf{K}_a$ are defined on the basis of conservativeness, which is independent of displacement. The other decompositions of stiffness are defined based on co-linearity or orthogonality of force with respect to displacement. It is therefore no surprise that the direction of displacement is necessary for the definitions of these stiffness components.

Appendix D

Ellipse Comparison Index

From Section 6.3.2, the definition of the Ellipse Comparison Index, ξ_J , is the amount of shifted area between the isopotential ellipses for $\mathbf{K}$ and $^J\mathbf{K}$, normalized on the area of the $^J\mathbf{K}$ ellipse. In order to calculate the shifted area we must first find the points of intersection between the $^J\mathbf{K}$ and $\mathbf{K}$ ellipses (see Figure 6.3.1). From (6.1.2) the radius for the $\mathbf{K}$ ellipse is

$$r = \sqrt{\frac{2E_p}{k_{xx}\cos^2\theta + (k_{xy} + k_{yx})\sin\theta\cos\theta + k_{yy}\sin^2\theta}} \quad (6.1.2)$$

and similarly the radius of the $^J\mathbf{K}$ ellipse is

$$r_J = \sqrt{\frac{2E_p}{^Jk_{xx}\cos^2\theta + (^Jk_{xy} + ^Jk_{yx})\sin\theta\cos\theta + ^Jk_{yy}\sin^2\theta}} \quad (D-1)$$

If an intersection exists for the same value of E_p , then we can find some θ for which $r = r_J$.

This equality in the above equations results in the condition

$$(k_{xx} - {}^Jk_{xx})\cos^2\theta + \{(k_{xy} - {}^Jk_{xy}) + (k_{yx} - {}^Jk_{yx})\}\sin\theta \cos\theta + (k_{yy} - {}^Jk_{yy})\sin^2\theta = 0 \quad (D-2)$$

But from (5.3.16) we know that ${}^G\mathbf{K} = \mathbf{K} - {}^J\mathbf{K}$, and so (D-2) reduces to

$${}^Gk_{xx} \cos^2\theta + ({}^Gk_{xy} + {}^Gk_{yx})\sin\theta \cos\theta + {}^Gk_{yy} \sin^2\theta = 0 \quad (D-3)$$

It is apparent from the radius equations above that the condition in (D-3) means that the ${}^G\mathbf{K}$ ellipse radius has become infinite at some value of θ , and hence ${}^G\mathbf{K}$ does not actually form an ellipse. Section 6.4 showed that this means that ${}^G\mathbf{K}$ is *unstable*. More specifically, to have an infinite radius, ${}^G\mathbf{K}$ must be a saddle. Dividing (D-3) by $\cos^2\theta$ and solving the quadratic, we get

$$\theta_{1,2} = \tan^{-1} \left(-\frac{{}^Gk_{xy}}{{}^Gk_{yy}} \pm \frac{1}{{}^Gk_{yy}} \sqrt{-\det {}^G\mathbf{K}} \right) \quad (D-4)$$

Note that there are four solution to (D-4) because two values can result from the arctan function, 180° apart. These correspond to the necessary four points of intersection of the ellipses (see Figure 6.3.1). Because of the symmetry of the ellipses, we only need two consecutive intersection points, and therefore only the principal angles of the arctan function should be chosen from (D-4). Equation (D-4) shows us that there are no intersection points unless the determinant of ${}^G\mathbf{K}$ is zero (resulting in two intersection points) or negative. This was also shown in Section 6.4 to be an indicator that ${}^G\mathbf{K}$ is a

saddle.

To find the area of the ${}^J\mathbf{K}$ ellipse, we take a small differential area

$$dA_J = (1/2 r_J)(r_J d\theta) = 1/2 r_J^2 d\theta \quad (D-5)$$

We integrate this from $-\pi/2$ to $\pi/2$ and double the resulting area, since the ellipse is symmetric:

$$\begin{aligned} A_J &= \int_{-\pi/2}^{\pi/2} r_J^2 d\theta \\ &= 2E_p \int_{-\pi/2}^{\pi/2} \frac{d\theta}{{}^J k_{xx} \cos^2 \theta + ({}^J k_{xy} + {}^J k_{yx}) \sin \theta \cos \theta + {}^J k_{yy} \sin^2 \theta} \\ &= 2E_p \left(\frac{1}{\sqrt{\det {}^J \mathbf{K}}} \tan^{-1} \left(\frac{{}^J k_{yy} \tan \theta + {}^J k_{xy}}{\sqrt{\det {}^J \mathbf{K}}} \right) \right) \Bigg|_{-\pi/2}^{\pi/2} \\ &= \frac{2\pi E_p}{\sqrt{\det {}^J \mathbf{K}}} \end{aligned} \quad (D-6)$$

The integration is taken from standard integration tables (e.g., [Beyer, 1991]). If ${}^G\mathbf{K}$ is unstable (but $\mathbf{K}$ is stable overall, otherwise it doesn't form an ellipse), the ellipses intersect at θ_1 , θ_2 , $\theta_1 + \pi$, and $\theta_2 + \pi$. Again, we can examine half the ellipse, from θ_1 to $\theta_1 + \pi$, and double the result. To find the difference in area we start with a differential amount

$$dA_{\text{shift}} = |dA_J - dA| = 1/2 |r_J^2 - r^2| d\theta \quad (D-7)$$

We use the absolute value because we do not know whether the $\mathbf{J}\mathbf{K}$ ellipse is outside the $\mathbf{K}$ ellipse at a given point, or vice versa, and we need the area to be positive. Otherwise, positive and negative areas can cancel, as would be the case of a purely rotated ellipse. We then integrate over the ranges of interest,

$$A_{\text{shift}} = \left| \int_{\theta_1}^{\theta_2} (r_J^2 - r^2) d\theta \right| + \left| \int_{\theta_2}^{\theta_1 + \pi} (r^2 - r_J^2) d\theta \right| \quad (\text{D-8})$$

We reversed the subtraction in the second interval to make note that the ellipses have reversed roles as to which one is outside the other over the interval. This reversal is not strictly necessary mathematically because of the absolute value signs.

The general form of the integral is [Beyer, 1991]

$$\int r^2 d\theta = \frac{2E_p}{\sqrt{\det \mathbf{K}}} \tan^{-1} \left(\frac{k_{yy} \tan \theta + k_{xy}}{\sqrt{\det \mathbf{K}}} \right) + \frac{n\pi}{\sqrt{\det \mathbf{K}}}; n = 0, 1, 2, \dots \quad (\text{D-9})$$

where $(2n-1)\pi/2 \leq \theta \leq (2n+1)\pi/2$, i.e., n indicates how many times the y -axis has been crossed going from 0 to θ . Note that θ_1 and θ_2 are chosen from (D-4) to be between $-\pi/2$ and $\pi/2$ (principal angles), where $n = 0$. But $\theta_1 + \pi$ will be between $\pi/2$ and $3\pi/2$, where $n=1$. Define

$$T_{ij} = \tan^{-1} \left(\frac{{}^i k_{yy} \tan \theta_j + {}^i k_{xy}}{\sqrt{\det {}^i \mathbf{K}}} \right); (i = J, A), (j = 1, 2) \quad (D-10)$$

The integral is then

$$A_{\text{shift}} = 2E_p \left\{ \left| \frac{T_{A2} - T_{A1}}{\sqrt{\det \mathbf{K}}} - \frac{T_{J2} - T_{J1}}{\sqrt{\det {}^J \mathbf{K}}} \right| + \left| \frac{T_{A2} - T_{A1}}{\sqrt{\det \mathbf{K}}} - \frac{T_{J2} - T_{J1}}{\sqrt{\det {}^J \mathbf{K}}} + \frac{\pi(\sqrt{\det \mathbf{K}} - \sqrt{\det {}^J \mathbf{K}})}{\sqrt{\det \mathbf{K} \det {}^J \mathbf{K}}} \right| \right\} \quad (D-11)$$

The ellipse comparison index is found by normalizing A_{shift} on A_J from (D-6),

$$\xi_J = \frac{1}{\pi} \left| \sqrt{\frac{\det {}^J \mathbf{K}}{\det \mathbf{K}}} (T_{A2} - T_{A1}) - (T_{J2} - T_{J1}) \right| + \frac{1}{\pi} \left| \sqrt{\frac{\det {}^J \mathbf{K}}{\det \mathbf{K}}} (T_{A2} - T_{A1}) - (T_{J2} - T_{J1}) + \pi \left(1 - \sqrt{\frac{\det {}^J \mathbf{K}}{\det \mathbf{K}}} \right) \right| \quad (D-12)$$

If we know that $\det {}^G \mathbf{K} \geq 0$, the ellipse comparison index simplifies. From (D-3) or (D-4) we know that this means the ${}^J \mathbf{K}$ and $\mathbf{K}$ ellipses do not intersect, and so one is completely inside the other (or they just touch at two points if $\det {}^G \mathbf{K} = 0$). From (6.3.3) we also know that $k_{\max} > {}^J k_{\max}$ and $k_{\min} > {}^J k_{\min}$ unless ${}^G k_{\max}$ and ${}^G k_{\min}$ are negative. They must both be positive or negative here, or else $\det {}^G \mathbf{K} < 0$, which we already considered in (D-12). Larger values of stiffness on the axis produce a smaller ellipse because of the

reciprocal relationship between stiffness and ellipse radius (as in (6.1.12)). We can therefore conclude that if $G_{k_{\max}}$ and $G_{k_{\min}}$ are positive, the $\mathbf{K}$ ellipse must be inside the ${}^J\mathbf{K}$ ellipse. The shifted area is

$$A_{\text{shift}} = A_J - A = 2\pi E_p \left(\frac{\sqrt{\det \mathbf{K}} - \sqrt{\det {}^J\mathbf{K}}}{\sqrt{\det \mathbf{K} \det {}^J\mathbf{K}}} \right) \quad (\text{D-13})$$

To get the ellipse comparison index, we normalize the shifted area on the ${}^J\mathbf{K}$ ellipse area,

$$\xi_J = \left| 1 - \sqrt{\frac{\det {}^J\mathbf{K}}{\det \mathbf{K}}} \right| \quad (\text{D-14})$$

If $G_{k_{\max}}$ and $G_{k_{\min}}$ are both negative, then the $\mathbf{K}$ ellipse must be bigger than the ${}^J\mathbf{K}$ ellipse. The shifted area in (D-13) would then be negative, and so the ellipse comparison index in (D-14) would be negative.

The range of values for this index is $0 \leq \xi_J \leq 1$. For values of $|\xi_J| \ll 1$, very little area is shifted and so it $G_{\mathbf{K}}$ can be considered negligible. For values of $\xi_J \approx 1$, most of the ellipse has shrunken down to a tiny $\mathbf{K}$ ellipse or the ellipse has grown significantly. Either of these cases would suggest that $G_{\mathbf{K}}$ is significant.