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The Group of an N-body Problem in Celestial Mechanics

by

James Handson Chalmers, B.A.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of
Mathematics and Statistics

Carleton University
Ottawa, Ontario
September, 1999

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in Celestial Mechanics**

Submitted by
James Handson Chalmers, B.A.Sc., M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

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Abstract

The N -body problem in celestial mechanics is studied for a force of attraction that is sufficiently general that initial value problems exist which do not exhibit uniqueness of solutions but are sufficiently limited that conservation of momentum, angular momentum and energy and invariance under orthogonal transformations are all obeyed. In this general setting and for arbitrary finite dimension of the space, a geometrical symmetry group is defined for a solution at any given time. Each member of the group permutes bodies of equal mass using related isometries in position and velocity space. It is shown that if the force of attraction is sufficiently pathological, the group can vary over time. However, if the attraction law yields uniqueness of solution, the group is preserved over time for the solutions. A family of coordinate frames, called *centroidal* is defined and is shown to characterize precisely those frames for which the position and velocity parts of the group reduce to identical orthogonal groups. It is seen that substantial reduction of the phase space can be accomplished for some problems using the group. Although others have studied many interesting examples of problems that involve such symmetry groups, the fact that each N -body problem has a group as shown herein has not been previously recognized.

For dimensions 2 and 3 the groups permitting non-zero angular momentum are found. In any finite dimensional space, if the force of attraction is strictly positive, the masses are positive and the group contains a hyperplane of reflection symmetry, then the solution leads either to or from a singularity or both. This result encompasses a large family of N -body problems. In dimensions 2 and 3 all groups containing this property are identified.

We next study the case of homogeneous forces of attraction to prove uniqueness theorems for *central configurations* (c.c.'s) for a whole family of symmetry groups in which the bodies form the vertices of concentric similar (*homothetic*) polygons, polyhedra and in general, polytopes. In these problems all bodies on any given polytope have the same mass but the masses of the bodies on different polytopes can be freely assigned independently of each other. The result is analogous to the Moulton-Euler theorem for collinear c.c.'s in that for each way of nesting the polytopes inside one another and for each way of assigning the (positive) masses for each polytope there is exactly one central configuration. Consequently, we obtain $n!$ c.c.'s for each type of polytope where n is the number of polytopes in the nest. This result is obtained for nested equilateral triangles in $\mathbf{R}^2$, regular tetrahedra in $\mathbf{R}^3$ regular pentatopes in $\mathbf{R}^4$ and in general regular $d + 1$ vertex simplices in $\mathbf{R}^d$. The same result is also obtained for nested squares in $\mathbf{R}^2$, octahedra in $\mathbf{R}^3$ and cross polytopes in $\mathbf{R}^d$.

The group is also used to study *non-homographic* solutions *à la Hill* in three

and four dimensions. Although these solutions cannot be written in the form of known functions, it is assured that neither collision nor non-collision singularities can occur. Furthermore, similar to Hill's study of the 3-body restricted problem, regions of space are determined which limit where the bodies can move based on the initial conditions.

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Dedication

To my wife, Joyce who has shown undying love and the patience of Job in supporting me in this endeavour.

to my mother, Mary who has given me love and encouragement for as long as I can remember and

to my dearly beloved father, the late James Rennie Chalmers who due to economic circumstances could not pursue his love of algebra.

to these three people I dedicate this work.

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Chapter 1

Historical Introduction

The N -body problem is one aspect of Newtonian mechanics in which the motion of N celestial bodies is determined by their mutual gravitational attraction. Each body is assumed to be much smaller than its distance from any other body. This is reasonable for planets whose diameters are of the order of tens of thousands of kilometers and are separated by hundreds of millions of kilometers. Also, it is accurate if the mass density of each planet depends only on the radial distance from the centre of the planet. One makes the approximation that all of the mass of each body is concentrated at its centre.

1.1 Definition of the Problem

If we number the N bodies and let m_k be the mass (assumed to be greater than zero) of body k , then the magnitude of the force between bodies i and j is $Gm_i m_j / d^2$ where d is the distance between them and G is the gravitational constant.

If we let $q_i(t)$ be the position of body i at time t and $q_{ij}(t) = q_j(t) - q_i(t)$ then $d = \|q_{ij}(t)\|$ where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^3$. Since the force exerted by body j on body i is in the direction of the vector $q_{ij}(t)$ it equals $(Gm_i m_j / \|q_{ij}(t)\|^3) q_{ij}(t)$.

Summing over all $j \neq i$ gives the total force F_i applied to body i by the other bodies. Applying Newton's law that the acceleration $\ddot{q}_i = F_i/m_i$ gives the o.d.e.

$$\ddot{q}_i = \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}(t)\|) q_{ij}(t) \quad (1.1)$$

where $s(r) = G/r^3$ and each dot over q represents differentiation with respect to time t (e.g., $\ddot{q}_i = d^2 q_i/dt^2$). Given a configuration of N points, $x_1, \dots, x_N \in \mathbf{R}^3$, we let $x \in \mathbf{R}^{3N}$ be the $3N \times 1$ column matrix partitioned as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}. \quad (1.2)$$

When two or more bodies share the same point in $\mathbf{R}^3$ they are said to be in **collision**. For $i \neq j$ we let $\Delta_{ij} = \{x \in \mathbf{R}^{3N} | x_i = x_j\}$ and we let $\Delta = \cup_{i \neq j} \Delta_{ij}$ and call it the **diagonal**. Then the diagonal is the set of points in $\mathbf{R}^{3N}$ for which two or more bodies are in collision. Defining $q(t) \in \mathbf{R}^{3N}$ by the above partition, we see that since $s(r)$ is not defined for $r = 0$, (1.1) is not defined for $q(t) \in \Delta$. Thus we let the domain on which the o.d.e. is defined be the (open, connected) set $D = \mathbf{R}^{3N} \setminus \Delta$. If $I = (t_-, t_+) \subseteq \mathbf{R}$ and $q : I \rightarrow D$ is such that $q_i(t)$ satisfies (1.1) $\forall t \in I, i = 1, \dots, N$ then q is said to be a **solution of the N body problem on I** . We define the mass vector $m = (m_1, \dots, m_N)^t \in (\mathbf{R}^+)^N$ where $\mathbf{R}^+ = (0, \infty)$. If $t_0 \in I$ and $q(t_0) = a \in D$, $\dot{q}(t_0) = b \in \mathbf{R}^{3N}$ then q is said to be a **solution of the initial value problem, (m, a, b, t_0) on I** .

1.2 Historical Development

The N -body problem has a huge history and many great names in mathematics have applied themselves to it. The body of work built up over the last three hundred

years is substantial and its history reads like a “who’s who” of mathematics. This historical introduction up to the Renaissance is based in large part on [Ku] and [Co] and for the eighteenth and nineteenth centuries on [SM, Ch. 1]. The problem of predicting the motion of the ‘wandering stars’ or *planeta* has been a subject of interest to philosophers, religious scholars, astronomers, physicists and engineers as well as mathematicians. The ancients saw the heavens as perfect compared to our Earthly realm with its decaying organic matter and human suffering. The motions of the Sun and planets were usually believed to be controlled by various gods. Newtonian mechanics and the N -body problem subsequently removed the need for such divine intervention to explain the observed phenomena. However, even today different scientific disciplines can view the problem differently. The modern physicist can argue that Newtonian mechanics is antiquated and a poor model because of relativity and quantum mechanics. Newtonian mechanics fails very badly to describe phenomena where bodies have very large masses (curvature of space-time), very high speeds (increased mass, Fitzgerald contraction and time dilatation) and very small particles (wave mechanics). Engineers who lay out the orbits for artificial satellites can argue that as far as they are concerned, it is good enough because it is so accurate for problems in which they are interested that the errors are too small to measure. They might argue instead that there is no point in studying such a subject. Given the initial state of a system they need only grind it out on a computer to determine its state at some future time. They might ask, “What is there to study?”. The mathematician on the other hand can ask questions that might never occur to and indeed, might not be understood by most physicists and engineers. He or she may feel free to see the problem purely as a differential equation to be studied in its own right and to be played with to obtain a deeper understanding.

Piecemeal efforts have have been made to expand the problem by varying the dimension of the underlying physical space and by using different functions governing

the force of attraction (e.g. inverse cube law in lieu of inverse square law). In this thesis we make an effort to pull these ideas together into a single framework. This has the effect of expanding an already huge problem but the reader not interested in the full scope should not be discouraged from continuing since there is often little added effort in treating the concepts in a more general setting. Our theory that formalizes the geometrical symmetry of an N -body problem fits naturally into the more general framework of higher dimensions and more general force functions. Although other results on singularities and central configurations simply do not extend to the more general setting, they give rise to some interesting surprises. Also it is a basic theme in mathematics to push the scope of a theorem as far as possible with the usual result of greater clarity and understanding. In this wider setting, some of the theory that is regarded as basic to the N -body problem has been rebuilt. Some new problems that arise out of this generalization are left unsolved. We begin by placing our work within the current body of knowledge on the subject and to do so we go back a very long time.

1.3 Ancient Astronomy

Most civilizations in recorded history have shown a keen interest in the celestial luminaries and not only the Sun and Moon. Around 1900 B.C. in Mesopotamia the appearance and disappearance of Venus were recorded, probably as omens or portents of the future. In the eighth century B.C. Babylonians, also concerned with omens, compiled complete records of celestial events, including eclipses and the retrograde motions of the wandering stars (planets). An observer of the night sky will first notice that the patterns formed by the stars do not change over time and that the whole pattern rotates westward at a constant rate. This led the early Egyptians and Babylonians to believe that the stars are attached to a rotating dome or vault. Before the fourth century B.C., Greeks observed that a departing ship's

hull disappears over the horizon before its sail and that it disappears sooner when viewed from a low point than from a hill. Based on this they concluded that the Earth is round and they produced a two-sphere model of the universe in which the Earth lies at the centre of a much larger rotating sphere containing the stars. This model did not account for the motion of the Sun, the Moon and the planets which also appear to move westward but more slowly than the stars. In addition, at certain times a planet will undergo retrograde motion in which it appears to move faster than the stellar background. In the fourth century B.C., the philosopher, Plato is reputed to be the first to ask how to explain the motion of the Sun, the Moon and the planets and his pupil Eudoxus gave the first answer by having rotating spheres mounted inside rotating spheres with the Earth as centre. Its simplest incarnation has a sphere holding the stars and rotating daily about the Earth's polar axis. A second sphere containing the sun is mounted inside that sphere on an axis tilted at $23\frac{1}{2}^{\circ}$ to the first axis. This model explains the motion of the sun through the seasons. More complicated models in this ingenious theory of homocentric spheres did a fairly good job of explaining planetary retrograde motion. The major criticism at the time was its failure to describe the change in apparent brightness of the planets which in this model remained at the same distance from the Earth.

This problem was overcome in the third century B.C., when Appolonius and Hipparchus replaced it with a model in which each planet is fixed to the rim of a circle, called an epicycle. Each epicycle rotates about its centre which is fixed to the rim of another rotating circle called a deferent, centred at the Earth. As the planet enters retrograde motion, it is on the inside of its epicycle, hence closer to the Earth and therefore brighter. Qualitatively, this agrees with the modern model in which a planet whose orbit is larger than the Earth's, say Mars, is in opposition hence closer to and fully illuminated by the Sun. The model also accounted for the time taken to journey around the ecliptic.

In the fourth century B.C., Heraclides of Pontus developed a model in which the

huge sphere containing the stars remains fixed and the much smaller Earth rotates to the east. This notion that the Earth moved was very difficult to accept and was contrary to various beliefs held at the time. For example, if you jump into the air on a moving planet you were expected to come down behind your point of departure. Nevertheless, Aristarchus of Samos took it a step further and placed the Sun at the centre with the Earth revolving about it. Alas, such revolutionary thinking was not too take hold so quickly. In the latter part of the second century B.C., Ptolemy returned to the Earth-centred, two-sphere universe with the epicycle-deferent system. It was more powerful and simpler than Eudoxus' spheres. His *Almagest* (or *Mathematiké Syntaxis*) was so influential that for the 17 centuries from Hipparchus to Copernicus, most astronomers' attempts to model the solar system involved some minor modification to this system [Ku, p. 66].

1.4 Astronomy of the Renaissance

When in 1515 the highly renowned Polish astronomer, Copernicus adopted the Sun centred model proposed by Aristarchus, European astronomers eagerly awaited his publication. To avoid opposition from the Roman church which argued from scripture that the Earth did not move, he postponed publication of "*De Revolutionibus Orbium Caelestium*" until 1543, the year he died. By making the work difficult for laymen to read and by dedicating it to the Pope he effectively delayed any opposition that would be mounted to allow it to be well circulated before it could be suppressed. Next the Dane, Tycho Brahe developed naked-eye equipment for his observatory that allowed him to measure planetary positions to within four minutes of arc reliably and some sets of measurements were good to within one minute. Unlike previous observers who took readings only during favourable configurations, Brahe diligently measured planetary positions on a regular basis. The full wealth of his observational data was not given to his German assistant Johannes Kepler until

after his death in 1601. However, this odd collaboration was probably the first combined effort of a experimental and theoretical physicist in the modern sense. Kepler, although weaned on the Copernican system, was critical of Copernicus' reticence to cast off certain Ptolemaic ideas. Since the observed position of Mars depends on the combined orbits of Earth and Mars, Kepler's task of finding a mathematical model to fit the data was extremely difficult. He tried eccentric circles and ovals and obtained a model that was far superior to any previous one in predicting planetary motions. However, he was still not satisfied with the errors that remained. Finally, after ten years of work he arrived at his first law that the planets move in elliptical orbits with the Sun at one focus, his now famous second law that a planet sweeps out equal areas in equal periods of time and his third law that states that the period of an orbit is proportional the $3/2$ power of the average orbital radius. He also speculated that the planets were held in their orbits by a central force, the 'anima motrix' from the Sun rather than some invisible Ptolemaic contraption of wheels within wheels. He incorrectly speculated that the force was inverse with the distance from the Sun and that it was magnetic in origin.

In 1608 a Dutch lens grinder, Hans Lippershey combined lenses to magnify distant objects [JW, Section 10.11, p. 178]. When Galileo Galilei heard of this invention, he improved upon it and directed it to the heavens. His observation of the phases of Venus indicated that it *did* go around the Sun contrary to the Ptolemaic model. He also observed Jupiter's four main satellites revolving about the planet like a miniture solar system and the vast number of stars in the Milky Way which had previously been unresolved. His measurements of the acceleration of gravity, both outdoors and in his laboratory, pushed physics beyond observational astronomy to a new level whereby an experimental event is designed, carried out and the results analysed to test a theory. His rebellious and impolitic nature led to his well known and much less favourable outcome with the Church than his predecessor, Copernicus but not before his work got out.

1.5 Newtonian Mechanics

The stage was now set for the entrance of Sir Isaac Newton who pulled it all together and set down the ordinary differential equation (o.d.e.) governing the motion of the celestial bodies under their mutual gravitational attraction. However, not only did he not have any theory of o.d.e.'s from which to launch his development, there wasn't even any calculus. In 1684, Robert Hooke, Edmund Halley and Sir Christopher Wren convinced themselves, based on an analogy with the emission of light that, contrary to Kepler, the force of attraction had to be inverse with the square of the distance from the Sun. However they were unable to show that this led to an elliptical orbit: so, Halley went to Cambridge to ask Newton what kind of an orbit the inverse square force causes and the immediate reply was 'elliptical'. Although Newton claimed that he had calculated it he had misplaced the proof. Halley's encouragement helped him return to the work which was published in 1687 under the title, "*Philosophiae Naturalis Principia Mathematica*" [N]. In this theory, the gravitational force is exerted not only by the Sun on the planets but by every particle of matter on every other particle of matter. He showed that this single universal force (a) keeps the planets in their orbits around the sun, (b) holds the satellites in their orbits, (c) causes falling objects to descend as observed, (d) holds objects on the Earth and (e) causes the tides. As a byproduct of his amazing achievement, he not only proved Kepler's second law but he showed that it depended only on the fact that the force between the Sun and a planet is equal and opposite and directed through the line joining the two. It does not depend on whether the force is inverse with the distance as Kepler had speculated or the square of the distance as Newton showed or whatever. It was a result of conservation of angular momentum. As an intellectual curiosity, Newton studied the effect of a force law that was directly proportional with distance and inverse with the cube of distance.

When Plato asked his students for an account of the motion of the planets,

inadvertently he was sowing a seed of thought which in the author's opinion, has had a profound effect on the world to this day. Kepler's three laws and his accurate determination of the planetary orbits was a major step towards resolving Plato's question. Newton carried the concept well beyond Plato's concerns. Suppose we are given the position and velocity at some time, of any finite number of point masses acting under gravity. Newtonian mechanics allows us to calculate their position and velocity at any time in the future up to any collisions. This suggested that if one knows the fundamental forces governing matter, the universe or an isolated part of it becomes like a machine whose future is predictable if its current state is known. This was a radical notion for a world whose explanations for physical phenomena were dominated largely by superstition, mysticism and divine intervention. Although quantum mechanics has limited this determinism in a fundamental way, even today for macroscopic phenomena, (i.e., those involving a large number of elementary particles) the predictability of the future state of a physical system based on its initial state runs throughout science and engineering. Quantum mechanics came much later and the mechanistic thinking permeated the technical world in terms of such things as clock making and automata (small toys made up to the end of the nineteenth century that perform a complicated sequence of tasks). Indeed, one can argue that the industrial revolution was in part, an outcome of this philosophical change. So, although the N -body problem does not enjoy the attention that it once commanded, in the author's opinion, it has had a substantial effect on economic and intellectual development over the past three centuries.

1.6 Homographic Solutions

Meanwhile, some mathematicians and physicists of the day actually tried applying this theory and it turned out to be much easier to say that the future is predictable than to predict it. A single body moves in a straight line at a constant speed and

two bodies trace out conic sections with their centre of mass as a focus (Keplerian motion). For more bodies it gets very difficult. In fact, most introductory texts in classical physics totally avoid the problem for more than two bodies. In 1687 Newton's Principia [N] gave solutions to the two body problem. Eighty years passed before progress was made in finding solutions in the three body problem. In the last half of the eighteenth century Euler [E1],[E2] and Lagrange [Lgr] found all homographic solutions in the collinear and non-collinear three body problem respectively and Laplace [Lpl] called their geometrical shapes, **central configurations** (c.c.'s). The general approach to central configurations was developed by Dziobek in 1900. In 1910 Moulton generalized Euler's result for N collinear bodies: namely, given N masses, for each arrangement of the bodies on a line there is exactly one central configuration up to similarity. We now clarify these notions.

If $c \in \mathbf{R}^3$, $r > 0$ $H \in \mathcal{O}(3)$, the group of orthogonal transformations on $\mathbf{R}^3$, and $g : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is given by $g(y) = c + rHy$ then g is called a **homography** or **similarity** on $\mathbf{R}^3$ and we denote $\mathcal{S}(3)$ as the group of all similarities on $\mathbf{R}^3$. If $x \in \mathbf{R}^{3N}$ is as in (1.2) p. 2, we extend g to a (diagonal) map $\tilde{g} : \mathbf{R}^{3N} \rightarrow \mathbf{R}^{3N}$ by

$$\tilde{g} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

where each $x_i \in \mathbf{R}^3$. If $H = I_3$ then g is called a **homothety**. If $x, y \in \mathbf{R}^{3N}$ and $y = \tilde{g}(x)$ for some $g \in \mathcal{S}(3)$, then y is said to be **homographic** or **similar** to x and we say that $y \sim x$. One normally thinks of the figure formed by the ordered set of points, $y_1, \dots, y_N$ as being homographic to the figure formed by $x_1, \dots, x_N$. If q is a solution of the N -body problem on I , $t_0 \in I$ and $q(t) \sim q(t_0) \forall t \in I$, then q is said to be a **homographic solution**. This means that for some $g : I \rightarrow \mathcal{S}(3)$, given by

$$g(t)(x) = c(t) + r(t)H(t)x, \quad (1.3)$$

where $c : I \rightarrow \mathbf{R}^3$, $H : I \rightarrow \mathcal{O}(3)$ and $r : I \rightarrow \mathbf{R}^+$, $g(t)$ is a one-parameter family of homographies and we have

$$q(t) = \widetilde{g(t)}(q(t_0)). \quad (1.4)$$

which means that

$$q_i(t) = c(t) + r(t) H(t) q_i(t_0) \quad (1.5)$$

for each $i = 1, \dots, N$, and $t \in I$. If $r \equiv 1$ (the constant function) then q is called a **relative equilibrium**. If $H \equiv I_3$ (also constant) then q is called a **homothetic solution**. If q and p are homographic solutions on the intervals, I_1 and I_2 respectively then it is clear that

$$\exists t_1 \in I_1, t_2 \in I_2 \text{ such that } q(t_1) \sim p(t_2) \Leftrightarrow \forall t_1 \in I_1, t_2 \in I_2, q(t_1) \sim p(t_2).$$

Furthermore, if either and hence both of these conditions holds, then we say that q and p are **homographically equivalent solutions** and write $q \approx p$. Clearly, $\approx$ is an equivalence relation. Define the equivalence class $\hat{q} = \{p | p \approx q\}$. Then for any mass vector, m in the three body problem, Euler's result gives three equivalence classes of collinear c.c.'s and Lagrange's gives one non-collinear class. However, if these solutions are given in $\mathbf{R}^2$ and one considers only direct symmetries: i.e., ones for which the orthogonal part, $H \in \mathcal{SO}(2)$, then there are two distinct Lagrange homographic classes and using this method of enumerating c.c.'s, we obtain five classes of c.c.'s where in the limiting case where the mass of the third body approaches zero, one c.c. is associated with each of the five Lagrange points in the restricted problem. With either method for counting c.c.'s, for any $m \in (\mathbf{R}^+)^3$, the totality of equivalence classes is finite and known for the three body problem. The question of how many equivalence classes of c.c.'s there are in the four body problem is still open. It is also an open question whether there exists some N and some $m \in (\mathbf{R}^+)^N$ giving rise to infinitely many classes.

Early in this century the understanding of homographic solutions was greatly improved by a theorem of Pizzetti which we shall now state. We first define a

solution to be **flat** if for any t there exists a plane containing all bodies and **planar** if in some inertial coordinate system there exists a plane which contains all bodies for all t .

Theorem 1 (Pizzetti, 1904) *[Pzt]* In the N -body problem

P1: if a homographic solution is non-flat then it is homothetic and

P2: if a homographic solution is flat then it is planar.

1.7 Appropriateness of the N -Body Model

Newtonian mechanics disagrees substantially with physical reality in the following cases:

1. when the bodies move at relativistic speeds (close to the speed of light).
2. when the masses are so great that the Euclidean model for space fails (curvature of space in general relativity) and
3. when the size and mass of the bodies is so small that quantum effects dominate (e.g. when the angular momentum of the bodies is only a few times Planck's constant).

However, the Newtonian model is so accurate that its error cannot be measured for most of our experience, including the solar system with a few exceptions. General relativity predicts a minor correction of the advance of the perihelion of the orbit of Mercury. (See e.g. [Pk. p.700]. See also [DMMS] for a mathematical analysis of such solutions).

The N -body problem also fails to explain the gradual increase in the distance between the Earth and the Moon. The gravitational field at the Earth due to the Moon is too great on the Moon's side of the Earth and too weak on the far side

to compensate for the Earth's centripetal acceleration toward's the centre of mass of the Earth-Moon system. So, ocean water piles up on these two opposite sides forming tides. The (daily) rotation of the Earth is much faster than the Moon's (monthly) orbital rate and the tides get ahead of the Moon. The Moon pulls them back slowing the Earth's rotation. This causes the angular momentum of the Moon to increase and it gradually grows farther from the Earth. The energy expended in raising the Moon to a higher orbit and in dissipating heat due to tidal action comes from the rotational energy of the Earth about its axis. Although this effect was explained by Newton in his theory, it is not explained by the N -body problem because of the point mass assumption. What other areas of the N -body problem fail to represent the physical reality of celestial motions? One area arises when two or more bodies collide. The centres of mass of two planets need not pass through a common point for them to collide. Indeed, given two "Jovian" planets: i.e., ones that comprise mostly liquid hydrogen and helium, their centres of mass would not even need to pass within the sum of their radii for them to collide since their shapes would distort under each other's influence. Although such catastrophic events have not been witnessed, the same could be said of two liquid core planets with thin lithospheres, unable to withstand the tidal forces of a close approach. In all of these cases, the N -body problem erroneously predicts that the bodies will pass by each other. In the case of a collision the N -body problem erroneously predicts that the velocities of two positive mass bodies tend to infinity and exceed the velocity of light as they pass inside the Schwartzchild radius. On the other hand, if the centres of mass of two bodies *do* collide, unlike physical reality for which time continues, the solution of the N -body problem stops; i.e., there is no future beyond the singularity unless one makes some 'regularizing transformation' of the variables in the equation which we shall discuss in the next section.

We have seen here many cases for which the N body model fails to describe the motion of celestial bodies under the action of their mutual gravitational attraction.

One of the basic goals of physics is to create mathematical models of the physical universe or some part of it and to test them to determine under what conditions they accurately describe physical reality. It is customary in such an assessment to declare that a particular model successfully describes reality under some condition or whether it ‘fails’. The reader unfamiliar with such terminology might read more into this than is intended. The term ‘fail’ here is not stated with any sort of acrimony but is a common expression used merely to point out where the model does not apply.

1.8 Singularities

We began this introduction by discussing the efforts made over many centuries to predict the motions of the planets and what a revelation it was to have theory that enabled one to do so, so accurately. We then went in the opposite direction to emphasize those areas where the N -body model fails to predict celestial events. Whatever its merits, once posed, the N body problem can be studied purely as a mathematical problem. As such it takes on a world of its own and gives rise to a wealth of fascinating solved and unsolved problems not likely to have been seen by those who helped create it. To see this we move to the late nineteenth century.

In the study of differential equations, a **singularity** is said to occur at some time τ if the solution cannot be extended beyond τ . In 1895, Paul Painlevé showed that for the N body problem, the distance from the solution $q(t)$ to the diagonal Δ (see section 1.1, p. 1) approaches zero as t approaches the singularity. A singularity is said to be a **collision singularity** if $q(t)$ approaches some point $q^* \in \Delta$ as $t \rightarrow \tau$. Since $q^* \in \Delta$ means that $q_i^* = q_j^*$ for some $i \neq j$, then two or more bodies must approach the same point q_i^* as $t \rightarrow \tau$, i.e., they collide and hence the name. Painlevé then showed that all singularities in the three body problem are collisions. He then made one of those rare intellectual leaps by asking whether there is a non-

collision singularity (i.e., some kind of singularity other than a collision) in the five body problem. Intuition might well lead people to guess that the only singularity is a collision; however, Painlevé's question helped stimulate research for another eighty years and yielded some surprising results. In 1908, von Zeipel showed that for a non-collision singularity to occur, the scalar moment of inertia of the system, $\sum_{i=1}^N m_i \|q_i(t)\|^2$, in barycentric coordinates, must become unbounded as $t \rightarrow \tau$. This means that some of the bodies must speed away to infinity in finite time.

To begin our discussion we look at two body collisions. From basic o.d.e. theory, a solution is a continuous function of both time and its initial conditions. One can find a sequence of initial values that do not lead to collision but converge on one that does. The corresponding sequence of solutions tends to the collision solution and the conic sections describing the orbits of the solutions become progressively flatter (the eccentricity of ellipses and hyperbolae approach unity and the 'latus rectum' goes to zero). The bodies loop around the centre of mass and come progressively closer to a path that is a straight line to the centre of mass followed by what might best be described as an 'elastic bounce' (i.e., one that conserves energy) back along the same path. Although the collision solution stops at the singularity one can make a transformation of position and time variables so that the "velocity" in the new variables does not go to infinity at the collision, the singularity becomes removable and the solution is replaced with an analytic function that extends beyond the singularity.

In 1858, Dirichlet told Kronecker that he could treat problems in mechanics using a step by step approximation method. However, he died without leaving any record of his results. Weierstrass directed his students, Kovalevski and Mittag-Leffler to try power series expansions as he suspected that this was Dirichlet's approach. At Mittag-Leffler's suggestion the King of Sweden and Norway offered a prize for anyone who could find a power series expansion for the coordinates of the N bodies, valid for all time. Henri Poincaré won the prize with an essay that did not solve the

problem but included a wealth of original ideas. In 1907-09 Sundman [Su2] solved it for the three body problem by regularizing two body collisions as just described for the two body problem. He obtained a series expansion for t and q in powers of the new variable that describes the entire motion. No corresponding result is known for $N > 3$ and Seigel has shown that most solutions involving triple collisions cannot be extended by this method. Sundman also showed that if all N bodies collide, the angular momentum of the system must be zero. He also showed [Su1] that in triple collisions in the three body problem, the ‘configuration’ of the three bodies approaches one of the central configurations found by Euler and Lagrange. (Here, by ‘configuration’ we mean the barycentric position of each body divided by the square root of the moment of inertia). Block [Blk] remarked that Sundman’s considerations extended to N -tuple collisions in the N -body problem (although one must interpret the result carefully).

In 1971, Easton [Est] extended solutions in the three body problem beyond collisions using a less stringent requirement that the function extending the solution need only be continuous, not necessarily regular. In 1974, McGehee [MG1] looked at triple collisions in the three body problem. By restricting his discussion to collinear solutions he could view the problem as one dimensional. This reduced the dimension of the phase space from eighteen to six. With further reductions due to barycentric coordinates and a constant energy surface, he found coordinate transformations that expanded or blew up the origin in position space. The phase space in the new coordinates was reduced to three dimensions bounded on the inside by a triple collision manifold (i.e., a surface on which all collisions converge) that is homeomorphic to a two-sphere with four holes. For some choices of masses, McGehee showed that triple collisions can be extended in the sense of Eastman. McGehee regularized all two body collisions and made the observation that for certain masses one can have two of the bodies depart in one direction as a coupled binary and the third body depart in the other direction. By having the three bodies come arbitrarily close to a

triple collision, one can have the binary depart as tightly coupled as desired with the speed at which the binary separates from the third body as high as desired. At the end of the paper he used this result to lay out a scheme for showing the existence of a collinear non-collision singularity in the five body problem: at least one where two body collisions, of which there need to be infinitely many, are presumed regularized and no longer ‘count’ as collisions. Subsequently, with Mather, McGehee showed the existence of such a solution in the collinear four body problem. This scheme was instrumental in Xia’s proof in 1988 of the existence of a non-collision singularity in the five body problem that was not encumbered by any binary collisions à la Mather and McGehee, proving that Painlevé’s original conjecture was correct.

1.9 Retrospective on the Original Problem

Let us retrace our steps to the original problem of determining the motions of the celestial bodies under their mutual gravitational attraction. Where do these singularities, non-collision singularities and the collision manifold fit? Clearly they don’t. As we saw, the N -body problem fails badly to describe planetary collisions. A non-collision singularity requires bodies to come arbitrarily close, again making the N -body problem a poor model. With this in mind one sees immediately that the flow of a solution along the collision manifold has no correspondence whatever to physical reality.

What has happened? We started with a problem in celestial mechanics that is very accurately modelled by the o.d.e. (1.1) in many realistic situations. As a mathematical problem it has proved very difficult and interesting in its own right, spawning a wealth of new problems of its own. It has stimulated the development of many areas of mathematics. Some ideas may still help practising astronomers, physicists and engineers. Nevertheless, in their attempts to understand the differential equation and its solutions, mathematicians have carried it freely into realms

that were not imagined when it was conceived. Over its three hundred years of development, mathematicians have not missed the fact that the theory they were developing belongs in a wider setting. Wintner, in his presentation of Pizzetti's theorem showed a keen interest in force functions other than the inverse square law of Newton and pointed out that the theorem doesn't generalize to the inverse cube law. Newton, himself studied various force laws in preparing his Principia to find the one that describes gravity and he seemed to play with different force laws as an intellectual exercise. Euler and Lagrange studied problems for which the underlying physical space could as easily be regarded as having dimension one or two and subsequently higher dimensions have been studied (e.g., Palmore [Pal]).

1.10 The Restricted Problem

In the last three centuries, much work has been done on the N -body problem where all masses are positive or in what is called **the restricted problem** where some masses are zero and the rest are positive. Engineers work with the restricted problem when they launch an artificial satellite whose mass is so small that it has no measurable effect on other celestial bodies and is presumed to be zero. However, zero mass bodies don't fit well into the Hamiltonian formalism of mechanical systems because the diagonal matrix $M = \text{diag}(m_1, \dots, m_N)$ is presumed to be invertible. If each $m_i > 0$ we can multiply both sides of (1.1) by m_i and obtain an equivalent o.d.e. of the form

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} \quad (1.6)$$

where

$$U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_j - q_i\|} \quad (1.7)$$

(See App. D for a discussion of the sign of U). However, in the restricted problem (1.1) cannot be recovered from (1.6) for any of the zero mass bodies. If we define

the $3N \times 3N$ diagonal mass matrix

$$\begin{aligned} M &= m_1 I_3 \oplus \cdots \oplus m_N I_3 \\ &= \text{diag}(m_1, m_1, m_1, \dots, m_N, m_N, m_N) \end{aligned}$$

and each $m_i \neq 0$ then we can use the invertibility of the matrix M to write the equations of motion as

$$\ddot{q} = M^{-1} \frac{\partial U}{\partial q} \quad (1.8)$$

Let $p_i(t) = m_i \dot{q}_i(t)$ and let $p = (p_1, \dots, p_N)^t$ as in (1.2). Define the Hamiltonian by the total energy.

$$H(q, p) = -U(q) + \frac{1}{2} \sum_{i=1}^N \frac{\|p_i\|^2}{m_i} \quad (1.9)$$

Then the reader can check that

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}. \quad (1.10)$$

This formalism fails when some of the masses are zero because in (1.8) the mass matrix is no longer invertible and in (1.9) we are trying to divide by zero. Thus, in the general setting the o.d.e. is given as in (1.1). p. 2.

Chapter 2

Summary of Results

In this chapter we shall give an overview of the main results obtained. Historically the N -body problem has been studied for various force functions (inverse square, inverse cube, etc.). We begin by defining the N -body problem with a force of attraction that is more general but still yields the conservation laws of momentum, angular momentum and energy as well as invariance under orthogonal transformations. Nevertheless, we permit the force function be sufficiently pathological that uniqueness of solutions is not guaranteed and we provide examples. We study the problem in more general dimensions and define the dimension of the solution at a given time. We show that even for the pathological force functions the dimension of a solution does not change over time.

Over the last century many special solutions of the N -body problem which make use of geometrical symmetry to simplify the problem have been studied. Some of these are central configurations: for example, [CeLl], [Di], [Lng], [Mkl] and [PW]. Examples involving non-homographic solutions are [Grv], [Sit] and [Xia]. Also groups of symmetries of o.d.e.'s have been studied in a general setting (see [Ar1, Ch. 1], [Ar2, Ch.3]). In this thesis we formalize this notion by defining a symmetry group for each initial value problem. The members of the group contain isometries for

position and velocity which share a common orthogonal part and which permute bodies of equal mass. We develop a structure theorem for the group that separates an inessential part from an essential part. Although the group is potentially uncountable, the essential part is finite. We then define centroidal coordinates and show that they are precisely the Galilean coordinate frames for which the position and velocity parts of the group are identical orthogonal groups (i.e., the group is simpler in such frames). We then show by counterexample that there exist sufficiently pathological force functions for which the group of some solution is not preserved over time. However, for force functions that give uniqueness of solutions we show that the group is preserved over the time of the solution.

We then develop a theorem that places restrictions on the angular momentum for a given group. In centroidal coordinates, if H is an orthogonal transformation in the group, x lies in $\text{fix } H$, the set of fixed points for H and y lies in the orthogonal complement of this fixed point set, then the component of angular momentum in the xy -plane is zero. As an immediate consequence of this, in $\mathbf{R}^3$ if the group contains two non-trivial rotations with different axes then the total angular momentum is zero. These results are used to identify those groups in dimensions 2 and 3 for which solutions may have non-zero angular momentum.

In any finite dimensional space where the force of attraction is strictly positive, the masses are positive and the group contains a reflection in a hyperplane (affine subspace of codimension one) we show that the solution either comes from a singularity, leads to a singularity or both. In dimensions 2 and 3 all groups with this property are identified. In $\mathbf{R}^4$ we give a family of relative equilibria that do not form central configurations (à la Palmore [Pal]) and provide uncountably many homographic configurations for each assignment of mass for the Newtonian force. We obtain such a family for each way of factoring a composite positive integer (the number of bodies) where both factors are greater than 1.

We then study forces of attraction that are homogeneous and use the theory of

the group to find a result that is analogous to Moulton's generalization of Euler's uniqueness theorem for collinear c.c.'s. Consider a polytope in $\mathbf{R}^d$ with centroid at the origin. (A polytope is a generalization of the notion of a line segment in $\mathbf{R}^1$, a polygon in $\mathbf{R}^2$, a polyhedron in $\mathbf{R}^3$, $\dots$, a polytope in $\mathbf{R}^d$.) Nest k of these objects inside one another homothetically so that the i th polytope has radius r_i . Place a body of mass m_i at each vertex of polytope, i . One asks the question whether a given polytope has the property that for each assignment of positive masses, $m_1, \dots, m_k$ and each way of arranging the nesting of the bodies, there is exactly one central configuration for a given moment of inertia (i.e., up to scaling the total configuration). We answer this question in the affirmative for all positive dimensions for the case of the regular simplex and the cross polytope. The regular simplex includes nested equilateral triangles in $\mathbf{R}^2$, nested regular tetrahedra in $\mathbf{R}^3$ and regular pentatopes in $\mathbf{R}^4$. The cross polytope includes nested squares in $\mathbf{R}^2$ and nested regular octahedra in $\mathbf{R}^3$. All results are of the following form: given some polytope of interest.

1. for a range of values of the parameter α defining the homogeneous force of attraction, s
2. for all values of the number of polytopes, $n \geq 1$ and
3. for all ways of assigning positive masses to the bodies on polytope, i ($i \leq n$)

we have shown that there is exactly one way of scaling the sizes of the polytopes to obtain a central configuration subject to the constraint that polytope $i + 1$ lies inside polytope i , ($0 < i < n$) and hence there are exactly $n!$ such c.c.'s where the arrangement of the polytopes in the nest is not subject to any constraint.

The examples are

1. the regular simplex for all dimensions, $d \geq 1$ and for all $\alpha \geq \min(2, d)$

2. the cross polytope for all dimensions, $d \geq 1$ and for all $\alpha \geq 2$.

For the dimensions of greatest interest these examples include

Description of Polytope Nested	Dimension of Space	Range of Values of α
line segment	1	$\alpha \geq 2$
equilateral triangle	2	$\alpha \geq 2$
regular tetrahedron	3	$\alpha \geq 3$
square	2	$\alpha \geq 2$
regular octahedron	3	$\alpha \geq 2$

In the case of the Newtonian (inverse square) law of attraction we use the group to find initial value problems which are not homographic but can be studied *à la Hill*. In three and four dimensions, we show that for the examples given their solutions can be extended to all of $\mathbf{R}$: i.e., they lead neither to nor from a singularity. The motion of the bodies is restricted to a region in configuration space determined by the initial conditions. This is done in the cases where the bodies form the vertices of an antiprism in one example and a configuration that is a *generalized Sitnikov motion* in another. A third example in $\mathbf{R}^4$ has the bodies forming the vertices of a rectangular product of regular polygons.

Chapter 3

The N-body Problem: A Larger Framework

In Section 1.5 the equation of motion is given for the Newtonian force in $\mathbf{R}^3$ with positive masses. We now generalize the problem

1. to permit the underlying physical space for the dynamical system to have dimension d for any $d \geq 1$ and
2. to allow a force law more general than the Newtonian one.

However, as we shall see, to ensure conservation of momentum and angular momentum we make the force between two bodies equal in magnitude, opposite in direction and directed along the line joining the two bodies. Body i has mass $m_i \in \mathbf{R}$ and the mass vector is $m = (m_1, \dots, m_N)^t \in \mathbf{R}^N$. At time t its position is $q_i(t) \in \mathbf{R}^d$. The differential equation governing the motion of the bodies is

$$\ddot{q}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}(t)\|) q_{ij}(t) \quad (3.1)$$

for each $i \leq N$ and for some continuous function $s : \mathbf{R}^+ \rightarrow \mathbf{R}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^d$, $q_{ij}(t) = q_j(t) - q_i(t)$ and as before, each dot over a function of t indicates differentiation; e.g., $\ddot{q}_i = d^2 q_i / dt^2$.

Definition 1 If $x \in \mathbf{R}^{dN}$ we partition the $dN \times 1$ column matrix x as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad (3.2)$$

where each $x_i \in \mathbf{R}^d$ is a $d \times 1$ column matrix and we simply write $x = (x_1, \dots, x_N)^t$. Using this notation we let $q(t) \in \mathbf{R}^{dN}$ be partitioned according to (3.2). Bodies i and j are said to be in collision if $q_i = q_j$. Let $\Delta_{ij} = \{q \in \mathbf{R}^{dN} | q_i = q_j\}$ and denote the diagonal $\Delta = \bigcup_{1 \leq i < j \leq n} \Delta_{ij}$. Then let the domain, $D = \mathbf{R}^{dN} \setminus \Delta$, the open set on which no two bodies occupy the same point in space; i.e., no two bodies are in collision. (Note that D is connected if either $d > 1$ or $N = 1$). If $I = (t_-, t_+)$ is an open interval of $\mathbf{R}$; i.e., $-\infty \leq t_- < t_+ \leq \infty$ and $q \in C^2(I, D)$ satisfies (3.1) $\forall t \in I, i \leq N$, then q is said to be a **solution of the N -body problem on I for the function s** . Given $a \in D$, $b \in \mathbf{R}^{dN}$ and an initial time $t_0 \in I$, if $q \in C^2(I, D)$ satisfies $q(t_0) = a$, $\dot{q}(t_0) = b$ and (3.1) then q is said to be a **solution of the initial value problem (m, a, b, t_0) on I** . We shall abbreviate the term 'initial value problem' with 'ivp'. The s function, is called **attractive** if $s(r) > 0$ for all $r \in (0, \infty)$ and **repulsive** if $s(r) < 0$ for all $r \in (0, \infty)$.

We have an autonomous second order o.d.e. in dN variables. By the continuity of s on $(0, \infty)$ the right side of (3.1) is a continuous function of q on D . By the standard method of reducing a second order o.d.e. to a first order one and by Peano's existence theorem a solution exists on some interval I containing t_0 .

Example 1 If $d = 3$, $s(r) = 1/r^3$ and $m \in (\mathbf{R}^+)^N$, i.e. all masses are positive, we have Newtonian mechanics in $\mathbf{R}^3$.

Example 2 If $d = 3$, $s(r) = 1/r^3$, some $m_i > 0$, some $m_i = 0$ but no $m_i < 0$ we have the restricted N -body problem for Newtonian mechanics in $\mathbf{R}^3$.

Example 3 The case $m = 0$ (all masses are zero) gives rectilinear motion, i.e., $q(t) = a + b(t - t_0)$ as the unique solution. Unless stated otherwise, we shall assume from now on that $m \neq 0$, i.e. that some masses are positive.

Example 4 Imagine a world in $\mathbf{R}^3$ where the s function is given by

$$s(r) = \begin{cases} \frac{6}{r} \left(1 - \frac{r}{2}\right)^{\frac{1}{3}}, & 0 < r < 2 \\ 0, & r \geq 2 \end{cases}$$

Then two bodies of unit mass with zero initial velocity and initial positions $q_1(0) = (1, 0, 0)^t$ and $q_2(0) = (-1, 0, 0)^t$ have a continuum of solutions

$$q_1(t) = -q_2(t) = \begin{cases} (1, 0, 0)^t, & 0 \leq t \leq \tau \\ (1 - (t - \tau)^3, 0, 0)^t, & \tau < t < \tau + 1 \end{cases}$$

one for each $\tau \in \mathbf{R}^+$. In quantum mechanics the location and momentum of a particle cannot be determined precisely and their values are limited by the uncertainty principle. The location and momentum are each given by a wavefunction, the square of whose modulus gives the probability that it has a particular value (see e.g. [PC, Ch. 1], [RKL, Ch. 4] or [WS, Ch.4]). This limitation on one's ability to determine the outcome of an experiment is not due any weakness of the equipment used but is a fundamental aspect of the theory. This is in contrast with the N -body problem for Newtonian mechanics where, given the initial state of a system one can determine the future state at any given time provided that one performs sufficiently detailed and accurate calculations. However, in the example given here we have 'created' a world based on our generalized N -body problem (as distinct from a quantum mechanical one) in which determinism fails due to the pathological nature of the force of attraction. This example points out that Newton's law of gravity needs a 'nicely' behaved s function to effect this determinism.

The following lemma is an elementary result applicable to autonomous second order o.d.e.'s with no first order term:

Lemma 2 *If q is a solution of the N -body problem on I , $\tau \in \mathbf{R}$ and*

$$r \in C^2(\tau - I, \mathbf{R}^{dN} \setminus \Delta)$$

given by $r(t) = q(\tau - t)$, then r is a solution of the N -body problem on $\tau - I$.

Proof. Fix $\tau \in \mathbf{R}$. $\forall t \in \tau - I$, we have $\tau - t \in I$. $\dot{r}(t) = -\dot{q}(\tau - t)$ and $\ddot{r}(t) = \ddot{q}(\tau - t)$. Substituting into (3.1) gives the result. $\blacksquare$

Remark 1 *In the case $d = 3$ we can denote $q_i = (x_i, y_i, z_i)^t$. In general we let $q_i = (q_i^1, \dots, q_i^d)^t$. We also denote $q_{ij}^k = q_j^k - q_i^k$, the k th entry in the difference vector q_{ij} ; i.e., $q_{ij} = (q_{ij}^1, \dots, q_{ij}^d)^t = (q_j^1 - q_i^1, \dots, q_j^d - q_i^d)^t$.*

Definition 2 *The usual inner product on $\mathbf{R}^{dN}$ is $(q, r) = r^t q = \sum_{i=1}^N \sum_{k=1}^d q_i^k r_i^k$, giving the Euclidean norm $\|q\|_{dN} = \sqrt{(q, q)}$.*

So,

$$\|q\|_{dN}^2 = \sum_{i=1}^N \|q_i\|^2$$

3.1 A First Order System and Uniqueness

Let us define a further partition

$$z(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \in (\mathbf{R}^{dN} \setminus \Delta) \times \mathbf{R}^{dN}. \quad (3.3)$$

Then q is a solution of (3.1) if and only if z is a solution of

$$\dot{z} = \begin{pmatrix} 0 & I_{dN} \\ F(q) & 0 \end{pmatrix} z \quad (3.4)$$

where F is partitioned as an $N \times N$ block of $d \times d$ matrices whose i, j th $d \times d$ matrix is

$$F_{ij}(q) = \begin{cases} m_j s(\|q_{ij}\|) I_d & \text{if } i \neq j \\ - \sum_{\substack{k=1 \\ k \neq i}}^N m_k s(\|q_{ik}\|) I_d & \text{if } i = j. \end{cases}$$

Example 4 points out the need for the following:

Theorem 3 *If s is locally Lipschitzian then the the right side of (3.4) is locally Lipschitzian, and the solution of the ivp (m, a, b, t_0) for the function s is unique.*

Proof. From (3.4), it is clear that we need only show that $F(q)q$ is locally Lipschitzian. Let K be any compact subset of the domain $D = \mathbf{R}^{dN} \setminus \Delta$. Let $l(q) = \min_{1 \leq i < j \leq N} \|q_{ij}\|_d$ and $u(q) = \max_{1 \leq i < j \leq N} \|q_{ij}\|_d$, $\forall q \in D$. Let $L = \inf_{q \in K} l(q)$ and $U = \sup_{q \in K} u(q)$. Since l and $u : D \rightarrow \mathbf{R}^+ = (0, \infty)$ are continuous and K is compact then $l(K)$ and $u(K)$ are compact subsets of $\mathbf{R}^+$ hence $0 < L \leq U < \infty$. Since s is continuous if we let $s_m = \sup_{x \in [L, U]} s(x)$, then s_m is finite. Since s is locally Lipschitzian $\exists J > 0$ such that $\forall x, y \in [L, U]$, $|s(x) - s(y)| \leq J|x - y|$, $\forall p, q \in K$,

$$\begin{aligned} \|s(\|q_{ij}\|) q_{ij} - s(\|p_{ij}\|) p_{ij}\| &= \|s(\|q_{ij}\|) (q_{ij} - p_{ij}) + [s(\|q_{ij}\|) - s(\|p_{ij}\|)] p_{ij}\| \\ &\leq |s(\|q_{ij}\|)| \cdot \|q_{ij} - p_{ij}\| + |s(\|q_{ij}\|) - s(\|p_{ij}\|)| \cdot \|p_{ij}\| \\ &\leq s_m \|q_{ij} - p_{ij}\| + J \|\|q_{ij}\| - \|p_{ij}\|\| U \\ &\leq (s_m + JU) \|q_{ij} - p_{ij}\| \\ &\leq 2(s_m + JU) \|q - p\|_{dN}. \end{aligned}$$

Thus, $s(\|q_{ij}\|) q_{ij}$, as a function of q is locally Lipschitzian. Since a linear combination of Lipschitz functions is Lipschitz, $F(q)q$ is Lipschitz as is the right side of (3.4). ■

Lemma 4 *If q is a solution of the N -body problem on I , if the s function is locally Lipschitzian and if for some $\tau \in I$, $\dot{q}(\tau) = 0$, then $\forall t \in I \cap (2\tau - I)$,*

$$q(t) = q(2\tau - t).$$

Furthermore, q can be extended to a solution $\tilde{q}$ defined on $I \cup (2\tau - I)$ and $\tilde{q}(t) = \tilde{q}(2\tau - t)$ on all of $I \cup (2\tau - I)$.

Proof. If we let $r : 2\tau - I \rightarrow \mathbf{R}^{dN} \setminus \Delta$ be given by $r(t) = q(2\tau - t)$ then by Lemma 2 p. 27, r is a solution of the N -body problem on $2\tau - I$, $\dot{r}(t) = -\dot{q}(2\tau - t)$, $\dot{r}(\tau) = -\dot{q}(\tau) = 0$ and $r(\tau) = q(\tau)$. $\therefore r$ and q are both solutions of the ivp $(m, q(\tau), 0, \tau)$ and by uniqueness must be equal on the set on which they are both defined, namely $I \cap (2\tau - I)$. $\therefore \forall t \in I \cap (2\tau - I)$, $q(2\tau - t) = r(t) = q(t)$ which completes the first part of the proof.

Let $\tilde{q}$ be the extension of q to all of $I \cup 2\tau - I$ given by

$$\tilde{q}(t) = \begin{cases} q(t), & \forall t \in I \\ r(t), & \forall t \in (2\tau - I) \setminus I \end{cases}$$

It is clear that we have two overlapping functions, q and r that agree on their common domain, both satisfy the o.d.e. (3.1) and so $\tilde{q}$ does also. $\forall t \in (2\tau - I) \setminus I$, we have $2\tau - t \in I$ and $\tilde{q}(t) = r(t) = q(2\tau - t) = \tilde{q}(2\tau - t)$. This covers all points not in $I \cap (2\tau - I)$. ■

3.2 The Potential Function

Even in our more general setting of the N body problem, the force between any two bodies is directed along the line joining them and its magnitude is a function of the distance between them. Such systems are conservative and lend themselves to the use of a potential function. We now give a development similar to that on page 18 of Chapter 1 for our more general setting.

Definition 3 Given the N body problem (m, a, b, t_0) for some continuous s function we let the **potential function** $u : \mathbf{R}^+ \rightarrow \mathbf{R}$ be the C^1 function

$$u(r) = - \int_{r_0}^r xs(x) dx \tag{3.5}$$

defined up to an additive constant.

Thus on $(0, \infty)$

$$u'(r) = -rs(r). \quad (3.6)$$

Let

$$U(q) = \sum_{i < j} m_i m_j u(\|q_{ij}\|) \quad (3.7)$$

(See App. D for a discussion of the sign of U). Then from (3.6), Remark 1 and the fact that

$$\frac{\partial \|q_{ij}\|}{\partial q_i^l} = \frac{\partial}{\partial q_i^l} \sqrt{\sum_{k=1}^d (q_j^k - q_i^k)^2} = -\frac{q_{ij}^l}{\|q_{ij}\|}$$

we have

$$\begin{aligned} \frac{\partial U}{\partial q_i^l} &= \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j [-\|q_{ij}\| s(\|q_{ij}\|)] \frac{\partial \|q_{ij}\|}{\partial q_i^l} \\ &= \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j s(\|q_{ij}\|) q_{ij}^l = m_i \ddot{q}_i^l \end{aligned}$$

and as in Chapter 1, (1.6), p. 18

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} \quad (3.8)$$

and for $s(r) = 1/r^3$ the U given here agrees with the previous one. However, in the restricted problem (3.1) cannot be recovered from (3.8) for any of the zero mass bodies. If we define the $dN \times dN$ diagonal mass matrix $M = m_1 I_d \oplus \cdots \oplus m_N I_d$ and if each $m_i \neq 0$ then we can write the equations of motion as

$$\ddot{q} = M^{-1} \frac{\partial U}{\partial q} \quad (3.9)$$

Let $p_i(t) = m_i \dot{q}_i(t)$ and let $p = (p_1, \dots, p_N)^t$ as in (3.2). As in Chapter 1, if all masses are non-zero, we can define the Hamiltonian by the total energy,

$$H(q, p) = -U(q) + \frac{1}{2} \sum_{i=1}^N \frac{\|p_i\|^2}{m_i} \quad (3.10)$$

Again we have an ode equivalent to (3.1).

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}. \quad (3.11)$$

As before, this formalism, (3.10) fails when some of the masses are zero. In the general setting the o.d.e. is given as in (3.1).

3.3 Integrals of the System

For Newtonian mechanics in $\mathbf{R}^3$ there are ten integrals of the system. (see [Brn]). We now give a discussion similar to what is found in [SM, pp 21.22] for our more general setting.

3.3.1 Barycentric Coordinates

If q is a solution of (m, a, b, t_0) let us denote **the first moment** by $\sum m_i q_i(t)$. From (3.1) on page 24

$$\begin{aligned} \frac{d^2}{dt^2} \sum_{i=1}^n m_i q_i(t) &= \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n m_i m_j s(\|q_{ij}(t)\|) q_{ij}(t) \\ &= \sum_{i < j} m_i m_j s(\|q_{ij}(t)\|) (q_{ij}(t) + q_{ji}(t)) = 0. \end{aligned}$$

Hence the first moment is a linear function of t ; say,

$$\sum m_i q_i(t) = \alpha + (t - t_0) \beta \quad (3.12)$$

for some $\alpha, \beta \in \mathbf{R}^d$. Clearly, $\alpha = \sum m_i a_i$ and $\beta = \sum m_i \dot{q}_i(t) = \sum m_i b_i$, the same for all $t \in I$. We say that the solution q is in **barycentric coordinates** or **centre of mass coordinates** if the first moment is zero for all $t \in I$, i.e., $\alpha = \beta = 0$. Thus we have shown

Lemma 5 *If q is a solution of the ivp (m, a, b, t_0) on I ,*

1. $\sum m_i q_i(t) = \sum m_i a_i + \sum m_i b_i(t - t_0)$ for all $t \in I$ and
2. q is in barycentric coordinates if and only if $\sum m_i a_i = \sum m_i b_i = 0$.

In the light of this we have

Definition 4 *The initial value problem (m, a, b, t_0) is said to be in **barycentric coordinates** if $\sum m_i a_i = \sum m_i b_i = 0$.*

Corollary 6 *If q is a solution of (m, a, b, t_0) then q is in barycentric coordinates if and only if (m, a, b, t_0) is in barycentric coordinates. If q and r are solutions of (m, a, b, t_0) then q is in barycentric coordinates if and only if r is.*

Remark 2 *This corollary would be vacuous were it not for cases like Example 4 on page 26 for which the solution is not unique. This example has a continuum of solutions all of which are in centre of mass coordinates.*

3.3.2 Conservation of Momentum

Definition 5 *For some $x, v \in \mathbf{R}^d$ and some open interval I of $\mathbf{R}$, if $q, r \in C^2(I, \mathbf{R}^{dN})$ satisfy*

$$r_i(t) = q_i(t) - x - (t - t_0)v. \quad (3.13)$$

*$\forall t \in I$ then we say that r is a **Galilean coordinate transformation** of the function q and denote it by $r = \mathcal{G}_{xv}(q)$. Let $\zeta : \mathbf{R}^d \rightarrow \mathbf{R}^{dN}$ be the linear map*

$$\zeta(x) = \begin{pmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{pmatrix} x = \begin{pmatrix} x \\ x \\ \vdots \\ x \end{pmatrix} \left\} N \text{ copies of } x$$

where I_d is the $d \times d$ identity matrix and for brevity we define $x' = \zeta(x)$. Given the ivp (m, a, b, t_0) and $x, v \in \mathbf{R}^d$, define the ivp

$$\mathcal{G}_{xv}(m, a, b, t_0) = (m, a - \zeta(x), b - \zeta(v), t_0) = (m, a - x', b - v', t_0).$$

The following lemma is elementary but important and a similar statement is found in [Mar. Eqn. 859, p. 302].

Lemma 7 *If $q \in C^2(I, \mathbf{R}^{dN})$ for some open interval, I of $\mathbf{R}$ and $x, v \in \mathbf{R}^d$, then q is a solution of (m, a, b, t_0) on I if and only if $\mathcal{G}_{xv}(q)$ is a solution of $\mathcal{G}_{xv}(m, a, b, t_0)$ on I .*

Proof. Let $r = \mathcal{G}_{xv}(q)$. Then $r_i(t) = q_i(t) - x - (t - t_0)v$, $r_{ij} = q_{ij}$ and $\ddot{r}_i = \ddot{q}_i$; so, r satisfies (3.1) on page 24 if and only if q does. Furthermore, it is clear that $q_i(t_0) = a_i \Leftrightarrow r_i(t_0) = a_i - x$ and $\dot{q}_i(t_0) = b_i \Leftrightarrow \dot{r}_i(t_0) = b_i - v$. ■

3.3.3 Angular Momentum and Energy

We now address the issue of conservation of angular momentum in arbitrary dimensions.

Notation Recall our notation of the remark on p. 27 in which $q_{ij}^k \equiv q_j^k - q_i^k$ is the k th coordinate of the difference vector q_{ij} .

For any body $i \leq N$ and spatial coordinates $k, l \leq d$, from (3.1), p. 24, the scalar expression,

$$\begin{aligned} m_i \begin{pmatrix} \ddot{q}_i^k & q_i^l - \ddot{q}_i^l & q_i^k \end{pmatrix} &= \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j s(\|q_{ij}\|) (q_{ij}^k q_i^l - q_{ij}^l q_i^k) \\ &= \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j s(\|q_{ij}\|) [(q_j^k q_i^l - q_i^k q_i^l) - (q_j^l q_i^k - q_i^l q_i^k)] \\ &= \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j s(\|q_{ij}\|) (q_j^k q_i^l - q_j^l q_i^k) \end{aligned}$$

Summing over i and expressing it as $\sum_{i < j}$ as was done with the first moment in Section 3.3.1 on page 31 gives

$$\begin{aligned} \sum_{i=1}^N m_i \left(\ddot{q}_i^k q_i^l - \ddot{q}_i^l q_i^k \right) &= \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N m_i m_j s(\|q_{ij}\|) (q_j^k q_i^l - q_j^l q_i^k) \\ &= \sum_{i < j}^N m_i m_j s(\|q_{ij}\|) (q_j^k q_i^l - q_j^l q_i^k + q_i^k q_j^l - q_i^l q_j^k) \end{aligned}$$

or

$$\sum_{i=1}^N m_i \left(\ddot{q}_i^k q_i^l - \ddot{q}_i^l q_i^k \right) = 0 \quad (3.14)$$

If we define the (obviously) skew-symmetric **angular momentum matrix**, $C(q(t))$ to be the $d \times d$ matrix whose l, k th entry is

$$C_{lk}(q(t)) = \sum_{i=1}^N m_i \left(\ddot{q}_i^k(t) q_i^l(t) - \ddot{q}_i^l(t) q_i^k(t) \right); \quad (3.15)$$

then clearly, dC_{lk}/dt is equal to the left side of (3.14); i.e., $dC_{lk}/dt = 0$ and hence

$$C(q(t)) = C(q(t_0)). \quad (3.16)$$

a constant matrix. We can also write C as

$$C = \sum_{i=1}^N m_i (q_i(t) \dot{q}_i^t(t) - \dot{q}_i(t) q_i^t(t))$$

$\forall t \in I$ where q_i^t and $\dot{q}_i^t$ are the transposes of the column vectors, q_i and $\dot{q}_i$.

Remark 3 *Note that this derivation has nothing to do with barycentric coordinates. It works with an arbitrary continuous s function. Hence it also works with systems for which the solutions are not unique. Thus, as with Corollary 6, if q and r are solutions of (m, a, b, t_0) (e.g., with non-uniqueness of solutions), then $C(q) = C(r)$. To visualize the lk component of angular momentum, perform the orthogonal projection of all positions and velocities onto the lk plane and calculate C_{lk} . The case*

$d = 3$ is special as Wintner noted since 2-planes have codimension 1 and hence in $\mathbf{R}^3$, C can be associated with a vector. In $\mathbf{R}^d$ the component of angular momentum in an arbitrary plane, uv where u and v are mutually orthogonal unit length vectors, is the inner product, (u, Cv) . Here the sense is from u to v : i.e., regarding u as the abscissa and v as the ordinate. Thus the basis $\{u, v\}$ for the plane is ordered and the angular momentum in the vu plane is the negative of this value.

Finally, conservation of energy is obtained from (3.1) in the usual manner, viz.,

$$\frac{dU}{dt} = \sum_{i=1}^N \left(\frac{\partial U}{\partial q_i} \cdot \dot{q}_i \right) = \sum_{i=1}^N \left(m_i \ddot{q}_i \cdot \dot{q}_i \right) = \frac{dT}{dt}$$

where the brackets $(,)$ denote inner product and the kinetic energy,

$$T = \frac{1}{2} \sum_{i=1}^N m_i \|\dot{q}_i\|^2. \quad (3.17)$$

So,

$$T - U = h \quad (3.18)$$

for a constant, h which is the total energy of the system. Thus equations (3.12), (3.15) and (3.18) give α , β , C and h which are

$$2d + d(d-1)/2 + 1 = \binom{d+2}{2}$$

integrals of the ode. For $d = 3$ this gives the well-known 10 integrals in $\mathbf{R}^3$.

3.4 The Affine Subspace of a Solution

In Def. 36 p. 207 and Def. 38 p. 209 App. A, we defined an affine subspace and the affine hull of a set of points in $\mathbf{R}^d$. We now extend that notation to our vectors in $\mathbf{R}^{dN}$ as a set of N points in $\mathbf{R}^d$ as in (3.2), p. 25.

3.4.1 Definitions

Definition 6 $\forall x, y \in \mathbf{R}^{dN}$ where $x = (x_1, \dots, x_N)^t$, let $X = \{x_1, \dots, x_N\}$ and define the linear subspace $\langle \Delta x \rangle = \langle \Delta X \rangle \equiv \langle X - X \rangle$ and the affine subspace, $\text{aff } x = \text{aff } X$. Furthermore, define the linear subspace, $\langle \Delta x, \Delta y \rangle \equiv \langle \Delta x \rangle + \langle \Delta y \rangle$ and the affine subspace, $\text{aff } (x, \Delta y) = \text{aff } x + \langle \Delta y \rangle$. If q is a solution of the ivp (m, a, b, t_0) on I and $t \in I$, we define the **affine subspace of the solution at time t** to be $\text{aff } (q(t), \Delta \dot{q}(t))$ and the **linear subspace of the solution q at time t** to be $\langle \Delta q(t), \Delta \dot{q}(t) \rangle$. We also call $\text{aff } (\dot{q}(t), \Delta q(t))$ the **velocity affine subspace at time t** . Furthermore, $\dim \langle \Delta q(t), \Delta \dot{q}(t) \rangle$ is called the **dimension of the solution q at time t** . Then $\langle \Delta q(t_0), \Delta \dot{q}(t_0) \rangle = \langle \Delta a, \Delta b \rangle$ and we call it the **linear subspace of the ivp** and $\dim \langle \Delta a, \Delta b \rangle$ is called the **dimension of the ivp** (m, a, b, t_0) .

Then in the above definition,

$$\langle \Delta x \rangle = (\text{aff } x) - x_1 = \left\{ \sum_{i=1}^N \alpha_i x_i \mid \sum_{i=1}^N \alpha_i = 0 \right\}$$

and

$$\text{aff } x = \left\{ \sum_{i=1}^N \alpha_i x_i \mid \sum_{i=1}^N \alpha_i = 1 \right\}.$$

By Lemma 82 on page 208 of Appendix A, $\text{aff } x$ is the intersection of all affine subspaces in $\mathbf{R}^d$ containing all x_i , $i = 1, \dots, N$ and $\langle \Delta x \rangle$ is the linear subspace of $\mathbf{R}^d$ which is the parallel transport of $\text{aff } x$ to the origin. If we fix $k \leq N$, (k arbitrary) clearly,

$$\langle \Delta x \rangle = \text{aff } x - x_k = \left\{ \sum_{\substack{i=1 \\ i \neq k}}^N \alpha_i x_{ki} \mid \alpha_i \in \mathbf{R} \right\} :$$

so, $\langle \Delta x \rangle$ is spanned by all of the difference vectors, $\{x_{ki}\}_{i \neq k}$ and $\langle \Delta q(t), \Delta \dot{q}(t) \rangle$ is the linear space spanned by all of the difference vectors $q_{ij}(t)$ together with their first derivatives. Also,

$$\text{aff } (q(t), \Delta \dot{q}(t)) = q_k(t) + \langle \Delta q(t) \rangle + \langle \Delta \dot{q}(t) \rangle = q_k(t) + \langle \Delta q(t), \Delta \dot{q}(t) \rangle \quad (3.19)$$

and

$$\dim \text{aff} (q(t), \Delta \dot{q}(t)) = \dim \langle \Delta q(t), \Delta \dot{q}(t) \rangle.$$

Similarly,

$$\dim \text{aff} (\dot{q}, \Delta q) = \dim \text{aff} (\Delta q, \Delta \dot{q}).$$

3.4.2 Uniqueness

We now show that if the s function in (3.1) gives unique solutions it is straightforward to see that the linear subspace and dimension of the solution are constants. Consider for a moment the case where the s function gives rise to uniqueness of solutions: e.g., if s is locally Lipschitzian. Let q be the unique solution of the ivp (m, a, b, t_0) on a maximal interval I . Perform a Galilean coordinate transformation so that the initial position and velocity of some body, (say body 1) is zero: i.e., let $\alpha = a - \zeta(a_1) = a - a'_1$ (D. 5 p. 32) and $\beta = b - \zeta(b_1) = b - b'_1$ and let r be the solution of the ivp (m, α, β, t_0) on I . Then $r_i(t) = q_i(t) - a_1 - (t - t_0)b_1$ and since each $\alpha_i = a_{1i}$ and each $\beta_i = b_{1i}$, the initial positions, α_i and initial velocities, β_i all lie in $\langle \Delta a, \Delta b \rangle$ which as we know is also $\langle \Delta \alpha, \Delta \beta \rangle$. Let $d' = \dim \langle \Delta \alpha, \Delta \beta \rangle \leq d$ and choose an orthonormal basis so that the first d' basis vectors span $\langle \Delta \alpha, \Delta \beta \rangle$. Then the last $d - d'$ coordinates of the initial positions, (α_k) and initial velocities, (β_k) relative to this basis are all zero and we can consider the initial value problem in the smaller (at least, not larger) space, $\mathbf{R}^{d'}$ by deleting the last $d - d'$ coordinates of each α_i and β_i . If u is the solution in the smaller space, $\mathbf{R}^{d'}$ on some maximal interval, I' then there exists a solution in the larger space, $\mathbf{R}^d$ of (m, α, β, t_0) on a maximal interval, I' obtained by appending to each $u_i(t)$, $d - d'$ zeroes. By uniqueness, this must be the only solution on that interval: i.e., $\forall k \leq N, t \in I'$.

$$r_k^l(t) = \begin{cases} u_k^l(t), & l \leq d' \\ 0, & d' < l \leq d \end{cases} \quad (3.20)$$

where we recall our notation (remark on p. 27) that r_k^l is the l th spatial coordinate of the position vector of body k with respect to the orthonormal basis chosen above. Furthermore, I' cannot be a proper subset of I because otherwise, u and hence r would have a singularity at some $\tau \in I$ which is not possible. Thus $I' = I$. From (3.20), $\langle \Delta r(t), \Delta \dot{r}(t) \rangle \leq \langle \Delta \alpha, \Delta \beta \rangle$ from which we have $\langle \Delta q(t), \Delta \dot{q}(t) \rangle = \langle \Delta r(t), \Delta \dot{r}(t) \rangle \leq \langle \Delta \alpha, \Delta \beta \rangle = \langle \Delta a, \Delta b \rangle$; i.e., the subspace of the solution does not increase in dimension with t . From (3.19) the affine subspace must not increase also. Since q is also the unique solution of the ivp $(m, q(t), \dot{q}(t), t) \forall t \in I$ and since t_0 can be viewed as just some point in I , it is clear by the same reasoning that $\langle \Delta a, \Delta b \rangle = \langle \Delta q(t_0), \Delta \dot{q}(t_0) \rangle \leq \langle \Delta q(t), \Delta \dot{q}(t) \rangle$; so that the affine subspace does not decrease also; i.e., the affine subspace is a constant. This argument is straightforward for 'nicely' behaved s functions that give unique solutions. However, do there exist s functions, sufficiently pathological and initial values for which there exists a solution, the affine subspace of which increases or decreases with t ? Happily, as the following theorem shows, no such solutions exist:

3.4.3 The General Case

Theorem 8 (Affine Subspace of the Solution Theorem) *The linear subspace and dimension of a solution of the N body problem are constants (i.e., independent of t). The affine subspace of the solution translates linearly with t and the velocity affine subspace is a constant. Specifically, if q is a solution of (m, a, b, t_0) on some interval I in $\mathbf{R}^d$, then $\forall t \in I, \forall x \in \text{aff}(a, \Delta b)$ and $\forall v \in \text{aff}(b, \Delta a)$,*

1. $\langle \Delta q(t), \Delta \dot{q}(t) \rangle = \langle \Delta a, \Delta b \rangle$,
2. $\dim \text{aff}(q(t), \Delta \dot{q}(t)) = \dim \langle \Delta q(t), \Delta \dot{q}(t) \rangle = \dim \langle \Delta a, \Delta b \rangle$,
3. $\text{aff}(q(t), \Delta \dot{q}(t)) = x + (t - t_0)v + \langle \Delta a, \Delta b \rangle$ and
4. $\text{aff}(\dot{q}(t), \Delta q(t)) = v + \langle \Delta a, \Delta b \rangle$.

What is meant by an affine subspace's being a linear function of t is that it is the sum of a fixed subspace and a linear function from $\mathbf{I} \rightarrow \mathbf{R}^d$. It could also be viewed as a linear function from I into a factor space, in this case, $\mathbf{R}^d / \langle \Delta a, \Delta b \rangle$.

Proof of theorem. *Show 1:* Suppose that q is a solution of (m, a, b, t_0) on $I = (t_-, t_+)$. Fix $\gamma \in \langle \Delta a, \Delta b \rangle^\perp = \langle \Delta a \rangle^\perp \cap \langle \Delta b \rangle^\perp$, the orthogonal complement of the linear subspace $\langle \Delta a, \Delta b \rangle$ in $\mathbf{R}^d$. We first show that $\gamma \in \langle \Delta q(t), \Delta \dot{q}(t) \rangle^\perp \forall t \in I$: i.e., $\langle \Delta q(t), \Delta \dot{q}(t) \rangle$ does not increase with t . Let $\tau \in (t_0, t_+)$. $\forall t \in [t_0, \tau]$, $\forall i \leq N$.

$$\begin{aligned} \dot{q}_i(t) &= \dot{q}_i(t_0) + \int_{t_0}^t \ddot{q}_i(u) du \\ &= b_i + \int_{t_0}^t \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}(u)\|) q_{ij}(u) du. \end{aligned}$$

from (3.1) p. 24. Applying this to the case $i = 1$, taking differences and writing q_{ij} as $q_{1j} - q_{1i}$ gives

$$\dot{q}_{1i}(t) = b_{1i} + \int_{t_0}^t \left[\sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}(u)\|) (q_{1j}(u) - q_{1i}(u)) - \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{1j}(u)\|) q_{1j}(u) \right] du$$

Since for each i, j ($i \neq j$), $q_{ij}(t) \neq 0$ on the compact $[t_0, \tau]$, the continuous $s(\|q_{ij}(t)\|)$ as a function of t on $[t_0, \tau]$ is bounded. Since there are finitely many such functions, $\exists K \geq 0$ such that $\forall t \in [t_0, \tau]$, $\forall i, j \leq N$, ($i \neq j$), $|m_j s(\|q_{ij}(t)\|)| \leq K$. Note also that since $\gamma \in \langle \Delta a \rangle^\perp$ and $\langle \Delta b \rangle^\perp$, the inner products $(a_{1i}, \gamma) = (b_{1i}, \gamma) = 0$. So,

$$|(\dot{q}_{1i}(t), \gamma)| \leq K \int_{t_0}^t \left\{ (N-1) |(q_{1i}(u), \gamma)| + 2 \sum_{j=1}^N |(q_{1j}(u), \gamma)| \right\} du$$

Summing over i gives

$$\sum_{i=1}^N |(\dot{q}_{1i}(t), \gamma)| \leq 3NK \int_{t_0}^t \sum_{i=1}^N |(q_{1i}(u), \gamma)| du. \quad (3.21)$$

Also, $q_{1i}(t) = a_{1i} + \int_{t_0}^t \dot{q}_{1i}(u) du$; so,

$$|(q_{1i}(t), \gamma)| \leq \int_{t_0}^t |(\dot{q}_{1i}(u), \gamma)| du \quad (3.22)$$

Now, let

$$y(t) = \sum_{i=1}^N \{|(q_{1i}(t), \gamma)| + |(\dot{q}_{1i}(t), \gamma)|\}$$

From (3.21) and (3.22),

$$0 \leq y(t) \leq \max(1, 3NK) \int_{t_0}^t y(u) du.$$

By Gronwall's inequality (see for example [Hal. p. 36]), $y(t) = 0, \forall t \in [t_0, \tau]$. Since $\tau \in (t_0, t_+)$ is arbitrary, $y(t) = 0, \forall t \in [t_0, t_+)$, i.e. $\forall i \leq N, q_{1i}(t), \dot{q}_{1i}(t) \in \langle \Delta a, \Delta b \rangle$ and $\langle \Delta q(t), \Delta \dot{q}(t) \rangle \subseteq \langle \Delta a, \Delta b \rangle$ for $t \in [t_0, t_+)$. We now simply reverse time to get the same result on $(t_-, t_0]$: Thus $\langle \Delta q(t), \Delta \dot{q}(t) \rangle \subseteq \langle \Delta a, \Delta b \rangle, \forall t \in I$. Conversely, for any $t \in I$, q is a solution of the ivp $(m, q(t), \dot{q}(t), t)$ on I and t_0 is a point in I . Applying the above proof to the case where the roles of t and t_0 are reversed, we see that $\langle \Delta a, \Delta b \rangle \subseteq \langle \Delta q(t), \Delta \dot{q}(t) \rangle$. Thus we have shown that $\forall t \in I, \langle \Delta q(t), \Delta \dot{q}(t) \rangle = \langle \Delta a, \Delta b \rangle$; i.e., the subspace of a solution is a constant.

Show 2: The dimension of a non-empty subspace equals the dimension of its parallel transport to the origin: so, $\dim \text{aff}(q(t), \Delta \dot{q}(t)) = \dim \langle \Delta q(t), \Delta \dot{q}(t) \rangle$. This together with part 1 gives the result for 2.

Show 3: We now show that the affine subspace of a solution is a linear function of t . Fix $x \in \text{aff}(a, \Delta b)$, $v \in \text{aff}(b, \Delta a)$ and $\gamma \in \langle \Delta a, \Delta b \rangle^\perp$. We can write

$$x = \sum_{i=1}^N (\alpha_i a_i + \beta_i b_i) \quad (3.23)$$

where $\sum_{i=1}^N \alpha_i = 1$ and $\sum_{i=1}^N \beta_i = 0$. Then let $\theta : I \rightarrow \mathbf{R}^d$ be given by

$$\theta(t) = \sum_{i=1}^N \alpha_i q_i(t) - x - (t - t_0) v \quad (3.24)$$

and let

$$\Lambda(t) = (\theta(t), \gamma). \quad (3.25)$$

Since $\forall t \in I$, $i, j \leq N$, $q_{ij}(t) \in \langle \Delta a, \Delta b \rangle$ as just shown in part 1.

$$\ddot{\theta}(t) = \sum_{i=1}^N \alpha_i \ddot{q}_i(t) = \sum_{i=1}^N \alpha_i \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}\|) q_{ij}(t) \in \langle \Delta a, \Delta b \rangle:$$

so, $\ddot{\Lambda}(t) = (\ddot{\theta}(t), \gamma) = 0$. Also from (3.23) and (3.24),

$$\theta(t_0) = \sum_{i=1}^N \alpha_i a_i - x = - \sum_{i=1}^N \beta_i b_i \in \langle \Delta b \rangle \subseteq \langle \Delta a, \Delta b \rangle$$

and

$$\dot{\theta}(t_0) = \sum_{i=1}^N \alpha_i b_i - v \in \text{aff}(b) - \text{aff}(b, \Delta a) = \text{aff}(b) - (\text{aff}(b) + \langle \Delta a \rangle) = \langle \Delta a, \Delta b \rangle:$$

so, $\dot{\Lambda}(t_0) = \Lambda(t_0) = 0$ and $\ddot{\Lambda}(t) \equiv 0$ on I : i.e., $\Lambda(t) \equiv 0$ on I . Since γ is an arbitrary member of $\langle \Delta a, \Delta b \rangle^\perp$ from (3.25), $\theta(t) \in \langle \Delta a, \Delta b \rangle$, $\forall t \in I$. Since $\sum_{i=1}^N \alpha_i = 1$, $\sum_{i=1}^N \alpha_i q_i(t) \in \text{aff}(q(t))$: so, from (3.24)

$$\begin{aligned} x + (t - t_0) v &\in \sum_{i=1}^N \alpha_i q_i(t) + \langle \Delta a, \Delta b \rangle \\ &= \text{aff}(q(t)) + \langle \Delta a, \Delta b \rangle \\ &= \text{aff}(q(t)) + \langle \Delta q(t), \Delta \dot{q}(t) \rangle \\ &= \text{aff}(q(t), \Delta \dot{q}(t)). \end{aligned}$$

Since an affine subspace is obtained by adding any one of its members to its parallel transport to the origin,

$$\text{aff}(q(t), \Delta \dot{q}(t)) = x + (t - t_0) v + \langle \Delta a, \Delta b \rangle.$$

Show 4: Similarly, since $\dot{\Lambda}(t) = (\dot{\theta}(t), \gamma) \equiv 0$, $\dot{\theta}(t) \in \langle \Delta a, \Delta b \rangle$ and differentiating (3.24) with respect to t gives

$$v = \sum_{i=1}^N \alpha_i \dot{q}_i(t) - \dot{\theta}(t) \in \sum_{i=1}^N \alpha_i \dot{q}_i(t) + \langle \Delta a, \Delta b \rangle = \text{aff}(\dot{q}(t), \Delta \dot{q}(t)).$$

So,

$$\text{aff}(\dot{q}(t), \Delta \dot{q}(t)) = v + \langle \Delta \dot{q}(t), \Delta \dot{q}(t) \rangle = v + \langle \Delta a, \Delta b \rangle.$$

■

3.4.4 Corollaries

We now state an obvious application to a case without uniqueness of solution:

Corollary 9 *If q and r are solutions of (m, a, b, t_0) for s in $\mathbf{R}^d$, their affine subspaces are the same and their linear subspaces are the same.*

Corollary 10 *If q is a solution of (m, a, b, t_0) on I then the following are equivalent:*

1. $0 \in \text{aff}(q(t), \Delta \dot{q}(t)), \forall t \in I$.
2. $\text{aff}(q(t), \Delta \dot{q}(t)) = \langle \Delta a, \Delta b \rangle, \forall t \in I$ and
3. $0 \in \text{aff}(a, \Delta b)$ and $0 \in \text{aff}(b, \Delta a)$.

Proof. *Show 1 $\Rightarrow$ 2:* Fix $t \in I$. Since by hypothesis, $0 \in \text{aff}(q(t), \Delta \dot{q}(t))$, from L. 81, p. 208, App. A, $\text{aff}(q(t), \Delta \dot{q}(t)) = \langle \Delta q(t), \Delta \dot{q}(t) \rangle = \langle \Delta a, \Delta b \rangle$ by the Plane of the Solution Theorem.

Show 2 $\Rightarrow$ 1: $0 \in \langle \Delta a, \Delta b \rangle = \text{aff}(q(t), \Delta \dot{q}(t)), \forall t \in I$.

Show (1 and 2) $\Rightarrow$ 3: Assume 1 and 2. Then by 1 for $t = t_0$, $0 \in \text{aff}(q(t_0), \Delta \dot{q}(t_0)) = \text{aff}(a, \Delta b)$. To show that $0 \in \text{aff}(b, \Delta a)$, consider any $t \in I$. $\forall x \in \text{aff}(a, \Delta b)$ and $\forall v \in \text{aff}(b, \Delta a)$,

$$\begin{aligned} \langle \Delta a, \Delta b \rangle &= \text{aff}(q(t), \Delta \dot{q}(t)) && \text{(from part 2)} \\ &= x + (t - t_0)v + \langle \Delta a, \Delta b \rangle && \text{(from part 3 of the above theorem).} \end{aligned}$$

In particular, as just shown we can choose $x = 0 \in \text{aff}(a, \Delta b)$ and $t \neq t_0$ from which we have $v \in \langle \Delta a, \Delta b \rangle$ and by L. 81, p. 208, App. A, $\text{aff}(b, \Delta a) = v + \langle \Delta a, \Delta b \rangle = \langle \Delta a, \Delta b \rangle$; i.e., $0 \in \text{aff}(b, \Delta a)$.

Show 3 $\Rightarrow$ 2: Again by Part 3 of the Plane of the Solution Theorem, we choose $x = 0 \in \text{aff}(a, \Delta b)$ and $v = 0 \in \text{aff}(b, \Delta a)$ to get $\text{aff}(q(t), \Delta \dot{q}(t)) = \langle \Delta a, \Delta b \rangle$. ■

In the light of the above theorem and corollary, we give the following:

Definition 7 *If q is a solution of the N body problem on I then we call $\langle \Delta q(t), \Delta \dot{q}(t) \rangle$, the **linear subspace of the solution** since it is independent of the choice of t . If 0 lies in the affine subspace of the solution $\forall t \in I$; i.e., $0 \in \text{aff}(q(t), \Delta \dot{q}(t))$, we say that the ivp is in **affine subspace of the solution coordinates** or simply **affine coordinates**.*

The reader may notice that the affine subspace of the solution is independent of the s function. Note also that there always exists some affine coordinate system (actually uncountably many for $N \geq 2$) for which the affine subspace passes through 0 for all $t \in I$.

Corollary 11 *The dimension of a solution is at most $2N - 2$.*

Proof. $\langle \Delta a, \Delta b \rangle$ is spanned by the vectors

$$\{q_{1j}(t_0) \mid j = 2, \dots, N\} \cup \{\dot{q}_{1j}(t_0) \mid j = 2, \dots, N\}.$$

■

Chapter 4

The Group of an Initial Value Problem

Before developing the group of an N body problem, we shall prepare the way with an example.

Notation. Let $\mathcal{E}(d)$ and $\mathcal{O}(d)$ be the group of isometries and the group of orthogonal transformations on $\mathbf{R}^d$, respectively. When the dimension, d is clear we simply write $\mathcal{E}$ and $\mathcal{O}$ respectively.

Example 5 Consider $\mathbf{R}^2$ with a body of unit mass at each of the N vertices of a regular polygon on the unit circle for $N \geq 2$, as shown in Figure 4-1 for the case $N = 5$. Suppose that the s function is locally lipschitzian so that the solution is unique on some interval (Thm. 3 p. 28). Let $H \in \mathcal{O}(2)$ be the counterclockwise rotation through the angle $2\pi/N$; i.e.,

$$H = \begin{pmatrix} \cos 2\pi/N & -\sin 2\pi/N \\ \sin 2\pi/N & \cos 2\pi/N \end{pmatrix}. \quad (4.1)$$

Let

$$a_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } b_0 = \begin{pmatrix} v \\ \omega \end{pmatrix}$$

Figure 4-1: Regular Polygon for Example 5

and let the initial position and velocity of body k be $a_k = H^k a_0$ and $b_k = H^k b_0$, respectively. Since $H^N = I_2$, the identity operator on $\mathbf{R}^2$, $a_N = a_0$ and $b_N = b_0$. Since $\sum_{k=1}^N H^k = 0_2$, the zero operator on $\mathbf{R}^2$ and the masses are all equal, $\sum_{k=1}^N m_k a_k = \sum_{k=1}^N H^k a_0 = 0$ and $\sum_{k=1}^N m_k b_k = \sum_{k=1}^N H^k b_0 = 0$; so, the ivp is barycentric. The radial component of the initial velocity, b_k is the inner product,

$$\left(b_k, \frac{a_k}{\|a_k\|} \right) = (H^k b_0, H^k a_0) = (b_0, a_0) = r.$$

If we let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the $\pi/2$ counterclockwise rotation, then since rotations in $\mathbf{R}^2$ commute, $HJ = JH$ and the (counterclockwise) tangential component of b_k

is

$$\left(b_k, J \frac{a_k}{\|a_k\|}\right) = (H^k b_0, J H^k a_0) = (H^k b_0, H^k J a_0) = (b_0, J a_0) = \omega.$$

The reader may anticipate correctly that this ‘pinwheel’ symmetry is retained over the solution. The formalization of this notion for this and less elementary examples is a major objective of this thesis.

4.1 Permutations

In the N body problem one is given a *set* of bodies to which an arbitrary ordering is applied to get a mass vector and position and velocity matrices. For example, the two body problem (m, a, b, t_0) models the same physical system as $(\hat{m}, \hat{a}, \hat{b}, t_0)$ where $\hat{m} = (m_2, m_1)'$, $\hat{a} = (a_2, a_1)'$ and $\hat{b} = (b_2, b_1)'$. Ordinarily, one is not concerned with such elementary concepts; however, we need to formalize this notion to develop the group of the N body problem. As usual, we denote S_N as the permutation group on the set $\{1, 2, \dots, N\}$, for $(N \geq 1)$. Let us denote the standard basis for $\mathbf{R}^N$ by $\{e_1, \dots, e_N\}$. $\forall \pi \in S_N$, consider the linear operator on $\mathbf{R}^N$, $\pi' : \mathbf{R}^N \rightarrow \mathbf{R}^N$, given by $\pi'(e_i) = e_{\pi(i)}$. Then $(\pi^{-1})' \pi'(e_i) = (\pi^{-1})'(e_{\pi(i)}) = e_{\pi^{-1}\pi(i)} = e_i$. Since a linear operator is determined by its effect on a basis, $(\pi^{-1})' \pi' = I_N$, the identity operator on $\mathbf{R}^N$. Therefore, $(\pi')^{-1} = (\pi^{-1})'$. We shall identify π' with its matrix relative to the standard basis. The j th column vector of π' is $\pi'(e_j) = e_{\pi(j)}$, which has a 1 in row $\pi(j)$ and a zero in all other rows. Therefore, entry (i, j) in the matrix π' is $\pi'_{ij} = \delta_{i, \pi(j)}$ where $\delta_{ij} = 1$ if $i = j$ and 0 if not (Kronecker delta). Also, $(\pi'^t \pi')_{ij} = e_i^t \pi'^t \pi' e_j = e_{\pi(i)}^t e_{\pi(j)} = \delta_{\pi(i)\pi(j)} = \delta_{ij}$ because π is bijective. Therefore, $\pi'^t \pi' = I_N$; so, π' is orthogonal. Also, if $\pi, \sigma \in S_N$, $\sigma' \pi'(e_i) = \sigma'(e_{\pi(i)}) = e_{\sigma\pi(i)} = (\sigma \circ \pi)'(e_i)$, true $\forall i = 1, \dots, N$; so, $\sigma' \pi' = (\sigma \circ \pi)'$. Finally, if $\pi, \sigma \in S_N$ and $\sigma' = \pi'$ then $\forall i \leq N$, $e_{\sigma(i)} = \sigma'(e_i) = \pi'(e_i) = e_{\pi(i)}$. Therefore, $\sigma(i) = \pi(i)$; so, $\sigma = \pi$. Thus we have shown:

Lemma 12 *The map $\psi : S_N \rightarrow \mathcal{O}(N)$, given by $\psi(\pi)(e_i) = e_{\pi(i)}$ is a group monomorphism and we shall denote the matrix of $\psi(\pi)$ relative to the standard basis by π' and call it **the permutation matrix for π** .*

$$\forall \pi \in S_N, y \in \mathbf{R}^N, y = (y_1, \dots, y_N)^t = \sum_{i=1}^N y_i e_i.$$

$$\pi' y = \sum_{i=1}^N y_i \pi' e_i = \sum_{i=1}^N y_i e_{\pi(i)} = \sum_{i'=1}^N y_{\pi^{-1}(i')} e_{i'}$$

where in the last step we substituted $i' = \pi(i)$ and used the fact that π is bijective to sum over i' . Therefore

$$\pi' y = (y_{\pi^{-1}(1)}, \dots, y_{\pi^{-1}(N)})^t$$

or $(\pi' y)_i = y_{\pi^{-1}(i)}$. Thus π' permutes the entries in y by moving the number in row $\pi^{-1}(i)$ to row i . Equivalently, the number in row i goes to row $\pi(i)$.

Definition 8 *We now define the mapping, $\phi : \mathcal{O}(N) \rightarrow \mathcal{O}(dN)$ given by*

$$\phi(A) = \begin{bmatrix} a_{11}I_d & \cdots & a_{1N}I_d \\ \vdots & & \vdots \\ a_{N1}I_d & \cdots & a_{NN}I_d \end{bmatrix}$$

where $A \in \mathcal{O}(N)$. If $\pi \in S_N$, for brevity we define $\pi'' = \phi(\pi') = \phi \circ \psi(\pi)$.

Note that the ij th $d \times d$ block in the partition of $\phi(A)^t \phi(A)$ is

$$\sum_{k=1}^N (a_{ki}I_d)(a_{kj}I_d) = \delta_{ij}I_d.$$

Thus, $\phi(A)$ is also orthogonal and ϕ does map $\mathcal{O}(N)$ into $\mathcal{O}(dN)$ as claimed. It is clear that ϕ is 1-1. Also, if $A, B \in \mathcal{O}(N)$, the ij th $d \times d$ block of the matrix $\phi(A)\phi(B)$ is $\sum_{k=1}^N a_{ik}b_{kj}I_d$. Thus, $\phi(AB) = \phi(A)\phi(B)$ and we have shown

Lemma 13 *The mapping $\phi : \mathcal{O}(N) \rightarrow \mathcal{O}(dN)$ given in Definition 8 is a group monomorphism.*

Then $\forall x \in \mathbf{R}^{dN}$ with the usual partition, $x = (x_1, \dots, x_N)^t$, $x_i \in \mathbf{R}^d$, given by (3.2) on page 25 and $\forall \pi \in S_N$, as shown above with scalars, we have

$$(\pi''x)_i = \sum_{j=1}^N \pi'_{ij} x_j = \sum_{j=1}^N \delta_{i,\pi(j)} x_j = \sum_{j=1}^N \delta_{\pi^{-1}(i),j} x_j = x_{\pi^{-1}(i)},$$

x_i is moved from the i th $d \times 1$ block of x to the $\pi(i)$ th block and

$$\pi''(x) = \phi(\pi')(x) = \begin{pmatrix} \overline{x_{\pi^{-1}(1)}} \\ \overline{x_{\pi^{-1}(2)}} \\ \vdots \\ \overline{x_{\pi^{-1}(N)}} \end{pmatrix}.$$

In Example 5, p. 44 for the case $N = 5$ and the permutation $\pi = (12345)$,

$$\pi' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \pi'' = \begin{bmatrix} 0_2 & 0_2 & 0_2 & 0_2 & I_2 \\ I_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & I_2 & 0_2 \end{bmatrix}$$

and the permuted initial position vector, $\pi''a = (a_5, a_1, a_2, a_3, a_4)^t$.

Theorem 14 *If $q \in C^2(I, \mathbf{R}^{dN})$ where I is an open interval of $\mathbf{R}$, $a \in \mathbf{R}^{dN} \setminus \Delta$, $b \in \mathbf{R}^{dN}$ and $\pi \in S_N$ then q is a solution of (m, a, b, t_0) on I if and only if $\pi''q$ is a solution of $(\pi'm, \pi''a, \pi''b, t_0)$ on I .*

Proof. Suppose that q is a solution of (m, a, b, t_0) on I . Let $r = \pi''q$. Clearly, $r(t_0) = \pi''q(t_0) = \pi''a$ and $\dot{r}(t_0) = \pi''\dot{q}(t_0) = \pi''b$. Thus, r satisfies the initial conditions for $(\pi'm, \pi''a, \pi''b, t_0)$. Also, $\forall t \in I$,

$$\frac{d^2}{dt^2} (\pi''q)_i = \ddot{q}_{\pi^{-1}(i)} = \sum_{\substack{j=1 \\ j \neq \pi^{-1}(i)}}^N m_j s(\|q_{\pi^{-1}(i)j}\|) q_{\pi^{-1}(i)j}$$

Let $j' = \pi(j)$. Then since π is a bijection,

$$\begin{aligned} \frac{d^2}{dt^2} (\pi'' q)_i &= \sum_{\substack{j'=1 \\ j' \neq i}}^N m_{\pi^{-1}(j')} s \left(\|q_{\pi^{-1}(i)\pi^{-1}(j')}\| \right) q_{\pi^{-1}(i)\pi^{-1}(j')} \\ &= \sum_{\substack{j'=1 \\ j' \neq i}}^N (\pi' m)_{j'} s \left(\|(\pi'' q)_{ij'}\| \right) (\pi'' q)_{ij'}. \end{aligned}$$

Thus,

$$\ddot{r}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^N (\pi' m)_j s(\|r_{ij}\|) r_{ij}$$

and r_i satisfies (3.1) on page 24 with m_j replaced by $(\pi m)_j$, i.e., r is a solution of $(\pi' m, \pi'' a, \pi'' b, t_0)$ on I . The converse is obtained by replacing q, π, m, a and b with $\pi'' q, \pi^{-1}, \pi' m, \pi'' a$ and $\pi'' b$ respectively and applying the above result to that case. ■

Theorem 14 says that relabelling the bodies gives essentially the same dynamical system and is a formalization of our comments given at the beginning of this section.

4.2 Isometries

Definition 9 Let $\chi : \mathcal{E}(d) \rightarrow \mathcal{E}(dN)$ be given by $[\chi(f)(x)]_i = f(x_i)$ where $f \in \mathcal{E}(d)$ and $x \in \mathbf{R}^{dN}$. For brevity, let $f' = \chi(f)$.

Thus $(f'(x))_i = f(x_i)$ and

$$f'(x) = \begin{pmatrix} \overline{f(x_1)} \\ \overline{f(x_2)} \\ \vdots \\ \overline{f(x_N)} \end{pmatrix}.$$

Clearly, $(fg)' = f'g'$ and $\chi(I_d) = I_{dN}$; i.e., the identity $\mathcal{E}(d)$ goes to the identity in $\mathcal{E}(dN)$. Also, if $\chi(f) = \chi(g)$, $f(x_1) = g(x_1)$, $\forall x_1 \in \mathbf{R}^d$; so, $f = g$ and we have shown:

Lemma 15 *The map, χ is a group monomorphism.*

We now show

Lemma 16 *For any $\pi \in S_N$ and $f \in \mathcal{E}(d)$, $\phi \circ \psi(\pi) \equiv \pi''$ commutes with $\chi(f) \equiv f'$ in $\mathcal{E}(dN)$.*

Proof. For any $x \in \mathbf{R}^{dN}$, $[\pi'' \circ f'(x)]_i = (f'(x))_{\pi^{-1}(i)} = f(x_{\pi^{-1}(i)})$ whereas $[f' \circ \pi''(x)]_i = f(\pi''(x)_i) = f(x_{\pi^{-1}(i)})$ as desired. ■

Note that if $H \in \mathcal{O}(d)$, H is an isometry and identifying H and H' with their matrices relative to the standard basis gives

$$H' = \chi(H) = H \oplus \cdots \oplus H = \begin{bmatrix} H & 0_d & \cdots & 0_d \\ 0_d & H & \cdots & 0_d \\ \vdots & \vdots & \ddots & \vdots \\ 0_d & 0_d & \cdots & H \end{bmatrix} \quad (4.2)$$

where 0_d is the $d \times d$ zero matrix.

In Example 5 on page 44 for $N = 5$ and H given in (4.1) p. 44, since $Ha_k = a_{k+1}$, $\chi(H)(a) = H'a = (a_2, a_3, a_4, a_5, a_1)^t = (\pi^{-1})'' a$ for $\pi = (12345)$.

Lemma 17 $\forall H \in \mathcal{O}(d)$, q is a solution of (m, a, b, t_0) on some interval I if and only if $H'q$ is a solution of $(m, H'a, H'b, t_0)$ on I .

Proof. Suppose that q is a solution of (m, a, b, t_0) on I . Clearly, $H'q$ satisfies the initial conditions for $(m, H'a, H'b, t_0)$. Since H is an isometry, $\|Hq_{ij}\| = \|q_{ij}\|$, and

$$\frac{d^2}{dt^2}(H'q)_i = H \ddot{q}_i = H \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|q_{ij}\|) q_{ij}$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^N m_j s(\|Hq_{ij}\|) Hq_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^N m_j s\left(\|(H'q)_{ij}\|\right) (H'q)_{ij}$$

and $H'q$ is a solution of $(m, H'a, H'b, t_0)$.

Sufficiency follows from the invertibility of H' and the proof of necessity, above. ■

The following theorem is more general but along the same lines as [Mar. Eqns 861-864, p. 303]. Recalling Definition 5 on Page 32 for Galilean coordinate transformations and the translation map $T_\gamma(y) = y - \gamma$ from Appendix A we have the following:

Theorem 18 *Given two isometries on $\mathbf{R}^d$, with the same orthogonal part: say, $f = T_\gamma H$ and $g = T_\delta H$, where $H \in \mathcal{O}(d)$, $\delta, \gamma \in \mathbf{R}^d$ and $q \in C^2(I, \mathbf{R}^{dN})$ with I , an open interval of $\mathbf{R}$, then q is a solution of (m, a, b, t_0) on I if and only if $\mathcal{G}_{\gamma\delta}(H'q)$ is a solution of $\mathcal{G}_{\gamma\delta}(m, H'a, H'b, t_0) = (m, f'a, g'b, t_0)$ on I .*

Proof.

The function q is a solution of (m, a, b, t_0) on I

$\Leftrightarrow H'q$ is a solution of $(m, H'a, H'b, t_0)$ on I (by Lemma 17)

$\Leftrightarrow \mathcal{G}_{\gamma\delta}(H'q)$ is a solution of $\mathcal{G}_{\gamma\delta}(m, H'a, H'b, t_0)$ on I (by Lemma 7 on p. 33). ■

Example 6 *Let us generalize our polygon Example 5 on Page 44 to N bodies forming the vertices of a regular N -gon. Let us also extend it so that the N -gon lies in the xy plane in $\mathbf{R}^3$ by placing a zero in the 3rd coordinate of each a_i and b_i . By choosing Newtonian mechanics ($s(r) = 1/r^3$) and letting the radial component of the initial velocity, $v = 0$, as the reader can check, setting $\omega = \left[\frac{1}{4N} \sum_{k=1}^{N-1} 1/(\sin \pi k/N) \right]^{1/2}$ gives the relative equilibrium solution*

$$q_k(t) = \begin{pmatrix} \cos(\omega t + 2\pi k/N) \\ \sin(\omega t + 2\pi k/N) \\ 0 \end{pmatrix}$$

$\forall t \in \mathbf{R}$ for the ivp $(m, a, b, 0)$. Let $\gamma = (0, 0, 0)^t$ and $\delta = (0, 0, 1)^t$. Now let

$$K = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the above theorem the solution of $\mathcal{G}_{\gamma, \delta}(m, K'a, K'b, 0)$ is $r = \mathcal{G}_{\gamma, \delta}(K'q)$: i.e.,

$$r_k(t) = \begin{pmatrix} \cos(\omega t + \theta + 2\pi k/N) \\ \sin(\omega t + \theta + 2\pi k/N) \\ t \end{pmatrix}.$$

The solution, r has the polygon initially rotated by the angle θ and spiralling up the z axis. If $N = 2$ and $\omega = 0$, the subspace of the solution r is the line $\{(u \cos \theta, u \sin \theta, 0) \mid u \in \mathbf{R}\}$ and the affine subspace of the solution is the line,

$$\text{aff}(r(t), \Delta \dot{r}(t)) = \{(u \cos \theta, u \sin \theta, t) \mid u \in \mathbf{R}\}.$$

Otherwise (for $N \geq 3$ or $\omega \neq 0$), the affine subspace of the solution at time t , is the plane, $z = t$; i.e., $\text{aff}(r(t), \Delta \dot{r}(t)) = \{(x, y, t) \mid x, y \in \mathbf{R}\}$ and the subspace for both q and r is the xy plane.

4.3 Isometries that Preserve the IVP

Now we shall develop the group of an initial value problem.

Definition 10 Given $a \in D = \mathbf{R}^{dN} \setminus \Delta$, $b \in \mathbf{R}^{dN}$, $\pi \in S_N$ and isometries $f, g \in \mathcal{E}(d)$ with the same orthogonal part, such that

$$\begin{aligned} \pi' m &= m \\ (\pi^{-1})'' a &= f' a \\ (\pi^{-1})'' b &= g' b \end{aligned}$$

then the triple (π, f, g) is said to **preserve the initial value problem** (m, a, b, t_0) , π is called the **permutation** for the triple and f and g are the respective **position** and **velocity isometries** for the triple.

The last two equations in the above definition are equivalent to $a_{\pi(i)} = f(a_i)$ and $b_{\pi(i)} = g(b_i)$, $\forall i = 1, \dots, N$ or that the isometry f maps the position vector for body i to the position vector for body $\pi(i)$. Similarly, the velocity vector of body i is mapped to that of body $\pi(i)$ by g , and bodies i and $\pi(i)$ have the same mass.

Theorem 19 *The set of all triples that preserve (m, a, b, t_0) forms a subgroup of the direct product group, $S_N \times \mathcal{E}(d) \times \mathcal{E}(d)$.*

Proof. Suppose that (π_1, f_1, g_1) and (π_2, f_2, g_2) preserve (m, a, b, t_0) . Then

$$(\pi_1 \pi_2)' m = \pi_1' \pi_2' m = \pi_1' m = m$$

by L. 12 p. 47. Furthermore,

$$\begin{aligned} [(\pi_1 \circ \pi_2)^{-1}]''(a) &= (\pi_2^{-1} \circ \pi_1^{-1})''(a) && (S_N \text{ is a group}) \\ &= (\pi_2^{-1})'' \circ (\pi_1^{-1})''(a) && (\text{group homomorphism } \phi \circ \psi) \\ &= (\pi_2^{-1})'' \circ f_1'(a) && ((\pi_1, f_1, g_1) \text{ preserves the ivp}) \\ &= f_1' \circ (\pi_2^{-1})''(a) && (\text{by Lemma 16 p. 50}) \\ &= f_1' \circ f_2''(a) && ((\pi_2, f_2, g_2) \text{ preserves the ivp}) \\ &= (f_1 \circ f_2)'(a) && (\text{by Lemma 15 p. 50}). \end{aligned}$$

Similarly, $[(\pi_1 \pi_2)^{-1}]''(b) = (g_1 g_2)'(b)$. Also f_1 and g_1 must have the same orthogonal part and f_2 and g_2 must have the same orthogonal part: so, by Corollary 72 on page 202 of Appendix A, $f_1 f_2$ and $g_1 g_2$ have the same orthogonal part. So, if (π_1, f_1, g_1) and (π_2, f_2, g_2) preserve (m, a, b, t_0) , then so too does $(\pi_1 \pi_2, f_1 f_2, g_1 g_2)$.

Also, if (π, f, g) preserves (m, a, b, t_0) then $\pi' m = m$ implies $m = (\pi^{-1})' m$. $(\pi^{-1})'' a = f' a$ implies, using commutativity of π'' and f' that $f' \pi'' a = \pi'' f' a =$

$\pi'' (\pi^{-1})'' a = a$; so, $\pi'' a = (f^{-1})' a$. Similarly $\pi'' b = (g^{-1})' b$. If $f = T_r H$, $f^{-1} = H^{-1} T_{-r} = T_{-H^{-1}r} H^{-1}$; so, the orthogonal parts of the inverses of f and g are the inverses of their orthogonal parts; i.e., f^{-1} and g^{-1} have the same orthogonal parts, too. Thus $(\pi^{-1}, f^{-1}, g^{-1})$ preserves (m, a, b, t_0) also. Finally, $(1_{S_N}, I_d, I_d)$ preserves any (m, a, b, t_0) where 1_{S_N} is the identity permutation and I_d is the identity map on $\mathbf{R}^d$. ■

4.4 Definition of the Group

Definition 11 We call the set of all triples that preserve (m, a, b, t_0) , **the group of the initial value problem**, denoted $G(m, a, b) \leq S_N \times \mathcal{E}(d) \times \mathcal{E}(d)$. Furthermore, if q is a solution of (m, a, b, t_0) on I and $t \in I$ then q is also a solution of the ivp $(m, q(t), \dot{q}(t), t)$ whose group is called **the group of the solution q at time t** ; i.e., $G(m, q(t), \dot{q}(t))$.

Given $(\rho, u, v) \in S_N \times \mathcal{E}(d) \times \mathcal{E}(d)$, note that the map $S_N \times \mathcal{E}(d) \times \mathcal{E}(d) \rightarrow S_N \times \mathcal{E}(d) \times \mathcal{E}(d)$ given by $(\pi, f, g) \mapsto (\rho, u, v) (\pi, f, g) (\rho, u, v)^{-1} = (\rho \pi \rho^{-1}, u f u^{-1}, v g v^{-1})$ is an inner automorphism.

Theorem 20 Given the N -body problem (m, a, b, t_0) and $a_0, b_0 \in \mathbf{R}^d$.

$$(1_{S_N}, T_{a_0}, T_{b_0}) G(m, a, b) (1_{S_N}, T_{-a_0}, T_{-b_0}) = G(m, a - a'_0, b - b'_0)$$

i.e., this inner automorphism carries the group of the ivp (m, a, b, t_0) to the group of its Galilean coordinate transformation $\mathcal{G}_{a_0 b_0}(m, a, b, t_0)$.

Proof. If we let $x = a - a'_0 \in \mathbf{R}^d \setminus \Delta$ and $y = b - b'_0 \in \mathbf{R}^d$, then, from Def. 5, p. 32, Ch. 3, $\forall f \in \mathcal{E}(d)$, $\pi \in S_N$, $i \leq N$, $T_{a_0} f T_{-a_0}(x_i) = T_{a_0} f(a_i - a_0 + a_0) = f(a_i) - a_0$.

$$\begin{aligned} \therefore T_{a_0} f T_{-a_0}(x_i) = x_{\pi(i)} &\Leftrightarrow f(a_i) - a_0 = a_{\pi(i)} - a_0 \\ &\Leftrightarrow f(a_i) = a_{\pi(i)}. \\ \therefore (\pi^{-1})'' x = (T_{a_0} f T_{-a_0})'(x) &\Leftrightarrow (\pi^{-1})'' a = f'(a) \end{aligned}$$

Similarly, $\forall g \in \mathcal{E}(d)$, $(\pi^{-1})'' y = (T_{b_0} g T_{-b_0})' (y) \Leftrightarrow (\pi^{-1})'' b = g' (b)$. Since the masses are the same for the two problems,

$$(\pi, f, g) \in G(m, a, b) \Leftrightarrow (\pi, T_{a_0} f T_{-a_0}, T_{b_0} g T_{-b_0}) \in G(m, x, y) = G(m, a - a'_0, b - b'_0)$$

and the result follows from

$$(\pi, T_{a_0} f T_{-a_0}, T_{b_0} g T_{-b_0}) = (1_{S_N}, T_{a_0}, T_{b_0}) (\pi, f, g) (1_{S_N}, T_{-a_0}, T_{-b_0}).$$

■

In Example 5 on Page 44, all bodies have equal mass: so, any $\pi \in S_N$ gives $\pi' m = m$. Since $H a_k = a_{k+1}$ and $H b_k = b_{k+1}$, $k = 0, \dots, N-1$, $(a_0 = a_N, b_0 = b_N)$ then for $\pi = (1, 2, \dots, N-1)$, $(\pi^{-1})' a_k = a_{\pi(k)} = H a_k$ or $(\pi^{-1})'' a = H' a$ and $(\pi^{-1})'' b = H' b$. Thus the triple, (π, H, H) preserves the ivp and clearly generates a cyclic group of order N . Is this the group? In $\mathbf{R}^2$ the symmetry group of our regular N -gon is isomorphic to $\mathcal{H}_2^N$, the dihedral group of order $2N$ generated by H and the reflection through the x axis. If we let $\sigma = (1, N-1)(2, N-2) \dots ([\frac{(N-1)}{2}], [\frac{(N+1)}{2}])$ where $[x]$ is the largest integer $\leq x$, (e.g., for $N = 5$, $\sigma = (1, 4)(2, 3)$) and if we let $R = \text{diag}(1, -1)$, (reflection through the x axis), then $(\sigma^{-1})'' a = R' a$. Note that RH^k is a reflection: so, $(RH^k)^2 = I_2$ and $H^k R H^k = R$. Then

$$\begin{aligned} (\sigma, R, R) \in G(m, a, b) &\Leftrightarrow b_{\sigma(k)} = R b_k, & \forall k \\ &\Leftrightarrow b_{N-k} = R b_k, & \forall k \\ &\Leftrightarrow H^{-k} b_0 = R H^k b_0, & \forall k \\ &\Leftrightarrow b_0 = (H^k R H^k) b_0, & \forall k \\ &\Leftrightarrow R b_0 = b_0 \\ &\Leftrightarrow \omega = 0. \end{aligned}$$

We conclude that if $\omega \neq 0$, $G(m, a, b) = \langle (\pi, H, H) \rangle \approx \mathcal{C}_N$, the cyclic group of order N . Otherwise, if $\omega = 0$, $G(m, a, b) = \langle (\pi, H, H), (\sigma, R, R) \rangle \approx \mathcal{H}_2^N$, the dihedral group of order, $2N$.

Extending the problem to $\mathbf{R}^3$ as was done in Example 6, gives for $\omega \neq 0$, $G(m, a, b) = \langle (\pi, H, H) \cdot (1_{S_N}, S, S) \rangle$ where S is the reflection through the xy plane ($S = \text{diag}(1, 1, -1)$). If $\omega = 0$ and $N = 2$, the affine subspace of the solution is the x axis and $G(m, a, b) = \langle (\pi, K, K) \rangle \times \text{stab}_{\mathcal{E}(d)} X$ where $K = \text{diag}(-1, 1, 1)$ is the reflection through the xy plane, $X = \{(x, 0, 0) | x \in \mathbf{R}\}$, the x axis; i.e., $\text{stab}_{\mathcal{E}(d)} X$ is the subgroup of $\mathcal{O}(3)$ that leaves the x axis fixed; i.e., a group isomorphic to $\mathcal{O}(2)$.

4.5 The Next Step

We now summarize where we shall proceed from here. A goal of this thesis is to find conditions under which the group of a solution is independent of t . When this happens the geometrical symmetry is preserved throughout the solution and it enables one to greatly reduce the dimension of the phase space. The bodies are partitioned by the permutation part of the group into 'orbits' (not to be confused with the orbit of a body as it moves through space over time). If the group is preserved over the whole solution, these orbits are preserved also and we need only solve for one 'seed' body in each orbit. Furthermore, the bodies are confined to move within a subspace of dimension potentially smaller than d , the dimension of the underlying physical space. These two factors can combine to greatly reduce the dimension of the phase space. This can in some cases (as we shall see) enable one to study problems that without knowledge of the symmetry would appear too complicated to consider.

We first provide a decomposition of the group into an 'essential' and an 'inessential' part. The former gives the non-trivial permutations of the bodies and the latter maps each position and velocity vector to itself. We shall also determine the fixed points for the groups of position and velocity isometries. This will enable us to perform a Galilean coordinate transformation that makes these position and velocity groups both orthogonal and identical to each other.

First we prepare the way in App. B by developing some theory in geometry that will make our work easier. Part of App. B contains material that can be found in geometry texts and it establishes some of the notation used in this work. However, the author has been unable to find some of the theory in the literature, in particular the ‘Structure Theorem for the Set-wise Stabilizer’ (Thm. 118 p. 227 App. B). Thus, we suggest that the reader will find the following chapter much easier having become familiar with App. B.

Chapter 5

Structure of the Group

In Appendix B we established some theory in geometry which we now apply to the group of an initial value problem. This chapter will be easier to follow if the reader first becomes familiar with the material in that appendix. A major result of App. B is the Structure Theorem for the Set-wise Stabilizer (Thm. 118, p. 227). Given a non-empty set, $A \subseteq \mathbf{R}^d$, consider $\mathcal{E}_{\{A\}}$, the set-wise stabilizer of A or what some authors call the complete symmetry group of A . This theorem shows that $\mathcal{E}_{\{A\}}$ is an internal direct product of two groups, the first being the essential part denoted, $\mathcal{E}_{\{A\}}^{\text{ess}}$ and the second group, the inessential part which is just the point-wise stabilizer, $\mathcal{E}_{(A)}$.

In this chapter we apply the structure theorem to the ivp (m, a, b, t_0) for an arbitrary s function. The result is to characterize the group of the ivp, $G(m, a, b)$ as an internal direct product of two groups, again an essential part and an inessential part. The inessential part is the group of all triples in $G(m, a, b)$ that map the initial position and velocity of each body to itself. The essential part, which we denote $E(m, a, b)$ is a finite subgroup of $S_N \times \mathcal{E} \times \mathcal{E}$ which is diagonal in the sense that the projection onto each of its 3 coordinates is a monomorphism. The finiteness of the position and velocity groups of $E(m, a, b)$ ensures the existence of a Galilean

coordinate frame in which these groups fix the origin: i.e., they are orthogonal groups and hence identical to each other. We then characterize all such coordinate frames as being what we call 'centroidal coordinates'.

In Section 4.4, p. 54, Ch. 4, we defined the group, $G(m, a, b)$ of an ivp (m, a, b, t_0) . Also, if q is a solution of this ivp on I and $t \in I$ we defined the group of the solution q at time t as $G(m, q(t), \dot{q}(t))$. This recognizes the elementary fact that q is also as solution of the ivp $(m, q(t), \dot{q}(t), t)$ on I . In this chapter, we develop theory about the group of the solution at a given time; however, we phrase it throughout as if $t = t_0$ in which case we refer to the ivp, $(m, q(t_0), \dot{q}(t_0), t_0) = (m, a, b, t_0)$. It is clearer and less cumbersome; however, it is elementary that such theory applies to the ivp $(m, q(t), \dot{q}(t), t)$ also.

We first establish some notation.

Notation: In what follows, most of the reference to Euclidean groups will be to $\mathcal{E}(d)$ where d is a fixed dimension. As was done in App. B, we shall simply write $\mathcal{E}$ and $\mathcal{T}$ for $\mathcal{E}(d)$ and $\mathcal{T}(d)$ respectively. Recall that in Section 3.4, p. 35, for any $x = (x_1, \dots, x_N)^t \in \mathbf{R}^{dN}$, we define $\text{aff } x \equiv \text{aff } \{x_1, \dots, x_N\}$. Loosely speaking, this notation associates the vector x with its set of components, $\{x_1, \dots, x_N\}$ under the partition given in (3.2), p. 25. We continue with this theme in the following:

Definition 12 *If $x \in \mathbf{R}^{dN}$ and $X = \{x_1, \dots, x_N\}$ is its decomposition given in (3.2), p. 25, we define $\mathcal{E}_{\{x\}} = \mathcal{E}_{\{X\}}$, $\mathcal{E}_{\{x\}}^{\text{ess}} = \mathcal{E}_{\{X\}}^{\text{ess}}$ and $\mathcal{E}_{(x)} = \mathcal{E}_{(X)}$ which we may also call $\text{stab}_{\mathcal{E}} x$. If $G = G_1 \times \dots \times G_k$ is a direct product of groups, for $i \leq k$, define the group homomorphism, $\Pi_i : G \rightarrow G_i$ by $\Pi_i(g_1, \dots, g_k) = g_i$ (projection onto the i th coordinate). If $H \leq G$, then $\Pi_i|_H$ is also a group homomorphism (by the correspondence theorem for groups; see e.g., [BJN, p.98]). In particular, given the ivp (m, a, b, t_0) for N bodies and the group $G(m, a, b) \leq S_N \times \mathcal{E} \times \mathcal{E}$, we define the homomorphisms,*

$$\xi_N = \Pi_1|_{G(m,a,b)},$$

$$\xi_a = \Pi_2|_{G(m,a,b)},$$

$$\xi_b = \Pi_3|_{G(m,a,b)};$$

i.e., $\xi_N(\pi, f, g) = \pi$, $\xi_a(\pi, f, g) = f$ and $\xi_b(\pi, f, g) = g$. We also define the images of these homomorphisms; namely,

$$G_N = \Pi_1(G(m, a, b)) = \xi_N(G(m, a, b)) \leq S_N.$$

$$G_a = \Pi_2(G(m, a, b)) = \xi_a(G(m, a, b)) \leq \mathcal{E}.$$

$$G_b = \Pi_3(G(m, a, b)) = \xi_b(G(m, a, b)) \leq \mathcal{E}.$$

Then from Section 4.3. p. 52. G_a and G_b are the respective groups of position and velocity isometries that preserve the ivp.

Given the N body problem (m, a, b, t_0) , with the usual partitions, $a = (a_1, \dots, a_N)^t$ and $b = (b_1, \dots, b_N)^t$, let $A = \{a_1, \dots, a_N\}$ and $B = \{b_1, \dots, b_N\}$. Then $\forall (\pi, f, g) \in G(m, a, b)$ and $\forall i \leq N$ recall from Section 4.3 p. 52 that $f(a_i) = a_{\pi(i)}$ and $g(b_i) = b_{\pi(i)}$; so, $f|_A$ and $g|_B$ are bijections; i.e., $f \in \mathcal{E}_{\{a\}}$ and $g \in \mathcal{E}_{\{b\}}$. So, from our Structure Theorem for the Set-wise Stabilizer (Thm. 118. p 227. App. B) we have the simple

Lemma 21 *Given the ivp (m, a, b, t_0) ,*

$$G_a \leq \mathcal{E}_{\{a\}} = \mathcal{E}_{\{a\}}^{\text{ess}} \times \mathcal{E}_{(a)} \text{ and} \quad (5.1)$$

$$G_b \leq \mathcal{E}_{\{b\}} = \mathcal{E}_{\{b\}}^{\text{ess}} \times \mathcal{E}_{(b)}. \quad (5.2)$$

Since A and B are finite, $\mathcal{E}_{\{a\}}^{\text{ess}}$ and $\mathcal{E}_{\{b\}}^{\text{ess}}$ are finite (Cor. 124. p. 231 App. B) and their respective subgroups obtained from G_a and G_b (Def. 44. p. 232), namely, $\Pi_{1,A}(G_a)$ and $\Pi_{1,B}(G_b)$, are also finite. Lemma 21 gives an easy way to decompose G_a and G_b according to the Structure Theorem. However, the linear subspaces,

Figure 5-1: IVP where Position and Velocity Symmetry Groups differ

$\langle \Delta a \rangle$ and $\langle \Delta b \rangle$ can be distinct. In this chapter, we would like to describe each of the two groups, G_a and G_b as a subgroup of a direct product in a manner that ties them more closely together: specifically, one that gives $\mathcal{E}_{(\text{aff}(a, \Delta b))}$ and $\mathcal{E}_{(\text{aff}(b, \Delta a))}$ as the respective second factors of the direct product. Such a result would, in affine subspace of the solution coordinates (Def. 7 p. 43 Ch. 3) make $\text{aff}(a, \Delta b)$ and $\text{aff}(b, \Delta a)$ equal to $\langle \Delta a, \Delta b \rangle$. Hence, the inessential parts of such a decomposition (see Def. 43, p. 228, App. B) for G_a and G_b would be the same.

Example 7 Consider the 2-body problem in $\mathbf{R}^2$ shown in Fig. 5-1. In this example, $m_1 = m_2$, $a_1 = -a_2 = (1, 0)$, $b_1 = -b_2 = (1, -1)$. $G(m, a, b) = \langle ((12), H, H) \rangle$, a group of order 2 where $H = \text{diag}(-1, -1)$, rotation of π radians. (This group is denoted $\mathcal{C}_2^2$ in App. C.) $\mathcal{E}_{\{a\}}^{\text{ess}}$ includes reflection through the y axis, $\text{diag}(-1, 1)$, which is not in G_a . The H of G_a is obtained by combining the $\text{diag}(-1, 1)$ from $\mathcal{E}_{\{a\}}^{\text{ess}}$ with $\text{diag}(1, -1)$ of $\mathcal{E}_{(a)}$. $\mathcal{E}_{\{b\}}^{\text{ess}} \times \mathcal{E}_{(b)}$ is broken down along some other axis of symmetry ($y = x$). Thus, describing G_a and G_b as subgroups of the internal direct products given in (5.1) and (5.2) fails to reflect the connection between the two

groups.

We would like to have a finite set of points from a and b that generates the affine subspaces $\text{aff } a\Delta b$ and $\text{aff } (b, \Delta a)$ in the same way that the finite sets, A and B generate $\text{aff } a$ and $\text{aff } b$ so that we can avail ourselves of Corollary 124, p. 231 and Theorem 128, p. 234, App. B. To do this, we make the following:

Definition 13 Given $x, y \in \mathbf{R}^{dN}$, partitioned as in (3.2) p. 25, given $y_0 \in \text{aff } y$ and $\tau \in \mathbf{R}$, if we let $z_i = x_i + \tau(y_i - y_0)$, $\forall i = 1, N$ and let $X = \{x_1, \dots, x_N\}$ and $Z = \{z_1, \dots, z_N\}$, then we define the finite set

$$\delta_{\tau, y_0}(x, y) = X \cup Z.$$

Note that we could have written $z = x + \tau(y - y'_0)$, (Def. 5, p. 32).

Lemma 22 Given $x, y \in \mathbf{R}^{dN}$, $y_0 \in \text{aff } y$ and $\tau \neq 0$, then

$$\text{aff } (x, \Delta y) = \text{aff } \delta_{\tau, y_0}(x, y).$$

Proof. Let z , X and Z be as in the above definition. Since $y_i = y_0 + z_i/\tau - x_i/\tau$,

$$\begin{aligned} \text{aff } (x, \Delta y) &= \left\{ \sum \alpha_i x_i + \sum \beta_i y_i \mid \sum \alpha_i = 1, \sum \beta_i = 0 \right\} \\ &= \left\{ \sum \delta_i x_i + \sum \gamma_i z_i \mid \sum \delta_i = 1, \sum \gamma_i = 0 \right\} \end{aligned}$$

where we have substituted $\delta_i = \alpha_i - \beta_i/\tau$ and $\gamma_i = \beta_i/\tau$. Since $\sum \delta_i + \sum \gamma_i = 1$, $\text{aff } x\Delta y \subseteq \text{aff } (X \cup Z) \equiv \text{aff } \delta_{\tau, y_0}(x, y)$.

Conversely, since any member of X lies in $\text{aff } x \subseteq \text{aff } (x, \Delta y)$ and $\forall i \leq N$, $x_i \in \text{aff } x$, $y_i, y_0 \in \text{aff } y$, $y_i - y_0 \in \langle \Delta y \rangle$, $\tau(y_i - y_0) \in \langle \Delta y \rangle$; so, $z_i \in \text{aff } (x, \Delta y)$. $\therefore X \cup Z \subseteq \text{aff } (x, \Delta y)$ and $\text{aff } (X \cup Z) \subseteq \text{aff } (x, \Delta y)$. $\therefore \text{aff } (x, \Delta y) = \text{aff } (X \cup Z) \equiv \text{aff } \delta_{\tau, y_0}(x, y)$. ■

5.1 Centroids and their Affine Subspaces

Given the ivp. (m, a, b, t_0) and letting $A = \{a_1, \dots, a_N\}$ recall from Def. 45. p. 233. App. B that since $G_a \leq \mathcal{E}_{\{a\}}$, we defined $\bar{a}_{i,G_a}$ to be the centroid of the orbit of a_i under the action of the group G_a and call $\bar{A}_{G_a}$ the *set* of centroids. Replacing each a_i with $\bar{a}_{i,G_a}$ allows us to define a *vector* of centroids for initial position and a corresponding one for velocities as follows:

Definition 14 *Given the ivp (m, a, b, t_0) , let $\bar{a}_i = \bar{a}_{i,G_a}$, and $\bar{b}_i = \bar{b}_{i,G_b}$. Thus*

$$\begin{aligned}\bar{a} &= (\bar{a}_{1,G_a}, \dots, \bar{a}_{N,G_a})^t \in \mathbf{R}^{dN} \text{ and} \\ \bar{b} &= (\bar{b}_{1,G_b}, \dots, \bar{b}_{N,G_b})^t \in \mathbf{R}^{dN}.\end{aligned}$$

Theorem 23 *Given the N -body problem, (m, a, b, t_0) , $\tau \in \mathbf{R}$, $a_0 \in \text{aff } \bar{a}$ and $b_0 \in \text{aff } \bar{b}$, then*

1. $G_a \leq \mathcal{E}_{\{a + \tau(b - b'_0)\}}$ and $G_b \leq \mathcal{E}_{\{b + \tau(a - a'_0)\}}$.
2. $G_a \leq \mathcal{E}_{\{\delta_{\tau,b_0}(a,b)\}}$ and $G_b \leq \mathcal{E}_{\{\delta_{\tau,a_0}(b,a)\}}$.
3. if $\tau \neq 0$ then $\text{aff } \delta_{\tau,b_0}(a,b) = \text{aff}(a, \Delta b)$ and $\text{aff } \delta_{\tau,a_0}(b,a) = \text{aff}(b, \Delta a)$ and
4. $G_a \leq \mathcal{E}_{\{\text{aff } a\}}$, $G_a \leq \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$, $G_b \leq \mathcal{E}_{\{\text{aff } b\}}$ and $G_b \leq \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}$.
5. $\forall (\pi, f, g) \in G(m, a, b)$, the affine subspaces, $\text{aff } a$ and $\text{aff}(a, \Delta b)$ are invariant under f and the affine subspaces, $\text{aff } b$ and $\text{aff}(b, \Delta a)$ are invariant under g . If H is the common orthogonal part of f and g , the linear subspaces, $\langle \Delta a \rangle$, $\langle \Delta b \rangle$ and $\langle \Delta a, \Delta b \rangle$ are invariant under H : i.e., $H \in \mathcal{E}_{\{\langle \Delta a \rangle\}} \cap \mathcal{E}_{\{\langle \Delta b \rangle\}} \cap \mathcal{E}_{\{\langle \Delta a, \Delta b \rangle\}}$.

Proof. *Show 1:* Let $c = a + \tau(b - b'_0)$. For any $f \in G_a$, $\exists \pi \in S_N$, $g \in \mathcal{E}$, such that $(\pi, f, g) \in G(m, a, b)$. Let H be the common orthogonal part of f and g . By

Lemma 21. $g \in \mathcal{E}_{\{b\}}$ and since $b_0 \in \text{aff } \bar{b}$, $g(b_0) = b_0$ (by Thm. 128, p. 234 App. B).
 $\forall i \leq N$, since $c_i = a_i + \tau(b_i - b_0)$,

$$\begin{aligned} f(c_i) &= f(a_i) + \tau(Hb_i - Hb_0) \\ &= a_{\pi(i)} + \tau(g(b_i) - g(b_0)) \\ &= a_{\pi(i)} + \tau(b_{\pi(i)} - b_0) \\ &= c_{\pi(i)}. \end{aligned}$$

$\therefore f \in \mathcal{E}_{\{c\}}$: i.e., $G_a \leq \mathcal{E}_{\{a + \tau(b - b'_0)\}}$.

Similarly, $G_b \leq \mathcal{E}_{\{b + \tau(a - a'_0)\}}$.

Show 2: By Def. 13, $\delta_{\tau, b_0}(a, b) = A \cup C$, where $A = \{a_1, \dots, a_N\}$, $C = \{c_1, \dots, c_N\}$ and $c_i = a_i + \tau(b_i - b_0)$. Also by Def. 12, $\mathcal{E}_{\{c\}} = \mathcal{E}_{\{C\}}$: so,

$$\begin{aligned} G_a &\leq \mathcal{E}_{\{A\}} \cap \mathcal{E}_{\{C\}} \quad (\text{by Part 1}) \\ &\leq \mathcal{E}_{\{A \cup C\}} \quad (\text{by Part 2 of L. 106, p. 221, Ch. B}) \\ &= \mathcal{E}_{\{\delta_{\tau, b_0}(a, b)\}} \quad (\text{Def. 13}). \end{aligned}$$

The result for G_b is similar.

Show 3: This is a direct application of Lemma 22 p. 62.

Show 4: A direct result of Part 1 for $\tau = 0$ and L. 109, p. 222 App. B is

$$G_a \leq \mathcal{E}_{\{a\}} \leq \mathcal{E}_{\{\text{aff } a\}} \text{ and } G_b \leq \mathcal{E}_{\{b\}} \leq \mathcal{E}_{\{\text{aff } b\}}.$$

We now choose some real $\tau \neq 0$, $a_0 \in \text{aff } \bar{a}$, $b_0 \in \text{aff } \bar{b}$, ($\tau = 1$, $a_0 = \bar{a}_1$ and $b_0 = \bar{b}_1$ will suffice) and for that choice, from Parts 2 and 3 and L. 109, p. 222

$$\begin{aligned} G_a &\leq \mathcal{E}_{\{\delta_{\tau, b_0}(a, b)\}} \leq \mathcal{E}_{\{\text{aff } \delta_{\tau, b_0}(a, b)\}} = \mathcal{E}_{\{\text{aff}(a, \Delta b)\}} \text{ and} \\ G_b &\leq \mathcal{E}_{\{\delta_{\tau, a_0}(b, a)\}} \leq \mathcal{E}_{\{\text{aff } \delta_{\tau, a_0}(b, a)\}} = \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}. \end{aligned}$$

Show 5: The first sentence of this part is equivalent to Part 4. This is because $G_a \leq \mathcal{E}_{\{\text{aff } a\}} \Leftrightarrow \forall f \in G_a, f(\text{aff } a) = \text{aff } a$; etc. That H lies in the set-wise stabilizer

of $\langle \Delta a \rangle$, $\langle \Delta b \rangle$ and $\langle \Delta a, \Delta b \rangle$ is a direct result of L. 108. p. 222 and the fact that $f \in \mathcal{E}_{\{\text{aff } a\}}$, $g \in \mathcal{E}_{\{\text{aff } b\}}$ and $f \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$. ■

Observe that, as was seen in the proof of Part 4 of this theorem, by setting $\tau = 0$ Part 1 subsumes L. 21 p. 60.

In Ex. 7. p. 61 and its Fig. 5-1, $\text{aff } a$ is just the x axis. $\langle \Delta b \rangle = \langle b_{12} \rangle = \langle (2, -2) \rangle = \{(x, -x) | x \in \mathbf{R}\}$ is the line $y = -x$ and $\text{aff}(a, \Delta b)$ is the whole xy plane with a similar result for $\text{aff}(b, \Delta a)$. Since $0 \in \text{aff } b$ we have $\forall \tau \neq 0$, $\delta_{\tau,0}(a, b) = \{a_1, a_2, a_1 + \tau b_1, a_2 + \tau b_2\}$. The centroids are $\bar{a}_{1,G_a} = \bar{a}_{2,G_a} = \bar{b}_{1,G_b} = \bar{b}_{2,G_b} = 0$. Part 1 of Thm. 23 and Thm. 118 p. 227 App. B tell us that $G_a \leq \mathcal{E}_{\{a\}} = \mathcal{E}_{\{a\}}^{\text{ess}} \times \mathcal{E}_{(a)}$ where $\mathcal{E}_{\{a\}}^{\text{ess}} = \langle \text{diag}(-1, 1) \rangle$ and $\mathcal{E}_{(a)} = \langle \text{diag}(1, -1) \rangle$. Part 2 of Thm. 23 and Thm. 118 p. 227 tell us that $G_a \leq \mathcal{E}_{\{\delta_{\tau,0}(a,b)\}} = \mathcal{E}_{\{\delta_{\tau,0}(a,b)\}}^{\text{ess}} \times \mathcal{E}_{(\delta_{\tau,0}(a,b))}$ where $\mathcal{E}_{\{\delta_{\tau,0}(a,b)\}}^{\text{ess}} = C_2^2$, the cyclic group of order 2 generated by a rotation of π radians and $\mathcal{E}_{(\delta_{\tau,0}(a,b))} = \{1_{\mathcal{E}}\}$, the trivial isometry group. Similarly, $G_b \leq \mathcal{E}_{\{\delta_{\tau,0}(b,a)\}} = \mathcal{E}_{\{\delta_{\tau,0}(b,a)\}}^{\text{ess}} \times \mathcal{E}_{(\delta_{\tau,0}(b,a))}$ where $\mathcal{E}_{\{\delta_{\tau,0}(b,a)\}}^{\text{ess}} = \mathcal{E}_{\{\delta_{\tau,0}(a,b)\}}^{\text{ess}} = C_2^2$ and $\mathcal{E}_{(\delta_{\tau,0}(b,a))} = \mathcal{E}_{(\delta_{\tau,0}(a,b))} = \{1_{\mathcal{E}}\}$. This latter internal direct product ties G_a and G_b more closely to each other in this example. As we shall see, this is true in general.

5.2 The Identity Permutation

In L. 21 p. 60 we showed that $G_a \leq \mathcal{E}_{\{a\}}$ and $G_b \leq \mathcal{E}_{\{b\}}$. However, having defined the finite sets, $\delta_{\tau,b_0}(a, b)$ and $\delta_{\tau,a_0}(b, a)$ which combine positions and velocities, we obtained a sharper result in Thm. 23 and showed that G_a and G_b are subgroups of the complete symmetry groups of these larger (albeit finite) sets. The following theorem has a long statement arising from a large number of combinations. For each of the position and velocity isometries there are two invariant affine subspaces. For each of these there are four equivalent statements arising out of Theorem 131. p. 236. Ch. B. Aside from this, the pattern is clear and the theorem is not complicated.

Theorem 24 Suppose that (π, f, g) preserves the ivp, (m, a, b, t_0) and that H is the common orthogonal part of f and g . Then

1. The following 12 statements are equivalent to each other:

- (a) $\pi = 1$
- (b1) $f|_{\text{aff } a}$ is a translation. (b2) $f \in \mathcal{E}_{(\text{aff } a)}$.
- (b3) $\forall x \in \text{aff } a, f = T_{-x}HT_x$. (b4) $H \in \mathcal{E}_{(\langle \Delta a \rangle)}$.
- (c1) $f|_{\text{aff}(a, \Delta b)}$ is a translation. (c2) $f \in \mathcal{E}_{(\text{aff}(a, \Delta b))}$.
- (c3) $\forall z \in \text{aff}(a, \Delta b), f = T_{-z}HT_z$. (c4) $H \in \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)}$.
- (d1) $g|_{\text{aff}(b, \Delta a)}$ is a translation. (d2) $g \in \mathcal{E}_{(\text{aff}(b, \Delta a))}$.
- (d3) $\forall z \in \text{aff}(b, \Delta a), g = T_{-z}HT_z$.

2. The following 4 statements are equivalent to each other:

- (e1) $g|_{\text{aff } b}$ is a translation. (e2) $g \in \mathcal{E}_{(\text{aff } b)}$.
- (e3) $\forall y \in \text{aff } b, g = T_{-y}HT_y$. (e4) $H \in \mathcal{E}_{(\langle \Delta b \rangle)}$.

Furthermore, if any and hence all of the statements in (1) are true, then all statements in (2) are true.

Note that $\Delta \text{aff}(a, \Delta b) = \Delta \text{aff}(b, \Delta a) = \langle \Delta a, \Delta b \rangle$ so that $\forall x \in \text{aff}(a, \Delta b), y \in \text{aff}(b, \Delta a), \text{aff}(a, \Delta b) = x + \langle \Delta a, \Delta b \rangle$ and $\text{aff}(b, \Delta a) = y + \langle \Delta a, \Delta b \rangle$. Similarly, $\Delta \text{aff}(\bar{a}, \Delta \bar{b}) = \Delta \text{aff}(\bar{b}, \Delta \bar{a}) = \langle \Delta \bar{a}, \Delta \bar{b} \rangle$ and $\forall a_0 \in \text{aff}(\bar{a}, \Delta \bar{b}), b_0 \in \text{aff}(\bar{b}, \Delta \bar{a}), \text{aff}(\bar{a}, \Delta \bar{b}) = a_0 + \langle \Delta \bar{a}, \Delta \bar{b} \rangle$ and $\text{aff}(\bar{b}, \Delta \bar{a}) = b_0 + \langle \Delta \bar{a}, \Delta \bar{b} \rangle$.

Proof. By Part 1 Thm. 23 for $\tau = 0$, $f \in G_a \leq \mathcal{E}_{\{a\}} = \mathcal{E}_{\{A\}}$ and $g \in G_b \leq \mathcal{E}_{\{b\}} = \mathcal{E}_{\{B\}}$, where $A = \{a_1, \dots, a_N\}$ and $B = \{b_1, \dots, b_N\}$. By Thm. 131, p. 236, App. B, since A and B are bounded, $b1 \Leftrightarrow b2 \Leftrightarrow b3 \Leftrightarrow b4$ and $e1 \Leftrightarrow e2 \Leftrightarrow e3 \Leftrightarrow e4$. Also, by Part 2 of that theorem, $f \in \mathcal{E}_{\{\delta_{\tau, b_0}(a, b)\}}$ and $g \in \mathcal{E}_{\{\delta_{\tau, a_0}(b, a)\}}$. Since $\delta_{\tau, b_0}(a, b)$ and $\delta_{\tau, a_0}(b, a)$ are finite sets, again by Thm. 131, p. 236, App. B, $c1 \Leftrightarrow c2 \Leftrightarrow c3 \Leftrightarrow c4$ and $d1 \Leftrightarrow d2 \Leftrightarrow d3 \Leftrightarrow c4$. Note that we have used $c4$ in our application of the fourth statement of this theorem to g as well as to f because

$$\Delta \text{aff}(a, \Delta b) = \langle \Delta a, \Delta b \rangle = \Delta \text{aff}(b, \Delta a).$$

Thus we have the equivalence of (c1) through (d3).

Show (a) $\Rightarrow$ (b2). Suppose $\pi = 1$. $\forall x \in \text{aff } a$, $x = \sum_{i=1}^N \alpha_i a_i$, where $\sum_{i=1}^N \alpha_i = 1$. Then $f(x) = \sum_{i=1}^N \alpha_i f(a_i) = \sum_{i=1}^N \alpha_i a_{\pi(i)} = \sum_{i=1}^N \alpha_i a_i = x$.

Show (b2) $\Rightarrow$ (a). Assume (b2). Then $\forall i \leq N$, $a_i = f(a_i) = a_{\pi(i)}$. Since a is not on the diagonal, $a_i = a_{\pi(i)} \Rightarrow i = \pi(i)$. Thus, $\pi = 1_{S_N}$ and we have the equivalence of (a) through (b4).

Since $\langle \Delta a \rangle \leq \langle \Delta a, \Delta b \rangle$, $\mathcal{E}_{(\langle \Delta a, \Delta b \rangle)} \leq \mathcal{E}_{(\langle \Delta a \rangle)}$; so, (c4) $\Rightarrow$ (b4). Thus, any one of (c1) through (d3) implies all of (a) through (b4).

Replacing a with b and f with g in the argument above where we showed (a) $\Rightarrow$ (b2) gives (a) $\Rightarrow$ (e2). Now suppose (b4). From this follows (a) from which follows (e2) hence (e4). $\therefore H \in \mathcal{E}_{(\langle \Delta a \rangle)}$ and $H \in \mathcal{E}_{(\langle \Delta b \rangle)}$. $\therefore H \in \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)}$. $\therefore$ (b4) $\Rightarrow$ (c4). This completes the proof of the equivalence of all statements in 1. the the equivalence of all statements in 2 and the fact that any statement in 1 implies any statement in 2. ■

Remark. It is not necessary for b to be off the diagonal. Consequently, the proof of (b2) $\Rightarrow$ (a), above does not carry over to velocities; i.e., no corresponding proof of (e2) $\Rightarrow$ (a) exists. In fact, it does not follow in general that if $H \in \mathcal{E}_{(\langle \Delta b \rangle)}$, or if $g \in \mathcal{E}_{(\text{aff } b)}$ then $\pi = 1$. As a counterexample consider any problem where the initial velocities are all zero. This gives $\text{aff } b = \{0\}$ which is left fixed by any g that is the velocity isometry for any triple preserving the ivp, including those for which the permutation is not the identity: thus in general, (e2) $\nRightarrow$ (a). However, if b is also off the diagonal and $g \in \mathcal{E}_{(\text{aff } b)}$ or $H \in \mathcal{E}_{(\langle \Delta b \rangle)}$ then it *does* follow as in the above theorem that $\pi = 1_{S_N}$; i.e., all statements a through e4 would be equivalent in that case.

Corollary 25 *Given the N -body problem (m, a, b, t_0) , the projections, ξ_a and ξ_b of $G(m, a, b)$ onto G_a and G_b , respectively, (as defined in Def. 12) are isomorphisms.*

Proof. $\forall (\pi, f, g) \in \ker \xi_a$. $f = 1_{\mathcal{E}}$ and from Part 1 of the above theorem, statement (c2) holds and is equivalent to statement (a) of the same theorem which gives $\pi = 1$. Since the orthogonal part of f is I_d , the orthogonal part of g is also I_d ; so, by (d3), we can choose any $z \in \text{aff}(b, \Delta a)$ which gives $g = T_{-z} I_d T_z = 1_{\mathcal{E}}$. $\therefore (\pi, f, g) = 1_{G(m,a,b)}$; i.e., $\ker \xi_a = 1_{G(m,a,b)}$ and ξ_a is 1-1. It is onto by the definition of G_a (D. 12 p. 59).

The argument for ξ_b is similar using (d2) and (c3) in place of (c2) and (d3) respectively from Theorem 24. ■

From Definition 12, p. 59, $\ker \xi_N = \{(\pi, f, g) \in G(m, a, b) \mid \pi = 1_{S_N}\}$ and by the fundamental isomorphism theorem for groups the quotient group, $G(m, a, b) / \ker \xi_N$ is isomorphic to the image of ξ_N which is a subgroup of S_N and hence finite. Suppose we are given an $x \in \text{aff}(a, \Delta b)$ and a $y \in \text{aff}(b, \Delta a)$. Then $\forall (1, f, g) \in \ker \xi_N$ where K is the common orthogonal part of f and g , by (c3) and (d3) of Part 1 of Theorem 24, $f = T_{-x} K T_x$ and $g = T_{-y} K T_y$ and by (c4), $K \in \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)}$. Conversely, $\forall K \in \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)}$, $\forall i \leq N$, $a_i - x \in \langle \Delta a, \Delta b \rangle$; $T_{-x} K T_x(a_i) = T_{-x} K(a_i - x) = T_{-x}(a_i - x) = a_i$. Similarly, $T_{-y} K T_y(b_i) = b_i$; so, $(1, T_{-x} K T_x, T_{-y} K T_y) \in G(m, a, b)$ and hence lies in $\ker \xi_N$. Thus, we have shown:

Corollary 26 *Given the irp, (m, a, b, t_0) , $\forall x \in \text{aff}(a, \Delta b)$, $y \in \text{aff}(b, \Delta a)$,*

$$\ker \xi_N = \{(1, T_{-x} K T_x, T_{-y} K T_y) \mid K \in \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)}\}.$$

5.3 The Group as a Direct Product

We combine Parts 2 and 3 of Thm. 23, p. 63 with the Structure Theorem 118, p. 227, App. B and L. 92 p. 213 App. A to get $\forall \tau \neq 0$, $a_0 \in \text{aff } \bar{a}$ and $b_0 \in \text{aff } \bar{b}$ that $G_a \leq \mathcal{E}_{\{\delta_{\tau, b_0}(a, b)\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff}(a, \Delta b))}$ and $G_b \leq \mathcal{E}_{\{\delta_{\tau, a_0}(b, a)\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff}(b, \Delta a))}$. By Cor. 26, above and Cor. 94, p. 214, App. A, $\forall x \in \text{aff}(a, \Delta b)$, $y \in \text{aff}(b, \Delta a)$,

$$\begin{aligned} \xi_a(\ker \xi_N) &= T_{-x} \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)} T_x = \mathcal{E}_{(\text{aff}(a, \Delta b))} \text{ and} \\ \xi_b(\ker \xi_N) &= T_{-y} \mathcal{E}_{(\langle \Delta a, \Delta b \rangle)} T_y = \mathcal{E}_{(\text{aff}(b, \Delta a))}. \end{aligned}$$

So, by the correspondence theorem for group homomorphisms (e.g., [BJN, p. 98]), since $\ker \xi_N \triangleleft G(m, a, b)$,

$$\mathcal{E}_{(\text{aff}(a, \Delta b))} \triangleleft G_a \text{ and} \quad (5.3)$$

$$\mathcal{E}_{(\text{aff}(b, \Delta a))} \triangleleft G_b. \quad (5.4)$$

Theorem 27 *Given the N -body problem (m, a, b, t_0) ,*

$$\begin{aligned} G_a &= \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \times \mathcal{E}_{(\text{aff}(a, \Delta b))} \text{ and} \\ G_b &= \left(G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} \right) \times \mathcal{E}_{(\text{aff}(b, \Delta a))} \end{aligned}$$

(internal direct products). Moreover, the groups $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}$ are finite. Also, given $\tau \neq 0$, $a_0 \in \text{fix}_{\mathbf{R}^d} G_a$, $b_0 \in \text{fix}_{\mathbf{R}^d} G_b$, if we let $(a, \delta b) = \delta_{\tau, b_0}(a, b)$ and $(b, \delta a) = \delta_{\tau, a_0}(b, a)$, then $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} = G_a \cap \mathcal{E}_{\{(a, \delta b)\}}^{\text{ess}}$ and $G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} = G_b \cap \mathcal{E}_{\{b \delta a\}}^{\text{ess}}$, independent of the choice of τ , a_0 and b_0 .

Proof. By Part 4 of Thm. 23 p. 63 and by the Structure Theorem for the Set-wise Stabilizer (p. 227, App. B), $G_a \leq \mathcal{E}_{\{\text{aff}(a, \Delta b)\}} = \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff}(a, \Delta b))}$. Thus, $\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \triangleleft \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$ and

$$G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \triangleleft G_a. \quad (5.5)$$

If $f \in G_a$ then $f = f_1 f_2$ where $f_1 \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $f_2 \in \mathcal{E}_{(\text{aff}(a, \Delta b))} \leq G_a$ by (5.3). $\therefore f_2^{-1} \in G_a$ and $f_1 = f \circ f_2^{-1} \in G_a$. $\therefore f_1 \in G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$.

$$\therefore f \in \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \cdot \mathcal{E}_{(\text{aff}(a, \Delta b))}$$

and $G_a \subseteq \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \cdot \mathcal{E}_{(\text{aff}(a, \Delta b))}$. Since $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $\mathcal{E}_{(\text{aff}(a, \Delta b))}$ are subgroups of G_a the reverse inclusion holds; so, $G_a = \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \cdot \mathcal{E}_{(\text{aff}(a, \Delta b))}$. This together with (5.3) and (5.5) and the fact that $\left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \cap \mathcal{E}_{(\text{aff}(a, \Delta b))} \leq \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \cap \mathcal{E}_{(\text{aff}(a, \Delta b))} = \{1_{\mathcal{E}}\}$ by the direct product structure of $\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$ gives $G_a = \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) \times \mathcal{E}_{(\text{aff}(a, \Delta b))}$, a direct product. A similar argument gives the result for G_b .

By Part 3. Thm. 23, p. 63, $\text{aff}(a, \delta b) = \text{aff}(a, \Delta b)$ and $\text{aff}(b, \delta a) = \text{aff}(b, \Delta a)$ and by Part 2 of the same theorem, $G_a \leq \mathcal{E}_{\{(a, \delta b)\}}$ and $G_b \leq \mathcal{E}_{\{(b, \delta a)\}}$. So, from Thm. 119, p. 228, App. B, $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} = G_a \cap \mathcal{E}_{\{(a, \delta b)\}}^{\text{ess}}$ and $G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} = G_b \cap \mathcal{E}_{\{(b, \delta a)\}}^{\text{ess}}$. Since $(a, \delta b)$ and $(b, \delta a)$ are finite sets, $\mathcal{E}_{\{(a, \delta b)\}}^{\text{ess}}$ and $\mathcal{E}_{\{(b, \delta a)\}}^{\text{ess}}$ are finite groups (Cor. 124, p. 231, App. B), from which we have $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} = G_a \cap \mathcal{E}_{\{(a, \delta b)\}}^{\text{ess}}$ and $G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} = G_b \cap \mathcal{E}_{\{(b, \delta a)\}}^{\text{ess}}$ are also finite. $\blacksquare$

5.3.1 Structure Theorem for the Group of an IVP

In the light of this theorem and Def. 43, p. 228, App. B, we make the following:

Definition 15 *Given the ivp, (m, a, b, t_0) , let*

$$E(m, a, b) = G(m, a, b) \cap (S_N \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \times \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}).$$

*Then $E(m, a, b)$ is called the **essential group** of the ivp and $\ker \xi_N$ is called the **inessential group**.*

Theorem 28 (Structure Theorem for the Group of an IVP) *The group of the ivp, (m, a, b, t_0) ,*

$$G(m, a, b) = E(m, a, b) \times \ker \xi_N,$$

an internal direct product of subgroups of $S_N \times \mathcal{E} \times \mathcal{E}$. Furthermore, the isomorphism ξ_a that carries $G(m, a, b)$ onto $G_a = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$, carries $E(m, a, b)$ onto $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $\ker \xi_N$ onto $\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$; i.e.,

$$\xi_a|_{E(m, a, b)} : E(m, a, b) \rightarrow G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \text{ and}$$

$$\xi_a|_{\ker \xi_N} : \ker \xi_N \rightarrow \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$$

are isomorphisms. Similarly, for $\xi_b : G(m, a, b) \rightarrow G_b = (G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}) \times \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}$,

$$\xi_b|_{E(m, a, b)} : E(m, a, b) \rightarrow G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} \text{ and}$$

$$\xi_b|_{\ker \xi_N} : \ker \xi_N \rightarrow \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}$$

Figure 5-2: Structure of the Group of an IVP

are isomorphisms. Also, $E(m,a,b)$ is a finite subgroup of $S_N \times \mathcal{E} \times \mathcal{E}$ and the projection onto each of its 3 coordinates is an isomorphism.

Proof. Since, $\mathcal{E}^{\text{ess}}_{\{\text{aff}(a,\Delta b)\}}$ and $\mathcal{E}^{\text{ess}}_{\{\text{aff}(b,\Delta a)\}}$ are subgroups of $\mathcal{E}$, $S_N \times \mathcal{E}^{\text{ess}}_{\{\text{aff}(a,\Delta b)\}} \times \mathcal{E}^{\text{ess}}_{\{\text{aff}(b,\Delta a)\}} \leq S_N \times \mathcal{E} \times \mathcal{E}$. Since $E(m,a,b)$ is the intersection of this group with $G(m,a,b)$, $E(m,a,b) \leq G(m,a,b)$. Also, $\ker \xi_N \triangleleft G(m,a,b)$; so,

$$E(m,a,b) \cdot \ker \xi_N \leq G(m,a,b).$$

Conversely, $\forall (\pi, f, g) \in G(m,a,b)$, by Thm. 27, $f = f_1 f_2$ where $f_1 \in G_a \cap \mathcal{E}^{\text{ess}}_{\{\text{aff}(a,\Delta b)\}}$ and $f_2 \in \mathcal{E}_{\{\text{aff}(a,\Delta b)\}}$ and $g = g_1 g_2$ where $g_1 \in G_b \cap \mathcal{E}^{\text{ess}}_{\{\text{aff}(b,\Delta a)\}}$ and $g_2 \in \mathcal{E}_{\{\text{aff}(b,\Delta a)\}}$. Since $f_2(a_i) = a_i$ and $g_2(b_i) = b_i$, $\forall i \leq N$, $(1_{S_N}, f_2, g_2) \in \ker \xi_N$; so, $(\pi, f_1, g_1) = (\pi, f, g)(1, f_2, g_2)^{-1} \in G(m,a,b)$ also. $\therefore f_1 \in G_a \cap \mathcal{E}^{\text{ess}}_{\{\text{aff}(a,\Delta b)\}}$ and $g_1 \in G_b \cap \mathcal{E}^{\text{ess}}_{\{\text{aff}(b,\Delta a)\}}$. $(\pi, f_1, g_1) \in E(m,a,b)$. $(\pi, f, g) = (\pi, f_1 g_1)(1, f_2, g_2) \in$

$$E(m, a, b) \cdot \ker \xi_N. \therefore G(m, a, b) \subseteq E(m, a, b) \cdot \ker \xi_N; \text{ i.e.,}$$

$$G(m, a, b) = E(m, a, b) \cdot \ker \xi_N.$$

To see that this is an internal direct product, since we know that $\ker \xi_N \triangleleft G(m, a, b)$, we now show $E(m, a, b) \triangleleft G(m, a, b)$. $\forall (\pi, f_1, g_1) \in E(m, a, b)$ and $\forall (1, f_2, g_2) \in \ker \xi_N$, since $f_1 \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $f_2 \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$, $f_1 f_2 = f_2 f_1$. Similarly, $g_1 g_2 = g_2 g_1$; so, $(1, f_2, g_2) E(m, a, b) = E(m, a, b) (1, f_2, g_2)$; i.e., $E(m, a, b) \triangleleft G(m, a, b)$, also. Finally, $E(m, a, b) \leq S_N \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \times \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}$ and $\ker \xi_N \leq \{1_{S_N}\} \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}} \times \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}$; so, $E(m, a, b) \cap \ker \xi_N = \{1_{S_N}\} \times \{1_{\mathcal{E}}\} \times \{1_{\mathcal{E}}\}$. $\therefore G(m, a, b) = E(m, a, b) \times \ker \xi_N$, an internal direct product.

Since ξ_a is 1-1 by Cor. 25 and by Def. 12, p. 59, G_a is the image, $\xi_a(G(m, a, b))$, then

$$G(m, a, b) = E(m, a, b) \times \ker \xi_N \approx G_a = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$$

(isomorphic). It is clear also from Cor. 26, p. 68, $\forall (1_{S_N}, T_{-x} K T_x, T_{-y} K T_y) \in \ker \xi_N$, $x \in \text{aff}(a, \Delta b)$, $y \in \text{aff}(b, \Delta a)$, that $\xi_a(1_{S_N}, T_{-x} K T_x, T_{-y} K T_y) = T_{-x} K T_x \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$. $\therefore \xi_a$ carries $\ker \xi_N$ isomorphically to $\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$. Also, $\forall (\pi, f, g) \in E(m, a, b) = G(m, a, b) \cap (S_N \times \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \times \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}})$, $f \in \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$; so, $\xi_a(\pi, f, g) \in G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$. $\therefore E(m, a, b) \approx G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$, a finite group (Thm. 22) then $E(m, a, b)$ is finite. A similar proof gives the result for ξ_b . $\blacksquare$

Definition 16 Given the ivp, (m, a, b, t_0) we define $E_a = G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ and $E_b = G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}$ and call them the **essential parts** of G_a and G_b respectively and $\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}$ and $\mathcal{E}_{\{\text{aff}(b, \Delta a)\}}$ are called their respective **inessential parts**.

Recall Corollary 10, p. 42, Ch. 3 and the subsequent definition for the affine subspace of the solution coordinates. In such coordinates, the affine subspace of the solution is fixed and contains the origin $\forall t$ ranging over the solution. This happens if and only if $0 \in \text{aff}(a, \Delta b)$ and $0 \in \text{aff}(b, \Delta a)$ by the abovementioned corollary

and this happens if and only if the inessential parts of G_a and G_b are subgroups of $\mathcal{O}$; i.e., we have shown:

Corollary 29 *The ivp (m, a, b, t_0) is in affine subspace of the solution coordinates if and only if the inessential parts of G_a and G_b are subgroups of $\mathcal{O}$.*

Of course when this happens, the inessential parts of G_a and G_b are identical. In Section 5.5, we establish the conditions under which the essential parts are subgroups of $\mathcal{O}$.

5.4 The Fixed Point Set for the Group

As a direct result of Theorem 128, p. 234 App. B and Thm. 23 p. 63 Part 1, we have the following:

Lemma 30 *Given the N -body problem (m, a, b, t_0) ,*

$$\text{fix}_{\text{aff } a} G_a = \text{aff } \bar{a} \text{ and } \text{fix}_{\text{aff } b} G_b = \text{aff } \bar{b}.$$

In other words, the set of fixed points for the group of position (resp., velocity) isometries that preserve the ivp, meets the affine subspace $\text{aff } a$ (resp., $\text{aff } b$) precisely in the affine subspace generated by the centroids of the orbits of the initial positions, a_i , (resp., initial velocities, b_i), $i = 1, N$ under the action of that group. This gives us part of what we seek here: however, we would like to know the *full* set of points left fixed by G_a and G_b ; i.e., the set in all of $\mathbf{R}^d$. We shall do exactly that but first we prepare the way with the following:

Lemma 31 *Given the N -body problem (m, a, b, t_0) , $\forall \tau \neq 0$, $a_0 \in \text{aff } \bar{a}$, $b_0 \in \text{aff } \bar{b}$,*

$$\begin{aligned} \text{aff } \overline{\delta_{\tau, b_0}(a, b)}_{G_a} &= \text{aff } (\bar{a}, \Delta \bar{b}) \text{ and} \\ \text{aff } \overline{\delta_{\tau, a_0}(b, a)}_{G_b} &= \text{aff } (\bar{b}, \Delta \bar{a}). \end{aligned}$$

Proof. Recall that by $\bar{a}$ and $\bar{b}$ we mean $\bar{a}_{G_a}$ and $\bar{b}_{G_b}$ (Def. 14. p. 63), respectively. Let $c = a + \tau(b - b'_0)$, $A = \{a_1, \dots, a_N\}$ and $C = \{c_1, \dots, c_N\}$. Then by Def. 13. p. 62, $\delta_{\tau, b_0}(a, b) = A \cup C$. So, $\forall i \leq N$, $c_i \in \text{aff } \delta_{\tau, b_0}(a, b) = \text{aff}(a, \Delta b)$ by Part 4 of Thm. 23. p. 63. $\therefore$ by Thm. 27. p. 69.

$$G_a c_i = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) (\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}) c_i = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) c_i.$$

By Thm. 28, $E(m, a, b)$ is finite; so, let $n = |E(m, a, b)|$ and let $E(m, a, b) = \{(\pi_k, f_k, g_k) | k = 1, n\}$. Also, by the same theorem, $E(m, a, b) \approx G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \approx G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}$; so, $G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} = \{f_k | k = 1, n\}$ and $G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}} = \{g_k | k = 1, n\}$. Since f_k and g_k have the same orthogonal part

$$f_k(c_i) = f_k(a_i + \tau(b_i - b_0)) = f_k(a_i) + \tau(g_k(b_i) - b_0)$$

where we have used the fact that $g_k(b_0) = b_0$ from L. 30 above. Similarly,

$$G_a a_i = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) (\mathcal{E}_{\{\text{aff}(a, \Delta b)\}}) a_i = (G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}) a_i$$

and $G_b b_i = (G_b \cap \mathcal{E}_{\{\text{aff}(b, \Delta a)\}}^{\text{ess}}) b_i$. $\therefore \bar{a}_i = \frac{1}{n} \sum_{k=1}^n f_k(a_i)$, $\bar{b}_i = \frac{1}{n} \sum_{k=1}^n g_k(b_i)$ and

$$\begin{aligned} \therefore \bar{c}_{i, G_a} &= \frac{1}{n} \sum_{k=1}^n f_k(c_i) \\ &= \frac{1}{n} \sum_{k=1}^n [f_k(a_i) + \tau(g_k(b_i) - b_0)] \\ &= \bar{a}_i + \tau(\bar{b}_i - b_0). \end{aligned}$$

Since $G_a \leq \mathcal{E}_{\{A\}}$ and $G_a \leq \mathcal{E}_{\{C\}}$ (Parts 1 and 2 of Thm. 23. p. 63) $\overline{\delta_{\tau, b_0}(a, b)}_{G_a} = \overline{A \cup C}_{G_a} = \overline{A}_{G_a} \cup \overline{C}_{G_a} = \delta_{\tau, b_0}(\bar{a}, \bar{b})$ by Def. 13. p. 62 and because $\overline{C}_{G_a} = \bar{c}_{G_a} = \bar{a} + \tau(\bar{b} - b'_0)$ as shown above. $\therefore \text{aff } \overline{\delta_{\tau, b_0}(a, b)}_{G_a} = \text{aff } \delta_{\tau, b_0}(\bar{a}, \bar{b}) = \text{aff}(\bar{a}, \Delta \bar{b})$ by L. 22. p. 62. The proof for $\delta_{\tau, a_0}(b, a)$ is similar. $\blacksquare$

Despite the fact that the isometry groups G_a and G_b are potentially uncountable, we have seen from Section 5.1 p. 63 that they have finite orbits and hence fixed points. We now sharpen L. 30 as promised above by finding all fixed points for each group.

Theorem 32 *Given the N -body problem (m, a, b, t_0) ,*

$$\text{fix}_{\mathbf{R}^d} G_a = \text{aff}(\bar{a}, \Delta \bar{b}) \text{ and } \text{fix}_{\mathbf{R}^d} G_b = \text{aff}(\bar{b}, \Delta \bar{a}),$$

two parallel, non-empty affine subspaces of $\mathbf{R}^d$.

Proof. Choose $\tau \neq 0$. $b_0 \in \text{aff } \bar{b}$.

$$\begin{aligned} \mathcal{E}_{(\delta_{\tau, b_0}(a, b))} &= \mathcal{E}_{(\text{aff } \delta_{\tau, b_0}(a, b))} \quad (\text{L. 92, p. 213}) \\ &= \mathcal{E}_{(\text{aff}(a, \Delta b))} \quad (\text{L. 22, p. 62}) \\ &\triangleleft G_a \quad (\text{Thm. 27, p. 69}). \end{aligned}$$

Then $\mathcal{E}_{(\delta_{\tau, b_0}(a, b))} \leq G_a \leq \mathcal{E}_{\{\delta_{\tau, b_0}(a, b)\}}$ where the second step follows from Part 3 of Thm. 23, p. 63. $\therefore$ by Thm. 130, p. 235, Ch. B, $\text{fix}_{\mathbf{R}^d} G_a = \text{aff } \overline{\delta_{\tau, b_0}(a, b)}_{G_a} = \text{aff}(\bar{a}, \Delta \bar{b})$ by Lemma 31. above. The proof for G_b is similar. $\blacksquare$

Example 8 *To see that $\text{fix}_{\mathbf{R}^d} G_a$ can be strictly larger than $\text{fix}_{\text{aff } a} G_a$ consider the 4 body problem in $\mathbf{R}^2$ given by $m = (1, 1, 1, 1)$ (all bodies of equal mass).*

$$\begin{aligned} a &= (a_1, a_2, a_3, a_4) = ((-1, 0), (1, 0), (-2, 0), (2, 0)) \\ b &= (b_1, b_2, b_3, b_4) = ((0, 1), (0, 1), (0, -1), (0, -1)). \end{aligned}$$

$G(m, a, b) = \langle (\pi, H, H) \rangle$ where $\pi = (12)(34)$ and $H = \text{diag}(-1, 1)$. (reflection in the y axis). The orbits are $\{1, 2\}$, $\{3, 4\}$ and

$$\begin{aligned} \bar{a} &= ((0, 0), (0, 0), (0, 0), (0, 0)) \\ \bar{b} &= ((0, 1), (0, 1), (0, -1), (0, -1)) \\ \text{aff } \bar{a} &= \{(0, 0)\} \\ \langle \Delta \bar{b} \rangle &= \langle (0, 1) \rangle, \text{ the } y \text{ axis and} \\ \text{aff}(\bar{a}, \Delta \bar{b}) &= \langle (0, 1) \rangle, \text{ also the } y \text{ axis.} \end{aligned}$$

Thus, by L. 30 and Thm. 32,

$$\text{fix}_{\text{aff } a} G_a = \text{aff } \bar{a} = \{(0, 0)\} \subsetneq \text{aff}(\bar{a}, \Delta \bar{b}) = \text{fix}_{\mathbf{R}^d} G_a.$$

We now provide an elementary lemma that determines how the fixed point sets are affected under a Galilean coordinate transformation.

Lemma 33 *Given the ivp (m, a, b, t_0) in $\mathbf{R}^d$ and given x and $y \in \mathbf{R}^d$, if we denote*

$$G_{a-x'} = \Pi_2(G(m, a - x', b - y')) \text{ and} \\ G_{b-y'} = \Pi_3(G(m, a - x', b - y'))$$

then

$$\begin{aligned} \text{aff}(\overline{a - x'}_{G_{a-x'}}, \Delta \overline{b - y'}_{G_{b-y'}}) &= \text{fix}_{\mathbf{R}^d} G_{a-x'} = -x + \text{fix}_{\mathbf{R}^d} G_a = -x + \text{aff}(\bar{a}_{G_a}, \Delta \bar{b}_{G_b}) \\ \text{aff}(\overline{b - y'}_{G_{b-y'}}, \Delta \overline{a - x'}_{G_{a-x'}}) &= \text{fix}_{\mathbf{R}^d} G_{b-y'} = -y + \text{fix}_{\mathbf{R}^d} G_b = -y + \text{aff}(\bar{b}_{G_b}, \Delta \bar{a}_{G_a}) \end{aligned}$$

Proof. The first and third equalities in each of the two lines above are direct applications of Thm. 32. Applying Thm. 20 p. 54 Ch 4 and taking the projection onto the second coordinate gives $T_x G_a T_{-x} = G_{a-x'}$. Thus

$$\begin{aligned} \text{fix } G_{a-x'} &= \text{fix } T_x G_a T_{-x} \\ &= -x + \text{fix } G_a \end{aligned}$$

by L. 93 p. 213 App. A. Similarly, projecting onto the third coordinate one obtains $\text{fix}_{\mathbf{R}^d} G_{b-y'} = -y + \text{fix}_{\mathbf{R}^d} G_b$. ■

5.5 Centroidal Coordinates

In light of the above theorem, we make the following:

Definition 17 *Given the N -body problem (m, a, b, t_0) , $\text{aff}(\bar{a}, \Delta \bar{b})$ is called the **centroidal position affine subspace of the ivp** and $\text{aff}(\bar{b}, \Delta \bar{a})$ is called the **centroidal velocity affine subspace of the ivp**. If $0 \in \text{aff}(\bar{a}, \Delta \bar{b})$ and $0 \in \text{aff}(\bar{b}, \Delta \bar{a})$ then the ivp is said to be in **centroidal coordinates**.*

Given the ivp. (m, a, b, t_0) , then $\forall a_0 \in \text{aff}(\bar{a}, \Delta\bar{b})$ and $b_0 \in \text{aff}(\bar{b}, \Delta\bar{a})$, the ivp.

$$\mathcal{G}_{a_0b_0}(m, a, b, t_0) = (m, a - a'_0, b - b'_0, t_0)$$

is in centroidal coordinates. These two problems are equivalent in the sense of L. 7, p. 33, Ch. 3; i.e., q is a solution of (m, a, b, t_0) if and only if $\mathcal{G}_{a_0b_0}(q)$ is a solution of $(m, a - a'_0, b - b'_0, t_0)$. Recall that $\mathcal{G}_{a_0b_0}(q)(t) \equiv q(t) - a'_0 - (t - t_0)b'_0$. Also their groups are equivalent in the sense that there is an inner automorphism between $G(m, a, b)$ and $G(m, a - a'_0, b - b'_0)$ (Thm. 20, p. 54, Ch. 4). Specifically,

$$G(m, a - a'_0, b - b'_0) = (1, T_{a_0}, T_{b_0}) G(m, a, b) (1, T_{-a_0}, T_{-b_0}).$$

From this definition, if (m, a, b, t_0) is already in centroidal coordinates, then $0 \in \text{aff}(\bar{a}, \Delta\bar{b})$, $0 \in \text{aff}(\bar{b}, \Delta\bar{a})$ from which we have $\text{aff}(\bar{a}, \Delta\bar{b}) = \text{aff}(\bar{b}, \Delta\bar{a}) = \langle \Delta\bar{a}, \Delta\bar{b} \rangle$; i.e., they are the same linear subspace of $\mathbf{R}^d$. Also, when this happens, since $\text{aff}(\bar{a}, \Delta\bar{b}) \subseteq \text{aff}(a, \Delta b)$, $0 \in \text{aff}(\bar{a}, \Delta\bar{b}) \leq \text{aff}(a, \Delta b) \leq \mathbf{R}^d$ and similarly $0 \in \text{aff}(\bar{b}, \Delta\bar{a}) \leq \text{aff}(b, \Delta a) \leq \mathbf{R}^d$; i.e., the problem is also in the affine coordinates (Def. 7, p. 43, Ch. 3).

At the end of Subsection 5.3.1, we showed that the necessary and sufficient conditions under which the inessential parts of G_a and G_b are subgroups of $\mathcal{O}$ (making them equal) is that the problem be in affine subspace of the solution coordinates. We now give the necessary and sufficient conditions for *all* of G_a and G_b to be subgroups of $\mathcal{O}$ and hence identical.

$$\begin{aligned} G_a, G_b \leq \mathcal{O} &\Leftrightarrow 0 \in \text{fix}_{\mathbf{R}^d} G_a \cap \text{fix}_{\mathbf{R}^d} G_b && (\text{L. 68, p. 201, App. A}) \\ &\Leftrightarrow 0 \in \text{aff}(\bar{a}, \Delta\bar{b}) \cap \text{aff}(\bar{b}, \Delta\bar{a}) && (\text{Thm. 32}) \\ &\Leftrightarrow (m, a, b, t_0) \text{ is in centroidal coords.} && (\text{Def. 17}). \end{aligned}$$

Also, if any and hence all of these statements are true and if $(\pi, f, g) \in G(m, a, b)$, then $f, g \in \mathcal{O}$, the translation parts of f and g are both the identity and since their orthogonal parts are equal, $f = g$ and we have shown:

Theorem 34 *Given the ivp, (m, a, b, t_0) , the following are equivalent:*

1. (m, a, b, t_0) is in centroidal coordinates
2. $0 \in \text{aff}(\bar{a}, \Delta \bar{b})$ and $0 \in \text{aff}(\bar{b}, \Delta \bar{a})$
3. $\text{aff}(\bar{a}, \Delta \bar{b}) = \text{aff}(\bar{b}, \Delta \bar{a}) = \langle \Delta \bar{a}, \Delta \bar{b} \rangle$
4. $0 \in \text{fix}_{\mathbf{R}^d} G_a$ and $0 \in \text{fix}_{\mathbf{R}^d} G_b$
5. $G_a \leq \mathcal{O}$ and $G_b \leq \mathcal{O}$.

If any and hence all of these statements are true then $G_a = G_b$. Moreover, $\forall (\pi, f, g) \in G(m, a, b)$, $f = g \in \mathcal{O}$.

Note that in centroidal coordinates the members of $G(m, a, b)$ have the form $(\pi, H, H) \cdot (1, K, K)$ where $H \in G_a \cap \mathcal{E}_{\{(\Delta a, \Delta b)\}}^{\text{ess}} = G_b \cap \mathcal{E}_{\{(\Delta a, \Delta b)\}}^{\text{ess}}$ and $K \in \mathcal{E}_{\{(\Delta a, \Delta b)\}}$. This theorem can be phrased as ‘in centroidal coordinates, the group is diagonal in its 2nd and 3rd coordinates’: i.e., one can speak of the isometry group of the ivp (the same for both positions and velocities). An historically important example of centroidal coordinates is offered in the following:

Theorem 35 *Given the N -body problem (m, a, b, t_0) the centre of mass coordinates are centroidal.*

Proof. Let $\mu = \sum_{i=1}^N m_i > 0$. As in (3.12), p. 31 let $\alpha = \sum_{i=1}^N m_i a_i$ and $\beta = \sum_{i=1}^N m_i b_i$. $\forall i \leq N$, let $G_a a_i = \{a_{i_1}, \dots, a_{i_k}\}$ be the orbit of body i where the a_{i_j} ’s are distinct. Then $|G_a a_i| = \left| \left(G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}} \right) a_i \right| = k$ and $\forall j \leq k$, $\exists! g_j \in G_a \cap \mathcal{E}_{\{\text{aff}(a, \Delta b)\}}^{\text{ess}}$ such that $g_j a_i = a_{i_j}$. Then $m_{i_j} = m_i$ and $\sum_{j=1}^k m_{i_j} a_{i_j} = m_i \sum_{j=1}^k a_{i_j} = m_i k \bar{a}_i = \sum_{j=1}^k m_{i_j} \bar{a}_{i_j}$, because $\bar{a}_{i_j} = \bar{a}_i$, $\forall j \leq k$. Summing over all orbits gives $\alpha = \sum_{i=1}^N m_i \bar{a}_i$. So, the barycentre $\alpha/\mu = \frac{1}{\mu} \sum_{i=1}^N m_i \bar{a}_i \in \text{aff } \bar{a}$ because $\frac{1}{\mu} \sum_{i=1}^N m_i = 1$. Similarly, $\beta/\mu \in \text{aff } \bar{b}$. Thus if (m, a, b, t_0) is barycentric, $0 = \sum_{i=1}^N m_i a_i \in \text{aff } \bar{a}$ and $0 = \sum_{i=1}^N m_i b_i \in \text{aff } \bar{b}$: i.e., the coordinates are centroidal. ■

arbitrary coordinates

affine subspace of the
solution coordinates

centroidal coordinates

centre of mass coordinates

Figure 5-3: Hierarchy of Coordinate Frames

5.6 Summary of Results on Coordinates

Since the centroid of a set of points lies in the affine hull of that set, it is elementary that $\text{aff}(\bar{a}, \Delta \bar{b})$ is an affine subspace of $\text{aff}(a, \Delta b)$. Similarly, $\text{aff}(\bar{b}, \Delta \bar{a}) \subseteq \text{aff}(b, \Delta a)$. Thus if a problem is in centroidal coordinates it is in affine coordinates. Also, from the above theorem if it is in barycentric (or centre of mass) coordinates it is in centroidal coordinate. This hierarchy of coordinate frames is summarized in Figure 5-3 where as we move from top to bottom in the diagram we go from the more general case to the more special.

Example 9 *To demonstrate this hierarchy of coordinate frames, consider, in $\mathbf{R}^3$, the 4 body problem where $m = (1, 1, 1, 1)$ (equal masses),*

$$-a_2 = a_1 = (2, 0, 0),$$

$$\begin{aligned}
a_3 &= (1, 1, 0) \\
a_4 &= (-1, 1, 0) \text{ and} \\
b_1 &= b_2 = b_3 = b_4 = 0.
\end{aligned}$$

The bodies form a trapezium in the xy -plane. The group, $G(m, a, b)$ has order 4 generated by (π, H, H) and $(1_{S_4}, K, K)$, where $\pi = (12)(34)$, $H = \text{diag}(-1, 1, 1)$, (reflection in the yz -plane), 1_{S_4} is the identity map in S_4 and $K = \text{diag}(1, 1, -1)$, (reflection in the xy -plane). The Galilean coordinate transformation, $\mathcal{G}_{\alpha\beta}$ puts the problem in

1. arbitrary coordinates for any choice of α and β ,
2. affine subspace of the solution coordinates for $\alpha = (x, y, 0)$ and $\beta = (u, v, 0)$ for any choice of x, y, u and v ,
3. centroidal coordinates for $\alpha = (0, y, 0)$ and $\beta = (0, v, 0)$ for any choice of y and v ,
4. and finally, barycentric coordinates for $\alpha = (0, \frac{1}{2}, 0)$ and $\beta = (0, 0, 0)$.

Since the ivp is given in centroidal coordinates $G_a = G_b = \{I_3, H, K, HK\} \leq \mathcal{O}$ (see Thm. 34, p. 78) despite the fact that it is not barycentric. Regarding the structure theorem for the group, note that the essential group, $E(m, a, b) = \langle(\pi, H, H)\rangle$ and the inessential group, $\ker \xi_N = \langle(1_{S_4}, K, K)\rangle$.

In Thm. 8 p. 38 Ch. 3 we showed that the affine subspace of a solution of any N -body problem has a constant dimension and translates linearly with time. This means that there exists a Galilean coordinate frame for which the origin lies inside the affine subspace of the solution for all time. As shown in this theorem, this holds true for any s function even those for which the solution is not unique. The necessary and sufficient condition for having such a coordinate frame is that $0 \in \text{aff}(a, \Delta b)$

and $0 \in \text{aff}(b, \Delta a)$: i.e., it is determined by the initial values. In the present chapter we have shown (Cor. 29 p. 73) that the inessential parts, $\mathcal{E}_{(\text{aff}(a, \Delta b))}$ and $\mathcal{E}_{(\text{aff}(b, \Delta a))}$ of both the position and velocity groups G_a and G_b respectively, collapse into identical subgroups of $\mathcal{O}$ if and only if the ivp is in these abovementioned affine subspace of the solution coordinates. The next obvious question is ‘in what coordinate frames do the *essential* parts of G_a and G_b collapse into identical orthogonal groups?’. In Thm. 34 p. 78 we showed that this happens precisely when we are in centroidal coordinates: i.e., when $0 \in \text{aff}(\bar{a}, \Delta \bar{b})$ and $0 \in \text{aff}(\bar{b}, \Delta \bar{a})$. Since these coordinates form a special class of affine subspace of the solution coordinates we see a hierarchy of coordinate frames that progressively simplify the group of the initial value problem.

In this chapter we have confined our discussion to the ivp and have studied the structure of the group of the ivp. Our results of Ch. 3 apply to the whole solution. Analogously, it is natural to ask given a solution, q on an interval, I of an ivp, (m, a, b, t_0) , whether all centroidal coordinate frames for the ivp remain centroidal over the whole solution or even whether there exists a coordinate frame in which a solution is centroidal for all time. That will be addressed more fully in the next chapter. Nevertheless, we can go to barycentric coordinates which comprise a special case of centroidal coordinates. In this case, such a coordinate frame stays barycentric and hence centroidal over the whole solution.

Chapter 6

Groups and Their Orbits

In the last chapter, we studied the structure of the group of an ivp. We showed that the group is a direct product of an essential part associated with the affine subspace of the solution and an inessential part associated with its orthogonal complement. The essential part is finite and is isomorphic to the permutation group of the bodies. We showed how centroidal coordinates simplify the group.

In this chapter we look at the individual orbits of bodies where the term ‘orbits’ here refers to orbits under the action of the group of the ivp and *not* the orbits that bodies trace out as they move through space over time. Specifically, two bodies lie in the same orbit if one is mapped to the other by some triple that preserves the ivp as defined in Section 4.3 p. 52. The group of any ivp partitions the bodies into orbits. By selecting one such orbit we can define a new ivp containing only the bodies in that orbit. Such an ivp is called a derived ivp since it is derived from a larger problem: i.e., one with more bodies. In this chapter we show how the group of any ivp can be determined by the group of its derived problems. We prove that up to a relabelling of the bodies, the group of an ivp is just the intersection of the groups of its derived problems.

We then study a new way of defining an ivp. Instead of starting with the ivp and

finding the group, we start with the group and some “seed” body. We show what is required for these “ $\mathcal{G}$ -orbit” ivp’s to be defined and how each N -body problem can be defined as such a problem.

6.1 The Derived IVP

The reader may have observed that the group of the ivp has nothing to do with the s function (from Eqn. (3.1) p. 24). There is another interesting aspect to it. Let us look at the distinct orbits in a and b under the action of the group. They correspond to the orbits of the set $\{1, \dots, N\}$ under the permutation group, G_N as defined in 12 p. 59. Now let us play the following little game. Let us discard some of these orbits and *all* bodies in each discarded orbit to obtain a new N' -body problem for some $N' < N$. We assign to each remaining body, the mass, initial position and initial velocity that it had before. We then ask, “what is the relationship between the new group and the old one?” In this section we shall answer that question.

6.1.1 A Motivating Example

We begin with an example in $\mathbf{R}^3$ of a cube and an octagonal prism.

Example 10 *The complete symmetry group of the cube is $\mathcal{W}^*$ (of order 48) and that of the regular octagonal prism is $(\mathcal{H}_3^8)^*$ of order 32 (see App. C). Suppose that a body of mass μ_1 is placed at each of the 8 vertices of a cube and that a body of mass μ_2 is placed at each of the 16 vertices of an octagonal prism both of these objects centred at the origin. Let the faces of the cube be parallel to the xy , xz and yz planes. Let the octagonal faces of the prism be parallel to the xy plane and let 2 of the rectangles be parallel to each of the xz and yz planes; i.e., we have oriented these geometrical objects so that their faces are aligned as best as possible. To be specific, set the vertices of the cube to be $(\pm 1, \pm 1, \pm 1)$ and the vertices of the regular*

octagonal prism to be $(\pm 1, \pm \tan \pi/8, \pm 1)$ and $(\pm \tan \pi/8, \pm 1, \pm 1)$. Let the initial velocities all be zero. Then this 24-body problem is in centroidal coordinates so that $G_a = G_b$. The vertices of the cube form one orbit under the action of the group and the octagonal prism's vertices form the second orbit. Note that reflection through the plane, $z = x$ lies in $\mathcal{W}^*$ but not $(\mathcal{H}_3^8)^*$, whereas reflection through the plane, $y = x \tan \pi/8$ lies in $(\mathcal{H}_3^8)^*$ but not in $\mathcal{W}^*$. Thus, both geometrical objects limit the group, $G_a (= G_b)$ and in fact $G_a = G_b = (\mathcal{H}_3^8)^* \cap \mathcal{W}^* = (\mathcal{H}_3^4)^*$ which is the symmetry group of the "square prism", whose height differs from the sides of the square (i.e., this prism is not a cube) and $|(\mathcal{H}_3^4)^*| = 16$. In this problem there are 2 orbits, 1 of order 16 and 1 of order 8. $\text{aff } \bar{a}$ is still $\{0\}$. In the solution of this 24-body problem, the octagons in the prism may not (and usually don't) remain regular depending on the s function. The cube ceases to be a cube and becomes a square prism whose four vertical edges need not be the same length as its eight horizontal edges.

6.1.2 Partitioning the Bodies into Orbits

As in the above example, any N -body problem, (m, a, b, t_0) gives rise to its group, $G(m, a, b)$ which partitions the set $\mathbf{N} = \{1, 2, \dots, N\}$ and the vectors m, a and b into corresponding orbits. For example, given some $j \leq N$ if we let $G_N j = \{j_1, \dots, j_k\}$ (i.e., the set of all bodies in the orbit of body j), then by Def. 10 p. 52, Ch. 4, $m_{j_1} = \dots = m_{j_k}$, $G_a a_j = \{a_{j_1}, \dots, a_{j_k}\}$ and $G_b b_j = \{b_{j_1}, \dots, b_{j_k}\}$. We now need some notation in order to work with these orbits. Let us start by numbering them based on their order of occurrence.

Definition 18 *Given the N -body problem (m, a, b, t_0) let n be the number of orbits under the action of the group $G(m, a, b)$. Inductively define the sequence, $i_1, \dots, i_n$ as follows: let $i_1 = 1$ and $\forall l$, where $2 \leq l \leq n$, let i_l be the smallest number not in the set, $\bigsqcup_{j=1}^{l-1} G_N i_j$; i.e., the first body not in one of the orbits already labelled. ($\bigsqcup$ indicates disjoint union). Furthermore, $\forall l \leq n$, let $k_l = |G_N i_l|$ (the number of*

bodies in orbit l) and let us label the l -th orbit, $G_N i_l$ as

$$\Omega_l = G_N i_l = \{i_{l,1}, \dots, i_{l,k_l}\}$$

where we choose $i_l = i_{l,1} < i_{l,2} < \dots < i_{l,k_l}$.

Define $P_l : \{1, \dots, k_l\} \rightarrow \Omega_l$ by

$$P_l(j) = i_{l,j},$$

define $w_l : (S_N)_{\{\Omega_l\}} \rightarrow S_{k_l}$ by

$$w_l(\pi) = P_l^{-1} \circ \pi \circ P_l,$$

define $\Psi_l : (S_N)_{\{\Omega_l\}} \times \mathcal{E} \times \mathcal{E} \rightarrow S_{k_l} \times \mathcal{E} \times \mathcal{E}$ by

$$\Psi_l(\pi, f, g) = (w_l(\pi), f, g).$$

$$\Pi_N = \bigcap_{l=1}^n (S_N)_{\{\Omega_l\}} \leq S_N \text{ and}$$

define $w : \Pi_N \rightarrow \prod_{l=1}^n S_{k_l}$ by

$$w(\pi) = (w_1(\pi), \dots, w_n(\pi)).$$

Clearly, $\mathbf{N} = \{1, 2, \dots, N\} = \coprod_{l=1}^n \Omega_l$ and $N = \sum_{l=1}^n k_l$. $P_l(j)$ is the j th body in orbit l . P_l^{-1} is a renumbering of the bodies in orbit l from 1 to k_l ; so, clearly, P_l is a bijection. Since S_N is a group acting on the set $\mathbf{N} = \{1, \dots, N\}$, Π_N is the subgroup of S_N comprising all permutations for which the set, Ω_l is an invariant subset for each $l = 1, \dots, n$. In other words, a permutation in Π_N permutes the numbers (bodies) in Ω_1 among themselves, Ω_2 among themselves, etc.

In Fig. for Example 10 above, $i_1 = 1$, $i_2 = 9$, $k_1 = 8$, $k_2 = 16$.

$$\Omega_1 = \{1, \dots, 8\}$$

$$\Omega_2 = \{9, \dots, 24\}$$

$$P_1(i) = i \quad (i \leq 8)$$

$$P_2(i) = i + 8 \quad (i \leq 16).$$

If $\pi \in S_{24} \setminus \{\Omega_2\}$ is the permutation $\pi = (9, 10, \dots, 16) (17, 18, \dots, 24)$ then despite the fact that $\pi \notin G_N(m, a, b)$ as the reader can verify, we still have that $\psi_2(\pi) \in S_{16}$ is given by

$$\begin{aligned}\psi_2(\pi) &= (1, 2, \dots, 8) (9, 10, \dots, 16) \\ \psi_1(\pi) &= 1_{S_8} \text{ and} \\ \psi(\pi) &= (1_{S_8} \cdot (1, 2, \dots, 8) (9, 10, \dots, 16)).\end{aligned}$$

Lemma 36 *Given the N -body problem (m, a, b, t_0) with n orbits in $G(m, a, b)$, then $\forall l \leq n$,*

1. ψ_l and Ψ_l are group homomorphisms and
2. ψ is a group isomorphism.

Proof. *Show 1:* To show that ψ_l is well defined, first note that $\forall j \leq k_l, \forall \pi \in (S_N)_{\{\Omega_l\}}, P_l(j) \in \Omega_l$; so, $\pi \circ P_l(j) \in \Omega_l$ and $P_l^{-1} \circ \pi \circ P_l(j) \leq k_l$. Also, being the composition of bijections, $P_l^{-1} \circ \pi|_{\Omega_l} \circ P_l = P_l^{-1} \circ \pi \circ P_l$ is a bijection on $\mathbf{k}_l = \{1, \dots, k_l\}$. Since $\psi_l(\rho\pi) = P_l^{-1} \rho \pi P_l = P_l^{-1} \rho P_l P_l^{-1} \pi P_l = \psi_l(\rho) \psi_l(\pi)$ and since a similar result holds for Ψ_l , ψ_l and Ψ_l are group homomorphisms.

Show 2: $\forall l \leq n, \Pi_N \leq (S_N)_{\{\Omega_l\}}$; so, $\psi_l|_{\Pi_N}$ is also a homomorphism. Since ψ is a direct product of homomorphisms, it is also a homomorphism. $\forall \pi \in \ker \psi, \forall l \leq n, \psi_l(\pi) = P_l^{-1} \circ \pi|_{\Omega_l} \circ P_l = 1$. Since $\pi|_{\Omega_l}$ is a bijection on Ω_l , $\pi|_{\Omega_l} = P_l P_l^{-1} = 1_{\Omega_l}$. Since $\mathbf{N} = \coprod_{l=1}^n \Omega_l$, $\pi = \pi|_{\mathbf{N}} = 1_{S_N}$. $\therefore \psi$ is 1-1. To show that ψ is onto, consider any $(\rho_1, \dots, \rho_n) \in \prod_{l=1}^n S_{k_l}$. Then, $\forall l \leq n, \rho_l \in S_{k_l}$; so, let $\pi \in \Pi_N$ be defined as follows: $\forall h \in \mathbf{N}, \exists! l \leq n$ such that $h \in \Omega_l$. Define $\pi(h) = P_l \rho_l P_l^{-1}(h)$. Clearly, $\pi|_{\Omega_l}$ is a bijection and since l is arbitrary, $\pi \in \Pi_N$. Also, $\psi_l(\pi) = P_l^{-1} P_l \rho_l P_l^{-1} P_l = \rho_l$; thus, $\psi(\pi) = (\rho_1, \dots, \rho_n)$ and ψ is onto; i.e., ψ is an isomorphism. ■

By Part 2 of the above lemma, $|\Pi_N| = |\prod_{l=1}^n S_{k_l}| = \prod_{l=1}^n |S_{k_l}| = \prod_{l=1}^n (k_l!)$.

6.1.3 Definition of the Derived IVP

Definition 19 Given the N -body problem (m, a, b, t_0) with n orbits under the action of $G(m, a, b)$ and as defined above, for any $l \leq n$, $\Omega_l = G_N i_l = \{i_{l,1}, \dots, i_{l,k_l}\}$, define

$$\begin{aligned}\mu_l &= (m_{i_{l,1}}, \dots, m_{i_{l,k_l}})^t \in \mathbf{R}^{k_l}, \\ x_l &= (a_{i_{l,1}}, \dots, a_{i_{l,k_l}})^t \in \mathbf{R}^{dk_l} \text{ and let} \\ y_l &= (b_{i_{l,1}}, \dots, b_{i_{l,k_l}})^t \in \mathbf{R}^{dk_l}.\end{aligned}$$

Define the **derived ivp for orbit l** to be the k_l -body problem, (μ_l, x_l, y_l, t_0) and denote its group as $G_l(m, a, b)$. Furthermore, we can make the obvious rearrangement and partition,

$$\begin{aligned}\mu &= (\mu_1, \dots, \mu_n)^t \in \mathbf{R}^N \\ x &= (x_1, \dots, x_n)^t \in \mathbf{R}^{dN} \\ y &= (y_1, \dots, y_n)^t \in \mathbf{R}^{dN}.\end{aligned}$$

The N -body problem (μ, x, y, t_0) is called the **arrangement and partition of the ivp (m, a, b, t_0) into its derived components**.

Of course all bodies in any orbit have the same mass: so, $m_{i_{l,1}} = m_{i_{l,2}} = \dots = m_{i_{l,k_l}} = m_{i_l}$. Note that in the above definition, the group of the k_l -body problem (μ_l, x_l, y_l, t_0) would normally be called $G(\mu_l, x_l, y_l)$ but we have given it the label, $G_l(m, a, b)$ because it arose from the derived problem for orbit l of (m, a, b, t_0) . In Example 10, there are two derived ivp's, one for the cube and one for the octagonal prism with respective isometry groups, W^* and D_8^* as discussed earlier.

6.1.4 An IVP and its Derived Problems

We are now in a position to answer the question posed at the beginning of this section. Given an N -body problem, we can find its group, $G(m, a, b)$. Having done

that, we can then partition the bodies into orbits under the action of that group. Each orbit (say orbit l), defines a new ivp, called its derived problem. It is obtained by throwing away the bodies in all the other orbits and reassigning the same initial positions and velocities to the bodies being kept. What is the relationship between $G(m, a, b)$ and the collection of groups, $G_l(m, a, b)$ for the derived problems? The formalism of the N -body problem causes us to renumber the bodies in orbit l when defining the derived problem. Aside from this technicality, we have the following happy result. First, the original or larger problem has a group that is (aside from this renumbering) a subgroup of each derived problem. Furthermore, we can recover the original group, $G(m, a, b)$, from the derived groups by simply (again, aside from this technicality) taking the intersection of the groups of the derived problems. Using Definitions 18 and 19 above, this result is stated more concisely as follows:

Theorem 37 *Given the N -body problem (m, a, b, t_0) with n orbits,*

$$G(m, a, b) = \bigcap_{l=1}^n \Psi_l^{-1}(G_l(m, a, b))$$

Proof. Fix $(\pi, f, g) \in G(m, a, b)$. Given the notation of Definitions 18 and 19, $\forall l \leq n$, $j \leq k_l$, $x_{l,j} = a_{i_{l,j}} \in \mathbf{R}^d$ is the initial position of body $i_{l,j}$ in (m, a, b, t_0) .

$$f(x_{l,j}) = f(a_{i_{l,j}}) = a_{\pi(i_{l,j})} = a_{\pi \circ P_l(j)} = x_{l, P_l^{-1} \circ \pi \circ P_l(j)}$$

where the last step arises because $\pi \circ P_l(j) \in \Omega_l$. Similarly, $g(y_{l,j}) = y_{l, P_l^{-1} \circ \pi \circ P_l(j)}$; so, since all masses in μ_l are the same and as shown above, $P_l^{-1} \circ \pi \circ P_l \in S_{k_l}$, then

$$(P_l^{-1} \circ \pi \circ P_l, f, g) \in G(\mu_l, x_l, y_l) = G_l(m, a, b).$$

$\therefore \Psi_l(\pi, f, g) \in G_l(m, a, b)$; i.e., $(\pi, f, g) \in \Psi_l^{-1}(G_l(m, a, b))$. Since l is arbitrary, $(\pi, f, g) \in \bigcap_{l=1}^n \Psi_l^{-1}(G_l(m, a, b))$; i.e., $G(m, a, b) \subseteq \bigcap_{l=1}^n \Psi_l^{-1}(G_l(m, a, b))$.

Conversely, $\forall (\pi, f, g) \in \bigcap_{l=1}^n \Psi_l^{-1}(G_l(m, a, b))$, $\forall h \leq N$, $\exists ! l \leq n$ and a $! j \leq k_l$ such that $h = i_{l,j} \in \Omega_l$. Then $(P_l^{-1} \circ \pi \circ P_l, f, g) = \Psi_l(\pi, f, g) \in G_l(m, a, b) =$

$$G(\mu_l, x_l, y_l).$$

$$f(a_h) = f(a_{i_{l,j}}) = f(a_{P_l(j)}) = f(x_j) = x_{P_l^{-1}\pi P_l(j)} = a_{\pi(i_{l,j})} = a_{\pi(h)}.$$

similarly, $g(b_h) = b_{\pi(h)}$. Since h is an arbitrary member of $\mathbf{N}$, $(\pi, f, g) \in G(m, a, b)$. ■

Note that since any derived problem, (μ_l, x_l, y_l, l_0) has only one orbit, the affine subspaces $\text{aff } \bar{x}_l$ and $\text{aff } \bar{y}_l$, spanned by the centroids of positions and velocities are singletons. Thus, $\text{aff } (\bar{x}_l, \Delta \bar{y}_l)$ and $\text{aff } (\bar{y}_l, \Delta \bar{x}_l)$ are also singletons. Furthermore, the centroidal affine subspace, $\text{aff } (\bar{a}, \Delta \bar{b})$ is just the affine hull of the centroids of these orbits. Similarly for $\text{aff } (\bar{b}, \Delta \bar{a})$. Thus we have the

Lemma 38 *The centroidal position (velocity) affine subspace of $G(m, a, b)$ is the affine hull of the centroidal position (velocity) affine subspaces of its derived problems.*

6.2 $\mathcal{G}$ -Orbit IVP's

We now study a new way of defining an N -body problem where instead of starting with the bodies and finding the group, we start with a group and some "seed" body to define the ivp. In essence, the seed body will generate a single group orbit.

Definition 20 *In $\mathbf{R}^d$, suppose we are given a group, $\mathcal{G} \leq \mathcal{O}(d)$, a mass $\mu \in \mathbf{R}^+$, an initial position $x \in \mathbf{R}^d$ and an initial velocity $y \in \mathbf{R}^d$ with the following properties:*

1. *the orbit $\mathcal{G}x$ is finite, say $|\mathcal{G}x| = N$,*
2. *$y \in \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} x$,*
3. *the centroids, $\bar{x}_{\mathcal{G}} = 0$ and $\bar{y}_{\mathcal{G}} = 0$ and*
4. *$\mathcal{G} = \bigcap_{\tau \in \mathbf{R}} \mathcal{E}_{\{\mathcal{G}(x+\tau y)\}}$.*

By Part 1 we can let $\mathcal{G}x = \{a_1, \dots, a_N\}$ where the a_i 's are all distinct. Then for each $i \leq N$ we can write $a_i = H_i x$ for some $H_i \in \mathcal{G}$. Given this we can let $b_i = H_i y$ and let

$$\begin{aligned} a &= (a_1, \dots, a_N) \in \mathbf{R}^{dN}, \\ b &= (b_1, \dots, b_N) \in \mathbf{R}^{dN} \text{ and} \\ m &= (\mu, \dots, \mu) \in \mathbf{R}^N. \end{aligned}$$

Then we say that the ivp (m, a, b, t_0) is the **$\mathcal{G}$ -orbit ivp** for the seed body with mass μ , initial position x , initial velocity y and initial time, t_0 .

If one ivp is obtained from another by relabelling the bodies then its solution is obtained from the other by a corresponding relabelling (Section 4.1 p. 46). Given $\mathcal{G}$, μ , x and y as above, we are free to arrange the bodies in any of $N!$ ways: so, the $\mathcal{G}$ -orbit ivp is a uniquely defined ivp only up to labelling the bodies. We now check that the $\mathcal{G}$ -orbit ivp is well-defined. First note from Part 2 of the definition and L. 78 p. 204 App. A, that $\mathcal{G}y$ is a finite set; so, it makes sense to speak of the centroid, $\bar{y}_{\mathcal{G}}$. We now check that each b_i is independent of the choice of H_i used to obtain a_i . Since $\forall L \in \mathcal{G}$

$$Lx = H_i x \Leftrightarrow H_i^{-1}L = \text{stab}_{\mathcal{G}}x \Leftrightarrow L \in H_i \text{stab}_{\mathcal{G}}x$$

then the set of all $L \in \mathcal{G}$ such that $Lx = H_i x$ is simply the left coset, $H_i \text{stab}_{\mathcal{G}}x$. So, any other representation of a_i has the form $a_i = H_i K x$ where $K \in \text{stab}_{\mathcal{G}}x \leq \text{stab}_{\mathcal{G}}y$ by Part 2 of the definition and Part 1, L. 77 p. 204 App. A. Thus any other choice for the initial velocity of body i is $\tilde{b}_i = H_i K y = H_i y = b_i$ as desired and the ivp (m, a, b, t_0) is uniquely defined up to rearrangement of the bodies. Also, since a is defined from the finite set, $\mathcal{G}x$ then a is not on the diagonal.

We now discuss the four statements given in the above definition. The first is required to generate finitely many bodies from the seed body. The second is essential

to give a uniquely defined problem: for, suppose contrary to the definition we had $y \notin \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} x$. Then again by Part 1, L. 77 p. 204 the orbit, $(\text{stab}_{\mathcal{G}} x) y$ would not just be the singleton $\{y\}$ and we could choose some $K \in \text{stab}_{\mathcal{G}} x$ such that $Ky \neq y$. Then we would have one body initially at $Kx = x$ with initial velocity $Ky \neq y$ and another also initially at x but with different velocity y . This means that the ivp would lie on the diagonal contrary to the definition of the N -body problem in (3.1) p. 24. Statement 3 of the definition places both position and velocity centroids of the bodies at the origin which we can do without loss of generality. Since there is a single orbit under the action of the group the problem is also in barycentric coordinates. Finally for statement 4, any subgroup of $\mathcal{E}$ is a subgroup of the complete symmetry group of any of its orbits (Cor. 110 p. 223): so, $\mathcal{G} \leq \cap_{\tau \in \mathbf{R}} \mathcal{E}_{\{\mathcal{G}(x+\tau y)\}}$. By maximizing $\mathcal{G}$ subject to this constraint we avoid the multiple ways of defining the same ivp as a $\mathcal{G}$ -orbit ivp for various choices of the group, $\mathcal{G}$ that could occur otherwise. For example, the set of vertices of the regular octahedron is the orbit of a single point under the action of any one of the following groups: $\mathcal{T}$, $\mathcal{T}^*$, $\mathcal{W}|\mathcal{T}$, $\mathcal{W}$ and $\mathcal{W}^*$ (see App. C for descriptions of these groups). Thus an ivp with 6 bodies of equal mass and zero initial velocity at each vertex of such a body could be a $\mathcal{G}$ -orbit ivp for any one of these groups if statement 4 were not included: however, under the definition, only the last group can be used.

Theorem 39 *Given $\mathcal{G} \leq \mathcal{O}(d)$, $\mu \in \mathbf{R}^+$, $x, y \in \mathbf{R}^d$ satisfying Def. 20 above, let m, a and b be as defined therein. Then $\forall t_0 \in \mathbf{R}$ the ivp (m, a, b, t_0) has one orbit, it is in centroidal coordinates and $G_a = G_b = \mathcal{G}$.*

Proof. Let $A = \{a_1, \dots, a_N\}$, $B = \{b_1, \dots, b_N\}$ and choose each H_i as in the definition. We first show that $\forall K \in \mathcal{G}$, $\exists \pi \in S_N$ such that $(\pi, K, K) \in G(m, a, b)$. Fix $K \in \mathcal{G}$. Then $\mathcal{G}A = \mathcal{G}\{H_i\}_{i=1}^N x = \mathcal{G}x = A$; so, $KA \subseteq A$. Since K being an isometry is 1-1 hence $K|_A$ is an injective map on the finite set, A : i.e., it is bijective. For each $i \leq N$ let $Ka_i = a_{\pi(i)}$. Then, $\pi \in S_N$, $\forall i \leq N$, $(KH_i)x = Ka_i = a_{\pi(i)}$.

From this and the definition of the b_i 's we must have $(KH_i)y = b_{\pi(i)}$. With $b_i = H_i y$ this gives $Kb_i = (KH_i)y = b_{\pi(i)}$. Thus, by D. 10 p. 52 (π, K, K) preserves the ivp (m, a, b, t_0) : i.e., it lies in $G(m, a, b)$ as desired. Since K is an arbitrary member of $\mathcal{G}$, $\mathcal{G} \leq G_a$ and $\mathcal{G} \leq G_b$. Furthermore, since $A = \mathcal{G}x \subseteq G_a x$ and $G_a \leq \mathcal{E}_{\{A\}}$ (by L. 21 p. 60) we must have $G_a x = A$: so, there is only one orbit. By Part 3 of the definition $\text{aff}_{\mathbf{R}^d} \bar{a} = \text{aff}_{\mathbf{R}^d} \bar{b} = \{0\}$ and the ivp is in centroidal coordinates. Finally, by Part 1 of Thm. 23 p. 63 Ch. 5 $\forall \tau \in \mathbf{R}$, $G_a \leq \mathcal{E}_{\{a+\tau b\}} = \mathcal{E}_{\{\mathcal{G}(x+\tau y)\}}$: so, $G_a \leq \cap_{\tau \in \mathbf{R}} \mathcal{E}_{\{\mathcal{G}(x+\tau y)\}} = \mathcal{G}$. Since we have shown above the reverse inclusion we have $G_a = \mathcal{G}$. We have also shown that the ivp is in centroidal coordinates: so, by 34 p. 78 Ch. 5, $G_a = G_b$ and the theorem is proved. ■

6.3 The Potential

Definition 21 *If q is a solution of the N -body problem we define the contribution of body i to the total potential to be*

$$U_i(q) = m_i \sum_{\substack{j=1 \\ j \neq i}}^N m_j u(\|q_{ij}\|). \quad (6.1)$$

Then clearly

$$U(q) = \frac{1}{2} \sum_{i=1}^N U_i(q). \quad (6.2)$$

Theorem 40 *If q is a solution of the N -body problem and bodies i and k lie on the same orbit under the action of the group then*

$$U_i(q) = U_k(q).$$

Proof. If bodies i and k lie on the same orbit then $\exists (\pi, f, g) \in G(m, a, b)$ such that $k = \pi(i)$, $m_k = m_i$ and

$$U_k = m_k \sum_{\substack{j=1 \\ j \neq k}}^N m_j u(\|q_{kj}\|)$$

If we substitute $j' = \pi^{-1}(j)$, then as j ranges over $\mathbf{N} \setminus \{k\}$, j' ranges over $\mathbf{N} \setminus \{i\}$, where we define $\mathbf{N} = \{1, 2, \dots, N\}$. Then

$$\begin{aligned}
U_k(q) &= m_k \sum_{\substack{j'=1 \\ j' \neq i}}^N m_{\pi(j')} u(\|q_{\pi(j')} - q_{\pi(i)}\|) \\
&= m_k \sum_{\substack{j'=1 \\ j' \neq i}}^N m_{j'} u(\|f(q_{j'}) - f(q_i)\|) \\
&= m_i \sum_{\substack{j'=1 \\ j' \neq i}}^N m_{j'} u(\|q_{ij'}\|) \\
&= U_i(q)
\end{aligned}$$

where we used the facts $m_k = m_{\pi(i)} = m_i$, $m_{\pi(j)} = m_{j'}$ and f is an isometry. ■

Corollary 41 *If q is the solution of a $\mathcal{G}$ -orbit i.e. then the total potential is*

$$U(q) = \frac{m^2 N}{2} \sum_{\substack{k=1 \\ k \neq i}}^N u(\|q_{ik}\|) \quad (6.3)$$

where m is the common mass of all bodies and the choice of i is arbitrary.

Proof. Fix $i \leq N$, the number of bodies. Then by the theorem, $U_k = U_i, \forall k \leq N$ and by (6.2)

$$U(q) = \frac{1}{2} \sum_{i=1}^N U_i(q) = \frac{N}{2} U_i(q)$$

which gives the result on substitution into (6.1). ■

Chapter 7

The Group of a Solution

Recall that in Appendix B we laid down some theory on symmetry groups in Euclidean spaces. That part of the work is in the realm of geometry and strictly speaking the mention of the N -body problem is not essential to its development. Nevertheless, everything in that appendix was done specifically to be applied to this problem. In Chapter 5 we made ample use of the theory from App. B and established the structure of the group of the ivp (m, a, b, t_0) . At no point did we venture into the solution $q(t)$ for t in some interval I . All references to positions used $q(t_0) = a$ and all references to velocities were to $\dot{q}(t_0) = b$. However, it is obvious that any such solution, q is also a solution of the ivp $(m, q(t), \dot{q}(t), t)$, $\forall t \in I$ and any theorems developed in Chapter 5 apply to this (more general) ivp. Furthermore, we can speak of the group $G(m, q(t), \dot{q}(t))$ as a function of t as was given in Def. 11, p. 54, Ch. 4. This leads us to the obvious question of how the group depends on t .

7.1 Variation of the Group over Time

We know from Thm. 34, p. 78, Ch. 5 that the group of an ivp has a simpler form when it is in centroidal coordinates ($G_a = G_b \leq \mathcal{O}$). However, unlike affine subspace

of the solution coordinates (Thm. 8, p. 38, Ch. 3), we have not yet established whether an ivp in centroidal coordinates (i.e., $0 \in \text{aff}(\bar{a}, \Delta\bar{b}) \cap \text{aff}(\bar{b}, \Delta\bar{a})$) stays in these coordinates over each solution. We now give an obvious extension of the ideas developed in Def. 6, p. 36, Ch. 3 and Def. 17, p. 76, Ch. 5.

Definition 22 *If q is a solution of (m, a, b, t_0) on I and $t \in I$, define $\text{aff}(\overline{q(t)}, \Delta\overline{\dot{q}(t)})$ to be the **centroidal position affine subspace** of the solution at time t and $\text{aff}(\overline{\dot{q}(t)}, \Delta\overline{\dot{q}(t)})$ to be the **centroidal velocity affine subspace** at time t . Also $\langle \Delta\overline{q(t)}, \Delta\overline{\dot{q}(t)} \rangle$ is called the **centroidal linear subspace** at time t . If $0 \in \overline{q(t)}\Delta\overline{\dot{q}(t)}$ and $0 \in \overline{\dot{q}(t)}\Delta\overline{\dot{q}(t)}$ then the solution q at time t is said to be in **centroidal coordinates**.*

Recall Theorem 8, p. 38, Ch. 3 where we showed that for any continuous s function and any solution q on I , the linear subspace of the solution, $\langle \Delta q, \Delta \dot{q} \rangle$ is a constant over I , that the position affine subspace of the solution, $\text{aff}(q, \Delta q)$ is a linear function of t added to $\langle \Delta q, \Delta \dot{q} \rangle$ and the velocity affine subspace, $\text{aff}(\dot{q}, \Delta \dot{q})$ is a constant vector plus $\langle \Delta q, \Delta \dot{q} \rangle$. Recall also that the centroidal position affine subspace $\text{aff}(\overline{q(t)}, \Delta\overline{\dot{q}(t)}) \subseteq \text{aff}(q(t), \Delta\dot{q}(t))$ and that $\text{aff}(\overline{\dot{q}(t)}, \Delta\overline{\dot{q}(t)}) \subseteq \text{aff}(\dot{q}(t), \Delta\dot{q}(t))$; i.e., the centroidal position and velocity affine subspaces move inside their corresponding solution affine subspaces. It would be a happy situation indeed, if we could obtain an analogous result for the centroidal subspace $\langle \Delta\overline{q(t)}, \Delta\overline{\dot{q}(t)} \rangle$: namely, that it is a constant over I and that $\text{aff}(\overline{q(t)}, \Delta\overline{\dot{q}(t)})$ moves linearly with t , etc. However, as we shall see, such is not the case in the general setting. We shall encounter solutions for which the centroidal affine subspace increases and decreases in dimension. Nevertheless, in the case where the s function gives rise to a unique solution, we shall have the pleasant result that the centroidal affine subspace of a solution is a constant and if we choose centroidal coordinates for the ivp we stay in centroidal coordinates over the whole solution.

One of the advantages of being able to transform to affine subspace of the solution coordinates is that the inessential parts of G_a and G_b , namely $\mathcal{E}_{(\text{aff}(q, \Delta \dot{q}))}$ and $\mathcal{E}_{(\text{aff}(\dot{q}, \Delta q))}$ respectively, become identical subgroups of $\mathcal{O}$ (Corollary 29, p. 73, Ch. 5). To focus on centroidal coordinates would further collapse the essential parts to identical orthogonal groups (Thm. 34, p. 78, Ch. 5). This would position us to attack the question of whether (in such coordinates), the group remains a constant over the solution. In fact, we need not even require that $\langle \overline{\Delta q(t)}, \overline{\Delta \dot{q}(t)} \rangle$ remain a constant. In this chapter we shall show that given any ivp we can always place it in barycentric coordinates which are centroidal and in such a frame there *exists* a solution that remains centroidal over the whole solution. Furthermore, out of this result will drop the corollary that if s gives uniqueness then every solution of a centroidal ivp remains centroidal for all t .

In this discussion we have not addressed what happens to the group as the centroidal affine subspace goes up and down in size: nevertheless, the reader now knows basically where we are headed.

7.2 Preservation of the Group

In the previous section we discussed some pathological cases. However, in the case where the s function gives unique solutions we can be given an ivp where the centroidal positional affine subspace is moving. This causes the group to contain a translational part that also varies with time. The following theorem addresses these situations and prepares the way for a definition of solutions that preserve the group by taking this into account.

Theorem 42 *Suppose we are given q , a solution of (m, a, b, t_0) on I and $b_0 \in \mathbf{R}^d$.*

1. Then the following are equivalent:

$$(a) \quad \forall t \in I, G(m, q(t), \dot{q}(t)) = G(m, a + (t - t_0)b_0, b),$$

(b) $\forall a_0 \in \mathbf{R}^d$ if we let $r = \mathcal{G}_{a_0 b_0}(q)$, then $\forall t \in I$.

$$G\left(m, r(t), \dot{r}(t)\right) = G\left(m, a - a'_0, b - b'_0\right);$$

i.e., the group of the solution, r is a constant.

(c) $\exists a_0 \in \mathbf{R}^d$ such that if we let $r = \mathcal{G}_{a_0 b_0}(q)$, then $\forall t \in I$.

$$G\left(m, r(t), \dot{r}(t)\right) = G\left(m, a - a'_0, b - b'_0\right).$$

2. Furthermore, if any and hence all of the above statements are true, then

(a) $\Pi_1(G(m, q(t), \dot{q}(t))) \leq S_N$ is a fixed permutation group $\forall t \in I$.

(b) $b_0 \in \text{aff}(\bar{b}, \Delta \bar{a})$ and

(c) if $a_0 \in \text{aff}(\bar{a}, \Delta \bar{b})$ and $r = \mathcal{G}_{a_0 b_0}(q)$ then r is in centroidal coordinates $\forall t \in I$

where by $\bar{a}$ and $\bar{b}$ we mean $\bar{a}_{G_n}$ and $\bar{b}_{G_n}$, respectively; i.e., in $G(m, a, b)$.

Before proving the theorem we note that if $r = \mathcal{G}_{a_0 b_0}(q)$: i.e., $r(t) = q(t) - a'_0 - (t - t_0)b'_0$ then r is a solution of $(m, a - a'_0, b - b'_0, t_0)$ on I (L. 7, p. 33, Ch. 3) and $G\left(m, r(t_0), \dot{r}(t_0)\right) = G\left(m, a - a'_0, b - b'_0\right)$.

Proof. For brevity, let $\Gamma_{a_0, b_0} = (1_{S_N}, T_{a_0 + (t - t_0)b_0}, T_{b_0}) \in S_N \times \mathcal{E} \times \mathcal{E}$.

Show $1(a) \Rightarrow 1(b)$: Assume 1(a). Fix $a_0 \in \mathbf{R}^d$ and let $r(t) = q(t) - a'_0 - (t - t_0)b'_0$ on I . Then $\dot{r}(t) = \dot{q}(t) - b'_0$ and $\forall t \in I$.

$$\begin{aligned} G\left(m, r(t), \dot{r}(t)\right) &= G\left(m, q(t) - a'_0 - (t - t_0)b'_0, \dot{q}(t) - b'_0\right) \\ &= \Gamma_{a_0, b_0} G\left(m, q(t), \dot{q}(t)\right) \Gamma_{a_0, b_0}^{-1} && (\text{Thm. 20, p. 54}) \\ &= \Gamma_{a_0, b_0} G\left(m, a + (t - t_0)b'_0, b\right) \Gamma_{a_0, b_0}^{-1} && (\text{by hypothesis}) \\ &= G\left(m, a - a'_0, b - b'_0\right) && (\text{Thm. 20, p. 54}) \end{aligned}$$

which is independent of t .

Show $1(b) \Rightarrow 1(c)$: This is obvious since $\mathbf{R}^d$ is non-empty.

Show $1(c) \Rightarrow 1(a)$: Given a_0 and r as in Part 1(c), $\forall t \in I$.

$$\begin{aligned}
 G\left(m, q(t), \dot{q}(t)\right) &= G\left(m, r(t) + a'_0 + (t - t_0) b'_0, \dot{r}(t) + b'_0\right) \\
 &= \Gamma_{a_0, b_0}^{-1} G\left(m, r(t), \dot{r}(t)\right) \Gamma_{a_0, b_0} \quad (\text{Thm. 20. p. 54}) \\
 &= \Gamma_{a_0, b_0}^{-1} G\left(m, a - a'_0, b - b'_0\right) \Gamma_{a_0, b_0} \quad (\text{by hypothesis}) \\
 &= G\left(m, a + (t - t_0) b'_0, b\right) \quad (\text{Thm. 20. p. 54})
 \end{aligned}$$

This completes the proof of equivalence.

Show 2(a). If Part 1 holds then from Part 1(a) and Thm. 20. p. 54 we have

$$\begin{aligned}
 \Pi_1\left(G\left(m, q(t), \dot{q}(t)\right)\right) &= \Pi_1\left(G\left(m, a + (t - t_0) b'_0, b\right)\right) \\
 &= \Pi_1\left(\left(1_{S_N}, T_{-(t-t_0)b_0}, 1_{\mathcal{E}}\right) G\left(m, a, b\right) \left(1_{S_N}, T_{(t-t_0)b_0}, 1_{\mathcal{E}}\right)\right) \\
 &= \Pi_1\left(G\left(m, a, b\right)\right).
 \end{aligned}$$

a constant, because $\Pi_1\left(\left(1_{S_N}, T_{-(t-t_0)b_0}, 1_{\mathcal{E}}\right)\right) = \Pi_1\left(\left(1_{S_N}, T_{(t-t_0)b_0}, 1_{\mathcal{E}}\right)\right) = 1_{S_N}$.

Show 2(b). $\forall a_0 \in \mathbf{R}^d$, $\forall g \in G_b = \Pi_3(G(m, a, b))$, $\exists \pi \in S_N$ and $f \in \mathcal{E}$ such that $(\pi, f, g) \in G(m, a, b)$. By Thm. 20 p. 54 Ch. 4.

$$(1_{S_N}, T_{a_0}, T_{b_0}) G(m, a, b) (1_{S_N}, T_{-a_0}, T_{-b_0}) = G(m, a - a'_0, b - b'_0);$$

so. $(\pi, T_{a_0} f T_{-a_0}, T_{b_0} g T_{-b_0}) \in G(m, a - a'_0, b - b'_0) = G(m, r(t), \dot{r}(t))$, $\forall t \in I$ by Part 1(b). Let $f = T_u H$ and $g = T_v H$. Then $\forall t \in I$, $i \leq N$, we have

$$T_{a_0} f T_{-a_0}(r_i(t)) = r_{\pi(i)}(t);$$

i.e., $H(r_i(t) + a_0) - a_0 - u = T_{a_0} f T_{-a_0}(r_i(t)) = r_{\pi(i)}(t)$. Differentiating gives

$$H \dot{r}_i(t) = \dot{r}_{\pi(i)}(t).$$

Also,

$$H(\dot{r}_i(t) + b_0) - v - b_0 = T_{b_0} g T_{-b_0}(\dot{r}_i(t)) = \dot{r}_{\pi(i)}(t).$$

Combining these two results gives $v = Hb_0 - b_0$. Thus, $g(b_0) = Hb_0 - v = b_0$. Since g is an arbitrary member of G_b , $b_0 \in \text{fix}_{\mathbf{R}^d} G_b = \text{aff}(\bar{b}, \Delta \bar{a})$.

Show 2(c). We are given that $a_0 \in \text{aff}(\bar{a}, \Delta \bar{b}) = \text{fix}_{\mathbf{R}^d} G_a$ by Thm. 32 p. 75 Ch. 5. By L. 93 p. 213 App. A, $\text{fix}_{\mathbf{R}^d} G_a = a_0 + \text{fix}_{\mathbf{R}^d} T_{a_0} G_a T_{-a_0}$ thus $0 \in \text{fix}_{\mathbf{R}^d} T_{a_0} G_a T_{-a_0} = \text{fix}_{\mathbf{R}^d} \Pi_2(G(m, a - a'_0, b - b'_0))$ by Thm. 20, p. 54 Ch. 4. Similarly, from 2(b) $b_0 \in \text{aff}(\bar{b}, \Delta \bar{a}) = \text{fix}_{\mathbf{R}^d} G_b$ and $0 \in \text{fix}_{\mathbf{R}^d} T_{b_0} G_b T_{-b_0} = \text{fix}_{\mathbf{R}^d} \Pi_3(G(m, a - a'_0, b - b'_0))$; so, $G(m, a - a'_0, b - b'_0) = G(m, r(t), \dot{r}(t))$ is in centroidal coordinates by Thm. 34 p. 78 Ch. 5. ■

Definition 23 Suppose that q is a solution of the ivp (m, a, b, t_0) on some interval, I and that $\exists a_0, b_0 \in \mathbf{R}^d$ such that if we let $r = \mathcal{G}_{a_0, b_0}(q)$ then $G(m, r(t), \dot{r}(t))$ is a constant ($= G(m, a - a'_0, b - b'_0)$) for all $t \in I$, then we say that the group, $G(m, q(t), \dot{q}(t))$ of the solution q is **preserved** over I .

Thus from the above theorem, q is preserved if any and hence all of the first three statements, 1(a), 1(b) and 1(c) in Thm. 42 above are true.

7.3 Solutions Whose Groups are Not Preserved

We now answer by a counterexample the question, whether for any s function the group of the N -body problem is preserved over each solution.

Example 11 Consider the s function given in Example 4 on page 26, Chapter 3 and the 3-body problem in $\mathbf{R}^1$ given by $a_1 = -2$, $a_2 = 0$, $a_3 = 2$, $b_1 = b_2 = b_3 = 0$ (written as scalars in $\mathbf{R}^1$) and $m_1 = m_2 = m_3 = 1$. Then $\forall \tau \geq 0$, the ivp $(m, a, b, 0)$

has among others, the solution,

$$\begin{aligned} q_1(t) &= -2, & t \geq 0 \\ q_2(t) &= \begin{cases} 0 & 0 \leq t \leq \tau \\ (t - \tau)^3 & \tau < t < \tau + 1 \end{cases} \\ q_3(t) &= \begin{cases} 2 & 0 \leq t \leq \tau \\ 2 - (t - \tau)^3 & \tau < t < \tau + 1. \end{cases} \end{aligned}$$

From the earlier example any two bodies can remain separated by the distance, 2 for some arbitrary time and then begin to fall towards each other, from which it is easy to see that this is a solution for this problem and we have a collision of bodies 2 and 3 at $t = \tau + 1$. For $0 \leq t \leq \tau$, the group of the solution, $q(t)$ is $G(m, q(t), \dot{q}(t)) = \langle (\pi, H, H) \rangle$, a group of order 2, where $\pi = (13)$ and $H = (-1)$. For $\tau + 1 > t > \tau$, $G(m, q(t), \dot{q}(t))$ is the trivial group. The centroidal affine subspace of the solution changes from $\{0\}$ for $0 \leq t \leq \tau$ to all of $\mathbf{R}^1$ for $\tau + 1 > t > \tau$.

We can reverse time in the above example as follows: let $r(t) = q(-t)$ on the interval $(-\tau - 1, t)$. Then r is a solution of the ivp (m, x, y, t_0) where $-\tau - 1 < t_0 < -\tau$, $x = q(-t_0)$ and $y = -\dot{q}(-t_0)$. In this case, the group increases at $t = -\tau$. Also both the centroidal affine subspace of the solution and the centroidal velocity affine subspace decrease. This means that had we chosen any other coordinate frame for this ivp we would have gone from centroidal coordinates to non-centroidal ones at time $-\tau$.

This answers some of the questions posed so far in this chapter and we have shown by counterexample

Theorem 43 *It is not true in general that the group of a solution q of the N -body problem is preserved over the solution. In fact there exist barycentric solutions of the N -body problem for which the group increases while the centroidal affine subspace of the solution and the centroidal velocity affine subspace decrease and some for which they do the reverse.*

This theorem paints a bleak picture for the usefulness of the group of an ivp. However, the reader is reminded that the s function used in the counterexample can be described as pathological. Before we develop some interesting results to show the power of the group, we continue to press this negative picture even further and offer the reader an idea of how much more elaborate the group transition can be made. Consider the same s function and the zero initial velocity problem where the initial configuration of the positions of the bodies is the orbit of some point ($\neq 0$) under the action of a finite subgroup of $\mathcal{O}$: e.g., any of the 5 Platonic solids or any of the 13 Archimedean solids for which the minimum distance between any two bodies is the critical distance for this s function which in this case is 2. Specifically, consider a small rhombicuboctahedron, the Archimedean solid whose vertices are given by

$$\left\{ \left(c_1 (1 + \sqrt{2}), c_2, c_3 \right), \left(c_1, c_2 (1 + \sqrt{2}), c_3 \right), \left(c_1, c_2, c_3 (1 + \sqrt{2}) \right) \mid c_i = \pm 1 \right\}$$

(see e.g., [CR, p. 105]). All rectangular faces are square with all edges equal to 2. The initial group is the symmetry group of the cube, $\mathcal{W}^*$. The reader is reminded that this is a group of order 48, generated by $-I_3$, the central inversion ($x \mapsto -x$) and the rotational group $\mathcal{W} \approx S_4$ which permutes the four lines joining each vertex of the cube with its antipode. Now, in one solution of this 24-body problem at time τ , the three vertices of any triangular face begin to fall towards their centroid. At this point, the reader is reminded that if we started with any vertex of a cube and adjoin to it the three vertices diagonally opposed on each of the 3 adjoining square faces of that vertex, we obtain a regular tetrahedron. Indeed, a cube comprises two such regular tetrahedra. Similarly, a small rhombicuboctahedron comprises two corresponding truncated tetrahedra. If we choose a solution in which all of the triangular faces of one such tetrahedron begin falling together at time τ and those triangular faces of the other to remain unmoved, then after time τ , the group changes to $\mathcal{W}[\mathcal{T}]$, a subgroup of $\mathcal{W}^*$ of index 2 that is the symmetry group of the regular tetrahedron. This hints at the wealth of ways that a solution of this ivp can

proceed and ivp's obtained by placing bodies at the vertices of the orbits of other finite subgroups of $\mathcal{O}(3)$.

7.4 Group Isomorphisms

Our notation of Chapter 5 says that for the ivp (m, a, b, t_0) , the groups of position and velocity isometries are $G_a = \Pi_2(G(m, a, b))$ and $G_b = \Pi_3(G(m, a, b))$, respectively (Def. 12, p. 59, Ch. 5). If q is a solution on I and $t \in I$, this notation means that q is also a solution of the ivp $(m, q(t), \dot{q}(t), t)$ on I . $G_{q(t)} = \Pi_2(G(m, q(t), \dot{q}(t)))$ and $G_{\dot{q}(t)} = \Pi_3(G(m, q(t), \dot{q}(t)))$.

We now extend Def. 12, p. 59, Ch. 5 as follows:

Definition 24 *If q is a solution of the N -body problem on I and $t \in I$, then for brevity, let $G(t) = G(m, q(t), \dot{q}(t))$ and $E(t) = E(m, q(t), \dot{q}(t))$. Recalling Π_i which maps $S_N \times \mathcal{E} \times \mathcal{E}$ to the i th coordinate ($i = 1, 3$), we define the group homomorphisms:*

$$\begin{aligned} \xi_N(t) &= \Pi_1|_{G(t)} & \chi_N(t) &= \Pi_1|_{E(t)} \\ \xi_q(t) &= \Pi_2|_{G(t)} & \chi_q(t) &= \Pi_2|_{E(t)} \\ \xi_{\dot{q}}(t) &= \Pi_3|_{G(t)} & \text{and } \chi_{\dot{q}}(t) &= \Pi_3|_{E(t)} \end{aligned}$$

and their images,

$$\begin{aligned} G_N(t) &= \Pi_1(G(t)) = \xi_N(G(t)) & E_N(t) &= \Pi_1(E(t)) = \chi_N(E(t)) \\ G_q(t) &= \Pi_2(G(t)) = \xi_q(G(t)) & E_q(t) &= \Pi_2(E(t)) = \chi_q(E(t)) \\ G_{\dot{q}}(t) &= \Pi_3(G(t)) = \xi_{\dot{q}}(G(t)) & E_{\dot{q}}(t) &= \Pi_3(E(t)) = \chi_{\dot{q}}(E(t)) \end{aligned}$$

Note from the Structure Theorem for the group (p. 70) that $\forall t \in I$, $\chi_N(t)$, $\chi_q(t)$ and $\chi_{\dot{q}}(t)$ are isomorphisms. Thus if $(\pi, f, g) = (\pi, f_1, g_1)(1, f_2, g_2) \in G(t)$ where $(\pi, f_1, g_1) \in E(t)$ then

$$\pi = \xi_N(t)(\pi, f, g) = \chi_N(t)(\pi, f_1, g_1)$$

$$\begin{aligned} f &= \xi_q(t)(\pi, f, g), & f_1 &= \chi_q(t)(\pi, f_1, g_1) \\ g &= \xi_{\dot{q}}(t)(\pi, f, g) & \text{and } g_1 &= \chi_{\dot{q}}(t)(\pi, f_1, g_1). \end{aligned}$$

Note also that $\xi_N(t)(G(t)) = \chi_N(t)(E(t))$. In the light of the fact that the group may vary over time, we make the following:

Definition 25 *Given q , a solution of the N -body problem (m, a, b, t_0) on I , $\pi \in S_N$ and $R \subseteq S_N$ we define*

$$\begin{aligned} I_{\pi, q} &= \{t \in I \mid \pi \in G_N(t)\} \text{ and} \\ I_{R, q} &= \bigcap_{\pi \in R} I_{\pi, q} = \{t \in I \mid R \subseteq G_N(t)\}. \end{aligned}$$

Then $I_{\pi, q}$ is the set of $t \in I$ at which π is the permutation for some triple that preserves $G(t)$. When it is clear that q is the solution we are discussing, we shall use I_π and I_R for $I_{\pi, q}$ and $I_{R, q}$ respectively. We can also write $I_\pi = \{t \in I \mid (\pi, f, g) \in G(t) \text{ for some } f, g \in \mathcal{E}\}$. Note that $I_R = I_{(R)}$.

7.5 Isometries Associated with a Permutation

Observe that $\forall t \in I_\pi$, since $\pi \in E_N(t)$, $\exists! f_\pi(t), g_\pi(t) \in \mathcal{E}$ such that $(\pi, f_\pi(t), g_\pi(t)) \in E(m, q(t), \dot{q}(t))$. In fact, $\chi_N(t)(\pi, f_\pi(t), g_\pi(t)) = \pi$ and since $\chi_N(t)$ is an isomorphism, $\chi_N(t)^{-1}(\pi)$ is unique $\forall t \in I_\pi$.

Definition 26 *Given a solution q of the N -body problem on I , $\forall \pi \in S_N$, $t \in I_\pi$, we can define $f_\pi, g_\pi : I_\pi \rightarrow \mathcal{E}$ by*

$$\begin{aligned} f_\pi(t) &= \chi_q(t) \circ \chi_N(t)^{-1}(\pi) \\ g_\pi(t) &= \chi_{\dot{q}}(t) \circ \chi_N(t)^{-1}(\pi). \end{aligned}$$

Thus we have shown

Lemma 44 *Given q a solution of the N -body problem on some interval, I given $\pi \in S_N$ and given $t \in I_{\pi,q}$, then $(\pi, f_\pi(t), g_\pi(t))$ is the unique member of $E(t)$ whose projection onto the first coordinate is π .*

We can also regard these maps as

$$f_\pi, g_\pi : I_\pi \times \mathbf{R}^d \rightarrow \mathbf{R}^d$$

and we shall sometimes write $f_\pi(t, x)$ for $f_\pi(t)(x)$, which is the isometry, $f_\pi(t)$ evaluated at the point, x . In this light we shall show that f_π and g_π are continuous.

Note that $\forall t \in I_\pi$, $(\pi, f_\pi(t), g_\pi(t)) \in E(m, q(t), \dot{q}(t))$: so,

$$\begin{aligned} f_\pi(t)(q_i(t)) &= q_{\pi(i)}(t) \text{ and} \\ g_\pi(t)(\dot{q}_i(t)) &= \dot{q}_{\pi(i)}(t). \end{aligned}$$

Recall from Def. 6, p. 36, Thm. 8, p. 38 and Def. 7, p. 43, all in Section 3.4 that the linear subspace $\langle \Delta q(t), \Delta \dot{q}(t) \rangle$ is independent of t . In the light of this, we now prove:

Lemma 45 *Given q , a solution of the N -body problem in $\mathbf{R}^d$ on some open interval, I and given $\tau \in I$, $\exists$ an interval, (τ_-, τ_+) such that $\tau \in (\tau_-, \tau_+) \subseteq I$ and $\exists i_1, \dots, i_k$ and $j_1, \dots, j_l$ where $k + l = \dim \langle \Delta q, \Delta \dot{q} \rangle$ such that $\forall t \in (\tau_-, \tau_+)$, the set*

$$\{q_{1,i_1}(t), \dots, q_{1,i_k}(t), \dot{q}_{1,j_1}(t), \dots, \dot{q}_{1,j_l}(t)\}$$

forms a basis for $\langle \Delta q, \Delta \dot{q} \rangle$.

Proof. Fix $\tau \in I$ and let $d' = \dim \langle \Delta q, \Delta \dot{q} \rangle$. Since by the affine subspace of the Solution Theorem, p. 38, Ch. 3, $\langle \Delta q, \Delta \dot{q} \rangle = \langle \Delta q(\tau), \Delta \dot{q}(\tau) \rangle = \langle \Delta q(\tau) \rangle + \langle \Delta \dot{q}(\tau) \rangle$, the set

$$\{q_{12}(\tau), \dots, q_{1N}(\tau), \dot{q}_{12}(\tau), \dots, \dot{q}_{1N}(\tau)\}$$

spans $\langle \Delta q, \Delta \dot{q} \rangle$. Thus $\exists i_1, \dots, i_k$ and $j_1, \dots, j_l$ such that

$$\{q_{1,i_1}(\tau), \dots, q_{1,i_k}(\tau), \dot{q}_{1,j_1}(\tau), \dots, \dot{q}_{1,j_l}(\tau)\}$$

forms a basis for $\langle \Delta q, \Delta \dot{q} \rangle$ where $k + l = d'$. Let $R = \{r_1, \dots, r_{d-d'}\}$ be a basis for $\langle \Delta q, \Delta \dot{q} \rangle^\perp$. Then

$$\{q_{1,i_1}(\tau), \dots, q_{1,i_k}(\tau), \dot{q}_{1,j_1}(\tau), \dots, \dot{q}_{1,j_l}(\tau)\} \cup R$$

forms a basis for $\mathbf{R}^d$. Replacing τ with t , we use these basis vectors to form the columns of the $d \times d$ matrix function.

$$X(t) \equiv [q_{1,i_1}(t) | \dots | q_{1,i_k}(t) | \dot{q}_{1,j_1}(t) | \dots | \dot{q}_{1,j_l}(t) | r_1 | \dots | r_{d-d'}].$$

Then $\det(X(\tau)) \neq 0$. By the continuity of $q, \dot{q}$ and the determinant function, $\det(X(t))$ is continuous and $\exists \tau_-, \tau_+ \in \mathbf{R}$ such that $\tau \in (\tau_-, \tau_+) \subseteq I$ and $\forall t \in (\tau_-, \tau_+)$, $\det(X(t)) \neq 0$. $\therefore \forall t \in (\tau_-, \tau_+)$, the columns of $X(t)$ are linearly independent and

$$\{q_{1,i_1}(t), \dots, q_{1,i_k}(t), \dot{q}_{1,j_1}(t), \dots, \dot{q}_{1,j_l}(t)\}$$

is a linearly independent set in the linear subspace, $\langle \Delta q, \Delta \dot{q} \rangle$ which has dimension, $d' = k + l$; i.e., it forms a basis as desired. ■

Theorem 46 *If q is a solution of the N -body problem on I and $\pi \in S_N$ then I_π is closed in I . Furthermore, if we regard f_π and g_π given in Def. 26 as maps*

$$f_\pi, g_\pi : I_\pi \times \mathbf{R}^d \rightarrow \mathbf{R}^d$$

then they are C^2 functions; i.e., they have an extension on an open interval containing each point in their domain which is C^2 .

Proof. $\forall t \in I_\pi$, by Lemma 44, f_π and g_π are uniquely defined on I_π and since $(\pi, f_\pi(t), g_\pi(t)) \in E(t)$, f_π and g_π have the same orthogonal part: viz. we can let

$$f_\pi(t) = T_{u(t)}H(t) \text{ and } g_\pi(t) = T_{v(t)}H(t) \quad (7.1)$$

where $H(t) \in \mathcal{O}$ and $u(t), v(t) \in \mathbf{R}^d$. Then $H(t)$, $u(t)$ and $v(t)$ are defined on I_π .

Let τ be any limit point of I_π in I . By the above lemma, $\exists(\tau_-, \tau_+) \subseteq I$ such that $\tau_- < \tau < \tau_+$ and $\exists i_1, \dots, i_k$ and $j_1, \dots, j_l$ such that $\forall t \in (\tau_-, \tau_+)$.

$$\{q_{1,i_h}(t) | h = 1, \dots, k\} \cup \{\dot{q}_{1,j_p}(t) | p = 1, \dots, l\}$$

forms a basis for the (fixed) subspace $\langle \Delta q, \Delta \dot{q} \rangle$. If $d' = \dim \langle \Delta q, \Delta \dot{q} \rangle$ then $k + l = d'$. We shall now extend f_π and g_π to all of (τ_-, τ_+) . Choose a basis $\{r_1, \dots, r_{d-d'}\}$ for $\langle \Delta q, \Delta \dot{q} \rangle^\perp$. Now fix $t \in (\tau_-, \tau_+)$. Define the $d \times d$ matrices,

$$X(t) \equiv [q_{1,i_1}(t) | \dots | q_{1,i_k}(t) | \dot{q}_{1,j_1}(t) | \dots | \dot{q}_{1,j_l}(t) | r_1 | \dots | r_{d-d'}]$$

and

$$Z(t) = [q_{\pi(1),\pi(i_1)}(t) | \dots | q_{\pi(1),\pi(i_k)}(t) | \dot{q}_{\pi(1),\pi(j_1)}(t) | \dots | \dot{q}_{\pi(1),\pi(j_l)}(t) | r_1 | \dots | r_{d-d'}].$$

By the lemma, the first d' column vectors of $X(t)$ form a basis for $\langle \Delta q, \Delta \dot{q} \rangle$ and the last $d - d'$ columns form a basis for $\langle \Delta q, \Delta \dot{q} \rangle^\perp$; thus $X(t)$ is invertible. Let

$$\tilde{H}(t) = Z(t) X^{-1}(t) \quad (7.2)$$

$$\tilde{u}(t) = \tilde{H}(t) q_1(t) - q_{\pi(1)}(t) \quad (7.3)$$

$$\tilde{v}(t) = \tilde{H}(t) \dot{q}_1(t) - \dot{q}_{\pi(1)}(t). \quad (7.4)$$

Note that $\tilde{H}$, $\tilde{u}$ and $\tilde{v}$ are defined on all of (τ_-, τ_+) . We now show that $\tilde{H}|_{I_\pi \cap (\tau_-, \tau_+)} = H$, $\tilde{u}|_{I_\pi \cap (\tau_-, \tau_+)} = u$ and $\tilde{v}|_{I_\pi \cap (\tau_-, \tau_+)} = v$. Let $\{e_1, \dots, e_d\}$ be the standard basis for $\mathbf{R}^d$.

Fix $t \in I_\pi \cap (\tau_-, \tau_+)$. $\forall h$ such that $1 \leq h \leq k$, column h in $X(t)$ is $q_{1,i_h}(t) = X(t) e_h$. Multiplying by X^{-1} gives $X^{-1}(t) q_{1,i_h}(t) = e_h$. Thus

$$\begin{aligned} \tilde{H}(t) q_{1,i_h}(t) &= Z(t) e_h = q_{\pi(1)\pi(i_h)}(t) \\ &= f_\pi(t)(q_{i_h}(t)) - f_\pi(t)(q_1(t)) \\ &= H(t) q_{1,i_h}(t). \end{aligned}$$

$\forall p \leq l$, we apply a similar process to column $p+k$ of $X(t)$: namely, $X(t) e_{p+k} = \dot{q}_{l+j_p}(t)$. This gives

$$\begin{aligned}\tilde{H}(t) \dot{q}_{l+j_p}(t) &= Z(t) X^{-1}(t) \dot{q}_{l+j_p}(t) = Z(t) e_{p+k} = \dot{q}_{\pi(1)\pi(j_p)}(t) \\ &= g_{\pi}(t) (\dot{q}_{j_p}(t)) - g_{\pi}(t) (\dot{q}_1(t)) \\ &= H(t) \dot{q}_{l+j_p}(t).\end{aligned}$$

Finally, $\forall w \leq d-d'$, since column $k+l+w = d' + w$ of both $X(t)$ and $Z(t)$ equals r_w , a similar argument gives

$$\tilde{H}(t) r_w = Z(t) X^{-1}(t) r_w = Z(t) e_{k+l+w} = r_w.$$

Since $(\pi, f_{\pi}(t), g_{\pi}(t)) \in E(m, q(t), \dot{q}(t)) \leq S_N \times \mathcal{E}_{\text{aff}(q, \Delta \dot{q})}^{\text{ess}} \times \mathcal{E}_{\text{aff}(\dot{q}, \Delta q)}^{\text{ess}}$ (Def. 15, p. 70, Ch. 5), then by Def. 42, p. 224, App. B, $H \in \mathcal{E}_{(\Delta q, \Delta \dot{q})^-}$; i.e., $H r_w = r_w$. $\forall w \in d-d'$. $\therefore \tilde{H}(t) r_w = H(t) r_w$. Thus the linear operators $\tilde{H}(t)$ and $H(t)$ agree on a basis and so they are equal: i.e., $\forall t \in I_{\pi} \cap (\tau_-, \tau_+)$,

$$\tilde{H}(t) = H(t) \tag{7.5}$$

Therefore from (7.3), (7.5) and (7.1),

$$\begin{aligned}\tilde{u}(t) &= H(t) q_1(t) - q_{\pi(1)}(t) \\ &= u(t) + f_{\pi}(t) (q_1(t)) - q_{\pi(1)}(t) \\ &= u(t)\end{aligned}$$

and from (7.4), (7.5) and (7.1),

$$\begin{aligned}\tilde{v}(t) &= H(t) \dot{q}_1(t) - \dot{q}_{\pi(1)}(t) \\ &= v(t) + g_{\pi}(t) (\dot{q}_1(t)) - \dot{q}_{\pi(1)}(t) \\ &= v(t).\end{aligned}$$

Thus we have shown that

$$\tilde{H}|_{I_\pi \cap (\tau_-, \tau_+)} = H, \quad (7.6)$$

$$\tilde{u}|_{I_\pi \cap (\tau_-, \tau_+)} = u \text{ and} \quad (7.7)$$

$$\tilde{v}|_{I_\pi \cap (\tau_-, \tau_+)} = v. \quad (7.8)$$

Since τ is a limit point of I_π in I , $\exists$ a sequence $\{t_n\}_{n=1}^\infty$ in I_π converging to τ . $\therefore \tau \in (\tau_-, \tau_+)$. $\exists N_0 \geq 1$ such that $t_n \in (\tau_-, \tau_+)$, $\forall n \geq N_0$. $\therefore q_i$ and $\dot{q}_i$ are C^1 . $X, X^{-1}, Z, \tilde{H}, \tilde{u}$ and $\tilde{v}$ are also C^1 . (Note that $X^{-1}X = I_d$; so, $(X^{-1})'X + X^{-1}X' = 0$ and $(X^{-1})' = -X^{-1}X'X^{-1}$ is continuous because each factor is continuous). Since, X is continuous and invertible on (τ_-, τ_+) , X^{-1} is continuous. $\therefore T_{\tilde{u}(t)}\tilde{H}(t)q_i(t) = Z(t)X^{-1}(t)q_i(t) - \tilde{u}(t)$ is a continuous function of t on (τ_-, τ_+) . So, $\lim_{n \rightarrow \infty} T_{\tilde{u}(t_n)}\tilde{H}(t_n)q_i(t_n) = T_{\tilde{u}(\tau)}\tilde{H}(\tau)q_i(\tau)$. Since $\forall n \geq N_0$, $t_n \in (\tau_-, \tau_+)$ by (7.6) through (7.8), $\tilde{H}(t_n) = H(t_n)$ and $\tilde{u}(t_n) = u(t_n)$; so,

$$\begin{aligned} T_{\tilde{u}(\tau)}\tilde{H}(\tau)q_i(\tau) &= \lim_{n \rightarrow \infty} T_{u(t_n)}H(t_n)q_i(t_n) \\ &= \lim_{n \rightarrow \infty} f(t_n)(q_i(t_n)) \\ &= \lim_{n \rightarrow \infty} q_{\pi(i)}(t_n) \\ &= q_{\pi(i)}(\tau) \end{aligned}$$

by the continuity of q . Similarly, $T_{\tilde{v}(\tau)}\tilde{H}(\tau)\dot{q}_i(\tau) = \dot{q}_{\pi(i)}(\tau)$. Also,

$$\tilde{H}(\tau)r_w = \lim_{n \rightarrow \infty} \tilde{H}(t_n)r_w = \lim_{n \rightarrow \infty} r_w = r_w.$$

Since $\{r_w\}_{w=1}^{d-d'}$ is a basis for $\langle \Delta q, \Delta \dot{q} \rangle^\perp$, $\tilde{H}(\tau) \in \mathcal{E}_{(\langle \Delta q, \Delta \dot{q} \rangle)^\perp}$ and

$$\left(\pi, T_{\tilde{u}(\tau)}\tilde{H}(\tau), T_{\tilde{v}(\tau)}\tilde{H}(\tau) \right) \in E(m, q(\tau), \dot{q}(\tau)):$$

i.e., $\tau \in I_\pi$ and H, u and v are defined at τ . $\therefore I_\pi$ contains its limit points in I ; i.e., I_π is closed in I (but not necessarily closed in $\mathbf{R}$). Since I_π is closed, every point in

I_π is a limit point. Thus, τ is an arbitrary point in I_π . Furthermore f_π and g_π have, as an extension on (τ_-, τ_+) , the functions,

$$\begin{aligned}\tilde{f}_\pi(t) &= T_{\tilde{u}(\tau)}\tilde{H}(\tau) \text{ and} \\ \tilde{g}_\pi(t) &= T_{\tilde{v}(\tau)}\tilde{H}(\tau)\end{aligned}$$

which are C^2 because $T_{\tilde{u}}$, $T_{\tilde{v}}$ and $\tilde{H}$ are all C^2 . ■

Corollary 47 *If q is a solution of the N -body problem (m, a, b, t_0) on I , $\tau \in I$ and $G(\tau) = G(m, q(\tau), \dot{q}(\tau))$ then $I_{G_N(\tau)}$, the set of all $t \in I$ for which $G_N(\tau) \leq G_N(t)$, is closed in I .*

Proof. $I_{G_N(\tau)}$ is the intersection of the closed sets I_π where $\pi \in G_N(\tau)$. ■

7.6 Solutions that Preserve the Group

This section lies at the heart of the group of the N -body problem.

Lemma 48 *If q is a solution of the N -body problem, (m, a, b, t_0) on I then $\forall i \leq N$,*

$$\ddot{q}_i(t_0) \in \Delta \text{fix}_{\text{aff } a} \text{stab}_{E_a} a_i \leq \Delta \text{fix}_{\text{aff}(a, \Delta b)} \text{stab}_{E_a} a_i.$$

Proof. Fix $i \leq N$. Let $\mathcal{H} = \text{stab}_{E_a} a_i \leq E_a \leq \mathcal{E}_{\{a\}}$. Since the essential group, E_a is finite, $\mathcal{H}$ is a finite group that partitions $\{a_1, \dots, a_N\}$ into orbits, one of which is the singleton $\{a_i\}$ because $\mathcal{H}a_i = \{a_i\}$. In (3.1), p. 24, Ch. 3 to obtain $\ddot{q}_i$ we shall group the sum on the right side according to the orbits. First, fix $j \neq i$. From [BJN, Thm. 4.7 p. 108], $|\mathcal{H}a_j| = [\mathcal{H} : \text{stab}_{\mathcal{H}} a_j]$ from which we have

$$|\mathcal{H}a_j| \cdot |\text{stab}_{\mathcal{H}} a_j| = |\mathcal{H}|. \quad (7.9)$$

As f ranges over $\mathcal{H}$, each left coset of the form, $f \cdot \text{stab}_{\mathcal{H}} a_j$, yields exactly one point in the orbit, $\mathcal{H}a_j$; namely, $(f \cdot \text{stab}_{\mathcal{H}} a_j)(a_j) = f(\{a_j\})$, a singleton: i.e., each point

in the orbit $\mathcal{H}a_j$ occurs exactly $\text{stab}_{\mathcal{H}}a_j$ times. All bodies in the orbit, $\mathcal{H}a_j$ have mass m_j (see D. 10 p. 52 Ch. 4). So, the contribution to the acceleration of body i at time t_0 , from the set of bodies in the orbit $\mathcal{H}a_j$ is from (3.1), p. 24, Ch. 3

$$\begin{aligned}
& \frac{m_j}{|\text{stab}_{\mathcal{H}}a_j|} \sum_{f \in \mathcal{H}} s(\|f(a_j) - a_i\|) [f(a_j) - a_i] \\
&= \frac{m_j |\mathcal{H}a_j|}{|\mathcal{H}|} \sum_{f \in \mathcal{H}} s(\|f(a_j) - f(a_i)\|) [f(a_j) - f(a_i)] \\
&= m_j s(\|a_{ij}\|) |\mathcal{H}a_j| \frac{1}{|\mathcal{H}|} \sum_{f \in \mathcal{H}} [f(a_j) - f(a_i)] \\
&= m_j s(\|a_{ij}\|) |\mathcal{H}a_j| [\theta_{\mathcal{H}}(a_j) - \theta_{\mathcal{H}}(a_i)].
\end{aligned}$$

where the first equality follows from (7.9) and the fact that $f \in \text{stab } a_i$, the second uses the fact that f is an isometry and we define $\theta_{\mathcal{H}} = \frac{1}{|\mathcal{H}|} \sum_{f \in \mathcal{H}} f$. By Cor. 134, p. 238, App. B, $\theta_{\mathcal{H}}(a_j) - \theta_{\mathcal{H}}(a_i)$ is the orthogonal projection of a_{ij} onto $\Delta \text{fix}_{\mathbf{R}^d} \mathcal{H}$. Since $\ddot{q}_i(t_0)$ is a linear combination of such terms it also lies in $\Delta \text{fix}_{\text{aff } a} \text{stab}_{E_a} a_i$.

$\therefore \text{aff } a \subseteq \text{aff}(a, \Delta b)$, $\text{fix}_{\text{aff } a} \mathcal{H} \subseteq \text{fix}_{\text{aff}(a, \Delta b)} \mathcal{H}$ hence $\Delta \text{fix}_{\text{aff } a} \mathcal{H} \subseteq \text{fix}_{\text{aff}(a, \Delta b)} \mathcal{H}$. ■

Let us suppose that q is a solution of the ivp, (m, a, b, t_0) which without loss of generality, we shall assume has been placed in centroidal coordinates. Thus it is in affine coordinates which means that the solution will stay inside $\langle \Delta a, \Delta b \rangle$ for all t . Also, $G_a = G_b \leq \mathcal{O}$ and we can speak of the group of orthogonal transformations for the ivp and $\forall i \leq N$, $\text{stab}_{E_a} a_i \leq \mathcal{O}$ hence fixes the origin: so,

$$\text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i = \Delta \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i \leq \mathbf{R}^d;$$

so, by L. 48 above,

$$\ddot{q}_i(t_0) \in \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i. \quad (7.10)$$

Also, $\forall H \in \text{stab}_{E_a} a_i$, since $Ha_i = a_i$ and $H \in E_a = E_b$, $\exists \pi \in S_N$ such that $(\pi, H, H) \in E(m, a, b)$. Since $a_{\pi(i)} = Ha_i = a_i$, then since $a \notin \Delta$, $\pi(i) = i$ and $Hb_i = b_i$. $\therefore b_i \in \text{fix}_{\langle \Delta a, \Delta b \rangle} H$. $\therefore H$ is an arbitrary member of $\text{stab}_{E_a} a_i$ then

$b_i \in \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i$. Furthermore, $\dot{q}_i(t_0) = b_i \in \langle \Delta a, \Delta b \rangle$; so,

$$\dot{q}_i(t_0) \in \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i. \quad (7.11)$$

Finally, it is elementary from Part 2 of L. 79 p. 205 App. A. since $q_i(t_0) = a_i \in \langle \Delta a, \Delta b \rangle$, that

$$q_i(t_0) \in \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i. \quad (7.12)$$

Note that just because $q_i(t_0)$, $\dot{q}_i(t_0)$ and $\ddot{q}_i(t_0)$ all lie in this affine subspace, it does not follow that $q_i(t)$ lies in this subspace $\forall t$. To see a counterexample, consider the 3-body problem in $\mathbf{R}^1$ given in Example 11 p. 99 for which $q_2(t_0) = 0$ and $H = -1$, reflection through 0, lies in the isometry group. Thus $\text{stab}_{G_a} q_i(t_0) = \text{stab}_{G_a} 0 = \langle H \rangle$ and $\text{fix}_{\text{stab}} q_2(t_0) = \{0\}$. However, $q_2(t)$ moves away from this point. Thus we have seen by counterexample the following:

Lemma 49 *It is not true in the general N -body problem for an arbitrary continuous s function, that if q is a solution on some interval I of the ivp, (m, a, b, t_0) in centroidal coordinates, then $q_i(t) \in \text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} a_i, \forall t \in I, i \leq N$.*

However, we shall show that given any ivp *there exists* a solution for which each body, i stays on this affine subspace, $\text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} a_i$. Thus for the cases where we have uniqueness, we shall be guaranteed this result for all solutions. We shall first discuss our method so that the reader knows where we are headed. Suppose we are given the ivp (m, a, b, t_0) . We want to find a solution $q(t)$ on some interval I , containing t_0 so that

1. $q_i(t) \in \text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} a_i, \forall t \in I, i \leq N$.
2. the solution is in centroidal coordinates for all t and
3. if $a_j = H a_i$ for some $i, j \leq N$, then $q_j(t) = H q_i(t), \forall t \in I$.

This last statement means that there exists a solution for which the group does not get any smaller and all this happens no matter how badly behaved the s function is and no matter how pathological the choice of masses is. It can however, happen that the group gets larger and there is nothing that we can do about this. To see this, consider Example 11 p. 99 with the solution $r(t) = q(-t)$ (reverse time) defined on the interval $-I$. As a solution of an ivp for some initial time, $t' < -t_0$, r is unique up to and including $t = -t_0$, at which point its group increases from trivial to a reflection group of order 2.

We first look at the group of the ivp and the orbits $\mathcal{O}_l$ of that group. We select one representative from each orbit: say $i_1, i_2, \dots, i_n$. Given $q_{i_1}(t), \dots, q_{i_n}(t)$ we shall find a solution q that satisfies the following condition: for any body $i_{l,j}$ in orbit number l , if $a_{i_{l,j}} = H(a_{i_l})$ for $H \in E_a (= E_b \leq \mathcal{O}$ because the ivp is centroidal), then $q_{i_{l,j}}(t) = Hq_{i_l}(t)$. In this way, every such solution can be recovered from the group of the ivp and from the solution for the representative bodies, $q_{i_1}, \dots, q_{i_n}$. Furthermore, if such a solution exists, we can write down the acceleration, $\ddot{q}_{i_l}(t)$ as a function of the $\{q_{i_1}, \dots, q_{i_n}\}$, representative bodies since by hypothesis, all position are determined from them: i.e., are function of them. Also, as we have shown, the resultant, $\ddot{q}_{i_l}(t) \in \text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} q_{i_l}(t) = \text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} a_{i_l}$. What this says is that we can orthogonally map the coordinates of each $\text{fix}_{\mathbf{R}^d} \text{stab}_{G_a} a_{i_l}$ into some new space and define a new variable, $x(t)$ in that new space so that each assignment of $(q_{i_1}, \dots, q_{i_n})$ has a unique x in the new space.

This suggests that we may be able to prove the existence of a solution for which each $q_i(t)$ stays in the linear subspace, $\text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_i$ ($\forall i = 1, \dots, N$). To accomplish this recall from Def. 18, p. 84, Ch. 5 that the set of bodies, $\{1, 2, \dots, N\}$ was partitioned by the action of the group, G_N into n distinct orbits for some $n \leq N$. We labelled these orbits, $\Omega_l = G_N i_l = \{i_{l,1}, \dots, i_{l,k_l}\}$, where $l = 1, n$. For each orbit, Ω_l , we shall first consider the differential equation for just one representative for that orbit: say, $q_{i_l}(t)$ in such a manner as to satisfy the requirement that $q_{i_l}(t) \in$

$\text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_{i_l}$ for all $t \in I$, the interval over which the solution q is defined. Then, $\forall j \in \Omega_l$, $a_j \in E_a a_{i_l}$ and $\exists! H \in E_a$, such that $a_j = H a_{i_l}$ by the Structure Theorem (28 p. 70). We shall set $q_j(t) = H q_{i_l}(t)$, $\forall t \in I$. We do this for all l to define all of q . We then test to see if $q(t)$ satisfies the equation (3.1), p. 24, on I . To begin this process, we let

$$W_l = \text{fix}_{\langle \Delta a, \Delta b \rangle} \text{stab}_{E_a} a_{i_l} \leq \mathbf{R}^d.$$

Thus W_l is an inner product space and any $y_l \in W_l$ inherits its inner product from its inner product in $\mathbf{R}^d$. Let $d_l = \dim W_l$ and let

$$\chi_l : W_l \mapsto \mathbf{R}^{d_l}$$

be an inner product space isomorphism: i.e., some orthogonal transformation. Let $W = \prod_{l=1}^n W_l$ be the inner product space with the norm $\|y\| = \sqrt{\sum_{l=1}^n \|y_l\|^2}$ where $y = (y_1, \dots, y_n)^t$ and each $y_l \in W_l$. Let

$$\chi : W \mapsto \mathbf{R}^{\sum_{l=1}^n d_l} \text{ be} \quad (7.13)$$

$$\chi = \chi_1 \times \dots \times \chi_n,$$

a direct product map. Then χ is also an inner product space isomorphism and we can consider the following autonomous o.d.e.: let $x \in \mathbf{R}^{\sum_{l=1}^n d_l}$ be partitioned as $x = (x_1, \dots, x_n)^t$. Where $x_l \in \mathbf{R}^{d_l}$ satisfies the initial condition

$$x_l(t_0) = \chi_l(a_{i_l}).$$

Let $\mathcal{H}_l = \text{stab}_{E_a} a_{i_l} \leq E_a$. Then $W_l = \text{fix}_{\langle \Delta a, \Delta b \rangle} \mathcal{H}_l \leq \mathbf{R}^d$. Let P_l be the orthogonal projection onto W_l . By the above lemma, $\ddot{q}_{i_l}(t_0) \in W_l$; so, $\ddot{q}_{i_l}(t_0) = P_l \ddot{q}_{i_l}(t_0)$. Now, the initial acceleration of body i_l is the sum of the contribution from all bodies in orbit $\mathcal{O}_j$ as j ranges over all orbits other than $\mathcal{O}_l$ plus the contribution from all bodies in orbit $\mathcal{O}_l$ other than body i_l . Again, as in the above lemma, as H ranges over E_a , $H a_{i_j}$ hits the same body $|\text{stab}_{E_a} a_{i_j}|$ times. Also as H ranges over E_a , $H a_{i_l}$

hits a_{i_l} precisely when $H \in \text{stab}_{E_a} a_{i_l} = \mathcal{H}_l$. In light of this we have that the initial acceleration of body i_l is

$$\begin{aligned} \ddot{q}_{i_l}(t_0) &= \sum_{\substack{j=1 \\ j \neq l}}^n \frac{m_{i_j}}{|\mathcal{H}_j|} \sum_{H \in E_a} s(\|Ha_{i_j} - a_{i_l}\|) P_l(Ha_{i_j} - a_{i_l}) \\ &\quad + \frac{m_{i_l}}{|\mathcal{H}_l|} \sum_{H \in E_a \setminus \mathcal{H}_l} s(\|Ha_{i_l} - a_{i_l}\|) P_l(Ha_{i_l} - a_{i_l}) \end{aligned} \quad (7.14)$$

where we have used $\ddot{q}_l(t_0) = P_l \ddot{q}_l(t_0)$ together with the linearity of P_l to apply P_l to each vector in the sum. Notice that $\ddot{q}_{i_l}(t_0)$ is expressed as a sum of terms involving only the n representatives $\{a_{i_j}\}_{j=1}^n$ of the n orbits $\{\Omega_j\}_{j=1}^n$ together with the number of the group $G(m, a, b)$. We shall make use of this with the isomorphism χ to define an o.d.e. in terms of variables $\{x_l(t)\}_{l=1}^n$. Consider x , a solution of the following o.d.e. Let the initial value of x be

$$x_l(t_0) = \chi_l(a_{i_l}) \quad (7.15)$$

$$\dot{x}_l(t_0) = \chi_l(b_{i_l}) \quad (7.16)$$

Equations (7.15) and (7.16) are well defined because from (7.11) and (7.12), $a_{i_l} = q_{i_l}(t_0)$ and $b_{i_l} = \dot{q}_{i_l}(t_0)$ lie in W_l . Let x satisfy the equation, $\forall l = 1, n$.

$$\begin{aligned} \ddot{x}_l(t) &= \sum_{\substack{j=1 \\ j \neq l}}^n \frac{m_{i_j}}{|\mathcal{H}_j|} \sum_{H \in E_a} s(\|H\chi_j^{-1}(x_j(t)) - \chi_l^{-1}(x_l(t))\|) \cdot \\ &\quad \chi_l P_l(H\chi_j^{-1}(x_j(t)) - \chi_l^{-1}(x_l(t))) \\ &\quad + \frac{m_{i_l}}{|\mathcal{H}_l|} \sum_{H \in E_a \setminus \mathcal{H}_l} s(\|(H - I_d)\chi_l^{-1}(x_l(t))\|) \cdot \\ &\quad \chi_l P_l(H - I_d)\chi_l^{-1}(x_l(t)). \end{aligned} \quad (7.17)$$

Since $P_l : \mathbf{R}^d \rightarrow W_l$ and $\chi_l : W_l \rightarrow \mathbf{R}^{d_l}$ then $\chi_l \circ P_l : \mathbf{R}^d \rightarrow \mathbf{R}^{d_l}$ is well defined. By the inner product space isomorphism, χ_j and the continuity of s , the r.h.s. of (7.17) is continuous and defined provided that

1. whenever $j \neq l$ and $H \in E_a$, $H\chi_j^{-1}x_j \neq \chi_l^{-1}x_l$ and
2. whenever $H \in E_a \setminus \text{stab}_{E_a} a_{i_l}$, $H\chi_l^{-1}x_l \neq \chi_l^{-1}x_l$.

At $t = t_0$, $\chi_j^{-1}x_j(t_0) = a_{i_j}$ and for $j \neq l$, i_j and i_l are on different orbits of G_N : i.e., a_{i_j} and a_{i_l} are on different orbits of E_a : so, $H\chi_j^{-1}x_j(t_0) = Ha_{i_j} \neq a_{i_l} = \chi_l^{-1}x_l(t_0)$. So, condition 1 is initially satisfied. Similarly, if $H \in E_a \setminus \text{stab}_{E_a} a_{i_l}$, then $H\chi_l^{-1}x_l(t_0) = Ha_{i_l} \neq a_{i_l} = \chi_l^{-1}x_l(t_0)$ and condition 2 is satisfied. Thus at $t = t_0$, the arguments of the function, s in (7.17) are all non-zero and the r.h.s. of the equation is well defined and continuous. Thus by the Cauchy existence criterion there exists a solution x on some interval I containing t_0 . $\forall l \leq n$, let

$$q_{i_l}(t) = \chi_l^{-1}(x_l(t)) \quad (7.18)$$

and $\forall j \leq n_l$, $a_{i_{l,j}} \in \Omega_l$, $\exists H_{l,j} \in E_a$ such that $a_{i_{l,j}} = H_{l,j}a_{i_l}$. In fact, the coset, $H_{l,j}(\text{stab}_{E_a} a_{i_l})$ contains all those members of E_a with this property. Let

$$q_{i_{l,j}}(t) = H_{l,j}q_{i_l}(t) = H_{l,j}\chi_l^{-1}(x_l(t)). \quad (7.19)$$

Since $\chi_l^{-1}x_l(t) \in W_l = \text{fix}_{(\Delta a, \Delta b)} \text{stab}_{E_a} a_{i_l}$, $(\text{stab}_{E_a} a_{i_l})\chi_l^{-1}x_l(t) = \{\chi_l^{-1}x_l(t)\}$. Thus any choice of the representative, $H_{l,j}\text{stab}_{E_a} a_{i_l}$ in the coset, $H_{l,j}$ has no effect on the value of $q_{i_{l,j}}(t)$. Since every $i \leq N$ lies in some orbit Ω_l , (7.19) defines a function $q : I \rightarrow \mathbf{R}^{dN}$. We now claim that $q(t)$ is a solution on I and the group $G(m, a, b) \leq G(m, q(t), \dot{q}(t))$.

Theorem 50 *Given the ivp (m, a, b, t_0) in centroidal coordinates for any continuous s function $\exists$ a solution q on some interval I given by equation (7.19) such that $\forall t \in I$,*

$$G(m, a, b) \leq G(m, q(t), \dot{q}(t)).$$

Proof. Given (m, a, b, t_0) let $x(t)$ be any solution of (7.17) with initial conditions (7.15) and (7.16) and let q be given by (7.19). We start by showing that the function

q satisfies the initial conditions for (m, a, b, t_0) . From (7.15) $\forall l \leq n$, $\chi_l^{-1}(x_l(t_0)) = \chi_l^{-1}\chi_l(a_{i_l}) = a_{i_l}$. $\therefore$ from (7.18), $q_{i_l}(t_0) = a_{i_l}$. Now for any $i_{l,j}$ in the orbit Ω_l ; i.e., $\forall j \leq n_l$, from (7.19)

$$q_{i_{l,j}}(t_0) = H_{l,j}q_{i_l}(t_0) = H_{l,j}a_{i_l} = a_{i_{l,j}}$$

by the choice of $H_{l,j}$. Similarly, from (7.18) and (7.16),

$$\dot{q}_{i_l}(t_0) = \chi_l^{-1}\dot{x}_l(t_0) = \chi_l^{-1} \circ \chi_l(b_{i_l}) = b_{i_l}$$

and $\forall j \leq n_l$,

$$\dot{q}_{i_{l,j}}(t_0) = H_{l,j}\dot{q}_{i_l}(t_0) = H_{l,j}b_{i_l}.$$

Again, since $b_{i_l} \in \text{fix}_{(\Delta a, \Delta b)} \text{stab}_{E_a} a_{i_l}$, $(\text{stab}_{E_a} a_{i_l})b_{i_l} = \{b_{i_l}\}$ and the choice of the representative for the coset, $H_{l,j}(\text{stab}_{E_a} a_{i_l})$ has no effect. Since (m, a, b, t_0) is centroidal, $H_{l,j} \in E_a = E_b$, $\exists (\pi_{l,j}, H_{l,j}, H_{l,j}) \in E(m, a, b)$. Then $a_{i_{l,j}} = H_{l,j}a_{i_l} = a_{\pi_{l,j}(i_l)}$. $\therefore \pi(i_l) = i_{l,j}$. So, $H_{l,j}b_{i_l} = b_{\pi_{l,j}(i_l)} = b_{i_{l,j}}$ also.

$$\therefore \dot{q}_{i_{l,j}}(t_0) = H_{l,j}\dot{q}_{i_l}(t_0) = b_{i_{l,j}}.$$

Since, $l = 1, \dots, n$, $j = 1, \dots, n_l$ ranges over all bodies, $i = 1, \dots, N$, we have that $q(t)$ satisfies the initial conditions for the ivp. (m, a, b, t_0) . We now check that q satisfies (3.1), p. 24. From (7.18) and (7.17),

$$\begin{aligned} \ddot{q}_{i_l}(t) &= \chi_l^{-1} \ddot{x}_l(t) \\ &= \sum_{\substack{p=1 \\ p \neq l}}^n \frac{m_{i_p}}{|\text{stab}_{E_a} a_{i_p}|} \sum_{H \in E_a} s(\|H\chi_p^{-1}(x_p(t)) - \chi_l^{-1}(x_l(t))\|) \\ &\quad P_l(H\chi_p^{-1}(x_p(t)) - \chi_l^{-1}(x_l(t))) \\ &\quad + \frac{m_{i_l}}{|\text{stab}_{E_a} a_{i_l}|} \sum_{H \in E_a \setminus \text{stab}_{E_a} a_{i_l}} s(\|(H - I_d)\chi_l^{-1}(x_l(t))\|) \\ &\quad P_l((H - I_d)\chi_l^{-1}(x_l(t))) \\ &= \sum_{\substack{p=1 \\ p \neq l}}^n \frac{m_{i_p}}{|\text{stab}_{E_a} a_{i_p}|} \sum_{H \in E_a} s(\|Hq_{i_p}(t) - q_{i_l}(t)\|) \end{aligned}$$

$$\begin{aligned}
& P_l (H q_{i_p} (t) - q_{i_l} (t)) \\
& + \frac{m_{i_l}}{|\text{stab}_{E_a} a_{i_l}|} \sum_{H \in E_a \setminus \text{stab}_{E_a} a_{i_l}} s (\| (H - I_d) q_{i_l} (t) \|) \\
& P_l (H - I_d) q_{i_l} (t).
\end{aligned}$$

Comparing this with (7.14), one sees that $\ddot{q}_{i_l}$ satisfies (3.1), p. 24, Ch. 3.

Now fix $l \leq n$ and $j' \leq n_l$. Substituting into the above equation, summing over $H' = H_{l,j'} H \in E_a$ and observing that $\|H q_{i_p} (t) - q_{i_p} (t)\| = \|(H' q_{i_p} (t) - H_{l,j'} q_{i_p} (t))\|$ we get

$$\begin{aligned}
\ddot{q}_{i_{l,j'}} (t) &= H_{l,j'} \ddot{q}_{i_l} (t) \\
&= \sum_{\substack{p=1 \\ p \neq l}}^n \frac{m_{i_p}}{|\text{stab}_{E_a} a_{i_p}|} \sum_{H' \in E_a} s \left(\left\| \left(H' q_{i_p} (t) - q_{i_{l,j'}} (t) \right) \right\| \right) \\
&\quad \left(H' q_{i_p} (t) - q_{i_{l,j'}} (t) \right) \\
&\quad + \frac{m_{i_l}}{|\text{stab}_{E_a} a_{i_l}|} \sum_{H \in E_a \setminus \text{stab}_{E_a} a_{i_l}} s (\| (H' - H_{l,j'}) q_{i_l} (t) \|) \\
&\quad H_{l,j'} (H - I_d) q_{i_l} (t) \\
&= \sum_{\substack{p=1 \\ p \neq l}}^n \frac{m_{i_p}}{|\text{stab}_{E_a} a_{i_p}|} \sum_{H' \in E_a} s \left(\left\| \left(H' q_{i_p} (t) - H_{l,j'} q_{i_l} (t) \right) \right\| \right) \\
&\quad \left(H' q_{i_p} (t) - H_{l,j'} q_{i_l} (t) \right) \\
&\quad + \frac{m_{i_l}}{|\text{stab}_{E_a} a_{i_l}|} \sum_{H \in E_a \setminus \text{stab}_{E_a} a_{i_l}} s (\| H_{l,j'} ((H - I_d) q_{i_l} (t)) \|) \\
&\quad (H - I_d) q_{i_{l,p}} (t).
\end{aligned}$$

which shows that $q_{i_{l,j'}}$ also satisfies the o.d.e. ■

Remark 4 In Theorem 50, we can extend I to some maximal interval and still satisfy the theorem. On the other hand we can use Thm. 46 which shows that $\forall \pi \in S_N \setminus G_N(m, a, b)$, I_π is closed. Therefore, the set on which $\pi \notin G_N(m, q(t), \dot{q}(t))$ is open. The finiteness of S_N gives that the set on which $G_N(m, q(t), \dot{q}(t)) =$

$G_N(m, a, b)$ is open. Therefore $\exists$ an open interval, I' containing t_0 sufficiently small that $\forall t \in I'$, $G(m, q(t), \dot{q}(t)) = G(m, a, b)$.

Corollary 51 *Given the ivp (m, a, b, t_0) in centroidal coordinates for any continuous s function $\exists$ an interval, I containing t_0 and a solution q on I such that $\forall t \in I$,*

$$G(m, q(t), \dot{q}(t)) = G(m, a, b).$$

7.7 Uniqueness

Corollary 52 *Given the ivp (m, a, b, t_0) in centroidal coordinates, if the s function is locally lipschitzian or such as to give rise to uniqueness of solutions and if q is the solution on some interval I , then $\forall t \in I$*

$$G(m, q(t), \dot{q}(t)) = G(m, a, b);$$

i.e., the group of the N -body problem is preserved over the solution.

Proof. $\forall t \in I$, from Thm. 50, $\exists$ a solution r on some interval I' such that $(m, q(t), \dot{q}(t), t)$ is centroidal on I and $G(m, a, b) \leq G(m, r(t), \dot{r}(t))$. Since the solution is unique, $r = q$ on $I \cap I'$. In the case where $I' \subsetneq I$, since $r = q$, it can be extended to all of I . Conversely, $q(t)$ is also a solution of the ivp $(m, q(t), \dot{q}(t), t_0)$ which is in centroidal coordinates by Thm. 50: so, again, $G(m, q(t), \dot{q}(t)) \leq G(m, a, b)$. Thus, they are equal. ■

This corollary enables us to observe that certain symmetries are preserved over the solution. It allows us to greatly reduce the number of variables in the differential equation. It also permits the following:

Definition 27 *If q is a solution of the N -body problem in centroidal coordinates on I for an s function that gives rise to uniqueness and $t \in I$, then we let $G(q) = G(m, q(t), \dot{q}(t))$, (the same $\forall t \in I$) and call it the **group of the solution**, q .*

Our theorem shows that for any set of masses, if the s function gives rise to uniqueness of solution then the centroidal affine subspace does not change size and moves linearly with t .

7.8 Summary of Results on the Group as Function of Time

Given an ivp we can place it in affine coordinates (D. 7 p. 43 Ch. 3) which give a linear subspace of $\mathbf{R}^d$ inside of which the bodies remain over the solution. Without loss of generality we may shrink the dimension of the space to fit this linear subspace: i.e., we may assume that the solution has dimension d . Consequently, the inessential part of the group of the ivp is trivial (D. 16 p. 72 Ch. 5): i.e., the group merely equals its essential part which is finite. Thus,

$$G(m, a, b) = E(m, a, b) \approx S_N \approx E_a (= G_a) \approx E_b (= G_b)$$

(Thm. 28 p. 70 Ch. 5). Furthermore, we may assume that we have placed the ivp in centroidal coordinates (D. 17 p. 76 Ch. 5) in which case $G_a = G_b \leq \mathcal{O}(d)$. Finally, if the s function is such as to give uniqueness of solutions (e.g. locally Lipschitzian) the group does not change over time and we can and shall identify the group of the solution with the finite subgroup, $E_a \leq \mathcal{O}(d)$.

Given an ivp we can find the group, perform these simplifications and then partition the bodies into their orbits under the action of the group (S. 6.1 p. 83 Ch. 6). We then select one body from each orbit. The inner product space isomorphism, (7.13) p. 113 yields an o.d.e. (7.17) of smaller dimension for x . The solution, x with (7.19) p. 115 gives q .

Chapter 8

Singularities, Symmetry and Angular Momentum

In this chapter we provide a few applications of the theory of the group.

8.1 Reflection Symmetry and Singularities

One of the appealing qualities of solutions that lead neither to nor from a singularity is that they largely reflect our experience with the solar system and the universe. Clearly, minor collisions occur with asteroids and meteoroids on a regular basis; nevertheless, the planets, their satellites and almost all stars have persisted throughout eons much greater than our lifetimes. If we are curious to find solutions that lead neither to nor from a singularity it behooves us to find those that do. Furthermore, singularities themselves comprise a fascinating study especially in the light of recent results on non-collision singularities [Xia], [Grv]. Thus, it is with great interest that we now make use of the group of the N -body problem to find a condition that ensures a singularity. Recall that a hyperplane is an affine subspace of codimension 1.

Theorem 53 *Suppose that q is a solution of the N -body problem on some time interval, I in $\mathbf{R}^d$ for some strictly positive (i.e., attractive) Lipschitz s function and for strictly positive masses. Suppose also that the essential group of q contains a reflection through a hyperplane. Then q leads to a singularity, from a singularity or both; i.e., the interval, I cannot be extended to all of $\mathbf{R}$.*

Proof. Let us assume, contrary to what we want to show, that the solution q can be (and has been) extended for all $t \in \mathbf{R}$. Since the s function is Lipschitz we have uniqueness of solutions and hence the group is preserved over time. We shall assume without loss of generality that the problem has been placed in centroidal coordinates. Let P be a plane of reflection symmetry for the essential group of the solution, q and let R be the reflection in P . Since the problem is in centroidal coordinates P is a linear subspace of codimension 1 and we can let the unit vector v span $P^\perp$. To say that R lies in the essential group means that there exists some permutation, $\pi \in S_N$ such that (π, R, R) lies in the essential group. By the structure theorem for the group (Thm. 28 p. 70) the essential group is isomorphic to each of its projections. Since R is not the identity map then neither is π and R must move some body's position; i.e., some body lies outside P for the whole solution. Since its image under R gives the position of another body there are bodies on both sides of P . We divide the bodies into three sets:

$$Q^+ = \{i \leq N \mid (q_i, v) > 0\} \neq \emptyset \quad (8.1)$$

$$Q^0 = \{i \leq N \mid (q_i, v) = 0\} \quad (8.2)$$

$$Q^- = \{i \leq N \mid (q_i, v) < 0\} \neq \emptyset, \quad (8.3)$$

where $\emptyset$ denotes the empty set and (8.1) and (8.3) include the fact that Q^+ and Q^- are non-empty as discussed above. Define the inner product

$$f(t) \equiv \sum_{i \in Q^-} m_i (q_i(t), v) \quad (8.4)$$

$\forall t \in \mathbf{R}$. Then f is the orthogonal projection of the first moment of the bodies 'above' the hyperplane, where by 'above' we mean on the same side of P as v and by (8.1)

$$f(t) > 0. \quad (8.5)$$

Then from the equation of motion (3.1) p. 24. since q is C^2 , f is also C^2 and

$$\begin{aligned} f''(t) &= \sum_{i \in Q^+} m_i \left(\ddot{q}_i(t) \cdot v \right) \\ &= \sum_{i \in Q^+} \sum_{j \in Q^0 \cup Q^-} m_i m_j s(\|q_{ij}(t)\|) (q_{ij}(t) \cdot v), \end{aligned} \quad (8.6)$$

where we have used the fact that

$$\sum_{i \in Q^+} \sum_{\substack{j \in Q^- \\ j \neq i}} m_i m_j s(\|q_{ij}(t)\|) q_{ij}(t) = 0$$

to eliminate these terms from the previous sum. In (8.6) $j \in Q^0 \cup Q^-$ and $i \in Q^+$: so, $(q_j, v) \leq 0$ and $(q_i, v) > 0$, by (8.1), (8.2) and (8.3). Thus, $(q_{ij}, v) < 0$ and we have a non-empty sum of strictly negative terms in (8.6). Therefore, $\forall t \in I$,

$$f''(t) < 0. \quad (8.7)$$

We now show that f must tend to zero either in future time or past time (or both). However, if $f \rightarrow 0$ the barycentre of the bodies in Q^+ must tend to P . Since all masses are positive the barycentre lies inside the convex hull of the bodies in Q^+ . Consequently, some body must meet the hyperplane, P and hence meet its image under P at least by that time (if in the future) or must have come from its image no sooner than that time (if in the past). Such an event is a singularity and will contradict the assumption that the solution can be extended for all time.

Case 1: Fix $t_0 \in I$. Suppose that

$$f'(t_0) \leq 0. \quad (8.8)$$

Then $\forall t \in [t_0, t_0 + 1]$

$$\begin{aligned} f'(t_0 + 1) &= f'(t_0) + \int_{t_0}^{t_0+1} f''(s) ds \\ &\leq f'(t_0) + L \\ &\leq L. \end{aligned}$$

where

$$L \equiv \sup_{[t_0, t_0+1]} f''(t) < 0$$

by (8.7) and the compactness of $[t_0, t_0 + 1]$. By (8.7) f' is decreasing: so, $\forall t \geq t_0 + 1$ we have $f'(t) \leq L$ and

$$\begin{aligned} f(t) &= f(t_0 + 1) + \int_{t_0+1}^t f'(s) ds \\ &\leq f(t_0 + 1) + L(t - t_0 - 1) \\ &\leq 0 \end{aligned}$$

$\forall t \geq t_1 \equiv t_0 + 1 - f(t_0 + 1) / L$. As discussed above, a singularity must occur at a time $\tau \leq t_1$, contradicting the hypothesis that q can be extended for all t .

Case 2: Otherwise, $f'(t_0) > 0$ for which one can consider the solution, $r(t) = q(-t)$ which has the same group as q . In light of its value at $t = -t_0$ we have $r'(t) = -q'(-t) = -q'(t_0)$ from which we get

$$\sum_{i \in Q'} m_i \left(\dot{r}_i(-t_0), v \right) = - \sum_{i \in Q'} m_i \left(\dot{q}_i(t_0), v \right) = -f'(t_0) < 0$$

which by Case 1 for the solution r leads to a singularity. Since r cannot be extended to $[-t_0, \infty)$, q cannot be extended to $(-\infty, t_0]$ in this case.

Thus in both cases the solution cannot be extended to all of $\mathbf{R}$. ■

This theorem is easily modified to include the case where there are bodies of positive mass in the hyperplane of reflection and all bodies not in this plane have zero mass. The proof involves simply choosing some zero-mass body not in the plane

of reflection and showing by the same technique above that the hypothesis that the solution can be extended for all t is contradicted by the fact that at some time in the future or past this body must meet the hyperplane of reflection (due to its attraction to the massive bodies in the hyperplane). Of course, when zero mass bodies collide only with each other their acceleration does not approach infinity; nevertheless, the solution does approach the diagonal.

8.1.1 Singularities in $\mathbf{R}^2$

The finite subgroups of $\mathcal{O}(2)$ are $\mathcal{C}_2^n$, the group generated by rotation of $2\pi/n$ radians $\forall n \geq 1$, and $\mathcal{H}_2^n$, the dihedral group of order $2n$ generated by reflection through two lines which intersect at an angle π/n (also $\forall n \geq 1$). Clearly, from the above theorem, all 2 dimensional solutions having $\mathcal{H}_2^n$ as their group lead either to or from a singularity (or both). The other family of groups, namely, $\mathcal{C}_2^n$ has an elementary but interesting special case to the contrary for the case of the Newtonian s function where there is exactly one orbit under the action of the group: i.e.,

Theorem 54 *For the Newtonian s function, $s(r) = 1/r^3$, all 2 dimensional solutions of $\mathcal{G}$ -orbit ivp's having $\mathcal{C}_2^n$ as their essential group can be extended to all of $\mathbf{R}$: i.e., they can have no singularities. They are also all homographic solutions.*

Proof. We may assume that the solution has been placed in $\mathbf{R}^2$ in centroidal coordinates where the solution has been expressed in polar coordinates. The seed body cannot be at the origin otherwise there would be only one body and the dimension of the solution would be zero. We can number the bodies starting at zero and as such the solution for body k ($0 \leq k \leq n-1$) is $(r(t), \theta(t) + 2\pi k/n)$ where r and θ are common to all bodies. The total angular momentum is $c_{xy} = nmr^2\dot{\theta}$ where m is the common mass of the bodies.

We now show that the angular momentum is non-zero for this problem. Suppose, contrary to what we want to prove, that the angular momentum is zero. Then

$\dot{\theta}(t) \equiv 0, \forall t$ and we can write $\theta(t) \equiv \theta_0$. Consequently, the velocity for body k in polar coordinates is $(\dot{r}, \theta_0 + 2\pi k/n)$; i.e., it has the same angle as the k th position vector. Then reflection through the line containing the ray, $\theta = \theta_0$ is a symmetry for the problem and the group is $\mathcal{H}_2^n$ contrary to what we are given. Thus the supposition is false and the angular momentum is non-zero. Since the orbit of a single body under the group, $\mathcal{C}_2^n$ is a regular n -gon the solution is homographic and any singularity must be a collision. Also since the angles, $\theta_0 + 2\pi k/n$ are all distinct any collision must occur at the origin ($r = 0$). However, this is a simultaneous collision which cannot happen unless the total angular momentum is zero (see e.g. [Wnt. S. 333 -335 p. 251-253]). ■

Note that this theorem is easily modified to include the case with one body at the origin and n bodies orbiting about it again with the group $\mathcal{C}_2^n$. However, the theorem fails in the case of the inverse cube law of attraction ($s(r) = 1/r^4$) which admits spiral collisions (see e.g. [Wnt. S. 374 bis p. 294 eqn (XII)]).

8.1.2 Singularities in $\mathbf{R}^3$

The finite subgroups of $\mathcal{O}(3)$ as listed in App. C are grouped into 3 classes:

1. subgroups of $\mathcal{SO}(3)$ (proper rotations)
2. groups of the form $\mathcal{K}^* \equiv \langle \mathcal{K}, -I_3 \rangle$ generated by $-I_3$ and a finite subgroup, $\mathcal{K}$ of $\mathcal{SO}(3)$ and
3. groups, designated $\mathcal{L}[\mathcal{H}]$, of the form $\mathcal{H} \cup \{-T \mid T \in \mathcal{L} \setminus \mathcal{H}\}$ where $\mathcal{L} \leq \mathcal{SO}(3)$, $\mathcal{L}$ is finite of even order and $[\mathcal{L} : \mathcal{H}] = 2$.

We now seek those finite subgroups of $\mathcal{O}(3)$ that contain reflections in planes. We first note that in $\mathbf{R}^3$ a reflection in a plane can be characterized by the composition of central inversion ($-I_3$) and a rotation of π radians about the orthogonal complement of that plane. Clearly there are no reflections in the first class which contains only

groups of rotations. By the above characterization of reflections in $\mathbf{R}^3$, some $\mathcal{K}^*$ in the second class ($\mathcal{K} \leq \mathcal{SO}(3)$) contains a reflection in a plane if and only if some member of $\mathcal{K}$ is a rotation of π radians. The groups with this property are:

- $(\mathcal{C}_3^{2k})^*$ ($k \geq 1$) since the cyclic group $\mathcal{C}_3^n$ contains a rotation of π if and only if n is even.
- $(\mathcal{H}_3^n)^*$ ($n \geq 2$) since $\mathcal{H}_3^n$ contains a rotation about the x -axis of π radians.
- $\mathcal{T}^*$.
- $\mathcal{W}^*$ and
- $\mathcal{I}^*$.

where the last three contain such reflections because each one intersects $\mathcal{SO}(3)$ in a rotational symmetry group of some Platonic solid which contains the rotation about the midpoint of any edge of π radians. In summary, every solution with a group in this second class contains a reflection in a plane except the group $(\mathcal{C}_3^{2k+1})^*$, for $k \geq 0$. The orbits of bodies under this group form antiprisms and more will be said about this later.

Finally, we consider the third class of groups listed above. Again from our characterization of reflections in $\mathbf{R}^3$, the group $\mathcal{L}]\mathcal{H}$ will have a reflection in a plane if and only if $\mathcal{L} \setminus \mathcal{H}$ contains a rotation of π radians. In the case $\mathcal{C}_3^{2n}]\mathcal{C}_3^n$, $\mathcal{C}_3^{2n}$ contains exactly one rotation of π radians and this will lie in $\mathcal{C}_3^n$ or $\mathcal{C}_3^{2n} \setminus \mathcal{C}_3^n$ according as n is even or odd, respectively. Thus, a reflection in a plane will occur only if n is odd; i.e., for the case $\mathcal{C}_3^{2(2k+1)}]\mathcal{C}_3^{2k+1}$. In $\mathcal{H}_3^n]\mathcal{C}_3^n$, a rotation of π about the x -axis always lies in $\mathcal{H}_3^n \setminus \mathcal{C}_3^n$; so, every such group contains a plane reflection. In $\mathcal{H}_3^{2n}]\mathcal{H}_3^n$ the coset, $\mathcal{H}_3^{2n} \setminus \mathcal{H}_3^n$ contains the rotation about the x -axis of π followed by a rotation about the z -axis of π/n which inverts the z -axis and fixes the point $(\cos \pi/2n, \sin \pi/2n, 0)$; i.e., it is a rotation of π radians. Finally, $\mathcal{W}]\mathcal{T}$ is the complete symmetry group of

the regular tetrahedron and contains a reflection through the plane passing through any two vertices and the midpoint of the opposite edge. In summary, for class 3 we have planes of reflection for the following groups:

- $C_3^{4k+2}]C_3^{2k+1}$, $k \geq 0$.
- $\mathcal{H}_3^n]C_3^n$, $k \geq 2$.
- $\mathcal{H}_3^{2n}]\mathcal{H}_3^n$ and
- $\mathcal{W}|\mathcal{T}$.

Note that every group in class 3 contains a plane of reflection except $C_3^{4k}]C_3^{2k}$. As in the case $(C_3^{2k+1})^*$ the orbits under these groups are antiprisms. Combining these results gives:

Theorem 55 *Suppose that q is a solution of the N -body problem on some time interval, I in $\mathbf{R}^3$ for some strictly positive (i.e., attractive) Lipschitz s function and for strictly positive masses. Suppose also that the essential group of q is one of the following:*

- $(C_3^{2k})^*$ for $k \geq 1$, $(\mathcal{H}_3^n)^*$ for $n \geq 2$, $\mathcal{T}^*$, $\mathcal{W}^*$, $\mathcal{I}^*$,
- $C_3^{4k+2}]C_3^{2k+1}$ for $k \geq 0$, $\mathcal{H}_3^n]C_3^n$ for $k \geq 2$, $\mathcal{H}_3^{2n}]\mathcal{H}_3^n$, or $\mathcal{W}|\mathcal{T}$.

Then q leads to a singularity, from a singularity or both; i.e., the interval, I cannot be extended to all of $\mathbf{R}$.

8.2 Angular Momentum and the Group

Many of the groups in $\mathbf{R}^3$ require that the total angular momentum is zero. Indeed each orbit under such groups has zero angular momentum as we shall see. Again we treat the case for arbitrary dimension.

Theorem 56 Suppose we are given the N -body problem, (m, a, b, t_0) in $\mathbf{R}^d$ in centroidal coordinates with some $(\pi, H, H) \in G(m, a, b)$. Let $K = \text{fix}_{\mathbf{R}^d} H$ and suppose that x and y are non-zero vectors with $x \in K$ and $y \in K^\perp$. Then the component of angular momentum of the bodies in any orbit under the group in the 2-plane spanned by x and y is zero. Consequently, the component of the total angular momentum of all N -bodies in this plane is also zero.

Proof. Suppose that x and y are as defined above. If π is the identity permutation then $K \supseteq \text{aff}(a, \Delta b) = \langle \Delta a, \Delta b \rangle$, the subspace of the solution. Since all $q_i(t)$ and $\dot{q}_i(t)$ lie inside $\langle \Delta a, \Delta b \rangle \subseteq K$ then $y \in K^\perp \subseteq \langle \Delta a, \Delta b \rangle^\perp$. Therefore, $q_i^y = \dot{q}_i^y = 0$, $q_i^x \dot{q}_i^y - q_i^y \dot{q}_i^x = 0$ and the theorem is true in this case.

Otherwise, assume that $\pi \neq 1_{S_N}$. Then by the Structure Theorem for the group (Thm. 28 p. 70), we can write

$$(\pi, H, H) = (\pi, H_\epsilon, H_\epsilon) (\pi, H_i, H_i)$$

where $(\pi, H_\epsilon, H_\epsilon) \in E(m, a, b)$, the essential group which is finite. Thus H_ϵ has finite order. Also since $\pi \neq 1_{S_N}$, $(\pi, H_\epsilon, H_\epsilon) \neq 1_{E(m, a, b)}$. Therefore, the order of $H_\epsilon = n$ for some finite $n > 1$.

Fix any body, $i \leq N$. since $\bar{a}_{i, H_\epsilon}$ and $\bar{b}_{i, H_\epsilon}$ are left fixed by H_ϵ , $\bar{a}_{i, H_\epsilon} \in K$ and $\bar{b}_{i, H_\epsilon} \in K$ (see defn. 45 p. 233 App. B). By Thm. 132 p. 237 App. B if we let

$$P = \frac{1}{n} \sum_{k=0}^{n-1} H_\epsilon^k$$

then P is the orthogonal projection onto $\text{fix}_{\mathbf{R}^d} \langle H \rangle = \text{fix}_{\mathbf{R}^d} H_\epsilon = K$. We now use a notation discussed in App. D where if v is a vector, v^x and v^y represent the component of that vector in the x and y direction, respectively. Since $x \in K$: i.e., x lies in the image of P then $\forall k \leq n$

$$(H_\epsilon^k a_i)^x = (P H_\epsilon^k a_i)^x = (H_\epsilon^k P a_i)^x = (H_\epsilon^k \bar{a}_{i, H_\epsilon})^x = \bar{a}_{i, H_\epsilon}^x$$

where the second step follows because by its definition, P commutes with H . Similarly, $\forall k \leq n$, $(H_\epsilon^k b_i)^x = \bar{b}_{i,H_\epsilon}^x$. Since a_i and b_i lie inside $\langle \Delta a, \Delta b \rangle$, $H_i a_i = a_i$. Therefore, $H a_i = H_\epsilon a_i$. Similarly, $H b_i = H_\epsilon b_i$. Therefore the component of angular momentum in the xy plane for the bodies in the orbit of i (namely, $i, \pi(i), \dots, \pi^{k-1}(i)$) is, using the fact that all bodies in the orbit of body, i have mass m_i ,

$$\begin{aligned}
c_{xy,i} &= \sum_{k=0}^{n-1} m_{\pi^k(i)} \left(a_{\pi^k(i)}^x b_{\pi^k(i)}^y - a_{\pi^k(i)}^y b_{\pi^k(i)}^x \right) \\
&= m_i \sum_{k=0}^{n-1} \left[(H_\epsilon^k a_i)^x (H_\epsilon^k b_i)^y - (H_\epsilon^k a_i)^y (H_\epsilon^k b_i)^x \right] \\
&= m_i \sum_{k=0}^{n-1} \left[\bar{a}_{i,H_\epsilon}^x (H_\epsilon^k b_i)^y - (H_\epsilon^k a_i)^y \bar{b}_{i,H_\epsilon}^x \right] \\
&= m_i \left[\bar{a}_{i,H_\epsilon}^x \sum_{k=0}^{n-1} (H_\epsilon^k b_i)^y - \bar{b}_{i,H_\epsilon}^x \sum_{k=0}^{n-1} (H_\epsilon^k a_i)^y \right] \\
&= m_i \left[\bar{a}_{i,H}^x n \bar{b}_{i,H}^y - \bar{b}_{i,H}^x n \bar{a}_{i,H}^y \right].
\end{aligned}$$

But $\bar{a}_{i,H}$ and $\bar{b}_{i,H}$ lie in K . $\therefore \bar{a}_{i,H}^y = \bar{b}_{i,H}^x = 0$. $\therefore c_{xy,i} = 0$ as desired.

Summing over all orbits gives the second result. ■

8.2.1 Application to $\mathbf{R}^2$

Corollary 57 *For the N -body problem in $\mathbf{R}^2$, if the group contains a reflection through a line the the angular momentum is zero.*

Proof. By the theorem, the component of angular momentum in the plane spanned by the axis of symmetry and its orthogonal complement is zero. However, this is the whole plane. ■

8.2.2 Application to $\mathbf{R}^3$

In $\mathbf{R}^3$, the angular momentum can be written

$$C = \begin{bmatrix} 0 & c_{xy} & -c_{zx} \\ -c_{xy} & 0 & c_{yz} \\ c_{zx} & -c_{yz} & 0 \end{bmatrix}.$$

One usually associates the angular momentum with the vector,

$$S = (c_{yz}, c_{zx}, c_{xy}).$$

Theorem 58 *If H is a non-trivial member of the rotation group of the N -body problem in $\mathbf{R}^3$ then the angular momentum vector, S is in the direction of the axis of rotation of H .*

Proof. We can choose an orthonormal basis such that the axis of rotation is the z axis. Then by the theorem, c_{zx} and c_{yz} are both zero. ■

This leads to the very important

Corollary 59 *If the group of an N -body problem in $\mathbf{R}^3$ contains two non-trivial rotations with different axes of rotation then the angular momentum of any orbit and hence that of the whole system is zero.*

This leads to:

Theorem 60 *For the N -body problem in $\mathbf{R}^3$, the only groups that admit non-zero angular momentum are $\mathcal{C}_3^n$, $(\mathcal{C}_3^n)^*$ and $\mathcal{C}_3^{2n}[\mathcal{C}_3^n]$.*

Proof. Of the finite subgroups of $\mathcal{SO}(3)$, the groups $\mathcal{H}_3^n$, $\mathcal{T}$, $\mathcal{W}$ and $\mathcal{I}$ all have more than one axis of rotational symmetry [GB, Section 2.4]. Thus, any subgroup of $\mathcal{O}(3)$ containing one of these as a subgroup also has this property. This leaves the groups $\mathcal{C}_3^n$, $(\mathcal{C}_3^n)^*$, $\mathcal{C}_3^{2n}[\mathcal{C}_3^n]$ and $\mathcal{H}_3^n[\mathcal{C}_3^n]$ as contenders for non-zero angular momentum.

We now show that if the group is $\mathcal{H}_3^n \setminus \mathcal{C}_3^n$ the angular momentum is zero. Recall that as defined in App. C the axis of rotation for the non-trivial members of $\mathcal{C}_3^n$ is the z axis. Since $\mathcal{C}_3^n \leq \mathcal{H}_3^n \setminus \mathcal{C}_3^n$ from Thm. 58, $c_{zx} = c_{yz} = 0$. Since $\mathcal{H}_3^n \setminus \mathcal{C}_3^n$ contains rotation of π about the x -axis, the reflection, R in the yz -plane lies in the group, $\mathcal{H}_3^n \setminus \mathcal{C}_3^n$. Then if u_x and u_y are the unit vectors in the x and y directions, respectively, $u_y \in \text{fix}_{\mathbf{R}^3} R$ and $u_x \in (\text{fix}_{\mathbf{R}^3} R)^\perp$ and from Thm. 56 p. 128 $c_{xy} = 0$. Thus if the group is $\mathcal{H}_3^n \setminus \mathcal{C}_3^n$ the angular momentum must be zero.

To see that the remaining groups *do* admit solutions with non-zero angular momentum note first that for the group $\mathcal{C}_3^n$, the total angular momentum in the xy -plane is just n times that of the seed bodies not on the z axis examples of which are elementary to construct. For the group $(\mathcal{C}_3^n)^*$ appending $-I_3$ to the group $\mathcal{C}_3^n$ merely doubles the angular momentum obtained for a given set of seed bodies. This follows from the fact that for each body with solution $q_i(t)$ there is a body of the same mass whose solution is $-q_i(t)$ having the same contribution to angular momentum since $(-q_i) \times (-\dot{q}_i) = q_i \times \dot{q}_i$ where $\times$ denotes the cross product. Finally, for $\mathcal{C}_3^{2n} \setminus \mathcal{C}_3^n$ and a given set of seed bodies, the bodies in the coset $\mathcal{C}_3^{2n} \setminus \mathcal{C}_3^n$ of the subgroup $\mathcal{C}_3^n$ in the group $\mathcal{C}_3^{2n}$ have the same angular momentum as those in $\mathcal{C}_3^n$ itself and as discussed above, central inversion does not change its value. Thus examples of non-zero angular momentum exist for $\mathcal{C}_3^{2n} \setminus \mathcal{C}_3^n$ as well. ■

If we define

$$H(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (8.9)$$

then $R^2(\theta) = H(2\theta)$ and if $\pi/\theta = n \in \mathbf{Z}^+$, a positive integer, then

$$\langle R^2(\theta) \rangle = \langle H(2\theta) \rangle = \langle H(2\pi/n) \rangle = C_3^n \quad (8.10)$$

and $R(\theta) = R(\pi/n)$ generates a finite group. Furthermore, for $n \geq 2$ the orbit under the action of this group of a point lying neither on the z axis nor on the xy -plane forms the vertices of an antiprism. The class as given in App. C into which $R(\pi/n)$ falls depends on whether n is even or odd. If n is odd: say, $n = 2k + 1$, ($k \geq 0$).

$$R^{2k+1}\left(\frac{\pi}{2k+1}\right) = R(\pi) = -I_3.$$

From this and (8.10) we see that the group is generated by $\langle R^2(\pi/(2k+1)) \rangle = C_3^{2k+1}$ and $-I_3$; i.e.,

$$(C_3^{2k+1})^* = \left\langle R\left(\frac{\pi}{2k+1}\right) \right\rangle.$$

If n is even: say $n = 2k$, ($k \geq 1$), by (8.10) C_3^{2k} is a subgroup of $\langle R(\pi/2k) \rangle$ of index 2 and the odd powers,

$$\begin{aligned} R^{2l+1}\left(\frac{\pi}{2k}\right) &= R\left(\frac{(2l+1)\pi}{2k}\right) \\ &= \text{diag}(1, 1, -1) H\left(\frac{(2l+1)\pi}{2k}\right) \\ &= -H(\pi) H\left(\frac{(2l+1)\pi}{2k}\right) \\ &= -H\left(\frac{(2l+2k+1)\pi}{2k}\right) \\ &= -H^{2(l+k)+1}\left(\frac{\pi}{2k}\right). \end{aligned}$$

Thus the odd powers of $R(\pi/2k)$ are $-I_3$ times the odd powers of $H(\pi/2k)$; so,

$$\begin{aligned} \left\langle R\left(\frac{\pi}{2k}\right) \right\rangle &= \left[\left\langle H\left(\frac{\pi}{2k}\right) \right\rangle \right] \left\langle H\left(\frac{\pi}{k}\right) \right\rangle \\ &= C_3^{4k} C_3^{2k} \end{aligned}$$

and we have shown:

Lemma 61 *If R is as given in (8.9) then $(C_3^{2k+1})^*$ and $C_3^{4k}[C_3^{2k}]$ are cyclic groups generated by $R(\pi/(2k+1))$ and $R(\pi/2k)$, respectively.*

8.3 The Rectangular Product of 2 Polygons in $\mathbf{R}^4$

We now discuss an application of the group in $\mathbf{R}^4$ for the Newtonian (inverse square) law of attraction. Coxeter [C1, S. 7.2 p. 124] gives a description of how to build up in $\mathbf{R}^4$, the rectangular product of two regular polygons. If one or both of the polygons is a digon (a line segment) we have a right prism and a rectangle respectively. In the case where it is the product of 2 equilateral triangles, it is called a simplotope of type (2, 2) [Som, p. 113]. The following analysis applies to all of these cases. For our purposes we need only concern ourselves with the movement of the vertices. We wish to consider a $\mathcal{G}$ -orbit ivp in $\mathbf{R}^4$ for Newtonian mechanics where, $\mathcal{G} = \langle H_{xy}, H_{zw} \rangle$, is the group generated by

$$H_{xy} = \begin{bmatrix} \cos \frac{2\pi}{n_1} & -\sin \frac{2\pi}{n_1} & 0 & 0 \\ \sin \frac{2\pi}{n_1} & \cos \frac{2\pi}{n_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8.11)$$

and

$$H_{zw} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \frac{2\pi}{n_2} & -\sin \frac{2\pi}{n_2} \\ 0 & 0 & \sin \frac{2\pi}{n_2} & \cos \frac{2\pi}{n_2} \end{bmatrix} \quad (8.12)$$

where $n_1 \geq 2$ and $n_2 \geq 2$. Clearly, this group is a direct product of two cyclic groups ($\approx C_2^k \times C_2^l$), one generated by a rotation "about the zw plane" or in the xy plane of the angle $2\pi/n_1$ and the other in the zw plane by an angle of $2\pi/n_2$. There will be $n_1 n_2$ bodies all forming a single orbit of some seed body under the action of the group. In this section we adopt a novel but appropriate notation for

numbering the bodies using a mixed radix notation. We begin with body 0 and we write the number j ($0 \leq j < n_1 n_2$) as follows: if $j = kn_2 + l$ where $0 \leq k < n_1$ and $0 \leq l < n_2$ then the choice of k and l is unique and we write the body number j as kl and we denote the solution for body j as $q_{kl}(t)$ where we shall find that $q_{kl}(t) = H_{xy}^k H_{zw}^l q(t)$. Let the seed body be body 0 which we label

$$q_{00}(t) = q(t) = (x(t), y(t), z(t), w(t)) \quad (8.13)$$

and let its mass be $1/n_1 n_2$. Then the total mass of all bodies is unity. We assume that the solution has dimension 4 and that the angular momentum has the form

$$C = \begin{bmatrix} 0 & c_{xy} & 0 & 0 \\ -c_{xy} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{zw} \\ 0 & 0 & -c_{zw} & 0 \end{bmatrix} \quad (8.14)$$

where $c_{xy} \neq 0$ and $c_{zw} \neq 0$. Let the initial position of the seed body be

$$a_{00} = (a_{00}^x, a_{00}^y, a_{00}^z, a_{00}^w) \quad (8.15)$$

and let its initial velocity be

$$b_{00} = (b_{00}^x, b_{00}^y, b_{00}^z, b_{00}^w). \quad (8.16)$$

Assume that

$$\begin{aligned} (a_{00}^x)^2 + (a_{00}^y)^2 &> 0 \text{ and} \\ (a_{00}^z)^2 + (a_{00}^w)^2 &> 0; \end{aligned} \quad (8.17)$$

i.e., the initial position of the seed body is neither inside the zw plane nor inside the xy plane. Thus, a_{00} is left fixed by neither generator, H_{xy} nor H_{zw} ; i.e., $\text{stab}_{\mathcal{G}} a_{00} = \{1_{\mathcal{G}}\}$, the trivial group and hence $|\mathcal{G}a_{00}| = [\mathcal{G} : \text{stab}_{\mathcal{G}} a_{00}] = n_1 n_2$. Then $\text{stab}_{\mathcal{G}} a_{00} = \{1_{\mathcal{G}}\}$ and any initial velocity, b_{00} for the seed body will lie in $\text{fix}_{\mathbf{R}^4} \text{stab}_{\mathcal{G}} a_{00} =$

fix $_{\mathbf{R}^4} I_4 = \mathbf{R}^4$ satisfying Statement 2 Def. 20 p. 89 Ch. 6. Hence the cardinality of the orbit, $|\mathcal{G}a_{00}| = |\mathcal{G}| = n_1 n_2$. Assume also that

$$\begin{aligned} c_1 &\equiv a_{00}^x b_{00}^y - a_{00}^y b_{00}^x \neq 0 \text{ and} \\ c_2 &\equiv a_{00}^z b_{00}^w - a_{00}^w b_{00}^z \neq 0. \end{aligned} \quad (8.18)$$

We shall see that $c_1 = c_{xy}$ and $c_2 = c_{zw}$, the angular momentum in the xy and zw planes, respectively. From the above discussion of the mixed radix numbering system for the bodies, we label the bodies as follows: let

$$\begin{aligned} a_{kl} &= H_{xy}^k H_{zw}^l a_{00} \text{ and} \\ b_{kl} &= H_{xy}^k H_{zw}^l b_{00}. \end{aligned} \quad (8.19)$$

Notice that H_{xy} and H_{zw} leave the zw and xy planes fixed, respectively: i.e., $(H_{zw} a_{00})^x = (H_{zw} a_{00})^y = a_{00}$ etc. In light of this we have

$$\begin{aligned} c_{xy} &= \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} m_{kl} (a_{kl}^x b_{kl}^y - a_{kl}^y b_{kl}^x) \\ &= \frac{1}{n_1 n_2} \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} [(H_{xy}^k H_{zw}^l a_{00})^x (H_{xy}^k H_{zw}^l b_{00})^y \\ &\quad - (H_{xy}^k H_{zw}^l a_{00})^y (H_{xy}^k H_{zw}^l b_{00})^x] \\ &= \frac{1}{n_1} \sum_{k=0}^{n_1-1} [(H_{xy}^k a_{00})^x (H_{xy}^k b_{00})^y - (H_{xy}^k a_{00})^y (H_{xy}^k b_{00})^x] \\ &= \frac{1}{n_1} \sum_{k=0}^{n_1-1} \left[\left(a_{00}^x \cos \frac{2\pi k}{n} - a_{00}^y \sin \frac{2\pi k}{n} \right) \left(b_{00}^x \sin \frac{2\pi k}{n} + b_{00}^y \cos \frac{2\pi k}{n} \right) \right. \\ &\quad \left. - \left(a_{00}^x \sin \frac{2\pi k}{n} + a_{00}^y \cos \frac{2\pi k}{n} \right) \left(b_{00}^x \cos \frac{2\pi k}{n} - b_{00}^y \sin \frac{2\pi k}{n} \right) \right] \\ &= a_0^x b_0^y - a_0^y b_0^x. \\ \therefore c_{xy} &= c_1. \end{aligned} \quad (8.20)$$

Similarly,

$$c_{zw} = a_0^z b_0^w - a_0^w b_0^z = c_2. \quad (8.21)$$

Let the permutation, $\pi_{10}(kl) = (k+1)l$ where the addition, $k+1$ is modulo n_1 and $\pi_{01}(kl) = k(l+1)$ where $l+1$ is modulo n_2 . Then $(\pi_{10}, H_{xy}, H_{xy})$ and $(\pi_{01}, H_{zw}, H_{zw}) \in G(m, a, b)$ and clearly, $\pi_{k'l'}(kl) = (k+k')(l+l')$ where again the two sums are modulo n_1 and n_2 in their respective orders. Then the solution for body kl at time, t is $q_{kl}(t) = H_{xy}^k H_{zw}^l q(t)$ for $0 \leq k < n_1$ and $0 \leq l < n_2$.

8.3.1 Double Polar Coordinates

We now place the solution in double polar coordinates; namely, let

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta \\ z &= \rho \cos \phi, w = \rho \sin \phi. \end{aligned} \quad (8.22)$$

Then

$$\tilde{q}_{kl}(t) = \left(r(t), \theta(t) + \frac{2\pi k}{n_1}, \rho(t), \phi(t) + \frac{2\pi l}{n_2} \right) \quad (8.23)$$

is the position of body kl in double polar coordinates. We now calculate the potential energy.

$$\begin{aligned} \|q_{00,kl}\| &\equiv \|q_{kl} - q_{00}\| \\ &= \left\| \left(r \left(\cos \frac{2\pi k}{n_1} - 1 \right), r \sin \frac{2\pi k}{n_1}, \rho \left(\cos \frac{2\pi l}{n_2} - 1 \right), \rho \sin \frac{2\pi l}{n_2} \right) \right\| \\ &= 2 \sqrt{r^2 \sin^2 \frac{\pi k}{n_1} + \rho^2 \sin^2 \frac{\pi l}{n_2}} \end{aligned} \quad (8.24)$$

where again the distances between the bodies do not depend on θ or ϕ . From this and (6.3) p. 93, we obtain the potential function,

$$\begin{aligned} \therefore U(r, \rho) &= \frac{1}{4n_1 n_2} \sum_{k=0}^{n_1-1} \sum_{l=1}^{n_2-1} \frac{1}{\sqrt{r^2 \sin^2 \frac{\pi k}{n_1} + \rho^2 \sin^2 \frac{\pi l}{n_2}}} \\ &\quad + \frac{1}{4n_1 n_2 r} \sum_{k=1}^{n_1-1} \frac{1}{\sin \frac{\pi k}{n_1}} \end{aligned} \quad (8.25)$$

$$\begin{aligned}
&= \frac{1}{4n_1 n_2} \sum_{k=1}^{n_1-1} \sum_{l=0}^{n_2-1} \frac{1}{\sqrt{r^2 \sin^2 \frac{\pi k}{n_1} + \rho^2 \sin^2 \frac{\pi l}{n_2}}} \\
&\quad + \frac{1}{4n_1 n_2 \rho} \sum_{l=1}^{n_2-1} \frac{1}{\sin \frac{\pi l}{n_2}}
\end{aligned} \tag{8.26}$$

where the two forms of the same expression will be useful for differentiating with respect to ρ and r . The development of this problem will be used in both the next subsection as well as in the next chapter.

8.3.2 Relative Equilibrium Solutions for the Group $C_2^k \times C_2^l$

Palmore showed that *central configurations* form a proper subset of the *relative equilibria* in $\mathbf{R}^4$ for the inverse cube law of attraction [Pal]. We now use the group of the N -body problem to find a family of ivp's in $\mathbf{R}^4$ each of which yields a *relative equilibrium solution*; i.e., one for which the distance between any two bodies remains fixed over time (see S. 1.6 p. 9 Ch. 1) for the Newtonian (inverse square law) force of attraction. We shall see that these solutions do not form a *central configuration* and that we have a continuum of homographic configurations for one assignment of masses. We obtain such a result for each way of factoring a composite positive integer where both factors are greater than 1. First we give a

Definition 28 *If q is a solution of the N -body problem in barycentric coordinates satisfying*

$$\ddot{q}_i(t_0) = -\lambda q_i(t_0)$$

*$\forall i \leq N$ for some real positive λ , we say that the configuration of the bodies at time t_0 is a **central configuration**.*

Let

$$a_{00} = (r_0, 0, \rho_0, 0)^t \text{ and} \tag{8.27}$$

$$b_{00} = (0, r_0 \omega_1, 0, \rho_0 \omega_2)^t \tag{8.28}$$

where $r_0 > 0$, $\rho_0 > 0$ satisfying (8.17) p. 134 and

$$r_0^2 + \rho_0^2 = 1 \quad (8.29)$$

and ω_1 and ω_2 will be chosen to give the relative equilibrium solution.

$$q_{00}(t) = L(t) a_{00} \quad (8.30)$$

where

$$L(t) \equiv \begin{bmatrix} \cos \omega_1 t & -\sin \omega_1 t & 0 & 0 \\ \sin \omega_1 t & \cos \omega_1 t & 0 & 0 \\ 0 & 0 & \cos \omega_2 t & -\sin \omega_2 t \\ 0 & 0 & \sin \omega_2 t & \cos \omega_2 t \end{bmatrix}. \quad (8.31)$$

If such a solution can be found, then from the group, $\forall k < n_1$ and $l < n_2$,

$$\begin{aligned} q_{kl}(t) &= H_{xy}^k H_{zw}^l q_{00}(t) \\ &= H_{xy}^k H_{zw}^l L(t) a_{00} \\ &= L(t) H_{xy}^k H_{zw}^l a_{00} \\ &= L(t) a_{kl}. \end{aligned}$$

where the third equality follows from the fact that L , H_{xy} and H_{zw} commute with each other. We can write

$$L(t) = L_1(t) \oplus L_2(t), \quad (8.32)$$

a direct sum decomposition of the linear operators on $\mathbf{R}^2$, given by

$$L_i(t) = \begin{bmatrix} \cos \omega_i t & -\sin \omega_i t \\ \sin \omega_i t & \cos \omega_i t \end{bmatrix},$$

for $i = 1, 2$. Then,

$$\ddot{L}(t) = (-\omega_1^2 L_1(t)) \oplus (-\omega_2^2 L_2(t)). \quad (8.33)$$

Note by symmetry, and the equation of motion (3.1) p. 24

$$\left(\ddot{q}_{00}(0) \right)^y = \left(\ddot{q}_{00}(0) \right)^w = 0 \quad (8.34)$$

and

$$\left(\ddot{q}_{00}\right)^x = \frac{1}{4n_1n_2} \sum_{k=1}^{n_1-1} \sum_{l=0}^{n_2-1} \frac{r_0 \sin^2 \frac{\pi k}{n_1}}{\left(r_0^2 \sin^2 \frac{\pi k}{n_1} + \rho_0^2 \sin^2 \frac{\pi l}{n_2}\right)^{\frac{3}{2}}} \quad (8.35)$$

$$\left(\ddot{q}_{00}\right)^z = \frac{1}{4n_1n_2} \sum_{k=0}^{n_1-1} \sum_{l=1}^{n_2-1} \frac{\rho_0 \sin^2 \frac{\pi l}{n_2}}{\left(r_0^2 \sin^2 \frac{\pi k}{n_1} + \rho_0^2 \sin^2 \frac{\pi l}{n_2}\right)^{\frac{3}{2}}}. \quad (8.36)$$

Since (8.33) gives

$$\frac{d^2}{dt^2} q_{00}(t) = -(\omega_1^2 x(t), \omega_1^2 y(t), \omega_1^2 z(t), \omega_1^2 w(t))$$

then

$$\ddot{q}_{00}(0) = -(\omega_1^2 r_0, 0, \omega_2^2 \rho_0, 0). \quad (8.37)$$

Thus a solution exists for each choice of r_0, ρ_0 subject to (8.29) by choosing ω_1 and ω_2 to satisfy

$$\omega_1^2 = \frac{1}{4n_1n_2} \sum_{k=1}^{n_1-1} \sum_{l=0}^{n_2-1} \frac{\sin^2 \frac{\pi k}{n_1}}{\left(r_0^2 \sin^2 \frac{\pi k}{n_1} + \rho_0^2 \sin^2 \frac{\pi l}{n_2}\right)^{\frac{3}{2}}} \text{ and} \quad (8.38)$$

$$\omega_2^2 = \frac{1}{4n_1n_2} \sum_{k=0}^{n_1-1} \sum_{l=1}^{n_2-1} \frac{\sin^2 \frac{\pi l}{n_2}}{\left(r_0^2 \sin^2 \frac{\pi k}{n_1} + \rho_0^2 \sin^2 \frac{\pi l}{n_2}\right)^{\frac{3}{2}}}. \quad (8.39)$$

Since r_0 and ρ_0 can be varied independently, we have a continuum of configurations giving rise to homographic solutions for each assignment of mass. Note also that the acceleration is not in proportion to the position in general: i.e., it is not a central configuration (c.c.). To see this note from (8.27) and (8.37) that to have a c.c. we would require $\omega_1^2 = \omega_2^2$. However, as we vary r_0 and ρ_0 subject to (8.29) note that by selecting the terms in (8.38) for $l = 0$ we get

$$\omega_1^2 > \frac{1}{4n_1n_2r_0^3} \sum_{k=1}^{n_1-1} \frac{1}{\sin^2 \frac{\pi k}{n_1}} \rightarrow \infty \text{ as } r_0 \rightarrow 0$$

whereas ω_2^2 tends to a finite limit as $\rho_0 \rightarrow 1$.

Chapter 9

Hill-like Problems

Central configurations and homographic solutions are central to the N -body problem because so little can be written down about non-homographic solutions. However, in 1878 G. W. Hill [Hl] studied non-homographic solutions in the *restricted* 3-body problem: i.e., one body of zero mass and two bodies (called primaries) of positive mass. The primaries can be thought of as the Earth and Moon and the zero mass or infinitesimal body could be an Earth launched artificial probe which has no effect upon movement of the primaries. Hill's fundamental work was published almost 80 years before the launch of Sputnik. In one example of his work, the primaries form a relative equilibrium and one chooses a rotating coordinate frame relative to which the positions of the primaries remain fixed. Although the solution for the zero mass body could not be written down, Hill found that he could place limits on the region of space that could be occupied by it for each set of initial conditions. Parameterizing the initial conditions yields surfaces (or curves) of zero velocity for the problem in 3 (or 2) dimensions. Such surfaces or curves define the boundaries of the regions of space outside of which the infinitesimal body does not venture. Discussion of Hill's problem is found in many good texts on celestial mechanics ([Mar. Chapter 9], [Sz. Ch. 4, esp. sections 4.7 and 4.8] and [Wnt. Section 489]).

In this chapter we use the group of the N -body problem to consider some ivp's whose solutions are also non-homographic but can be studied, à la Hill. That is, we are able to find contour curves that place limits on the movements of the bodies based on the initial conditions. In our examples as in Hill's problem the force of attraction is Newtonian but unlike his problem all masses are strictly positive; so, not even the solution of the primaries can be written down. In each problem we specify the number of bodies, their masses, the total angular momentum and the group of the ivp. For different values of the total energy and subject to these restrictions, we obtain contour curves that restrict where the bodies can move in space. Our examples all have the curious feature that, because of the choice of the group and the linear subspaces of the seed bodies, no collision can occur in future or past time: i.e., the solution can be extended for all $t \in \mathbf{R}$.

9.1 The Rotating Antiprism

In the first example, we consider a $\mathcal{G}$ -orbit ivp (see Section 6.2 p. 89) in which bodies of equal mass occupy the vertices of a rotating antiprism in $\mathbf{R}^3$. There is a single orbit (i.e., a single seed body) for the group and the angular momentum is non-zero. Let the number of bodies be $2n$ where $n \geq 2$. The problem is in barycentric coordinates which of course is centroidal. Hence the position and velocity isometries for any member of the group are equal and are orthogonal (i.e., linear) operators. The group is cyclic and is generated by the orthogonal transformation.

$$H = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} & 0 \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (9.1)$$

It is clear that H has order, $2n$.

9.1.1 Definition of the Antiprism Problem

We define a family of ivp's as follows: for convenience we number the bodies starting at 0 and let the initial time, $t_0 = 0$. Let the initial position of the seed body be $a_0 = (a_0^x, a_0^y, a_0^z) = q_0(0)$ and let its initial velocity be $b_0 = (b_0^x, b_0^y, b_0^z) = \dot{q}_0(0)$ where

$$(a_0^x)^2 + (a_0^y)^2 > 0 \quad (9.2)$$

(i.e., the initial position of the seed body, a_0 is not on the z axis). We shall also define a parameter, c and require it to be non-zero: namely,

$$c \equiv a_0^x b_0^y - a_0^y b_0^x \neq 0. \quad (9.3)$$

We shall see that c is the total angular momentum in the xy plane (or equivalently in the z direction). We remark at the outset that our Hill functions will be created by varying the total energy, h of the system while the angular momentum will be held constant. For the time being we shall also require that

$$(a_0^z)^2 + (b_0^z)^2 > 0 \quad (9.4)$$

(i.e., the solution will not be confined to the xy plane for all time). In this way the problem will have dimension 3 (see Defn. 6 p. 36 and Thm. 8 p. 38) and the group will be uniquely defined. Later we shall include solutions in which the bodies are confined to the xy plane in order to fill in gaps in our Hill functions. Such solutions have dimension 2 and have a larger group.

Let the mass of the seed body (and hence all bodies) be $1/2n$. For $i \geq 1$, let

$$a_i = H^i a_0 \text{ and} \quad (9.5)$$

$$b_i = H^i b_0. \quad (9.6)$$

Let the solution for the seed body be given by

$$q_0(t) = (x(t), y(t), z(t)). \quad (9.7)$$

Then from the theory of the group, the solution for body i is

$$q_i(t) = (x_i(t), y_i(t), z_i(t)) = H^i q_0(t); \quad (9.8)$$

so, the solution for all $2n$ bodies is an immediate consequence of the solution for the seed body: i.e., we may concentrate our attention on the solution, (x, y, z) of the seed body. We now check that this is a well-defined $\mathcal{G}$ -orbit ivp (Def. 20 p. 89 Ch. 6). Clearly, $H^{2n} = I_3$, the identity map on $\mathbf{R}^3$ and $\langle H \rangle$ is a cyclic group of order $2n$ satisfying requirement 1 of that definition. Since a_i does not lie on the z axis by (9.2) then $\forall i$ where $1 \leq i \leq 2n-1$, $H^i a_0 \neq a_0$ and $\text{stab}_{\langle H \rangle} a_0$ is the trivial group. Thus $\text{fix}_{\mathbf{R}^d} \text{stab}_{\langle H \rangle} a_0$ is all of $\mathbf{R}^3$ and $b_0 \in \text{fix}_{\mathbf{R}^d} \text{stab}_{\langle H \rangle} a_0$ whatever the choice of b_0 (thus satisfying requirement 2 of that definition). It is clear from (9.1) that $\sum_{i=0}^{2n-1} H^i = 0_3$, the zero operator in $\mathbf{R}^3$ and hence the centroid of any orbit is 0 (Thm. 101 p. 217 App. A) satisfying requirement 3 of the definition. We have established that this is a $\mathcal{G}$ -orbit ivp for *some* group $\mathcal{G}$. We now consider Part 4 of Def. 20 which requires that no larger group than $\langle H \rangle$ preserve the ivp. Since we have a well-defined problem we can determine the total angular momentum. Since H^2 fixes the z axis the only possible non-zero component of angular momentum is c_{xy} by Thm. 58 p. 130 Ch. 8 and we can write

$$C = \begin{pmatrix} 0 & c_{xy} & 0 \\ -c_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9.9)$$

where by the definition in Section 3.3.3 p. 33 and equations (9.1), (9.5), (9.6) and (9.9)

$$\begin{aligned} c_{xy} &= \sum_{k=0}^{2n-1} m_k (a_k^x b_k^y - a_k^y b_k^x) \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} [(H^k a_0)^x (H^k b_0)^y - (H^k a_0)^y (H^k b_0)^x] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2n} \sum_{k=0}^{2n-1} \left(a_0^x \cos \frac{\pi k}{n} - a_0^y \sin \frac{\pi k}{n} \right) \left(b_0^x \sin \frac{\pi k}{n} + b_0^y \cos \frac{\pi k}{n} \right) \\
&\quad - \left(a_0^x \sin \frac{\pi k}{n} + a_0^y \cos \frac{\pi k}{n} \right) \left(b_0^x \cos \frac{\pi k}{n} - b_0^y \sin \frac{\pi k}{n} \right) \\
&= a_0^x b_0^y - a_0^y b_0^x.
\end{aligned}$$

$$\therefore c_{xy} = c \tag{9.10}$$

by (9.3) and c equals the total angular momentum in the xy plane as claimed above.

The proper rotations in $\langle H \rangle$ are $\langle H \rangle \cap \mathcal{SO}(3) = \langle H^2 \rangle \approx C_n$ which is the cyclic group of order n generated by a rotation about the z axis through an angle of $2\pi/n$. This is the only rotational axis possible because the angular momentum is non-zero (Cor. 59 p. 130 Ch. 8). The only angle of rotation smaller than $2\pi/n$ about the z axis that could possibly permute the a_i 's would be π/n . However, from (9.4) this fails to permute both the a_i 's and the b_i 's. Thus no new rotations preserve the ivp. Since the index of the rotational subgroup of any finite subgroup of $\mathcal{O}(3)$ is either 1 or 2 and $[\langle H \rangle : C_n]$ is already 2, there can be no more members of the group that have determinant equal to -1 and hence $\langle H \rangle$ is the whole group and part 4 of the definition is satisfied. Thus we have a well-defined $\langle H \rangle$ -orbit ivp.

In order for a collision to occur we would require that $H^i q_0(t) = q_0(t)$ for some $i = 1, \dots, 2n-1$. This could only happen if $q_0^x(t) = q_0^y(t) = 0$; i.e., n bodies collide on the z axis at $(0, 0, z)$ and another n bodies collide at $(0, 0, -z)$. This includes but is not restricted to the case where $z = 0$ for which we have a simultaneous collision of all bodies at 0. Although the theorem that the total angular momentum must be zero for a simultaneous collision requires all $2n$ bodies to collide at the same point we shall see that in this case a collision is impossible because of the Hill-like function restriction.

9.1.2 Cylindrical Coordinates

From (9.1), (9.7) and (9.8) $x_k^2 + y_k^2 = x^2 + y^2$ and $z_k = (-1)^k z$ whenever $0 \leq k < 2n$. Let $\tilde{q}_k$ be the solution q_k in cylindrical coordinates: i.e., let $\tilde{q}_k(t) = (r_k(t), \theta_k(t), z_k(t))$ where

$$\begin{aligned} r_k(t) &= r(t) \cos \theta_k(t) \\ y_k(t) &= r_k(t) \sin \theta_k(t) \end{aligned} \quad (9.11)$$

and define $r = r_0$ and $\theta = \theta_0$. Then

$$x = r \cos \theta, \quad y = r \sin \theta \quad (9.12)$$

and from (9.1), (9.8) and (9.11), $r_k^2 = x_k^2 + y_k^2 = x^2 + y^2 = r^2$; also, $\cos \theta_k = x_k/r = (H^k q_0)^x / r = (x \cos \frac{\pi k}{n} - y \sin \frac{\pi k}{n}) / r = \cos(\theta + \frac{\pi k}{n})$. Similarly, $\sin \theta_k = \sin(\theta(t) + \frac{\pi k}{n})$: so,

$$\begin{aligned} r_k(t) &= r(t) \\ \theta_k(t) &= \theta(t) + \frac{\pi k}{n} \\ z_k(t) &= (-1)^k z(t): \text{ so,} \end{aligned} \quad (9.13)$$

$$\tilde{q}_k(t) = \left(r(t), \theta(t) + \pi k/n, (-1)^k z(t) \right). \quad (9.14)$$

We now calculate the potential energy for the system as a function of (r, θ, z) . To do this we first get the distances between pairs of bodies. From (9.13) the difference vectors,

$$\|q_{0,2k}\| = 2r \sin \frac{\pi k}{n} \text{ and} \quad (9.15)$$

$$\|q_{0,2k+1}\| = 2r \sqrt{\sin^2 \frac{\pi}{2n} (2k+1) + \left(\frac{z}{r}\right)^2}. \quad (9.16)$$

Note that the distances between pairs of bodies do not depend on θ . They and hence the potential energy depend only on r and z . For the Newtonian potential,

$u(x) = 1/x$. Since we have a $\mathcal{G}$ -orbit ivp. from (6.3) p. 93 Ch. 6. where $N = 2n$ and $m = 1/(2n)$, writing the potential as a function of r and z and using (9.15) and (9.16) gives

$$U(r, z) = \frac{P_n}{2r} + \frac{1}{8nr} \sum_{k=0}^{n-1} \frac{1}{\sqrt{\left(\frac{z}{r}\right)^2 + \sin^2 \frac{\pi}{2n} (2k+1)}} \quad (9.17)$$

where we define

$$P_n = \frac{1}{4n} \sum_{k=1}^{n-1} \frac{1}{\sin \frac{\pi k}{n}}. \quad (9.18)$$

As in the derivation of (9.10) we have with conservation of angular momentum

$$\begin{aligned} c &= c_{xy}(t) \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} [q_k^x(t) \dot{q}_k^y(t) - q_k^y(t) \dot{q}_k^x(t)] \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} \left[(H^k q_0(t))^x (H^k \dot{q}_0(t))^y - (H^k q_0(t))^y (H^k \dot{q}_0(t))^x \right] \\ &= x(t) \dot{y}(t) - \dot{x}(t) y(t) \\ \therefore c &= r^2 \dot{\theta} \end{aligned} \quad (9.19)$$

from (9.12). The total kinetic energy is (since all bodies have the same kinetic energy)

$$K = \frac{1}{2} \sum_{i=0}^{2n-1} m_i \dot{q}_i^2 = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

where the last step comes from $\dot{x}^2 + \dot{y}^2 = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta)^2 + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)^2 = \dot{r}^2 + r^2 \dot{\theta}^2$. Applying (9.19) gives

$$K = \frac{1}{2} \left(\dot{r}^2 + \dot{z}^2 + \frac{c^2}{r^2} \right) \quad (9.20)$$

Thus the total energy of the system is

$$h = \frac{1}{2} \left(\dot{r}^2 + \dot{z}^2 + \frac{c^2}{r^2} \right) - U(r, z). \quad (9.21)$$

Recognizing that $\dot{r}^2 + \dot{z}^2 \geq 0$ permits us to observe that the only places to which the bodies can move must satisfy

$$h \geq \frac{c^2}{2r^2} - U(r, z). \quad (9.22)$$

This equation will be the source of the Hill function.

9.1.3 Spherical Coordinates

Although we shall return to cylindrical coordinates to obtain the contour plots, we shall temporarily use spherical coordinates to obtain some critical values of energy and r . One sees from (9.17) that $U(r, z)$ is homogeneous of degree -1 in r and z . As we shall see if we convert to spherical coordinates now, not only will it be computationally easier but it will reveal asymptotic behaviour of our Hill function. Let

$$z = \rho \cos \phi \quad (9.23)$$

$$r = \rho \sin \phi.$$

Then ϕ is the angle between the z axis and the vector $q = (x, y, z)$ and

$$0 < \phi < \pi. \quad (9.24)$$

Define

$$f(\phi) = \frac{1}{2}P_n + \frac{1}{8n} \sum_{k=0}^{n-1} \frac{1}{\sqrt{\cot^2 \phi + \sin^2 \frac{\pi}{2n} (2k+1)}} \quad (9.25)$$

then from (9.17) and (9.23)

$$U(r, z) = \frac{f(\phi)}{r}. \quad (9.26)$$

From (9.22) and (9.26) we have

$$h \geq \frac{c^2}{2r^2} - \frac{f(\phi)}{r}.$$

or

$$hr^2 + f(\phi)r - \frac{c^2}{2} \geq 0 \quad (9.27)$$

Note the interplay between the spherical and cylindrical coordinates. At any time we can get back to spherical coordinates by substituting into (9.23). Equation (9.27) is a quadratic inequality in r , given h , c and ϕ . It gives us our Hill region for a given ϕ , the boundaries of which are as follows: if $h < 0$ the two roots are both positive and the quadratic is only positive between them; namely,

$$r = \frac{-f(\phi) \pm \sqrt{f^2(\phi) + 2hc^2}}{2h}. \quad (9.28)$$

If $h = 0$, the Hill region is bounded below at $r = c^2/2f(\phi)$ and there is no upper bound; i.e., total escape is possible. If $h > 0$ one of the roots is negative and the Hill region is bounded below at

$$r = \frac{-f(\phi) + \sqrt{f^2(\phi) + 2hc^2}}{2h}.$$

Set

$$r_1 = \begin{cases} \frac{-f(\phi) + \sqrt{f^2(\phi) + 2hc^2}}{2h}, & \text{if } h \neq 0 \\ \frac{c^2}{2f(\phi)}, & \text{if } h = 0. \end{cases} \quad (9.29)$$

and set

$$r_2 = \frac{-f(\phi) - \sqrt{f^2(\phi) + 2hc^2}}{2h}. \quad (9.30)$$

From (9.29) in all cases $r_1 > 0$ and we have the following restrictions on r :

$$\begin{aligned} h < 0: & \quad 0 < r_1 \leq r \leq r_2 \\ h = 0: & \quad 0 < r_1 \leq r < \infty \\ h > 0: & \quad 0 < r_1 \leq r < \infty. \end{aligned} \quad (9.31)$$

Note also that for $h = 0$,

$$r_1 = \lim_{h \rightarrow 0} \frac{-f(\phi) + \sqrt{f^2(\phi) + 2hc^2}}{2h}.$$

This resolves an earlier claim that r can never go to zero; i.e., we can never have collisions for this family of ivp's with non-zero angular momentum.

9.1.4 Negative Total Energy

If $h < 0$, clearly, solutions only occur if the discriminant in (9.28) is positive: i.e.,

$$f^2(\phi) \geq -2hc^2. \quad (9.32)$$

This means that for a given value of h and c , the contour function defining the region in which the bodies must move is confined to those values of ϕ for which (9.32) is satisfied: i.e., $f(\phi) \geq c\sqrt{-2h}$. Note that as ϕ increases from 0 to $\pi/2$, $z/r = \cot \phi$ decreases from ∞ to 0 and $f(\phi)$ increases from

$$\lim_{\phi \rightarrow 0^+} f(\phi) = \frac{P_n}{2}$$

to

$$f\left(\frac{\pi}{2}\right) = \frac{1}{8n} \sum_{k=1}^{n-1} \frac{1}{\sin \frac{\pi k}{n}} + \frac{1}{8n} \sum_{k=0}^{n-1} \frac{1}{\sin \frac{\pi}{2n} (2k+1)} = \frac{1}{8n} \sum_{l=1}^{2n-1} \frac{1}{\sin \frac{\pi l}{n}} = P_{2n}.$$

Since $\cot(\pi - \phi) = -\cot \phi$,

$$f(\pi - \phi) = f(\phi) \quad (9.33)$$

and since $\phi \in (0, \pi)$, then by (9.24),

$$\lim_{\phi \rightarrow 0^+} f(\phi) = \frac{1}{2}P_n < f(\phi) \leq P_{2n} = f\left(\frac{\pi}{2}\right). \quad (9.34)$$

Since $\cot \phi$ is strictly decreasing on $(0, \pi/2]$, from (9.25) we can see that $f(\phi)$ is strictly increasing on $(0, \pi/2]$ and only adopts its maximum value at $\phi = \pi/2$.

Minimum Energy Condition

We now determine how negative h can become and still yield a solution. Substituting the second inequality of (9.34) into (9.27) gives

$$hr^2 + P_{2n}r - \frac{c^2}{2} \geq hr^2 + f(\phi)r - \frac{c^2}{2} \geq 0. \quad (9.35)$$

Since we are currently treating the case where $h < 0$, in order to have any solution whatever, the discriminant of the first quadratic in this display must be positive: namely, $P_{2n}^2 + 2hc^2 \geq 0$. Let

$$h_0 = -\frac{P_{2n}^2}{2c^2}. \quad (9.36)$$

From (9.35) with $h = h_0$ the first quadratic must be zero forcing the second quadratic to be zero also. Hence, $f(\phi)$ adopts its maximum value, P_{2n} which only occurs for $\phi = \pi/2$: i.e., all bodies are confined to the xy plane for all time. So, $z(t) \equiv 0$ and from (9.35)

$$r(t) \equiv r_0 = -\frac{P_{2n}}{2h_0} = \frac{c^2}{P_{2n}}. \quad (9.37)$$

This minimum energy case is a relative equilibrium where the bodies are confined to the xy -plane.

It is interesting to compare these results with Wintner [Wnt. (13) S. 334 p. 252] which gives

$$J'' - 2h - \frac{J'^2}{4J} \geq \frac{C^2}{J}.$$

He is studying simultaneous collisions and since it is more general we can expect it to be less precise when applied to this example. Nevertheless, our (9.36) and (9.37) can be written as $h_0 = -c^2/2r_0^2$ for which $z = 0$. In our case, the total mass is unity and $J = r^2 + z^2$. For the minimum energy condition, $z = 0$ and we can write

$$h \geq h_0 = -\frac{c^2}{2r_0^2} > -\frac{c^2}{2J}$$

which is what we obtain from Wintner's (13) for a relative equilibrium (J is constant).

Minimum Energy for Polygonal Escape

We now consider the minimum energy at which the two regular n -gons forming the top and bottom planes of the antiprism can just escape from each other. Unlike *total disintegration* à la Pollard [Pl2] in which all mutual distances between pairs of bodies

become infinite, in this case the two polygons, as we shall see, move infinitely far apart but do not become infinitely large. Thus, z will become unbounded but not r . This will require ϕ to approach 0 or π . Note that total escape can occur without ϕ approaching either limiting value; however, this requires that $r \rightarrow \infty$ causing total disintegration à la Pollard for which the potential energy, $U(r, z)$ tends to 0. Such a case needs $h \geq 0$ and from (9.31) it is clear that any $h \geq 0$ permits r to become unbounded. In the case studied here we shall see that $h < 0$.

In any solution in which $z \rightarrow \infty$ while r remains bounded, $\phi \rightarrow$ either 0 or π and $f(\phi) \rightarrow P_n/2$ by (9.33) and (9.34). Then the limiting value of the quadratic in (9.27) as $\phi \rightarrow 0$ is non-negative also; i.e.,

$$hr^2 + \frac{P_n}{2}r - \frac{c^2}{2} \geq 0. \quad (9.38)$$

All values of $r(t)$ in the solution must satisfy this inequality because all values of $f(\phi) > P_n/2$. Again, since we seek a minimum value of $h < 0$ the leading coefficient of the quadratic is negative and solutions only occur to the inequality if the discriminant, $P_n^2/4 + 2hc^2 \geq 0$; i.e., $h \geq -P_n^2/8c^2$. Then the minimum energy permitting the two polygons to escape from one another is h_e , where

$$h_e = -\frac{P_n^2}{8c^2} \quad (9.39)$$

and we see that it is negative as claimed above. From (9.38) the value that r approaches as $z \rightarrow \infty$ for this energy, h_e which we denote as r_e is given by

$$r_e = -\frac{P_n}{4h_e} = \frac{2c^2}{P_n}. \quad (9.40)$$

9.1.5 The Choice of Angular Momentum

Consider two choices of angular momentum, c_α and c_β , where $c_\alpha = kc_\beta$. If we let their corresponding energies be h_α and h_β where $h_\alpha = h_\beta/k^2$ then from (9.29), (9.30) and (9.31) the curves for r_α and r_β for these values of c and h from (9.31)

are related by $r_\alpha(\phi) = k^2 r_\beta(\phi)$; i.e., the contour curves for different values of c are merely scaled versions of one another. So, we shall simply choose a value of c that causes the relative equilibrium solution to have unit radius; namely, from (9.37)

$$c = \sqrt{P_{2n}}, \quad (9.41)$$

$$r_0 = 1 \quad (9.42)$$

$$h_0 = -\frac{P_{2n}}{2} \quad (9.43)$$

where the last line in the display comes from (9.36). Note also that the minimum value of energy for polygonal escape is by (9.39)

$$h_\epsilon = -\frac{P_n^2}{8P_{2n}} \quad (9.44)$$

and the corresponding limiting r value as the polygons move infinitely far apart is

$$r_\epsilon = \frac{2P_{2n}}{P_n}. \quad (9.45)$$

For $n = 2$, $P_2 = 1/8$, $P_1 = (1 + 2\sqrt{2})/16$ and

$$\frac{r_{\epsilon,2}}{r_{0,2}} = 1 + 2\sqrt{2} \quad (9.46)$$

where, when we are discussing the antiprism problem for different values of n we use the terms $r_{0,n}$ and $r_{\epsilon,n}$ for r_0 and r_ϵ respectively.

Note also that

$$\lim_{n \rightarrow \infty} \frac{r_{\epsilon,n}}{r_{0,n}} = 2 \lim_{n \rightarrow \infty} \frac{P_{2n}}{P_n} = 2. \quad (9.47)$$

9.1.6 Example Diagrams

Figure 9-1 shows a graph of the Hill curves for the antiprism for $n = 2$ with energies, $h = -0.09, -0.05, -0.03, -0.02, -0.015, -0.012$ and -0.01 where the curve for $h = -0.09$ is the tiny circle enclosing $r_0 = 1$ and for $h = -0.01$ the curve is split into an upper and a lower part extending beyond the graph. The minimum energy

Figure 9-1: Antiprism for $n = 2$.

for polygonal escape, h_e gives a curve that extends to $z = \pm\infty$. For the digon (i.e., $n = 2$) $h_e = -(2\sqrt{2} - 1)/224 \cong -0.00816262$. Since this is less than -0.01 this largest curve shown in this figure is bounded. The horizontal line in the figure is $r = r_e$ is asymptotic to the two halves of the curve for $h = h_e$ (not shown).

Figure 9-2 shows the Hill curves for $n = 6$ and h ranging over $-0.15, -0.1, -0.05, -0.03, h_e \cong -0.0279163$.

Note that all bodies forming the vertices of the antiprism move in $\mathbf{R}^3$ inside the surface of revolution defined by its Hill curve.

9.1.7 The Equations of Motion for r and z

In summary we have chosen a value of angular momentum in (9.41) for which the minimum energy solution is a relative equilibrium with all bodies on the unit circle in the xy plane centred at the origin. As we increase the total energy from this minimal

Figure 9-2: Antiprism for $n = 6$

value closed curves form in the rz plane of increasing size as seen in the figures. If q is a solution for some value of h that is not confined to the xy plane then it is a non-homographic solution and we have seen that the movement of the bodies is confined to the region defined by the curve corresponding to h ; hence, the similarity to Hill's problem. Furthermore, if a body and equivalently in this case, if all bodies simultaneously reach the boundary of the Hill curve, their component of velocity in the r and z directions are zero. The component of velocity in the θ direction equals $r\dot{\theta} = c/r$ from (9.19); i.e., it is determined by the value of r . The equations of motion for r and z , as we shall see, depend only on r and z . Furthermore, θ is an immediate consequence of any solution of the differential equation governing r and z by the integral.

$$\theta(t) = \theta_0 + c \int_{t_0}^t \frac{1}{r(\tau)^2} d\tau. \quad (9.48)$$

Thus, we can reduce the antiprism problem to one of determining the solution for r and z . In this light the boundary curves can be thought of as "zero-velocity" curves and hence exhibit another similarity to Hill functions.

Converting (9.17) p. 146 to Cartesian coordinates and taking the gradient gives the acceleration of body 0. The reason for this is as follows: when we have a $\mathcal{G}$ -orbit problem the potential, U is written as a function of the coordinates of the seed body only. When we take its gradient we are measuring the rate of change of potential as we vary $N = 2n$ bodies' positions simultaneously. The resultant is $2n$ times the force applied to the seed (or any) body. Thus the force applied to the seed body is $\nabla U/2n$. Since the mass of the bodies is $1/2n$, the factors of $2n$ cancel and the acceleration is ∇U ; namely,

$$\begin{aligned} \ddot{q}_0(t) = & - \left\{ \frac{P_n}{2r^3} + \frac{1}{8n} \sum_{k=0}^{n-1} \frac{\sin^2 \frac{\pi}{2n} (2k+1)}{[(x^2 + y^2) \sin^2 \frac{\pi}{2n} (2k+1) + z^2]^{\frac{3}{2}}} \right\} (x, y, 0) \\ & - \frac{1}{8n} \sum_{k=0}^{n-1} \frac{1}{[(x^2 + y^2) \sin^2 \frac{\pi}{2n} (2k+1) + z^2]^{\frac{3}{2}}} (0, 0, z). \end{aligned} \quad (9.49)$$

This then is the o.d.e. governing the motion of the antiprism. We now separate $\ddot{q}_0$ into its components in the r , θ and z directions. Differentiating (9.12) twice w.r.t. t gives

$$\begin{aligned} \ddot{x} &= \ddot{r} \cos \theta - 2 \dot{r} \dot{\theta} \sin \theta - r \ddot{\theta} \sin \theta - r \dot{\theta}^2 \cos \theta \text{ and} \\ \ddot{y} &= \ddot{r} \sin \theta + 2 \dot{r} \dot{\theta} \cos \theta + r \ddot{\theta} \cos \theta - r \dot{\theta}^2 \sin \theta. \end{aligned}$$

From these equations we calculate the component of acceleration of body 0 in the r direction: viz.,

$$\ddot{x} \cos \theta + \ddot{y} \sin \theta = \ddot{r} - r \dot{\theta}^2 \quad (9.50)$$

Whereas, the component in the θ direction is

$$-\ddot{x} \sin \theta + \ddot{y} \cos \theta = 2 \dot{r} \dot{\theta} + r \ddot{\theta} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0 \quad (9.51)$$

from (9.19) and conservation of angular momentum. Substituting the left side of this last display into (9.49) also gives zero; i.e., in obtaining the component of $\ddot{q}_0$ in the θ direction, we have merely rederived the conservation of angular momentum. Thus, the o.d.e. governing the motion of the antiprism is given by substituting (9.50) into (9.49) and applying (9.19)

$$\ddot{r} = \frac{c^2}{r^3} - \frac{P_n}{2r^2} - \frac{r}{8n} \sum_{k=0}^{n-1} \frac{\sin^2 \frac{\pi}{2n} (2k+1)}{\left[r^2 \sin^2 \frac{\pi}{2n} (2k+1) + z^2 \right]^{\frac{3}{2}}} \text{ and} \quad (9.52)$$

$$\ddot{z} = -\frac{z}{8n} \sum_{k=0}^{n-1} \frac{1}{\left[r^2 \sin^2 \frac{\pi}{2n} (2k+1) + z^2 \right]^{\frac{3}{2}}} \quad (9.53)$$

where the second equation is just the z component of $\ddot{q}_0$ from (9.49). These are the equations of motion for r and z and are independent of θ . Given a solution for $r(t)$ we obtain θ from (9.48). Thus as claimed above, we can study the problem as an o.d.e. in only r and z for which the phase space $(r, z, \dot{r}, \dot{z})$ has dimension 4.

For any given value of h when (r, z) lies inside the zero velocity curves $\dot{r}^2 + \dot{z}^2 \neq 0$ and we can define v and ψ by

$$\begin{aligned} \dot{r} &= v \cos \psi \\ \dot{z} &= v \sin \psi. \end{aligned}$$

Then the phase space is determined by (r, z, v, ψ) . Combining (9.21) p. 146 with the fact that $v^2 = \dot{r}^2 + \dot{z}^2$ gives

$$v = \sqrt{2[h + U(r, z)] - \frac{c^2}{r^2}};$$

i.e., v is a function of r and z . Thus the phase space is completely determined by (r, z, ψ) where (r, z) is confined to a closed subspace of $\mathbf{R}^+ \times \mathbf{R}$ and $\psi \in [-\pi, \pi)$. When a solution reaches the boundary of the Hill function, its velocity, v goes to zero and then reverses direction whereupon the solution retraces its earlier path as always happens in an o.d.e. like (9.52), (9.53) and (3.1) which are autonomous

second order equations with no first derivative term. Note in this case that $\dot{\theta}$ does not go to zero and does not reverse direction: however, we have removed θ from (9.52) and (9.53) and can still make the claim for r and z . What happens in the (r, z, ψ) space is that the path reaches the boundary for some value of $\psi = \psi_0$ and returns with $\psi = \psi_0 + \pi$. Thus we must topologically identify the pairs

$$\{(r, z, \psi), (r, z, \psi + \pi) \mid (r, z) \text{ lie on the Hill boundary} \}.$$

In the case where $h < h_c$, the Hill function is compact and the flow in (r, z, ψ) space is in a compact space that is homeomorphic to a solid torus whose boundary is pasted to itself as just described.

9.2 Generalized Sitnikov Motion for C_n ($n \geq 2$)

Our second example of Hill-like problems again has the following properties in common with the rotating antiprism problem from the previous section

- all bodies have positive mass.
- the group of the ivp is cyclic.
- the angular momentum is non-zero.
- collisions are not possible and
- the problem is barycentric which means that the isometry groups for position and velocity are orthogonal and equal.

However, unlike the previous problem it is not a $\mathcal{G}$ -orbit ivp since there are two distinct orbits under the action of the group. In this case for some choice of $n \geq 2$, the orthogonal group is isomorphic to C_n , the cyclic group of order n generated by

a rotation about the z axis through an angle of $2\pi/n$: i.e., generated by

$$H = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} & 0 \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9.54)$$

There are two seed bodies, one on the z axis whose orbit is obviously a singleton and one not on the z axis whose orbit generates n bodies forming a rotating regular n -gon. This is a generalization of the 3-body problem studied by Sitnikov[Sit].

9.2.1 Definition of the Sitnikov Problem

Again we define a family of ivp's and again we count the bodies starting at 0 with the initial time, $t_0 = 0$. Let the seed body for the n bodies not on the z axis be body 0 with its initial position $a_0 = (a_0^x, a_0^y, a_0^z) = q_0(0)$ and let its initial velocity be $b_0 = (b_0^x, b_0^y, b_0^z) = \dot{q}_0(0)$ where

$$(a_0^x)^2 + (a_0^y)^2 > 0 \quad (9.55)$$

Let $a_k = H^k a_0$ and $b_k = H^k b_0$ for $0 < k \leq n-1$. Then bodies 0 through $n-1$ form the vertices of the regular n -gon. Let $a_n = (0, 0, a_n^z)$ and $b_n = (0, 0, b_n^z)$: so, body n moves along the z axis. Let the total mass of all bodies be 1. Let the mass of body 0 (and hence of bodies 1 through $n-1$) be α/n and the mass of body n be β . Then, to have the total mass equal to unity we require

$$\alpha + \beta = 1 \quad (9.56)$$

where

$$0 < \alpha < 1 \text{ and } 0 < \beta < 1. \quad (9.57)$$

We shall also define a parameter, c and require it to be non-zero: namely,

$$c \equiv \alpha (a_0^x b_0^y - a_0^y b_0^x) \neq 0. \quad (9.58)$$

As in the earlier problem we shall see that c is the total angular momentum in the xy plane (or equivalently in the z direction). For the time being we shall also require that

$$(a_0^z)^2 + (b_0^z)^2 > 0 \quad (9.59)$$

(i.e., the solution will not be confined to the xy plane for all time). In this way the problem will have dimension 3 (see Defn. 6 p. 36 and Thm. 8 p. 38) and the group will be uniquely defined. Later we shall include solutions in which the bodies are confined to the xy plane in order to fill in gaps in our Hill functions. Such solutions have dimension 2 and have a larger group. As in the last problem the isometry group of this problem is a cyclic group, in this case $\langle H \rangle$. Let the solution for the seed body not on the z axis be given by

$$q_0(t) = (x(t), y(t), z(t)). \quad (9.60)$$

Let us denote the solution for body k , $q_k(t) = (x_k(t), y_k(t), z_k(t))$, $0 \leq k \leq n$. Since $\text{fix}_{\mathbb{R}^3} H$ is the z axis, body n is confined to move along that axis; i.e., $q_n(t) = (0, 0, z_n(t))$. By the barycentric condition,

$$\alpha z(t) + \beta z_n(t) = 0. \quad (9.61)$$

From this and the theory of the group, the solution for body i is

$$q_k(t) = \begin{cases} H^k q_0(t) & \text{if } 0 \leq k < n \\ (0, 0, -\frac{\alpha}{\beta} z(t)) & \text{if } k = n: \end{cases} \quad (9.62)$$

so, the solution for all $n + 1$ bodies is an immediate consequence of the solution for the seed body, q_0 for the bodies not on the z axis; i.e., we may concentrate our attention on the solution, (x, y, z) of this seed body.

Again as in the last problem we use cylindrical coordinates and let

$$x = r \cos \theta, y = r \sin \theta \quad (9.63)$$

and from (9.56) and (9.61)

$$\|q_{0n}\| = \sqrt{(z - z_n)^2 + r^2} = \sqrt{\frac{z^2}{\beta^2} + r^2}$$

Using (6.3) from Cor. 41 p. 93 Ch. 6 for the mutual potential of the bodies off the z axis, we obtain the potential function.

$$\begin{aligned} U(r, z) &= \frac{n}{2} \left(\frac{\alpha}{n}\right)^2 \sum_{k=1}^{n-1} \frac{1}{2r \sin \frac{\pi k}{n}} + \frac{n \frac{\alpha}{n} \beta}{\sqrt{\frac{z^2}{\beta^2} + r^2}} \\ \therefore U(r, z) &= \frac{\alpha^2 P_n}{r} + \frac{\alpha \beta^2}{\sqrt{z^2 + \beta^2 r^2}}. \end{aligned} \quad (9.64)$$

where P_n is given by (9.18). Again we define the angular momentum matrix by (9.9) and observe that the calculations in the displays for (9.10) and (9.19) are very close to this case. Here α is the total mass in the orbit of body 0 and a similar computation yields from (9.58)

$$c_{xy} = \alpha (a_0^x b_0^y - a_0^y b_0^x) = c = \alpha r^2 \dot{\theta}. \quad (9.65)$$

Then as claimed c is the total angular momentum. Differentiating (9.61) gives $\dot{z}_n = -\alpha \dot{z}/\beta$. We use this with (9.65) to get the kinetic energy.

$$\begin{aligned} K &= \frac{1}{2} \beta \dot{z}_n^2 + \frac{1}{2} \alpha \left(\dot{z}^2 + \dot{r}^2 + r^2 \dot{\theta}^2 \right) \\ &= \frac{\alpha}{2} \left(\frac{\dot{z}^2}{\beta} + \dot{r}^2 + \frac{c^2}{\alpha^2 r^2} \right). \end{aligned}$$

From which we have

$$h = \frac{\alpha}{2} \left(\frac{\dot{z}^2}{\beta} + \dot{r}^2 + \frac{c^2}{\alpha^2 r^2} \right) - U(r, z). \quad (9.66)$$

As before we use spherical coordinates; i.e.,

$$\begin{aligned} z &= \rho \cos \phi \\ r &= \rho \sin \phi. \end{aligned} \quad (9.67)$$

We let

$$f(\phi) = \alpha^2 P_n + \frac{\alpha \beta^2}{\sqrt{\cot^2 \phi + \beta^2}}. \quad (9.68)$$

Then from (9.64)

$$f(\phi) = rU(r, z). \quad (9.69)$$

Since $\alpha > 0$, $\beta > 0$, $\dot{z}^2 \geq 0$, and $\dot{r}^2 \geq 0$, from (9.66) we have $h \geq c^2/2\alpha r^2 - f(\phi)/r$ or

$$hr^2 + f(\phi)r - \frac{c^2}{2\alpha} \geq 0. \quad (9.70)$$

Again we have Hill-like functions limiting regions in configuration space where the solution can go. As the reader has probably observed, the analysis is almost identical to that for the antiprism except that $f(\phi)$ is a different function and now we have a new parameter, mass which affects the result. In a manner similar to that of the previous problem, we set

$$r_1 = \begin{cases} \frac{-f(\phi) + \sqrt{f^2(\phi) + 2hc^2/\alpha}}{2h}, & \text{if } h \neq 0 \\ \frac{c^2}{2\alpha f(\phi)}, & \text{if } h = 0, \text{ and} \end{cases} \quad (9.71)$$

$$r_2 = \frac{-f(\phi) - \sqrt{f^2(\phi) + 2hc^2/\alpha}}{2h} \quad (9.72)$$

Then from (9.71) in all cases $r_1 > 0$ and we have the following restrictions on r :

$$\begin{aligned} h < 0: & \quad 0 < r_1 \leq r \leq r_2 \\ h = 0: & \quad 0 < r_1 \leq r < \infty \\ h > 0: & \quad 0 < r_1 \leq r < \infty. \end{aligned} \quad (9.73)$$

Note also that for $h = 0$,

$$r_1 = \lim_{h \rightarrow 0} \frac{-f(\phi) + \sqrt{f^2(\phi) + 2hc^2/\alpha}}{2h}.$$

Again as claimed r can never go to zero; so, we do not have collisions.

9.2.2 Negative Total Energy

If $h < 0$, clearly, solutions only occur if the discriminant in (9.70) is positive: i.e.,

$$f^2(\phi) \geq -\frac{2hc^2}{\alpha}. \quad (9.74)$$

This means that for a given value of h and c , the contour function defining the region in which the bodies must move is confined to those values of ϕ for which (9.74) is satisfied; i.e., $f(\phi) \geq c\sqrt{-2h/\alpha}$. As before, we have

$$f(\pi - \phi) = f(\phi) \quad (9.75)$$

and by (9.68)

$$\lim_{\phi \rightarrow 0^+} f(\phi) = \alpha^2 P_n < f(\phi) \leq \alpha(\alpha P_n + \beta) = f\left(\frac{\pi}{2}\right). \quad (9.76)$$

Also, as before $f(\phi)$ is strictly increasing on $(0, \pi/2]$ and only adopts its maximum value at $\phi = \pi/2$.

Minimum Energy Condition

We now determine how negative h can become and still yield a solution. Substituting the second inequality of (9.76) into (9.70) gives

$$hr^2 + \alpha(\alpha P_n + \beta)r - \frac{c^2}{2\alpha} \geq hr^2 + f(\phi)r - \frac{c^2}{2\alpha} \geq 0. \quad (9.77)$$

Since we are currently treating the case where $h < 0$, in order to have any solution whatever, the discriminant of the first quadratic in this display must be positive: namely, $\alpha^2(\alpha P_n + \beta)^2 + 2hc^2/\alpha \geq 0$. Let

$$h_0 = -\frac{\alpha^3(\alpha P_n + \beta)^2}{2c^2}. \quad (9.78)$$

Then from (9.77) with $h = h_0$ the first quadratic factors into a square and adopts its maximum value of zero forcing the second quadratic to be at most zero also. Hence,

$f(\phi)$ adopts its maximum value, $\alpha(\alpha P_n + \beta)$ which only occurs for $\phi = \pi/2$; i.e., all bodies are confined to the xy plane for all time. So, $z(t) \equiv 0$ and from (9.77) the only value of r giving a solution is

$$r(t) \equiv r_0 = -\frac{\alpha(\alpha P_n + \beta)}{2h_0} = \frac{c^2}{\alpha^2(\alpha P_n + \beta)}. \quad (9.79)$$

This minimum energy case is a relative equilibrium where the bodies are confined to the xy plane.

Minimum Energy for Polygonal Escape

As in the earlier problem we now consider the minimum energy at which the polygon and the seed body on the z axis can just escape from each other. Again we see that the minimum energy for polygonal escape is negative and for such an escape $z \rightarrow \infty$ while r remains bounded, $\phi \rightarrow$ either 0 or π and $f(\phi) \rightarrow \alpha^2 P_n$ by (9.75) and (9.76). Then the limiting value of the quadratic in (9.70) as $\phi \rightarrow 0$ is non-negative also; i.e.,

$$hr^2 + \alpha^2 P_n r - \frac{c^2}{2\alpha} \geq 0. \quad (9.80)$$

All values of $r(t)$ in the solution must satisfy this inequality because all values of $f(\phi) > \alpha^2 P_n$. Again, since we seek a minimum value of $h < 0$ the leading coefficient of the quadratic is negative and solutions only occur to the inequality if the discriminant, $\alpha^4 P_n^2 + 2hc^2/\alpha \geq 0$; i.e., $h \geq -\alpha^5 P_n^2/2c^2$. Then the minimum energy permitting the two polygons to escape from one another is h_e , where

$$h_e = -\frac{\alpha^5 P_n^2}{2c^2} \quad (9.81)$$

and we see that it is negative as claimed above. From (9.80) the value that r approaches as $z \rightarrow \infty$ for this energy, h_e which we denote as r_e is given by

$$r_e = -\frac{\alpha^2 P_n}{2h_e} = \frac{c^2}{\alpha^3 P_n}.$$

9.2.3 The Choice of Angular Momentum

Consider two choices of angular momentum, c_α and c_β , where $c_\alpha = kc_\beta$. If we let their corresponding energies be h_α and h_β where $h_\alpha = h_\beta/k^2$ then from (9.71), (9.72) and (9.73) the curves for r_α and r_β for these values of c and h from (9.73) are related by $r_\alpha(\phi) = k^2 r_\beta(\phi)$; i.e., the contour curves for different values of c are merely scaled versions of one another. So, we shall simply choose a value of c that causes the relative equilibrium solution to have unit radius: namely, from (9.79)

$$\begin{aligned} c &= \alpha \sqrt{\alpha P_n + \beta}, \\ r_0 &= 1 \\ h_0 &= -\frac{\alpha(\alpha P_n + \beta)}{2} \end{aligned}$$

where the third display came from (9.78). Note also that the minimum value of energy for polygonal escape is by (9.81)

$$h_c = -\frac{\alpha^3 P_n^2}{2(\alpha P_n + \beta)}$$

and the corresponding limiting r value as the polygons move infinitely far apart is

$$r_\epsilon = \frac{\alpha P_n + \beta}{\alpha P_n}.$$

For $n = 2$, $P_2 = 1/8$ and

$$\frac{r_{\epsilon,2}}{r_{0,2}} = \frac{\alpha + 8\beta}{\alpha}$$

where, when we are discussing the problem for different values of n we use the terms $r_{0,n}$ and $r_{\epsilon,n}$ for r_0 and r_ϵ respectively.

Note also that

$$\lim_{n \rightarrow \infty} \frac{r_{\epsilon,n}}{r_{0,n}} = 1.$$

9.2.4 Example Figures

Figure 9-3 shows the Hill curves for Sitnikov motion in the case where $n = 2$ and the total mass of the two bodies forming the polygon (in this case a digon) is the same

Figure 9-3: Sitnikov Motion for $\alpha = \beta = 1/2$ and $n = 2$

as the mass of the body on the z axis. The values of h plotted in the graph are (from small to large) $-1/64$, $-1/128$, $-1/256$, $-1/384$ and $-1/512$. Since $h_e = -1/576$ the largest curve shown is bounded but its energy is very close to escape. Again the horizontal line shown is $r = r_c$, which in this case is 9.

9.3 The Rectangular Product of 2 Polygons in $\mathbf{R}^4$

In Section 8.3 we developed the problem for the rectangular product of 2 polygons in $\mathbf{R}^4$. We now apply that to find a Hill-like function for this problem. As before we have

$$c_{xy} = r^2 \dot{\theta} \text{ and} \quad (9.82)$$

$$c_{zw} = \rho^2 \dot{\phi}. \quad (9.83)$$

The kinetic energy is

$$\begin{aligned}
K &= \frac{1}{2n_1n_2} \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} \dot{q}_{kl}^2 \\
&= \frac{1}{2n_1n_2} \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) \\
&= \frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right)
\end{aligned}$$

From (9.82) and (9.83) we get

$$K = \frac{1}{2} \left(\dot{r}^2 + \frac{c_{xy}^2}{r^2} + \dot{\rho}^2 + \frac{c_{zw}^2}{\rho^2} \right). \quad (9.84)$$

Thus the total energy of the system is

$$h = \frac{1}{2} \left(\dot{r}^2 + \frac{c_{xy}^2}{r^2} + \dot{\rho}^2 + \frac{c_{zw}^2}{\rho^2} \right) - U(r, \rho). \quad (9.85)$$

Since $\dot{r}^2 + \dot{\rho}^2 \geq 0$

$$h \geq \frac{c_{xy}^2}{2r^2} + \frac{c_{zw}^2}{2\rho^2} - U(r, \rho) \equiv f(r, \rho). \quad (9.86)$$

This defines the Hill curve.

9.3.1 Example: $n_1 = n_2 = 2$

If we let $n = 2$ the simplotope reduces to a familiar object: namely, the rectangle.

We also assume that

$$c_{xy} = c_{zw} = c,$$

i.e., the angular momentum in the xy and zw planes are the same. From (8.25)

$$U(r, \rho) = \frac{1}{16} \left(\frac{1}{r} + \frac{1}{\rho} + \frac{1}{\sqrt{r^2 + \rho^2}} \right) \quad (9.87)$$

and

$$h \geq \frac{c^2}{2} \left(\frac{1}{r^2} + \frac{1}{\rho^2} \right) - \frac{1}{16} \left(\frac{1}{r} + \frac{1}{\rho} + \frac{1}{\sqrt{r^2 + \rho^2}} \right) \equiv f(r, \rho). \quad (9.88)$$

Since $c^2 > 0$, then for any value of h , both r and ρ must be bounded away from 0 because otherwise $f(r, \rho) \rightarrow \infty$ as either r or $\rho \rightarrow 0$. Thus the solution cannot tend to the diagonal, Δ and after a result due to Painlevé (see [Wnt. S. 409 p. 326]) we cannot have a singularity. To see that the solution cannot tend to the diagonal, consider any two distinct bodies with positions, $q_{k_1 l_1}$ and $q_{k_2 l_2}$. Then by our numbering system for the bodies, $0 \leq k_1, k_2 < n_1$ and $0 \leq l_1, l_2 < n_2$. Since the bodies are distinct either $k_1 \neq k_2$ or $l_1 \neq l_2$ or both. Let $P_{xy} = \text{diag}(1, 1, 0, 0)$ and $P_{zw} = \text{diag}(0, 0, 1, 1)$, the respective orthogonal projections onto the xy and zw planes. Then,

$$\begin{aligned}
\|q_{k_1 l_1} - q_{k_2 l_2}\|^2 &= \|H_{xy}^{k_1} H_{zw}^{l_1} q - H_{xy}^{k_2} H_{zw}^{l_2} q\|^2 \\
&= \|(H_{xy}^{k_1 - k_2} H_{zw}^{l_1 - l_2} - I_4) q\|^2 \\
&= \|q_{k_1 - k_2, l_1 - l_2} - q\|^2 \\
&= 2 \sqrt{r^2 \sin^2 \frac{\pi(k_1 - k_2)}{n_1} + \rho^2 \sin^2 \frac{\pi(l_1 - l_2)}{n_2}} \\
&> 0
\end{aligned}$$

where the last equality comes from (8.24). Since r and ρ are bounded below the solution cannot approach the diagonal and we cannot have a singularity.

Minimum Energy Condition

As in the antiprism and the Sitnikov motion we derive h_0 , r_0 and h_e and choose a value of c making $r_0 = 1$. For $h = h_0$, $r = r_0$ and $\rho = \rho_0$ we have from (9.88)

$$\left. \frac{\partial f}{\partial r} \right|_{(r, \rho) = (r_0, \rho_0)} = -\frac{c^2}{r_0^3} + \frac{1}{16} \left(\frac{1}{r_0^2} + \frac{r}{(r_0^2 + \rho_0^2)^{3/2}} \right) = 0 \text{ and} \quad (9.89)$$

$$\left. \frac{\partial f}{\partial \rho} \right|_{(r, \rho) = (r_0, \rho_0)} = -\frac{c^2}{\rho_0^3} + \frac{1}{16} \left(\frac{1}{\rho_0^2} + \frac{\rho}{(r_0^2 + \rho_0^2)^{3/2}} \right) = 0. \quad (9.90)$$

(9.89) $\times r_0^3$ gives

$$-c^2 + \frac{1}{16} \left(r_0 + \frac{r_0^4}{(r_0^2 + \rho_0^2)^{3/2}} \right) = 0 \quad (9.91)$$

(9.90) $\times \rho_0^3$ gives

$$-c^2 + \frac{1}{16} \left(\rho_0 + \frac{\rho_0^4}{(r_0^2 + \rho_0^2)^{3/2}} \right) = 0$$

Subtracting gives

$$r_0 - \rho_0 + \frac{r_0^4 - \rho_0^4}{(r_0^2 + \rho_0^2)^{3/2}} = 0$$

Since $r_0 > 0$ and $\rho_0 > 0$ we must have

$$r_0 = \rho_0.$$

From (9.91)

$$c^2 = \frac{r_0}{16} \left(1 + \frac{1}{2\sqrt{2}} \right) = \frac{r_0}{64} (4 + \sqrt{2})$$

As in the other problems we choose a value of angular momentum that makes

$$r_0 = \rho_0 = 1.$$

This gives

$$c^2 = \frac{4 + \sqrt{2}}{64}$$

and from (9.88)

$$h_0 = \frac{4 + \sqrt{2}}{64} - \frac{1}{16} \left(2 + \frac{1}{\sqrt{2}} \right) = -\frac{4 + \sqrt{2}}{64}.$$

Minimum Energy for Digonal Escape

Here we let on of r or ρ tend to ∞ but not the other. We shall let $\rho \rightarrow \infty$ for which

$$U(r, \rho) \rightarrow \frac{1}{16r}.$$

To make this happen we must have $h \geq c^2/2r^2 - 1/16r$ or

$$hr^2 + \frac{r}{16} - \frac{c^2}{2} \geq 0.$$

The minimum value of h giving such solutions occurs for the discriminant, $1/256 + 2hc^2 = 0$ for which

$$h = h_\epsilon = -\frac{1}{512c^2} = -\frac{1}{8(4 + \sqrt{2})} = -\frac{4 - \sqrt{2}}{112}$$

for which

$$r = r_\epsilon = -\frac{1}{32h_\epsilon} = 16c^2 = \frac{4 + \sqrt{2}}{4}.$$

Of course, $\rho_\epsilon = r_\epsilon$.

Example Figure

Figure 9-4 shows the curves, $h = -1/16$, $-1/24$, and $-1/32$. The abscissa and ordinate are r and ρ respectively. The horizontal and vertical lines are $\rho = \rho_\epsilon$ and $r = r_\epsilon$ respectively. These curves differ from the first two cases in the following respect: The antiprism and the Sitnikov motion curves become (2 dimensional) surfaces of revolution in $\mathbf{R}^3$ when rotated about the z axis. For the rotating rectangle in $\mathbf{R}^4$ studied here, the boundaries (zero velocity curves) become, on rotation about both the (ρ, ϕ) and (r, θ) planes, 3-dimensional hypersurfaces, bounding a 4-dimensional region to which the bodies are confined in their motion. When $h < h_\epsilon$ as it is for the three curves shown, the hypersurface is a compact manifold and of course the bodies are confined to move within a compact set in $\mathbf{R}^4$.

Figure 9-4: Rotating Rectangle in $\mathbf{R}^4$

Chapter 10

Nested Polytopes

In 1767, Euler [E1] showed that in the collinear 3-body problem for Newtonian mechanics, for each assignment of positive masses and for each way of arranging the bodies on a line there is exactly one *central configuration* (c.c.). In 1910, Moulton [Mn1], [Mn2] generalized this to N collinear bodies showing that the total number of c.c.'s in this case up to homographic equivalence is $N!/2$. In dimensions greater than one the collinear problems form a very special type of N -body problem. In this chapter, we also study c.c.'s for some very special types of N -body problems and obtain some analogous results. In these problems the bodies lie at the vertices of polytopes of a certain class nested inside one another. All polytopes in the nest for a given problem are homothetic to each other and share a common centroid. All bodies on the same polytope have the same mass. Under the group of the ivp the vertices of each polytope form one orbit and for any solution each body is constrained to move radially; i.e., it is confined to move along the line through its initial point and the common centroid. Our goal in this chapter is to find problems of this type such that for each assignment of masses and for each way of arranging the polytopes in the nest (i.e., ordering the polytopes by size) there is exactly one way of scaling the sizes of the polytopes to give a c.c..

10.1 Definition of the Problem

We begin by making a

Definition 29 *If q is a solution of the N -body problem in barycentric coordinates satisfying*

$$\ddot{q}_i(t_0) = -\lambda q_i(t_0)$$

*$\forall i \leq N$ for some real positive λ , we say that the configuration of the bodies at time t_0 is a **central configuration**.*

We assume that the s function as defined in (3.1) p. 24 is homogeneous of degree $-\alpha$: i.e., let

$$s(x) = \frac{1}{x^\alpha} \quad (10.1)$$

where $x \in \mathbf{R}_+$. We now reiterate a well-known result: namely,

Theorem 62 *If q is a central configuration for a homogeneous s function with the initial condition $\dot{q}_i(t_0) = -\omega q_i(t_0)$, $\forall i$ then q is a homothetic solution.*

Proof. Consider the scalar o.d.e.

$$x'' = -\frac{\lambda}{x^{\alpha-1}}$$

with initial conditions, $x(t_0) = 1$ and $\dot{x}(t_0) = -\omega$. Then there is a unique solution for x . We claim the solution for q is

$$q_i(t) = x(t) q_i(t_0)$$

which is obviously homothetic. Clearly, q satisfies the initial conditions. To verify that this function, q is the solution, note that for q , the r.h.s. of (3.1) p. 24 is

$$\sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_j}{\|q_{ij}(t)\|^\alpha} q_{ij}(t) = \frac{1}{x^{\alpha-1}} \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_j}{\|q_{ij}(t_0)\|^\alpha} q_{ij}(t_0) = \frac{1}{x^{\alpha-1}} \ddot{q}_i(t_0) = -\frac{\lambda}{x^{\alpha-1}} q_i(t_0)$$

whereas the l.h.s. of the same equation is

$$\ddot{q}_i(t) = -\frac{\lambda}{x^{\alpha-1}} q_i(t_0)$$

also. Thus this is the unique solution for q . ■

Thus the c.c.'s that we find in this chapter all yield homographic solutions. Without loss of generality we shall assume that the origin is the common centroid of the nested polytopes. Thus we consider a finite orthogonal group, $\mathcal{G} \leq \mathcal{O}(d)$ and a unit length vector, $v \in \mathbf{R}^d$ such that $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v = \langle v \rangle$, the one dimensional space spanned by v . If we place n seed bodies along the ray, $\{kv \mid k > 0\}$ then the multiple body $\mathcal{G}$ -orbit ivp grown from these seeds will have a solution for which the seed bodies all stay on that ray. If the seed body of mass $m_i > 0$ is at $r_i v$ where $0 < r_n < \dots < r_1$ then the orbit, $\mathcal{G}(r_i v)$ forms the set of vertices of a polytope. Let $p = |\mathcal{G}v|$. Then each polytope has p vertices. The points in the orbit, $\mathcal{G}v$ all lie on the unit $(d-1)$ sphere, S^{d-1} . The vertices on polytope i lie on $r_i S^{d-1}$, the unit sphere scaled by the factor r_i . The n polytopes are homothetic to each other and are nested inside one another. (Of course in $\mathbf{R}^3$ the polytope is a polyhedron.) Our goal in this chapter is to find some examples for which a group, $\mathcal{G}$ operates on $\mathbf{R}^d$ (for some d) and some vector $v \in \mathbf{R}^d$ such that for each way of arranging the bodies along the ray, there is exactly one such $\mathcal{G}$ -orbit ivp giving rise to a c.c.

Recall the stabilizer of v in $\mathcal{G}$ denoted as $\mathcal{G}_v$ or $\text{stab}_{\mathcal{G}} v$. $\forall L \in \mathcal{G}_v$, $Lv = v$ and we have $|\mathcal{G}_v|$ members of $\mathcal{G}$ which map v to the same point. Recall also that

$$|\mathcal{G}| = |\mathcal{G}v| |\text{stab}_{\mathcal{G}} v| \tag{10.2}$$

Thus we have an $N = np$ -body problem with n seed bodies for the group, $\mathcal{G}$. Each seed body generates one of the n polytopes. All bodies on polytope i have the same mass, m_i and the polytopes are homotheties of each other. For each assignment of mass, m_i to the bodies of polytope i we wish to determine whether there is exactly one way of scaling the sizes of the polytopes to obtain a c.c. subject to the condition

that polytope, i lies outside polytope $i + 1$, for each i ranging from 1 to $n - 1$. For those cases for which this happy result ensues, we shall have exactly $n!$ c.c.'s if we include all ways of arranging the polytopes in the nest.

10.2 A Specific Type of Nesting

As mentioned above there is a special class of such nestings which we intend to study and these are given by the following

Definition 30 *Suppose that we are given some finite non-trivial group, $\mathcal{G} \leq \mathcal{E}(d)$ and some vector, $v \in \mathbf{R}^d$ such that*

1. $\dim \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v = 1$
2. $\mathcal{G} = \mathcal{E}_{\{\mathcal{G}v\}}$ and
3. $\text{aff } \mathcal{G}v = \mathbf{R}^d$.

*Then the vector, v is said to be **ray-nestable** for $\mathcal{G}$.*

Part 1 in this definition says that $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v$ is a one dimensional affine subspace passing through v ; i.e., this line is a maximally stabilized subset of $\mathbf{R}^d$ for the action of the group, $\mathcal{G}$ (see D. 35 p. 206 App. A). For any $\mathcal{G} \leq \mathcal{E}(d)$ and $v \in \mathbf{R}^d$ we always have $\mathcal{G} \leq \mathcal{E}_{\{\mathcal{G}v\}}$ by L. 105 p. 221 App. B (i.e., a group is always a subgroup of the setwise stabilizer of any of its orbits). In the case where v is ray-nestable, part 3 of the definition says that the orbit $\mathcal{G}v$ forms the vertices of a polytope of dimension d and by part 2 that polytope has a complete symmetry group that is no larger than $\mathcal{G}$.

In what follows we shall assume that the centroid of the orbit $\mathcal{G}v$ lies at the origin so that $\mathcal{G} \leq \mathcal{O}(d)$. In this case $\text{stab}_{\mathcal{G}} v$ fixes the origin: so, $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v$ is

a linear subspace of dimension 1 by Part 1: so, $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v = \langle v \rangle$. We shall also assume that

$$\|v\| = 1. \quad (10.3)$$

Consider the $|\mathcal{G}v|$ -body problem (i.e., only one polytope) where each body has unit mass and the initial velocity of the body at Lv for some $L \in \mathcal{G}$ is b_0Lv where $b_0 \in \mathbf{R}$. Then the problem is in centroidal coordinates, $\mathcal{G}$ is the isometry group of the ivp and the acceleration of the body at Lv is $-\lambda Lv$ where λ is common to all bodies. Thus this initial value problem is a central configuration. In $\mathbf{R}^3$, this fact is not new and is mentioned in Cedo and Llibre [CeLl] for the 5 Platonic solids, the cuboctahedron and the icosidodecahedron. For each of these polyhedra the vertices are the orbits of one such vertex under the action of their complete symmetry group and satisfy (1) through (3) of the above definition. Note that the remaining eleven Archimedean solids do *not* satisfy requirement (1) above and are excluded from discussion herein. This is not to say that one cannot study the nesting of such objects but that our current approach does not apply to such objects. Moving from the configurations studied here to such considerations is analogous to moving from the collinear problems to the problem of c.c.'s in the plane and $\mathbf{R}^3$.

10.3 Coxeter Groups

We now give some important examples of ray-nestable vectors. In this paragraph we review some definitions from the subject of finite reflection groups (see e.g. [GB, Ch. 4]) and the reader familiar with them may want to skip to the theorem that follows. In $\mathbf{R}^d$ a finite subgroup $\mathcal{G} < \mathcal{O}(d)$ generated by reflections in hyperplanes is called a finite reflection group. If in addition, $\text{fix}_{\mathbf{R}^d} \mathcal{G} = 0$, it is called a Coxeter group. For each hyperplane, P of reflection for $\mathcal{G}$ we choose some non-zero vector r in $P^\perp$, and label $\pm r$ as roots. The (finite) set, Δ of all such roots is called the root system. Fix t , a non-zero vector whose orthogonal complement does not contain any

of the roots. The set of roots whose inner product with t is positive is called Δ^+ . A minimal subset of Δ^+ for which linear combinations with positive coefficients yield all members of Δ^+ is called a t -base. It turns out that there is exactly one t -base and it is a basis for the space. Let the t -base be $\Pi = \{r_1, r_2, \dots, r_d\}$ and define the dual basis, $\Pi^* = \{s_1, s_2, \dots, s_d\}$ to be the unique set of vectors satisfying, $(r_i, s_j) = \delta_{ij}$. If the t -base is not the disjoint union of two non-empty orthogonal subsets the Coxeter group is called irreducible.

Theorem 63 *If $\mathcal{G}$ is an irreducible Coxeter group in $\mathbf{R}^d$, $\Pi = \{r_1, r_2, \dots, r_d\}$ is a t -base for $\mathcal{G}$ and $\{s_1, s_2, \dots, s_d\}$ is the dual base for Π then each s_i ($i \leq d$) is ray-nestable for some group, $\mathcal{G}'$ containing $\mathcal{G}$.*

Proof. Each s_i lies at the intersection of the $d - 1$ hyperplanes.

$$r_1^\perp, r_2^\perp, \dots, r_{i-1}^\perp, r_{i+1}^\perp, \dots, r_d^\perp$$

which is one dimensional. Since the reflection in each of these hyperplanes is a member of $\mathcal{G}$, $\text{stab}_{\mathcal{G}} s_i$ contains all of these reflections: so, $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} s_i = \langle s_i \rangle$, the one dimensional subspace of $\mathbf{R}^d$ spanned by s_i .

Since $\mathcal{G}$ is Coxeter, $\text{fix } \mathcal{G} = 0$: so, $0 = \overline{s_i}_{\mathcal{G}} \in \text{aff } \mathcal{G} s_i$ and hence $\text{aff } \mathcal{G} s_i \leq \mathbf{R}^d$. If $\text{aff } \mathcal{G} s_i \subsetneq \mathbf{R}^d$ then some $r_j \notin \text{aff } \mathcal{G} s_i \cup (\text{aff } \mathcal{G} s_i)^\perp$ otherwise $\mathcal{G}$ would not be irreducible. Since reflection along r_j would take s_i out of $\text{aff } \mathcal{G} s_i$ contrary to the assumption and we have that $\text{aff } \mathcal{G} s_i = \mathbf{R}^d$.

Let $\mathcal{G}' = \mathcal{E}_{\{\mathcal{G} s_i\}}$. Every group lies in the setwise stabilizer of any of its orbits: so, $\mathcal{G} \subseteq \mathcal{G}'$. Then $\text{stab}_{\mathcal{G}'} s_i \supseteq \text{stab}_{\mathcal{G}} s_i$, hence $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}'} s_i \subseteq \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} s_i = \langle s_i \rangle$. Since $s_i \in \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}'} s_i$ we have $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}'} s_i = \langle s_i \rangle$: i.e., dimension 1. $\text{aff } \mathcal{G}' s_i \supseteq \text{aff } \mathcal{G} s_i = \mathbf{R}^d$ and we have the result. ■

In many cases $\mathcal{G}' = \mathcal{G}$: however, note that the midpoint of any edge of the regular tetrahedron serves as such an s_i for the group $\mathcal{W}|\mathcal{T}$ but its orbit under this group is the regular octahedron whose complete symmetry group is the strictly larger $\mathcal{W}^*$.

Thus to find nested polytopes, suitable for analysis here, we can use the irreducible Coxeter graphs [GB, Thm. 5.1.7 and Fig. 5.3]. Each node is associated with a root whose dual give us the ray-nestable vector. Its orbit under the action of the Coxeter group is the polytope.

10.4 The Problem of Nested Polytopes

Returning to the problem, we let r_i be the radius of the $(d - 1)$ sphere containing the vertices of polytope, i . We shall assume that

$$0 < r_n < r_{n-1} < \cdots < r_1. \quad (10.4)$$

Let

$$p = |\mathcal{G}v| \quad (10.5)$$

the number of vertices on the polytope. Since n is the number of polytopes, the total number of bodies is

$$N = pn. \quad (10.6)$$

$\forall L \in \mathcal{G}$, there is a body of mass m_i initially at r_iLv . The initial velocities are all assumed (for simplicity) to be zero. (They could in general be proportional to the initial position). Recall that our potential function,

$$u(x) = - \int_{x_0}^x \xi s(\xi) d\xi \quad (10.7)$$

which with (10.1) gives

$$\frac{du(x)}{dx} = - \frac{1}{x^{\alpha-1}}, \quad (10.8)$$

$$\frac{d^2u(x)}{dx^2} = \frac{\alpha - 1}{x^\alpha}. \quad (10.9)$$

Using the fact that u is only defined up to a constant of integration, from (10.1) and (10.7) we can write

$$u(x) = \begin{cases} x^{2-\alpha}/(\alpha - 2) & \text{if } \alpha \neq 2 \\ -\ln x & \text{if } \alpha = 2. \end{cases} \quad (10.10)$$

As discussed above for the case of the single polytope, since $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v = \langle v \rangle$, the bodies initially at $r_1 v, \dots, r_n v$ remain on the ray ρv ($\rho > 0$) for all time up to a collision and if the body initially at $r_i v$ has solution $r_i(t) v$, ($r_i(t) > 0$) then the body initially at $r_i L v$ has solution $r_i(t) L v$. Thus we have a c.c. for the nest of polytopes if and only if

$$\ddot{r}_i(0) = -\lambda r_i(0), \quad \forall i \leq n. \quad (10.11)$$

i.e., we need only test for one body on each polytope. This will greatly simplify the Lagrangian function describing the constrained extrema that give the c.c.'s.

10.5 The Potential Function

We start by defining a potential, V which is a function of the scalars, $r_1, \dots, r_n$. We then show that $(\partial V / \partial r_i) v$ is the force applied to body i by all other bodies. Setting $r = (r_1, \dots, r_n)^t \in \mathbf{R}_+^n$, we let

$$\begin{aligned} V(r) &= \frac{1}{|\mathcal{G}_v|} \sum_{1 \leq l < l' \leq n} m_l m_{l'} \sum_{L \in \mathcal{G}} u(\|(r_l I_d - r_{l'} L) v\|) \\ &\quad + \frac{1}{2|\mathcal{G}_v|} \sum_{l=1}^n m_l^2 \sum_{L \in \mathcal{G}^*} u(r_l \|(I_d - L) v\|) \end{aligned} \quad (10.12)$$

where

$$\mathcal{G}^* = \mathcal{G} \setminus \mathcal{G}_v. \quad (10.13)$$

We shall now show that the force applied to the body at $r_i v$ by all other bodies is $(\partial V / \partial r_i) v$. First if $L \in \mathcal{G}$ since $\|v\| = 1$ by (10.3) p. 175 we define

$$c_L = \cos \theta_L = (v, Lv). \quad (10.14)$$

Then since L is orthogonal, $\|Lv\| = 1$ and

$$\begin{aligned} \|(r_i I_d - r_j L) v\|^2 &= r_i^2 - 2r_i r_j \cos \theta_L + r_j^2 \\ &= \|(r_i L - r_j I_d) v\|^2. \end{aligned} \quad (10.15)$$

From (10.15) we see that we can switch the operator L associated with r_j over to r_i without changing the norm. Using this, from (10.12) we have

$$\begin{aligned} \frac{\partial V(r)}{\partial r_i} &= \frac{1}{|\mathcal{G}_v|} \sum_{\substack{j=1 \\ j \neq i}}^n m_i m_j \sum_{L \in \mathcal{G}} \frac{\partial}{\partial r_i} u(\|(r_i I_d - r_j L) v\|) \\ &\quad + \frac{m_i^2}{|\mathcal{G}_v|} \sum_{L \in \mathcal{G}} \frac{1}{2} \frac{\partial}{\partial r_i} u(r_i \|(I_d - L) v\|). \end{aligned} \quad (10.16)$$

Note that in the second line of (10.16) we can rewrite as follows:

$$\begin{aligned} \frac{\partial}{\partial r_i} u(r_i \|(I_d - L) v\|) &= \frac{\partial}{\partial r_i} u(\|(r_i I_d - r_i L) v\|) \\ &= \frac{\partial}{\partial r_i} \{u(\|(r_i I_d - x L) v\|) + u(\|(x I_d - r_i L) v\|)\} \Big|_{x=r_i} \\ &= 2 \frac{\partial}{\partial r_i} u(\|(r_i I_d - x L) v\|) \Big|_{x=r_i} \end{aligned} \quad (10.17)$$

where in the last step we used (10.15).

$$\begin{aligned} \therefore \frac{\partial V(r)}{\partial r_i} &= \frac{1}{|\mathcal{G}_v|} \sum_{\substack{j=1 \\ j \neq i}}^n m_i m_j \sum_{L \in \mathcal{G}} \frac{\partial}{\partial r_i} u(\|(r_i I_d - r_j L) v\|) \\ &\quad + \frac{m_i^2}{|\mathcal{G}_v|} \sum_{L \in \mathcal{G}} \frac{\partial}{\partial r_i} u(\|(r_i I_d - x L) v\|) \Big|_{x=r_i}. \end{aligned} \quad (10.18)$$

We now show that $(\partial V / \partial r_i) v$ is the force applied to the body at the point $r_i v$ by all other bodies. First, the direction must lie in the linear space $\langle v \rangle$ by Part 1 of Def. 30 p. 174. Thus we need only show that the component in the v direction is $\partial V / \partial r_i$. Each term in the first sum is the component of the force applied by the body at $r_i L v$ for each $j \neq i$ and as L ranges over $\mathcal{G}$. Since each body is picked up in the sum $|\mathcal{G}_v|$ times, the factor $1/|\mathcal{G}_v|$ accounts for this. Then the first sum gives the v component of force applied by all bodies on all polytopes other than polytope i . The second sum accounts for the force applied by all bodies on polytope i other than the one at $r_i v$. In the determination of the force we must hold all bodies at

$r_i L v$ fixed while we do the partial derivative with respect to r_i . Hence we use the dummy variable x . Inspection again shows that the sum is the desired component of force. Hence (10.18) does give the correct force. Now we simplify further as follows:

$$\begin{aligned} \frac{\partial}{\partial r_i} u(\|(r_i I_d - xL) v\|) &= \frac{u'(\|(r_i I_d - xL) v\|)}{2 \|(r_i I_d - xL) v\|} \frac{\partial}{\partial r_i} \|(r_i I_d - xL) v\|^2 \\ &= s(\|(r_i I_d - xL) v\|) (v, (r_i I_d - xL) v) \\ &= s(\|(r_i I_d - xL) v\|) (r_i - c_L x) \end{aligned}$$

Applying this gives

$$\begin{aligned} \frac{\partial V(r)}{\partial r_i} &= \frac{1}{|\mathcal{G}_v|} \sum_{\substack{j=1 \\ j \neq i}}^n m_i m_j \sum_{L \in \mathcal{G}} s(\|(r_i I_d - r_j L) v\|) (r_i - c_L r_j) \\ &\quad + \frac{m_i^2}{|\mathcal{G}_v|} \sum_{L \in \mathcal{G}} s(\|(r_i I_d - r_i L) v\|) (r_i - c_L r_i) \end{aligned} \quad (10.19)$$

10.6 Central Configurations

We have a c.c. if and only if $\forall i \leq n$,

$$\frac{\partial V(r)}{\partial r_i} = -\lambda m_i r_i \quad (10.20)$$

for some $\lambda \in \mathbf{R}_+$ and the c.c.'s will all occur (up to homothety) at the local extrema of the function

$$F(r, \lambda) = V(r) + \frac{\lambda}{2} \left(\sum_{i=1}^n m_i r_i^2 - 1 \right). \quad (10.21)$$

As in the Euler-Moulton problem, for each arrangement of the polytopes in the nest, i.e.: $\forall r$ satisfying (10.4) we can let $r_{n+1} = 0$ and define

$$\begin{aligned} D &= \{r \in \mathbf{R}_+^n \mid 0 < r_n < r_{n-1} < \cdots < r_1\}, \\ \Delta_i^+ &= \{r \in \mathbf{R}_+^n \mid r_i = r_{i+1}\} \quad \forall i = 1, \dots, n \text{ and} \\ \Delta^+ &= \bigcup_{1 \leq i \leq n} \Delta_i^+ \end{aligned} \quad (10.22)$$

From (10.10) p. 177, if $\alpha \geq 2$, $u(x) \rightarrow \infty$ as $x \rightarrow 0+$. Since the polytopes have more than 1 body $\exists$ some $L \in \mathcal{G}^* = \mathcal{G} \setminus \mathcal{G}_v$. As $r \rightarrow \Delta_n^+$, $r_n \rightarrow r_{n+1}+$; i.e., $r_n \rightarrow 0+$, $(r_n I_d - r_n L) v \rightarrow 0$; so, $u(\|(r_n I_d - r_n L) v\|) \rightarrow \infty$ and $V(r) \rightarrow \infty$ due to terms in the second line in (10.12) p. 178. Hence Δ_n^+ is part of the diagonal. For $1 \leq i \leq n-1$, as $r \rightarrow \Delta_i^+$, $r_i - r_{i+1} \rightarrow 0+$ and $u(\|(r_i I_d - r_{i+1} L) v\|) \rightarrow \infty$ whenever $L \in \mathcal{G}_v$; i.e., whenever $Lv = v$. In all cases,

$$\lim_{r \rightarrow \Delta^-} V(r) = \infty. \quad (10.23)$$

In $\mathbf{R}^n$ the $(n-1)$ dimensional ellipsoid

$$\sum m_i r_i^2 = 1$$

meets the open pencil D , in a manifold without boundary that is connected and bounded by Δ^+ . From (10.23) there exists a local minimum for V on this manifold; i.e., $\exists$ a c.c. for each arrangement of the polytopes.

10.7 Hessians

We now attempt to address the general question:

For some $\mathcal{G} \leq \mathcal{O}(d)$ and some $v \in \mathbf{R}^d$ defined above, for which values of $\alpha \geq 2, n \geq 1$ is the Hessian of $V(r)$ positive definite on all of D ?

Such cases will give rise, not only to existence but uniqueness of c.c.'s for each arrangement of the polytopes. We first define the function $q_c : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ by

$$q_c(x, y) \equiv \sqrt{x^2 - 2cxy + y^2} \quad (10.24)$$

where the parameter, $c \in [-1, 1]$. Then using (10.15)

$$\begin{aligned} q_{c_L}(r_l, r_{l'}) &= [r_l^2 - 2r_l r_{l'} \cos \theta_L + r_{l'}^2]^{1/2} \\ &= \|(r_l I_d - r_{l'} L) v\|. \end{aligned} \quad (10.25)$$

Consider the zeroes of the function $q_c(x, y)$ defined on $\mathbf{R}_+^2$. $\forall x, y > 0, c \in [-1, 1]$

$$q_c(x, y) = \sqrt{(x - cy)^2 + (1 - c^2)y^2}.$$

So, clearly $q_c = 0$ only if $c = \pm 1$. Also since $x, y > 0$, in the case where $c = -1$, $x - cy > 0$ and $q_c > 0$; so, it is clear that

$$q_c(x, y) = 0 \Leftrightarrow (c = 1) \text{ and } (x = y).$$

Since u is defined on $\mathbf{R}^+$ we can define $u \circ q_c(x, y)$ whenever $q_c \neq 0$; i.e., when $x \neq y$ or $c \neq 1$. If $L \in \mathcal{G}^*$; i.e., $L \notin \mathcal{G}_v$ then $Lv \neq v$ and $c_L \equiv (v, Lv) < 1$ and $q_{c_L}(x, x) > 0$. Thus we can define for $x \neq y$

$$f(x, y) \equiv \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (u \circ q_{c_L})(x, y) \quad (10.26)$$

and

$$\tilde{f}(x) \equiv \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}^*} (u \circ q_{c_L})(x, x). \quad (10.27)$$

Then from (10.12), (10.25) and (10.26), we have

$$V(r) = D(r) + |\mathcal{G}_v| \sum_{1 \leq l < l' \leq n} m_l m_{l'} f(r_l, r_{l'}) \quad (10.28)$$

where

$$D(r) = \frac{|\mathcal{G}_v|}{2} \sum_{l=1}^n m_l^2 \tilde{f}(r_l), \quad (10.29)$$

Note from (10.24) for $x > 0$,

$$q_{c_L}(x, x) = x\sqrt{2(1 - c_L)} > 0. \quad (10.30)$$

Substituting (10.30) into (10.27) and then into (10.29) gives with (10.2) p. 173,

$$D(r) = \frac{1}{2|\mathcal{G}_v|} \sum_{l=1}^n m_l^2 \sum_{L \in \mathcal{G}^*} u\left(r_l \sqrt{2(1 - c_L)}\right) \quad (10.31)$$

Since each term in $D(r)$ depends only on some single r_l , the off-diagonal entries in the Hessian of $D(r)$ are all zero. To obtain the diagonal entries in $D''(r)$ first note that

$$\begin{aligned}\frac{\partial^2}{\partial r_i^2} u\left(r_i \sqrt{2(1-c_L)}\right) &= u''\left(r_i \sqrt{2(1-c_L)}\right) 2(1-c_L) \\ &= \frac{(\alpha-1)}{r_i^\alpha [2(1-c_L)]^{\alpha/2-1}}.\end{aligned}$$

where the second line comes from (10.9) p. 177. So, from (10.31)

$$\frac{\partial^2}{\partial r_i^2} D(r) = \frac{m_i^2(\alpha-1)}{|\mathcal{G}_v| r_i^\alpha 2^{\alpha/2}} \sum_{L \in \mathcal{G}^*} \frac{1}{(1-c_L)^{\alpha/2-1}}.$$

As mentioned earlier, the orbit $\mathcal{G}v$ is not just the singleton $\{v\}$; so $\exists$ some $L \in \mathcal{G}^* = \mathcal{G} \setminus \mathcal{G}_v$ and for that L , $Lv \neq v$ because $L \notin \mathcal{G}_v$. Thus $c_L = (v, Lv) < 1$ which gives $\partial^2 D(r) / \partial r_i^2 > 0$, $\forall i \leq n$. Thus the Hessian of $D(r)$ is positive definite. Note also that in (10.28), $V(r)$ equals $D(r)$ plus a linear combination with positive coefficients, of functions, $f(r_l, r_{l'})$ each of which depends only on some r_l and $r_{l'}$ ($l \neq l'$). Let us denote $f''(x, y)$ as the 2×2 matrix,

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Let $\Lambda''(r)$ be the $n \times n$ Hessian matrix obtained by regarding $f(r_l, r_{l'})$ as a function of $r = (r_1, \dots, r_n)$. Then the i, j th entry in the matrix, $\Lambda''(r)$ is

$$\Lambda_{ij}''(r) = \frac{\partial^2}{\partial r_i \partial r_j} f(r_l, r_{l'}).$$

This means that for $l < l'$

$$\begin{aligned}\Lambda_{ll}''(r) &= f_{xx}(r_l, r_{l'}) \\ \Lambda_{ll'}''(r) &= \Lambda_{l'l}''(r) = f_{xy}(r_l, r_{l'}) \text{ and} \\ \Lambda_{l'l'}''(r) &= f_{yy}(r_l, r_{l'}).\end{aligned}$$

All other entries in the matrix are zero. We can partition this sparse matrix as follows:

$$\Lambda^{ll'}(r) = \begin{array}{cc} & \begin{array}{cc} \text{column } l & \text{column } l' \end{array} \\ \begin{array}{c} \text{row } l \\ \text{row } l' \end{array} & \left[\begin{array}{ccccc} \ddots & \vdots & & \vdots & \\ \cdots & f_{xx}(r_l, r_{l'}) & \cdots & f_{xy}(r_l, r_{l'}) & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & f_{xy}(r_l, r_{l'}) & \cdots & f_{yy}(r_l, r_{l'}) & \cdots \\ & \vdots & & \vdots & \ddots \end{array} \right] \end{array}$$

where all entries other than those indicated are zero. Then $\forall w = (w_1, \dots, w_n)^t \in \mathbf{R}^n$,

$$(w, \Lambda^{ll'}(r) w) = w^t \Lambda^{ll'}(r) w = (w_l, w_{l'}) f''(r_l, r_{l'}) \begin{pmatrix} w_l \\ w_{l'} \end{pmatrix}.$$

Thus, $\Lambda^{ll'}(r)$ is positive semi-definite $\forall r \in \mathbf{R}_+^n \setminus \Delta^+$ iff $f''(x, y)$ is positive semi-definite $\forall x > y > 0$. From (10.28) p. 182 we can write the Hessian of $V(r)$ as

$$V''(r) = D''(r) + |\mathcal{G}r| \sum_{1 \leq l < l' \leq n} m_l m_{l'} \Lambda^{ll'}(r).$$

This is the sum of the positive definite Hessian, $D''(r)$ and a linear combination with positive coefficients of the matrices $\Lambda^{ll'}(r)$. Each $\Lambda^{ll'}(r)$ is positive semi-definite iff $f''(r_l, r_{l'})$ is positive semi-definite. Thus $\forall w = (w_1, \dots, w_n) \neq 0$, if $f''(x, y)$ is positive semi-definite $\forall 0 < y < x$ then $\forall r (w, \Lambda^{ll'} w) \geq 0$, $(w, Dw) > 0$ and in this case

$$(w, V'' w) = (w, D'' w) + |\mathcal{G}r| \sum_{1 \leq l < l' \leq n} m_l m_{l'} (w, \Lambda^{ll'} w) > 0.$$

Thus we have shown

Lemma 64 *If $f''(x, y)$ is positive semi-definite $\forall 0 < y < x$ then $V''(r)$ is positive definite $\forall 0 < r_n < \dots < r_1$ and in that case there is only one c.c. for each way of arranging the polytope nest.*

Note from (10.26) p. 182 that f and hence the matrices $\Lambda^{u'}$ do not depend on the masses. They depend only on

1. the parameter α in the homogeneous s function (10.1) p. 172 and
2. the geometry of the polytope which determines both the symmetry group $\mathcal{G}$ and the set of cosines $c_L = (v, Lv)$, as L ranges over $\mathcal{G}$.

Thus when this technique gives us uniqueness of c.c.'s it does so for all assignments of positive masses. We now look at the function, $f(x, y)$ for $0 < y < x$ to see when it is positive semi-definite.

$$\frac{\partial}{\partial x} q_c(x, y) = \frac{x - cy}{q_c(x, y)} \quad (10.32)$$

and

$$\frac{\partial}{\partial y} q_c(x, y) = \frac{y - cx}{q_c(x, y)}. \quad (10.33)$$

From (10.8) p. 177 and (10.32),

$$\frac{\partial}{\partial x} u \circ q_c(x, y) = -\frac{x - cy}{q_c(x, y)^\alpha} \quad (10.34)$$

$$\therefore \frac{\partial^2}{\partial x^2} u \circ q_c(x, y) = \frac{\alpha(x - cy)^2}{q_c(x, y)^{\alpha+2}} - \frac{1}{q_c(x, y)^\alpha}.$$

We can rewrite this as

$$\frac{\partial^2}{\partial x^2} u \circ q_c(x, y) = -\frac{\alpha(1 - c^2)y^2}{q_c(x, y)^{\alpha+2}} + \frac{\alpha - 1}{q_c(x, y)^\alpha} \quad (10.35)$$

Similarly,

$$\frac{\partial^2}{\partial y^2} u \circ q_c(x, y) = -\frac{\alpha(1 - c^2)x^2}{q_c(x, y)^{\alpha+2}} + \frac{\alpha - 1}{q_c(x, y)^\alpha} \quad (10.36)$$

From (10.34) and (10.33) we have

$$\frac{\partial^2}{\partial x \partial y} u \circ q_c(x, y) = \frac{\alpha(x - cy)(y - cx)}{q_c(x, y)^{\alpha+2}} + \frac{c}{q_c(x, y)^\alpha}$$

Again we can rearrange this as

$$\frac{\partial^2}{\partial x \partial y} u \circ q_c(x, y) = \frac{\alpha(1-c^2)xy}{q_c(x, y)^{\alpha+2}} - \frac{(\alpha-1)c}{q_c(x, y)^\alpha} \quad (10.37)$$

Given $h : [-1, 1] \rightarrow \mathbf{R}$, we define the expectation operator

$$E(h) = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} h((v, Lv)) = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} h(c_L) \quad (10.38)$$

For simplicity, we shall write h as a function of the parameter c . For example, $E(1-c^2)$ means $E(h)$ where $h(c) = 1-c^2$. Then from (10.26) p. 182, (10.35), (10.36) and (10.37) we have

$$\frac{\partial^2}{\partial x^2} f(x, y) = -\alpha y^2 E\left(\frac{1-c^2}{q_c^{\alpha+2}}\right) + (\alpha-1) E\left(\frac{1}{q_c^\alpha}\right) \quad (10.39)$$

$$\frac{\partial^2}{\partial y^2} f(x, y) = -\alpha x^2 E\left(\frac{1-c^2}{q_c^{\alpha+2}}\right) + (\alpha-1) E\left(\frac{1}{q_c^\alpha}\right) \quad (10.40)$$

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \alpha xy E\left(\frac{1-c^2}{q_c^{\alpha+2}}\right) - (\alpha-1) E\left(\frac{c}{q_c^\alpha}\right) \quad (10.41)$$

where for brevity we have written q_c in lieu of $q_c(x, y)$.

$$\begin{aligned} \therefore \det(f''(x, y)) &= -\alpha(\alpha-1) E\left(\frac{1-c^2}{q_c^{\alpha+2}}\right) E\left(\frac{x^2 + y^2 - 2cxy}{q_c^\alpha}\right) \\ &\quad + (\alpha-1)^2 \left\{ E\left(\frac{1}{q_c^\alpha}\right)^2 - E\left(\frac{c}{q_c^\alpha}\right)^2 \right\} \end{aligned}$$

where we have used the fact that the products of the first terms cancel and we have used the linearity of the expectation operator in the cross terms.

$$\begin{aligned} \therefore \det(f''(x, y)) &= -\alpha(\alpha-1) E\left(\frac{1-c^2}{q_c^{\alpha+2}}\right) E\left(\frac{1}{q_c^{\alpha-2}}\right) \\ &\quad + (\alpha-1)^2 E\left(\frac{1+c}{q_c^\alpha}\right) E\left(\frac{1-c}{q_c^\alpha}\right) \end{aligned} \quad (10.42)$$

Thus $\forall x \neq y$, if $[f_{xx} \geq 0$ or $f_{yy} \geq 0]$ and $\det f'' \geq 0$ from (10.39), (10.40) and (10.42) respectively, then f'' is positive semi-definite and by Lemma 64 p. 184 above there

is exactly one c.c. for each arrangement of the polytopes in the nest. Conversely, if $f_{yy} < 0$ we know that this method will not show uniqueness. Let us substitute

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad (10.43)$$

then

$$\begin{aligned} q_c(x, y) &= \sqrt{x^2 - 2cxy + y^2} \\ &= r\sqrt{1 - c \sin 2\theta} \end{aligned} \quad (10.44)$$

We need only consider the region, $0 < y < x$; i.e., $0 < \theta < \pi/4$. We further substitute

$$\sigma = \sin 2\theta. \quad (10.45)$$

Then we need only consider the region,

$$0 < \sigma < 1. \quad (10.46)$$

Note from (10.44), $q_c(x, y)$ is linear in r ; so, from (10.42), $\det(f'')$ is homogeneous of degree -2α in r . Thus using these equations with (10.45), since

$$\frac{q_c(x, y)}{r} = \sqrt{1 - c\sigma}$$

we have that $r^{2\alpha} f''(x, y)$ can be expressed as a function of θ (or σ) only. Consequently, we can define

$$\begin{aligned} G(\sigma) &= r^{2\alpha} \det(f''(x, y)) \\ &= -\alpha(\alpha - 1) E\left(\frac{1 - c^2}{(1 - c\sigma)^{\frac{\alpha}{2} + 1}}\right) E\left(\frac{1}{(1 - c\sigma)^{\frac{\alpha}{2} - 1}}\right) \\ &\quad + (\alpha - 1)^2 E\left(\frac{1 + c}{(1 - c\sigma)^{\frac{\alpha}{2}}}\right) E\left(\frac{1 - c}{(1 - c\sigma)^{\frac{\alpha}{2}}}\right). \end{aligned} \quad (10.47)$$

We now determine $E(c^2)$ as a function of the dimension of the polytope. Note that

$$\frac{y^2}{r^2} = \sin^2 \theta = \frac{1 - \cos 2\theta}{2} = \frac{1 - \sqrt{1 - \sigma^2}}{2}.$$

From this and again using the homogeneity of f_{xx} in (10.39) we let

$$\begin{aligned} F_1(\sigma) &= r^\alpha f_{xx}(x, y) \\ &= -\frac{\alpha(1 - \sqrt{1 - \sigma^2})}{2} E\left(\frac{1 - c^2}{(1 - c\sigma)^{\frac{\alpha}{2} + 1}}\right) + (\alpha - 1) E\left(\frac{1}{(1 - c\sigma)^{\frac{\alpha}{2}}}\right) \end{aligned} \quad (10.48)$$

Similarly

$$\frac{x^2}{r^2} = \cos^2 \theta = \frac{1 + \cos 2\theta}{2} = \frac{1 + \sqrt{1 - \sigma^2}}{2}.$$

So we let

$$\begin{aligned} F_2(\sigma) &= r^\alpha f_{yy}(x, y) \\ &= -\frac{\alpha(1 + \sqrt{1 - \sigma^2})}{2} E\left(\frac{1 - c^2}{(1 - c\sigma)^{\frac{\alpha}{2} + 1}}\right) + (\alpha - 1) E\left(\frac{1}{(1 - c\sigma)^{\frac{\alpha}{2}}}\right) \end{aligned} \quad (10.49)$$

We now define

$$\Gamma_c(\sigma) = \frac{1}{\sqrt{1 - c\sigma}}. \quad (10.50)$$

For brevity we shall write Γ_c for $\Gamma_c(\sigma)$. Then

$$F_1(\sigma) = (\alpha - 1) E(\Gamma_c^\alpha) - \frac{\alpha}{2} (1 - \sqrt{1 - \sigma^2}) E((1 - c^2) \Gamma_c^{\alpha+2}) \quad (10.51)$$

$$F_2(\sigma) = (\alpha - 1) E(\Gamma_c^\alpha) - \frac{\alpha}{2} (1 + \sqrt{1 - \sigma^2}) E((1 - c^2) \Gamma_c^{\alpha+2}) \quad (10.52)$$

and

$$\begin{aligned} G(\sigma) &= (\alpha - 1)^2 E((1 + c) \Gamma_c^\alpha) E((1 - c) \Gamma_c^\alpha) \\ &\quad - \alpha(\alpha - 1) E((1 - c^2) \Gamma_c^{\alpha+2}) E(\Gamma_c^{\alpha-2}). \end{aligned} \quad (10.53)$$

Fixing $\sigma \in (0, 1)$ and letting the parameter c vary we have from (10.50)

$$\frac{\partial \Gamma_c(\sigma)}{\partial c} = \frac{1}{2} \sigma \Gamma_c^3(\sigma) > 0.$$

Therefore $\Gamma_c(\sigma)$ is a monotone increasing function of c . $\forall \sigma \in (0, 1)$. Note also that $\Gamma_0(\sigma) = 1$. $\forall \sigma \in (0, 1)$.

In summary if $G(\sigma) \geq 0$ and either $F_1(\sigma)$ or $F_2(\sigma) \geq 0$ for all $\sigma \in (0, 1)$ then we have the desired uniqueness. Since $F_1 > F_2$ we need only show that $F_1 \geq 0$ and $G \geq 0$. Conversely, if we wish to show that this technique fails, we need only show that $F_2 < 0$ or that $G < 0$ for some $\sigma \in (0, 1)$. We now proceed to show that $F_2 < 0$ for some small enough σ whenever $\alpha < d$.

Theorem 65 *Given a vector $v \in \mathbf{R}^d$ that is ray-nestable for some finite $\mathcal{G} \leq \mathcal{O}(d)$ and given E , F_2 and $V(r)$ as defined in (10.38) p. 186. (10.49) above and (10.12) p. 178, respectively then*

1. $\forall x \in \mathbf{R}^d \setminus \{0\}$, the set $\mathcal{G}x$ spans $\mathbf{R}^d$.

2. $E(c^2) = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (v, Lv)^2 = 1/d$.

3. $\lim_{\sigma \rightarrow 0^+} F_2(\sigma) \begin{cases} < 0 & \text{if } \alpha < d \\ = 0 & \text{if } \alpha = d \\ > 0 & \text{if } \alpha > d. \end{cases}$

4. If $\alpha < d$, $\exists$ some $(x, y) \in \mathbf{R}_+^2$ for which $f''(x, y)$ is **not** positive semi-definite.

5. Given $\alpha < d$, then $\forall n \geq 2$, $\exists$ some $m, r \in \mathbf{R}_+^n$, such that $V''(r)$ is **not** positive definite.

Proof. *Show 1:* Choose any non-zero vector $x \in \mathbf{R}^d$. Since $x \neq 0$ and $\mathcal{G}v$ spans all of $\mathbf{R}^d$ by hypothesis, $\exists L \in \mathcal{G}$ such that $Lv \notin \langle x \rangle^\perp$; i.e., $(x, Lv) \neq 0$; Let $w = Lv$. We first show that $\overline{\mathcal{G}_w x}$ is a non-zero vector in $\langle w \rangle$.

$$\begin{aligned} \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} w &= \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} Lv \\ &= L \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v \quad (\text{P. 2 of L. 98 p. 215 App. A}) \\ &= L \langle v \rangle \quad (\text{since } v \text{ is ray-nestable}) \\ &= \langle Lv \rangle \\ &= \langle w \rangle. \end{aligned}$$

The centroid of the orbit $(\text{stab}_G w) x \equiv \mathcal{G}_w x$ is left fixed by every member of $\mathcal{G}_w$ (L. 99 p. 216 App. B): i.e., $\overline{\mathcal{G}_w x} \in \text{fix}_{\mathbf{R}^d} \text{stab}_G w = \langle w \rangle$. $\forall K \in \text{stab}_G w$. $(w, Kx) = (K^{-1}w, x) = (w, x) \neq 0$ by the choice of w .

$$\begin{aligned} \therefore (w, \overline{\mathcal{G}_w x}) &= \frac{1}{|\mathcal{G}_w|} \sum_{K \in \mathcal{G}_w} (w, Kx) \\ &= \frac{1}{|\mathcal{G}_w|} \sum_{K \in \mathcal{G}_w} (w, x) \\ &= (w, x) \neq 0. \end{aligned}$$

Thus $\overline{\mathcal{G}_w x}$ is a non-zero vector in $\langle w \rangle$ and $w = Lv$ is a scalar multiple of $\overline{\mathcal{G}_{Lv} x}$: i.e., a linear combination of the vectors in $\mathcal{G}_{Lv} x \subseteq \mathcal{G}x$: i.e., $Lv \in \langle \mathcal{G}x \rangle$. $\therefore \mathcal{G}v = \mathcal{G}Lv \subseteq \langle \mathcal{G}x \rangle$. However, $\mathcal{G}v$ spans all of $\mathbf{R}^d$: so, $\langle \mathcal{G}x \rangle = \mathbf{R}^d$: i.e., $\forall x \neq 0$, $\mathcal{G}x$ spans all of $\mathbf{R}^d$ also.

Show 2: Consider the quadratic form

$$g(x) = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} x' (Lv' L') x = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (x, Lv)^2.$$

($x \in \mathbf{R}^d$). Clearly, $g(x) \geq 0$. Also, given $x \neq 0$, since $\mathcal{G}v$ spans $\mathbf{R}^d$, $\exists$ some $L \in \mathcal{G}$ such that $(x, Lv) > 0$. $\therefore g$ is positive definite. Let x_0 be a maximal unit vector for g : i.e., $\|x_0\| = 1$ and $\forall y \in S^{d-1}$, the unit $(d-1)$ sphere,

$$g(x_0) \geq g(y).$$

$\forall H \in \mathcal{G}$, since $H^{-1} = H^t$,

$$\begin{aligned} g(Hx_0) &= \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (x_0, H^{-1}Lv)^2 \\ &= \frac{1}{|\mathcal{G}|} \sum_{L' \in H^{-1}\mathcal{G}=\mathcal{G}} (x_0, L'v)^2 = g(x_0) \end{aligned}$$

Since the set $\mathcal{G}x_0 = \{Hx_0 | H \in \mathcal{G}\}$ spans $\mathbf{R}^d$ by part 1, the eigenspace associated with the largest eigenvalue is all of $\mathbf{R}^d$: i.e., g is totally degenerate, i.e., $g(x) = \lambda \|x\|^2$ for some $\lambda > 0$.

Let $\{e_1, \dots, e_d\}$ be the standard orthogonal basis for $\mathbf{R}^d$. Then

$$\begin{aligned} 1 &= \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} \|v\|^2 = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (Lv, Lv) \\ &= \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} \left(\sum_{i=1}^d (Lv, e_i) e_i, Lv \right) \\ &= \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} \sum_{i=1}^d (Lv, e_i) (e_i, Lv) = \sum_{i=1}^d g(e_i) = \lambda d. \end{aligned}$$

and

$$E(c^2) = \frac{1}{|\mathcal{G}|} \sum_{L \in \mathcal{G}} (v, Lv)^2 = g(v) = 1/d.$$

Show 3: From (10.49) p. 188 if

$$\begin{aligned} \lim_{\sigma \rightarrow 0^+} F_2(\sigma) &= -\alpha E(1 - c^2) + (\alpha - 1) E(1) \\ &= -\alpha \left(1 - \frac{1}{d}\right) + (\alpha - 1) \\ &= \frac{\alpha}{d} - 1 = \frac{\alpha - d}{d} \end{aligned}$$

and the result follows.

Show 4: Assume $\alpha < d$. By the continuity of F_2 on $(0, 1)$, $F_2(\sigma) < 0$ for some sufficiently small σ . Recalling the substitution $\sigma = \sin 2\theta$ and $x = r \cos \theta$, $y = r \sin \theta$, for sufficiently small θ or sufficiently small $y/x = \tan \theta$, $f_{yy}(x, y) < 0$. Thus $f''(x, y)$ is not positive definite $\forall 0 < y < x$.

Show 5: Given $\mathcal{G} \leq \mathcal{O}(d)$, $\alpha < d$ and $n \geq 2$ our objective here is to find a choice of positive masses and sizes for the polytopes so that $V''(r)$ is not positive definite. By Part 3 of this theorem we can choose σ sufficiently small that $F_2(k) < 0$, $\forall k \in (0, \sigma]$. Choose

$$r_n = \left(\frac{1 - \sqrt{1 - \sigma^2}}{\sigma} \right) r_{n-1} = r_{n-1} \tan \theta$$

where $\theta = \frac{1}{2} \arcsin \sigma$ (i.e. $\sigma = \sin 2\theta$) as in the earlier discussion. From (10.28) p. 182

$$\frac{\partial^2 V}{\partial r_n^2}(r) = |\mathcal{G}_v| \sum_{j=1}^{n-1} m_j m_n f_{yy}(r_j, r_n) + \frac{|\mathcal{G}_v|}{2} m_n^2 \tilde{f}''(r_n).$$

Then $\forall j < n-1$, we have

$$\frac{r_n}{r_j} \leq \frac{r_n}{r_{n-1}} = \tan \theta.$$

Thus

$$(r_j^2 + r_n^2)^{\alpha/2} f_{yy}(r_j, r_n) = F_2 \left(\sin \left(2 \arctan \frac{r_n}{r_j} \right) \right) < F_2(\sigma) < 0$$

by the monotonicity of both $\sin 2\gamma$ and $\arctan \delta$ for $\gamma \in (0, \pi/4)$ and $\delta \in (0, 1)$.

Thus, the first $n-2$ terms in the sum for $\partial^2 V / \partial r_n^2$ are negative. Ignoring them, we can write

$$\frac{\partial^2 V}{\partial r_n^2}(r) < m_n^2 |\mathcal{G}_v| \left[\frac{m_{n-1}}{m_n} f_{yy}(r_{n-1}, r_n) + \frac{1}{2} \tilde{f}''(r_n) \right].$$

Since $f_{yy}(r_{n-1}, r_n) < 0$ we can choose m_{n-1}/m_n sufficiently large that $\partial^2 V / \partial r_n^2 < 0$.

■

From Part 5 of the above theorem, we conclude that, given $n \geq 2$, we can only show uniqueness of c.c.'s for all assignment of masses $(m_1, \dots, m_n) \in \mathbf{R}_+^n$ using the suggested technique if $\alpha \geq d$. It may still turn out for some assignment of masses sufficiently close to each other, that we can use this Hessian technique to show uniqueness but we confine our discussion here to the simpler case.

10.8 Examples

Now that the theory has been established we can apply it to some special cases.

10.8.1 The Regular Simplex

For each value of $d \geq 1$ consider $d+1$ vertices on the unit $(d-1)$ -sphere in $\mathbf{R}^d$ such that the distance between all pairs of vertices is the same. They form a regular

d -simplex in $\mathbf{R}^d$. We show here for each $d \geq 1$ and $\alpha \geq \min(2, d)$ that the vertices are ray-nestable for their set-wise stabilizer (or complete symmetry group) and that for each $m = (m_1, \dots, m_n) \in \mathbf{R}_+^n$ there is exactly one c.c. for each arrangement of the nest. To obtain some idea of the scope of this example we give familiar special cases.

1. In $\mathbf{R}^1$ this is a line segment with vertices at ± 1 . This is a special case of the collinear problems dealt with by Moulton [Mn2] although α is generalized to any value greater than or equal to 2. The group is the group of order 2 generated by reflection through 0.
2. In $\mathbf{R}^2$ we have the equilateral triangle with vertices at $(1, 0)$ and $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. The group,

$$\left\langle \text{diag}(1, -1), \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ \sin(2\pi/3) & -\cos(2\pi/3) \end{pmatrix} \right\rangle$$

has order 6.

3. In $\mathbf{R}^3$ it can be represented by the regular tetrahedron with vertices

$$\frac{1}{\sqrt{3}} \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$$

and with the group $\mathcal{W}|T$ of order 24.

4. In $\mathbf{R}^4$ we have a regular pentatope.

We begin by specifying the $(d + 1)$ vertices of the d -simplex in $\mathbf{R}^d$ on the unit $(d - 1)$ -sphere.

Definition 31 $\forall d \geq 1$ we inductively define the set of bodies

$$\Pi_d = \{x_k^d = (x_{k,1}^d, \dots, x_{k,d}^d) \in \mathbf{R}^d \mid k = 1, \dots, d + 1\} \subset S^{d-1}$$

1. For $d = 1$, $x_1^1 = 1$, $x_{-1}^1 = -1$.

2. Given Π_d , we define Π_{d+1} by

$$x_1^{d+1} = e_1^{d+1} = (1, 0, \dots, 0) \in \mathbf{R}^{d+1}$$

and $\forall k \geq 1$

$$x_{k+1}^{d+1} = -\frac{1}{d+1}e_1^{d+1} + \sqrt{1 - \frac{1}{(d+1)^2}} (0, x_{k,1}^d, \dots, x_{k,d}^d).$$

For this definition, we need to verify that the members of Π_d really do lie on the unit sphere in $\mathbf{R}^d$. The following lemma gets this and the distance between bodies.

Lemma 66 $\forall d \geq 1$, for Π_d defined above

1. $\forall k \leq d+1$, $\|x_k^d\| = 1$.

2. $\forall k, k' \leq d+1$, $k \neq k'$,

$$\|x_k^d - x_{k'}^d\|^2 = \frac{2(d+1)}{d}.$$

Proof. We use induction on d . For $d = 1$, by Part 1 of the definition $\Pi_1 = \{\pm 1\}$ and we see by inspection that Parts 1 and 2 of the lemma are satisfied.

Now assume 1 and 2 are true for some $d \geq 1$. Define Π_{d+1} from Part 2 of the definition. Then clearly $\|x_1^{d+1}\| = \|e_1^{d+1}\| = 1$. Also $\forall k \geq 1$, by part 2 of the definition,

$$\|x_{k+1}^{d+1}\|^2 = \frac{1}{(d+1)^2} + \left(1 - \frac{1}{(d+1)^2}\right) \|x_k^d\|^2 = 1$$

because $\|x_k^d\| = 1$ by the induction hypothesis. Thus Part 1 of the lemma is satisfied.

For Part 2, note that

$$\begin{aligned} \|x_{k+1}^{d+1} - x_1^{d+1}\|^2 &= \left(1 + \frac{1}{d+1}\right)^2 + \left(1 - \frac{1}{(d+1)^2}\right) \|x_k^d\|^2 \\ &= 1 + \frac{2}{d+1} + \frac{1}{(d+1)^2} + 1 - \frac{1}{(d+1)^2} \\ &= \frac{2(d+2)}{d+1} \end{aligned}$$

agreeing with Part 2 for dimension $d + 1$. $\forall k \neq k'$, by the induction hypothesis and Part 2 of the definition

$$\begin{aligned} \|x_{k+1}^{d+1} - x_{k'+1}^{d+1}\|^2 &= \left(1 - \frac{1}{(d+1)^2}\right) \|x_k^d - x_{k'}^d\|^2 \\ &= \frac{d^2 + 2d}{(d+1)^2} \left(\frac{2(d+1)}{d}\right) \\ &= \frac{2(d+2)}{d+1} \end{aligned}$$

satisfying Part 2 for dimension $d + 1$. ■

We now show

Lemma 67 *For the regular d -simplex in $\mathbf{R}^d$ if $\mathcal{G} = \mathcal{E}(d)_{\{\Pi_d\}}$ then $\mathcal{G} \approx S_{d+1}$ and $\forall v \in \Pi_d$, $\text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{G}} v = \langle v \rangle$; i.e., v is ray-nestable in $\mathcal{G}$.*

Proof. Since Π_d comprises $d + 1$ independent points in $\mathbf{R}^d$, $\text{aff}_{\mathbf{R}^d} \Pi_d = \mathbf{R}^d$. Thus the ineffective part of $\mathcal{G}$ is trivial. $\therefore \mathcal{G}$ is finite. Since all distances are the same, each permutation π on the $d + 1$ bodies of Π_d generates a ! orthogonal transformation, say $H_\pi \in \mathcal{G}$. Setting $v = e_1^d$ as usual, we have $\text{stab}_{\mathcal{G}} v \approx S_d$ obtained by permuting the bodies $\{2, \dots, d + 1\}$ which moves every point in $\mathbf{R}^d$ not on $\langle v \rangle$. ■

We now show the uniqueness of c.c.'s for each nesting of such bodies. We need to show that $F_1(\sigma) \geq 0$ and $G(\sigma) \geq 0$, $\forall \sigma \in (0, 1)$. Thus we have a probability density table:

c	n_c	P_c
1	1	$\frac{1}{d+1}$
$-\frac{1}{d}$	d	$\frac{d}{d+1}$

The entries (c) in the first column are the values (Lv, v) for $L \in \mathcal{G}$. The second column contains the number of vertices, n_c , on the polytope with this value. The last column, P_c , contains the fraction of vertices with this value. Using this table, we can simplify the calculation of the expectation as follows: $\forall$ function $h : [-1, 1] \rightarrow \mathbf{R}$,

$$E(h) = \frac{h(1)}{d+1} + \frac{d}{d+1} h\left(-\frac{1}{d}\right).$$

We first show that $F_1(\sigma) > 0$ for all $\sigma \in (0, 1)$. From (10.51) p. 188,

$$F_1(\sigma) = \frac{\alpha - 1}{d + 1} \left[\Gamma_1^\alpha + d \Gamma_{-\frac{1}{d}}^\alpha \right] - \frac{\alpha(1 - \sqrt{1 - \sigma^2})}{2(d + 1)} \left[0 + d \left(1 - \frac{1}{d^2} \right) \Gamma_{-\frac{1}{d}}^{\alpha+2} \right].$$

Note from (10.50) p. 188 that

$$\frac{1}{\sqrt{1 - \sigma}} \equiv \Gamma_1(\sigma) > 1 > \Gamma_{-\frac{1}{d}}(\sigma) \equiv \frac{1}{\sqrt{1 + \sigma/d}}$$

and $-(1 - \sqrt{1 - \sigma^2}) > -1$.

$$\begin{aligned} \therefore \frac{F_1(\sigma)}{\Gamma_{-\frac{1}{d}}^\alpha} &> \frac{\alpha - 1}{d + 1} (1 + d) - \frac{\alpha(d^2 - 1)}{2(d + 1)d} \\ &= \alpha - 1 - \frac{\alpha(d - 1)}{2d} \\ &= \frac{\alpha}{2} - 1 + \frac{\alpha}{2d}. \end{aligned}$$

Since $\alpha \geq 2$, $F_1(\sigma) > 0$.

We now show that $G(\sigma) > 0, \forall \sigma$. From (10.53) p. 188

$$\begin{aligned} \frac{d + 1}{\alpha - 1} G(\sigma) &= (\alpha - 1) \left[2\Gamma_1^\alpha + d \left(1 - \frac{1}{d} \right) \Gamma_{-\frac{1}{d}}^\alpha \right] \left[0 + d \left(1 + \frac{1}{d} \right) \Gamma_{-\frac{1}{d}}^\alpha \right] \\ &\quad - \alpha \left[0 + d \left(1 - \frac{1}{d^2} \right) \Gamma_{-\frac{1}{d}}^{\alpha+2} \right] \left[\Gamma_1^{\alpha-2} + d \Gamma_{-\frac{1}{d}}^{\alpha-2} \right] \\ &= (\alpha - 1) 2\Gamma_1^\alpha (d + 1) \Gamma_{-\frac{1}{d}}^\alpha - \frac{\alpha}{d} (d + 1) (d - 1) \Gamma_{-\frac{1}{d}}^{\alpha+2} \Gamma_1^{\alpha-2} \\ &\quad + (d^2 - 1) \Gamma_{-\frac{1}{d}}^{2\alpha} [\alpha - 1 - \alpha] \\ &\geq (d + 1) \Gamma_1^\alpha \Gamma_{-\frac{1}{d}}^\alpha \left[2(\alpha - 1) - \frac{\alpha(d - 1)}{d} - (d - 1) \right] \\ &= (d + 1) \Gamma_1^\alpha \Gamma_{-\frac{1}{d}}^\alpha \left[\alpha - 1 + \frac{\alpha}{d} - d \right] \\ &\geq 0 \end{aligned}$$

because $\alpha \geq d$. Since $F_1(\sigma) > 0$ and $G(\sigma) > 0, \forall \sigma \in (0, 1)$, we have uniqueness.

This means that the nested equilateral triangles give uniqueness of c.c.'s $\forall \alpha \geq 2$.

Also, the nested regular tetrahedra in $\mathbf{R}^3$ give uniqueness $\forall \alpha \geq 3$.

10.8.2 Cross Polytopes

Consider the polytope in $\mathbf{R}^d$ whose $2d$ vertices are $\{\pm e_1, \pm e_2, \dots, \pm e_d\}$. Again this provides an example for each value of $d \geq 1$.

1. For $d = 1$ this is a repeat of the previous example of the collinear problem with symmetry about the origin as treated originally by Moulton.
2. For $d = 2$ this is a square with vertices at $\pm(1, 0)$ and $\pm(0, 1)$.
3. For $d = 3$ this is a regular octahedron with vertices at $\pm(1, 0, 0)$, $\pm(0, 1, 0)$ and $\pm(0, 0, 1)$.

In the general case we set $v = e_1$. This $2d$ -body problem has $2(d-1)$ bodies at $\pm e_j$, where $j > 1$. If $e_j = Lv$ for some $L \in \mathcal{G}^*$ then $(Lv, v) = (\pm e_j, e_1) = 0$. The remaining vertex is at $-e_1$ and $(-e_1, e_1) = -1$. Thus we have the probability density table.

c	n_c	P_c
1	1	$1/2d$
0	$2(d-1)$	$(d-1)/d$
-1	1	$1/2d$

From (10.51) p. 188

$$F_1(\sigma) = \frac{\alpha - 1}{2d} [\Gamma_1^\alpha + (2d - 2) + \Gamma_{-1}^\alpha] - \frac{\alpha(1 - \sqrt{1 - \sigma^2})}{4d} [0 + (2d - 2) + 0].$$

Ignoring the positive terms in Γ gives

$$\begin{aligned} 2dF_1(\sigma) &= (\alpha - 1) [\Gamma_1^\alpha + (2d - 2) + \Gamma_{-1}^\alpha] - \frac{\alpha}{2} (1 - \sqrt{1 - \sigma^2}) (2d - 2) \\ &> (2d - 2) \left[\alpha - 1 - \frac{\alpha}{2} \right] > 0. \end{aligned}$$

We now show that $G(\sigma)$ is positive for all σ . From (10.53) p. 188

$$\frac{(2d)^2 G(\sigma)}{\alpha - 1} = (\alpha - 1) [2\Gamma_1^\alpha + (2d - 2) + 0] [0 + (2d - 2) + 2\Gamma_{-1}^\alpha]$$

$$\begin{aligned}
& -\alpha [0 + (2d - 2) + 0] [\Gamma_1^{\alpha-2} + (2d - 2) + \Gamma_{-1}^{\alpha-2}] . \\
& = -(2d - 2)^2 + (2d - 2) [2(\alpha - 1) (\Gamma_1^\alpha + \Gamma_{-1}^\alpha) - \alpha (\Gamma_1^{\alpha-2} + \Gamma_{-1}^{\alpha-2})] \\
& \quad + 4(\alpha - 1) \Gamma_1^\alpha \Gamma_{-1}^\alpha .
\end{aligned}$$

Also, Γ_c is monotone increasing in c : so, $\Gamma_1^\alpha - \Gamma_{-1}^\alpha > 0$ and

$$\begin{aligned}
\Gamma_1^{\alpha-2} + \Gamma_{-1}^{\alpha-2} &= \Gamma_1^\alpha (1 - \sigma) + \Gamma_{-1}^\alpha (1 + \sigma) \\
&= \Gamma_1^\alpha + \Gamma_{-1}^\alpha - \sigma (\Gamma_1^\alpha - \Gamma_{-1}^\alpha) \\
&< \Gamma_1^\alpha + \Gamma_{-1}^\alpha .
\end{aligned}$$

Using these results and dividing through by $2d - 2$ gives

$$\begin{aligned}
\therefore \frac{(2d)^2 G(\sigma)}{(2d - 2)(\alpha - 1)} &> -(2d - 2) + [2(\alpha - 1) - \alpha] (\Gamma_1^\alpha + \Gamma_{-1}^\alpha) + \frac{2(\alpha - 1)}{d - 1} \Gamma_1^\alpha \Gamma_{-1}^\alpha \\
&= -2(d - 1) + (\alpha - 2) (\Gamma_1^\alpha + \Gamma_{-1}^\alpha) + \frac{2(\alpha - 1)}{d - 1} \Gamma_1^\alpha \Gamma_{-1}^\alpha .
\end{aligned}$$

Note from (10.50) p. 188 that for $k > 0$

$$\begin{aligned}
\Gamma_1^k + \Gamma_{-1}^k &= (1 - \sigma)^{-k/2} + (1 + \sigma)^{-k/2} . \\
\therefore \frac{\partial}{\partial \sigma} (\Gamma_1^k + \Gamma_{-1}^k) &= \frac{k}{2} (1 - \sigma)^{-(\frac{k+2}{2})} - \frac{k}{2} (1 + \sigma)^{-(\frac{k+2}{2})} > 0 . \\
\therefore \Gamma_1^k(\sigma) + \Gamma_{-1}^k(\sigma) &\text{ is an increasing function of } \sigma \text{ for } k > 0, \sigma \in (0, 1) .
\end{aligned}$$

$$\therefore \Gamma_1^\alpha + \Gamma_{-1}^\alpha > \lim_{\sigma \rightarrow 0} \Gamma_1^\alpha + \Gamma_{-1}^\alpha = 2 .$$

We also have

$$\Gamma_{-1} \Gamma_1 = \frac{1}{\sqrt{(1 - \sigma)(1 + \sigma)}} = \frac{1}{\sqrt{1 - \sigma^2}} > 1 .$$

We combine this with the fact that $\alpha \geq d$ to obtain

$$\begin{aligned}
\frac{(2d)^2 G(\sigma)}{(2d - 2)(\alpha - 1)} &> -2(\alpha - 1) + (\alpha - 2) (\Gamma_1^\alpha + \Gamma_{-1}^\alpha) + 2\Gamma_1^\alpha \Gamma_{-1}^\alpha \\
&> -2(\alpha - 1) + (\alpha - 2) 2 + 2 = 0 .
\end{aligned}$$

Thus the Hessian is positive definite in this case and we have exactly one c.c. for each arrangement of the polytopes.

10.9 Summary of Results on Nested Polytopes

All results are of the following form: given some polytope of interest.

1. for a range of values of the parameter α defining the homogeneous force of attraction, s
2. for all values of the number of polytopes, $n \geq 1$ and
3. for all ways of assigning positive masses to the bodies on polytope, i ($i \leq n$)

we have shown that there is exactly one way of scaling the sizes of the polytopes to obtain a central configuration subject to the constraint that polytope $i + 1$ lies inside polytope i , ($0 < i < n$) and hence there are exactly $n!$ such c.c.'s where the arrangement of the polytopes in the nest is not subject to any constraint.

The examples are

1. the regular simplex for all dimensions, $d \geq 1$ and for all $\alpha \geq \min(2, d)$
2. the cross polytope for all dimensions, $d \geq 1$ and for all $\alpha \geq 2$.

For the dimensions of greatest interest these examples include

Description of Polytope Nested	Dimension of Space	Range of Values of α
line segment	1	$\alpha \geq 2$
equilateral triangle	2	$\alpha \geq 2$
regular tetrahedron	3	$\alpha \geq 3$
square	2	$\alpha \geq 2$
regular octahedron	3	$\alpha \geq 2$

Appendix A

Isometries on $\mathbf{R}^d$

This appendix contains some basics on isometries on Euclidean spaces that can prepare the reader for the development of the group of an N body problem without having to obtain a reference text. In addition, some notation is provided and some of the theorems and lemmas are tailored specifically for use elsewhere in the thesis. Most of the material is addressed in [Yal] for spaces of dimension ≤ 3 . Also included is a brief review of groups acting on sets that can be found in most abstract algebra texts (see for example, [Jcb, Section 1.12]). The term, fixator of a group acting on a set is found in [Gi, Chapter 6].

An isometry on $\mathbf{R}^d$ is a map $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ such that $\|f(x) - f(y)\| = \|x - y\|$. Clearly f is injective. Substitution into the cosine law for triangles; i.e.,

$$(x - a, y - a) = \frac{1}{2} (\|x - a\|^2 + \|y - a\|^2 - \|x - y\|^2),$$

$\forall a, x, y \in \mathbf{R}^d$ gives $(f(x) - f(a), f(y) - f(a)) = (x - a, y - a)$. If $\{e_k\}_{k=1}^d$ is the standard basis for $\mathbf{R}^d$, then $(f(e_k) - f(0), f(e_{k'}) - f(0)) = (e_k, e_{k'}) = \delta_{k,k'}$. So, $\{f(e_k) - f(0)\}_{k=1}^d$ is an orthonormal basis for $\mathbf{R}^d$ and any $y \in \mathbf{R}^d$ can be expressed as $y = f(0) + \sum_{k=1}^d \alpha_k (f(e_k) - f(0))$ for some $\alpha_k \in \mathbf{R}$. If we let $x = \sum \alpha_k e_k$ then $(f(x) - f(0), f(e_k) - f(0)) = (x, e_k) = \alpha_k = (y - f(0), f(e_k) - f(0))$, true for all k . Thus $y = f(x)$, f is onto and hence a bijection. If $f(0) = 0$, $\{f(e_k)\}_{k=1}^d$ is

an orthonormal basis and for $x = \sum \alpha_k e_k$, $f(x) = \sum \alpha_k f(e_k)$; so, an isometry that fixes the origin is a linear map that preserves lengths; i.e., an orthogonal transformation. Conversely, if f is orthogonal, it fixed 0 and preserves length and we have shown

Lemma 68 *If f is an isometry on $\mathbf{R}^d$ then*

1. *f is a bijection*
2. *$\forall a, x, y \in \mathbf{R}^d$, $(f(x) - f(a), f(y) - f(a)) = (x - a, y - a)$.*
3. *f fixes the origin if and only if f is an orthogonal transformation.*

Remark 5 *We denote $\mathcal{E}(d)$ as the group of isometries on $\mathbf{R}^d$, called the Euclidean group, $\mathcal{O}(d)$ as the group of orthogonal transformations on $\mathbf{R}^d$ and $\mathcal{T}(d)$, the group of translations on $\mathbf{R}^d$. It is clear that $\mathcal{O}(d)$ and $\mathcal{T}(d)$ are subgroups of $\mathcal{E}(d)$. When the dimension, d of the space is clear, we shall simply write $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{T}$ for $\mathcal{E}(d)$, $\mathcal{O}(d)$ and $\mathcal{T}(d)$ respectively.*

Definition 32 *If $x_0 \in \mathbf{R}^d$ we define the translation function $T_{x_0} : \mathbf{R}^d \rightarrow \mathbf{R}^d$ by $T_{x_0}(x) = x - x_0$.*

$T_{x_0}^{-1} = T_{-x_0}$ and we have:

Lemma 69 *Every isometry on $\mathbf{R}^d$ is the product of a translation and an orthogonal transformation and this decomposition is unique.*

Proof. $\forall$ isometry f on $\mathbf{R}^d$, let $x_0 = f(0)$ and $C = T_{x_0}f$. Then $C(0) = f(0) - x_0 = 0$; so, by Lemma 68 Part 3, $C \in \mathcal{O}(d)$ and $f = T_{-x_0}C$ has the desired form. Suppose also that $f = T_{-y_0}C_2$ for some $y_0 \in \mathbf{R}^d$ and $C_2 \in \mathcal{O}(d)$. Then $y_0 = T_{-y_0}C_2(0) = f(0) = x_0$ and $C_2 = T_{y_0}f = T_{x_0}f = C$ and the choice of x_0 and C is unique. ■

Definition 33 If f is an isometry on $\mathbf{R}^d$ and $f = T_y H$ is its unique decomposition according to Lemma 69 then T_y is called its **translation part** and H is called its **orthogonal part**.

Lemma 70 $\forall y \in \mathbf{R}^d, H \in \mathcal{O}(d), HT_y = T_{Hy} H$.

Proof. $\forall x \in \mathbf{R}^d, (HT_y)x = H(x - y) = Hx - Hy = (T_{Hy}H)x$. ■

From this we have

Lemma 71 The orthogonal part of the composition of two isometries is the composition of their orthogonal parts.

Proof. $(T_{y_1} H_1)(T_{y_2} H_2) = T_{y_1 + H_1 y_2} (H_1 H_2)$ ■

Corollary 72 Given two isometries f_1 and g_1 with the same orthogonal part and f_2 and g_2 with the same orthogonal part, then $f_1 f_2$ and $g_1 g_2$ have the same orthogonal part.

Lemma 73 $\mathcal{T}(d) \triangleleft \mathcal{E}(d)$.

Proof. $\forall f \in \mathcal{E}(d)$ and $T_x \in \mathcal{T}(d)$, $f = T_y H$ for some $y \in \mathbf{R}^d$, $H \in \mathcal{O}(d)$ (Lemma 69) and $f T_x f^{-1} = T_y H T_x H^{-1} T_{-y} = T_{Hx} \in \mathcal{T}(d)$: since x is arbitrary, $f \mathcal{T}(d) f^{-1} \subseteq \mathcal{T}(d)$. ■

A.1 Groups Acting on Sets

The reader is reminded that a group G is said to act on a set S if there is a homomorphism $\phi : G \rightarrow \text{Sym}(S)$, where $\text{Sym}(S)$ is the symmetric group on S , i.e. the group of bijective transformations on S with composition as the binary operator. Equivalently, G acts on S if there is a map $G \times S \rightarrow S$ denoted $(g, s) \mapsto gs$ such that $\forall g_1, g_2 \in G, s \in S, g_1(g_2 s) = (g_1 g_2)s$ and $1_G s = s$ where 1_G is the identity in

G . The connection between the two equivalent definitions is that $\phi(g)(s) = gs$ (see e.g. [Jcb, Section 1.12, p. 69]). If G acts on S and $H \leq G$ then H acts on S by restriction. If the map ϕ is a monomorphism, we say that G acts **effectively** on S .

Since $\mathcal{E}(d) \leq \text{Sym}(\mathbf{R}^d)$, the action $\phi : \mathcal{E}(d) \rightarrow \text{Sym}(\mathbf{R}^d)$ is the embedding $\phi(f) = f$. Consequently, $\mathcal{E}(d)$ acts effectively on $\mathbf{R}^d$ and since $\mathcal{O}(d) \leq \mathcal{E}(d)$, $\mathcal{O}(d)$ also acts effectively on $\mathbf{R}^d$.

Definition 34 If G acts on S , $H \leq G$, $A \subseteq S$, $g \in G$ and $s \in S$ then the **orbit** of s under the action of H is $Hs = \{hs \mid h \in H\}$, the **stabilizer** of s in H is $H_s = \{h \in H \mid hs = s\}$ and the **fixator** of g in A is $A_g = \{a \in A \mid ga = a\}$. We extend these notions to sets; viz., define the **pointwise stabilizer** of the set, A to be $H_{(A)} = \bigcap_{a \in A} H_a$, if $R \subseteq G$, define the **fixator** of the set, R in A to be $A_{(R)} = \bigcap_{r \in R} A_r$ and define the orbit, $HA = \bigcup_{a \in A} Ha = \{ha \mid h \in H, a \in A\}$. We also denote the pointwise stabilizer of A in H ; i.e., $H_{(A)}$ as $\text{stab}_H A$, the fixator of g in A as $\text{fix}_A g$ and the fixator of R in A ; i.e., $A_{(R)}$ as $\text{fix}_A R$.

We use the usual notation that the intersection over the empty set, ϕ is the universal set; i.e., $H_{(\phi)} = H$ and $A_{(\phi)} = A$ and that the union over the empty set is the empty set; viz. the orbit $H\phi = \phi$. It is elementary that H_s is a subgroup of H and the intersection of such subgroups, $H_{(A)} \leq H$. Since the intersection taken over a larger index set is as small or smaller, we have the elementary

Lemma 74 Suppose G is a group acting on a set, S .

1. If $A' \subseteq A \subseteq S$ and $H \leq G$ then $H_{(A')} \supseteq H_{(A)}$.
2. If $R' \subseteq R \subseteq G$ and $A \subseteq S$ then $A_{(R')} \supseteq A_{(R)}$.

Denote $\langle R \rangle$ as the subgroup generated by R .

Lemma 75 If G acts on S , $R \subseteq G$ and $A \subseteq S$ then $A_{(R)} = A_{(\langle R \rangle)}$.

Proof. Since $R \subseteq \langle R \rangle$, $A_{(R)} \supseteq A_{(\langle R \rangle)}$. Conversely, $\forall x \in A_{(R)}$ and $\forall g \in \langle R \rangle$, $g = g_1^{r_1} \cdots g_k^{r_k}$ for some $g_i \in R, r_i \in \mathbf{Z}$. $\forall i \leq k$, $g_i x = x$, $g_i^{r_i} x = x$ and by induction on k , we have $gx = x$; so, $x \in A_g$ and since $g \in \langle R \rangle$ is arbitrary, $x \in A_{(\langle R \rangle)}$. Therefore, $A_{(R)} \subseteq A_{(\langle R \rangle)}$. ■

The following discussion to the end of this section contains multiple applications of the stabilizer and fixator and is expressed in the alternate notation to avoid excessive use of subscripts on subscripts.

Lemma 76 *If G acts on S , $A \subseteq S$ and $H \subseteq G$ then $A \subseteq \text{fix}_S H \Leftrightarrow H \subseteq \text{stab}_G A$.*

Proof.

$$\begin{aligned}
A \subseteq \text{fix}_S H &\Leftrightarrow a \in \text{fix}_S H && \forall a \in A \\
&\Leftrightarrow a \in \text{fix}_S h && \forall a \in A, \forall h \in H \\
&\Leftrightarrow ha = a && \forall a \in A, \forall h \in H \\
&\Leftrightarrow h \in \text{stab}_G a && \forall a \in A, \forall h \in H \\
&\Leftrightarrow h \in \text{stab}_G A && \forall h \in H \\
&\Leftrightarrow H \subseteq \text{stab}_G A.
\end{aligned}$$

■

Corollary 77 *If G acts on S and $x, y \in S$ and $f, g \in G$ then*

1. $y \in \text{fix}_S \text{stab}_G x \Leftrightarrow \text{stab}_G x \leq \text{stab}_G y$ and
2. $f \in \text{stab}_G \text{fix}_S g \Leftrightarrow \text{fix}_S g \subseteq \text{fix}_S f$.

Proof. Apply the above lemma where in part 1 we set $A = \{y\}$ and $H = \text{stab}_G x$ and in part 2 we choose $H = \{f\}$ and $A = \text{fix}_S g$. ■

Lemma 78 *If G acts on S and the orbit Gx is finite for some $x \in S$ then $\forall y \in \text{fix}_S \text{stab}_G x$, the orbit Gy is finite. Moreover, $|Gy| \leq |Gx|$.*

Proof. Choose any $y \in \text{fix}_S \text{stab}_G x$. By Part 1 of the above corollary $\text{stab}_G x \leq \text{stab}_G y \leq G$. Since Gx is finite $[G : \text{stab}_G x] = |Gx| = n$ (see e.g. [BJN. Thm. 4.7 p. 108]) for some $n < \infty$. Since each left coset in $\text{stab}_G y$ contains at least one left coset in $\text{stab}_G x$ there must be finitely many of them and $|Gy| = [G : \text{stab}_G y] \leq [G : \text{stab}_G x] = |Gx|$ as desired. ■

Lemma 79 *If G acts on S , $H \subseteq G$, $A \subseteq S$ then*

1. $H \subseteq \text{stab}_G \text{fix}_S H$.
2. $A \subseteq \text{fix}_S \text{stab}_G A$.
3. $\text{fix}_S H = \text{fix}_S \text{stab}_G \text{fix}_S H$ and
4. $\text{stab}_G A = \text{stab}_G \text{fix}_S \text{stab}_G A$.

Simply phrased, ‘when you take the fixator of the stabilizer or the stabilizer of the fixator, the result is never smaller but this non-decreasing process stops after the first application.’ In the standard notation these four statements read:

$$H \subseteq G_{(S(H))}, A \subseteq S_{(G(A))}, S_{(H)} = S_{\left(G_{(S(H))}\right)} \text{ and } G_{(A)} = G_{\left(S_{(G(A))}\right)}.$$

Proof. *Show 1:* $\forall g \in H$ and $x \in \text{fix}_S H$, $gx = x$. $\therefore g \in \text{stab}_G x$. Since x is an arbitrary member of $\text{fix}_S H$ the result follows.

Show 2: $\forall a \in A$, $\forall g \in \text{stab}_G A$, $ga = a$. $\therefore a \in \text{fix}_S g$. Since g is an arbitrary member of $\text{stab}_G A$, $a \in \text{fix}_S \text{stab}_G A$.

Show 3: Substituting $A = \text{fix}_S H$ in part 2 gives $\text{fix}_S H \subseteq \text{fix}_S \text{stab}_G \text{fix}_S H$. However, since the fixator taken over a larger set is smaller, taking the fixator of both sides of part 1 gives the reverse inclusion.

Show 4: This proof is similar to part 3, reversing the roles of A and H . ■

Definition 35 If G acts on S then a set, $H \subseteq G$, (resp. $A \subseteq S$) is said to be a **maximally stabilized subset** of G (resp. S) if $H = \text{stab}_G B$ for some $B \subseteq S$ (resp. $A = \text{fix}_S R$ for some $R \subseteq G$).

Clearly, the maximally stabilized subsets of G , being stabilizers are subgroups of G but not all subgroups are maximally stabilized. In fact, if $H \subseteq G$, then $\text{stab}_G \text{fix}_S H$, being a subgroup of G containing H , contains $\langle H \rangle$, the group generated by H . However, they are not necessarily equal. For example, in the case of the Euclidean group, $\mathcal{E}(2)$, acting on $\mathbf{R}^2$, any rotation through some angle not equal to a multiple of 2π radians has $\{0\}$ as its fixator which has $\mathcal{O}(2)$ as its stabilizer. However, the group generated by such a rotation is at most countable and cannot be all of $\mathcal{O}(2)$.

If X is a set, we denote $\mathcal{P}(X)$ as the family of all subsets of X (the power set).

Theorem 80 If G acts on S and we let

$$\begin{aligned}\Gamma &= \{\text{stab}_G A \mid A \subseteq S\} \subseteq \mathcal{P}(G) \quad \text{and} \\ \Sigma &= \{\text{fix}_S H \mid H \subseteq G\} \subseteq \mathcal{P}(S),\end{aligned}$$

then there is a 1-1 correspondence between Γ , the family of **maximally stabilized subgroups** of G and Σ , the family of **maximally stabilized subsets** of S given by $\text{stab}_G A \mapsto \text{fix}_S \text{stab}_G A$.

Proof. If we let $f : \Gamma \rightarrow \Sigma$ be given by $f(\text{stab}_G A) = \text{fix}_S \text{stab}_G A$ and $g : \Sigma \rightarrow \Gamma$ be $g(\text{fix}_S H) = \text{stab}_G \text{fix}_S H$ then it is clear from parts 3 and 4 of L. 79 that $g \circ f = \text{id}_\Gamma$ and $f \circ g = \text{id}_\Sigma$ (the identity maps on Γ and Σ , respectively). ■

Example 12 If E is a splitting field over a field F , we call $\text{aut } E$, the group of automorphisms of E/F . Then $\text{aut } E \leq \text{Sym}(E)$ and hence acts on E . If K is an intermediate field; i.e., $F \leq K \leq E$ and the Galois group, $\text{Gal}(E/K)$ is the subgroup of automorphisms of E that leave each member of K fixed which in the notation of

groups acting on sets is just $\text{stab}_{\text{Aut } E} K$. Part of the fundamental theorem of Galois theory can be rephrased as 'The maximally stabilized subsets of the group, $\text{Gal}(E/F)$ acting on E are the Galois groups, $\text{Gal}(E/K)$ where K is a subfield of E containing F .' The 1-1 correspondence in Thm. 80 is precisely the one between the set of subfields containing F and the subgroups of $\text{Gal}(E/K)$.

A.2 Affine Subspaces in $\mathbf{R}^d$

We now introduce a generalization of the notions of point, line and plane to higher dimensions (see e.g. [Bst, Ch. 1]).

Definition 36 If $Y \subseteq \mathbf{R}^d$, we define $\langle Y \rangle$ to be the linear span of Y in $\mathbf{R}^d$. If X is a subset in $\mathbf{R}^d$, define

$$\Delta X = X - X = \{x - y | x, y \in X\}. \quad (\text{A.1})$$

An **affine subspace**, U in $\mathbf{R}^d$ is the solution set of a system of linear equations and if U is non-empty, we call ΔU the **parallel transport of U to the origin**.

Thus, $\langle Y \rangle = \{\sum_{i=1}^n \alpha_i y_i | n \geq 0, y_i \in Y \text{ and } \alpha_i \in \mathbf{R}, i \leq n\}$ and $\langle \emptyset \rangle = \{0\}$. Also, an affine subspace, U has the form $U = \{x \in \mathbf{R}^d | Ax = c\}$ for some $A \in L(\mathbf{R}^d, \mathbf{R}^k)$, $k \geq 1$ (i.e., A is a $k \times d$ matrix) and $c \in \mathbf{R}^k$. If U is a non-empty affine subspace and $y \in \Delta U$ then $y = x - x_0$ for some $x, x_0 \in U$; i.e., $Ay = 0$ and $\Delta U = \{y \in \mathbf{R}^d | Ay = 0\}$; so, ΔU is the nullspace of A ; i.e., a linear subspace of $\mathbf{R}^d$. Conversely, if $W \leq \mathbf{R}^d$ (say, the solution space of $Ay = 0$ for some $A \in L(\mathbf{R}^d, \mathbf{R}^k)$, $k \geq 1$), $x_0 \in \mathbf{R}^d$ and if we let $c = Ax_0$, then $x \in x_0 + W \Leftrightarrow Ax = c$; i.e., $x_0 + W$ is an affine subspace. Now, suppose that $U = \{z \in \mathbf{R}^d | Az = c\}$ and $x \in U$. Clearly, $\Delta U \supseteq U - x$. Conversely, $\forall y \in \Delta U$, $Ay = 0$, $A(x + y) = c$. So, $x + y \in U$, $y \in U - x$; i.e., $\Delta U \subseteq U - x$ and we have shown

Lemma 81 *If U is a non-empty affine subspace then ΔU is a linear subspace and $\Delta U = U - x, \forall x \in U$. Conversely, if $W \leq \mathbf{R}^d$ (linear subspace) and $x \in \mathbf{R}^d$ then $x + W$ is an affine subspace.*

A point, line and plane are affine subspaces of dimension 0, 1 and 2 respectively. In $\mathbf{R}^d$ an affine subspace of dimension $(d - 1)$ is called a hyperplane.

Definition 37 *Given $A \subseteq \mathbf{R}^d$, an **affine combination** of the points in A is a linear combination of the form*

$$\sum_{j=1}^k \alpha_j x_j \quad (\text{A.2})$$

where each $x_i \in A$ and

$$\sum_{j=1}^k \alpha_j = 1. \quad (\text{A.3})$$

Lemma 82 1. *If $U \subseteq \mathbf{R}^d$, U is an affine subspace if and only if any affine combination of points from U is in U .*

2. *The intersection of affine subspaces of $\mathbf{R}^d$ is an affine subspace.*

Proof. (1): If U is affine: say, the solution space of $Ax = c, x_i \in U, i = 1, k$ and if $\sum \alpha_i = 1$, then $A \sum \alpha_i x_i = \sum \alpha_i Ax_i = \sum \alpha_i c = c$; i.e., the affine combination $\sum \alpha_i x_i \in U$. Conversely, suppose that U is a set containing all affine combinations in U . Then, if $U = \emptyset$, U is affine; otherwise, let $x_0 \in U$, then $\forall x_1, \dots, x_k \in U$ and $\forall \alpha_i \in \mathbf{R}, i = 1, k$, choose $\alpha_0 = -\sum_{i=1}^k \alpha_i$. By hypothesis, $x_0 + \sum_{i=1}^k \alpha_i (x_i - x_0) = (1 + \alpha_0) x_0 + \sum_{i=1}^k \alpha_i x_i \in U$. Therefore, an arbitrary linear combination of members of $U - x_0$; namely, $\sum_{i=1}^k \alpha_i (x_i - x_0) \in U - x_0$; i.e., $U - x_0 \leq \mathbf{R}^d$ and U is affine (by L. 81).

(2): If $\{U_\beta | \beta \in \Lambda\}$ are affine subspaces and $x_i \in \cap_{\beta \in \Lambda} U_\beta$ then any affine combination of x_i lies in each U_β and hence in the intersection. ■

Definition 38 If $A \subseteq \mathbf{R}^d$, we define the **affine hull** of A or the **affine subspace spanned by A** , denoted $\text{aff } A$ to be the intersection of all affine subspaces containing A . The set $X = \{x_0, x_1, \dots, x_k\} \subseteq \mathbf{R}^d$ is said to be **affinely independent** if the only solution of $\sum_{j=0}^k \alpha_j x_j = 0$ subject to the constraint $\sum_{j=0}^k \alpha_j = 0$ is $\alpha_0 = \alpha_1 = \dots = \alpha_k = 0$. The **dimension** of a non-empty affine subspace is defined to be the dimension of its parallel transport to the origin.

It is easy to show that X in the above definition is an independent set if and only if $\{x_1 - x_0, \dots, x_k - x_0\}$ is linearly independent. Furthermore, when this happens, $\text{aff } X$ and $\langle \Delta X \rangle$ have dimension k . Then $\forall i \leq k$,

$$\text{aff } X - x_i = \left\{ \sum_{j=0}^k \alpha_j (x_j - x_i) \mid \alpha_j \in \mathbf{R} \right\}$$

where the restriction $\sum \alpha_j = 1$ has been removed because α_i is arbitrary. Thus $\text{aff } X = x_i + \langle X - x_i \rangle$ is an affine subspace containing each x_i and $\langle \Delta X \rangle = \text{aff } X - x_i = \langle X - x_i \rangle$. Note that the empty set is an affine subspace. It is also easy to show that

Lemma 83 Given $X = \{x_0, x_1, \dots, x_k\} \subseteq \mathbf{R}^d$, $k \geq 0$, X or the affine hull of X is

$$\text{aff } X = \left\{ \sum_{j=0}^k \alpha_j x_j \mid \sum \alpha_j = 1 \right\}.$$

Lemma 84 Given A , a non-empty subset of $\mathbf{R}^d$, $\exists k \leq d$ and an independent set, $X = \{x_0, \dots, x_k\} \subseteq A$ such that $\text{aff } X = \text{aff } A$.

Proof. We inductively form an independent sequence $x_0, x_1, \dots$ such that $X_n = \{x_0, \dots, x_n\} \subseteq A$ as follows: choose any $x_0 \in A$ and given $X_n = \{x_0, \dots, x_n\}$ such that $A \setminus \text{aff } X_n \neq \emptyset$, choose $x_{n+1} \in A \setminus \text{aff } X_n$. Since $x_{n+1} \notin \text{aff } X_n = x_0 + \langle \Delta X_n \rangle$ and $x_{n+1} - x_0 \notin \langle \Delta X_n \rangle = \text{span} \{x_i - x_0 \mid i = 1, n\}$ then $\{x_i - x_0 \mid i = 1, n+1\}$ is linearly independent and X_{n+1} is independent. The dimension of $X_n = n$. Since

$\dim \mathbf{R}^d = d$ is finite, then for some $k \leq d$, the process must stop: i.e., $A \subseteq \text{aff } X_k$ (otherwise we could have chosen $x_{k+1} \in A \setminus \text{aff } X_k$ and continued). Since $X_k \subseteq A$, any affine subspace that contains A contains X_k and hence contains $\text{aff } X_k$ (by D. 38). Also $\text{aff } X_k$ is an affine subspace that contains A . Thus X_k is the desired set X : i.e., $\text{aff } X_k = \text{aff } A$. ■

Definition 39 A mapping $f : \mathbf{R}^d \rightarrow \mathbf{R}^c$ is called an *affine mapping* if it preserves affine combination; i.e., whenever $\sum_{j=1}^k \alpha_j = 1$,

$$f\left(\sum_{j=1}^k \alpha_j x_j\right) = \sum_{j=1}^k \alpha_j f(x_j).$$

Lemma 85 *Isometries are affine maps.*

Proof. By Lemma 69 p. 201, if g is an isometry on $\mathbf{R}^d$, $g = T_r H$, where $r \in \mathbf{R}^d$, $H \in \mathcal{O}(d)$.

$$g\left(\sum_{i=1}^k \alpha_i x_i\right) = H\left(\sum \alpha_i x_i\right) - r = \sum \alpha_i (Hx_i - r) = \sum \alpha_i g(x_i).$$

■

Lemma 86 *Isometries map non-empty affine subspaces onto affine subspaces of the same dimension.*

Proof. Let U be an affine subspace in $\mathbf{R}^d$ and $g \in \mathcal{E}$. If $\dim U = 0$, U is a singleton and its image under g is a singleton: i.e., it has dimension 0 also. Otherwise, let $\dim U = k \geq 1$, let $\{y_1, \dots, y_k\}$ be a basis for ΔU and let $x \in U$. Let $x_i = x + y_i \in U$, $i \leq k$, let $x_{k+1} = x$ and let $X = \{x_1, \dots, x_{k+1}\}$. Then X is an independent set in U , $U = \text{aff } X$ and

$$\begin{aligned} a \in \text{aff}_U X &\Leftrightarrow a = \sum \alpha_i x_i, \text{ where } \sum \alpha_i = 1 \\ &\Leftrightarrow g(a) = \sum \alpha_i g(x_i), \sum \alpha_i = 1 \quad (\text{Lemma 85}) \\ &\Leftrightarrow g(a) \in \text{aff}(g(X)). \end{aligned}$$

Thus, $g(U) = \text{aff}(g(X))$ is an affine subspace. If $g = T_r H$, $r \in \mathbf{R}^d$, $H \in \mathcal{O}(d)$, the set $\{g(x_i) - g(x)\}_{i=1}^k = H\{y_i\}_{i=1}^k$ forms a basis for $\Delta g(U)$ and so $g(U)$ has the same dimension as U . ■

A.3 The Action of $\mathcal{E}(d)$ on $\mathbf{R}^d$

Lemma 87 *If $\mathcal{G} \subseteq \mathcal{E}(d)$ and U is an affine subspace in $\mathbf{R}^d$, $\text{fix}_U \mathcal{G}$ is an affine subspace. (The fixator in an affine subspace, of a set of isometries is itself an affine subspace.)*

Proof. If $\text{fix}_U \mathcal{G}$ is empty then it is an affine subspace by definition. Otherwise, by Lemma 84 we can find an independent set $X = \{x_1, \dots, x_k\} \subseteq \text{fix}_U \mathcal{G}$ such that $\text{aff } X \supseteq \text{fix}_U \mathcal{G}$. For any $x \in \text{aff } X$, $x = \sum \alpha_i x_i$ where $\sum_{i=1}^k \alpha_i = 1$. $\forall f \in \mathcal{G}$, $f(x_i) = x_i$; so, since isometries are affine maps, (L. 85), $f(x) = \sum \alpha_i f(x_i) = \sum \alpha_i x_i = x$. Since $X \subseteq U$, $x \in \text{aff } X \subseteq U$. Thus $x \in U_f$ and since $f \in \mathcal{G}$ is arbitrary, $x \in \text{fix}_U \mathcal{G}$. $\therefore \text{aff } X \subseteq \text{fix}_U \mathcal{G}$; so, $\text{aff } X = \text{fix}_U \mathcal{G}$; i.e., $\text{fix}_U \mathcal{G}$ is affine. ■

Lemma 88 *If U is an affine subspace in $\mathbf{R}^d$, $U = \text{fix}_{\mathbf{R}^d} \text{stab}_{\mathcal{E}(d)} U$.*

Proof. If $U = \emptyset$ then $\text{stab}_{\mathcal{E}} U = \mathcal{E}$ and $\text{fix}_{\mathbf{R}^d} \mathcal{E} = \emptyset$; so, it holds in this case. Otherwise, by Lemma 79, $U \subseteq \text{fix} \cdot \text{stab } U$. To show the reverse inclusion we find an isometry that fixes every point in U and moves every other point in $\mathbf{R}^d$. Let $p: \mathbf{R}^d \rightarrow \mathbf{R}^d$ be the map that carries x to the point in U closest to x . Consider the map $f(x) = 2p(x) - x$. To see that f is an isometry, $\forall x, y \in \mathbf{R}^d$

$$\begin{aligned} f(x) - f(y) &= -(x - y) + 2[p(x) - p(y)] \\ \|f(x) - f(y)\|^2 &= \|x - y\|^2 + 4(p(x) - p(y), p(x) - p(y) - x + y) \end{aligned}$$

Since, $p(x) - x$ and $p(y) - y$ are orthogonal to the affine subspace U that contains $p(x)$ and $p(y)$, the last term is zero and we have an isometry. Also f obviously

fixes every point of U : i.e., $f \in \text{stab } U$; so, $\text{fix} \cdot \text{stab } U \subseteq \text{fix}(f)$. Furthermore, if $f(x) = x$, $x = p(x)$, $x \in U$ and $\text{fix}(f) \subseteq U$. $\therefore \text{fix} \cdot \text{stab}(U) \subseteq U$: i.e., they are equal. ■

If $H \in \mathcal{O}(d)$ then $x \in \mathbf{R}_H^d \equiv \text{fix}_{\mathbf{R}^d} H \Leftrightarrow Hx = x \Leftrightarrow x$ is in the eigenspace for the eigenvalue $+1$. If $\mathcal{A} \subseteq \mathcal{O}(d)$, $\mathbf{R}_{(\mathcal{A})}^d \equiv \text{fix}_{\mathbf{R}^d} \mathcal{A}$ is the intersection of these eigenspaces $\forall H \in \mathcal{A}$. Recall that if T is a linear operator on a linear space V , $W \leq V$ and $T(W) \subseteq W$ then we say that W is an invariant subspace for T .

Corollary 89 *The family of maximally stabilized subsets of $\mathbf{R}^d$ under the action of $\mathcal{E}(d)$ is precisely the family of affine subspaces.*

Proof. By D. 35, p. 206, any maximally stabilized subset has the form $\text{fix}_{\mathbf{R}^d} \mathcal{G}$ where $\mathcal{G} \subseteq \mathcal{E}$ which is an affine subspace by L. 87. Conversely, by L. 88 any affine subspace is the fixator of some subgroup of $\mathcal{E}$ and is hence maximally stabilized. ■

Note that the corresponding maximally stabilized subgroups of $\mathcal{E}(d)$ are merely the stabilizers of these affine subspaces.

Lemma 90 *If S is a non-empty subset of $\mathbf{R}^d$ and $0 \in S$, then $\text{aff } S = \langle S \rangle$.*

Proof. It is obvious that $\text{aff } S \subseteq \langle S \rangle$. Conversely, if $x \in \langle S \rangle$, for some $n \geq 1$, $\alpha_i \in \mathbf{R}$, $s_i \in S$, $x = \sum_{i=1}^n \alpha_i s_i = \sum_{i=1}^{n+1} \alpha_i s_i$ where we choose $s_{n+1} = 0 \in S$ and $\alpha_{n+1} = -\sum_{i=1}^n \alpha_i$. Then $\sum_{i=1}^{n+1} \alpha_i = 1$ and hence $x \in \text{aff } S$. ■

Lemma 91 *If W is an invariant subspace for $H \in \mathcal{O}(d)$, then $H(W) = W$, $H(W^\perp) = W^\perp$; i.e., H can be written as a direct sum decomposition of a part in W and a part in $W^\perp$. Equivalently, $H = H_1 H_2 = H_2 H_1$ where $H_1 \in \mathcal{O}(d)_{(W^\perp)}$ and $H_2 \in \mathcal{O}(d)_{(W)}$.*

Proof. Since H is injective and $\dim W < \infty$, $H|_W$ is onto: so, $H(W) = W$. Since $H|_W$ is bijective, $\forall y \in W$, let $y' = H^{-1}y \in W$. $\forall x \in W^\perp$, $(Hx, y) = (Hx, Hy') = (x, y') = 0$. Since $y \in W$ is arbitrary, $Hx \in W^\perp$; i.e., $H(W^\perp) \subseteq W^\perp$

and as above, $H(W^\perp) = W^\perp$. Thus, $H = H|_W \oplus H|_{W^\perp}$ is an orthogonal direct sum decomposition. If we let I_W and $I_{W^\perp}$ be the identity maps on W and $W^\perp$ respectively and let $H_1 = H|_W \oplus I_{W^\perp}$ and $H_2 = I_W \oplus H|_{W^\perp}$, then any $z \in \mathbf{R}^d$ has the form, $z = x + y$ where $x \in W^\perp$ and $y \in W$. Then $H_1 x = x$; so, $H_1 \in \mathcal{O}_{(W^\perp)}$. Similarly, $H_2 \in \mathcal{O}_{(W)}$. Also, $H_1 \circ H_2 = H|_W \oplus H|_{W^\perp} = H$. Similarly, $H_2 H_1 = H$. ■

In Lemma 75, we showed that the fixator of a subset of a group acting on a set is the fixator of the subgroup generated by that subset. In the special case of the Euclidean group acting on $\mathbf{R}^d$ we provide an analogous lemma where the roles of group and set are reversed. In this case we show that the stabilizer of a set of points in $\mathbf{R}^d$ is the stabilizer of the affine subspace generated by that set.

Lemma 92 *If $S \subseteq \mathbf{R}^d$ and $G \leq \mathcal{E}$, then $G_{(S)} = G_{(\text{aff } S)}$. (In a Euclidean group the pointwise stabilizer of a set is the pointwise stabilizer of its affine hull).*

Proof. By Lemma 84 we can let $S' = \{s_1, \dots, s_k\} \subseteq S$ such that $\text{aff } S' = \text{aff } S$, (a finite subset of S that generates the same affine subspace). Since the intersection taken over a larger set is smaller, $G_{(\text{aff } S)} \leq G_{(S)}$. Conversely, if $g \in G_{(S)}$ then $gs_i = s_i, \forall i$ and $\forall x \in \text{aff } S, x = \sum \alpha_i s_i$ where $\sum \alpha_i = 1$. Then by Lemma 85, $g(x) = \sum \alpha_i g(s_i) = \sum \alpha_i s_i = x$. $\therefore g \in G_x$. $\because x \in \text{aff } S$ is arbitrary, $g \in G_{(\text{aff } S)}$. $\therefore G_{(S)} \subseteq G_{(\text{aff } S)}$; so, they are equal. ■

Lemma 93 *Given $x \in \mathbf{R}^d$,*

1. $\forall A \subseteq \mathbf{R}^d, \mathcal{E}_{(A-x)} = T_x \mathcal{E}_{(A)} T_{-x}$
2. $\forall G \leq \mathcal{E}, (\text{fix}_{\mathbf{R}^d} G) - x = \text{fix}_{\mathbf{R}^d} (T_x G T_{-x})$.

Proof. We first establish some identities on the fixator and stabilizer of single points. $\forall y \in \mathbf{R}^d, g \in G$,

$$y \in x + \text{fix}_{\mathbf{R}^d} g \Leftrightarrow T_x y \in \text{fix}_{\mathbf{R}^d} g \Leftrightarrow T_x y = g T_x y \Leftrightarrow g \in \mathcal{E}_{y-x}.$$

Also.

$$\begin{aligned}
T_x y = g T_x y &\Leftrightarrow y = (T_{-x} g T_x) y \\
&\Leftrightarrow y \in \text{fix}_{\mathbf{R}^d} T_{-x} g T_x \\
&\Leftrightarrow T_{-x} g T_x \in \mathcal{E}_y \\
&\Leftrightarrow g \in T_x \mathcal{E}_y T_{-x}.
\end{aligned}$$

These identities give two results: namely,

$$\mathcal{E}_{y-x} = T_x \mathcal{E}_y T_{-x} \text{ and} \quad (\text{A.4})$$

$$x + \text{fix}_{\mathbf{R}^d} g = \text{fix}_{\mathbf{R}^d} T_{-x} g T_x. \quad (\text{A.5})$$

Show (1): From (A.4), applying the definition of the stabilizer of a set, we get

$$\mathcal{E}_{(A-x)} = \bigcap_{y \in A} \mathcal{E}_{(y-x)} = \bigcap_{y \in A} T_x \mathcal{E}_y T_{-x} = T_x \mathcal{E}_{(A)} T_{-x}.$$

Show (2): From (A.5), similarly for the fixator of a set

$$x + \text{fix}_{\mathbf{R}^d} G = \bigcap_{y \in G} x + \text{fix}_{\mathbf{R}^d} g = \bigcap_{y \in G} \text{fix}_{\mathbf{R}^d} (T_{-x} g T_x) = \text{fix}_{\mathbf{R}^d} (T_{-x} G T_x).$$

Replacing x with $-x$ gives the result. ■

Corollary 94 *If U is a non-empty affine subspace in $\mathbf{R}^d$ and $x \in U$,*

$$\mathcal{E}_{(U)} = T_{-x} \mathcal{E}_{(\Delta U)} T_x.$$

Proof. From Part 1 above $\mathcal{E}_{(\Delta U)} = \mathcal{E}_{(U-x)} = T_{-x} \mathcal{E}_{(U)} T_x$. ■

Corollary 95 *Given $x \in \mathbf{R}^d$, the map, $f \mapsto T_{-x} f T_x$, $\forall f \in \mathcal{E}$ is an inner automorphism of $\mathcal{E}$ that carries $\mathcal{O}$ onto $\mathcal{E}_x$: i.e., $\mathcal{E}_x = T_{-x} \mathcal{O} T_x$ is isomorphic to $\mathcal{O}$.*

Proof. Choose $U = \{x\}$. Then, $\Delta U = \{0\}$ and $\mathcal{E}_{(\Delta U)} = \mathcal{O}$ and the result follows directly from Corollary 94. ■

Lemma 96 *If $H \in \mathcal{O}$ and $x \in \text{fix}(H)$ then $T_x H = H T_x$.*

Proof. If $Hx = x$, by Lemma 70, $HT_x = T_{Hx}H = T_xH$. ■

Definition 40 If U is a non-empty affine subspace in $\mathbf{R}^d$, define $U^\perp = \Delta U^\perp$, the orthogonal complement of the linear subspace ΔU .

Lemma 97 If A is a non-empty subset in $\mathbf{R}^d$ and $x \in \mathbf{R}^d$.

1. $\text{aff}(A - x) = (\text{aff } A) - x$.
2. $\Delta \text{aff}(A - x) = \Delta \text{aff } A$ and
3. $(\text{aff}(A - x))^\perp = (\text{aff } A)^\perp$.

Proof. Show (1):

$$\begin{aligned} \text{aff}(A - x) &= \left\{ \sum \alpha_i (a_i - x) \mid a_i \in A, \sum \alpha_i = 1 \right\} \\ &= \left\{ \sum \alpha_i a_i \mid a_i \in A, \sum \alpha_i = 1 \right\} - x \\ &= \text{aff } A - x. \end{aligned}$$

Show (2): $\Delta \text{aff}(A - x) = \text{aff}(A - x) - \text{aff}(A - x) = \text{aff } A - x - (\text{aff } A - x) = \Delta \text{aff } A$.

Show (3): $(\text{aff}(A - x))^\perp \equiv (\Delta \text{aff}(A - x))^\perp = (\Delta \text{aff } A)^\perp \equiv (\text{aff } A)^\perp$, where the second equality is from Part 2. ■

Lemma 98 If $\mathcal{G} \leq \mathcal{O}$ and $x \in \mathbf{R}^d$ and $H \in \mathcal{G}$ then

1. $\text{stab}_{\mathcal{G}} Hx = H(\text{stab}_{\mathcal{G}} x)H^{-1}$ and
2. $\text{fix stab } Hx = H \text{ fix stab } x$.

Proof. Show 1: If $L \in \text{stab } x$ then

$$(HLH^{-1})(Hx) = HLx = Hx$$

$$\begin{aligned}\therefore H L H^{-1} &\in \text{stab } Hx \\ \therefore H(\text{stab } x) H^{-1} &\subseteq \text{stab } Hx.\end{aligned}$$

Conversely, if $L \in \text{stab } Hx$, then

$$L H x = H x \text{ or } (H^{-1} L H) x = x$$

or $H^{-1} L H \in \text{stab } x$ or $L \in H(\text{stab } x) H^{-1}$. $\therefore \text{stab } Hx \subseteq H(\text{stab } x) H^{-1}$.

Show 2: If $v \in \text{fix stab } Hx$ then $\forall H L H^{-1} \in \text{stab } Hx = H(\text{stab } x) H^{-1}$ where $L \in \text{stab } x$ we have $H L H^{-1} v = v$ and $L(H^{-1} v) = H^{-1} v$. Since L is an arbitrary member of $\text{stab } x$, $H^{-1} v \in \text{fix stab } x$ or $v \in H \text{ fix stab } x$. $\therefore \text{fix stab } Hx \subseteq H \text{ fix stab } x$. Conversely, $\forall H y \in H \text{ fix stab } x$; i.e., $\forall y \in \text{fix stab } x$, $\forall L \in \text{stab } Hx$, since $y \in \text{fix stab } x$, $H^{-1} L H \in \text{stab } x$. $\therefore H^{-1} L H y = y$ and $L(H y) = H y$. Since L is an arbitrary member of $\text{stab } Hx$, $H y \in \text{fix stab } Hx$. ■

A.4 Centroids

In geometry, one has a theorem that says that a finite subgroup of $\mathcal{E}$ has a fixed point [C1, Section 3.41, p. 44], [C2, Section 15.41, p. 273], [Yal, p. 90, Thm. 3.5]. One text [Mlr, Thm. 2.3, p. 24] requires the group G to be discrete (i.e., one for which a bounded set contains only finitely many points of a G -orbit) and a non-empty bounded set to be invariant under G . These elementary theorems are almost but not quite what we need for our potentially infinite groups of position and velocity isometries that preserve an ivp. We first remind the reader that the **centroid** of a finite set, $X = \{x_1, \dots, x_n\} \subseteq \mathbf{R}^d$ is defined to be $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$. Since $\sum_{i=1}^n \frac{1}{n} = 1$, $\bar{x} \in \text{aff } X$.

Lemma 99 *If $G \leq \mathcal{E}$ and G has a finite orbit, then the centroid of that orbit is a fixed point of G ; i.e., if $|Gx| < \infty$ for some $x \in \mathbf{R}^d$, $\bar{x} \in \text{fix}_{\text{aff } Gx} G$.*

Proof. Suppose $Gx = \{x_1, \dots, x_n\}$ for some $x \in \mathbf{R}^d$. Then the centroid, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \in \text{aff } Gx$. $\forall g \in G$, $\{g(x_1), \dots, g(x_n)\} = g(Gx) = (gG)x = Gx$. Since isometries are affine maps (L. 85, p. 210) $g(\bar{x}) = \frac{1}{n} \sum g(x_i) = \frac{1}{n} \sum x_i = \bar{x}$. Thus $\bar{x} \in \text{fix}_{\text{aff } Gx} G$. ■

The abovementioned result given in geometry texts follows from this theorem, namely:

Corollary 100 *A finite subgroup of isometries has a fixed point.*

Proof. If G is finite then $\forall x \in \mathbf{R}^d$, Gx is finite. ■

Note that if f and g are isometries, by $f + g$ we mean the function given by $(f + g)(x) \equiv f(x) + g(x)$.

Theorem 101 *If G is a finite subgroup of $\mathcal{E}$ the function*

$$\begin{aligned} \theta_G &: \mathbf{R}^d \rightarrow \mathbf{R}^d \text{ given by} \\ \theta_G &= \frac{1}{|G|} \sum_{g \in G} g \end{aligned}$$

maps every x in $\mathbf{R}^d$ to the centroid of its orbit Gx .

Proof. Since G is finite we can let $n = |Gx|$ and $Gx = \{x_1, \dots, x_n\}$. Then $\exists g_1, \dots, g_n \in G$ such that $\forall i \leq n$, $x_i = g_i x$. Since the x_i 's are distinct, $\forall j, l \leq n$,

$$\begin{aligned} j = l &\Leftrightarrow x_j = x_l \\ &\Leftrightarrow g_j x = g_l x \\ &\Leftrightarrow g_l^{-1} g_j x = x \\ &\Leftrightarrow g_l^{-1} g_j \in G_x \\ &\Leftrightarrow g_j G_x = g_l G_x. \end{aligned}$$

Thus the cosets $\{g_j G_x | j = 1, n\}$ are distinct. Also, $\forall g \in G$, $gx = g_i x$ for some $i \leq n$: so, $g \in g_i G_x$ and we can write $G = \cup_{i=1}^n g_i G_x = \{g_i h | i = 1, n \text{ and } h \in G_x\}$

and $|G| = |Gx| \cdot |G_x|$.

$$\begin{aligned} \therefore \theta_G(x) &= \frac{1}{|G|} \sum_{g \in G} g(x) = \frac{1}{|Gx| \cdot |G_x|} \sum_{j=1}^n \sum_{h \in G_x} g_j h(x) \\ &= \frac{1}{|Gx|} \sum_{j=1}^n g_j(x) = \frac{1}{n} \sum_{j=1}^n x_j = \bar{x}, \end{aligned}$$

the centroid. ■

A.5 Some Group Theory

The author cannot find this little lemma in his texts: so, it is included here for completeness:

Lemma 102 *If $H = H_1 \times H_2 \leq G$ is an internal direct product and $g \in G$ then*

$$g^{-1}Hg = (g^{-1}H_1g) \times (g^{-1}H_2g).$$

also an internal direct product of subgroups of G .

Proof. It is elementary that $g^{-1}H_i g \leq g^{-1}Hg \leq G$, $i = 1, 2$. As sets, $(g^{-1}H_1g) \cdot (g^{-1}H_2g) = g^{-1}Hg = g^{-1}H_2H_1g = (g^{-1}H_2g) \cdot (g^{-1}H_1g)$, $\forall f \in g^{-1}Hg$, $f = g^{-1}h_1h_2g$, where $h_i \in H_i$. Then, $f^{-1}(g^{-1}H_1g)f = g^{-1}h_2^{-1}h_1^{-1}gg^{-1}H_1gg^{-1}h_1h_2g = g^{-1}h_2^{-1}H_1h_2g = g^{-1}H_1g$ because $H_1 \triangleleft H$. $\therefore g^{-1}H_1g \triangleleft g^{-1}Hg$. A similar argument using the fact that $h_1h_2 = h_2h_1$ gives $g^{-1}H_2g \triangleleft g^{-1}Hg$. Also $(g^{-1}H_1g) \cap (g^{-1}H_2g) = g^{-1}(H_1 \cap H_2)g = \{1_G\}$. ■

Appendix B

Symmetry Groups in $\mathbf{R}^d$

In this appendix we develop some theory which strictly speaking belongs in the realm of geometry but is tailored specifically to the development of the group of the N -body problem. Appendix A contains an introduction to groups acting on sets, material that can be found in introductory algebra texts (e.g., [Jcb], [Gi]). It also contains the application to isometries of groups acting on sets. We now develop some basic theory on this subject that applies directly to the problem at hand but which the author has been unable to find elsewhere. We begin with a notion similar to global and pointwise invariance discussed in [Yal, p 11, Thm 1.8].

Remark 6 *In this appendix we shall use almost exclusively $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{T}$ for the Euclidean, orthogonal and translation groups acting on $\mathbf{R}^d$, respectively in place of $\mathcal{E}(d)$, $\mathcal{O}(d)$ and $\mathcal{T}(d)$, where it is clear that the dimension of the Euclidean space is d . We shall also use $H_{(A)}$ for the point-wise stabilizer of A in H in place of $\text{stab}_H A$.*

B.1 The Set-wise Stabilizer

Given a homomorphism $\phi : G \rightarrow \text{Sym } S$ defining the action of a group G on a set S and given $H \leq G$ then $\phi|_H$ is also a homomorphism that defines an action of H on

S and one says that ‘ H acts on S by restriction’ (see App. A, Section A.1, p. 202). We now give another kind of action by restriction. Given such a group, G acting on S suppose, for some $A \subseteq S$ and $H \leq G$, that $\forall h \in H, \phi(h)(A) = A$. Let

$$\begin{aligned}\zeta : H &\rightarrow \text{Sym } A \quad \text{be given by} \\ \zeta(h) &= \phi(h)|_A.\end{aligned}$$

Since $\phi(h) \in \text{Sym } S$, $\zeta(h) = \phi(h)|_A$ is injective and since the image, $\phi(h)(A) = h(A) = A$ by hypothesis, then $\zeta(h) : A \rightarrow A$ is also a bijection; so, ζ is a map from H to $\text{Sym } A$ as claimed; i.e., ζ is well defined and we can say that H acts on A by restriction. In the light of this and the definition of the point-wise stabilizer of A in H : i.e., $\text{stab}_H A = H_{(A)}$ (Def. 34, p. 203, App. A), we make the following

Definition 41 *If a group G acts on a set S , $H \leq G$ and $A \subseteq S$, we define the **set-wise stabilizer** of A in H to be*

$$H_{\{A\}} = \{h \in H | h(A) = A\}. \quad (\text{B.1})$$

If $A = \{a\}$, a singleton then clearly the set and point-wise stabilizers are the same and we simply write, H_a for $H_{(A)} = H_{\{A\}}$.

In some texts, $H_{\{A\}}$ is also called the **complete symmetry group** of A in H [MLr] in particular where the group is an isometry group on a Euclidean space about which we shall have more to say in the next section.

Lemma 103 *If a group G acts on a set S , $H \leq G$ and $A \subseteq S$, then $H_{\{A\}}$ is a subgroup of H that contains every subgroup of H that acts on A by restriction; i.e., $\exists$ a group homomorphism $\zeta : H_{\{A\}} \rightarrow \text{Sym } A$ given by $\zeta(h)(a) = ha$, $\forall h \in H_{\{A\}}$, $a \in A$ and in particular,*

$$H_{(A)} \leq H_{\{A\}} \leq G.$$

Proof. In the above definition we used the notation $h(A) = \phi(h)(A)$ for the homomorphism, ϕ defining the group action. Clearly $1_H \in H_{\{A\}}$ and $\forall g \in H_{\{A\}}$, since $\phi(g)|_A$ is a bijection, $g^{-1}(A) = A$: i.e., $g^{-1} \in H_{\{A\}}$. Also, if $g_1 g_2 \in H_{\{A\}}$, $(g_1 g_2)(A) = g_1(g_2(A)) = g_1(A) = A$: so, $g_1 g_2 \in H_{\{A\}}$. Thus, $H_{\{A\}} \leq H$ and clearly contains every other subgroup of H that acts on A by restriction. Since for any $g \in H_{\{A\}} \equiv \text{stab}_H A$, g leaves every member of A fixed, $g(A) = A$ and $H_{\{A\}}$ acts on A by restriction. ■

Clearly if $H \leq G$ and $A \subseteq S$ then $H_{\{A\}} = H \cap G_{\{A\}}$ from which it is obvious that

Lemma 104 *If G acts on S , $A \subseteq S$ and $H \leq K \leq G$ then $H_{\{A\}} \leq K_{\{A\}}$*

Lemma 105 *If G acts on S , $H \leq G$ and $A \subseteq S$ then $H \leq G_{\{HA\}}$.*

Proof. $\forall h \in H$, $h(HA) = \{hHx | x \in A\} = \{Hx | x \in A\} = HA$. $\therefore h \in G_{\{HA\}}$. Since h is an arbitrary member of H , $H \leq G_{\{HA\}}$. ■

In particular, if A is a singleton, say $A = \{x\}$, then $H \leq G_{\{Hx\}}$.

Lemma 106 *If G acts on S and $A, B \subseteq S$, then*

1. $G_{\{A\}} = G_{\{S \setminus A\}}$.
2. $G_{\{A\}} \cap G_{\{B\}} \leq G_{\{A \cup B\}}$ and
3. $G_{\{A\}} \cap G_{\{B\}} \leq G_{\{A \cap B\}}$.

Proof. *Show 1:* If ϕ is the homomorphism defining the group action then $\forall g \in G$, $\phi(g)$ is a bijection on S : so, $g(A) = A \Leftrightarrow g(S \setminus A) = S \setminus A$.

Show 2: If $g \in G_{\{A\}} \cap G_{\{B\}}$, then $g(A \cup B) = g(A) \cup g(B) = A \cup B$. $\therefore g \in G_{\{A \cup B\}}$.

Show 3: Similarly, if $g \in G_{\{A\}} \cap G_{\{B\}}$, then since $\phi(g)$ is 1-1, $g(A \cap B) = g(A) \cap g(B) = A \cap B$: so, $g \in G_{\{A \cap B\}}$. ■

B.2 Application to Isometries

As mentioned in Section A.3, since $\mathcal{E} \leq \text{Sym}(\mathbf{R}^d)$, the group action is an embedding; i.e., there is an isomorphic copy of $\mathcal{E}$ in $\text{Sym}(\mathbf{R}^d)$ and each isometry in $\mathcal{E}$ is carried to itself under the map defining the group action. Thus if ϕ is the group homomorphism defining the action and $x \in \mathbf{R}^d$ then what we mean by $\phi(f)(x)$, which for groups acting on sets is usually written fx , is just $f(x)$. In the case of $\mathcal{E}$ acting on $\mathbf{R}^d$ if $A \subseteq \mathbf{R}^d$, $\mathcal{E}_{\{A\}}$ is the group of isometries for which A is globally invariant. Yale [Yal. p.11] denotes it $[G, A; gi]$ and Miller [Mlr. Section 2.3, p. 23] calls it the *complete symmetry group* of A . If U is an affine subspace in $\mathbf{R}^d$, then by Lemma 86, p. 210, App. A, $g(U) \subseteq U \Leftrightarrow g(U) = U$; hence we have shown,

Lemma 107 *If U is an affine subspace in $\mathbf{R}^d$,*

$$\mathcal{E}_{\{U\}} = \{g \in \mathcal{E} | g(U) \subseteq U\}. \quad (\text{B.2})$$

Recalling Def. 36, p. 207 that for $A \subseteq \mathbf{R}^d$, $\Delta A = A - A$ we have:

Lemma 108 *If $A \subseteq \mathbf{R}^d$ and $f \in \mathcal{E}_{\{A\}}$ where H is the orthogonal part of f then $H \in \mathcal{E}_{\{\Delta A\}}$.*

Proof. $\forall z = x - y \in \Delta A$ where $x, y \in A$. $H z = H x - H y = f(x) - f(y) \in \Delta A$. ■

Note that the above lemma holds even if A is empty.

Lemma 109 *If $G \leq \mathcal{E}$ and A is a non-empty subset of $\mathbf{R}^d$ then*

$$G_{\{A\}} \leq G_{\{\text{aff } A\}}.$$

Proof. $\forall g \in G_{\{A\}}$ and $\forall x \in \text{aff } A$, $x = \sum_{i=1}^n \alpha_i a_i$ where $\sum_{i=1}^n \alpha_i = 1$ and each $a_i \in A$, $\forall i \leq n$, $g(a_i) \in A$ and by L. 85, p. 210, App. A, $g(x) = \sum \alpha_i g(a_i) \in \text{aff } A$; so, $g(\text{aff } A) \subseteq \text{aff } A$ and $g \in G_{\{\text{aff } A\}}$; hence $G_{\{A\}} \leq G_{\{\text{aff } A\}}$. ■

Figure B-1: Stabilizer of an Affine Subspace

Corollary 110 *If $G \leq \mathcal{E}$ and A is a non-empty subset of $\mathbf{R}^d$ then*

$$G \leq \mathcal{E}_{\{GA\}} \leq \mathcal{E}_{\{\text{aff } GA\}}.$$

Proof. This is a direct result of Lemmas 105 and 109. ■

Applying L. 103 p. 220 to an affine subspace, U in $\mathbf{R}^d$ gives $\mathcal{E}_{(U)} \leq \mathcal{E}_{\{U\}}$. We now show the related result:

Lemma 111 *If U is an affine subspace in $\mathbf{R}^d$, $\mathcal{E}_{(U^\perp)} \leq \mathcal{O}_{\{U\}} \leq \mathcal{E}_{\{U\}}$.*

Proof. Recall from D. 40 p. 215 App. A that $U^\perp \equiv (\Delta U)^\perp$. Since $0 \in U^\perp$, every member of $\mathcal{E}_{(U^\perp)}$ fixes the origin: so, $\mathcal{E}_{(U^\perp)} \leq \mathcal{O}$ (L. 68, p. 201, App. A). Let $H \in \mathcal{E}_{(U^\perp)}$ and let $x \in U$. Let x_0 be the point in U closest to 0. (See Figure B-1). Then $x_0 \in U^\perp$ and $Hx_0 = x_0$. Since $H(U^\perp) = U^\perp$, $H(\Delta U) = \Delta U$ (L. 91 p. 212 App. A). $\therefore (Hx) - x_0 = H(x - x_0) \in \Delta U$; so, $Hx \in x_0 + \Delta U = U$; i.e., $H(U) \subseteq U$ and $H \in \mathcal{O}_{\{U\}}$ (L. 107). Since $\mathcal{O} \leq \mathcal{E}$, $\mathcal{O}_{\{U\}} \leq \mathcal{E}_{\{U\}}$ (L. 104, p. 221). ■

Recall that $\mathcal{T}$ is the group of translations on $\mathbf{R}^d$ (see App. A).

Lemma 112 *If U is a non-empty affine subspace of $\mathbf{R}^d$ and $y \in \mathbf{R}^d$ then*

$$\mathcal{T}_{\{U-y\}} = \mathcal{T}_{\{U\}} = \mathcal{T}_{\{\Delta U\}} = \{T_x | x \in \Delta U\} \quad (\text{B.3})$$

Proof. $T_x \in \mathcal{T}_{\{U-y\}} \Leftrightarrow U-y-x \equiv T_x(U-y) = U-y \Leftrightarrow x \in \Delta U$. $\therefore \mathcal{T}_{\{U-y\}} = \{T_x | x \in \Delta U\}$, independent of the choice of y . In particular, for $y = 0$ and $y \in U$ we have $\mathcal{T}_{\{U-y\}} = \mathcal{T}_{\{U\}}$ and $\mathcal{T}_{\{\Delta U\}}$ respectively, which gives the result. ■

Definition 42 *If A is a non-empty subset of $\mathbf{R}^d$, we define*

$$\mathcal{E}_{\{A\}}^{\text{ess}} = \left(\mathcal{T}_{\{\text{aff } A\}} \mathcal{E}_{(\text{aff } A -)} \right) \cap \mathcal{E}_{\{A\}}. \quad (\text{B.4})$$

Since $\mathcal{E}_{\{A\}} \leq \mathcal{E}_{\{\text{aff } A\}}$ (L. 109), $\mathcal{E}_{\{A\}} = \mathcal{E}_{\{\text{aff } A\}} \cap \mathcal{E}_{\{A\}}$ and

$$\mathcal{E}_{\{A\}}^{\text{ess}} = \left(\mathcal{T}_{\{\text{aff } A\}} \mathcal{E}_{(\text{aff } A -)} \right) \cap \mathcal{E}_{\{\text{aff } A\}} \cap \mathcal{E}_{\{A\}} = \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \cap \mathcal{E}_{\{A\}}. \quad (\text{B.5})$$

Note also that $\mathcal{T}$ is an abelian subgroup of $\mathcal{E}$ and forms a linear space for the scalar product, $kT_x \equiv T_{kx}$ for $k \in \mathbf{R}$, $x \in \mathbf{R}^d$. Clearly, $\mathbf{R}^d \approx \mathcal{T}$ (vector space isomorphism) under the map $x \mapsto T_x$. If $W \leq V \leq \mathbf{R}^d$ (linear subspaces) then $\mathcal{T}_{\{W\}}$ and $\mathcal{T}_{\{V\}}$ are the images of W and V , respectively under this isomorphism and this gives the following elementary

Lemma 113 *If $W \leq V \leq \mathbf{R}^d$, $\mathcal{T}_{\{W\}} \leq \mathcal{T}_{\{V\}} \leq \mathcal{T}$ (linear subspaces).*

In this Lemma, since $\mathcal{T}_{\{W\}}$ is a linear subspace of $\mathcal{T}$ it is also a subgroup under vector addition which in this space, means the composition of (translation) functions. With Lemma 73, p. 202 this gives $\mathcal{T}_{\{W\}} \triangleleft \mathcal{T} \triangleleft \mathcal{E}$ (but *not* in general, $\mathcal{T}_{\{W\}} \triangleleft \mathcal{E}$).

Now consider any affine subspace, U . Since $\mathcal{E}_{\{U-\}} \leq \mathcal{E}_{\{U\}}$ (L. 111) and $\mathcal{T}_{\{U\}} \leq \mathcal{E}_{\{U\}}$ (L. 104) then $\mathcal{T}_{\{U\}} \mathcal{E}_{\{U-\}} \subseteq \mathcal{E}_{\{U\}}$. This product, $\mathcal{T}_{\{U\}} \mathcal{E}_{\{U-\}}$ of subgroups of $\mathcal{E}_{\{U\}}$ will be a subgroup if and only if $\mathcal{T}_{\{U\}} \mathcal{E}_{\{U-\}} = \mathcal{E}_{\{U-\}} \mathcal{T}_{\{U\}}$, (see e.g. [BJN, p. 74, Thm. 3.5]). To show this, we consider any $H \in \mathcal{E}_{\{U^\perp\}} = \mathcal{O}_{\{U^\perp\}}$ and $T_y \in \mathcal{T}_{\{U\}}$. Then $y \in \Delta U$ (L. 112) and $Hy \in \Delta U$ (L. 91, p. 212, App. A), $T_{Hy} \in \mathcal{T}_{\{U\}}$ (L. 112)

and $HT_y = T_{Hy}H \in \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)}$ and we have $\mathcal{E}_{(U^\perp)}\mathcal{T}_{\{U\}} \subseteq \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)}$. Similarly, any $T_yH \in \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)}$ satisfies $T_yH = HT_{H^{-1}y} \in \mathcal{E}_{(U^\perp)}\mathcal{T}_{\{U\}}$.

$$\therefore \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)} = \mathcal{E}_{(U^\perp)}\mathcal{T}_{\{U\}}.$$

Thus, $\mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)} \leq \mathcal{E}_{\{U\}}$; hence $\mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)} = \left(\mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)}\right) \cap \mathcal{E}_{\{U\}} = \mathcal{E}_{\{U\}}^{\text{ess}}$ and we have shown

Lemma 114 *If U is a non-empty affine subspace of $\mathbf{R}^d$,*

$$\mathcal{E}_{\{U\}}^{\text{ess}} = \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)} \leq \mathcal{E}_{\{U\}}.$$

Then from (B.5) we have

Corollary 115 *For any non-empty set $A \subseteq \mathbf{R}^d$,*

$$\mathcal{E}_{\{A\}}^{\text{ess}} = \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \cap \mathcal{E}_{\{A\}} \leq \mathcal{E}_{\{A\}}.$$

Lemma 116 *If U is an affine subspace in $\mathbf{R}^d$ and A is a non-empty subset of U , then $\mathcal{E}_{\{A\}}^{\text{ess}} \leq \mathcal{E}_{\{U\}}^{\text{ess}}$.*

Proof. Since $\langle \Delta A \rangle \leq \langle \Delta U \rangle = \Delta U$, we have from Lemmas 112 and 113

$$\mathcal{T}_{\{\text{aff } A\}} = \mathcal{T}_{\{\langle \Delta A \rangle\}} \leq \mathcal{T}_{\{\Delta U\}} = \mathcal{T}_{\{U\}}.$$

Combining Def. 40, p. 215 with $\langle \Delta A \rangle \leq \langle \Delta U \rangle$ we get

$$\text{aff } A^\perp = \langle \Delta A \rangle^\perp \supseteq \Delta U^\perp = U^\perp,$$

which gives $\mathcal{E}_{(\text{aff } A^\perp)} \leq \mathcal{E}_{(U^\perp)}$ (the point-wise stabilizer of a larger set is smaller).

$$\therefore \mathcal{E}_{\{A\}}^{\text{ess}} \leq \mathcal{T}_{\{\text{aff } A\}}\mathcal{E}_{(\text{aff } A^\perp)} \leq \mathcal{T}_{\{U\}}\mathcal{E}_{(U^\perp)} = \mathcal{E}_{\{U\}}^{\text{ess}}$$

from L. 114. ■

B.3 The Symmetry Group as a Direct Product

In this section, we shall provide a theorem that is important in determining the group of an initial value problem but we ease our way into it with the following special case:

Lemma 117 *If U is a non-empty affine subspace of $\mathbf{R}^d$, then*

$$\mathcal{E}_{\{U\}} = \mathcal{E}_{\{U\}}^{\text{ess}} \times \mathcal{E}_{(U)}.$$

an internal direct product of two subgroups of $\mathcal{E}$.

Proof. First show that $\mathcal{E}_{\{U\}} = \mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)} = \mathcal{E}_{(U)} \mathcal{E}_{\{U\}}^{\text{ess}} \leq \mathcal{E}$: By L. 103, p. 220 $\mathcal{E}_{(U)} \leq \mathcal{E}_{\{U\}} \leq \mathcal{E}$ and by L. 114, $\mathcal{E}_{\{U\}}^{\text{ess}} \leq \mathcal{E}_{\{U\}}$; so, $\mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)} \subseteq \mathcal{E}_{\{U\}}$. We now show the reverse inclusion. Fix $g = T_r H \in \mathcal{E}_{\{U\}}$ where $r \in \mathbf{R}^d$ and $H \in \mathcal{O}$. $\forall v \in \Delta U$, $v = x - y$ for some $x, y \in U$. Since $g \in \mathcal{E}_{\{U\}}$, $g(x)$ and $g(y) \in U$ and $Hv = H(x - y) = g(x) - g(y) \in \Delta U$. $\therefore H(\Delta U) \subseteq \Delta U$. By Lemma 91, p. 212, App. A, $H = H_1 H_2 = H_2 H_1$ where $H_1 \in \mathcal{E}_{(U^+)}$, $H_2 \in \mathcal{E}_{(\Delta U)}$. Let x_0 be the point in U closest to 0. Then by inspection

$$g = (T_{-g(x_0)+x_0} H_1) (T_{-x_0} H_2 T_{x_0}). \quad (\text{B.6})$$

One sees this by observing that the right side of (B.6) has H , the orthogonal part of g , as its orthogonal part and when evaluated at x_0 gives $g(x_0)$. By Lemma 94, p. 214, App. A, $T_{-x_0} H_2 T_{x_0} \in \mathcal{E}_{(U)}$. Also since $x_0, g(x_0) \in U$, $\therefore x_0 - g(x_0) \in \Delta U$. $T_{-g(x_0)+x_0} H_1 \in \mathcal{T}_{\{U\}} \mathcal{E}_{(U^+)} = \mathcal{E}_{\{U\}}^{\text{ess}}$ by L. 114. $\therefore g \in \mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)}$, $\mathcal{E}_{\{U\}} \subseteq \mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)}$ and we have shown that, as sets, $\mathcal{E}_{\{U\}} = \mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)}$. Since $\mathcal{E}_{\{U\}} \leq \mathcal{E}$ (L. 103, p. 220) then $\mathcal{E}_{\{U\}} = \mathcal{E}_{\{U\}}^{\text{ess}} \mathcal{E}_{(U)} = \mathcal{E}_{(U)} \mathcal{E}_{\{U\}}^{\text{ess}} \leq \mathcal{E}$ (see for example [BJN, Thm. 3.5, p. 74]).

Show that every member of $\mathcal{E}_{\{U\}}^{\text{ess}}$ commutes with every member of $\mathcal{E}_{(U)}$, hence, $\mathcal{E}_{\{U\}}^{\text{ess}}$ and $\mathcal{E}_{(U)} \triangleleft \mathcal{E}_{\{U\}}$: Again, let x_0 be the point in U closest to 0. $\forall f = T_x H_1 \in \mathcal{E}_{\{U\}}^{\text{ess}}$ (by Def. 42), $x \in \Delta U$ and $H_1 \in \mathcal{E}_{(U^+)}$. By Corollary 94, p. 214, Ch. A, any

$g \in \mathcal{E}_{(U)}$ has the form $g = T_{-x_0} H_2 T_{x_0}$ where $H_2 \in \mathcal{E}_{(\Delta U)}$. $\therefore H_2 x = x$ which means $T_x H_2 = H_2 T_x$ and likewise, since $x_0 \in U^\perp$, $T_{x_0} H_1 = H_1 T_{x_0}$ and $T_{-x_0} H_1 = H_1 T_{-x_0}$. Also, since $H_1 H_2 = H_2 H_1$, we have $fg = T_x H_1 T_{-x_0} H_2 T_{x_0} = T_{-x_0} H_2 T_{x_0} T_x H_1 = gf$.

We now show that $\mathcal{E}_{\{U\}}^{\text{ess}} \cap \mathcal{E}_{(U)} = \{1_{\mathcal{E}}\}$: Let $f = T_x H \in \mathcal{E}_{\{U\}}^{\text{ess}} \cap \mathcal{E}_{(U)}$. Since $T_x H \in \mathcal{E}_{\{U\}}^{\text{ess}}$, $x \in \Delta U$ and $H \in \mathcal{E}_{(U)}$. Also, since $f \in \mathcal{E}_{(U)}$, $H \in \mathcal{E}_{(\Delta U)}$. $\therefore H = I_d = 1_{\mathcal{E}}$. Since $U \neq \emptyset$, $\mathcal{E}_{(U)} \cap \mathcal{T} = \{1_{\mathcal{E}}\}$ because any translation that fixes a point is the identity map. $\therefore x = 0$ and $f = 1_{\mathcal{E}}$. The result follows from the definition of an internal direct product. $\blacksquare$

The above lemma is subsumed by the following generalization in which we replace U with an arbitrary non-empty set, A .

Theorem 118 (Structure Theorem for the Set-wise Stabilizer) *If A is a non-empty subset of $\mathbf{R}^d$,*

$$\mathcal{E}_{\{A\}} = \mathcal{E}_{\{A\}}^{\text{ess}} \times \mathcal{E}_{(A)},$$

an internal direct product of two subgroups of $\mathcal{E}$.

Proof. Since $A \subseteq \text{aff } A$

$$\begin{aligned} \mathcal{E}_{(\text{aff } A)} &= \mathcal{E}_{(A)} && \text{(by L. 92, p. 213, App. A)} \\ &\leq \mathcal{E}_{\{A\}} && \text{(by L. 103, p. 220)} \\ &\leq \mathcal{E}_{\{\text{aff } A\}} && \text{(by L. 109, p. 222)} \\ &= \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff } A)} && \text{(by L. 117, above).} \\ &= \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \times \mathcal{E}_{(A)} \end{aligned}$$

a direct product of subgroups of $\mathcal{E}$.

Thus $\mathcal{E}_{\{A\}} \leq \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff } A)}$ and any $g \in \mathcal{E}_{\{A\}}$ has the form $g = g_1 g_2$ where $g_1 \in \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$ and $g_2 \in \mathcal{E}_{(\text{aff } A)}$. Thus $g_2^{-1} \in \mathcal{E}_{(\text{aff } A)} \leq \mathcal{E}_{\{A\}}$ (from the first two steps in the above display); so, $g_1 = g g_2^{-1} \in \mathcal{E}_{\{A\}}$. $\therefore g_1 \in \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \cap \mathcal{E}_{\{A\}} = \mathcal{E}_{\{A\}}^{\text{ess}}$ (by C. 115) and $\mathcal{E}_{\{A\}} \subseteq \mathcal{E}_{\{A\}}^{\text{ess}} \mathcal{E}_{(\text{aff } A)}$. Conversely, by C. 115, $\mathcal{E}_{\{A\}}^{\text{ess}} \leq \mathcal{E}_{\{A\}}$. $\therefore \mathcal{E}_{\{A\}}^{\text{ess}} \mathcal{E}_{(\text{aff } A)} \subseteq \mathcal{E}_{\{A\}}$ and $\mathcal{E}_{\{A\}} = \mathcal{E}_{\{A\}}^{\text{ess}} \mathcal{E}_{(\text{aff } A)}$. So, from L. 116,

$$\mathcal{E}_{\{A\}}^{\text{ess}} \leq \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \triangleleft \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \times \mathcal{E}_{(\text{aff } A)}$$

and every member of $\mathcal{E}_{\{A\}}^{\text{ess}}$ lies inside $\mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$ and hence commutes with every member of $\mathcal{E}_{\{\text{aff } A\}}$; so, $\mathcal{E}_{\{A\}}^{\text{ess}} \mathcal{E}_{\{\text{aff } A\}}$ is also a direct product. ■

In the light of the above structure theorem we make the following:

Definition 43 $\mathcal{E}_{\{A\}}^{\text{ess}}$ is called the **essential part** of the set-wise stabilizer, $\mathcal{E}_{\{A\}}$ and the point-wise stabilizer, $\mathcal{E}_{(A)}$ is also called the **inessential part** of $\mathcal{E}_{\{A\}}$.

Theorem 119 If A is a non-empty subset of $\mathbf{R}^d$ and $G \leq \mathcal{E}_{\{A\}}$, then

$$G \cap \mathcal{E}_{\{A\}}^{\text{ess}} = G \cap \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}.$$

Proof.

$$\begin{aligned} G \cap \mathcal{E}_{\{A\}}^{\text{ess}} &= G \cap \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \cap \mathcal{E}_{\{A\}} \quad \text{by C. 115, p. 225} \\ &= G \cap \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \end{aligned}$$

because $G \cap \mathcal{E}_{\{A\}} = G$. ■

Theorem 120 If A is a non-empty subset of $\mathbf{R}^d$, $x \in \mathbf{R}^d$,

1. $\mathcal{E}_{\{\text{aff } (A-x)\}}^{\text{ess}} = \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$,
2. $\mathcal{E}_{\{A-x\}} = T_x \mathcal{E}_{\{A\}} T_{-x}$,
3. $\mathcal{E}_{\{A-x\}}^{\text{ess}} = T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x}$,
4. $\mathcal{E}_{(A-x)} = T_x \mathcal{E}_{(A)} T_{-x}$ and
5. $\mathcal{E}_{\{A-x\}} = \left(T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x} \right) \times \left(T_x \mathcal{E}_{(A)} T_{-x} \right)$, an internal direct product.

Proof. Show (1):

$$\begin{aligned} \mathcal{E}_{\{\text{aff } (A-x)\}}^{\text{ess}} &= \mathcal{T}_{\{\langle \Delta(A-x) \rangle\}} \mathcal{E}_{(\text{aff } (A-x))^-} \quad (\text{L. 114, p. 225}) \\ &= \mathcal{T}_{\{\langle \Delta A \rangle\}} \mathcal{E}_{(\langle \Delta A \rangle)^-} \quad (\text{L. 97, p. 215, App. A}) \\ &= \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}. \quad (\text{L. 114, p. 225}) \end{aligned}$$

Show (2):

$$\begin{aligned}
f \in \mathcal{E}_{\{A-x\}} &\Leftrightarrow f(A-x) = A-x \\
&\Leftrightarrow T_{-x}fT_x(A) = A \\
&\Leftrightarrow T_{-x}fT_x \in \mathcal{E}_{\{A\}} \\
&\Leftrightarrow f \in T_x\mathcal{E}_{\{A\}}T_{-x}.
\end{aligned}$$

Show (3): Let $x = x_1 + x_2$ where $x_1 \in \langle \Delta A \rangle$, $x_2 \in \langle \Delta A \rangle^\perp$. $\forall f = T_r H \in \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$:
i.e., $r \in \langle \Delta A \rangle$, $H \in \mathcal{E}_{(\langle \Delta A \rangle)^\perp}$, $Hx_2 = x_2$ and $Hx_1 \in \langle \Delta A \rangle$.

$$\begin{aligned}
\therefore T_x f T_{-x} &= T_x T_r H T_{-x} = T_{x_1+x_2+r-Hx_1-Hx_2} H \\
&= T_{x_1+r-Hx_1} H \in \mathcal{T}_{\{\langle \Delta A \rangle\}} \mathcal{E}_{(\langle \Delta A \rangle)^\perp} = \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}.
\end{aligned}$$

$\therefore T_x \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} T_{-x} \subseteq \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$. Replacing x with $-x$ gives the reverse inclusion: so,

$$T_x \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} T_{-x} = \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} = \mathcal{E}_{\{\text{aff } (A-x)\}}^{\text{ess}} \quad (\text{B.7})$$

by Part (1): i.e., $\mathcal{T} \leq \mathcal{N}(\mathcal{E}_{\{\text{aff } A\}}^{\text{ess}})$, the normalizer of $\mathcal{E}_{\{\text{aff } A\}}^{\text{ess}}$.

$$\begin{aligned}
\therefore \mathcal{E}_{\{A-x\}}^{\text{ess}} &= \mathcal{E}_{\{\text{aff } (A-x)\}}^{\text{ess}} \cap \mathcal{E}_{\{A-x\}} \quad (\text{Def. 42, p. 224 and L. 114}) \\
&= \left(T_x \mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} T_{-x} \right) \cap (T_x \mathcal{E}_{\{A\}} T_{-x}) \quad (\text{Eqn. (B.7) and Part 2}) \\
&= T_x \left(\mathcal{E}_{\{\text{aff } A\}}^{\text{ess}} \cap \mathcal{E}_{\{A\}} \right) T_{-x} \\
&= T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x}.
\end{aligned}$$

Show (4): This is Part 1 of L. 93, p. 213, App. A.

Show (5): This is just the Structure Theorem applied to the set $A-x$, combined with Parts 3 and 4. It can also be obtained with Part 2 and L. 102, p. 218, App. A. ■

Theorem 121 If A is a non-empty subset of $\mathbf{R}^d$ and $x \in \langle \Delta A \rangle^\perp$ then $\mathcal{E}_{\{A\}}^{\text{ess}} = \mathcal{E}_{\{A-x\}}^{\text{ess}}$.

Proof. $\forall f \in \mathcal{E}_{\{A\}}^{\text{ess}}$, $y \in A$, let $f = T_r H$, $r \in \langle \Delta A \rangle$, $H \in \mathcal{E}_{(\langle \Delta A \rangle)^-}$. $\mathcal{E}_{\{A-x\}}^{\text{ess}} = T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x}$ by Part 3, Thm. 120, above. $\therefore T_x f T_{-x} = T_{x+r-Hx} H = T_r H = f$ because $Hx = x$. $\therefore \mathcal{E}_{\{A-x\}}^{\text{ess}} = T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x} = \mathcal{E}_{\{A\}}^{\text{ess}}$. ■

Theorem 122 *If A is a non-empty subset of $\mathbf{R}^d$ and $x \in \text{fix}_{\text{aff } A} \mathcal{E}_{\{A\}}^{\text{ess}}$, then*

1. $\mathcal{E}_{\{A-x\}}^{\text{ess}} \leq \mathcal{E}_{(\langle \Delta A \rangle)^-} \leq \mathcal{O}$.
2. $T_x \mathcal{E}_{\{A\}} T_{-x} = \mathcal{E}_{\{A-x\}} \leq \mathcal{E}_{(\langle \Delta A \rangle)^-} \times \mathcal{E}_{(\langle \Delta A \rangle)} \leq \mathcal{O}$.

Proof. *Show (1):* $\forall f \in \mathcal{E}_{\{A\}}^{\text{ess}}$, $T_x f T_{-x}(0) = T_x f(x) = T_x(x) = 0$. $\therefore T_x f T_{-x} \in \mathcal{O}$ and hence equals the orthogonal part of f which is in $\mathcal{E}_{(\langle \Delta A \rangle)^-}$ by Def. 42, p. 224; i.e., $T_x f T_{-x} \in \mathcal{E}_{(\langle \Delta A \rangle)^-}$. Since $T_x f T_{-x}$ is an arbitrary member of $T_x \mathcal{E}_{\{A\}}^{\text{ess}} T_{-x} = \mathcal{E}_{\{A-x\}}^{\text{ess}}$, by Part 3 of Theorem 120 $\mathcal{E}_{\{A-x\}}^{\text{ess}} \leq \mathcal{E}_{(\langle \Delta A \rangle)^-}$.

Show (2): This follows directly from Part 2 of Theorem 120, the Structure Theorem, Part 1 of this theorem and the fact that $\text{aff}(A-x) = \langle \Delta A \rangle$. ■

In the next section we shall see how, when A is finite, such an x is assured.

B.4 Homomorphic Images of $\mathcal{E}_{\{A\}}$

As mentioned in Section A.3, since $\mathcal{E} \leq \text{Sym}(\mathbf{R}^d)$, the group action is an embedding; i.e., there is an isomorphic copy of $\mathcal{E}$ in $\text{Sym}(\mathbf{R}^d)$ and each isometry in $\mathcal{E}$ is carried to itself under the map defining the group action. Thus the group action is a monomorphism; i.e., the action is effective. If A is a non-empty subset of $\mathbf{R}^d$, clearly the restriction of this embedding to $\mathcal{E}_{\{A\}}$ is also 1-1. Consider the homomorphism ζ defined at the beginning of this chapter. $\zeta : \mathcal{E}_{\{A\}} \rightarrow \text{Sym } A$ is given by $\zeta(f) = f|_A$. The group action defined by ζ of $\mathcal{E}_{\{A\}}$ on A is not (in general) effective. To see this, consider the case where A is a subset of the x axis in $\mathbf{R}^3$. Any rotation about or reflection through this axis leaves A fixed; so, the kernel of ζ is non-trivial. Indeed,

the kernel of ζ is precisely the group of isometries whose restriction to A is the identity map; i.e., the pointwise stabilizer, $\mathcal{E}_{(A)}$. Thus,

$$\ker \zeta = \mathcal{E}_{(A)} \triangleleft \mathcal{E}_{\{A\}}.$$

Let

$$\text{nat}_1 : \mathcal{E}_{\{A\}} / \mathcal{E}_{(A)} \rightarrow \zeta(\mathcal{E}_{\{A\}})$$

be the natural map that carries the coset,

$$f \cdot \mathcal{E}_{(A)} \mapsto \zeta(f)$$

which, by the fundamental isomorphism theorem for groups, makes

$$\mathcal{E}_{\{A\}} / \mathcal{E}_{(A)} \approx \zeta(\mathcal{E}_{\{A\}}) :$$

i.e., nat_1 is a group isomorphism.

Since by our Structure Theorem (118), $\mathcal{E}_{\{A\}} = \mathcal{E}_{\{A\}}^{\text{ess}} \times \mathcal{E}_{(A)}$, an internal direct product, $\mathcal{E}_{\{A\}}^{\text{ess}} \approx \mathcal{E}_{\{A\}} / \mathcal{E}_{(A)}$. If we let the natural map,

$$\text{nat}_2 : \mathcal{E}_{\{A\}} / \mathcal{E}_{(A)} \rightarrow \mathcal{E}_{\{A\}}^{\text{ess}}$$

be given as follows: if $f = f_1 f_2 \in \mathcal{E}_{\{A\}}$ where $f_1 \in \mathcal{E}_{\{A\}}^{\text{ess}}$ and $f_2 \in \mathcal{E}_{(A)}$ then

$$f \cdot \mathcal{E}_{(A)} \mapsto f_1.$$

then the composition map, $\text{nat}_1 \circ \text{nat}_2^{-1} : \mathcal{E}_{\{A\}}^{\text{ess}} \rightarrow \zeta(\mathcal{E}_{\{A\}})$ being the composition of isomorphisms, is an isomorphism. Thus $\mathcal{E}_{\{A\}}^{\text{ess}}$ is isomorphic to $\zeta(\mathcal{E}_{\{A\}})$, a subgroup of $\text{Sym } A$ and we have shown:

Theorem 123 *The map $\zeta|_{\mathcal{E}_{\{A\}}^{\text{ess}}} : \mathcal{E}_{\{A\}}^{\text{ess}} \rightarrow \text{Sym } A$ given by $\zeta(f) = f|_A$ is a group monomorphism; i.e., the action of $\mathcal{E}_{\{A\}}^{\text{ess}}$, the essential part of $\mathcal{E}_{\{A\}}$, on A is effective.*

Corollary 124 *If A is a non-empty finite subset of $\mathbf{R}^d$, $\mathcal{E}_{\{A\}}^{\text{ess}}$ is finite. In fact, $\mathcal{E}_{\{A\}}^{\text{ess}}$ is isomorphic to a subgroup of S_n where $n = |A|$.*

Figure B-2: Homomorphisms for $\mathcal{E}_{\{A\}}$

Proof. $\text{Sym } A$ is finite and $\zeta|_{\mathcal{E}_{\{A\}}^{\text{ess}}}$ is an injective map from $\mathcal{E}_{\{A\}}^{\text{ess}}$ into $\text{Sym } A$. ■

Corollary 125 *If $A \subseteq \mathbf{R}^d$ is non-empty and $f, g \in \mathcal{E}_{\{A\}}^{\text{ess}}$ such that $f|_A = g|_A$ then $f = g$ on all of $\mathbf{R}^d$.*

Proof. If f and g agree on A , the monomorphism $\text{nat}_1 \circ \text{nat}_2^{-1}$ maps them to the same member of $\text{Sym } A$ and hence they are the same. ■

This corollary means that the members of $\mathcal{E}_{\{A\}}^{\text{ess}}$ are determined by their effect of A . In the light of the structure theorem (118 p. 227), we make the following:

Definition 44 *If $A \subseteq \mathbf{R}^d$, $f \in \mathcal{E}_{\{A\}}$ and $f = f_1 f_2$ is its unique decomposition where $f_1 \in \mathcal{E}_{\{A\}}^{\text{ess}}$ and $f_2 \in \mathcal{E}_{(A)}$ we define $\Pi_{i,A} : \mathcal{E}_{\{A\}} \rightarrow \mathcal{E}_{\{A\}}$ by $\Pi_{i,A}(f) = f_i$, ($i = 1, 2$).*

Thus $\Pi_{i,A}$ is the group homomorphism ‘projection onto the i th coordinate’ in the decomposition of $\mathcal{E}_{\{A\}}$ into the internal direct product given in Theorem 118 and for

any $f \in \mathcal{E}_{\{A\}}$, $f = f_1 f_2 = \Pi_{1,A}(f) \circ \Pi_{2,A}(f)$. If $G \leq \mathcal{E}_{\{A\}}$, the map $\Pi_{1,A}$ is defined on all of G and we can speak of $\Pi_{1,A}(G)$.

$$\Pi_{1,A}(G) \leq \mathcal{E}_{\{A\}}^{\text{ess}} \text{ and } \Pi_{2,A}(G) \leq \mathcal{E}_{\{A\}} \quad (\text{B.8})$$

Lemma 126 *If $A \subseteq \mathbf{R}^d$ and $G \leq \mathcal{E}_{\{A\}}$ then*

$$1. \forall f \in G, f|_{\text{aff } A} = \Pi_{1,A}(f)|_{\text{aff } A}$$

$$2. \forall x \in \text{aff } A, Gx = (\Pi_{1,A}(G))x$$

$$3. \text{fix}_{\text{aff } A} G = \text{fix}_{\text{aff } A} (\Pi_{1,A}(G)).$$

Proof. *Show 1:* Fix $x \in \text{aff } A$ and $f \in G$. Then $f(x) = \Pi_{1,A}(f) \circ \Pi_{2,A}(f)(x) = \Pi_{1,A}(f)(x)$ because $\Pi_{2,A}(f) \in \mathcal{E}_{\{A\}} = \mathcal{E}_{(\text{aff } A)} \leq \mathcal{E}_x$. $\therefore f|_{\text{aff } A} = \Pi_{1,A}(f)|_{\text{aff } A}$.

Show 2: $\forall x \in \text{aff } A, Gx = \{f(x) | f \in G\} = \{\Pi_{1,A}(f)(x) | f \in G\}$ by Part 1. $\therefore Gx = (\Pi_{1,A}(G))x$.

Show 3: $\forall x \in \text{aff } A$,

$$\begin{aligned} x \in \text{fix}_{\text{aff } A} G &\Leftrightarrow Gx = \{x\} \\ &\Leftrightarrow (\Pi_{1,A}(G))x = \{x\} \quad \text{by Part 2} \\ &\Leftrightarrow x \in \text{fix}_{\text{aff } A} (\Pi_{1,A}(G)). \end{aligned}$$

■

B.5 Centroids

Definition 45 *Given a finite, non-empty set, $X \subseteq \mathbf{R}^d$ and $G \leq \mathcal{E}_{\{X\}}$, if $x \in X$ we define $\bar{x}_G$ to be the centroid of the orbit, Gx and define $\bar{X}_G = \{\bar{x}_G | x \in X\}$ and call it the **set of centroids** of the orbits of X under G .*

When there is no confusion about the group G we simply write $\bar{x}$ and $\bar{X}$ for $\bar{x}_G$ and $\bar{X}_G$, respectively. Since $G \leq \mathcal{E}_{\{X\}}$, $\forall x \in X$, $Gx \subseteq X$. If we let $k = |Gx|$ and the orbit, $Gx = \{x_1, \dots, x_k\}$ then

$$\bar{x}_G = \frac{1}{k} \sum_{j=1}^k x_j \in \text{aff } Gx \subseteq \text{aff } X.$$

Thus $\bar{X}_G \subseteq \text{aff } X$: so, $\text{aff } \bar{X}_G \subseteq \text{aff } X$ and we have shown:

Lemma 127 *If $X \subseteq \mathbf{R}^d$ is non-empty and finite and $G \leq \mathcal{E}_{\{X\}}$, then $\text{aff } \bar{X}_G \subseteq \text{aff } X$.*

Lemma 127 will be subsumed by the following theorem which although given purely as a theorem in geometry, will be the key to characterizing the fixed point set of the group of position and velocity isometries of an N -body problem. It describes an interplay between a finite set X in $\mathbf{R}^d$ and a subgroup, G of its set-wise stabilizer. The theorem shows that the set of fixed points in $\text{aff } X$ is precisely the plane generated by the centroids of the orbits in X under G .

Theorem 128 *If X is a finite, non-empty subset of $\mathbf{R}^d$ and $G \leq \mathcal{E}_{\{X\}}$ then*

$$\text{fix}_{\text{aff } X} G = \text{aff } \bar{X}_G.$$

Proof. Since $G \leq \mathcal{E}_{\{X\}}$, $\forall x \in X$, $Gx \subseteq X$ (finite). Let $\bar{x}$ be the centroid of Gx . Then $\bar{x} \in \text{aff } \bar{X}_G \subseteq \text{aff } X$ by Lemma 127. By Lemma 99, $\bar{x} \in \text{fix}_{\text{aff } X} G$. Thus $\bar{X}_G \subseteq \text{fix}_{\text{aff } X} G$. Since $\text{fix}_{\text{aff } X} G$ is affine (L. 87, p. 211, App. A), $\text{aff } \bar{X}_G \subseteq \text{fix}_{\text{aff } X} G$ (by L. 82, p. 208, App. A).

To show the reverse inclusion, suppose that $X = \{x_1, \dots, x_n\}$ and choose any $y \in \text{fix}_{\text{aff } X} G$. Since $y \in \text{aff } X$, $y = \sum_{i=1}^n \alpha_i x_i$ where $\sum_{i=1}^n \alpha_i = 1$. Since $G \leq \mathcal{E}_{\{X\}}$, then $\forall i \leq n$, $Gx_i \subseteq X$. By Part 3 of Lemma 126, p. 233, $\text{fix}_{\text{aff } X} G = \text{fix}_{\text{aff } X} G_1$.

where $G_1 = (\Pi_{1,X}(G)) \leq \mathcal{E}_{\{X\}}^{\text{ess}}$. Since X is finite, $\mathcal{E}_{\{X\}}^{\text{ess}}$ is finite (Corollary 124, p. 231); so, G_1 is finite. $\therefore y \in \text{fix}_{\text{aff } X} G_1$.

$$\theta_{G_1}(y) = \frac{1}{|G_1|} \sum_{g \in G_1} g(y) = \frac{1}{|G_1|} \sum_{g \in G_1} y = y.$$

On the other hand, by Theorem 101

$$\begin{aligned} \theta_{G_1}(y) &= \frac{1}{|G_1|} \sum_{g \in G_1} g(y) \\ &= \frac{1}{|G_1|} \sum_{g \in G_1} \sum_{i=1}^n \alpha_i g(x_i) \\ &= \sum_{i=1}^n \alpha_i \theta_{G_1}(x_i) \\ &= \sum_{i=1}^n \alpha_i \bar{x}_{i,G_1}. \end{aligned}$$

By Part 2 of Lemma 126, $Gx_i = G_1x_i$; so, $\bar{x}_{i,G_1} = \bar{x}_{i,G}$ and we have $y = \theta_{G_1}(y) = \sum_{i=1}^n \alpha_i \bar{x}_{i,G} \in \text{aff } \bar{X}_G$ and since y is an arbitrary member of $\text{fix}_{\text{aff } X} G$, $\text{fix}_{\text{aff } X} G \subseteq \text{aff } \bar{X}_G$; i.e., these affine subspaces are equal. ■

Theorem 128 with Cor. 124, p. 231 tells us that if $X \subseteq \mathbf{R}^d$ is finite and non-empty, the essential part of its set-wise stabilizer, $\mathcal{E}_{\{X\}}^{\text{ess}}$ is finite and partitions X into a set of orbits each of which has a centroid. The set of centroids has an affine hull that is precisely the set of fixed points for this essential group, $\mathcal{E}_{\{X\}}^{\text{ess}}$.

Corollary 129 *If X is a finite, non-empty subset of $\mathbf{R}^d$ then $\forall x_0 \in \text{aff } \bar{X}_{\mathcal{E}_{\{X\}}^{\text{ess}}}$,*

$$\begin{aligned} \mathcal{E}_{\{X-x_0\}}^{\text{ess}} &\leq \mathcal{E}_{(\langle \Delta, X \rangle)^-} \leq \mathcal{O} \text{ and} \\ T_{x_0} \mathcal{E}_{\{X\}} T_{-x_0} &= \mathcal{E}_{\{X-x_0\}} \leq \mathcal{E}_{(\langle \Delta, X \rangle)^-} \times \mathcal{E}_{(\langle \Delta, X \rangle)} \leq \mathcal{O}. \end{aligned}$$

Proof. This follows directly from Thm. 122, p. 230 and Thm. 128. ■

Theorem 130 *If X is a finite, non-empty subset of $\mathbf{R}^d$ and*

$$\mathcal{E}_{(X)} \leq G \leq \mathcal{E}_{\{X\}},$$

then $\text{fix}_{\mathbf{R}^d} G = \text{aff } \overline{X}_G$.

Proof.

$$\begin{aligned} \text{aff } X &= \text{fix}_{\mathbf{R}^d} \mathcal{E}_{(\text{aff } X)} \quad (\text{by L. 88, p. 211, App. A}) \\ &= \text{fix}_{\mathbf{R}^d} \mathcal{E}_{(X)} \quad (\text{by L. 92, p. 213, App. A}) \\ &\supseteq \text{fix}_{\mathbf{R}^d} G \quad (\text{by L. 74, p. 203, App. A}) \end{aligned}$$

because $\mathcal{E}_{(X)} \leq G$ (the intersection taken over a larger set is as small or smaller).
 $\therefore \text{fix}_{\mathbf{R}^d} G = \text{fix}_{\text{aff } X} G = \text{aff } \overline{X}_G$ by Thm. 128 p. 234. ■

B.6 Bounded Sets

Theorem 131 *Suppose that X is a non-empty bounded subset of $\mathbf{R}^d$, $f \in \mathcal{E}_{\{X\}}$ and H is the orthogonal part of f . Then the following are equivalent:*

1. $f|_{\text{aff } X}$ is a translation.
2. $f|_{\text{aff } X}$ is the identity map; i.e., $f \in \mathcal{E}_{(\text{aff } X)}$.
3. $\forall x \in \text{aff } X, f = T_{-x} H T_x$.
4. $H|_{\langle \Delta X \rangle}$ is the identity map; i.e., $H \in \mathcal{E}_{(\langle \Delta X \rangle)}$.

Proof. We can write $f = T_r H$ for some $r \in \mathbf{R}^d$ (L. 69, p. 201, App. A).

Show (1) $\Rightarrow$ (2): Suppose $f|_{\text{aff } X} = T_x$ for some $x \in \mathbf{R}^d$. If $x \neq 0$, let $M = \inf_{y \in X} (x, y)$, (the inner product of x with y). By the Cauchy-Schwartz inequality, $\forall y \in X, (y, x) \geq -\|x\| \|y\| \geq -\|x\| \sup_{y \in X} \|y\|$. Thus, $M \geq -\|x\| \sup_{y \in X} \|y\| > -\infty$ because X is bounded. Since $\|x\|^2 > 0$ we can choose $y' \in X$ such that $(y', x) < M + \|x\|^2$. Then $(T_x y', x) = (y' - x, x) = (y', x) - \|x\|^2 < M$. $\therefore T_x y' \notin X$.

$\therefore f(X) = T_x X \neq X$ contrary to f 's membership in $\mathcal{E}_{\{X\}}$. Thus $x = 0$ and the only translation in $\mathcal{E}_{\{X\}}$ is $T_0 = I_d$, the identity operator.

Show (2) $\Rightarrow$ (3): Assume (2). $\forall x \in \text{aff } X, y \in \mathbf{R}^d, f(x) = x$. So, $f(y) = f(y) - f(x) + x = H(y - x) + x = (T_{-x} H T_x) y$; i.e., $f = T_{-x} H T_x$.

Show (3) $\Rightarrow$ (2): $\forall x \in \text{aff } X, f(x) = (T_{-x} H T_x)(x) = T_{-x} H(0) = T_{-x}(0) = x$.

Show (2) $\Rightarrow$ (4): Suppose that $f \in \mathcal{E}_{(\text{aff } X)}$. Let $v \in \langle \Delta X \rangle$. Then $v = y - x$ for some $x, y \in \text{aff } X$. So, $f(x) = x, f(y) = y$ and as above $Hv = Hy - Hx = f(y) - f(x) = y - x = v$.

Show (4) $\Rightarrow$ (1): Assume that $H|_{\langle \Delta X \rangle}$ is the identity. Fix x_0 in X . Then, $\forall x \in \text{aff } X, x - x_0 \in \langle \Delta X \rangle$; so, $f(x) - f(x_0) = H(x - x_0) = x - x_0$ and $f(x) = T_{x_0 - f(x_0)} x$; i.e., $f|_{\text{aff } a} = T_{x_0 - f(x_0)}$ as desired. This completes the proof of the theorem. ■

B.7 Finite Subgroups of $\mathcal{O}$

We finish this appendix with a result that will be used in the development of the group of a solution. The following theorem is similar to a discussion given in [Hu, Ch. 3].

Theorem 132 *If $\mathcal{G}$ is a finite subgroup of $\mathcal{O}$, $\theta_{\mathcal{G}} \equiv \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H$ is the orthogonal projection onto $\text{fix}_{\mathbf{R}^d} \mathcal{G}$.*

Proof. Clearly, $\theta_{\mathcal{G}}$, being a linear combination of linear operators is linear. Since $\mathcal{G}$ is a group, $\forall H \in \mathcal{G}, H\mathcal{G} = \mathcal{G}H = \mathcal{G}$ and $\forall H' \in \mathcal{G}$,

$$H'\theta_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H'H = \frac{1}{|\mathcal{G}|} \sum_{H'' \in H'\mathcal{G} = \mathcal{G}} H'' = \theta_{\mathcal{G}}. \quad (\text{B.9})$$

Similarly, $\theta_{\mathcal{G}}H' = \theta_{\mathcal{G}}$. So,

$$\theta_{\mathcal{G}}^2 = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H\theta_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} \theta_{\mathcal{G}} = \theta_{\mathcal{G}}.$$

$\therefore \theta_{\mathcal{G}}$ is an idempotent linear operator; i.e., a linear projection. Since $\mathcal{G}$ is a group, the mapping that takes each $H \in \mathcal{G}$ to H^{-1} is a bijection on $\mathcal{G}$ and since $H \in \mathcal{O}$ satisfies $H^{-1} = H^t$, we have $\mathcal{G} = \{H^{-1} | H \in \mathcal{G}\} = \{H^t | H \in \mathcal{G}\}$. $\therefore \theta_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H^t = \left(\frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H \right)^t = \theta_{\mathcal{G}}^t$; so, $\theta_{\mathcal{G}}$ is a linear self-adjoint projection operator on $\mathbf{R}^d$; i.e., an orthogonal projection.

To show that $\text{fix}_{\mathbf{R}^d} \mathcal{G}$ is the image of $\theta_{\mathcal{G}}$, consider any $x \in \text{fix}_{\mathbf{R}^d} \mathcal{G}$. $\therefore \theta_{\mathcal{G}} x = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} Hx = \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} x = x$. $\therefore x \in \theta_{\mathcal{G}}(\mathbf{R}^d)$, the image of $\theta_{\mathcal{G}}$; so, $\text{fix}_{\mathbf{R}^d} \mathcal{G} \subseteq \theta_{\mathcal{G}}(\mathbf{R}^d)$. Conversely, $\forall x \in \theta_{\mathcal{G}}(\mathbf{R}^d)$, $\exists y \in \mathbf{R}^d$ such that $x = \theta_{\mathcal{G}} y$. Then $\forall H \in \mathcal{G}$, using (B.9), $Hx = H\theta_{\mathcal{G}} y = \theta_{\mathcal{G}} y = x$. $\therefore x \in \text{fix}_{\mathbf{R}^d} \mathcal{G}$ and $\theta_{\mathcal{G}}(\mathbf{R}^d) \subseteq \text{fix}_{\mathbf{R}^d} \mathcal{G}$; hence $\theta_{\mathcal{G}}(\mathbf{R}^d) = \text{fix}_{\mathbf{R}^d} \mathcal{G}$; i.e., $\theta_{\mathcal{G}}$ is the orthogonal projection onto $\text{fix}_{\mathbf{R}^d} \mathcal{G}$. ■

Corollary 133 *If G is a finite subgroup of $\mathcal{E}$, $\theta_G \equiv \frac{1}{|G|} \sum_{g \in G} g$ maps each point, $y \in \mathbf{R}^d$ to the point in $\text{fix}_{\mathbf{R}^d} G$ closest to y .*

Proof. Since G is finite it has a fixed point, say x . The inner automorphism, ρ_x on $\mathcal{E}$ given by $\rho_x(g) = T_x g T_{-x}$ maps $\mathcal{E}_x$ (and hence $G \leq \mathcal{E}_x$) to $\mathcal{O}$ (Cor. 94, p. 214, App. A). Thus $G' = \rho_x(G) = T_x G T_{-x}$ is a finite subgroup of $\mathcal{O}$ and by the above theorem.

$$\theta_{G'} = T_x \theta_G T_{-x} = \frac{1}{|G|} \sum_{g \in G} T_x g T_{-x}$$

is the orthogonal projection onto the linear subspace, $\text{fix}_{\mathbf{R}^d} G' = (\text{fix}_{\mathbf{R}^d} G) - x = \Delta \text{fix}_{\mathbf{R}^d} G$ (L. 93, Part 2, p. 213). Since the automorphism preserves distances, $\theta_{G'}$ maps a point in $\mathbf{R}^d$ to the point in $\text{fix}_{\mathbf{R}^d} G$ closest to it. ■

Corollary 134 *If G is a finite subgroup of $\mathcal{E}$ and $y, z \in \mathbf{R}^d$ then*

$$\theta_G(z) - \theta_G(y) = P(z - y)$$

where P is the orthogonal projection onto $\Delta \text{fix}_{\mathbf{R}^d} G$.

Proof. $\forall x \in \text{fix}_{\mathbf{R}^d} G$ let $\mathcal{G} = T_x G T_{-x}$. Clearly $\mathcal{G}$ fixes the origin: so, $\mathcal{G} \leq \mathcal{O}$ and by Thm. 132 $\theta_{\mathcal{G}}$ is the orthogonal projection onto $\text{fix}_{\mathbf{R}^d} \mathcal{G} = \text{fix}_{\mathbf{R}^d} T_x G T_{-x} = -x + \text{fix } G$ (L. 93, part, 2, p. 213). $\therefore \text{fix } \mathcal{G} = \Delta \text{fix } \mathcal{G} = \Delta \text{fix } G$. Now,

$$\begin{aligned}
 \theta_G(z) - \theta_G(y) &= \frac{1}{|G|} \sum_{f \in G} f(z) - f(y) \\
 &= \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} (T_{-x} H T_x(z) - T_{-x} H T_x(y)) \\
 &= \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} (Hz - Hx + x) - (Hy - Hx + x) \\
 &= \frac{1}{|\mathcal{G}|} \sum_{H \in \mathcal{G}} H(z - y)
 \end{aligned}$$

which by Thm. 132 is the orthogonal projection onto the subspace, $\text{fix}_{\mathbf{R}^d} \mathcal{G} = \Delta \text{fix}_{\mathbf{R}^d} \mathcal{G} = \Delta \text{fix}_{\mathbf{R}^d} G$. ■

Appendix C

Finite Subgroups of $\mathcal{O}(2)$ and $\mathcal{O}(3)$

Following [GB, Ch. 1] we list the finite subgroups of $\mathcal{O}(2)$ and $\mathcal{O}(3)$ to provide a standard notation for reference in the main body of the thesis. Any subgroup of $\mathcal{O}(2)$ or $\mathcal{O}(3)$ is geometrically equivalent to one listed below: i.e., there is an inner automorphism or conjugacy between any such subgroup and one listed below.

C.1 Finite Subgroups of $\mathcal{O}(2)$

These are

- $\mathcal{C}_2^n$ for $n \geq 1$, the finite group of order n generated by a rotation through an angle of $2\pi/n$ or
- $\mathcal{H}_2^n$ for $n \geq 1$, the dihedral group of order $2n$ generated by reflections through two lines separated by an angle of π/n .

C.2 Finite Subgroups of $\mathcal{O}(3)$

In this section, if $\mathcal{K} \leq \mathcal{O}(3)$ we denote

$$-\mathcal{K} \equiv (-I_3) \mathcal{K} = \{-T | T \in \mathcal{K}\},$$

which is the usual set multiplication of the element $-I_3$, called central inversion, with the group $\mathcal{K}$. The finite subgroups of $\mathcal{O}(3)$ fall into 3 classes .

Class I: Finite subgroups of $\mathcal{SO}(3)$ (rotations). They are

- $\mathcal{C}_3^n$, (the cyclic group of order n , $n \geq 1$) generated by a rotation about the z axis of $2\pi/n$ radians.
- $\mathcal{H}_3^n$ (the dihedral group of order $2n$, $n \geq 2$) generated by a rotation about the z axis of $2\pi/n$ radians and a rotation about the x axis of π radians.
- $\mathcal{T}$ (the tetrahedral group $\approx A_4$).
- $\mathcal{W}$ (the octahedral group $\approx S_4$) and
- $\mathcal{I}$ (the icosahedral group $\approx A_5$).

Class II: Take any $\mathcal{K} \leq \mathcal{SO}(3)$ from Class I above and define $\mathcal{K}^* \equiv \langle \mathcal{K}, -I_3 \rangle = \mathcal{K} \cup (-\mathcal{K})$; i.e., $\mathcal{K}^*$ is the group generated by $\mathcal{K}$ and the scalar $-I_3$, the central inversion. This gives

- $(\mathcal{C}_3^n)^*$, ($n \geq 1$).
- $(\mathcal{H}_3^n)^*$, ($n \geq 2$)
- $\mathcal{T}^*$.
- $\mathcal{W}^*$, the complete symmetry group of the regular octahedron (or cube) in $\mathcal{O}(3)$ and
- $\mathcal{I}^*$, the complete symmetry group of the regular icosahedron (or dodecahedron) in $\mathcal{O}(3)$

Class III: Given any $\mathcal{L} \leq \mathcal{SO}(3)$ of even order and a subgroup, $\mathcal{K} \leq \mathcal{L}$ of index 2 in $\mathcal{L}$, define $\mathcal{L}|\mathcal{K} \equiv \mathcal{K} \cup -T\mathcal{K}$, where $T \in \mathcal{L} \setminus \mathcal{K}$. This gives the groups

- $C_3^{2n}]C_3^n$, $n \geq 1$.
- $\mathcal{H}_3^n]C_3^n$, $n \geq 2$.
- $\mathcal{H}_3^{2n}]\mathcal{H}_3^n$, $n \geq 2$ and
- $\mathcal{W}|\mathcal{T}$, the complete symmetry group of the regular tetrahedron in $\mathcal{O}(3)$.

There are no others.

Appendix D

Notation

In this thesis the author has attempted to find a notation that follows as much as possible the historical precedent in this field while addressing spaces of arbitrary dimension. For example, if $q(t)$ is a solution of the N -body problem in $\mathbf{R}^3$ we sometimes write, $q_i(t) = (x_i(t), y_i(t), z_i(t))$ or simply $q_i = (x_i, y_i, z_i)$. In $\mathbf{R}^d$ we need to index the coordinates, namely, $q_i = (q_i^1, q_i^2, \dots, q_i^d)$. This should not cause confusion with exponentiation since q_i is a vector and cannot be raised to an arbitrary power. It also avoids multiple subscripts which can be confused with the historically accepted $q_{ij} = q_j - q_i$. It is also used in some literature [Wnt. Sections 373 and 374]. The initial position, $a_i = q_i(t_0)$ is written as $a_i = (a_i^x, a_i^y, a_i^z)$ in $\mathbf{R}^3$ with a similar notation for the initial velocity, b_i . In $\mathbf{R}^d$ it is written $a_i = (a_i^1, a_i^2, \dots, a_i^d)$. In $\mathbf{R}^d$, the component of angular momentum in the ij plane ($1 \leq i, j \leq d$) is c_{ij} ; whereas, in $\mathbf{R}^3$ we use c_{xy} etc.

For subgroups of $\mathcal{E}(d)$ we use upper case calligraphic symbols such as $\mathcal{T}$, $\mathcal{O}$, $\mathcal{H}_3^n$ etc.

Physicists and engineers also study gravity and Hamiltonian systems. It is customary in such fields to define the potential energy of a system. In our (1.7) p. 18 the function, $U(q)$ is the negative of the function generally defined in these scientific

disciplines. The energy required to pull all pairs of bodies infinitely far apart is U ; so, the potential energy of the system relative to such a state is $-U$. However, the notation used here conforms to a large body of literature studying the mathematical foundations of celestial mechanics. If the reader accustomed to the other notation can keep in mind these facts confusion should not result.

In column 3 below, App, Ch, D, E, L, R and S mean appendix, chapter, definition, equation, lemma, remark and section respectively.

Symbol	Meaning	Defined in
m_j	mass of body, j	E 3.1 p 24 Ch 3
m	$= (m_1, \dots, m_N)^t \in \mathbf{R}^N$, mass vector	E 3.1 p 24 Ch 3
x	$(x_1, \dots, x_N) \in \mathbf{R}^{dN}$, $x_i \in \mathbf{R}^d$	D 1 p 25 Ch 3
y'	$\underbrace{(y, \dots, y)}_{N \text{ copies}} \in \mathbf{R}^{dN}$ for $y \in \mathbf{R}^d$	D 5 p 32 Ch 3
q_{ij}	$q_j - q_i \in \mathbf{R}^d$, difference vector	
q_i^k	the k th coordinate of $q_i \in \mathbf{R}^d$	R 1 p 27 Ch 3
q_{ij}^k	the k th coordinate of the difference vector $q_{ij} = q_j - q_i \in \mathbf{R}^d$	R 1 p 27 Ch 3
$\langle A \rangle$	linear subspace spanned by the set A	D 36 p 207 App A
$\text{aff } A$	affine subspace spanned by the set A	D 38 p 209 App A
$\text{aff } x$	$= \text{aff } \{x_1, \dots, x_N\}$, $x \in \mathbf{R}^{dN}$	D 38 p 209 App A
ΔA	$= A - A$. Set difference, where $A \subseteq \mathbf{R}^d$	D 36 p 207 App A
(x, y)	inner product	
$\leq$	subgroup or linear subspace	
$\triangleleft$	normal subgroup	
$\langle \Delta x \rangle$	linear subspace spanned by $\{x_{1i}\}_{i=2}^N$	D 6 p 36 Ch 3

Symbol	Meaning	Defined in
$\langle \Delta x, \Delta y \rangle$	$= \langle \Delta x \rangle + \langle \Delta y \rangle$	D 6 p 36 Ch 3
$\text{aff}(x, \Delta y)$	$= \text{aff } x + \langle \Delta y \rangle$	D 6 p 36 Ch 3
$\langle \Delta q, \Delta \dot{q} \rangle$	$= \langle \Delta q(t) \cdot \Delta \dot{q}(t) \rangle$, the same $\forall t$	D 7 p 43 Ch 3
q_{ij}	$= q_i - q_j \in \mathbf{R}^d$	E 3.1 p 24 Ch 3
Δ_{ij}	$\{q \mid q_i = q_j\} \subseteq \mathbf{R}^{dN}$	D 1 p 25 Ch 3
Δ	$= \cup_{i < j} \Delta_{ij} \subseteq \mathbf{R}^d$, (closed subset)	D 1 p 25 Ch 3
$\mathcal{P}(S)$	power set of S	p. 206 App A
	family of subsets of the set S	
$\emptyset$	the empty set	
$\coprod$	disjoint union	
$\Leftrightarrow$	is equivalent to	
$\rightarrow$	surjective mapping	
o.d.e.	ordinary differential equation	
ivp	initial value problem	D 1 p 25 Ch 3
$\mathbf{R}^+$ or $\mathbf{R}_+$	$(0, \infty)$, the positive real numbers	
$\mathcal{E}(d)$	Euclidean Group on $\mathbf{R}^d$ (isometries)	
$\mathcal{S}(d)$	Similarity group on $\mathbf{R}^d$ (homographies)	
$\mathcal{O}(d)$	Orthogonal (transformation) group	
$T_x(y)$	$= y - x$, $x, y \in \mathbf{R}^d$, a translation	D 32 p 201 App A
$\mathcal{T}(d)$	Translation group $= \{T_x \in \mathcal{E}(d) \mid x \in \mathbf{R}^d\}$	R 5 p 201 App A
$\mathcal{O}, \mathcal{E}, \mathcal{S}, \mathcal{T}$	$\mathcal{O}(d), \mathcal{E}(d), \mathcal{S}(d)$ and $\mathcal{T}(d)$ (resp.), d known	
$H_{\{A\}}$	the setwise stabilizer for the set $A (\subseteq S)$ in the group, $H (\leq G)$ where G acts on S .	D 41 p 220 App B
$H_{(A)}$	the pointwise stabilizer, also called $\text{stab}_H A$	D 34 p 203 App A

Symbol	Meaning	Defined in
$U^\perp$	$= \Delta U^\perp$, orthogonal complement of the linear subspace ΔU in $\mathbf{R}^d$, where U is an affine subspace.	D 40 p 215 App B
$\mathcal{E}_{\{A\}}^{\text{ess}}$	the essential part of $\mathcal{E}_{\{A\}}$	D 42 p 224 App B
$\mathcal{E}_{(A)}$	the inessential part of $\mathcal{E}_{\{A\}}$	D 42 p 224 App B
$\Pi_{1,A}$	projection onto $\mathcal{E}_{\{A\}}^{\text{ess}}$	D 44 p 232 App B
$\Pi_{2,A}$	projection onto $\mathcal{E}_{(A)}$	D 44 p 232 App B
$\overline{X}_G$	the set of centroids of the finite set $X \subseteq \mathbf{R}^d$ under $G \leq \mathcal{E}_{\{X\}}$.	D 45 p 233 App B
θ_G	$= (1/ \mathcal{G}) \sum_{H \in \mathcal{G}} H$	T 132 p 237 A B
s	$:\mathbf{R}^+ \rightarrow \mathbf{R}$, continuous force function	D 1 p 25 Ch 3
$u(r)$	$= - \int_{r_0}^r x s(x) dx$, potential function	D 3 p 29 Ch 3
$U(q)$	$\sum_{i < j} m_i m_j u(\ q_{ij}\)$, the negative of the potential energy	E 3.7 p 30 Ch 3
T	kinetic energy	E 3.17 p 35 Ch 3
h	$T - U$, total energy	E 3.18 p 35 Ch 3
$C(q(t))$	angular momentum matrix	E 3.16 p 34 Ch 3
$\text{Sym}(S)$	Symmetric group on S	S A.1 p 202 App A
$\mathbf{n}$	$= \{1, \dots, n\}$	
$S_{\mathbf{n}}$	$\text{Sym}(\mathbf{n})$, permutation group on $\mathbf{n}$	
π	$\in S_N$	S 4.1 p 46 Ch 4
π'	$N \times N$ permutation matrix for π	L 12 p 47 Ch 4
π''	$dN \times dN$ permutation matrix for π	D 8 p 47 Ch 4
$f'(x)$	$= (f(x_1), \dots, f(x_N))^t$ where $x = (x_1, \dots, x_N)^t \in \mathbf{R}^{dN}$, $f \in \mathcal{E}(d)$	D 9 p 49 Ch 4

Symbol	Meaning	Defined in
$\text{diag}(x_1, \dots, x_n)$	diagonal matrix whose (i, i) th entry is x_i	
A^t	transpose of the matrix A	
δ_{ij}	Kronecker delta	
$H_1 \oplus \dots \oplus H_n$	direct sum of linear operators	
$\ q\ _{dN}$	Euclidean norm in $\mathbf{R}^{dN}$	D 5 p 32 Ch 3
$\mathcal{G}_{rv}(q(t))$	Galilean coordinate transformation of a solution q	D 5 p 32 Ch 3
$\mathcal{G}_{rv}(m, a, b, t_0)$	$= (m, a - x', b - v', t_0)$. Galilean coordinate transformation of an ivp	D 5 p 32 Ch 3
H'	$= \text{diag}(\underbrace{H, \dots, H}_{N \text{ times}}) \in L(\mathbf{R}^{dN}, \mathbf{R}^{dN})$ where $H \in L(\mathbf{R}^d, \mathbf{R}^d)$	E 4.2 p 50 Ch 4
(π, f, g)	$\in S_N \times \mathcal{E}(d) \times \mathcal{E}(d)$, a triple that preserves an ivp. π, f and g are called the permutation, the position isometry and the velocity isometry for that triple, respectively.	D 10 p 52 Ch 4
$G(m, a, b)$	group of the ivp (m, a, b, t_0)	D 11 p 54 Ch 4
$\mathcal{E}_{\{x\}}, \mathcal{E}_{\{x\}}^{\text{ess}}$	$\mathcal{E}_{\{X\}}, \mathcal{E}_{\{X\}}^{\text{ess}}$, where $X = \{x_1, \dots, x_N\}$, $x = (x_1, \dots, x_N)^t \in \mathbf{R}^{dN}$.	D 12 p 59 Ch 5
$\mathcal{E}_{\{x\}}$	$\mathcal{E}_{\{X\}} = \text{stab}_{\mathcal{E}} X$ (also called $\text{stab}_{\mathcal{E}} x$)	D 12 p 59 Ch 5
ξ_N, ξ_a, ξ_b	projection of $G(m, a, b)$ onto 1st, 2nd and 3rd coordinates, resp.	D 12 p 59 Ch 5
G_N, G_a, G_b	resp. images these projection maps	D 12 p 59 Ch 5

Symbol	Meaning	Defined in
$\delta_{\tau, y_0}(x, y)$	a finite subset in $\mathbf{R}^d$ whose affine hull is $\text{aff } x\Delta y$	D 13 p 62 Ch 5
$\bar{x}_G$	centroid of a (finite) orbit of $x \in \mathbf{R}^d$ under the action of group, G	D 45 p 233 App B
$\bar{X}_G$	set of centroids under group action	D 45 p 233 App B
$\bar{a}$	vector of centroids, $\bar{a}_i = \bar{a}_{i, G_{i_1}}$	D 14 p 63 Ch 5
$E(m, a, b)$	essential group of the ivp (m, a, b, t_0)	D 15 p 70 Ch 5
$\mathcal{O}_l$	$= G_N i_l = \{i_{l,1}, \dots, i_{l,k_l}\}$, orbit l	D 18 p 84 Ch 5
P_l	$P_l(j) = i_{l,j}$, a renumbering of orbit l	D 18 p 84 Ch 5
ψ_l, Ψ_l	group homomorphism associated with orbit l .	D 18 p 84 Ch 5
Π_N	$= \bigcap_{l=1}^n (S_N)_{\mathcal{O}_l} \leq S_N$, contains the group G_N . The group of permutations on $\mathbf{N}$ for which each orbit is globally invariant	D 18 p 84 Ch 5
$G_l(m, a, b)$	group of the derived ivp for orbit l	D 18 p 84 Ch 5
$\xi_N(t)$	projection of $G(m, q(t), \dot{q}(t))$ onto 1st or permutation coordinate	D 24 p 102 Ch 7
$G_N(t)$	image of $\xi_N(t)$	D 24 p 102 Ch 7
$\xi_a(t)$	projection of $G(m, q(t), \dot{q}(t))$ onto 2nd or position isometry coordinate	D 24 p 102 Ch 7
$G_a(t)$	image of $\xi_a(t)$	D 24 p 102 Ch 7
$\xi_b(t)$	projection of $G(m, q(t), \dot{q}(t))$ onto 3rd or velocity isometry coordinate	D 24 p 102 Ch 7
$G_b(t)$	image of $\xi_b(t)$	D 24 p 102 Ch 7

Symbol	Meaning	Defined in
$\text{aff}(\bar{q}(t), \Delta \bar{q}(t))$	centroidal affine subspace of the solution at time t	D 22 p 95 Ch 7
$\text{aff}(\bar{\dot{q}}(t), \Delta \bar{\dot{q}}(t))$	centroidal velocity affine subspace of the solution at time t	D 22 p 95 Ch 7
I_π	set of all t for which the perm. $\pi \in G_N$	D 25 p 103 Ch 7
I_R	$\cap_{\pi \in R} I_\pi$. over some set R of permutations	D 25 p 103 Ch 7

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