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**Error Estimates for Finite-Element Solutions
of Elliptic Boundary Value Problems**

by

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The undersigned hereby recommend to
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of Elliptic Boundary Value Problems**

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Abstract

This thesis presents a comprehensive analysis of the Duality Method in error analysis. This method is applied to problems of solid mechanics. In this analysis a procedure for computing a statically admissible stress-field is described. The statically admissible stress-fields are computed by using the data of the problem (prescribed traction and body-force) and the kinematically admissible solutions. The error-indicator is defined by the difference between the kinematically and statically admissible stress-fields, measured in the strain energy norm, over the elements. The relative value of the error-indicator with respect to the strain energy of the deformed body defines the local error-measure. The error-bound for the approximate solution of the problem obtained by the finite-element method is computed by the summation of the local error-measures over all elements in the mesh. The Duality Method is determined to be a robust method of error analysis.

In addition to the analysis of the statically admissible stress-field, this thesis innovates several approximations to statically admissible stress-fields. The approximate stress-fields are compared with statically admissible stress-fields by their features. These stress-fields are used to define a number of error-measures which are shown to be reliable tools to control the accuracy of the finite-element solutions. Some of these error-measures are implemented in a computer program and tested for some problems*.

To compare the results obtained with the results computed by using methods in the literature, the error-estimates presented by Zienkiewicz and Zhu are implemented in the computer program and the same test problems are solved. Then the errors in the solutions are estimated using this method. In addition, the exact errors in the solutions are computed for problems for which exact solutions exist. The computed results are presented and compared with each other. The accuracy and reliability of the error-measures presented in this work supports the viability of the methods of error analysis.

* Four methods are implemented in a computer program.

Also, this thesis presents a summary of the Residual Method in error analysis. Correction indicator, Error-indicators, and error-estimators corresponding to this method are quoted. The overall analysis of this method indicates that the error-measures computed by this method are less reliable than those from the duality method.

In the last part, the presented error-bounds and error-estimates are summarized. Their advantages and limitations from the perspectives of accuracy, complexity, and computational cost are stated. For comparison purposes, the method of error analysis by Zienkiewicz and Zhu, and the Residual Method are included in this argument.

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Nomenclature

x, y, z	= Global coordinates
ξ, η, ζ	= Natural coordinates
$\frac{\partial}{\partial \xi}$	= Partial derivative with respect to ξ
Ω	= Physical domain
$\partial\Omega$	= Boundary of domain
$\partial_1\Omega$	= Part of the boundary of domain over which the displacement boundary conditions are prescribed
$\partial_2\Omega$	= Part of the boundary of domain over which the traction boundary conditions are prescribed
ω_h	= Discretized domain
ω, ω_j	= Element (subdomain)
$\partial\omega$	= Boundaries of element
b	= Element-face
$\mathcal{F}$	= Set containing the element-faces b
$\eta_{\omega_j}^b$	= Function defined on element-face b
Γ	= The set of element boundaries
ρ_h	= Set containing all elements connected to a node
k	= Set of all nodes in the interior of mesh
k_1	= Set of all nodes lying on the boundary $\partial_1\Omega$
k_2	= Set of all nodes lying on the boundary $\partial_2\Omega$
$\bar{k}$	= Set of all nodes in the mesh
l	= Number of elements in the mesh
m	= Number of Gauss points contributing to a node
N_i	= Basis function associated with the node i
x_i, y_i, z_i	= Components of the <i>nodal force due to tractions</i> associated with the node i
τ	= Traction-vector computed from the kinematically

	admissible stress field
a_i, b_i, c_i	= Vectors defined from the basis function associated with the node i
N^*	= Bubble function
∇u	= Gradient of displacement
div	= Divergence
B	= The gradient operator from displacement field to strain field
B^T	= Divergence
$B(\cdot, \cdot)$	= Bilinear form
$\bar{T}, \bar{T}_{(n)}$	= Prescribed tractions over the boundaries of domain
u	= Displacement field
$\tilde{u}_f$	= Exact displacement field
$\bar{u}$	= Prescribed displacement at the boundary $\partial_1 \Omega$ of domain.
h	= Discretization subscript, i.e., a measure of the element size
u_h	= Finite element displacement vector
$[K]$	= Stiffness matrix
$\{R\}$	= Load vector
P_i, Q_i, R_i	= Right hand side in equations (2.33), (2.34), and (2.35)
$[J]$	= Jacobian matrix
$[M]$	= The rectangular coefficient matrix in the linear system (2.38)
$[G]$	= Coefficient matrix in equations (2.74)
$[H]$	= Coefficient matrix defined by (2.67)
$[\Lambda]$	= Operator from gradient to divergence defined by (2.71)
$\{\Delta R\}$	= Load vector in equations (5.4)
σ	= Stress vector
$\delta \sigma$	= Stress variable in terms of bubble functions
λ	= Eigenvalue
$\tilde{\sigma}_f$	= Natural state of stress (the exact solution)
σ_h	= Kinematically admissible stress field
σ_h^c	= Continuous stress field
$\bar{\sigma}_h$	= Quasi-statically admissible stress field

$\bar{\sigma}$	= Mean-SAS field
σ_h°	= Weak-statically admissible stress field
$\{\sigma\}$	= Vector of nodal stresses in element
ϵ	= Strain vector
$\hat{\sigma}_h$	= Statically admissible stress-field
E	= Modulus of elasticity
ν	= Poisson's ratio
C, C^{-1}	= Material properties matrices
$f, \bar{f}$	= Body forces
T	= Traction vector and Transposition operator
$L^p(\Omega)$	= Lebesgue space
$H^1(\Omega)$	= Space of generalized derivatives (Sobolev space)
$H(\text{div}, \Omega)$	= Induced Sobolev space
$\mathbb{N}, \mathbb{N}_0$	= Spaces of admissible displacements
χ_f, χ_0	= Spaces of statically admissible stresses
$\chi_{\bar{f}}$	= Space of quasi-statically admissible stress fields
Σ, Σ_0	= Spaces of kinematically admissible stress fields
$(\cdot, \cdot)$	= Energy inner product
$\ \cdot\ $	= Energy norm
$ \cdot $	= Usual L^2 norm
$\forall$	= For all
$\in$	= Belongs to
n	= Outward unit vector normal to the surface
$[K]$	= Coefficient matrix in equations (2.79)
$\mathcal{N}$	= Notation for nullspace
$[Z]$	= Matrix made by the basis vectors of a nullspace
$\{b\}$	= Right hand side in equations (2.79)
ϕ	= Stress coefficient in expression (2.55)
$\{X\}, \{\delta X\}$	= Stress gradients with respect to global coordinate system
$\{\Xi\}, \{\delta \Xi\}$	= Stress gradients with respect to natural coordinate system
ρ	= Density

θ	= Angular velocity
$\ e\ ^2$	= Relative error
ε	= Error-indicator in the Residual Method
$\bar{\varepsilon}$	= Error Estimator in the Residual Method.

Acronyms

FEM	= Finite Element Method
BC	= Boundary Conditions
SBC	= Stress (Traction) Boundary Conditions
DBC	= Displacement Boundary Conditions
KAS	= Kinematically Admissible Stress
SAS	= Statically admissible Stress
Mean – SAS	= Mean-Statically Admissible Stress
Weak – SAS	= Weak-Statically Admissible Stress
Quasi – SAS	= Quasi-Statically Admissible Stress
PDE	= Partial Differential Equations

1. Scope of the Research

1.1 Background

Many engineering problems are presented in terms of differential equations whose solutions, for several hundred years, have been limited to those of a closed-form type. Such problems are restricted to those cases in which physical properties are idealized and/or the geometry of the domains is simplified. For more general problems, however, numerical techniques have been used. The application of these techniques in practice enhances the varieties of the methods of solutions.

With the introduction in the 1960's of digital computers and the finite-element method, engineers were able to solve more general (realistic) problems. Since then, the finite-element method has become increasingly important and popular in engineering analysis. It can be argued that the introduction of this method was the most significant change in engineering analysis in this century.

Several finite-element techniques have been developed to provide solutions to problems. These methods are summarized as follows:

1- Displacement Method. This is the primal method for stress analysis in which all unknowns, i.e., stresses and strains are described in terms of the *displacement* vector. Therefore the displacement vector is the unknown in this method. According to this method, components of the displacement vector are continuous within the domain and across such external boundaries where the displacement boundary conditions are prescribed. Therefore the displacement fields computed by using this method satisfy the kinematic constraints. Because of this characteristic, the displacement and stress-fields obtained by using this method are called kinematically admissible fields.

2- Force Method. In this method the *stress* field is unknown and all other variables are expressed in terms of this variable. The stress-fields are assumed to satisfy the equations of equilibrium in the interior of the domain and the element-tractions (stress-vectors) computed from the stress-fields to be in agreement with the prescribed tractions on the external boundaries. The stress-fields computed by using this method are called

statically admissible stress-fields because they satisfy the equations of equilibrium in the interior and on the external boundaries of the domain where the traction boundary conditions are prescribed.

3- Mixed Methods. In these methods a displacement field which is not kinematically admissible and a stress-field which is not statically admissible, constitute the unknowns. In some mixed methods, so called hybrid, the computed stress-fields whose element-tractions are discontinuous across the boundaries of elements and on the external boundaries, satisfy the equations of equilibrium only in the interior of elements and the displacements are interpolated along the boundaries of elements. Since the computed displacements do not satisfy the kinematic constraints exactly, they are not kinematically admissible. Also, because the stress-fields do not satisfy the static conditions exactly, they are not statically admissible fields.

4- Dual Method. In this method, again, two separate fields of displacements and stresses are the unknowns. Unlike the Mixed Methods, in this method the complete kinematic and static conditions are retained for displacement and stress-fields simultaneously. Essentially, the displacement fields are required to satisfy the continuity conditions; therefore they are kinematically admissible. In addition, the stress-fields are required to satisfy the equations of equilibrium in the interior and the conformity with the prescribed tractions on the external boundaries over which the traction boundary conditions are prescribed; thus the computed stress-field is statically admissible. To date, the dual method has been the most complicated procedure with respect to theoretical and practical analyses; but since both kinematically and statically admissible fields are generated by using this method, this becomes very useful.

Among the above approaches, the displacement method is the most popular in engineering environments. In the displacement finite-element formulation, the energy functional operates on a function space in which the appropriate derivatives of every function must be square integrable. This condition characterizes the restriction operator in the function space. The restriction operator in the function space associated with this formulation is a simple operator, which selects the functions whose first order partial derivatives can possess finite jumps; therefore the domain of the functional is relatively large. More-

over, the simplicity of the restriction operator allows the use of isoparametric elements. Besides the foregoing assertions, the finite dimensional subspaces of the function space retain the characteristics (restrictions) of the infinite dimensional function space in this formulation. Furthermore, convergence to the exact solution is assured by mesh refinement [1]. The advantages of the displacement finite-element method have made it desirable in engineering applications.

The displacement finite-element method yields an approximate solution that approaches a true solution by increasing the number of discrete variables. Because the finite-element solution is assumed to represent the physical behavior of the structure under consideration, the validity of the computed solution is an important issue. In other words, the reliability of the finite-element solution must be ensured. In some cases, users compare the computed solution with the one obtained from a more refined mesh to evaluate the accuracy of the finite-element solution. This method of evaluation is costly because the re-analysis (re-meshing) of the problem with more degrees of freedom is performed which is more expensive than the computed original solution.

Unlike the method of re-analysis, local error-measures present an economical way of judging the acceptability of the finite-element solutions. Local error-measures are computed from the data and the existing finite-element solutions. These error-measures provide information on the accuracy of the solution within the subdomains. If in any region the quality of the finite-element solution is found unsatisfactory, the mesh may be refined over the same local area leaving the mesh in the rest of the domain unchanged; therefore the re-analysis (re-meshing) of the whole problem becomes unnecessary.

Local error-measures are indispensable tools in adaptive finite-element methods. The full integration of finite-element analysis into computer-aided-design systems requires the capability of developing a reliable automatic finite-element modeling technique. Reliable models are those which satisfy some preassigned level of accuracy. The finite-element algorithm must include some means of automatically checking the accuracy of results and their improvement by an adaptive procedure. Without a local error-measure, no efficient way of controlling the accuracy of the finite-element solution can be given.

1.2 Error-Measures for Adaptive Finite-Element Method

In an adaptive method the mesh at any stage is embedded in a more refined one. In this method, a local error-measure is evaluated; then mesh refinement for more accurate solution is carried out adaptively such that the results computed from coarser mesh are used to calculate a more accurate solution.

Several methods of error analysis for adaptive finite-element solutions of boundary value problems have been published. These techniques are as follows [2]:

- 1- Element Residual Method,
- 2- Subdomain Residual Method,
- 3- Interpolation Method,
- 4- Post Processing Method,
- 5- Duality Method.

Among these methods the Duality Method which 'exploits notions of duality theory in convex optimization' is the most reliable method in error analysis. The reliable error-bounds which are computed by using this method utilize the kinematically and the statically admissible stress-fields. One of the intentions in this work is to present a procedure to compute the statically admissible stress-field from the kinematically admissible stress-field. Moreover, the computation of error bound from these stress fields is another objective in this work.

1.3 Aims of the Thesis

This thesis focuses on algorithms for computing local error-measures for finite-element solutions of elliptic boundary value problems. Four issues are considered:

- (a) Study of the literature on local error-estimates.

The method of local error-estimates is an integral part of adaptive finite-element techniques. Local error-measures are defined from the data of the problem and the computed finite-element solutions. In the previous section, the methods of error-estimates, commonly used in practice, have been presented. In the following sections a summary of the

published works on these methods are delineated.

Zienkiewicz *et al* [3][4] [5][6][7] have presented methods to calculate error-estimates. Their techniques are based on the Element Residual Method. These error-measures are computed from the residuals in the equations of equilibrium. The local error-measures are defined in terms of the variations of error in the equations of equilibrium, measured in the energy norm, due to the introduction of one additional degree of freedom in the finite-element solution. In these methods, the error in the system will diminish asymptotically upon introducing a number of additional degrees of freedom in the solution. Although the error-estimates based on the Residual Method are used in many applications, they have some drawbacks; the error-estimates based on the residual calculations are not reliable since they do not indicate the true error in the solution. Moreover, experience shows that validation of finite-element results by this method tends to be slow and expensive [8].

A set of three error-estimates for finite-element solutions of elliptic partial differential equations have been presented by Bank and Weisner [9]. They have defined four subspaces of weak forms in their analysis among which the subspace of finite-element solutions constitutes one of the weak forms. A very complicated mathematical procedure has been applied in which several variables and constants are defined. Their error-estimates, which are expressed in the energy norm, are different for coarse and fine meshes. These estimates are based on solving local Neumann problems in each element. The estimates differ in how they enforce consistency of the Neumann problems. They have shown empirically that as the mesh size decreases, under suitable assumptions, two of these error estimates approach upper bounds on the norm of the exact error; and all three error-estimates are within a factor of three of the norm of the true error. The complexity and multiplicity of these error-estimates in application are the drawbacks of these error-estimates and reduce the viability of these error-estimates.

Recently, Zienkiewicz and Zhu [10] proposed an estimate of the error in finite-element solutions of linear elastic problems. Initially, in this analysis, the values of stresses at the nodes are computed by an averaging technique or by a projection method from the sampling points to the nodes. Then a continuous stress-field is constructed over the domain by interpolation between the nodes, using the same basis functions used for the

displacements. In the case of one-dimensional linear elastic problems, they have shown by a numerical example that this stress-field in C^0 is a good approximation to the stress-field over the entire body; then they compute the error-measures by using the finite-element solution and this stress-field. This is a simple approach and can be implemented in any finite-element code [10]. Since the stress-fields are neither exact nor belong to the function space of statically admissible stress fields, the error-measures do not produce any bound for the error incurred in the finite-element solution. As these error-measures captures a major portion of the error in the stress-field computed by the displacement finite-element method [4], they are more reliable than the results of error analysis proposed by Zienkiewicz *et al* [3][4][5][6][7]; however, the theoretical analysis and the numerical experiments show that none of these error-measures are totally reliable.

Stein and Bischoff, Brand and Plank [11] proposed a method to compute the dual norm of the residuum. In this approach an upper bound for the potential energy of the system has been derived elaborately. The criteria for mesh refinement, as suggested by this analysis, are functions of jumps in the normal tractions on the interelement and global boundaries of elements, a measure describing the size of the boundaries of elements, and an unknown constant associated with the physical domain. A true error-bound can be calculated using this formulation provided that the aforementioned constant can be determined precisely.

Applying the Duality Method, Fraeijns de Veubeke [12] and Sander [13] proposed an approach for solving the approximate primal and dual problems; and giving an upper and lower bound of the energy. In other words, they showed that using the compatibility and the equilibrium requirements exclusively yield lower and upper bounds on the strain energy of the exact solution. The difficulty is the implementation of the equilibrium method. Also, this method requires too much computational effort [14]. 'The method suggested by Babuska and Rheinboldt [15] [16] works toward the same end: the optimization of finite-element grids, and presents similar features [14]'. This method is reliable in error analysis, but its computational cost prohibits the application of this method.

Also, the Duality Method in error analysis was applied to finite-element analysis by Ladeveze and Leguillon [14] to compute the error-bound for finite-element solutions. This

method makes use of the hypercircle theory of Synge, [17][18], in which the statically and kinematically admissible stress-fields are defined. These stress-fields may be used to determine the lower and upper bounds on the strain energy of the exact solution of the problem. Theoretically, this method is similar to the method presented by Fraeijs de Veubeke [12] and Sander [13], but the equilibrium finite-element solution is not obtained by solving the dual problem for the statically admissible field; they define the spaces of admissible as restrictions to the finite-element solutions such that the dual problem is solved as a local problem using the data of the problem (prescribed traction boundary conditions and body-force) and the kinematically admissible fields computed from the displacement finite-element method. Their procedure to construct statically admissible stress-fields consists of two steps: in the first step, the *nodal forces due to tractions* on the boundaries of the elements are computed. These nodal forces are such that each element is in equilibrium with respect to its loading condition. In other words, the total forces from tractions and body forces over each element vanishes. The computation of the *nodal forces due to tractions* has been done in a manner that ensures the continuity of tractions across the boundaries of elements. Moreover, the conformity of the associated stress-fields with the traction boundary conditions is guaranteed. In the second step, the statically admissible stress-field over the interior of the elements is constructed by using the computed *nodal forces due to tractions* which are now known on the element boundaries. In this process, the element is subdivided into a number of subelements and a small finite-element problem is solved for the statically admissible stress-field over the element. This method of error analysis was applied in the computer program to control the accuracy of finite-element solution of two-dimensional plane stress problems in elastoplasticity [19]; in this calculation, constant stress triangular element were used. Also, the application of the method has been extended to linear tetrahedron elements in three-dimensional spaces for problems of elastoplastic and dynamic analysis [20]. The method of calculating the statically admissible field for linear triangular and tetrahedron elements, described by Ladeveze *et al*, is limited to the cases in which the stress distributions are assumed linearly varying in physical space. Therefore the body-forces are assumed constants to match the stress distributions. However, for body-forces described by higher order polynomials, this method does not generate the statically admissible stress-field. Also, when other elements

are considered, this procedure for computing $\hat{\sigma}_h$ (statically admissible stress-field) could lead to a substantial effort in numerical calculations [20].

Gibert [21] has implemented the above method of error analysis into a finite-element program to determine error-estimates for finite-element solutions. He examined the accuracy of the computed stress-fields in a highly heterogeneous structures such as the case for inhomogeneous structures, obtained by coupling substructures with quite different stiffness moduli near the coupling boundary. The underlying theory of this analysis is given in [21].

In general, error-measures should possess certain characteristics in order to be usable as a tool in the adaptive finite-element method. Some of the desirable characteristics for error-estimates are: simple to code, low computational cost, independent of the nature of problem, reliable, accurate, unique in coarse and fine mesh, unique in h -refinement (grid refinement) and p -refinement (increasing the order of polynomial approximation within a selected number of elements), and physically and mathematically meaningful. An investigation of the existing methods in error-estimates shows that among the error-estimate approaches those methods which only indicate the error in the finite-element solution such as those using residual calculations are the least reliable and those using the duality method are the most reliable.

(b) Developing new procedures.

The central theme of this thesis is the mathematical analysis of the error-bound using the Duality Method. A comprehensive analysis of an efficient method of computing the statically admissible stress-field and error-bound is the main goal. Some other stress fields which are theoretically less involved are introduced. These stress fields are approximations to the statically admissible stress-field. (These stress-fields are defined in appendix 1.) The stress-fields are applied to compute various new error estimates. (Appendix 2 presents these error-measures.) An existing error-measure in the Post Processing Method (Error estimate employed by Zienkiewicz and Zhu) is analyzed with respect to the theoretical aspects of error-bound and statically admissible fields. Also, a summary of error analysis using the Element Residual Method is presented.

(c) Structuring this knowledge to allow the rational choice of error-estimates on the basis of reliability, accuracy, software development cost, and computational cost.

The methods of error-analysis are investigated based on the following aspects:

- i- the accuracy of the stress-fields,
- ii- the reliability and accuracy of the error measures calculated by these methods,
- iii- implementation cost,
- iv- computational cost.

The comparison of the methods of error analysis allows users to choose the appropriate method for their analysis.

(d) Implementing and testing of error-measures.

Four methods of error analysis including the approximate SAS field and the approximate error-bound are implemented in a computer program. These methods are:

- 1- Approximate Error-Bound computed from the KAS field and the mean-SAS fields;
- 2- Estimated Error-Bound applying the KAS and the quasi-SAS fields;
- 3- Error Estimates using the KAS and the quasi-SAS fields;
- 4- Error Estimates presented by Zienkiewicz and Zhu.

The implementation of these methods will provide us with an explicit knowledge of the associated cost.

For testing purposes, a set of problems are solved by using the finite-element method and the errors in the solutions are estimated by employing those methods of error-analysis which are implemented in the computer program. For evaluation purposes, the exact errors in some of the test problems, for which the exact solutions are known, are computed and compared with the estimated ones. For problems in which no exact solutions exist, the solutions are compared with those which are obtained with an extremely fine mesh. These are taken as the exact solutions. The accuracy and the computational cost associated with the error-measures can be determined in relation to the results of these test problems.

For further comparison, the method of error analysis employed by Zienkiewicz and Zhu will be applied in the software program and the same test problems are solved.

2. Error-Bound Using Statically Admissible Stress-Fields

2.1 Basic Ideas

In continuum mechanics, the displacement finite-element method is usually used to solve for the displacement field u ; then the stress-field σ is computed from the displacement field through the linear transformation:

$$\sigma_h = C^{-1} B u_h$$

In these equations, the subscript h denotes the finite-element approximations to the functions, B is the gradient operator such that $Bu = 1/2(\nabla u + (\nabla u)^T)$, and C is the tensor of physical properties which is assumed to be positive definite. In explicit representation, the variables, σ , B , and u are defined by:

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

The computed stress-field σ_h , which is called the kinematically admissible stress (KAS) field, need not satisfy the equations of equilibrium in the interior of domain, i.e.,

$$\text{div } \sigma_h + f \neq 0$$

for the body-force f . In addition, the traction vector computed from the KAS field can be non-equilibrating field across the boundaries of elements; moreover, the traction vector need not satisfy the prescribed traction boundary conditions. If the traction vector computed from a stress-field has a jump across the boundaries of elements, the associated stress-field may not be statically admissible.

One of the objectives of this section is to present a comprehensive analysis of function spaces for kinematically and statically admissible fields. These fields will be studied with

respect to finite dimensional subspaces. Procedures will be presented to compute the SAS field, using the prescribed data (prescribed traction boundary conditions and body-forces) and the finite-element solution to problems of solid mechanics, over linear tetrahedral and hexahedral elements.

Another objective of this section is to apply the Duality Method in error analysis by computing the error-bound associated with the solutions of problems of solid mechanics computed by using the finite-element method. In the Duality Method, which is the most reliable technique in error analysis, the KAS and SAS fields are defined to compute an upper bound, in the energy norm, for the error incurred in the finite-element solution. Moreover, the kinematically and statically admissible stress-fields can determine upper and lower bounds on the energy of the exact solution of the problem.

2.2 Mathematical Formulation of the Problem

Let (u, σ) be a displacement field and a stress-field respectively, on an open bounded subset Ω of $\mathbb{R}^3$ with the boundary $\partial\Omega$. For generality, we assume mixed boundary conditions prescribed on two complementary subsets $\partial_1\Omega$ and $\partial_2\Omega$ of $\partial\Omega$. For instance, let the displacement $\bar{u}$ be specified on $\partial_1\Omega$ and the traction $\bar{T}_{(n)}$ on $\partial_2\Omega$. We would like to compute the solutions (u, σ) by using the finite-element method, for the following system of differential equations:

$$\begin{cases} \operatorname{div} \sigma + f = 0 & \text{(equations of equilibrium)} \\ \epsilon = Bu & \text{(equations of continuity)} \\ \sigma = C^{-1}\epsilon & \text{(constitutive law)} \end{cases} \quad (2.1)$$

subject to the boundary conditions

$$u|_{\partial_1\Omega} = \bar{u} \quad n^T \cdot \sigma|_{\partial_2\Omega} = \bar{T}_{(n)} \quad (2.2)$$

Also, we would like to compute the error for such solutions. In the above equations (2.1), f, C , and B are defined as before; $n = (n_x, n_y, n_z)$ is the unit vector in the outward normal direction on $\partial\Omega$ and n^T denotes its transpose.

Naturally, the system of partial differential equations (2.1) with boundary conditions (2.2) defines a dual problem with solutions (u, σ) . This is a saddle point problem which can be solved by the Mini-Max (minimum-maximum) principle. In order to describe a

dual problem, the function spaces of kinematically and statically admissible fields and their correspondence are introduced. These function spaces are presented in the following section.

2.3 Function Spaces of Admissible Fields

To solve for a weak solution of a system of differential equations (2.1) subject to boundary conditions (2.2), the two spaces of inhomogeneous and homogeneous kinematically admissible displacements, denoted by $\aleph$ and $\aleph_0$ respectively, are defined by:

$$\aleph = \{u \in [H^1(\Omega)]^3, u|_{\partial_1\Omega} = \bar{u}\} \quad (2.3)$$

$$\aleph_0 = \{u \in [H^1(\Omega)]^3, u|_{\partial_1\Omega} = 0\} \quad (2.4)$$

In the above spaces, $H^1(\Omega)$ denotes the Sobolev space. This space is defined by:

$$H^1(\Omega) = \{u | \nabla u \in L^2(\Omega)\}$$

and by the space $L^2(\Omega)$ we mean:

$$L^2(\Omega) = \{u | \int_{\Omega} |u|^2 d\Omega < \infty\}$$

The mathematical details of these spaces are found in [22]. The spaces of stresses, Σ and Σ_0 , are constructed from spaces $\aleph$ and $\aleph_0$, respectively, through the constitutive law. These spaces are expressed by:

$$\Sigma = \{\sigma | \sigma \in L^2(\Omega), \sigma = C^{-1}[Bu], u \in \aleph\} \quad (2.5)$$

$$\Sigma_0 = \{\sigma | \sigma \in L^2(\Omega), \sigma = C^{-1}[Bu], u \in \aleph_0\} \quad (2.6)$$

These are called the spaces of inhomogeneous and homogeneous kinematically admissible stress (KAS) fields respectively. Also, the spaces of inhomogeneous and homogeneous statically admissible stress (SAS) fields, χ and χ_0 , are defined by:

$$\chi = \{\sigma | \sigma \in H(\text{div}, \Omega), \text{div } \sigma + f = 0 \text{ in } \Omega, n^T \cdot \sigma = \bar{T}_{(n)} \text{ on the boundary } \partial_2\Omega\} \quad (2.7)$$

$$\chi_0 = \{\sigma | \sigma \in H(\text{div}, \Omega), \text{div } \sigma = 0 \text{ in } \Omega, n^T \cdot \sigma = 0 \text{ on the boundary } \partial_2\Omega\} \quad (2.8)$$

assuming $\bar{T}_{(n)}$ and f are square integrable, i.e., $\bar{T}_{(n)}, f \in L^2(\Omega)$. In the SAS spaces χ and χ_0 , we refer to $H(\text{div}, \Omega)$ as the induced Sobolev space defined by:

$$H(\text{div}, \Omega) = \{\sigma \mid \text{div } \sigma \in L^2(\Omega)\}$$

The norm of the stress vector σ , in the function spaces, is defined in the strain energy measure by the following inner product:

$$(\sigma, \sigma) = \int_{\Omega} \sigma^T C \sigma \, d\Omega \quad (2.9)$$

The kinematically and statically admissible stress spaces of Σ, Σ_0, χ , and χ_0 , respectively, are the fundamental concepts in this analysis. One of the objectives of this section is to obtain a finite-element solution of problem (2.1) subject to boundary conditions (2.2) such that the solution satisfies conditions (2.7); in other words, derive a statically admissible stress-field from the kinematically admissible stress-field through a sequence of operations. The second objective is to apply the SAS field to compute the errors in the finite-element solutions.

2.4 Geometry of the Function Spaces

Since the power of the method of error-bound lies in an important characteristic of the function spaces, i.e., the mutual orthogonality of the function spaces, first we prove this fundamental property; then we shall present the expression to compute the error-bound. The Lemma which expresses the fundamental property of the function spaces is:

Lemma 2.1: The homogeneous kinematically and homogeneous statically admissible stress spaces, Σ_0 and χ_0 , are mutually orthogonal with respect to inner product (2.9).

This lemma can be proven in this manner;

Let: $\sigma_0 \in \chi_0$ and $s_0 \in \Sigma_0$, then $s_0 = C^{-1} B u_0$, for some $u_0 \in \mathfrak{N}_0$. Using inner product

(2.9);

$$\begin{aligned}
(\sigma_0, s_0) &= \int_{\Omega} \sigma_0^T C s_0 \, d\Omega \\
&= \int_{\Omega} \sigma_0^T B u_0 \, d\Omega \\
&= \int_{\partial\Omega} (\sigma_0^T \cdot n) u_0 \, ds - \int_{\Omega} u_0 B^T \sigma_0 \, d\Omega \\
&= 0
\end{aligned} \tag{2.10}$$

In the above proof, the Green's identity [23], and the definition of χ_0 and Σ_0 have been used. It can be concluded from this lemma that the set containing the intersection of the two spaces of the KAS and SAS fields possesses a unique solution $(\tilde{u}_f, \tilde{\sigma}_f)$ which is the exact solution of problem (2.1) subject to the prescribed displacement and traction boundary conditions. The geometry of the statically and kinematically admissible spaces and the exact solution are presented in Figure (2.1).

Figure (2.1)

Geometry of the function spaces and the exact solution

The following lemma, which utilizes lemma 2.1, describes the error-bound inequality.

Lemma 2.2: The difference between a kinematically admissible stress-field and a statically admissible stress-field associated with the problem in norm (2.9) is an upper bound for the difference between the kinematically admissible stress-field and the exact solution of the problem in that norm.

The proof for this lemma is given as follows:

Let: $\tilde{\sigma}_f$ denote the exact stress solution;
 σ_h denote a kinematically admissible solution;
 $\hat{\sigma}_h$ denote a statically admissible solution of the problem.

Since $(\sigma_h - \tilde{\sigma}_f, \hat{\sigma}_h - \tilde{\sigma}_f) \in \mathbb{N}_0 \times \chi_0$ and by virtue of lemma 2.1: $\mathbb{N}_0 \perp \chi_0$, then:

$$\begin{aligned} \|\hat{\sigma}_h - \sigma_h\|^2 &= \|\hat{\sigma}_h - \tilde{\sigma}_f + \tilde{\sigma}_f - \sigma_h\|^2 \\ &= \|\hat{\sigma}_h - \tilde{\sigma}_f\|^2 + \|\sigma_h - \tilde{\sigma}_f\|^2 \end{aligned} \quad (2.11)$$

It can be seen from equation (2.11) that:

$$\|\sigma_h - \tilde{\sigma}_f\|^2 \leq \|\sigma_h - \hat{\sigma}_h\|^2 \quad (2.12a)$$

$$\|\hat{\sigma}_h - \tilde{\sigma}_f\|^2 \leq \|\sigma_h - \hat{\sigma}_h\|^2 \quad (2.12b)$$

and the lemma is proved.

This lemma expresses the results of the theory of hypercircle developed by Synge [17]. This theory expresses that: the exact state of stress associated with a boundary value problem lies on or inside the hypercircle constructed on the vector joining the KAS field and the SAS field as its diameter. The center of the hypercircle is determined by bisecting the diameter of the hypercircle. The center of the hypercircle enables one to estimate the lower and upper bounds for the error incurred in any approximate stress-field associated with the problem.

2.5 Method of Solution

Let $(\tilde{u}_f, \tilde{\sigma}_f)$ be the exact solutions of problem (2.1) and (2.2), then with the definitions of the function spaces defined in (2.3)-(2.8), this problem is reformulated as follows: Find the displacement and stress-fields $(\tilde{u}_f, \tilde{\sigma}_f)$ such that:

$$\tilde{\sigma}_f = C^{-1} B \tilde{u}_f \quad \text{in which } \tilde{u}_f \in \mathbb{N} \text{ and } \tilde{\sigma}_f \in \chi \quad (2.13)$$

In other words, given arbitrary displacements and stress-fields $(u, \sigma) \in \mathbb{N} \times \chi$, find the fields $(u_0, \sigma_0) \in \mathbb{N}_0 \times \chi_0$ such that:

$$\sigma + \sigma_0 = C^{-1} B(u + u_0)$$

In order to solve for the displacement and stress fields (u, σ) , one may apply the dual method. But, because of complexity involved in this method, the displacement formulation is more frequently used. This formulation yields a kinematically admissible displacement field from which the kinematically admissible stress-field is computed. A weak form of the equations of equilibrium in terms of displacements is employed as follows:

$$\begin{aligned} \tilde{u}_f &\in u + \mathbb{N}_0 \quad \text{for } u \in \mathbb{N} \\ \int_{\Omega} (B\tilde{u}_f)^T C^{-1} (Bv) d\Omega &= \int_{\Omega} f v d\Omega + \int_{\partial_2 \Omega} \bar{T}_{(n)} v ds, \quad \forall v \in \mathbb{N}_0 \\ \tilde{\sigma}_f &= C^{-1} B\tilde{u}_f \end{aligned} \quad (2.14)$$

which admits a unique solution. The above formulation leads to the displacement finite-element method.

Let $\mathbb{N}_h \subset \mathbb{N}$ and $\mathbb{N}_{0h} \subset \mathbb{N}_0$ denote the subspaces of kinematically admissible fields. Let ϖ_h be a mesh obtained by the discretization of the domain Ω with respect to these subspaces. If u_h denote the finite-element approximation to the displacement field u , then in relation to the finite dimensional subspaces, equations of equilibrium (2.14) are:

$$\begin{aligned} u_h &\in \bar{u}_h + \mathbb{N}_{0h} \quad \text{for } \bar{u}_h \in \mathbb{N}_h \\ \int_{\Omega} (Bu_h)^T C^{-1} (Bv_h) d\Omega &= \int_{\Omega} f v_h d\Omega + \int_{\partial_2 \Omega} \bar{T}_{(n)} v_h ds, \quad \forall v_h \in \mathbb{N}_{0h} \subset \mathbb{N}_0 \\ \sigma_h &= C^{-1} Bu_h \quad (\text{within each element } \omega) \end{aligned} \quad (2.15)$$

which possess a unique solution u_h . The computed stress-field using u_h is the kinematically admissible stress-field. We wish to compute the counterpart of this stress-field, i.e., the statically admissible stress-field and apply inequality (2.12) to compute the error-bound associated with the stress-field σ_h .

2.6 Necessary and Sufficient Conditions for SAS Fields

Let ϖ_h be the discretized domain Ω with the discretized boundary $\partial\varpi_h$. Suppose $\hat{\sigma}_h$ is a stress-field computed from a finite-element method, over the discretized domain ϖ_h ,

such that $\hat{\sigma}_h \in C^1$ in the interior of the elements; then referring to the space of SAS fields (2.7), in order for $\hat{\sigma}_h$ to be statically admissible, it must satisfy the following equation:

$$\int_{\Omega} (\operatorname{div} \hat{\sigma}_h + f)v \, d\Omega + \int_{\partial_2 \Omega} (n^T \cdot \hat{\sigma}_h - \bar{T}_{(n)})w \, ds = 0 \quad \forall v, w \in \mathcal{N}_0 \quad (2.16)$$

If the mesh contains l finite-elements ω_j with element-boundaries $\partial\omega_j$, noting the relations $n^T \cdot \hat{\sigma} = \hat{T}_{(n)}$, the above equations indicate that the statically admissible stress field at the element level must satisfy the following conditions:

$$\sum_l \left(\int_{\omega_j} (\operatorname{div} \hat{\sigma}_h + f)v \, d\Omega + \int_{\partial\omega_j} \hat{T}_{(n)}w \, ds - \int_{\partial_2 \Omega} \bar{T}_{(n)}w \, ds \right) = 0 \quad \forall v, w \in \mathcal{N}_0 \quad (2.17)$$

Consider the discretized domain ϖ_h , and let p be a point in the interior of the element ω_j ; there are functions v in $\mathcal{N}_0$ which are non-negative at p and zero outside ω_j and on $\partial\omega_j$ (boundaries of ω_j). By considering the set of all such functions it can be deduced that $\int_{\omega_j} (\operatorname{div} \hat{\sigma}_h + f)v \, d\Omega$ must be zero at p . Since p is an arbitrary point of the interior of ω_j , it follows that $\int_{\omega_j} (\operatorname{div} \hat{\sigma}_h + f)v \, d\Omega$ vanishes identically inside ω_j for all elements ω_j . This discussion implies that one of the necessary conditions for a stress field to be statically admissible, is that it must satisfy the equations of equilibrium within the element, ω_j , i.e.

$$\operatorname{div} \hat{\sigma}_h + f = 0 \quad (2.18)$$

In addition, from (2.17) other conditions must be true on the boundaries of elements. In order to study the necessary conditions on the boundaries of elements, let us define the set Γ of all boundaries of elements in the mesh by:

$$\Gamma = \bigcup_{\omega_j \in \varpi_h} \partial\omega_j$$

in which there are two typical classes of element-boundaries:

- 1- the element-boundaries (element-faces) b_{int} which lie entirely in the interior of the mesh;
- 2- the element-boundaries (element-faces) b_{ext} which lie on the external boundaries $\partial\varpi_h$ of the mesh.

Therefore the set Γ can be written as:

$$\Gamma = \{b_{int} \cup b_{ext}\}$$

Now, considering conditions (2.18) and upon rearranging the sum over the boundaries of elements, condition (2.17) reduces to

$$\sum_{b_{int}} \int_{b_{int}} (\hat{T}_{(n)}^+ - \hat{T}_{(n)}^-) w \, ds + \sum_{b_{ext}} \int_{b_{ext}} (\hat{T}_{(n)}^+ - \bar{T}_{(n)}) w \, ds = 0 \quad \forall w \in \mathbb{N}_0 \quad (2.19)$$

where b_{int} and b_{ext} are the interelement boundaries and the boundaries of elements exposed to the external boundaries respectively. Note that the $(\hat{T}_{(n)}^+ - \hat{T}_{(n)}^-)$ and $(\hat{T}_{(n)}^+ - \bar{T}_{(n)})$ are the jumps at the interelement and external boundaries. Similar to the above discussion, if q is a point on the interelement boundary b_{int} , and ϵ is a neighbourhood of q , it is possible to find w in $\mathbb{N}_0$ which is non-zero at q and vanishes outside ϵ . By considering the set of all such functions w the jump integral $\int_{b_{int}} (\hat{T}_{(n)}^+ - \hat{T}_{(n)}^-) w \, ds$ must vanish at q . Again since q is an arbitrary point on the boundary b_{int} , the above jump integral vanishes at every point of all interelement boundaries. This implies that the tractions across the boundaries of elements must be continuous. A similar discussion can be made for the boundaries of elements lying on $\partial_2\Omega$. From the above analysis, the following theorem for the statically admissible stress-field is stated:

Theorem 2.1: If $\hat{\sigma}_h$ be statically admissible stress-field, then:

- (1) $\text{div } \hat{\sigma}_h + f = 0$ in the interior of each element ω_j ;
- (2) $n^T \cdot \hat{\sigma}_h$ is single valued (continuous) across the boundaries of elements in the interior of Ω ; in other words, the element-tractions are in equilibrium across the boundaries of elements.
- (3) $n^T \cdot \hat{\sigma}_h = \bar{T}_{(n)}$ on the boundaries $\partial_2\Omega$.

Theorem 2.1 can be restated as follows: the necessary and sufficient conditions for a stress-field $\hat{\sigma}_h$ to be statically admissible is to satisfy the following conditions:

- 1- The stress-field must satisfy the equations of equilibrium in the interior of the elements.
- 2- The stress-vectors (element-tractions) must be single valued (continuous) across the boundaries of elements.
- 3- On the external boundaries, the stress-vectors (element-tractions) must be in equilibrium with the prescribed traction boundary conditions.

When constructing the SAS field, the above conditions must be imposed on the stress-

fields. The main task is to compute the *nodal forces due to tractions* on the element-faces. The *nodal forces due to tractions*, $\int_{\partial\omega_j} T \cdot a_i \, ds$, are defined by integrating the traction vectors multiplied by the basis function associated with the nodes, over the boundaries of elements: $\int_{\partial\omega_j} T \cdot a_i \, ds = \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega$. This representation ensures the continuity of tractions across the boundaries of elements. In addition, the *nodal forces due to tractions* are such that the element-tractions are in equilibrium with the body-force within each element. If the *nodal forces due to tractions* are computed, then the calculation of the stress-field within the elements can be performed locally from these *nodal forces due to tractions*.

2.7 Constructing the SAS Fields

2.7.1 Basic Definitions

In order to construct a statically admissible stress-field, one must define a stress-field which satisfies the foregoing conditions. Ladeveze *et al* [20] and [19], constructed the SAS field $\hat{\sigma}_h$ in two steps: In the first step, the *nodal forces due to tractions* are computed such that each element is in equilibrium with respect to the applied tractions and body-forces. The method of computing the *nodal forces due to tractions*, as will be explained subsequently, ensures the continuity of the tractions across the common boundaries of the adjacent elements. In the second step, the SAS field in each element is computed by using the *nodal forces due to tractions*, which are known from the previous step. This stress-field is required to satisfy the equations of equilibrium in the interior of elements and $n^T \cdot \sigma = T$ on the element-boundary.

To compute the *nodal force due to tractions*, let Γ be a set containing the boundaries of elements; define a vector-valued function T over the set Γ of the boundaries of elements such that $T = \bar{T}_{(n)}$ on the external boundaries $\partial_2\Omega$. This requires a sign convention to define the orientation of the boundaries of elements. To incorporate the sign associated with the element-tractions T , define a function $\eta_{\omega_j}^b$ over the boundaries of elements such that this function takes the values from the set $\{-1, +1\}$ according to the following conventions:

- (i) If the boundary (element-face) b lies on $\partial\Omega$, i.e., the external boundary b_{ext} , then:

$$\eta_{\omega_j}^b = +1.$$

- (ii) If the boundary (element-face) b is the common boundary between the two elements ω_j and ω_k , then $\eta_{\omega_j}^b$ and $\eta_{\omega_k}^b$ have opposite signs.

Now we will explicitly state the requirements that T must satisfy. Integrate the equations of equilibrium:

$$\text{div } \hat{\sigma}_h + f = 0$$

over the element ω_j , then apply the Green's identity, and make use of the fact that on the boundaries $\partial\omega_j$ we must have:

$$n^T \cdot \hat{\sigma}_h = \eta_{\omega_j} T$$

it follows that the condition:

$$\int_{\omega_j} f \, d\Omega + \int_{\partial\omega_j} \eta_{\omega_j} T \, ds = 0 \quad (2.20)$$

must be true for the applied tractions and body-force. This equation, which is called the solvability condition, characterizes the equilibrium of the external forces on the element ω_j . In addition to condition (2.20), the element-traction T must be continuous across the boundaries b . Furthermore, the element-traction T must be equal to the prescribed traction boundary conditions $\bar{T}$ on the external boundary $\partial_2\Omega$.

To determine the element-tractions T , one must solve equation (2.20). If we let $x^b = \int_b T \, ds$; then equations (2.20) can be written as:

$$\sum_{b \subseteq \partial\omega_j} \eta_{\omega_j}^b x^b = \int_{\omega_j} f \, d\Omega \quad \text{for each element } \omega_j$$

This is a linear system which has, in general, infinitely many solutions. Unfortunately, the coefficient matrix is not structured in any very favourable way, so we would like to find a better way of determining T [24].

In order to compute the element-traction T , one must make the discrete problem more specific referring it to a particular family of elements. Ladeveze *et al* have solved this problem for a mesh containing constant strain elements. Hughes [24] has employed this method over linear quadrilateral elements. In this study, an attempt is made to solve this problem over a mesh containing linear hexahedral elements by modifying the method applied by Hughes [24]. We will assume that T is a bilinear function over each face of the

elements. This means that one can specify T on an element-face by four vectors expressing the values of T at the nodes of the element-face. Now T is specified separately on each element-face; and, as it will be seen, it is advantageous to specify T on an element-face using a different representation than the nodal values [24]. This representation would describe the *nodal force due to tractions* on the nodes.

The *nodal force due to tractions* must be computed for every node in the mesh. There are three types of nodes within the mesh: internal nodes, nodes lying on the external boundaries $\partial_1\Omega$, and the nodes lying on the external boundaries $\partial_2\Omega$. In the following section the procedure for computing the *nodal force due to tractions* is described. In this calculation, it is assumed that the node under consideration lies in the interior of the mesh. It can be seen that for the case in which the node lies on the external boundary $\partial_1\Omega$, a similar calculation, with a minor modification, is applicable. Finally, the case in which the node is lying on the external boundary $\partial_2\Omega$, the *nodal force due to tractions* is computed directly over this boundary. A general algorithm will be presented to calculate the *nodal force due to tractions*. This algorithm includes the three categories of nodes in the mesh.

2.7.2 Computing the Nodal Force Due to Tractions

The methodology for computing the *nodal force due to tractions* is based on the procedure applied by Ladeveze *et al* [20] over constant strain elements, and the modified method by Hughes [24] on linear quadrilateral elements. Ladeveze *et al*, [14], have defined a space of statically admissible field as a restriction to the finite-element solutions. In this space the nodal forces due to the application of statically admissible stress-field equals those computed using the kinematically admissible stress-fields. In mathematical terminology, this space is described by [14]:

$$\hat{\chi}_f = \{\sigma \in \hat{\sigma}_h + \chi_0, \int_{\omega_j} B_h^T(\sigma - \sigma_h) d\Omega = 0, \forall i \in \bar{k} \text{ (set of nodes)}, \forall \omega_j \in \varpi_h\} \quad (2.21)$$

This definition allows one to calculate the SAS field as a local problem. We will apply the same space of statically admissible stress-field in our calculations.

Assuming the mesh, ϖ_h , consists of linear hexahedral elements. The boundary $\partial\omega_j$ of the element ω_j is made up of six boundaries b_i , ($i = 1, 2, \dots, 6$) which are called "element-

faces". Each element-face, b_i , contains four corner nodes. Let i denote an interior node in the mesh; let P_h be a set containing eight elements, $\omega_j, \omega_k, \dots$ such that every element in the set P_h contains node i , and every two elements in this set share an element-face. Define another set $\mathcal{F}$ consisting of all element-faces which contain the node i . This set, which possesses twelve members, is expressed by:

$$\mathcal{F} = \{b_1, b_2, \dots, b_{12}\}$$

Figure (2.2) illustrates the geometry of the elements in the set P_h , the node i , and the element-faces b_m ($m = 1, 2, \dots, 12$).

Figure (2.2)

Geometry of the set of elements containing the node i

If Γ is the set of all boundaries of elements in the mesh as defined before, ($\Gamma = \{b_{int} \cup b_{ext}\}$), and T denotes the traction vector defined over the set Γ such that $T = T_{(n)}$ over the external boundaries $\partial_2\Omega$, we want to compute the *nodal force due to tractions* T . Let ω_j be an element in $\mathcal{P}_h$, and N_i denote the basis function associated with the node i . Define three vectors: a_i , b_i , and c_i associated with the node i in element ω_j such that:

$$a_i = \begin{bmatrix} N_i \\ 0 \\ 0 \end{bmatrix} \quad b_i = \begin{bmatrix} 0 \\ N_i \\ 0 \end{bmatrix} \quad c_i = \begin{bmatrix} 0 \\ 0 \\ N_i \end{bmatrix} \quad (2.22)$$

Within the element ω_j , the error in the equation of equilibrium:

$$\text{div } \hat{\sigma}_h + f = 0$$

must be orthogonal to the vectors a_i , b_i , and c_i with respect to energy inner product (2.9). This condition is expressed by

$$\begin{cases} \int_{\omega_j} (\text{div } \hat{\sigma}_h + f) \cdot a_i \, d\Omega = 0 \\ \int_{\omega_j} (\text{div } \hat{\sigma}_h + f) \cdot b_i \, d\Omega = 0 \\ \int_{\omega_j} (\text{div } \hat{\sigma}_h + f) \cdot c_i \, d\Omega = 0 \end{cases}$$

Using the divergence theorem over the above equations, we get:

$$\begin{cases} - \int_{\omega_j} \hat{\sigma}_h B a_i \, d\Omega + \int_{\omega_j} f \cdot a_i \, d\Omega + \int_{\partial\omega_j} (n_{\omega_j}^T \cdot \hat{\sigma}_h) \cdot a_i \, ds = 0 \\ - \int_{\omega_j} \hat{\sigma}_h B b_i \, d\Omega + \int_{\omega_j} f \cdot b_i \, d\Omega + \int_{\partial\omega_j} (n_{\omega_j}^T \cdot \hat{\sigma}_h) \cdot b_i \, ds = 0 \\ - \int_{\omega_j} \hat{\sigma}_h B c_i \, d\Omega + \int_{\omega_j} f \cdot c_i \, d\Omega + \int_{\partial\omega_j} (n_{\omega_j}^T \cdot \hat{\sigma}_h) \cdot c_i \, ds = 0 \end{cases} \quad (2.23)$$

Recall theorem (2.1) on SAS fields, which gives:

$$n_{\omega_j}^T \cdot \hat{\sigma}_h = \eta_{\omega_j} T \quad (2.24)$$

in which η_{ω_j} is defined as before. Making use of equation (2.24) in equations (2.23) and rearranging these equations, we get:

$$\begin{cases} \int_{\partial\omega_j} \eta_{\omega_j} T \cdot a_i \, ds = \int_{\omega_j} \hat{\sigma}_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot b_i \, ds = \int_{\omega_j} \hat{\sigma}_h B b_i \, d\Omega - \int_{\omega_j} f \cdot b_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot c_i \, ds = \int_{\omega_j} \hat{\sigma}_h B c_i \, d\Omega - \int_{\omega_j} f \cdot c_i \, d\Omega \end{cases} \quad (2.25)$$

In order to compute the left hand side of these equations, the integrals:

$$\int_{\omega_j} \hat{\sigma}_h B a_i \, d\Omega, \quad \int_{\omega_j} \hat{\sigma}_h B b_i \, d\Omega, \quad \text{and} \quad \int_{\omega_j} \hat{\sigma}_h B c_i \, d\Omega$$

must be known. As was mentioned before, Ladeveze *et al*, [20], have defined the space of SAS fields as a restriction to the finite-element solution. In the extension space they have assumed that:

$$\int_{\omega_j} \partial_h B a_i \, d\Omega = \int_{\omega_j} \sigma_h B a_i \, d\Omega \quad (2.26)$$

$$\int_{\omega_j} \partial_h B b_i \, d\Omega = \int_{\omega_j} \sigma_h B b_i \, d\Omega \quad (2.27)$$

$$\int_{\omega_j} \partial_h B c_i \, d\Omega = \int_{\omega_j} \sigma_h B c_i \, d\Omega \quad (2.28)$$

in which σ_h is the known KAS field. Substituting equations (2.26)-(2.28) into equations (2.25), the following system of equations is derived:

$$\begin{cases} \int_{\partial\omega_j} \eta_{\omega_j} T \cdot a_i \, ds = \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot b_i \, ds = \int_{\omega_j} \sigma_h B b_i \, d\Omega - \int_{\omega_j} f \cdot b_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot c_i \, ds = \int_{\omega_j} \sigma_h B c_i \, d\Omega - \int_{\omega_j} f \cdot c_i \, d\Omega \end{cases} \quad (2.29)$$

which can be solved for the unknown *nodal force due to tractions*. The integrals in the left hand side of these equations can be written as the sum of the integrals over the element-faces b . So for each node i and for each element ω_j that intersects the support of a_i , b_i , or c_i , we have:

$$\begin{cases} \sum_{b \subseteq \partial\omega_j} \eta_{\omega_j}^b x_i^b = P_i^{\omega_j} \\ \sum_{b \subseteq \partial\omega_j} \eta_{\omega_j}^b y_i^b = Q_i^{\omega_j} \\ \sum_{b \subseteq \partial\omega_j} \eta_{\omega_j}^b z_i^b = R_i^{\omega_j} \end{cases} \quad (2.30)$$

in which the variables are defined by:

$$\begin{aligned} x_i^b &= \int_b T \cdot a_i \, d\Omega, \\ y_i^b &= \int_b T \cdot b_i \, d\Omega, \\ z_i^b &= \int_b T \cdot c_i \, d\Omega, \\ P_i^{\omega_j} &= \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega \\ Q_i^{\omega_j} &= \int_{\omega_j} \sigma_h B b_i \, d\Omega - \int_{\omega_j} f \cdot b_i \, d\Omega \\ R_i^{\omega_j} &= \int_{\omega_j} \sigma_h B c_i \, d\Omega - \int_{\omega_j} f \cdot c_i \, d\Omega \end{aligned} \quad (2.31)$$

In these expressions, x_i^b , y_i^b , and z_i^b are the components of the nodal force on the node i due to tractions over the element-face b . It can be shown that the sum of all the a_i^j 's, b_i^j 's, and c_i^j 's at any point of the domain are the constants:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

respectively; therefore it follows that any T which satisfies equations (2.30), must satisfy conditions (2.20).

Now we wish to set up the system of equations to solve for the unknowns, x_i^b , y_i^b , and z_i^b . According to (2.30), for fixed node i , there is a set of three equations from each element ω_j . For the set P_h of 8 elements containing the node i there are, therefore, 24 equations such that in every equation only three non-zero terms exist because a_i , b_i , and c_i are zero on the boundaries not incident upon the node i . In general, x_i^b is non-zero only for 12 element-faces which are contained in the set $\mathcal{F}$. Only three of these faces appear in any of equations (2.30). The entire system of equations is block diagonal. According to Figure (2.2), the block corresponding to the node i associated with the vector a_i is written as:

$$\begin{aligned} \omega_1 : \quad & \eta_{\omega_1}^{b_1} x_i^{b_1} + \eta_{\omega_1}^{b_4} x_i^{b_4} + \eta_{\omega_1}^{b_9} x_i^{b_9} = P_i^{\omega_1} \\ \omega_2 : \quad & \eta_{\omega_2}^{b_1} x_i^{b_1} + \eta_{\omega_2}^{b_2} x_i^{b_2} + \eta_{\omega_2}^{b_{10}} x_i^{b_{10}} = P_i^{\omega_2} \\ \omega_3 : \quad & \eta_{\omega_3}^{b_2} x_i^{b_2} + \eta_{\omega_3}^{b_3} x_i^{b_3} + \eta_{\omega_3}^{b_{11}} x_i^{b_{11}} = P_i^{\omega_3} \\ \omega_4 : \quad & \eta_{\omega_4}^{b_3} x_i^{b_3} + \eta_{\omega_4}^{b_4} x_i^{b_4} + \eta_{\omega_4}^{b_{12}} x_i^{b_{12}} = P_i^{\omega_4} \\ \omega_5 : \quad & \eta_{\omega_5}^{b_5} x_i^{b_5} + \eta_{\omega_5}^{b_8} x_i^{b_8} + \eta_{\omega_5}^{b_9} x_i^{b_9} = P_i^{\omega_5} \\ \omega_6 : \quad & \eta_{\omega_6}^{b_5} x_i^{b_5} + \eta_{\omega_6}^{b_6} x_i^{b_6} + \eta_{\omega_6}^{b_{10}} x_i^{b_{10}} = P_i^{\omega_6} \\ \omega_7 : \quad & \eta_{\omega_7}^{b_6} x_i^{b_6} + \eta_{\omega_7}^{b_7} x_i^{b_7} + \eta_{\omega_7}^{b_{11}} x_i^{b_{11}} = P_i^{\omega_7} \\ \omega_8 : \quad & \eta_{\omega_8}^{b_7} x_i^{b_7} + \eta_{\omega_8}^{b_8} x_i^{b_8} + \eta_{\omega_8}^{b_{12}} x_i^{b_{12}} = P_i^{\omega_8} \end{aligned} \quad (2.32)$$

which is a system of eight equations in 12 unknowns: $x_i^{b_1}, x_i^{b_2}, \dots, x_i^{b_{12}}$. This system of equations is written in matrix form as follows:

$$\begin{bmatrix} \eta_{\omega_1}^{b_1} & & & & & & & & & & \\ & \eta_{\omega_1}^{b_4} & & & & & & & & & \\ & & \eta_{\omega_2}^{b_9} & & & & & & & & \\ \eta_{\omega_2}^{b_1} & \eta_{\omega_2}^{b_2} & & & & & & & & & \\ & \eta_{\omega_3}^{b_2} & \eta_{\omega_3}^{b_3} & & & & & & & & \\ & & \eta_{\omega_4}^{b_3} & \eta_{\omega_4}^{b_4} & & & & & & & \\ & & & \eta_{\omega_5}^{b_5} & \eta_{\omega_5}^{b_8} & \eta_{\omega_5}^{b_9} & & & & & \\ & & & \eta_{\omega_6}^{b_5} & \eta_{\omega_6}^{b_6} & & \eta_{\omega_6}^{b_{10}} & & & & \\ & & & & \eta_{\omega_7}^{b_6} & \eta_{\omega_7}^{b_7} & & \eta_{\omega_7}^{b_{11}} & & & \\ & & & & & \eta_{\omega_8}^{b_7} & \eta_{\omega_8}^{b_8} & & \eta_{\omega_8}^{b_{12}} & & \\ & & & & & & & & & & \end{bmatrix} \begin{bmatrix} x_i^{b_1} \\ x_i^{b_2} \\ x_i^{b_3} \\ x_i^{b_4} \\ x_i^{b_5} \\ x_i^{b_6} \\ x_i^{b_7} \\ x_i^{b_8} \\ x_i^{b_9} \\ x_i^{b_{10}} \\ x_i^{b_{11}} \\ x_i^{b_{12}} \end{bmatrix} = \begin{bmatrix} P_i^{\omega_1} \\ P_i^{\omega_2} \\ P_i^{\omega_3} \\ P_i^{\omega_4} \\ P_i^{\omega_5} \\ P_i^{\omega_6} \\ P_i^{\omega_7} \\ P_i^{\omega_8} \end{bmatrix} \quad (2.33)$$

Applying the same calculation as was described before and upon using the vectors b_i or c_i instead of the vector a_i , a system of equations similar to equations (2.33) is obtained. The appropriate system of equations are given as:

$$\begin{bmatrix} \eta_{\omega_1}^{b_1} & & & & & & & & & & & \\ & \eta_{\omega_1}^{b_4} & & & & & & & & & & \\ & & \eta_{\omega_2}^{b_1} & & & & & & & & & \\ & & & \eta_{\omega_2}^{b_2} & & & & & & & & \\ & & & & \eta_{\omega_3}^{b_2} & & & & & & & \\ & & & & & \eta_{\omega_3}^{b_3} & & & & & & \\ & & & & & & \eta_{\omega_4}^{b_3} & & & & & \\ & & & & & & & \eta_{\omega_4}^{b_4} & & & & \\ & & & & & & & & \eta_{\omega_5}^{b_5} & & & \\ & & & & & & & & & \eta_{\omega_5}^{b_8} & & \\ & & & & & & & & & & \eta_{\omega_5}^{b_9} & \\ & & & & & & & & & & & \eta_{\omega_6}^{b_{10}} \\ & & & & & & & & & & & & \eta_{\omega_3}^{b_{11}} \\ & & & & & & & & & & & & & \eta_{\omega_4}^{b_{12}} \\ & & & & & & & & & & & & & & \eta_{\omega_6}^{b_{10}} \\ & & & & & & & & & & & & & & & \eta_{\omega_7}^{b_{11}} \\ & & & & & & & & & & & & & & & & \eta_{\omega_8}^{b_{12}} \\ & & & & & & & & & & & & & & & & & \eta_{\omega_8}^{b_8} \end{bmatrix} \begin{bmatrix} y_i^{b_1} \\ y_i^{b_2} \\ y_i^{b_3} \\ y_i^{b_4} \\ y_i^{b_5} \\ y_i^{b_6} \\ y_i^{b_7} \\ y_i^{b_8} \\ y_i^{b_9} \\ y_i^{b_{10}} \\ y_i^{b_{11}} \\ y_i^{b_{12}} \end{bmatrix} = \begin{bmatrix} Q_i^{\omega_1} \\ Q_i^{\omega_2} \\ Q_i^{\omega_3} \\ Q_i^{\omega_4} \\ Q_i^{\omega_5} \\ Q_i^{\omega_6} \\ Q_i^{\omega_7} \\ Q_i^{\omega_8} \end{bmatrix} \quad (2.34)$$

these components are determined from the following expressions:

$$\begin{aligned} P_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot a_i \, ds \\ Q_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot b_i \, ds \\ R_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot c_i \, ds \end{aligned} \quad (2.37)$$

in which b_m ($m = 1, 2, \dots, 12$) indicates the common boundary of the adjacent elements ω_j and ω_k . To minimize the system (2.36) with respect to the average tractions (2.37), the following notations are introduced:

$\{x_i\}$ - Unknown vector of the *nodal force due to tractions* associated with the SAS fields.

$\{x_i^o\}$ - Vector of the *nodal force due to tractions* associated with the averaged tractions using the KAS fields.

$[M]$ - Rectangular coefficient matrix in linear system (2.36);

$\{P_i^w\}$ - Column vector in system (2.36);

$[Z]$ - Matrix whose columns are the bases of nullspace of the coefficient matrix $[M]$.

$\{P_i^b\}$ - Column vector of the *nodal force due to averaged tractions* from the KAS fields, on the element-faces.

Applying these notations, equations (2.36) is written as:

$$[M]\{x_i\} = \{P_i^w\} \quad (2.38)$$

Now we wish to compute the unknown vector $\{x_i\}$ that minimizes the distance between $\{x_i\}$ to $\{x_i^o\}$. In other words, we want to minimize the functional:

$$\|x_i - x_i^o\|^2 \quad (2.39)$$

Hughes [24] has shown that in order to minimize functional (2.39), the vector $\{x_i - x_i^o\}$ must be orthogonal to the nullspace of the coefficient matrices $[M]$.

Let $[Z] = (z_1, z_2, \dots, z_k)$ be the nullspace $\mathcal{N}(M)$ in which $z_1, z_2, \dots, z_k$ are the basis vectors which span this space, then according to the above orthogonality condition we must have:

$$z_k^T \{x_i - x_i^o\} = 0 \quad \forall z_k \in Z \quad (2.40)$$

This condition provides a sufficient number of equations to solve the system (2.38) to determine the unknown vector $\{x_i\}$ uniquely. The complete system of equations are written as:

$$\begin{cases} [M]\{x_i\} = \{P_i^w\} \\ [Z]^T\{x_i - x_i^o\} = 0 \end{cases} \quad (2.41)$$

In order to express conditions (2.40) in terms of equations, the nullspace of the coefficient matrix $[M]$ must be built. The nullspace, $\mathcal{N}(M)$, is defined to span the subspace for the set of all vectors $\{X\}$ that satisfies the equations:

$$[M]\{X\} = 0$$

Using the Singular Value Decomposition Method [25], a set of basis vectors for the nullspace $\mathcal{N}(M)$ is derived as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ +1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ +1 \end{bmatrix} \begin{bmatrix} +1 \\ +1 \\ 0 \\ 0 \\ -1 \\ 0 \\ +1 \\ 0 \\ -1 \\ 0 \\ +1 \\ 0 \end{bmatrix} \begin{bmatrix} +1 \\ 0 \\ 0 \\ 0 \\ -1 \\ +1 \\ +1 \\ 0 \\ -1 \\ +1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ +1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.42)$$

A set of sophisticated routines are provided in reference [25] to compute these basis vectors. LINPACK [26], also, presents the software to compute the nullspace of a rectangular matrix.

Applying the expressions (2.37) and (2.42) into (2.40), we get:

$$\begin{bmatrix}
 & & +1 & -1 & & +1 \\
 +1 & +1 & & -1 & +1 & -1 & +1 \\
 +1 & & -1 & +1 & +1 & & -1 & +1 \\
 -1 & -1 & -1 & +1 & & & &
 \end{bmatrix}
 \begin{bmatrix}
 x_i^{b_1} \\
 x_i^{b_2} \\
 x_i^{b_3} \\
 x_i^{b_4} \\
 x_i^{b_5} \\
 x_i^{b_6} \\
 x_i^{b_7} \\
 x_i^{b_8} \\
 x_i^{b_9} \\
 x_i^{b_{10}} \\
 x_i^{b_{11}} \\
 x_i^{b_{12}}
 \end{bmatrix}
 =
 \begin{bmatrix}
 P_i^{b_5} - P_i^{b_8} + P_i^{b_{12}} \\
 P_i^{b_1} + P_i^{b_2} - P_i^{b_5} + P_i^{b_7} - P_i^{b_9} + P_i^{b_{11}} \\
 P_i^{b_1} - P_i^{b_5} - P_i^{b_6} + P_i^{b_7} - P_i^{b_9} + P_i^{b_{10}} \\
 -P_i^{b_1} - P_i^{b_2} - P_i^{b_3} + P_i^{b_4}
 \end{bmatrix}
 \quad (2.43)$$

which is combined with equations (2.36) to yield the system of equations:

$$\begin{bmatrix}
 +1 & & +1 & & +1 \\
 -1 & +1 & & & +1 \\
 & -1 & +1 & & +1 \\
 & & -1 & -1 & +1 \\
 & & & +1 & +1 & -1 \\
 & -1 & +1 & & -1 \\
 & & -1 & +1 & -1 \\
 & & & -1 & -1 & -1 \\
 & +1 & & -1 & +1 \\
 +1 & +1 & -1 & +1 & -1 & +1 \\
 +1 & & -1 & +1 & +1 & -1 & +1 \\
 -1 & -1 & -1 & +1 & & &
 \end{bmatrix}
 \begin{bmatrix}
 x_i^{b_1} \\
 x_i^{b_2} \\
 x_i^{b_3} \\
 x_i^{b_4} \\
 x_i^{b_5} \\
 x_i^{b_6} \\
 x_i^{b_7} \\
 x_i^{b_8} \\
 x_i^{b_9} \\
 x_i^{b_{10}} \\
 x_i^{b_{11}} \\
 x_i^{b_{12}}
 \end{bmatrix}
 =
 \begin{bmatrix}
 P_i^{\omega_1} \\
 P_i^{\omega_2} \\
 P_i^{\omega_3} \\
 P_i^{\omega_4} \\
 P_i^{\omega_5} \\
 P_i^{\omega_6} \\
 P_i^{\omega_7} \\
 P_i^{\omega_8} \\
 P_i^{b_5} - P_i^{b_8} + P_i^{b_{12}} \\
 P_i^{b_1} + P_i^{b_2} - P_i^{b_5} + P_i^{b_7} - P_i^{b_9} + P_i^{b_{11}} \\
 P_i^{b_1} - P_i^{b_5} - P_i^{b_6} + P_i^{b_7} - P_i^{b_9} + P_i^{b_{10}} \\
 -P_i^{b_1} - P_i^{b_2} - P_i^{b_3} + P_i^{b_4}
 \end{bmatrix}
 \quad (2.44)$$

The above system consists of 12 equations in 12 unknowns which determine the components of the *nodal force due to tractions*. This system of equations corresponds to the internal node i presented in Figure (2.2).

In order to determine the other components y and z of the *nodal force due to tractions* on this node, the same coefficient matrix can be used. Employing this matrix, the column vectors associated with these components must be calculated and substituted into equations (2.44). The solutions to these systems yield the appropriate components of the *nodal force due to tractions*.

In order to perform the computations in an efficient manner, the same coefficient matrix calculated for the internal node i can be used for all other internal nodes. To make use of this coefficient matrix for other internal nodes, all characteristics of the method including the numbering schemes of the elements and element-faces in the patch P_h must be carried over the other internal nodes. For the second and subsequent internal nodes, only the relevant column vectors need to be computed. Therefore this method can be effectively used to compute the *nodal force due to tractions* over the nodes interior to the mesh.

So far the method for computing the *nodal force due to tractions* has been described for the node i lying entirely in the interior of the mesh. If the node i lies on the external boundary $\partial_1\Omega$, a similar procedure can be applied to compute the *nodal force due to tractions*. In this case the number of elements and element-faces containing the node vary depending upon the configuration of the boundary.

Finally, if the node i lies on the external boundaries $\partial_2\Omega$, the values of the *nodal force due to tractions* $\bar{T}_{(n)}$ are computed explicitly by:

$$\begin{aligned} x_i^b|_{\partial_2\Omega} &= \int_b \bar{T}_{(n)} \cdot a_i \, d\Omega \\ y_i^b|_{\partial_2\Omega} &= \int_b \bar{T}_{(n)} \cdot b_i \, d\Omega \\ z_i^b|_{\partial_2\Omega} &= \int_b \bar{T}_{(n)} \cdot c_i \, d\Omega \end{aligned} \tag{2.45}$$

and assigned to the node.

An algorithm for implementing the above method for computing the *nodal force due to tractions* is presented in the sequel. This algorithm includes the nodes lying on the external boundaries as well.

2.7.3 Algorithm for Computing the Nodal Force Due to Traction

The following methodology may be employed to compute the *nodal force due to traction* applied on the nodes within a mesh containing linear hexahedral elements. The *nodal force due to traction* computed by using this algorithm could determine the SAS fields. The application of this procedure is not limited to the nodes lying in the interior of the mesh, but it includes the boundary nodes as well. The algorithm consists of the following steps:

- 1- Compute the stresses at the Gauss points by using the displacement finite-element method.
- 2- Define a continuous KAS (kinematically admissible stress) field within each element by using a trilinear function and the data of the Gauss points in that element.
- 3- Make a list of all internal and external nodes in the mesh.
- 4- Make the list of all element-faces in the mesh. This includes the faces exposed to the external boundaries.
- Set up the system of linear equations to solve for the components of the unknown *nodal forces due to traction*, on the nodes.
- 5- Take a node from the list of nodes which has been prepared in item 3.
- 6- Check if the node lies in the interior of the mesh, then there are two possibilities to evolve the system of equations to solve for the components of the *nodal force due to traction* over the internal nodes:
 - a- A straightforward approach is to implement system (2.44) and continue the procedure from the step 24.
 - b- Apply the step by step procedure to construct this system of equations as follows:
- 7- Retrieve the patch of 8 elements containing this node and number them in any order. (For example, the numbering scheme applied in Figure (2.2).) The same numbering order should be applied when dealing with the other internal nodes.

Note: In this calculation, the total number of elements containing an internal node is

assumed to be 8. However, this number can vary in various discretizations.

- 8- Retrieve all 12 element-faces connected to this node. Number these element-faces in any manner. (For example, follow the numbering order according to Figure (2.2).) This numbering scheme should be applied for the other internal nodes.

Note: Depending upon the discretization, the number of element-faces containing a node can vary.

- 9- Define and initialize a square matrix $[A]$ of size 12. Also, define a column vector of size 12 components.
- 10- Take the first element from the patch of elements retrieved in item 7.
- 11- Retrieve the three element-faces of this element which contain the node, and read their numbers.
- 12- Choose a row within the matrix $[A]$, associated with the element number in the patch, and identify the corresponding entries in this row in connection with the above element-face numbers and mark these entries. (These entries would take the non-zero values in this row.)
- 13- Take another element from the patch, until the list consisting of 8 elements containing that node is exhausted, and Go To item 11 and repeat items 11 and 12.
- 14- Go through the columns in the coefficient matrix, one by one, from the top to the bottom, and search for the marks.
- 15- Substitute +1 for the first mark found, in every column, and the second one by -1. These substitutions assign the values to the function η^b , defined on an element-face between the two elements.
- 16- Apply the Singular Value Decomposition Method, [26], to compute the basis vectors which span the nullspace of above rectangular matrix $[A]$. Then define a rectangular matrix $[Z]$ by using the determined basis vectors for its columns.
- 17- Retrieve the rectangular matrix $[A]$ which has been generated in item 15.
- 18- Add the matrix $[Z]^T$ to the retrieved matrix $[A]$ to construct the coefficient matrix of full rank. The modified matrix $[A]$ is called 'the augmented coefficient matrix'.

- Compute the *nodal force due to averaged tractions* from the KAS fields. The procedures for doing this are as follows:

- 19- Loop over the element-faces connected to the node.
- 20- In each element-face, derive the values for η^b as substituted in item 15.
- 21- Compute the *nodal force due to averaged tractions* on this node by using the KAS fields from the neighbouring elements having this element-face in common. This computation makes use of the expressions:

$$\begin{aligned} P_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot a_i \, ds \quad x - \text{component} \\ Q_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot b_i \, ds \quad y - \text{component} \quad (m = 1, 2, \dots) \\ R_i^{bm} &= \frac{1}{2} \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm} + \eta_{\omega_k}^{bm} \tau_{\omega_k}^{bm}) \cdot c_i \, ds \quad z - \text{component} \end{aligned}$$

in which $\tau_{\omega_j}^{bm} = n_{b_m}^T \cdot \sigma_h|_{\omega_j}$. σ_h is the KAS field in the element ω_j . The computed values are stored in the column vector $\{P_i^{bm}\}$ which is defined before.

- 22- Pre-multiply the column vector $\{P_i^{bm}\}$, computed from averaged tractions, by $[Z]^T$ which has been stored previously. Store the results in the lowest part of the column vector, $\{P_i^{bm}\}$. (This supersedes the existing values which have been computed and stored in step 21.)
- 23- Compute the remaining entries by using the expressions:

$$\begin{aligned} P_i^{\omega_j} &= \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega \quad x - \text{component} \\ Q_i^{\omega_j} &= \int_{\omega_j} \sigma_h B b_i \, d\Omega - \int_{\omega_j} f \cdot b_i \, d\Omega \quad y - \text{component} \quad (\omega_j = 1, 2, \dots, 8) \\ R_i^{\omega_j} &= \int_{\omega_j} \sigma_h B c_i \, d\Omega - \int_{\omega_j} f \cdot c_i \, d\Omega \quad z - \text{component} \end{aligned}$$

and store them in the column vector $\{P_i^{bm}\}$ from the top. (These values supersede the existing values in these positions.) The development of the system of equations is completed at this stage.

- 24- Solve the above system of 12 equations in 12 unknowns to determine the x - component of the unknown *nodal force due to tractions* at that node.

- 25- To determine the other components of the *nodal force due to tractions*, apply items, 21-24 with the appropriate values, $\{Q_i^{bm}\}$ and $\{R_i^{bm}\}$, computed in the same manner for $\{P_i^{bm}\}$. These quantities are calculated from the above expressions and substituted for the vector $\{P_i^{bm}\}$ in the system of equations.
- 26- Take another node from the list of nodes, prepared in item 3. If this node is an internal node, Go To step 21. Otherwise, Go To item 27. Repeat this until the list of internal nodes is exhausted.
- 27- Retrieve the patch of elements containing this node and number the elements in an order.
- 28- Retrieve all element-faces connected to this node. Number these element-faces in an order.
- 29- Define and initialize a square matrix $[B]$ with the number of rows and columns equal to the number of element-faces connected to this node. Also, a column vector should be defined compatible with the number of rows in the coefficient matrix $[B]$.
- 30- Take one element from the patch of elements identified in 27.
- 31- Retrieve the three element-faces of this element which contain the node, and read their numbers.
- 32- Choose a row within the matrix $[B]$, associated with the element number in the patch; identify the corresponding entries of this row in connection with the above element-face numbers and mark these entries. These entries would have the non-zero entries in this row.
- 33- Go To item 30 until the list of elements containing the current node is exhausted.
- 34- Go through every column in this coefficient matrix from the top to the bottom, search for the marks.
- 35- Substitute the first mark found, in every column, by +1 and if there exist the second one substitute by -1. This convention ensures that: $\eta = +1$ on the external boundaries.
- 36- Apply Singular Value Decomposition Method to construct the basis vectors for the nullspace of this rectangular matrix obtained in item 35.

- 37- Form matrix $[Z]$ using the basis vectors of the nullspace for its columns.
- 38- Retrieve the rectangular matrix $[B]$ which has been stored in item 35.
- 39- Add the matrix $[Z]^T$ to this retrieved matrix $[B]$ to construct the coefficient matrix of full rank. This version of the matrix $[B]$ is called 'the augmented coefficient matrix $[B]'$ '.

- Compute the *nodal force due to the averaged tractions* from the KAS fields. The procedure for doing this is as follows:

- 40- Loop over the element-faces connected to the node;
 - a- If an element-face is the common face of the two neighbouring elements, then: compute the *nodal force due to averaged tractions* on this node using the KAS fields from the neighbouring elements. The same expressions, presented for internal nodes, can be used.
 - b- If this element-face is exposed to the external boundary, then make use of the expressions:

$$\begin{aligned}
 P_i^{bm} &= \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm}) \cdot a_i \, ds \\
 Q_i^{bm} &= \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm}) \cdot b_i \, ds \\
 R_i^{bm} &= \int_{b_m} (\eta_{\omega_j}^{bm} \tau_{\omega_j}^{bm}) \cdot c_i \, ds
 \end{aligned}$$

to compute the nodal force due to tractions from the KAS fields. Note that: $\eta = +1$ on the element-faces over the external boundaries; τ is defined as before. Place the computed values in the column vector $\{P_i^{bm}\}$.

- 41- Premultiply the vector computed in 40 by the rectangular matrix $[Z]^T$. Store the results in the lower part of the column vector. These values replace the existing ones in the bottom part of the column vector $\{P_i^{bm}\}$.
- 42- Compute the remaining components, needed to complete the column vector, $\{P_i^{bm}\}$

by using the equations:

$$\begin{aligned} P_i^{\omega_j} &= \int_{\omega_j} \sigma_h B a_i d\Omega - \int_{\omega_j} f \cdot a_i d\Omega \quad x - \text{component} \\ Q_i^{\omega_j} &= \int_{\omega_j} \sigma_h B b_i d\Omega - \int_{\omega_j} f \cdot b_i d\Omega \quad y - \text{component} \quad (\omega_j = 1, 2 \dots) \\ R_i^{\omega_j} &= \int_{\omega_j} \sigma_h B c_i d\Omega - \int_{\omega_j} f \cdot c_i d\Omega \quad z - \text{component} \end{aligned}$$

in which all variables are known. Insert the values, in order, from the top of the column vector. The construction of the system of equation is completed at this step.

- 43- If some of the element-faces under consideration are subjected to the prescribed traction boundary conditions, then compute the *nodal force due to tractions* on that element-face by using the expressions (2.45) which is:

$$\begin{aligned} x_i^{b_m} |_{\partial_2 \Omega} &= \int_{b_m} \bar{T}_{(n)} \cdot a_i d\Omega \\ y_i^{b_m} |_{\partial_2 \Omega} &= \int_{b_m} \bar{T}_{(n)} \cdot b_i d\Omega \\ z_i^{b_m} |_{\partial_2 \Omega} &= \int_{b_m} \bar{T}_{(n)} \cdot c_i d\Omega \end{aligned}$$

and impose these boundary conditions over the system of equations. The procedure for applying the boundary conditions is as follow:

- (i) Multiply the value of the assigned variable $x_i^{b_m}$ by the relevant column vector in the augmented coefficient matrix $[B]$.
 - (ii) Subtract the product computed in item (i) from the column vector in the right hand side of the system of equations. Then replace the column vector in the right hand side by the results of subtraction.
 - (iii) Eliminate this variable from the column vector of unknowns in this system of equations. Also, remove the pertinent row and column to this variable from the augmented coefficient matrix and right hand side.
- 44- Solve the system of equations obtained in above for the components of the unknown *nodal force due to tractions*.
- 45- Repeat the steps 40-44 for the other components of the *nodal force due to tractions* on the node i , by using the appropriate column vectors computed for these components.

Note: If no boundary conditions are prescribed on any element-face exposed to the external boundaries, then it is assumed that the traction boundary conditions with components of zero magnitude are in effect. In this case, the zero nodal force due to zero tractions is imposed to the system of equations.

Following the above instructions, the various components of the *nodal force due to tractions*, for all nodes in the mesh, are computed. These quantities are used to calculate the SAS field within the elements.

2.7.4 Computing the Stress-Fields from the Nodal Force Due to Traction

Recall the previous sections in which the *nodal forces due to tractions* on the element-faces have been calculated. The computed *nodal force due to tractions* ensures the continuity of tractions across the boundaries of elements. These tractions are in equilibrium with respect to the body-force, i.e., they satisfy solvability condition (2.20). The next problem is to derive the stress-fields, within the element, from the determined *nodal forces due to tractions*. These stress-fields must be computed such that they satisfy the equations of equilibrium in the interior of the elements, and produce the same *nodal force due to tractions* as computed before, at each node. This is a problem which can be solved within the elements; thus the problem of finding the SAS field has been reduced to a sequence of problems one on each element.

In mathematical terminology, we seek for the stress-field $\hat{\sigma}_h$ such that:

$$(1) \operatorname{div} \hat{\sigma}_h + f = 0 \text{ in the interior of each element } \omega; \quad (2.46a)$$

$$(2) n_\omega^T \cdot \hat{\sigma}_h = T \text{ on the boundaries } \partial\omega \text{ of element.} \quad (2.46b)$$

Any stress-field, $\hat{\sigma}_h$, satisfying the static admissibility conditions (2.46a) and (2.46b) satisfies the conditions required by theorem (2.1). If all of the elements in the mesh look alike, and if the class of functions T is restricted to a finite dimensional subspace, then this problem can be solved once on the "model" element; its application to each element, ω_j , of the mesh involves a linear transformation from the "model" element solution [24]. The problem of computing the stress-field from the *nodal force due to tractions* has been solved by Ladeveze *et al.*, [27], for the cases of triangular and quadrilateral elements in two-dimensional and tetrahedral element in three-dimensional spaces. The SAS fields in their calculations, are limited to the case of linear stress distributions and constant body-forces. In the case of two-dimensional problems, they have subdivided the "model" element into a finite number of subelements which are linear triangles, and computed a linear stress-field over the subelements. We will present a method which can be used to compute a linear stress-field on the "model" element directly without subdividing into subelements.

2.7.5 SAS Field for Piecewise Constant Body-Force over Triangular Elements

An alternative method has been developed by the author to compute the stress-field by using the prescribed *nodal force due to tractions* and a piecewise constant body-force on a linear triangular element. Recall equations of equilibrium (2.46a), for a body-force of constant distribution over the element ω , *divergence* of the stress-field must be a constant. This condition implies that the stress distribution must be linear over the element; therefore the tractions vary linearly along the boundaries of elements as well. This methodology implies that this method generates the SAS field for a constant body-force.

To compute the stress-field $\hat{\sigma}_h$, first, the stress-field is expressed by a linear function with unknown constant coefficients:

$$\hat{\sigma}_h = a + b\xi + c\eta$$

then this stress-field is substituted in the system of equations expressing the *nodal force due to tractions* in terms of stress-fields and the equations of equilibrium as follows:

$$\begin{cases} \int_b (n_\omega^T \cdot \hat{\sigma}_h) \cdot a_i ds = \eta_\omega^b x_i^b \\ \int_b (n_\omega^T \cdot \hat{\sigma}_h) \cdot b_i ds = \eta_\omega^b y_i^b \\ \int_b (n_\omega^T \cdot \hat{\sigma}_h) \cdot c_i ds = \eta_\omega^b z_i^b \\ -\text{div } \hat{\sigma}_h = f \end{cases} \quad (2.47)$$

In equations (2.47) b denotes the element-face, and a_i , b_i , and c_i denote the vectors of basis functions in x , y , and z directions, respectively, defined by equations (2.22). For a constant body-force applied over a linear triangular element, the space of stress-fields and the space of tractions are spanned by the same basis functions; moreover, $\text{div } \hat{\sigma}_h$ lies in the space of body-forces. This indicates that the right-hand-side of equations (2.47) can be represented as a linear combination of the column vectors of the coefficient matrix in the left-hand-side. In other words, the right-hand-side lies in the column space of the coefficient matrix of the left-hand-side of system of equations (2.47). Furthermore, system (2.47) could lead to the solvability conditions (2.20). Therefore system (2.47) always possesses a unique solution, and a linear stress-field can be computed over the elements, uniquely, from the *nodal force due to tractions*.

This method of calculating stress-field has been studied in detail by the author for the case of a plane-stress problem over a linear triangular element. In this case, the stress-field has been expressed over the element by a linear functions with 9 unknown coefficients; then substituted in equations (2.47). The first set of equations in (2.47) (expressing the nodal force due to tractions) supplies 12 relationships within the unknown coefficients, four on each node. In addition, the second expression, the equation of equilibrium produces another 2 equations; totally 14 equations are obtained which determine the 9 unknown coefficients in the stress variables, uniquely. The explicit derivation of the system of equations is not presented here. A similar system will be derived explicitly for a linear hexahedral element subsequently.

The advantage of this method for stress calculation is that no subdivision is required; the stress-field which is a linear function and conforms with a constant body-force is computed directly on the triangular "model" element. The generalization of this method to three-dimensional space for tetrahedral elements should be straightforward. In this case, the six components of the stress-field are expressed by linear functions over the element. Then the stress-field is substituted in equations (2.47) from which the 24 unknown stress-coefficients are determined.

The stress-field computed by the application of this method is statically admissible for a constant body-force distribution over a linear triangular element. The features of this stress-field may be written as:

- 1- The element-tractions T are such that they satisfy the solvability condition (2.20), i.e.,

$$\int_{\omega_j} f \, d\Omega + \int_{\partial\omega_j} \eta_{\omega_j} T \, ds = 0$$

for linear tractions and constant body-force. In other words, elements are in equilibrium with respect to the applied forces.

- 2- The forces exerted on each node are in equilibrium because:

$$\int_{\partial\omega} T \cdot a_i \, ds + \int_{\omega} f \cdot a_i \, d\Omega = \int_{\omega} \hat{\sigma}_h B a_i \, d\Omega$$

- 3- This stress-field satisfies the equations of equilibrium:

$$\text{div } \hat{\sigma}_h + f = 0$$

in the interior of elements.

- 4- The traction vector is continuous (equilibrating) across the boundaries of elements. Moreover, the stress-fields satisfy the tractions on the boundaries of elements:

$$n^T \cdot \hat{\sigma}_h = T$$

- 5- On the external boundaries $\partial_2\Omega$, the stress-field $\hat{\sigma}_h$ conforms to the traction boundary conditions:

$$n^T \cdot \hat{\sigma}_h|_{\partial_2\Omega} = \bar{T}$$

For body-forces of linear and other distributions, this method does not generate a statically admissible stress-field. With these body-forces, it is evident that, the equation of equilibrium are not satisfied within the elements. The error in the equation of equilibrium induces an unbalanced force on the nodes. These errors destroy the static admissibility of the stress-field.

2.7.6 Mean-SAS Field over Linear Hexahedral Element

According to the above discussions, the method of constructing the SAS field over linear triangular elements for the case of constant body-force should be straightforward. Since other elements, like quadrilaterals in two-dimensional and hexahedral elements in three-dimensional spaces are frequently used in engineering practice, an explicit derivation of the SAS field on these elements becomes necessary. Because the computation of the *nodal force due to tractions* has been performed for linear hexahedral element, the same element is applied to compute the stress-field. Unlike the method, presented by Ladeveze *et al* [27], in which the stress and the body-force distributions are restricted to the linear and constant distributions, respectively, in this method of calculating the stresses, no such limitations are imposed.

In this derivation, the "model" element ω is a rectangular linear hexahedron which is illustrated in Figure (2.3).

The element ω has six element-faces, b , and eight corner nodes each of which is contained in three element-faces. The element-faces and nodes in this element are numbered according to Figure (2.3). There are nine quantities associated with each node i in the

Figure (2.3)

Illustration of a linear hexahedron

element ω . These quantities are components of the *nodal force due to tractions*. These are designated by:

$$x_i^{b_j}, x_i^{b_k}, x_i^{b_l}, y_i^{b_j}, y_i^{b_k}, y_i^{b_l}, z_i^{b_j}, z_i^{b_k}, \text{ and } z_i^{b_l}$$

in which b_j, b_k , and b_l are the element-faces incident to the node i . The above quantities are defined by expressions (2.31), and they are known from the solutions of systems of equations (2.33), (2.34), and (2.35). For eight nodes in the element there are, therefore, 72 known quantities; because of the symmetric property of the stress-tensor, only 48 of these data are independent.

The state of stress within the element ω can be uniquely determined from the prescribed *nodal force due to tractions* on the element-faces incident to the nodes of the

element. Let n_{b_j} , n_{b_k} , and n_{b_l} denote the unit vectors normal to the element-faces: b_j , b_k , and b_l containing a node, i ; the system of equations which satisfies the state of stresses at the node i , is written as:

$$\begin{cases} \int_{b_j} (n_{\omega}^T \cdot \hat{\sigma}_h) \cdot a_i ds = \eta_{\omega}^{b_j} x_i^{b_j} \\ \int_{b_k} (n_{\omega}^T \cdot \hat{\sigma}_h) \cdot a_i ds = \eta_{\omega}^{b_k} x_i^{b_k} \\ \int_{b_l} (n_{\omega}^T \cdot \hat{\sigma}_h) \cdot a_i ds = \eta_{\omega}^{b_l} x_i^{b_l} \\ \text{similarly for } y_i \text{ and } z_i. \end{cases} \quad (2.48)$$

Let $\{\sigma\}$ denote the vector of nodal stresses in the element, i.e.,

$$\{\sigma\} = \begin{bmatrix} (\sigma_{zz})_1 \\ (\sigma_{yy})_1 \\ (\sigma_{zz})_1 \\ (\sigma_{zy})_1 \\ (\sigma_{yz})_1 \\ (\sigma_{zz})_1 \\ \vdots \\ (\sigma_{zz})_8 \\ (\sigma_{yy})_8 \\ (\sigma_{zz})_8 \\ (\sigma_{zy})_8 \\ (\sigma_{yz})_8 \\ (\sigma_{zz})_8 \end{bmatrix} \quad (2.49)$$

Let N_i be the standard shape function associated with the node i of element; this is written as:

$$N_i(\xi, \eta, \zeta) = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta) \quad (2.50)$$

The state of stress within the element is expressed by:

$$\bar{\sigma} = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) \sigma_i = [N]\{\sigma\} \quad (2.51)$$

in which σ_i are the values of stresses at the node i . Since there are 6 components of stresses associated with each node, the total number of 48 degrees of freedom exist in expression (2.51) which must be determined. Substitution of expression (2.51) into equations (2.48) supplies 72 equations with 48 unknowns. As was mentioned earlier, only 48 independent

equations exist because the state of stress within the element is defined precisely from 48 known quantities. Moreover, the column vector in the right-hand-side of equations (2.48) lies in the column space of the coefficient matrix of system (2.48) which expresses the unknown nodal values of stresses. In other words, the spaces of the *nodal force due to tractions* and stresses are spanned by the same basis functions. This characteristic implies that system (2.48) admits a unique solution for nodal stresses.

In order to determine the nodal values of stresses, it is advantageous to resolve system of equations (2.48) over three directions: x , y , and z . Because each component of the stress-field is interpolated within the element by the same set of shape functions, the system of equations which solve the x component of the *nodal force due to tractions*, can serve to solve the unknowns in the other directions as well.

Let us resolve expression (2.51) in x direction, we get:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & \dots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & \dots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & \dots & 0 & 0 & N_8 \end{bmatrix} \begin{bmatrix} (\sigma_{xx})_1 \\ (\sigma_{xy})_1 \\ (\sigma_{zx})_1 \\ \vdots \\ (\sigma_{xx})_8 \\ (\sigma_{xy})_8 \\ (\sigma_{zx})_8 \end{bmatrix} \quad (2.52)$$

Applying this system to the first equation in system (2.48), it yields:

$$\left\{ \int_{b_j} [n_x^{b_j} \ n_y^{b_j} \ n_z^{b_j}] \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & \dots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & \dots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & \dots & 0 & 0 & N_8 \end{bmatrix} \cdot a_i \ ds \right\} \begin{bmatrix} (\sigma_{xx})_1 \\ (\sigma_{xy})_1 \\ (\sigma_{zx})_1 \\ \vdots \\ (\sigma_{xx})_8 \\ (\sigma_{xy})_8 \\ (\sigma_{zx})_8 \end{bmatrix} = x_i^{b_j} \quad (2.53)$$

which introduces one equation between the unknowns. Similarly, equation (2.53) can be integrated over the other element-faces, b_k and b_l , incident to the node i to define two more equations. Repeating the above calculations for the remaining nodes, using expressions

(2.52) and the appropriate a_i , here 24 equations are obtained. These equations are solved for 24 unknowns which express the nodal values of stresses in x direction associated with this element.

A system of equations, similar to (2.53), can be applied to calculate the nodal values of stresses associated with the other directions y and z . In these cases, the coefficient matrix of system (2.53) remains unchanged. The changes are applied on the column vector in the right-hand-side of the system. If x_i is replaced by y_i or z_i , the solution from the system would be the nodal stresses in these directions, respectively.

To compute the nodal values of stresses in the other elements, the above system (2.53) is applicable in a similar fashion.

In summary, the following procedure may be employed to compute the nodal values of stresses from the *nodal force due to tractions* given within an element.

- 1- Take a node in the element.
- 2- Retrieve the three element-faces connected to this node.
- 3- Integrate system (2.53) over each element-face.
- 4- Repeat steps 2 and 3 for all nodes in the element.
- 5- Solve the system of 24 equations in 24 unknowns for the x component of nodal stresses in this element.
- 6- Solve a modified system of (2.53) for the other unknown nodal stresses.

Note that the solution for other elements of the mesh (assuming they are the same type) are obtained by a linear transformation from the above computed solution.

Following the foregoing instructions, the nodal values of stresses for all nodes in the mesh are obtained accordingly.

With the knowledge of the nodal values of stresses, the continuous stress-field σ in the interior of the element is constructed by using expression (2.51). The stress-field σ which is constructed according to this method possesses certain features, which may be written as follows:

- 1- The element-tractions T are such that solvability conditions (2.20) are satisfied, i.e.,

$$\int_{\omega_j} f \, d\Omega + \int_{\partial\omega_j} \eta_{\omega_j} T \, ds = 0$$

- 2- The forces applied on each node are in equilibrium, because:

$$\int_{\partial\omega} T \cdot a_i \, ds + \int_{\omega} f \cdot a_i \, d\Omega = \int_{\omega} \bar{\sigma} B a_i \, d\Omega$$

- 3- Equations of equilibrium:

$$\operatorname{div} \bar{\sigma} + f \neq 0$$

are not satisfied in the interior of element; but this equation is satisfied within the element in an integral sense, i.e.,

$$\int_{\omega} (\operatorname{div} \bar{\sigma} + f) \, d\Omega = 0$$

- 4- The traction vector is continuous across the boundaries of elements. Moreover, the stress-fields satisfy the tractions on the boundaries of elements:

$$n^T \cdot \bar{\sigma} = T$$

- 5- On the external boundaries $\partial_2\Omega$, the stress-field $\bar{\sigma}_h$ conforms to the prescribed traction boundary conditions:

$$n^T \cdot \bar{\sigma} = \bar{T}$$

Because this stress-field does not satisfy the equations of equilibrium, it is not a statically admissible stress-field. Nevertheless, the equation of equilibrium is satisfied within the element in an integral sense. In addition, the tractions on the boundaries of elements are in equilibrium with the body-forces. Moreover, the computed stress-field conforms to the traction boundary conditions. Therefore it can be expected that this stress-field is a close approximation to the statically admissible stress-field. This stress-field satisfies all features of SAS field except the equation of equilibrium in a pointwise sense. Because this equation is satisfied in an integral sense, this stress-field is called a mean-SAS field.

2.7.7 SAS Field over Rectangular Linear Hexahedral Element

In order to compute SAS field for body-forces with linear and higher-order polynomials distributions, a set of stress variables with unknown coefficients is superimposed on the stress-field $\bar{\sigma}$ computed from the nodal values of stresses. The interpolation functions associated with these stress variables are such that they vanish on the boundaries of the elements. The shape function N^* is defined by:

$$N^*(\xi, \eta, \zeta) = (1 - \xi^2)(1 - \eta^2)(1 - \zeta^2) \quad (2.54)$$

Applying this knowledge, the stress-field $\hat{\sigma}_h$ is interpolated within the elements by the expression:

$$\begin{aligned} \hat{\sigma}_h(\xi, \eta, \zeta) = & \sum_{i=1}^8 N_i(\xi, \eta, \zeta) \sigma_i + N^*(\xi, \eta, \zeta) \phi + \sum_{j=1}^{\infty} \xi^j N^*(\xi, \eta, \zeta) \phi_j^{(\xi)} + \\ & \sum_{k=1}^{\infty} \eta^k N^*(\xi, \eta, \zeta) \phi_k^{(\eta)} + \sum_{l=1}^{\infty} \zeta^l N^*(\xi, \eta, \zeta) \phi_l^{(\zeta)} \end{aligned} \quad (2.55)$$

in which $\phi, \phi_j^{(\xi)}, \dots$ are the unknown stress-coefficients. In this expression, $N_i(\xi, \eta, \zeta)$ are the standard isoparametric shape functions associated with the vertices of linear hexahedron which interpolate the nodal values of stresses σ_i to the interior of element. Likewise, the nodeless shape function $N^*(\xi, \eta, \zeta)$, called the bubble function, is defined by expression (2.54). The main feature of this shape function is that it vanishes on the boundaries of the element. Therefore the stress variables constructed, in the interior of the element, by using the multiples of this shape function do not affect the state of stresses $\bar{\sigma}$ at the nodes. Consequently, the prescribed nodal force due to tractions, on the element-faces, remain unchanged upon superposition of the stress variables associated with these shape functions over the stress-fields $\bar{\sigma}$ interpolated from the nodes.

In order to compute the unknown coefficients $\phi, \phi_j^{(\xi)}, \dots$, define the stress-field $\hat{\sigma}_h$ as the summation of the two stress-fields as follows:

$$\hat{\sigma}_h = \bar{\sigma} + \delta\sigma \quad (2.56)$$

in which:

$$\bar{\sigma} = \sum_1^8 N_i(\xi, \eta, \zeta) \sigma_i \quad (2.57)$$

$$\delta\sigma = N^*(\xi, \eta, \zeta)\phi + \sum_{j=1}^{\infty} \xi^j N^*(\xi, \eta, \zeta) \phi_j^{(\xi)} + \sum_{k=1}^{\infty} \eta^k N^*(\xi, \eta, \zeta) \phi_k^{(\eta)} + \sum_{l=1}^{\infty} \zeta^l N^*(\xi, \eta, \zeta) \phi_l^{(\zeta)} \quad (2.58)$$

We wish to determine the stress-field $\hat{\sigma}_h$ such that:

$$\begin{cases} \text{div } \hat{\sigma}_h + f = 0 & \text{in the interior of element } \omega \\ n_{\omega}^T \cdot \hat{\sigma}_h = n_{\omega}^T \cdot \bar{\sigma} & \text{on the boundary } \partial\omega \end{cases} \quad \begin{matrix} (2.59a) \\ (2.59b) \end{matrix}$$

More specifically, the unknown coefficients, $\phi, \phi_j^{(\xi)}, \dots$, must be determined such that conditions (2.59a) and (2.59b) are fulfilled. The fact that condition (2.59b) is maintained on the element-faces, is obvious from the characteristic of the bubble function involved in every term in the expression $\delta\sigma$. It remains to impose, still, the condition (2.59a). To impose this condition, substitute for $\hat{\sigma}_h$ in (2.59a) from equation (2.56), this gives:

$$\text{div } (\bar{\sigma} + \delta\sigma) + f = 0$$

The linearity of the operator div implies that:

$$\text{div } \delta\sigma = -\text{div } \bar{\sigma} - f \quad (2.60)$$

In order to compute the correction term $\delta\sigma$, from the equations (2.60), one needs to evaluate the expression $\text{div } \bar{\sigma}$. In addition, $\text{div } \delta\sigma$ must be explicitly expressed in terms of the unknown stress coefficients $\phi, \phi_j^{(\xi)}, \dots$. Upon substitution for these terms, equations (2.60) admit a solution to the unknown stress coefficients $\phi, \phi_j^{(\xi)}, \dots$.

To evaluate the expression $\text{div } \bar{\sigma}$, initially, the stress gradient vector $\{X\}$ associated with the stress-field $\bar{\sigma}$ should be computed. Then $\text{div } \bar{\sigma}$ is calculated from the determined stress gradient vector using an appropriate operator which will be introduced later.

Let us define the two stress gradient vectors $\{X\}$ and $\{\Xi\}$ containing the derivatives of the stress components with respect to the global and natural coordinates, respectively.

These vectors are written as:

$$\{X\} = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} \\ \frac{\partial \sigma_{xx}}{\partial y} \\ \frac{\partial \sigma_{xx}}{\partial z} \\ \vdots \\ \frac{\partial \sigma_{xx}}{\partial x} \\ \frac{\partial \sigma_{xx}}{\partial y} \\ \frac{\partial \sigma_{xx}}{\partial z} \end{bmatrix} \quad \{\Xi\} = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial \xi} \\ \frac{\partial \sigma_{xx}}{\partial \eta} \\ \frac{\partial \sigma_{xx}}{\partial \zeta} \\ \vdots \\ \frac{\partial \sigma_{xx}}{\partial \xi} \\ \frac{\partial \sigma_{xx}}{\partial \eta} \\ \frac{\partial \sigma_{xx}}{\partial \zeta} \end{bmatrix} \quad (2.61)$$

Define $[J]$ as the square Jacobian matrix satisfying:

$$\{\Xi\} = [J]\{X\} \quad (2.62)$$

The configuration of the Jacobian matrix $[J]$ appears through the explicit computation of derivatives with respect to the global and natural coordinates. It is known that the two coordinate systems are transformed into each other through the functions:

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \quad (2.63)$$

The relationship between the two gradient vectors $\{X\}$ and $\{\Xi\}$ are established by using the chain rule:

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial \xi} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \xi} \\ \frac{\partial \sigma_{xx}}{\partial \eta} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \eta} \\ \frac{\partial \sigma_{xx}}{\partial \zeta} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \zeta} \\ \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \\ \frac{\partial \sigma_{xx}}{\partial \xi} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \xi} \\ \frac{\partial \sigma_{xx}}{\partial \eta} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \eta} \\ \frac{\partial \sigma_{xx}}{\partial \zeta} = \frac{\partial \sigma_{xx}}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial \sigma_{xx}}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial \sigma_{xx}}{\partial z} \frac{\partial z}{\partial \zeta} \end{cases}$$

Using the notations of the equations (2.62) into these derivatives, the Jacobian matrix $[J]$

is written as:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & & & \vdots & \ddots & & & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (2.64)$$

which can be evaluated by using the coordinates transformation equations (2.63). In order to express the stress gradient vector $\{X\}$ in terms of the stress-field $\bar{\sigma}$, the derivatives of the stress components $\bar{\sigma}$ with respect to the natural coordinates (ξ, η, ζ) must be known. These derivatives are computed using the expression (2.57):

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial \xi} = \sum \frac{\partial N_i(\xi, \eta, \zeta)}{\partial \xi} \sigma_i \\ \frac{\partial \sigma_{xx}}{\partial \eta} = \sum \frac{\partial N_i(\eta, \eta, \zeta)}{\partial \eta} \sigma_i \\ \frac{\partial \sigma_{xx}}{\partial \zeta} = \sum \frac{\partial N_i(\zeta, \eta, \zeta)}{\partial \zeta} \sigma_i \\ \vdots \\ \frac{\partial \sigma_{xx}}{\partial \xi} = \sum \frac{\partial N_i(\xi, \eta, \zeta)}{\partial \xi} \sigma_i \\ \frac{\partial \sigma_{xx}}{\partial \eta} = \sum \frac{\partial N_i(\eta, \eta, \zeta)}{\partial \eta} \sigma_i \\ \frac{\partial \sigma_{xx}}{\partial \zeta} = \sum \frac{\partial N_i(\zeta, \eta, \zeta)}{\partial \zeta} \sigma_i \end{cases} \quad (2.65)$$

Employing the vector of nodal stresses $\{\sigma\}$, the natural derivatives (2.65) is represented, in matrix form, by:

$$\{\Xi\} = [H]\{\sigma\} \quad (2.66)$$

In expression (2.66), matrix $[H]$ which transforms the nodal stresses σ_i to the gradient of stress-field, in natural coordinates, is represented in the form of:

Applying the stress transformation matrix $[H]$ and the nodal stress vector $\{\sigma\}$ into equations (2.62), this system reduces to:

$$[H]\{\sigma\} = [J]\{X\} \quad (2.68)$$

from which the stress gradient vector $\{X\}$ is computed by:

$$\{X\} = [J]^{-1}[H]\{\sigma\} \quad (2.69)$$

In this equation it is assumed that the Jacobian matrix of the coordinates transformation is invertible.

Now in order to compute the expression $\text{div } \bar{\sigma}$, the matrix operator $[\Lambda]$ is introduced. This matrix operator is defined such that it transforms the stress gradient vector $\{X\}$ to $\text{div } \bar{\sigma}$, i.e.,

$$\text{div } \bar{\sigma} = [\Lambda]\{X\} \quad (2.70)$$

The matrix operator $[\Lambda]$ is defined by:

$$[\Lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad (2.71)$$

The equations (2.69), (2.70), and (2.71) are combined to compute $\text{div } \bar{\sigma}$:

$$\text{div } \bar{\sigma} = [\Lambda][J]^{-1}[H]\{\sigma\} \quad (2.72)$$

According to equation (2.70), the computation of $\text{div } \bar{\sigma}$ is straightforward. The nodal values of stresses and the geometrical characteristics of the element are the basic data in this calculation. The matrix $[\Lambda]$ is introduced in the system only once. The coordinates transformation matrix $[J]$ and the stress transformation matrix $[H]$ remain, analytically, unchanged for the typical element under consideration.

In addition to the above calculations, in order to solve equations (2.60) it is necessary to express $\text{div } \delta \sigma$ in terms of the unknown stress coefficient $\phi, \phi_j^{(\epsilon)}, \dots$. To do this, expansion (2.58) must be used. In order to simplify the calculation, expression (2.58) is truncated after the first term and applied in the calculations. The methodology to describe the expression $\text{div } \delta \sigma$ in terms of the unknown stress coefficient vector ϕ is similar

to the one which was applied to express $\text{div } \bar{\sigma}$ in terms of nodal values of stresses. To compute $\text{div } \delta\sigma$, the stress gradient vector with respect to the natural coordinate system must be calculated. This vector contains the unknown coefficient vector ϕ . Let $\{\delta X\}$ and $\{\delta \Xi\}$ denote the stress gradient vectors with respect to the global and natural coordinates respectively; a transformation equation between the stress gradient, similar to (2.62), may be written as follows:

$$\{\delta \Xi\} = [J]\{\delta X\} \quad (2.73)$$

in which $[J]$ is the Jacobian matrix (2.64). In equations (2.73), the vector $\{\delta \Xi\}$ is a function of the unknown stress coefficients ϕ . As was mentioned earlier, for simplicity, this vector is computed by using the first term in expression (2.58) as:

$$\delta\sigma = N^*(\xi, \eta, \zeta)\phi$$

The derivatives of $\delta\sigma$ with respect to the natural coordinates are:

$$\{\delta \Xi\} = \begin{bmatrix} \frac{\partial \delta\sigma_{xx}}{\partial \xi} \\ \frac{\partial \delta\sigma_{xx}}{\partial \eta} \\ \frac{\partial \delta\sigma_{xx}}{\partial \zeta} \\ \vdots \\ \frac{\partial \delta\sigma_{xx}}{\partial \xi} \\ \frac{\partial \delta\sigma_{xx}}{\partial \eta} \\ \frac{\partial \delta\sigma_{xx}}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N^*}{\partial \xi} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial N^*}{\partial \eta} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial N^*}{\partial \zeta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N^*}{\partial \xi} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N^*}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N^*}{\partial \zeta} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial N^*}{\partial \xi} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial N^*}{\partial \eta} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial N^*}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} \quad (2.74)$$

in which we set:

$$\{\phi\} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix}$$

which is the vector of the unknown coefficients. Symbolically, expressions (2.74) are written as:

$$\{\delta\Xi\} = [G]\{\phi\} \quad (2.75)$$

Replacing the vector $\{\delta\Xi\}$ in (2.73) by the definition given in equation (2.75) and computing the vector $\{\delta X\}$, we can obtain the equation

$$\{\delta X\} = [J]^{-1}[G]\{\phi\} \quad (2.76)$$

Following a similar calculation procedure which has been carried out for calculating $\text{div } \bar{\sigma}$, $\text{div } \delta\sigma$ becomes:

$$\begin{aligned} \text{div } \delta\sigma &= [\Lambda]\{\delta X\} \\ &= [\Lambda][J]^{-1}[G]\{\phi\} \end{aligned} \quad (2.77)$$

in which the operator $[\Lambda]$ is defined by (2.71). Applying expressions (2.72) and (2.77) into equations (2.60), this becomes:

$$[\Lambda][J]^{-1}[G]\{\phi\} = -[\Lambda][J]^{-1}[H]\{\sigma\} - f \quad (2.78)$$

In this equation, f denotes the vector of body-forces. It is evident that system (2.78) contains three equations in six unknowns. To solve these equations for the unknown vector $\{\phi\}$, we can apply a collocation method. In this case, equation (2.78) must be evaluated at a finite number of points, within the element. The Gauss points are suitable for this purpose. For $2 \times 2 \times 2$ integration scheme, there are eight Gauss points in the element; therefore a set of 24 equations are generated from equations (2.78) which are evaluated at these Gauss points. The outcome system is an overdetermined system of linear equations in the form of:

$$[K]\{\phi\} = \{b\} \quad (2.79)$$

System of equations (2.79) can be solved by a least squares method to determine the stress coefficient vector $\{\phi\}$. From equations (2.79), the system of normal equations is defined by:

$$[K]^T[K]\{\phi\} = [K]^T\{b\}$$

which determines the stress coefficient $\{\phi\}$ in a least square sense. The computed coefficients $\{\phi\}$ associated with the bubble functions in the expansion of $\delta\sigma$, expresses the

stress variables over the element in the form of:

$$\hat{\sigma}_h = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) \sigma_i + N^*(\xi, \eta, \zeta) \phi \quad (2.80)$$

The stress distribution (2.80) is a tri-quadratic function in the natural coordinates. Thus *divergence* of this stress-field may be matched with a body-force of polynomial of finite order other than zero. Therefore this method is capable of computing the stresses which match with the body-forces of linear and higher order polynomials. For body-forces of linear and higher order polynomials, this stress-field is expected to be more accurate than the stress-fields computed by using the method attributed to Ladeveze *et al*, [27], for linear triangular elements because the stress-field computed using the method employed by Ladeveze *et al* captures the body-force of constant distribution only. The method presented by Ladeveze *et al* however does not produce SAS field for body-forces of linear and higher-order polynomials.

The superposition of the bubble functions over the mean-SAS field satisfy the equations of equilibrium in the interior of elements. The nodal forces due to this stress-field are in equilibrium with respect to the nodal forces computed from the external forces. In addition, the tractions are continuous (equilibrating) across the boundaries of elements. Also, the stress-field conforms to the prescribed traction boundary conditions. Regarding these characteristics, this is the only method which computes the SAS fields for body-forces of linear and higher order polynomials. The features of the SAS fields computed for linear triangular elements may also be applied for this stress-field.

The stress field computed through expression (2.80) consists of two separate stress fields: the mean-SAS field $\bar{\sigma}$, and the correction term $\delta\sigma$. It is clear that a negligible effort is required to express the continuous stress-field $\bar{\sigma}$, over the element, from the computed nodal values of stresses. In other words, the mean-SAS field is computed very easily from the nodal values of stresses. On the other hand, the foregoing analysis shows that the computation of the stress coefficients associated with the bubble functions, for the correction term $\delta\sigma$, can be expensive because it involves a cumbersome calculations. The superposition of the correction term $\delta\sigma$ over the stress-field $\bar{\sigma}$ computed from the nodal stresses may increase the accuracy of the stress-field, but the corresponding implementation and computational costs are expected to be high. If the accuracy of the stress-field is

a critical issue, then the application of bubble functions may be an appropriate approach to compute an accurate stress-field. However, for error analysis purposes, the application of the mean-SAS field seems to be more efficient.

An alternative approach may be used to compute the SAS field. In this method the stress-field within the element is described in terms of stress-functions [28]. The stress-functions are defined along the axes; x, y , and z separately. Then, the stress-fields within the element are determined by the combinations of the second derivatives of the stress-functions [28]. Because the computation of the second derivatives introduces complexities in the calculations, this method is not recommended by the author; so the method of stress-function is not presented in this methodology.

2.8 Error Estimator and Error Indicator

Recall lemma (2.2) in which the quantity $\|\hat{\sigma}_h - \sigma_h\|^2$, i.e., the difference between the KAS and the SAS fields, measured in the energy norm, defines an upper bound on the error incurred in the finite-element solution. To compute this error measure, the KAS and the SAS fields must be defined. The KAS field is known from the finite-element solution. In addition, the SAS field may be computed by using the methods presented in this section. Applying these stress-fields in the above expression $\|\hat{\sigma}_h - \sigma_h\|^2$, this expression is evaluated in relation to energy inner product (2.9) as follows:

$$\|\hat{\sigma}_h - \sigma_h\|^2 = \int_{\Omega} (\hat{\sigma}_h - \sigma_h)^T C (\hat{\sigma}_h - \sigma_h) d\Omega \quad (2.81)$$

which provides an upper bound, in the energy norm, for the absolute error incurred in the finite-element solution. The error indicator is defined by the element contribution of the above expression. The local error-measure is defined by the element contribution of expression (2.81) normalized by the total strain energy of the deformed body. Thus the local error-measure is written, using expression (2.81), as:

$$\|e\|_{\omega}^2 = \int_{\omega} \left[\frac{(\hat{\sigma}_h - \sigma_h)^T C (\hat{\sigma}_h - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (2.82)$$

with

$$\|\sigma_h\|^2 = \int_{\Omega} \sigma_h^T C \sigma_h d\Omega$$

The expression (2.82) provides an upper bound on the relative error in the finite-element solution. Expression (2.82), which indicates the error, defines a local error-measure as well.

For the mean-SAS field $\bar{\sigma}$, the local error measures are defined in the same manner as defined for SAS field. The local error measure using the mean-SAS field $\bar{\sigma}$ is computed from the expression:

$$\|e\|_{\omega}^2 = \int_{\omega} \left[\frac{(\bar{\sigma} - \sigma_h)^T C (\bar{\sigma} - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (2.83)$$

which provides an approximate error-bound for the finite-element solutions.

The bound on relative error-measure is defined by the summation of all local error measures, computed from (2.82), for all elements in the mesh as follows:

$$\|e\|^2 = \sum_{\omega \in \varpi_h} \|e\|_{\omega}^2 \quad (2.84)$$

Similarly, the local error measure (2.83) provides an approximate bound on relative error-measure through the application of the expression (2.84).

Expressions (2.82), (2.83), and (2.84) may be used in adaptive finite-element analysis as criteria for mesh refinement.

In summary, three methods of error-analysis are presented in this part:

Method A: Error-bound computed from the KAS and the SAS fields for body-forces of linear and higher order polynomials defined over rectangular linear quadrilateral and hexahedral elements.

The construction of the SAS field using the KAS field includes an involved procedure which consists of the following calculations. This stress-field may be used to compute the error-bounds.

- 1- Compute the *nodal forces*, at each node, due to *tractions* over the element-faces, for each element, such that their associated element-tractions are in equilibrium with respect to the body-force and traction boundary conditions or force-boundary conditions.
- 2- Determine the nodal values of stresses by using the computed *nodal forces due to tractions* and the expression relating those nodal forces to stress-fields.

- 3- Generate the continuous stress-field within the element by using the standard interpolation technique from the nodes to the interior of the elements.
- 4- Superimpose a sufficient number of stress variables in terms of bubble functions with unknown coefficients on the continuous stress-field.
- 5- Solve the equations of equilibrium to compute the unknown coefficients, associated with the bubble functions in the stress distributions. The SAS field is determined by applying these coefficients to the stress-field.
- 6- Compute the error-bound, by evaluating the error indicator (2.82) and error estimator (2.84). Appendix 3 presents numerical analysis of this method for a test problem in one-dimensional space.

Method B : Error-Bound computed from the kinematically and the statically admissible stress-fields for piecewise constant body-force, defined over linear triangular and tetrahedral elements.

The computation of SAS field and error-bound for this class of problems may be performed through the application of the following instructions:

- 1- Compute the *nodal forces, at each node, due to tractions* on the element-faces, in the same fashion as described in method A.
- 2- Express the stress-fields over the elements by a complete polynomial of order one with unknown coefficients.
- 3- Equate the computed *nodal forces due to tractions* on each node with *nodal forces due to tractions* calculated from the internal stresses in terms of unknown coefficients. This forms a number of equations between the unknown coefficients.
- 4- Substitute the stress-field with unknown coefficients in the equations of equilibrium. This substitution provides another set of equations in terms of unknown coefficients.
- 5- Solve the equations derived from steps 3 and 4 for the unknown coefficients. The SAS field is determined by using these stress coefficients in the stress distributions.
- 6- Compute the error-bound, by evaluating error indicator (2.82) and error estimator (2.84) using the KAS and the SAS fields.

Method C : Approximate Error-Bound computed from the KAS field and the mean-SAS fields.

The construction of the mean-SAS field from the KAS field includes the following calculations. This stress-field may be used to compute the approximate error-bound.

- 1- Compute the *nodal forces, at each node, due to tractions* over the element-faces, for each element, such that their associated element-tractions are in equilibrium with the body-force and force-boundary conditions. The procedure is identical to the computation of the *nodal force due to tractions* described in method A.
- 2- Determine the nodal values of stresses by using the computed *nodal forces due to tractions* and the expression relating those nodal forces to stress-field.
- 3- Generate the continuous stress-field by using the standard interpolation technique from the nodes to the interior of elements. This continuous stress-field is called the mean-SAS field.
- 4- Compute the approximate error-bound, by evaluating the error indicator (2.83) and error estimator (2.84).

2.9 Implementation

In order to employ the above methods of error analysis to the applications, these methods (A and C) are being implemented in software. Presently, the Method C has been tested for a number of small problems with constant stress distributions through a body.

Method B has been, initially, implemented in a computer program by Ladeveze *et al*, and tested for a problem of solid mechanics in plane-stress case.

The method of stress calculation from a given set of *nodal forces due to tractions*, has been examined by the author. In this calculation, a stress-field has been computed manually over a linear triangular element in plane stress case.

3. Error-Estimate Using Quasi-SAS Fields

3.1 Introduction

In the previous section, methods have been presented to compute the error-bounds for the finite-element solution of problem (2.1) subject to boundary conditions (2.2). Some of these methods make use of the statically admissible stress-fields $\hat{\sigma}_h$. It has been shown that for the exact solutions $(\tilde{u}_f, \tilde{\sigma}_f)$ and a statically admissible stress-field $\hat{\sigma}_h$, there exists: $\hat{\sigma}_h - \tilde{\sigma}_f \in \chi_0$ in which χ_0 is the space of homogeneous SAS fields. Furthermore, it was found that for a kinematically admissible stress-field σ_h , the inequality:

$$\|\sigma_h - \tilde{\sigma}_f\|^2 \leq \|\sigma_h - \hat{\sigma}_h\|^2 \quad (3.1)$$

holds. Also, an alternative method was presented to compute an approximate SAS field and an approximate error-bound for the finite-element solutions of elliptic boundary value problems. All of the stress-fields presented in the previous section are such that their element-tractions are in equilibrium with the body-forces. The error-bounds computed from the SAS fields are absolutely reliable.

This section innovates an error-estimate which is computed from a family of stress-fields which are called quasi-statically admissible stress (quasi-SAS) fields. Unlike the previous stress-fields, the new stress-field does not satisfy the equations of equilibrium. Moreover, the associated tractions do not satisfy solvability condition (2.20). This stress-field which does not retain many features of the SAS field, will be described subsequently. The quasi-SAS field will be used to estimate the error in the finite-element solution of the problem. Since the quasi-SAS field is not statically admissible, it is not expected that the computed error-estimate, using this stress-field, produces an upper bound for the error in the finite-element solution.

3.2 Quasi-SAS Field

The quasi-SAS field is a continuous stress-field, in the interior of a domain, which conforms to the traction boundary conditions on the boundaries of the domain. This stress-field can be computed from the KAS field by using a smoothing technique. In

addition, this stress-field must conform to the prescribed traction boundary conditions on the external boundary of the domain. In mathematical terminology, a stress-field σ_h is quasi-statically admissible if:

- 1- $\bar{\sigma}_h$ is in C^1 in the interior of each element; (3.2a)

- 2- $\bar{\sigma}_h \cdot n$ is single valued (continuous) across the boundaries of elements; (3.2b)

- 3- $\bar{\sigma}_h \cdot n = \bar{T}_{(n)}$ on the boundary $\partial_2 \Omega$. (3.2c)

The stress-field which satisfies the conditions (3.2a-c), differs from the SAS field which satisfies conditions 1-3 of theorem 2.1 in that the quasi-SAS field does not satisfy the equations of equilibrium in the interior of the elements. The space χ_f containing such stresses are defined by:

$$\chi_f = \{\bar{\sigma}_h \mid \bar{\sigma}_h \in H(\text{div}, \Omega), \text{div } \bar{\sigma}_h + f \neq 0 \text{ in } \Omega, n^T \cdot \bar{\sigma}_h = \bar{T}_{(n)} \text{ on the boundary } \partial_2 \Omega\} \quad (3.3)$$

in which $H(\text{div}, \Omega)$ denotes the induced Sobolev space defined by:

$$H(\text{div}, \Omega) = \{\sigma \mid \text{div } \sigma \in L^2(\Omega)\}$$

Furthermore, the relaxation of the equations of equilibrium in the quasi-SAS fields violates the solvability equation (2.20) which is essential for the static admissibility of a stress-field.

The features associated with the quasi-statically admissible stress-field may be written as follows:

- 1- The quasi-SAS field is in C^1 in the interior of each element.
- 2- On the external boundaries, the stress-vectors (element-tractions) computed from these stress-fields conform to the prescribed traction boundary conditions, i.e.,

$$n^T \cdot \bar{\sigma}_h = \bar{T} \quad (3.4)$$

- 3- The elements in the mesh are *not* in equilibrium with respect to the element-tractions computed from this stress-field and the body-force, i.e.,

$$\int_{\partial \omega} n_\omega^T \cdot \bar{\sigma}_h \, ds + \int_\omega f \, d\Omega \neq 0 \quad (3.5)$$

- 4- The nodal forces computed from the internal stresses and external forces, on every node, are *not* in equilibrium, i.e.,

$$\int_{\partial\omega} (n_{\omega}^T \cdot \bar{\sigma}_h) \cdot a_i \, ds + \int_{\omega} f \cdot a_i \, d\Omega \neq \int_{\omega} \bar{\sigma}_h B a_i \, d\Omega \quad (3.6)$$

- 5- The quasi-statically admissible stress-field does *not* satisfy the equation of equilibrium* in the interior of elements, i.e.,

$$\operatorname{div} \bar{\sigma}_h + f \neq 0 \quad (3.7)$$

- 6- On the boundaries of elements, the stress-vectors (element-tractions) computed from the quasi-SAS field are *not* matched with the stress-vectors (element-tractions) $T = n^T \cdot \bar{\sigma}_h$, i.e.,

$$n^T \cdot \bar{\sigma}_h \neq T \quad (3.8)$$

In addition to the above imperfect characteristics of the quasi-SAS field $\bar{\sigma}_h$, it can be shown from the space (3.3) that: $\bar{\sigma}_h - \tilde{\sigma}_f \notin \chi_0$; therefore the error-bound:

$$\|\sigma_h - \tilde{\sigma}_f\|^2 \not\leq \|\sigma_h - \bar{\sigma}_h\|^2 \quad (3.9)$$

does not hold.

Despite these deficiencies, the quasi-SAS fields may be useful, in practice, for estimating the error in the finite-element solution of problem (2.1) and (2.2). A measure of error in the equations of equilibrium can be estimated. This measure indicates the accuracy of the quasi-SAS field. In the following section, a procedure for constructing the quasi-statically admissible stress-field is presented. Then this stress-field is used to compute local and global error-measures.

3.2.1 Procedure for Constructing the Quasi-SAS Fields

The purpose of this section is to develop a procedure for constructing the quasi-statically admissible stress-fields from the kinematically admissible stress-fields. The displacement finite-element method computes the kinematically admissible stress-field whose element-tractions can be discontinuous on the interelement and global boundaries; moreover, it does not need to satisfy the equations of equilibrium in the interior. In fact, the

* This equation may be satisfied in a finite number of points.

displacement finite-element solution violates restrictions 1-3 of theorem 2.1 (SAS field) given in the previous sections. Nevertheless, in the displacement finite-element method the compatibility equations are assured, but the equation of equilibrium is satisfied approximately [30]. Numerical computations have shown that a major contribution to the error in the stresses computed from the displacement finite-element method arises from the discontinuities of the traction vectors (stress-vectors) across the interelement and global boundaries [31]. These results signify the definition of the quasi-SAS fields. According to the foregoing definition, the quasi-SAS field σ_h is a continuous stress field that satisfies the traction boundary conditions on the external faces $\partial_2\Omega$. Therefore the construction of the quasi-SAS fields may involve two major steps:

- I- To define a continuous stress field, $\sigma \in H(\text{div}, \Omega)$, over the interior and on the boundaries of a domain, using the displacement finite-element solution. Barlow has shown that the optimal Gauss point data are most accurate within the domain [32]. Making use of this important feature in the displacement finite-element method and employing a suitable smoothing operator in the interior of the domain, a stress field in C^0 can be constructed. The computations involve the determination of the stress values at all the nodes in the mesh. Various procedures have been used to obtain the nodal values of stresses, and in the present context any such procedure can be employed. Zienkiewicz suggests that a simple bilinear (trilinear) extrapolation from the Gauss points should be made [30]. Other examples are given by Hinton *et al* [33]. Margolin *et al* proposed a method to construct *divergence* of flux at the centroid from the centroidal temperatures associated with the adjacent elements [34]. In this procedure, first, the temperature gradients are computed at the nodes using the centroidal temperatures of the elements containing the node; then, *divergence* of flux is computed at the center from the nodal temperature gradients. However, in this work an average value of the stresses from the neighbouring Gauss points, computed from the displacement finite-element method, is assigned to the nodes, i.e.,

$$\bar{\sigma}_j = \frac{1}{m} \sum_{\omega_j=1}^m \{\sigma_j^{\omega_j}\} \quad (3.10)$$

In this expression, m is the number of Gauss points contributing to the node j ; and $\bar{\sigma}_j$ is the vector of stresses at the node j . The continuous stress field in the interior

of the element is constructed using the same shape functions $[N]$ as are used for the displacement interpolation. The expression describing the continuous stress field $\bar{\sigma}_h$ is written as:

$$\sigma_h^c = [N][\bar{\sigma}_j] \quad (3.11)$$

The accuracy of the continuous stress field depends upon the accuracy of the Gauss point data and the nature of the smoothing operator.

- II- To smooth the stress fields over the external boundaries $\partial_2\Omega$ in such a way that they become compatible with the traction boundary conditions on the external boundaries $\partial_2\Omega$; in mathematical term the conditions, $n^T \cdot \bar{\sigma}_h = \bar{T}_{(n)}$ on the external boundaries must be fulfilled. The methodology to assign the traction boundary conditions is given as follows:

A- Determine the stress vectors at the center of each element face, s , which is subjected to the traction boundary conditions. This can be done by interpolating from the nodal values of stresses (computed previously) to the center of the element face. In this work the average value of the stress components from the nodes lying on the element face is assigned to the center of that element-face.

B- At the center of this element face, define a traction vector on the element face, using the stress vector at the center and the relation: $T_s = n_s^T \cdot \bar{\sigma}_h$. The normal and tangential components of the traction vector can be obtained from the computed traction at the center of the element-face. In this work, first, a local system of coordinates, which is directed to the normal and tangential directions to the face, is specified at the center of the element-face; then, using the stress transformation technique: $\bar{\sigma}_h = [r]\bar{\sigma}_h[r]^T$, the stress tensor at the center of the element face is transformed with respect to the above local system of coordinates. In the transformation expression, $[r]$ denotes the matrix of direction cosines of the local coordinate system with respect to the global system. The appropriate components of the traction vector are chosen from the stress tensor in the local coordinate system and replaced by the

similar components from the prescribed traction boundary conditions. This stress-field is called the corrected stress-field.

C- Transform the corrected stress-field in local coordinates back to the original coordinates through stress transformation technique.

D- The stress vector at the center of element-faces, which are conformed to the traction boundary conditions are projected to the nodes contained in the element-faces.

The process of assigning the traction boundary conditions to the external boundaries includes some approximations which arise from various projections. Even with the presence of these approximations, the stress-field improves substantially over the boundary $\partial_2\Omega$. Some numerical results are presented which support this assertion.

A comprehensive instruction for computing the quasi-SAS field is given in the following section.

3.2.2 Algorithm for Implementation of the Quasi-SAS Field

There are two major steps involved in this development; in the first step, the nodal stresses that define the continuous stress field over the domain are computed, and in the second step the nodal stresses on the external boundaries $\partial_2\Omega$ are corrected so that they are in equilibrium with the prescribed traction boundary conditions.

Step I- Computation of the nodal stresses:

- 1- Calculate the Gauss point stresses using the displacement finite-element method.
- 2- Calculate nodal stress components by projecting from the neighbouring Gauss point stress components. In this work we simply average the nearest Gauss point stresses and assign this average value to the node.

Step II- Assignment of the traction boundary conditions:

- 3- For each node on the external boundary over which the traction boundary conditions are prescribed:
 - Loop over the external faces of elements containing the node.
 - For each external face containing the node, compute the stress components at the

center of the external face, by averaging the nodal stresses of the same external face.

- Define an orthogonal system of coordinates at the center of this external face such that one of the coordinates is the outward normal to the face.
 - Find the stress tensor at the center of the face with respect to the local coordinates through coordinates transformation (stress tensor in local coordinates).
- 4- Retrieve the traction boundary conditions. There are three possibilities for the traction boundary conditions:
- a- If the external face is subjected to a non-zero traction boundary condition, the traction is defined, and no further calculations are required at this stage- Go to step 5.
 - b- If the traction boundary conditions are prescribed at the nodes, interpolate them to the centroid of the external-face.
 - c- If no boundary conditions are specified on an external face, we consider that the traction boundary conditions are such that the boundary tractions are zero.
- 5- Equate the three stress components of the local stress tensor with the traction components at the centre of each external face (corrected local stress tensor).
- 6- Transform the corrected local stress tensor at the center of each external face back to the global coordinate system.
- 7- Compute the stress tensor for every external node on the boundary $\partial_2\Omega$ by averaging the stress components at the centers of the external faces containing that node.

The nodal values of the stresses computed by using the above algorithm are the values of the quasi-statically admissible stress fields at these points. The continuous quasi-statically admissible stress field can be constructed using the shape functions $[N]$ and the nodal values $\bar{\sigma}_j$, according to: $\bar{\sigma}_h = [N]\{\bar{\sigma}_j\}$.

The above algorithm has been implemented in a finite-element program and successfully tested; the results are applied to compute the error estimates. We will show, by

numerical results, that this error-estimate is more accurate than the error-estimate computed from the continuous stress-field without assigning the traction boundary conditions

3.3 Error Estimator and Error Indicator

Recall expression (3.9) in which the quantity $\|\sigma_h - \bar{\sigma}_h\|^2$, i.e., the difference between the quasi-SAS field from the KAS field, measured in the strain energy norm, is used to estimate the absolute error in the finite-element solution σ_h . This quadratic error-measure is defined by:

$$\|\bar{\sigma}_h - \sigma_h\|^2 = \int_{\Omega} (\bar{\sigma}_h - \sigma_h)^T C (\bar{\sigma}_h - \sigma_h) d\Omega \quad (3.12)$$

The element contribution of expression (3.12) defines the error-indicator. The local error-measure is defined by the element contribution of the absolute error (3.12) normalized by the total strain energy of the deformed body. The local error-measure is expressed by:

$$\|e\|_{\omega}^2 = \int_{\omega} \left[\frac{(\bar{\sigma}_h - \sigma_h)^T C (\bar{\sigma}_h - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (3.13)$$

with

$$\|\sigma_h\|^2 = \int_{\Omega} \sigma_h^T C \sigma_h d\Omega$$

The error-estimator is defined by the relative value of $\|\bar{\sigma}_h - \sigma_h\|^2$ with respect to the total strain energy of the loaded body. In mathematical form, the error-estimator is defined by:

$$\|e\|^2 = \sum_{\omega \in \omega_h} \|e\|_{\omega}^2 \quad (3.14)$$

In practical applications, first, the local error-measure (3.13) is computed; then the error-estimator is computed by the summation of the local error-measures from all elements in the mesh.

3.4 Numerical Results

In order to examine the error indicator (3.13) and the error-estimator (3.14), first, these expressions have been implemented in a computer program. Then some problems, for which the exact solutions or other polynomial solutions are defined, have been solved by the finite-element method. Next, the error in the finite-element solutions of these

problems are estimated by using this method of error analysis. In addition, the exact errors are calculated from the difference between the finite-element solutions versus the exact solutions. For those problems in which the exact solutions are not available, an accurate finite-element solution computed from a very fine mesh serves as an exact solution for comparison purposes.

The computer code uses 4-noded quadrilateral elements and 8-noded hexahedrons. The use of these elements for the method has been incidental, and the method can be implemented in other computer codes that use other elements without losing its characteristics.

3.4.1 Examples:

Test Problem 1.

Title : Uniaxial tensile test with $\sigma_{yy} = \text{constant}$.

Purpose : Since the finite-element solution is exact, therefore, this problem will examine how close we can get to the exact error which is zero.

The first test case is a plate of rectangular shape under the application of a uniform tensile force of intensity σ_0 on two opposite edges as shown in Figure (3.1).

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Traction Boundary Conditions : Normal tractions, 100 MPa are prescribed along the top and bottom edges, Figure (3.1).

Displacement Boundary Conditions : Displacements are prescribed at four points illustrated in Figure (3.2) to constrain the rigid body modes.

The mesh consists of 16 of 4-noded quadrilateral elements to calculate the KAS field as shown in Figure (3.2). To compute the quasi-SAS field, first, the stresses are calculated at 2×2 Gauss points. The Gauss point stresses are averaged and assigned to the nodes. In addition, the traction boundary conditions are assigned to the top and bottom boundaries.

It can be seen that the exact stress distribution is constant throughout the domain [37]. This exact solution is computed by the displacement finite-element method. There-

Figure (3.1)

Square plate with a uniform edge tension.

fore the computed quasi-SAS field is the exact stress field which is a SAS field as well.

The stress distributions and the error-estimate are determined as:

$$\begin{cases} \sigma_{xx} = 0.98820E - 14. & \text{round off error.} \\ \sigma_{yy} = 100. & \text{MPa} \\ \sigma_{xy} = -0.19557E - 14. & \text{round off error.} \end{cases}$$

Estimated error=0.0

Test Problem 2.

Title : Square plate with a parabolic edge tension.

Purpose : To compare the error-estimate obtained from the present algorithm to the error-estimate computed using another polynomial solution.

As the second example, test problem 1 was solved once again, but this time the edge tension was parabolically distributed.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Figure (3.2)
Geometry of the mesh for KAS field calculations.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Traction Boundary Conditions : Normal tractions of parabolic distribution are prescribed to the top and bottom edges, Figure (3.3).

Displacement Boundary Conditions : Displacements are prescribed at four points illustrated in the Figure (3.2) to constrain the rigid body modes.

In Figure (3.3) the illustration of the isotropic square plate of side length L is presented. The plate is loaded on two opposite edges by a parabolic tensile traction of maximum value $\sigma_0 = 100 \text{ MPa}$ at the middle as follows:

$$\sigma_{yy}|_{y=\pm(\frac{L}{2})} = \left[1 - 4\left(\frac{x}{L}\right)^2\right] \sigma_0 \quad (3.15)$$

The plate was discretized into 16 of four-noded quadrilateral elements over which the finite-element solution was obtained. The quasi-SAS field has been calculated using the same procedures as for the previous problem. Furthermore, the error-measures are calculated using expressions (3.13) and (3.14). The three-dimensional representation of these errors versus the elements is exhibited in Figure (3.4).

Figure (3.3)
Square plate loaded by the parabolic tensile
traction applied in the y -direction

The total relative error associated with this discretization is estimated to be 0.01725. Unfortunately there is no exact solution for this problem, but a semi-analytical solution for stresses has been developed by the author in terms of a series expansion using the Ritz method [37] as follows:

$$\begin{aligned}
 \sigma_{xx} &= 4 \frac{2.72222316}{L^6} \sigma_0 \left(x^2 - \frac{L^2}{4} \right)^2 \left(3y^2 - \frac{L^2}{4} \right) + \dots \\
 \sigma_{yy} &= \left[1 - 4 \frac{x^2}{L^2} \right] \sigma_0 + 4 \frac{2.72222316}{L^6} \sigma_0 \left(y^2 - \frac{L^2}{4} \right)^2 \left(3x^2 - \frac{L^2}{4} \right) + \dots \\
 \sigma_{xy} &= -16 \frac{2.72222316}{L^6} \sigma_0 xy \left(x^2 - \frac{L^2}{4} \right) \left(y^2 - \frac{L^2}{4} \right) + \dots
 \end{aligned} \tag{3.16}$$

for the maximum stress, $\sigma_0 = 100$ MPa, at the midpoint on top and bottom boundaries. The relative error in the finite-element solution in comparison with this semi-analytical

Figure (3.4)

Three-dimensional representation of errors versus the elements

solution is found to be 0.03842994. As test problem No. 3, the same problem was solved once again, but this time the domain was discretized into 64 linear quadrilateral elements. The error-estimate and the exact error computed and by comparing to the solution (3.16) are 0.00677346 and 0.00948447 respectively. These values violate the notion of error-bound. The decreasing characteristic of the errors upon mesh refinement indicates that

the error-estimates are behaving well.

Test Problem 4.

Title : Pure in-plane bending of a square plate.

Purpose : To compare the error-estimate obtained from the present algorithm to the exact error computed from the exact solution.

The geometrical configuration of this example is shown in Figure (3.5). This is the pure bending of an isotropic square plate of side length L under a normal stress of maximum value $\sigma_0 = 100$ MPa, varying linearly over the edge, given by:

$$\sigma_{xz}|_{x=\pm \frac{L}{2}} = 2 \left(\frac{y}{L} \right) \sigma_0 \quad (3.17)$$

with the origin of the coordinates x, y at the plate center.

Solution Specifications :

Partial Differential Equations Type : Linear elasticity, Plane Stress.

Material Properties : $E = 210$ GPa, $\nu = 0.3$.

Traction Boundary Conditions : Linearly varying normal traction applied to the vertical sides, Figure (3.5).

Displacement Boundary Conditions : Displacements are prescribed at four points illustrated in the Figure (3.2) to constrain the rigid body motion.

As before, this problem is solved for the displacement and the stress-fields (u_h, σ_h) by the displacement finite-element method using 16 quadrilateral elements. The quasi-SAS field is constructed from the associated kinematically admissible stress field. Moreover, the local error-measures are computed using the expression (3.13). Using expression (3.14), the error-estimator in the solution is determined to be 0.03611.

The exact solution for the displacements is given, [38]:

$$\begin{aligned} u &= 2 \left(\frac{\sigma_0}{E} \right) \frac{xy}{L} \\ v &= - \left(\frac{\sigma_0}{E} \right) \frac{x^2 + \nu y^2}{L} \end{aligned} \quad (3.18)$$

The exact errors are calculated by comparing the kinematically admissible stress-field to the above exact solution. The exact error which is normalized by the total strain energy of the body is determined as 0.03998911.

Figure (3.5)

Geometry of square plate and loading condition
for the case of in-plane pure bending problem

To perform the test problem No. 5, once again the plate was discretized using 64 elements and was solved in the same manner. The results of the error analysis using this method and the exact error using the exact solution (3.18) were found to be 0.01044 and 0.00820643 respectively. Comparison of these two values indicates that this method of error analysis does not characterize error-bound for this problem.

In Figure (3.6) the block representation of the local error-measures are presented.

Figure (3.6)
Block representation of errors versus the elements
for in-plane bending problem

Test Problem 6.

Title : Rotating disk with a circular hole

Purpose : To compare the error-estimate computed from the present approach with the exact error calculated from the exact solution. For this problem the body force is generated from the centrifugal force.

The error analysis for rotating circular disks is of particular importance. The disk which was considered in this problem has interior and exterior radii of $a = 0.8$ m and $b = 1.5$ m respectively, Figure (3.7); this disk is loaded by the centrifugal force as the body force.

$a = 0.8\text{m}$
 $b = 1.5\text{m}$

Figure (3.7)

**Illustration of rotating disk
with a circular hole**

Figure (3.8)

Geometry of the mesh

Solution Specifications :

Partial Differential Equations Type : Linear elasticity, Plane Stress.

Material Properties and forces : $E = 210 \text{ GPa}$, $\nu = 0.3$, Angular velocity, $\theta = 104.7 \text{ rad/sec.}$, Centrifugal force is modelled as body force.

Traction Boundary Conditions : No tractions are prescribed.

Displacement Boundary Conditions : Displacements are prescribed at four points illustrated in the Figure (3.8) to constrain the rigid body motion.

The above disk is discretized into 128 quadrilateral elements to solve for the kinematically admissible stress-field as shown in Figure (3.8). By the application of the method presented in this work, the error in the kinematically admissible stress field is estimated to be 0.00641425. The exact stress field for this problem is given with respect to a cylindrical coordinates system [37] in the form of:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\theta^2 \left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2 \right) \\ \sigma_\theta = \frac{3+\nu}{8}\rho\theta^2 \left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2 \right) \\ \sigma_{r\theta} = 0 \end{cases} \quad (3.19)$$

where ν, ρ, θ are: Poisson's ratio, density, and angular velocity of the disk respectively, and r is the associated radius of an arbitrary point on the disk. With respect to the exact solution (3.19), the exact error in the kinematically admissible stresses which is calculated by the displacement finite-element method, is determined to be 0.00605266. As is clear, the exact error is less than the estimated error; so the error-estimate is reliable in this case.

Test Problem 7.

Title : L-shaped domain plane stress and error analysis

Purpose : To compare the error-estimates associated with coarse and fine meshes

The L-shaped is another interesting problem. This is a standard problem which has been used in many studies. The geometry and the physical boundary conditions associated with this problem are illustrated in Figure (3.9).

Solution Specifications :

Partial Differential Equations Type : Linear elasticity, Plane Stress.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Traction Boundary Conditions : Normal tractions are prescribed on the left edge as shown in Figure (3.9).

Displacement Boundary Conditions : Displacements are prescribed on the two other edges as shown on the Figure (3.9) to constrain the rigid body motion.

Figure (3.9)
Geometrical configuration and loading conditions
for a thin L-shaped domain

Initially, the L-shaped domain is discretized into 27 elements of uniform four-noded quadrilaterals to determine the stress field using displacement finite-element method. Then the quasi-SAS field has been constructed employing the presented method. The estimated relative error in this mesh, using this scheme, is 0.08947.

Since there is no exact solution available for this problem, the analysis is repeated using 768 elements of four-noded quadrilaterals; the solution associated with this procedure, which is counted as the test problem No. 8, has been employed in problem No. 7 as a more accurate solution. The error-measures are calculated using the same procedure as before. The error-estimator here is found to be 0.01175. The block representation of the error distribution, for this problem, has been illustrated in the Figure (3.10). As is evident from this Figure, a major portion of error is accumulated at the vicinity of the singularity about which the stress gradients vary rapidly. This solution is considered as a more accurate solution for comparison purposes, and can be used to examine the accuracy of the solution obtained from the coarse mesh. Comparison of the error-measures obtained from the two analyses confirms the validity of this method because the error-estimates

diminish upon mesh refinement.

Figure (3.10)

Block representation of errors versus the elements
for plane stress distribution in a thin L-shaped domain

A resolution test chart featuring various patterns of vertical and horizontal lines. The patterns are arranged in a grid-like fashion, with some patterns having numerical values next to them. The values include 1.0, 1.1, 1.25, 1.4, 1.6, 1.8, 2.0, 2.2, 2.5, 2.8, 3.2, 3.6, 4.0, 4.5, 5.0, 5.6, 6.3, 7.1, 8.0, 9.0, 10.0, 11.2, 12.5, 14.0, 16.0, 18.0, 20.0, 22.5, 25.0, 28.0, 32.0, 36.0, 40.0, 45.0, 50.0, 56.0, 63.0, 71.0, 80.0, 90.0, 100.0, 112.0, 125.0, 140.0, 160.0, 180.0, 200.0, 225.0, 250.0, 280.0, 320.0, 360.0, 400.0, 450.0, 500.0, 560.0, 630.0, 710.0, 800.0, 900.0, 1000.0, 1120.0, 1250.0, 1400.0, 1600.0, 1800.0, 2000.0, 2250.0, 2500.0, 2800.0, 3200.0, 3600.0, 4000.0, 4500.0, 5000.0, 5600.0, 6300.0, 7100.0, 8000.0, 9000.0, 10000.0, 11200.0, 12500.0, 14000.0, 16000.0, 18000.0, 20000.0, 22500.0, 25000.0, 28000.0, 32000.0, 36000.0, 40000.0, 45000.0, 50000.0, 56000.0, 63000.0, 71000.0, 80000.0, 90000.0, 100000.0, 112000.0, 125000.0, 140000.0, 160000.0, 180000.0, 200000.0, 225000.0, 250000.0, 280000.0, 320000.0, 360000.0, 400000.0, 450000.0, 500000.0, 560000.0, 630000.0, 710000.0, 800000.0, 900000.0, 1000000.0, 1120000.0, 1250000.0, 1400000.0, 1600000.0, 1800000.0, 2000000.0, 2250000.0, 2500000.0, 2800000.0, 3200000.0, 3600000.0, 4000000.0, 4500000.0, 5000000.0, 5600000.0, 6300000.0, 7100000.0, 8000000.0, 9000000.0, 10000000.0, 11200000.0, 12500000.0, 14000000.0, 16000000.0, 18000000.0, 20000000.0, 22500000.0, 25000000.0, 28000000.0, 32000000.0, 36000000.0, 40000000.0, 45000000.0, 50000000.0, 56000000.0, 63000000.0, 71000000.0, 80000000.0, 90000000.0, 100000000.0, 112000000.0, 125000000.0, 140000000.0, 160000000.0, 180000000.0, 200000000.0, 225000000.0, 250000000.0, 280000000.0, 320000000.0, 360000000.0, 400000000.0, 450000000.0, 500000000.0, 560000000.0, 630000000.0, 710000000.0, 800000000.0, 900000000.0, 1000000000.0, 1120000000.0, 1250000000.0, 1400000000.0, 1600000000.0, 1800000000.0, 2000000000.0, 2250000000.0, 2500000000.0, 2800000000.0, 3200000000.0, 3600000000.0, 4000000000.0, 4500000000.0, 5000000000.0, 5600000000.0, 6300000000.0, 7100000000.0, 8000000000.0, 9000000000.0, 10000000000.0, 11200000000.0, 12500000000.0, 14000000000.0, 16000000000.0, 18000000000.0, 20000000000.0, 22500000000.0, 25000000000.0, 28000000000.0, 32000000000.0, 36000000000.0, 40000000000.0, 45000000000.0, 50000000000.0, 56000000000.0, 63000000000.0, 71000000000.0, 80000000000.0, 90000000000.0, 100000000000.0, 112000000000.0, 125000000000.0, 140000000000.0, 160000000000.0, 180000000000.0, 200000000000.0, 225000000000.0, 250000000000.0, 280000000000.0, 320000000000.0, 360000000000.0, 400000000000.0, 450000000000.0, 500000000000.0, 560000000000.0, 630000000000.0, 710000000000.0, 800000000000.0, 900000000000.0, 1000000000000.0, 1120000000000.0, 1250000000000.0, 1400000000000.0, 1600000000000.0, 1800000000000.0, 2000000000000.0, 2250000000000.0, 2500000000000.0, 2800000000000.0, 3200000000000.0, 3600000000000.0, 4000000000000.0, 4500000000000.0, 5000000000000.0, 5600000000000.0, 6300000000000.0, 7100000000000.0, 8000000000000.0, 9000000000000.0, 10000000000000.0, 11200000000000.0, 12500000000000.0, 14000000000000.0, 16000000000000.0, 18000000000000.0, 20000000000000.0, 22500000000000.0, 25000000000000.0, 28000000000000.0, 32000000000000.0, 36000000000000.0, 40000000000000.0, 45000000000000.0, 50000000000000.0, 56000000000000.0, 63000000000000.0, 71000000000000.0, 80000000000000.0, 90000000000000.0, 100000000000000.0, 112000000000000.0, 125000000000000.0, 140000000000000.0, 160000000000000.0, 180000000000000.0, 200000000000000.0, 225000000000000.0, 250000000000000.0, 280000000000000.0, 320000000000000.0, 360000000000000.0, 400000000000000.0, 450000000000000.0, 500000000000000.0, 560000000000000.0, 630000000000000.0, 710000000000000.0, 800000000000000.0, 900000000000000.0, 1000000000000000.0, 1120000000000000.0, 1250000000000000.0, 1400000000000000.0, 1600000000000000.0, 1800000000000000.0, 2000000000000000.0, 2250000000000000.0, 2500000000000000.0, 2800000000000000.0, 3200000000000000.0, 3600000000000000.0, 4000000000000000.0, 4500000000000000.0, 5000000000000000.0, 5600000000000000.0, 6300000000000000.0, 7100000000000000.0, 8000000000000000.0, 9000000000000000.0, 10000000000000000.0, 11200

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS
STANDARD REFERENCE MATERIAL 1010a
(ANSI and ISO TEST CHART No. 2)

Test Problem 9.

Title : Error analysis in a cube subjected to a uniform prescribed displacements on opposite faces.

Purpose : To test the statically admissible stress fields and error-bounds.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Displacement Boundary Conditions : Displacements of 0.001 m are prescribed at the external nodes on the faces, Figure (3.11).

Figure (3.11)

Geometry of mesh and loading conditions

The first test problem in three-dimensional space deals with the stress and error analysis in a cube loaded by forces of constant magnitude over the appropriate faces, Figure (3.11). The objective is to compute the error-measures for this problem.

These problems constitute a form of the patch test on a cube containing 64 elements for each of the components of the stress tensor as a constant with other terms equal to

zero. This patch test was exactly satisfied to machine accuracy; therefore, the statically admissible stress fields for these problems are assured.

Since, for these problems, the KAS fields and the SAS fields are identical, therefore, they constitute the exact solutions; the method computes an error-measure of zero magnitude.

Test Problem 10.

Title : Error analysis of a cube loaded by a point force.

Purpose : To investigate the behavior of the error-estimate in the presence of point force.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Traction Boundary Conditions : No tractions but a point force is prescribed.

Displacement Boundary Conditions : Displacements are prescribed on the $Y Z$ face to constrain the rigid body modes.

In order to justify the usefulness of the present error analysis scheme, a cube of finite dimensions loaded by a point force applied at the corner of a cube is investigated. The geometry of the cube and the physical load are shown in the Figure (3.12).

First, the KAS and the quasi-SAS fields are computed. Then the local error-measures are computed. The variation of the local error-measures versus the element numbers is presented in the Figure (3.13). It is evident that the maximum error occurs in the element subjected to the point load. The error in the finite element solution is computed to be 0.7075, using error-indicator (3.13) and error-estimator (3.14).

Figure (3.12)
Cube loaded by a point force

Test Problem 11.

Title : Error analysis in a three-dimensional L – shaped domain.

Purpose : To compute the the error-estimate in an ideal problem.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Displacement Boundary Conditions : Displacements of 0.001 m are prescribed at the external nodes on the face orthogonal to x -axis, Figure (3.14).

The thickness of the L – shaped domain is divided into four subdivisions. The mesh contains a total of 432 hexahedral elements. First, the KAS field associated with the loading conditions was calculated using the displacement finite-element method; then, the quasi-SAS field has been derived from it using the algorithm which was discussed earlier. The error-estimator associated with this discretization is estimated to be 0.00198365.

As test problem No. 12 and for further assessment of the capability of the method, problem No. 11 was solved, this time using 1728 elements of the same type; the error-estimator in the mesh is estimated to be: 0.00106221 which is smaller than the similar

Figure (3.13)
Error distribution versus the element numbers

Geometry of Mesh :

Figure (3.14)

Geometry of mesh and loading conditions

results of the previous test case.

Test Problem 13.

Title : Error analysis in a thick rectangular domain discretized by non-uniform mesh.

Purpose : To compute the error-estimate in the case of non-uniform skewed elements, Kershaw mesh.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity.

Material Properties : $E = 210 \text{ GPa}$, $\nu = 0.3$.

Displacement Boundary Conditions : Displacements of 0.001 m are prescribed at the external nodes on the face orthogonal to the x -axis, Figure (3.15).

The domain is discretized through the thickness into two elements. The mesh contains a total of 648 hexahedrons. First, the KAS field was calculated using the displacement finite-element method; then the quasi-SAS field was derived from the KAS field using the described method. Figure (3.15) represents the distribution of σ_{xx} associated with this

Figure (3.15)

Geometry of the mesh and the distribution of the quasi-SAS field stress-field. The total relative error associated with this discretization is estimated to be 0.0000070514. The exact error in the solution was computed by comparing the KAS field with the exact solution; this error was found to be 0.0000089832. As it is known, the exact error is larger than the estimated error. This comparison indicates that this method of error analysis does not generate a bound on the exact error in the solution of this problem.

The results of the test problems are tabulated in the following table (3.1). Next section compares the current results and those computed using the continuous stress-fields without matching with the prescribed natural boundary conditions.

Table (3.1): Results of error-estimates using the quasi-SAS field

No.	Description	Mesh	$\frac{\ \sigma_h - \sigma_h\ ^2}{\ \sigma_h\ ^2}$	$\frac{\ \sigma_h - \sigma_{exact}\ ^2}{\ \sigma_h\ ^2}$
1	Plane stress for rectangular plate loaded by uniform edge tension	16	0.00000	0.00000
2	Plane stress for rectangular plate loaded by parabolic edge tension	16	0.01725	0.03842994 *
3	Plane stress for rectangular plate loaded by parabolic edge tension	64	0.00677346	0.00948447 *
4	Plane stress for rectangular plate loaded by in-plane bending	16	0.03611	0.03998911
5	Plane stress for rectangular plate loaded by in-plane bending	64	0.01044	0.00820643
6	Plane stress in rotating disk	128	0.00641425	0.00605266
7	Thin L-shaped domain plane stress analysis (coarse mesh)	27	0.08947	-
8	Thin L-shaped domain plane stress analysis (fine mesh)	768	0.01175	-
9	A cube subjected to a constant displacements on the boundaries (6 cases)	64	0.000000000	0.0
10	Stress distribution in a cube loaded by a point force	22	0.7075	-
11	Stress analysis in thick L-shaped domain. (coarse mesh)	432	0.00198365	-
12	Stress analysis in thick L-shaped domain (fine mesh).	1728	0.00106221	-
13	Kershaw Challenge (skewed mesh) (finite thickness)	648	0.0000070514	0.0000089832

* Polynomial solutions developed by the author.

3.5 Comparison with the Alternative Method

In order to demonstrate the advantages of this method of error analysis, the present results are compared with those obtained from the method of Zienkiewicz and Zhu [10]. In their method, they compare the KAS field computed by the finite-element method with the continuous stress-field which does not satisfy the traction boundary conditions. First, the method of error analysis applied by Zienkiewicz and Zhu [10] was implemented in the computer code, solving selected test problems for which the exact solutions are available; then, the errors involved in the solutions were estimated. The error-estimates obtained with the method of Zienkiewicz and Zhu, the exact solution, and the present author are shown in table (3.2), columns 4, 5 and 6, respectively.

Table (3.2): Comparison of the error-estimates with the method of Zienkiewicz and Zhu

No.	Description	Mesh	$\frac{\ \sigma_h - \sigma_h^c\ ^2}{\ \sigma_h\ ^2}$	$\frac{\ \sigma_h - \sigma_{exact}\ ^2}{\ \sigma_h\ ^2}$	$\frac{\ \sigma_h - \sigma_h\ ^2}{\ \sigma_h\ ^2}$
1	Plane stress for rectangular plate loaded by parabolic edge tension	16	0.015001612	0.03842994 *	0.01725
2	Plane stress for rectangular plate loaded by parabolic edge tension	64	0.0058199156	0.00948447 *	0.00677346
3	Plane stress for rectangular plate loaded by in-plane bending	16	0.030313002	0.03998911	0.03611
4	Plane stress for rectangular plate loaded by in-plane bending	64	0.0074110242	0.00820643	0.01044
5	Plane stress in rotating disk	128	0.0041564632	0.00605266	0.00641425

Comparison of the error-estimates computed by Zienkiewicz and Zhu's method [10] with the exact errors indicates that the method employed by Zienkiewicz and Zhu does

* Polynomial solutions developed by the author.

not produce an error-bound; in other words, the predicted error-estimates are less than the true errors. The error-estimates calculated by using the quasi-SAS fields, however, are greater than the error-estimates computed by Zienkiewicz and Zhu's method; the error-estimates computed from the quasi-SAS fields are closer to the exact errors in these problems. In addition, in some cases (see the results), the error-estimates computed from the quasi-SAS fields are error-bounds. It should be noted that an accurate error-measure is crucial in an adaptive analysis. Therefore the method of error analysis using the quasi-SAS field, which produces more accurate error-estimates, is more reliable than the method presented by Zienkiewicz and Zhu.

In summary:

a method of error-estimation has been presented in this section. In this method, first, the quasi-SAS field has been constructed; then this stress-field has been employed to calculate the error-estimates. The procedure is as follows:

- 1- Compute the stress-field at the Gauss points using the displacement finite-element method.
- 2- Compute the nodal values of stresses by averaging the data from the nearest Gauss points to the nodes.
- 3- Interpolate the nodal values of stresses to the interior of elements using the usual shape functions.
- 4- Compute the error-indicator (3.13) and error-estimator (3.14).

4. Estimated Error-Bound Using Quasi-SAS Fields

4.1 Introduction

In the previous section, the quasi-SAS field was introduced. Then this stress-field was applied to estimate the error in the finite-element solution of problem (2.1) and (2.2). The total value of error in the solution was estimated by $\|\sigma_h - \bar{\sigma}_h\|^2$, measured in the energy norm. Because the quasi-SAS field is not statically admissible (except in the case of constant stress distribution), the error-estimate $\|\sigma_h - \bar{\sigma}_h\|^2$, computed by using the quasi-SAS field, does not produce an error-bound in most cases.

The objective of this section is to determine the geometry of the space of quasi-SAS fields with respect to the spaces of kinematically and statically admissible fields, and to estimate the error in the quasi-SAS field in comparison with the statically admissible stress-field. Thus the error in the equation of equilibrium (lack of static admissibility) is added to the error in the finite-element solution estimated by using the quasi-SAS field to compute a more reliable error-measure for a finite-element solution. If an exact value or an upper bound for the error due to the lack of static admissibility can be obtained, then the summation of this error-measure with the error in the solution estimated by using the quasi-SAS field can be used to define an error-measure, which may be called 'estimated error-bound', for the finite-element solutions. However, as will be explained, the exact value or an upper bound for the error due to the lack of static admissibility is difficult to obtain. For this reason, an approximation to the lack of static admissibility will be computed, in the strain energy norm. Therefore addition of this approximate error to the error-estimate computed from the quasi-SAS field provides an error-measure, which may be called 'approximate estimated error-bound', for the finite-element solution. The geometry of the quasi-SAS field and the determination of the approximate estimated error-bound are performed in the subsequent sections.

4.2 Geometry of a Quasi-SAS Field

As was mentioned earlier, a quasi-SAS field is a continuous stress-field which conforms to the prescribed traction boundary conditions. This continuous stress-field is constructed

within the domain by using a smoothing technique and the Gauss point values of stresses which are computed using the displacement finite-element method. The quasi-SAS field, obtained in this manner, does not satisfy the equation of equilibrium in the interior of elements:

$$\operatorname{div} \bar{\sigma}_h + f \neq 0 \quad (4.1)$$

(The equation of equilibrium may be satisfied, only, in a finite number of points within the domain.) The error in this equation introduces several deficiencies in this stress-field. The features of the quasi-SAS field describing these defects were presented in the previous section.

We would like to study the geometry of a quasi-SAS field with respect to the SAS field and estimate the error in that stress-field due to the lack of static admissibility. In other words, we wish to estimate the error in the equation of equilibrium (4.1).

Let Σ denotes the space of KAS fields, and χ_f be the space of SAS field associated with the same problem; then, as is known from lemma 2.1, these spaces are orthogonal to each other with respect to the energy inner product (2.9). The intersection of these spaces characterizes the exact solutions $(\tilde{u}_f, \tilde{\sigma}_f)$ of this problem, Figure(4.1).

Since for a quasi-SAS field $\bar{\sigma}_h$ there exists a body-force $\bar{f}$ such that: $\bar{f} = -\operatorname{div} \sigma_h$, then a space χ_f can be defined which is the space of statically admissible stress-fields with respect to the body-force $\bar{f}$; this space is defined by:

$$\chi_f = \{\bar{\sigma}_h \mid \bar{\sigma}_h \in H(\operatorname{div}, \Omega), \operatorname{div} \bar{\sigma}_h + \bar{f} = 0 \text{ in } \Omega, n^T \cdot \bar{\sigma}_h = \bar{T}_{(n)} \text{ on the boundary } \partial_2 \Omega\} \quad (4.2)$$

Associated with the determined body-force $\bar{f}$, the exact solutions $(\tilde{u}_f, \tilde{\sigma}_f)$ may be characterized. If $\bar{\sigma}_h$ and $\tilde{\sigma}_f$ belong to χ_f , then the difference $\bar{\sigma}_h - \tilde{\sigma}_f$ belongs to the homogeneous space of statically admissible stress-field, χ_0 . Also, for a kinematically admissible displacement u_f , the difference $u_f - \tilde{u}_f$ belongs to the homogeneous space of kinematically admissible displacements $\mathcal{N}_0$ from which the homogeneous space of kinematically admissible stress-fields, Σ_0 , is constructed. Lemma 2.1 states that the two subspaces Σ_0 and χ_0 are mutually orthogonal with respect to the energy norm. This orthogonal property implies that the spaces of kinematically admissible stress-fields Σ and quasi-statically admissible stress-fields χ_f are orthogonal to each other with respect to the energy norm. The

Figure (4.1)

Geometry of the function spaces, Σ , χ_f , and $\chi_{\bar{f}}$; and the exact solutions

geometrical representation of the three spaces: Σ , χ_f , and $\chi_{\bar{f}}$ are given in Figure (4.1). This Figure, also, shows the positions of the exact solutions for the body-forces f and $\bar{f}$ within the admissible function spaces.

4.3 Estimated Error-Bound

Consider the triangle whose vertices, within the function spaces, are defined by the solutions: $\hat{\sigma}_h$, $\bar{\sigma}_h$, and σ_h ; the inequality

$$\|\hat{\sigma}_h - \sigma_h\| \leq \|\hat{\sigma}_h - \bar{\sigma}_h\| + \|\bar{\sigma}_h - \sigma_h\| \quad (4.3)$$

holds within this triangle. Figure (4.1) illustrates the position of this triangle within the spaces. Inequality (4.3) states that the summation of the distances between the quasi-SAS field to the kinematically admissible stress-field and the quasi-SAS field to a statically admissible stress-field specifies an upper bound for the distance between the kinematically admissible stress-field to the statically admissible stress-field. The difference between the

quasi-SAS field to the kinematically admissible stress-field can be explicitly computed. However, the other term $\|\hat{\sigma}_h - \bar{\sigma}_h\|$ can not be easily computed.

Referring to Figure (4.1), it can be deduced that the distance between the two exact solutions $(\tilde{u}_f, \tilde{\sigma}_f)$ and $(\tilde{u}_f, \tilde{\sigma}_f)$, measured in the norm (2.9), can describe the error committed by using the quasi-SAS field $\bar{\sigma}_h$ instead of a SAS field $\hat{\sigma}_h$, i.e., the error due to the lack of static admissibility. The expression:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\| \leq \frac{|f - \bar{f}|}{\lambda^{\frac{1}{2}}} \quad (4.4)$$

has been derived by Hughes, [24], to compute an upper bound, in the energy norm, for the error in the equation of equilibrium. In expression (4.4), the function $|\cdot|$ is the usual norm in L^2 space defined by:

$$|f - \bar{f}|^2 = \int_{\Omega} (f - \bar{f})^T (f - \bar{f}) d\Omega \quad (4.5)$$

‘The parameter λ , in expression (4.4), denotes the least eigenvalue of the expression $B^T C B$. The expression (4.4) defines an upper bound for the error in the solution resulting from an error in the body-forces f and $\bar{f}$. The upper bound (4.4) is a little disconcerting because it says that to get an upper bound for $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|$, one must find a lower bound on the least eigenvalue of the problem. The computation of the lower bound on the least eigenvalue is difficult and expensive. If one is content with a rough estimate for this quantity, a technique similar to LINPACK, [26], can be used to estimate the condition number of the stiffness matrix for this purpose [24].’

Alternatively, an approximate method to estimate the error-measure $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|$ due to the lack of static admissibility may be defined. In this method, one should express the error $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|$ in terms of the energy inner product (2.9) as follows:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 = \int_{\Omega} (\tilde{\sigma}_f - \tilde{\sigma}_f)^T C (\tilde{\sigma}_f - \tilde{\sigma}_f) d\Omega \quad (4.6)$$

Let $\tilde{\sigma}_f = C^{-1} B \tilde{u}_f$ and $\tilde{\sigma}_f = C^{-1} B \tilde{u}_f$; substitute these values in expression (4.6), this becomes:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 = \int_{\Omega} (\tilde{\sigma}_f - \tilde{\sigma}_f)^T B (\tilde{u}_f - \tilde{u}_f) d\Omega \quad (4.7)$$

Using the divergence theorem on this expression, it becomes:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 = - \int_{\Omega} [\text{div} (\tilde{\sigma}_f - \tilde{\sigma}_f)]^T (\tilde{u}_f - \tilde{u}_f) d\Omega + \sum_{\omega} \int_{\partial\omega} [(\tilde{\sigma}_f - \tilde{\sigma}_f) \cdot n] (\tilde{u}_f - \tilde{u}_f) ds$$

in which the second term becomes zero because no jumps in the tractions exist. Recall from the equations of equilibrium that: $f = -\text{div } \tilde{\sigma}_f$ and $\bar{f} = -\text{div } \tilde{\sigma}_f$, the above expression becomes:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 = - \int_{\Omega} (f - \bar{f})^T (\tilde{u}_f - \tilde{u}_f) d\Omega \quad (4.8)$$

To compute expression (4.8), in addition to $\bar{f}$, the difference between the exact displacement fields $\tilde{u}_f$ and $\tilde{u}_f$, i.e.,

$$\Delta u = \tilde{u}_f - \tilde{u}_f$$

must be calculated. In most cases, this cannot be computed; therefore the exact value for $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2$ can not be obtained. For this reason, the estimated error-bound can not be calculated. The finite-element solutions u_h and $\bar{u}_h$, corresponding to the body-forces f and $\bar{f}$ are used to substitute for the exact displacements from which the difference between the exact displacement fields is approximated by the difference between the finite-element values,

$$\Delta u_h = u_h - \bar{u}_h \quad (4.9)$$

Substituting from (4.9) into expression (4.8), an approximation to the error-measure due to the lack of static admissibility can be computed. To determine the value of the Δu_h , the finite element displacement u_h is known; but $\bar{u}_h$ has to be computed. The cost of this calculation is quite reasonable compared with other procedures. In this calculation the load vector must be calculated in such a way that to incorporate the body-force $\bar{f}$.

Another approximate method to estimate $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2$, which involves less computations, is described below. In this method, expression (4.7) is used in which the difference $\tilde{\sigma}_f - \tilde{\sigma}_f$ (of the exact stresses associated with the body-forces f and $\bar{f}$) is replaced by $\hat{\sigma}_h - \bar{\sigma}_h$, where $\hat{\sigma}_h \in \chi_f$ and $\bar{\sigma}_h \in \chi_f$ are statically admissible stress-fields, not necessarily the exact stresses, for the body-forces $\bar{f}$ and f , respectively. In a similar manner as before, substitute their finite-element values. After substituting in (4.7), an expression which approximates the error $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2$ is introduced by:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 \approx \int_{\Omega} [(\hat{\sigma}_h - \bar{\sigma}_h)^T B (u_h - \bar{u}_h)] d\Omega \quad (4.10)$$

Now, recall the space of extension to the SAS fields proposed by Ladeveze *et al* [14], and extract the relations:

$$\int_{\omega} (\hat{\sigma}_h - \sigma_h) B(N_i) d\Omega = 0 \quad \forall i \in k, \forall \omega \in \omega_h \quad (4.11)$$

in which σ_h is the KAS field for the body-force f , and N_i denotes the basis functions at the node i . Using this definition, expression (4.10) reduces to:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 \approx \int_{\Omega} (\sigma_h - \bar{\sigma}_h)^T B(u_h - \bar{u}_h) d\Omega \quad (4.12)$$

Applying the strain-displacement relationship and the constitutive law, an approximation to the expression $\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2$ is obtained as follows:

$$\|\tilde{\sigma}_f - \tilde{\sigma}_f\|^2 \approx \int_{\Omega} (\sigma_h - \bar{\sigma}_h)^T C(\sigma_h - \bar{\sigma}_h) d\Omega \quad (4.13)$$

which describes the error due to the lack of static admissibility. Knowing the error in the equation of equilibrium by expression (4.13), the inequality (4.3) is used to define an error-measure for the finite-element solution (u_h, σ_h) .

Note that expression (4.13) represents an approximate value for the error in the equation of equilibrium. This expression looks like the one describing the error between the kinematically admissible stress-field to the quasi-SAS field. Therefore the extra calculation is not needed to determine the error in the quasi-SAS field; the calculated error-measure using the KAS field and the quasi-SAS field can be used to estimate this error.

The error-measure computed from inequality (4.3) in relation to (4.13) has been implemented in a computer program and tested for the same test problems as in the previous section. The computed error-measures have been compared with the exact errors in the solutions of some problems for which the exact solutions are known. The error-measures computed by using this method are found to be greater than the exact errors in all cases. The magnitude of these error-measures are at least three times larger than the magnitudes of the associated exact errors. The author believes that this error estimate is inaccurate; therefore it is inefficient in application. Because of this undesired characteristic of this error-measure, a new local error-measure is introduced which is expected to be more accurate. Using this local error-measure, the error-estimator is computed.

4.4 Error Estimator and Error Indicator

Because the error-measure (4.3) is inaccurate, a modification of this error-estimate is defined, below, which equals the sum of the squares of the two expressions in the right hand side of inequality (4.3). The error indicator, the local error-measure, and the error estimator corresponding to this error-measure will be introduced in this section. The new error-measures, also, have been implemented in a computer program and tested. The results are presented in table (4.1). This table shows that the computed error-estimates using the new measures are relatively close to the corresponding exact-errors.

The error-indicator is a numerical measure which indicates the error within the mesh. The error-indicator may be defined by the expression $2\|\bar{\sigma}_h - \sigma_h\|_\omega^2$ over the elements in the mesh.

The local error-measure may be defined by the element contribution to the error-measure $2\|\bar{\sigma}_h - \sigma_h\|^2$ normalized by the total strain energy of the deformed body. Based on this definition, the local error-measure is expressed by:

$$\|e\|_\omega^2 = 2 \int_\omega \left[\frac{(\bar{\sigma}_h - \sigma_h)^T C (\bar{\sigma}_h - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (4.14)$$

with

$$\|\sigma_h\|^2 = \int_\Omega \sigma_h^T C \sigma_h d\Omega$$

The expression (4.14) describes a relative value of the quantity used to define the error-indicator. While this value represents the local error-measure, it can be applied to indicate the error within the mesh as well.

The error-estimator is defined by the relative value of $2\|\bar{\sigma}_h - \sigma_h\|^2$ with respect to the total strain energy of the loaded body. In other words, the summation of the local error-measures in the mesh is used to define the error-estimator. In mathematical form, the error-estimator $\|e\|^2$ in terms of the local error-measure $\|e\|_\omega^2$ is described by:

$$\|e\|^2 = \sum_{\omega \in \mathcal{W}_h} \|e\|_\omega^2 \quad (4.15)$$

Note that expression (4.14) defines a local error-measure whose value is twice as the value of the local error-measure defined in the previous section. This indicates that the costs of

implementation and computation in the present method of error analysis are the same as those associated with the 'error-estimate using quasi-SAS field', presented in the previous section. But the comparison of the numerical results of the test problems show that the error-measures (4.14) and (4.15) are more reliable than the error-measures introduced in that section because these are larger than the corresponding exact-errors.

4.5 Numerical Results

The local error-measure (4.14) and the error estimator (4.15) have been implemented in a finite-element program and the same test problems solved in the previous section, were solved by the finite-element method. In addition, the errors in the solutions were estimated using these error-measures. Furthermore, the exact errors in the solutions were calculated. These are presented in the table (4.1). It can be seen from the table (4.1) that the estimated errors are larger than the corresponding exact errors. The magnitude of the estimated errors are within factor three of the exact errors.

Table (4.1): Results of the estimated error-bound using the quasi-SAS field

No.	Description	Mesh	$\frac{2\ \sigma_h - \sigma_h\ ^2}{\ \sigma_h\ ^2}$	$\frac{\ \sigma_h - \sigma_{exact}\ ^2}{\ \sigma_h\ ^2}$
1	Plane stress for rectangular plate loaded by uniform edge tension	16	0.00000	0.00000
2	Plane stress for rectangular plate loaded by parabolic edge tension	16	0.03449688	0.03842994 *
3	Plane stress for rectangular plate loaded by parabolic edge tension	64	0.01354691	0.00948447 *
4	Plane stress for rectangular plate loaded by in-plane bending	16	0.07222294	0.03998911
5	Plane stress for rectangular plate loaded by in-plane bending	64	0.02088954	0.00820643
6	Plane stress in rotating disk	128	0.01282850	0.00605266
7	Thin L-shaped domain plane stress analysis (coarse mesh)	27	0.17893661	-
8	Thin L-shaped domain plane stress analysis (fine mesh)	768	0.02349043	-
9	A cube subjected to a constant displacements on the boundaries (6 cases)	64	0.000000000	0.0
10	Stress distribution in a cube loaded by a point force	22	1.415	-
11	Stress analysis in thick L-shaped domain. (coarse mesh)	432	0.0039673	-
12	Stress analysis in thick L-shaped domain (fine mesh).	1728	0.00212442	-
13	Kershaw Challenge (skewed mesh) (finite thickness)	648	0.000014103	0.0000089832

* Polynomial solutions developed by the author.

4.6 Discussion of Results

The first test problem deals with the analysis of stresses and the corresponding error in an isotropic square plate with a uniform distributed tensile force applied on opposite edges. The exact solution for this problem is a constant stress distribution in the domain which is continuous on the boundaries. The finite-element solution calculates the same stress-field in the interior of the domain discretized using a uniform mesh of rectangular elements. Therefore the error-measures of zero magnitude are computed from a comparison of finite-element and the exact solutions. Ultimately, the error-measure computed using this method of error analysis is found to be zero. It can be claimed that the estimated value of error for this problem is a valid error-measure.

In the second test problem the same plate was considered, and a tensile force of a parabolic distribution was applied on the opposite edges of the plate. The error-measure was computed using this approach which shows to be in agreement with the error computed by using an alternative polynomial solution. Once again the same problem was solved using a finer mesh, for which the results of the error analysis are found satisfactory.

The results of the error analysis for the plane stress analysis in a plate with in-plane bending boundary conditions, for coarse and fine meshes, are found to be valid in comparison with the error-measures computed using the exact solutions. In these cases, the exact errors are found to be smaller than the estimated ones. These results are presented in the table (4.1).

The rotating disc was another interesting test problem which was examined in this work. In this problem the applied load is the centrifugal force resulting from rotation. For this problem an exact solution is available with which the finite-element solution can be compared. The computed error-measure, using this method, is found to be an upper bound for the computed error using the exact solution.

The standard test problem of an L-shaped domain was considered and solved for plane stress analysis using coarse and fine meshes. The error-estimates associated with these analyses are relatively satisfactory because as the mesh size decreases, the estimated errors diminish. Similar results are obtained in the case of the L-shaped problem in three

dimensional space.

Another test problem, considered here, is the cube containing 64 hexahedral 8-noded elements. The body was subjected to a constant prescribed displacements on opposite faces. The stress distributions according to exact solution are constants; the same solution is calculated by using the finite-element method. Therefore the exact error is zero. The error-measures in the present algorithm compute zero values which are in conformity with the exact errors using the exact solutions.

Finally, the Kershaw challenge problem (the plate of finite thickness discretized by irregular elements of 8-noded hexahedral elements) was solved to demonstrate the application of this method of error analysis on the mesh of non-uniform skewed elements. In the case in which the body is loaded by a uniform prescribed displacements on opposite faces, the exact stress-field for this problem is a constant [37]. However, for a non-uniform mesh, the displacement finite-element method generates a stress-field which is non-constant; this occurs because of element distortion. The quasi-SAS field computed from this stress-field appears to be non-uniform as well. This results in a non-zero error-estimate; the value of which may be small. This estimated error is larger than the exact error, thus it is a bound for the exact error.

The aforementioned results of the error analysis for the test problems in which the computed error-measures for a finer mesh are smaller than those using a coarser mesh, support the validity of this method of error analysis. The convergence of the finite-element solutions upon mesh refinement implies this assertion.

The general overview of these results show that all of the error-measures calculated by the application of this method are larger than the exact errors within reasonable ratios; therefore the reliability of this error analysis should be established from the theoretical analysis as well as the numerical results. The method is capable of handling body forces of any configuration, and expected to produce a reliable error-measure for these problems as well.

In summary:

A method to estimate the error in the finite-element solution of the elliptic boundary value

problem has been presented. This method involves the following steps:

- 1- Compute the Gauss point values of stresses using the displacement finite-element method.
- 2- Compute a continuous stress-field within the domain using the Gauss point stresses and a smoothing operator.
- 3- Assign the prescribed traction boundary conditions on the boundaries $\partial_2\Omega$ of the domain. The procedure for this assignment, has been delineated in the previous section.
- 4- Compute the local error-measure and error estimator by employing the expressions (4.14) and (4.15), respectively.

5. Error Estimate Using Weak-SAS Fields

5.1 Introduction

In this section a continuous stress-field is introduced to approximate the exact stress-field. This stress-field, which is called 'weak-SAS field', is denoted by σ_h° and can be employed to estimate the error in the finite-element solution $\|\sigma_h - \tilde{\sigma}_f\|^2$. The error-measure using the weak-SAS field is defined by $\|\sigma_h - \sigma_h^\circ\|^2$. Since the weak-SAS field σ_h° may not be statically admissible, the error-measure $\|\sigma_h - \sigma_h^\circ\|^2$ does not specify an error-bound, i.e.,

$$\|\sigma_h - \tilde{\sigma}_f\|^2 \not\leq \|\sigma_h - \sigma_h^\circ\|^2 \quad (5.1)$$

However, some numerical results indicate that the error-measure $\|\sigma_h - \sigma_h^\circ\|^2$ can be a useful error-estimate.

5.2 Weak-SAS Fields

As was mentioned in the previous sections, the difference between the statically and the quasi-statically admissible stress-fields is that the quasi-statically admissible stress-field does not satisfy the equation of equilibrium in the interior of a domain. This violates the solvability conditions (2.20) for individual elements; in other words, the elements in the mesh are not in equilibrium with respect to the applied tractions and the body-forces. Moreover, the forces exerted on the nodes from external sources and internal stresses are not in equilibrium. Despite these deficiencies, it may be shown that an accurate stress-field can be computed, from the quasi-SAS field, using the iterative method proposed by Loubignac *et al* [35]. Loubignac *et al* have employed a continuous stress-field in an iterative method. They have shown, by numerical examples, that enforcing the equilibrium within the forces applied on the nodes, iteratively, leads to a nearly exact stress-field in the interior of the domain. The main feature of the stress-field, obtained from the iterative method, in comparison to the applied continuous stress-field, is that the iterated stress-field satisfies the equation of equilibrium, in the interior, in a weak sense within a finite-dimensional function space. Thus, if the state of stress on the external boundaries is matched with the prescribed boundary tractions, the stress-field should be a good approximation to the

SAS field. This stress-field which satisfies the equation of equilibrium in a weak sense and conforms to the prescribed boundary conditions is called weak-SAS field.

In mathematical terminology, the stress-field σ_h^o computed by using this method satisfies the static admissibility conditions with respect to a finite dimensional subspace. These conditions are described in the following equations:

$$\int_{\Omega} (\text{div } \sigma_h^o + f) \cdot v_h \, d\Omega + \int_{\partial_2 \Omega} (n^T \cdot \sigma_h^o - T_{(n)}) \cdot w_h \, ds = 0 \quad \forall v_h, w_h \in \mathbb{N}_{0h} \subset \mathbb{N}_0 \quad (5.2)$$

in which $\mathbb{N}_{0h}$ is the finite dimensional subspace associated with the finite-element discretization. We wish to compute the stress-field σ_h^o using the displacement finite-element solutions and an iterative method. There are two steps involved in the calculations of the weak-SAS field which satisfies the conditions (5.1). In the first step, a continuous stress-field is computed, from the KAS fields using an iterative method. This stress-field is such that it satisfies the equation of equilibrium in a weak sense. The procedure followed here is similar to the iterative method proposed by Loubignac *et al* [35]. In this method, initially, the nodal displacements are computed, then a continuous displacement field is defined over the domain using the shape functions. Thus the kinematically admissible fields, (u_h, σ_h) , are calculated from the displacement through the equation:

$$\sigma_h = C^{-1} B u_h \quad (5.3)$$

Next, the nodal values of stresses are computed by averaging the stresses at the nearest Gauss points from the elements containing the node. The mathematical formulation of this processes is given by expression (3.10). Subsequently, a continuous stress-field $\sigma_h^c = [N]\{\bar{\sigma}_j\}$ is constructed over the elements using the shape functions $[N]$ which were applied for the displacement interpolation. In this expression, $\bar{\sigma}_j$ is the nodal value of stresses at the node j .

Loubignac *et al* used the iterative procedure:

$$[K]\{\Delta u^{(i+1)}\} = \{\Delta R^{(i)}\} \quad (5.4)$$

in which $\{\Delta R^{(i)}\}$ is the vector of unbalanced forces at the nodes. This vector is defined by:

$$\{\Delta R^{(i)}\} = \{R\} - \sum_{\omega} \int_{\omega} [B]^T [N] \{\sigma_j^{(i)}\} \, d\Omega \quad (5.5)$$

In these expressions, $[K]$ and $\{R\}$ are the usual stiffness matrix and load vector associated with the discretization. In equation (5.4) the expression $\Delta u^{(i+1)}$ is the $(i+1)th$ correction to the nodal displacement vector $u^{(i)}$. This correction vector is calculated and added to the displacement field which was calculated in $(i)th$ iteration. The most recent displacement value, $u^{(i)} + \Delta u^{(i+1)}$, is used to calculate a new KAS field. Similar procedure as before is applied to compute the new load vector $\{\Delta R^{(i+1)}\}$. Then, equation (5.4) is reapplied until the correction vector becomes sufficiently small. This iterative processes is similar to the modified Newton-Raphson method, but used in solving a linear problem instead of a non-linear one [36]. As this scheme converges, then:

$$\{\Delta R^{(i+1)}\} \rightarrow 0$$

which means that the forces on the nodes diminish by iteration. Loubignac *et al* [35] have shown by numerical examples that the scheme converges to the exact solution. In their calculations, they have used five iterations. The stress-field computed in the last iteration satisfies the equations of equilibrium in a weak sense with respect to the finite dimensional subspace. The first step in which the stress-field is corrected in the interior, iteratively, is completed when the scheme converges. In the second step, the stresses are corrected at the external boundaries $\partial_2 \Omega$ such that they conform to the prescribed boundary tractions. In other words, the boundary tractions are assigned to the boundaries. The method to assign the traction boundary conditions, was explained in the previous sections. Following these two steps, the result would be the stress-field σ_h^o which is expected to be a good approximation to the exact stress-field. This stress-field is called the weak-SAS field. The features of the weak-SAS field may be written as follows:

- 1- The elements in the mesh should be approximately in equilibrium with respect to the element-tractions computed from the weak-SAS field and the body-forces, i.e.,

$$\int_{\partial\omega} n^T \cdot \sigma_h^o ds + \int_{\omega} f d\Omega \approx 0 \quad (5.6)$$

for the body-force f .

- 2- The nodal forces computed from the internal stresses and external forces on every node are nearly in equilibrium, i.e.,

$$\int_{\partial\omega} (n^T \cdot \sigma_h^o) \cdot a_i ds \approx \int_{\omega} \sigma_h^o B a_i d\Omega - \int_{\omega} f \cdot a_i d\Omega \quad (5.7)$$

- 3- The stress-fields computed by using this method do not satisfy the equations of equilibrium* in the interior of elements, i.e.,

$$\operatorname{div} \sigma_h^\circ + f \neq 0 \quad (5.8)$$

However, the equations of equilibrium is satisfied in a weak sense within a finite dimensional subspace.

- 4- The associated stress-vectors (element-tractions) computed from the determined stress fields are single valued across the boundaries of elements because the weak-SAS field is continuous in the interior of the domain. But the weak-SAS field does not satisfy the element tractions $T = n^T \cdot \sigma_h$, i.e.,

$$n^T \cdot \sigma_h^\circ \neq T \quad (5.9)$$

- 5- On the external boundaries $\partial_2 \Omega$, the stress-vectors computed from this stress-field conform to the prescribed boundary tractions:

$$n^T \cdot \sigma_h^\circ = \bar{T} \quad (5.10)$$

In comparison with the quasi-SAS field, this stress-field should be a closer approximation to the SAS field. The above features are similar to those of the quasi-SAS fields in some ways; but the weak-SAS field should be more accurate. Because the unbalanced forces on the nodes diminish by iteration, the equations of equilibrium may be satisfied in a weak sense within the finite dimensional subspace. Moreover, the numerical results obtained by Loubignac *et al* indicate that the weak-SAS field is a nearly exact stress-field at some points within the domain. In addition, the summation of the element-tractions and the body-forces over each element should be approximately zero. Because these features are not precisely true for the quasi-SAS field, one can conclude that this stress-field is less accurate than the weak-SAS field.

5.3 Error Estimator and Error Indicator

The error-indicator which indicates the error within the mesh, may be defined by the expression $\|\sigma_h^\circ - \sigma_h\|_\omega^2$ over the elements in the mesh.

* This equation may be satisfied at a finite number of points.

The local error-measure may be defined by the element contribution to the error-measure $\|\sigma_h^\circ - \sigma_h\|^2$ normalized by the total strain energy of the deformed body. Based on this definition, the local error-measure is expressed by:

$$\|e\|_\omega^2 = \int_\omega \left[\frac{(\sigma_h^\circ - \sigma_h)^T C (\sigma_h^\circ - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (5.11)$$

with

$$\|\sigma_h\|^2 = \int_\Omega \sigma_h^T C \sigma_h d\Omega$$

Expression (5.11) describes a relative value of the error-indicator. This local error-measure can indicate the error within the mesh as well.

The error-estimator is defined by the relative value of $\|\sigma_h^\circ - \sigma_h\|^2$ with respect to the total strain energy of the loaded body. In other words, the summation of the local error-measures in the mesh defines the error-estimator. In mathematical form, the error-estimator $\|e\|^2$ in terms of the local error-measure $\|e\|_\omega^2$ is described by:

$$\|e\|^2 = \sum_{\omega \in \mathcal{T}_h} \|e\|_\omega^2 \quad (5.12)$$

5.4 Implementation

This method of stress and error analysis is not implemented in a computer program. The cost of implementation seems to be reasonable, but the computational cost is, apparently, high. Loubignac *et al* have implemented this iterative method, but they applied a continuous stress-field in the interior without correcting the stress-field on the boundaries. The cost of including the traction boundary conditions seems to be relatively high. Because of the costs, the implementation of this error estimator is not recommended.

5.5 Conclusions from Theoretical Analysis

The weak-SAS field σ_h° , which is computed from the quasi-SAS field by using the iterative technique, yields an error estimate which should be a reliable error measure. This error estimate, strictly speaking, is not an error bound, but since the weak-SAS field satisfies the equation of equilibrium in the interior of domain in a weak sense and its tractions are single valued functions across the interelement and global boundaries, the

computed error estimates should approximate the exact errors reasonably well. However, the disadvantage of this method is that the computational cost is remarkably high. Among the continuous stress-fields presented in this context, this stress-field is expected to be most accurate.

In summary:

A method for estimating error in the finite-element solution has been presented in this part. In this method the error in the KAS field is measured against the weak-SAS field. This stress-field and the associated error-measures are determined according to the followings:

- 1- Compute the continuous stress-field in the same manner as in the construction of quasi-SAS field.
- 2- Iterate the continuous stress-field until the preassigned accuracy is obtained.
- 3- Assign the prescribed tractions on the external boundaries. This conforms the stress-field with the traction boundary conditions.
- 4- Compute the local error-measure from expression (5.11) and the error-estimator from expression (5.12) using the weak-SAS field.

6. Analysis of the Error-Estimates Presented by Zienkiewicz and Zhu

6.1 Introduction

An alternative method to estimate the error in the finite-element solution of problem (2.1) and (2.2) was proposed by Zienkiewicz and Zhu [10]. In this method the error in the finite-element solution is measured versus a continuous stress-field within the domain.

The stress-field computed by using the displacement formulation can possess discontinuous element-tractions; moreover, the KAS field does not satisfy the equations of equilibrium in the interior of elements. Some error-measures in terms of the interelement jump $\mathbf{n}^T \cdot \boldsymbol{\sigma}_h$ have been used in many applications to estimate the error in the finite-element solution. Szabo, [31], has shown by numerical examples that the error in the finite-element solution is mainly due to the discontinuity of the tractions across the boundaries of elements.

The method of error-estimation suggested by Zienkiewicz and Zhu makes use of the fact that a continuous stress-field, obtained by an averaging process or by similar means, should capture the error due to the discontinuity of tractions across the boundaries of elements. However, the averaging technique introduces approximations in the stress-fields; therefore the stresses are not totally reliable. In the following section, the continuous stress-field is discussed, and a procedure for constructing this stress-field is described. Also, a comparison with the SAS field is made.

6.2 The Continuous Stress-Field

This stress-field is the simplest one from analytical points of view. The method of constructing this stress-field is as follows:

- 1- Compute the Gauss point values of stresses by using the displacement finite-element method.
- 2- Compute the nodal values of stresses by averaging the values from the nearest Gauss points. Other projection methods can be used, but in this work the averaging technique

is used as:

$$\bar{\sigma}_j = \frac{1}{m} \sum_{\omega_j=1}^m \{\sigma_j^{\omega_j}\} \quad (6.1)$$

In this expression, m is the number of Gauss points contributing to the node j ; and $\bar{\sigma}_j$ is the vector of stresses at the node j .

3- Construct the continuous stress-field σ_h^e by using the nodal values of stresses. In this process, the same basis functions used for interpolating the displacements are employed as:

$$\sigma_h^e = [N][\bar{\sigma}_j] \quad (6.2)$$

The features associated with the continuous stress-field may be written as follows:

1- The elements in the mesh are *not* in equilibrium with respect to the element-tractions computed from the nodal stresses and the body-forces, i.e.,

$$\int_{\partial\omega} n^T \cdot \sigma_h^e ds + \int_{\omega} f d\Omega \neq 0 \quad (6.3)$$

for the body-force f .

2- The nodal forces computed from the internal stresses and external forces on the nodes are *not* in equilibrium, i.e.,

$$\int_{\partial\omega} (n^T \cdot \sigma_h^e) \cdot a_i ds \neq \int_{\omega} \sigma_h^e B a_i d\Omega - \int_{\omega} f \cdot a_i d\Omega \quad (6.4)$$

3- The stress-fields computed by using this method do *not* satisfy the equations of equilibrium* in the interior of elements, i.e.,

$$\text{div } \sigma_h^e + f \neq 0 \quad (6.5)$$

4- On the boundaries of elements, the stress-vectors differ from the stress-vector computed from the statically admissible stress-field $T = n^T \cdot \hat{\sigma}_h$, i.e.,

$$n^T \cdot \sigma_h^e \neq T \quad (6.6)$$

5- On the external boundaries, the stress-vectors computed from these stress-fields do *not* conform to the prescribed traction boundary conditions, i.e.,

* This equation may be satisfied at a finite number of points.

$$n^T \cdot \sigma_h^e \neq \bar{T} \quad (6.7)$$

The stress-field having the above features does not belong to any function space, defined in this work. It is obvious that all features corresponding to the SAS field is violated by this stress-field. The only common feature with the SAS field is that the element tractions are single value functions. Since the continuous stress-field does not satisfy the prescribed traction boundary conditions, this stress-field does not belong to the space of quasi-SAS field. Also, it may not belong to the space of kinematically admissible stress-fields.

6.3 Error Estimator and Error Indicator

The error-indicator may be defined by the expression $\|\sigma_h^e - \sigma_h\|_\omega^2$ over the elements in the mesh.

The local error-measure may be defined by the element contribution to the error-measure $\|\sigma_h^e - \sigma_h\|^2$ normalized by the total strain energy of the deformed body. Based on this definition, the local error-measure is expressed by:

$$\|e\|_\omega^2 = \int_\omega \left[\frac{(\sigma_h^e - \sigma_h)^T C (\sigma_h^e - \sigma_h)}{\|\sigma_h\|^2} \right] d\Omega \quad (6.7)$$

with

$$\|\sigma_h\|^2 = \int_\Omega \sigma_h^T C \sigma_h d\Omega$$

Expression (6.7) describes a relative value of the error used to define the error-indicator. This value represents the local error-measure; also, it can be applied as error-indicator within the mesh as well.

The error-estimator is defined by the relative value of $\|\sigma_h^e - \sigma_h\|^2$ with respect to the total strain energy of the loaded body. In other words, the summation of the local error-measures in the mesh, is used to define the error-estimator. In mathematical term, the error-estimator $\|e\|^2$ with respect to the local error-measure $\|e\|_\omega^2$ is expressed by:

$$\|e\|^2 = \sum_{\omega \in \mathcal{T}_h} \|e\|_\omega^2 \quad (6.8)$$

The configuration of the error-measures (6.7) and (6.8) are similar to the other ones presented in the previous analyses.

6.4 Numerical Results

In order to obtain practical experience on this method of error-analysis, the method has been implemented in a computer program. A number of problems for which the exact solutions are known, were solved using the finite-element method. In addition, the error in the solutions were estimated by utilizing this method of error-analysis. Next, the exact errors associated with the solutions were determined employing the corresponding exact solutions. Comparison of the estimated errors using Zienkiewicz and Zhu's method with the exact errors indicates that the estimated errors are underestimated. The results are presented in table (6.1). The error-estimates obtained by Zienkiewicz and Zhu' method, the exact solution, and the estimated error-bound are shown in columns 4, 5 and 6, respectively.

Table (6.1): Comparison of the estimated error-bound with the method of Zienkiewicz and Zhu

No.	Description	Mesh	$\frac{\ \sigma_A - \sigma_A^c\ ^2}{\ \sigma_A\ ^2}$	$\frac{\ \sigma_A - \sigma_{exact}\ ^2}{\ \sigma_A\ ^2}$	$\frac{2\ \sigma_A - \sigma_A\ ^2}{\ \sigma_A\ ^2}$
1	Plane stress for rectangular plate loaded by parabolic edge tension	16	0.015001612	0.03842994 *	0.03449688
2	Plane stress for rectangular plate loaded by parabolic edge tension	64	0.0058199156	0.00948447 *	0.01354691
3	Plane stress for rectangular plate loaded by in-plane bending	16	0.030313002	0.03998911	0.07222294
4	Plane stress for rectangular plate loaded by in-plane bending	64	0.0074110242	0.00820643	0.02088954
5	Plane stress in rotating disk	128	0.0041564632	0.00605266	0.01282850

As this table shows, the method of estimated error-bound produces the error-measures which are larger than the corresponding exact errors. This characteristic implies that the method of error analysis by Zienkiewicz and Zhu generates less accurate error-measures; therefore they are less reliable.

The method of error analysis presented by Zienkiewicz and Zhu is the least expensive method in practice. The definition of the continuous stress-field is very simple. The cost of implementation is very low. Also, a least computational cost is required in comparison with the other methods presented in this work. However, the accuracy and the reliability of this method is substantially less than the alternative approaches presented in this context.

In summary:

a simple method to estimate the error in the finite-element solutions of elliptic boundary value problems is presented in this section. The cost of implementation and computation associated with this method is minimum within the presented methods. This method generates the least accurate stress-field of all stress-fields presented in this research. Also,

* Polynomial solutions developed by the author.

the error-measure computed by using this stress-field is least accurate; therefore it is least reliable. The implementation of this method consists of the following steps:

- 1- Compute the Gauss point stresses using the finite-element method.
- 2- Assign the averaged stresses from the nearest Gauss points to the nodes.
- 3- Construct the continuous stress-field by interpolating the nodal stresses to the interior of elements, using the usual shape functions.
- 4- Compute the local error-measure and the error-estimator using expressions (6.7) and (6.8), respectively.

7. Error Estimates Using Residual Methods

7.1 Introduction

The methods of error analysis presented in the preceding sections are based on the comparison between the finite-element solution (kinematically admissible fields) with the SAS fields, mean-SAS field, weak-SAS field, quasi-SAS field, and the continuous stress-field. These stress-fields which are smooth are expected to be closer approximations to the true stress-field compared to the KAS field. The Duality Method in error analysis employs a statically admissible stress-field to compute a bound on the error in the finite-element solution. Various approximations to the statically admissible stress-field were introduced from which the error-measures are computed. The calculated error-measures have been compared with each other for reliability, cost, and effectiveness. The error-measures computed using the Duality Method are found most reliable. But the cost of implementation is substantially high. On the other hand, the error-measures in the Post Processing Method presented by Zienkiewicz and Zhu [10] have been shown to have the lowest computational and implementation cost; the accuracy and reliability of this error-measure is lowest among the foregoing approaches.

In the Residual Method no other stress-field is compared to the KAS field. The error in the finite-element solution is measured with respect to the residual in the equation of equilibrium and the discontinuity of the stress-vectors (element tractions) across the boundaries of elements. This discontinuity includes the jump in the stress-vector and the prescribed traction boundary conditions across the external boundaries $\partial_2\Omega$.

7.2 Residual Method

The Residual Method is one of the oldest methods in error analysis. This method is attributed to Zienkiewicz *et al* in collaboration with Babuska, [3][4][5][6][7]. In this method the finite-element solution is evaluated based on the error in the equation of equilibrium, the discontinuity of the stress-vector across the boundaries of element, and the discontinuity of the stress-vector and the traction boundary conditions across the external boundaries.

Three sources of error are inherent to the kinematically admissible stress-field σ_h which is computed by the displacement finite-element method:

1- The kinematically admissible stress-field σ_h does not satisfy the equation of equilibrium* in the interior of elements, i.e.,

$$\operatorname{div} \sigma_h + f = r \neq 0 \quad (7.1)$$

in which r denotes the residual in the equation of equilibrium due to the substitution of the KAS field in this equation.

2- On the boundaries of elements the stress-vector $n^T \cdot \sigma_h$ (element-traction) can be multiple valued functions, i.e.,

$$n^T \cdot \sigma_h^+ - n^T \cdot \sigma_h^- \neq 0 \quad (7.2)$$

in which σ_h^+, σ_h^- are the stress tensors in the elements adjacent to the common boundaries of elements. Expression (7.2) defines a discontinuity (jump) in the stress-vector across the boundaries of elements.

3- On the external boundaries $\partial_2 \Omega$ the stress-vector $n^T \cdot \sigma_h$ is not compatible with the prescribed traction $\bar{T}$, i.e.,

$$n^T \cdot \sigma_h - \bar{T} \neq 0 \quad (7.3)$$

which describes a jump in the stress-vector with respect to the prescribed traction boundary conditions across the boundaries $\partial_2 \Omega$.

The above three characteristics are the properties of the stress-field obtained by the displacement finite-element method. In the Residual Method, the error in the equation of equilibrium and the jumps in the stress-vectors across the boundaries of elements are used to estimate the error in the finite-element solution. Numerical tests have been performed to estimate the relative magnitude of these errors. Szabo *et al*, [31], have shown empirically that a major portion of the error in the finite-element solution is due to the discontinuity of the stress-vectors across the boundaries of elements. The author verified this result, also. This observation can support the application of the continuous stress-field in the interior of a domain constructed from the KAS fields to estimate the error in the finite-element

* This equation may be satisfied at a finite number of points.

solution. The error estimates computed by using the continuous stress-field should possess the same accuracy as the error-measures defined by the jump of stress-vectors across the boundaries of elements. Based on this argument, it can be claimed that the error estimates presented by Zienkiewicz and Zhu [10] and those computed from the jump of stress-vectors on the boundaries of elements produce similar accuracy.

The contributions from the foregoing sources of errors are summed up to represent the error in the finite-element solution as:

$$\text{Error} = \sum_{\omega} \int_{\omega} (\text{div } \sigma_h + f) d\Omega + \sum_{b_{int}} \int_{\partial\omega} (n^T \cdot \sigma_h^+ - n^T \cdot \sigma_h^-) ds + \sum_{b_{ext}} \int_{\partial_2\Omega} (n^T \cdot \sigma_h - \bar{T}) ds \quad (7.4)$$

in which b_{int} and b_{ext} are the internal and external boundaries, respectively. Zienkiewicz *et al* [6] have computed an error-measure using the residual in the equation of equilibrium. They have applied it to a problem in elasticity in one-dimensional space. In this case, they have ignored the integrals of the jumps of the stress-vectors over the internal and external boundaries; the remaining term is the residual in the equation of equilibrium.

Let u_{n_k} denote the finite-element solution associated with n number of basis functions and u_{n+1_k} is the solution corresponding to $n + 1$ basis functions; then the error in the finite-element solution u_{n_k} is defined by:

$$\begin{aligned} e &\approx u_{n+1_k} - u_{n_k} \\ &\approx u_{n_k} + N_{n+1}u_{n+1} - u_{n_k} \end{aligned} \quad (7.5)$$

in which N_{n+1} is the $(n + 1)th$ basis function and u_{n+1} is the degree of freedom associated with the basis function N_{n+1} . The energy norm of this error is defined by:

$$\|e\|^2 = u_{n+1} \int_{\Omega} r N_{n+1} d\Omega \quad (7.6)$$

in which r is the residual in the equation of equilibrium. In this calculation, it is advantageous to apply the hierarchical basis functions. By employing these functions the unknown coefficient u_{n+1} is determined explicitly without using the previously calculated degrees of freedom. Zienkiewicz *et al*, [3][4][5][6][7], have defined several error-estimates by using expression (7.6). The error-measures in the Residual Method defined by Zienkiewicz *et al* depend on the size of element, order of element, and the coefficient of the leading

term in the differential equation. The error-measure due to the residual in the equation of equilibrium for one dimensional problem is described by:

$$\|e\|^2 = \sum_{\omega} C_{\omega} \int_{\omega} r^2 d\Omega \quad (7.7)$$

in which C_{ω} is the coefficient in terms of the foregoing parameters:

$$C_{\omega} = \frac{h_{\omega}^2}{12a_{min}p^2}$$

In this expression, h_{ω} is a measure describing the size of element, p denotes the order of element, and a_{min} is the minimum value of a (coefficient of the leading term in the differential equation) over the element ω .

In the case of two-dimensional and three-dimensional problems, beside the error in the equation of equilibrium, the error-measures must include the contributions due to the jumps in the stress-vectors across the boundaries of elements. Zienkiewicz *et al* [6] have derived error-measures for two-dimensional problems by applying the error-estimates derived for the problems in one-dimensional space. Zienkiewicz *et al* [6] proposed an error-measure for the bilinear element as:

$$\epsilon^2 = \frac{h^2}{24a} \int_{\omega} r^2 d\Omega + \frac{h}{24a} \int_{b_{int}} J^2 ds \quad (7.8)$$

in which the coefficients are functions of element size, element order, and the coefficient of the leading term in the differential equation. Γ is the set containing all internal and external boundaries, i.e., $\Gamma = \{b_{int} \cup b_{ext}\}$. The details may be found in reference [6]. In the following section some of the error-indicator and error-estimators defined in the Residual Method are presented.

7.3 Correction Indicator

In many cases the error-measure (7.6) is not necessarily an efficient error-estimate. Although this error-measure behaves asymptotically, there are inefficiencies when moving from a particular mesh to a refined one. The principal problem is that the error, although it may be large, can also be orthogonal or almost orthogonal to the test functions added to the

refined mesh [3]. This is a serious shortcoming of these error-estimates. In mathematical term:

$$\int r N_{p+1} d\Omega = 0 \quad (7.9)$$

might be true. In equation (7.9) r is assumed to be the error in the equation of equilibrium and the error due to the discontinuity of stress-vectors across the boundaries of elements. To overcome this problem, the correction indicator is introduced. The idea here is to measure the decrease in error of the solution obtained by adding any particular new degree of freedom [3].

If we assume that we have a solution to the problem in terms of n parameters, and introduce another parameter hierarchically, then the error corrected by this refinement can be approximated by:

$$e_{n+1} = u_{n+1h} - u_{nh}$$

It is known that $e_{n+\infty}$ would represent the actual error in the solution. If we expand u_{n+1} and u_n in terms of their shape functions, we have

$$\begin{aligned} u_{nh} &= \sum_{i=1}^n u_i^* N_i \\ u_{n+1h} &= \sum_{i=1}^{n+1} u_i N_i \end{aligned} \quad (7.10)$$

For the hierarchical basis functions: $u_i^* = u_i \quad i = 1, \dots, n$; then

$$e_{n+1} = N_{n+1} u_{n+1} \quad (7.11)$$

so the stiffness matrix and load vector reduce to:

$$\begin{bmatrix} K_n & K_{n,n+1} \\ K_{n+1,n} & K_{n+1,n+1} \end{bmatrix} \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} \quad (7.12)$$

Knowing u_n from the previous coarse mesh solution, i.e., u_n^* , we are able to compute u_{n+1} as follows:

$$u_{n+1} = \frac{1}{K_{n+1,n+1}} (f_{n+1} - K_{n+1,n} u_n^*) \quad (7.13)$$

Therefore the error e_{n+1} becomes:

$$e_{n+1} = \frac{(f_{n+1} - K_{n+1,n} u_n^*)}{K_{n+1,n+1}} N_{n+1} \quad (7.14)$$

We now have to evaluate the norm of the function e_{n+1} to measure the decrease in error in that norm. The energy norm is appropriate. The error-indicator is measured with this norm as follows:

$$\|e_{n+1}\|^2 = \left\| \frac{(f_{n+1} - K_{n+1,n} u_n^*)}{K_{n+1,n+1}} N_{n+1} \right\|^2$$

which expresses approximately the value of the energy which will be cancelled by the introduction of the new degree of freedom $(n + 1)$ in the solution. Alternatively this expression can be written as:

$$\|e_{n+1}\|^2 = \frac{(f_{n+1} - K_{n+1,n} u_n^*)^2}{K_{n+1,n+1}^2} \|N_{n+1}\|^2 \quad (7.15)$$

The energy norm of the function N_{n+1} is computed by:

$$\|N_{n+1}\|^2 = B(N_{n+1}, N_{n+1}) = K_{n+1,n+1} \quad (7.16)$$

in which the function $B(\cdot, \cdot)$ is the bilinear form defined by:

$$B(u, v) = \int_{\Omega} [Bu]^T C^{-1} Bv \, d\Omega$$

in which C^{-1} is the tensor of material property, and B , in the right hand side of the bilinear form, is the gradient operator which maps the displacement vector to the strain vector. Applying (7.16) in (7.15), it is found that:

$$\|e_{n+1}\|^2 = \frac{(f_{n+1} - K_{n+1,n} u_n^*)^2}{K_{n+1,n+1}} \quad (7.17)$$

Renaming the above quantity $\|e_{n+1}\|^2$ and manipulating its right hand side, yields:

$$\eta_{n+1}^2 = \frac{[\int_{\Omega} f \cdot N_{n+1} \, d\Omega - B(N_{n+1}, u_{n_h})]^2}{B(N_{n+1}, N_{n+1})} \quad (7.18)$$

Zienkiewicz *et al*, [5], noted that expression (7.18) can be written as:

$$\eta_{n+1}^2 = \frac{[\int_{\Omega} r N_{n+1} \, d\Omega]^2}{K_{n+1,n+1}} \quad (7.19)$$

This form describes the energy due to the introduction of the next degree of freedom in the solution.

7.4 Error-Estimator

In principal, correction indicators evaluated for all possible new degrees of freedom, simultaneously, and summed should provide an accurate measure of the total error in the energy norm occurring in the solution at the refinement stage reached. Unfortunately, this is impracticable as only the next (or next few) levels may be studied without increasing the cost out of proportion [3]. Zienkiewicz *et al* [3] have presented three different types of estimators, two of which are based on direct extensions of the indicators (7.19). If expression (7.19) is considered, the residual may be orthogonal or nearly orthogonal to the basis function in question and that the bulk of error may lie in higher order terms. Therefore if the correction indicators are added together, an underestimated error is obtained. However, by applying the Cauchy-Schwarz inequality to the appropriate term in that expression, it is found that:

$$\left(\int_{\Omega} N_{n+1} r \, d\Omega \right)^2 \leq \int_{\Omega} N_{n+1}^2 \, d\Omega \int_{\Omega} r^2 \, d\Omega \quad (7.20)$$

and an error-estimate, in the energy norm, associated with the $(n+1)$ th degree of freedom is suggested by the following expression:

$$\epsilon_{n+1}^2 = \frac{\int_{\Omega} N_{n+1}^2 \, d\Omega \int_{\Omega} r^2 \, d\Omega}{K_{n+1,n+1}} \quad (7.21)$$

The residual term in this expression may consist of two separate parts: the error in the equation of equilibrium in the interior of elements and the error due to the discontinuity of the stress-vectors across the boundaries of elements. The discontinuity of the stress-vectors constitutes a major part of the error-estimate [31]. Regarding this consideration, error-estimator (7.21) is presented by, [3],:

$$\epsilon = \left(\frac{\sum \int_{\Omega} N_{n+1}^2 \, d\Omega \int_{\Omega} r^2 \, d\Omega + \sum \int_{\Gamma} N_{n+1}^2 \, ds \int_{\Gamma} J^2 \, ds}{K_{n+1,n+1}} \right)^{\frac{1}{2}} \quad (7.22)$$

for the residual r in the equation of equilibrium and the discontinuity J in the stress-vector across the boundaries of elements. Other expressions to define error-estimates are presented by Zienkiewicz *et al* [3].

7.5 Effectiveness of the Error-Measures

The Residual Method is an inefficient method in error analysis. The error-measures derived for a one-dimensional problem described by a simple second order ordinary differential equation are modified to apply in two-dimensional problems. The application of the error-indicator is costly because the error indicator must be evaluated for all degrees of freedom corresponding to the next refinement stage to indicate which of these should be introduced [3]. The coefficients in the error-measures are not tenable as they are determined from some simplified problems; therefore the error-measures are not reliable. Because there is no notion of SAS field, there is no concept of error-bound involved in this error analysis scheme. In general, the Residual Method is the least effective technique among the methods of error analysis presented in this work.

In summary:

The Residual Method in error analysis is described in this section. In order to implement the error-measure, the following processes must be accomplished:

- 1- Compute the KAS field at the Gauss points within the elements.
- 2- Compute the residual in the equation of equilibrium using the continuous stress-field.
- 3- Compute the jumps on the boundaries of elements.
- 4- Determine the basis function associated with the next degree of freedom, i.e., N_{n+1} . The next degree of freedom can be introduced in the model by *h-refinement* or *p-refinement* technique.
- 5- Compute the error-indicator (7.22) by using the obtained information.

8. Conclusions and Further Research

This research has developed several techniques to compute errors in finite-element solutions of elliptic boundary-value problems. Some of these methods make use of the statically admissible stress (SAS) and the kinematically admissible stress (KAS) fields to compute error-bounds. Those methods employing kinematically and statically admissible stress fields are mathematically rigorous and their error-measures are absolutely reliable. The rest of the methods of error analysis presented in this work are more reliable in comparison with most of the existing ones. A comparison of the results obtained in this work with the results calculated from the method employed by Zienkiewicz and Zhu supports this claim.

8.1 Proposed Methods

Method A: Error-Bound computed from the KAS and the SAS fields for body-forces of high-order polynomials defined over rectangular linear quadrilateral and hexahedron elements.

The construction of the SAS field from the KAS field includes an involved procedure which consists of the following calculations.

- 1- Compute the *nodal forces due to tractions* over the element-faces for each element such that their associated element-tractions are in equilibrium with respect to the body-force and traction boundary conditions or force-boundary conditions. (The computation of the *nodal forces due to tractions* over the element-faces is done in a manner that ensures the continuity (equilibrium) of the element-tractions across the element-boundaries and global boundaries.)
- 2- Determine the nodal values of stresses by using the computed *nodal forces due to tractions* and the expression relating those nodal forces to the stress-field.
- 3- Generate a continuous stress-field by the usual interpolation technique from the nodes to the interior of the element. In addition, superimpose a sufficient number of stress variables with unknown coefficients on the continuous stress-field in such a way that *divergence* of the stress-field matches the body-force pointwise. The stress variables are defined in

such a way that they vanish on the boundaries of the element (bubble functions). The unknown coefficients associated with these stress variables are determined in such a way that the equation of equilibrium are satisfied between the stress-field and the body-force in the interior of elements. The stress-fields computed, in this manner, obey the static admissibility conditions which have been derived earlier. These stress-fields, therefore, are statically admissible. Their intrinsic features may be described as follows:

- 1- Every element in the mesh is in equilibrium with respect to the tractions and the body-forces.
- 2- The nodal forces computed from the internal stresses and external forces on every node are in equilibrium.
- 3- The stress-fields computed by using this method satisfy the equation of equilibrium within the interior of elements.
- 4- The associated element-tractions (stress-vector) computed from the determined stress-fields are single valued across the boundaries of elements. However, the stress-field may be discontinuous in the interior of the domain.
- 5- On the external boundaries, the stress-vectors computed from these stress-fields conform to the prescribed traction boundary conditions.

For body-forces of linear and higher order polynomials, this method is the only one which computes the SAS field and error-bound. If the body-force is expressed by a finite order polynomial over the interior of the elements, the stress-field in terms of bubble functions is computed in such a way that those stresses match with the body-force exactly. In addition, since the remaining conditions are true, the SAS field and error-bound are computed precisely.

This method of error and stress analysis is expected to be inefficient in applications. Although it computes the error-bound, the theoretical analysis for stress calculations is complicated. Moreover, programming the procedure is costly. Also, the computational cost of stress calculation associated with this analysis is expected to be high. This method of stress and error analysis has not been implemented yet.

Method B: Error-Bound computed from the kinematically and the statically

admissible stress-fields for piecewise constant body-force defined over linear triangular and tetrahedron elements.

Like the previous method, the SAS field is computed over these elements. In this calculation, first the *nodal forces due to tractions* on the element faces are computed in the same fashion as described in part A. Then express the stress-fields over the elements by complete polynomials of order one with unknown coefficients. In order to determine the unknown coefficients in the stress-fields, equate the computed *nodal forces due to tractions* with those calculated from the internal stresses in terms of unknown coefficients on each node. This condition provides a set of equation with the unknown stress-coefficients. In addition, the stress-fields are required to satisfy the equation of equilibrium in the interior of elements. This creates another set of equations in terms of the same unknowns. These two sets of equations determine the unknown coefficients from which the stress-fields are defined. The features of the statically admissible stress-fields described in the previous part A are maintained within the stress-fields computed in this method B.

The above method of stress calculation differs from the method proposed by Ladeveze *et al*, [20], in such a way that in our method the stress-field is computed on the elements directly whereas in the method employed by Ladeveze *et al*, first elements are subdivided into a number of linear subelements, then a small finite-element problem is solved on these subelements. Because of subdividing the elements into subelements in the former method, the technique of computing the stress-field suggested in this work could be more efficient in practice.

The methods employed by Ladeveze *et al* and by this work, both produce SAS fields for piecewise constant body-forces. The linear stress distributions in the interior of elements satisfy the equation of equilibrium for a constant body-force. Also, the traction vectors are continuous across the element boundaries. Moreover, the tractions conform to the prescribed traction boundary conditions. Thus for problems in which the stress-fields and the body-force distributions are linear and constant respectively, the statically admissible stress-fields are determined by using this method.

For body-forces of higher order polynomials, this method neither generates the SAS field nor the error-bound. Although the tractions across the element and global boundaries

maintain the necessary conditions for static admissibility, the equation of equilibrium in the interior of elements are not satisfied. Therefore the computed stress-field is not statically admissible and the error-measure computed from this stress field is not bounded.

This method of error analysis is effective in practice for problems with constant body-force. Since in these cases the SAS fields are described by low order polynomials, the theoretical analysis of the stresses would not be involved. Also, the implementation of this method in a computer program is expected to require a medium cost. Moreover, the computational cost of this method should be reasonable. Ladeveze *et al* have implemented this method of error analysis in a software program and tested it. In this procedure they have employed their own method of stress calculations on linear triangular and tetrahedron elements.

Method C: Approximate Error-Bound computed from the KAS field and the mean-SAS fields.

In order to construct the mean-SAS fields, first the *nodal forces due to tractions* over the element-boundaries are computed in the same manner as in part A. Then calculate the nodal values of stresses from the *nodal forces due to tractions* which are computed over the element-boundaries. Next, construct the continuous stress-fields by using the nodal stresses and the same interpolation functions as described in part A.

The stress-field computed in this method, differs from the one constructed in part A in some details. The stress-field in the present method does not contain the additional stress variables in terms of bubble functions; it is simply made up with the continuous stress-field interpolated from the nodal values of stresses. All of the features described in the method A except the equation of equilibrium are satisfied within the stress-field computed by using this method. Here the equation of equilibrium satisfy the stress-fields and body-force in an integral sense.

This method computes the mean-SAS field and an approximate error-bound which are nearly a SAS field and an error-bound respectively. Because the equation of equilibrium are not satisfied within the elements, the exact static admissibility and error-bound are not obtained. However, since the element-boundary-tractions are in equilibrium with the

body-forces, the computed stress-field should approximate the SAS field closely and the error estimate computed by using this stress-field should approximate the error-bound therefore reasonably well.

The author considers this method of stress and error analysis as the most effective balance between cost and accuracy. The programming and computations related to the *nodal forces due to tractions* and nodal stresses are expected to be performed with a reasonable cost. The stress-field is simply interpolated from the nodes to the interior of elements by using the standard shape functions. This characteristic indicates that the implementation of the method in software will be relatively simple. Also, the computation of the stress-field from the nodal values needs a minor effort. These characteristics signify the efficiency and usefulness of the method. The implementation of this method of stress and error calculation in a computer program is in progress. Presently, a class of problems with constant stress distributions is tested using this method.

Method D: Estimated Error-Bounds applying the KAS and the quasi-SAS fields.

The quasi-SAS field is a continuous stress-field which conforms to the traction boundary conditions. This stress-field is constructed as described below. In the first stage, the Gauss point stresses are calculated by using the displacement finite-element method. Then, the nodal values of stresses are computed by averaging the data of the nearest Gauss points to the nodes and assigned the averaged values to the nodes. To generate a continuous stress-field over the domain, the nodal values of stresses are interpolated within the interior of elements using the same shape functions applied in the calculations of displacements. In the second stage, the prescribed traction boundary conditions are assigned to the continuous stress-field over the external boundaries. The computed stress-field, in this manner, is called quasi-SAS field. The features corresponding to the quasi-SAS field may be described as follows:

- 1- The elements in the mesh are *not* in equilibrium with respect to the element-tractions computed from the nodal stresses and the body-forces.
- 2- The nodal forces computed from the internal stresses and external forces, on every node,

are *not* in equilibrium.

3- The stress-fields computed by using this method do *not* satisfy the equation of equilibrium* in the interior of elements.

4- Because the stress-field is continuous in the interior of the domain, the associated element-tractions (stress-vectors) computed from the determined stress-fields are single valued across the element-boundaries.

5- On the external boundaries, the stress-vectors computed from these stress-fields conform to the prescribed traction boundary conditions.

Comparing to the SAS field, the quasi-SAS field is inaccurate stress-field. Because the equation of equilibrium are not satisfied within the interior of elements, the quasi-SAS field is in error whose value varies between zero to 50% of the error in the finite-element solution. These errors have been measured versus the exact solutions for a number of test problems. The results are illustrated in the previous sections. Although the continuity of tractions across the element-boundaries and the conformity with the traction boundary conditions are remarkable features of quasi-SAS field, the accuracy of this stress field is reduced because the equation of equilibrium is not satisfied and the element-tractions are not in equilibrium with the body-force.

It has been demonstrated that an estimated error-bound can be computed by using the quasi-SAS field. This estimated error-bound is expressed by the summation of the distance between the KAS field to the quasi-SAS field, measured in the strain energy norm, plus the exact value or an estimated upper-bound for the error in equation of equilibrium in the same norm. The error in the equation of equilibrium can be described by the orthogonal distance between the space of quasi-SAS field to the space of SAS field in the strain energy norm. The sum of the two distances, measured in energy norm, can be used to define an estimated upper bound for the distance between the kinematically admissible stress-field to the statically admissible stress-field; therefore it describes an estimated error-bound for the exact error in the finite-element solution.

To evaluate the estimated error-bound, two separate expressions must be numerically

* The equation of equilibrium may be satisfied in a finite number of points.

computed. The first term, i.e., the distance between the kinematically to the quasi-SAS fields can be explicitly measured by using the norm; however, an estimated upper bound for the second term can not be computed easily because the estimated upper bound depends on the eigenvalues of the system which are difficult to estimate. Thus in this method, we accept an approximate value for the estimated upper bound. In this approximation the finite-element displacement fields are substituted for their exact displacement fields in the expression representing the estimated upper bound. In addition to the error introduced by the above substitutions, the finite-element displacements are replaced in terms of their stress-functions introducing another source of error in the calculation of the estimated upper bound. The summation of the two values associated with the above distances is used to determine the error-estimates.

This method of stress and error analysis which requires relatively low cost in applications has been implemented in a computer program by the author and tested for some problems. The numerical results of those test problems for which the exact solutions are known show that this error analysis scheme provides error-estimates whose values are greater than the exact errors in the solutions. Based on these observations, this method can be used to compute reasonable error-estimates for other problems of the same nature. The method is successfully applied to the problems of solid mechanics using linear quadrilateral element in two dimensional and linear hexahedron element in three dimensional spaces.

Method E: Error Estimate applying the KAS and the weak-SAS fields.

The weak-SAS field is computed from the quasi-SAS field using an iterative method as follows. First the quasi-SAS field is constructed as described in section D. Then the quasi-SAS field is applied in an iterative method (for example, the iterative method presented by Loubignac *et al*) in which this stress-field is iterated until the unbalanced nodal forces (see the features of the quasi-SAS field) from the internal stresses and external forces vanish. Loubignac *et al* have shown that the iterative method converges. However, the iterated quasi-SAS field may not converge to the SAS field. This iterated stress-field which is expected to be more accurate than the quasi-SAS field is called a weak-SAS field. The features associated with this stress-field may be described as follows:

- 1- The elements in the mesh should be approximately in equilibrium with respect to the element-tractions computed from the nodal stresses and the body-forces.
- 2- The nodal forces computed from the internal stresses and external forces on every node are nearly in equilibrium.
- 3- The stress-fields computed by using this method do not satisfy the equation of equilibrium* in the interior of elements.
- 4- The associated stress-vectors (element-tractions) computed from the determined stress-fields are single valued across the boundaries of elements because the mean-SAS field is continuous in the interior of the domain.
- 5- On the external boundaries, the stress-vectors computed from these stress-fields conform to the prescribed traction boundary conditions.

Apparently, these features are similar to those of the quasi-SAS fields in some ways; but the weak-SAS field should be more accurate. Because the unbalanced forces on the nodes diminish by iteration, the equation of equilibrium may be satisfied in a weak sense within the finite dimensional subspace. Moreover, the numerical results obtained by Loubignac *et al* indicate that the weak-SAS field is a nearly exact stress-field at some points within the domain. In addition, the summation of the element-tractions and the body-forces over each element should be approximately zero. Because these features are not precisely true for the quasi-SAS field, one can conclude that this stress-field is less accurate than the weak-SAS field.

The analysis of error by using the weak-SAS field may not be desirable in application. The implementation and computational costs of this method of error analysis seems to be so high which reduce the viability of the method. Moreover, because the associated stress-fields are not SAS fields, the error measures are not error-bounds; they are not therefore totally reliable. This method of error analysis has not been implemented.

Method F: Error Estimates using the KAS and the quasi-SAS fields.

In this method, the difference between the KAS field from the quasi-SAS field measured in energy norm is used to define the error-estimate. The method of constructing the

* The equation of equilibrium may be satisfied in a finite number of points.

quasi-SAS field and its features have been defined in part D. This error estimate differs from that in part D in such a way that the present error estimate does not contain the estimated upper bound for the distance between the spaces of quasi-SAS and SAS fields. Also, this error measure differs from the error estimate presented by Zienkiewicz and Zhu such that the continuous stress-field defined by Zienkiewicz and Zhu for their error estimate calculations does not satisfy the traction boundary conditions while the quasi-SAS field is a continuous stress-field which conforms to the traction boundary conditions. Because the quasi-SAS field conforms to the traction boundary conditions, it is a more accurate stress-field in comparison to the continuous stress-field employed by Zienkiewicz and Zhu; therefore the error estimates are computed more accurately using it.

Since the stress-field applied in this method of error analysis is a crude approximation to the statically admissible stress-field the error estimates calculated by using this stress-field are rough approximations to the error-bounds. Even though this is true in general, some numerical examples are known in which the quasi-SAS field creates an error-bound. The numerical results associated with this error-analysis scheme support this claim. The error estimates computed from the quasi-SAS fields are less reliable in comparison to the error measures mentioned earlier. However, the quasi-SAS field and the associated error estimates are readily computed which is the advantage of this method. This scheme for error analysis has been implemented in a computer program by the author and tested; the error measures are not the most reliable but their low computational cost and accuracy are remarkably low.

8.2 Existing Methods

Method G : Error Estimates presented by Zienkiewicz and Zhu.

These error estimates are calculated by using the KAS field versus the continuous stress-field obtained from the KAS field. The procedure for constructing this continuous stress-field is described below. First the Gauss point stresses, which are computed by the displacement finite-element method, are averaged and assigned to the nearest nodes. Then the continuous stress-field over the elements is generated by interpolating the stress-field from the nodes to the interior points of the elements. This interpolated stress-field is the

continuous stress-field proposed by Zienkiewicz and Zhu for error estimation. The features of this continuous stress-field may be written as follows:

- 1- The elements in the mesh are *not* in equilibrium with respect to the element-tractions computed from the nodal stresses and the body-forces.
- 2- The nodal forces computed from the internal stresses and external forces on the nodes are *not* in equilibrium.
- 3- The stress-fields computed by using this method do *not* satisfy the equation of equilibrium* in the interior of elements.
- 4- Because the stress-field is continuous across the boundaries and in the interior of elements, the associated stress-vectors (element-tractions) computed from the determined stress-fields are single valued across the boundaries of elements.
- 5- On the external boundaries, the stress-vectors computed from these stress-fields do *not* conform to the prescribed traction boundary conditions.

The continuous stress-field has only one characteristic in common with the SAS field; the continuity of the tractions across the boundaries of elements is the common feature. Several deficiencies exist in the characteristics of the continuous stress-field which reduce the quality of this stress field; this stress field seems to be the least accurate among all of the other stress-fields that approximate a SAS field.

This method of error analysis produces least accurate error-measure among all of the above methods. The lack of accuracy in error estimates arises from the low accuracy involved in the continuous stress-field. This method yields an error estimate which is not, obviously, an error-bound. This is the main drawback of this method of error analysis.

The method of error analysis employed by Zienkiewicz and Zhu is the most practical approach in finite-element analysis. The theory of the continuous stress-field is very simple to understand. Moreover, the implementation in a software program needs the least effort in comparison to the other procedures. In addition, the computational cost of this method of stress and error calculation is very low which is the main advantage of this method.

The author has implemented this method of error analysis in a finite-element program

* The equation of equilibrium may be satisfied in a finite number of points.

and tested it by analyzing some problems for which the exact errors can be computed. In comparison to the exact errors, these error measures are underestimated. The results are also compared with the other error-estimates obtained in this work from which the method employed by Zienkiewicz and Zhu is found to be inaccurate. These observations indicate that accuracy has been traded off for economy and simplicity in this method.

Method H : Error Estimates presented by Zienkiewicz *et al* in collaboration with Babuska.

This error analysis scheme deals with the KAS field only. In this class of error estimates, mainly, the residual values are employed to define error measures. The error in the equation of equilibrium in the interior of the elements and/or the normal jumps on the element-boundaries are computed in this class of error estimates. To justify the validity of these error estimates, some additional variables like, the correction indicators, and the effectivity index have been defined. The main disadvantage associated with this class of error estimates is that they do not generate the error-bounds.

8.3 Attributes of the Methods of Error Analysis

There are four separate issues associated with the above methods:

- i- the accuracy of the stress-fields,
- ii- the reliability and accuracy of the error measures calculated by using these methods,
- iii- implementation cost,
- iv- computational cost.

Method A generates the SAS field and error-bound for body-forces of polynomials of order linear and higher over rectangular linear quadrilateral and hexahedron elements. In this method, the implementation cost is expected to be too high and the stress-field is expected to be computed relatively expensively. But this method is expected to be reliable and accurate for stress and error calculations.

Method B creates the SAS fields and error-bounds for piecewise constant body-force over triangular and tetrahedron elements. The implementation and computational costs of this method are expected to be medium. The stress-fields are accurate up to linear order;

therefore the constant body-force is exactly captured. For body-forces of linear and higher order polynomials, this method does not produce the statically admissible stress-field or error-bound.

Method C generates neither the exact statically admissible field nor the exact error-bound. This method is expected to yield a good approximation for stress-fields because the corresponding tractions are in equilibrium with the body forces. In addition, it is expected to be the most efficient approach for the stress and the error calculations. Based on the theoretical analysis, the error measures calculated by this method are reliable and can be considered as good approximations to the true error-bounds.

Methods D and F compute the quasi-SAS field and error estimates. With respect to the SAS field, the computed stress-field is not accurate; therefore the error estimates are not an error-bound. The implementation and computational costs of these methods are fair.

Method E computes the weak-SAS field and error estimates. Obviously, the cost of implementation is higher than the methods D and F. In addition, the computational cost in this method does not sound to be reasonable. While in this method the costs are apparently high, the stress-field and error measures are not reliable in comparison to the SAS fields and the error-bounds.

Method G computes the continuous stress-field and error estimates which have the lowest accuracy. The implementation of this method is straightforward. Moreover, the computational cost is lowest among the other methods. This method of stress and error calculation has been employed in practice because of its simplicity and economy.

Method H deals with the KAS field only; therefore there is no notion of statically admissible stress-field or error-bound in this method. In addition, it is not clear that this method is efficient in various aspects including computational cost. Despite these deficiencies, this method has been applied in many finite-element programs.

The analysis of procedures described in this thesis indicates that the Duality Method is the most reliable method of error analysis; however, many adaptive finite-element programs use the alternative approaches. Depending on the circumstances and considering

the results of error analysis presented in this work, the users can choose the method of error estimation most appropriate for their requirements.

8.4 Further Research

A critical component to the development of a reliable adaptive finite-element method is the definition of an accurate error measure. This work has presented several error measures including the error-bounds using the kinematically and statically admissible stress-fields. In addition to the results and conclusions based on the above theoretical analysis the following developments are in progress:

The implementation of the procedures A and C for calculating the statically admissible stress-fields and error-bound are under consideration. These methods are applied to linear problems of solid mechanics for linear quadrilaterals and hexahedron finite-elements. Upon completion of the implementation, the procedure will be tested for numerical errors.

In addition, the following objectives for the further investigations are recommended:

The notion of the statically admissible fields and the error-bounds should be extended to the other areas of applications in which the governing differential equations are of elliptic type, for instance the steady state response of systems.

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Appendix 1

Definitions of the Stress-Fields

Kinematically Admissible Stress-Field: The stress-field whose displacement-field satisfies the continuity conditions in the interior and on the boundary of domain, where the displacement boundary conditions (DBC) are prescribed.

Statically Admissible Stress-Field: The stress-field which satisfies the static equilibrium conditions in the interior and on the boundaries of domain. In other words, the statically admissible stress-field satisfies the equation of equilibrium in the interior of a domain; moreover the associated element-tractions are single valued function across the boundaries of interior elements. In addition, the stress-vector satisfies the traction boundary conditions (SBC) on the external boundaries of the domain.

Mean-Statically Admissible Stress-Field: A stress-field whose element-tractions are in equilibrium with respect to the body-forces; moreover, the element-tractions are single-valued functions across the boundaries of elements. In addition, the mean-SAS field satisfies the equation of equilibrium in an integral sense in the interior of elements.

Weak-Statically Admissible Stress-Field: A stress-field which is continuous in the interior of a domain. It satisfies the traction boundary conditions on the external boundaries. The associated element-tractions are single valued functions, across the boundaries of elements. The equation of equilibrium is satisfied in a weak sense with respect to the finite dimensional subspace.

Quasi-Statically Admissible Stress-Field: A stress-field which is continuous in the interior of a domain and satisfies the prescribed traction boundary conditions on the external boundaries.

Appendix 2

Definitions of the Error-Measures

Error-Bound: An upper bound on the error between a computed solution and the exact solution is called an error-bound. In this thesis, the difference between a KAS (kinematically admissible stress) field and a SAS (statically admissible stress) field, measured in the strain energy norm, defines the error-bound. This term can be applied when the SAS field is computed for constant body-force over a triangular mesh, and for body-force of higher order polynomial over a mesh containing quadrilateral elements with bubble-functions.

Approximate Error-Bound: The difference between the KAS field and the mean-SAS field measured in the energy norm.

Estimated Error-Bound: Error-measure which may be computed by using a KAS field and a quasi-SAS field plus an exact value or an upper bound for the error in the equation of equilibrium (error due to the lack of static admissibility), measured in the energy norm.

Approximate Estimated Error-Bound: Error-measure which can be computed from a quasi-SAS field plus an approximate value for the error in the equation of equilibrium (approximate value of error due to the lack of static admissibility), measured in the energy norm.

Error-Estimate: Error-measures computed in any of the following schemes:

- a) Error-measures computed by using the KAS field and an iterated quasi-SAS field (Weak-SAS field).
- b) Error-measures computed by using the KAS field and a quasi-SAS field.
- c) Error-measures introduced by Zienkiewicz and Zhu by comparing the KAS field versus their continuous stress-field.

d) Error-measures presented by Babuska and Zienkiewicz using the Residual Method.

Relative Error: The relative values of the above error-measures normalized with respect to the total strain energy of the deformed body.

Error-Indicator: A measure which indicates the error.

Local Error-measure: Error-measure associated with an element in a mesh.

Error-Estimator: Error-measure which designate a total error in a mesh. This is defined as the summation of contributions from all elements in the mesh.

Exact Error: Error-measure computed by using a KAS field versus the associated exact stress-field.

Note: The term 'Error-measure' may be used as a general term in error analysis.

Appendix 3

Error-Bound in One-Dimensional Space

This Appendix analyzes a simple one-dimensional two-element problem in detail. The purpose is to provide an exposition of the method of computing an error-bound.

A method to compute an error-bound for the finite-element solutions of the problems of solid mechanics has been presented in section 2. The determination of this error-bound begins by computing the kinematically admissible stress-field $\sigma_h = C^{-1}Bu_h$ by using the usual displacement finite-element method. Next, a statically admissible stress-field is constructed. Finally, the error-bound is computed from the difference between the kinematically and statically admissible stress-fields.

The process of computing the statically admissible stress-field consists of the following calculations:

- 1- Compute the *nodal forces due to tractions* over the element-faces for each element such that their associated element-tractions are in equilibrium with respect to the body-force and traction boundary conditions; in other words, the element tractions and body forces are in equilibrium, i.e., they satisfy the solvability condition.
- 2- Define a stress-field in the FEM trial function space with unknown coefficients over each element. In this case, the two node elements define a piecewise linear field for σ_{xx} .
- 3- The unknown coefficients in the above stress-field are determined using the *nodal forces due to tractions* computed before, and the expression relating these nodal forces and the stress-field. The stress-field computed in this manner is the mean-statically admissible stress-field.
- 4- A sequence of stress variables, which are bubble functions, with unknown coefficients are superimposed on the mean-statically admissible stress-field in such a way that *divergence* of the stress-field matches the body-force pointwise. Then the unknown coefficients associated with these stress variables (bubbles) are determined in such a way that the equation of equilibrium is satisfied pointwise in the interior of each element. This stress-field is the

statically admissible stress-field. If an approximate solution is adequate, the sequence can be truncated. In this case, the equation of equilibrium may be satisfied in a weak sense to determine those unknown coefficients.

This is an exposition of the aforementioned method in practise. In this demonstration a problem in one-dimensional space is solved by the finite-element method to compute the kinematically admissible stress-field. Then the mean-statically admissible stress-field is constructed using the data of the problem (body force and tractions), and the kinematically admissible stress-field. Moreover, in order to compute the statically admissible stress-field, the bubble functions are superimposed upon the mean-statically admissible stress-field. The mean-statically admissible and the statically admissible stress-fields are plotted and compared with the exact stress-field.

Finally, the error in the stress-field computed by the finite-element method is estimated using the mean-statically admissible stress-field, the statically admissible stress-field, the continuous stress-field defined by Zienkiewicz and Zhu, and the quasi-statically admissible stress-field, separately. The quantities computed in these manners are compared with each other and with the true error in the finite-element solution. The error-estimate of Zienkiewicz and Zhu is shown to be a first approximation to the actual error in the solution.

Test Problem:

Title : Uniaxial tensile test with linear body-force and tractions.

Purpose : To compute the kinematically and statically admissible stress-field and error-bound.

This test case consists of a bar of uniform cross-sectional area of 1.0 cm^2 . A uniform traction, $\bar{T}_{(n)}$, is applied to one end as shown in Figure (A3.1). The body-force is assumed to be a linear function over the domain.

Solution Specifications :

Partial Differential Equations (PDE) Type : Linear elasticity:
$$\begin{cases} \text{div } \sigma + f = 0 \\ \epsilon = \frac{\partial u}{\partial x} \\ \sigma = E\epsilon \end{cases}$$

Material Property (modulus of elasticity) : $E = 1.0 \text{ MPa}$.

Body – Force : A body-force, $f = x \text{ N/cm}^3$, of linear distribution is applied to the domain.

Traction Boundary Conditions : Normal tractions, 100.0 Pa, are applied to one end of the bar, Figure (A3.1).

Displacement Boundary Conditions : Displacement is prescribed at the other end of the bar, as illustrated in Figure (A3.1), to constrain the rigid body modes.

Geometry of Mesh : There are two linear elements in the mesh.

Figure (A3.1)
Geometry of the Uniform Bar.

Finite Element Calculations:

The finite-element equilibrium equation is:

$$[K]\{u\} = \{f\}$$

in which $[K]$ denotes the stiffness matrix, $\{u\}$ is the vector of nodal displacements, and $\{f\}$ characterizes the vector of nodal forces.

In the above equation of equilibrium of nodal forces the terms in the stiffness matrix and load vector are determined by:

$$k_{ij} = E \sum_{\omega} \int_{\omega} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \cdot \text{area } dx$$

$$f_i = \sum_{\omega} \int_{\omega} f \cdot N_i \cdot \text{area } dx - \int_{\partial_2 \Omega} T_{(n)} N_i \, ds$$

In these expressions, N_i and $T_{(n)}$ denote the shape functions and the traction boundary conditions respectively.

- Element Calculation:

Element #1:

The shape functions are:

$$N1 = \frac{1}{40}(40 - x)$$

$$N2 = \frac{1}{40}x$$

The finite element equilibrium equations for element #1 are:

$$\begin{bmatrix} 25000. & -25000. \\ -25000. & 25000. \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 266.667 \\ 533.333 \end{bmatrix}$$

Element #2:

The shape functions are:

$$N1 = \frac{1}{40}(80 - x)$$

$$N2 = \frac{1}{40}(x - 40)$$

The finite element equilibrium equations for element #2 are:

$$\begin{bmatrix} 25000. & -25000. \\ -25000. & 25000. \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1066.667 \\ 1433.333 \end{bmatrix}$$

- The global system of equilibrium equations are given by:

$$\begin{bmatrix} 25000. & -25000. & 0 \\ -25000. & 50000. & -25000. \\ 0 & -25000. & 25000. \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 266.667 \\ 1600. \\ 1433.333 \end{bmatrix}$$

- Impose the Displacement Boundary Condition $u|_{x=0} = u_1 = 0$ and solve the system of equations for the unknowns u_2 and u_3 . Then apply these values to compute the kinematically admissible stress-fields.

- For element #1:

The displacement is described by:

$$u_h = 0.00303332x \quad (x \text{ and } u \text{ given in cm})$$

The kinematically admissible stress-field for this element is computed by:

$$\begin{aligned}\sigma_h &= E \frac{\partial u}{\partial x} \\ &= 10^6 * 0.00303332 \\ &= 3033.32 \text{ (Pa)}\end{aligned}$$

- For element #2:

The displacement is described by:

$$u_h = 0.00303332(80. - x) + 0.00446665(-40. + x) \quad (x \text{ and } u \text{ given in cm})$$

The kinematically admissible stress-field for this element is determined to be:

$$\begin{aligned}\sigma_h &= E \frac{\partial u}{\partial x} \\ &= 10^6 * 0.00143333 \\ &= 1433.33 \text{ (Pa)}\end{aligned}$$

Compute the Statically Admissible Stress-Field $\hat{\sigma}_h$

To compute the statically admissible stress-field, assume each element is a free body. The nodal forces due to kinematically admissible stress-field and body force are active. Compute the *nodal forces associated with each node due to tractions* over the boundaries of element such that their corresponding element-tractions are in equilibrium with respect to the body-force. At internal boundaries such as node 2, the element-traction must be equilibrating. At the boundaries with prescribed traction boundary conditions, the element-traction takes the value of the applied traction.

In order to compute the statically admissible stress-field, first, the *nodal forces due to tractions* are equated to the nodal forces due to kinematically admissible stress-field subtracted from the nodal forces due to the body-force. In other words, equations (2.29) are used; these equations are quoted for problems in three-dimensional space:

$$\begin{cases} \int_{\partial\omega_j} \eta_{\omega_j} T \cdot a_i \, ds = \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot b_i \, ds = \int_{\omega_j} \sigma_h B b_i \, d\Omega - \int_{\omega_j} f \cdot b_i \, d\Omega \\ \int_{\partial\omega_j} \eta_{\omega_j} T \cdot c_i \, ds = \int_{\omega_j} \sigma_h B c_i \, d\Omega - \int_{\omega_j} f \cdot c_i \, d\Omega \end{cases} \quad (2.29)$$

The appropriate equation for computing the *nodal force* on node i in one-dimensional space is written as:

$$\int_{\partial\omega_j} \eta_{\omega_j} T \cdot a_i \, ds = \int_{\omega_j} \sigma_h B a_i \, d\Omega - \int_{\omega_j} f \cdot a_i \, d\Omega$$

which is reduced to the following equation:

$$\eta_{\omega_j} \sigma \cdot n \cdot N_i \cdot \text{area} = \text{Nodalforce at the node } i = \int_{\omega_j} \sigma_h \frac{\partial N_i}{\partial x} \text{area} \, dx - \int_{\omega_j} f \cdot N_i \cdot \text{area} \, dx$$

in which n denotes the unit vector in the direction of outward normal to the boundaries of elements. η_{ω_j} is a function defined on the boundaries of elements. This function takes the value from the set $\{-1, +1\}$. (Remark: usually, we do not compute the element-tractions, explicitly; the nodal forces computed according to the above rule ensures the continuity of element-tractions across the boundaries of the adjacent elements.) The differences between the *nodal forces due to kinematically admissible stress-fields and body forces* are computed as:

- For element #1:

$$\begin{aligned} \text{Nodalforce_1} &= \int_0^{40} 3033.32 \frac{-1}{40} * 1.0 \, dx - \int_0^{40} x \frac{1}{40} (40 - x) * 1.0 \, dx \\ &= -3300.0 \end{aligned}$$

$$\begin{aligned} \text{Nodalforce_2} &= \int_0^{40} 3033.32 \frac{1}{40} * 1.0 \, dx - \int_0^{40} x \frac{1}{40} x * 1.0 \, dx \\ &= 2500.0 \end{aligned}$$

- For element #2:

$$\begin{aligned} \text{Nodalforce_1} &= \int_{40}^{80} 1433.33 \frac{-1}{40} * 1.0 \, dx - \int_{40}^{80} x \frac{1}{40} (80 - x) * 1.0 \, dx \\ &= -2500.0 \end{aligned}$$

$$\begin{aligned} \text{Nodalforce_2} &= \int_{40}^{80} 1433.33 \frac{1}{40} * 1.0 \, dx - \int_{40}^{80} x \frac{1}{40} (x - 40) * 1.0 \, dx \\ &= 100.0 \end{aligned}$$

These nodal forces are equilibrated to the *nodal forces due to tractions* by the expression:

$$\eta_{\omega_j} T_i \cdot \text{area} = \text{Nodalforce at the node } i$$

- Compute the linear stress-field:

$$\bar{\sigma} = a + bx$$

for each node in each element as follows:

- For element #1:

$$\begin{cases} -1 * (a + 0 * b) * 1 * 1.0 = -3300.0 \\ +1 * (a + 40 * b) * 1 * 1.0 = 2500.0 \end{cases}$$

- This system is solved for the unknowns $a = 3300.0$ and $b = -20.0$ to define the linear stress-field:

$$\bar{\sigma} = 3300.0 - 20.0x$$

- For element #2:

$$\begin{cases} -1 * (a + 40 * b) * 1 * 1.0 = -2500.0 \\ +1 * (a + 80 * b) * 1 * 1.0 = 100.0 \end{cases}$$

- This system is solved for the unknowns $a = 4900.0$ and $b = -60.0$ to define the linear stress-field:

$$\bar{\sigma} = 4900.0 - 60.0x$$

The graphical representations of these stress-fields are given in Figure (A3.2). This is called the *mean-statically admissible stress-field*. One can show that the mean-statically admissible stress-field satisfies the solvability condition:

$$\int_{\partial\omega} \bar{\sigma} \cdot n \, ds + \int_{\omega} f \cdot \text{area} \, dx = 0$$

or equivalently:

$$\int_{\omega} (\text{div } \bar{\sigma} + f) \cdot \text{area} \, dx = 0$$

- For element #1:

$$\int_0^{40} (-20.0 + x) \cdot \text{area} \, dx = [-20.0x + \frac{1}{2}x^2]_0^{40} = -800.0 + 800.0 = 0$$

- For element #2:

$$\int_{40}^{80} (-60.0 + x) \cdot \text{area} \, dx = [-60.0x + \frac{1}{2}x^2]_{40}^{80} = -4800.0 + 3200.0 + 2400.0 - 800.0 = 0$$

- Next we compute a statically admissible stress-field by superimposing the bubble functions upon the mean-statically admissible stress-field, $\bar{\sigma}$, within the elements. In this calculation the expression (2.55) for eight node bricks:

$$\begin{aligned} \hat{\sigma}_h(\xi, \eta, \varsigma) = & \sum_{i=1}^8 N_i(\xi, \eta, \varsigma) \sigma_i + N^*(\xi, \eta, \varsigma) \phi + \sum_{j=1}^{\infty} \xi^j N^*(\xi, \eta, \varsigma) \phi_j^{(\xi)} + \\ & \sum_{k=1}^{\infty} \eta^k N^*(\xi, \eta, \varsigma) \phi_k^{(\eta)} + \sum_{l=1}^{\infty} \varsigma^l N^*(\xi, \eta, \varsigma) \phi_l^{(\varsigma)} \end{aligned} \quad (2.55)$$

defined in three-dimensional space is used. This expression is simplified in one-dimensional case.

- For element #1:

The bubble function is defined to be zero on the element boundaries by:

$$\text{bubble function} \equiv x(40 - x)$$

-Define the statically admissible stress-field $\hat{\sigma}_h$ by

$$\hat{\sigma}_h = 3300.0 - 20x + x(40 - x)\phi$$

with the unknown coefficient ϕ . In order to compute the statically admissible stress-field, this stress-field must satisfy the equation of equilibrium pointwise, i.e.,

$$\text{div } \hat{\sigma}_h + f = -20.0 + (40 - 2x)\phi + x \equiv 0$$

which determines the unknown coefficient ϕ :

$$\phi = \frac{1}{2}$$

- The statically admissible stress-field for this element is given by

$$\hat{\sigma}_h = 3300.0 - \frac{1}{2}x^2$$

- For element #2:

The bubble function is defined by

$$\text{bubble function} \equiv (40 - x)(80 - x)$$

-Define the statically admissible stress-field $\hat{\sigma}_h$ by

$$\hat{\sigma}_h = 4900.0 - 60x + (40 - x)(80 - x)\phi$$

Again, this stress-field must satisfy the equation of equilibrium

$$\text{div } \hat{\sigma}_h + f = -60 - (120 - 2x)\phi + x \equiv 0$$

which determines the unknown coefficient ϕ :

$$\phi = -\frac{1}{2}$$

- The statically admissible stress-field for this element is given by

$$\hat{\sigma}_h = 3300.0 - \frac{1}{2}x^2$$

The graphical form of these stress-fields are presented in the Figure (A3.2).

Compute the Exact Solution:

For comparison purposes the exact displacement and stress-field are computed by solving the system of differential equation of equilibrium, the continuity equation, and the constitutive law:

$$\begin{cases} \text{div } \sigma + f = 0 \\ \epsilon = \frac{\partial u}{\partial x} \\ \sigma = E\epsilon \end{cases}$$

subject to the boundary conditions: $u|_{x=0} = 0$ and $\sigma \cdot n|_{x=80} = 100$.

- It can be shown that the exact displacement and stress distributions are described by:

$$\begin{aligned} \tilde{u} &= \frac{-1}{6E}x^3 + \frac{3300}{E}x \\ \tilde{\sigma} &= 3300.0 - \frac{1}{2}x^2. \end{aligned}$$

respectively. In this case the exact stress-field equals the statically admissible stress-field. The graph of the exact stress-field is shown in the Figure (A3.2).

Compute the Error in the Finite-Element Stress-Field

- Apply the energy inner product (2.9) to calculate the energy of the error vector defined from the kinematically admissible stress-field to the mean-statically admissible stress-field, statically admissible stress-field, and the exact stress-field.

$$(\sigma, \sigma) = \int_{\Omega} \sigma^T \epsilon \, d\Omega \quad (2.9)$$

- Calculate the error between the kinematically admissible and the mean-statically admissible stress-fields. This is called the approximate error-bound.

$$\begin{aligned} \|\sigma_h - \bar{\sigma}\|^2 &= \frac{1}{E} \int_0^{40} (\bar{\sigma} - \sigma_h)^2 \cdot \text{area} \, dx + \\ &\quad \frac{1}{E} \int_{40}^{80} (\bar{\sigma} - \sigma_h)^2 \cdot \text{area} \, dx \\ &= 10^{-6} \int_0^{40} (3300.0 - 20.0x - 3033.32)^2 \cdot 1.0 \, dx + \\ &\quad 10^{-6} \int_{40}^{80} (4900.0 - 60x - 1433.33)^2 \cdot 1.0 \, dx \\ &= 22.7555 \quad \text{N.cm} \end{aligned}$$

- Similarly the error between the kinematically and the statically admissible stress-fields are computed. This is called the error-bound.

$$\begin{aligned} \|\sigma_h - \hat{\sigma}_h\|^2 &= \frac{1}{E} \int_0^{40} (\hat{\sigma}_h - \sigma_h)^2 \cdot \text{area} \, dx + \\ &\quad \frac{1}{E} \int_{40}^{80} (\hat{\sigma}_h - \sigma_h)^2 \cdot \text{area} \, dx \\ &= 10^{-6} \int_0^{40} \left(3300.0 - \frac{1}{2}x^2 - 3033.32\right)^2 \cdot 1.0 \, dx + \\ &\quad 10^{-6} \int_{40}^{80} \left(3300.0 - \frac{1}{2}x^2 - 1433.33\right)^2 \cdot 1.0 \, dx \\ &= 21.6178 \quad \text{N.cm} \end{aligned}$$

- The Absolute-error computed by using the mean-statically admissible stress-field is larger than the Error-Bound computed by employing the kinematically admissible stress-field and the statically admissible stress-field. In mathematical terms the following relationships happen to hold between the actual error and the estimated ones in this problem:

$$\begin{array}{lll} \|\bar{\sigma} - \sigma_h\|^2 & = \|\sigma_h - \hat{\sigma}_h\|^2 & \leq \|\sigma_h - \bar{\sigma}\|^2 \\ 21.6178 & = 21.6178 & < 22.7555 \end{array}$$

This indicates that the error-measure computed by using the mean-statically admissible stress-field can be a reliable error-estimate as well.

Error-Estimates by Zienkiewicz and Zhu

In this method a continuous stress-field is computed. First, the average values of stresses from the nearest Gauss-points are assigned to the nodes. Then the nodal values of stresses are interpolated to the interior of the elements using the usual shape functions. The continuous stress-field introduced by Zienkiewicz and Zhu is plotted in the Figure (A3.3). It is clear that this continuous stress-field does not satisfy the equation of equilibrium within individual elements nor does it satisfy the prescribed traction boundary conditions. This stress-field is computed inexpensively.

The error between the kinematically admissible stress-field and the continuous stress-field is computed in the same way as before using the continuous stress-field:

$$\begin{aligned}\|\sigma_h - \sigma_h^c\|^2 &= \frac{1}{E} \int_0^{80} (\sigma_h^c - \sigma_h)^2 \cdot \text{area} \, dx \\ &= 10^{-6} \int_0^{80} (3033.32 - 20.0x - 3033.32)^2 \cdot 1.0 \, dx \\ &= 17.0663 \text{ N.cm}\end{aligned}$$

Comparing this error-estimate to the exact error computed previously, it is known that the method of error-estimates by Zienkiewicz and Zhu underestimates the actual error in the solution, i.e.,

$$17.0663 < 21.6178.$$

Error-Estimates Using Quasi-Statically Admissible Stress-Field

This stress-field is a continuous stress-field which conforms to the prescribed traction boundary conditions over the external boundaries where the tractions are prescribed. It is simple and numerically cheap to correct the continuous stress-field of Zienkiewicz and Zhu to satisfy the traction boundary conditions. In one-dimensional space, simply set the stresses to the traction boundary conditions on the external boundaries over which the tractions are specified. In two and three-dimensional spaces the procedure given in the previous sections is used to impose the relationship $\sigma \cdot n = \bar{T}_{(n)}$. The quasi-statically admissible stress-field is plotted in the Figure (A3.3) for the one-dimensional test case discussed in this appendix.

The error between the kinematically admissible stress-field and the quasi-statically admissible stress-field is computed in a similar fashion as before using the energy norm as follows:

$$\begin{aligned}
 \|\sigma_h - \bar{\sigma}_h\|^2 &= \frac{1}{E} \int_0^{40} (\bar{\sigma}_h - \sigma_h)^2 \cdot \text{area } dx + \\
 &\quad \frac{1}{E} \int_{40}^{80} (\bar{\sigma}_h - \sigma_h)^2 \cdot \text{area } dx \\
 &= 10^{-6} \int_0^{40} (3033.32 - 20.0x - 3033.32)^2 \cdot 1.0 \, dx + \\
 &\quad 10^{-6} \int_{40}^{80} (55.8331x - 109.166(x - 40) - 1433.33)^2 \cdot 1.0 \, dx \\
 &= 26.5478 \quad \text{N.cm}
 \end{aligned}$$

This error-estimate shows that the quasi-statically admissible stress-field has produced an error-estimate which is greater than the actual error in the solution.

$$26.5478 > 21.6178.$$

This result and the error-estimates computed by using this stress-field, mentioned in the previous sections, indicate that the error-estimates using the quasi-statically admissible stress-field should be more reliable than the error-estimates computed by using the method of Zienkiewicz and Zhu. Nevertheless, theoretical analysis shows that these error-measures are, generally, less reliable in comparison to the error-measures computed by using the mean-statically admissible and the statically admissible stress-fields.

Figure (A3.2)

Stress Analysis in a Bar Subjected to a Linear Bodyforce I

Figure (A3.3)

Stress Analysis in a Bar Subjected to a Linear Body Force II

END

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FIN