The Evaluation of T -stress Using Boundary Element Method

by

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ABSTRACT

A two-dimensional numerical study employing the boundary element method (BEM) of elastic T -stress for test specimens and thick-walled cylinders is presented in this thesis. The T -stress, together with the stress intensity factor K , provide a more reliable two-parameter prediction of fracture in linear elastic fracture mechanics. In this study, both the direct displacement approach and a contour integral approach, in conjunction with BEM analysis, have been developed to obtain the T -stress. A wide range of relative crack-tip element sizes, crack length ratios and different loading conditions are considered. The technique is first validated with analytical and finite element solutions in the literature for an infinite plate and three test specimens. Some general guidelines for proper mesh design are developed. The pressured thick-walled cylinder problem is one of much practical importance in industry. When cracked in the radial direction, it is normally considered to be as a 'low-constraint' geometry. T -stress solutions for this problem under internal pressure and crack face loading conditions are obtained. Different radius ratios and relative crack lengths are considered. Using the BEM results as reference T -stress solutions, the T -stress weight functions for thick-walled cylinder have also been derived. Results of the study show that the two-dimensional BEM analysis, together with weight function method, can be used to provide a quick and accurate estimate of T -stress for 2-D crack problems.

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Contents

Acceptance Sheet	ii
Abstract	iii
Acknowledgements	iv
Contents	v
List of Tables	viii
List of Figures	xiii
Nomenclature	xviii

Chapter 1

Introduction	1
--------------------	---

Chapter 2

Elastic T -stress in Fracture Mechanics	5
2.0 Introduction	5
2.1 Elastic T -stress	6
2.1.1 Conventional Single Parameter Linear Elastic Fracture Mechanics	6
2.1.2 Two-parameter LEFM and T -stress	7
2.1.3 The Significance of T -stress	8
2.1.4 Methods of Calculating T -stress	10

2.2	Boundary Element Method (BEM) for Fracture Mechanics	11
2.2.1	Direct Approach for T -stress Evaluation Using BEM.....	14
2.2.2	Contour Integral Approach for T -stress Evaluation Using BEM	16
2.3	Weight Function Method	20
2.3.1	Weight Function for Stress Intensity Factors	20
2.3.2	Weight Function for T -stress	22
2.4	Closing Remarks	24

Chapter 3

Test Problems and Validations	31
3.0 Introduction	31
3.1 Crack in an Infinite Plate	32
3.1.1 Problem Definition and Numerical Modelling	32
3.1.2 Results Discussion	33
3.2 Central Cracked Plate (CCP) Specimen	33
3.2.1 Problem Definition	33
3.2.2 Numerical Modelling	34
3.2.3 Results and Discussions	35
3.3 Double Edge Cracked Plate (DECP) Specimen	37
3.3.1 Problem Definition and Numerical Modelling	37
3.3.2 Results and Discussions	38
3.4 Single Edge Cracked Plate (SECP) Specimen	39

3.4.1	Problem Definition and Numerical Modelling	39
3.4.2	Results and Discussions	40
3.5	Conclusion	40
 Chapter 4		
Elastic <i>T</i>-stress Solutions for Cracked Thick-Walled Cylinders		79
4.0	Introduction	79
4.1	Problem Definition	80
4.2	Numerical Modelling	81
4.3	BEM Results and Discussions	81
4.4	Weight Function	83
4.5	Conclusion	86
 Chapter 5		
Conclusions and Recommendations		110
5.0	Conclusions	110
5.1	Recommendations	112
 References		113

LIST OF TABLES

3.1	Comparison of T -stresses from BEM direct approach with exact solutions for the central crack in an infinite plate under remote tension.	42
3.2	Comparison of T -stresses from BEM contour integral approach with exact solutions for the central crack in an infinite plate under remote tension.	42
3.3	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.2$	43
3.4	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.3$	43
3.5	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.4$	44
3.6	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.5$	44
3.7	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.6$	45
3.8	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang) for central crack plate specimen under different contour geometry and size; $a/w=0.4$, remote tension.	45
3.9	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under constant crack face loading.	46

3.10	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under linear crack face loading.	46
3.11	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under parabolic crack face loading.	47
3.12	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under cubic crack face loading.	47
3.13	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.2$	48
3.14	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.3$	48
3.15	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.4$	49
3.16	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.5$	49
3.17	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.6$	50
3.18	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under constant crack face loading.	50
3.19	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under linear crack face loading.	51

3.20	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under parabolic crack face loading.	51
3.21	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under cubic crack face loading.	52
3.22	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.2$	52
3.23	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.3$	53
3.24	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.4$	53
3.25	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.5$	54
3.26	Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.6$	54
3.27	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under constant crack face loading.	55
3.28	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under linear face loading.	55
3.29	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under parabolic crack face loading.	56

3.30	Comparison of T-stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under cubic crack face loading.	56
4.1	Comparison of normalized stress intensity factor solutions, $K^*_I=K_I/\sigma_o\sqrt{\pi a}$, from present BEM analysis and solutions by Wu and Carlsson (1991): thick-walled cylinder under internal pressure, $\sigma_o=P$	88
4.2	Comparison of normalized stress intensity factor solutions, $K^*_I=K_I/\sigma_o\sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under constant crack face loading, $n=0$	89
4.3	Comparison of normalized stress intensity factor solutions, $K^*_I=K_I/\sigma_o\sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under constant crack face loading, $n=1$	90
4.4	Comparison of normalized stress intensity factor solutions, $K^*_I=K_I/\sigma_o\sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under constant crack face loading, $n=2$	91
4.5	Comparison of normalized stress intensity factor solutions, $K^*_I=K_I/\sigma_o\sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under constant crack face loading, $n=3$	92
4.6	Comparison of normalized T-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=1.5$).	93

4.7	Comparison of normalized T -stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=1.75$).	93
4.8	Comparison of normalized T -stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.0$).	94
4.9	Comparison of normalized T -stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.25$).	94
4.10	Comparison of normalized T -stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.5$).	95
4.11	Normalized T -stresses for thick-walled cylinder under crack face loading ($R_o/R_i=1.5$).	95
4.12	Normalized T -stresses for thick-walled cylinder under crack face loading ($R_o/R_i=1.75$).	95
4.13	Normalized T -stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.0$).	96
4.14	Normalized T -stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.25$)	96
4.15	Normalized T -stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.5$)	96
4.16	The coefficients of V_1 : normalized T -stress empirical equation under constant crack face stress distribution.	97
4.17	The coefficients of V_2 : normalized T -stress empirical equation under linear crack face stress distribution.	97

LIST OF FIGURES

2.1	Three basic modes of fracture.	25
2.2	Three-dimensional coordinate system for the region at the crack tip.	25
2.3	Linear elastic body R with boundary S in a two-dimensional coordinate system.	26
2.4	Discretized solution domain R, with boundary S.	26
2.5	A quadratic isoparametric line element.	27
2.6	Node symbols for quarter-point crack tip element.	27
2.7	Integration paths and coordinate definitions.	28
2.8	Weight Function for one-dimensional cracks.	29
2.9	<i>T</i> -stress weight function.	30
3.1	A large square plate with a small crack under remote tension; $H=W$, $a/W=0.01$	57
3.2	Boundary element mesh used for large plate with small crack.	58
3.3	Comparison of exact solution and BEM results (direct approach): for infinite plate with a central crack under remote tension.	59
3.4	Comparison of exact solution and BEM results (contour integral approach): infinite plate with a central crack under remote tension.	59
3.5	The central crack plate (CCP) specimen.	60
3.6	Crack face loading distributions: $\sigma = \sigma_o(1 - \frac{x_1}{a})^n$	61
3.7	Typical boundary element mesh used for test specimen.	62
3.8	Percentage discrepancies of <i>T</i> -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for CCP.	63

3.9	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for CCP.	63
3.10	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for CCP.	64
3.11	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for CCP.	64
3.12	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.6$) for CCP.	65
3.13	Comparison of T -stress between BEM contour integral analysis and FEM results for central cracked plate ($a/W=0.4$) under different contour geometry and size.	65
3.14	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under constant crack face loading.	66
3.15	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under linear crack face loading.	66
3.16	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under parabolic crack face loading.	67
3.17	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under cubic crack face loading.	67
3.18	The double edge crack plate (DECP) specimen.	68
3.19	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for DECP.	69

3.20	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for DECP.	69
3.21	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for DECP.	70
3.22	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for DECP.	70
3.23	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.6$) for DECP.	71
3.24	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under constant crack face loading.	71
3.25	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under linear crack face loading.	72
3.26	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under parabolic crack face loading.	72
3.27	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under cubic crack face loading.	73
3.28	The single edge cracked plate (SECP) specimen	74
3.29	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for SECP.	75

3.30	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for SECP.	75
3.31	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for SECP.	76
3.32	Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for SECP.	76
3.33	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under constant crack face loading.	77
3.34	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under linear crack face loading.	77
3.35	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under parabolic crack face loading.	78
3.36	Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under cubic crack face loading.	78
4.1	A thick-walled cylinder with a single radial crack.	98
4.2	Typical boundary element mesh for thick-walled cylinder.	99
4.3	Comparison of normalized BEM T -stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=1.5$).	100
4.4	Comparison of normalized BEM T -stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=1.75$).	101

4.5	Comparison of normalized BEM T -stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.0$).	102
4.6	Comparison of normalized BEM T -stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.25$).	103
4.7	Comparison of normalized BEM T -stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.5$).	104
4.8	Comparison of normalized BEM T -stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under internal pressure.	105
4.9	Comparison of normalized BEM T -stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face constant loading.	106
4.10	Comparison of normalized BEM T -stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face linear loading.	107
4.11	Comparison of normalized BEM T -stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face parabolic loading.	108
4.12	Comparison of normalized BEM T -stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face cubic loading.	109

NOMENCLATURE

a	- crack length
A_i	- coefficients of least square fitting for V_1
B_i	- coefficients of least square fitting for V_2
D_1, D_2	T -stress weight function parameters
E	- Young's modulus
h	-rectangular contour half length
H	- plate height; general elastic modulus
$ J(\xi) $	- Jacobian transformation
K	- stress intensity factor
K_I	- mode I stress intensity factor
K_{IC}	- fracture toughness
K_r	- reference stress intensity factor
l	- crack tip element size
M	- number of elements
M_1, M_2, M_3	- coefficients of weight function for stress intensity factor
n	- power index for stress distribution
n_j	- unit outward normal component
N	- number of nodes
$N^{(i)}$	- shape functions
p	- internal pressure
P	- load point in BEM analysis
$P^{d(b,c)}$	- c^{th} node of the b^{th} element

Q	far field load; field point in BEM analysis
r	contour radius; distance between load and field point; r-coordinate
R	BEM solution domain
R_i	outside radius of a thick walled cylinder
R_o	inside radius of a thick walled cylinder
S	boundary of a domain; far field force
t_i	tractions
T_{ij}	traction fundamental solutions
u_i	displacements
U_{ij}	displacement fundamental solutions
V_1, V_2	normalized T -stress for crack face constant and linear loading cases
W	plate width; thickness of a thick-walled cylinder
x_i	Cartesian coordinates
X_i	body force per unit volume
β	T -stress biaxial ratio
δ_{ij}	Kronecker symbol
λ	biaxial load ratio
σ_o	nominal stress
σ_{ij}	stress tensor
θ	angular coordinates
ν	Poisson's ratio
ξ	natural coordinates for isoparametric element
Γ	integral contour
Ω	BEM domain
μ	shear modulus

CHAPTER 1

INTRODUCTION

In conventional failure assessment of crack problems using linear elastic fracture mechanics (LEFM), comparison is made of the calculated stress intensity factor with the fracture toughness of the material, which is usually measured from 'high constraint' standard test specimens, e.g. deeply cracked three-point bending specimen and compact tension specimen. Over the years, questions have arisen if the fracture toughness measured from these specimens can be applied to cracked structures with 'low constraint' characteristics, i.e. where there is low stress triaxiality in the vicinity of the crack tip. The analytical, numerical and experimental studies over the past thirty years on the parameter known as T -stress, and the subsequent inclusion of this parameter together with the stress intensity factor in linear elastic fracture mechanics analysis, are direct responses to the question of transferability of standard laboratory test data to life prediction in service structures.

In linear elastic fracture mechanics, the stress intensity factor, K , characterizes the magnitude of stress field in the vicinity of a crack tip, and is used to predict the onset of fracture. This theory is also true for elastic-plastic problem, if yielding is well contained, i.e. if small-scale yielding conditions (Rice, 1967, 1968) are satisfied. Larsson and Carlsson (1973) conducted finite element analysis for four different test specimens and found that the

value of the fracture toughness, K_{IC} , showed some discrepancies among different specimen geometries even though the load level is far below the ASTM limit. These discrepancies were later explained by the inclusion of the second term of Williams' series expansion in their analysis. Williams' series expansion provides the asymptotic stress field solution in the vicinity of the crack tip. Rice (1974) referred to this second non-singular term as the elastic T -stress, or simply T -stress. The effects of T -stress on fracture toughness, crack growth rate and crack growth instability have been investigated by, for example, Larsson and Carlsson (1973), Leever et al. (1976, 1979), and Cotterell and Rice (1980). The later work by Bilby et al (1986) indicated the relationship between T -stress and crack tip constraint. Recently, the application of two-parameter fracture mechanics, i.e. the T -stress and stress intensity factor K , to include the constraint effect in the failure assessment procedure has been established (Ainsworth et al., 2000). Based on this correlation between constraint parameter T -stress and fracture toughness properties of the material, fracture predictions of cracked structures are now more reliable and the two-parameter fracture mechanics are getting widely accepted as general criteria to predict service life of cracked components.

In order to apply this two-parameter fracture mechanics methodology, it is important to provide accurate stress intensity factor solutions and T -stress solutions for the crack problems of practical interests. Although several handbooks (e.g. Rooke and Cartwright (1976) and Tada et al (1984)) devoted solely to stress intensity factors have been published, the available solutions for T -stress are very limited. Several analytical and numerical methods were developed to obtain the T -stress for different test specimens. The closed-form T -stress solutions for infinite plate crack problems have been analytically obtained by Chen (2000) using complex potential theory. Based on Eshelby's theorem, the interaction integral

method, in conjunction with finite element method has been used by Wang (2002) to obtain the T -stress solutions for several commonly used test specimens under remote and crack face loading conditions.

The boundary integral equation method, commonly known as the boundary element method (BEM), is an alternative to finite element method (FEM); it has distinct advantages in fracture mechanics analysis. Only the boundary of the solution domain needs to be modeled, which reduces the numerical dimensionality of the physical problem. Some researchers have explored formulas (e.g. Tan, 1987; Aliabadi and Rooke, 1991) to obtain stress intensity factors using BEM. Unfortunately, the BEM solutions for T -stress are still very limited in literature. Recent work conducted by Tan and Wang (2003) has derived a direct approach, analogous to the displacement formula of stress intensity factor solution for plane crack problems. Sladek et al (1997) developed an integral formula to calculate the T -stress on a contour away from the crack tip. The present study is the expansion of the two above-mentioned works to develop the T -stress solutions for 'low constraint' geometries, e.g. cracked thick-walled cylinders under different loading conditions.

In practice, the loading conditions of cracked components are usually quite complex. This can be due to the existence of residual stresses, stress concentrations, or thermal stresses. The weight function (WF) method, mathematically, the Green's function method, is one of the most efficient methods to derive stress intensity factor solutions for complex non-linear stress distributions. This method has been extended by Wang (2002) to derive the T -stress solutions under complex stress conditions in his recent work using the finite element results. However, the weight function analysis of T -stress using the boundary element results as reference T -stress solutions has not yet been carried out previously. Hence, to derive the

weight functions for cracked thick-walled cylinders using BEM solutions is a focus of the present study as well.

In this thesis, the theoretical background of the T -stress, the boundary element method (BEM) and weight Function (WF) method will be briefly reviewed in Chapter 2, along with the derivations of direct displacement formula and contour integral formula for two-dimensional elastic T -stress solutions. In order to provide confidence in the numerical models and the results obtained, the technique of boundary element method used will be validated in Chapter 3. This is done by comparing results obtained using the BEM with exact solutions and finite element solutions, for an infinite plate and three types of specimens, respectively. Both remote tension and crack face loading conditions are considered. Some general guidelines for the appropriate mesh design in the determination of T -stress solutions will also be developed. The main physical problem for the various loading conditions investigated in this study is the cracked thick-walled cylinder. The T -stress solutions for this geometry are of practical importance in industry; but, they are not currently available in literature. In Chapter 4, the BEM T -stress solutions for thick-walled cylinders with a wide range of radius ratios and relative crack lengths will be determined. The weight functions for the T -stress will also be obtained and verified. Finally, some conclusions from the present study and the recommendations for future work will be presented in Chapter 5.

CHAPTER 2

ELASTIC *T*-STRESS IN FRACTURE MECHANICS

2.0 Introduction

In this chapter, a review of conventional theories of linear elastic fracture mechanics (LEFM) is presented first. The need to involve an additional parameter, namely, the *T*-stress, in fracture mechanics analysis to include the constraint effect is then discussed. The *T*-stress is defined and various methods of calculating this quantity are briefly discussed. The second part of this chapter reviews the boundary integral equation (BIE) method, also commonly known as boundary element method (BEM), for two-dimensional, linear, isotropic elastostatics problems. The BEM is then applied to LEFM analysis to obtain *T*-stress solutions; two approaches presented here are the direct displacement approach and the contour integral approach. In the last part of this chapter, the weight function method is discussed. It is useful for obtaining solutions with non-uniform stress distributions. These may arise in situations with thermal stresses, residual stresses and stress concentrations.

2.1 Elastic *T*-stress

For a cracked body, there are three basic modes of deformation. These are characterized by the displacements of upper and lower crack surfaces, and are shown in Fig. 2.1. Mode I is called the opening mode, where the two crack surfaces are pulled apart. For mode II, the two crack surfaces slide over each other in plane, it is called the sliding mode. In mode III, the tearing mode, the two crack surfaces tear apart out of plane. In this thesis, only mode I crack problems will be discussed.

2.1.1 Conventional Single Parameter Linear Elastic Fracture Mechanics

Brittle fracture is a major mode of failure in components and structures containing cracks. Crack growth in brittle material often takes place very fast and with serious consequences. Therefore, it is important to define an appropriate procedure for predicting the onset of brittle fracture in cracked specimen.

Linear elastic fracture mechanics (LEFM) proposes that for brittle materials under mode I loading, a single parameter, the critical stress intensity factor K_{IC} (also known as fracture toughness) is sufficient for predicting fracture. The deformation at crack tip is governed by the intensity of the stress field, which is characterized by the stress intensity factor K_I . The conventional failure criterion for mode I fracture is as follows:

$$K_I \geq K_{IC} \quad (2.1)$$

This LEFM consideration still can be true for an elastic-plastic material if the plastic zone size is small compared to other geometric dimensions of the problem, such as the crack length a . This is the so-called ‘small scale yielding approximation’ by Rice, et al. (1968). Then the crack problem can be solved by assuming that the near-tip stress field of the elastic-

plastic crack problem is given by the singular term in the elastic stress solution of Williams' (1957) series expansion,

$$\sigma_{ij} = Ar^{-\frac{1}{2}}f_{ij}(\theta) + Br^0g_{ij}(\theta) + Cr^{\frac{1}{2}}h_{ij}(\theta) + \dots \quad (2.2)$$

where r and θ are polar coordinates centred at the crack tip, see Fig. 2.2. The functions $f_{ij}(\theta)$, $g_{ij}(\theta)$ and $h_{ij}(\theta)$ contain trigonometric functions of the angular location relative to crack tip. Parameters A , B and C are proportional to the applied load. As $r \rightarrow 0$, the leading term dominates and exhibits $1/\sqrt{r}$ singularity, while the higher order terms remain finite values or approach zero. Therefore, the stress field near the crack tip is expressed in the following form by only involving the first singular term of equation (2.2):

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) \quad (2.3)$$

where K_I is the stress intensity factor. It characterizes the magnitude of stresses in the vicinity of crack tip and the onset of failure when linear elastic conditions are dominant. Solutions for K_I are available in the literature (e.g. Rooke and Cartwright, 1974) for a wide range of crack geometries and loading conditions.

2.1.2 Two-Parameter LEFM and T-stress

Over the past thirty years, increasing attention has been put to study the effect of higher order terms of Williams' (1957) series expansion on the initiation of mode I fracture under predominantly linear elastic deformation. Larsson and Carlsson (1973) carried out investigation on elastic-plastic problem for cracks in different type of specimens using finite element method (FEM) and found that the solutions for the stress state near the crack tip can not be related to Eqn. (2.3) through the stress intensity factor K_I alone, even when the

requirements for 'small scale yielding' are all met. They noted that the discrepancies could be resolved if the first non-singular term of Williams' series expansion (Eqn. (2.2)) is included in the near tip stress solution, which is

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + T_{ij} \delta_{i1} \delta_{j1} \quad (2.4)$$

In fact, there is only one non-singular term in this expansion for the elastic stress field in mode I, namely, T_{11} . This represents the uniform stress acting parallel to the crack plane. In the notation of Rice (1974), this second term of Williams series expansion is denoted as the *T*-stress or elastic *T*-stress. *T*-stress is directly proportional to the load applied to the cracked structure and depends on the geometrical parameters.

To normalize the effect of the *T*-stress relative to the stress intensity factor in mode I, Leevers and Radon (1982) proposed a dimensionless parameter called the biaxiality ratio β , which is,

$$\beta = (T\sqrt{\pi a}) / K_I \quad (2.5)$$

where a is crack depth and K_I is mode I stress intensity factor.

2.1.3 The Significance of *T*-stress

Over the years, a significant amount of analytical, numerical and experimental studies have attempted to describe the important influence of *T*-stress on material fracture prediction. It has been observed by a number of researchers that fracture toughness K_{IC} , crack propagation rate and crack growth direction can depend on the magnitude and sign of *T*-stress.

Larsson and Carlsson (1973) showed the differences of fracture toughness K_{IC} between two groups of specimens (one is compact tension and three-point bending

specimens, the other is centre-cracked and double edge cracked specimens), although for specimens in the same group would differ very little within the valid loading scope. Similar observations were made by Hall (1971) in his experimental studies of cracked aluminium and titanium specimens. Results of finite element analysis performed by Shih and German (1981) on three-point bending specimen and centre-cracked specimen also showed the same phenomenon.

The stress field variation and fracture toughness discrepancies due to geometric differences are usually caused by the different 'constraint' effects in various specimen geometries. For so-called 'high constraint' geometries, such as compact tension and three-point bending specimens, a single parameter (K_I in LEFM) was able to fully describe the stress field and fracture properties; whereas for 'low constraint' geometries, such as centre cracked and double edge cracked specimens, the fracture toughness estimated would be lower than its actual value.

The constraint effect could be better incorporated into fracture mechanics evaluation by the elastic T -stress. Work by Bilby et al (1986) indicated that the sign and magnitude of T -stress could substantially change the level of crack tip stress triaxiality, hence influence crack tip constraint. Positive T -stress enhances the level of crack tip stress triaxiality and leads to high crack tip constraint; while negative T -stress reduce the level of crack tip stress triaxiality and leads to the loss of crack tip constraint. The works by Betegon and Hancock (1991), Du and Hancock (1991), O'Dowd and Shih (1991) and Wang (1993) further indicated that the T -stress, in addition to the stress intensity factor K_I , provides a more appropriate two-parameter characterization of plain strain elastic-plastic crack tip fields in a variety of crack configurations and loading conditions.

The effect of T -stress on crack propagation rate was studied by Leever et al. (1979) through a series of experiments using a square, biaxially loaded centre cracked plate specimen. The variation of elastic T -stress was controlled by varying the ratio of the orthogonal boundary loads, λ . When λ changes from 0 to 2, the steady-state crack propagation rate was reduced by a factor of three or more.

Leevers et al. (1976), Cotterell and Rice (1980) investigated the influence of the sign of the T -stress evaluated in front of a flat crack on crack growth directions. They showed that if the T -stress was negative (or compressive) the crack tends to grow along the crack plane; and if the T -stress was positive the crack deviates from its initial plane of growth. Recent work by Selvarathinam and Goree (1998) indicated the crack direction instability did not take place immediately after the T -stress become positive but at a considerably higher positive value. They define a critical T -stress (T_{crit}) at which the crack begin to turn, and it can be estimated from the corresponding β value on their K_{IC} versus β diagram.

2.1.4 Methods of calculating T -stress

Similar to the stress intensity factor solutions, it is important to provide accurate T -stress solutions for various crack configuration and loading types. Only limited work has been conducted over the past years to evaluate the T -stress for cracked geometries. For instance, Larsson and Carlsson (1973) employed stress substitution method by substituting the elastic finite element results into a 'boundary layer' formulation. Leever and Radon (1982) incorporated the eigenfunctions into a variational formulation in elasticity and computed the coefficients of these eigenfunctions directly. Kfour (1986), Nakamura and Parks (1992), and Wang (2002) use the interaction integral method, based on Eshelby's

theorem, in conjunction with the finite element method (FEM). Fett (2002) has used the Boundary Collocation Method (BCM) to evaluate *T*-stresses for various crack geometries.

2.2 Boundary Element Method (BEM) for Fracture Mechanics

The boundary element method (BEM) is an alternative to the finite element method (FEM) for the numerical analysis of many engineering problems. The differences between these two methods arise from the fact that FEM is based on partial differential equations, whereas the BEM is based on integral equations. The BEM analysis has the distinct advantage that only the boundary of the problem domain needs to be numerically modeled. By discretizing only the boundary of the solution domain, the problem can be solved with much simpler mesh designs and data preparation. The BEM analysis is efficient in dealing with problems with high stress gradients such as those concentrated stresses near crack tips. As a result, the BEM method has become widely used in the elastic fracture mechanics analysis of engineering components, in particular, for obtaining stress intensity factor solutions. Unfortunately, only very limited work (Sladek et al., 1997; Tan and Wang, 2002) have been done to evaluate the *T*-stress using BEM analysis.

The analytical formulation of the BEM begins from the differential equations for equilibrium and converts them into integral equations by the use of a fundamental solution and a reciprocal theorem. For the sake of brief, the indicial notation will be used hereafter. In this notation, subscript indices (1,2,3) are used to replace the coordinates variables in the (*x,y,z*) co-ordinate system, so the *x,y* and *z* become x_1 , x_2 and x_3 .

In this thesis, only the two-dimensional isotropic, linear elastostatic problem would be considered. Define an isotropic linear body R , with boundary S shown in Fig. 2.3. Starting with the Navier's equations of equilibrium, they are transformed into integral equations with the use of fundamental solutions and Betti-Rayleigh's reciprocal work theorem. Following the usual limiting process, the final boundary integral equation for elastostatics in the absence of body force can be written as, (see Tan (1987))

$$C_{ji}(P)u_i(P) + \int_S u_i(Q)T_{ji}(P,Q)dS(Q) = \int_S t_i(Q)U_{ji}(P,Q)dS(Q) \quad (2.6)$$

where $U_{ji}(P,Q)$ and $T_{ji}(P,Q)$ are the Kelvin's fundamental solutions for displacement and traction to Navier's equation, respectively. More specifically, $U_{ji}(P,Q)$ is the displacement in the x_j -direction at $Q(x)$ due to the application of a unit concentrated load in the x_i -direction at $P(x)$. Similarly, $T_{ji}(P,Q)$ is the traction in the x_j -direction at $Q(x)$ due to the application of a unit concentrated load in the x_i -direction at $P(x)$. $u_i(P,Q)$ and $t_i(P,Q)$ are the unknown displacements and tractions respectively; and:

$$C_{ji}(P) = \lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon} T_{ji}(P,Q)dS(Q) \quad (2.7)$$

where $C_{ji}(P)$ depends on the local geometry of the surface at point P , and S_ε is the boundary of the small region of exclusion around P .

Equation (2.6) is an integral equation relating the boundary tractions to the boundary displacements. In general, it is difficult to analytically solve this equation. Therefore, a numerical scheme must be employed. To solve the integral equation numerically, the boundary S of the solution domain is represented by a sequence of M line elements, as shown in Fig. 2.4. Following the experience of finite element studies, the quadratic isoparametric element, as shown in Fig. 2.5, is employed to describe the variation of the element geometry and functions. The numerical procedure has been presented by Tan (1987). The quadratic

element is defined by three equally spaced nodes with intrinsic coordinates $\xi=-1$, $\xi=0$ and $\xi=1$ respectively. The shape functions for these elements are

$$\begin{aligned} N^{(1)}(\xi) &= -\frac{1}{2}\xi(1-\xi) \\ N^{(2)}(\xi) &= 1-\xi^2 \\ N^{(3)}(\xi) &= \frac{1}{2}\xi(1+\xi) \end{aligned} \quad -1 \leq \xi \leq 1 \quad (2.8)$$

where the superscripts on the shape function $N^{(i)}(\xi)$ represents the local nodes of the element.

The geometric parameters, displacements and tractions can be expressed in terms of the shape functions and nodal values as follows

$$\begin{aligned} x_j(\xi) &= N^c(\xi)x_j^c \\ u_j(\xi) &= N^c(\xi)u_j^c \\ t_j(\xi) &= N^c(\xi)t_j^c \end{aligned} \quad c = 1, 3 \quad (2.9)$$

Substituting Eqn. (2.9) into Eqn. (2.6), the discretized form of the boundary integral equation can then be written as

$$\begin{aligned} & C_{ji}(P^a)u_i(P^a) + \sum_{b=1}^M \sum_{c=1}^3 u_i(P^{d(b,c)}) \int_S T_{ji}(P^a, Q) N^c(\xi) |J(\xi)| dS \\ &= \sum_{b=1}^M \sum_{c=1}^3 t_i(P^{d(b,c)}) \int_S U_{ji}(P^a, Q) N^c(\xi) |J(\xi)| dS \\ & P^a = 1, N \end{aligned} \quad (2.10)$$

this equation gives a total of $4M$ linear algebraic equations for the unknown displacements and tractions at the boundary nodes of the problem domain, where

M =number of element in the mesh;

N =number of nodes ($2M$);

$p^{d(b,c)} = c^{\text{th}}$ node of the b^{th} element;

$b = 1, M$;

$c = 1, 3$; and

$|J(\zeta)| = \text{Jacobian transformation.}$

Standard matrix methods can then be used to solve the algebraic equations for the unknown displacements and tractions on the boundary S .

2.2.1 Direct Approach for *T*-stress Evaluation Using BEM

A simple formula for obtaining the elastic *T*-stress using boundary element method (BEM) fracture mechanics analysis is presented by Tan and Wang (2002). It is obtained by comparing the displacement variation on the ‘quarter-point crack-tip boundary element’ with the classical field solutions, which will be explained below.

The displacements and stress field in the vicinity of crack tip are proportional to $\sqrt{r}$ and $1/\sqrt{r}$, respectively. In order to obtain the correct variation of the field parameters, alternative shape functions have been introduced by Yamada, et al. (1979), Blackburn (1977) and Akin (1976) respectively. Henshell and Shaw (1975) and Barsoum (1976) have shown, however, that by moving the mid-side node of a quadratic element to a quarter point position as shown in Fig. 2.6, the desired $\sqrt{r}$ variation for displacement can be achieved.

Let $x^A=0$, $x^B=L/4$ and $x^C=L$ (for $y=0$) where L is the length of the element. Superscripts denote the nodes in Fig. 2.6. Substituting Eqn. (2.8) into Eqn. (2.9), the coordinate x becomes

$$x = \frac{1}{2}\xi(1+\xi) \cdot L + \frac{1}{4}(1-\xi^2) \quad (2.11)$$

Expressing ξ in terms of x , result in the following:

$$\xi = -1 + 2\sqrt{\frac{x}{L}} = -1 + 2\sqrt{\frac{r}{L}} \quad (2.12)$$

Substituting Eqn. (2.12) again into Eqn. (2.9), the displacements along a quarter-point element can be written as

$$u_i = u_i^A + (-3u_i^A + 4u_i^B - u_i^C)\sqrt{\frac{r}{L}} + 2(u_i^A - 2u_i^B + u_i^C) \cdot \frac{r}{L} \quad (2.13)$$

Williams' (1957) series expansion (Eqn. (2.2)) represents the classical solution for the elastic stress in the vicinity of a crack-tip. The corresponding displacement can be written, for plane strain, as

$$u_1 = \frac{K_I}{4\mu} \sqrt{\frac{r}{2\pi}} f_1(\theta) + (1-\nu^2) \cdot \frac{T \cdot r}{E} \cos \theta \quad (2.14a)$$

$$u_2 = \frac{K_I}{4\mu} \sqrt{\frac{r}{2\pi}} f_2(\theta) - \nu(1+\nu) \cdot \frac{T \cdot r}{E} \sin \theta \quad (2.14b)$$

where μ is the shear modulus, E is the Young's modulus and ν is poisson's ratio.

The displacement formula for the computation of the stress intensity factor K_I have been obtained by equating the coefficient of the $\sqrt{r}$ term in Eqn. (2.13) to the classical solutions in Eqn. (2.14a, b). An analogous formula can be obtained for elastic T -stress by comparing the coefficient of r term in Eqn. (2.13) with that in Eqn. (2.14a). The displacements of the quarter-point crack-tip element on the crack face with $\theta=180^\circ$ can be written as

$$2(u_i^A - 2u_i^B + u_i^C) \frac{1}{L} = -(1-\nu^2) \cdot \frac{T}{E} \quad (2.15)$$

which results in

$$T = -\frac{2E}{(1-\nu^2) \cdot L} (u_i^A - 2u_i^B + u_i^C) \quad (2.16)$$

the *T*-stress can then be evaluated directly from BEM computed nodal displacements on crack face quarter point element.

2.2.2 Contour Integral Approach for *T*-stress Evaluation Using BEM

An alternative approach of obtaining the elastic *T*-stress was introduced by Sladek, et al. (1997). This is based on Betti-Rayleigh's reciprocal work theorem and some auxiliary fields. A contour integral formula is obtained which can evaluate *T*-stress along a closed path remote from the crack-tip, thus circumventing the more difficult task of establishing the accurate crack-tip fields and eliminating the numerical errors for near field solution associated with the crack-tip stress singularity. The required field variables along the integration path are obtained from the BEM analysis.

Consider a cracked isotropic, elastic domain *R* shown in Fig. 2.7 enclosed by the boundary *S*. Inside this domain, a closed integration path composed of Γ_o , Γ_c^+ and Γ_c^- is considered. For two sets of equilibrium states of the sub-domain, using Gauss's divergence theorem, Hooke's law and strain-displacement relation equations, Betti-Rayleigh's reciprocal work theorem can be written as

$$\int_{\Gamma} (\sigma_{ij} u_i' n_j - \sigma_{ij}' u_i n_j) \cdot d\Gamma = \int_{\Omega} (X_i' u_i - X_i u_i') \cdot d\Omega \quad (2.17)$$

where X_i and X_i' are body forces in two load states respectively; and n_j is the outward normal at the contour Γ of integration sub-domain Ω .

Because of the stress singularity at the crack-tip, a small circular region bounded by Γ_ε in the vicinity of the crack-tip has to be excluded, as shown in Fig. 2.7. The radius ε is considered to be very small and is shrunk to zero in the limiting process. Contour

$\Gamma = \Gamma_o + \Gamma_c^+ + \Gamma_c^- - \Gamma_\varepsilon$ is a closed integration path in the anticlockwise direction.

With no loss of generality, assume that in Eqn. (2.17), the primed state corresponds to an auxiliary field. The non-primed state corresponds to the unknown field. Assume an auxiliary field where $\sigma_{ij}' = 0$ on crack faces and body forces $X_i' = 0$. Eqn. (2.17) can then be written as

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (\sigma_{ij}' u_i n_j - \sigma_{ij} u_i' n_j) \cdot d\Gamma \\ &= \int_{\Gamma_o} (\sigma_{ij}' u_i n_j - \sigma_{ij} u_i' n_j) \cdot d\Gamma - \int_{\Gamma_c^+} (\sigma_{ij}^+ u_i' n_j^+ - \sigma_{ij}^- u_i' n_j^-) \cdot d\Gamma - \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_\varepsilon} X_i u_i' \cdot d\Omega \end{aligned} \quad (2.18)$$

where n_j^+ and n_j^- are outward normal vectors on upper and lower crack face respectively, and $n_j^+ = -n_j^-$. For small equilibrium stress loads, assume $\sigma_{ij}^+ = \sigma_{ij}^-$. Using the familiar relationship between traction and stresses, $t_i = \sigma_{ij} \cdot n_j$, Eqn. (2.18) can be re-written as

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (t_i' u_i - t_i u_i') \cdot d\Gamma \\ &= \int_{\Gamma_o} (t_i' u_i - t_i u_i') \cdot d\Gamma - \int_{\Gamma_c^+} 2t_i^+ u_i' \cdot d\Gamma - \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_\varepsilon} X_i u_i' \cdot d\Omega \end{aligned} \quad (2.19)$$

By choosing different auxiliary fields, the Eqn. (2.19) can be employed to derive integral formulas for computing various fracture parameters. For instance, Bueckner (1989) selected singular auxiliary fields to derive the integral expression for stress intensity factor K_I . In order to obtain a non-vanishing contribution of the elastic *T*-stress, and at the same time, eliminate the contribution due to singular integrand term, a special auxiliary field solution has to be found. Sladek, et al. (1997) suggested an auxiliary field that has one order higher singularity than that used by Kfoury (1986). The auxiliary displacements and tractions

are proportional to r^{-1} and r^{-2} respectively; they are actually obtained by differentiating auxiliary field proposed by Kfoury, with respect to x_I . The exact expression of the auxiliary field is,

$$\begin{aligned}
 u_1'(r, \theta) &= -\frac{f}{\pi E r} (1 - \nu^2) \cdot \left(\cos \theta - \frac{1}{1 - \nu} \sin^2 \theta \cos \theta \right) \\
 u_2'(r, \theta) &= -\frac{f}{2\pi E r} (1 + \nu) \cdot (1 - 2\nu - \cos 2\theta) \cdot (-\sin \theta) \\
 \sigma_{11}'(r, \theta) &= \frac{f}{\pi r^2} \cos^2 \theta \cdot (\cos^2 \theta - 3 \sin^2 \theta) \\
 \sigma_{12}'(r, \theta) &= \frac{f}{\pi r^2} \sin 2\theta \cdot \cos 2\theta \\
 \sigma_{22}'(r, \theta) &= \frac{f}{\pi r^2} \sin^2 \theta \cdot (3 \cos^2 \theta - \sin^2 \theta)
 \end{aligned} \tag{2.20}$$

where f is a static point force (see Fig.2.7) applied to the crack tip in the direction parallel to the crack plane.

The unknown asymptotic displacements and stresses can be divided into two parts

$$u_i = u_i^s + u_i^T \tag{2.21}$$

$$\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^T \tag{2.22}$$

where terms with superscript s are associated with the singular stress field and they correspond to first terms of Eqn. (2.14a,b) and Eqn. (2.3), respectively; those with superscript T are

$$\sigma_{ij}^T = T \cdot \delta_{i1} \delta_{j1} \tag{2.23}$$

$$u_i^T = \frac{T \cdot r}{E} [\delta_{i1} (1 - \nu^2) \cos \theta - \delta_{i2} \nu (1 + \nu) \sin \theta] \tag{2.24}$$

Substituting the above auxiliary field solution and second term of Williams series expansion (T -stress term) into the left hand side of Eqn. (2.19), gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (t_i' u_i^T - t_i^T u_i') \cdot d\Gamma = \frac{1-\nu^2}{E} T \cdot f \quad (2.25)$$

Substituting Eqn. (2.20) and the singular terms u_i^S and t_i^S into the left hand side of Eqn. (2.19), gives a vanishing contribution as

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (t_i' u_i^S - t_i^S u_i') \cdot d\Gamma = 0 \quad (2.26)$$

By further substituting Eqn. (2.25) and Eqn. (2.26) into Eqn. (2.19), the integral representation of T -stress can finally be derived as follows

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} (t_i' u_i - t_i u_i') \cdot d\Gamma - \frac{2E}{f(1-\nu^2)} \int_{\Gamma_0^+} t_i^+ u_i' \cdot d\Gamma \quad (2.27)$$

when body forces are absent.

Furthermore, it can be proved that for mode I cracked problems, even with non-zero crack face traction in the x_2 direction, the second term of Eqn. (2.27) is always zero. So the general contour integral formula for T -stress, in the absence of body force, can then be expressed as

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} (t_i' u_i - t_i u_i') \cdot d\Gamma \quad (2.28)$$

It would be more explicit if Eqn. (2.20) is substituted into the above equation. The T -stress expression is as follows

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} \frac{f}{\pi a^2} (F_{ij}(\theta) u_i - t_i \frac{F_{ij}(\theta)}{E}) \cdot d\Gamma \quad (2.29)$$

where the function $F_{ij}(\theta)$ contain trigonometric functions of the angular location.

The above integral expression can be numerically evaluated by using Gaussian quadrature scheme. The corresponding nodal parameters u_i and t_i along the integration path can be obtained from BEM analysis.

2.3 Weight Function Method

Complex loading conditions are often encountered in engineering applications due to the effects of residual stresses, stress concentration, thermal stresses and other non-linear stress fields. The weight function method has been established to be one of the most powerful techniques to obtain the stress intensity factors for such problems. Once the weight function is known for a particular cracked geometry, the stress intensity factor can be obtained for any arbitrary stress distribution by appropriate integration. Recently, this technique has also been extended to obtain the *T*-stress under complex loading situations by Fett (1997) and Wang (2002). In this section, the theoretical background of the weight function method for stress intensity factor is reviewed, then the weight function methods for *T*-stress calculation is summarized.

2.3.1 Weight function for stress intensity factors

Bueckner (1970) first demonstrated that for a cracked body as shown in Fig.2.8(a), loaded by stress field S , the stress intensity factor for this problem is equivalent to the same cracked body loaded by a crack face pressure, $\sigma(x)$, as shown in 2.8(b). The stress distribution $\sigma(x)$ is the stress distribution acting on the prospective crack face for the

uncracked body. Therefore the calculation of stress intensity factor for a specific geometry under any applied load can be converted to the calculation of same cracked body with the corresponding crack pressure $\sigma(x)$ acting on the crack face. The stress intensity factor for a cracked body with loading on the crack face can be calculated by integrating the product of the weight function $m(x,a)$ and the stress distribution $\sigma(x)$ on the crack face,

$$K = \int_0^a \sigma(x) \cdot m(x,a) dx \quad (2.30)$$

The weight function $m(x,a)$ depends only on the geometry of the cracked body. Once the weight function has been determined, the stress intensity factor for this specific geometry can be obtained for any stress distribution $\sigma(x)$ from Eqn. (2.30).

Mathematically, the weight function $m(x,a)$ is a generalization of Green's function for the present stress intensity factor problem. A Green's function is defined as the response of a system to an impulse. Correspondingly, the weight function is the stress intensity factor due to a pair of unit point loads acting on the crack face as shown in Fig. 2.8(c).

The determination of weight function $m(x,a)$ was simplified by Rice (1972) using the following relationship:

$$m(x,a) = \frac{H}{K_r} \cdot \frac{\partial u_r(x,a)}{\partial a} \quad (2.31)$$

where H is the generalized elastic modulus, which equals to Young's modulus E for plane stress or $E/(1-\nu^2)$ for plane strain. In order to determine the weight function from Eqn. (2.31), a reference intensity factor K_r for a specific geometry and applied stress system has to be obtained in addition to the corresponding crack opening displacements field $u_r(x,a)$. Due to the difficulties in obtaining crack opening displacements function $u_r(x,a)$, several authors have proposed approximate expressions for the crack opening displacements (Petroski and

Achenbach, 1978) and weight functions. Shen and Glinka (1991) found that the weight function of the following form could be used to approximate the weight functions for several one-dimensional edge and central cracks,

$$m(x, a) = \frac{2}{\sqrt{2\pi(a-x)}} \cdot \left[1 + M_1 \left(1 - \frac{x}{a}\right)^{1/2} + M_2 \left(1 - \frac{x}{a}\right) + M_3 \left(1 - \frac{x}{a}\right)^{3/2} \right] \quad (2.32)$$

where M_1 , M_2 and M_3 are constant depending on the geometry of the cracked body. By using this general expression, the determination of weight function reduces to the determination of three parameters M_1 , M_2 and M_3 . If two reference stress intensity factors are known, these three parameters can be obtained with the help of an additional condition.

2.3.2 Weight function for *T*- stress

For mode I crack problems, using the superposition method, it has been demonstrated by Wang (2002) that for a cracked body as shown in Fig. 2.9(a), loaded by stress field Q , the *T*-stress is the superposition of *T*-stresses for two cases. The first is the *T*-stress for the same cracked body loaded by the crack face pressure $\sigma(x)$, which is induced by the stress field Q in the uncracked body, as shown in Fig. 2.9(b). The second case is the *T*-stress in the uncracked body under stress field Q , as shown in Fig. 2.9(c). Therefore the *T*-stress for the problem 2.9(a) can be calculated from the summation of the *T*-stress for these two sub-problems,

$$T_{remote\ loading} = T_{crack\ face\ pressure} + T_{uncracked} \quad (2.33)$$

Note the regular stress field (Fig. 2.9(c)) has no singularity at the crack tip, the corresponding stress intensity factor is zero. However, the *T*-stress has a finite value. Wang (2002) has shown that the corresponding *T*-stress for the regular field is,

$$T_{uncracked} = (\sigma_x - \sigma_y)_{crack\ tip} \quad (2.34)$$

The T -stress for a cracked body with loading applied to the crack face can again be calculated by integrating the product of the weight function $t(x,a)$ and the stress distribution $\sigma(x)$ on the crack face,

$$T_{\text{crack face pressure}} = \int_0^a \sigma(x) \cdot t(x,a) dx \quad (2.35)$$

where $t(x,a)$ is the weight function for T -stress. Similar to the weight functions for the stress intensity factor, the weight function for T -stress is also only dependent on crack geometry and is independent of the loading conditions. The weight function $t(x,a)$, is the Green's function for the T -stress. It represents the T -stress at the crack tip for a pair of unit loads acting along the crack face as the location x as shown in Fig. 2.9(d).

Substituting Eqn. (2.34) and Eqn. (2.35) into Eqn. (2.33), the T -stress for the cracked body as shown in Fig. 2.9(a), loaded by a stress field Q , is obtained,

$$T_{\text{remote loading}} = \int_0^a \sigma(x) \cdot t(x,a) dx + (\sigma_x - \sigma_y)|_{\text{crack tip}} \quad (2.36)$$

Eqn. (2.36) provides a very efficient way for T -stress calculation for arbitrary load situations. Once the weight function $t(x,a)$ is determined, and a stress analysis of uncracked body is conducted to obtain $(\sigma_x - \sigma_y)|_{\text{crack tip}}$ and $\sigma(x)$, the corresponding T -stress can be calculated for any arbitrary loading conditions. Several authors have proposed different approaches to obtain weight functions for T -stress. For example, Sham (1991) has provided numerical methods for the determinations of weight functions for T -stress. Fett (1997), Hooton. et al. (1998) and Wang (2002) proposed different methods for the derivation of weight functions from reference T -stress solutions. Here, the approach of using the two-term approximation by Wang (2002) is adopted.

Based on the analogy to the stress intensity factor, the following two-term weight function expression has been proved by Wang (2002) as an excellent approximation for one-dimensional *T*-stress weight functions,

$$t(x, a) = \frac{2}{\pi a} \cdot \left[D_1 \left(1 - \frac{x}{a}\right)^{1/2} + D_2 \left(1 - \frac{x}{a}\right)^{3/2} \right] \quad (2.37)$$

where D_1 and D_2 can be determined from two reference *T*-stress solutions. This two-term weight function formula is used in this thesis in conjunction with BEM results to obtain *T*-stress for non-linear distributed load.

2.4 Closing Remarks

The theoretical background of *T*-stress, the boundary element method (BEM) and the weight function method have been reviewed. The formulations for the direct approach and contour integral approach for determining the *T*-stress and for the weight function predictions have also been derived in this chapter.

Fig. 2.1 Three basic modes of fracture

Fig. 2.2 Three-dimensional coordinate system for the region at the crack tip

Fig. 2.3 Linear elastic body R with boundary S in a two-dimensional coordinate system

Fig. 2.4 Discretized solution domain R , with boundary S

Fig. 2.5 A quadratic isoparametric line element

Fig. 2.6 Node symbols for quarter-point crack tip element

Fig. 2.7 Integration paths and coordinate definitions

Fig.2.8 Weight Function for one-dimensional cracks

Fig. 2.9 *T*-stress weight function

CHAPTER 3

TEST PROBLEMS AND VALIDATIONS

3.0 Introduction

The purpose of this chapter is to verify the techniques for calculating T -stress using BEM and to develop some mesh design guidelines for numerical modelling. This chapter presents the solutions of the two-dimensional BEM T -stress analysis, including the direct approach and the contour integral approach as described in Chapter 2, to three test specimens for different relative crack length and relative crack tip element sizes. The specimens are the central cracked plate (CCP) specimen, the double edge cracked plate (DECP) specimen and single edge cracked plate (SECP) specimen. In addition, the case of a crack in an infinite plate was also analysed where exact solutions are available. All BEM analyses were carried out using the computer program BIE2DSV, which was developed by Professor C.L Tan at Carleton University. Two separate Fortran subroutines for calculating the T -stress directly based on the above two BEM approaches were developed. These T -stress solutions are compared with the available exact solution (e.g. Chen, 2000) and other solutions obtained by Wang (2002) using the two dimensional FEM analysis.

3.1 Crack in an Infinite Plate

3.1.1 Problem Definition and Numerical Modelling

In order to verify the boundary element model, the T -stress was obtained for a crack in an infinite plate with different relative crack tip element size l/a and different relative contour radius r/a under remote tension, which is applied perpendicular to the crack face (see Fig.3.1). Plane strain condition is assumed. For the numerical analysis, a large square plate with relative crack length $a/W=0.01$ was employed to model the infinite plate problem. Only one-quarter of the plate needs to be modelled by virtue of symmetry. The two-dimensional standard quadratic isoparametric elements were used to model the boundary of the solution domain, except for the crack tip elements. Conventional and traction singular quarter-point elements were used at the crack tip. The size of the boundary elements was gradually increased away from crack tip. The BEM mesh is shown in Fig. 3.2. For the contour integral approach, an extra closed integration path was needed. The integration can be numerically evaluated using Gaussian quadrature scheme. In this case, eight discretized circular elements and the second order Gaussian quadrature were used in the numerical integration. Relatively small boundary elements were chosen for elements near the integral contour to reduce the numerical errors caused by the variation of the kernel of the integral equations. The boundary conditions applied to the numerical model are that the elements at the uncracked ligament were constrained in the x_2 direction but were free to displace along x_1 -direction; whereas the elements along the vertical plane of symmetry were constrained in the x_1 -direction but were free to displace along x_2 -direction.

3.1.2 Results and Discussion

The results were compared with the exact solution by Chen (2000) using complex potential theory. The T -stress results of these calculations are presented in Table 3.1 and 3.2; and plotted graphically in Figs. 3.3 and 3.4, respectively. From the comparison of the BEM results and the exact solution, it can be seen that by using the direct approach, the T -stress percentage error is around 1% when the relative crack tip element size (l/a) is in the range of 0.025 to 0.1. For the BEM contour integral approach, the discrepancies are around 0.2% if the relative contour radius (r/a) is equal to 0.5. These results validate the effectiveness of the BEM model and indicate that contour integral approach is a more accurate and reliable approach in BEM T -stress analysis.

3.2 Central Cracked Plate (CCP) Specimen

3.2.1 Problem Definition

Two-dimensional BEM analysis, and using the direct approach and contour integral approach for evaluating the T -stress, is carried out for a rectangular plate containing a central flat crack. The geometry and coordinate system used are shown in Fig. 3.5. The geometry of the rectangular plate in the x_1 - x_2 plane was defined by height-to-width ratio, H/W , and relative crack length a/W .

For the purpose of comparison with other numerical results in the literature, the plane strain condition and a fixed H/W ratio of three is assumed in the analysis. The relative crack length was varied from $a/W = 0.1$ to $a/W = 0.6$. Different load conditions were investigated. First, a remote tension, σ_o , in the direction perpendicular to crack face was applied. Next, the crack face loading conditions with the following stress distributions:

$$\sigma(x_1) = \sigma_o \left(1 - \frac{x_1}{a}\right)^n \quad (3.1)$$

were analysed, where σ_o is the nominal stress, a is the crack length, and n equals 0,1,2 or 3. These four types of loads are shown in Fig. 3.6.

3.2.2 Numerical Modelling

For this specific two-dimensional geometry, because of the symmetry, only one-quarter of the specimen needs to be modelled. A typical BEM model is presented in Fig. 3.7. The boundary condition applied is exactly the same as the one described in the infinite plate case.

The present analysis employed 2-D quadratic isoparametric elements over the entire boundary. For the elements adjacent to the crack tip, conventional quarter-point and traction-singular quarter-point elements are employed to simulate the singular stress field at the crack tip. It has been noted that relative crack tip element size, l/a , has a major effect on the accuracy of the BEM direct approach analysis, and this was to be examined as well.

For the contour integral approach, an integration path, Γ_o , with a specific relative radius r/a was applied to calculate the T -stress. The integration can be calculated by discretizing the circular path into a number of elements and employing Gaussian quadrature scheme. To verify the independence of the path of the contour, rectangular contours with different relative height ratio h/a as shown in Fig. 3.7 were also tested, and the results were compared with circular contour integral solutions with the same r/a ratio. In the contour integral approach, besides the crack tip element size, the relative contour radius r/a may also have an effect on the accuracy of BEM analysis.

Following the previous work of Tan and Wang (2003), a crack tip element size, l , of 5% of the crack length a was tested first. Then a wide range of crack tip element size was employed to validate the accuracy of the results. The meshing guideline for the elements away from crack tip element is that the size of the elements range from approximately $1\frac{1}{2}$ to 2 times the size of the previous element, except for those boundary elements adjacent to the integral contour. In those specific locations, special consideration should be given to eliminate the effect of rapid integrand variation. For this reason, a relative small mesh size was chosen to reduce numerical errors.

3.2.3 Results and Discussions

For verification of the BEM analysis and mesh design, the T -stresses for centrally cracked finite width plate with different l/a and different r/a ratios under remote tension were calculated and compared with the corresponding FEM results from Wang (2002) who used the interaction integral method.

For the remote tension loading case, the percentage discrepancies between BEM solutions for the direct approach and the FEM results (Wang, 2002) for normalized T -stress (T/σ_0) are presented in Table 3.3 to 3.7 and also plotted graphically in Figs. 3.8 to 3.12, respectively. The range of relative crack length $a/W=0.2$ to 0.6 were chosen to correspond to the same range of FEM calculation by Wang (2002). Also shown are the corresponding BEM results for the normalized stress intensity factor, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$, and comparison with those given by Rooke (1976). The comparison indicated that the BEM T -stress solutions are more sensitive to crack tip element size when compared to the stress intensity factor solutions. For example, when $a/W=0.4$, the T -stress percentage discrepancies changed about 2.5% when

crack tip element size varied from $l/a=0.05$ to $l/a=0.2$; while the percentage discrepancies for the stress intensity factor only changed by about 1%, with the same crack tip element size variation. These figures also indicated an unusual discrepancy when the crack tip element size is less than $l/a=0.025$; this has been reported elsewhere (Aliabadi and Rooke (1991)). Generally speaking, in order to get a same T -stress percentage discrepancy level compared to stress intensity factor solutions, a relatively small crack tip element size should be employed in BEM modelling. There is a trade-off between calculation efficiency and accuracy. A crack tip element size of 5% of crack length is generally adopted in the present work and was proved to be effective and sufficiently accurate for all of the test cases.

To validate the BEM contour integral model, different contour geometries (circular and rectangular) and different contour sizes were employed to test the accuracy and contour independence of this approach. As an example, a specimen with relative crack size $a/W=0.4$ under remote tension was investigated. All the calculation results are presented in Table 3.8 and Fig. 3.13. The discrepancies of the results for different contour sizes and geometries for the T -stress were less than 1%. Comparing the BEM results with those obtained by Wang (2002), the biggest percentage difference was 1.8% when using the contour integral approach. Generally, by choosing a circular contour with a radius equal to half of the crack length, excellent agreement, with difference of less than 1%, can be obtained in BEM with the contour integral analysis.

The crack face loading case is also an important test problem for the cracked specimen. Weight functions can be derived using crack face loading results as reference T -stresses, and more complicated stress distribution case can then be calculated by integrating with the corresponding weight function. For uniform, linear, parabolic and cubic crack face

stress distribution indicated by Eqn. (3.1), the corresponding BEM contour integral T -stress solutions with the relative crack length range a/W from 0.1 to 0.6 are shown in Table 3.9 to 3.12 and Figs. 3.14 to 3.17, respectively. These results are compared with the FEM results from Wang (2002). It can be seen that except for those points for shallow crack length, which involved the normalized T -stress values very close to zero, the percentage difference between the BEM contour integral solutions and Wang's (2002) FEM solutions are all less than 3.1%.

3.3 Double Edge Cracked Plate (DECP) Specimen

3.3.1 Problem Definition and Numerical Modelling

The geometry and coordinate system used for double edge cracked plate (DECP) specimen are shown in Fig. 3.18. The geometry of the rectangular plate was defined by height-to-width ratio, H/W , and relative crack length a/W . As before, plane strain conditions and a fixed H/W ratio of three were assumed in all the analysis and the relative crack length was varied from $a/W = 0.1$ to $a/W = 0.6$.

The rectangular plate was subjected, first, to remote tension, σ_0 , in the direction perpendicular to crack face; and then to the crack face loading conditions. For the DECP specimen, again, due to symmetry, only one-quarter of the specimen needs to be modelled. The BEM model is exactly the same as that for the CCP specimen in Fig. 3.7, except for the boundary conditions. The boundary condition applied is that the elements in the uncracked ligament were constrained in the x_2 direction but were allowed to displace along x_1 direction. All the elements in the x_2 -direction at $x_1 = W$ were constrained in the x_1 direction because of symmetry, but were free to move in x_2 direction.

It has been verified in Section 3.2.2 that relative crack tip element size, l/a , has an effect on the accuracy of BEM direct analysis. Following the same procedure for the CCP specimen, a wide range of crack tip element sizes was employed to validate the accuracy of the results. For the contour integral approach, as concluded in Section 3.2.2, circular integration paths, Γ_o , with r/a ratios equals to 0.5 were used to calculate the T -stress

3.3.2 Results and Discussions

The T -stress results obtained from the BEM analysis were compared with the corresponding FEM results by Wang (2002). For validation of the BEM T -stress approach and mesh design, the T -stresses for doubled edge cracked finite width plate under remote tension was first calculated and compared with the corresponding FEM results by Wang (2002). A range of relative crack tip element size l/a was considered, as before, for a given relative crack length a/W which was varied from 0.2 to 0.6. The percentage discrepancies between the BEM solutions using the direct approach and the FEM results (Wang, 2002) for the normalized T -stress, T/σ_o , are presented in Tables 3.13 to 3.17. These results are also shown graphically in Figs. 3.19 to 3.23, respectively. The corresponding results for the normalized stress intensity factor, $\frac{K_I}{\sigma_o \sqrt{\pi a}}$, are also shown. The BEM T -stress solutions can

be seen to be more sensitive to the crack tip element size when compared with the case of the CCP specimen. The percentage discrepancies for the T -stress changed about 30% when the crack tip element size varied from $l/a=0.01$ to $l/a=0.4$; while those for the stress intensity factor only by changed about 3%, for the same crack tip element size variation. It was also observed that the percentage discrepancy increases when the crack tip element size is below $l/w=0.025$. Thus, a crack tip element size of 5% of crack length is generally adopted. For

crack face loading cases, using BEM contour integral analysis, the T -stress under four loading conditions, that is, uniform, linear, parabolic and cubic crack face stress distribution indicated by Eqn. (3.1), were calculated and compared with FEM results from Wang (2002). The corresponding results for the different relative crack lengths treated, namely, $a/W=0.1$ to 0.6, are shown in Tables 3.18 to 3.21 and Figs. 3.24 to 3.27, respectively. It can be seen that excellent agreements were generally obtained. The biggest percentage difference between the BEM contour integral solutions and the FEM solutions is only 2.3%.

3.4 Single Edge Cracked Plate (SECP) Specimen

3.4.1 Problem Definition and Numerical Modelling

Figure 3.28 shows the geometry of the single edge cracked plate (SECP) specimen analysed. As in the previous specimen treated, H/W was fixed at three and the relative crack length a/W was varied from 0.1 to 0.6. Plane strain conditions were again assumed and the load cases considered were the same as that considered for the previous test specimen. By virtue of symmetry, only one half of the specimen needs to be modelled. The BEM models used was exactly the same as those used for CCP and DECP specimens, except for the boundary conditions. The boundary conditions applied were as follows: the elements in the uncracked ligament were constrained in the x_2 direction, but were free to displace in the x_1 direction. All the elements along the x_2 direction were free to displace in both x_1 and x_2 direction. To preclude rigid body motion in the x_1 direction, one node along the plane of symmetry was constrained in that direction.

Again, following the procedures used for previous specimens, a wide range of crack tip element sizes was tested to validate the accuracy of the results. For the contour integral

approach, as treated in the previous specimens, circular integration paths with r/a ratio equals to 0.5 was used to calculate the T -stress.

3.4.2 Results and Discussions

Again, the T -stresses for single edge cracked finite wide plate with different relative crack tip element size l/a under remote tension was first calculated and compared with the corresponding FEM results by Wang (2002) to verify the BEM T -stress approach and mesh design. The percentage discrepancies between the BEM solutions using the direct approach and the FEM results (Wang (2002)) for normalized T -stress, T/σ_o , with a/W varying from 0.2 to 0.6 are presented in Tables 3.22 to 3.26. These results are also shown graphically in Figs. 3.29 to 3.32, respectively. The corresponding results for the normalized stress intensity factor, $\frac{K_I}{\sigma_o \sqrt{\pi a}}$, are also shown. As before, these figures show that the T -stress is sensitive to the relative size of the crack-tip element. For crack face loading cases, by using a r/a ratio of 0.5, the T -stress under four loading conditions were again calculated and compared with FEM results by Wang (2002). The corresponding results with the a/W ratio from 0.1 to 0.6 were shown in Tables 3.27 to 3.30 and Figs. 33 to 36, respectively. Except for a few points, good agreement is generally obtained; the biggest percentage difference was 8.3%.

3.5 Conclusion

The two-dimensional BEM technique has been used for calculating the T -stress in an infinite plate and three finite-width plate specimens. The relative crack tip element length ratio l/a and relative contour radius ratio r/a were recognized as important factors which affect the accuracy of BEM modelling. The direct approach provides an easy method to

extract T -stress. The contour integral approach has been proved to be a more accurate and reliable approach in BEM T -stress analysis. Some general meshing guidelines, i.e., the crack tip element size equals to 5% of crack length for direct approach and contour radius equals to half of crack length for contour integral approach have been established. The BEM analyses were validated with exact and FEM solutions (Wang, 2002) and excellent agreements were generally achieved.

l/a	Exact	BEM (Direct Approach)	% Difference
0.010	-1.000	-1.043	4.31
0.025	-1.000	-1.002	0.20
0.050	-1.000	-0.988	1.18
0.075	-1.000	-0.995	0.49
0.100	-1.000	-0.992	0.80
0.200	-1.000	-0.978	2.20
0.400	-1.000	-0.956	4.38

Table 3.1 Comparison of T -stresses from BEM direct approach with exact solutions for the central crack in an infinite plate under remote tension

l/a	Exact	BEM (Contour Integral Approach)	% Difference
0.250	-1.000	-0.988	1.23
0.375	-1.000	-0.993	0.68
0.500	-1.000	-0.998	0.19
0.625	-1.000	-0.998	0.18
0.750	-1.000	-0.999	0.11

Table 3.2 Comparison of T -stresses from BEM contour integral approach with exact solutions for the central crack in an infinite plate under remote tension

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-1.075	-1.029	-4.47	1.009	1.02	1.08
0.025	-1.031	-1.029	-0.19	1.027	1.02	0.69
0.050	-1.022	-1.029	-0.68	1.036	1.02	1.57
0.075	-1.016	-1.029	-1.26	1.037	1.02	1.67
0.100	-1.013	-1.029	-1.55	1.038	1.02	1.76
0.200	-0.998	-1.029	-3.01	1.037	1.02	1.67
0.400	-0.975	-1.029	-5.25	1.057	1.02	3.63

Table 3.3 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.2$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-1.142	-1.070	-6.73	1.048	1.051	0.29
0.025	-1.065	-1.070	-0.47	1.051	1.051	0.00
0.050	-1.061	-1.070	-0.84	1.067	1.051	1.52
0.075	-1.056	-1.070	-1.31	1.068	1.051	1.62
0.100	-1.051	-1.070	-1.78	1.072	1.051	2.00
0.200	-1.036	-1.070	-3.18	1.080	1.051	2.76
0.400	-1.011	-1.070	-5.51	1.099	1.051	4.57

Table 3.4 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.3$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-1.254	-1.139	-10.10	1.098	1.099	0.091
0.025	-1.137	-1.139	-0.18	1.095	1.099	0.364
0.050	-1.128	-1.139	-0.97	1.117	1.099	1.638
0.075	-1.122	-1.139	-1.49	1.118	1.099	1.729
0.100	-1.114	-1.139	-2.19	1.121	1.099	2.002
0.200	-1.099	-1.139	-3.51	1.127	1.099	2.548
0.400	-1.067	-1.139	-6.32	1.159	1.099	5.460

Table 3.5 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.4$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-1.299	-1.254	-3.59	1.140	1.176	3.061
0.025	-1.250	-1.254	-0.32	1.186	1.176	0.850
0.050	-1.238	-1.254	-1.28	1.200	1.176	2.041
0.075	-1.229	-1.254	-1.99	1.202	1.176	2.211
0.100	-1.222	-1.254	-2.55	1.203	1.176	2.296
0.200	-1.196	-1.254	-4.63	1.206	1.176	2.551
0.400	-1.154	-1.254	-7.97	1.222	1.176	3.912

Table 3.6 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central cracked plate, under remote tension; $a/w=0.5$

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-1.564	-1.448	-8.01	1.269	1.292	1.780
0.025	-1.445	-1.448	-0.21	1.303	1.292	0.851
0.050	-1.418	-1.448	-2.07	1.316	1.292	1.858
0.075	-1.404	-1.448	-3.04	1.320	1.292	2.167
0.100	-1.392	-1.448	-3.87	1.328	1.292	2.786
0.200	-1.350	-1.448	-6.77	1.326	1.292	2.632
0.400	-1.278	-1.448	-11.74	1.335	1.292	3.328

Table 3.7 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the central crack plate specimen under remote tension; $a/w=0.6$.

r/a or l/a	Rectangular Contour	Circular Contour	FEM (Wang (2002))	% Diff. for Rectangular Contour	% Diff. for Circular Contour
0.025	-1.119	-1.121	-1.139	1.79	1.61
0.050	-1.131	-1.130	-1.139	0.75	0.79
0.075	-1.135	-1.136	-1.139	0.35	0.31

Table 3.8 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang) for central crack plate specimen under different contour geometry and size; $a/w=0.4$, remote tension.

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	-0.0079	-0.007	-----
0.2	-0.0289	-0.0289	0.0
0.3	-0.0692	-0.0702	1.4
0.4	-0.1349	-0.1394	3.2
0.5	-0.2499	-0.2536	1.5
0.6	-0.4419	-0.4484	1.4

Table 3.9 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under constant crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	-0.0099	-0.004	-----
0.2	-0.0203	-0.0171	18.5
0.3	-0.0431	-0.0406	6.2
0.4	-0.0811	-0.0802	1.1
0.5	-0.1465	-0.1456	0.6
0.6	-0.2538	-0.2557	0.7

Table 3.10 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under linear crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	-0.0089		-----
0.2	-0.0100	-0.0116	14.0
0.3	-0.0305		-----
0.4	-0.0571	-0.0554	3.1
0.5	-0.1022		-----
0.6	-0.1763	-0.1771	0.5

Table 3.11 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under parabolic crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	-0.0081		-----
0.2	-0.0082	-0.0087	5.9
0.3	-0.0235		-----
0.4	-0.0448	-0.0421	6.4
0.5	-0.0787		-----
0.6	-0.1347	-0.1350	0.2

Table 3.12 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate, under cubic crack face loading

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.554	-0.531	-4.33	1.094	1.127	2.939
0.025	-0.530	-0.531	-0.19	1.109	1.127	1.600
0.050	-0.515	-0.531	-3.01	1.122	1.127	0.518
0.075	-0.503	-0.531	-5.27	1.123	1.127	0.358
0.100	-0.491	-0.531	-7.53	1.122	1.127	0.518
0.200	-0.447	-0.531	-15.82	1.138	1.127	0.910
0.400	-0.359	-0.531	-32.39	1.134	1.127	0.564

Table 3.13 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge cracked plate specimen under remote tension; $a/w=0.2$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.565	-0.536	-5.41	1.111	1.124	1.16
0.025	-0.527	-0.536	-1.68	1.108	1.124	1.42
0.050	-0.517	-0.536	-3.54	1.124	1.124	0.00
0.075	-0.505	-0.536	-5.78	1.125	1.124	0.09
0.100	-0.493	-0.536	-8.02	1.128	1.124	0.36
0.200	-0.448	-0.536	-16.42	1.130	1.124	0.53
0.400	-0.359	-0.536	-33.02	1.135	1.124	0.95

Table 3.14 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge crack plate specimen under remote tension; $a/w=0.3$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.627	-0.543	-15.47	1.130	1.132	0.18
0.025	-0.540	-0.543	-0.55	1.131	1.132	0.09
0.050	-0.525	-0.543	-3.31	1.144	1.132	1.06
0.075	-0.512	-0.543	-5.71	1.146	1.132	1.24
0.100	-0.500	-0.543	-7.92	1.145	1.132	1.15
0.200	-0.455	-0.543	-16.21	1.146	1.132	1.24
0.400	-0.364	-0.543	-32.97	1.153	1.132	1.86

Table 3.15 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge crack plate specimen under remote tension; $a/w=0.4$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.688	-0.553	-24.41	1.170	1.161	0.78
0.025	-0.547	-0.553	-1.08	1.170	1.161	0.78
0.050	-0.534	-0.553	-3.44	1.182	1.161	1.81
0.075	-0.521	-0.553	-5.79	1.184	1.161	2.01
0.100	-0.509	-0.553	-7.96	1.178	1.161	1.46
0.200	-0.464	-0.553	-16.09	1.189	1.161	2.41
0.400	-0.373	-0.553	-32.55	1.201	1.161	3.45

Table 3.16 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge crack plate specimen under remote tension; $a/w=0.5$

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.615	-0.566	-8.66	1.206	1.225	1.55
0.025	-0.592	-0.566	-4.59	1.238	1.225	1.06
0.050	-0.544	-0.566	-3.89	1.250	1.225	2.04
0.075	-0.530	-0.566	-6.36	1.254	1.225	2.37
0.100	-0.518	-0.566	-8.48	1.248	1.225	1.88
0.200	-0.474	-0.566	-16.25	1.255	1.225	2.45
0.400	-0.384	-0.566	-32.16	1.270	1.225	3.67

Table 3.17 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the double edge crack plate specimen under remote tension; $a/w=0.6$.

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	0.4611	0.4719	2.3
0.2	0.4598	0.4687	1.9
0.3	0.4590	0.4640	1.1
0.4	0.4560	0.4570	0.2
0.5	0.4485	0.4468	0.4
0.6	0.4412	0.4340	1.7

Table 3.18 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under constant crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	0.3176	0.3234	1.8
0.2	0.3140	0.3214	2.3
0.3	0.3140	0.3193	1.7
0.4	0.3145	0.3167	0.7
0.5	0.3104	0.3118	0.4
0.6	0.3100	0.3065	1.1

Table 3.19 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under linear crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang(2002))	% Difference
0.1	0.2480		
0.2	0.2430	0.2482	2.1
0.3	0.2428		
0.4	0.2435	0.2454	0.8
0.5	0.2413		
0.6	0.2426	0.2393	1.4

Table 3.20 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under parabolic crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	0.2071		
0.2	0.1991	0.2028	1.8
0.3	0.1989		
0.4	0.2025	0.2010	0.7
0.5	0.1995		
0.6	0.2004	0.1969	1.8

Table 3.21 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate, under cubic crack face loading

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.637	-0.592	-7.60	1.357	1.373	1.17
0.025	-0.601	-0.592	-1.52	1.364	1.373	0.66
0.050	-0.581	-0.592	-1.86	1.376	1.373	0.22
0.075	-0.568	-0.592	-4.05	1.380	1.373	0.51
0.100	-0.556	-0.592	-6.08	1.377	1.373	0.29
0.200	-0.510	-0.592	-13.85	1.381	1.373	0.58
0.400	-0.419	-0.592	-29.22	1.384	1.373	0.80

Table 3.22 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.2$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.672	-0.596	-12.67	1.634	1.665	1.86
0.025	-0.613	-0.596	-2.89	1.645	1.665	1.20
0.050	-0.607	-0.596	-1.88	1.669	1.665	0.24
0.075	-0.591	-0.596	-0.91	1.672	1.665	0.42
0.100	-0.586	-0.596	-1.63	1.677	1.665	0.72
0.200	-0.546	-0.596	-8.46	1.672	1.665	0.42
0.400	-0.459	-0.596	-23.07	1.674	1.665	0.54

Table 3.23 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.3$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.645	-0.578	-11.63	2.085	2.113	1.33
0.025	-0.618	-0.578	-6.94	2.108	2.113	0.24
0.050	-0.602	-0.578	-4.08	2.123	2.113	0.47
0.075	-0.595	-0.578	-2.89	2.130	2.113	0.80
0.100	-0.589	-0.578	-1.90	2.126	2.113	0.62
0.200	-0.566	-0.578	-2.01	2.128	2.113	0.71
0.400	-0.504	-0.578	-12.80	2.123	2.113	0.47

Table 3.24 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/w=0.4$.

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.584	-0.451	-29.49	2.791	2.843	1.83
0.025	-0.495	-0.451	-9.76	2.822	2.843	0.74
0.050	-0.501	-0.451	-11.09	2.831	2.843	0.42
0.075	-0.514	-0.451	-13.97	2.836	2.843	0.25
0.100	-0.523	-0.451	-15.96	2.835	2.843	0.28
0.200	-0.550	-0.451	-21.95	2.829	2.843	0.49
0.400	-0.555	-0.451	-23.06	2.802	2.843	1.44

Table 3.25 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/W=0.5$

l/a	T -stress BEM	T -stress FEM (Wang (2002))	% Difference	K_I BEM	K_I (Rooke(1976))	% Difference
0.010	-0.075	0.027	---	3.976	4.051	1.85
0.025	-0.167	0.027	---	4.009	4.051	1.04
0.050	-0.192	0.027	---	4.009	4.051	1.04
0.075	-0.190	0.027	---	4.037	4.051	0.35
0.100	-0.302	0.027	---	4.033	4.051	0.44
0.200	-0.460	0.027	---	3.998	4.051	1.31
0.400	-0.635	0.027	---	3.876	4.051	4.32

Table 3.26 Comparison of T -stresses and stress intensity factors from present BEM direct approach and other numerical methods for the single edge crack plate specimen under remote tension; $a/W=0.6$

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	0.4347	0.4012	8.3
0.2	0.4127	0.4050	1.9
0.3	0.4069	0.4037	0.8
0.4	0.4253	0.4217	0.9
0.5	0.5777	0.5491	5.2
0.6	1.0421	1.0405	0.2

Table 3.27 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under constant crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% Difference
0.1	0.3019	0.2813	7.3
0.2	0.2863	0.2831	1.1
0.3	0.2841	0.2828	0.5
0.4	0.2950	0.2953	0.1
0.5	0.3943	0.3775	4.5
0.6	0.7010	0.7019	0.1

Table 3.28 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under linear crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang(2002))	% Difference
0.1	0.2367		
0.2	0.2231	0.2205	1.2
0.3	0.2218		
0.4	0.2294	0.2300	0.3
0.5	0.3031		
0.6	0.5319	0.5328	0.2

Table 3.29 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under parabolic crack face loading

Relative Crack Length (a/W)	BEM	FEM (Wang (2002))	% difference
0.1	0.1982		
0.2	0.1836	0.1813	1.3
0.3	0.1827		
0.4	0.1910	0.1889	1.1
0.5	0.2487		
0.6	0.4298	0.4300	0.0

Table 3.30 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate, under cubic crack face loading

Fig 3.1 A large square plate with a small crack under remote tension;
 $H=W$, $a/W=0.01$

Fig. 3.2 Boundary element mesh used for large plate with small crack

Fig. 3.3 Comparison of exact solution and BEM results (direct approach): for infinite plate with a central crack under remote tension

Fig. 3.4 Comparison of exact solution and BEM results (contour integral approach): infinite plate with a central crack under remote tension

Fig. 3.5 The central crack plate (CCP) specimen

Fig. 3.6 Crack face loading distributions: $\sigma = \sigma_o(1 - \frac{x_1}{a})^n$

Fig3.7 Typical boundary element mesh used for test specimen

Fig 3.8 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for CCP

Fig 3.9 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for CCP

Fig 3.10 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for CCP

Fig 3.11 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for CCP

Fig 3.12 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.6$) for CCP

Fig 3.13 Comparison of T -stress between BEM contour integral analysis and FEM results for central cracked plate ($a/W=0.4$) under different contour geometry and size

Fig 3.14 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under constant crack face loading

Fig 3.15 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under linear crack face loading

Fig 3.16 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under parabolic crack face loading

Fig 3.17 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for central cracked plate under cubic crack face loading

Fig 3.18 The double edge crack plate (DECP) specimen

Fig 3.19 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for DECP

Fig 3.20 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for DECP

Fig 3.21 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for DECP

Fig 3.22 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for DECP

Fig 3.23 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.6$) for DECP

F3.24 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under constant crack face loading

Fig 3.25 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under linear crack face loading

Fig 3.26 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under parabolic crack face loading

Fig 3.27 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for double edge cracked plate under cubic crack face loading

Fig 3.28 The single edge cracked plate (SECP) specimen

Fig 3.29 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.2$) for SECP

Fig 3.30 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.3$) for SECP

Fig 3.31 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.4$) for SECP

Fig 3.32 Percentage discrepancies of T -stress and stress intensity factor (K_I) between current BEM direct analysis and other numerical results ($a/W=0.5$) for SECP

Fig 3.33 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under constant crack face loading

Fig 3.34 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under linear crack face loading

Fig 3.35 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under parabolic crack face loading

Fig 3.36 Comparison of T -stress between present BEM contour integral approach and FEM results (Wang, 2002) for single edge cracked plate under cubic crack face loading

CHAPTER 4

ELASTIC *T*-STRESS SOLUTIONS FOR CRACKED THICK-WALLED CYLINDERS

4.0 Introduction

In practice, failure of thick-walled pressurized cylinders (e.g. gun tubes, pipes and pressure vessels) is often due to the presence of internal or external cracks. A long internal single radial crack may be treated as a through crack in two-dimensions and is commonly seen in such components. Generally, cracked cylinders under internal pressure can be considered as “low-constrained” geometries. Thus the fracture toughness measured from “high-constrained” test specimens, such as compact tension and three-point bending specimens may be conservative when applied to this geometry. Therefore, accurate *T*-stress solutions for thick-walled cylinder with internal radial crack are needed to reliably predict the failure loads. Unfortunately, these solutions are not available in the literature at present. In Chapter 3, *T*-stress solutions for different plate specimens were calculated using the BEM technique and have been verified with FEM results (Wang, 2002). It is the purpose of this chapter to extend the BEM analysis to thick-walled cylinders to obtain the *T*-stress solutions using both the direct approach as well as the contour integral approach, as before. At the same time, the stress intensity factor solutions were also calculated and compared to the published

results in the literature (Andrasic and Parker (1984) and Wu and Carlsson (1991)) to verify the accuracy of the BEM model.

To facilitate the *T*-stress calculations for more complex loading conditions, for example, high residual stresses, stress concentrations or thermal stress, the weight function for *T*-stress needs to be developed. In this chapter, the two-term weight function approximation proposed by Wang (2002) is used to obtain the *T*-stress in cracked thick-walled cylinders. The BEM solutions under constant and linear crack face loading conditions are fitted with empirical formulas, which are then used to obtain the coefficients of weight functions. Finally, the weight functions are verified using *T*-stress solutions for internal pressure, parabolic and cubic stress distributions on the crack face obtained directly.

4.1 Problem Definition

A thick-walled cylinder with an internal single radial crack, as shown in Fig. 4.1, was analysed in this study. The geometry of the cylinder in the x_1 - x_2 plane was defined by relative crack length a/W and R_o/R_i ratio, where $W = R_o - R_i$. As before, plane strain conditions are assumed in the analysis.

In this study, the relative crack length was varied with $a/W = 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6 . For a given a/W ratio, the following radius ratios were analysed, $R_o/R_i = 1.5, 1.75, 2.0, 2.25$ and 2.5 to investigate the effect of wall thickness on the *T*-stress. Different load conditions were also investigated. Uniform internal pressure, p , was first applied, then, as before, the four types of basic crack face loading conditions corresponding to $n=0$ to 3 in the following stress distribution:

$$\sigma(x_1) = \sigma_o \left(1 - \frac{x_1}{a}\right)^n \quad (4.1)$$

were analysed, where σ_o is the nominal stress, a is the crack length, they correspond to the uniform, linear, parabolic and cubic crack face loading, respectively.

4.2 Numerical Modelling

Because of symmetry, only one-half of the physical problem needs to be modelled. A typical BEM model is presented in Fig. 4.2. The following boundary conditions were applied: along the horizontal plane of symmetry, elements in the uncracked ligament were constrained in the x_2 direction, but were free to displace in the x_1 direction. Also, one node along the plane of symmetry was constrained in x_1 direction to prevent rigid body motion in that direction. As before, quarter-point elements were used for those elements adjacent to the crack-tip. For the contour integral approach to obtain the *T*-stress, an integration path, Γ_o , with relative radius $r/a=0.5$ as shown in Fig. 4.2 was employed for both internal pressure and crack face loading conditions.

4.3 BEM Results and Discussions

For the purpose of validating the BEM model of thick-walled cylinder, the stress intensity factors were first calculated. The stress intensity factor were normalized as follows

$$K_I^* = \frac{K_I}{\sigma_o \sqrt{\pi a}} \quad (4.2)$$

where σ_o is the nominal stress for crack face loading case, and equals p in the internal pressure case. The normalized stress intensity factor results under internal pressure are presented in

Table 4.1 and compared with the solutions from Wu and Carlsson (1991). The largest discrepancy was found to be only 1.2%. The normalized stress intensity factors for crack face loading, i.e. constant, linear, parabolic and cubic load cases, with the wall thickness ratio $R_o/R_i = 1.5, 1.75, 2.0$ or 2.5 were also obtained and are shown in Table 4.2 to Table 4.5, respectively. In these tables, comparison between the BEM stress intensity factor solutions and those from Andrasic and Parker (1984) using modified mapping collocation (MMC) technique are also presented. The percentage differences are generally less than 2%. The excellent agreement between the current BEM stress intensity factor solutions and these numerical results in the literatures demonstrated the reliability of the current BEM model for thick-walled cylinder analysis.

T -stress solutions obtained in the analysis were normalized as follows:

$$T^* = \frac{T}{\sigma_o} \quad (4.3)$$

where σ_o has the same meaning as in Eqn. 4.2. For the internal pressure case, the normalized values of T^* are obtained using both direct approach and contour integral approach and listed in Table 4.6 to Table 4.10 for the different radius ratios $R_o/R_i = 1.5, 1.75, 2.0, 2.25$ or 2.5 . It can be seen that the discrepancies of the solutions obtained using this two approaches varied from 5% to 10%. From model verification in previous chapter, the contour integral solutions can be considered as more stable and accurate results. For the crack face loading cases, the normalized T -stress solutions under uniform, linear, parabolic and cubic crack face loading are presented in Table 4.11 to Table 4.15 for the radius ratio $R_o/R_i = 1.5, 1.75, 2.0, 2.25$ or 2.5 , respectively. These normalized T -stress solutions are also shown in Figs. 4.3 to 4.7. It can be seen that all the T -stress values for each of the different crack face loading conditions show a trend of decreasing value with the relative crack length from $a/W = 0.1$ to 0.6 . This suggests

that deeply cracked thick-walled cylinders have lower constraint effects at the crack tip when compared with those with shallow cracks under crack face loading. From these figures, it can also be seen that, compared to non-linear crack face loading (e.g. parabolic and cubic) conditions, the *T*-stress curves under the uniform and linear crack face loading conditions show relatively steeper declines when the crack length grows. This implies that the cracked cylinders with lower constraint conditions can be more easily obtained under uniform and linear crack face loading conditions increasing crack length.

In order to better demonstrate the effects of radius ratio, R_o/R_i , on the *T*-stress solutions, these *T*-stress data for internal pressure and the four types of crack face loading conditions were re-organized and re-plotted in Fig. 4.8 to Fig. 4.12, respectively. The *T*-stress solutions under crack face loading showed a decreasing trend when R_o/R_i varied from 1.5 to 2.5. On the other hand, the *T*-stress results for internal pressure showed a trend of increasing value when R_o/R_i increased from 1.5 to 2.5. These figures also showed that the effect of radius ratio R_o/R_i on *T*-stress is more evident when the relative crack length increased.

4.4 Weight Function

The following two-term weight function expression

$$t(x,a) = \frac{2}{\pi a} \cdot \left[D_1 \left(1 - \frac{x}{a}\right)^{1/2} + D_2 \left(1 - \frac{x}{a}\right)^{3/2} \right] \quad (4.4)$$

proposed by Wang (2002) was proved to be sufficiently accurate in predicting *T*-stress in two-dimensional crack problems. Based on this expression and previous BEM solutions for constant and linear loading conditions, weight functions for *T*-stress in cracked thick-walled cylinder under internal pressure, parabolic and cubic loading conditions and different radius

ratios $R_o/R_i = 1.5, 1.75, 2.0, 2.25$ and 2.5 were derived for the relative crack length range of $0.1 \leq a/W \leq 0.6$.

Two reference solutions for *T*-stress are used to determine the coefficients D_1 and D_2 in Eqn. (4.4): constant and linear crack face stress distributions corresponding to $n=0$, and $n=1$ in Eqn. (4.1), respectively. The BEM *T*-stress solutions were approximated by simple polynomials using least square fitting with error less than 1%. The range of applicability for these curve-fits is $0.1 \leq a/W \leq 0.6$. The coefficients D_1 and D_2 are then obtained by using the following procedure:

- The *T*-stress results for constant crack face stress distribution

$$\sigma(x) = \sigma_o \quad (4.5)$$

are:

$$T_{1r} = \sigma_o V_1 \quad (4.6)$$

where V_1 represents normalized *T*-stress for constant loading condition using the following polynomial:

$$V_1 = A_0 + A_1\left(\frac{a}{W}\right) + A_2\left(\frac{a}{W}\right)^2 + A_3\left(\frac{a}{W}\right)^3 + A_4\left(\frac{a}{W}\right)^4 + A_5\left(\frac{a}{W}\right)^5 \quad (4.7)$$

the coefficients A_0 to A_5 for different radius ratio, R_o/R_i , are fitted and presented in Table 4.16.

- The *T*-stress results for a linear decreasing crack face stress distribution

$$\sigma(x) = \sigma_o \left(1 - \frac{x}{a}\right) \quad (4.8)$$

are:

$$T_{2r} = \sigma_o V_2 \quad (4.9)$$

where V_2 represents normalized *T*-stress for linear loading condition and can be approximated as follows:

$$V_2 = B_0 + B_1\left(\frac{a}{W}\right) + B_2\left(\frac{a}{W}\right)^2 + B_3\left(\frac{a}{W}\right)^3 + B_4\left(\frac{a}{W}\right)^4 + B_5\left(\frac{a}{W}\right)^5 \quad (4.10)$$

Similarly, the coefficients B_0 to B_5 for different radius ratio, R_o/R_i , are obtained and listed in Table 4.17.

- By substituting Eqns. (4.5), (4.6), (4.8) and (4.9) into Eqn. (2.35), two equations with two unknown parameters D_1 and D_2 were established. For crack face loading case, the uncracked T -stress term in Eqn. (2.34) is zero because there is no external loading is applied. Thus the parameters in the weight function expression were determined to be as follows

$$D_1 = \frac{-105\pi}{16}V_2 + \frac{75}{16}V_1 \quad (4.11)$$

$$D_2 = \frac{175\pi}{16}V_2 - \frac{105}{16}V_1 \quad (4.12)$$

The weight functions can then be determined from Eqn. (2.37).

The weight function derived above can be validated using BEM results for crack face non-linear stress distributions and internal pressure cases in previous sections. Using Eqn. (2.35), weight function T -stresses were calculated for the following parabolic and cubic decreasing stress field:

$$\sigma(x) = \sigma_o \left(1 - \frac{x}{a}\right)^2 \quad (4.13)$$

and

$$\sigma(x) = \sigma_o \left(1 - \frac{x}{a}\right)^3 \quad (4.14)$$

The weight function T -stress predictions and the BEM T -stress data for the above stress distributions under different radius ratio, $R_o/R_i=1.5, 1.75, 2.0, 2.25$ and 2.5 , are shown in Fig.

4.3 to 4.7 and Fig. 4.9 to 4.12 to compare with BEM results. The differences between the BEM *T*-stress solutions and the weight function predictions are generally less than 5%.

Equation. (2.36) was used to calculate the *T*-stress values for the internal pressure case. It must be noted that the uncracked *T*-stress is not zero in this case. The stress distribution in an uncracked cylinder under internal pressure, *p*, can be expressed using Lamé's equations (Timoshenko, 1956) as follows:

$$\sigma_{rr}(x) = p \cdot \frac{R_1^2}{R_2^2 - R_1^2} - \frac{p}{R_2^2 - R_1^2} \frac{R_1^2 \cdot R_2^2}{r^2} \quad (5.12)$$

$$\sigma_{\theta\theta}(x) = p \cdot \frac{R_1^2}{R_2^2 - R_1^2} + \frac{p}{R_2^2 - R_1^2} \frac{R_1^2 \cdot R_2^2}{r^2} \quad (5.13)$$

The uncracked *T*-stress in Eqn. (2.34) can then be expressed as follows:

$$T_{uncracked} = (\sigma_x - \sigma_y) \Big|_{\substack{x=a \\ y=0}} = (\sigma_{rr} - \sigma_{\theta\theta}) \Big|_{\substack{r=a \\ \theta=0}} - \frac{2 \cdot p}{R_2^2 - R_1^2} \frac{R_1^2 \cdot R_2^2}{r^2} \quad (5.14)$$

The integral Equation (2.36) can be evaluated by a numerical integration with the above Lamé's stress distributions equations. The weight function *T*-stress solutions for different radius ratios were compared with the contour integral BEM results obtained in previous sections, as before. The results are plotted in Fig. 4.8. The discrepancies between these two sets of solutions are generally less than 2%.

4.5 Conclusion

Two-dimensional BEM *T*-stress analysis of thick-walled cylinders with different R_o/R_i ratios and loading conditions were analysed for a range of relative crack lengths, a/W , varying from 0.1 to 0.6. These results are new and have not been available previously in the literature.

They can be used for failure assessment analysis of cracked thick-walled cylinders. For internal pressure case, *T*-stress solutions show a typical “low constraint” effect. The *T*-stress weight functions for cracked thick walled cylinders with different radius ratio have also been derived. The weight function technique has been verified by using the BEM *T*-stress solutions for crack face loading conditions and internal pressure case. All the comparisons demonstrate the reliability of weight function method in engineering *T*-stress analysis.

Ro/Ri	a/W	K^*_I (BEM)	K^*_I (Wu and Carlsson (1991))	K^*_I % Diff.
1.5	0.1	2.875	2.894	0.6
	0.2	2.984	3.006	0.7
	0.3	3.246	3.250	0.1
	0.4	3.584	3.575	0.3
	0.5	3.988	3.944	1.1
	0.6	4.383	4.342	1.0
1.75	0.1	2.085	2.078	0.4
	0.2	2.073	2.090	0.8
	0.3	2.136	2.161	1.2
	0.4	2.276	2.271	0.2
	0.5	2.425	2.407	0.7
	0.6	2.567	2.563	0.2
2	0.1	1.692	1.683	0.5
	0.2	1.617	1.625	0.5
	0.3	1.604	1.620	1.0
	0.4	1.652	1.647	0.3
	0.5	1.710	1.695	0.9
	0.6	1.769	1.765	0.2
2.25	0.1	1.453	1.447	0.3
	0.2	1.338	1.342	0.3
	0.3	1.286	1.295	0.7
	0.4	1.288	1.283	0.4
	0.5	1.305	1.294	0.8
	0.6	1.328	1.325	0.2
2.5	0.1	1.289	1.283	0.5
	0.2	1.149	1.157	0.7
	0.3	1.074	1.087	1.2
	0.4	1.052	1.048	0.4
	0.5	1.047	1.038	0.8
	0.6	1.053	1.054	0.1

Table 4.1 Comparison of normalized stress intensity factor solutions, $K^*_I = K_I / \sigma_o \sqrt{\pi a}$, from present BEM analysis and solutions by Wu and Carlsson (1991): thick-walled cylinder under internal pressure, $\sigma_o = p$.

Ro/Ri	a/W	K^*_I (BEM)	K^*_I (Andrasic and Parker (1984))	K^*_I % Diff.
1.5	0.1	1.148	1.147	0.1
	0.2	1.237	1.248	0.9
	0.3	1.387	1.389	0.2
	0.4	1.572	1.571	0.0
	0.5	1.788	1.773	0.9
	0.6	2.008	2.001	0.3
1.75	0.1	1.129	1.127	0.2
	0.2	1.189	1.200	0.9
	0.3	1.286	1.300	1.1
	0.4	1.427	1.427	0.0
	0.5	1.577	1.563	0.9
	0.6	1.725	1.718	0.4
2	0.1	1.111	1.109	0.2
	0.2	1.149	1.156	0.6
	0.3	1.219	1.232	1.0
	0.4	1.327	1.326	0.1
	0.5	1.441	1.428	0.9
	0.6	1.557	1.553	0.3
2.5	0.1	1.079	1.074	0.5
	0.2	1.087	1.097	0.9
	0.3	1.125	1.139	1.2
	0.4	1.198	1.196	0.2
	0.5	1.279	1.270	0.7
	0.6	1.369	1.365	0.3

Table 4.2 Comparison of normalized stress intensity factor solutions, $K^*_I = K_I / \sigma_o \sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under constant crack face loading, $n=0$.

Ro/Ri	a/W	K_I^* (BEM)	K_I^* (Andrasic and Parker (1984))	K_I^* % Diff.
1.5	0.1	0.453	0.453	0.1
	0.2	0.509	0.514	1.0
	0.3	0.594	0.601	1.1
	0.4	0.719	0.714	0.7
	0.5	0.847	0.840	0.8
	0.6	0.984	0.981	0.3
1.75	0.1	0.442	0.441	0.2
	0.2	0.480	0.485	1.1
	0.3	0.540	0.547	1.1
	0.4	0.630	0.625	0.8
	0.5	0.715	0.708	0.9
	0.6	0.804	0.801	0.4
2	0.1	0.430	0.430	0.2
	0.2	0.456	0.459	0.7
	0.3	0.499	0.505	1.1
	0.4	0.567	0.562	0.9
	0.5	0.629	0.623	0.9
	0.6	0.698	0.696	0.2
2.5	0.1	0.411	0.410	0.2
	0.2	0.418	0.421	0.9
	0.3	0.441	0.447	1.3
	0.4	0.487	0.481	1.1
	0.5	0.528	0.524	0.8
	0.6	0.578	0.577	0.2

Table 4.3 Comparison of normalized stress intensity factor solutions, $K_I^* = K_I / \sigma_o \sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under linear crack face loading, $n=1$.

Ro/Ri	a/W	K_I^* (BEM)	K_I^* (Andrasic and Parker (1984))	K_I^* % Diff.
1.5	0.1	0.292	0.291	0.2
	0.2	0.335	0.335	0.1
	0.3	0.402	0.398	0.9
	0.4	0.482	0.480	0.4
	0.5	0.577	0.572	0.8
	0.6	0.676	0.677	0.1
1.75	0.1	0.283	0.282	0.3
	0.2	0.314	0.314	0.0
	0.3	0.362	0.359	0.8
	0.4	0.417	0.416	0.4
	0.5	0.480	0.476	0.9
	0.6	0.544	0.542	0.3
2	0.1	0.275	0.274	0.2
	0.2	0.297	0.295	0.6
	0.3	0.331	0.328	0.9
	0.4	0.372	0.370	0.5
	0.5	0.418	0.414	1.0
	0.6	0.466	0.465	0.2
2.5	0.1	0.261	0.260	0.2
	0.2	0.270	0.268	0.6
	0.3	0.289	0.286	0.7
	0.4	0.314	0.311	0.7
	0.5	0.344	0.341	0.8
	0.6	0.378	0.377	0.2

Table 4.4 Comparison of normalized stress intensity factor solutions, $K_I^* = K_I / \sigma_o \sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under parabolic crack face loading, $n=2$.

R_o/R_i	a/W	K^*_I (BEM)	K^*_I (Andrasic and Parker (1984))	K^*_I % Diff.
1.5	0.1	0.219	0.216	1.6
	0.2	0.248	0.250	0.8
	0.3	0.296	0.299	1.0
	0.4	0.372	0.365	1.9
	0.5	0.443	0.437	1.4
	0.6	0.518	0.516	0.5
1.75	0.1	0.212	0.209	1.4
	0.2	0.232	0.235	1.3
	0.3	0.266	0.269	1.3
	0.4	0.319	0.314	1.8
	0.5	0.366	0.361	1.4
	0.6	0.413	0.413	0.2
2	0.1	0.206	0.203	1.3
	0.2	0.218	0.219	0.3
	0.3	0.243	0.245	0.9
	0.4	0.284	0.278	2.0
	0.5	0.317	0.311	1.8
	0.6	0.352	0.351	0.2
2.5	0.1	0.195	0.192	1.5
	0.2	0.197	0.197	0.3
	0.3	0.210	0.212	1.0
	0.4	0.237	0.232	2.3
	0.5	0.258	0.254	1.6
	0.6	0.282	0.282	0.2

Table 4.5 Comparison of normalized stress intensity factor solutions, $K^*_I = K_I / \sigma_o \sqrt{\pi a}$, from present BEM calculation and solutions by Andrasic and Parker (1984): thick-walled cylinder under cubic crack face loading, $n=3$.

a/W	<i>T</i> -stress BEM (Contour Integral Approach)	<i>T</i> -stress BEM (Direct Approach)	<i>T</i> -stress % Diff.
0.1	-2.1145	-2.1739	2.7
0.2	-1.9909	-2.2250	10.5
0.3	-1.8947	-2.0921	9.4
0.4	-1.7837	-1.9282	7.5
0.5	-1.7502	-1.8832	7.1
0.6	-1.8308	-2.0066	8.8

Table 4.6 Comparison of normalized *T*-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=1.5$).

a/W	<i>T</i> -stress BEM (Contour Integral Approach)	<i>T</i> -stress BEM (Direct Approach)	<i>T</i> -stress % Diff.
0.1	-1.7377	-1.7765	2.2
0.2	-1.5577	-1.7239	9.6
0.3	-1.4274	-1.5560	8.3
0.4	-1.3372	-1.4338	6.7
0.5	-1.3329	-1.4130	5.7
0.6	-1.4362	-1.5274	6.0

Table 4.7 Comparison of normalized *T*-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=1.75$).

a/W	<i>T</i> -stress BEM (Contour Integral Approach)	<i>T</i> -stress BEM (Direct Approach)	<i>T</i> -stress % Diff.
0.1	-1.5310	-1.5591	1.8
0.2	-1.3161	-1.4496	9.2
0.3	-1.1714	-1.2753	8.1
0.4	-1.0874	-1.1630	6.5
0.5	-1.0841	-1.1432	5.2
0.6	-1.1695	-1.2313	5.0

Table 4.8 Comparison of normalized *T*-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.0$).

a/W	<i>T</i> -stress BEM (Contour Integral Approach)	<i>T</i> -stress BEM (Direct Approach)	<i>T</i> -stress % Diff.
0.1	-1.3929	-1.4178	1.8
0.2	-1.1541	-1.2693	9.1
0.3	-1.0025	-1.0919	8.2
0.4	-0.9209	-0.9843	6.4
0.5	-0.9129	-0.9614	5.0
0.6	-0.9786	-1.0264	4.7

Table 4.9 Comparison of normalized *T*-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.25$).

a/W	<i>T</i> -stress BEM (Contour Integral Approach)	<i>T</i> -stress BEM (Direct Approach)	<i>T</i> -stress % Diff.
0.1	-1.2898	-1.3143	1.9
0.2	-1.0340	-1.1364	9.0
0.3	-0.8793	-0.9582	8.2
0.4	-0.7990	-0.8541	6.5
0.5	-0.7859	-0.8276	5.0
0.6	-0.8355	-0.8751	4.5

Table 4.10 Comparison of normalized *T*-stresses, T/σ_o , between BEM contour integral approach and BEM direct approach for different relative crack lengths, a/W ; thick-walled cylinder under internal pressure loading ($R_o/R_i=2.5$).

Crack Face Loading: n	Relative Crack Length, a/W					
	0.1	0.2	0.3	0.4	0.5	0.6
$n=0$	0.4210	0.3781	0.3432	0.2952	0.2271	0.1161
$n=1$	0.2912	0.2617	0.2392	0.2103	0.1720	0.1148
$n=2$	0.2281	0.2040	0.1849	0.1661	0.1395	0.1018
$n=3$	0.1909	0.1681	0.1543	0.1396	0.1182	0.0895

Table 4.11 Normalized *T*-stresses for thick-walled cylinder under crack face loading ($R_o/R_i=1.5$).

Crack Face Loading: n	Relative Crack Length, a/W					
	0.1	0.2	0.3	0.4	0.5	0.6
$n=0$	0.4174	0.3615	0.3103	0.2365	0.1286	-0.0384
$n=1$	0.2864	0.2500	0.2173	0.1722	0.1092	0.0170
$n=2$	0.2225	0.1949	0.1686	0.1377	0.0932	0.0299
$n=3$	0.1848	0.1604	0.1409	0.1166	0.0811	0.0327

Table 4.12 Normalized *T*-stresses for thick-walled cylinder under crack face loading ($R_o/R_i=1.75$).

Crack Face Loading: n	Relative Crack Length, a/W					
	0.1	0.2	0.3	0.4	0.5	0.6
$n=0$	0.4091	0.3430	0.2783	0.1868	0.0578	-0.1312
$n=1$	0.2803	0.2374	0.1961	0.1399	0.0640	-0.0417
$n=2$	0.2173	0.1852	0.1529	0.1137	0.0598	-0.0133
$n=3$	0.1801	0.1525	0.1280	0.0971	0.0544	-0.0016

Table 4.13 Normalized *T*-stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.0$).

Crack Face Loading: n	Relative Crack Length, a/W					
	0.1	0.2	0.3	0.4	0.5	0.6
$n=0$	0.3994	0.3243	0.2485	0.1453	0.0054	0.1918
$n=1$	0.2736	0.2248	0.1764	0.1129	0.0304	-0.0801
$n=2$	0.2120	0.1755	0.1382	0.0935	0.0348	-0.0416
$n=3$	0.1756	0.1447	0.1161	0.0806	0.0344	-0.0241

Table 4.14 Normalized *T*-stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.25$).

Crack Face Loading: n	Relative Crack Length, a/W					
	0.1	0.2	0.3	0.4	0.5	0.6
$n=0$	0.3892	0.3058	0.2213	0.1105	-0.0345	-0.2335
$n=1$	0.2667	0.2123	0.1583	0.0902	0.0047	-0.1065
$n=2$	0.2067	0.1661	0.1247	0.0765	0.0158	-0.0611
$n=3$	0.1711	0.1371	0.1051	0.0688	0.0191	-0.0396

Table 4.15 Normalized *T*-stresses for thick-walled cylinder under crack face loading ($R_o/R_i=2.5$).

R_o/R_i	A_0	A_1	A_2	A_3	A_4	A_5
1.5	0.5370	-1.9123	10.1725	-30.5708	43.2500	-24.9167
1.75	0.5504	-2.1282	10.7629	-31.8875	43.2083	-24.4167
2	0.5516	-2.2326	10.9358	-32.7458	43.9167	-24.4167
2.25	0.1630	6.5110	-61.3712	239.6875	-436.3750	295.7500
2.5	0.5432	-2.3066	10.4629	-32.0917	43.7083	-24.3333

Table 4.16 The coefficients of V_1 : normalized T -stress empirical equation under constant crack face stress distribution

R_o/R_i	B_0	B_1	B_2	B_3	B_4	B_5
1.5	0.3684	-1.2482	6.3754	-18.5375	25.4583	-14.0833
1.75	0.3696	-1.3100	6.4158	-18.7167	24.9167	-13.6667
2	0.3700	-1.3874	6.6237	-19.6667	26.1250	-14.1667
2.25	0.3666	-1.4011	6.3929	-19.2458	25.7083	-13.9167
2.5	0.5432	-2.3066	10.4629	-32.0917	43.7083	-24.3333

Table 4.17 The coefficients of V_2 : normalized T -stress empirical equation under linear crack face stress distribution

Fig. 4.1 A thick-walled cylinder with a single radial crack

Fig4.2 Typical boundary element mesh for thick-walled cylinder

Fig 4.3 Comparison of normalized BEM *T*-stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=1.5$).

Fig. 4.4 Comparison of normalized BEM *T*-stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=1.75$).

Fig. 4.5 Comparison of normalized BEM *T*-stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.0$).

Fig. 4.6 Comparison of normalized BEM *T*-stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.25$).

Fig. 4.7 Comparison of normalized BEM *T*-stresses and weight function predictions for thick-walled cylinder with internal single radial crack under different crack face loading ($R_o/R_i=2.5$).

Fig. 4.8 Comparison of normalized BEM *T*-stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under internal pressure

Fig. 4.9 Comparison of normalized BEM *T*-stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face constant loading

Fig. 4.10 Comparison of normalized BEM *T*-stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face linear loading

Fig. 4.11 Comparison of normalized BEM *T*-stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face parabolic loading

Fig. 4. 12 Comparison of normalized BEM *T*-stresses and weight function predictions for different R_o/R_i ratio: thick-walled cylinder with internal single radial crack under crack face cubic loading

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.0 Conclusions

In this study, the two-dimensional linear elastic analysis of crack problems has been performed using the boundary element method (BEM) for the purpose of calculating the fracture mechanics parameter: T -stress. The problems considered involve an infinite plate; three cracked plate specimens, namely, central cracked plate, single edge cracked specimen and double edge cracked specimen; and a thick-walled cylinder with a radial crack. The T -stress weight functions for cracked thick-walled cylinders with different radius ratios have been derived as well.

Brief reviews of the principles of linear elastic fracture mechanics (LEFM), the two-parameter approach in fracture mechanics analysis, the boundary element method (BEM) and the weight function method have been presented in Chapter 2. The derivation of the formulas for the direct approach and the contour integral approach has also been presented and they were implemented into the BEM T -stress calculation. The validations of the BEM technique for T -stress analysis have been conducted in Chapter 3 by comparing the results for the infinite plate case and the three test specimens with the corresponding analytical and finite element method (FEM) results, respectively. A range of crack tip element size and relative

crack length were treated for each of test specimens in the BEM analysis. Different loading conditions have also been considered for each geometry case. The accuracy of T -stress solutions using the direct displacement approach and the contour integral approach was analysed and some general guidelines for mesh design have been developed. Numerical solutions of the T -stress for the problem of a thick-walled cylinder with a radial crack have been obtained and presented in Chapter 4. As before, both the direct approach and contour integral approach were used and a wide range of radius ratios and relative crack lengths were considered. The BEM models used to obtain T -stress solutions was verified by comparing the stress intensity factor solutions obtained from these models with available stress intensity factor solutions in the literature. For more complex stress distributions near the crack tip, the weight function (WF) technique has been conducted to determine the T -stress. The BEM T -stress results have been used as reference T -stress in deriving the weight function solutions. The accuracy of weight function technique was verified by comparing to corresponding T -stress results obtained directly by the BEM.

The following general conclusions can be made from the analyses conducted in this study. First, the crack tip element size is a critical parameter which affects the accuracy of the BEM direct displacement approach T -stress solutions. Generally, a crack tip element size of 5% of the crack length can limit the numerical error of the T -stress obtained to be within 5%. Second, because the computations in the contour integral approach are carried out away from crack tip, it is numerically more stable and accurate than direct approach in BEM T -stress analysis. The results are generally independent of the contour size and geometry. By choosing a circular contour with radius which equals half of the crack length, the numerical results for T -stress were generally less than 2% in discrepancy with those in the literature

using the finite element method and with the BEM direct approach. Third, the thick-walled cylinder exhibits typical 'low constraint' characteristics under internal pressure loads. For decreasing crack face loading conditions, the 'low constraint' effect is more obvious with deeper cracks. This indicates that using the two-parameter fracture mechanics, which include the constraint effect, will likely lead to less conservative life predictions in service.

5.1 Recommendations for future work

In this study, both the direct approach and contour integral approach have been used in conjunction with BEM analysis to obtain the T -stress solutions. It should be noted that the direct displacement approach was not used to analysis the T -stress under crack face loading condition. Also, the traction formula is very efficient in stress intensity factor analysis, but the corresponding direct approach analogous to it cannot be derived easily. Future work could also be carried out to evaluate the T -stress in three-dimensional crack problems, which is more commonly found in practice.

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