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SOME RESULTS ON INFINITE
GROUP RINGS

by

JOHN LAWRENCE

Submitted to the Faculty of Graduate Studies & Research
in Partial Fulfilment of the Requirements for
Doctor of Philosophy
Department of Mathematics
Carleton University
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November, 1975

✓
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The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis "Some Results on Infinite Group Rings" submitted by John William Lawrence in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

This thesis is concerned mainly with primitive and semiperfect group rings and the application of methods to other problems in ring theory.

In chapter one we characterize the coefficient ring of a primitive group ring. One of the results of this section is that if $R[G]$ is primitive for some group G , then $R[H]$ is primitive for some suitable free group H .

Our work in chapter one leads naturally to the definition of a strongly prime ring. This class of rings includes all domains, prime Goldie rings, simple rings, and free products of algebras. These are studied in chapter two.

In chapter three, we apply the methods of the first two chapters to the study of primitive rings.

Chapter four introduces the notion of a valuation on a module. Methods are developed to study right invertible elements in tensor products of algebras.

In chapter five we apply our previous theorem to the study of several problems in group rings and tensor products. For example, we show that the tensor product of two algebras is semilocal only if one of the algebras is algebraic.

In the final chapter, we examine semiperfect and semilocal group rings. We extend results of Passman and Valette, and answer a conjecture of Goursaud in the negative.

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For my interest in infinite group rings, I owe thanks to three people in particular. Thanks to the fathers of (the relatively new) theory of infinite group rings, Ian Connell and Donald Passman. Thanks to Edward Formanek whose beautiful papers on group rings inspired much of the work in this thesis.

I owe special thanks to David Handelman. Although he has objected to my interest in group rings, he has remained a friend and stimulating colleague.

NOTATION AND DEFINITIONS

Most of the notation in this thesis is taken from the books of Herstein, Jacobson, Lambek and Passman.

To several broad classes of algebraic objects we associate several capital letters. Thus P and T usually denote rings, G and H usually denote groups, F and K fields, and M usually denotes a module. On top of this, we have the following notation.

From set theory $\text{card } S$ - the cardinality of the set S .

From ring theory $P(R)$ - the prime radical of R .

$J(R)$ - the Jacobson radical of R .

$A \otimes_F B$ - the tensor products of algebras over a field.

$Z_l(R)$ - the left singular ideal of R .

$A *_F B$ - the free product (coproduct) of algebras over a field .

$R[X]$, the polynomial ring.

$R[[X]]$, the power series ring.

$Q_l(R)$ - the maximal (complete) left quotient ring of R .

$M_n(R)$ - the $n \times n$ matrices over R .

N - the natural numbers.

Z - the integers .

Q - the rational numbers.

R - the real numbers .

$C(R)$ - the centre of R .

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From group theory $H \leq G - H$ is a subgroup of G .

$H \trianglelefteq G - H$ is a normal subgroup of G .

$G^{(n)}$ - the n 'th derived group of G .

$G * H$ - the free product (coproduct) of groups.

From group rings $R[G]$ - the group ring of the group G over the ring R .

$w(H)$ - the right ideal of $R[G]$ generated by the set

$\{(1-h)h : h \in H\}$, where $H \leq G$.

The following is a list of terms used in this thesis. When definitions are given in the thesis, we give the page number; in other cases we give a reference from the bibliography.

algebraic algebra - [Jacobson (1964), p. 19]

augmentation ideal - [Passman (1971c) (1974c)]

bounded strongly prime p.22

centre of a ring - [Lambek (1966) p.17]

comaximal left ideal - p.1.3

complete ring of quotients - [Lambek (1966) p.94]

derived group = commutator subgroup - [Magnus, Karrass, Solitar (1966)]

enough valuations - p.5.8

essential extension - p.2.6

essential submodule - [Lambek (1966), p.60]

filtration - p.4.8

finitely approximable - p. 6.5

inductive class - p.3.7

insulator - of an element p.2.1

- of an ideal p.2.1

involution - p.3.9

Jacobson radical - [Lambek (1966), p.57]
kernel of a valuation - p.4.2
lifting idempotents - [Lambek (1966) p.72]
local idempotent - p. 6.1
local ring - p.5.4
locally finite algebra - [Jacobson (1964), p.240]
locally finite group - [Herstein (1968), p.62]
monoid (semigroup with identity) [Magnus, Karrass, Solitar (1966)]
Morita invariant [Faith (1973)]
nil ideal-[Lambek (1966), p.70]
Ore domain - [Herstein (1968), p.170]
p-group -[Connell (1963), p.697]
p-left ideal - p.1.3
polycyclic group -[Roseblade (1973)]
polynomial identity (PI) algebra -[Jacobson (1964), p.222]
prime ring - p.1.1
primitive idempotent - p.6.1
(von Neumann) regular ring -[Lambek (1966) p.67]
semilocal ring-p5.4
semiperfect ring-p6.1
simple ring - [Lambek (1966), p.52]
singular ideal - p.1.13
socle-p.2.7
solvable group-[Magnus, Karrass, Solitar, (1966)]
*-subring - p.3.9
strongly prime ring - p.1.5

tensor product of algebras - [Jacobson (1964) p.102]

torsion group - [Herstein (1968), p.62]

uniformly strongly prime - p.2.2

valuation, max-extension - p.4.10

valuation, maximal - p.4.2.

valuation, on a module, archimedean - p.4.2

valuation, on a module, nonarchimedean - p.4.2

valuation, on a ring - p.4.1

weakly primitive - p.1.4

0. INTRODUCTION

This thesis divides into two main sections, the first half motivated by the study of primitive group rings and the second half motivated by the study of semiperfect and semilocal group rings.

The work on primitive group rings leads to the further study of prime and primitive rings. One of the basic notions of the first three chapters is the insulator. An element with an insulator is a generalization of a nonzero-divisor in a ring. A generalization of the insulator is given in chapter one and is used to characterize the coefficient ring of a primitive group ring. In chapter two we look at the class of rings in which every non-zero element has an insulator-strongly prime rings. The class of strongly prime rings is relatively large in the class of prime rings and includes all domains and simple rings. There are interesting connections between the class of strongly prime rings and the classes of simple, primitive and prime Goldie rings. In the third chapter we exploit some of the connections.

The work on semi perfect group rings leads to a further study of right invertible elements in tensor products of algebras. Thus, in turn, motivates the definition of a valuation on a module. It is well-known that if R is a commutative ring, then $1 + \alpha x$ is invertible in $R[x]$ only if α is nilpotent. Chapter Four is an attempt to generalize the ideas behind the proof of this. This allows us to show a close relation between tensor products or group rings with certain properties and algebraic or locally finite algebras. For example, we show that the tensor product of two algebras is semilocal only if one of the algebras is algebraic.

In the final chapter we examine semiperfect group rings. We extend results of Passman and Valette and answer a conjecture of Goursaud in the negative.

1. THE COEFFICIENT RING OF A PRIMITIVE GROUP RING

The long open problem of the existence of nontrivial examples of primitive group rings was only recently settled in the affirmative. [Formanek, Snider (1972)]. In their paper Formanek and Snider showed that if F is any field and G any group, then there exists a group H containing G such that the group ring $F[H]$ is primitive. Since then, two other papers have been published [Passman (1973)], [Formanek (1973)] on primitive group rings. All three papers deal with group rings, where for the most part, the coefficient ring is assumed to be a field.

In this section, we look at the more general case of primitive group rings, where the coefficient ring is arbitrary. The main theorem (Theorem 1.10) of this section comes from a paper by the author [Lawrence (1975)].

Recall, that a ring is prime if the product of two nonzero ideals is nonzero ; a ring is (left) primitive if it has a faithful irreducible (left) module. If R is a ring and G a group, the group ring $R[G]$ is the R -algebra with basis $\{g \mid g \in G\}$ and multiplication induced by the group operation. Basic properties of the group ring (with arbitrary coefficient ring) are given in [Lambek (1966)] .

In this section we characterize R when $R[G]$ is primitive. In this way, we define a new class of rings which includes as a subclass the strongly prime rings of the next chapter. We will show later that these results are of interest in constructing examples of primitive rings.

If $R[G]$ is primitive, what can be said about R ? First, since all primitive rings are prime, the following theorem allows us to say that R is prime.

THEOREM 1.1. [Connell (1963)]. The group ring $R[G]$ is prime if and only if R is prime and G has no nontrivial finite normal subgroups.

We cannot, however, conclude that R must be primitive. This follows from the following.

THEOREM 1.2 [Formanek (1973)]. Let R be a domain (not necessarily commutative) and $G = A * B$ a free product of non-trivial groups A and B (except $A = B = \mathbb{Z}_2$). If $\text{card } G \geq \text{card } R$, then the group ring $R[G]$ is primitive.

Thus the class of rings which are coefficient rings of primitive group rings includes all domains. However, a commutative domain is primitive if and only if it is a field.

The cardinality condition in the above theorem is of interest and somewhat unexpected. Using theorems of Roseblade [Roseblade (1973)], Passman has given examples of groups where the group ring is primitive only if the field is 'large enough' [Passman (1973)].

The above makes it all the more difficult to characterize primitive group rings in the case where we have an arbitrary coefficient ring. To the author's knowledge, this has not been done even in the abelian (group) case. For this reason, we have settled on a more modest task. If $R[G]$ is primitive, what can be said about R ? It is somewhat surprising that

we get a reasonably decent answer to the question.

In fact, we do something more general than that stated above. First we characterize primitive rings as those having a proper 'comaximal left ideal' (to be defined). If I is a comaximal left ideal of the group ring $R[G]$, we characterize the left ideal $R \cap I$ of R . Thus a ring R will be the coefficient ring of a primitive group ring if and only if it has a proper left ideal of the above form. We will show that if $R[G]$ is primitive, and the free product $A*B$ (of groups) 'suitably large', then $R[A*B]$ is primitive.

DEFINITION 1.3. A left ideal I of a ring R is said to be a comaximal left ideal if for every nonzero ideal J of R , $I + J = R$.

It was pointed out in [Formanek, Snider (1972)] that a ring is primitive if and only if it contains a proper comaximal left ideal.

DEFINITION 1.4. A left ideal I of a ring R is said to be a P-left ideal if it has the following property: for every nonzero ideal J of R , there exists a finite subset $S(J) \subset J$ such that

$$\{a \in R \mid a.S(J) \subset I\} \subset I.$$

For example, in $\mathbb{Z}$, (0) is the only proper P-left ideal.

PROPOSITION 1.5.

1. A ring is (left) primitive if and only if it has a proper comaximal left ideal.
2. Every comaximal left ideal is a P-left ideal.

3. (0) is a comaximal left ideal if and only if the ring is simple.
4. (0) is maximal as an ideal in any proper P-left ideal.
5. In a regular ring every P-left ideal is a comaximal left ideal.

PROOF. 1. This is well known; for example see [Formanek, Snider (1972)] .

2. Suppose that I is a comaximal left ideal and J a nonzero ideal. Then there exists $b \in J$ such that $b-1 \in I$. If $ab \in I$, then, since $ab - a \in I$, $a \in I$. We then let $S(J) = \{b\}$.

3. If J is a nonzero ideal of R , then $(0) + J = R$; whence $J = R$.

4. Let I be a P-left ideal. If $J \neq (0)$ is an ideal such that $J \subset I$, then $aS(J) \subset I$ for all a ; hence, $R = I$.

5. For each nonzero ideal J , the right ideal generated by the set $S(J)$ is generated by an idempotent $e_J \neq 0$. If I is a P-left ideal, then, since $(1 - e_J)e_J = 0$, we have $(1 - e_J)S(J) \subset I$; whence $1 - e_J \in I$. As $e_J \in J$, $I + J = R$, and so I is a comaximal left ideal.

DEFINITION 1.6. A ring is said to be (left) weakly primitive if it has a proper P-left ideal.

Examples of weakly primitive rings include, by the above theorem, all primitive rings. In any domain, (0) is a P-left ideal; whence, any domain is weakly primitive. By Proposition 1.5, if R is a weakly primitive regular ring, then R is primitive. Finally, let's point out that all weakly primitive rings are prime. To see this, note that if I is a proper P-left ideal, then I contains no nontrivial ideals of R . Thus, if J is a nonzero ideal, its left annihilator must be zero, for it is

an ideal contained in I .

It is not known whether or not every left weakly primitive ring is right weakly primitive, however, this seems unlikely.

DEFINITION 1.7. A ring is said to be (left) strongly prime if (0) is a P-left ideal.

In contrast to weakly primitive rings, it is not difficult to construct left strongly prime rings that are not right strongly prime. An example will be given later. By our earlier remarks, all domains are strongly prime.

We now come to the main theorems of this section.

THEOREM 1.8. Suppose R is a ring and G a monoid. If I is a P-left ideal of the monoid ring $R[G]$, then $I \cap R$ is a P-left ideal of R .

PROOF. Let J be a nonzero ideal of R . Then $J[G]$ is an ideal of $R[G]$; hence, there exists a finite set $S(J[G]) \subset J[G]$ satisfying the conditions of the definition (of a P-left ideal). Let $S(J)$ be the set of coefficients of terms occurring in $S(J[G])$. If $a \in R$ and $aS(J) \subset I \cap R$, then $aS(J[G]) \subset I$; hence, $a \in I \cap R$. Thus, $I \cap R$ is a P-left ideal.

THEOREM 1.9.

1. Let R be a ring, and let $G = A * B$ be a free product of groups with $\text{card } A = \infty$, $\text{card } B \geq 2$ and $\text{card } G \geq \text{card } R$. If M is a P-left ideal of R , then there exists a comaximal left ideal I' of $R[G]$ such that $R \cap I' = M$.

2. Let R be a ring and X a set of indeterminates with $\text{card } X \geq \max(\text{card } R, \infty)$. If M is a P-left ideal of R , then there exists a comaximal left ideal I' of $R[X]$ (the free monoid ring on the set X) such that $R \cap I' = 1$.

NOTE. Our method of proof is somewhat similar to the proof of the main theorem in [Formanek (1973)]. The theorem could be further generalized to certain free products of monoids; however, we prefer to prove it in the most interesting cases. The proof of 2 appeared in [Lawrence (a)] It is easier to prove than 1. In this section we prove only 1.

PROOF. We may suppose that $\text{card } A \geq \text{card } B$, hence $\text{card } A = \text{card } G = \text{card } R[G]$.

An element $g = a_1 b_1 a_2 b_2 \dots a_n b_n a_{n+1}$ of G , in reduced form is said to have length $2n + 1$, denoted by $\ell(g)$. In a similar fashion, length is defined for group elements starting with or ending with $b \in B$. A product of two elements g and g' of G is said to be pure if $\ell(gg') = \ell(g) + \ell(g')$. For completeness, define $\ell(1) = 0$.

Suppose that M is a P-left ideal of R . If $M = R$, then the result is obvious. We therefore assume that M is proper (so R is weakly primitive).

From each nonzero ideal J of $R[G]$, choose $\alpha = \alpha(J) \in J$ ($\neq 0$) such that α has minimal support. Suppose

$$(1) \quad \alpha = \sum r(\alpha, g)g,$$

where $r(a, g) \in R$ and $g \in G$. Let $g(a)$ be an element of maximal length in the support of a . Thus $g(a)$ has coefficient $r(a, g(a))$. Consider the ideal of R , $I(a) = \langle r(a, g(a)) \rangle$, and suppose that $S(I(a)) = \{s_1, s_2, \dots, s_n\}$. (Here the set $S(I(a))$ is given with respect to M). For each $s_j \in S(I(a))$, let $a(s_j)$ be an element of RaR with the coefficient of $g(a)$ being s_j . Thus, using our notation $a(s_j) = \sum r(a(s_j), g)g$, and the support of $a(s_j)$ is the same as the support of a .

Fix $b \in B - \{1\}$, and let $W: (R[G] - \{0\}) \times N \rightarrow A - \{1\}$ be a bijection. Given a as above, let $T(a, j)$ equal

$$\begin{aligned} & bw(a, j_1)ba(s_j)bw(a, j_1)b + w(a, j_2)bw(a, j_2)ba(s_j) \\ & bw(a, j_2) + bw(a, j_3)ba(s_j)w(a, j_3)bw(a, j_3) \\ & + w(a, j_4)bw(a, j_4)ba(s_j)w(a, j_4)b + bw(a, j_5)a(s_j)bw(a, j_5) \\ & bw(a, j_5) + w(a, j_6)bw(a, j_6)a(s_j)bw(a, j_6)b \\ & + bw(a, j_7)a(s_j)w(a, j_7)bw(a, j_7)b + w(a, j_8)bw(a, j_8)a(s_j)w(a, j_8)bw(a, j_8), \end{aligned}$$

where $W(a, j_k)$ is chosen so that it is not equal to any factor in the reduced form of any element in the support of a ($k = 1, 2, \dots, 8$), and

$$j_1 < j_2 < \dots < j_8 < (j+1)_1.$$

Finally, let $H(a) = \sum_j T(a, j)$, a finite sum (whose cardinality is that of the finite set $S(I(a))$).

Note that if $\beta \in R[G]$, and $r \in R$ occurs as the coefficient of some term in the expansion of $\beta T(a, j)$, then r is the coefficient of a pure product in this expansion.

Let M' be the left ideal of $R[G]$ generated by the set $\{H(\alpha) + 1\}$, where we have chosen some α from each nonzero ideal in $R[G]$. Now let $I' = M' + M[G]$.

Clearly $I' \cap R \supseteq M$. Also, if J is a nonzero ideal of $R[G]$, then $\alpha(J) \in J$; hence, $H(\alpha) \in J$. Thus $1 = -H(\alpha) + H(\alpha) + 1 \in J + I'$, and we conclude that I' is a comaximal left ideal of $R[G]$. It remains to show that $R \cap I' \subseteq M$.

If $R \cap I' \not\subseteq M$, then there exists $\beta_1, \beta_2, \beta_3, \dots, \beta_m \in R[G]$, $\gamma \in M[G]$, and $\delta \in R \setminus M$ such that

$$(2) \quad \gamma + \sum_{i=1}^m \beta_i [H(\alpha_i) + 1] = \delta$$

Since $\delta \notin M$, we must have either

$$(3) \quad r(\beta_i, \gamma) r(\alpha_i(s_j), z) \notin M,$$

for some $i = 1, 2, \dots, m$, and some $s_j \in S(I(\alpha_i))$

or (3') $r(\beta_i, \gamma) \notin M$, for some $i = 1, 2, \dots, m$.

If (3') occurs, then $r(\beta_i, \gamma) \cdot r(\alpha_i(s_j), g(\alpha_i)) \notin M$, for some j , since the set $\{r(\alpha_i(s_j), g(\alpha_i))\}_{j=1}^n$ equals the set $S(J_i)$. (We

are now using the definition of a P -left ideal). Thus we may assume that (3) holds. Choose y and z in (3) so that $\ell(y) + \ell(z)$ is maximal in all the products in the expansion of (2) for which (3) holds. We now have $r = r(\beta_i, \gamma) \cdot r(\alpha_i(s_j), z)$ as the coefficient, in the expansion of (2) of a group element x , with $\ell(x) = \ell(y) + \ell(z) + 6$.

In order to arrive at a contradiction, we make two observations on arbitrary products in (2). First, if r' is the coefficient of a group element x' in the expansion of $\beta_1.T(\alpha_1, j')$, then r' is the coefficient of x'' (in the same expansion), where x'' ends in $W(\alpha_1, j'_k)b$ or $W(\alpha_1, j'_k)$, and $\ell(x'') \geq \ell(x')$. Therefore, by the maximality of $\ell(x)$ with respect to (3), $x' = x$ with $(i, j) \neq (i', j')$ implies that $r' \in M$. Second, if $r(\beta_1, y')r(\alpha_1, z') \notin M$ is the coefficient in the expansion of $\beta_1.T(\alpha_1, j)$, of a group element x' , and $\ell(x')$ is maximal with this coefficient, then $\ell(x') = \ell(y') + \ell(z') + 6$. Thus we may suppose that

$$x = y \text{ --- } z \text{ --- } W(\alpha_1, j_k) ,$$

or

$$x = y \text{ --- } z \text{ --- } W(\alpha_1, j_k)b$$

and

$$x' = y' \text{ --- } z' \text{ --- } w(\alpha_1, j'_k)$$

or

$$x' = y' \text{ --- } z' \text{ --- } W(\alpha_1, j'_k)b ,$$

where both x and x' are in reduced form; hence, $x = x'$ implies $k = k'$. Let us suppose, therefore, that (for example)

$$x = ybW(\alpha_1, j_1)bzbW(\alpha_1, j_1)b$$

and

$$x' = y'bw(\alpha_1, j_1)bz'bw(\alpha_1, j_1)b ,$$

where both are in reduced form and neither z nor z' contains $w(\alpha_1, j_1)$ as a factor in their reduced form. Therefore, if $x = x'$ we must have $y = y'$ and $z = z'$.

By our second observation, we see that x occurs as an element in the support of $\beta_1 T(\alpha_1, j)$ and has coefficient $r \notin M$. (Here we use the maximality of $\ell(y) + \ell(z)$, hence of $\ell(x)$.) Now rx must cancel with a sum of other terms in the expansion of (2); therefore, by the maximality of $\ell(x)$ (with respect to (3)), and since all the coefficients of y are in M , we see that $r'x$ ($r' \notin M$) does not occur in the expansion of $\beta_1 T(\alpha_1, j')$, for $(i, j) \neq (i', j')$.

We conclude that x is in the support of β_1 , for some i' , and $r(\beta_1, x) \notin M$. Then for some j' ,

$$(4) \quad r(\beta_1, x) \cdot r(\alpha_1, (s_{j'},)) \cdot g(\alpha_1,)) \notin M.$$

(Here we are once again using the fact that M is a P-left ideal.) However, $\ell(y) + \ell(z) < \ell(x) \leq \ell(x) + \ell(g(\alpha_1,))$, thus (4) contradicts the fact that y and z were chosen such that $\ell(y) + \ell(z)$ was maximal with respect to (3). The contradiction completes the proof.

COROLLARY 1.9.1. A ring R is the coefficient ring of a primitive group ring if and only if it is weakly primitive.

PROOF. If G is a group such that $R[G]$ is primitive, let M be a proper comaximal left ideal of $R[G]$. Then $R \cap M$ is a proper P-left ideal of R (by the theorem); hence, R is weakly primitive. Conversely, if R is weakly primitive, let M be a proper P-left ideal of R . Let G be a non-abelian free group with $\text{card } G \geq \text{card } R$. By the

previous theorem, there is a comaximal left ideal I of $R[G]$ such that $R \cap I = M$. Since M is proper, I is proper in $R[G]$; whence, $R[G]$ is primitive.

We now summarize our previous results in the following

THEOREM 1.10. For any ring, the following are equivalent:

1. R is weakly primitive
2. If $G = A * B$ is a free product of groups, with $\text{card } A = \infty$, $\text{card } B \geq 2$, $\text{card } G \geq \text{card } R$, then $R[G]$ is primitive.
3. If X is a set of indeterminates with $\text{card } X \geq \text{card } R$, then the free monoid ring $R[X]$ is primitive.
4. There exists a monoid G such that the monoid ring $R[G]$ is primitive.

PROOF. This is clear from the previous results.

PROPOSITION 1.11. A strongly prime von Neumann regular ring is simple.

PROOF. As R is strongly prime, (0) is a P-left ideal. As R is regular, every P-left ideal is a comaximal left ideal (Proposition 1.5); and thus, (0) is a comaximal left ideal. By the same proposition, R is then simple.

We will give another, more direct proof of the above proposition in the next section.

PROPOSITION 1.12. The property 'weakly primitive' is Morita invariant.

PROOF. It is sufficient to show that if R is weakly primitive, then $M_n(R)$ is weakly primitive, and if R is weakly primitive and $0 \neq e = e^2 \in$

and $ReR = R$, then eRe is weakly primitive.

1. Suppose that R is weakly primitive and e is a nonzero idempotent of R . Let X be a set of indeterminates, with $\text{card } X \geq \text{card } R$. Then $(eRe)[X] \cong eR[X]e$. Since $R[X]$ is primitive (by Theorem 1.10), $eR[X]e$ and hence, $(eRe)[X]$ are primitive, for primitivity is Morita invariant. Thus eRe is weakly primitive.

2. Using a similar argument, we have $(M_n(R))[X] \cong M_n(R[X])$. Since the right side is primitive, so is the left side. Hence $M_n(R)$ is weakly primitive.

A natural question that arises is whether the cardinality condition $\text{card } X > \text{card } R$ is necessary in Theorem 1.9. That is, does there exist a weakly primitive ring R such that the free monoid ring $R[X]$ is primitive if and only if $\text{card } X \geq \text{card } R$? We now construct a simple example.

Let Ω be any uncountable cardinal and let Y be a set of Ω indeterminates. Let $\mathbb{Q}[Y]$ (to be denoted by $\mathbb{Q}[\Omega]$) be the commutative polynomial ring over the rationals.

PROPOSITION 1.13 Let X be a set of indeterminates. Then the free monoid ring $(\mathbb{Q}[\Omega])[X]$ is primitive if and only if $\text{card } X \geq \Omega$.

PROOF. Suppose that $\Omega > \text{card } X$ and yet $(\mathbb{Q}[\Omega])[X]$ is primitive. Let M be a left ideal comaximal with every nonzero ideal. For each y_i there exists $a_i \in \langle y_i \rangle$ such that $a_i + 1 \in M$. Since $\text{card } \{a_i\} = \Omega$ and

card $X < \Omega$, card $\{\text{supp } (a_i)\} < \Omega$. Hence, there is an infinite sequence $a_{i_1}, a_{i_2}, \dots$ such that :

1. $a_{i_j} \in \langle Y_{i_j} \rangle$, $a_{i_j} + 1 \in M$, $j = 1, 2, \dots$,
2. Y_{i_j} does not occur in any coefficient of a_{i_k} , $k < j$, $j = 1, 2, \dots$
3. $\text{supp } (a_{i_j}) = \text{supp } (a_{i_2}) = \dots$.

By linear dependence, there is $b_i \in \mathbb{Q}[\Omega]$, $i = 1, 2, \dots, k$, for some k , and $b_k \neq 0$ such that

$$b_1 a_{i_1} + \dots + b_k a_{i_k} = 0.$$

We may assume that k is chosen minimal so that all of the above hold. Using 1 and 2, we see that $b_j \in \langle Y_{i_k} \rangle$ for all $j < k$; and therefore if we assume $\text{GCD}(b_1, \dots, b_k) = 1$, $b = b_1 + \dots + b_k \neq 0$. Then

$$b = b_2(a_{i_1} + 1) + \dots + b_k(a_{i_k} + 1) \in M,$$

so $b \in M$. This is a contradiction, since we cannot have $\langle b \rangle + M = (\mathbb{Q}[\Omega])[X]$ and M proper. This contradiction completes the proof of the theorem.

We now look at an example of a prime semiprimitive ring which is not weakly primitive. This example was originally constructed by Osofsky [Osofsky (1967)] as an example of a prime semiprimitive ring with nonzero singular ideal.

DEFINITION 1.14. The (left) singular ideal of a ring is the set of elements which annihilate some essential left ideal on the right. It is a two-sided ideal of the ring.

Let $F = Z_2[X, Y_i]$, $i = 1, 2, 3, \dots$, be the free Z_2 -algebra.

An arbitrary monomial may be represented in the form

$$m = x^{i_1} y_{j_1} x^{i_2} \dots y_{j_{n-1}} x^{i_n}, \quad i_k \geq 0, \quad j_k \geq 1,$$

repetitions allowed. Let I be the ideal of F generated by monomials which satisfy

$$\sum i_k > (\max \{j_k\}) (\text{the number of times } y_{\max \{j_k\}} \text{ appears}).$$

Let $R = F/I$. This is Osofsky's example.

THEOREM 1.15 [Osofsky (1967)]. R is a prime semiprimitive ring.

THEOREM 1.16. R is not weakly primitive.

PROOF. Let G be a group and suppose that the group ring $R[G]$ is primitive.

Thus there exists a proper comaximal left ideal M . There exists $a \in \langle x \rangle$

such that $a-1 \in M$. Choose h so large that if y_i occurs in a , then

$h > i$. Then there exists $b \in \langle y_h \rangle$ such that $b-1 \in M$. Let n be a

positive integer and consider $a^n b$. By our choice of h , $(\max \{j_k\})$

(the number of times $y_{\max \{j_k\}}$ appears) is independent of n in any

monomial of $a^n b$. However, in such a monomial, x occurs at least n

times. Hence, for sufficiently large n , $a^n b = 0$. Therefore,

$$1 = a^n (b-1) + \left[\sum_{i=0}^{n-1} a^i \right] (a-1) \in M,$$

contradicting the fact that M is proper.

We conclude this section with a survey of several more elementary properties of weakly primitive rings.

PROPOSITION 1.17. The free monoid ring $R[X]$, with $\text{card } X \geq \text{card } R$, is weakly primitive if and only if it is primitive.

PROOF. If $R[X]$ is weakly primitive, so is R . Hence $R[X]$ is primitive. The converse is equally obvious.

PROPOSITION 1.18.

1. An over left ideal of a comaximal left ideal is a comaximal left ideal.
2. The intersection of a finite number of P -left ideals is a P -left ideal.
3. A P -left ideal which is maximal as a left ideal is a comaximal left ideal.

PROOF. 1. This is clear.

2. Suppose $I_1, I_2, \dots, I_k$ are P -left ideals. Let $S_i(J)$ be the set associated with the ideal $J(\neq(0))$ with respect to I_i .

Set $S(J) = \bigcup_{i=1}^n S_i(J)$. Then

$$\left\{ a \in R \mid a.S(J) \subset \bigcap_{i=1}^n I_i \right\} \subset \bigcap_{i=1}^n I_i.$$

3. If I is a P -left ideal which is maximal as a left ideal, then (0) is maximal as an ideal in I (Prop. 1.5.4). Thus, if $J(\neq(0))$ is an ideal, then $I + J = R$.

PROPOSITION 1.19. Let R be a ring and G a cancellative ordered monoid. If M is a P -left ideal of R , then $M[G]$ is a P -left ideal of $R[G]$.

PROOF. Let $J(\neq(0))$ be an ideal of $R[G]$. Choose $0 \neq a \in J$ where $a = r_1 q_1 + \dots + r_n q_n$, $r_n \neq 0$, $q_1 < q_2 < \dots < q_n$. Let $S(\langle r_n \rangle) = \{s_{n_1}, s_{n_2}, \dots, s_{n_k}\} \subset R$. We now select $S(J) = \{a_1, a_2, \dots, a_k\} \subset J$, where $a_j = t_{j1} q_1 + \dots + s_{jn_j} q_n$. It is easy to show that

$$\{B \in R[G] \mid BS(J) \subset M[G]\} \subset M[G].$$

COROLLARY 1.19.1. Let R be a strongly prime ring and G a cancellative ordered monoid. Then the monoid ring $R[G]$ is strongly prime.

2. STRONGLY PRIME RINGS

Strongly prime rings were first studied by R. Rubin [Rubin (1973)] under the name 'absolutely torsion free' rings. Rubin defined them in terms of the kernel functors of O. Goldman [Goldman (1969)]. A ring R is absolutely torsion-free if for every kernel functor σ on $\text{Mod } R$, $\sigma(R) = (0)$ or R . Further work on absolutely torsion free rings was done in [Viola-Prioli (1973)]. Independently of Rubin (and Viola-Prioli), the author studied strongly prime rings in [Lawrence (1973)] using the definition given in this thesis. D. Handelman and the author then wrote a paper on strongly prime rings [Handelman-Lawrence (1975)]. Most of the material from this section is taken from the joint paper.

DEFINITION 2.1. Suppose R is a ring and $r \in R$. Then $\text{ann}_r r = \{x \in R \mid xr = 0\}$. Similarly $\text{ann}_l r = \{x \in R \mid rx = 0\}$. These are called, respectively, the left and right annihilators of r .

DEFINITIONS 2.2. Let R be a ring and r a nonzero element of R . A left insulator of r is a finite set $S(r) \subset R$ such that $\bigcap_{s \in S(r)} \text{ann}_l sr = (0)$.

If I is a nonzero ideal of R , then a left insulator of I is a finite set $S(I) \subset I$ such that $\bigcap_{s \in S(I)} \text{ann}_l s = (0)$.

PROPOSITION 2.3. For any ring R , the following are equivalent:

1. (0) is a P-left ideal of R ,
2. every nonzero element has a left insulator,
3. every nonzero ideal has a left insulator.

PROOF. $1 \Rightarrow 3$. Assume that (0) is a P-left ideal. Let $S(J)$ be the set associated with the nonzero ideal J . Then $\{r \in R \mid rS(J) = 0\} = \{0\}$ so $S(J)$ is a left insulator for J .

$3 \Rightarrow 2$. If $a \in R$ is a nonzero element, then $\langle a \rangle$ the ideal of R generated by a has a left insulator, say $\{ \sum_{i=1}^n b_{i,j} a c_{i,j} \}$. The set $\{b_{i,j}\}_{i,j}$ is finite and is a left insulator for a .

$2 \Rightarrow 1$. Let J be a nonzero two-sided ideal, and let a be a nonzero element of J . Let $S(a)$ be the insulator of a . Then

$$\{x \in R \mid xS(a)a = 0\} = \{0\},$$

so (0) is a P-left ideal, where $S(J) = S(a)a$.

To each left strongly prime ring R we associate a natural number or the symbol ∞ as follows. If there exists an integer n such that every non-zero element has an n -element insulator, then $SP(R)$ is the minimal integer with this property. Otherwise we let $SP(R) = \infty$.

If R is a domain, then $SP(R) = 1$. If R is an $n \times n$ matrix ring over a division ring, then $SP(R) = n$. If R is strongly prime and $SP(R) \neq \infty$, then R is said to be bounded strongly prime. If there is a finite set $I \subset R$ which is a left insulator for every (nonzero) element of the ring R , then R is said to be uniformly strongly prime.

Not every left strongly prime ring is right strongly prime (see Example 2.15). Henceforth, the term 'strongly prime' will mean 'left strongly prime'.

PROPOSITION 2.4. If a prime ring R satisfies D.C.C. on left annihilators, then R is strongly prime,

PROOF. Take a nonzero element $r \in R$, and let $S, S_2, \dots$ be a sequence of finite subsets of R defined inductively by:

1. $S_1 = \{1\}$
2. If $S_k = \{s_1, \dots, s_k\}$ and $\bigcap_{i=1}^k \text{ann}_l s_i r = I \neq (0)$,

choose s_{k+1} such that $\text{ann } s_{k+1} r \neq I$. Then let $S_{k+1} =$

$S_k \cup \{s_{k+1}\}$. Otherwise, let $S_{k+1} = S_k$. Since R satisfies

DCC on left annihilators, there exists n such that $S_{n+1} = S_n$. Then

S_n is a left insulator for r .

Since every prime Goldie ring satisfies DCC on left annihilators, we see that every prime (right or left) Goldie ring is (right and left) strongly prime.

THEOREM 2.5. Strong primeness is a Morita invariant property.

PROOF. 1. If R is strongly prime and $0 \neq e \in R$ is an idempotent, then we claim that eRe is strongly prime. For if $0 \neq ere$ has a left insulator S in R , then $eSe = \{exe \mid x \in S\}$ is a left insulator of ere in eRe .

2. Let $\{e_{ij}\}_{i,j=1}^n$ be a set of matrix units of the $n \times n$ matrix ring $M_n(R)$. Suppose that R is strongly prime. If $0 \neq \alpha = \sum_{i,j} r_{ij} e_{ij} \in M_n(R)$, we may suppose that $r_{xy} \neq 0$. We then let $S(\alpha) = \{se_{ij} \mid 1, j=1, 2, \dots, n, s \in S(r_{xy})\}$. This is a left insulator of α in $M_n(R)$.

A prime ring has no nonzero nilpotent ideals and, on the other hand, a domain or prime Goldie ring has no nil ideals. We get an intermediate result for strongly prime ring.

PROPOSITION 2.6. The locally nilpotent (Levitzki) radical of a strongly prime ring is zero.

PROOF. If R has a nonzero locally nilpotent ideal, then so does $R[G]$, for any group G . Thus $R[G]$ is not primitive, contradicting Corollary 1.9.1.

We have already seen (Proposition 1.11) that a strongly prime regular ring is simple. Also, every simple ring is strongly prime. This is a special case of Kaplansky's conjecture that every prime regular ring is primitive. Partial answers have been given in [Fisher-Snider (1974)] and [Goodearl (1973a)].

The (left) singular ideal $Z_\ell(R)$ of a ring R is the set of elements which annihilate essential left ideals on the right. A ring is said to be (left) nonsingular if $Z_\ell(R) = (0)$.

PROPOSITION 2.7. If R is strongly prime, then $Z_\ell(R) = (0)$.

PROOF. If the assertion is not correct, suppose that $0 \neq r \in Z_\ell(R)$.

Let $S(r) = \{s_1, \dots, s_n\}$ be a left insulator of R . Since $Z_\ell(R)$ is an ideal there exist essential left ideals $E_1, \dots, E_n$ such that

$E_i s_i r = 0$, $i = 1, 2, \dots, n$. Thus $(\bigcap E_i) S(r) r = 0$; hence, $\bigcap E_i = (0)$.

But the intersection of a finite number of essential ideals is essential; hence, $\bigcap E_i \neq (0)$, a contradiction.

It is not true that the left singular ideal of a right strongly prime ring is necessarily zero (Example 2.18).

A natural question which arises is the question of subrings and factor rings of strongly prime rings. How much can we say about them?

THEOREM 2.8. Every prime ring is a subring and a factor ring of some strongly prime ring. More generally, a ring is a subring of some strongly prime ring if and only if it is torsion-free over its prime subring (the prime subring is the subring generated by 1).

The above theorem will follow from the examination of another class of strongly prime rings - free products of algebras.

Let A and B be F -algebras with basis $S = \{a_i\}$ and $S' = \{b_j\}$, respectively. The free product of A and B over the field F is the F -algebra with basis $\{a_{i_1} b_{j_1} a_{i_2} b_{j_2} \dots a_{i_k} b_{j_k} : a_{i_l} \in S, b_{j_l} \in S', k = 1, 2, \dots\}$. We denote the free product by $A_F^* B$. [(See Cohn (1959)].

PROPOSITION 2.9. Let A and B be nontrivial F -algebras

$(\dim_F A > 1, \dim_F B > 1)$. Then the free product $A_F^* B$ is uniformly strongly prime.

PROOF. Choose a basis of A containing 1, and a basis of B containing 1. Let a and b be elements not equal to 1 in the basis of A and B respectively. Then the set $\{a, b, ba, ab\}$ is an insulator for every element in the free product.

It can be shown that in 'most' free products, every element has a one element insulator.

We are now in a position to prove Theorem 2.8. If R is a ring torsion-free over its prime subring, then by inverting the central elements we may suppose that R is an algebra over a field F . We then let $S = R *_F F[x]$, be the free product of R with the polynomial ring $F[x]$. R is a subring of S , and S is strongly prime.

COROLLARY 2.8.1. Every prime ring is a subring of a coefficient ring of a primitive group ring.

If M and N are (left) R -modules, then M is said to be a (left) essential extension of N if N is an essential (large) submodule of M . [Lambek (1966) p.60] . If R is a subring of a ring S , then S is said to be an essential extension of R if ${}_R S$ is an essential extension of ${}_R R$, that is if S is an essential extension of R as an R -module.

It is well-known that an essential extension of a prime ring is prime.

THEOREM 2.10. An essential extension S of a strongly prime ring R is strongly prime. Moreover, $SP(S) \leq SP(R)$.

PROOF. If $0 \neq s \in S$, then there exists $r \in R$ such that $0 \neq rs \in R$. Since R is strongly prime, there is a left insulator $S(rs)$ of rs in R . Then $S(rs)r$ is a left insulator for s in S . Clearly $\text{card} \{S(rs)r\} \leq \text{card} \{S(rs)\}$; whence, $SP(S) \leq SP(R)$.

An important corollary is the following (first proved by Goldman and Rubin, [Rubin (1973)]).

THEOREM 2.11. The complete ring of quotients (maximal quotient ring) of a strongly prime ring is a simple regular ring.

PROOF. If R is strongly prime, then $Z_c(R) = (0)$, and so $Q(R)$ the (left) complete quotient ring is regular and is an essential extension of R [Lambek (1966)]. Thus $Q(R)$ is strongly prime and in fact, since it is regular, simple.

The above theorem allows us to describe the class of rings which are subrings of simple rings.

THEOREM 2.12. A ring is a subring of some simple ring if and only if it is torsion-free over its prime subring.

PROOF. Since a simple ring is prime, it is torsion-free over its centre; thus, every subring of a simple ring is torsion-free over its prime subring.

Conversely, if R is torsion-free over its prime subring, then R can be embedded in a strongly prime ring S (Theorem 2.8). R is then a subring of the complete ring of quotients of S , a simple ring (Theorem 2.11).

Recall that the left socle of a ring R is the sum of the minimal left ideals of R . In a semiprime ring, the right socle equals the left socle and is called the socle.

PROPOSITION 2.13. A strongly prime ring with nonzero socle is completely reducible.

PROOF. If R is a strongly prime ring with nonzero socle, then $Q(R)$ the complete ring of quotients of R is a full linear ring [Lambek (1966)p.98

However, $Q(R)$ is simple and so is a full matrix ring over a division ring. It follows that $\text{Soc}(R) = R$; whence R is completely reducible.

We now examine conditions under which a group ring $R[G]$ is strongly prime.

THEOREM 2.14. 1. If R is strongly prime and $G = A * B$ is a free product of nontrivial groups, then the group ring $R[G]$ is strongly prime.

2. If $R[G]$ is strongly prime, then R is strongly prime and G has no locally finite normal subgroups (other than $\langle 1 \rangle$).

PROOF. 1. Suppose $0 \neq \alpha = \sum r_i g_i \in R[G]$. Let S be an insulator for r_1 in R . Choose $a \in A - \{1\}$ and $b \in B - \{1\}$. Then $Sa \cup Sb \cup Sab \cup Sba$ is a left insulator for α in $R[G]$. (We assume that g_1 has maximal length).

2. If $R[G]$ is strongly prime, then (0) is a P-left ideal of $R[G]$. Therefore (0) is a P-left ideal of R ; whence, R is strongly prime. Now suppose that G has a nontrivial locally finite normal subgroup N . Choose $g \in N - \{1\}$. We show that $g - 1$ has no left insulator in $R[G]$. For if $\{s_i\}$ is an insulator for $g - 1$, then $T = \{g_k\} = \bigcup \text{supp}(s_i)$ is also an insulator for $g - 1$. Let H be the subgroup generated by $\{g_k g g_k^{-1} \mid g_k \in T\}$. Since H is a finitely generated subgroup of the locally finite group N , it is finite. Hence, there exists a nonzero $\beta \in R[G]$ such that $\beta(g_k g g_k^{-1} - 1) = 0$ for all k , [Lambek (1966), p.154]. But $(g_k g g_k^{-1} - 1)g_k = g_k(g - 1)$, so $\beta\{g_k\}(g - 1) = 0$, a contradiction.

It would be interesting to know if the converse of Theorem 2.14.2 is true, at least for some reasonable class of groups such as solvable groups. The converse is true for the class of polycyclic groups. The problem is closely related to the zero-divisor problem for group rings, that is, if R is a domain and G a torsion-free group, is $R[G]$ a domain?

We will now look at several examples of strongly prime rings.

EXAMPLE 2.15. We construct a ring that is left strongly prime but not right strongly prime.

Let Z_2 denote the field with two elements, and let $D = Z_2[X_i]$ where i runs through the positive integers) be the free Z_2 -algebra. Let I be the two-sided ideal of D generated by monomials of the form $X_i X_j X_k$ with $i < j < k$. Set $R = D/I$.

We show that R is left strongly prime. If m is a nonzero monomial in R of the form $X_{j_1} X_{j_2} \dots X_{j_k}$, then $S(m) = \{X_{j_1} X_{j_1}\}$ is a left insulator for m . If $r = m_1 + \dots + m_n$ is a sum of nonzero monomials in R , choose m_1 a monomial of maximal degree among the m_i . Then $S(m_1)$ is a left insulator for r .

However, X_1 has no right insulator: for a finite subset $\{r_j\}$ of R we can find h sufficiently large so that $X_1 r_j X_h X_{h+1} = 0$ for all j . Thus R is not right strongly prime.

EXAMPLE 2.16. We construct a strongly prime ring that is not bounded.

PROPOSITION 2.17. Let $R = M_n(F)$, where F is a field. Let $\{e_{ij}\}$

be a set of matrix units, and let $r = e_{11} + e_{22} + \dots + e_{mm}$, with $m \leq n$.

Suppose $\{s_i\}_{i=1}^k$ is an insulator for r in R . Then $mk \geq n$.

PROOF. Each $s_i r$ must be of the form

$$\left[\begin{array}{c|c} R_1 & 0 \\ \hline (n \times m) & (n \times n-m) \end{array} \right]$$

For each i we have m equations (each column of R_1) and we thus have a total of km equations. If $n > km$, we can find nonzero t in R such that $ts_i r = 0$ for all i , hence, $km \geq n$.

Let R be the direct limit of matrix rings over a field via the embedding $M_{2^n}(F) \rightarrow M_{2^{n+1}}(F)$ given by

$$A \rightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

Since R is a direct limit of simple rings, it is simple, hence strongly prime. Any insulator of e_{11} in $M_n(F)$ has cardinality at least n , and by the preceding proposition, this does not decrease in the direct limit. Thus the cardinalities of

$$\left\{ \text{images of } e_{11} \in M_n(F) \right\}_{n=1}^{\infty}$$

are unbounded.

EXAMPLE 2.18. We construct a strongly prime ring that is right singular (the right singular ideal is nonzero).

Let $F = \mathbb{Z}_2[x, y_j], j=1,2,\dots,$ be a free $\mathbb{Z}_2$ -algebra. Monomials m can be written in the form

$$m = x^{i_1} y_{j_1} x^{i_2} \dots y_{j_n} x^{i_{n+1}}.$$

Let I be the ideal of F generated by monomials satisfying:

1. $j_n = \max \{ j_k \}, k=1,2,\dots,n,$
2. $j_n \leq 2i_{n+1}$
3. $j_n < \sum_{k=1}^{n+1} i_k.$

Set $R = F/I$. This is the desired example. Given a nonzero element $\alpha \in R$, $\{y_t\}$ is a left insulator of α for suitably large t . On the other hand if $k = \text{degree of } X \text{ in } \alpha$, then $\alpha y_t x^{t-k} \neq 0$ and $x \alpha y_t x^{t-k} = 0$. Thus x is in the right singular ideal of R .

In [Handelman-Lawrence (1975)], the authors asked if a strongly prime ring could have a nonzero nil ideal (see Proposition 2.6). This was answered in the affirmative by Lance Small. His example is sketched in [Goodearl-Handelman-Lawrence (1974)].

If $M_n(D)$ is an $n \times n$ matrix ring over a division ring, then $SP(M_n(D)) = n$. In fact, for any ring R , $SP(M_n(R)) \leq n$, as can be seen from the proof of Theorem 2.5. Equality, in the case where R is a

division ring, follows from Proposition 2.17. Using the Faith-Utumi Theorem, we can extend this result to prime Goldie rings.

THEOREM 2.19. Let R be a prime Goldie ring which is a (left) order in an $n \times n$ matrix ring over a division ring. Then $SP(R) = n$.

In [Handelman-Lawrence (1975)], the authors conjectured a partial converse to this theorem. This conjecture was proved by Goodearl.

THEOREM 2.20. [Goodearl-Handelman-Lawrence (1974)]. Suppose that R is bounded strongly prime and $SP(R) > 1$. Then R is a prime (left) Goldie ring.

We will not reproduce Goodearl's proof in this thesis. We will, however, look at several corollaries.

PROPOSITION 2.21. Let R be a strongly prime ring. The following are equivalent:

1. R is prime (left) Goldie,
2. There exists $m > 1$ such that $SP(R) < SP(M_m(R))$,
3. For every $m > 1$, $SP(R) < SP(M_m(R))$.

PROOF. $1 \Rightarrow 2$. Use Theorem 2.19.

$2 \Rightarrow 3$. As $SP(M_m(R)) > SP(R)$, $M_m(R)$ is prime Goldie (Theorem 2.20). Use Theorem 2.19.

$3 \Rightarrow 1$. As $\infty > SP(M_2(R)) \geq 2$, $M_2(R)$ is prime Goldie, thus, so is R .

PROPOSITION 2.22. If R is bounded strongly prime, then R has no non-zero nil ideals.

PROOF. If $SP(R) > 1$, then the result follows from the fact that R is a prime Goldie ring [Fisher (1971)]. If $SP(R) = 1$, the result follows trivially.

PROPOSITION 2.23. If R is bounded strongly prime and the right zero divisors of R form a right ideal, then R is right weakly primitive.

PROOF. We may assume that R is not right strongly prime (for then the result is obvious). Hence, we may assume that every nonzero element of R has a one element left insulator. Let M be the proper right ideal of right zero divisors. We will prove that M is a P-right ideal and from this it will follow that R is right weakly primitive. If J is a nonzero ideal of R , choose $0 \neq t \in J$ and let $S(J) = S(t)t$, where $S(t)$ is a one element insulator of t . If $S(J)r \in M$, then $S(J)r$ is a right zero divisor, so there exists $0 \neq u \in R$ such that $uS(J)r = 0$. But $uS(J) \neq 0$; hence, r is a right zero divisor, thus, $r \in M$. Hence, $\{r \in R \mid S(J)r \in M\} \subset M$.

3. APPLICATIONS TO PRIMITIVE RINGS

In this chapter we apply some of the results of previous chapters to construct examples of primitive rings. Of these, the most interesting is the construction of singular primitive rings. In [Osofsky (1967)] an example of a singular semiprimitive ring was constructed. Faith conjectured the existence of singular primitive rings [Faith (1967, p. 128)]. This was answered in the affirmative by the author [Lawrence (1974)]. In this chapter we give two examples of singular primitive rings. The simplest example is nonsingular on one side. The second example is primitive and singular on both sides.

Kaplansky proved that every commutative domain is the centre of some primitive ring [Jacobson (1964), p.36]. We generalize this by showing that every prime ring can be embedded in a primitive ring so that their centres coincide.

We show that the class of weakly primitive rings is not inductive. In fact, we show that a countable union of strongly prime rings need not be weakly primitive. As a corollary, we have an example to show that the class of primitive rings is not countably inductive.

THEOREM 3.1. Let R be a prime ring with centre $C(R)$. Then R can be embedded in a primitive ring T such that $C(R) = C(T)$.

PROOF. There exists a strongly prime ring S containing R such that $C(R) = C(S)$. Let $T = S[X]$ be the free monoid ring with $\text{card } X \geq \text{card } S$. Then $C(T) = C(S)$ and T is primitive (Theorem 1.10).

THEOREM 3.2. There exists a ring which is right and left primitive, right singular and left nonsingular.

PROOF. Let R be Example 2.18. R is left strongly primitive and right singular. Let $T = R[W]$ be the free monoid ring with $\text{card } W \geq \text{card } R$.

1. T is left strongly prime (Corollary 1.19.1). Hence T is left nonsingular, (Proposition 2.7).
2. R is left strongly prime, so T is left primitive (Theorem 1.10).
3. As R is right singular, T is right singular [Burgess (1969)].
4. If $0 \neq r \in R$ annihilates a nonzero monomial of R on the right, then $\sum_n x^n r = 0$, for sufficiently large n . We now use a variation on Proposition 2.23 in order to prove that R is right weakly primitive.

Let $M = \{r \in R \mid r \text{ annihilates a nonzero monomial of } R \text{ on the right}\}$. By the above observation, M is a right ideal of R . If $(0) \neq J$ is an ideal of R , choose $0 \neq s \in J$ with minimal support. Now s has left insulator $\{y_t\}$ and if $y_t s r \in M$, then $\sum_n x^n y_t s r = 0$ for sufficiently large n . But $\sum_n x^n y_t s \neq 0$, thus $r \in M$. Let $S(J) = \{y_t s\}$. We now see that M is a P-left ideal of R ; therefore, $R[W]$ is right primitive.

The above example is not quite satisfactory. This is because the ring is singular on only one side. We now give a more interesting example.

Let $F = Z_2[X, Y_j]$, $j = 1, 2, \dots$, be the free algebra over Z_2 in noncommuting indeterminates. An arbitrary monomial of F can be written as

$$m = x^{i_1} y_{j_1} x^{i_2} \dots y_{j_{n-1}} x^{i_n}, \quad i_{\underline{}} \geq 0, j_{\underline{}} \geq 1,$$

Define

$$c(m) = \sum i_k = \text{degree of } X \text{ in } m,$$

$$d(m) = \sum i_k + n-1 = \text{degree of } m,$$

$$e(m) = \max_{j_k} \{ j_k \text{ times the number of times } Y_{j_k} \text{ appears in } m \}, \text{ if } m \text{ contains a } Y \text{ term,}$$

$$= 0, \text{ otherwise.}$$

Let I be the ideal of F generated by monomials m in F such that $c(m) > e(m) \geq 1$. Let $R = F/I$. We claim that this is the desired example. I is 'homogeneous' in the sense that a sum of distinct monomials of F is in I if and only if each monomial is in I . This allows us to speak of monomials in R . Given $(0 \neq) f = m_1 + \dots + m_n$, a sum of distinct (nonzero) monomials in R , define

$$c(f) = \max \{ c(m_i) \},$$

$$d(f) = \max \{ d(m_i) \},$$

$$e(f) = \max \{ e(m_i) \}.$$

PROPOSITION 3.3. The singular ideal of R is nonzero.

PROOF. The proof is similar to the proof of Lemma 4 in [Osofsky (1967)].

Suppose $(0 \neq) f \in R$ is a sum of nonzero monomials as above. Let

$s = \min \{ c(m_i) \}$. Choose $t > \max \{ d(f), e(f) \}$. Then $g = x^{t-s} Y_t f \neq 0$

and $gX = 0$. Let E be the left ideal of R generated by all the g 's

as f runs through all nonzero elements of R . We claim that E is an essential left ideal. If $(0) \neq J$ is a left ideal of R , let $f \in J$ be a nonzero element of J . Then $0 \neq g \in J \cap E$, thus E is essential. But $EX = 0$, so $X \in Z_\ell(R)$. This completes the proof of the proposition.

Since R is countable, we can order the nonzero elements $f_2, f_3, \dots$. We start the numbering at 2 in order to simplify the notation in certain subsequent statements. Given f_n in this sequence, choose $j_n > 1$ large enough so that $Y_{j_n} f_n Y_{j_n} \neq 0$, and let

$$g_n = Y_{j_n}^{i_n} Y_{j_n} f_n Y_{j_n} Y_{j_n}^{i_n} Y_{j_n} \neq 0,$$

where $i_n > 2 \max \{d(f_n), e(f_n)\}$. For convenience, choose $j_{n+1} > j_n$,

and $i_{n+1} > i_n$. Suppose $g_n = a(n, 1) + \dots + a(n, k_n)$, where $a(n, i)$

is a nonzero monomial in R , $i = 1, 2, \dots, k_n$. Let A be the Z_2 -subalgebra of R generated by all the $a(n, i)$.

LEMMA 3.4. The following hold in A .

1. If m_1, m_2 are any monomials of R and $a(n, i), a(m, j)$ are generators of A , then $0 \neq m_1 a(n, i) = m_2 a(m, j)$ implies $i = j$ and $m = n$.
2. If $a_1, \dots, a_n$ are generators of A , then

$$\prod_{i=1}^n a_i \neq 0.$$

3. A is a free $\mathbb{Z}_2$ -algebra on the given generators.

4. If $a \in A$ and $b \in R$ and b is a sum of monomials, none of which is in A , then $ba \in A$ implies $ba = 0$.

PROOF. 1. Suppose the products are nonzero. They are equal if and only if they are identical. Since $a(n, i)$ ends with y_n , we conclude that $m = n$. Now

$$m_1 a(n, i) = m_1 y_1^{i_1} y_{j_n} b y_{j_n} y_1^{i_n} y_n, \quad n \neq 1, j_n \neq 1.$$

As $i_n > d(b)$, we can 'decide' which generator occurs at the end of the product.

$$2. \quad \prod a_k = y_1^{i_1} y_{j_k} b y_{j_k} y_1^{i_k} y_k, \quad \text{where } i_k > 2 \max \{d(b_k), e(b_k)\}.$$

If this product is zero, then some segment of it must be a generator of I , say

$$m = y_{j_s} b y_{j_s} y_1^{i_s} y_s \dots y_1^{i_t} y_{j_t} b y_{j_t} y_t.$$

If $s \neq t$, then $c(m) \leq \sum_{k=s}^t c(b_k)$ and $e(m) \geq \sum_{k=s}^t i_k$, and so, $e(m) > c(m)$,

a contradiction. If the generator contains only one b_k (or part of one), the result is obvious.

3. This follows from 1 and 2 if $a_{i_1} \dots a_{i_k} = a_{j_1} \dots a_{j_n}$,

then 2 implies that this product is nonzero, and using 1 inductively,

we obtain $a_{j_n} = a_{i_k}$ etc.

4. Since both I and A are generated by monomials, we will have finished if we can prove the result assuming both a and b are monomials. Suppose $a = a_{i_1} \dots a_{i_k}$ and $0 \neq ba = a_{j_1} \dots a_{j_n}$. Using 1, we conclude that $b \in A$, a contradiction. If $a = 1$, the result is obvious.

THEOREM 3.5. R is (left) primitive.

PROOF. Let M be the left ideal of R generated by the set $\{g_n + 1\}$, $n = 2, 3, \dots$. If $(0) \neq J$ is a two-sided ideal of R , then $f_n \in J$, for some n . Hence, $g_n \in J$. Then $1 \in M + J$, thus M is comaximal. If M is not proper, then for some $\{r_i\} \subset R$ and some n , we have

$$r_2(g_2 + 1) + \dots + r_n(g_n + 1) = 1.$$

Let $r_i = s_i + t_i$, where each monomial of s_i is not in A and each monomial of t_i is in A . Then

$$\sum (s_i + t_i)(g_i + 1) = 1,$$

and so

$$\sum s_i(g_i + 1) = 0,$$

by the previous lemma, and

$$\sum t_i(g_i + 1) = 1.$$

The latter equation gives a non-trivial relation in the free algebra A , and thus, is impossible. Since we arrive at a contradiction by assuming that M is not proper, the theorem is proved.

We now turn our attention to the question of inductivity of the class of primitive rings. Recall that a class R of rings is inductive if for every ordered chain $\{R_i\}_{i \in I} \subset R$, where I is totally ordered and $R_i \subset R_j$ if $i \leq j$, $\bigcup_{i \in I} R_i \in R$. An example due to Formanek and Passman shows that the class of primitive rings is not in general inductive. In their example the set I is uncountable.

Let Ω be the set of countable ordinals. For each ordinal α let $R_\alpha = \mathbb{Q}[Y_\beta; \beta < \alpha]$ be the commutative polynomial ring. Clearly, R_α is a countable domain for each α and $\bigcup R_\alpha$ is uncountable. Let X be a countable set of indeterminates and let $R_\alpha[X]$ be the free monoid ring. By Theorem 1.10, $R_\alpha[X]$ is primitive, while, by Proposition 1.13 $(\bigcup R_\alpha)[X] \cong U(R_\alpha[X])$ is not primitive.

We now give an example of a countable chain of primitive rings whose union is not primitive or even weakly primitive.

PROPOSITION 3.6. There exists a countable ascending chain of strongly prime rings whose union is not weakly primitive.

PROOF. Let $D = Z_2[W, Y_j], j = 1, 2, \dots$ be the free Z_2 -algebra and let $D_n = Z_2[W, Y_j], j = 1, 2, \dots, n$. Let I be the ideal of D generated by monomials of the form

$$m = WY_{j_1} Y_{j_2} \dots Y_{j_\ell} W^{i_1} Y_{j_{\ell+1}} \dots W^{i_{n-\ell}} Y_{j_n} W^{i_{n-\ell+1}}$$

where

1. $n > 1$,
2. $j_\ell = \max \{j_k\}, k = 1, 2, \dots, n$,
3. (degree of W in m) $> j_\ell$ (number of times Y_{j_ℓ} appears in m),
4. $i_{n+1-\ell} > 0, i_k \geq 0, k = 1, 2, \dots, n-\ell$

Set $R = D/I$ and $R_n = D_n/I \cap D_n$.

We first show that for all n , R_n is strongly prime.

Choose $0 \neq a \in R_n$, and let $h = 2$ (degree of W in a). Then $\{Y_n^h\}$ is an insulator of a in R_n . To see this note that it is an insulator of a monomial of maximal degree in R .

We now show that R is not weakly primitive. Suppose M is a P -left ideal of R . Choose h such that if Y_j occurs as an indeterminate in any term in $S(\langle W \rangle)$, then $h > j$. It is easily proven that for sufficiently large t ,

$$S(\langle WY_h \rangle) \cdot S(\langle W \rangle)^t = \{0\}.$$

By definition, for any ideal I ,

$$\{a \in R \mid a \cdot S(I) \subset M\} \subset M.$$

Hence, by induction, $M = R$. Thus R has no proper P -left ideal.

COROLLARY 3.6.1. There exists a countable ascending chain of primitive rings whose union is not weakly primitive.

PROOF. We use the usual argument applied to the previous theorem, that is, look at certain suitable free monoid rings.

Recall that an involution $*$ on a ring R is a map $*$: $R \rightarrow R$ satisfying:

1. $(a + b)^* = a^* + b^*$,
2. $(a^*)^* = a$,
3. $(ab)^* = a^*b^*$.

An element $a \in R$ is symmetric if $a = a^*$ and skew symmetric if $a^* = -a$.

If R is a subring of the ring T , can an involution on R be extended to T ? In general, the answer is 'no'. In [Rowen (1974)] it was proved that an involution on a semiprime PI ring can be extended to the complete ring of quotients. The following propositions show the limitation of the extension of involutions to the complete ring of quotients.

PROPOSITION 3.7. Let S^* be a ring with involution with $*$ -subring $R(R^* = R)$. If ${}_R S_R$ is essential over R_R , then ${}_R S$ is essential over ${}_R R$. Also, if S_R is R -injective, then ${}_R S$ is R -injective.

PROOF. The first claim is clear. In order to prove the second claim, we use the Baer Criterion. Let I be a left ideal of R and let $\pi : I \rightarrow {}_R S$ be an R -module map. It is sufficient to show that there exists $s \in S$ such that $\pi(m) = ms$ for all $m \in I$. Now I^* is a right ideal of R , and $\pi^* : I^* \rightarrow S_R$ defined by $\pi^*(n) = (\pi(n^*))^*$ is a right

R -module map. Since S_R is injective, there exists $s^* \in S_R$ such that $\pi^*(m) = s^*m$, for all $m \in I^*$. Thus $\pi^*(m^*) = s^*m^* = (ms)^*$, for all $m \in I$. Hence, $\pi(m) = ms$ for all $m \in I$.

COROLLARY 3.7.1. Let V be an infinite dimensional F -vector space. Then $\text{End}_F V$ is a ring without involution.

PROOF. $\text{End}_F V$ is right but not left self-injective.

COROLLARY 3.7.2. Let R be a (right) nonsingular ring with right complete ring of quotients Q . If Q is a ring with involution $*$ and R is a $*$ -subring, then Q is the left complete quotient ring of R .

PROOF. Q_R is injective and essential over R_R , thus R^Q is injective and left essential. Thus Q is the left complete quotient ring of R [Lambek (1966)].

Given an algebra with involution, there exists a simple regular algebra containing the given algebra, to which the involution can be extended [Lawrence (b)]. We prove a weakened form of this result.

PROPOSITION 3.8. An algebra with involution can be embedded in a primitive algebra with involution.

PROOF. Let A be an F -algebra with involution, and let $R = A *_F F[X]$ denote the coproduct of A with the polynomial ring $F[X]$ amalgamating F . We extend the involution on A to R by letting $(a_{i_1} X a_{i_2} X \dots X a_{i_n})^* = a_{i_n}^* X \dots X a_{i_2}^* X a_{i_1}^*$, for all $a_{i_j} \in A$. Now let Y be a set of indeterminates

with $\text{card } Y \geq \text{card } R$, and let $B = R[Y]$ be the free monoid ring. The involution on R can be extended to $R[Y]$ by mapping each indeterminate identically. Also, B is primitive.

4. VALUATIONS ON MODULES AND TENSOR PRODUCTS

In the following chapters, we shift our attention to an examination of the Jacobson radical of tensor products, and related problems. The idea for the method comes from a paper of Formanek [Formanek (1974)].

The main question asked is: if $1-\alpha$ is invertible in a tensor product or group ring, can we conclude that α is algebraic? For example, if $\alpha(r) = r(1-g) \in \mathbb{Q}[C]$, where $\langle g \rangle = C$ is the infinite cyclic group, and $r \in \mathbb{Q}$ then $1-\alpha(r)$ is invertible only if $r = 1$ or 0 . Clearly, $\alpha(r)$ is algebraic only if $r = 0$. Thus, with one exception, $1-\alpha(r)$ invertible implies that $\alpha(r)$ is algebraic. We will see shortly that the exception occurs for topological reasons.

Recall that a non-archimedean valuation v on a ring R is a map $v: R \rightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ = non-negative real numbers) satisfying:

1. $v(1) = v(-1) = 1$, $v(r) = 0$ if and only if $r = 0$,
2. $v(r+s) \leq \max \{v(r), v(s)\}$, for all $r, s \in R$,
3. $v(rs) = v(r)v(s)$, for all $r, s \in R$.

If (2) is replaced by the weaker condition (2') $v(r+s) \leq v(r) + v(s)$, for all $r, s \in R$, then v is said to be an archimedean valuation. The absolute value on the rational numbers is an example of an archimedean valuation.

Note that (1) and (3) imply that R is a domain. Every domain has a 'trivial' valuation defined by $v(r) = 1$ if and only if $r \neq 0$. If R is a unique factorization domain and $x \in R$ is a prime, then every nonzero element r is a product of the form $r = x^n y$, where $n \geq 0$ and x does not divide y . Define the ' x -adic' valuation on R by $v(r) = 2^{-n}$

It is easily checked that v satisfies the conditions of a nonarchimedean valuation.

Let R be a ring with valuation v (archimedean or non-archimedean), and let M be a left R -module. A nonarchimedean valuation on M is a map $\mu : R \rightarrow R^+$ satisfying:

1. $\mu(m+m') \leq \max \{ \mu(m), \mu(m') \}$, for all $m, m' \in M$,
2. $\mu(rm) = v(r) \cdot \mu(m)$, for all $r \in R, m \in M$.

If (1) is replaced by the weaker condition

$$1'. \mu(m+m') \leq \mu(m) + \mu(m'),$$

μ is said to be archimedean.

It is too restrictive to insist that $\mu(m) = 0$ if and only if $m = 0$.

Let $A(M)$ denote the submodule of M generated by the set

$\{m \in M : \exists 0 \neq r \in R \text{ such that } rm = 0\}$. If μ is a valuation on M ,

let $\ker \mu = \{m \in M : \mu(m) = 0\}$. It is clear that $\ker \mu \supseteq A(M)$. If

$\ker \mu = A(M)$, then μ is said to be maximal.

Given a ring R with valuation v and a left R -module M , let

$V(M)$ denote the set of nonarchimedean valuations on M and $U(M)$ denote

the set of archimedean valuations on M . Clearly $V(M) \subseteq U(M)$. We now

define operations on $U(M)$ which gives it a structure similar to that of

a module. If $\mu, \mu' \in U(M)$ define $\mu + \mu'$ by $(\mu + \mu')(m) = \max \{ \mu(m), \mu'(m) \}$.

Also define, for $r \in R$, μr by $\mu r(m) = \mu(rm)$.

PROPOSITION 4.1. Let R be a ring with valuation v and let M be a left R -module.

1. If $\mu, \mu' \in U(M)$, then $\mu + \mu' \in U(M)$.
2. If $\mu \in U(M)$, $r \in R$, then $ur \in U(M)$.
3. If $\mu, \mu' \in V(M)$, then $\mu + \mu' \in V(M)$.
4. If $\mu \in V(M)$, $r \in R$, then $ur \in V(M)$.

PROOF. 1. We have $(\mu + \mu')(m + m') = \max \{ \mu(m + m'), \mu'(m + m') \}$
 $\leq \max \{ \mu(m) + \mu(m'), \mu'(m) + \mu'(m') \}$
 $\leq \max \{ \mu(m), \mu'(m) \} + \max \{ \mu(m'), \mu'(m') \} = (\mu + \mu')(m) + (\mu + \mu')(m').$
 Also, $(\mu + \mu')(rm) = \max \{ \mu(rm), \mu'(rm) \} = \max \{ \forall(r)\mu(m), \forall(r)\mu'(m) \}$
 $= \forall(r) \max \{ \mu(m), \mu'(m) \} = \forall(r)(\mu + \mu')(m).$

2. We have $(\mu r)(m + m') = \mu(rm + rm')$
 $\leq \mu(rm) + \mu(rm') = (ur)(m) + (ur)(m').$
 Also, $(ur)(r'm) = u(rr'm) = \forall(r)\forall(r')\mu(m)$
 $= \forall(r')\forall(r)\mu(m) = \forall(r')(\mu r)(m).$

3. We have $(\mu + \mu')(m) = \max \{ \mu(m + m'), \mu'(m + m') \}$
 $\leq \max \{ \max \{ \mu(m), \mu(m') \}, \max \{ \mu'(m), \mu'(m') \} \}$
 $= \max \{ (\mu + \mu')(m), (\mu + \mu')(m') \}.$
 Also, $(\mu + \mu')(rm) = \max \{ \mu(rm), \mu'(rm) \}$
 $= \forall(r) \max \{ \mu(m), \mu'(m) \} = \forall(r)(\mu + \mu')(m).$

4. The proof is similar to the proof of 2.

We can generalize the above addition of valuations.

PROPOSITION 4.2. Let R be a ring with valuation v and let M be a

left R -module. Suppose that $\{\mu_i\}_{i \in I}$ is a set of valuations on M such that for all $m \in M$, $\sup_{i \in I} \{\mu_i(m)\} < \infty$. Then $\mu = \sup \{\mu_i\}$

defined by $\mu(m) = \sup_{i \in I} \{\mu_i(m)\}$ is a valuation on M .

PROOF. The proof is similar to the proof of the previous proposition.

PROPOSITION 4.3. Let $\mu \in V(M)$ be a nonarchimedean valuation on a module M . If $\mu(m) > \mu(m')$, then $\mu(m + m') = \mu(m)$.

PROOF. This is a well-known result for nonarchimedean valuations on rings.

We have

$$\begin{aligned} \mu(m) &\leq \max \{ \mu(m + m'), \mu(-m') \} \\ &= \max \{ \mu(m + m'), v(-1) \mu(m') \} \\ &= \mu(m + m') \leq \max \{ \mu(m), \mu(m') \} = \mu(m). \end{aligned}$$

Thus the inequalities are all equalities.

Now take two R -modules M and N , where R is a ring with valuation v . Suppose we have a homomorphism

$$\pi : M \longrightarrow N.$$

If μ is a valuation on N we can 'lift' it back to M by defining $\pi^{-1}\mu(m) = \mu(\pi(m))$. If $\mu \in V(N)$ then $\pi^{-1}\mu \in V(M)$. We give a short proof.

If $m, m' \in M$, then $\pi^{-1}\mu(m+m') = \mu(\pi(m+m')) = \mu(\pi m + \pi m')$

$$\leq \max \{ \mu(\pi m), \mu(\pi m') \} = \max \{ \pi^{-1}\mu(m), \pi^{-1}\mu(m') \}.$$

Also, if $m \in M$ and $r \in R$, then $\pi^{-1}\mu(rm) = \mu(\pi(rm)) = \mu(r\pi m)$.

$$= v(r)\mu(\pi m) = v(r)\pi^{-1}\mu(m).$$

In a similar fashion, we can show that if $\mu \in U(N)$, then

$$\pi^{-1}\mu \in U(M).$$

It is clear that if π is an epimorphism, then every valuation on N 'comes from' some valuation on M , namely μ comes from $\pi^{-1}\mu$. What can we say about the dual situation? Suppose π is a monomorphism. Can we then say that every valuation on M is the 'restriction' of some valuation on N ? In general the answer is no. For a fairly large class of domains, the answer is 'yes'.

THEOREM 4.4. Suppose R is a domain with valuation v . Then any valuation on any R -submodule M of any R -module N can be extended to N if and only if R is a left Ore domain. Moreover, if R is a left Ore domain, N a left R -module and M a submodule of N , then a maximal valuation on M may be extended to a maximal valuation on N .

PROOF. First suppose that R is a left Ore domain. Let μ be a valuation on M . By ordering the graphs of possible extensions of μ and by use of Zorn's Lemma, we may suppose that there is no extension of μ to μ' on a submodule N' properly containing M .

It then remains to show that $M = N$. Let $M' = \{n \in N : \exists 0 \neq r \in R \text{ such that } rn \in M\}$. If $m \in M'$ and r and r' are two nonzero elements of R such that $rm \in M$ and $r'm \in M$, then $v(r)\mu(r'm) = v(r')\mu(rm)$. To see this, note that r and r' have a common left multiple; hence, there exists $0 \neq a, b \in R$ such that $ar = br'$. Thus $v(a)\mu(rm) = v(b)\mu(r'm)$ and

$$v(a)v(r) = v(b)v(r'), \text{ proving our claim. } M'$$

is a left R -submodule of N (since R is a left Ore domain). We extend μ to μ' on M' by defining $\mu'(m) = \mu(rm)/v(r)$, where $0 \neq r \in R$ and $rm \in M$. We claim that μ' is a valuation on M' . If μ is nonarchimedean,

we claim that μ' is nonarchimedean. First, $\mu'(sm) = \mu(r'sm)/\nu(r')$, where $r'sm \in M$. Choose $0 \neq a, b \in R$ such that $ar = bs$. Then $\mu'(sm) = \mu(bsm)/\nu(b) = \mu(arm)/\nu(b) = \mu(rm)\nu(a)/\nu(b) = \nu(r)\mu'(m)\nu(a)/\nu(b) = \nu(s)\mu'(m)$. Also, $\mu'(m + m') = \mu(r(m + m'))/\nu(r)$, where $rm \in M$, $rm' \in M$. Thus $\mu'(m + m') \leq \max\{\mu(rm), \mu(rm')\}/\nu(r) = \max\{\mu(rm)/\nu(r), \mu(rm')/\nu(r)\} = \max\{\mu'(m), \mu'(m')\}$. Thus μ' is nonarchimedean. Similarly if μ is archimedean, then μ' is archimedean. Now, $\ker \mu' = \{m \in M : \mu'(m) = 0\} = \{m \in M : \exists 0 \neq r \in R \text{ such that } \mu(rm) = 0, \text{ where } rm \in M\}$. Since R is a left Ore domain, if $\ker \mu = A(M)$, then $\ker \mu' = A(M')$.

Since, by hypothesis, we have only trivial extensions of μ , we may suppose that $M = M'$. Now assume that $M \neq N$. Take $t \in N \setminus M$. Clearly $Rt \cap M = 0$, and $rt = 0$ implies $r = 0$. Define a valuation μ'' on Rt by $\mu''(rt) = \nu(r)$. By our previous remarks, this is well-defined and is a valuation on Rt . Now define a valuation μ' on $M \oplus Rt$ by $\mu' = \mu + \mu''$. By Proposition 4.1, this is a valuation on $M \oplus Rt$. If μ and ν are nonarchimedean, then μ' is non-archimedean. $\ker \mu' = \{m \in M + Rt : \mu'(m) = 0\}$. It is easily checked that if $\ker \mu = A(M)$, then $\ker \mu' = A(M \oplus Rt)$.

Thus, on the assumption that $M \neq N$, we have properly extended μ . This contradiction completes the proof that if R is a left Ore domain, valuations on submodules may be extended to valuations on modules. We have also shown that a maximal valuation may be extended to a maximal valuation. It remains to show that if valuations can, in general, be extended, then R is a left Ore domain.

Let ν be the trivial valuation on a domain R . Choose $0 \neq x, y \in R$. Let $R' = R[z]$ and $M = R'/[Rxz + R(yz-1)]$ be a left R -module. We

claim that R is not an R -submodule of M via the canonical map.

For, suppose we could extend v to a valuation μ on M . Then $\mu(z) = v(x)\mu(z) = \mu(xz) = 0$; however, $yz = 1$, so $\mu(1) = v(1) = 0$, a contradiction. Thus there exists $0 \neq r \in R$ such that $r = axz + byz - b$, where $a, b \in R$. Comparing degrees of z , we see that $ax = -by$. If $ax = 0$, then $b = 0$, so $r = 0$ a contradiction. Thus x and y have a common left multiple; hence, R is a left Ore domain. This completes the proof of the theorem.

COROLLARY 4.4.1. Suppose that A is an F -algebra with $a \in A$ not algebraic over F . Let $R = F[a]$. Then there is a valuation v on R^A such that v is trivial on F and $v(a) = 2^{-1}$.

PROOF. Let v be the a -adic valuation on R . Since R is an Ore domain, this valuation can be extended to R^A .

We now look at two examples which show that extension properties do not come easily.

EXAMPLE 4.5. We give an example of a valuation on a commutative domain which cannot be extended, as a ring valuation, to an over-domain. Let

$Q[x, y]$ be the free Q -algebra on two non-commuting variables. Let $D = Q[x, y] / \langle xy + (y-1)x \rangle$. It can be shown [Cohn (1958)] that D is a domain with $Q[\bar{y}] \cong Q[y]$ and $\bar{x} \neq 0$. Let v be the y -adic valuation on $Q[\bar{y}]$. If this could be extended to D , then since $\overline{xy} + \overline{yx} = \bar{x}$, we would have $v(\bar{x}) \leq \max \{v(\overline{xy}), v(\overline{yx})\} = v(\overline{xy}) = v(\bar{x})v(\bar{y}) < v(\bar{x})$, a contradiction.

EXAMPLE 4.6. We give an example of a filtration on a submodule which cannot be extended to a filtration on the module. (Recall that with a filtration, we replace (2) by the condition $\mu(rm) \leq \nu(r) + \mu(m)$; (1) remains as before). Let $R = \mathbb{Z}_2[x_i]$, $i = 1, 2, 3, \dots$, be the commutative polynomial

ring in countably many variables. Let $M = \bigoplus_{j=1}^{\infty} Ry_j$ be an R -module

with R acting on M by $x_i y_j = 0$ for all i, j . Now let

$$M' = [M \oplus Rz] / \sum_{i=1}^{\infty} R(x_i z - y_i).$$

We claim that M is an R -submodule of M' via the canonical map.

For suppose $r_1 y_1 + \dots + r_k y_k \in \sum R(x_j z - y_j)$, in $M \oplus Rz$, where $r_i \in R$.

We may suppose that $r_i \in \mathbb{Z}_2$. Thus $\sum r_i y_i = \sum s_j (x_j z - y_j)$, where

$s_j \in R$. Let $s_j = t_j + r_j'$, where r_j' is the zero degree part of s_j .

Then $\sum (t_j + r_j') (x_j z - y_j) = \sum r_i y_i$. But $t_j y_j = 0$, so $r_i' = r_i$ for all i , and

$$\sum (t_j + r_j') x_j z = 0.$$

Taking the first degree part (in $R[x_i]$), we have $\sum r_j' x_j z = 0$, which implies that $r_j' = 0$ for all j . Thus $r_i = 0$ for all i . This proves our claim. Let ν be the trivial valuation on R , and let μ be the

filtration on M defined by $\mu(y_n) = n$ and $\mu\left(\sum_{i=1}^k y_i\right) = \sup\{\mu(y_i)\}$.

If μ could be extended to M' , then $\mu(\bar{z}) = \nu(x_n) \mu(\bar{z}) \geq \mu(\overline{x_n z}) = \mu(\bar{y}_n) = n$ for all n . This is impossible since $\mu(\bar{z})$ is a real number.

We now use our work on valuations to examine zero divisors and invertible elements in group rings and tensor products.

PROPOSITION 4.7. 1. Suppose that T is a ring, R a subring and ν a nonarchimedean (archimedean) valuation on R^T . Let G be a group.

If $\alpha = \sum t_i q_i \in T[G]$, define $\mu(\alpha) = \max \{\nu(t_i)\}$. Then μ is a non-archimedean (archimedean) valuation on $R(T[G])$ which extends ν . If $\beta \in R[G]$, then $\mu(\beta)\mu(\alpha) \geq \mu(\beta\alpha)$, if ν is nonarchimedean, and $k\mu(\beta)\mu(\alpha) \geq \mu(\beta\alpha)$, where $k = \text{card supp } \beta$, if ν is archimedean.

2. Suppose that A is an F -algebra with subalgebra R , and ν is a nonarchimedean valuation on R^A which is trivial on F . Let B be an F -algebra and let $\{b_i\}$ be an F -basis of B . Take $\alpha \in A \otimes_F B$ with $\alpha = \sum (a_i \otimes b_i)$, $a_i \in A$, and define $\mu(\alpha) = \max \{\nu(a_i)\}$. Then μ is a nonarchimedean valuation on $R(A \otimes B)$. Moreover, if $\beta \in R \otimes_F B$, then $\mu(\beta\alpha) \leq \mu(\beta)\mu(\alpha)$.

PROOF. 1. First suppose that ν is nonarchimedean. Let $\alpha = \sum t_i q_i$ and $\gamma = \sum t'_i q_i$. If $r \in R$, then $\mu(r\alpha) = \max \{\nu(rt_i)\} = \nu(r) \max \{\nu(t_i)\} = \nu(r)\mu(\alpha)$. It is also easily seen that $\mu(\alpha + \gamma) \leq \max\{\mu(\alpha), \mu(\gamma)\}$. If

$$\beta = \sum r_j q_j, \quad r_j \in R, \quad \text{then} \quad \mu(\beta\alpha) = \mu\left(\sum r_j t_i q_j q_i\right) = \mu\left(\sum \left(\sum r_c t_c\right) q_d\right)$$

$$= \max \{ \nu \left(\sum r_c t_d \right) \} \leq \max \{ \nu(r_c t_d) \} \leq \max \{ \nu(r_c) \} \cdot \max \{ \nu(t_d) \} = \mu(\beta) \mu(\alpha).$$

Now suppose that T is archimedean. Let $\alpha = \sum t_i g_i \in T[G]$ and

$$\beta = \sum r_j g_j \in R[G]. \text{ Then } \mu(\beta\alpha) = \mu \left(\sum \left(\sum r_c t_d \right) g \right), \text{ where each}$$

$$\text{sum } \sum r_c t_d \text{ consists of at most } k \text{ terms. Thus } \mu(\beta\alpha) = \max \left\{ \nu \left(\sum r_c t_d \right) \right\}$$

$$\leq k \max \{ \nu(r_c) \cdot \nu(t_d) \} \leq k \mu(\beta) \mu(\alpha).$$

2. The proof of this is similar to the proof of 1. Suppose

$$\alpha = \sum (a_i \otimes b_i) \text{ and } \beta = \sum (r_j \otimes b_j), \quad r_j \in R. \text{ Let } b_j b_i = \sum f_{jik} b_k,$$

$$f_{jik} \in F. \text{ Then } \mu(\beta\alpha) = \mu \left(\sum_{i,j} (r_j a_i \otimes \sum_k f_{jik} b_k) \right) = \mu \left(\sum_k \left(\sum_{i,j} f_{jik} r_j a_i \right) \otimes b_k \right)$$

$$= \max_k \left\{ \nu \left(\sum f_{jik} r_j a_i \right) \right\} \leq \max_k \left\{ \nu(f_{jik} r_j a_i) \right\} \leq \max \left\{ \nu(r_j a_i) \right\} = \max \left\{ \nu(r_j) \nu(a_i) \right\}$$

$$\leq \mu(\beta) \mu(\alpha).$$

The above extension μ is called the max-extension of ν on $T[G]$.

PROPOSITION 4.8. Suppose that T is a ring, R a subring, and ν is a nonarchimedean (archimedean) valuation on R with $\ker \nu = 0$. Let μ be the max-extension of ν on $T[G]$. Suppose $\alpha \in R[G]$ and $\mu(\alpha) < 1$ ($k\mu(\alpha) < 1$, where $k = \text{card supp } \alpha$), then $1 - \alpha$ is not a zero divisor in $T[G]$.

PROOF. Suppose that ν is nonarchimedean. If $\beta \in T[G]$ and $(1 - \alpha)\beta = 0$, then $\beta = \alpha\beta$; hence, $\mu(\beta) = \mu(\alpha\beta) = \mu(\alpha)\mu(\beta)$. Thus $\beta = 0$.

The above proposition can be generalized to the tensor product case if we assume that ν is a nonarchimedean valuation which is trivial

on the base field.

THEOREM 4.9.1. Let T be a ring, R a subring, and v a nonarchimedean [archimedean] valuation on R . Let G be a group and suppose

$\alpha = \sum t_i q_i \in R[G]$. Let $\partial(n) = 1 + \alpha + \alpha^2 + \dots + \alpha^n$. Suppose that

- i. $1-\alpha$ is right invertible with inverse β
- ii. $\mu(\alpha) = s < 1$, ($\mu = \text{max-extension of } v$).

Suppose $g \in G$. If there exists $m \in \mathbb{N}$ such that $v(\text{coefficient of } g \text{ in } \partial(m)) > v(\text{coefficient of } g \text{ in } \alpha^n)$ for all $n > m$ [if there exists m such that $v(\text{coefficient of } g \text{ in } \partial(m)) > k^m s^{m+1} (1-ks)^{-1}$, where $k = \text{card supp } \alpha$ and $1-ks > 0$], then $g \in \text{supp } \beta$.

2. Suppose that the same conditions held, where T is an F -algebra and A is an F -algebra. If v is a nonarchimedean valuation on R , trivial on F , then a similar conclusion holds for $T \otimes_F A$.

PROOF: 1. First assume that v is nonarchimedean. If $g \notin \text{supp } \beta$, then $\mu(\beta - \partial(m)) \geq v(\text{coefficient of } g \text{ in } \partial(m)) = h > 0$. Now if $n > m$, $\mu(\beta - \partial(n)) = \mu(\beta - \partial(m) - (\partial(n) - \partial(m))) \geq \mu(\beta - \partial(m))$. Clearly $\mu(\alpha^n) \leq s^n$ for all n . Thus $s^{n+1} \geq \mu(\alpha^{n+1}) = \mu((1-\alpha)(\beta - \partial(n))) = \mu((\beta - \partial(n)) - \alpha(\beta - \partial(n)))$. But $\mu(\alpha) < 1$, so, by Proposition 4.7, $\mu(\alpha(\beta - \partial(n))) < \mu(\beta - \partial(n))$; whence $s^{n+1} \geq \mu(\beta - \partial(n)) \geq h$, for all n . This is a contradiction, since $s < 1$ and $h > 0$. Thus $g \in \text{supp } \beta$.

Now suppose that v is archimedean. We have $\mu(\alpha^n) \leq k^{n+1} s^n$, for all n . If $g \notin \text{supp } \beta$, $\sigma = \mu(\beta - \partial(m)) > k^m s^{m+1} (1-ks)^{-1}$. We now calculate $k^m s^{m+1} \geq \mu(\alpha^{m+1}) = \mu((1-\alpha)(\beta - \partial(m))) = \mu(\beta - \partial(m) - \alpha(\beta - \partial(m)))$

$\geq \mu(\beta - \partial(m)) - \mu(\alpha(\beta - \partial(m))) \geq 0 - k\sigma = \sigma(1 - ks) > k^m s^{m+1}$, a contradiction.

We conclude that $g \in \text{supp } \beta$.

2. The proof of this is similar to the proof of 1.

COROLLARY 4.9.1. Assume all the conditions of (1) or (2) in the previous theorem. Assume also that v is nonarchimedean (and is trivial on F in the case (2)). Suppose that if $g \in \text{supp } \alpha^n$, then $v(\text{coefficient of } g \text{ in } \alpha^n) = s^n$, for all $g \in G$ (respectively in the basis of A).

Then $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is finite.

PROOF. Each element in $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is in $\text{supp } \beta$, a finite set.

5. APPLICATIONS TO GROUP RINGS AND TENSOR PRODUCTS

We now apply the results of the previous section. We first generalize a theorem of Amitsur characterizing the Jacobson radical of tensor products. We then apply this to examine local and semilocal tensor products. Finally, we look at some miscellaneous applications to group rings.

1. GENERALIZATION OF A THEOREM OF AMITSUR

In [Amitsur (1957)] the following was proved.

THEOREM 5.1 [Amitsur (1957)] . Suppose A is an F -algebra and K a purely transcendental extension of F ($F \neq K$). Then $J(K \otimes_F A) \cap A = N$ is a nil ideal of A ; moreover, $J(K \otimes_F A) = K \otimes_F N$. (Here A is identified with $1 \otimes_F A$).

We now generalize the first part of Amitsur's theorem and look at some corollaries.

THEOREM 5.2. Suppose A and B are F -algebras and A is non-algebraic. Then $J(A \otimes_F B) \cap B$ is a nil ideal of B .

PROOF. Let $a \in A$ be transcendental over F , and suppose that $\alpha \in J(A \otimes_F B) \cap B$. Let $R = F[a]$ and consider the a -adic valuation on R . Extend this valuation to the R -module ${}_R A$ (Cor. 4.4.1). Now $1 - a\alpha$ satisfies the conditions of Corollary 4.9.1. Hence $\bigcup_{n=1}^{\infty} \text{supp}(\alpha a)^n$ is finite. Since $a\alpha = \alpha a$, $\bigcup_{n=1}^{\infty} \text{supp} \alpha^n$ is infinite; whence, α is algebraic.

Since a is in the Jacobson radical, it is nilpotent [Jacobson (1964) p.19].

COROLLARY 5.2.1. Suppose that A is a non-algebraic F -algebra and G a group. Then $J(A[G]) \cap F[G]$ is a nil ideal of $F[G]$.

PROOF. Use the well known isomorphism $A[G] \cong A \otimes_F F[G]$. [Jacobson (1964)].

COROLLARY 5.2.2. Suppose that F is a field and G and H two groups. If $F[G]$ is not algebraic, then $J(F[G \times H]) \cap F[H]$ is a nil ideal of $F[H]$.

PROOF. Use the well known isomorphism

$$F[G \times H] \cong F[G] \otimes_F F[H]$$

The above corollary generalizes a theorem in [Wallace (1967)].

Wallace proved it in the case where G is not torsion.

The following theorem gives a partial characterization of algebraic group rings.

THEOREM 5.3 [Herstein (1970)]. Let F be a field and G a group. If G is locally finite, then $F[G]$ is algebraic; if $F[G]$ is algebraic, then G is torsion. If $\text{char } F = 0$, then $F[G]$ is algebraic if and only if G is locally finite.

Let G be a group. We denote the direct sum (restricted direct product) of countably many copies of G by $\sum_{i=1}^{\infty} G$. This is the direct limit of $G \longrightarrow G \times G \longrightarrow G \times G \times G \longrightarrow \dots$

PROPOSITION 5.4. Let $\{G_i, i \in I\}$ be an infinite collection of groups

such that either all G_i are locally finite or infinitely many $F[G_i]$ are not algebraic (F is a field). Then $J\left(F\left[\sum_{i \in I} G_i\right]\right)$ is a nil ideal of $F\left[\sum_{i \in I} G_i\right]$.

PROOF. If all G_i are locally finite, then $F\left[\sum_{i \in I} G_i\right]$ is algebraic, hence its Jacobson radical is nil. Suppose then that infinitely many of the $F[G_i]$ are not algebraic and suppose further that $\alpha \in J\left(F\left[\sum_{i \in I} G_i\right]\right)$.

Then there is a normal subgroups $L, H \triangleleft \sum_{i \in I} G_i$ such that

- (a) $\sum_{i \in I} G_i \cong H \times L$,
- (b) $\alpha \in F[L]$
- (c) $F[H]$ is not algebraic.

Clearly $\alpha \in J((F[H])[L]) \cap F[L]$, a nil ideal.

COROLLARY 5.4.1. Let G be any group. Then $Q\left[\sum_{i=1}^{\infty} G\right]$ is semiprimitive.

PROOF. If G is not locally finite, then $Q[G]$ is not algebraic.

In [Woods (1974)] it was shown that if $H \triangleleft G$ and G/H is a locally finite group, then $J(F[H]) \subseteq J(F[G])$. By Corollary 5.2.2, this cannot be generalized without solving the more general problem of the structure of the radical in group rings. To see this, suppose L is a non-locally finite group such that if $G/H_0 \cong L$, then $J(F[H]) \subseteq J(F[G])$.

We claim that for all H , $Q[H]$ is semiprimitive. For $Q[L]$ is not algebraic; hence, $J(Q[L \times H]) \cap Q[H]$ is a nil ideal, whence, zero [Amitsur (1959)]. But $L \times H/H \cong L$, so $J(Q[H]) \subset J(Q[L \times H]) \cap Q[H] = (0)$

It is conjectured that the Jacobson radical of $Q[G]$ is zero for any group G [Kaplansky (1970)], and more generally that the radical of any group ring (whose coefficient is a field) is nil. Thus the augmentation ideal of a local group ring should be nil.

PROPOSITION 5.5. Suppose that G is a group which contains a direct product of two copies of itself. If $F[G]$ is local (where F is a field), then the augmentation ideal is nil.

PROOF. By hypothesis $G \times G \hookrightarrow G$; hence, since $F[G]$ is local, so is $F[G \times G]$. If $F[G]$ is algebraic, then the augmentation ideal is nil [Jacobson (1964) p.19]. If $F[G]$ is not algebraic, then $w(F[G]) = J(F[G \times G]) \cap F[G]$ is a nil ideal (Theorem 5.2.2).

II. LOCAL AND SEMILOCAL TENSOR PRODUCTS.

DEFINITION 5.6. A ring R is said to be local if $\bar{R} = R/J(R)$ is a division ring [Lambek (1966), p.75]. A ring R is said to be semilocal if $\bar{R} = R/J(R)$ is Artinian (thus completely reducible).

In a recent paper, Sweedler gave necessary and sufficient conditions for the tensor product of two commutative algebras to be local. In this section we generalize Sweedler's results to give partial answers to when the tensor product of noncommutative algebras is local or semilocal.

THEOREM 5.7 [Sweedler (1975)]. Let A and B be commutative F -algebras. Then $A \otimes_F B$ is local if and only if:

1. $\overline{A} \otimes_F \overline{B}$ is local,
2. A and B are local,
3. either A or B is F -algebraic.

LEMMA 5.8. Let A and C be F -algebras such that $A \otimes_F C$ is local. If B is a division subring of A containing F , then $B \otimes_F C$ is local. Conversely, if A is a locally finite division ring such that for every finitely generated division subring B containing F , $B \otimes_F C$ is local, then $A \otimes_F C$ is local.

PROOF. Suppose $a \in B \otimes_F C$ is invertible in $A \otimes_F C$. Then it is invertible in $B \otimes_F C$. This follows from the fact that $B \otimes_F C$ is a direct summand of $A \otimes_F C$ as a $B \otimes_F C$ -module. Suppose first that $A \otimes_F C$ is local. Choose $x \in B \otimes_F C$. We must show that either x or $1 + x$ is invertible in $B \otimes_F C$. But x or $1 + x$ is invertible in $A \otimes_F C$; hence, x or $1 + x$ is invertible in $B \otimes_F C$. This proves the first half of the lemma. The proof of the second half is equally obvious.

COROLLARY 5.8.1. Let A and B be F -algebras such that $A \otimes_F B$ is local. Then A is local.

PROOF. Since $A \otimes_F B$ is local, $A \otimes_F F \cong A$ is local.

The following lemma is a useful necessary criterion for a ring to be semilocal.

LEMMA 5.9 [Woods (1974)]. Let R be a semilocal ring. Given $x \in R$ define a sequence $\{x_n\}$ inductively by

$$x_1 = x, \quad x_{n+1} = x_n - x_n^2.$$

Then, for some n , $1 - x_n$ is right invertible.

THEOREM 5.10. Let A and B be local F -algebras such that A is locally finite and $\bar{A} \otimes_F \bar{B}$ is local. Then $A \otimes_F B$ is local.

PROOF. As $J(A)$ is locally nilpotent, $J(A) \otimes B$ is a nil ideal of $A \otimes B$ and so is contained in $J(A \otimes B)$. The problem thus reduces to showing that $\bar{A} \otimes B$ is local. By Lemma 5.8, we may assume that $\bar{A}$ is finite dimensional over F . But in this case $\bar{A} \otimes J(B) \subseteq J(A \otimes B)$ [Passman (1971), Theorem 16.3]; hence, it is sufficient to show that $\bar{A} \otimes \bar{B}$ is local, which is one of the given conditions.

Our next theorem shows that in certain cases, 'local finiteness' in the above theorem is necessary. Let $A = F[[x, y]]$ be the power series ring in two noncommuting indeterminates. A is local and $\bar{A}$ is isomorphic to F .

THEOREM 5.11. Suppose $A = F[[x, y]]$ and B is an F -algebra. If $A \otimes_F B$ is local, then B is locally finite.

PROOF. If $0 \neq \alpha \in A$, then $\alpha = \sum_{i=0}^{\infty} z_i$, where z_i is homogeneous of degree i . Let $n(\alpha) = \min\{j \mid z_j \neq 0\}$. Define $v(\alpha) = z_{n(\alpha)}$. Then v is a valuation on A . Let $\{b_i\}$ be an F -basis for B and let μ be the max-extension of v on $A(A \otimes B)$. Now choose $d_1, d_2, \dots, d_n \in \{b_i\}$, and let $a_i = x x \dots x y \dots y$ be a monomial in A consisting of i x 's followed by $n+1-i$ y 's. Clearly $F[a_1, a_2, \dots, a_n]$ is a free F -algebra contained in A . If $A \otimes B$ is local, then $a_1 \otimes 1, \dots, a_n \otimes 1 \in J(A \otimes B)$;

hence, $1 - \delta = 1 - \sum_{i=1}^n (a_i \otimes d_i)$ is invertible in $A \otimes B$. Since

$F[a_1, a_2, \dots, a_n]$ is a free algebra, every element in $\text{supp}(d_{i_1} d_{i_2} \dots d_{i_n})$ is contained in $\text{supp } \delta^n$ (where $d_{i_j} \in \{d_1, \dots, d_n\}$). By Theorem 4.9 $\bigcup_{n=1}^{\infty} \text{supp } \delta^n$ is finite. Therefore, $d_1, \dots, d_n$ generate a finite dimensional subalgebra of B .

THEOREM 5.12. Suppose A and B are nonalgebraic F -algebras. Let $x \in A$ and $y \in B$ be transcendental. Then the element $1 - f_1(x \otimes y) - \dots - f_k(x \otimes y)^k$ is right invertible in $A \otimes_F B$ if and only if $f_1 = f_2 = \dots = f_k = 0$ ($f_i \in F$).

PROOF. Let $\alpha = \sum_{i=1}^k f_i(x \otimes y)^i$. Extend the set $\{1, y, y^2, \dots\}$ to an

F -basis S of B . Since $x \in A$ is transcendental, we have an x -adic valuation v on $R = F[x]$. Extend this valuation to ${}_R A$ and let μ be the max-extension of v on ${}_R(A \otimes B)$, using the basis S . For all $n \in \mathbb{N}$, let $\delta(n) = 1 + \alpha + \dots + \alpha^n$. Let β be the right inverse of $1 - \alpha$.

For all sufficiently large n , say $n \geq \ell$, $y^n \notin \text{supp } \beta$. Consider

$$\delta(\ell) = \sum (a_i \otimes s_i), \quad a_i \in A, \quad s_i \in S. \quad \text{We write this as two sums}$$

$$\sum (a_i \otimes s_i) = \sum' (a_{i'} \otimes s_{i'}) + \sum'' (a_{i''} \otimes s_{i''}),$$

where $\{i'\} \cup \{i''\} = \{i\}$ and $\{i'\} \cap \{i''\} = \emptyset$, and where $v(a_{i'}) > 2^{-\ell}$, $\forall i'$, and $v(a_{i''}) \leq 2^{-\ell}$, $\forall i''$. Let $\Pi = \sum' (a_{i'} \otimes s_{i'})$ and $\psi_\ell = \sum'' (a_{i''} \otimes s_{i''})$. Clear

Π and ψ_ℓ are both elements of $F[x] \otimes F[y]$ and $\mu(\psi_\ell) \leq 2^{-\ell}$.

Now consider $\delta(n)$, where $n > \ell$. Since $\text{supp } \Pi \subset \{1, y, y^2, \dots, y^{\ell-1}\}$, $\text{supp } \Pi \cap \text{supp } (\delta(n) - \delta(\ell)) = \emptyset$. Hence, $\delta(n) = \Pi + \psi_n$, where $\psi_n \in F[x] \otimes F[y]$.

We now claim that $y^i, \ell \leq i < n$ does not occur in $\text{supp } \delta(n)$. For if it did, $v(\text{coefficient of } y^i \text{ in } \delta(n)) > 0$; hence, $y^i \in \text{supp } \beta$, a contradiction. Thus, if $y^i \in \text{supp } \psi_n$, then $i \geq n$. Now this implies that $\mu(\psi_n) \leq 2^{-n}$.

Putting together our main points, we have shown that for all $n > \ell$,

$\delta(n) = \Pi + \psi_n$, where Π and ψ_n are elements of $F[x] \otimes F[y]$, Π is independent of n , and $\mu(\psi_n) \leq 2^{-n}$.

We now restrict our attention to $F[x] \otimes F[y]$. There is a natural embedding of this into the power series ring $F[[x, y]]$. In the power series ring $1 - \alpha$ is invertible with inverse $\beta' = 1 + \alpha + \alpha^2 + \dots$. By our previous work, $\beta' = \Pi$ which implies that $1 - \alpha$ is invertible in $F[x] \otimes F[y]$. It is clear that $\alpha = 0$, our desired result.

We now generalize (3) in Theorem 5.7.

THEOREM 5.13. Suppose A and B are F -algebras. If $A \otimes_F B$ is semilocal, then either A is F -algebraic or B is F -algebraic.

PROOF. If not, choose $x \in A$ and $y \in B$ transcendental over F . Use Lemma 5.9 and Theorem 5.12 to obtain a contradiction.

In Chapter 6, we will use the above theorem to extend results on semilocal group rings.

III. OTHER APPLICATIONS

A ring R is said to have enough valuations if there is a set of valuations (archimedean or nonarchimedean) $\{v_i \mid i \in I\}$ such that:

1. For all real $r > 0$, the set $\{x \in R \mid v_i(x) \geq r \text{ for all } i \in I\}$ is finite,
2. For each $0 \neq r \in R$, there exists a natural number N with $N \geq v_i(r) \geq N^{-1}$, for all i .

The rational field $\mathbb{Q}$ is a field with enough valuations.

If $\alpha \in R[G]$, let $S(\alpha, n) = \{r \in R \mid r\alpha = \alpha r, 1 - r\alpha \text{ is invertible in } R[G], \text{ and } \text{card supp}(1 - r\alpha)^{-1} \leq n\}$

THEOREM 5.14. Let R be a ring with enough valuations and suppose

$T = \bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is infinite. Then, for all n , the set $S(\alpha, n)$ is finite.

PROOF. If not, assume that $S(\alpha, n) \supset \{r_1, r_2, \dots\}$. Choose $g_1, \dots, g_{n+1} \in T$ and suppose that g_i occurs first in $\text{supp } \alpha^{m_i}$ and has coefficient a_i .

Choose $b > 0$ such that $v_j(a_i) \geq b$, $i = 1, 2, \dots, n+1$, for all j . For any $r \in R$, $v_j((ra)^{m_i}) \geq v_j(r) + m_i b$. Thus $v_j(1 + (ra)^{m_1} + \dots + (ra)^{m_{n+1}}) \geq v_j(r) + m_i b$.

We now choose $r \in S(\alpha, n)$ such that $b > k \frac{m_i}{v_j(r) L^{m_i+1}} (1 - kv_j(r)L)^{-1}$,

where $v_j(\alpha) \leq L$. Set $s = v_j(r)L$. Then $v_j(ra) \leq v_j(r)L = s$, and

$b > k \frac{m_i}{v_j(r)L^{m_i+1}} (1 - ks)^{-1}$. But $v_j(1 + (ra)^{m_1} + \dots + (ra)^{m_{n+1}}) \geq v_j(r) + m_i b >$

$k \frac{m_i}{s^{m_i+1}} (1 - ks)^{-1}$. By Theorem 4.9, $g_i \in \text{supp}(1 - r\alpha)^{-1}$, $i = 1, 2, \dots, n+1$, a

contradiction.

Herstein asked if a group ring $F[G]$ in which every element is either invertible or a zero divisor is locally finite (implying that G is locally finite) [Herstein (1970) p.36]. Also open is the question of

algebraic group rings. If $F[G]$ is F -algebraic, is G locally finite [Herstein (1970) p.36]? Herstein answered the second question in the case where F has characteristic zero, (see Theorem 5.3). Recently Formanek answered the first question in the same case.

THEOREM 5.15

(a) [Formanek (1974)]. Let F be a field of characteristic zero and let G be a group. If every element of $F[G]$ is invertible or a zero divisor, then G is locally finite.

(b) Let F be a field of infinite transcendence degree and let G be a group. If every element of $F[G]$ is invertible or a zero divisor, then $F[G]$ is algebraic

PROOF. (a) Take $\{g_1, g_2, \dots, g_n\} \subset G$. Let v be the absolute value on $\mathbb{Q}$ and extend this to F . Let $\alpha = g_1 + \dots + g_n$. By Proposition 4.8, $1 - n^{-2}\alpha$ is not a zero divisor; hence, $1 - n^{-2}\alpha$ is invertible. By Theorem 4.9 $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is finite; whence, $\{g_1, \dots, g_n\}$ generates a finite subgroup G .

It follows that G is locally finite.

(b) Suppose $\alpha \in F[G]$. Since F has infinite transcendence degree, there exists a subfield K of F such that F/K is transcendental and $\alpha \in K[G]$. Take $x \in F \setminus K$ transcendental over K . As before, $1 - x\alpha$ is not a zero divisor in $K(x)[G]$, thus it is invertible. Using the x -adic valuation extended to $K(x)^F$, we conclude that $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is finite. Therefore α is algebraic.

6. SEMIPERFECT GROUP RINGS

A group ring is said to be semiperfect if it is semilocal and idempotents can be lifted modulo the radical [Lambek (1966), p.73]. An idempotent is said to be primitive if it cannot be written as a sum of two nonzero orthogonal idempotents. An idempotent $e \in R$ is local if eRe is a local ring.

THEOREM 6.1. [Mueller (1970)]. A ring is semiperfect if and only if 1 is a sum of orthogonal local idempotents.

Two central problems concerning semiperfect group rings have been attacked in recent years. First, if $F[G]$ is semiperfect, is G locally finite? Second, if $F[G]$ is semiperfect, is G a finite extension of a p -group? Virtually no progress has been made on the first problem, although it is known that G must be a torsion group [Woods (1974)]. The first question is by far the deeper of the two. In fact, Goursaud and Passman have independently proved that an affirmative answer to the first question implies an affirmative answer to the second question.

THEOREM 6.2 [Goursaud (1974)], [Passman (1974 c)]. Let G be a locally finite group and F a field of characteristic p . If $F[G]$ is semiperfect, then G is a finite extension of a p -group.

Goursaud assumed only that F was a division ring (skew field) of characteristic p . Passman's proof required F to be commutative.

In [Goursaud (1973)] it was conjectured that if F is a skew field of characteristic p , then $F[G]$ is semiperfect if and only if G is a finite extension of a p -group.

However, the second question has been answered in the affirmative in a case where an answer to the first question is unknown.

THEOREM 6.3 [Valette (1972)]. Let G be a group and F an uncountable field. If $F[G]$ is semiperfect, then it is algebraic. If, in addition, F is algebraically closed, then G is a finite extension of a p -group.

In this section we give a strong negative answer to Goursaud's conjecture (Theorem 6.13). We also extend Valette's Theorem (Proposition 6.5).

PROPOSITION 6.4. Suppose that G is a group and F is a field transcendental over a subfield K . If $F[G]$ is semiperfect, then $K[G]$ is algebraic.

PROOF. This is an immediate consequence of Theorem 5.13.

The following proposition is of the well-known variety.

PROPOSITION 6.5. A ring is semilocal if and only if the Jacobson radical is the intersection of finitely many maximal left ideals.

PROOF. Suppose that R is semilocal. Then in $R/J(R)$ there is a finite set of maximal left ideals whose intersection is zero; hence, $J(R)$ is the intersection of finitely many maximal left ideals. Conversely, suppose that $J(R)$ is the intersection of finitely many maximal left ideals. Then in $R/J(R)$, (0) is the intersection of finitely many maximal left ideals, say $M_1, \dots, M_k$. Thus $R/J(R)$ can be embedded (as an R -module) in the direct product $R/M_1 \times \dots \times R/M_k$, a completely reducible module. Therefore $R/J(R)$ is completely reducible.

THEOREM 6.6 [Rosenberg, Zelinsky (1957)].

Let A and B be F -algebras. If $A \otimes_F B$ is semilocal, then A is semilocal.

PROOF. If A is not semilocal, then there exists an infinite sequence of maximal left ideals of A say $I_1, I_2, \dots$ with $\bigcap_{i=1}^k I_i \neq \bigcap_{i=1}^{k+1} I_i$ for all k (Proposition 6.5). Thus $\bigcap_{i=1}^k I_i \otimes B \neq \bigcap_{i=1}^{k+1} I_i \otimes B$. Let $J_k = \bigcap_{i=1}^k I_i \otimes B$.

Clearly $J_k \supseteq J(A \otimes B)$, thus $J_1/J(A \otimes B) \supsetneq J_2/J(A \otimes B) \supsetneq \dots$ is a proper descending chain of left ideals of $A \otimes B / J(A \otimes B)$, a contradiction.

THEOREM 6.7. Suppose G is a group and F a field transcendental over the algebraic closure of its prime subfield. If $F[G]$ is semiperfect, then G is a finite extension of a p -group.

PROOF. Let K be the algebraic closure of the prime subfield of F . By Proposition 6.4, $K[G]$ is algebraic, and by the previous theorem, $K[G]$ is semilocal. As K is algebraically closed, $K[G] / J(K[G])$ is a P.I. algebra. Let $\bar{\cdot} : G \rightarrow K[G] / J(K[G])$ be the canonical map. Then $\text{Ker}(\bar{\cdot}) = \{g \in G : 1-g \in J(K[G])\}$. By [Procesi (1965)], $G/\text{Ker}(\bar{\cdot})$ is locally finite; hence, $G/\text{Ker}(\bar{\cdot})$ is a finite extension of a p -group. [Goursaud (1973)]. But $\text{Ker}(\bar{\cdot})$ is a p -group; whence, G is a finite extension of p -group.

COROLLARY 6.7.1. Let F be an algebraically closed field of infinite transcendence degree. Let G be a group and H a subgroup of G . If $F[G]$ is semiperfect, then $F[H]$ is semiperfect.

PROOF. Since $F[G]$ is algebraic and F algebraically closed, $F[G] / J(F[G])$ is finite dimensional over F . Since $J(F[H]) \supseteq F[H] \cap J(F[G])$ [Passman (1971) p. 19], $F[H] / J(F[H])$ is Artinian. Also, $J(F[H])$ is nil [Jacobson (1964), p.19];

hence, idempotents can be lifted modulo the radical [Lambek (1966), p.72].

COROLLARY 6.7.2. Let F be an algebraically closed field of infinite transcendence degree. Let G be a finitely generated p -group. If $F[G]$ is semiperfect, then $G^{(1)} \neq G$ and $G/G^{(1)}$ is finite ($G^{(1)}$ = derived group of G).

PROOF. Looking at the proof of Theorem 6.7 we see that $G/\text{Ker}(-)$ is locally finite, hence, finite. If $G/\text{Ker}(-)$ is not trivial, we will have finished. If $\text{Ker}(-) = G$, then $J(F[G]) = w(F[G])$. Since $w(F[G])$ is finitely generated as a right ideal, $w(F[G])^2 \neq w(F[G])$, and so $G^{(1)} \neq G$.

PROPOSITION 6.8. Let G and H be groups, and let F be an algebraically closed field. If $F[G \times H]$ is semiperfect, then either G is a finite extension of a p -group or H is a finite extension of a p -group.

PROOF. $F[G]$ and $F[H]$ are homomorphic images of $F[G \times H]$, hence, semiperfect [Bass (1960)]. The rest of the proof is similar to the proof of Theorem 6.7.

PROPOSITION 6.9. Let F be an algebraically closed field of infinite transcendence degree, and let $\text{char } F = p$. There exists a p -group G such that $F[G]$ is not semiperfect.

PROOF. Our proof is similar to [Lihtman (1963)]. In this paper Lihtman showed that there existed p -groups such that $F[G]$ was never local for any field F . Let G be a finitely generated p -group such that $G = G^{(1)}$ [Magnus, Karrass, Solitar (1966), p.379]. By Corollary 6.5.2, $F[G]$ is not semiperfect.

A group G is said to be finitely approximable if $\bigcap_{n=1}^{\infty} G^{(n)} = \langle 1 \rangle$.

($G^{(n)}$ is the n -th derived group).

PROPOSITION 6.10. Suppose that G is a finitely generated, finite extension of a finitely approximable p -group H . Let F be a skew field of characteristic p . If $F[G]$ is semiperfect, then $F[H]$ is local.

PROOF. We claim that for all $h \in H$, $1-h \in J(F[G])$. If not, since $F[G]$ is semiperfect, there exists $\beta \in F[G]$ and $\delta \in J(F[G])$ such that $0 \neq \alpha^2 = \alpha(1-h)\beta + \delta$. Choose n large enough so that the support of α remains distinct modulo $H^{(n)}$. Since $H^{(n)}$ is characteristic in H , it is normal in G . Considering the canonical map $\bar{}: F[G] \rightarrow F[G/H^{(n)}]$, we see that $\bar{0} \neq \bar{\alpha}^2 = \bar{\alpha}(1-h)\bar{\beta} + \bar{\delta}$. However $(1-h)\bar{\beta} \in J(F[G/H^{(n)}])$ [Passman (1971), p.84]; hence, $\bar{\alpha} \in J(F[G/H^{(n)}])$, a contradiction. Thus for all $h \in H$, $1-h \in J(F[G])$. As $J(F[H]) \supseteq J(F[G]) \cap F[H]$ we conclude that $F[H]$ is local.

PROPOSITION 6.11 [Goursaud (1973)]. Let $F_1, \dots, F_n$ be a finite set of skew fields, and $g_1, \dots, g_k$ a finite set of elements of the group G . Suppose that $F_i[G]$ is semiperfect for each i . Then there is a finitely generated subgroup H of G such that $g_j \in H$, for each j , and $F_i[H]$ is semiperfect, for each i .

THEOREM 6.12. Let K be a field. There exists a skew field F containing K such that:

1. if every element of $F[G]$ is invertible or a zero divisor, then G is locally finite,
2. if $F[G]$ is local, then G is locally finite.

PROOF: Let $K(x)$ be the rational function field. Let $A = K(x)[y]$ be the skew polynomial ring with $yx = x^2y$ [Cohn (1971)], that is, the endomorphism is given by $x \mapsto x^2$. This is a left but not right Ore domain. Let F be its left quotient field. Let v be the y -adic valuation on A . Extend v to F . Since $w = xy$ and y have no common right multiple, $K[w, y]$ is a free subalgebra of F [Jategaonkar (1969)]. Suppose that $g_1, g_2, \dots, g_t$ is a finite set of elements of G . Consider the element $\alpha = ww \dots wyg_1 + \dots + twyy \dots yg_t$, where the n -th monomial $ww \dots wy \dots y$ consists of $t+1-n$ w 's followed by n y 's. Since $k[ww \dots wy, \dots, wyy \dots y]$ is a free algebra, $\text{supp } \alpha^n$ contains every word generated by the set $g_1, \dots, g_t$ of length n ; hence, $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n = \text{sgp } \langle g_1, \dots, g_t \rangle$ the semigroup generated by the set. If $g \in \text{supp } \alpha^n$, then $v(\text{coefficient of } g \text{ in } \alpha^n) = 2^{-(t+1)n}$. Also, $v(ww \dots wy) = \dots = v(wyy \dots y) = 2^{-(t+1)}$. If we could show that $1-\alpha$ was invertible, then we could use Corollary 4.9.1 to conclude that $\bigcup_{n=1}^{\infty} \text{supp } \alpha^n$ is finite, and this would yield our desired result. We consider two cases.

CASE 1. Suppose that every element of $F[G]$ is invertible or a zero divisor. As $\mu(\alpha) < 1$ (μ being the max-extension of v on $F[G]$), $1-\alpha$ is not a zero divisor (Proposition 4.8), thus it is invertible.

CASE 2. Suppose that $F[G]$ is local. Let $w(\)$ be the augmentation map. Then $w(1-\alpha) = 1-ww \dots wy - \dots - wyy \dots y \neq 0$. Hence, $1-\alpha$ is invertible. This completes the proof of the theorem.

We now give our negative answer to Goursaud's Conjecture.

THEOREM 6.13. Let G be a group and p a prime. Then $F[G]$ is semi-perfect for all skew fields F of characteristic p if and only if G is a finite extension of a locally finite p -group.

PROOF. The 'if' part has already been proved by Goursaud. Also, by Valette's Theorem, we know that G is a finite extension of a p -group H . It remains to show that H is locally finite. Let K be an algebraically closed field of infinite transcendence degree and of characteristic p . Let F be a skew field containing K such that if $F[N]$ is local, for some group N , then N is locally finite (Theorem 6.12). If G is not locally finite, then, by Proposition 6.11, we may consider it to be finitely generated and infinite, and assume that both $K[G]$ and $F[G]$ are semiperfect. Now H is finitely generated and, by induction, $H^{(n)} \neq H^{(n+1)}$ for all n (Corollary 6.7.1 and Corollary 6.7.2). Let $A = H / \bigcap_{n=1}^{\infty} H^{(n)}$ and $A' = G / \bigcap_{n=1}^{\infty} H^{(n)}$. Clearly, A is an infinite, finitely generated, finitely approximable p -group, and A' is a finite extension of A . But $F[A']$ is semiperfect (being a homomorphic image of a semiperfect ring); whence, $F[A]$ is local (Proposition 6.10). This contradicts the fact that A is infinite, for A must also be locally finite and hence finite.

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