EVALUATION OF CERTAIN INFINITE SERIES USING THEOREMS OF JOHN, RADEMACHER AND KRONECKER

by

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Abstract

John's theorem relates the sum of a certain infinite series involving a real-valued periodic function of period 1, which is of bounded variation on $[0,1]$, to a Riemann integral. A detailed proof of this theorem based on the proof of Rademacher is given. John's theorem is then used to determine the sum of some interesting infinite series. For example it is shown that the sum of

$$\frac{1}{2}-\frac{1}{3}+\frac{2}{4}-\frac{2}{5}+\frac{2}{6}-\frac{2}{7}+\frac{3}{8}-\frac{3}{9}+\frac{3}{10}-\frac{3}{11}+\frac{3}{12}-\frac{3}{13}+\frac{3}{14}-\frac{3}{15}+\frac{4}{16}-\frac{4}{17}+\cdots$$

is Euler's constant, see Theorem 1.6.2. Rademacher's theorem is the extension of John's theorem to algebraic number fields and this theorem is applied to determine the sum of further infinite series such as

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{4}+\frac{2}{5}-\frac{1}{8}+\frac{1}{9}-\frac{2}{10}+\cdots=\frac{\pi\log 2}{4},$$

see Theorem 2.4.2. Finally, Kronecker's limit formula is used to determine the sums of the infinite series

$$\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\frac{(-1)^m}{am^2+bmn+cn^2},\quad \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\frac{(-1)^n}{am^2+bmn+cn^2},\quad \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\frac{(-1)^{m+n}}{am^2+bmn+cn^2},$$

for any primitive, positive-definite, integral, binary quadratic form $ax^2+bxy+cy^2$.

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This thesis was written in LaTeX. George Gratzer's book "Math into LaTeX" has been an extremely useful reference.

Lastly I am solely responsible for any errors and shortcomings left in this thesis.

Notation

$\mathbb{Z}$ = domain of integers

$\mathbb{N}$ = set of positive integers

$\mathbb{Q}$ = field of rational numbers

$\mathbb{R}$ = field of real numbers

$\gcd(m,n)$ = greatest common divisor of $m, n \in \mathbb{Z},\ (m,n) \neq (0,0)$

γ = Euler's constant $= 0.5772156649\ldots = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i} - \log n\right)$

$[x]$ = greatest integer $\leq x$ $(x \in \mathbb{R})$

$\{x\}$ = fractional part of $x = x - [x]$ $(x \in \mathbb{R})$

$[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ $(a, b \in \mathbb{R},\ a < b)$

$\{x_0, x_1, \ldots, x_n\}$ = partition of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$

$\Delta x_k = x_k - x_{k-1}$ $(k = 1, 2, \ldots, n)$

$\wp[a,b]$ = set of all partitions of $[a,b]$

$\left(\frac{d}{n}\right)$ = Legendre-Jacobi-Kronecker symbol $(d \in \mathbb{Z}, d \equiv 0, 1 \pmod 4, n \in \mathbb{N})$

$H(d)$ = form class group of discriminant d

$h(d)$ = form class number = order of form class group $H(d)$

f = conductor of discriminant d

$$\Delta = \frac{d}{f^2}$$

$$\sum_{m,n=-\infty}^{\infty}{}' f(m,n) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} f(m,n)$$

Contents

Introduction

The aim of this thesis is to use three theorems, namely John's theorem, Rademacher's extension of John's theorem, and Kronecker's limit formula, to determine the sums of certain infinite series.

In 1934 Fritz John [20] showed that if $f(x)$ is a real-valued function defined for all real x such that

(1) $f(x)$ is periodic of period 1,

(2) $f(x)$ is of bounded variation for $0 \leq x \leq 1$,

and $c = \dfrac{p}{q}$ is a rational number > 1 with $q > 0$ and $\gcd(p, q) = 1$, then the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n(c)}{n} f\left(t - \frac{\log n}{\log c}\right)$$

converges for all real values of t, where

$$a_n(c) = \begin{cases} 0, & \text{if } p \nmid n,\ q \nmid n, \\ -p, & \text{if } p \mid n,\ q \nmid n, \\ q, & \text{if } p \nmid n,\ q \mid n, \\ q-p, & \text{if } p \mid n,\ q \mid n, \end{cases}$$

and its sum is

$$\log c \int_0^1 f(x)\, dx.$$

Since $f(x)$ is of bounded variation on $[0,1]$, $f(x)$ is Riemann integrable on $[0,1]$, and so $\int_0^1 f(x)\, dx$ exists. In 1936 Hans Rademacher [27] proved John's theorem with (2) replaced by the weaker condition

(2)′ $f(x)$ is Riemann integrable on $[0,1]$.

We give an exposition of Rademacher's proof of John's theorem expanding on the details where necessary in Section 1.3.

Included in the volume [28] featuring his "lost" notebook are some fragments of papers by Ramanujan. In particular, on pages 274 and 275 in [28], there is the beginning of a manuscript that probably was to focus on integrals related to Euler's constant γ. Berndt and Bowman [8] have presented Ramanujan's work in this fragment and have related it to other theorems in the literature. In particular they prove [8, Lemma 2.5, p. 21] an integral given by Ramanujan for Euler's constant γ, namely,

$$\gamma = \int_0^1 \left(\frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} x^{n^k - 1}\, dx,$$

and use it to obtain (among others) a series representation of γ due to Vacca [33], namely

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\log n}{\log 2}\right].$$

It is our purpose to show that Vacca's formula for γ, as well as the evaluation of other particular infinite series, follows easily from John's theorem. We just mention one such evaluation. We show in Section 1.5 how Liang and Todd's evaluation [24] of the infinite series $\sum_{n=1}^{\infty}(-1)^n \frac{\log^k n}{n}$, where k is a positive integer, can be deduced easily from John's theorem.

In his 1936 paper, Rademacher also extended John's theorem to algebraic number fields. In Chapter 2 we use Rademacher's theorem to obtain the evaluation of certain infinite series involving binary quadratic forms. In the case of positive-definite forms, we show for example that

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+mn+2n^2} = -\frac{2\pi}{\sqrt{7}}\log 2,$$

where the sum is ordered by increasing values of $m^2+mn+2n^2$, follows from Rademacher's theorem. In the case of indefinite forms we prove for example that

$$\sum_{\substack{m,n=-\infty \\ m+n\sqrt{2}>0 \\ 1\leq\left|\frac{m+n\sqrt{2}}{m-n\sqrt{2}}\right|<3+2\sqrt{2}}}^{\infty} \frac{(-1)^m}{|m^2-2n^2|} = -\frac{\pi}{\sqrt{2}}\log 2.$$

In Chapter 3 we use Kronecker's limit formula to evaluate the infinite series

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{am^2+bmn+cn^2},$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{am^2+bmn+cn^2},$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{am^2+bmn+cn^2},$$

for any positive-definite, primitive, integral, binary quadratic form $ax^2 + bxy + cy^2$. As an illustration of these results, we show that

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+19n^2} = -\frac{4\pi}{\sqrt{19}}\log\theta,$$

where θ is the unique real root of the cubic equation $x^3 - 2x - 2 = 0$.

Chapter 1

John's Theorem

1.1 Functions of bounded variation

John's theorem, which is the main topic of this chapter, expresses an infinite series involving a periodic function of bounded variation as a Riemann integral. We begin by reviewing the basic properties of functions of bounded variation. We follow the treatment given by Apostol in [4, pp. 165-169], see also [30, pp. 117-121].

Throughout this section, a and b denote real numbers with $a < b$.

Definition 1.1.1. *Let f be a real-valued function defined on the closed interval $[a, b]$. If, for every pair of points x and y in $[a, b]$, $x < y$ implies $f(x) \leq f(y)$, then f is said to be increasing on $[a, b]$. If $x < y$ implies $f(x) < f(y)$ then f is said to be strictly increasing on $[a, b]$. Decreasing and*

strictly decreasing functions are similarly defined.

Definition 1.1.2. *A real-valued function f defined on the closed interval $[a, b]$ is said to be monotonic if it is increasing on $[a, b]$ or decreasing on $[a, b]$.*

Definition 1.1.3. *If $[a, b]$ is a finite interval, then a finite set of points*

$$P = \{x_0, x_1, \ldots, x_n\}$$

satisfying the inequalities $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$ is called the kth subinterval of P and we write $\Delta x_k = x_k - x_{k-1}$, so that

$$\sum_{k=1}^{n} \Delta x_k = b - a.$$

The collection of all possible partitions of $[a, b]$ is denoted by $\wp[a, b]$.

Definition 1.1.4. *Let f be a real-valued function defined on $[a, b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, we set $\Delta f_k = f(x_k) - f(x_{k-1})$, $k = 1, 2, \ldots, n$. If there exists a positive number M such that*

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

for all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$.

We now give some theorems that help us to decide when a function is of bounded variation.

Theorem 1.1.1. *Let f be a real-valued function defined on $[a, b]$. If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$.*

Proof. As f is monotonic on $[a,b]$, by Definition 1.1.2 f is either increasing or decreasing on $[a,b]$. Suppose f increasing on $[a,b]$. Then, for any partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$, we have $f(x_k) - f(x_{k-1}) \geq 0$ giving $\Delta f_k \geq 0$. Now

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} \Delta f_k = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = f(b) - f(a)$$

so $\sum_{k=1}^{n} |\Delta f_k| \leq M$ with $M = f(b) - f(a)$. Thus f is of bounded variation on $[a,b]$. If f is decreasing on $[a,b]$ then $-f$ is increasing on $[a,b]$ and the above argument again shows that f is of bounded variation on $[a,b]$. □

Theorem 1.1.2. *Let f be a real-valued function defined on $[a,b]$. If f is continuous on $[a,b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \leq A$ for all x in (a,b), then f is of bounded variation on $[a,b]$.*

Proof. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Applying the mean value theorem to the subinterval $[x_{k-1}, x_k]$, we have

$$\Delta f_k = f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}) = f'(t_k)\Delta x_k$$

for some $t_k \in (x_{k-1}, x_k)$. Then

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} |f'(t_k)\Delta x_k| = \sum_{k=1}^{n} |f'(t_k)|\Delta x_k \leq A \sum_{k=1}^{n} \Delta x_k.$$

Hence $\sum_{k=1}^{n} |\Delta f_k| \leq M$ with $M = A(b-a)$ so f is of bounded variation on $[a,b]$. □

The function

$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is of bounded variation on $[0,1]$ as f is continuous on $[0,1]$ and

$$f'(x) = \begin{cases} \sin\dfrac{1}{x} + 2x\cos\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$|f'(x)| \leq 3.$$

We note that f' need not be bounded for f to be of bounded variation. For example, $f(x) = x^{1/3}$ is monotonic so f is of bounded variation in every finite interval (Theorem 1.1.1). However, $f'(x) = \frac{1}{3x^{2/3}} \to +\infty$ as $x \to 0$.

Not all continuous functions are of bounded variation. Consider the function

$$f(x) = \begin{cases} x\sin\left(\dfrac{\pi}{x}\right), & 0 < x \leq 2, \\ 0, & x = 0, \end{cases}$$

which is continuous on $[0,2]$, and the partition

$$P = \left\{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{5}, \frac{2}{3}, 2\right\}.$$

Here

$$x_0 = 0, x_k = \frac{2}{2n-(2k-1)}, k = 1, 2, \ldots, n.$$

For $k = 2, \ldots, n$ we have,

$$\begin{aligned} \Delta f_k &= f(x_k) - f(x_{k-1}) \\ &= \frac{2}{2n-(2k-1)} \sin\left(\frac{(2n-(2k-1))\pi}{2}\right) \\ &\quad - \frac{2}{2n-(2k-3)} \sin\left(\frac{(2n-(2k-3))\pi}{2}\right) \\ &= \frac{2}{2n-2k+1}(-1)^{n-k} - \frac{2}{2n-2k+3}(-1)^{n-k+1} \\ &= (-1)^{n-k}\left(\frac{2}{2n-2k+1} + \frac{2}{2n-2k+3}\right). \end{aligned}$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta f_k| &\geq \sum_{k=2}^{n} |\Delta f_k| \\
&= \sum_{k=2}^{n} \left(\frac{2}{2n-2k+1} + \frac{2}{2n-2k+3} \right) \\
&> \sum_{k=2}^{n} \frac{2}{2n-2k+1} \\
&> \sum_{k=2}^{n} \frac{1}{n-k+1} \\
&= \sum_{k=1}^{n-1} \frac{1}{k},
\end{aligned}$$

so f is not of bounded variation on $[0,2]$ as $\lim_{n\to+\infty} \sum_{k=1}^{n-1} \frac{1}{k} = +\infty$.

Theorem 1.1.3. *Let f be a real-valued function on $[a,b]$. If f is of bounded variation on $[a,b]$, say $\sum |\Delta f_k| \leq M$ for all partitions of $[a,b]$, then f is bounded on $[a,b]$. In fact, $|f(x)| \leq |f(a)| + M$ for $x \in [a,b]$.*

Proof. Let $x \in (a,b)$ and consider the partition $P = \{a, x, b\}$. As f is of bounded variation on $[a,b]$, we have

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M,$$

which gives $|f(x) - f(a)| \leq M$. Now

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)| \leq M + |f(a)|$$

as required.

If $x = a$, the inequality holds trivially. If $x = b$, the partition $\{a, b\}$ gives $|f(b)| \leq |f(b) - f(a)| + |f(a)| \leq M + |f(a)|$. Hence $|f(x)| \leq |f(a)| + M$ for all $x \in [a, b]$. □

The converse of Theorem 1.1.3 is not true. The function

$$f(x) = \begin{cases} x \sin \dfrac{\pi}{x}, & 0 < x \leq 2, \\ 0, & x = 0, \end{cases}$$

is bounded on $[0, 2]$ as $|f(x)| \leq |x| \leq 2$ but is not of bounded variation on $[0, 2]$.

Definition 1.1.5. *Let f be a real-valued function defined on $[a, b]$. Let f be of bounded variation on $[a, b]$, and let $\sum(P)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. The number*

$$V_f = V_f(a, b) = \sup \left\{ \sum(P) | P \in \wp[a, b] \right\}$$

is called the total variation of f on the interval $[a, b]$.

Theorem 1.1.4. *Let f and g be real-valued functions defined on $[a, b]$. Suppose further that f and g are both of bounded variation on $[a, b]$. Then so are their sum, difference and product. Also, we have*

$$V_{f \pm g} \leq V_f + V_g \text{ and } V_{fg} \leq AV_f + BV_g,$$

where $A = \sup\{|g(x)| \, | x \in [a, b]\}$ and $B = \sup\{|f(x)| \, | x \in [a, b]\}$.

Proof. We just prove the second of these two inequalities. Consider $h(x) =$

$f(x)g(x)$. Then for each partition P of $[a,b]$, we have

$$\begin{aligned}
|\Delta h_k| &= |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= |f(x_k)g(x_k) - f(x_{k-1})g(x_k)| \\
&\qquad\qquad + |f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&\leq A|\Delta f_k| + B|\Delta g_k|.
\end{aligned}$$

Hence $h = fg$ is of bounded variation and $V_{fg} \leq AV_f + BV_g$. □

There is a close relation between monotonic functions and functions of bounded varition. We have already seen that monotonic functions are always of bounded variation (Theorem 1.1.1), and we will see shortly that functions of bounded variation can always be written in terms of monotonic functions (Theorem 1.1.8). Being of bounded variation is a stronger condition than monotonicity, indeed, the sum or product of monotonic functions need not be monotonic. For example, x and $-x^2$ are monotonic on $[0,1]$, but $x - x^2$ is not; and x is monotonic on $[-1,1]$ but x^2 is not.

It should be noted that the reciprocal of a function of bounded variation is not necessarily of bounded variation. For instance, suppose $f(x) \to 0$ as $x \to c$. Then $1/f$ is not bounded in any interval containing c. Hence by Theorem 1.1.3, $1/f$ is not of bounded variation on such an interval. Excluding functions whose values get arbitrarily close to zero let us extend Theorem 1.1.4 to quotients.

Theorem 1.1.5. *Let f be of bounded variation on $[a,b]$ and assume that f is bounded away from zero, that is there exists a positive number m such that*

$0 < m \leq |f(x)|$ *for all* x *in* $[a, b]$. *Then* $g = 1/f$ *is also of bounded variation on* $[a, b]$, *and* $V_g \leq \frac{V_f}{m^2}$.

Proof. We have

$$|\Delta g_k| = \left|\frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})}\right| = \left|\frac{f(x_{k-1}) - f(x_k))}{f(x_k)f(x_{k-1})}\right| = \left|\frac{\Delta f_k}{f(x_k)f(x_{k-1})}\right| \leq \frac{|\Delta f_k|}{m^2}.$$

□

Theorem 1.1.6. *Let* f *be a real-valued function defined on* $[a, b]$. *Let* f *be of bounded variation on* $[a, b]$, *and assume that* $c \in (a, b)$. *Then* f *is of bounded variation on* $[a, c]$ *and on* $[c, b]$ *and we have*

$$V_f(a, b) = V_f(a, c) + V_f(c, b).$$

Proof. Consider partitions P_1 of $[a, c]$ and P_2 of $[c, b]$. Then $P_0 = P_1 \cup P_2$ is a partition of $[a, b]$. Denote by $\sum(P)$ the sum $\sum |\Delta f_k|$ corresponding to a partition P. Then

$$\sum(P_1) + \sum(P_2) = \sum(P_0) \leq V_f(a, b).$$

Hence $\sum(P_1)$ and $\sum(P_2)$ are bounded by $V_f(a, b)$ so f is of bounded variation on $[a, c]$ and $[c, b]$. Now

$$\sum(P_1) + \sum(P_2) \leq V_f(a, b) \implies V_f(a, c) + V_f(c, b) \leq V_f(a, b).$$

For the reverse inequality, consider a partition $P = \{x_0, x_1, \ldots, x_n\} \in \wp[a, b]$ and $P_0 = P \cup \{c\}$ the partition obtained by adjoining the point c. If $c \in [x_{k-1}, x_k]$ then

$$|f(x_k) - f(x_{k-1})| \leq |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|$$

giving $\sum(P) \leq \sum(P_0)$. The points of P_0 in $[a, c]$ give a partition P_1 of $[a, c]$ and the points in $[c, b]$ give a partition P_2 of $[c, b]$. Now

$$\sum(P) \leq \sum(P_0) = \sum(P_1) + \sum(P_2) \leq V_f(a, c) + V_f(c, b).$$

This shows that $V_f(a, c) + V_f(c, b)$ is an upper bound for every sum $\sum(P)$. Since this cannot be smaller than the least upper bound, we have

$$V_f(a, b) \leq V_f(a, c) + V_f(c, b).$$

The asserted equality now follows from the two inequalities. □

Theorem 1.1.7. *Let f be a real-valued function defined on $[a, b]$ and let f be of bounded variation on $[a, b]$. Let V be defined on $[a, b]$ as follows: $V(x) = V_f(a, x)$ if $a < x \leq b$, $V(a) = 0$. Then:*

(i) *V is an increasing function on $[a, b]$.*

(ii) *$V - f$ is an increasing function on $[a, b]$.*

Proof. For $a < x < y \leq b$, write $V_f(a, y) = V_f(a, x) + V_f(x, y)$. Then

$$V(y) - V(x) = V_f(a, y) - V_f(a, x) = V_f(x, y) \geq 0.$$

Hence (i) holds.

Consider $D(x) = V(x) - f(x)$ if $x \in [a, b]$. For $a \leq x < y \leq b$,

$$\begin{aligned} D(y) - D(x) &= V(y) - V(x) - [f(y) - f(x)] \\ &= V_f(x, y) - [f(y) - f(x)]. \end{aligned}$$

By the definition of $V_f(x, y)$, we have $f(y) - f(x) \leq V_f(x, y)$ so $D(y) - D(x) \geq 0$ and (ii) holds. □

Theorem 1.1.8. *Let f be a real-valued function defined on $[a, b]$. Then*

$$f \text{ is of bounded variation on } [a,b] \iff \begin{array}{c} f \text{ can be expressed as the difference} \\ \text{of two increasing functions.} \end{array}$$

Proof. ($\Longrightarrow$) Suppose f is of bounded variation on $[a, b]$. Write $f = V - D$ where D is the function from the proof of Theorem 1.1.7. Now V and $D = V - f$ are increasing by Theorem 1.1.7.

($\Longleftarrow$) Suppose $f = g - h$ where g, h are increasing. Then g, h are monotonic and so are of bounded variation by Theorem 1.1.1. Now $g - h$ is of bounded variation by Theorem 1.1.4, and so f is of bounded variation. □

This representation is not unique! If $f = f_1 - f_2$ is one representation of f as the difference of two increasing functions, then $f = (f_1 + g) - (f_2 + g)$, where g is an arbitrary increasing function, is another representation of f. If g is *strictly* increasing, so are $f_1 + g$ and $f_2 + g$, therefore Theorem 1.1.8 holds if "increasing" is replaced by "strictly increasing".

Theorem 1.1.9. *Let f be a real-valued function defined on $[a, b]$ and let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then every point of continuity of f is also a point of continuity of V. The converse also holds.*

Proof. ($\Longleftarrow$) V is monotonic as it is increasing, hence the right- and left-hand limits $V(x+)$ and $V(x-)$ exist for each point x in (a, b). By Theorem 1.1.8, the same is true of $f(x+)$ and $f(x-)$.

If $a < x < y \leq b$, then $0 \leq |f(y) - f(x)| \leq V(y) - V(x)$ by the definition of $V_f(x, y)$. Letting $y \to x$, we see $0 \leq |f(x+) - f(x)| \leq V(x+) - V(x)$. Similarly, $0 \leq |f(x) - f(x-)| \leq V(x) - V(x-)$. These inequalities imply that a point of continuity of V is also a point of continuity of f.

($\Longrightarrow$) Let f be continuous at the point c in (a, b). Given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $0 < |f(x) - f(c)| < \epsilon/2$. For the same ϵ, there exists a partition P of $[c, b]$, say $P = \{x_0, x_1, \ldots, x_n\}$, $x_0 = c$, $x_n = b$ such that

$$V_f(c, b) - \frac{\epsilon}{2} < \sum_{k=1}^{n} |\Delta f_k|.$$

Adding more points to the partition can only increase the sum $\sum |\Delta f_k|$, so we may assume $0 < x_1 - x_0 < \delta$. This means that

$$|\Delta f_1| = |f(x_1) - f(c)| < \frac{\epsilon}{2}.$$

As $\{x_1, x_2, \ldots, x_n\}$ is a partition of $[x_1, b]$, we have

$$V_f(c, b) - \frac{\epsilon}{2} < \frac{\epsilon}{2} + \sum_{k=2}^{n} |\Delta f_k| \leq \frac{\epsilon}{2} + V_f(x_1, b).$$

Hence

$$V_f(c, b) - V_f(x_1, b) < \epsilon.$$

But $0 \leq V(x_1) - V(c) = V_f(a, x_1) - V_f(a, c) = V_f(c, x_1) = V_f(c, b) - V_f(x_1, b) < \epsilon$. Hence $0 < x_1 - c < \delta$ implies $0 \leq V(x_1) - V(c) < \epsilon$. This shows $V(c+) = V(c)$. Similarly, $V(c-) = V(c)$. □

Theorem 1.1.10. *Let f be continuous on $[a, b]$. Then*

f is of bounded variation on $[a, b]$	$\Longleftrightarrow$	*f can be expressed as the difference of two increasing continuous functions.*

Proof. Write $f = V - D$ as in Theorem 1.1.7. Then V and D are increasing by Theorem 1.1.7, and are continuous by Theorem 1.1.9. □

As in Theorem 1.1.8, Theorem 1.1.10 holds if "increasing" is replaced by "strictly increasing".

Theorem 1.1.11.

$$\begin{array}{c} f \textit{ is of bounded variation} \\ \textit{on } [a,b] \end{array} \implies \textit{the Riemann integral } \int_a^b f(x)\,dx \textit{ exists.}$$

Theorem 1.1.11 is proved in [4, pp. 207-212].

The converse of Theorem 1.1.11 does not hold, that is, a function may be Riemann integrable without being of bounded variation. Consider

$$f(x) = \begin{cases} \dfrac{1}{x^{1/2}}, & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

Then f is defined on $[0,1]$. As

$$\int_\epsilon^1 f(x)\,dx = \int_\epsilon^1 \frac{1}{x^{1/2}}\,dx = [2x^{1/2}]_\epsilon^1 = 2 - 2\epsilon^{1/2},$$

we have $\int_0^1 f(x)\,dx = 2$ so the Riemann integral exists. Consider the partition P of $[0,1]$ with $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\right\}$. Then $\Delta f_1 = f(x_1) - f(x_0) = n^{\frac{1}{2}}$. Suppose f is of bounded variation on $[0,1]$. Then there exists a positive number M such that

$$n^{1/2} = |\Delta f_1| \leq \sum_{k=1}^{n} |\Delta f_k| \leq M \quad (\forall n),$$

but this fails whenever $n > M^2$. Hence f is not of bounded variation on $[0,1]$.

1.2 John's theorem

First we give John's theorem, which was stated and proved by John in 1934 in [20].

Theorem 1.2.1. *Let f be a real-valued function on $(-\infty, +\infty)$ which is periodic of period 1 and of bounded variation on $[0,1]$. If $c > 1$ is a given rational number, then*

$$\sum_{n=1}^{\infty} \frac{a_n(c)}{n} \; f\left(\frac{\log n}{\log c}\right) = \log c \int_0^1 f(y)\, dy, \tag{1.2.1}$$

where $a_n(c)$ is defined as follows: set $c = p/q$, where p and q are coprime integers with $q > 0$, then

$$a_n(c) = \begin{cases} 0, & \text{if } p \nmid n,\ q \nmid n, \\ -p, & \text{if } p \mid n,\ q \nmid n, \\ q, & \text{if } p \nmid n,\ q \mid n, \\ q-p, & \text{if } p \mid n,\ q \mid n. \end{cases} \tag{1.2.2}$$

Our formulation is slightly different from John's original statement which is as follows:

Theorem 1.2.2. *Let g be a real-valued function on $(-\infty, +\infty)$ which is periodic of period 1 and of bounded variation on $[0,1]$. If $c > 1$ is a given rational number, then*

$$\sum_{n=1}^{\infty} \frac{a_n(c)}{n} g\left(t - \frac{\log n}{\log c}\right) = \log c \int_0^1 g(y)\, dy, \tag{1.2.3}$$

for the same definition of $a_n(c)$ and for any $t \in \mathbb{R}$.

We show that the two versions of the theorem are equivalent.

Proof. (Theorem 1.2.1 $\Longrightarrow$ Theorem 1.2.2) Let g be a real-valued function on $(-\infty, +\infty)$ which is periodic of period 1 and of bounded variation on $[0,1]$. Let $c > 1$ be a given rational number, and let $t \in \mathbb{R}$.

Set $f(x) = g(t-x)$. Then f is a real-valued function of $(-\infty, \infty)$. Note that f is periodic of period 1 as $f(x+1) = g(t-(x+1)) = g(t-x-1) = g(t-x) = f(x)$.

Let $\{x_0 = 0, x_1, \ldots, x_n = 1\}$ be a partition of $[0,1]$.

If $t \in \mathbb{Z}$ we set $x'_k = 1 - x_{n-k}$, $k = 0, 1, \ldots, n$, so that $\{x'_0 = 0, x'_1, \ldots, x'_n = 1\}$ is a partition of $[0,1]$. Since g is of bounded variation on $[0,1]$, there exists $M > 0$ such that

$$\sum_{k=1}^{n} |g(x'_k) - g(x'_{k-1})| \leq M.$$

Hence

$$\sum_{k=1}^{n} |g(1 - x_{n-k}) - g(1 - x_{n-k+1})| \leq M.$$

As g is periodic with period 1 and $t \in \mathbb{Z}$, we have

$$\sum_{k=1}^{n} |g(t - x_{n-k}) - g(t - x_{n-k+1})| \leq M.$$

Next, as $f(x) = g(t-x)$, we deduce that

$$\sum_{k=1}^{n} |f(x_{n-k}) - f(x_{n-k+1})| \leq M,$$

that is

$$\sum_{k=1}^{n} |f(x_{n-k+1}) - f(x_{n-k})| \leq M.$$

Changing the summation variable from k to $l = n - k + 1$, we obtain

$$\sum_{l=1}^{n} |f(x_l) - f(x_{l-1})| \leq M,$$

so that f is of bounded variation on $[0, 1]$.

If $t \notin \mathbb{Z}$, then

$$x_0 = 0 < t - [t] < 1 = x_n.$$

Let m be the unique integer $\in \{0, 1, \ldots, n-1\}$ such that

$$x_m \leq t - [t] < x_{m+1}.$$

Set

$$\begin{aligned} x'_0 &= 0, \\ x'_1 &= t - [t] + 1 - x_n, \\ &\vdots \\ x'_{n-m} &= t - [t] + 1 - x_{m+1}, \\ x'_{n-m+1} &= 1. \end{aligned}$$

Then $\{x'_0, x'_1, \ldots, x'_{n-m+1}\}$ is a partition of $[0, 1]$. Since g is of bounded variation on $[0, 1]$, there exists $M_1 > 0$ such that

$$\sum_{k=1}^{n-m+1} |g(x'_k) - g(x'_{k-1})| \leq M_1.$$

Hence

$$\begin{aligned} &\sum_{k=2}^{n-m} |g(t - [t] + 1 - x_{n+1-k}) - g(t - [t] + 1 - x_{n+2-k})| \\ &\qquad = \sum_{k=2}^{n-m} |g(x'_k) - g(x'_{k-1})| \leq M_1. \end{aligned}$$

As $f(x) = g(t-x)$ we deduce that

$$\sum_{k=2}^{n-m} |f([t]-1+x_{n+1-k}) - f([t]-1+x_{n+2-k})| \leq M_1.$$

As f is periodic of period 1, we obtain

$$\sum_{k=2}^{n-m} |f(x_{n+1-k}) - f(x_{n+2-k})| \leq M_1.$$

Changing the summation variable from k to $l = n-k+1$, we obtain

$$\sum_{l=m+1}^{n-1} |f(x_l) - f(x_{l+1})| \leq M_1,$$

that is,

$$\sum_{l=m+1}^{n-1} |f(x_{l+1}) - f(x_l)| \leq M_1,$$

equivalently

$$\sum_{k=m+2}^{n} |f(x_k) - f(x_{k-1})| \leq M_1.$$

As g is of bounded variation on $[0,1]$, by Theorem 1.1.3 g is bounded on $[0,1]$. As g is periodic of period 1, g is bounded on $(-\infty,\infty)$. As $f(x) = g(t-x)$, f is bounded on $(-\infty,\infty)$, say $|f(x)| \leq K$ for all $x \in \mathbb{R}$. Hence

$$\begin{aligned}
\sum_{k=m+1}^{n} |f(x_k) - f(x_{k-1})| &= |f(x_{m+1}) - f(x_m)| + \sum_{k=m+2}^{n} |f(x_k) - f(x_{k-1})| \\
&\leq |f(x_{m+1})| + |f(x_m)| + M_1 \\
&\leq 2K + M_1.
\end{aligned}$$

Similarly by forming a partition from $x_0, x_1, \ldots, x_m$ we can bound $\sum_{k=1}^{m} |f(x_k) - f(x_k - 1)|$, proving that f is of bounded variation on $[0,1]$.

Lastly using Theorem 1.2.1, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n(c)}{n} g\left(t - \frac{\log n}{\log c}\right) &= \sum_{n=1}^{\infty} \frac{a_n(c)}{n} f\left(\frac{\log n}{\log c}\right) \\
&= \log c \int_0^1 f(y)\, dy \\
&= \log c \int_0^1 g(t-y)\, dy \\
&= -\log c \int_t^{t-1} g(z)\, dz \quad (\text{with } z = t - y) \\
&= \log c \int_{t-1}^{t} g(z)\, dz \\
&= \log c \int_0^1 g(y)\, dy.
\end{aligned}$$

(Theorem 1.2.2 $\Longrightarrow$ Theorem 1.2.1) Let f be a real valued function on $(-\infty, \infty)$ which is periodic of period 1 and of bounded variation on $[0, 1]$. Let $c > 1$ be a given rational number.

Set $g(x) = f(-x)$. Then g is a real valued function on $(-\infty, \infty)$. Note that g is periodic of period 1 as $g(x+1) = f(-(x+1)) = f(-x-1) = f(-x) = g(x)$.

Let $x_0 = 0, x_1, \ldots, x_n = 1$ be a partition of $[0, 1]$. Then

$$x_0' = 1 - x_n, x_1' = 1 - x_{n-1}, \ldots, x_n' = 1 - x_0$$

is also a partition of $[0, 1]$.

Since f is of bounded variation on $[0, 1]$, there exists $M > 0$ such that

$$\sum_{k=1}^{n} |f(x_k') - f(x_{k-1}')| \leq M \text{ and } \sum_{k=1}^{n} |f(1 - x_{n-k}') - f(1 - x_{n-k+1}')| \leq M.$$

As $f(1-x'_{n-k}) = f(-x_{n-k}) = g(x_{n-k})$, we have

$$\sum_{k=1}^{n} |g(x_{n-k}) - g(x_{n-k+1})| \leq M.$$

Setting $j = n-k+1$ gives $\sum_{j=1}^{n} |g(x_{j-1}) - g(x_j)| \leq M$, and, as we may change the order due to the absolute value,

$$\sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \leq M,$$

so g is of bounded variation on $[0,1]$.

Lastly, setting $t = 0$ in Theorem 1.2.2, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n(c)}{n} f\left(\frac{\log n}{\log c}\right) &= \sum_{n=1}^{\infty} \frac{a_n(c)}{n} g\left(t - \frac{\log n}{\log c}\right) \\
&= \log c \int_0^1 g(y)\, dy \\
&= \log c \int_0^1 f(-y)\, dy \\
&= \log c \int_0^1 f(1-y)\, dy \\
&= -\log c \int_0^1 f(z)\, dz \quad (\text{with } z = 1-y) \\
&= \log c \int_0^1 f(z)\, dz,
\end{aligned}$$

as required. □

1.3 Proof of John's theorem

Let $c > 1$ be a rational number. Set $c = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Set $\lambda = \dfrac{1}{\log c}$. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha \leq 1$. Let $M \in \mathbb{N}$.

We begin by listing a few simple properties of c, p, q, λ, α and M that we will need in the proof of John's theorem.

Lemma 1.3.1. (i) $p > q$,

(ii) $\log c > 0$,

(iii) $\lambda, \lambda^{-1} > 0$,

(iv) $\alpha + \lambda \log Mq \leq \lambda \log Mp$,

(v) $\{\lambda \log Mp\} = \{\lambda \log Mq\}$,

(vi) $\dfrac{\log q}{\log p - \log q} = \dfrac{\log p}{\log p - \log q} - 1$.

Proof. (i) As $c = p/q > 1$ and $q > 0$ we have $p > q$.

(ii) As $c > 1$ we have $\log c > \log 1 = 0$.

(iii) By (ii) we have

$$\lambda = \frac{1}{\log c} > 0, \ \frac{1}{\lambda} = \log c > 0.$$

(iv) We have

$$\begin{aligned}
\alpha \leq 1 \quad &\Longrightarrow \quad \alpha \leq \lambda\lambda^{-1} \Longrightarrow \alpha \leq \lambda \log c \\
&\Longrightarrow \quad \alpha \leq \lambda \log \frac{p}{q} \Longrightarrow \alpha \leq \lambda \log \frac{Mp}{Mq} \\
&\Longrightarrow \quad \alpha \leq \lambda(\log Mp - \log Mq) \\
&\Longrightarrow \quad \alpha + \lambda \log Mq \leq \lambda \log Mp.
\end{aligned}$$

(v) Also

$$\begin{aligned}
\{\lambda \log Mp\} &= \{\lambda \log Mp - \lambda \log Mq + \lambda \log Mq\} \\
&= \{\lambda \log \frac{p}{q} + \lambda \log Mq\} \\
&= \{\lambda \cdot \lambda^{-1} + \lambda \log Mq\} \\
&= \{1 + \lambda \log Mq\} \\
&= \{\lambda \log Mq\}.
\end{aligned}$$

(vi) Finally

$$\frac{\log q}{\log p - \log q} = \frac{-(\log p - \log q) + \log p}{\log p - \log q} = -1 + \frac{\log p}{\log p - \log q}.$$

□

Next we recall Euler's constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right) = 0.5772156649\ldots.$$

We will need the following estimate.

Lemma 1.3.2. *Let $x, y \in \mathbb{R}$ be such that $x > y > 0$. Then*

$$\sum_{\substack{m \in \mathbb{N} \\ y < m \leq x}} \frac{1}{m} = \log\left(\frac{x}{y}\right) + O\left(\frac{1}{y}\right),$$

as $y \to +\infty$, where the constant implied by the O-symbol is absolute.

Proof. The asymptotic formula

$$\sum_{\substack{m\in\mathbb{N}\\ m\le y}} \frac{1}{m} = \log y + \gamma + O\left(\frac{1}{y}\right),\ y \to \infty$$

is well-known, see for example [3, p. 55]. The implied constant is absolute. Thus for $x > y > 0$ we have

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N}\\ y<m\le x}} \frac{1}{m} &= \sum_{\substack{m\in\mathbb{N}\\ m\le x}} \frac{1}{m} - \sum_{\substack{m\in\mathbb{N}\\ y\le m}} \frac{1}{m} \\ &= \left(\log x + \gamma + O\left(\frac{1}{x}\right)\right) - \left(\log y + \gamma + O\left(\frac{1}{y}\right)\right) \\ &= \log\left(\frac{x}{y}\right) + O\left(\frac{1}{y}\right),\end{aligned}$$

as $y \to \infty$. □

We now make use of Lemmas 1.3.1 and 1.3.2 to prove the following result.

Lemma 1.3.3.

$$\lim_{M\to\infty} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le\{-\lambda\log m\}<\alpha}} \frac{1}{m} = \alpha\lambda^{-1}.$$

Proof. We have

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le\{-\lambda\log m\}<\alpha}} \frac{1}{m} &= \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le-\lambda\log m-[-\lambda\log m]<\alpha}} \frac{1}{m} \\ &= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le-\lambda\log m-[-\lambda\log m]<\alpha\\ [-\lambda\log m]=-\ell}} \frac{1}{m}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<-\lambda\log Mp+1}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le-\lambda\log m+\ell<\alpha\\ [-\lambda\log m]=-\ell}} \frac{1}{m}\\
&= \sum_{\substack{\ell\in\mathbb{N}\\ \lambda\log Mq<\ell<\lambda\log Mp+1}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ 0\le-\lambda\log m+\ell<\alpha}} \frac{1}{m}\\
&= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ e^{\frac{\ell-\alpha}{\lambda}}<m\le e^{\frac{\ell}{\lambda}}}} \frac{1}{m}.
\end{aligned}
$$

We now break the sum over ℓ into 4 subsums S_1, S_2, S_3, S_4 according to the sizes of $e^{\frac{\ell-\alpha}{\lambda}}$ and $e^{\frac{\ell}{\lambda}}$ relative to Mp and Mq. We have

$$
\sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp\\ e^{\frac{\ell-\alpha}{\lambda}}<m\le e^{\frac{\ell}{\lambda}}}} \frac{1}{m} = S_1+S_2+S_3+S_4,
$$

where

$$
\begin{aligned}
S_1 &= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1\\ e^{\frac{\ell-\alpha}{\lambda}}<Mq\\ Mp\le e^{\frac{\ell}{\lambda}}}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp}} \frac{1}{m},\\
S_2 &= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1\\ Mq\le e^{\frac{\ell-\alpha}{\lambda}}\\ Mp\le e^{\frac{\ell}{\lambda}}\\ e^{\frac{\ell-\alpha}{\lambda}}<Mp}} \sum_{\substack{m\in\mathbb{N}\\ e^{\frac{\ell-\alpha}{\lambda}}<m\le Mp}} \frac{1}{m},\\
S_3 &= \sum_{\substack{\ell\in\mathbb{Z}\\ \lambda\log Mq<\ell<\lambda\log Mp+1\\ e^{\frac{\ell-\alpha}{\lambda}}<Mq\\ e^{\frac{\ell}{\lambda}}<Mp}} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le e^{\frac{\ell}{\lambda}}}} \frac{1}{m},
\end{aligned}
$$

$$S_4 = \sum_{\substack{\ell\in\mathbb{Z} \\ \lambda\log Mq<\ell<\lambda\log Mp+1 \\ Mq\le e^{\frac{\ell-\alpha}{\lambda}} \\ e^{\frac{\ell}{\lambda}}<Mp}} \sum_{\substack{m\in\mathbb{N} \\ e^{\frac{\ell-\alpha}{\lambda}}<m\le e^{\frac{\ell}{\lambda}}}} \frac{1}{m}.$$

We note that $S_2 = 0$ unless $e^{\frac{\ell-\alpha}{\lambda}} < Mp$. Also in the sum S_3 we observe that $Mq < e^{\frac{\ell}{\lambda}}$ as $\lambda\log Mq < \ell$.

First we examine S_1. By Lemma 1.3.2 we have

$$\begin{aligned} S_1 &= \sum_{\substack{\ell\in\mathbb{Z} \\ \lambda\log Mq<\ell<\lambda\log Mp+1 \\ \lambda\log Mp\le\ell<\alpha+\lambda\log Mq}} \left(\log\left(\frac{p}{q}\right)+O\left(\frac{1}{M}\right)\right) \\ &= 0, \end{aligned}$$

as $\alpha + \lambda\log Mq \le \lambda\log Mp$ by Lemma 1.3.1 (iv).

Next we determine S_2. By Lemma 1.3.2 we have

$$\begin{aligned} S_2 &= \sum_{\substack{\ell\in\mathbb{Z} \\ \lambda\log Mq<\ell<\lambda\log Mp+1 \\ \alpha+\lambda\log Mq\le\ell<\alpha+\lambda\log Mp \\ \lambda\log Mp\le\ell}} \left(\log\left(\frac{Mp}{e^{\frac{\ell-\alpha}{\lambda}}}\right)+O\left(\frac{1}{e^{\frac{\ell-\alpha}{\lambda}}}\right)\right) \\ &= \sum_{\substack{\ell\in\mathbb{Z} \\ \lambda\log Mp\le\ell<\lambda\log Mp+\alpha}} \left(\log\left(\frac{Mp}{e^{\frac{\ell-\alpha}{\lambda}}}\right)+O\left(\frac{1}{e^{\frac{\ell-\alpha}{\lambda}}}\right)\right), \end{aligned}$$

by Lemma 1.3.1 (iv). We consider 3 cases according as $\{\lambda\log Mp\} = 0$, $0 < \{\lambda\log Mp\} \le 1-\alpha$, or $1-\alpha < \{\lambda\log Mp\} < 1$.

If $\{\lambda\log Mp\} = 0$ then $\lambda\log Mp \in \mathbb{Z}$ so $\ell = \lambda\log Mp$. Thus

$$\begin{aligned} S_2 &= \log\left(\frac{Mp}{e^{(\lambda\log Mp-\alpha)/\lambda}}\right)+O\left(\frac{1}{e^{(\lambda\log Mp-\alpha)/\lambda}}\right) \\ &= \frac{\alpha}{\lambda}+O\left(\frac{1}{M}\right). \end{aligned}$$

If $0 < \{\lambda \log Mp\} \leq 1-\alpha$ then $[\lambda \log Mp]+1 \geq \lambda \log Mp+\alpha$ so $\lambda \log Mp \notin \mathbb{Z}$ and no such integer ℓ exists so $S_2 = 0$.

If $1-\alpha < \{\lambda \log Mp\} < 1$ then $\lambda \log Mp \notin \mathbb{Z}$ and $[\lambda \log Mp]+1 < \lambda \log Mp + \alpha$ and $\ell = [\lambda \log Mp]+1$. Then

$$\begin{aligned} S_2 &= \log\left(\frac{Mp}{e^{([\lambda \log Mp]+1-\alpha)/\lambda}}\right) + O\left(\frac{1}{e^{([\lambda \log Mp]+1-\alpha)/\lambda}}\right) \\ &= \log\left(\frac{Mp}{e^{(\lambda \log Mp-\{\lambda \log Mp\}+1-\alpha)/\lambda}}\right) + O\left(\frac{1}{e^{(\lambda \log Mp-\{\lambda \log Mp\}+1-\alpha)/\lambda}}\right) \\ &= \log\left(e^{\frac{\alpha-1+\{\lambda \log Mp\}}{\lambda}}\right) + O\left(\frac{1}{M}\right) \\ &= \frac{\alpha-1+\{\lambda \log Mp\}}{\lambda} + O\left(\frac{1}{M}\right). \end{aligned}$$

Hence

$$S_2 = \begin{cases} \dfrac{\alpha}{\lambda} + O\left(\dfrac{1}{M}\right), & \{\lambda \log Mp\} = 0, \\ 0, & 0 < \{\lambda \log Mp\} \leq 1-\alpha, \\ \dfrac{\alpha-1+\{\lambda \log Mp\}}{\lambda} + O\left(\dfrac{1}{M}\right), & 1-\alpha < \{\lambda \log Mp\}. \end{cases}$$

Now we turn to the evaluation of S_3. We have by Lemma 1.3.2

$$\begin{aligned} S_3 &= \sum_{\substack{\ell \in \mathbb{Z} \\ \lambda \log Mq \leq \ell < \lambda \log Mp+1 \\ \ell < \alpha + \lambda \log Mq \\ \ell < \lambda \log Mp}} \left(\log\left(\frac{e^{\frac{\ell}{\lambda}}}{Mq}\right) + O\left(\frac{1}{M}\right)\right) \\ &= \sum_{\substack{\ell \in \mathbb{Z} \\ \lambda \log Mq \leq \ell < \alpha + \lambda \log Mq}} \left(\log\left(\frac{e^{\frac{\ell}{\lambda}}}{Mq}\right) + O\left(\frac{1}{M}\right)\right), \end{aligned}$$

by Lemma 1.3.1 (iv).

Here we consider 3 cases according as $\{\lambda \log Mq\} = 0$, $0 < \{\lambda \log Mq\} \leq 1-\alpha$, or $1-\alpha < \{\lambda \log Mq\} < 1$.

If $\{\lambda \log Mq\} = 0$ then $\lambda \log Mq \in \mathbb{Z}$ and we have

$$S_3 = \log\left(\frac{e^{\frac{\lambda \log Mq}{\lambda}}}{Mq}\right) + O\left(\frac{1}{M}\right) = O\left(\frac{1}{M}\right).$$

If $0 < \{\lambda \log Mq\} \leq 1 - \alpha$ then $[\lambda \log Mq] + 1 \geq \alpha + \lambda \log Mq$ so $\lambda \log Mq \notin \mathbb{Z}$ and no such integer ℓ exists so $S_3 = 0$.

If $1 - \alpha < \{\lambda \log Mq\} < 1$ then $[\lambda \log Mq] + 1 < \lambda \log Mq + \alpha$ and we have

$$\begin{aligned} S_3 &= \log\left(\frac{e^{\frac{[\lambda \log Mq]+1}{\lambda}}}{Mq}\right) + O\left(\frac{1}{M}\right) \\ &= \frac{[\lambda \log Mq] + 1}{\lambda} - \log Mq + O\left(\frac{1}{M}\right) \\ &= \frac{1 - \{\lambda \log Mq\}}{\lambda} + O\left(\frac{1}{M}\right). \end{aligned}$$

Hence

$$S_3 = \begin{cases} O\left(\frac{1}{M}\right), & \{\lambda \log Mq\} = 0, \\ 0, & 0 < \{\lambda \log Mq\} \leq 1 - \alpha, \\ \dfrac{1 - \{\lambda \log Mq\}}{\lambda} + O\left(\dfrac{1}{M}\right), & 1 - \alpha < \{\lambda \log Mq\}, \end{cases}$$

which is equivalent to

$$S_3 = \begin{cases} O\left(\frac{1}{M}\right), & \{\lambda \log Mp\} = 0, \\ 0, & 0 < \{\lambda \log Mp\} \leq 1 - \alpha, \\ \dfrac{1 - \{\lambda \log Mq\}}{\lambda} + O\left(\dfrac{1}{M}\right), & 1 - \alpha < \{\lambda \log Mp\}, \end{cases}$$

as $\{\lambda \log Mp\} = \{\lambda \log Mq\}$, by Lemma 1.3.1 (v).

Lastly we consider S_4. By Lemma 1.3.2 we have

$$\begin{aligned} S_4 &= \sum_{\substack{\ell\in\mathbb{Z} \\ \lambda\log Mq\le\ell<\lambda\log Mp+1 \\ \alpha+\lambda\log Mq\le\ell<\lambda\log Mp}} \left(\log\left(\frac{e^{\ell/\lambda}}{e^{\frac{\ell-\alpha}{\lambda}}}\right)+O\left(\frac{1}{e^{\frac{\ell-\alpha}{\lambda}}}\right)\right) \\ &= \sum_{\substack{\ell\in\mathbb{Z} \\ \alpha+\lambda\log Mq\le\ell<\lambda\log Mp}} \left(\frac{\alpha}{\lambda}+O\left(\frac{1}{M}\right)\right). \end{aligned}$$

We consider 3 cases according as $\{\lambda\log Mp\}=0$, $0<\{\lambda\log Mp\}\le 1-\alpha$, or $1-\alpha<\{\lambda\log Mp\}<1$.

If $\{\lambda\log Mp\}=0$ then $\lambda\log Mp\in\mathbb{Z}$ and no such integer ℓ exists so $S_4=0$.

If $0<\{\lambda\log Mp\}\le 1-\alpha$ then $\ell=[\lambda\log Mp]$ and

$$S_4=\frac{\alpha}{\lambda}+O\left(\frac{1}{M}\right).$$

If $1-\alpha<\{\lambda\log Mp\}<1$ then no such integer ℓ exists and $S_4=0$.

Hence

$$S_4=\begin{cases} 0, & \{\lambda\log Mp\}=0, \\ \dfrac{\alpha}{\lambda}+O\left(\dfrac{1}{M}\right), & 0<\{\lambda\log Mp\}\le 1-\alpha, \\ 0, & 1-\alpha<\{\lambda\log Mp\}<1. \end{cases}$$

Putting it all together, we now look at the required sum $S_1+S_2+S_3+S_4$ in 3 cases according as $\{\lambda\log Mp\}=0$, $0<\{\lambda\log Mp\}\le 1-\alpha$, or $1-\alpha<\{\lambda\log Mp\}<1$.

If $\{\lambda \log Mp\} = 0$ then

$$S_1 = 0, \quad S_2 = \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right), \quad S_3 = O\left(\frac{1}{M}\right), \quad S_4 = 0,$$

so

$$S_1 + S_2 + S_3 + S_4 = \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right).$$

If $0 < \{\lambda \log Mp\} \leq 1 - \alpha$ then

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = 0, \quad S_4 = \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right),$$

so

$$S_1 + S_2 + S_3 + S_4 = \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right).$$

If $1 - \alpha < \{\lambda \log Mp\} < 1$ then

$$\begin{aligned} S_1 &= 0, \quad S_2 = \frac{\alpha - 1 + \{\lambda \log Mp\}}{\lambda} + O\left(\frac{1}{M}\right), \\ S_4 &= 0, \quad S_3 = \frac{1 - \{\lambda \log Mp\}}{\lambda} + O\left(\frac{1}{M}\right), \end{aligned}$$

so

$$\begin{aligned} S_1 + S_2 + S_3 + S_4 &= \frac{\alpha - 1 + \{\lambda \log Mp\} + 1 - \{\lambda \log Mp\}}{\lambda} + O\left(\frac{1}{M}\right), \\ &= \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right). \end{aligned}$$

Hence in all three cases we have

$$S_1 + S_2 + S_3 + S_4 = \frac{\alpha}{\lambda} + O\left(\frac{1}{M}\right)$$

so

$$\lim_{M \to \infty} \sum_{\substack{m \in \mathbb{N} \\ Mq < m \leq Mp \\ 0 \leq \{-\lambda \log m\} < \alpha}} \frac{1}{m} = \frac{\alpha}{\lambda}.$$

as required. □

For $\alpha \in [0,1]$ we define a special step function ϕ_α as follows: for $y \in \mathbb{R}$ we set

$$\phi_0(y) = 0, \quad y \in \mathbb{R},$$
$$\phi_1(1) = 1, \quad y \in \mathbb{R},$$

and for $\alpha \neq 0, 1$

$$\phi_\alpha(y) = \begin{cases} 1, & 0 \leq \{y\} < \alpha, \\ 0, & \alpha \leq \{y\} < 1. \end{cases} \tag{1.3.1}$$

Lemma 1.3.4.

$$\lim_{M\to\infty} \sum_{\substack{m\in\mathbb{N} \\ Mq<m\leq Mp}} \frac{1}{m}\, \phi_\alpha(-\lambda \log m) = \alpha\lambda^{-1}.$$

Proof. As $\phi_0(y) = 0$ for all y, Lemma 1.3.4 is trivially true for $\alpha = 0$ and we may assume that $0 < \alpha \leq 1$.

By the definition of ϕ_α we have

$$\sum_{\substack{m\in\mathbb{N} \\ Mq<m\leq Mp}} \frac{1}{m}\, \phi_\alpha(-\lambda \log m) = \sum_{\substack{m\in\mathbb{N} \\ Mq<m\leq Mp \\ 0\leq\{-\lambda\log m\}<\alpha}} \frac{1}{m},$$

and the asserted result follows by applying Lemma 1.3.3. □

Lemma 1.3.5. *Let f be a periodic step function of period* 1. *Then*

$$\lim_{M\to\infty} \sum_{\substack{m\in\mathbb{N} \\ Mq<m\leq Mp}} \frac{1}{m}\, f(-\lambda \log m) = \lambda^{-1} \int_0^1 f(y)\, dy.$$

Proof. As f is a periodic step function of period 1, f can be built up as a linear combination of a finite number of the step functions $\phi_\alpha(y)$ with different parameters α. Hence the asserted result holds by repeated application of Lemma 1.3.4. □

Lemma 1.3.6. *Let f be a periodic function of period 1, which is Riemann integrable on $[0,1]$. Then*

$$\lim_{M\to\infty} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\leq Mp}} \frac{1}{m} f(-\lambda \log m) = \lambda^{-1} \int_0^1 f(y)\, dy.$$

Proof. Let $\epsilon > 0$. As $f(y)$ is a Riemann integrable function on $[0,1]$ and is periodic of period 1, there exist two step-functions $\phi(y)$ and $\Phi(y)$ of period 1 such that

$$\phi(y) \leq f(y) \leq \Phi(y) \tag{1.3.2}$$

and

$$\int_0^1 (\Phi(y) - \phi(y))\ dy < \epsilon, \tag{1.3.3}$$

see for example [5, Sections 1.1, 1.2, pp. 11-26]. Set

$$\begin{aligned} S_M(\phi) &= \sum_{\substack{m\in\mathbb{N}\\ Mq<m\leq Mp}} \frac{1}{m}\phi(-\lambda \log m), \\ S_M(\Phi) &= \sum_{\substack{m\in\mathbb{N}\\ Mq<m\leq Mp}} \frac{1}{m}\Phi(-\lambda \log m). \end{aligned}$$

Then, by Lemma 1.3.5, we have

$$\begin{aligned} \lim_{M\to\infty} S_M(\phi) &= \lambda^{-1} \int_0^1 \phi(y)\, dy, \\ \lim_{M\to\infty} S_M(\Phi) &= \lambda^{-1} \int_0^1 \Phi(y)\, dy, \end{aligned} \tag{1.3.4}$$

and from (1.3.2) we have

$$S_M(\phi) \leq \sum_{\substack{m\in\mathbb{N}\\ Mq<m\leq Mp}} \frac{1}{m} f(-\lambda \log m) \leq S_M(\Phi).$$

From this and
$$\begin{aligned}
\lambda^{-1}\int_0^1 \phi(y)\,dy &\le \liminf_{M\to\infty} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp}} \frac{1}{m} f(-\lambda\log m)\\
\lambda^{-1}\int_0 \phi(y)\,dy &\le \liminf_{M\to\infty} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp}} \frac{1}{m} f(-\lambda\log m)\\
&\le \limsup_{M\to\infty} \sum_{\substack{m\in\mathbb{N}\\ Mq<m\le Mp}} \frac{1}{m} f(-\lambda\log m)\\
&\le \lambda^{-1}\int_0^1 \Phi(y)\,dy,
\end{aligned}$$

and applying (1.3.3) gives the asserted result. □

We now have the results needed to prove John's theorem (Theorem 1.2.2).

Proof. We follow the proof of John's theorem given by Rademacher in [27, pp. 170–173], but expanding on the details where necessary. We note that the proof only requires g to be Riemann integrable (recall from Theorem 1.1.11 that functions of bounded variation are always Riemann integrable).

We wish to study the expression

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{a_n(c)}{n}\, g(t-\lambda\log n)$$

for a Riemann integrable function g and $a_n(c)$ as defined in (1.2.2). We begin by showing that we need only consider such N which are divisible by pq.

For $0 < R < pq$, as $a_n(c)$ and g are bounded, we have

$$\left|\sum_{n=Npq+1}^{Npq+R} \frac{a_n(c)}{n} g(t-\lambda\log n)\right| \le A \sum_{n=Npq+1}^{Npq+R} \frac{1}{n}$$

for some constant A.

By Lemma 1.3.2 we have

$$\begin{aligned}\sum_{n=Npq+1}^{Npq+R} \frac{1}{n} &= \sum_{Npq<n\leq Npq+R} \frac{1}{n} \\ &= \log\left(\frac{Npq+R}{Npq}\right) + O\left(\frac{1}{N}\right) \\ &= \log\left(1+\frac{R}{Npq}\right) + O\left(\frac{1}{N}\right).\end{aligned}$$

Hence

$$\lim_{N\to\infty} \sum_{n=Npq+1}^{Npq+R} \frac{1}{n} = \log 1 = 0,$$

proving our claim.

Now set

$$S_M(g) = \sum_{n=1}^{Mpq} \frac{a_n(c)}{n} g(t-\lambda\log n).$$

Then, using the definition of $a_n(c)$, we obtain

$$\begin{aligned}S_M(g) &= \sum_{\substack{1\leq n\leq Mpq \\ q|n}} \frac{q}{n}\, g(t-\lambda\log n) - \sum_{\substack{1\leq n\leq Mpq \\ p|n}} \frac{p}{n}\, g(t-\lambda\log n) \\ &= \sum_{m=1}^{Mp} \frac{1}{m}\, g(t-\lambda\log mq) - \sum_{m=1}^{Mq} \frac{1}{m}\, g(t-\lambda\log mp).\end{aligned}$$

Using Lemma 1.3.1 (vi), we have

$$\lambda\log mq = \frac{\log m+\log q}{\log p-\log q} = \frac{\log m}{\log p-\log q} + \frac{\log p}{\log p-\log q} - 1,$$

which gives

$$g(t-\lambda\log mq) = g\left(t - \frac{\log m}{\log p-\log q} - \frac{\log p}{\log p-\log q}\right),$$

as g is periodic of period one. As

$$g(t-\lambda\log mp) = g\left(t - \frac{\log m}{\log p-\log q} - \frac{\log p}{\log p-\log q}\right),$$

we deduce that

$$\begin{aligned} S_M(g) &= \sum_{m=Mq+1}^{Mp} \frac{1}{m}\, g\left(t - \frac{\log m}{\log p - \log q} - \frac{\log p}{\log p - \log q}\right) \\ &= \sum_{m=Mq+1}^{Mp} \frac{1}{m}\, g(t - \lambda \log m - \lambda \log p). \end{aligned}$$

We set $t = \lambda \log p$, for if (1.2.3) is true for any special value t_0, it is also true for any other t, as $g(x)$ and $g(x - t_0 + t)$ regarded as functions of x are both periodic and Riemann integrable.

Hence all we need to show is

$$\lim_{M\to\infty} S_M(g) = \lim_{M\to\infty} \sum_{m=Mq+1}^{Mp} \frac{1}{m}\, g(-\lambda \log m) = \lambda^{-1} \int_0^1 g(y)\, dy.$$

But this was shown in Lemma 1.3.6, and the proof is complete. □

1.4 Evaluation of certain infinite series using John's theorem

We begin by choosing $f(x) \equiv 1$ $(x \in \mathbb{R})$. Clearly f satisfies the conditions of John's theorem.

Theorem 1.4.1. *Let p and q be coprime integers with $p > q > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{a_n(p/q)}{n} = \log \frac{p}{q},$$

where a_n is defined in (1.2.2).

Taking $p = 2$ and $q = 1$, we obtain the well known result

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

With $p = 3$ and $q = 1$ we have

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots = \log 3.$$

With $p = 3$ and $q = 2$ we obtain

$$\frac{2}{2} - \frac{3}{3} + \frac{2}{4} - \frac{1}{6} + \frac{2}{8} - \frac{3}{9} + \frac{2}{10} - \frac{1}{12} + \cdots = \log \frac{3}{2}.$$

These series are all of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n},$$

where $\{a_n\}$ is a repeating sequence of integers. Such a series is called a harmonic-type series. The harmonic series itself demonstrates that not all harmonic-type series converge. Lesko [23] has recently established a necessary and sufficient condition for a harmonic-type series to converge, namely

A harmonic-type series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ *with repeating coefficients* $a_1, a_2, \ldots, a_k$ *converges if and only if* $\sum_{i=1}^{k} a_i = 0$.

Series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{k^n},$$

where $\{a_n\}$ is a repeating sequence of integers and k is a given real number, have been treated by Longuet-Higgins [25].

For an integer $b > 1$ it is convenient to define

$$\epsilon_n = \epsilon_n(b) = \begin{cases} b-1, & \text{if } b \mid n, \\ -1, & \text{if } b \nmid n, \end{cases} \tag{1.4.1}$$

as in [8, p. 22] and [12, p. 261]. Clearly

$$a_n(b) = -\epsilon_n(b), \qquad \epsilon_n(2) = (-1)^n. \tag{1.4.2}$$

Taking $p = b$ and $q = 1$ in Theorem 1.4.1 we obtain

Corollary 1.4.1. *Let b be an integer > 1. Then*

$$\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} = -\log b.$$

Corollary 1.4.1 is well-known. It appears for example in [16, Problem 31], [19], [22, p. 136].

We next use John's theorem to evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left\{ \frac{\log n}{\log 2} \right\}^k$$

for any $k \in \mathbb{N}$. Recall that $\{y\} = y - [y]$ is the fractional part of the real number y.

Theorem 1.4.2. *For $k \in \mathbb{N}$, we have*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \frac{\log n}{\log 2} \right\}^k = -\frac{1}{k+1} \log 2.$$

Proof. In John's theorem we take $c = p/q = 2/1$ and $f(x) = \{x\}^k$ $(x \in \mathbb{R})$. Then f is real valued on $(-\infty, \infty)$ and is periodic of period 1.

By (1.4.2) we have

$$a_n(2) = -\epsilon_n(2) = -(-1)^n = (-1)^{n-1}.$$

Set

$$\begin{aligned} g(x) &= x^k, \quad x \in [0,1], \\ h(x) &= \begin{cases} 0, & x \in [0,1), \\ 1, & x = 1. \end{cases} \end{aligned}$$

Then g is of bounded variation on $[0,1]$ as it is monotonic on $[0,1]$ and h is obviously of bounded variation. Now $f = g - h$ so f is of bounded variation on $[0,1]$ by Theorem 1.1.4.

Then, by John's theorem, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left\{ \frac{\log n}{\log 2} \right\}^k &= \log 2 \int_0^1 \{y\}^k \, dy \\ &= \log 2 \int_0^1 y^k \, dy \\ &= \log 2 \left[\frac{y^{k+1}}{k+1} \right]_0^1 \\ &= \frac{1}{k+1} \log 2, \end{aligned}$$

and the asserted result follows. □

Taking $k = 1$ in Theorem 1.4.2 we obtain

Corollary 1.4.2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \frac{\log n}{\log 2} \right\} = -\frac{1}{2} \log 2.$$

Next we use John's theorem to evaluate an infinite series which includes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \frac{\log n}{\log 2} \right\}^k$$

as the special case $b = 2$.

Theorem 1.4.3. *Let $k \in \mathbb{N}$. Let $b \in \mathbb{N}$ be such that $b > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{ \frac{\log n}{\log b} \right\}^k = -\frac{1}{k+1} \log b.$$

Proof. In John's theorem we take $c = b$. By (1.4.2), we see that

$$a_n(c) = -\epsilon_n(b).$$

We choose $f(x) = \{x\}^k$ $(x \in \mathbb{R})$ so that f is real-valued on $(-\infty, \infty)$, is periodic of period 1, and of bounded variation on $[0, 1]$.

Then, by John's theorem, we have

$$\sum_{n=1}^{\infty} \frac{-\epsilon_n(b)}{n} \left\{ \frac{\log n}{\log b} \right\}^k = \log b \int_0^1 \{y\}^k \, dy = \frac{1}{k+1} \log b,$$

and the asserted result follows. □

Finally we use John's theorem to evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{ \frac{\log(n/a)}{\log b} \right\}^k$.

Theorem 1.4.4. *Let $k \in \mathbb{N}$. Let a and b be integers with $a \geq 1$ and $b > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{ \frac{\log(n/a)}{\log b} \right\}^k = -\frac{1}{k+1} \log b.$$

Proof. We use John's theorem in the form of Theorem 1.2.2. We choose

$$g(x) = \{-x\}^k, \quad c = b, \quad t = \frac{\log a}{\log b}.$$

Then

$$a_n(c) = -\epsilon_n(b)$$

and John's theorem gives

$$\sum_{n=1}^{\infty} \frac{-\epsilon_n(b)}{n} \left\{ \frac{\log n}{\log b} - \frac{\log a}{\log b} \right\}^k = \log b \int_0^1 \{-y\}^k \, dy,$$

that is

$$\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{ \frac{\log(n/a)}{\log b} \right\}^k = -\frac{1}{k+1} \log b,$$

as asserted. □

1.5 The generalized Euler constants

Euler's constant γ is defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) = 0.5772156649\ldots. \tag{1.5.1}$$

It is well known that

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \tag{1.5.2}$$

as $x \to +\infty$, see for example [3, p. 55].

The generalized Euler constants γ_k $(k = 0, 1, 2, \ldots)$ are defined by

$$\gamma_k = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{\log^k j}{j} - \frac{\log^{k+1} n}{k+1} \right), \quad k = 0, 1, 2, \ldots, \tag{1.5.3}$$

see [12, p. 259], where we understand $\log^0 1 = 1$. Clearly

$$\gamma_0 = \gamma. \tag{1.5.4}$$

From [3, p. 70] we know that

$$\sum_{n \le x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + \gamma_1 + O\left(\frac{\log x}{x}\right), \tag{1.5.5}$$

as $x \to +\infty$.

We next use the asymptotic formulas (1.5.2) and (1.5.5) to evaluate the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\log n}{n}$, which will be needed in the next section.

Theorem 1.5.1.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\log n}{n} = -\gamma \log 2 + \frac{1}{2}\log^2 2.$$

Proof. Let $N \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=1}^{2N}(-1)^{n-1}\frac{\log n}{n} - \sum_{n=1}^{2N}\frac{\log n}{n} &= \sum_{n=1}^{2N}((-1)^{n-1}-1)\frac{\log n}{n} \\
&= -2\sum_{n=1}^{N}\frac{\log 2n}{2n} \\
&= -\sum_{n=1}^{N}\frac{\log 2n}{n} \\
&= -\sum_{n=1}^{N}\frac{\log 2 + \log n}{n} \\
&= -\log 2\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{\log n}{n}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{2N}(-1)^{n-1}\frac{\log n}{n} &= \sum_{n=1}^{2N}\frac{\log n}{n} - \log 2\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{\log n}{n} \\
&= \frac{1}{2}\log^2(2N) + \gamma_1 + O\left(\frac{\log 2N}{2N}\right) \\
&\quad - \log 2\left(\log N + \gamma + O\left(\frac{1}{N}\right)\right) \\
&\quad - \left(\frac{1}{2}\log^2 N + \gamma_1 + O\left(\frac{\log N}{N}\right)\right) \\
&= \left(\frac{1}{2}(\log 2 + \log N)^2 - \log 2\log N - \frac{1}{2}\log^2 N\right) \\
&\quad + (\gamma_1 - \gamma\log 2 - \gamma_1) + O\left(\frac{\log N}{N}\right) \\
&= \left(\frac{1}{2}\log^2 2 + \log 2\log N + \frac{1}{2}\log^2 N - \log 2\log N - \frac{1}{2}\log^2 N\right) \\
&\quad - \gamma\log 2 + O\left(\frac{\log N}{N}\right) \\
&= \frac{1}{2}\log^2 2 - \gamma\log 2 + O\left(\frac{\log N}{N}\right).
\end{aligned}$$

Letting $N \to +\infty$ we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\log n}{n} = -\gamma\log 2 + \frac{1}{2}\log^2 2,$$

which gives the asserted result. □

Theorem 1.5.1 can be found in [12, p. 263] and [15, p. 288].

Our next result provides a generalization of Theorem 1.5.1.

Theorem 1.5.2. *Let $b > 1$ be an integer. Then*

$$\sum_{n=1}^{\infty}\epsilon_n(b)\frac{\log n}{n} = \gamma\log b - \frac{1}{2}\log^2 b.$$

Proof. For $x \geq 1$ we have

$$\begin{aligned}
\sum_{n\leq x} \epsilon_n(b)\frac{\log n}{n} &= \sum_{n\leq x}(\epsilon_n(b)+1)\frac{\log n}{n} - \sum_{n\leq x}\frac{\log n}{n} \\
&= b\sum_{\substack{n\leq x\\ b|n}}\frac{\log n}{n} - \sum_{n\leq x}\frac{\log n}{n} \\
&= b\sum_{n\leq x/b}\frac{\log(bn)}{bn} - -\sum_{n\leq x}\frac{\log n}{n} \\
&= \log b\sum_{n\leq x/b}\frac{1}{n} + \sum_{n\leq x/b}\frac{\log n}{n} - \sum_{n\leq x}\frac{\log n}{n} \\
&= \log b\left(\log\frac{x}{b} + \gamma + O\left(\frac{1}{x/b}\right)\right) \\
&\quad + \left(\frac{1}{2}\log^2 x/b + \gamma_1 + O\left(\frac{\log x/b}{x/b}\right)\right) \\
&\quad - \left(\frac{1}{2}\log^2 x + \gamma_1 +)\left(\frac{\log x}{x}\right)\right) \\
&= \log b\log x - \log^2 b + \gamma\log b \\
&\quad -\frac{1}{2}\log b(2\log x - \log b) + O\left(\frac{\log x}{x}\right) \\
&= -\frac{1}{2}\log^2 b + \gamma\log b + O\left(\frac{\log x}{x}\right).
\end{aligned}$$

Letting $x \to +\infty$ we obtain

$$\sum_{n=1}^{\infty}\epsilon_n(b)\frac{\log n}{n} = \gamma\log b - \frac{1}{2}\log^2 b.$$

□

Theorem 1.5.1 is the special case $b = 2$ of Theorem 1.5.2.

In 1972 Liang and Todd [24] proved the following extension of Theorem

1.5.1,

$$\sum_{n=1}^{\infty}(-1)^n\frac{\log^k n}{n} = \sum_{v=0}^{k-1}\binom{k}{v}(\log 2)^{k-v}\gamma_v - \frac{\log^{k+1} 2}{k+1}.$$

We generalize Liang and Todd's result as follows.

Theorem 1.5.3. *Let $b > 1$ be an integer. Let $k \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty}\epsilon_n(b)\frac{\log^k n}{n} = \sum_{v=0}^{k-1}\binom{k}{v}(\log b)^{k-v}\gamma_v - \frac{\log^{k+1} b}{k+1}.$$

Proof. Let $N \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty}\epsilon_n(b)\frac{\log^k n}{n} &= \lim_{N\to\infty}\sum_{n=1}^{Nb}\epsilon_n(b)\frac{\log^k n}{n} \\
&= \lim_{N\to\infty}\left\{\sum_{\substack{n=1\\ b|n}}^{Nb}(b-1)\frac{\log^k n}{n} + \sum_{\substack{n=1\\ b\nmid n}}^{Nb}(-1)\frac{\log^k n}{n}\right\} \\
&= \lim_{N\to\infty}\left\{\sum_{\substack{n=1\\ b|n}}^{Nb}(b)\frac{\log^k n}{n} - \sum_{n=1}^{Nb}(-1)\frac{\log^k n}{n}\right\} \\
&= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{\log^k nb}{n} - \sum_{n=1}^{Nb}\frac{\log^k n}{n}\right\} \\
&= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{(\log n + \log b)^k}{n} - \sum_{n=1}^{Nb}\frac{\log^k n}{n}\right\} \\
&= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n}\sum_{s=0}^{k}\binom{k}{s}\log^s n\log^{k-s} b - \sum_{n=1}^{Nb}\frac{\log^k n}{n}\right\} \\
&= \lim_{N\to\infty}\left\{\sum_{s=0}^{k}\binom{k}{s}\log^{k-s} b\sum_{n=1}^{N}\frac{\log^s n}{n} - \sum_{n=1}^{Nb}\frac{\log^k n}{n}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\left(\sum_{s=0}^{k}\binom{k}{s}\log^{k-s} b\left(\sum_{n=1}^{N}\frac{\log^s n}{n}-\frac{\log^{s+1} N}{s+1}\right)\right.\\
&\qquad +\sum_{s=0}^{k}\binom{k}{s}\log^{k-s} b\frac{\log^{s+1} N}{s+1}\\
&\qquad \left.-\left(\sum_{n=1}^{Nb}\frac{\log^k n}{n}-\frac{\log^{k+1} Nb}{k+1}\right)-\frac{\log^{k+1} Nb}{k+1}\right)\\
&= \sum_{s=0}^{k}\binom{k}{s}\log^{k-s} b\,\gamma_s-\gamma_k\\
&\qquad +\lim_{N\to\infty}\left(\sum_{s=0}^{k}\binom{k}{s}\log^{k-s} b\frac{\log^{s+1} N}{s+1}-\frac{\log^{k+1} Nb}{k+1}\right)\\
&= \sum_{s=0}^{k-1}\binom{k}{s}\log^{k-s} b\,\gamma_s\\
&\qquad +\lim_{N\to\infty}\left(\frac{1}{k+1}\sum_{s=0}^{k}\binom{k+1}{s+1}\log^{k-s} b\log^{s+1} N-\frac{\log^{k+1} Nb}{k+1}\right)\\
&= \sum_{s=0}^{k-1}\binom{k}{s}\log^{k-s} b\,\gamma_s\\
&\qquad +\lim_{N\to\infty}\left(\frac{1}{k+1}(\log b+\log N)^{k+1}-\frac{\log^{k+1} b}{k+1}-\frac{\log^{k+1} Nb}{k+1}\right)\\
&= \sum_{s=0}^{k-1}\binom{k}{s}\log^{k-s} b\,\gamma_s\\
&\qquad +\lim_{N\to\infty}\left(\frac{\log^{k+1} Nb}{k+1}-\frac{\log^{k+1} b}{k+1}-\frac{\log^{k+1} Nb}{k+1}\right)\\
&= \sum_{s=0}^{k-1}\binom{k}{s}\log^{k-s} b\,\gamma_s-\frac{\log^{k+1} b}{k+1}.
\end{aligned}
$$

□

Theorem 1.5.3 can also be deduced from Corollary 1 and Proposition 4 in [12, pp. 260-261].

Before proceeding we recall Euler's summation formula.

Theorem 1.5.4. *If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then*

$$\begin{aligned}\sum_{y<n\leq x} f(n) &= \int_y^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt \\ &\quad + f(x)([x]-x) - f(y)([y]-y).\end{aligned}$$

Proof. See [3, p. 54]. □

Theorem 1.5.5. *Let $k \in \mathbb{N}$. Then*

$$\sum_{n\leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} n}{k+1} + \gamma_k + O\left(\frac{\log^k x}{x}\right),$$

as $x \to \infty$.

Proof. In Theorem 1.5.4 we choose $y = 1$ and $f(x) = \frac{\log^k x}{x}$. Clearly

$$f'(x) = \frac{k\log^{k-1} x - \log^k x}{x^2}$$

is continuous on $[1, x]$. Then Euler's summation formula gives

$$\begin{aligned}\sum_{1<n\leq x} \frac{\log^k n}{n} &= \int_1^x \frac{\log^k t}{t}\,dt + \int_1^x (t-[t])\left(\frac{k\log^{k-1} t - \log^k t}{t^2}\right)dt \\ &\quad + \frac{\log^k x}{x}([x]-x).\end{aligned}$$

Now

$$\int_1^x \frac{\log^k t}{t}\,dt = \left[\frac{\log^{k+1} t}{k+1}\right]_1^x = \frac{\log^{k+1} x}{k+1};$$

$$\begin{aligned}
&\int_1^x (t-[t])\left(\frac{k\log^{k-1}t-\log^k t}{t^2}\right)\,dt \\
&\quad= \int_1^\infty (t-[t])\left(\frac{k\log^{k-1}t-\log^k t}{t^2}\right)\,dt \\
&\qquad\qquad -\int_x^\infty (t-[t])\left(\frac{k\log^{k-1}t-\log^k t}{t^2}\right)\,dt \\
&\quad= A_k + O\left(\int_x^\infty \frac{\log^k t - k\log^{k-1}t}{t^2}\,dt\right) \\
&\quad= A_k + O\left(\left[\frac{-\log^k t}{t}\right]_x^\infty\right) \\
&\quad= A_k + O\left(\frac{\log^k x}{x}\right),
\end{aligned}$$

for x sufficiently large and a constant A_k; and

$$\frac{\log^k x}{x}([x]-x) = O\left(\frac{\log^k x}{x}\right).$$

Thus

$$\sum_{n\le x}\frac{\log^k n}{n} = \frac{\log^{k+1}x}{k+1} + A_k + O\left(\frac{\log^k x}{x}\right).$$

As

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{\log^k n}{n} - \frac{\log^{k+1}x}{k+1}\right) = \gamma_k,$$

we deduce that $A_k = \gamma_k$, so that

$$\sum_{n\le x}\frac{\log^k n}{n} = \frac{\log^{k+1}x}{k+1} + \gamma_k + O\left(\frac{\log^k x}{x}\right),$$

as asserted. □

1.6 Dr. Vacca's series for γ

Euler's constant γ is defined by means of a limit, namely

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).$$

Vacca [33] in 1909 was the first person to show that γ can be expressed by means of an infinite series, namely

$$\gamma = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right]. \tag{1.6.1}$$

The series (1.6.1) has become known as "Dr. Vacca's series for γ". We obtain (1.6.1) by deducing it from Corollary 1.4.2 and Theorem 1.5.1. For other proofs of (1.6.1), see Addison [1], Bauer [6], Gerst [13], Koecher [21], and Sandham [31].

Theorem 1.6.1.

$$\gamma = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right].$$

Proof. As $[y] = y - \{y\}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right] &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\log n}{\log 2}-\left\{\frac{\log n}{\log 2}\right\}\right)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\log n}{\log 2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left\{\frac{\log n}{\log 2}\right\}\\
&= \frac{1}{\log 2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\log n-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left\{\frac{\log n}{\log 2}\right\}\\
&= \frac{1}{\log 2}\left(\gamma\log 2-\frac{1}{2}\log^2 2\right)-\left(-\frac{1}{2}\log 2\right)
\end{aligned}$$

by Theorem 1.5.1 and Corollary 1.4.2 respectively. Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\log n}{\log 2}\right] = \gamma - \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \gamma,$$

as asserted. □

As a consequence of Theorem 1.6.1 we have the following alternative series for γ.

Theorem 1.6.2.

$$\gamma = \sum_{n=1}^{\infty} n \left(\frac{1}{2^n} - \frac{1}{2^n+1} + \cdots - \frac{1}{2^{n+1}-1}\right).$$

Proof. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} n \left(\frac{1}{2^n} - \frac{1}{2^n+1} + \cdots - \frac{1}{2^{n+1}-1}\right) &= \sum_{n=1}^{\infty} n \sum_{m=2^n}^{2^{n+1}-1} \frac{(-1)^m}{m} \\
&= \sum_{n=1}^{\infty} \sum_{m=2^n}^{2^{n+1}-1} \frac{(-1)^m}{m} n \\
&= \sum_{n=1}^{\infty} \sum_{m=2^n}^{2^{n+1}-1} \frac{(-1)^m}{m} \left[\frac{\log m}{\log 2}\right] \\
&= \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \left[\frac{\log m}{\log 2}\right] \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left[\frac{\log m}{\log 2}\right] \\
&= \gamma,
\end{aligned}$$

by Theorem 1.6.1. □

Our next result generalizes the Vacca series for γ.

Theorem 1.6.3. *Let b be an integer > 1. Then*

$$\gamma = \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log n}{\log b}\right].$$

Proof. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log n}{\log b}\right] &= \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left(\frac{\log n}{\log b} - \left\{\frac{\log n}{\log b}\right\}\right) \\
&= \frac{1}{\log b} \sum_{n=1}^{\infty} \epsilon_n(b) \frac{\log n}{n} - \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{\frac{\log n}{n}\right\} \\
&= \frac{1}{\log b} \left(\gamma \log b - \frac{1}{2} \log^2 b\right) - \left(-\frac{1}{2} \log b\right) \\
&= \gamma.
\end{aligned}$$

□

Theorem 1.6.3 is due to Berndt and Bowman [8, p. 22, Theorem 2.6]. Our proof is much simpler than that of Berndt and Bowman, which involves complicated integrals.

Exactly as we proved Theorem 1.6.2, we can prove the following result as a consequence of Theorem 1.6.3.

Theorem 1.6.4. *Let b be an integer > 1. Then*

$$\gamma = \sum_{n=1}^{\infty} n \left(\frac{1}{b^n} - \frac{1}{b^n + 1} + \cdots - \frac{1}{b^{n+1} - 1}\right).$$

The following generalization of Theorem 1.6.3 was proved recently by Berndt and Bowman [8, Theorem 2.8, p. 23].

Theorem 1.6.5. *Let a and b be integers with $a \geq 1$ and $b > 1$. Then*

$$\gamma = \sum_{n=a}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] + \sum_{n=1}^{a-1} \frac{1}{n} - \log a.$$

Proof. We have

$$\begin{aligned}
&\sum_{n=a}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{n=a}^{\infty} \frac{\epsilon_n(b)}{n} \left(\frac{\log(n/a)}{\log b} - \left\{\frac{\log(n/a)}{\log b}\right\}\right) \\
&= \frac{1}{\log b} \sum_{n=1}^{\infty} \epsilon_n(b) \frac{\log n}{n} - \frac{\log a}{\log b} \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} - \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left\{\frac{\log(n/a)}{\log b}\right\} \\
&= \frac{1}{\log b} \left(\gamma \log b - \frac{1}{2} \log^2 b\right) - \frac{\log a}{\log b}(-\log b) - (-\frac{1}{2} \log b) \\
&= \gamma - \frac{1}{2} \log b + \log a + \frac{1}{2} \log b \\
&= \gamma + \log a.
\end{aligned}$$

Hence

$$\begin{aligned}
\gamma &= \sum_{n=1}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] - \log a \\
&= \sum_{n=a}^{\infty} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] + \sum_{n=1}^{a-1} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] - \log a.
\end{aligned}$$

Finally

$$\begin{aligned}
&\sum_{n=1}^{a-1} \frac{\epsilon_n(b)}{n} \left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{n=1}^{a-1} \frac{\epsilon_n(b)+1}{n} \left[\frac{\log(n/a)}{\log b}\right] - \sum_{n=1}^{a-1} \frac{1}{n} \left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{\substack{n=1 \\ b|n}}^{a-1} \frac{b}{n} \left[\frac{\log(n/a)}{\log b}\right] - \sum_{n=1}^{a-1} \frac{1}{n} \left[\frac{\log(n/a)}{\log b}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le n<a/b} \frac{1}{n}\left[\frac{\log(bn/a)}{\log b}\right] - \sum_{1\le n<a} \frac{1}{n}\left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{1\le n<a/b} \frac{1}{n}\left(1+\left[\frac{\log(n/a)}{\log b}\right]\right) - \sum_{1\le n<a} \frac{1}{n}\left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{1\le n<a/b} \frac{1}{n} - \sum_{a/b\le n<a} \frac{1}{n}\left[\frac{\log(n/a)}{\log b}\right] \\
&= \sum_{1\le n<a/b} \frac{1}{n} - \sum_{a/b\le n<a} \frac{1}{n}(-1) \\
&= \sum_{1\le n<a} \frac{1}{n} \\
&= \sum_{n=1}^{a-1} \frac{1}{n}.
\end{aligned}
$$

□

Again our proof of Theorem 1.6.5 is simpler than that of Berndt and Bowman [8, Theorem 2.8]. Berndt and Bowman remark that this theorem is "apparently equivalent to" a theorem of Glaisher [14] without giving any details.

In Theorem 1.5.3 we evaluated the infinite series

$$\sum_{n=1}^{\infty} \epsilon_n(b)\frac{\log^k n}{n},\ b\,(>1)\in\mathbb{Z},\ k\in\mathbb{N}.$$

We now have enough information to estimate the sum

$$\sum_{n\le x} \epsilon_n(b)\frac{\log^k n}{n},\ b\,(>1)\in\mathbb{Z},\ k\in\mathbb{N},$$

for large x.

Theorem 1.6.6. *Let $b, k \in \mathbb{N}$ with $b > 1$. Then*

$$\sum_{n \le x} \epsilon_n(b) \frac{\log^k n}{n} = \sum_{s=0}^{k-1} \gamma_s \binom{k}{s} \log^{k-s} b - \frac{\log^{k+1} b}{k+1} + O\left(\frac{\log^k x}{x}\right),$$

as $x \to \infty$.

Proof. Let $b, k \in \mathbb{N}$ with $b > 1$, and $x \in \mathbb{R}$. Then

$$\begin{aligned}
\sum_{n \le x} \epsilon_n(b) \frac{\log^k n}{n} &= \sum_{n \le x} (1 + \epsilon_n(b)) \frac{\log^k n}{n} - \sum_{n \le x} \frac{\log^k n}{n} \\
&= b \sum_{\substack{n \le x \\ b \mid n}} \frac{\log^k n}{n} - \sum_{n \le x} \frac{\log^k n}{n} \\
&= b \sum_{n \le x/b} \frac{\log^k bn}{n} - \sum_{n \le x} \frac{\log^k n}{n} \\
&= \sum_{n \le x/b} \frac{(\log b + \log n)^k}{n} - \sum_{n \le x} \frac{\log^k n}{n} \\
&= \sum_{n \le x/b} \frac{1}{n} \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b + \log^s n - \sum_{n \le x} \frac{\log^k n}{n} \\
&= \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b \sum_{n \le x/b} \frac{\log^s n}{n} - \sum_{n \le x} \frac{\log^k n}{n} \\
&= \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b \left(\frac{\log^{s+1} x/b}{s+1} + \gamma_s + O\left(\frac{\log^s x}{x}\right)\right) \\
&\qquad - \left(\frac{\log^{k+1} x/b}{k+1} + \gamma_k + O\left(\frac{\log^k x}{x}\right)\right) \\
&= \frac{1}{k+1} \sum_{s=0}^{k} \binom{k+1}{s+1} \log^{k-s} b \log^{s+1} x/b + \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b\, \gamma_k \\
&\qquad - \frac{\log^{k+1} x}{k+1} - \gamma_k + O\left(\frac{\log^k x}{x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k+1}(\log b + \log x/b)^{k+1} - \frac{1}{k+1}\log^{k+1} b \\
&\qquad + \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b\, \gamma_s - \frac{\log^{k+1} x}{k+1} - \gamma_k + O\left(\frac{\log^k x}{x}\right) \\
&= \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b\, \gamma_s - \frac{1}{k+1}\log^{k+1} b - \gamma_k + O\left(\frac{\log^k x}{x}\right) \\
&= \sum_{s=0}^{k} \binom{k}{s} \log^{k-s} b\, \gamma_s - \frac{1}{k+1}\log^{k+1} b + O\left(\frac{\log^k x}{x}\right),
\end{aligned}
$$

as asserted. □

Chapter 2

Rademacher's Theorem

2.1 Notation

The following notation is used throughout this chapter.

Let K be an algebraic number field of degree n over the rational number field $\mathbb{Q}$. The ring of integers of K is denoted by O_K. The number of roots of unity in K is denoted by $w(K)$, the discriminant of K by $d(K)$ and the regulator of K by $R(K)$. The number of real fields among the conjugate fields of K is denoted by r and the number of nonreal fields by $2s$ so that $n = r + 2s$. The structure constant of the field K is the quantity

$$\kappa = \frac{2^{r+s}\pi^s R(K)}{w(K)\sqrt{|d(K)|}}. \tag{2.1.1}$$

Two nonzero ideals A and B of O_K are said to be equivalent, written $A \sim B$,

if there exist $\alpha(\neq 0) \in O_K$ and $\beta(\neq 0) \in O_K$ such that

$$<\alpha> A =<\beta> B. \tag{2.1.2}$$

Clearly $\sim$ is an equivalence relation on the set of nonzero ideals of O_K. The equivalence class containing A is denoted by $[A]$. Since the quotient field of O_K is K [2, Theorem 6.1.5, p. 111], (2.1.2) is equivalent to the existence of $c(\neq 0) \in K$ such that

$$A = cB. \tag{2.1.3}$$

The norm of an element $c \in K$ is denoted by $N(c)$ [2, p. 222] and the norm of an ideal A of O_K by $N(A)$ [2, p. 143]. The greatest common divisor of A and B is denoted by (A, B). The following result will be important in the extension of John's theorem to algebraic number fields due to Rademacher [27, Theorem, p. 173].

Theorem 2.1.1. *Let $c\ (\neq 0) \in K$. Then there exist unique nonzero ideals A and B of O_K such that*

$$A = cB, \qquad (A, B) =< 1 > .$$

Proof. As $c\ (\neq 0) \in K$ there exist $\alpha(\neq 0) \in O_K$ and $b \in \mathbb{N}$ such that

$$c = \alpha/b,$$

see for example [2, Theorem 4.2.6, p. 85]. Let $P_1, \ldots, P_m$ be the set of prime ideals which divide either $< \alpha >$ or $< b >$ (or both). Then there exist nonnegative integers $a_1, \ldots, a_m$ and nonnegative integers $b_1, \ldots, b_m$ such that

$$< \alpha >= P_1^{a_1} \cdots P_m^{a_m}, \qquad < b >= P_1^{b_1} \cdots P_m^{b_m}.$$

Reorder $P_1, \ldots, P_m$ so that $a_i \geq b_i$ for $i = 1, 2, \ldots, k$ and $a_i < b_i$ for $i = k+1, \ldots, m$. Set

$$A = P_1^{a_1 - b_1} \cdots P_k^{a_k - b_k}, \qquad B = P_{k+1}^{b_{k+1} - a_{k+1}} \cdots P_m^{b_m - a_m}.$$

Then

$$P_1^{a_1} \cdots P_m^{a_m} = < \alpha > = < bc > = c < b > = cP_1^{b_1} \cdots P_m^{b_m}$$

so

$$P_1^{a_1 - b_1} \cdots P_k^{a_k - b_k} = cP_{k+1}^{b_{k+1} - a_{k+1}} \cdots P_m^{b_m - a_m},$$

that is,

$$A = cB.$$

Since the only prime ideals dividing A are $P_1, \ldots, P_k$, the only prime ideals dividing B are $P_{k+1}, \ldots, P_m$, and $P_1, \ldots, P_m$ are distinct, it follows that

$$(A, B) = < 1 >,$$

which establishes the existence of A and B.

Suppose A' and B' are nonzero ideals of O_K such that

$$A' = cB', \qquad (A', B') = < 1 > .$$

Then

$$A'B = (cB')B = (cB)B' = AB'.$$

Hence

$$A \mid A'B.$$

But $(A, B) =< 1 >$, so

$$A \mid A',$$

say

$$A' = AM$$

for some ideal M of O_K. Then

$$AMB = A'B = AB',$$

so

$$B' = BM.$$

Hence

$$< 1 >= (A', B') = (AM, BM) = M(A, B) = M < 1 >= M.$$

Thus

$$A' = A < 1 >= A, \qquad B' = B < 1 >= B,$$

which establishes the uniqueness of A and B. □

Example 2.1.1. *Let $K = \mathbb{Q}(\sqrt{-5})$. We choose $c = \frac{1+\sqrt{-5}}{2} \in K$. We determine ideals A and B of $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$ such that*

$$A = \left(\frac{1+\sqrt{-5}}{2}\right) B, \qquad (A, B) =< 1 > .$$

We have

$$N(1+\sqrt{-5}) = 6 = 2 \cdot 3, \qquad N(2) = 4 = 2^2.$$

The prime ideal factorizations of $< 2 >$ and $< 3 >$ are

$$< 2 >= P^2, \qquad < 3 >= P_1 P_2,$$

where

$$P =< 2, 1+\sqrt{-5} >=< 2, 1-\sqrt{-5} >,$$

$$P_1 =< 3, 1+\sqrt{-5} >, \qquad P_2 =< 3, 1-\sqrt{-5} >,$$

see for example [2, p. 265]. *Clearly* $P \neq P_1$, $P \neq P_2$, $P_1 \neq P_2$, $N(P) = 2$ *and* $N(P_1) = N(P_2) = 3$. *Now*

$$\begin{aligned} < 1+\sqrt{-5} > &= < 1+\sqrt{-5} >< 1-\sqrt{-5}, 2, 3, 1+\sqrt{-5} > \\ &= < 6, 2(1+\sqrt{-5}), 3(1+\sqrt{-5}), (1+\sqrt{-5})^2 > \\ &= < 2, 1+\sqrt{-5} >< 3, 1+\sqrt{-5} > \\ &= PP_1 \end{aligned}$$

and

$$< 2 >= P^2,$$

so we choose (*guided by the choice in the proof of Theorem* 2.1.1)

$$A = P_1, \qquad B = P.$$

Clearly $(A, B) = 1$ *and*

$$\begin{aligned} cB &= \frac{1+\sqrt{-5}}{2} B =< 1+\sqrt{-5} >< 2 >^{-1} B \\ &= PP_1P^{-2}P = P_1 = A. \end{aligned}$$

If $c \in K$ is such that $|N(c)| > 1$ so that $c \neq 0$, by Theorem 2.1.1 there exist unique nonzero ideals A and B of O_K such that

$$A = cB, \qquad (A, B) = 1,$$

and we define by analogy with (1.2.2) the arithmetic function $a_I(c)$ for any nonzero ideal I of O_K by

$$a_I(c) = \begin{cases} 0, & \text{if } A \nmid I,\ B \nmid I, \\ -N(A), & \text{if } A \mid I,\ B \nmid I, \\ N(B), & \text{if } A \nmid I,\ B \mid I, \\ N(B) - N(A), & \text{if } A \mid I,\ B \mid I. \end{cases} \tag{2.1.4}$$

2.2 Rademacher's extension of John's theorem

The following extension of John's theorem was proved by Rademacher [27, Theorem, p. 173] in 1936.

Theorem 2.2.1. *Let M be a nonzero ideal of O_K. Let $c \in K$ be such that $|N(c)| > 1$. If $f(x)$ is Riemann integrable on $[0,1]$ and of period 1, then*

$$\sum_{I \in [M]} \frac{a_I(c)}{N(I)} \; f\left(\frac{\log N(I)}{\log |N(c)|}\right) = \kappa \log |N(c)| \int_0^1 f(y)\, dy,$$

where the sum is over nonzero ideals I of O_K equivalent to M and the summands are arranged according to increasing $N(I)$.

The above series inherits its convergence from the ordering of $N(I)$. Note also that we only require the function $f(x)$ to be Riemann integrable on its period, which is a looser requirement than being of bounded variation.

2.3 Rademacher's theorem for algebraic number fields

In this section, we consider the special case of Theorem 2.2.1 when $f(x)$ is identically 1 and K is assumed to contain an integral element c of norm ± 2. The latter condition ensures that $a_I(c) = \pm 1$ for every nonzero ideal I of O_K.

Theorem 2.3.1. *Let K be an algebraic number field such that there exists $c \in O_K$ with $|N(c)| = 2$. Let M be a nonzero ideal of O_K. Then*

$$\sum_{I \in [M]} \frac{\epsilon(c, I)}{N(I)} = -\frac{2^{r+s}\pi^s R(K)}{w(K)\sqrt{|d(K)|}} \log 2,$$

where the sum is over all nonzero ideals I of O_K equivalent to M and the summands are arranged according to increasing $N(I)$, and

$$\epsilon(c, I) = \begin{cases} 1, & \text{if } < c > | \, I, \\ -1, & \text{if } < c > \nmid I. \end{cases} \tag{2.3.1}$$

Proof. We take

$$A = < c >, \qquad B = < 1 >,$$

so that

$$c = A/B, \qquad (A, B) = 1.$$

Also

$$N(A) = N(< c >) = |N(c)| = 2.$$

For any nonzero ideal I of O_K we have

$$\begin{aligned} a_I(c) &= \begin{cases} N(B) = 1, & \text{if } <c> \nmid I, \\ N(B) - N(A) = 1 - 2 = -1, & \text{if } <c> \mid I, \end{cases} \\ &= -\epsilon(c, A). \end{aligned}$$

Also let $f(x) \equiv 1$ $(x \in \mathbb{R})$. Then $f(x)$ is Riemann integrable on $[0,1]$ with $\int_0^1 f(x)\,dx = 1$ and has period 1 as $f(x+1) = 1 = f(x)$ $(x \in \mathbb{R})$. Hence, by Rademacher's theorem, we have

$$\sum_{I \in [M]} \frac{a_I(c)}{N(I)} = \kappa \log 2$$

so that

$$\sum_{I \in [M]} \frac{\epsilon(c, I)}{N(I)} = -\frac{2^{r+s}\pi^s R(K)}{w(K)\sqrt{|d(K)|}} \log 2$$

completing the proof. □

2.4 Rademacher's theorem for imaginary quadratic fields

Let K be an imaginary quadratic field. Then there exists a unique squarefree integer $m < 0$ such that $K = \mathbb{Q}(\sqrt{m})$, see for example [2, Theorem 5.4.1, p. 95]. In this case

$$n = 2, \qquad r = 0, \qquad s = 1,$$

$$w(K) = \begin{cases} 2, & \text{if } m \neq -1, -3, \\ 4, & \text{if } m = -1, \\ 6, & \text{if } m = -3, \end{cases}$$

$$d(K) = \begin{cases} m, & \text{if } m \equiv 1 \pmod 4, \\ 4m, & \text{if } m \not\equiv 1 \pmod 4, \end{cases}$$

$$O_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{m}, & \text{if } m \equiv 1 \pmod 4, \\ \mathbb{Z} + \left(\frac{1+\sqrt{m}}{2}\right), & \text{if } m \not\equiv 1 \pmod 4, \end{cases}$$

$$R(K) = 1,$$

see for example [2, Theorem 5.4.3, p. 98; Theorem 5.4.2, p. 96; Definition 13.7.1, p. 380]. The structure constant κ of K is given by

$$\kappa = \begin{cases} \frac{\pi}{4}, & \text{if } m = -1, \\ \frac{\pi}{3\sqrt{3}}, & \text{if } m = -3, \\ \frac{\pi}{\sqrt{|m|}}, & \text{if } m \equiv 1 \pmod 4,\ m \neq -3, \\ \frac{\pi}{2\sqrt{|m|}}, & \text{if } m \not\equiv 1 \pmod 4,\ m \neq -1. \end{cases} \tag{2.4.1}$$

Rademacher's theorem for imaginary quadratic fields, a special case of Theorem 2.3.1, is as follows:

Theorem 2.4.1. *Let K be an imaginary quadratic field such that there exists $c \in O_K$ with $N(c) = 2$. Let m be the unique squarefree negative integer such that $K = \mathbb{Q}(\sqrt{m})$. Let M be a nonzero ideal of O_K. Then*

$$\sum_{I \in [M]} \frac{\epsilon(c, I)}{N(I)} = -\kappa \log 2,$$

where the sum is over nonzero ideals I of O_K equivalent to M and the summands are arranged according to increasing $N(I)$, $\epsilon(c, I)$ is given by (2.3.1) *and κ is given by* (2.4.1).

We next determine those imaginary quadratic fields K such that O_K contains an element c of norm 2.

If $K = \mathbb{Q}(\sqrt{m})$ with $m \not\equiv 1 \pmod 4$ we seek $c = x + y\sqrt{m}$ $(x, y \in \mathbb{Z})$ such that $x^2 + |m|y^2 = 2$. If $|m| \geq 3$ there are no integers x and y satisfying this equation. If $|m| = 2$, that is $m = -2$, we can take $x = 0, y = 1$. If $|m| = 1$, that is $m = -1$, we can take $x = y = 1$. Thus we have the two possibilities

$$K = \mathbb{Q}(\sqrt{-1}), \quad c = 1 + i,$$

$$K = \mathbb{Q}(\sqrt{-2}), \quad c = \sqrt{-2}.$$

If $K = \mathbb{Q}(\sqrt{m})$ with $m \equiv 1 \pmod 4$ we seek $c = \dfrac{x + y\sqrt{m}}{2}$ $(x, y \in \mathbb{Z}, x \equiv y \pmod 2)$ such that $\dfrac{x^2 + |m|y^2}{4} = 2$, that is $x^2 + |m|y^2 = 8$. If $|m| \geq 9$ there are no integers x and y satisfying this equation. For $|m| \leq 8$ the eligible $m \equiv 1 \pmod 4$ are $m = -3$ and $m = -7$. If $m = -3$ the equation $x^2 + 3y^2 = 8$ has no solutions in integers x and y. If $|m| = -7$ the equation $x^2 + 7y^2 = 8$ has the solution $x = y = 1$. This gives the single possibility

$$K = \mathbb{Q}(\sqrt{-7}), \quad c = \frac{1 + \sqrt{-7}}{2}.$$

We examine these three possibilities in the next three subsections.

2.4.1 $K = \mathbb{Q}(\sqrt{-1})$, $c = 1 + i$

With $K = \mathbb{Q}(\sqrt{-1})$, we have $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-1} = \mathbb{Z} + \mathbb{Z}i$. As $h(K) = 1$, see for example [10, p. 151], O_K is a principal ideal domain (in fact, O_K is a

Euclidean domain, see for example [2, p. 33]). Here by (2.4.1) the structure constant is $\kappa = \frac{\pi}{4}$.

We choose

$$c = 1 + i$$

so that

$$N(c) = (1+i)(1-i) = 2.$$

Thus for an arbitrary ideal I of $O_K = \mathbb{Z} + \mathbb{Z}i$ we have

$$\epsilon(c, I) = \begin{cases} 1, & \text{if } < 1+i > | \, I, \\ -1, & \text{if } < 1+i > \nmid I. \end{cases}$$

We next determine a necessary and sufficient condition for an ideal I of O_K to be divisible by $< 1+i >$. As O_K is a principal ideal domain, we have $I =< a + bi >$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} < 1+i > | \, I \quad &\Longleftrightarrow \quad < 1+i > | < a+bi > \\ &\Longleftrightarrow \quad 1+i \mid a+bi \\ &\Longleftrightarrow \quad \exists \, c, d \in \mathbb{Z} \text{ such that } a+bi = (1+i)(c+di) \\ &\Longleftrightarrow \quad \exists \, c, d \in \mathbb{Z} \text{ such that } a = c-d, b = c+d \\ &\Longleftrightarrow \quad a \equiv b \pmod 2. \end{aligned}$$

Thus

$$\epsilon(c, I) = \epsilon(1+i, < a+bi >) = (-1)^{a+b}.$$

As $h(K) = 1$ every nonzero ideal of O_K is equivalent to $< 1 >$ so we choose

$$M =< 1 >$$

so that with $K^* = K\backslash\{0\}$

$$\begin{aligned}[M] &= \{\alpha M \subseteq O_K | \alpha \in K^*\} \\ &= \{(a+bi) < 1 > \subseteq O_K | a, b \in \mathbb{Q},\ (a,b) \neq (0,0)\} \\ &= \{< a+bi > | a, b \in \mathbb{Z},\ (a,b) \neq (0,0)\}.\end{aligned}$$

Hence, by Theorem 2.4.1 we have

$$\sum_{\substack{I \neq 0 \\ I = <a+bi> \\ a,b \in \mathbb{Z}}} \frac{(-1)^{a+b}}{a^2+b^2} = -\frac{\pi}{4} \log 2.$$

Now

$$< a+bi > = < a'+b'i > \iff a+bi = \theta(a'+b'i),$$

for some unit θ of O_K. Since the only units in O_K are $\pm 1, \pm i$, we deduce that

$$\sum_{\substack{I \neq 0 \\ I = <a+bi> \\ a,b \in \mathbb{Z}}} \frac{(-1)^{a+b}}{a^2+b^2} = \frac{1}{4} \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} \frac{(-1)^{a+b}}{a^2+b^2}.$$

Hence we have proved the following result.

Theorem 2.4.2.

$$\sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} \frac{(-1)^{a+b}}{a^2+b^2} = -\pi \log 2.$$

We emphasize that the sum in Theorem 2.4.2 is ordered according to increasing values of a^2+b^2 so that

$$-\frac{4}{1} + \frac{4}{2} + \frac{4}{4} - \frac{8}{5} + \frac{4}{8} - \frac{4}{9} + \frac{8}{10} - \cdots = -\pi \log 2.$$

As a check on our calculations, we derive Theorem 2.4.2 in another way.

Alternate proof of Theorem 2.4.2. We have

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{(-1)^{a+b}}{a^2+b^2} &= \sum_{n=1}^{\infty} \sum_{\substack{a,b\in\mathbb{Z}\\a^2+b^2=n}} \frac{(-1)^{a+b}}{a^2+b^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{\substack{a,b\in\mathbb{Z}\\a^2+b^2=n}} 1\end{aligned}$$

as

$$a+b \equiv a^2+b^2 = n \pmod 2.$$

Now, by a classical result (see for example [17, pp. 115-120], [35]), we have

$$\sum_{\substack{a,b\in\mathbb{Z}\\a^2+b^2=n}} 1 = 4\sum_{d|n}\left(\frac{-4}{d}\right),$$

so that

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{(-1)^{a+b}}{a^2+b^2} &= 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{d|n}\left(\frac{-4}{d}\right) \\ &= 4\sum_{d,e=1}^{\infty}\frac{(-1)^{de}}{de}\left(\frac{-4}{d}\right) \\ &= 4\sum_{\substack{d=1\\2\nmid d}}^{\infty}\frac{\left(\frac{-4}{d}\right)}{d}\sum_{e=1}^{\infty}\frac{(-1)^e}{e} \\ &= -4\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\right)\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right) \\ &= -4\left(\frac{\pi}{4}\right)(\log 2) \\ &= -\pi\log 2.\end{aligned}$$

as desired. □

We note that Theorem 2.4.2 agrees with the final formula in [27] and with the value of $\sigma_2(1)$ given in [37, p. 192].

2.4.2 $K = \mathbb{Q}(\sqrt{-2})$, $c = \sqrt{-2}$

With $K = \mathbb{Q}(\sqrt{-2})$, we have $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$. As $h(K) = 1$ (see for example [10, p. 151]), O_K is a principal ideal domain (in fact, O_K is a Euclidean domain, see for example [2, p. 33]). Here by (2.4.1) the structure constant is $\kappa = \dfrac{\pi}{2\sqrt{2}}$.

We choose

$$c = \sqrt{-2}$$

so that

$$N(c) = (\sqrt{-2})(-\sqrt{-2}) = 2.$$

Thus for an arbitrary ideal I of $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$, we have

$$\epsilon(c, I) = \begin{cases} 1, \text{if } < \sqrt{-2} > | \, I, \\ -1, \text{if } < \sqrt{-2} > \nmid I. \end{cases}$$

We next determine a necessary and sufficient condition for an ideal I of O_K be to divisible by $< \sqrt{-2} >$. As O_K is a principal ideal domain we have $I =< a + b\sqrt{-2} >$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} < \sqrt{-2} > | \, I &\iff < \sqrt{-2} > | < a + b\sqrt{-2} > \\ &\iff \sqrt{-2} \mid a + b\sqrt{-2} \\ &\iff \sqrt{-2} \mid a \\ &\iff 2 \mid a. \end{aligned}$$

Thus

$$\epsilon(c, I) = \epsilon(\sqrt{-2}, < a + b\sqrt{-2} >) = (-1)^a.$$

As $h(K) = 1$, every nonzero ideal of O_K is equivalent to $< 1 >$ so we choose

$$M =< 1 >$$

so that

$$\begin{aligned}
[M] &= \{\alpha M \subseteq O_K | \alpha \in K^*\} \\
&= \{(a + b\sqrt{-2}) < 1 > \subseteq O_K | a, b \in \mathbb{Q},\ (a, b) \neq (0, 0)\} \\
&= \{< a + b\sqrt{-2} > | a, b \in \mathbb{Z},\ (a, b) \neq (0, 0)\}.
\end{aligned}$$

Hence, by Theorem 2.4.1, we have

$$\sum_{\substack{I \neq <0> \\ I = <a+b\sqrt{-2}> \\ a,b \in \mathbb{Z}}} \frac{(-1)^a}{a^2 + 2b^2} = -\frac{\pi}{2\sqrt{2}} \log 2.$$

Now

$$< a + b\sqrt{-2} >=< a' + b'\sqrt{-2} > \iff a + b\sqrt{-2} = \theta(a' + b'\sqrt{-2}),$$

for some unit θ of $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$. But the only units of $\mathbb{Z} + \mathbb{Z}\sqrt{-2}$ are ± 1 so

$$\sum_{\substack{I \neq <0> \\ I = <a+b\sqrt{-2}> \\ a,b \in \mathbb{Z}}} \frac{(-1)^a}{a^2 + 2b^2} = \frac{1}{2} \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} \frac{(-1)^a}{a^2 + 2b^2}.$$

Hence we have proved the following result.

Theorem 2.4.3.

$$\sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} \frac{(-1)^a}{a^2 + 2b^2} = -\frac{\pi}{\sqrt{2}} \log 2.$$

As a check on our calculations, we derive Theorem 2.4.3 in another way.

Alternate proof of Theorem 2.4.3. We have (as $a \equiv a^2 \equiv a^2+2b^2 \pmod 2$)

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{(-1)^a}{a^2+2b^2} &= \sum_{n=1}^{\infty} \sum_{\substack{a,b\in\mathbb{Z}\\a^2+2b^2=n}} \frac{(-1)^a}{a^2+2b^2}\\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{\substack{a,b\in\mathbb{Z}\\a^2+2b^2=n}} 1.\end{aligned}$$

Now

$$\sum_{\substack{a,b\in\mathbb{Z}\\a^2+2b^2=n}} 1 = 2\sum_{d|n}\left(\frac{-8}{d}\right)$$

(see for example [11, Theorem 64, p. 78], [36]) so that

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{(-1)^{a+b}}{a^2+2b^2} &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{d|n}\left(\frac{-8}{d}\right)\\ &= 2\sum_{d,e=1}^{\infty} \frac{(-1)^{de}}{de}\left(\frac{-8}{d}\right)\\ &= 2\sum_{\substack{d,e=1\\2\nmid d}}^{\infty} \frac{(-1)^{de}}{de}\left(\frac{-8}{d}\right)\\ &= 2\sum_{e=1}^{\infty} \frac{(-1)^e}{e} \sum_{d=1}^{\infty} \frac{\left(\frac{-8}{d}\right)}{d}\\ &= -2(\log 2)\, L(1,-8),\end{aligned}$$

where the Dirichlet L-series $L(1,D)$ for an arbitrary discriminant D is given by

$$L(1,D) = \sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right)}{n}.$$

For $D < 0$ by Dirichlet's class number formula we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right)}{n} = \frac{2\pi h(D)}{w(D)\sqrt{|D|}},$$

see for example [17, Theorem 10.1, p. 321]. With $D = -8$, as $w(-8) = 2$, $\sqrt{|D|} = \sqrt{8} = 2\sqrt{2}$ and $h(-8) = 1$, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{-8}{n}\right)}{n} = \frac{\pi}{2\sqrt{2}}.$$

Thus

$$\sum_{\substack{a,b\in\mathbb{Z} \\ (a,b)\neq(0,0)}} \frac{(-1)^a}{a^2+2b^2} = -2(\log 2)\frac{\pi}{2\sqrt{2}} = -\frac{\pi \log 2}{\sqrt{2}}.$$

We note that Theorem 2.4.3 agrees with the value of $\sigma_1(\sqrt{2})$ given in [37, p. 192].

2.4.3 $K = \mathbb{Q}(\sqrt{-7})$, $c = \dfrac{1+\sqrt{-7}}{2}$

With $K = \mathbb{Q}(\sqrt{-7})$, we have $O_K = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-7}}{2}\right)$. As $h(K) = 1$ (see for example [10, p. 151]), O_K is a principal ideal domain (in fact, O_K is a Euclidean domain, see for example [2, p. 34]). Here by (2.4.1) the structure constant is $\kappa = \dfrac{\pi}{\sqrt{7}}$.

We choose

$$c = \frac{1+\sqrt{-7}}{2}$$

so that

$$N(c) = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right) = \frac{1+7}{4} = \frac{8}{4} = 2.$$

Thus for an arbitrary ideal I of $O_K = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-7}}{2}\right)$, we have

$$\epsilon(c, I) = \begin{cases} 1, \text{if } \left\langle \frac{1+\sqrt{-7}}{2} \right\rangle \mid I, \\ -1, \text{if } \left\langle \frac{1+\sqrt{-7}}{2} \right\rangle \nmid I. \end{cases}$$

We next determine a necessary and sufficient condition for an ideal I of O_K to be divisible by $\left\langle \frac{1+\sqrt{-7}}{2} \right\rangle$. As O_K is a principal ideal domain we have $I = \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}
\left\langle \frac{1+\sqrt{-7}}{2} \right\rangle \Big| I &\iff \left\langle \frac{1+\sqrt{-7}}{2} \right\rangle \Big| \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle \\
&\iff \frac{1+\sqrt{-7}}{2} \Big| a + b\left(\frac{1+\sqrt{-7}}{2}\right) \\
&\iff \frac{1+\sqrt{-7}}{2} \Big| a \\
&\iff \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right) \Big| a\left(\frac{1-\sqrt{-7}}{2}\right) \\
&\iff 2 \Big| a\left(\frac{1-\sqrt{-7}}{2}\right) \\
&\iff 2 \mid a.
\end{aligned}$$

Thus

$$\epsilon(c, I) = \epsilon\left(\left(\frac{1+\sqrt{-7}}{2}\right), \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle\right) = (-1)^a.$$

As $h(K) = 1$, every nonzero ideal of O_K is equivalent to $< 1 >$ so we choose

$$M =< 1 >$$

so that

$$\begin{aligned}
[M] &= \{\alpha M \subseteq O_K | \alpha \in K^*\} \\
&= \{(x + y\sqrt{-7}) < 1 > \subseteq O_K | x, y \in \mathbb{Q},\ (x,y) \neq (0,0)\} \\
&= \left\{ \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle | a, b \in \mathbb{Z},\ (a,b) \neq (0,0) \right\}.
\end{aligned}$$

As

$$\begin{aligned}
& N\left(a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right) \\
&= \left(a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right)\left(a + b\left(\frac{1-\sqrt{-7}}{2}\right)\right) \\
&= a^2 + ab + 2b^2,
\end{aligned}$$

by Theorem 2.4.1 we have

$$\sum_{\substack{I \neq <0> \\ I = \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle \\ a,b \in \mathbb{Z}}} \frac{(-1)^a}{a^2 + ab + 2b^2} = -\frac{\pi}{\sqrt{7}} \log 2.$$

Now

$$\begin{aligned}
\left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle &= \left\langle a' + b'\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle \\
\iff a + b\left(\frac{1+\sqrt{-7}}{2}\right) &= \theta\left(a' + b'\left(\frac{1+\sqrt{-7}}{2}\right)\right),
\end{aligned}$$

where θ is a unit of $O_K = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-7}}{2}\right)$. But the only units of $\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-7}}{2}\right)$ are ± 1 so

$$\sum_{\substack{I \neq <0> \\ I = \left\langle a + b\left(\frac{1+\sqrt{-7}}{2}\right)\right\rangle \\ a,b \in \mathbb{Z}}} \frac{(-1)^a}{a^2 + ab + 2b^2} = \frac{1}{2} \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} \frac{(-1)^a}{a^2 + ab + 2b^2}.$$

Hence we have proved the following result.

Theorem 2.4.4.

$$\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{(-1)^a}{a^2+ab+2b^2} = -\frac{2\pi}{\sqrt{7}}\log 2.$$

As the parity of a is not directly related to the parity of $a^2+ab+2b^2$, it is not possible to give a proof of Theorem 2.4.4 along the lines of the alternative proofs of Theorems 2.4.2 and 2.4.3 given earlier.

2.5 Rademacher's theorem for real quadratic fields

Let K be an real quadratic field. Then there exists a unique squarefree integer $m > 0$ such that $K = \mathbb{Q}(\sqrt{m})$, see for example [2, Theorem 5.4.1, p. 95]. In this case

$$n = 2, \qquad r = 2, \qquad s = 0,$$

$$w(K) = 2,$$

$$d(K) = \begin{cases} m, & \text{if } m \equiv 1 \pmod 4, \\ 4m, & \text{if } m \not\equiv 1 \pmod 4, \end{cases}$$

$$O_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{m}, & \text{if } m \equiv 1 \pmod 4, \\ \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{m}}{2}\right), & \text{if } m \not\equiv 1 \pmod 4, \end{cases}$$

$$R(K) = \log\eta, \text{where } \eta \text{ is the fundamental unit}(>1) \text{ of } K,$$

see for example [2, Theorem 5.4.2, p. 96; Theorem 13.7.1, p. 380]. Thus, as $\kappa = \frac{2^2 \log \eta}{2\sqrt{|d(K)|}}$, we have

$$\kappa = \begin{cases} \dfrac{2\log\eta}{\sqrt{m}}, & \text{if } m \equiv 1 \pmod 4, \\ \dfrac{\log\eta}{\sqrt{m}}, & \text{if } m \not\equiv 1 \pmod 4. \end{cases} \tag{2.5.1}$$

Rademacher's theorem for real quadratic fields, a special case of Theorem 2.3.1 is as follows:

Theorem 2.5.1. *Let K be a real quadratic field such that there exists $c \in O_K$ with $|N(c)| = 2$. Let m be the unique squarefree positive integer such that $K = \mathbb{Q}(\sqrt{m})$. Let M be a nonzero ideal of O_K. Then*

$$\sum_{I \in [M]} \frac{\epsilon(c, I)}{N(I)} = -\kappa \log 2,$$

where the sum is over nonzero ideals I of O_K equivalent to M and the summands are arranged according to increasing $N(I)$, $\epsilon(c, I)$ is given by (2.3.1) *and κ is given by* (2.5.1).

Definition 2.5.1. *An integer α of a real quadratic field is said to be primary if*

$$\alpha > 0, \quad 1 \leq \left|\frac{\alpha}{\alpha'}\right| < \eta^2,$$

where α' denotes the conjugate of α and η (> 1) is the fundamental unit of the field.

Theorem 2.5.2. *Every nonzero real quadratic integer has exactly one associate which is primary.*

Theorem 2.5.2 is proved in Cohn [10, Theorem 3, p. 146].

In Section 2.4 we saw that there are only three imaginary quadratic fields K containing an integral element c of norm ± 2. In contrast each of the infinitely many real quadratic fields $\mathbb{Q}(\sqrt{p})$ (p (prime) $\equiv 3 \pmod 4$) contains such an element.

2.5.1 $K = \mathbb{Q}(\sqrt{2})$, $c = \sqrt{2}$

With $K = \mathbb{Q}(\sqrt{2})$, we have $O_K = \mathbb{Z}+\mathbb{Z}\sqrt{2}$. As $h(K) = 1$ (see for example [10, p. 271]), O_K is a principal ideal domain (in fact, O_K is a Euclidean domain [2, Theorem 2.2.6, p. 35]). The fundamental unit is $\eta_1 = 1+\sqrt{2}$ (see for example [2, Example 13.7.1, p. 381]), so the regulator is $R(K) = \log(1+\sqrt{2})$. Here by (2.5.1) the structure constant is $\kappa = \dfrac{\log(1+\sqrt{2})}{\sqrt{2}}$.

We choose

$$c = \sqrt{2}$$

so that

$$N(c) = (\sqrt{2})(-\sqrt{2}) = -2.$$

Thus for an arbitrary ideal I of $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{2}$, we have

$$\epsilon(c, I) = \begin{cases} 1, \text{if } \langle\sqrt{2}\rangle \mid I, \\ -1, \text{if } \langle\sqrt{2}\rangle \nmid I. \end{cases}$$

We next determine a necessary and sufficient condition for an ideal I of O_K to be divisible by $\langle\sqrt{2}\rangle$. As O_K is a principal ideal domain we have

$I =< a + b\sqrt{2} >$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} < \sqrt{2} >| I &\iff < \sqrt{2} >|< a + b\sqrt{2} > \\ &\iff \sqrt{2} \mid a + b\sqrt{2} \\ &\iff \sqrt{2} \mid a \\ &\iff 2 \mid a. \end{aligned}$$

Thus

$$\epsilon(c, I) = \epsilon(\sqrt{2}, < a + b\sqrt{2} >) = (-1)^a.$$

As $h(K) = 1$, every nonzero ideal of O_K is equivalent to $< 1 >$ so we choose

$$M =< 1 >$$

so that

$$\begin{aligned} [M] &= \{\alpha M \subseteq O_K | \alpha \in K^*\} \\ &= \{(a + b\sqrt{2}) < 1 > \subseteq O_K | a, b \in \mathbb{Q},\ (a, b) \neq (0, 0)\} \\ &= \left\{< a + b\sqrt{2} > | a, b \in \mathbb{Z},\ (a, b) \neq (0, 0).\right\} \end{aligned}$$

As $N(< \alpha >) = |N(\alpha)|$, we have

$$\begin{aligned} N(< a + b\sqrt{2} >) &= |N(a + b\sqrt{2})| \\ &= |(a + b\sqrt{2})(a - b\sqrt{2})| \\ &= |a^2 - 2b^2|, \end{aligned}$$

and so by Theorem 2.5.1, we have

$$\sum_{\substack{I \neq <0> \\ I = <a+b\sqrt{2}> \\ a,b \in \mathbb{Z}}} \frac{(-1)^a}{|a^2 - 2b^2|} = -\frac{\log(1 + \sqrt{2})}{\sqrt{2}} \log 2.$$

We next determine a unique generator $a+b\sqrt{2}$ for the ideal I. By Theorem 2.5.2 we set $a+b\sqrt{2}$ in the range

$$a+b\sqrt{2}>0, \quad 1 \leq \left|\frac{a+b\sqrt{2}}{a-b\sqrt{2}}\right| < (1+\sqrt{2})^2,$$

that is,

$$a+b\sqrt{2}>0, \quad 1 \leq \left|\frac{a+b\sqrt{2}}{a-b\sqrt{2}}\right| < 3+2\sqrt{2},$$

to get a unique generator.

Hence we have proved the following result.

Theorem 2.5.3.

$$\sum_{\substack{a,b\in\mathbb{Z} \\ a+b\sqrt{2}>0 \\ 1\leq\left|\frac{a+b\sqrt{2}}{a-b\sqrt{2}}\right|<3+2\sqrt{2}}} \frac{(-1)^a}{|a^2-2b^2|} = -\frac{\log(1+\sqrt{2})}{\sqrt{2}}\log 2.$$

2.5.2 $K=\mathbb{Q}(\sqrt{p})$, p (prime) $\equiv 3 \pmod 4$

Let p (prime) $\equiv 3 \pmod 4$ and let $K=\mathbb{Q}(\sqrt{p})$. Then $O_K=\mathbb{Z}+\mathbb{Z}\sqrt{p}$. We note that O_K is not necessarily a principal ideal domain. Here we have $n=2$, $r=2$, $s=0$, $w(K)=2$, $d(K)=4p$ and $R(K)=\log\eta$, where η is the fundamental unit of K.

If $p\equiv 3 \pmod 8$ then $x^2-py^2=-2$ is solvable in integers x,y, and if $p\equiv 7 \pmod 8$ then $x^2-py^2=+2$ is solvable in integers x,y, see for example [2, Exercises 14, 15, p. 297] respectively. This gives rise to an element of norm $(-1)^{\frac{p+1}{4}}2$ in $K=\mathbb{Q}(\sqrt{p})$ whenever $p\equiv 3 \pmod 4$. Let $c=x+y\sqrt{p}\in O_K$

be such an element of norm $(-1)^{\frac{p+1}{4}}2$. As $x^2 - py^2 = (-1)^{\frac{p+1}{4}}2$ we have $x^2 + y^2 \equiv 2 \pmod 4$ so that $x \equiv y \equiv 1 \pmod 2$.

Let I be a nonzero principal ideal of O_K. Then, by Theorem 2.5.2, there exists a unique element $a + b\sqrt{p}$ of O_K such that $I =< a + b\sqrt{p} >$ and

$$a + b\sqrt{p} > 0, \qquad 1 \leq \left|\frac{a + b\sqrt{p}}{a - b\sqrt{p}}\right| < \eta^2.$$

We have

$$\epsilon(c, I) = \epsilon(c, < a + b\sqrt{p} >) = \begin{cases} 1, & \text{if } c \mid a + b\sqrt{p}, \\ -1, & \text{if } c \nmid a + b\sqrt{p}. \end{cases}$$

Now, as $x \equiv y \equiv 1 \pmod 2$, we obtain

$$\begin{aligned} c \mid a + b\sqrt{p} &\iff x + y\sqrt{p} \mid a + b\sqrt{p} \\ &\iff 2 \mid (a + b\sqrt{p})(x - y\sqrt{p}) \\ &\iff 2 \mid (ax - pby) - (ay - bx)\sqrt{p} \\ &\iff 2 \mid (a + b) - (a + b)\sqrt{p} \\ &\iff 2 \mid a + b \end{aligned}$$

so that

$$\epsilon(c, I) = \epsilon(x + y\sqrt{p}, < a + b\sqrt{p} >) = (-1)^{a+b}.$$

Also

$$\begin{aligned} N(I) &= N(< a + b\sqrt{p} >) = |N(a + b\sqrt{p})| \\ &= |(a + b\sqrt{p})(a - b\sqrt{p})| = |a^2 - pb^2|. \end{aligned}$$

Hence, by Theorem 2.3.1 we have

Theorem 2.5.4.

$$\sum_{\substack{a,b\in\mathbb{Z}\\ a+b\sqrt{p}>0\\ 1\le\left|\frac{a+b\sqrt{p}}{a-b\sqrt{p}}\right|<\eta^2}} \frac{(-1)^{a+b}}{|a^2-pb^2|} = -\frac{(\log 2)(\log\eta)}{\sqrt{p}}.$$

2.6 Rademacher's theorem for real cubic fields with two nonreal embeddings

Let K be a real cubic field with two nonreal conjugate fields. Then $n = 3$, $r = 1$ and $s = 1$. The only roots of unity in K are ± 1 so $w(K) = 2$ (see for example [2, Theorem 13.5.3, p. 367]). The cubic field K has a unique fundamental unit η (> 1) (see for example [2, Theorem 13.4.2, p. 362]). The regulator $R(K)$ of K is

$$R(K) = |\log\eta| = \log\eta,$$

(see for example [2, Definition 13.7.1, p. 380]). The structure constant of K is

$$\kappa = \frac{2^{r+s}\pi^s R(K)}{w(K)\sqrt{|d(K)|}} = \frac{2\pi\log\eta}{\sqrt{|d(K)|}}. \tag{2.6.1}$$

Theorem 2.6.1. *Let K be a real cubic field with two nonreal embeddings. Let $\eta > 1$ be the unique fundamental unit of O_K, so that all units of O_K are given by $\pm\eta^n$ ($n \in \mathbb{Z}$). Given $\alpha \in O_K\backslash\{0\}$ there exists a unique associate β of α such that*

$$\beta > 0, \quad 1 \le \frac{\beta}{|N(\beta)|^{1/2}} < \eta.$$

Proof. The associates of α are $\pm\eta^n\alpha$ ($n \in \mathbb{Z}$). If $\alpha > 0$, we choose $\beta = +\eta^n\alpha > 0$, and if $\alpha < 0$, we choose $\beta = -\eta^n\alpha > 0$. Thus β is a positive associate of α.

Clearly

$$\beta = |\beta| = \eta^n|\alpha| = \operatorname{sgn}(\alpha)\eta^n.$$

Denoting the conjugates of β by β' and β'', we obtain

$$\beta' = \pm\left(\eta'\right)^n\alpha', \quad \beta'' = \pm\left(\eta''\right)^n\alpha''.$$

Hence

$$\beta'\beta'' = \left(\epsilon'\epsilon''\right)^n\alpha'\alpha''.$$

As η is a unit of O_K we have

$$\eta\eta'\eta'' = N(\eta) = \pm 1.$$

Now $\eta > 1$ and $\eta'' = \overline{\eta'}$ so that $\eta'\eta'' = \eta'\overline{\eta'} = |\eta'|^2 > 0$ and thus

$$\eta\eta'\eta'' = 1.$$

Hence

$$\eta'\eta'' = \frac{1}{\eta}.$$

Thus

$$\beta'\beta'' = \frac{\alpha'\alpha''}{\eta^n}$$

and so

$$\left|\frac{\beta}{\beta'\beta''}\right| = \frac{\beta}{|\beta'\beta''|} = \frac{|\alpha|\eta^{2n}}{|\alpha'\alpha''|}.$$

Now $\dfrac{|\alpha|\eta^{2n}}{|\alpha'\alpha''|}$ is a strictly increasing function of $n \in \mathbb{Z}$ as $\eta > 1$ such that

$$\lim_{n\to-\infty} \frac{|\alpha|\eta^{2n}}{|\alpha'\alpha''|} = 0, \quad \lim_{n\to+\infty} \frac{|\alpha|\eta^{2n}}{|\alpha'\alpha''|} = +\infty.$$

Hence there exists a unique $m \in \mathbb{Z}$ such that

$$\frac{|\alpha|\eta^{2(m-1)}}{|\alpha'\alpha''|} < 1 \leq \frac{|\alpha|\eta^{2m}}{|\alpha'\alpha''|},$$

that is

$$1 \leq \frac{|\alpha|\eta^{2m}}{|\alpha'\alpha''} < \eta^2.$$

Thus $\beta = \operatorname{sgn}(\alpha)\eta^n$ is the unique associate of α such that

$$\beta > 0, \quad 1 \leq \left|\frac{\beta}{\beta'\beta''}\right| < \eta^2.$$

Finally we note that

$$\frac{\beta}{\beta'\beta''} = \frac{\beta^2}{\beta\beta'\beta''} = \frac{\beta^2}{N(\beta)}$$

so that the condition $1 \leq \left|\frac{\beta}{\beta'\beta''}\right| < \eta^2$ is equivalent to

$$1 \leq \frac{\beta}{|N(\beta)|^{1/2}} < \eta$$

giving the asserted result. □

Theorem 2.6.2. *Let K be a real cubic field with two nonreal conjugate fields. Suppose that there exists $c \in O_K$ with $|N(c)| = 2$. Then*

$$\sum_I \frac{\epsilon(c, I)}{N(I)} = -\frac{2\pi(\log\eta)(\log 2)}{\sqrt{d(K)}},$$

where the sum is over all nonzero principal ideals I of O_K and the summands are arranged according to increasing $N(I)$, and $\epsilon(c, I)$ is given by (2.3.1).

We now apply Theorem 2.6.2 to the cubic field $K = \mathbb{Q}(\sqrt[3]{2})$ which is a real cubic field with two nonreal conjugate fields $\mathbb{Q}(\omega\sqrt[3]{2})$ and $\mathbb{Q}(\omega^2\sqrt[3]{2})$, where $\omega \in \mathbb{C}$ is such that $\omega \neq 1$, $\omega^3 = 1$.

2.6.1 $K = \mathbb{Q}(\sqrt[3]{2})$, $c = \sqrt[3]{2}$

Let $K = \mathbb{Q}(\sqrt[3]{2})$. Then $d(K) = -108$, see for example [2, Table 1, p. 177]. As $h(K) = 1$ (see for example [2, Table 9, p. 329]), $O_K = \{u + v\sqrt[3]{2} + w(\sqrt[3]{2})^2 | u, v, w \in \mathbb{Z}\}$ is a principal ideal domain. Let $c = \sqrt[3]{2} \in O_K$. Then $N(c) = 2$. We have $n = 2$, $r = 1$, $s = 1$, $w(K) = 2$, $d(K) = -108$. The fundamental unit is $\eta = 1 + \sqrt[3]{2} + (\sqrt[3]{2})^2$ [2, Table 11, p. 375], so the regulator is $R(K) = \log(1 + \sqrt[3]{2} + (\sqrt[3]{2})^2)$. Hence

$$\kappa = \frac{2\pi \log(1 + \sqrt[3]{2} + (\sqrt[3]{2})^2)}{\sqrt{|-108|}} = \frac{\pi \log(1 + \sqrt[3]{2} + (\sqrt[3]{2})^2)}{3\sqrt{3}}.$$

Let I be a principal ideal of O_K. Then, by Theorem 2.6.1, there exists a unique $\beta = x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2 \in O_K$ such that $I =< \beta >$ and

$$\beta > 0, \qquad 1 \leq \frac{\beta}{|N(\beta)|^{1/2}} < \eta,$$

that is,

$$x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2 > 0, \quad 1 \leq \frac{x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2}{(x^3 + 2y^3 + 4z^3 - 6xyz)^{1/2}} < 1 + \sqrt[3]{2} + (\sqrt[3]{2})^2,$$

as

$$N\left(x + y\sqrt[3]{2} + z\sqrt[3]{2}^2\right) = x^3 + 2y^3 + 4z^3 - 6xyz > 0.$$

We have

$$\begin{aligned} \epsilon(c, I) &= \epsilon(c, < \beta >) \\ &= \begin{cases} 1, & \text{if } < c > | < \beta >, \\ -1, & \text{if } < c > \nmid < \beta >, \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 1, & \text{if } c \mid \beta, \\ -1, & \text{if } c \nmid \beta, \end{cases} \\
&= \begin{cases} 1, & \text{if } \sqrt[3]{2} \mid x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2, \\ -1, & \text{if } \sqrt[3]{2} \nmid x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2, \end{cases} \\
&= \begin{cases} 1, & \text{if } \sqrt[3]{2} \mid x, \\ -1, & \text{if } \sqrt[3]{2} \nmid x, \end{cases} \\
&= \begin{cases} 1, & \text{if } 2 \mid x, \\ -1, & \text{if } 2 \nmid x, \end{cases} \\
&= (-1)^x.
\end{aligned}
$$

Hence, by Theorem 2.3.1 we have

Theorem 2.6.3.

$$
\sum_{\substack{x,y,z\in\mathbb{Z} \\ x+y\sqrt[3]{2}+z(\sqrt[3]{2})^2>0 \\ 1\le\frac{x+y\sqrt[3]{2}+z(\sqrt[3]{2})^2}{(x^3+2y^3+4z^3-6xyz)^{1/2}}<1+\sqrt[3]{2}+(\sqrt[3]{2})^2}} \frac{(-1)^x}{x^3+2y^3+4z^3-6xyz} = -\frac{\pi(\log 2)\log(1+\sqrt[3]{2}+(\sqrt[3]{2})^2)}{3\sqrt{3}}.
$$

Chapter 3

Kronecker's Theorem

3.1 Dedekind eta function

In this section we give some properties of the Dedekind eta function that we will need in the next section.

Let H denote the upper half of the complex plane, that is

$$H = \{z \in \mathbb{C} | z = x + iy,\ x, y \in \mathbb{R},\ y > 0\}.$$

For $z \in H$ the Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = e^{\pi i z/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z}). \tag{3.1.1}$$

This infinite product converges absolutely and uniformly in every compact subset of H, see for example [32, p. 15]. Thus $\eta(z)$ is analytic in H. Since

none of the factors of the convergent infinite product is zero for $z \in H$ it follows that $\eta(z) \neq 0$ for $z \in H$, see [32].

Theorem 3.1.1.

$$|\eta(x+iy)| = |\eta(-x+iy)|$$

for all $x, y \in \mathbb{R}$ *with* $y > 0$.

Proof. Let $x, y \in \mathbb{R}$ with $y > 0$. Then

$$\begin{aligned}\eta(\pm x+iy) &= e^{\frac{\pi i(\pm x+iy)}{12}} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m(\pm x+iy)}\right) \\ &= e^{\frac{-\pi y}{12}} e^{\frac{\pm \pi i x}{12}} \prod_{m=1}^{\infty} \left(1 - e^{-2\pi m y} e^{\pm 2\pi i m x}\right)\end{aligned}$$

so that

$$|\eta(\pm x+iy)| = e^{\frac{-\pi y}{12}} \prod_{m=1}^{\infty} \left|1 - e^{-2\pi m y} e^{\pm 2\pi i m x}\right|.$$

Now

$$\left|1 - e^{-2\pi y} e^{2\pi i m x}\right| = \left|\overline{1 - e^{-2\pi y} e^{2\pi i m x}}\right| = \left|1 - e^{-2\pi y} e^{-2\pi i m x}\right|$$

so that

$$|\eta(x+iy)| = |\eta(-x+iy)|,$$

as asserted. □

The fundamental transformation formulae of $\eta(z)$ are [32, pp. 17-18], [34, Vol. 3, p. 113]

$$\eta(z+1) = e^{\pi i/12}\eta(z), \quad \eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\,\eta(z), \tag{3.1.2}$$

where the branch of the squareroot is taken so that it has the value 1 at $z = i$.

It follows from (3.1.2) that if $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ satisfy $\alpha\delta - \beta\gamma = 1$ then

$$\eta\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \epsilon\sqrt{\gamma z + \delta}\,\eta(z), \tag{3.1.3}$$

where $\epsilon = \epsilon(\alpha, \beta, \gamma, \delta)$ and $|\epsilon| = 1$, see [32, Prop. 3, p. 17].

Hence

$$\left|\eta\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)\right|^4 = |\gamma z + \delta|^2\,|\eta(z)|^4. \tag{3.1.4}$$

Now let $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ be two positive-definite, integral, binary quadratic forms of discriminant d (so that $a, a' > 0, c, c' > 0, d < 0$) which are equivalent, so that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with $\alpha\delta - \beta\gamma = 1$, say

$$a'x^2 + b'xy + c'y^2 = a(\delta x + \beta y)^2 + b(\delta x + \beta y)(\gamma x + \alpha y) + c(\gamma x + \alpha y)^2.$$

Then

$$a' = a\delta^2 + b\delta\gamma + c\gamma^2$$

and

$$b' = 2a\delta\beta + b\gamma\beta + b\delta\alpha + 2c\gamma\alpha.$$

Set

$$z = \frac{b + \sqrt{d}}{2a} \in H.$$

Then

$$\frac{\alpha z + \beta}{\gamma z + \delta} = \frac{b' + \sqrt{d}}{2a'} \in H$$

and

$$|\gamma z + \delta|^2 = \frac{a'}{a}.$$

Thus, by (3.1.4), we have

$$\left|\eta\left(\frac{b'+\sqrt{d}}{2a'}\right)\right|^4 = \frac{a'}{a}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|^4. \tag{3.1.5}$$

We note that

$$\eta(iy) \in \mathbb{R}^+, \quad e^{-\pi i/24}\eta\left(\frac{1+iy}{2}\right) \in \mathbb{R}^+ \tag{3.1.6}$$

for $y > 0$, so that

$$\eta(iy) = |\eta(iy)|, \quad \eta\left(\frac{1+iy}{2}\right) = e^{\pi i/24}\left|\eta\left(\frac{1+iy}{2}\right)\right|. \tag{3.1.7}$$

Example 3.1.1. *We consider the form $x^2 + 2y^2$ of discriminant -8. Here $a = 1, b = 0, c = 2, d = -8$. As $x^2 + 2y^2$ is equivalent to $2x^2 + y^2$ (since $2x^2 + y^2 = (0x + (-1)y)^2 + 2(1x + 0y^2)$) we have $a' = 2, b' = 0, c' = 1, d' = d = -8$. Thus by* (3.1.5) *we have*

$$\left|\eta\left(\frac{\sqrt{-8}}{4}\right)\right|^4 = 2\left|\eta\left(\frac{\sqrt{-8}}{2}\right)\right|^4.$$

Then by (3.1.7) *we deduce*

$$\eta\left(\frac{\sqrt{-2}}{2}\right) = 2^{1/4}\eta(\sqrt{-2}).$$

Taking logarithms of (3.1.5) we obtain

$$\log a - 4\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = \log a' - 4\log\left|\eta\left(\frac{b'+\sqrt{d}}{2a'}\right)\right|. \tag{3.1.8}$$

In particular, as $ax^2 + bxy + cy^2$ and $cx^2 - bxy + ay^2$ are equivalent forms (since $ax^2 + bxy + cy^2 = c(0x + 1y)^2 - b(0x + 1y)(-1x + 0y) + a(-1x + 0y)^2$), we have by (3.1.8) and Theorem 3.1.1

$$\begin{aligned} \log a - 4\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| &= \log c - 4\log\left|\eta\left(\frac{-b+\sqrt{d}}{2c}\right)\right| \\ &= \log c - 4\log\left|\eta\left(\frac{b+\sqrt{d}}{2c}\right)\right|. \end{aligned} \tag{3.1.9}$$

We close this section by noting the following two important relationships satisfied by the Dedekind eta function, which follow from the theory of theta functions, see for example [18, p. 275]: for $z \in H$

$$\eta\left(\frac{z}{2}\right)\eta\left(\frac{1+z}{2}\right)\eta(2z) = e^{\pi i/24}\eta^3(z), \tag{3.1.10}$$

$$\eta\left(\frac{z}{2}\right)^8 + 16\eta(2z)^8 = e^{-\pi i/3}\eta\left(\frac{1+z}{2}\right)^8. \tag{3.1.11}$$

3.2 Weber's functions

For $z \in H$ Weber's functions $f(z), f_1(z), f_2(z)$ are defined in terms of the Dedekind eta function by

$$f(z) = e^{\frac{-\pi i}{24}}\eta\left(\frac{1+z}{2}\right), \tag{3.2.1}$$

$$f_1(z) = \frac{\eta\left(\frac{z}{2}\right)}{\eta(z)}, \tag{3.2.2}$$

$$f_2(z) = \sqrt{2}\,\frac{\eta(2z)}{\eta(z)}, \tag{3.2.3}$$

see [34, Vol. 3, p. 114]. From (3.1.10), (3.1.11), (3.2.1), (3.2.2) and (3.2.3) we obtain

$$\sqrt{2} = f(z)f_1(z)f_2(z) \tag{3.2.4}$$

and

$$f(z)^8 = f_1(z)^8 + f_2(z)^8, \tag{3.2.5}$$

see for example [34, Vol. 3, p. 114].

From (3.1.6) and (3.2.1) - (3.2.3) we see that if $m \in \mathbb{N}$ then $f(\sqrt{-m})$, $f_1(\sqrt{-m})$, $f_2(\sqrt{-m}) \in \mathbb{R}^+$.

Example 3.2.1. *From Example* 3.1.1 *and* (3.2.2) *we deduce*

$$f_1(\sqrt{-2}) = 2^{1/4}$$

in agreement with the value given in [34, Vol. 3, p. 721].

3.3 Kronecker's limit formula

Throughout this section, $ax^2+bxy+cy^2$ is a positive-definite, integral binary quadratic form of discriminant $d = b^2 - 4ac$.

We saw in Theorems 2.4.2, 2.4.3 and 2.4.4 that Rademacher's theorem allowed us to evaluate the sums

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{(-1)^{m+n}}{m^2+n^2}, \qquad \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{(-1)^{m}}{m^2+2n^2}, \qquad \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{(-1)^{m}}{m^2+mn+2n^2}.$$

However it does not seem possible to use Rademacher's theorem to evaluate the more general sums

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{am^2+bmn+cn^2}, \quad \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2}, \quad \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{am^2+bmn+cn^2},$$

where the dash (′) indicates that the term $(m,n)=(0,0)$ is omitted. For these we use a theorem of Kronecker, known as Kronecker's limit formula, which asserts that as $s \to 1^+$ we have

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s} = \frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + K(a,b,c) + O(s-1), \quad (3.3.1)$$

see for example [32, Theorem 1, p. 14], where $d = b^2 - 4ac < 0$ and

$$K(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log|d|}{\sqrt{|d|}} + \frac{2\pi}{\sqrt{|d|}} \log a - \frac{8\pi}{\sqrt{|d|}} \log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|. \quad (3.3.2)$$

By (3.1.9) we see that

$$K(a,b,c) = K(c,b,a). \quad (3.3.3)$$

Theorem 3.3.1. (i) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{am^2+bmn+cn^2} = 2K(4a,2b,c) - K(a,b,c),$$

(ii) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{am^2+bmn+cn^2} = 2K(a,2b,4c) - K(a,b,c),$$

(iii) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{am^2+bmn+cn^2} = -\frac{2\pi\log 4}{\sqrt{|d|}} - 2K(4a,2b,c) - 2K(a,2b,4c) + 2K(a,b,c).$$

Proof. (i) Let $s > 1$. Then

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty}{}' \frac{(1+(-1)^m)}{(am^2+bmn+cn^2)^s} &= \sum_{\substack{m,n=-\infty \\ m \text{ even}}}^{\infty}{}' \frac{2}{(am^2+bmn+cn^2)^s} \\ &= \sum_{m,n=-\infty}^{\infty}{}' \frac{2}{(4am^2+2bmn+cn^2)^s}.\end{aligned}$$

Hence, as $4am^2 + 2bmn + cn^2$ has discriminant

$$(2b)^2 - 4(4a)c = 4b^2 - 16ac = 4(b^2 - 4ac) = 4d,$$

we obtain, by Kronecker's limit formula

$$\begin{aligned}&\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{(am^2+bmn+cn^2)^s} \\ &= 2\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(4am^2+2bmn+cn^2)^s} - \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s} \\ &= 2\left(\frac{2\pi}{\sqrt{|4d|}} \cdot \frac{1}{s-1} + K(4a,2b,c) + O(s-1)\right) \\ &\qquad - \left(\frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + K(a,b,c) + O(s-1)\right) \\ &= 2K(4a,2b,c) - K(a,b,c) + O(s-1).\end{aligned}$$

Letting $s \to 1^+$ we obtain

$$\lim_{s\to 1^+} \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{(am^2+bmn+cn^2)^s} = 2K(4a,2b,c) - K(a,b,c).$$

As $\displaystyle\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{(am^2+bmn+cn^2)}$ converges, we obtain the asserted result.

(ii) We have by part (i) and (3.3.3)

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2} &= \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{an^2+bnm+cm^2} \\
&= \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{cm^2+bmn+an^2} \\
&= 2K(4c,2b,a) - K(c,b,a) \\
&= 2K(a,2b,4c) - K(a,b,c).
\end{aligned}$$

(iii) Let $s > 1$. We have

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(1+(-1)^m)(1+(-1)^n)}{(am^2+bmn+cn^2)^s} &= \sum_{\substack{m,n=-\infty \\ m,n \text{ even}}}^{\infty}{}' \frac{4}{(am^2+bmn+cn^2)^s} \\
&= \frac{1}{4^{s-1}} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{(am^2+bmn+cn^2)^s} &= \left(\frac{1}{4^{s-1}} - 1\right) \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s} \\
&\quad - \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{(am^2+bmn+cn^2)^s} \\
&\quad - \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{(am^2+bmn+cn^2)^s}.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{4^{s-1}} - 1 &= e^{-(s-1)\log 4} - 1 \\
&= (1-(s-1)\log 4 + O((s-1)^2)) - 1 \\
&= -(s-1)\log 4 + O((s-1)^2)
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\frac{1}{4^{s-1}}-1\right)\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s} \\
&= \left(-(s-1)\log 4 + O((s-1)^2)\right)\left(\frac{2\pi}{\sqrt{|d|}}\frac{1}{s-1} + K(a,b,c) + O(s-1)\right) \\
&= \frac{-2\pi\log 4}{\sqrt{|d|}} + O(s-1).
\end{aligned}$$

Hence letting $s \to 1^+$ and appealing to (i) and (ii) we obtain

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{am^2+bmn+cn^2} &= \frac{-2\pi\log 4}{\sqrt{|d|}} - (2K(4a,2b,c) - K(a,b,c)) \\
&\quad -(2K(a,2b,4c) - K(a,b,c))
\end{aligned}$$

as asserted. □

Appealing to Theorem 3.3.1 and (3.3.2), we obtain the following theorem.

Theorem 3.3.2. (i) $\displaystyle\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{am^2+bmn+cn^2} = -\frac{8\pi}{\sqrt{|d|}}\log\left|f_1\left(\frac{b+\sqrt{d}}{2a}\right)\right|.$

(ii) $\displaystyle\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2} = -\frac{8\pi}{\sqrt{|d|}}\log\left|f_2\left(\frac{b+\sqrt{d}}{2a}\right)\right|.$

(iii) $\displaystyle\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{am^2+bmn+cn^2} = -\frac{8\pi}{\sqrt{|d|}}\log\left|f\left(\frac{b+\sqrt{d}}{2a}\right)\right|.$

Proof. (i) By (3.3.2) we have

$$K(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi\log|d|}{\sqrt{|d|}} + \frac{2\pi\log a}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|$$

and

$$\begin{aligned}
K(4a,2b,c) &= \frac{4\pi\gamma}{\sqrt{4|d|}} - \frac{2\pi\log 4|d|}{\sqrt{4|d|}} + \frac{2\pi\log 4a}{\sqrt{4|d|}} - \frac{8\pi}{\sqrt{4|d|}}\log\left|\eta\left(\frac{2b+\sqrt{4d}}{8a}\right)\right| \\
&= \frac{2\pi\gamma}{\sqrt{|d|}} - \frac{\pi\log 4}{\sqrt{|d|}} - \frac{\pi\log|d|}{\sqrt{|d|}} + \frac{\pi\log 4}{\sqrt{|d|}} + \frac{\pi\log a}{\sqrt{|d|}} \\
&\qquad - \frac{4\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right| \\
&= \frac{2\pi\gamma}{\sqrt{|d|}} - \frac{\pi\log|d|}{\sqrt{|d|}} + \frac{\pi\log a}{\sqrt{|d|}} - \frac{4\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right|
\end{aligned}$$

so that

$$\begin{aligned}
&2K(4a,2b,c) - K(a,b,c) \\
&\quad = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi\log|d|}{\sqrt{|d|}} + \frac{2\pi\log a}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right| \\
&\qquad - \frac{4\pi\gamma}{\sqrt{|d|}} + \frac{2\pi\log|d|}{\sqrt{|d|}} - \frac{2\pi\log a}{\sqrt{|d|}} + \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| \\
&\quad = \frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right|}.
\end{aligned}$$

Then, appealing to Theorem 3.3.1 (i) and (3.2.2), we obtain

$$\begin{aligned}
\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{am^2+bmn+cn^2} &= \frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right|} \\
&= -\frac{8\pi}{\sqrt{|d|}}\log\left|f_1\left(\frac{b+\sqrt{d}}{2a}\right)\right|.
\end{aligned}$$

(ii) By (3.3.2) we have

$$K(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi\log|d|}{\sqrt{|d|}} + \frac{2\pi\log a}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|$$

and

$$\begin{aligned} K(a,2b,4c) &= \frac{4\pi\gamma}{\sqrt{4|d|}} - \frac{2\pi\log 4|d|}{\sqrt{4|d|}} + \frac{2\pi\log a}{\sqrt{4|d|}} - \frac{8\pi}{\sqrt{4|d|}}\log\left|\eta\left(\frac{2b+\sqrt{4d}}{2a}\right)\right| \\ &= \frac{2\pi\gamma}{\sqrt{|d|}} - \frac{\pi\log 4}{\sqrt{|d|}} - \frac{\pi\log|d|}{\sqrt{|d|}} + \frac{\pi\log a}{\sqrt{|d|}} - \frac{4\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right| \end{aligned}$$

so that

$$\begin{aligned} &2\,K(a,2b,4c) - K(a,b,c) \\ &= \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi\log 4}{\sqrt{|d|}} - \frac{2\pi\log|d|}{\sqrt{|d|}} + \frac{2\pi\log a}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right| \\ &\quad -\frac{4\pi\gamma}{\sqrt{|d|}} + \frac{2\pi\log|d|}{\sqrt{|d|}} - \frac{2\pi\log a}{\sqrt{|d|}} + \frac{8\pi}{\sqrt{|d|}}\log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| \\ &= -\frac{4\pi\log 2}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|}. \end{aligned}$$

Then, appealing to Theorem 3.3.1(ii) and to (3.2.3), we obtain

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2} &= -\frac{4\pi}{\sqrt{|d|}}\log 2 - \frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|} \\ &= -\frac{8\pi}{\sqrt{|d|}}\log\left|f_2\left(\frac{b+\sqrt{d}}{2a}\right)\right|. \end{aligned}$$

(iii) From parts (i) and (ii) we have

$$\begin{aligned} &-2K(4a,2b,c) - 2K(a,2b,4c) + 2K(a,b,c) \\ &\quad = -(2K(4a,2b,c) - K(a,b,c)) - (2K(a,2b,4c) - K(a,b,c)) \\ &\quad = -\frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right|} - \left(-\frac{4\pi\log 2}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}}\log\frac{\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|}\right) \end{aligned}$$

$$= \frac{2\pi \log 4}{\sqrt{|d|}} + \frac{8\pi}{\sqrt{|d|}} \log \frac{\left|\eta\left(\frac{b+\sqrt{d}}{a}\right)\right| \left|\eta\left(\frac{b+\sqrt{d}}{4a}\right)\right|}{\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|^2}.$$

Appealing to Theorem 3.3.1(iii), and to (3.2.2), (3.2.3), and (3.2.4), we obtain

$$\begin{aligned}
\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{am^2+bmn+cn^2} &= \frac{8\pi}{\sqrt{|d|}} \log 2^{-\frac{1}{2}} \left|f_1\left(\frac{b+\sqrt{d}}{2a}\right)\right| \left|f_2\left(\frac{b+\sqrt{d}}{2a}\right)\right| \\
&= \frac{8\pi}{\sqrt{|d|}} \log \left(\frac{\sqrt{2}}{\left|f_1\left(\frac{b+\sqrt{d}}{2a}\right)\right| \left|f_2\left(\frac{b+\sqrt{d}}{2a}\right)\right|}\right)^{-1} \\
&= -\frac{8\pi}{\sqrt{|d|}} \log \left|f\left(\frac{b+\sqrt{|d|}}{2a}\right)\right|,
\end{aligned}$$

completing the proof. □

In the next theorem we reprove Theorem 2.4.3 using Theorem 3.3.2.

Theorem 3.3.3.

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+2n^2} = -\frac{\pi}{\sqrt{2}} \log 2.$$

Proof. From Example 3.2.1 we have

$$f_1\left(\sqrt{-2}\right) = 2^{1/4}.$$

Taking $a = 1, b = 0, c = 2, d = b^2 - 4ac = -8$ in Theorem 3.3.2 (i) we obtain

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+2n^2} = -\frac{8\pi}{\sqrt{8}} \log \left|f_1\left(\sqrt{-2}\right)\right|$$

$$\begin{aligned} &= -\frac{4\pi}{\sqrt{2}}\log\left(2^{1/4}\right) \\ &= -\frac{\pi}{\sqrt{2}}\log 2 \end{aligned}$$

as required. □

Taking $a = 1, b = 0, c = \lambda \in \mathbb{N}$, so that $d = b^2 - 4ac = -4\lambda < 0$, in Theorem 3.3.2 we recover a result of Zhang and Williams [37, Theorem 3, p. 189] proved in 1999. We give the results in terms of Ramanujan's functions

$$g_\lambda = 2^{-1/4} f_1(\sqrt{-\lambda}), \quad G_\lambda = 2^{-1/4} f(\sqrt{-\lambda}),$$

see [29, p. 27].

Theorem 3.3.4. (i) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+\lambda n^2} = -\frac{\pi}{\sqrt{\lambda}}\log\left(2g_\lambda^4\right).$$

(ii) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+\lambda n^2} = \frac{\pi}{\sqrt{\lambda}}\log\left(g_\lambda^4 G_\lambda^4\right).$$

(iii) $$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+\lambda n^2} = -\frac{\pi}{\sqrt{\lambda}}\log\left(2G_\lambda^4\right).$$

Proof. (i) By Theorem 3.3.2(i) we have

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+\lambda n^2} &= -\frac{8\pi}{2\sqrt{\lambda}}\log\left|f_1(\sqrt{-\lambda})\right| \\ &= -\frac{4\pi}{\sqrt{\lambda}}\log f_1(\sqrt{-\lambda}) \\ &= -\frac{4\pi}{\sqrt{\lambda}}\log(2^{1/4}g_\lambda) \\ &= -\frac{\pi}{\sqrt{\lambda}}\log(2g_\lambda^4). \end{aligned}$$

(ii) By Theorem 3.3.2(ii) and (3.2.4) we have

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{m^2+\lambda n^2} &= -\frac{8\pi}{2\sqrt{\lambda}} \log\left|f_2(\sqrt{-\lambda})\right| \\
&= -\frac{4\pi}{\sqrt{\lambda}} \log f_2(\sqrt{-\lambda}) \\
&= -\frac{4\pi}{\sqrt{\lambda}} \log \frac{\sqrt{2}}{f(\sqrt{-\lambda})f_1(\sqrt{-\lambda})} \\
&= -\frac{4\pi}{\sqrt{\lambda}} \log \left(2^{-1/2} f(\sqrt{-\lambda})f_1(\sqrt{-\lambda})\right)^{-1} \\
&= \frac{4\pi}{\sqrt{\lambda}} \log \left(2^{-1/4} f(\sqrt{-\lambda}) 2^{-1/4} f_1(\sqrt{-\lambda})\right) \\
&= \frac{4\pi}{\sqrt{\lambda}} \log \left(g_\lambda G_\lambda\right) \\
&= \frac{\pi}{\sqrt{\lambda}} \log \left(g_\lambda^4 G_\lambda^4\right).
\end{aligned}$$

(iii) By Theorem 3.3.2(ii) we have

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2+\lambda n^2} &= -\frac{8\pi}{2\sqrt{\lambda}} \log\left|f(\sqrt{-\lambda})\right| \\
&= -\frac{4\pi}{\sqrt{\lambda}} \log f(\sqrt{-\lambda}) \\
&= -\frac{4\pi}{\sqrt{\lambda}} \log \left(2^{1/4} G_\lambda\right) \\
&= -\frac{\pi}{\sqrt{\lambda}} \log \left(2 G_\lambda^4\right),
\end{aligned}$$

completing the proof. □

The first four values in Table VI of [34, Vol. 3, p. 721] are

$$\begin{aligned}
f\left(\sqrt{-1}\right) &= 2^{1/4}, \\
f_1\left(\sqrt{-2}\right) &= 2^{1/4},
\end{aligned}$$

$$\begin{aligned} f\left(\sqrt{-3}\right) &= 2^{1/3}, \\ f_1\left(\sqrt{-4}\right) &= 2^{3/8} \text{ (with a typo corrected).} \end{aligned}$$

From (3.2.4) and (3.2.5) we deduce

$$\begin{aligned} f_1\left(\sqrt{-1}\right) &= 2^{1/8}, & f_2\left(\sqrt{-1}\right) &= 2^{1/8} \\ f\left(\sqrt{-2}\right) &= 2^{1/8}\left(\sqrt{2}+1\right)^{1/8}, & f_2\left(\sqrt{-2}\right) &= 2^{1/8}\left(\sqrt{2}-1\right)^{1/8}, \\ f_1\left(\sqrt{-3}\right) &= 2^{1/12}\left(2+\sqrt{3}\right)^{1/8} & f_2\left(\sqrt{-3}\right) &= 2^{1/12}\left(2-\sqrt{3}\right)^{1/8} \\ f\left(\sqrt{-4}\right) &= 2^{1/16}\left(1+\sqrt{2}\right)^{1/4} & f_2\left(\sqrt{-4}\right) &= 2^{1/16}\left(1-\sqrt{2}\right)^{1/4} \end{aligned}$$

Hence

$$\begin{aligned} g_1 &= 2^{-1/8}, & G_1 &= 1, \\ g_2 &= 1, & G_2 &= 2^{-1/8}\left(\sqrt{2}+1\right)^{1/8}, \\ g_3 &= 2^{-1/6}\left(2+\sqrt{3}\right)^{1/8}, & G_3 &= 2^{1/12}, \\ g_4 &= 2^{1/8}, & G_4 &= 2^{-3/16}\left(1+\sqrt{2}\right)^{1/4}. \end{aligned}$$

Appealing to Theorem 3.3.4 we obtain the following series evaluations.

Theorem 3.3.5.

$$\begin{aligned} \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+n^2} &= -\frac{\pi}{2}\log 2, \\ \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+n^2} &= -\frac{\pi}{2}\log 2, \\ \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+n^2} &= -\pi\log 2. \end{aligned}$$

Theorem 3.3.6.

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+2n^2} = -\frac{\pi}{\sqrt{2}}\log 2$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+2n^2} = -\frac{\pi}{2\sqrt{2}}\log 2 + \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+2n^2} = -\frac{\pi}{2\sqrt{2}}\log 2 - \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

Theorem 3.3.7.

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+3n^2} = -\frac{\pi}{3\sqrt{3}}\log 2 - \frac{\pi}{2\sqrt{3}}\log\left(2+\sqrt{3}\right),$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+3n^2} = -\frac{\pi}{3\sqrt{3}}\log 2 + \frac{\pi}{2\sqrt{3}}\log\left(2+\sqrt{3}\right),$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+3n^2} = -\frac{4\pi}{3\sqrt{3}}\log 2.$$

Theorem 3.3.8.

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+4n^2} = -\frac{3\pi}{4}\log 2,$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+4n^2} = -\frac{\pi}{8}\log 2 + \frac{\pi}{2}\log\left(1+\sqrt{2}\right),$$

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+4n^2} = -\frac{\pi}{8}\log 2 - \frac{\pi}{2}\log\left(1+\sqrt{2}\right).$$

3.4 Final results

The evaluation of Weber's functions $|f(z)|$, $|f_1(z)|$ and $|f_2(z)|$ at quadratic irrationalities $z = \frac{b+\sqrt{d}}{2a}$ $(d < 0)$ has recently been carried out by Muzaffar and Williams [26]. Putting their evaluation together with Theorem 3.3.2, we obtain the evaluation of the infinite series

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{am^2+bmn+cn^2}, \quad \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2}, \quad \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{am^2+bmn+cn^2},$$

for a positive-definite, primitive, integral binary quadratic form $ax^2+bxy+cy^2$ of discriminant d (< 0).

In order to state the result of Muzaffar and Williams it is necessary to introduce some notation. Let d be the discriminant of a positive-definite, primitive, integral binary quadratic form $ax^2+bxy+cy^2$ so that $d = b^2-4ac \equiv 0$ or $1 \pmod 4$, $a > 0$, $c > 0$, $d < 0$. The conductor f of d is the largest integer such that $d/f^2 \equiv 0$ or $1 \pmod 4$. We set $\Delta = d/f^2$. The set of classes of positive-definite, primitive, integral, binary quadratic forms of discriminant $d(< 0)$ under the action of the modular group

$$\Gamma = \left\{ \begin{bmatrix} r & s \\ t & u \end{bmatrix} \middle| \, r, s, t, u \in \mathbb{Z},\ ru - st = 1 \right\}$$

is denoted by $H(d)$. It is well known that $H(d)$ is a finite abelian group with respect to Gaussian composition, see for example [9]. The group $H(d)$ is called the form class group. The order of $H(d)$ is the class number $h(d)$. We write $[a, b, c]$ for the class containing the form $ax^2 + bxy + cy^2$. The identity

of $H(d)$ is the class

$$I = \begin{cases} \left[1, 0, \dfrac{-d}{4}\right], & \text{if } d \equiv 0 \pmod 4, \\ \\ \left[1, 1, \dfrac{1-d}{4}\right], & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

The inverse of the class $K = [a, b, c] \in H(d)$ is the class $K^{-1} = [a, -b, c] \in H(d)$. Let $A_1, \ldots A_s$ be generators of $H(d)$ chosen so that

$$h_1 h_2 \cdots h_s = h(d), \quad 1 < h_1 \mid h_2 \mid \cdots \mid h_s, \quad \operatorname{ord}(A_i) = h_i (i = 1, \ldots, s).$$

For $K = A_1^{k_1} \cdots A_s^{k_s} \in H(d)$ and $L = A_1^{\ell_1} \cdots A_s^{\ell_s} \in H(d)$ we define

$$\chi(K, L) = e^{2\pi i \left(\frac{k_1 \ell_1}{h_1} + \cdots + \frac{k_s \ell_s}{h_s}\right)}.$$

If p is a prime with $\left(\frac{d}{p}\right) = 1$, we let x_1 and x_2 be the two solutions of $x^2 \equiv d \pmod{4p}$, $0 \leq x < 2p$, with $x_1 < x_2$. We define the class $K_p \in H(d)$ by

$$K_p = \left[p, x_1, \frac{x_1^2 - d}{4p}\right]$$

so that

$$K_p^{-1} = \left[p, x_2, \frac{x_2^2 - d}{4p}\right].$$

For $K (\neq I) \in H(d)$ the Bernays constant [7, Teil I, §3, §4, pp. 36-68] is defined by

$$j(K, d) = \lim_{s \to 1^+} \prod_{\substack{p \\ \left(\frac{d}{p}\right) = 1}} \left(1 - \frac{\chi(K, K_p)}{p^s}\right) \left(1 - \frac{\chi(K^{-1}, K_p)}{p^s}\right).$$

It is known that $j(K, d)$ is a nonzero real number such that $j(K, d) = j(K^{-1}, d)$ [26, Lemma 7.6]. For $K \in H(d)$ and $n \in \mathbb{N}$ we define $H_K(n)$

to be the number of integers h satisfying

$$0 \le h < 2n, \quad h^2 \equiv d \pmod{4n}, \quad \left[n, h, \frac{h^2-d}{4n}\right] = K.$$

Further for $n \in \mathbb{N}$ and $K \in H(d)$ we set

$$Y_K(n) = \sum_{L \in H(d)} \chi(K, L) H_L(n).$$

For $K \in H(d)$ and a prime p we set

$$A(K, d, p) = \sum_{d=0}^{\infty} \frac{Y_K(p^d)}{p^d}.$$

Then we define for $K \in H(d)$

$$\ell(K, d) = \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p}\right) \prod_{p \mid f} A(K, d, p).$$

We also set

$$t_1(d) = \prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{1}{p^2}\right).$$

Finally for $K \in H(d)$ we define

$$E(K, d) = \frac{\pi\sqrt{|d|}\, w(d)}{48\, h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} \ell(L, d).$$

We are now ready to state the result of Muzaffar and Williams [26, Theorem 2]. It is convenient to write $f_0(z)$ for $f(z)$. The power of 2 occurring in the nonzero rational number r is denoted by $\nu_2(r)$, so that

$$\begin{aligned} \nu_2(24) &= \nu_2(2^3 \cdot 3) = 3, \\ \nu_2\left(\frac{20}{3}\right) &= \nu_2(2^2 \cdot 3^{-1} \cdot 5) = 2, \\ \nu_2\left(\frac{7}{24}\right) &= \nu_2(2^{-3} \cdot 3^{-1} \cdot 7) = -3. \end{aligned}$$

Theorem 3.4.1. *Let $K = [a, b, c] \in H(d)$. Set*

$$q_0 = a + b + c, \quad q_1 = c, \quad q_2 = a,$$

$$\lambda_i = \begin{cases} 1, & \text{if } q_i \equiv 2 \pmod 4, \\ 1, & \text{if } q_i \equiv 0 \pmod 4, b \equiv 1 \pmod 2 \\ 1/2, & \text{if } q_i \equiv 0 \pmod 4, b \equiv 0 \pmod 2 \\ 2, & \text{if } q_i \equiv 1 \pmod 2, \end{cases}$$

for $i = 0, 1, 2$,

$$\begin{aligned} M_0 &= \left[2a\lambda_0, \lambda_0(2a+b), \frac{\lambda_0}{2}(a+b+c)\right] \in H(\lambda_0^2 d), \\ M_1 &= \left[2a\lambda_1, \lambda_1 b, \frac{\lambda_1}{2}c\right] \in H(\lambda_1^2 d), \\ M_2 &= \left[\frac{\lambda_2}{2}a, \lambda_2 b, 2\lambda_2 c\right] \in H(\lambda_2^2 d), \end{aligned}$$

$$m_i = 2 - 2^{1-\nu_2(\lambda_i)} = \begin{cases} 0, & \text{if } \lambda_i = 1, \\ 1, & \text{if } \lambda_i = 2, \\ -2, & \text{if } \lambda_i = 1/2, \end{cases}$$

for $i = 0, 1, 2$. Then

$$\left| f_i\left(\frac{b+\sqrt{d}}{2a}\right)\right| = \left(\frac{2}{\lambda_i}\right)^{1/4} 2^{m_i \frac{1-(\Delta/2)}{2-(\Delta/2)} 2^{-2-\nu_2(f)}} e^{E(K,d)-E(M_i,\lambda_i^2 d)}$$

for $i = 0, 1, 2$.

From Theorems 3.3.2 and 3.4.1 we obtain the following theorem.

Theorem 3.4.2. *With the notation of Theorem* 3.4.1,

(i) $$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{am^2+bmn+cn^2}$$
$$= -\frac{8\pi}{\sqrt{|d|}}\left\{\frac{1}{4}\log\left(\frac{2}{\lambda_1}\right) + m_1\frac{(1-(\frac{\Delta}{2}))}{(2-(\frac{\Delta}{2}))}2^{-2-\nu_2(f)}\log 2 + E(K,d) - E(M_1,\lambda_1^2 d)\right\}.$$

(ii) $$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{am^2+bmn+cn^2}$$
$$= -\frac{8\pi}{\sqrt{|d|}}\left\{\frac{1}{4}\log\left(\frac{2}{\lambda_2}\right) + m_2\frac{(1-(\frac{\Delta}{2}))}{(2-(\frac{\Delta}{2}))}2^{-2-\nu_2(f)}\log 2 + E(K,d) - E(M_2,\lambda_2^2 d)\right\}.$$

(iii) $$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{am^2+bmn+cn^2}$$
$$= -\frac{8\pi}{\sqrt{|d|}}\left\{\frac{1}{4}\log\left(\frac{2}{\lambda_0}\right) + m_0\frac{(1-(\frac{\Delta}{2}))}{(2-(\frac{\Delta}{2}))}2^{-2-\nu_2(f)}\log 2 + E(K,d) - E(M_0,\lambda_0^2 d)\right\}.$$

We conclude this thesis with an example illustrating Theorem 3.4.2. We take

$$a = 1,\ b = 0,\ c = 19$$

so that

$$d = -76,\ f = 2,\ \Delta = -19,\ \left(\frac{\Delta}{2}\right) = -1,\ \nu_2(f) = 1,$$
$$K = [1,0,19] \in H(-76),$$
$$q_0 = 20,\ q_1 = 19,\ q_2 = 1,$$
$$\lambda_0 = \frac{1}{2},\ \lambda_1 = 2,\ \lambda_2 = 2,$$
$$M_0 = [1,1,5] \in H(-19),$$

$$m_0 = 2 - 2^{1-\nu_2(\lambda_0)} = 2 - 4 = -2.$$

Muzaffar and Williams [26] have shown that

$$E(K,d) = \log\left(\frac{\theta}{2^{1/3}}\right), \; E(M_0, -19) = 0,$$

where θ is the unique real root of $x^3 - 2x - 2 = 0$. Thus, by Theorem 3.4.2(iii), we have

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2+19n^2} &= -\frac{8\pi}{\sqrt{76}}\left(\frac{1}{4}\log 4 - 2\cdot\frac{2}{3}\cdot\frac{1}{8}\log 2 + \log\left(\frac{\theta}{2^{1/3}}\right)\right) \\
&= -\frac{4\pi}{\sqrt{19}}\left(\frac{1}{2}\log 2 - \frac{1}{6}\log 2 + \log\theta - \frac{1}{3}\log 2\right) \\
&= -\frac{4\pi}{\sqrt{19}}\log\theta.
\end{aligned}$$

We have proved

Theorem 3.4.3.

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2+19n^2} = -\frac{4\pi}{\sqrt{19}}\log\theta.$$

Similarly we can show from Theorem 3.4.2(i), that

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m}}{m^2+58n^2} = -\frac{\pi}{\sqrt{58}}\log(27+5\sqrt{29}),$$

a result given by Zucker and Robertson in [38].

We finish by remarking that the series in Theorem 3.4.2 are related to series of the form $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(\sqrt{k}\,\pi n)}$, $k \in \mathbb{N}$, see [35], [36], [37].

Conclusion

We conclude this thesis by mentioning a possible line of future research. We have noted that certain infinite series evaluations such as

$$\begin{aligned}
\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+3n^2} &= -\frac{\pi}{3\sqrt{3}}\log 2 - \frac{\pi}{2\sqrt{3}}\log\left(2+\sqrt{3}\right), \\
\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^n}{m^2+3n^2} &= -\frac{\pi}{3\sqrt{3}}\log 2 + \frac{\pi}{2\sqrt{3}}\log\left(2+\sqrt{3}\right), \\
\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^{m+n}}{m^2+3n^2} &= -\frac{4\pi}{3\sqrt{3}}\log 2,
\end{aligned}$$

see Theorem 3.3.7, follow from Kronecker's limit formula but do not appear to be capable of deduction from Rademacher's theorem. It is therefore natural to conjecture that there is a generalization of Rademacher's theorem which gives the above results as special cases. Much remains to be investigated.

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