1

CANADIAN THESES ON MICROFICHE

I.S.B.N.

THESES CANADIENNES SUR MICROFICHE

National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Ottawa, Canada
K1A 0N4

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE**

National Library
of Canada

Bibliothèque nationale
du Canada

Canadian Theses Division

Division des thèses canadiennes

Ottawa, Canada
K1A 0N4

65778

0-315-17869-8

PERMISSION TO MICROFILM — AUTORISATION DE MICROFILMER

• Please print or type — Écrire en lettres moulées ou dactylographier

Full Name of Author — Nom complet de l'auteur

MUHAMMAD ASLAM CHAUDHRY	Aslam PAKISTAN
-------------------------	-------------------

Date of Birth — Date de naissance	Country of Birth — Lieu de naissance
-----------------------------------	--------------------------------------

NOV 8, 1955	PAKISTAN
-------------	----------

Permanent Address — Résidence fixe

93-C VEHARI, PAKISTAN

Title of Thesis — Titre de la thèse

DISTRIBUTIONAL HILBERT TRANSFORM
AND BOUNDARY VALUE PROBLEMS

University — Université

CARLETON UNIVERSITY OTTAWA, CANADA

Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée

Ph.D. (Mathematics)

Year this degree conferred — Année d'obtention de ce grade	Name of Supervisor — Nom du directeur de thèse
--	--

Jan 6, 1984

Dr. J. N. PANDEY

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de microfilmer cette thèse et de prêter ou de vendre des exemplaires du film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

Date	Signature
------	-----------

Jan 18, 1984

Muhammad Aslam

DISTRIBUTIONAL HILBERT TRANSFORM AND BOUNDARY
VALUE PROBLEMS

by

M. Aslam Chaudhry

A thesis submitted to the
Faculty of Graduate Studies
and Research in partial ful-
fillment of the requirements
for the degree of Doctor of
Philosophy

Department of Mathematics and Statistics.

Carleton University

Ottawa, Canada

December, 1983

© copyright
1983, M.A. Chaudhry

Ther
Ph.D.
1784
C44.

(1)

The undersigned hereby recommend to
The Faculty of Graduate Studies and Research
acceptance of the thesis, submitted by:

M. Aslam Chaudhry,

B.Sc. Government College of Science 1976

M.Sc. University of Punjab 1979

M.Sc. Carleton University 1981

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy.

Kenneth S. Williams

.....
Chairman, Department of Mathematics
and Statistics

J. N. Pandey.....
Thesis Supervisor

Armin H. Zemanian.....
External Examiner

Carleton University

December, 1983

ABSTRACT

The present dissertation consists of three chapters, each of which deals with a specific problem of distributional transformation and its applications to a variety of problems.

In chapter 1, the classical Hilbert-transform is proved to be a homeomorphism from the test space $\mathcal{D}_{L_p}(\mathbb{R})$ ($1 < p < \infty$) onto itself. The Hilbert-transform of generalized functions $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is then defined by:

$$\langle Hf, \varphi \rangle \triangleq \langle f, -H\varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R})$$

where $H\varphi$ is the classical Hilbert-transform of φ defined by:

$$[H\varphi](x) \triangleq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(t) dt}{t - x}.$$

It is proved that H is an isomorphism from $(\mathcal{D}_{L_p}(\mathbb{R}))'$ onto itself. The Riesz-Titchmarsh classical inversion formula for the Hilbert-transform, interpreting the limits in the distributional sense, is proved to be valid for the class of distributions in $(\mathcal{D}_{L_p}(\mathbb{R}))'$, ($1 < p < \infty$). The notion of an approximate Hilbert-transform $H_{\eta}[f](x)$, ($\eta > 0$) of the generalized function $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is introduced. It is shown that the numerical valued function $H_{\eta}[f](x)$ converges to the Hilbert-transform Hf of f in $(\mathcal{D}_{L_p}(\mathbb{R}))'$ as $\eta \rightarrow 0+$. Applications of these results in solving

some singular integral equations and the Dirichlet boundary value problem are also shown.

In chapter 2, the problem of the Cauchy representation of generalized-functions in the space $(\mathcal{D}_{L_p}(\mathbb{R}))'$ is solved. Our method guarantees the validity of the Cauchy representation of certain generalized-functions with non-compact support. A particular example of this Cauchy representation is shown. The distributional representation of a certain class of analytic functions is also discussed. This class of such analytic functions is considerably larger than that discussed by Mitrovic' [18]. Applications of these results in solving distributional boundary value problems are also shown.

In chapter 3, an attempt has been made to find n -dimensional analogues of the results discussed in chapter 1. For this purpose, the test-space $X(\mathbb{R}^n)$ is introduced. It consists of all the tensor products of test functions in $\mathcal{D}(\mathbb{R})$ of the form

$$\varphi = \sum_{\mu=1}^k (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}), \varphi_{\mu,1} \in \mathcal{D}(\mathbb{R}).$$

It is shown that $X(\mathbb{R}^n)$ is dense in the test space $\mathcal{D}_{L_p}(\mathbb{R}^n)$.

The Hilbert transform of test functions in $X(\mathbb{R}^n)$ is defined by:

$$H\varphi \triangleq \sum_{\mu=1}^k ((H_1(\varphi_{\mu,1})) \otimes (H_2(\varphi_{\mu,2})) \otimes \dots \otimes (H_n(\varphi_{\mu,n}))),$$

(iv)

where $H_1(\varphi_{\mu,1})$ is the classical Hilbert-transform of $\varphi_{\mu,1}$.

It is shown that $H: X(\mathbb{R}^n) \rightarrow \mathcal{D}_{L_p}(\mathbb{R}^n)$ is a continuous linear embedding. The Hilbert transform of test functions in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ is defined by:

$$[H\varphi] \stackrel{\Delta}{=} \lim_{\mu \rightarrow \infty} [H\varphi_{\mu}]$$

where $\{\varphi_{\mu}\}_{\mu=1}^{\infty}$ is a sequence in $X(\mathbb{R}^n)$ converging to φ in $\mathcal{D}_{L_p}(\mathbb{R}^n)$.

It is proved that H is a homeomorphism from $\mathcal{D}_{L_p}(\mathbb{R}^n)$ onto itself and $H^{-1} = \left(\frac{-1}{\pi^2}\right)^n H$. The Hilbert transform of the generalized functions f in $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ is then defined by:

$$\langle Hf, \varphi \rangle = \langle f, (-1)^n H\varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n).$$

It is then seen that H is an isomorphism from $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ onto

itself and satisfies $\left(\frac{-1}{\pi^2}\right)^n H^2 f = f \quad \forall f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$. The

notion of harmonic functions in the space $\mathbb{R}^{n+1,+} = \{(x,y): x \in \mathbb{R}^n, y > 0\}$

is introduced. Applications of these results to the solution of

some singular integral equations are shown. The Dirichlet problem

for the space $\mathbb{R}^{n+1,+}$ is also solved.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Professor J.N. Pandey for his kindness during my Ph.D. studies, his constant encouragement and constructive criticism in the preparation of the thesis.

I would like to thank Professor M. Rahman and Dr. B.C. Mortimer for valuable conversations on several occasions. Also, I would like to thank Professors C.W.L. Garner and K.S. Williams for their help in making my stay in Canada such a pleasant one. Special thanks to Mrs. Susan Jameson for her careful and accurate typing of the thesis.

Finally, I would like to express my thanks to the Government of Pakistan for financial support through a Quaid-e-Azam Scholarship. Carleton University and the Ontario Graduate Scholarship program also deserve my thanks for their financial support.

CONTENTS

	Page
<u>INTRODUCTION</u>	1
 CHAPTER 1	 12
(1.1) Introduction	12
(1.2) Notation and Preliminaries	13
(1.3) The Testing Function Space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ and its Dual $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$	16
(1.4) The Generalized Hilbert Transform	24
(1.5) Applications to Integral Equations	27
(1.6) An Approximate Hilbert Transform	30
 CHAPTER 2	 39
(2.1) Introduction	39
(2.2) Analytic Representation of Generalized Functions	47
(2.3) Distributional Representation of Analytic Functions	56
(2.4) Hilbert Transform of Schwartz Distributions	62
(2.5) Existence and Uniqueness of the Solution to Dirichlet Boundary Value Problems	63
(2.6) The Hilbert Problem for Generalized Functions	67

	Page
CHAPTER 3	71
(3.1) Introduction	71
(3.2) The Test Space $X(\mathbb{R}^n)$	73
(3.3) The Hilbert-Transform of Test Functions in $X(\mathbb{R}^n)$	76
(3.4) The Hilbert-Transform of Test-Functions in $\mathcal{D}_{L_p}(\mathbb{R}^n)$	79
(3.5) Hilbert Transform of Generalized Functions in $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$	83
(3.6) Applications to Integral Equations in $\mathbb{R}^n$	86
(3.7) The Generalized Dirichlet Boundary Value Problem in $\mathbb{R}^n$	88
BIBLIOGRAPHY	93

INTRODUCTION

The theory of Fourier transforms of distribution of slow growth as developed by Laurent Schwartz [28], is quite simple, and has a wide variety of applications. But there does not exist a corresponding neat and simple theory for the Hilbert-transform of generalized-functions (distributions) having wide applications. One of our objectives is to develop such a theory for the Hilbert-transform of generalized-functions and to indicate its applicability as a powerful tool in solving certain distributional boundary value problems and singular and non-singular integral equations.

The extensions to generalized functions of different integral-transformations have become an active area of research for the last thirty years. In the late 1960's, A.H. Zemanian extended the Laplace, Mellin, Hankel, K, Weierstrass and Convolution transforms to a certain class of generalized functions. The work of Zemanian [37] on integral transformations of generalized functions and those of Pandey and others [24] - [27] on different generalized integral transformations are worth mentioning.

The Hilbert-transforms for distributions of various sub-spaces of $(\mathcal{D})'$ were investigated by a number of authors in [2], [3] [17] - [19]. Mitrovic in [17] - [19] has extended the Hilbert-

transform to $(0_Q)'$ [3] and Orton in [22] - [23] to an arbitrary element of $(\mathcal{D})'$. The method followed by Orton [22] - [23] is dependent upon the analytic representation of distributions [3].

It is a straightforward application of some of our results developed in chapter 2 and as such it is not very constructive. Moreover, the method developed by her cannot be applied with sufficient ease to applied problems which involve computation of Hilbert-transforms of distributions having non-compact support. The work done by Mitrovic [17] - [19] concerning the Hilbert transform of distributions of $(0_Q)'$ suffers from similar drawbacks. However, the analytic representation $F(z)$ of an arbitrary $f \in (0_Q)'$ is straightforward i.e. under suitable restrictions on F we have

$$F(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle, \quad \text{Im } z \neq 0. \quad (1)$$

The corresponding computation of the Hilbert-transform of an arbitrary $f \in (0_Q)'$ is not straightforward.

In problems of physics [20] sometimes we need to find harmonic functions $U(x,y)$ in the region $y > 0$ whose limit as $y \rightarrow 0+$ does not exist in a pointwise sense, but it does exist in a distributional sense. The theory of Hilbert-transform of generalized functions that we are going to develop will provide an answer to the existence and uniqueness of this problem.

It is known that if $f \in L_p(\mathbb{R})$, $1 < p < \infty$, then its Hilbert-transform

$$F(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt \quad (2)$$

$$\lim_{\epsilon \rightarrow 0+} \int_{|t-x|>\epsilon} \frac{f(t)}{t-x} dt$$

belongs to the space $L_p(\mathbb{R})$ [31, p. 132]. Therefore, if g is another function and belongs to $L_q(\mathbb{R})$ where $1/p + 1/q = 1$ and $G(x)$ is its Hilbert-transform, then the integrals

$$\int_{-\infty}^{\infty} F(x) g(x) dx \quad \text{and} \quad \int_{-\infty}^{\infty} G(x) f(x) dx$$

both exist and

$$\int_{-\infty}^{\infty} F(x) g(x) dx = - \int_{-\infty}^{\infty} f(t) G(t) dt \quad (3)$$

[33, p. 170]

In adjoint notation this result can be written as:

$$\langle Hf, g \rangle = - \langle f, Hg \rangle \quad (4)$$

where Hf stands for the Hilbert transform of f in the classical and the Cauchy principal value sense. We will show in chapter 1 that the classical Hilbert-transform H is a homeomorphism from

$\mathcal{D}_{L_p}(\mathbb{R})$, $p > 1$, onto itself and then define the Hilbert transform Hf of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ as the generalized function in $(\mathcal{D}_{L_p}(\mathbb{R}))'$ by the relation:

$$\langle Hf, \varphi \rangle = \langle f, -H\varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}) \quad (5)$$

This is an analogue of the classical-result (3) proved in [31, p. 132].

The relation (5) enables us to compute the Hilbert-transform of any $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ and thus, in view of the inversion formula

$$\frac{-1}{\pi^2} H^2 f = f \quad \forall f \in (\mathcal{D}_{L_p}(\mathbb{R}))', \quad p > 1 \quad (6)$$

which we will prove in chapter 1, it becomes quite simple to apply our technique to solve some singular integral equations in $(\mathcal{D}_{L_p}(\mathbb{R}))'$, $p > 1$. To our knowledge, none of the theories of the Hilbert-transform of distributions including those of Orton and Mitrovic will be so well suited to solve singular integral equations. Besides, contrary to Orton and Mitrovic and others, we will be interpreting the limits in distributional-boundary value problems in $(\mathcal{D}_{L_p}(\mathbb{R}))'$ (and not in $(\mathcal{D})'$). The weak convergence of our solutions to boundary-value problems follows as a simple corollary. There are several ways of extending an integral transform to generalized functions. One of them is the direct method which consists of embedding the kernel $K(x,y)$ of the integral transform into the testing function space of generalized functions, Zemanian [37]. However, the kernel $K(x,y) = \frac{1}{x-y}$ in the case of Hilbert-transform is not infinitely differentiable at $x = y$ and as such we cannot embed it into the testing function space $\mathcal{D}_{L_p}$. For this reason we define an approximate Hilbert-transform of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ by the relation:

$$H_{\eta}[f](x) = \langle f(t), \frac{t-x}{(t-x)^2 + \eta^2} \rangle, \quad \eta > 0 \quad (7)$$

We will prove in chapter 1 that the generalized Hilbert transform Hf is also the weak limit as $\eta \rightarrow 0+$ of the numerical-valued approximate Hilbert-transform $H_{\eta}[f](x)$ of the generalized function f . Applications of these results in solving the Dirichlet problem with distributional boundary values are also shown.

Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$. If there exists an analytic function $F(z)$ analytic in the z -plane, except possibly on the real axis $\{z: \text{Im } z = 0\}$ such that

$$\lim_{\epsilon \rightarrow 0+} \int_{-\infty}^{\infty} [F(x+i\epsilon) - F(x-i\epsilon)] \varphi(x) dx = \langle f, \varphi \rangle \quad (8)$$

$\forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R})$, then $F(z)$ is called the analytic representation of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$. The Cauchy-representations of analytic functions (in terms of their boundary values) are important tools for the mathematical study of physical phenomena. The rôle of such representations is even more important when we consider the generalized functions. We will prove in chapter 2 that for any $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$, $p > 1$, the analytic-function $F(z)$, $\text{Im } z \neq 0$ defined by

$$F(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle \quad (9)$$

is an analytic representation of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$. We will also

solve the problem of distributional representation of analytic-functions [23], for the space $(\mathcal{D}_{L_p}(\mathbb{R}))'$; that is, given an analytic function $F(z)$ in the region $\text{Im } z \neq 0$ and satisfying the uniform bound

$$\sup_{\substack{-\infty < x < \infty \\ y \geq \delta > 0}} |F(x \pm iy)| \leq A_\delta < \infty \quad (10)$$

and the uniform asymptotic order

$$|F(x \pm iy)| = o(1) \quad \text{as } y \rightarrow \infty \quad (11)$$

such that the distributional limits (in $(\mathcal{D}_{L_p}(\mathbb{R}))'$) of $F(x \pm iy)$ as $y \rightarrow 0+$ exist and equal $f^\pm(x)$ respectively, then we wish to prove that for $\text{Im } z \neq 0$

$$F(z) = \frac{1}{2\pi i} \langle f^+(t) - f^-(t), \frac{1}{t-z} \rangle \quad (12)$$

The representation (9) will be exploited in proving the uniqueness and the existence of the solution to the Dirichlet boundary value problem in the space $(\mathcal{D}_{L_p}(\mathbb{R}))'$. We will also use the representation (9) in extending the notion of the Hilbert-transform to $(\mathcal{D}(\mathbb{R}))'$ and in characterizing the space $\mathcal{H}(\mathcal{D})$.

We consider analytic-functions in the complex z -plane. If $F(z)$ is analytic in the upper half plane $\text{Im } z > 0$, this will be indicated by $F^+(z)$. Similarly, $F^-(z)$ denotes an analytic-function which is holomorphic in the lower half plane $\text{Im } z < 0$. The limits

of $F^+(z)$ and $F^-(z)$ on the real axis are indicated by $f^+(x)$ and $f^-(x)$. Thus we have

$$\left. \begin{aligned} f^+(x) &= \lim_{y \rightarrow 0+} F(x + iy) \\ f^-(x) &= \lim_{y \rightarrow 0+} F(x - iy) \end{aligned} \right\} \quad (13)$$

In chapter 2, we shall also solve the following problem, called the Hilbert-problem.

Let $f \in (\mathfrak{A}_{L_p}(\mathbb{R}))'$. Then the functions $F^+(z)$ and $F^-(z)$ are to be sought such that

$$f^+(x) + f^-(x) = f(x) \quad \text{in } (\mathfrak{A}_{L_p}(\mathbb{R}))'. \quad (14)$$

In chapter 3, we shall extend the notion of the Hilbert-transform and its inversion formula to the spaces $\mathfrak{A}_{L_p}(\mathbb{R}^n)$ and $(\mathfrak{A}_{L_p}(\mathbb{R}^n))'$. The Riesz-Titchmarsh methods of the Hilbert-transform of the $L_p(\mathbb{R})$ spaces were developed by Zygmund and Calderon [7], [8], [38] who extended some of the theorems of the Hilbert-transform of ordinary functions to a very wide class of n -dimensional transforms which arise in Potential theory [7], [8]. Their generalization is the following:

Letting $K(x) = x^{-1}$, (2) may be written

$$[H\varphi](x) = P \int_{-\infty}^{\infty} K(t-x) \varphi(t) dt. \quad (15)$$

We note that the kernel $k(x) = x^{-1}$ is not integrable near the points $x = 0$, $x = \infty$, and that $n = 1$ is the only number such that x^{-n} is simultaneously not integrable at 0 and ∞ .

Moreover, the property

$$k(x) + k(-x) = 0$$

is essential for the existence of $[H\phi](x)$ because of the interference of positive and negative values. Hence, if we want to consider the plane $\mathbb{R}^2$ as an analogue of the kernel $k(x) = x^{-1}$, it is natural to take the kernels $k(z)$ of the form:

$$k(z) = \frac{w(\theta)}{|z|^2}, \quad z = |z| e^{i\theta} \quad (16)$$

where the function $w(\theta)$ does not depend upon $|z|$ and satisfies

$$\int_0^{2\pi} w(\theta) d\theta = 0 \quad (17)$$

The last condition corresponds to the condition

$$k(x) + k(-x) = 0$$

of $k(x) = x^{-1}$, and $|z|^2$ is the denominator of $k(z)$ corresponding to the fact that $n = 2$ is the only number such that $|z|^{-n}$ is not integrable, both at $z = 0$ and $|z| = \infty$, over the plane $\mathbb{R}^2$. To each kernel $k(z)$, $z \in \mathbb{C}$, with the above properties, corresponds a two dimensional Hilbert-transform

$$(H\varphi)(z) = P \int \int_{\mathbb{R}^2} k(u-z)\varphi(u)du. \quad (18)$$

Similar definitions are made in the n -dimensional case, $n \geq 2$. However, Zygmund and Calderon did not consider the double Hilbert transform defined by:

$$\begin{aligned} [H\varphi](x) &= P \int \int_{-\infty}^{\infty} \frac{\varphi(t_1, t_2) dt_1 dt_2}{(t_1 - x_1)(t_2 - x_2)} \\ &= \lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} \int \int_{\substack{|t_1 - x_1| > \epsilon \\ |t_2 - x_2| > \delta}} \frac{\varphi(t_1, t_2) dt_1 dt_2}{(t_1 - x_1)(t_2 - x_2)} \end{aligned} \quad (19)$$

Of course, it is important to note the difference between the simple transform (15) of two dimensional kernel $k(z)$ and the double transform (16) of the one dimensional-kernel $k(x) = x^{-1}$. There is only one double transform (16) but an infinity of 2-dimensional transforms (15).

It seems difficult to extend the notion of the Hilbert-transform and its inversion formula to the spaces $\mathcal{D}_{L_p}(\mathbb{R}^n)$ and $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ if the Hilbert-transform of $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ is defined by

$$[H\varphi](x) \stackrel{\Delta}{=} P \int_{\mathbb{R}^n} \frac{\varphi(t_1, \dots, t_n) dt}{\prod_{i=1}^n (t_i - x_i)} \quad (20)$$

$$\stackrel{\Delta}{=} \lim_{\substack{(\max \epsilon_i) \rightarrow 0 \\ 1 \leq i \leq n}} \int \dots \int_{\substack{|t_i - x_i| > \epsilon_i}} \frac{\varphi(t_1, \dots, t_n) dt}{\prod_{i=1}^n (t_i - x_i)}$$

However, we shall introduce the space $X(\mathbb{R}^n)$ of test functions φ in $\mathcal{D}(\mathbb{R}^n)$ of the form

$$\varphi = \sum_{\mu=1}^Y (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}) \quad (21)$$

where $\varphi_{\mu,i} \in \mathcal{D}(\mathbb{R})$.

We shall induce the relative topology on $X(\mathbb{R}^n)$ induced by the topology of $\mathcal{D}(\mathbb{R}^n)$. We will prove that $X(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$, and then the Hilbert-transform of the test-functions in $X(\mathbb{R}^n)$ will be defined by:

$$[H\varphi](x) = \sum_{\mu=1}^Y P \int_{\mathbb{R}^n} \frac{(\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(t) dt}{\prod_{i=1}^n (t_i - x_i)} \quad (22)$$

It will be shown that the Hilbert-transform (19) is continuous linear embedding of $X(\mathbb{R}^n)$ into $\mathcal{D}_{L_p}(\mathbb{R}^n)$ and satisfy the inversion formula

$$\left(\frac{-1}{\pi}\right)^n H^2 \varphi = \varphi \quad \forall \varphi \in X(\mathbb{R}^n) \quad (23)$$

The Hilbert-transform of test-functions in $\mathcal{D}_{L_p}(\mathbb{R}^n)$, then will be defined by:

$$[H\varphi] = \lim_{\mu \rightarrow \infty} [H\varphi_{\mu}] \quad (24)$$

where $(\varphi_{\mu})_{\mu=1}^{\infty}$ is a sequence in $X(\mathbb{R}^n)$, converging to φ in $\mathcal{D}_{L_p}(\mathbb{R}^n)$.

It will be proved that the Hilbert-transform (21) is a homeomorphism from $\mathcal{D}_{L_p}(\mathbb{R}^n)$ onto itself and satisfy the inversion formula (20). The Hilbert transform of the generalized functions $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ will be defined by:

$$\langle Hf, \varphi \rangle \stackrel{\Delta}{=} \langle f, (-1)^n H\varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n) \quad (25)$$

Using the definition (22) and the inversion formula (20), we prove that

$$\left(\frac{-1}{\pi^2}\right)^n H^2 f = f \quad \forall f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$$

which proves the fact that the generalized Hilbert-transform H as defined in (22) is an isomorphism of $(\mathcal{D}_{L_p}(\mathbb{R}))'$ onto itself. Applications of these results in solving some singular integral equations are also shown. We will generalize the idea of harmonic functions in the upper half plane to the space $\mathbb{R}^{n+1,+} = \{(x,y): x \in \mathbb{R}^n, y > 0\}$ and solve the Dirichlet problem in $\mathbb{R}^{n+1,+}$ with distributional boundary conditions.

CHAPTER 1

(1.1) INTRODUCTION

For $p > 1$ let $\mathfrak{D}_{L_p}(\mathbb{R})$ be Schwartz testing function space consisting of C^∞ -functions defined on the real line which, along with derivatives of any order, belongs to $L_p(\mathbb{R})$. It is shown that the classical Hilbert-transform H defined by

$$(H\varphi)(x) = p \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-x} dt \quad (1.1)$$

is a homeomorphism from $\mathfrak{D}_{L_p}(\mathbb{R})$ onto itself. As a consequence of this result, we define the Hilbert-transform Hf of a generalized function $f \in (\mathfrak{D}_{L_p}(\mathbb{R}))'$ by the relation

$$\langle Hf, \varphi \rangle = \langle f, -H\varphi \rangle \quad \forall \varphi \in \mathfrak{D}_{L_p}(\mathbb{R}). \quad (1.2)$$

Using the definition (1.2) and the Riesz-Titchmarsh classical inversion formula for the Hilbert-transform, we prove that

$$-\frac{1}{\pi^2} H^2 f = f \quad \forall f \in (\mathfrak{D}_{L_p}(\mathbb{R}))' \quad (1.3)$$

which proves the fact that the generalized Hilbert-transform H is an isomorphism from $(\mathfrak{D}_{L_p}(\mathbb{R}))'$ onto itself.

Further it is shown that the generalized Hilbert-transform Hf is also the weak limit as $\eta \rightarrow 0^+$ of the numerical valued approximate Hilbert-transform $H_\eta[f](x)$ of the generalized function f defined by

$$H_\eta[f](x) = \left\langle f(t), \frac{(t-x)}{(t-x)^2 + \eta^2} \right\rangle \quad (1.4)$$

Applications of these results in solving some singular integral equations in the space $(\mathcal{D}_{L_p}(\mathbb{R}))'$ are also shown.

(1.2) NOTATION AND PRELIMINARIES

We shall introduce here some notations which will be useful in what follows. Suppose, as usual, that real and complex n -dimensional Euclidean spaces are denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively. An element of $\mathbb{R}^n$ or $\mathbb{C}^n$ is an n -tuple $x = (x_1, \dots, x_n)$ or $z = (z_1, \dots, z_n)$. We write $z = x + iy$ for $z \in \mathbb{C}^n$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $z_j = x_j + iy_j$ ($j = 1, 2, \dots, n$). We also write

$$\begin{aligned}
 |x| &= \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} & |z| &= \left(\sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} \\
 D^\alpha &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} & |\alpha| &= \alpha_1 + \alpha_2 + \dots + \alpha_n \quad (1.5) \\
 x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} & \alpha! &= \alpha_1! \alpha_2! \alpha_3! \dots \alpha_n!
 \end{aligned}$$

One should distinguish implicitly here between $|x|$ and $|\alpha|$.

If Ω is an open sub-set of $\mathbb{R}^n$, we denote by $C^k(\Omega)$ $0 \leq k \leq \infty$, the class of k -times continuous differentiable functions on Ω .

If φ is a function defined on $\mathbb{R}^n$, the closure of the set $\{x: \varphi(x) \neq 0\}$ is called the support of the function φ and denoted by $\text{supp } \varphi$.

Space of Test Functions $\mathcal{D}(\mathbb{R}^n)$

$\mathcal{D}(\mathbb{R}^n)$ is the Schwartz testing function space. It consists of all C^∞ -functions on $\mathbb{R}^n$ that have compact support. Convergence is defined as follows: A sequence $\{\varphi_\mu\}_{\mu=1}^\infty$ is said to converge to zero in $\mathcal{D}(\mathbb{R}^n)$ iff all φ_μ have the support contained in a fixed compact sub-set of $\mathbb{R}^n$ and if all φ_μ , as well as their derivatives, converge uniformly to zero.

Space of Distributions $\mathcal{D}(\mathbb{R}^n)'$

A functional f on $\mathcal{D}(\mathbb{R}^n)$ is an operation that associates with every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ a complex number. We denote this associated number by $\langle f, \varphi \rangle$. A functional on $\mathcal{D}(\mathbb{R}^n)$ is linear if:

- (i) $\langle f, \varphi_1 + \varphi_2 \rangle = \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle$ for every $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n)$
- (ii) $\langle f, \alpha \varphi \rangle = \alpha \langle f, \varphi \rangle$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and every $\alpha \in \mathbb{C}$.

A functional f is continuous if $\langle f, \varphi_\mu \rangle$ converges to zero for every sequence $\{\varphi_\mu\}_{\mu=1}^\infty$ of functions $\varphi_\mu \in \mathcal{D}(\mathbb{R}^n)$ that converges to zero in $\mathcal{D}(\mathbb{R}^n)$. A distribution, as defined by Schwartz, is a continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$. The space of all distributions is denoted by $\mathcal{D}(\mathbb{R}^n)'$ and is the dual space of $\mathcal{D}(\mathbb{R}^n)$. A distribution f is said to be zero in an open sub-set Ω of $\mathbb{R}^n$ if $\langle f, \varphi \rangle = 0$ for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ whose support is contained in Ω . The support of a distribution is the complement of the maximal open set in which f is zero.

Space of Distributions $\mathcal{E}(\mathbb{R}^n)'$

Let $\mathcal{E}(\mathbb{R}^n)$ be the space of all C^∞ -functions on $\mathbb{R}^n$ with arbitrary support. A sequence $\{\varphi_\mu\}_{\mu=1}^\infty$, $\varphi_\mu \in \mathcal{E}(\mathbb{R}^n)$ converges to zero if the φ_μ converge to zero uniformly in every

compact sub-set of $\mathbb{R}^n$ and the same is true for the derivatives of any order. The dual space of $E(\mathbb{R}^n)$, that is, the space of continuous linear functionals on $E(\mathbb{R}^n)$, is denoted by $(E(\mathbb{R}^n))'$. Schwartz has shown [28] that $(E(\mathbb{R}^n))'$ consists of those distributions in $(\mathcal{D}(\mathbb{R}^n))'$ that have compact support.

(1.3) TESTING FUNCTION SPACE $\mathcal{D}_{L_p}(\mathbb{R}^n)$ ($1 \leq p < \infty$)

An infinitely differentiable function φ defined over $\mathbb{R}^n$ is said to belong to $\mathcal{D}_{L_p}(\mathbb{R}^n)$ if $D^\alpha \varphi(x)$ belongs to $L_p(\mathbb{R}^n)$ for each $|\alpha| = 0, 1, 2, \dots$; We define a sequence of norms on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ as follows:

For each $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ we define

$$\begin{aligned} \|\varphi\|_m &= \sup_{|\alpha| \leq m} \left(\int_{\mathbb{R}^n} |D^\alpha \varphi(x)|^p dx \right)^{1/p} \\ &= \sup_{|\alpha| \leq m} \left(\int_{\mathbb{R}^n} \left| \frac{\partial^{|\alpha|} \varphi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p dx \right)^{1/p} \end{aligned}$$

$$m = 0, 1, 2, \dots, \quad (1.6)$$

It is easily seen that $\|\cdot\|_m$ ($m = 0, 1, 2, \dots$) are compatible

norms. Also, for each $\varphi \in \mathfrak{L}_p(\mathbb{R}^n)$

$$\|\varphi\|_1 \leq \|\varphi\|_2 \leq \dots \leq \|\varphi\|_m \leq \dots ; \quad (1.7)$$

The collection of all the open sets of the form

$$U_{m,\epsilon} = \{ \varphi \in \mathfrak{L}_p(\mathbb{R}^n) : \|\varphi\|_m < \epsilon \} , \quad \epsilon > 0$$

forms the basic 0-neighbourhood system. A sequence $\{\varphi_\mu\}_{\mu=1}^\infty$,

$\varphi_\mu \in \mathfrak{L}_p(\mathbb{R}^n)$ converges to zero if and only if $\|\varphi_\mu\|_m \rightarrow 0$ as

$\mu \rightarrow \infty$ for each $m = 0, 1, 2, 3, \dots$. The sequence $\{\varphi_\mu\}_{\mu=1}^\infty$ is

Cauchy if, for each $m = 0, 1, 2, \dots$ $\|\varphi_\nu - \varphi_\mu\|_m \rightarrow 0$ as $\nu, \mu \rightarrow \infty$.

Let $\mathfrak{L}_p^{(m)}(\mathbb{R}^n)$ be the completion of the space $\mathfrak{L}_p(\mathbb{R}^n)$ with

respect to the norm $\|\cdot\|_m$. Then it follows from (1.7) that

$$\mathfrak{L}_p^{(1)}(\mathbb{R}^n) \supseteq \mathfrak{L}_p^{(2)}(\mathbb{R}^n) \supseteq \dots \supseteq \mathfrak{L}_p^{(m)}(\mathbb{R}^n) \supseteq \dots \supseteq \mathfrak{L}_p(\mathbb{R}^n) \quad (1.8)$$

In view of the results [10, pp. 184-185] we have

$$\mathfrak{L}_p(\mathbb{R}^n) = \bigcap_{m=1}^\infty \mathfrak{L}_p^{(m)}(\mathbb{R}^n) \quad (1.9)$$

so that $\mathcal{D}_{L_p}(\mathbb{R}^n)$ is a complete countably normed space. We can define the same topology on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ by introducing a sequence of semi-norms $\{\gamma_m\}_{m=0}^\infty$ on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ as follows:

For each $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ we define

$$\begin{aligned} \gamma_m(\varphi) &= \left(\int_{-\infty}^{\infty} |D^\alpha \varphi(x)|^p dx \right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} \left| \frac{\partial^{|\alpha|} \varphi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p dx \right)^{1/p} \end{aligned} \quad (1.10)$$

where $m = |\alpha| = 0, 1, 2, \dots$;

Since γ_0 is a norm, the collection of semi-norms $\{\gamma_m\}_{m=0}^\infty$ is separating [37, p. 8]. Therefore, the space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ is a countably multinormed space. We denote by $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$, the space of all continuous linear functionals on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ i.e. $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ is the dual of $\mathcal{D}_{L_p}(\mathbb{R}^n)$. We say that a sequence $\{\varphi_\mu\}_{\mu=1}^\infty$ converges to φ in $\mathcal{D}_{L_p}(\mathbb{R}^n)$, if for each $m=0, 1, 2, \dots$,

$$\gamma_m(\varphi_\mu - \varphi) \rightarrow 0 \text{ as } \mu \rightarrow \infty.$$

The topology of $\mathcal{D}_{L_p}(\mathbb{R}^n)$ is generated by the semi-norms $\{\gamma_m\}_{m=0}^\infty$

in the usual manner, Zemanian [37, pp. 8-14]. It coincides with the

topology on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ defined by the system of norms (1.2). In the sequel, we shall describe the convergence in the space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ with the help of the semi-norms $\{\gamma_m\}_{m=0}^{\infty}$. The Schwartz testing-space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ [10; p. 87] and if $1 \leq p < q$ we have

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}_{L_p}(\mathbb{R}^n) \subset \mathcal{D}_{L_q}(\mathbb{R}^n) \quad (1.11)$$

More specifically, if $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ then $\varphi_{\mu} = \varphi e_{\mu} \in \mathcal{D}(\mathbb{R}^n)$

where $e_{\mu}(x) = e_1\left(\frac{x}{\mu}\right)$ ($\mu = 1, 2, 3, \dots$) and e_1 is a function in $\mathcal{D}(\mathbb{R}^n)$ such that

$$e_1(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

(The existence of such a function $e_1(x)$ is proved in Zemanian [36, p.32].)

It can easily be seen that $\forall m = 0, 1, 2, \dots$

$$\gamma_m(\varphi_{\mu} - \varphi) \rightarrow 0 \quad \text{as } \mu \rightarrow \infty.$$

As a consequence of the relation (1.11) we have

$$(E)' \subset (\mathcal{O}_{L_q})' \subset (\mathcal{O}_{L_p})' \subset (\mathcal{O})' \quad (1 \leq p < q) \quad (1.12)$$

where $(E)'$ is the space of distributions with compact support. We state here, the structure formula for $f \in (\mathcal{O}_{L_p}(\mathbb{R}^n))'$ which can be proved in view of the similar results proved for the spaces $(K(M_p))'$ in [12, pp. 109-110].

THEOREM (1.1) (Structure Formula)

IF $f \in (\mathcal{O}_{L_p}(\mathbb{R}^n))'$ ($1 < p < \infty$) then f is equal to a finite linear combination of the derivatives of functions in $L_q(\mathbb{R}^n)$, that is, for each $f \in (\mathcal{O}_{L_p}(\mathbb{R}^n))'$

$$\langle f, \varphi \rangle = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha \varphi(x) f_{|\alpha|}(x) dx \quad \forall \varphi \in \mathcal{O}_{L_p}(\mathbb{R}^n) \quad (1.13)$$

where $f_{|\alpha|}$'s are functions in $L_q(\mathbb{R}^n)$, $1/p + 1/q = 1$.

NOTE:

Throughout this and the next section, the independent variables x and t will be assumed to be one-dimensional. However, the n -dimensional analogue of some of the results proved in these sections will be discussed in Chapter 3.

THEOREM (1.2)

Let H be a mapping defined by (1.1). Then for $1 < p < \infty$,
 H is a linear homeomorphism from $\mathcal{D}_{L_p}(\mathbb{R})$ onto itself and
 $H^{-1} = -\frac{1}{\pi^2} H$.

PROOF:

If $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ then denoting $(H\varphi)(x)$ by $F(x)$, we have

$$F(x) = P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-x} dt.$$

Using the transformation $t-x=\tau$, it can be shown that

$$F(x) = P \int_{-\infty}^{\infty} \frac{\varphi(\tau+x)}{\tau} d\tau \quad (1.14)$$

Now

$$\begin{aligned} F(x) &= P \int_{-N}^N \left[\frac{\varphi(\tau+x)}{\tau} - \frac{\varphi(x)}{\tau} \right] d\tau + \left(\int_{-\infty}^{-N} + \int_N^{\infty} \right) \frac{\varphi(\tau+x)}{\tau} d\tau \\ &= \int_{-N}^N \psi(\tau, x) d\tau + \left(\int_{-\infty}^{-N} + \int_N^{\infty} \right) \frac{\varphi(\tau+x)}{\tau} d\tau \end{aligned}$$

where

$$\begin{aligned} \psi(\tau, x) &= \frac{\varphi(\tau+x) - \varphi(x)}{\tau} \quad \text{if } \tau \neq 0 \\ &= \varphi'(x) \quad \text{if } \tau = 0 \end{aligned}$$

Clearly, $\psi(\tau, x) \in C^\infty(\mathbb{R})$.

Now, consider the integral

$$I(x) = \int_N^\infty \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau.$$

Using Holder's inequality for integrals we note that

$$|I(x)| \leq \frac{1}{q-1} N^{-1/p} \|\varphi^{(1)}\|_{L_p} \rightarrow 0$$

as $N \rightarrow \infty$ uniformly for all x , $1/p + 1/q = 1$. Similarly, we can show that the integral $\int_{-\infty}^{-N} \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau$ converges uniformly

for all real x . Therefore, using the results on interchange of

order of integration and differentiation, we have

$$\begin{aligned} F^{(1)}(x) &= \int_{-N}^N \frac{\partial \psi}{\partial x} d\tau + \left(\int_{-\infty}^{-N} + \int_N^\infty \right) \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau \\ &= \lim_{\epsilon \rightarrow 0+} \left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^N \right) \frac{\partial \psi}{\partial x} d\tau + \left(\int_{-\infty}^{-N} + \int_N^\infty \right) \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau \\ &= P \int_{-N}^N \left[\frac{\varphi^{(1)}(\tau+x)}{\tau} - \frac{\varphi^{(1)}(x)}{\tau} \right] d\tau + \left(\int_{-\infty}^{-N} + \int_N^\infty \right) \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau \\ &= P \int_{-\infty}^\infty \frac{\varphi^{(1)}(\tau+x)}{\tau} d\tau \\ &= P \int_{-\infty}^\infty \frac{\varphi^{(1)}(t)}{t-x} dt \quad (\tau+x=t) \\ &= H[\varphi^{(1)}](x). \end{aligned}$$

Since $\varphi^{(1)} \in L_p(\mathbb{R})$, $F^{(1)} \in L_p(\mathbb{R})$ Titchmarsh [31, pp. 132-133].

Using a similar technique and the method of induction we can prove that

$$(H\varphi)^{(k)} = H[\varphi^{(k)}] \quad (k = 0, 1, 2, \dots) \quad (1.15)$$

Therefore, $H\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$.

The linearity of H is trivial. To prove the continuity of H in the topology of $\mathcal{D}_{L_p}(\mathbb{R})$ assume that

$$\varphi_\mu \xrightarrow{\mathcal{D}_{L_p}(\mathbb{R})} \varphi$$

Therefore, for each $(m = 0, 1, 2, 3, \dots)$

$$\begin{aligned} \gamma_m(\varphi_\mu - \varphi) &\rightarrow 0 \quad \text{as } \mu \rightarrow \infty \\ \Rightarrow \|\varphi_\mu^{(m)} - \varphi^{(m)}\|_{L_p} &\rightarrow 0 \quad \text{as } \mu \rightarrow \infty \end{aligned}$$

By virtue of the fact that

$$\|H\varphi\|_{L_p} \leq M_p \|\varphi\|_{L_p} \quad (1.16)$$

where M_p is a constant independent of φ [31, p. 132] and using

(1.15) we note that

$$\gamma_m (H\varphi_\mu - H\varphi) \leq M_p \|\varphi_\mu^{(m)} - \varphi^{(m)}\|_{L_p} \rightarrow 0$$

as $\mu \rightarrow \infty$ ($\forall m = 0, 1, 2, 3, \dots$), so that H is continuous.

The fact that H is one-to-one and onto follows by virtue of the Riesz-Titchmarsh classical inversion formula

$$-\frac{1}{\pi^2} H^2 \varphi = \varphi \quad (1.17)$$

Therefore, H^{-1} exists and $H^{-1} = -\frac{1}{\pi^2} H$ is continuous.

(1.4) GENERALIZED HILBERT TRANSFORM.

The generalized Hilbert-transform of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is defined to be a distribution $Hf \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ such that

$$\langle Hf, \varphi \rangle = \langle f, -H\varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}) \quad (1.18)$$

where $H\varphi$ is the classical Hilbert-transform of φ defined by

(1.1) In other words, we can say that the Hilbert-transform

Hf of a generalized function $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is a generalized

function in $(\mathcal{D}_{L_p}(\mathbb{R}))'$ which assigns the same number to an element $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ as f assigns to $-H\varphi$ for all $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$, $1 < p < \infty$.

Note that the functional Hf defined by (1.18) belongs to $(\mathcal{D}_{L_p}(\mathbb{R}))'$

in view of theorems 1, Zemanian [37, pp. 25-31]. The generalized function generated by $f(x) \in L_q(\mathbb{R})$, where $q > 1$, and $1/p + 1/q = 1$ is denoted by f and is defined by the relation

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}) \quad (1.19)$$

The fact that $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is obvious by the inequality

$$|\langle f, \varphi \rangle| \leq \|f\|_{L_q} \cdot \|\varphi\|_{L_p}. \quad [37, \text{pp. 53-54}].$$

Denoting the Hilbert-transform of the regular generalized-function f by Hf and using the definition (1.18) as well as (3) on p.3, it follows that the generalized Hilbert-transform Hf of $f \in L_q(\mathbb{R})$, $q > 1$ is the same as the regular generalized function generated by $P \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt$.

Using (1.17) and (1.18) as well as Zemanian [37, pp. 27-28] we prove

COR (1.1)

$$-\frac{1}{\pi^2} H^2 f = f \quad \forall f \in (\mathcal{O}_{L_p}(\mathbb{R}))' \quad 1 < p < \infty \quad (1.20)$$

COR (1.2)

The operator $H: (\mathcal{O}_{L_p}(\mathbb{R}))' \rightarrow (\mathcal{O}_{L_p}(\mathbb{R}))'$, $1 < p < \infty$

defined as in (1.18) is an isomorphism.

Let us find the Hilbert transform of the singular-distribution

$\delta(x) \in (\mathcal{O}_{L_p}(\mathbb{R}))'$. By definition

$$\begin{aligned} \langle H\delta, \varphi \rangle &= \langle \delta(x), -H\varphi \rangle \\ &= \langle \delta(x), -P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-x} dt \rangle \\ &= -P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t} dt \\ &= \langle -PV\left(\frac{1}{t}\right), \varphi \rangle \end{aligned}$$

$$\text{Therefore,} \quad H\delta = -PV\left(\frac{1}{t}\right), \quad (1.21)$$

and using (1.20) we have

$$H[PV\left(\frac{1}{t}\right)] = \pi^2 \delta. \quad (1.22)$$

(1.5) APPLICATIONS TO INTEGRAL EQUATIONSEXAMPLE (1.1)

Let us find the distributional solution y to the integral equation:

$$Hy = P \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt + \delta(x) \quad (1.23)$$

in the space $(\mathcal{D}_L(\mathbb{R}))'$ for $f \in L_q(\mathbb{R})$ where $p, q > 1$ satisfying the relation $1/p + 1/q = 1$. The relation (1.23) can be written as

$$Hy = Hf + \delta(x)$$

operating both sides by the operator H we get

$$-\pi^2 y = -\pi^2 f - PV\left(\frac{1}{t}\right) \quad [\text{By (1.20) - (1.22)}]$$

or
$$y = f + \frac{1}{2\pi} PV\left(\frac{1}{t}\right)$$

EXAMPLE (1.2)

Consider the operator equation

$$y + Hy = f \quad (1.24)$$

where $f \in (\mathcal{O}_{L_p}(\mathbb{R}))'$. Operating on both sides of (1.24) by H we get,

$$\begin{aligned} Hy - \pi^2 y &= Hf \\ \Rightarrow f - y - \pi^2 y &= Hf \\ \Rightarrow y &= \frac{1}{1+\pi^2} [f - Hf] . \end{aligned}$$

Let us define the derivative f' of a generalized-function

$f \in (\mathcal{O}_{L_p}(\mathbb{R}))'$, $1 < p < \infty$ by the relation

$$\langle f', \varphi \rangle = \langle f, -\varphi^{(1)} \rangle \quad \forall \varphi \in \mathcal{O}_{L_p}(\mathbb{R})$$

Zemanian [37, p. 30].

i.e. f' is an element of $(\mathcal{O}_{L_p}(\mathbb{R}))'$ which assigns the same number to $\varphi \in \mathcal{O}_{L_p}(\mathbb{R})$ as f assigns to $-\varphi^{(1)}$.

THEOREM (1.3)

If $f \in (\mathcal{O}_{L_p}(\mathbb{R}))'$, then

$$(Hf)^k = H[f^k] \quad (\forall k = 1, 2, 3, \dots) \quad (1.25)$$

where $(Hf)^k$ stands for the k th derivative of the generalized function Hf .

PROOF:

We will prove the result for $k = 1$ and the general result can be proved by induction on k . For $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ we have

$$\begin{aligned}
 \langle Hf', \varphi \rangle &= \langle f', -H\varphi \rangle \\
 &= \langle f, \frac{d}{dx} [H\varphi](x) \rangle \\
 &= \langle f, H[\varphi^{(1)}(t)] \rangle \\
 &= \langle Hf, -\frac{d}{dt} \varphi \rangle \\
 &= \langle (Hf)', \varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}).
 \end{aligned}$$

Therefore,

$$Hf' = (Hf)'.$$

This completes the proof of the theorem (1.3).

EXAMPLE (1.3)

Let us now find the distribution $y(t)$ satisfying the differential equation

$$\frac{dy}{dt} = H\delta'(t) \quad \text{in } (\mathcal{D}_{L_p}(\mathbb{R}))' \quad 1 < p < \infty.$$

The given differential equation can be written as

$$\frac{d}{dt} [y - H\delta] = 0 \quad (1.26)$$

Since the only constant generalized function in $(\mathcal{D}'_L(\mathbb{R}))'$, $p > 1$ is the zero distribution [10], therefore,

$$\begin{aligned} y - H\delta &= 0 \\ \Rightarrow y - H\delta &= -PV\left(\frac{1}{t}\right) \end{aligned}$$

(1.6) APPROXIMATE HILBERT TRANSFORM

There are essentially two ways of extending an integral transform to generalized functions. One is the direct method which consists in embedding the kernel $K(x,y)$ of the integral transform into the testing function space of generalized functions, Zemanian [37]. The other method consists in using the method of adjoints discussed in the previous sections. In this method, the testing space Φ is mapped continuously, by the integral transform concerned, into another testing function space Ψ . The adjoint mapping then defines the generalized integral transform for the element of the dual of Ψ . The transform defined in this way is, in general, not

a numerical valued function but a generalized-function, namely, an element of the dual of Φ . The kernel $K(x,y) = \frac{1}{x-y}$ in case of Hilbert-transform is not infinitely differentiable at $x=y$ and as such we cannot embed $K(x,y)$ into the testing function space $\mathcal{D}_L$. For this reason, we define an approximate Hilbert-transform of $f \in (\mathcal{D}_L(\mathbb{R}))'$ by the relation:

$$[H_\eta f](x) = \langle f(t), \frac{t-x}{(t-x)^2 + \eta^2} \rangle \equiv F_\eta(x), \quad \eta > 0 \quad (1.27)$$

We shall prove that the generalized Hilbert-transform Hf is also the weak limit as $\eta \rightarrow 0$ of the numerical valued approximate Hilbert-transform $[H_\eta f](x)$ of the generalized function f . Applications of these results in solving the Dirichlet problem with distributional boundary values are also shown.

Let $f \in (\mathcal{D}_L(\mathbb{R}))'$ ($1 < p < \infty$) and x be a real. Consider the numerical valued function $F_\eta(x)$ defined by (1.27). Let

$$K_\eta(x,t) = \frac{t-x}{(t-x)^2 + \eta^2} \quad (1.28)$$

Then for each fixed x and $\eta > 0$, $K_\eta(x,t)$ is a C^∞ -function.

Moreover,

$$\|K_\eta(x, -)\|_{L_p}^p = \int_{-\infty}^{\infty} \left| \frac{t-x}{(t-x)^2 + \eta^2} \right|^p dt.$$

By the shift properties of the integrals we have

$$\begin{aligned} \|K_\eta(x, -)\|_{L_p}^p &= \int_{-\infty}^{\infty} \left| \frac{t}{t^2 + \eta^2} \right|^p dt \\ &= 2 \int_0^{\infty} \left| \frac{t}{t^2 + \eta^2} \right|^p dt. \end{aligned} \quad (1.29)$$

Substituting $t = \eta \tan \theta$ in (1.29) we get

$$\|K_\eta(x, -)\|_{L_p}^p = \frac{2}{\eta^{p-1}} \int_0^{\pi/2} \sin^p \theta \cdot \cos^{p-2} \theta d\theta. \quad (1.30)$$

The integral (1.30) is convergent for all $p > 1$. Therefore, for each fixed x and $\eta > 0$, $K_\eta(x, t) \in \mathcal{D}_{L_p}(\mathbb{R})$ ($1 < p < \infty$). Hence, the function $F_\eta(x)$ as defined in (1.27) exists. Moreover, using the same technique as used in proving [27, Theorem 1], we can show that

$$\frac{\partial^k}{\partial x^k} F_\eta(x) = \left\langle f(t), \frac{\partial^k}{\partial x^k} \frac{(t-x)}{(t-x)^2 + \eta^2} \right\rangle \quad (1.31)$$

THEOREM (1.4)

(a) Let $f \in (\mathcal{W}_L^p(\mathbb{R}))'$, $p > 1$ and $F_\eta(x)$ be the approximate Hilbert-transform of f as defined by (1.27) then for each

$(k = 0, 1, 2, \dots;)$ $F_\eta^{(k)}(x) \in L_q(\mathbb{R})$ where $q > 1$ and $1/p + 1/q = 1$.

(b) The regular generalized function generated by $F_\eta(x)$ belongs to $(\mathcal{W}_L^p(\mathbb{R}))'$ and

$$\int_{-\infty}^{\infty} F_\eta(x) \varphi(x) dx = \langle f(t), \int_{-\infty}^{\infty} \frac{(t-x)\varphi(x)}{(t-x)^2 + \eta^2} dx \rangle \quad (1.32)$$

PROOF:

(a) We will sketch the proof of the fact that $F_\eta(x) \in L_q(\mathbb{R})$, $q > 1$ (i.e. for $k = 0$) as the proof of the cases $(k = 1, 2, 3, \dots)$ can be given in a similar fashion by using (1.31). In view of the structure formula (1.13) there exists a non-negative finite integer γ and functions $f_i \in L_q(\mathbb{R})$ such that:

$$F_\eta(x) = \sum_{i=1}^{\gamma} \int_{-\infty}^{\infty} f_i(t) \left(-\frac{\partial}{\partial t} \right)^{i-1} \frac{(t-x)}{(t-x)^2 + \eta^2} dt. \quad (1.33)$$

Now, using the results [31, p. 134] and the fact that

$$\left| \frac{t-x}{(t-x)^2 + \eta^2} \right| \leq \frac{1}{2\eta} \quad (\eta \text{ real } \neq 0),$$

it follows that each of the terms in the summation (1.33), when treated as a function of x , is an element of $L_q(\mathbb{R})$ for all fixed $\eta \neq 0$. Thus $F_\eta(x) \in L_q(\mathbb{R})$.

(b) Since $F_\eta(x) \in L_q(\mathbb{R})$, it follows that the regular generalized-function generated by $F_\eta(x)$ belongs to $(\mathcal{D}_{L_p}(\mathbb{R}))'$, i.e. the map $\varphi \rightarrow \int_{-\infty}^{\infty} F_\eta(x)\varphi(x)dx$ defines a continuous linear functional on $\mathcal{D}_{L_p}(\mathbb{R})$. Now, using the structure-formula for $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$, we have

$$\int_{-\infty}^{\infty} F_\eta(x)\varphi(x)dx = \int_{-\infty}^{\infty} \varphi(x) \left(\sum_{i=1}^Y \int_{-\infty}^{\infty} f_i(t) \left(-\frac{\partial}{\partial t} \right)^{i-1} \left[\frac{t-x}{(t-x)^2 + \eta^2} \right] dt \right) dx.$$

Using Fubini's theorem and the integration by parts, we get

$$\begin{aligned} \int_{-\infty}^{\infty} F_\eta(x)\varphi(x)dx &= \int_{-\infty}^{\infty} \sum_{i=1}^Y f_i(t) \left(-\frac{\partial}{\partial t} \right)^{i-1} \int_{-\infty}^{\infty} \frac{\varphi(x)(t-x)}{(t-x)^2 + \eta^2} dx dt \\ &= \sum_{i=1}^Y \langle f_i(t), \left(-\frac{\partial}{\partial t} \right)^{i-1} \int_{-\infty}^{\infty} \frac{\varphi(x)(t-x)}{(t-x)^2 + \eta^2} dx \rangle \\ &= \langle f(t), - \int_{-\infty}^{\infty} \frac{(x-t)\varphi(x)}{(x-t)^2 + \eta^2} dx \rangle \\ &= \langle f(t), -[H_\eta \varphi](t) \rangle \\ &= \langle f, -H_\eta \varphi \rangle. \end{aligned} \tag{1.34}$$

Since $\varphi \in L_p(\mathbb{R})$, in view of the results proved by Riesz and Titchmarsh [31, pp. 132-137], $[H_\eta \varphi] \in L_p$. Again, since $\varphi \rightarrow 0$ as $x \rightarrow \pm\infty$, using the technique of integration by parts, we can see that

$$\frac{d^k}{dx^k} (H_\eta \varphi) = (H_\eta \varphi^{(k)}) \in L_p(\mathbb{R}) \quad (\forall k = 0, 1, 2, 3, \dots) \quad (1.35)$$

Therefore, $[H_\eta \varphi] \in \mathcal{D}_{L_p}(\mathbb{R})$ and hence, the expression on the right hand side of (1.34) is meaningful. Using the relation (1.34) and the results [31, p. 136] we can see that $(H_\eta \varphi)(x) \rightarrow (H\varphi)(x)$ as $\eta \rightarrow 0+$, in the topology of $\mathcal{D}_{L_p}(\mathbb{R})$. Therefore, letting $\eta \rightarrow 0+$ in (1.34) we get

$$\begin{aligned} \lim_{\eta \rightarrow 0+} \langle H_\eta f, \varphi \rangle &= \langle f, -H\varphi \rangle \\ &= \langle Hf, \varphi \rangle \quad [\text{By definition}]. \end{aligned}$$

We have now proved

THEOREM (1.5)

For $1 < p < \infty$ let $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$. For each $\eta > 0$ let $H_\eta f$ be its approximate Hilbert-transform as defined by (1.27). Then the regular distribution $H_\eta f$ as generated by $(H_\eta f)(x)$ tends to Hf in the weak topology of $(\mathcal{D}_{L_p}(\mathbb{R}))'$ as $\eta \rightarrow 0+$.

DEFINITION:

Let $U(x, y)$ be a function defined for real x and $y > 0$. A generalized function $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ is called the distributional limit of $U(x, y)$ at $y = 0$, written as

$$\lim_{y \rightarrow 0} U(x, y) = U(x, 0+) = f(x)$$

if for every $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$

$$\lim_{y \rightarrow 0+} \int_{-\infty}^{\infty} U(x, y) \varphi(x) dx = \langle f, \varphi \rangle. \quad (1.36)$$

THEOREM (1.6)

Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$, $(1 < p < \infty)$. Then for all $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$

$$\langle \langle f(t), \frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle, \varphi(x) \rangle \rightarrow \langle f, \varphi \rangle$$

as $\eta \rightarrow 0+$, that is, $\langle f(t), \frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle \rightarrow f$ in $(\mathcal{D}_{L_p}(\mathbb{R}))'$.

PROOF:

Using the structure formula (1.13) for $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ we have

$$\begin{aligned} & \langle \langle f(t), \frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle, \varphi(x) \rangle \\ &= \sum_{i=1}^Y \langle \langle f_i(t), (-\frac{\partial}{\partial t})^{1-1} (\frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle, \varphi(x) \rangle \end{aligned}$$

where $f_i \in L_q(\mathbb{R})$, $q > 1$, $1/p + 1/q = 1$.

Using Fubini's theorem and the definition of the derivative of the generalized functions, Zemanian [37, p. 30] we get

$$\begin{aligned}
 \langle\langle f(t), \frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle, \varphi(x) \rangle &= \sum_{i=1}^Y \langle f_1^{(i-1)}(t), \langle \frac{\eta}{\pi[(t-x)^2 + \eta^2]}, \varphi(x) \rangle \rangle \\
 &= \langle f(t), \langle \frac{\eta}{\pi[(t-x)^2 + \eta^2]}, \varphi(x) \rangle \rangle
 \end{aligned}
 \tag{1.37}$$

However,

$$\frac{\eta}{\pi} \int \frac{dx}{(t-x)^2 + \eta^2} = 1 \quad \forall \eta > 0 \quad [3, p.73]$$

and for each fixed t as well as $\eta > 0$

$$\frac{\eta}{\pi[(t-x)^2 + \eta^2]} = O\left(\frac{1}{x^2}\right) \text{ as } |x| \rightarrow \infty.$$

Therefore,

by using the result, Zemanian [36, pp. 44-45]

as well as the relation (1.11) we have

$$\langle \frac{\eta}{\pi[(t-x)^2 + \eta^2]}, \varphi(x) \rangle \rightarrow \varphi(t) \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R})
 \tag{1.38}$$

Thus, by letting $\eta \rightarrow 0+$ in (1.37), we get

$$\lim_{\eta \rightarrow 0+} \langle\langle f(t), \frac{\eta}{\pi[(t-x)^2 + \eta^2]} \rangle, \varphi(x) \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R})
 \tag{1.39}$$

EXAMPLE (1.4)

Let Ω_+ be the upper half plane. If $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$, $1 < p < \infty$, let us find a solution to the distributional Dirichlet problem.

$$\Delta^2 u \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0 \quad (1.39)$$

$$\lim_{y \rightarrow 0+} U(x, y) = U(x, 0+) = f(x) \text{ in } (\mathcal{D}_{L_p}(\mathbb{R}))'$$

An obvious solution is

$$U(x, y) = \langle f(t), \frac{y}{\pi[(t-x)^2 + y^2]} \rangle, \quad y > 0.$$

It is easily shown that $U(x, y)$ is a harmonic-function in the region Ω_+ and satisfies the boundary condition (1.39) in view of the theorem (1.6).

CHAPTER 2

(2.1) INTRODUCTION

Let $f \in (\mathcal{D}_{L_P}(\mathbb{R}))'$. If there exists a complex-valued function $F(z)$ analytic in the z -plane except possibly on the real axis $\{z: \text{Im } z = 0\}$ such that

$$\lim_{\epsilon \rightarrow 0+} \int_{-\infty}^{\infty} [F(x+i\epsilon) - F(x-i\epsilon)] \varphi(x) dx = \langle f, \varphi \rangle \quad (2.1)$$

for all $\varphi \in \mathcal{D}_{L_P}(\mathbb{R})$, then $F(z)$ is called the analytic representation of $f \in (\mathcal{D}_{L_P}(\mathbb{R}))'$. The Cauchy integral representation of

analytic-functions (in terms of their boundary values) is an important tool for the mathematical study of physical phenomena. The role of such representations is even more important when we consider the generalized functions. The analytic-representation of a distribution f with compact support is given by:

$$\hat{f}(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle \quad (2.2)$$

$\hat{f}(z)$ is called the Cauchy representation of $f \in (\mathcal{E}(\mathbb{R}))'$.

However, the Cauchy representation (2.2) is not valid for every Schwartz distribution $f \in (\mathcal{D}(\mathbb{R}))'$ because $(t-z)^{-1} \notin \mathcal{D}(\mathbb{R})$.

The distributional representation (2.4) of the analytic function $F(z)$ satisfying (2.3) and (2.3a) is arrived at in a way similar to that arrived at by Mitrovic for the spaces $(\mathcal{O}_\alpha(\mathbb{R}))'$ [18]. The representation (2.4) will be exploited in proving the uniqueness of the solution to a Dirichlet boundary value-problem in the space $(\mathcal{O}_{L_p}(\mathbb{R}))'$. We will also characterize the space

$$H(\mathcal{O}) = \{ \varphi : \varphi \equiv H[\psi] \equiv P \int_{-\infty}^{\infty} \frac{\psi(t)}{t-x} dt ; \psi \in \mathcal{O} \}$$

by using the Cauchy representation of the generalized functions and extend the notion of the Hilbert transform to Schwartz-distributions. Lastly, the Hilbert problem for the generalized-functions will be solved in a very constructive way.

THEOREM (2.1)

Let $f \in (\mathcal{O}_{L_p}(\mathbb{R}))'$, $p > 1$ and $F(z)$ be complex-valued function defined in the region $\Omega = \{z : \operatorname{Im} z \neq 0\}$ by:

$$F(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle \quad (2.5)$$

Then $F(z)$ is the Cauchy representation of f .

PROOF:

To prove the analyticity of $F(z)$ we note that

$$\frac{1}{h}[F(z+h) - F(z)] = \frac{1}{2\pi i} \langle f(t), \frac{1}{h} \left[\frac{1}{t-z-h} - \frac{1}{t-z} \right] \rangle.$$

Now $h^{-1}[(t-z-h)^{-1} - (t-z)^{-1}]$ converges to $(t-z)^{-2}$ in the sense of $\mathcal{D}_{L_p}(\mathbb{R})$. More specifically, if

$$\Delta_h(t, z) = [h^{-1}\{(t-z-h)^{-1} - (t-z)^{-1}\} - (t-z)^{-2}].$$

Then for $\text{Im } z \neq 0$,

$$\left\| \frac{d^k}{dt^k} \Delta_h(-, z) \right\|_{L_p} \leq h \cdot O\left(\frac{1}{|y|^{k+3}}\right) \rightarrow 0 \text{ as } h \rightarrow 0$$

($k = 0, 1, 2, 3, \dots$)
($y = \text{Im } z$)

so that

$$\gamma_k(\Delta_h(-, z)) \rightarrow 0 \quad (\forall k = 0, 1, 2, \dots).$$

Therefore,

$$\frac{d}{dz} F(z) = \langle f(t), \frac{1}{(t-z)^2} \rangle \quad \forall z \in \Omega.$$

Thus, $F(z)$ is analytic in the region Ω .

Let us define

$$I(\epsilon) = \int_{-\infty}^{\infty} [F(x+i\epsilon) - F(x-i\epsilon)] \varphi(x) dx. \quad (2.6)$$

Since $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ and for each fixed value of $\epsilon > 0$

$F(x \pm i\epsilon) \in L_q(\mathbb{R})$ [Theorem (1.4)], the integral $I(\epsilon)$ exists for all $\epsilon > 0$. Now

$$\begin{aligned} I(\epsilon) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \varphi(x) \left\langle f(t), \left[\frac{1}{t-x-i\epsilon} - \frac{1}{t-x+i\epsilon} \right] \right\rangle dx \\ &= \int_{-\infty}^{\infty} \varphi(x) \left\langle f(t), \frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \right\rangle dx. \end{aligned}$$

Using the structure formula (1.13) for f we get

$$I(\epsilon) = \sum_{j=1}^r (-1)^j \int_{-\infty}^{\infty} \left\langle f_j(t), \frac{\partial^j}{\partial t^j} \left[\frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \right] \right\rangle \varphi(x) dx$$

where $f_j \in L_q(\mathbb{R})$ ($1 \leq j \leq r$).

However,

$$\begin{aligned} \frac{\partial^j}{\partial t^j} \left[\frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \right] &= (-1)^j \frac{\partial^j}{\partial x^j} \left[\frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \right] \\ &\quad (\forall j = 1, 2, 3, \dots) \end{aligned}$$

Therefore,

$$I(\epsilon) = \sum_{j=1}^Y (-1)^j \int_{-\infty}^{\infty} \langle f_j(t), (-1)^j \frac{\partial^j}{\partial x^j} \left[\frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \right] \rangle \varphi(x) dx \quad (2.7)$$

Using Fubini's theorem, and the integration by parts, we get

$$I(\epsilon) = \sum_{j=1}^Y (-1)^j \langle f_j(t), \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{\varphi^{(j)}(x) dx}{[(t-x)^2 + \epsilon^2]} \rangle \quad (2.7)$$

However,

$$\frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{\varphi^{(j)}(x) dx}{[(t-x)^2 + \epsilon^2]} \rightarrow \varphi^{(j)}(t) \quad [\text{By (1.38)}]$$

in $\mathcal{D}_{L_p}(\mathbb{R})$ as $\epsilon \rightarrow 0+$.

Hence, letting $\epsilon \rightarrow 0+$ in (2.7) we get

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} I(\epsilon) &= \sum_{j=1}^Y (-1)^j \langle f_j(t), \varphi^{(j)}(t) \rangle \\ &= \langle f(t), \dot{\varphi}(t) \rangle \quad [\text{by (1.13)}] . \end{aligned}$$

$$\forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}) .$$

Therefore, $F(z)$ is the Cauchy representation of $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$.

EXAMPLE (2.1)

Dirac's δ -function is defined by:

$$\langle \delta(t), \varphi(t) \rangle = \varphi(0) \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}) .$$

The Cauchy representation of $\delta(t)$ is given by

$$F(z) = \frac{1}{2\pi i} \langle \delta(t), \frac{1}{t-z} \rangle = -\frac{1}{2\pi i} \frac{1}{z} . \quad (2.8)$$

The n th derivative of δ -function is defined by:

$$\langle \delta^{(n)}(t), \varphi(t) \rangle = (-1)^n \varphi(0) .$$

The Cauchy representation of $\delta^n(t)$ is,

$$F_n(z) = \frac{(-1)^n n!}{2\pi i} \frac{1}{z^{n+1}} \quad (2.9)$$

EXAMPLE (2.2)

Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ be a distribution whose support consists of an isolated point a . Then, according to Zemanian [36, p. 98], f is a finite linear-combination of Dirac's δ -function and its derivatives, i.e.

$$f(t) = \sum_{n=1}^N a_n \delta^{(n-1)}(t-a) .$$

The Cauchy representation of f is given by:

$$F(z) = \sum_{n=1}^{\infty} a_n \frac{(-1)^n (n-1)!}{2\pi i} \frac{1}{(z-a)^n} \quad (2.10)$$

EXAMPLE (2.3)

$$\text{Let } f(t) = \sum_{n=1}^{\infty} a_n \delta(t-n)$$

where a_n are appropriate constants (for example $\sum_{n=1}^{\infty} |a_n| < \infty$). Then,

$$\text{supp } f = \{1, 2, 3, \dots\}$$

is a non-compact set and, therefore, $f \notin (E(\mathbb{R}))'$. However,

for every $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ the map:

$$\varphi \rightarrow \langle f, \varphi \rangle = \sum_{n=1}^{\infty} a_n \varphi(n) \quad (2.11)$$

is well-defined since, $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Therefore, the series (2.11) is convergent for all $\varphi \in \mathcal{D}_{L_p}(\mathbb{R})$ and $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$. The Cauchy representation of f is given by:

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n}{2\pi i} \left(\frac{1}{z-n} \right). \quad (2.12)$$

It is now clear from the example (2.3) that we have extended the notion of the Cauchy-representation of generalized functions to the wider class $(\mathcal{D}_{L_p}(\mathbb{R}))'$ than that discussed in [3] and [18]-[19].

(2.2) DISTRIBUTIONAL REPRESENTATION OF ANALYTIC FUNCTIONS

To every generalized function $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ we proved that there corresponds a complex-valued function (the Cauchy representation of f) given by:

$$F(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle, \quad \text{Im } z \neq 0.$$

The function $F(z)$ is analytic everywhere except possibly at the supp f and vanishes as $|z| \rightarrow \infty$, with $\text{Im } z \neq 0$. The converse of the above statement is, however, not true in general. It can be seen by a counter-example [3]. The function $e^{\frac{1}{z}} - 1$ is analytic for $z \neq 0$ and tends to zero as $|z| \rightarrow \infty$. Suppose the contrary, i.e. the desired generalized function exists. Then, since $e^{\frac{1}{z}} - 1$ is analytic everywhere except at $z = 0$, the generalized function f must be concentrated at the origin. By Zemanian [36,p.98], f is a finite linear combination of Dirac's - distribution and its derivatives. Thus f has the Cauchy representation

$$F(z) = \sum_{n=1}^N \frac{a_n}{z^n}, \quad (2.13)$$

where a_n are certain constants. On the other hand, we have

$$e^{\frac{1}{z}} - 1 = \sum_{n=1}^{\infty} \frac{(1/n!)}{z^n}. \quad (2.14)$$

Hence there does not exist any $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ such that

$$F(z) = \sum_{n=1}^{\infty} \frac{(1/n!)}{z^n}.$$

In this section we shall classify a certain class of analytic-functions having distributional representation. Our class of analytic-functions having such type of representations is much larger than that discussed by Mitrovic [18].

Using the structure formula for $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$, $1 < p < \infty$, we note that,

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle \\ &= \frac{1}{2\pi i} \sum_{j=1}^{\infty} (-1)^{j-1} (j-1)! \int_{-\infty}^{\infty} \frac{f_1(t) dt}{(t-z)^j}. \end{aligned} \quad (2.15)$$

A typical term on the right hand side of (2.15) is $\int_{-\infty}^{\infty} \frac{f_1(t) dt}{(t-z)^j}$,

and its contribution to $F(z)$ is a constant multiple of

$\int_{-\infty}^{\infty} (t-z)^{-j} f_j(t) dt$. Its m th derivative is a constant multiple of $\int_{-\infty}^{\infty} (t-z)^{-j-m} f_j(t) dt$. Since $(t-z)^{-j-m} \in \mathcal{D}_{L_p}(\mathbb{R})$ ($1 < p < \infty$)

$\forall z \in \{z: \operatorname{Im} z \neq 0\}$, these integrals are bounded functions of z for

$\{z: |\operatorname{Im} z| \geq \delta > 0\}$. More specifically, the term $\int_{-\infty}^{\infty} (t-z)^{-1} f_1(t) dt$

determines the order of the function $F(z)$. Using Holder's

inequality for integrals we get,

$$\left| \int_{-\infty}^{\infty} (t-z)^{-1} f_1(t) dt \right| \leq \|f_1\|_{L_q} \| (t-z)^{-1} \|_{L_p} . \quad (2.16)$$

However,

$$\| (t-z)^{-1} \|_{L_p}^p = \int_{-\infty}^{\infty} \left(\frac{1}{(t-x)^2 + y^2} \right)^{p/2} dt .$$

Using the shift property of the infinite integrals we have

$$\| (t-z)^{-1} \|_{L_p}^p = \int_{-\infty}^{\infty} \left(\frac{1}{t^2 + y^2} \right)^{p/2} dt .$$

Using the transformation $t = y \tan \theta$, we get

$$\| (t-z)^{-1} \|_{L_p}^p = \frac{2}{y^{p-1}} \int_0^{\pi/2} \cos^{p-2} \theta d\theta . \quad (2.17)$$

The integral in (2.17) converges for all $p > 1$. Therefore,

$$\| (t-z)^{-1} \|_{L_p}^p = O\left(\frac{1}{y^\delta}\right) . \quad (2.18)$$

where $\delta = \frac{p-1}{p} > 0$.

In view of (2.16), as well as (2.18) we note that

$$F(z) = O\left(\frac{1}{y^\delta}\right) , \quad y \rightarrow \infty . \quad (2.19)$$

THEOREM (2.2)

Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ ($1 < p < \infty$), and

$$F(z) = \frac{1}{2\pi i} \langle f(t), \frac{1}{t-z} \rangle, \quad \text{Im } z \neq 0.$$

Then,

$$\lim_{\epsilon \rightarrow 0+} F(x \pm i\epsilon) = f^{\pm}(x) \in (\mathcal{D}_{L_p}(\mathbb{R}))'.$$

PROOF:

We prove the existence of the limit,

$$\lim_{\epsilon \rightarrow 0+} F(x + i\epsilon) = f^+(x)$$

as the other part can be proved similarly.

We note that,

$$\begin{aligned} F(x + i\epsilon) &= \frac{1}{2\pi i} \langle f(t), \frac{(t-x)}{(t-x)^2 + \epsilon^2} + i \frac{\epsilon}{(t-x)^2 + \epsilon^2} \rangle \\ &= \frac{1}{2\pi i} \langle f(t), \frac{t-x}{(t-x)^2 + \epsilon^2} \rangle + \frac{1}{2} \langle f(t), \frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \rangle \\ &= \frac{1}{2\pi i} H_{\epsilon}[f](x) + \frac{1}{2} \langle f(t), \frac{\epsilon}{\pi[(t-x)^2 + \epsilon^2]} \rangle \end{aligned} \quad (2.21)$$

where $H_{\epsilon}[f]$ is the approximate Hilbert-transform of $f[\S:(1.5)]$.

Now, letting $\epsilon \rightarrow 0+$ in (2.21) and using the theorems (1.5) and (1.6)

we get,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} F(x+i\epsilon) &= \frac{1}{2\pi i} H[f](x) + \frac{1}{2} f(x) \\ &= f^+(x) \end{aligned} \quad (2.21)$$

where,

$$f^+ = \frac{1}{2\pi i} H[f] + \frac{1}{2} f \in (\mathcal{O}_{L_p}(\mathbb{R}))' \quad [\text{By Cor (1.2)}]$$

This completes the proof of the theorem.

Let us now reverse the problem. Let $F(z)$ be analytic in the region $\{z: \text{Im } z \neq 0\}$ and satisfying the relation (2.3), and the uniform asymptotic order (2.3a), such that $F(x \pm i\epsilon) \in L_q(\mathbb{R})$ for fixed $\epsilon > 0$ and

$$\lim_{\epsilon \rightarrow 0^+} F(x \pm i\epsilon) = f^\pm(x) \text{ in } (\mathcal{O}_{L_p}(\mathbb{R}))'.$$

Can we find $f \in (\mathcal{O}_{L_p}(\mathbb{R}))'$ satisfying

$$F(z) = \langle f(t), \frac{1}{t-z} \rangle, \quad \text{Im } z \neq 0? \quad (2.22)$$

The answer is in the affirmative. We first solve the problem for the functions analytic in the upper half plane because the problem for the functions analytic in the lower half plane can be solved in a similar fashion.

It can be easily seen by definition that,

$$\frac{1}{2\pi i} \langle f^+(t), \frac{1}{t-z} \rangle = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{F(t+i\epsilon)}{t-z} dt. \quad (2.23)$$

Since $F(t+i\epsilon) \in L_q(\mathbb{R})$ and $(t-z)^{-1} \in L_p(\mathbb{R})$ ($\text{Im } z > 0$) the integral in (2.23) converges for $\text{Im } z > 0$. In order to calculate the integral, let us consider the closed half disc

$$B_r^+(0) = \{z: |z| \leq r, \text{Im } z > 0\}.$$

in the upper half plane such that $z \in \text{Int } B_r^+(0)$. Clearly, the function $\xi \rightarrow \frac{F(\xi+i\epsilon)}{\xi-z}$ ($\text{Re } \xi = t$) is analytic in $B_r^+(0) \setminus \{z\}$ and continuous on the boundary of $B_r^+(0)$. Hence, applying the Cauchy integral-formula to this function, and letting $r \rightarrow \infty$, we get:

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t+i\epsilon)}{t-z} dt = \begin{cases} F(z+i\epsilon) & \text{for } \text{Im } z > 0 \\ 0 & \text{for } \text{Im } z < 0 \end{cases} \quad (2.24)$$

Now $(t-z)^{-1} \in L_p(\mathbb{R})$ therefore, from (2.23) we have

$$\frac{1}{2\pi i} \langle F(t+i\epsilon), \frac{1}{t-z} \rangle = F(z+i\epsilon), \quad z \in \text{Int } B_r^+(0). \quad (2.25)$$

Since $F(z)$ is holomorphic in the region $\text{Im } z > 0$, letting $\epsilon \rightarrow 0+$ in (2.25) we have,

$$\frac{1}{2\pi i} \langle f^+(t), \frac{1}{t-z} \rangle = F(z)$$

Therefore, (2.2) holds with

$$f(t) = \frac{1}{2\pi i} f^+(t) \quad (2.26)$$

We now conclude the following

THEOREM (2.3)

Let $F(z)$ be a complex-valued function of the complex-variable $z = x+iy$ analytic in the region $\Omega_+ = \{z: \text{Im } z > 0\}$ such that

(i) (a) for fixed $y > 0$, $q = p/(p-1)$, $p > 1$, we have

$$F(x+iy) \in L_q(\mathbb{R})$$

$$(b) \lim_{y \rightarrow 0+} F(x+iy) = f^+(x) \text{ in } (L_p(\mathbb{R}))' \quad (2.27)$$

$$(ii) F(z) = o(1) \text{ as } |y| \rightarrow \infty \text{ (uniformly for all } x) \quad (2.28)$$

$$(iii) \sup_{\substack{-\infty < x < \infty \\ y \geq \delta}} |f(z)| = A_\delta < \infty, \quad \delta > 0. \quad (2.29)$$

Then,

$$F(z) = \langle f^+(t), \frac{1}{2\pi i} \frac{1}{t-z} \rangle \quad \forall z \in \Omega_+ \quad (2.30)$$

$$= 0 \text{ if } \text{Im } z < 0.$$

As a consequence of the above theorem we have,

COR (2.1)

Let Ω_+ be the region as defined in theorem (2.3). Then the analytic-function $F(z)$, $z \in \Omega_+$, satisfying [2.27] $\rightarrow$ [2.30] is uniquely determined by $f^+(x)$.

Now we state the theorem for the distributional representation of the function $F(z)$ analytic in the lower half plane.

THEOREM (2.4)

Let $f(z)$ be a complex-valued function of the complex variable $z = x + iy$ analytic in the region $\Omega_- = \{z: \text{Im } z < 0\}$ such that,

(i) (a) for fixed $y > 0$, $q = p/(p-1)$, $p > 1$ we have

$$F(x - iy) \in L_q(\mathbb{R})$$

$$(b) \lim_{y \rightarrow 0^+} F(x - iy) = f^-(x) \text{ in } (L_p(\mathbb{R}))'$$

(ii) $F(z) = o(1)$ as $y \rightarrow \infty$

$$(iii) \sup_{\substack{-\infty < x < \infty \\ y \geq \delta}} |F(z)| = A_\delta < \infty, \quad \delta > 0.$$

Then,

$$F(z) = \frac{1}{2\pi i} \langle f^-(t), \frac{1}{t-z} \rangle \quad \forall z \in \Omega_-$$

$$= 0 \quad \text{if } \text{Im } z > 0.$$

The proof is quite similar to the one given for the theorem (2.3).

However, in this case, we consider the closed half disc

$B_r^-(0) = \{z: |z| \leq r, \operatorname{Im} z < 0\}$ in the lower half plane. Then

for any $z \in \operatorname{Int} B_r^-(0)$ we get,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t - i\varepsilon)}{t - z} dt = -F(z - i\varepsilon). \quad (2.30)$$

For fixed $z \in \operatorname{Int} B_r^-(0)$, $(t-z)^{-1} \in \mathcal{O}_{L_P}(\mathbb{R})$, therefore, we have

$$\begin{aligned} \frac{1}{2\pi i} \langle F(t - i\varepsilon), \frac{1}{t-z} \rangle &= -F(z - i\varepsilon). \\ &= 0 \text{ for } \operatorname{Im} z \notin \Omega_-. \end{aligned}$$

Letting $\varepsilon \rightarrow 0+$ and using the analyticity of $F(z)$ in Ω_- we get,

$$F(z) = -\langle f^-(t), \frac{1}{t-z} \rangle \quad \forall z \in \Omega_-.$$

As a consequence of the theorems (2.3) and (2.4) we have

THEOREM (2.5)

Let $F(z)$ be complex-valued function of complex-variable

$z = x+iy$ analytic in the region $\Omega = \{z: \operatorname{Im} z \neq 0\}$. Further,

assume that $f(z)$ satisfies the hypothesis of theorem (2.3) and (2.4)

$F(x \pm i0) = f^\pm(x) \in (\mathcal{O}_{L_P}(\mathbb{R}))'$: Then,

$$F(z) = \langle f(t), \frac{1}{t-z} \rangle$$

where

$$f = \frac{1}{2\pi i} [f^+ - f^-] \in (\mathcal{O}_{L_P}(\mathbb{R}))'.$$

(2.3) THE HILBERT TRANSFORM OF SCHWARTZ DISTRIBUTIONS

Let $\mathcal{D}(\mathbb{R})$ be the Schwartz space of C^∞ -functions defined on $\mathbb{R}$ with compact support. Let $H(\mathcal{D})$ be the space of all C^∞ -functions on $\mathbb{R}$ such that $\psi \in H(\mathcal{D})$ iff there exists $\varphi \in \mathcal{D}(\mathbb{R})$ such that

$$\psi(x) = P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-x} dt.$$

According to (1.15)

$$\psi^{(k)}(x) = P \int_{-\infty}^{\infty} \frac{\varphi^{(k)}(t)}{t-x} dt = P \int_{-a}^a \frac{\varphi^{(k)}(t)}{t-x} dt,$$

where the support of φ is contained in $[-a, a]$. Clearly,

$$\psi^{(k)}(x) = O\left(\left|\ln \left|\frac{x-a}{x+a}\right|\right|\right) \text{ as } |x| \rightarrow \infty. \quad (2.31)$$

If we assume that ψ is a C^∞ -function defined on $\mathbb{R}$ satisfying the asymptotic order (2.31), then does it necessarily imply that

$\psi \in H(\mathcal{D})$, i.e. is ψ the Hilbert-transform of some $\varphi \in \mathcal{D}(\mathbb{R})$?

We shall attempt to prove this by adding some more conditions on ψ to (2.31) and by using the analytic representation of distributions, discussed in the previous sections.

Now consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{D}_{L_p}(\mathbb{R}) & \xrightarrow{H} & \mathcal{D}_{L_p}(\mathbb{R}) \\
 \uparrow i & \nearrow & \uparrow j \\
 \mathcal{D}(\mathbb{R}) & \xrightarrow{H} & H(\mathcal{D})
 \end{array}$$

Since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}_{L_p}(\mathbb{R})$ [10, p. 87], and

$H: \mathcal{D}_{L_p}(\mathbb{R}) \rightarrow \mathcal{D}_{L_p}(\mathbb{R})$ is a homeomorphism [Theorem 1.2], i and j are the natural embeddings of $\mathcal{D}(\mathbb{R})$ and $H(\mathcal{D})$ into $\mathcal{D}_{L_p}(\mathbb{R})$ provided we induce a sub-space topology on $H(\mathcal{D})$. In fact, we have the following,

THEOREM (2.6)

- (a) $H(\mathcal{D})$, with the sub-space topology on it, is dense in $\mathcal{D}_{L_p}(\mathbb{R})$.
- (b) Convergence of a sequence in $H(\mathcal{D})$ implies its convergence in $\mathcal{D}_{L_p}(\mathbb{R})$.

PROOF:

- (a) Let $\psi \in \mathcal{D}_{L_p}(\mathbb{R})$. In view of the inversion formula [31]

$$-H^2 f / \pi^2 = f \quad \forall f \in L_p(\mathbb{R}).$$

it follows that there exists $\theta \in \mathfrak{D}_{L_P}(\mathbb{R})$, such that $[H\theta](x) = \psi(x)$.

Since $\mathfrak{D}(\mathbb{R})$ is dense in $\mathfrak{D}_{L_P}(\mathbb{R})$ [10, p. 87], there exists a sequence $\{\theta_\mu\}_{\mu=1}^\infty$ in $\mathfrak{D}(\mathbb{R})$ tending to θ in $\mathfrak{D}_{L_P}(\mathbb{R})$ as $\mu \rightarrow \infty$. Now

$$\begin{aligned} \gamma_k(\psi - H\theta_\mu) &= \|(H\theta)^{(k)} - (H\theta_\mu)^{(k)}\|_{L_P} \\ &= \|H(\theta^{(k)} - \theta_\mu^{(k)})\|_{L_P} \\ &\leq C_p \|\theta^{(k)} - \theta_\mu^{(k)}\|_{L_P} \rightarrow 0 \text{ as } \mu \rightarrow \infty. \quad [31, p.132] \end{aligned}$$

$$(\forall k = 0, 1, 2, 3, \dots).$$

Therefore, $H(\mathfrak{D})$ is dense in $\mathfrak{D}_{L_P}(\mathbb{R})$. This completes the proof of (a).

- (b) If $\{\varphi_\mu\}_{\mu=1}^\infty$ is a sequence in $H(\mathfrak{D})$ converging to zero in $H(\mathfrak{D})$ as $\mu \rightarrow \infty$, then there exists a sequence $\{\psi_\mu\}_{\mu=1}^\infty$ in $\mathfrak{D}(\mathbb{R})$ such that $\varphi_\mu = H[\psi_\mu]$ and $\psi_\mu \xrightarrow{\mathfrak{D}(\mathbb{R})} 0$. Since convergence in $\mathfrak{D}(\mathbb{R})$ implies the convergence in $\mathfrak{D}_{L_P}(\mathbb{R})$, it follows that

$$\begin{aligned} \gamma_k(\varphi_\mu) &= \gamma_k(H\psi_\mu) \\ &\leq C_p \|\psi_\mu^{(k)}\|_{L_P} \rightarrow 0 \text{ as } \mu \rightarrow \infty. \end{aligned}$$

We have not yet described the space $H(\mathbb{D})$ and its topology in an intrinsic way. We now give an intrinsic definition of the space $H(\mathbb{D})$, and its topology.

Let $\varphi(t) \in \mathcal{D}(\mathbb{R})$ be such that $\text{supp } \varphi \subseteq [-a, a]$. Let φ be the regular distribution generated by $\varphi(x)$. Then φ has a compact support. Let us define,

$$\tilde{\varphi}(z) = \frac{1}{2\pi i} \langle \varphi(t), \frac{1}{t-z} \rangle \quad (2.32)$$

According to theorem (2.1), $\tilde{\varphi}(z)$ is the Cauchy representation of φ . Since φ has a compact support, it follows that

$$\tilde{\varphi}(z) = O\left(\frac{1}{|z|}\right), \quad |z| \rightarrow \infty \quad (2.33)$$

Moreover, $\tilde{\varphi}(z)$ is analytic everywhere except possibly on the compact set $K = [-a, a]$. Therefore, the limits $\tilde{\varphi}^+(x)$ and $\tilde{\varphi}^-(x)$ exist in $(\mathcal{D}'_L(\mathbb{R}))'$ and by theorem (2.5),

$$\tilde{\varphi}(z) = \frac{1}{2\pi i} \langle \tilde{\varphi}^+(t) - \tilde{\varphi}^-(t), \frac{1}{t-z} \rangle$$

Thus,

$$\tilde{\varphi}^+(t) - \tilde{\varphi}^-(t) = \varphi(t) \quad (2.34)$$

Now from (2.32)

$$\tilde{\varphi}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-z} dt = \frac{1}{2\pi i} \int_{-a}^a \frac{\varphi(t)}{t-z} dt \quad (2.35)$$

Letting $\text{Im } z \rightarrow 0+$ in (2.35) and using [2.32] - [2.34] we get,

$$\bar{\phi}^+(x) = \frac{1}{2\pi i} P \int_{-a}^a \frac{\varphi(t)}{t-x} dt + \frac{1}{2} \varphi(x) \text{ in } (\mathcal{D}_L(\mathbb{R}))' \quad (2.36)$$

Similarly, letting $\text{Im } z \rightarrow 0-$ in (2.35) we get

$$\bar{\phi}^-(x) = \frac{1}{2\pi i} P \int_{-a}^a \frac{\varphi(t)}{t-x} dt - \frac{1}{2} \varphi(x) \text{ in } (\mathcal{D}_L(\mathbb{R}))' \quad (2.37)$$

From (2.36) and (2.37) we get

$$\begin{aligned} \bar{\phi}^+(x) + \bar{\phi}^-(x) &= \frac{1}{\pi i} P \int_{-a}^a \frac{\varphi(t)}{t-x} dt \\ &= \frac{1}{\pi i} H[\varphi](x) \\ &= \frac{\psi(x)}{\pi i} \end{aligned} \quad (2.38)$$

So $\psi(x)$ defined by (2.38) belongs to $H(\mathcal{D})$. Therefore, from

(2.35) $\rightarrow$ (2.38), it follows that

$$\bar{\phi}^+ - \bar{\phi}^- \in \mathcal{D}(\mathbb{R}) \Rightarrow \bar{\phi}^+ + \bar{\phi}^- \in H(\mathcal{D}) \quad (2.39)$$

The relation (2.39) therefore, characterizes the space $H(\mathcal{D})$ with respect to the analytic-function $\bar{\phi}(z)$ defined by (2.32). We

now state:

THEOREM (2.7)

For every element $\psi \in H(\mathbb{Q})$, there exists one and only one analytic function $\phi(z)$ analytic everywhere on the complex-plane except possibly on a compact sub-set K of $\mathbb{R}$ such that

$$(i) \quad \phi(z) = O\left(\frac{1}{|z|}\right), \quad |z| \rightarrow \infty.$$

(ii) Weak distributional limits ϕ^+ and ϕ^- of $\phi(x+iy)$ both exist as $y \rightarrow 0^+$ such that $\phi^+(x) - \phi^-(x) = \omega(x)$ where $\omega \in \mathcal{D}(\mathbb{R})$ and then, $\psi(x) = \frac{1}{\pi i} H(\omega)(x)$.

We now define the convergence in $H(\mathbb{Q})$ as follows:

A sequence $\{\psi_\mu\}_{\mu=1}^\infty$ in $H(\mathbb{Q})$ converges to zero in $H(\mathbb{Q})$ iff

(i) all the complex-functions $\phi_\mu(z)$ that generate $\psi_\mu(x)$ in accordance with theorem (2.7) are analytic outside a fixed interval $[-a, a]$. This enforces that the corresponding functions $\omega_\mu(x)$ satisfying

$$H\phi_\mu = \psi_\mu$$

have their supports contained in $[-a, a]$.

(ii) $\gamma_k(\psi_\mu) \rightarrow 0$ as $\mu \rightarrow \infty$ for each $k = 0, 1, 2, 3, \dots$

By (ii) we get

$$\begin{aligned} \gamma_k(\omega_\mu) &= \gamma_k(H\phi_\mu) \\ &\leq C_p \gamma_k(\psi_\mu) \rightarrow 0 \text{ as } \mu \rightarrow \infty. \end{aligned}$$

Therefore, for each $(k = 0, 1, 2, 3, \dots)$, $\varphi_{\mu}^{(k)} \rightarrow 0$ uniformly on $\mathbb{R}$ [28].

In view of (i), all of the φ_{μ} have the same compact support so that $\varphi_{\mu} \rightarrow 0$ in $\mathcal{D}(\mathbb{R})$. Therefore, the mode of convergence of the sequence $\{\varphi_{\mu}\}_{\mu=1}^{\infty}$ in $H(\mathcal{Q})$ to zero can be intrinsically defined by (i) and (ii).

REMARK (2.1)

In view of the inversion formula (1.17), it can be seen that H is a linear homeomorphism from $\mathcal{Q}$ onto $H(\mathcal{Q})$ and the mapping $H: H(\mathcal{Q}) \rightarrow \mathcal{D}(\mathbb{R})$ is linear and continuous.

(2.4) HILBERT TRANSFORM OF SCHWARTZ DISTRIBUTIONS

The generalized Hilbert-transform Hf of $f \in (\mathcal{Q}(\mathbb{R}))'$ is defined to be an ultradistribution $Hf \in (H(\mathcal{Q}))'$ such that

$$\langle Hf, \varphi \rangle = \langle f, -H\varphi \rangle \quad \forall \varphi \in H(\mathcal{Q}).$$

If $g \in (H(\mathcal{Q}))'$, its Hilbert-transform Hg is defined to be a Schwartz distribution by the relation

$$\langle Hg, \varphi \rangle = \langle g, -H\varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

The inversion formula (1.20) and the result (1.25) can be proved in the similar fashion for the space $(\mathcal{Q}(\mathbb{R}))'$.

(2.5) EXISTENCE AND UNIQUENESS OF THE SOLUTION TO DIRICHLET BOUNDARY VALUE PROBLEM

DEFINITION:

A harmonic-function $w(x,y)$ defined in the upper half plane

$\Omega_+ = \{z: \operatorname{Im} z > 0\}$ is said to belong to the space $\mathcal{H}$ iff

(i) (a) $w(x,y)$ for fixed $y > 0$ belongs to $L_q(\mathbb{R})$,

$$q = \frac{p}{p-1}, \quad p > 1$$

(b) $w(x,y)$ converges to a distributional limit in $(\mathcal{D}'_{L_p})'$

as $y \rightarrow 0+$.

$$(ii) \quad \sup_{\substack{-\infty < x < \infty \\ y \geq \delta > 0}} |w(x,y)| = A_\delta < \infty$$

(iii) $w(x,y) = o(1)$ as $y \rightarrow \infty$ uniformly with respect to x .

THEOREM (2.8)

Let $\mathcal{H}$ be the space of harmonic-functions defined above.

Assume that $U(x,y)$ and $V(x,y)$ are conjugate harmonic-functions

belonging to the space $\mathcal{H}$ and converging to the weak distributional

limits f and g respectively in $(\mathcal{D}'_{L_p}(\mathbb{R}))'$ as $y \rightarrow 0+$, then we

have:

$$\frac{1}{\pi} Hf = -g$$

$$\frac{1}{\pi} Hg = f$$

(2.40)

and

$$U(x,y) = \frac{1}{\pi} \left\langle f(t), \frac{y}{(t-x)^2 + y^2} \right\rangle \quad (2.41)$$

$$V(x,y) = \frac{1}{\pi} \left\langle f(t), \frac{(t-x)}{(t-x)^2 + y^2} \right\rangle \quad (2.42)$$

More clearly, U and V as defined by (2.41) and (2.42) are the only harmonic functions belonging to $\mathcal{H}$ satisfying

$$\begin{aligned} \lim_{y \rightarrow 0+} U(x,y) &= f \\ \lim_{y \rightarrow 0+} V(x,y) &= g = -\frac{1}{\pi} Hf \end{aligned} \quad (2.43)$$

PROOF:

Let U and V both belong to $\mathcal{H}$ and let f and g be their distributional limits respectively in $(\mathcal{D}'_L(\mathbb{R}))'$ as $y \rightarrow 0+$. Then the theorem (2.5) is applicable to the function $F(z) = U(x,y) + iV(x,y)$ ($y > 0$) such that

$$F(z) = \left\langle f(t) + ig(t), \frac{1}{2\pi i} \frac{1}{t-z} \right\rangle \quad (2.44)$$

$$\begin{aligned} \text{or } F(z) &= \frac{1}{2\pi i} \left\langle f(t) + ig(t), \frac{t-x}{(t-x)^2 + y^2} + i \frac{y}{(t-x)^2 + y^2} \right\rangle \\ &= \frac{1}{2\pi i} \left\langle f(t) + ig(t), \frac{t-x}{(t-x)^2 + y^2} \right\rangle + \frac{1}{2\pi} \left\langle f(t) + ig(t), \frac{y}{(t-x)^2 + y^2} \right\rangle \\ &= \frac{1}{2\pi i} H_y[f + ig] + \frac{1}{2} \left\langle f(t) + ig(t), \frac{1}{2\pi} \frac{y}{[(t-x)^2 + y^2]} \right\rangle \end{aligned} \quad (2.45)$$

where $H_y[f+ig]$ is the approximate Hilbert-transform of $f+ig$.

Letting $y \rightarrow 0+$ in (2.45) and using the theorems (1.5) and (1.6)

we note that

$$\begin{aligned} f+ig &= \frac{1}{2\pi i} H[f+ig] + \frac{1}{2} [f+ig] \\ \Rightarrow f+ig &= \frac{-1}{\pi} H[f+ig] . \end{aligned} \quad (2.46)$$

Since f and g are real functionals (i.e. functionals which assign real numbers to real valued functions of $\mathcal{D}_L(\mathbb{R})$) equating the real and imaginary parts in both sides of (2.46) we get the reciprocity relation (2.40).

Again equating the real and imaginary parts of both sides of (2.44) and using the definition of the Hilbert-transform of generalized-functions and the reciprocity relation (2.40) we get

$$U(x,y) = \frac{1}{\pi} \left\langle f(t), \frac{y}{(t-x)^2 + y^2} \right\rangle$$

$$V(x,y) = \frac{1}{\pi} \left\langle f(t), \frac{t-x}{(t-x)^2 + y^2} \right\rangle$$

The relation for $V(x,y)$ can be written in the form

$$V(x,y) = \frac{1}{\pi} \left\langle g(t), \frac{y}{(t-x)^2 + y^2} \right\rangle$$

It can easily be verified that U and V defined above are conjugate harmonic functions. An appeal to the structure formula for $f \in (\omega_L(\mathbb{R}))'_p$ and the result (2.18) shows that

$$U(x,y) = O\left[\frac{1}{y^{\frac{p-1}{p}}}\right], \quad y \rightarrow \infty$$

$$V(x,y) = O\left[\frac{1}{y^{\frac{p-1}{p}}}\right], \quad y \rightarrow \infty$$

uniformly for all $x \in \mathbb{R}$ and

$$\sup_{\substack{-\infty < x < \infty \\ y \geq \delta > 0}} |U(x,y)| = A_\delta < \infty$$

$$\sup_{\substack{-\infty < x < \infty \\ y \geq \delta > 0}} |V(x,y)| = B_\delta < \infty$$

Therefore, the functions U and V as defined above belong to the space $\mathcal{H}$. The uniqueness follows from COR(2.1).

EXAMPLE (2.1)

Consider the Dirichlet boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega_+$$

$$\lim_{y \rightarrow 0+} U(x,y) = -PV\left(\frac{1}{x}\right) \quad \text{in } (\omega_L(\mathbb{R}))'_p$$

The problem can be written as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \quad \text{in } \Omega_+$$

$$U(x, 0+) = H\delta$$

The obvious solution is:

$$\begin{aligned} U(x, y) &= \langle H\delta, \frac{y}{(t-x)^2 + y^2} \rangle \\ &= \langle \delta, \frac{t-x}{(t-x)^2 + y^2} \rangle \\ &= \frac{-x}{x^2 + y^2} \in \mathcal{H} \end{aligned}$$

(2.6) THE HILBERT PROBLEM FOR GENERALIZED FUNCTIONS

DESCRIPTION:

Let $g(x)$ be a given function of real variable x defined on the real line. Let $g^+(z)$ and $g^-(z)$ be analytic-functions holomorphic in the upper half plane ($\text{Im } z > 0$), and the lower half plane ($\text{Im } z < 0$) respectively. Assume that $g^+(x)$ and $g^-(x)$ are the limits of $g^+(z)$ and $g^-(z)$ respectively, i.e.

$$g^+(x) = \lim_{y \rightarrow 0+} g^+(x+iy) \quad (2.47)$$

$$g^-(x) = \lim_{y \rightarrow 0-} g^-(x+iy)$$

Our object is now to find analytic functions $g^+(z)$ and $g^-(z)$ as described before, satisfying the relation:

$$g^+(x) + g^-(x) = g(x). \quad (2.48)$$

In many problems of mathematical-physics a slightly more difficult problem appears viz.,

$$g^+(x) + k(x)g^-(x) = g(x) \quad (2.49)$$

where $k(x)$ is a known function defined on the real axis. By factorizing $k(x)$ as:

$$k(x) = k^+(x)k^-(x) \quad (2.50)$$

we can write the problem (2.49) as:

$$\frac{g^+(x)}{k^+(x)} + k^-(x)g^-(x) = \frac{g(x)}{k^+(x)} \quad (2.51)$$

However, the factorization (2.50) is also a problem of the form (2.48)

because, at least formally, the relation (2.50) can be written as:

$$\ln k(x) = \ln k^+(x) + \ln k^-(x) \quad (2.52)$$

In [16, pp. 157-166], Lauwerier has considered the simpler problem (2.48) and has called it Hilbert-problem. In [16, p. 158], Lauwerier gives the solution to the Hilbert-problem in the form:

$$\begin{aligned} g^+(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t-z} dt + P(z), \quad \text{Im } z > 0 \\ g^-(z) &= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t-z} dt - P(z), \quad \text{Im } z < 0 \end{aligned} \quad (2.53)$$

where $P(z)$ is an arbitrary polynomial and $g(x)$ is the given function.

We consider the Hilbert-problem (2.48) when g is the generalized function belonging to the generalized function space $(\mathcal{D}_{L_p}(\mathbb{R}))'$, $p > 1$.

In order to restrict the class of solutions of (2.48), we further

assume that $g^+(z)$ and $g^-(z)$ are of order $(|z|^m)$ as $|z| \rightarrow \infty$.

Solutions to the Hilbert-problem (2.48) when $g \in (\mathcal{D}_{L_p}(\mathbb{R}))'$ and the limits are being interpreted in the weak distributional sense (or in $(\mathcal{D}_{L_p}(\mathbb{R}))'$) can be written

$$\begin{aligned} g^+(z) &= \frac{1}{2\pi i} \left\langle g(t), \frac{1}{t-z} \right\rangle + P(z), \quad \text{Im } z > 0 \\ g^-(z) &= \frac{-1}{2\pi i} \left\langle g(t), \frac{1}{t-z} \right\rangle - P(z), \quad \text{Im } z < 0 \end{aligned} \quad (2.54)$$

where $P(z)$ is an arbitrary polynomial of order m and $\langle g(t), \frac{1}{t-z} \rangle$ is the Hilbert transform of g with complex parameter z .

Proof:

The analyticity of the functions $g^+(z)$ and $g^-(z)$ in the upper and lower half planes respectively follows from the proof of the theorem (2.1). As a matter of fact, $g^+(z)$ and $g^-(z)$ are analytic outside $\text{supp } g$. We can now write:

$$g^+(z) = \frac{1}{2\pi i} \langle g(t), \frac{t-x}{(t-x)^2 + y^2} \rangle + \frac{1}{2\pi} \langle g(t), \frac{y}{(t-x)^2 + y^2} \rangle + P(z),$$

$$y > 0. \quad (2.55)$$

However,

$$\frac{1}{2\pi i} \langle g(t), \frac{t-x}{(t-x)^2 + y^2} \rangle = \frac{1}{2\pi i} H_y[g] \rightarrow \frac{1}{2\pi i} H[g]$$

as $y \rightarrow 0+$ [By theorem (1.5)] and

$$\frac{1}{2\pi} \langle g(t), \frac{y}{(t-x)^2 + y^2} \rangle \rightarrow \frac{1}{2} g(x) \text{ as } y \rightarrow 0+$$

[By theorem (1.6)]. Therefore, letting $y \rightarrow 0+$ in (2.55) we get,

$$g^+(x) = \frac{1}{2\pi i} H[g](x) + \frac{1}{2} g(x) + P(x). \quad (2.56)$$

Similarly,

$$g^-(x) = \frac{-1}{2\pi i} H[g](x) + \frac{1}{2} g(x) - P(x). \quad (2.57)$$

Adding (2.56) and (2.57) we get

$$g^+(x) + g^-(x) = g(x).$$

CHAPTER 3

(3.1) INTRODUCTION

In this chapter we shall attempt to find the n -dimensional analogue of the results proved in chapter one. It seems difficult to extend the notion of the Hilbert transform and its inversion-formula to the spaces $\mathcal{D}_L(\mathbb{R}^n)$ and $(\mathcal{D}_L(\mathbb{R}^n))'$ directly, that is, if we define the Hilbert transform of the test function

$\varphi \in \mathcal{D}_{L_P}(\mathbb{R}^n)$ by

$$H[\varphi](x) \stackrel{\Delta}{=} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \frac{\varphi(t_1, t_2, \dots, t_n) dt}{\prod_{j=1}^n (t_j - x_j)} \quad (3.1)$$

$$\stackrel{\Delta}{=} \lim_{\substack{\max_j \epsilon_j \rightarrow 0+ \\ (1 \leq j \leq n)}} \int \dots \int \frac{\varphi(t_1, t_2, \dots, t_n) dt}{\prod_{j=1}^n (t_j - x_j)} \quad |t_j - x_j| > \epsilon_j$$

However, we shall introduce the space $X(\mathbb{R}^n)$ of test functions φ in $\mathcal{D}(\mathbb{R}^n)$

such that

$$\varphi = \sum_{\mu=1}^k (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}) \quad (3.2)$$

where $\varphi_{\mu,1}(x_1) \in \mathcal{D}(\mathbb{R})$ and k is finite.

We define the direct product $\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}$ by:

$$(\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(x) = \varphi_{\mu,1}(x_1) \cdot \varphi_{\mu,2}(x_2) \dots \varphi_{\mu,n}(x_n).$$

We will introduce the sub-space topology on $X(\mathbb{R}^n)$ and by using the Weierstrass approximation theorem [13, pp. 68-71], it will be proved that $X(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$. The Hilbert-transform of test functions in $X(\mathbb{R}^n)$ is then defined by:

$$[H\varphi](x) \triangleq \sum_{\mu=1}^k \int_{\mathbb{R}^n} \frac{(\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(t) dt}{\prod_{i=1}^n (t_i - x_i)} \quad (3.3)$$

It will be proved that the Hilbert-transform defined by (3.3) is a continuous embedding of $X(\mathbb{R}^n)$ into $\mathcal{D}_{L_p}(\mathbb{R}^n)$. Using the linearity property of the embedding H as well as the results proved in

[1, pp. 48-61] we will define the Hilbert-transform of the test-functions in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ by

$$[H\varphi](x) \triangleq \lim_{\mu \rightarrow \infty} [H\varphi_{\mu}](x) \quad (3.4)$$

where $\{\varphi_{\mu}\}_{\mu=1}^{\infty}$ is a sequence in $X(\mathbb{R}^n)$ converging to φ in the sense of convergence in $\mathcal{D}_{L_p}(\mathbb{R}^n)$. We will prove that the Hilbert-transform defined by (3.4) is a homeomorphism from $\mathcal{D}_{L_p}(\mathbb{R}^n)$ onto itself.

We will define the Hilbert-transform Hf of $f \in (\mathcal{S}_{L_p}(\mathbb{R}^n))'$ as a generalized-function in $(\mathcal{S}_{L_p}(\mathbb{R}^n))'$ by the relation:

$$\langle Hf, \varphi \rangle \stackrel{\Delta}{=} \langle f, (-1)^n H\varphi \rangle \quad \forall \varphi \in \mathcal{S}_{L_p}(\mathbb{R}^n) \quad (1 < p < \infty)^p \quad (3.5)$$

The formula (3.5) enables us to compute the Hilbert-transform of any $f \in (\mathcal{S}_{L_p}(\mathbb{R}^n))'$. In view of the inversion formula:

$$\left(-\frac{1}{2}\right)^n \frac{H^2 f}{\pi} = f \quad \forall f \in (\mathcal{S}_{L_p}(\mathbb{R}^n))', \quad p > 1 \quad (3.6)$$

which will be proved in the sequel, it becomes quite simple to apply our technique to solve some singular integral equations in $(\mathcal{S}_{L_p}(\mathbb{R}^n))'$.

To our knowledge, there does not exist any theory of Hilbert-transforms of distributions, including that of Horvath, [14, pp. 122-123], to solve singular integral equations and the operator equations involving the Hilbert-operator in the space $(\mathcal{S}_{L_p}(\mathbb{R}^n))'$. Applications of our results in solving Dirichlet boundary value problem in the space $\mathbb{R}^{n+1,+}$ will be shown.

(3.2) THE TEST SPACE $X(\mathbb{R}^n)$

The space $X(\mathbb{R}^n)$ consists of all test-functions

$\varphi(x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$ of the form:

$$\varphi = \sum_{\mu=1}^r (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}) \quad (3.6)$$

where $\varphi_{\mu,i} \in \mathcal{D}(\mathbb{R})$ ($\forall i = 1, 2, \dots, n$ and $\forall \mu = 1, 2, \dots, r$) and r is some finite positive integer. We introduce the relative topology on $X(\mathbb{R}^n)$ induced by the topology of the Schwartz-testing function space $\mathcal{D}(\mathbb{R}^n)$.

LEMMA (3.1):

The test space $X(\mathbb{R}^n)$ is dense in $\mathcal{D}(\mathbb{R}^n)$.

PROOF:

Let $\varphi(x_1, x_2, \dots, x_n) \in \mathcal{D}(\mathbb{R}^n)$ have support contained in

$$B(a) = \{x \in \mathbb{R}^n : |x_i| \leq a, 1 \leq i \leq n\}, \quad a > 0$$

By Weierstrass approximation theorem [13, pp. 68-71], for $\epsilon = 1/m$

there exists a polynomial $P_m(x_1, x_2, \dots, x_n)$ such that in

$B(2a) = \{x \in \mathbb{R}^n : |x_i| \leq 2a, 1 \leq i \leq n\}$; it differs from

$\varphi(x_1, x_2, \dots, x_n)$ by less than ϵ together with its derivatives up to order m .

Further, let $e(t)$ ($-\infty < t < \infty$) be fixed test-function in $\mathcal{D}(\mathbb{R})$ equal to unity for $t \leq a$ and zero for $t \geq 2a$, Zemanian [36, p.32].

Then as $\epsilon \rightarrow 0$, the sequence $\{P_m(x) \cdot e(x_1) \cdot e(x_2) \dots e(x_n)\}_{m=1}^{\infty}$

converges to $\varphi(x)$ in $\mathcal{D}(\mathbb{R}^n)$. Thus $X(\mathbb{R}^n)$ is dense in $\mathcal{D}(\mathbb{R}^n)$.

COR (3.1):

$X(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$.

PROOF:

Let $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ [10, p. 87], there exists a sequence $\{\psi_\mu\}_{\mu=1}^\infty$ in $\mathcal{D}(\mathbb{R}^n)$ such that

$$\psi_\mu \xrightarrow{\mathcal{D}_{L_p}(\mathbb{R}^n)} \varphi \text{ as } \mu \rightarrow \infty. \quad (3.7)$$

$X(\mathbb{R}^n)$ is dense in $\mathcal{D}(\mathbb{R}^n)$ by Lemma (3.1) therefore, for each fixed μ there exists $\varphi_{\mu,\nu} \in X(\mathbb{R}^n)$ such that

$$\varphi_{\mu,\nu} \xrightarrow{\mathcal{D}(\mathbb{R}^n)} \psi_\mu \text{ as } \nu \rightarrow \infty.$$

However, the convergence in $\mathcal{D}(\mathbb{R}^n)$ implies the convergence in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ [10, p.87], it follows that

$$\varphi_{\mu,\nu} \xrightarrow{\mathcal{D}_{L_p}(\mathbb{R}^n)} \psi_\mu \text{ as } \nu \rightarrow \infty. \quad (3.8)$$

Now we have the following situation.

$$\begin{array}{ccc}
 \varphi_{1,\nu} & \xrightarrow{\mathfrak{A}_{L_p}(\mathbb{R}^n)} & \psi_1 \\
 \varphi_{2,\nu} & \xrightarrow{\quad\quad\quad} & \psi_2 \\
 \varphi_{3,\nu} & \xrightarrow{\quad\quad\quad} & \psi_3 \\
 \vdots & & \vdots \\
 \varphi_{\mu,\nu} & \xrightarrow{\mathfrak{A}_{L_p}(\mathbb{R}^n)} & \psi_{\mu} \\
 \vdots & & \vdots \\
 \varphi_{\mu} & \xrightarrow{\mathfrak{A}_{L_p}(\mathbb{R}^n)} & \varphi \text{ as } \mu \rightarrow \infty.
 \end{array}$$

and

Since the space $\mathfrak{A}_{L_p}(\mathbb{R}^n)$ is metrizable with metric defined by:

$$\rho(\varphi, \psi) = \sum_{m=1}^{\infty} 2^{-m} \frac{\gamma_m(\varphi - \psi)}{1 + \gamma_m(\varphi - \psi)}$$

Therefore, by [28ap.8] there exists a sub-sequence $(\varphi_{\mu_i, \nu_i})_{i=1}^{\infty}$ in $X(\mathbb{R}^n)$ such that,

$$\varphi_{\mu_i, \nu_i} \xrightarrow{\mathfrak{A}_{L_p}(\mathbb{R}^n)} \varphi \text{ as } i \rightarrow \infty.$$

Thus $X(\mathbb{R}^n)$ is dense in $\mathfrak{A}_{L_p}(\mathbb{R}^n)$.

(3.3) THE HILBERT TRANSFORM OF TEST FUNCTIONS IN $X(\mathbb{R}^n)$

Let $\varphi = \sum_{\mu=1}^r (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})$ be a test function

in $X(\mathbb{R}^n)$. We define the Hilbert-transform $H\varphi$ of φ on $\mathbb{R}^n$ by:

$$[H\varphi](x) \triangleq \sum_{\mu=1}^r \int_{\mathbb{R}^n} \frac{(\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(t) dt}{(t_1 - x_1)(t_2 - x_2) \dots (t_n - x_n)}$$

$$\begin{aligned}
 &\triangleq \sum_{\mu=1}^r \lim_{\max(\epsilon_1, \dots, \epsilon_n) \rightarrow 0} \int \dots \int \frac{(\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(t) dt}{(t_1 - x_1)(t_2 - x_2) \dots (t_n - x_n)} \\
 &\quad \quad \quad |t_1 - x_1| > \epsilon_1 \\
 &\quad \quad \quad |t_2 - x_2| > \epsilon_2 \\
 &\quad \quad \quad \vdots \\
 &\quad \quad \quad |t_n - x_n| > \epsilon_n
 \end{aligned}$$

(3.12)

Let us denote by $H_j[\varphi_{\mu,j}] = \hat{\varphi}_{\mu,j}$, the classical Hilbert-transform of $\varphi_{\mu,j}$ defined by (1.1). Then, it follows from (3.12) that

$$\begin{aligned} [H\varphi](x) &\triangleq \sum_{\mu=1}^r (H_1(\varphi_{\mu,1}) \otimes H_2(\varphi_{\mu,2}) \otimes \dots \otimes H_n(\varphi_{\mu,n}))(x) \\ &\triangleq \sum_{\mu=1}^r (\hat{\varphi}_{\mu,1} \otimes \hat{\varphi}_{\mu,2} \otimes \dots \otimes \hat{\varphi}_{\mu,n})(x). \end{aligned} \quad (3.13)$$

Since $\hat{\varphi}_{\mu,j} \in \mathfrak{L}_p(\mathbb{R})$ ($\forall \mu = 1, 2, \dots, r$ and $\forall j = 1, 2, \dots, n$)

[Theorem (1.2)], it follows from (3.13) that:

$$H\varphi \in \mathfrak{L}_p(\mathbb{R}^n) \text{ for all } \varphi \in X(\mathbb{R}^n).$$

Thus the Hilbert transform, as defined in (3.12) is a linear mapping of $X(\mathbb{R}^n)$ into $\mathfrak{L}_p(\mathbb{R}^n)$ i.e.

$$X(\mathbb{R}^n) \xrightarrow{H} \mathfrak{L}_p(\mathbb{R}^n).$$

Applying the operator H on both sides of (3.12) we get

$$[H^2\varphi](t) = \sum_{\mu=1}^r ((\hat{\hat{\varphi}}_{\mu,1} \otimes \hat{\hat{\varphi}}_{\mu,2} \otimes \dots \otimes \hat{\hat{\varphi}}_{\mu,n})(t)).$$

However,

$$\hat{\hat{\varphi}}_{\mu,j} = \left(-\frac{1}{\pi^2}\right) \varphi_{\mu,j} \quad [\text{By (1.17)}].$$

Therefore,

$$\begin{aligned} [H^2 \varphi](t) &= \left(-\frac{1}{2}\right)^n \sum_{\mu=1}^r ((\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n})(t)) \\ &= \left(-\frac{1}{2}\right)^n \varphi(t) \quad \forall \varphi \in X(\mathbb{R}^n) \end{aligned} \quad (3.14)$$

To prove the continuity of the map H , assume that

$$\varphi_{\mu} \xrightarrow{X(\mathbb{R}^n)} 0 \quad \text{as } \mu \rightarrow \infty.$$

Since the convergence in $X(\mathbb{R}^n)$ implies the convergence in $\mathfrak{d}_{L_p}(\mathbb{R}^n)$, it follows that

$$\varphi_{\mu} \xrightarrow{\mathfrak{d}_{L_p}(\mathbb{R}^n)} 0 \quad \text{as } \mu \rightarrow \infty. \quad (3.15)$$

Suppose that,

$$\varphi_{\mu} = \sum_{i=1}^r (\varphi_{\mu,i,1} \otimes \varphi_{\mu,i,2} \otimes \dots \otimes \varphi_{\mu,i,n})$$

where $\varphi_{\mu,i,j} \in \mathfrak{d}(\mathbb{R}) \quad \forall \mu, i, \& j$.

The convergence in (3.15) to zero now implies that $(\forall |k| = 0, 1, 2, 3, \dots)$

$$\gamma_k(\varphi_{\mu}) = \sum_{i=1}^r (\gamma_{k_1}(\varphi_{\mu,i,1}) \otimes \gamma_{k_2}(\varphi_{\mu,i,2}) \otimes \dots \otimes \gamma_{k_n}(\varphi_{\mu,i,n})) \rightarrow 0,$$

as $\mu \rightarrow \infty$.

(3.16)

Since the map $\varphi \rightarrow \hat{\varphi}$ is a homeomorphism from $\mathcal{D}_{L_p}(\mathbb{R})$ onto itself Theorem [1.2] and

$$\gamma_{k_j}(\varphi_{\mu,i,j}) \leq C_{p,j} \gamma_{k_j}(\varphi_{\mu,i,j}) \quad [31, p.132].$$

Therefore, $(\forall |k| = 0, 1, 2, 3, \dots)$.

$$\gamma_k(\varphi_{\mu}) \leq C_{p,1} \cdot C_{p,2} \cdot \dots \cdot C_{p,n} \gamma_k(\varphi_{\mu}) \rightarrow 0, \text{ as } \mu \rightarrow \infty. \quad (3.17)$$

As a matter of fact, we have proved the following

THEOREM (3.1):

Let H be a mapping defined by (3.12). Then H is a linear continuous injection of $X(\mathbb{R}^n)$ into $\mathcal{D}_{L_p}(\mathbb{R}^n)$.

(3.4) THE HILBERT TRANSFORM OF FUNCTIONS IN $\mathcal{D}_{L_p}(\mathbb{R}^n)$

We are now able to define the Hilbert-transform of test-functions in $\mathcal{D}_{L_p}(\mathbb{R}^n)$. Consider the following diagram.

Since $X(\mathbb{R}^n)$ is dense in $\mathcal{D}_{L_p}(\mathbb{R}^n)$ therefore, the continuous injection $H: X(\mathbb{R}^n) \longrightarrow \mathcal{D}_{L_p}(\mathbb{R}^n)$ extends uniquely to the whole space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ by the limiting process [1, pp. 48-61] as follows:

For $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$, there exists a sequence $\{\varphi_\mu\}_{\mu=1}^\infty$ in $X(\mathbb{R}^n)$ such that,

$$\varphi_\mu \xrightarrow{\mathcal{D}_{L_p}(\mathbb{R}^n)} \varphi \quad \text{as } \mu \rightarrow \infty.$$

We define $H\varphi$ on $\mathbb{R}^n$ by the relation:

$$[H\varphi](x) = \lim_{\mu \rightarrow \infty} [H\varphi_\mu](x). \quad (3.18)$$

The existence of the limit (3.18) follows from the continuity of the operator H and the completeness of $\mathcal{D}_{L_p}(\mathbb{R}^n)$. To prove the uniqueness of the limit, assume that there exist two sequences $\{\varphi_\mu\}_{\mu=1}^\infty$ and $\{\varphi_\nu\}_{\nu=1}^\infty$ in $X(\mathbb{R}^n)$ which converge to φ in $\mathcal{D}_{L_p}(\mathbb{R}^n)$. Then, by definition, we have $(\forall |k| = 0, 1, 2, 3, \dots)$

$$\gamma_k(\varphi - \varphi_\mu) \rightarrow 0 \quad \text{as } \mu \rightarrow \infty \quad (3.19)$$

$$\text{and } \gamma_k(\varphi - \varphi_\nu) \rightarrow 0 \quad \text{as } \nu \rightarrow \infty.$$

Now, using the inequalities (3.17) as well as the triangular inequality of the semi-norms we get

$$\begin{aligned}
\gamma_k(H\varphi_\mu - H\varphi_\nu) &= \gamma_k(H(\varphi_\mu - \varphi_\nu)) \\
&\leq C_{p,1} \cdot C_{p,2} \cdot \dots \cdot C_{p,n} \gamma_k(\varphi_\mu - \varphi_\nu) \\
&\leq C_{p,1} \cdot C_{p,2} \cdot \dots \cdot C_{p,n} [\gamma_k(\varphi_\mu - \varphi) + \gamma_k(\varphi - \varphi_\nu)] \rightarrow 0
\end{aligned}$$

as $\mu \rightarrow \infty$ for all $|k| = 0, 1, 2, 3, \dots$.

Thus the limit (3.18) exists and is unique.

As a consequence of (3.12)-(3.19) we have now proved

THEOREM (3.2):

Let H be the mapping defined by (3.18). Then for $1 < p < \infty$, H is linear homeomorphism from $\mathfrak{D}_{L_p}(\mathbb{R}^n)$ onto itself and

$$H^{-1} = \left(-\frac{1}{2}\right)^n H \quad (3.20)$$

THEOREM (3.3):

Let $\varphi \in \mathfrak{D}_{L_p}(\mathbb{R}^n)$, then $D^k[H\varphi] = H[D^k\varphi]$ for all $|k| = 0, 1, 2, 3, \dots$.

PROOF:

We will prove the result for the test-functions in $X(\mathbb{R}^n)$ and the result for the test-functions in $\mathfrak{D}_{L_p}(\mathbb{R}^n)$ can be proved by the limiting process since D^k is also a continuous mapping of $\mathfrak{D}_{L_p}(\mathbb{R}^n)$ into $\mathfrak{D}_{L_p}(\mathbb{R}^n)$.

Let

$$\varphi = \sum_{\mu=1}^r (\varphi_{\mu,1} \otimes \varphi_{\mu,2} \otimes \dots \otimes \varphi_{\mu,n}) \in X(\mathbb{R}^n).$$

Then

$$\begin{aligned} D^k[H\varphi](x) &= \sum_{\mu=1}^r \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \left([H_1(\varphi_{\mu,1})] \otimes [H_2(\varphi_{\mu,2})] \otimes \dots \otimes [H_n(\varphi_{\mu,n})] \right) (x) \\ &= \sum_{\mu=1}^r \left(\frac{\partial^{k_1}}{\partial x_1^{k_1}} [H_1(\varphi_{\mu,1})] \otimes \frac{\partial^{k_2}}{\partial x_2^{k_2}} [H_2(\varphi_{\mu,2})] \otimes \dots \otimes \frac{\partial^{k_n}}{\partial x_n^{k_n}} [H_n(\varphi_{\mu,n})] \right) (x) \end{aligned}$$

However,

$$\frac{\partial^{k_1}}{\partial x_1^{k_1}} [H_1(\varphi_{\mu,1})(x_1)] = H_1^{(k_1)}[\varphi_{\mu,1}(x_1)] \quad [\text{By (1.15)}]$$

$$\forall i = 1, 2, 3, \dots, n.$$

Therefore,

$$\begin{aligned} D^k[H\varphi](x) &= \sum_{\mu=1}^r (H_1^{(k_1)}[\varphi_{\mu,1}] \otimes H_2^{(k_2)}[\varphi_{\mu,2}] \otimes \dots \otimes H_n^{(k_n)}[\varphi_{\mu,n}]) (x) \\ &= H \left(\sum_{\mu=1}^r (\varphi_{\mu,1}^{(k_1)} \otimes \varphi_{\mu,2}^{(k_2)} \otimes \dots \otimes \varphi_{\mu,n}^{(k_n)}) (x) \right) \\ &= H[D^k\varphi] \quad \forall \varphi \in X(\mathbb{R}^n). \end{aligned} \tag{3.21}$$

This completes the proof of the theorem.

(3.5) HILBERT TRANSFORM OF GENERALIZED FUNCTIONS IN $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$

If $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$, then the Hilbert transform Hf of f is defined by the relation:

$$\langle Hf, \varphi \rangle = \frac{1}{i} \langle f, (-1)^n H_p \varphi \rangle \quad \forall \varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n) \quad (3.22)$$

where H_p is the Hilbert transform φ defined by (3.18). In other words, we can say that the Hilbert-transform Hf of a generalized-function $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ is a generalized-function in $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ which assigns the same number to an element $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ as f assigns to $(-1)^n H_p \varphi$ for all $\varphi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$, ($1 < p < \infty$).

Note that the functional Hf defined by (3.22) belongs to $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ in view of the results proved by Zemanian [37, pp. 25-31].

In view of the inversion formula, (3.20) and the relation (3.21) we have:

COR (3.2):

The operator $H: (\mathcal{D}_{L_p}(\mathbb{R}^n))' \rightarrow (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ ($1 < p < \infty$)

defined as in (3.22) is an isomorphism and

$$\left(-\frac{1}{\pi}\right)^n H^2 f = f \quad \forall f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))' \quad (3.23)$$

Let us, find the Hilbert-transform of the singular-distribution

$\delta = \delta(x_1, x_2, \dots, x_n)$ in $(\mathcal{D}_L(\mathbb{R}^n))'$. By (3.22) we have

$$\langle H\delta, \varphi \rangle = \langle \delta(x), (-1)^n H\varphi \rangle.$$

Let

$$\varphi_\mu = \sum_{i=1}^r (\varphi_{\mu,i,1} \otimes \varphi_{\mu,i,2} \otimes \dots \otimes \varphi_{\mu,i,n})$$

be a sequence in $X(\mathbb{R}^n)$ such that

$$\varphi_\mu \xrightarrow[\mathcal{D}_L(\mathbb{R}^n)]{P} \varphi \text{ as } \mu \rightarrow \infty.$$

Then, by (3.18) we get

$$\begin{aligned} \langle H\delta, \varphi \rangle &= \lim_{\mu \rightarrow \infty} (-1)^n \sum_{i=1}^r \langle \delta(x), (\hat{\varphi}_{\mu,i,1} \otimes \hat{\varphi}_{\mu,i,2} \otimes \dots \otimes \hat{\varphi}_{\mu,i,n})(x) \rangle \\ &= \lim_{\mu \rightarrow \infty} (-1)^n \sum_{i=1}^r P \int_{\mathbb{R}^n} \frac{(\varphi_{\mu,i,1} \otimes \varphi_{\mu,i,2} \otimes \dots \otimes \varphi_{\mu,i,n})(t)}{t_1 \cdot t_2 \dots t_n} \\ &= (-1)^n \langle PV(\frac{1}{t_1 t_2 \dots t_n}), \lim_{\mu \rightarrow \infty} \sum_{i=1}^r (\varphi_{\mu,i,1} \otimes \varphi_{\mu,i,2} \otimes \dots \otimes \varphi_{\mu,i,n})(t) \rangle \\ &= (-1)^n \langle PV(\frac{1}{t_1 t_2 \dots t_n}), \varphi \rangle \quad \forall \varphi \in \mathcal{D}_L(\mathbb{R}^n). \end{aligned}$$

Therefore,

$$[H\delta](t) = (-1)^n PV(\frac{1}{t_1 t_2 \dots t_n}) \quad (3.24)$$

and by using the inversion formula (3.23) we get

$$\left(H \left[FV \left(\frac{1}{t_1 t_2 \dots t_n} \right) \right] \right) = (\pi^2)^n \delta(x_1, x_2, \dots, x_n). \quad (3.25)$$

DEFINITION:

Let $f \in (\mathcal{O}_{L_P}(\mathbb{R}^n))'$, then we define the generalized function,

$D^k f \in (\mathcal{O}_{L_P}(\mathbb{R}^n))'$ by:

$$\begin{aligned} \langle D^k f, \varphi \rangle &= (-1)^{|k|} \langle f, D^k \varphi \rangle \quad \forall \varphi \in \mathcal{O}_{L_P}(\mathbb{R}^n) \\ &(|k| = 1, 2, 3, \dots) \end{aligned}$$

THEOREM (3.4):

If $f \in (\mathcal{O}_{L_P}(\mathbb{R}^n))'$, then $D^k[Hf] = H[D^k f]$, $\forall |k| = 0, 1, 2, 3, \dots$.

PROOF:

For $\varphi \in \mathcal{O}_{L_P}(\mathbb{R}^n)$ we have

$$\begin{aligned} \langle H[D^k f], \varphi \rangle &= \langle D^k f, (-1)^n H\varphi \rangle \\ &= (-1)^{|k|} \langle f, (-1)^n D^k[H\varphi] \rangle \\ &= (-1)^{|k|} \langle f, (-1)^n H[D^k \varphi] \rangle \quad [\text{By (3.21)}] \\ &= (-1)^{|k|} \langle Hf, D^k \varphi \rangle \\ &= \langle D^k[Hf], \varphi \rangle \quad \forall \varphi \in \mathcal{O}_{L_P}(\mathbb{R}^n) \end{aligned}$$

Therefore,

$$H[D^k f] = D^k[Hf] \quad \forall f \in (\mathcal{O}_{L_P}(\mathbb{R}^n))'.$$

(3.6) APPLICATIONS TO INTEGRAL EQUATIONSEXAMPLE (3.1):

Let us find the distributional solution to the integral equation:

$$H[y] = Hf(t) + \delta(x_1, x_2, \dots, x_n) \quad (3.26)$$

in the space $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ for $f \in L_q(\mathbb{R}^n)$, where $p, q > 1$ and $1/p + 1/q = 1$, the expression $Hf(t)$ is the same as that defined in (3.4).

The relation (3.25) can be written as in chapter 2 form as:

$$H[y] = Hf + \delta(x_1, \dots, x_n) \quad (3.27)$$

operating on both sides by the operator H and using (3.23)-(3.24) we get

$$\begin{aligned} (-\pi^2)^n y &= (-\pi^2)^n f + (-1)^n \text{PV} \left(\frac{1}{t_1 \cdot t_2 \cdot \dots \cdot t_n} \right) \\ &= y = f + \left(\frac{1}{\pi^2} \right)^n \text{PV} \left(\frac{1}{t_1 \cdot t_2 \cdot \dots \cdot t_n} \right) \in (\mathcal{D}_{L_p}(\mathbb{R}^n))' \end{aligned}$$

is the distributional solution to (3.26).

EXAMPLE (3.2):

Let us consider the solution to the integral-equation (operator equation)

$$y(x) = Hy + f(x) \quad (3.28)$$

where $f \in (\omega_{L_p}(\mathbb{R}^n))'$, $p > 1$, where H is as defined in (3.4).

Operating on both sides of (3.28) by H we get

$$\begin{aligned} Hy &= H^2 y + Hf \\ \Rightarrow y - f &= (-\pi^2)^n y + Hf \\ \Rightarrow [1 - (-\pi^2)^n] y &= f + Hf \\ \Rightarrow y &= \frac{1}{[1 - (-\pi^2)^n]} [f + Hf] \in (\omega_{L_p}(\mathbb{R}^n))' \end{aligned}$$

EXAMPLE (3.3):

Let $P(D)$ be a polynomial in the operator D defined by (1.1).

Let us find the distribution $y(t)$ satisfying the differential equation

$$P(D)y = H[P(D)\delta] \text{ in } (\omega_{L_p}(\mathbb{R}^n))', \quad 1 < p < \infty.$$

The given differential equation can be written as

$$P(D)[y - H\delta] = 0 \quad (3.29)$$

22
OF / DE

Since the only polynomial in $(\mathcal{O}_L(\mathbb{R}^n))'$, $p > 1$, is the zero polynomial [10], therefore,

$$y - H\delta = 0$$

$$\Rightarrow y = (-1)^n \text{PV} \left(\frac{1}{t_1 \cdot t_2 \cdot \dots \cdot t_n} \right).$$

(3.7) GENERALIZED DIRICHLET BOUNDARY VALUE PROBLEMS

Let $\mathbb{R}^{n+1,+} = \{(x,y): x \in \mathbb{R}^n \text{ and } y > 0\}$. We call $U(x,y)$ a harmonic-function in an open region of $\mathbb{R}^{n+1,+}$, if it is infinitely differentiable at each point of the region and it satisfies:

$$U_{yy}(x,y) + \sum_{i=1}^n U_{x_i x_i}(x,y) = 0. \quad (3.30)$$

NOTE:

Our definition of harmonic-functions is quite different from the one given in [3, p. 148] of n -harmonic-functions.

Now, consider the Poisson kernel for $\mathbb{R}^{n+1,+}$ given by

$$P_y(x) = \frac{1}{C_n} \frac{y}{[y^2 + |x|^2]^{(n+1)/2}}, \quad y > 0$$

where

$$C_n = \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}.$$

It has the following properties:

$$(3.31) \quad P_y(x) > 0 \quad \forall y > 0$$

$$(3.32) \quad \int_{\mathbb{R}^n} P_y(x) dx = 1 \quad \text{for each } y > 0$$

$$(3.33) \quad U_f(x,y) \text{ defined by:}$$

$$U_f(x,y) = \int_{\mathbb{R}^n} f(x-t) P_y(t) dt,$$

is harmonic in the space $\mathbb{R}^{n+1,+}$ and

$$(a) \quad \|U_f(x,y)\|_{L_q} \leq \|f(x)\|_{L_q}, \quad y > 0$$

$$(b) \quad U_f(x,y) \text{ converges to } f(x) \text{ in the norm of } L_q(\mathbb{R}^n), \text{ as } y \rightarrow 0+.$$

Proofs of (3.31)-(3.33) are to be found in [34, pp. 25-62].

THEOREM (3.5):

Let $f \in (\mathcal{O}_{L_p}(\mathbb{R}^n))'$ ($1 < p < \infty$). Then, for all $\varphi \in \mathcal{O}_{L_p}(\mathbb{R}^n)$

$$\langle \langle f(t), P_y(x-t) \rangle, \varphi(x) \rangle \rightarrow \langle f, \varphi \rangle \quad \text{as } y \rightarrow 0+$$

$$\text{i.e. } \langle f(t), P_y(x-t) \rangle \rightarrow f(x) \text{ in } (\mathcal{O}_{L_p}(\mathbb{R}^n))' \text{ as } y \rightarrow 0+.$$

PROOF:

Using the structure formula (1.13) for $f \in (\mathcal{O}_{L_p}(\mathbb{R}^n))'$ we have

$$I(y) = \langle \langle f(t), P_y(x-t) \rangle, \varphi(x) \rangle = \sum_{|\alpha|=0}^r (-1)^{|\alpha|} \langle \langle f_{|\alpha|}(t), D_t^\alpha P_y(x-t) \rangle, \varphi(x) \rangle \quad (3.34)$$

where $f_{|\alpha|}$'s are in $L_q(\mathbb{R}^n)$.

However,

$$\frac{\partial}{\partial t_1} P_y(x-t) = \frac{-\partial}{\partial x_1} P_y(x-t) = \frac{1}{C_n} \cdot \frac{-(n+1)(x_1 - t_1)y}{(y^2 + |x-t|^2)^{(n+3)/2}}$$

or more generally,

$$D_t^\alpha [P_y(x-t)] = (-1)^{|\alpha|} D_x^\alpha P_y(x-t). \quad (3.35)$$

Therefore, we can write (3.34) as

$$I(y) = \sum_{|\alpha|=0}^r \langle \langle f_{|\alpha|}(t), D_x^\alpha P_y(x-t) \rangle, \varphi(x) \rangle.$$

Using Fubini's theorem and the integration by parts we get:

$$\begin{aligned} I(y) &= \sum_{|\alpha|=0}^r \langle f_{|\alpha|}(t), \langle P_y(x-t), \varphi^{(\alpha)}(x) \rangle \rangle \\ &= \sum_{|\alpha|=0}^r \langle f_{|\alpha|}(t), \int_{\mathbb{R}^n} P_y(x-t) \varphi^{(\alpha)}(x) dx \rangle \quad (3.36) \end{aligned}$$

Let

$$U_\varphi(x, y) = \int_{\mathbb{R}^n} P_y(x-t) \varphi^{(\alpha)}(x) dx.$$

Then, it follows from (3.33) that for each fixed $y > 0$, $U_\varphi(x, y)$ belongs to $C^\infty(\mathbb{R}^{n+1}, +)$ and

$$\|U_\varphi(x, y)\|_{L_p} \leq \|\varphi^{(\alpha)}(x)\|_{L_q} < \infty, \quad y > 0 \quad (3.37)$$

and

$$\|U_\varphi(x, y) - \varphi^{(\alpha)}(x)\|_{L_q} \rightarrow 0 \quad \text{as } y \rightarrow 0+ \quad (3.38)$$

However,

$$D_t^\beta U_\varphi(t, y) = \int_{\mathbb{R}^n} P_y(x-t) \varphi^{(\alpha+\beta)}(x) dx \quad (\forall |\beta| = 0, 1, 2, 3, \dots)$$

Therefore,

$$\gamma_\beta(U_\varphi(t, y) - \varphi^{(\alpha)}(t)) \rightarrow 0 \quad \text{as } y \rightarrow 0+ \quad \forall |\beta| = 0, 1, 2, 3, \dots$$

Thus,

$$U_\varphi(t, y) \xrightarrow{\mathcal{D}_L(\mathbb{R}^n)} \varphi^{(\alpha)}(t) \quad \text{as } y \rightarrow 0+,$$

$$\text{i.e.} \quad \int_{\mathbb{R}^n} P_y(x-t) \varphi^{(\alpha)}(x) dx \rightarrow \varphi^{(\alpha)}(t) \quad \text{as } y \rightarrow 0+. \quad (3.39)$$

From (3.36) and (3.39), we get,

$$\begin{aligned} \lim_{y \rightarrow 0+} I(y) &= \sum_{|\alpha|=0}^{\infty} \langle f_{|\alpha|}(t), \varphi^{(\alpha)}(t) \rangle \\ &= \langle f, \varphi \rangle \quad \forall f \in \mathcal{D}_L(\mathbb{R}^n). \end{aligned}$$

Therefore,

$$\langle f(t), P_y(x-t) \rangle \rightarrow f(x) \text{ as } y \rightarrow 0^+ . \quad (3.40)$$

EXAMPLE (3.4):

Consider the Dirichlet boundary value problem:

$$U_{yy}(x,y) + \sum_{i=1}^n U_{x_i x_i}(x,y) = 0, \quad y > 0 \quad (3.41)$$

$$\lim_{y \rightarrow 0^+} U(x,y) = U(x,0^+) = f(x) \text{ in } (\omega_{L_p}(\mathbb{R}^n))' \quad (3.42)$$

Let us define a function $U(x,y)$ by:

$$U(x,y) = \langle f(t), P_y(x-t) \rangle, \quad y > 0$$

Using the structure formula (1.13) for f we get

$$\begin{aligned} U(x,y) &= \sum_{|\alpha|=0}^r (-1)^{|\alpha|} \langle f_{|\alpha|}(t), D_t^\alpha P_y(x-t) \rangle, \\ &= \sum_{|\alpha|=0}^r (-1)^{|\alpha|} U_\alpha(x,y) \end{aligned}$$

where

$$U_\alpha(x,y) = \langle f_{|\alpha|}(t), D_t^\alpha P_y(x-t) \rangle.$$

According to (3.33), each $U_\alpha(x,y)$ is harmonic in $\mathbb{R}^{n+1,+}$, therefore, $U(x,y)$ is harmonic. In view of the theorem (3.5) $U(x,y)$ satisfies the boundary condition (3.42).

BIBLIOGRAPHY

1. Akhiezer, N.I. and Glazman, I.M. Theory of Linear operators in Hilbert spaces, Vols. I and II, Ungar, New York, 1963-1965.
2. Beltrami, E.J. and Wohlers, M.R. Distributional boundary values theorems and Hilbert transforms. Arch. Rational Mech., Anal., Vol. 18 (1965), 304-309.
3. Bremermann, H.J. Distributions, Complex Variables, and Fourier transforms, Addison Wesley, 1965.
4. Bremermann, H.J. On analytic continuation, multiplication and Fourier Transformations of Schwartz Distributions. J. Math. Phys. 2, 240-258, (1961).
5. Bruijn, N.G. de. A theory of generalized functions, with applications to Wigner distributions and Weyl correspondence. Nieuw Archief von Wiskunde (3) 21 (1973), 205-280.
6. Butzer, P.L. and Trebels, W. (1968). Hilbert transforms, fractional integration and differentiation. Bull. Am. Math. Soc. 74, 106-110.
7. Calderon, A.P. (1950). Singular Integrals. Bull. Am. Math. Soc. 72, 427-465.

8. Calderon, A.P. and Zygmund, A. (1956). On Singular Integrals.
Am. J. Math. 78, 289-309.
9. Cotlar, M. A unified theory of Hilbert transforms and ergodic theorems. Rev. Mat. Cuyana 1, 105-167.
10. Friedman, A. Generalized Functions and Partial Differential Equations, Prentice-Hall, Englewood Cliffs, N.J. (1963).
11. Gelfand, I.M. and Shilov, G.E. Generalized Functions, Vols. 1 and 3. Academic Press, New York, 1964, 1967.
12. Gelfand, I.M. and Shilov, G.E. Generalized Functions, Vol. 2, Moscow, 1958.
13. Gunning, Robert C., and Rossi, H. Analytic functions of several complex-variables. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1965).
14. Horvath, J. Hilbert transforms of distributions in $\mathbb{R}^n$. Proc. International Congress of Mathematics, Amsterdam, 1954, pp. 122-123.
15. Jones, D.S. Generalized Functions. McGraw-Hill Publishing Company, (1966).
16. Lauwerier, H.A. The Hilbert problem for generalized functions. Arch., Rational Mech. Anal., 13 (1963), pp. 157-166.

17. Mitrovic, D. A Hilbert distributional boundary value problem.
Mathematica Balkanica I (1971), pp. 177-180.
18. Mitrovic, D. A distributional representation of analytic functions. Mathematica Balkanica 479 (1974), pp. 437-440.
19. Mitrovic, D. Some distributional boundary-value problems.
Mathematica Balkanica 2 (1972), pp. 161-164.
20. Newcomb, W.R. Hilbert transforms-distribution theory. Stanford Electronics Laboratories, Technical report No. 2250-1 (February, 1962).
21. Okikiolu, G.O. Aspects of the theory of Bounded Integral operators in L^p - spaces, Academic Press (1971).
22. Orton, M. Hilbert transforms, Plemelj relations and Fourier transforms of distributions. SIAM J. Math. Anal., Vol. 4, November, 1973, 656-667.
23. Orton, M. Hilbert boundary value problems - A distributional approach. Proc. Royal Soc., Edinburgh 76A (1977), 193-208.
24. Pandey, J.N. and Chaudhry, M.A. Hilbert transform of generalized functions and applications. Canadian J. Maths. (to appear).
25. Pandey, J.N. and Chaudhry, M.A. The Hilbert transform of Schwartz distributions II. Proc. of AMS, (to appear).

26. Pandey, J.N. and Hughes, E. An approximate transform and its inversion. *Tohoku Mathematical J.*, Vol. 28 #4, pp. 497-509.
27. Pandey, J.N. On the Stieltjes transform of generalized functions. *Proc. Camb. Phil. Soc.* (1972), Vol. 71, pp. 85-95.
28. Schwartz, L. *Theories des Distributions*. Vol. I, II. Hermann, Paris, 1957, 1959.
- 28.a) Shilov, Georgi E. *Generalized functions and Partial Differential Equations*. Science Publishers, Inc. (1968).
29. Taibleson, M.H. On the theory of Lipschitz spaces of distributions on Euclidean n -spaces I & II. *J. Math. Mech.*, 13, 407-479 and 14, 827-839.
30. Titchmarsh, E.C. *Theory of functions*. Oxford Univ. Press, 2nd ed., 1939.
31. Titchmarsh, E.C. *Introduction to the theory of Fourier-integrals*. Clarendon Press, Oxford, 1936; 2nd ed., 1948.
32. Treves, F. *Distributions and general theory of differential operators*. Technical report 1, University of California, Berkeley, January, 1960.
33. Tricomi, F. *Integral equations*. Interscience publishers, New York, 1957.
34. Weiss, G. and Stein, M.E. On the theory of harmonic-functions of several variables I. The theory of HP spaces, *Acta. Math.* 103 (1960), 25-62.

35. Wigner, E. On the quantum correction for thermodynamic equilibrium. Phys. Rev., 40 (1932), 749-759.
36. Zemanian, A.H. Distribution theory and transform analysis. Mc Graw-Hill, New York, 1965.
37. Zemanian, A.H. Generalized integral transformations. Interscience Publishers, New York, 1968.
38. Zygmund, A. Trigonometric series, Vol. I, 2nd ed., Cambridge University Press, Cambridge, 1968.

END

0 1 1 1 1 8 5

FIN