NOTE TO USERS

This reproduction is the best copy available.

UMI[®]

Simulating the Effects of Ship Motion on Postural Stability Using Articulated Dynamic Models

by

Joyce C. McKee, B.Eng.

Carleton University

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Master of Applied Science

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
Carleton University

Ottawa, Ontario

March 2, 2004

© Copyright

2004 - Joyce C. McKee

Library and
Archives Canada

Bibliothèque et
Archives Canada

Published Heritage
Branch

Direction du
Patrimoine de l'édition

395 Wellington Street
Ottawa ON K1A 0N4
Canada

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence

ISBN: 0-612-97427-8

Our file Notre référence

ISBN: 0-612-97427-8

The author has granted a non-exclusive license allowing the Library and Archives Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque et Archives Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

In compliance with the Canadian Privacy Act some supporting forms may have been removed from this thesis.

Conformément à la loi canadienne sur la protection de la vie privée, quelques formulaires secondaires ont été enlevés de cette thèse.

While these forms may be included in the document page count, their removal does not represent any loss of content from the thesis.

Bien que ces formulaires aient inclus dans la pagination, il n'y aura aucun contenu manquant.

Canada

PAGINATION ERROR.

TEXT COMPLETE.

ERREUR DE PAGINATION.

LE TEXTE EST COMPLET.

Abstract

An articulated dynamic postural stability model applicable in a moving environment has been created for predicting motion-induced interruptions of humans on ships. The model is a combination of two planar models, a simple inverted pendulum in the sagittal plane and a four-bar linkage in the frontal plane. The dynamics of both planar models were derived by hand and then linearized for the purpose of controller design. Single-input-single-output controllers were applied to each model actuating ankle joint torque. The controllers were designed such that the closed-loop poles had the damping ratios and undamped natural frequencies that match experimental data produced at the Naval Biodynamics Laboratory (NBDL). The fully nonlinear form of the dynamic models were implemented using ADAMS multibody dynamics software. The ship motions used in the experiments represented motions at the flight deck of an FFG7 class frigate operating in sea state five at a ship speed of 10 knots and ship heading of 45° relative to the principal wave direction. The sagittal and frontal plane models were then validated individually against a second set of data with a heading of 105° relative to the principal wave direction. The combined model was then run for a sample set of simulated ship motions having the same degrees of freedom as those included in the NBDL experiments, and the results discussed. It was found that articulated dynamic models offer the potential for improved prediction of MIIs over alternative non-articulated models.

Acknowledgements

First and foremost I would like to thank my thesis supervisor Dr. Rob Langlois for his guidance, support and patience throughout this project. His willingness to make himself available at any time was invaluable.

Valuable contributions were also made by Jim Colwell at DRDC-Atlantic for providing the experimental data; Kevin McTaggart at DRDC-Atlantic for supplying the SHIPMO program used in the creation of the ship motion; Indal for providing SHIPSIM; and DND and the Navy for providing financial support as well as a leave of absence from military duty.

Dedicated to my family,
for always being supportive, no matter which road I decide to take.

List of Symbols

Subscripts:

x, y, z	x component, y component, z component
$1, 2, 3$	four-bar linkage segment identification numbers

Overscripts:

$\cdot$	(dot) first time derivative
$\ddot{}$	(double-dot) second time derivative
$\rightarrow$	vector

Variables:

a	half of the minor axis of an ellipsoidal cylinder
A_i	wave amplitude of the i component
A, B, C	assignment variable, assignment matrix
b	half of the major axis of an ellipsoidal cylinder
c_c	critical damping
D_i	displacement of the ship in the i direction
F_r	total frictional force
F_L	total lateral force
F_N	total normal force
F_T	total tangential force
F_i	force in the i direction
h	height of the centre of mass above ground
$\overline{H}_{1/3}$	significant wave height
I_i	mass moment of inertia about the i axis

I_G	mass moment of inertia about G
k_{min}	minimum spring constant
K	controller gain matrix
L	height of centre of mass (G) above point A
L_i	height of centre of mass for link i above rotational point
m	total mass of the human
m_i	mass of individual segment i
M_A	moment applied to the inverted pendulum at position A
R_X, R_Y, R_Z	kinematic position variables expressed in global coordinates
R_x, R_y, R_z	kinematic position variables expressed in local coordinates
r_G	position of the centre of mass (G) for the inverted pendulum
r_A	position of the location of the human on deck
$r_{G/A}$	position vector of G with respect to A
r_{ij}	position of i for link j used in the four-bar linkage equations
r_{jijk}	position of joint i with respect to joint k used in four-bar linkage
$S_\zeta(\omega_n)$	wave spectral ordinate
$S_{B\zeta}(\omega)$	spectral ordinate for Bretschneider wave slope spectrum
$S_{J\zeta}(\omega)$	spectral ordinate for JONSWAP wave spectrum
$\bar{T}$	average wave period
t	time
t_s	settling time
u	controller signal
W	half of the width of the human stance
<i>Weight</i>	mass of the individual body segments
x, y, z	position of the human on the deck used in the inverted pendulum equations

v_G	velocity of G for the inverted pendulum
v_{ij}	velocity of i for link j used in four-bar linkage equations
x_1, x_2	state variables
y	output of the system

Matrices:

$[C]$	matrix relating the independent and dependent generalized speeds
$\{F_k\}$	force acting on body k
$[I]$	inertial property matrix
$[J]$	constraint Jacobian matrix
$\{Q^e\}$	generalized external force vector
$\{Q^*\}$	generalized inertia force vector
$\{u\}$	generalized speed vector
$[V]$	reduced partial velocity matrix

Greek Variables

Δ_{max}	linear spring displacement
ϵ_n	random phase angle associated with the n th wave
η_i	angle of the ship about the i axis
μ	coefficient of friction used by Graham
μ_{static}	coefficient of static friction
ω_i	frequency of wave component i
ω_n	n th wave frequency in Fourier series
	representation of an irregular wave record
$\delta\omega$	wave frequency interval
θ_i	phase angle of the i th component

θ	position of the pendulum with respect to vertical
$\zeta(t)$	wave amplitude at time t
$\bar{\zeta}$	mean wave amplitude
ζ_{n0}	amplitude of sine wave component n in a Fourier series representation of an irregular wave
ζ	damping ratio

Contents

Acceptance	ii
Abstract	iii
Acknowledgements	iv
List of Symbols	vi
Contents	x
List of Figures	xiii
List of Tables	xvi
1 Introduction	1
1.1 Background	2
1.1.1 Testing	2
1.1.2 Simulation of the Naval Environment	2
1.2 Biomechanical Models	4
1.2.1 Quiet Standing	4
1.2.2 Control	9
1.2.3 Gait Analysis	9

1.3	Objectives	10
1.4	Thesis Overview	11
2	Mathematical Modelling	12
2.1	ADAMS	12
2.2	Ship Motion	13
2.3	ADAMS Ship Model	21
2.4	Ship Motion Validation	21
2.5	Implementation of Graham's Postural Stability Model in ADAMS	23
2.6	Validation	27
2.6.1	Simple Motion Model	27
2.6.2	Motion Induced Interruption (MII) Model	32
3	Biomechanics Articulated Model	38
3.1	ADAMS Biomechanical Models	38
3.2	Human Characteristics	39
3.3	Governing Dynamic Equations for the Articulated Models	42
3.3.1	Inverted Pendulum	42
3.3.2	Four-Bar Linkage	44
3.4	Controller Design	48
3.4.1	Inverted Pendulum	48
3.4.2	Four-Bar Linkage	52
3.5	Pole Selection	53
3.5.1	Inverted Pendulum	54
3.5.2	Four-Bar Linkage	55
4	Validation and Sample Results	59
4.1	Validation	59

4.2	Sample Results	63
5	Conclusion and Recommendations	65
5.1	Conclusion	65
5.2	Recommendations	66
5.2.1	Experimental Data	67
5.2.2	Dynamic Model	68
	References	70
	Appendices	74
A	MII Results	75
B	Inertial Properties of Biomechanical Model Segments	77
C	Equations of Motion for the Four-Bar linkage	81

List of Figures

1.1	Graham's assumed model for a person [2]	3
1.2	Planes of the human body	5
1.3	Model of a basic inverted pendulum [14]	5
1.4	Model of a multi-segmented inverted pendulum [15]	6
2.1	Ship Coordinate System	15
2.2	Flowchart summarizing steps in creating realistic ship motion in ADAMS .	16
2.3	Calculation of heave energy spectrum in encounter frequency and wave frequency domains [3]	17
2.4	JONSWAP and Bretschneider wave energy spectra: significant wave height 4.0 metres; modal period 10 seconds.	18
2.5	Ship and placement of subject	22
2.6	Block used to represent the average male for Graham's model	24
2.7	(a) Example of spheres used in Models 1, 2, and 3 (b) Example of the linear springs used in Model 4	25
2.8	Schematic of block used in calculation of tipping and sliding	29
2.9	Model 1 tipping validation.	30
2.10	Model 4 tipping validation.	31
2.11	Model 1 sliding validation.	32
2.12	Model 4 sliding validation.	33

2.13	Comparison of tipping from Graham's and ADAMS results for sea state 5 .	35
2.14	Comparison of sliding MIIs as a function of μ for sea state 5, 105 degrees, and 5 knots	36
2.15	Comparison of sliding MII frequency as determined from Graham's and ADAMS results for $\mu = 0.5$ and sea state 6	37
3.1	Model of the human in ADAMS	40
3.2	Schematic illustration of the inverted pendulum model	42
3.3	Schematic illustration of the four-bar linkage model	45
3.4	Graham's (+), experimental (x), and inverted pendulum models (*) plot of tipping events as a function of time for the tuning data set	57
3.5	Graham's (+), experimental (x), and four-bar linkage models (*) plot of tipping events as a function of time for the tuning data set	58
4.1	Graham's model (+), experimental (x), and inverted pendulum (*) model results of tipping events as a function of time for the validation data set . .	60
4.2	Graham's model (+), experimental (x), and four-bar linkage (*) model re- sults of tipping events as a function of time for the validation data set . . .	61
4.3	Comparison of tipping results for Graham's and the combined biomechanical model for sea state 4	63
4.4	Comparison of tipping results for Graham's and the combined biomechanical model for sea state 5	64
A.1	Comparison of tipping from Graham's and ADAMS results for sea state 4 .	75
A.2	Comparison of tipping from Graham's and ADAMS results for sea state 6 .	76
A.3	Comparison of tipping from Graham's and ADAMS results for sea state 7 .	76
B.1	Location of individual human segments	78

C.1 Values for the four-bar linkage	82
---	----

List of Tables

2.1	Comparison of features included in various rigid body models	24
2.2	Comparison of the tipping and sliding validation calculation and simulation results	28
2.3	Values of relevant data for sea state [29]	33
3.1	Body segment lengths defined by Diffrient et al. [30]	40
3.2	Comparison of the joint constraints	41
3.3	Angular joint limits (degrees)	41
3.4	Values used in applying joint limitations	42
3.5	Comparison of MIIs in the NBDL experimental data	53
4.1	Comparison of MIIs in the NBDL experimental data	62
4.2	Comparison of major incidents between the NBDL and the biomechanical models	62

Chapter 1

Introduction

Ship motion has the undesirable effect of making it more difficult for humans to move in a controlled and coherent manner while onboard ships, thus causing routine tasks to be impaired, hazardous, or in severe cases impossible to perform. A means of quantifying the severity of such impairment is through the detection of Motion Induced Interruptions (MIIs). MIIs are defined by Baitis, Woolaver and Beck [1] as occasions when a member of the crew would have to stop working at the current task and hold onto some convenient anchorage to prevent loss of balance. Graham, Baitis and Meyers [2] suggested that MIIs occur if the person in question tends to slip on the deck or topple over as the result of ship motion. Currently the probability of occurrence of MIIs is most often computed using Graham's model [2]. Graham's model is quasi-static and the person is assumed to react in the same way as a rigid block having geometrical and inertial properties representative of a human. Graham's model has been shown to correlate with experimental data though it tends to overestimate the number of MII occurrences because of the rigid body assumption. In practice, a person may change their stance, crouch, or sit down if they anticipate an imminent MII.

1.1 Background

There are two ways in which human postural stability on ships data can be generated: experimentally or computationally. Experiments can be performed on a motion platform or on a ship on the open sea. Computational results can be obtained by dynamically simulating the ship in a computer generated environment, and placing a virtual human on the ship.

1.1.1 Testing

Ideally all testing for ships would be done using sea trials, but that is not always possible as indicated by Lloyd [3]. Trials involving ships are irregular due to the cost, and the testing of ships in this manner is not effective in the design process as the ship must be built before the trial can be conducted. It is therefore generally necessary to conduct testing of humans with live subjects using a motion platform. Results from some such trials can be found in the literature [4, 5, 6, 7]. In these trials humans were placed on a moving platform with motions in up to 5 degrees of freedom [6]. In all the referenced trials, the subject or an independent observer, indicated when the subject experienced MIIs. Although testing humans in this way gives realistic values, it is time consuming and expensive, as specialized facilities are required, setup and calibration of the platforms is necessary, and the number of individuals that can be tested per day is very limited as shown by Baitis et al. [4]. For these reasons the naval community, when possible, turns to computer models to predict relevant factors.

1.1.2 Simulation of the Naval Environment

Simulation of dynamic systems has become an integral part of the modern design process, as it allows engineers to predict the behaviour of a system before it has been built. In order

Figure 1.1: Graham's assumed model for a person [2]

to effectively simulate humans in a naval environment, the movements of the ship must first be simulated for the required environment. In order to predict these motions, Graham et al. [2] indicate that three elements are required: a statistical description of the wind and wave environment, a computer program to determine the ship response, and the seakeeping criteria, which will indicate the effects of the motions. At this point in time, databases of environmental conditions have been created using hindcasting [8, 9]. Computer programs, such as SHIPMO [10], have been developed and are in widespread use, that can generate Response Amplitude Operators (RAOs) that characterize the response of the ship to the sea. Using the RAOs and knowing the wave conditions that the ship is exposed to, the ship response spectrum can be calculated by using such programs as SHIPSIM [11]. Once the ship response spectrum is calculated, it can be used to calculate the motions at any point on a ship as a function of time.

The current human model contains no articulations, and was developed by Graham et al. [2]. This model is the current standard for prediction of MIIs in the naval community. The model uses a stationary human, as shown in Figure 1.1. This model uses the linear and angular accelerations at the point of interest on the ship to calculate the total number

of MIIs. The model was originally created in the time domain [12], but has since been modified so that it works in the frequency domain [13], thus allowing faster calculations. The mathematical equations for this model are presented in Section 2.6.2.

1.2 Biomechanical Models

The area of biomechanical simulation is a developing field and the focus of much research. Two of the current major research areas are “quiet standing” and “gait analysis”. Humans are generally unstable during upright standing, due to the high ratio of centre of gravity height to stance width, as can be seen in Figure 1.1. This is true both in the fore/aft direction as well as side to side.

1.2.1 Quiet Standing

Generally quiet standing is considered to be standing on a non-moving surface. This is a large area of research due to the fact that most people will be exposed to this type of balance during normal everyday activities. It is also useful in analysis of the elderly, and slip and fall events. The development of simulations in this area requires two main components, a dynamic model and a control system.

Models of humans used for quiet standing are generally divided into two planar models, the sagittal plane and the frontal plane. The sagittal plane is the plane that divides the body down the centre dividing the body into left and right halves, as seen in Figure 1.2. Models in this plane move in the front (anterior) and back (posterior) directions. The frontal plane is the plane that divides the body into anterior and posterior sections. Models in this plane move in the side to side directions.

Greater amounts of work have been done in the sagittal plane, due to the fact that humans tend to be more unstable, during quiet standing, in this plane. Models in the

Figure 1.2: Planes of the human body

Figure 1.3: Model of a basic inverted pendulum [14]

Figure 1.4: Model of a multi-segmented inverted pendulum [15]

sagittal plane are generally built on the principle of an inverted pendulum, as shown in Figure 1.3. The inverted pendulum models use an inverted pendulum with a rotation point at the ankles, and in the case of a single link, the centre of mass is located as a point mass equivalent to the entire mass of the body, not including the feet. For multi-link inverted pendulum models, the typical configuration is for the two links to represent the legs and the upper body, with rotations at the ankles, and hips. In the case of a three-link model the links represent the upper-body, the upper legs and lower legs, with rotations at the hips, knees, and ankles. Experiments have been conducted on a single inverted pendulum, such as those done by Loram et al. [16], in which they were comparing the magnitudes of sway and ankle impedance in order to determine if the magnitude of sway experienced by the subject is controlled by the ankle impedance. They showed that the subjects could significantly reduce the mean sway amplitude, although not the frequency by giving full attention to the goal of reducing the sway amplitude. The single-link inverted pendulum was also used by Pai et al. [14] in order to determine step initiation. In this case, the

dynamic model proved to be more accurate than a static model. Other models, based on multi-link inverted pendulums, as shown in Figure 1.4, have also been used. Four of the major assumptions made about the dynamics of inverted pendulum models, discussed by Soetanto [15], are identified below.

1. The model is planar, which reduces the complexity.
2. The body segments are rigid, although this is not specifically true, since the centre of mass of each segment moves as the corresponding muscle is flexed. This causes the moment of inertia of each section to change as a function of time. By ignoring this small change it greatly reduces the complexity of the model.
3. There is bilateral symmetry. With this assumption the parameters characterizing each segment, such as the segment length, mass, and muscle length and force, are the same for the left and right sides of the body. The initial positions of the left and right body segments are the same, and the muscle forces and torques are identical on the left and right sides.
4. There are no excessive motions of the upper body segments. This assumes that the upper body segments, the head-arm-trunk (HAT), are represented as one link. This is generally necessary since, as found by Kobetic et al. [17], voluntary upper-body movements interfere with the controller's ability to control the lower body.

Models in the frontal plane are generally four-bar linkages, with the links being the legs, the ground and the upper body (HAT). Hemami et al. [18] indicate that although many studies have been devoted to human gait and locomotion in the sagittal plan, studies and information about frontal plane locomotion tends to be sparse and scattered. Experiments were conducted in order to determine if the two planar models could be decoupled in regards to the control of the individual models [19]. In this human study it was concluded

that the postural responses result from a control strategy that transforms multivariable control problems into two decoupled univariable control problems, one in each plane. This is a key observation as it allows control systems to be developed somewhat independently in each plane, and applied to non-planar models. Generally, the above models are controlled using forms of state feedback, which assume that the nervous system is able to derive the exact system states from sensory information. Van der Kooij et al. [20] suggest that this is not strictly so, since there are neural time delays, and sensory dynamics are not included. They suggest that to fully model the control used by a human for a three-link inverted pendulum the following kinematic and dynamic variables must be monitored:

- the angles of the required joints (in this case the ankle, knee and hip);
- the ground reaction force in the normal and tangential directions to the sole of the foot;
- the centre of pressure of the foot and ground interface;
- the rotational and translational accelerations sensed by the semicircular canals and the otolith organs; and
- the position and velocity of the head in relation to a visual reference frame (of which two are described, one relative to the support base and one relative to the fixed world).

As can be seen from this list, many sensors are involved in determining the positions of the human body relative to the environment. This becomes important in the selection of the method of controlling human postural models.

1.2.2 Control

Control of human stance has been investigated using many different strategies, ranging from linear control to adaptive control. Linear controllers are widely used, and have been highly developed. A case of such linear control is described by de Vlugt et al. [21] who use a linear feedback control model to describe the basic intrinsic and reflexive postural control mechanisms of the central nervous system for relatively small variations of the variables around an equilibrium position. Hemami et al. [18] also used a linear control model, but in their case used both feedback gains as well as feedforward gains. The use of both feedback and feedforward linear control is also used by Kuo [22]. Kuo indicates that the feedback portion of the controller is used to react to direct inputs experienced, while the feedforward portion of the controller is responsible for movements based on previous experiences. It has also been suggested by van der Kooij et al. [23] that an adaptive estimator model of human spatial orientation can be used. In this study, the adaptive model dynamically weights sensory error signals, that is, it weights the difference between expected and actual sensory signals as a function of the environmental conditions. In order to do this a Kalman filter is used. Nonlinear controls have also been used, as reported by Micheau et al. [24]. This study proposes that the musculo-skeletal system can be represented by a nonlinear system, with time-delay in feedback. It can therefore be seen that many methods of control have been used in simulating human stance control.

1.2.3 Gait Analysis

The analysis of human gait, as with quiet standing, requires two main interrelated components - a dynamic model and a control system. Many different types of dynamic models have been used in order to simulate human gait. These range from a simple inverted pendulum, as used by Onyshko and Winter [25], to a three-dimensional multi-segment

representation of the lower limb used by Pandy and Berme [26]. Much of the research in this area is done in the sagittal plane. Once a dynamic model has been chosen, the control of the model can take many different forms. Different types of control have been used, such as was discovered by Bauby and Kuo [27]. They found that lateral balance must be stabilized with visual-vestibular feedback, but fore-aft dynamic stability requires little or no feedback. Their dynamic model is simple with no knees or upper-body, however they admit that if an upper body was added it would require an adaptive controller.

1.3 Objectives

Up to this point, articulated dynamic modelling of humans in the naval environment has not been significantly developed. The specific objective for this project was to develop and refine dynamic postural stability models. The intent was to develop a simulation of an articulated human, to which ship motion could be applied and that could be used for predicting MII frequency as a function of ship operating conditions. This was to be done using two planar models, with appropriate control methodology. The models were to be simple models as far as humans are concerned, as they still would not have the total number of articulated joints that humans actually have, but instead the number of joints would be limited to six. The models were to be tuned and validated using experimental data from the Naval Biodynamics Laboratory (NBDL) and discrepancies between the simulation and the experimental data were to be discussed thoroughly. An attempt was to be made to fix any problems found in the simulation, but if they could not be fixed within the scope of this project, recommendations for future courses of action would be suggested.

1.4 Thesis Overview

This chapter presented a brief history of ship motion generation, existing human models as applied in the ship environment, and current biomechanical models as they are used in a stationary environment as an introduction to the problem that is the focus of this project. A number of topics relevant to the project have been reviewed and the motivation and objectives for the project were stated. Chapter 2 describes the mathematical models used in the generation of the ship motion, validation of the use of the computer program ADAMS for developing articulated postural stability models, and the implementation of the existing rigid-body model (Graham's model). Chapter 3 describes the mathematical modelling of the articulated biomechanical models. These include: the development of the two models governing dynamic equations, the selection of the control strategy, and the selection of the controller poles based on experimental data. Chapter 4 contains validation of the controller poles based on experimental data and results produced by the biomechanical models to simulated ship motion. Chapter 5 contains concluding remarks with recommendations for future work.

Chapter 2

Mathematical Modelling

The models developed and implemented in this chapter are used as the basis for the development of the more sophisticated articulated models in Chapter 3. It was first necessary to model the ship motions at the points of interest. This was done through the simulation of ship motions that were then applied to a representative ship model in ADAMS, as described in Section 2.1. It was then necessary to develop a method of interfacing the human model and the ship, that allows detection of the onset of an MII. After the models interface had been developed, it was validated to ensure that the modelling approach chosen in ADAMS produces results that are consistent with other approaches. Finally, Graham's model was implemented as a benchmark.

2.1 ADAMS

ADAMS, that derives from Automatic Dynamic Analysis of Mechanical Systems, is mechanical system simulation software that can be used to create detailed simulations of the full-motion behaviours of complex mechanical systems [28]. It allows for modelling, analysis and visualization of the system response. Solution capabilities include kinematic, static, quasi-static and dynamic types. The simulation software also supports external

user-created subroutines for different components within the software. ADAMS is an appropriate method for the simulation of human motion as it allows for interactive simulations with visuals and allows for the calculation of the dynamics of the model to be done internally, thereby allowing successive dynamic model refinements to be implemented efficiently.

2.2 Ship Motion

The motion of a ship at sea is governed by the ship design and by the wave conditions that the ship experiences. A ship generally is capable of motion in six degrees of freedom - three linear motions and three angular motions. The overall movement of the ship with time results from the random influence of the sea. The operating conditions of the ship, in terms of heading relative to the principal sea direction and speed, as well as the sea conditions dictate the respective amplitudes of the six types of motion. Linear motions that the ship experiences occur in the three perpendicular directions as seen in Figure 2.1, and are given the names surge, sway and heave. Surge is motion in the horizontal plane of the ship parallel to the direction of travel, and can be in the forward or aft (backwards) directions. Sway is motion in the horizontal plane of the ship perpendicular to the direction of travel, and can be in the port (left) or starboard (right) directions. Heave is motion in the vertical direction, or up and down, as experienced by an individual on the ship. Positive motions for surge, sway and heave are forward, port and up respectively. Linear ship motions are usually evaluated at the ship's 'centre of rotation' that is defined as the intersection of a vertical line through the centre of mass and the waterplane. Linear motions can be evaluated at any point on the ship but result from a combination of both linear and angular displacements for points other than the centre of rotation. The angular motions that the ship experiences occur about three mutually perpendicular axes, in the ship frame

of reference. These motions are given the names roll, pitch and yaw. Roll is rotation about the longitudinal centreline of the ship or “surge” axis, and causes the port and starboard portions of the ship to rise or fall. Pitch is rotation about the “sway” axis, and causes the bow and stern portions of the ship to rise or fall. Yaw is rotation around the “heave” axis, and causes the bow and stern portions of the ship to move to port and starboard in the horizontal plane. Positive angular motions are defined by the right-hand rule applied about the positive linear axes. Simple ship motions sources, varying from constant rotation to sinusoidal motions are useful in development of the basic dynamic models as they can be repetitive and of constant amplitude. However, these simple motions do not represent realistic ship behaviours. Actual sea trials are the most accurate method of determining the motion of a ship. However, in most cases sea trial data is not available for the exact or range of conditions of interest, and therefore computed ship motions must be used. The most widely-used approach for generating ship motion is from a frequency domain representation of the response of the ship to the sea. This approach works well for small to moderate ship motions, and seas up to and including sea state five or six in the case of frigates. Beyond this limit, accurate computed motions must be developed using nonlinear ship simulation software. Accurate analysis of human stability over a wide range of sea conditions relies on the availability of both linear and nonlinear ship motion sources.

Ship Motion Generation for Use With ADAMS

To introduce linear ship motion into ADAMS, it was necessary to develop a ship motion generation subroutine. The generation of realistic ship motion was done through a multi-step process as shown in Figure 2.2. Each step in the process (block's in Figure 2.2) will be described sequentially.

Figure 2.1: Ship Coordinate System

Calculation of RAOs

The preliminary step in the sequence detailed by Figure 2.2 is the calculation of the ship's RAOs. The first component that is necessary in the calculation of the complete ship motion is a set of transfer functions between the waves and the ship motion for each degree of freedom. The transfer functions, that are the square roots of the RAOs, describe the 6-degree-of-freedom responses of the ship to unit amplitude sinusoidal waves as a function frequency. The RAOs are a function of the ship's properties, which include mass, geometry, and the operating conditions (for example ship heading, speed and sea state). By multiplying a set of RAOs by the wave energy spectrum, the ship response spectrum for each degree of freedom can be calculated, as shown in Figure 2.3. Note in Figure 2.3 that the calculation of the ship response spectrum can be performed in either the wave frequency domain or the encounter frequency domain which are defined by Lloyd in [3]. Experience suggests that, for numerical reasons, it is generally preferable to perform the evaluation in the wave frequency domain.

For this research, the RAOs were calculated using SHIPMO [10], a hydrodynamics

Figure 2.2: Flowchart summarizing steps in creating realistic ship motion in ADAMS

code based on strip theory. Strip theory calculations are described in detail by Lloyd [3].

Wave Energy Spectrum (Sea Spectrum)

The ocean surface is made up of irregular waves, that is waves that do not follow a purely sinusoidal pattern. Irregular wave elevation can be represented mathematically by a Fourier series defined by Lloyd [3] as:

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_{n0} \cos(\omega_n t + \epsilon_n) \quad (2.1)$$

where:

$\zeta(t)$ is the wave amplitude at time t ;

$\bar{\zeta}$ is the mean wave amplitude;

ζ_{n0} is the amplitude of component n ;

ω_n is the n th wave frequency; and

Figure 2.3: Calculation of heave energy spectrum in encounter frequency and wave frequency domains [3]

ϵ_n is the random phase angle associated with the n th wave.

Using Equation 2.1, a wave energy spectrum can be created. The wave energy spectrum details the relative magnitude of the component sine waves making up the irregular waves. This method is used to develop sea spectra from experimental data. The spectral ordinate, $S_\zeta(\omega_n)$ is given by

$$S_\zeta(\omega_n) = \frac{\zeta_{n0}^2}{2\delta\omega} \quad (2.2)$$

where $\delta\omega$ is the frequency interval.

If experimental data is not available then an idealized wave spectrum can be used. For the open ocean, the Bretschneider wave energy spectrum is used, and for coastal (littoral) waters the JONSWAP spectrum is used [3]. Both are shown in Figure 2.4 for typical assumed values of significant wave height and wave modal period.

The Bretschneider spectrum is defined as

Figure 2.4: JONSWAP and Bretschneider wave energy spectra: significant wave height 4.0 metres; modal period 10 seconds.

$$S_{B\zeta}(\omega) = \frac{A}{\omega^5} \exp\left(\frac{-B}{\omega^4}\right) \quad (2.3)$$

where ω is the wave frequency, and

$$A = 172.75 \left(\frac{\overline{H}_{1/3}^2}{\overline{T}^4} \right) \quad (2.4)$$

and

$$B = \frac{691}{\overline{T}^4} \quad (2.5)$$

where $\overline{H}_{1/3}$ is the significant wave height and $\overline{T}$ is the average wave period.

The JONSWAP spectrum is defined as

$$S_{J\zeta}(\omega) = 0.658CS_{B\zeta}(\omega) \quad (2.6)$$

where the factor C is given by

$$C = 3.3^J \quad (2.7)$$

and

$$J = \exp \left[\frac{-1}{2\gamma^2} \left(\frac{\omega T_0}{2\pi} - 1 \right)^2 \right] \quad (2.8)$$

where T_0 is the modal period corresponding to the peak of the wave spectrum, and $\gamma = 0.07$ for $\omega < \frac{2\pi}{T_0}$ and $\gamma = 0.09$ for $\omega >> \frac{2\pi}{T_0}$.

SHIPSIM

The proprietary program SHIPSIM, developed by Indal Technologies [11], combines the sea spectrum with RAOs to produce ship response spectra as illustrated in Figure 2.3 for each component of the ship motion. It can be used to generate extended periods of representative ship motion from which ship motion statistics can be calculated. The program is also capable of generating ship motion time histories from sine waves or experimental motion data files. SHIPSIM is capable of outputting a complete set of kinematic data as a function of time, as well as the ship response spectra used by the MOTSUB ship motion evaluation subroutine.

Motion Subroutine - MOTSUB

The ship created in ADAMS is forced to move by a six-degree-of-freedom motion generator, which in turn is driven by the subroutine MOTSUB that is interfaced to the ADAMS software. In this subroutine, the ship response spectrum is input and used to create

displacement, velocity, and acceleration time histories for each degree of freedom with the linear motions applicable at the ship's centre of rotation such that

$$Displacement = \sum_{i=1}^{40} A_i \cos(\omega_i t + \theta_i) \quad (2.9)$$

$$Velocity = \sum_{i=1}^{40} -A_i \omega \sin(\omega_i t + \theta_i) \quad (2.10)$$

$$Acceleration = \sum_{i=1}^{40} -A_i \omega^2 \cos(\omega_i t + \theta_i) \quad (2.11)$$

where A is the wave amplitude of the i th component, ω_i is the wave frequency of the i th component, and θ_i is the phase angle of the i th component. Note that in Equations 2.9 through 2.11, 40 solution components are superimposed. While this number is variable, 40 components have been used in this thesis, which is consistent with accepted practice.

For points of interest away from the centre of rotation, the linear motions are effectively amplified. For example, when a ship heads directly into a set of waves, that is a heading of 0 degrees with respect to the principal wave direction, as the position of interest is moved either forwards or aft, the magnitude of the heave is affected by the rotational angle of the pitch, as shown in Equation 2.12.

$$D_{z\,total} = D_z + R_x \sin(\eta_y) \quad (2.12)$$

where D_z is the displacement in the z direction, R_x is the position in the x direction, and η_y is the ship pitch angle.

Non-linear and Experimental Motion

Non-linear ship motion generated by other programs may be included in the form of experimental motion inputs. In this case, the data is imported directly into ADAMS, then curve fit using cubic splines. This process was used in applying the experimental motion inputs described in Section 3.5 used to select the poles of the articulated models and subsequently for model validation discussed in Section 4.1.

2.3 ADAMS Ship Model

The model created for use in ADAMS consists of a block which represents the ship and has a length of 110 metres, a width of 16 metres and a height of 9.16 metres. The ship motion is governed by the motion generator, that is attached at the centre of rotation of the ship, and calls the MOTSUB subroutine if realistic ship motion is desired, or simply calculates the motions based on simple mathematics, for example a sinusoidal pattern during model development and validation. Figure 2.5 shows the ship with the sample subject described subsequently, placed at the stern to indicate the relative size of the ship to the average North American male. This ship model will provide a motion base for the human models developed and implemented in this thesis.

2.4 Ship Motion Validation

Validation of the developed ship motion subroutine was accomplished by comparing the output from SHIPSIM with the measured output from the ADAMS model. The location of the test point for both programs was $P = (-49.0, 1.5, 3.0)$ metres relative to the ship centre of rotation where the coordinates correspond to surge, sway, and heave respectively. This point represents a point on the flight deck of a typical frigate.

Figure 2.5: Ship and placement of subject

SHIPSIM outputs are in local ship coordinates, whereas ADAMS outputs the location of the points in global coordinates. It was therefore necessary to transform one of the outputs to the other's coordinate system. In this case the local coordinates output from SHIPSIM were transformed into global coordinates using Equation 2.13.

$$\begin{Bmatrix} R_X \\ R_Y \\ R_Z \end{Bmatrix} = \begin{bmatrix} c\theta_y c\theta_z & -c\theta_y s\theta_z & s\theta_y \\ c\theta_x s\theta_z + s\theta_x s\theta_y c\theta_z & c\theta_x c\theta_z - s\theta_x s\theta_y s\theta_z & -s\theta_x c\theta_y \\ s\theta_x s\theta_z - c\theta_x s\theta_y c\theta_z & s\theta_x c\theta_z + c\theta_x s\theta_y s\theta_z & c\theta_x c\theta_y \end{bmatrix} \begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} \quad (2.13)$$

where $\{R_X, R_Y, R_Z\}^T$ represents kinematic variables expressed in global coordinates, $\{R_x, R_y, R_z\}^T$ represent variables expressed in local coordinates, c and s represent cosine and sine respectively and θ_x , θ_y , and θ_z represent the angles in the roll, pitch, and yaw degrees of freedom with the order of rotations described by the Bryant angle convention (roll followed by pitch followed by yaw in transforming from global to local coordinates).

2.5 Implementation of Graham's Postural Stability Model in ADAMS

Graham's postural stability model is a model created in order to determine the effects of ship motion on operational performance [2]. Graham's model calculates the MIIs of a human based on the human being a solid object, which reacts in a similar manner to inanimate objects, that is with no ability to actively control stability.

The initial model created in ADAMS was created using basic ADAMS model components. The model's basic features were created to mimic Graham's existing sliding and tipping model [2], and to verify that ADAMS can be used for such an application. In this model, the human is represented by a block, as seen in Figure 2.6, and will be referred to as the human. The block's dimensions are: height (h) of 1.75 metres, width of $\frac{1}{4}h$ metres, and length of $0.17h$ metres. These parameter values were chosen to ensure consistent parameters between Graham's model, the ADAMS model, and an average male. Four variations of Graham's model were ultimately created and are described below. A summary comparison of the model's features is shown in Table 2.1, where visual indicates that an individual watching a graphical simulation of the model will actually see the model perform the indicated action. The final model used was developed through an iterative process, starting with a model that was able to visually tip and slide and ending with the current model that does not visually tip or slide, but runs every efficiently. The series of models will be described sequentially.

Model 1

In the first model, four spheres were added to the bottom corner vertices of the human. The human was then placed so that the spheres sit on a second block, indicated in Figure 2.7b by the lower horizontal line. The block, lower horizontal line, that the human sits on

Figure 2.6: Block used to represent the average male for Graham's model

Table 2.1: Comparison of features included in various rigid body models

	Visual Tipping	Visual Sliding	Calculated Tipping	Calculated Sliding	Speed of Simulation
Model 1	yes	yes	no	no	very slow
Model 2	yes	no	no	yes	slow
Model 3	yes	no	yes	yes	slow
Model 4	no	no	yes	yes	fast

represents the deck, but is much smaller, only slightly larger than the human block, and therefore allows for faster calculations. This second block was then locked to the ship such that the interactions between the spheres and the second block are at the desired vertical location on the ship, essentially coplanar with the actual ship deck. Contact forces were used to create the reaction forces between the human and the block. The contact force type used was the most general one, that allows friction between the human and the deck to be included in the model.

The value of the contact force can be output to a file, where tip is indicated by any matching pair of contacts in the z direction becoming 0, which indicates that the sphere has lifted off the support block. In order to detect sliding, two measures were created using an internal ADAMS function that calculates the distance between two markers, one attached to the human, and one to the support block. The markers measure the movement

Figure 2.7: (a) Example of spheres used in Models 1, 2, and 3 (b) Example of the linear springs used in Model 4

of the human with respect to the support block in the x and y directions.

Model 2

In the second model, the frictional component of the contact forces at the spheres were replaced by a joint that removes three degrees of freedom (surge, sway and yaw) from the human block relative to the ship. The joint was added in order to increase the speed of the internal calculations done by ADAMS, as the calculation of the frictional force was encumbering. In doing so, the model is still free to visually tip, but is no longer free to visually slide. In this model, the sliding occurrences must be calculated using Equation 2.14.

$$F_L < \mu_{static} F_N \quad (2.14)$$

where F_L is the magnitude of the total lateral force acting on the spheres, F_N is the total normal force acting at the spheres, and μ_{static} is the static friction of the deck. When this

equation becomes true, it indicates that an MII due to sliding has occurred. This model improved the simulation time, as the frictional forces no longer needed to be calculated and the numerical integration encountered fewer discontinuities.

Model 3

In Model 1, the block is allowed to visually tip and slide, while in Model 2 the block is allowed to visually tip. Although this can be useful, it is not practical as the models must be reset after each instance of tipping or sliding in Model 1, and each instance of tipping in Model 2. Resetting the models becomes a time consuming matter, so Model 3 was created to eliminate the problem. In Model 3, a righting moment was applied to the human to return the human to the deck when tipping occurs. The value of the moment created was a function of the angle between the human and the deck and the angular velocity of the human relative to the deck. The function that was used allows the moment to be turned on as a function of angular displacement as well as multiply the angle by a value, thus increasing the moment as a function of the angular displacement and velocity from the support block. This allows for the simulation to continue for the desired simulation time, whereas with Models 1 and 2 the simulation must be restarted with the time set to begin after the previous tip had occurred. This model worked well in the development of the model with simple motions applied, but not well with random motions. The values that are multiplied by the displacement need to be correctly picked in order to react with the proper amount of force. The block then becomes an inverted pendulum problem.

Model 4

In the fourth model, the contact forces were replaced by stiff linear springs as shown in Figure 2.7b. In this model, the human hangs by these stiff linear springs from the deck of the ship, as can be seen in Figure 2.7b where the upper marker is attached to the

deck, and the lower marker to the corner of the human. This eliminates the need for the righting forces used in Model 3, and decreases the simulation time. In this model, tipping is detected when the force in two of the springs attached to the vertices of the corresponding edge are positive (in compression). The stiffness and damping coefficients were chosen using Equations 2.15 and Equation 2.16 respectively.

$$k_{min} = \frac{mg}{\Delta_{max}} \quad (2.15)$$

$$c_c = 2\sqrt{k_{min}m} \quad (2.16)$$

where m is the mass of the human, g is acceleration due to gravity, Δ_{max} is the linear displacement of the spring, k_{min} is the minimum spring constant, and c_c is the critical damping.

2.6 Validation

The validation of the simple tipping and sliding block model was done in two stages, the first stage involved using slowly applied monotonically-increasing angular motion to determine the accuracy of the ADAMS model in terms of whether the block slid and tipped at the correct angles, and the second stage involved applying realistic ship motion to the simulation to validate the frequency of sliding and tipping against an independent implementation of Graham's model.

2.6.1 Simple Motion Model

Validation of the tipping and sliding calculations done in ADAMS was completed in two separate steps. The first was to validate tipping, and the second sliding. Both Model 1 and

Table 2.2: Comparison of the tipping and sliding validation calculation and simulation results

	Model 1 Tipping	Model 4 Tipping	Model 1 Sliding	Model 4 Sliding
Calculated	11.1 sec	11.1 sec	7.9 sec	7.9 sec
ADAMS	11.1 sec	11.2 sec	7.9 sec	7.9 sec

Model 4 were subjected to the same testing procedure, as they were the models predicted to be of most use, Model 1 for its visual appeal, and Model 4 for its speed of calculations. For both tipping and sliding validation, the models were subjected to linearly increasing angular displacement about the x axis of the ship block defined by Equation 2.17.

$$\theta_x = C t \quad (2.17)$$

where θ_x is the angular displacement about the x axis, C is a small constant (approximately 0.025 so that inertial effects do not effect the results), and t is time. The models were run for 16 seconds for the testing with simple motion.

Tipping

Verification of tipping was carried out by first theoretically calculating when the block should tip, and then comparing it with the two ADAMS models (Model 1 and Model 4). Consideration of moments about the corner of the block allows the angle at which the man would theoretically tip to be expressed as

$$\theta = \arctan\left(\frac{W}{H}\right) \quad (2.18)$$

where $H = \frac{Height}{2} + r_{sphere}$ for Model 1 and $H = \frac{Height}{2}$ for Model 4 and W is half of the width of the base as defined in Figure 2.8. By combining Equations 2.17 and 2.18 the time at which the blocks should tip was calculated.

Figure 2.8: Schematic of block used in calculation of tipping and sliding

The theoretical and calculated tipping and sliding times for a C of 0.025 rad/s are shown in Table 2.2. The method used to calculate tipping in each model was different, as different components were used as indicators. In the case of Model 1 the value of the 2 contact forces on the right side of the human block became 0 when the block began to tip, as seen in Figure 2.9. In the case of Model 4, a tip was indicated by the force in a spring becoming positive, as seen in Figure 2.10.

Sliding

The verification of sliding for Model 1 was done by first calculating the maximum amount of friction that would allow the block to slide before tipping. The contact force due to gravity for the man is calculated as:

Figure 2.9: Model 1 tipping validation.

$$F = mg \quad (2.19)$$

where g is the acceleration due to gravity and m is the mass of the human. In the case of a rotated object, the force can be divided into normal (F_N) and tangential (F_T) components such that

$$F_N = \cos(\theta)gm \quad (2.20)$$

$$F_T = \sin(\theta)gm \quad (2.21)$$

where θ is the angle between the deck and horizontal. The frictional force is the force that opposes the motion and allows the human to remain stationary with respect to the block. It has a maximum value of

Figure 2.10: Model 4 tipping validation.

$$F_r = \mu_{static} F_N \quad (2.22)$$

Sliding occurs when Equation 2.21 becomes greater than Equation 2.22. By combining these equations, the value for μ that would allow sliding to occur before tipping can be calculated from

$$\mu \leq \tan \theta \quad (2.23)$$

It was known from the previous calculations that the block tips at an angle of 0.278 rad, and therefore μ must be less than 0.285. For that reason, a test value of $\mu = 0.20$ was chosen. Using this value the block should theoretically slide as indicated in Table 2.2. In the case of Model 1, sliding was indicated by a change in the block's velocity relative to the deck, as indicated in Figure 2.11. In the case of Model 4 sliding was indicated by F_r becoming less than F_T . This is shown in Figure 2.12. From the simple sliding and tipping results presented in Table 2.2 it was concluded that both Models 1 and 4 produce correct

results for the tests considered. Therefore, based on the reduced run time and the large number of simulation cases involved in the second phase of testing, Model 4 was selected for further evaluation.

Figure 2.11: Model 1 sliding validation.

2.6.2 Motion Induced Interruption (MII) Model

Model 4 was given the inertial properties of an elliptical cylinder such that

$$(I_x, I_y, I_z) = m \left(\frac{h^2}{3} + \frac{b^2}{4}, \frac{h^2}{3} + \frac{a^2}{4}, \frac{4(a^2 + b^2)}{3\pi} \right) \quad (2.24)$$

where I_x , I_y and I_z are the moment of inertia, h is the height of the human's centre of gravity above the deck, $2a$ is the minor axis of an ellipsoidal cylinder, and $2b$ is the major axis of the same cylinder. The equation is then simplified by Graham et al. [2] such that

$$(I_x, I_y, I_z) = m \left(\frac{h^2}{3}, \frac{h^2}{3}, 0 \right) \quad (2.25)$$

Figure 2.12: Model 4 sliding validation.

Table 2.3: Values of relevant data for sea state [29]

Sea state	Significant Wave Height (meters)	Modal Period (seconds)
4	2.5	8.8
5	4.0	9.7
6	6.0	12.4
7	9.0	15.0

The model was run for 4 sea states (4 to 7), with 13 ship headings relative to the principal wave direction (0° to 180° in 15° intervals), and 5 ship speeds (5 to 25 knots in 5 knot intervals). Each simulation run represented 30 minutes of operation in the stated ship operating and sea condition. Relevant sea state values can be found in Table 2.3.

The validation of the model's predicted MIIs due to tipping was accomplished by comparing the results calculated from the ADAMS model with the results calculated by the following criteria established by Graham for tips to port and starboard respectively.

$$-\frac{1}{3}h\ddot{\eta}_x - \ddot{D}_y - g\eta_x - \frac{l}{h}\ddot{D}_z > \frac{l}{h}g \quad (2.26)$$

$$-\frac{1}{3}h\ddot{\eta}_x + \ddot{D}_y + g\eta_x - \frac{l}{h}\ddot{D}_z > \frac{l}{h}g \quad (2.27)$$

In this work an inequality is used to indicate that when the statement becomes true, that is if the inequality is satisfied, an MII occurs.

Equations 2.26 and 2.27 are reproduced from Graham et al. [2] and form the core of Graham's tipping model. To ensure consistency with the ADAMS model, the required data for the equations were exported from ADAMS and used in the evaluation of Equations 2.26 and 2.27.

An example of a set of results obtained from the two models, Grahams and the ADAMS model (Model 4), is found in Figure 2.13, where the results from the ADAMS model are plotted from 0° to 180°, and the results from Graham's model are plotted from 180° to 360°. Due to the ships symmetry, port and starboard headings are expected to produce identical results for a given model. The plot shows how the number of MIIs occurring per minute (radial axis), vary as a function of speed and ships heading (circumferential axis). The complete set of results for sea states 4 through 7 are included in Appendix A. Approximate symmetry between the left and right halves of the plots confirm that both models predict similar numbers of MIIs.

Validation of the model's sliding MIIs was done by comparing the results calculated from the ADAMS model with the results calculated from Graham's equations. A slide to port was calculated by Equation 2.28 for the ADAMS model, and by Equation 2.29 for Graham's model

$$F_y > -\mu F_N \quad (2.28)$$

$$-\ddot{\eta}_y - g\eta_x - \mu_s \ddot{\eta}_z > \mu_s g \quad (2.29)$$

Figure 2.13: Comparison of tipping from Graham's and ADAMS results for sea state 5

Similarly, a slide to starboard was calculated by Equation 2.30 for the ADAMS model and by Equation 2.31 for Graham's model

$$F_y < \mu F_N \quad (2.30)$$

$$\ddot{\eta}_y + g\eta_x - \mu_s \ddot{\eta}_z > \mu_s g \quad (2.31)$$

To ensure consistency with the ADAMS model, the required data for these equations as well was exported from ADAMS.

A comparison of the sliding results as a function of μ for a sea state 5, a heading of 105 degrees and a speed of 5 knots is shown in Figure 2.14. Due to the dependency of sliding on the value of μ , which in turn depends on the conditions on board the ship, a value of

$\mu = 0.5$ was chosen as a representative value for a relatively slippery deck and is plotted for sea state 6 in Figure 2.15.

Figure 2.14: Comparison of sliding MIIs as a function of μ for sea state 5, 105 degrees, and 5 knots

Figure 2.15: Comparison of sliding MII frequency as determined from Graham's and ADAMS results for $\mu = 0.5$ and sea state 6

Chapter 3

Biomechanics Articulated Model

The articulated models developed in this chapter are based on more realistic human kinematics, active control, and properties than the model described in the previous chapter. Two planar models are developed: an inverted pendulum model in the sagittal plane, and a four-bar linkage model in the frontal plane. These models are chosen as they are the simplest models that can be applied to the human body in the two planes. The dynamics of the models are then derived in order to design single-input-single-output controllers. These controllers are then tuned and validated based on experimental data.

3.1 ADAMS Biomechanical Models

The biomechanical model built in ADAMS uses three main components: the ship, which is constructed and controlled as described in Section 2.2; the support block, which in this case is based on the block developed in Model 4 of Section 2.5 and is used as the method of interaction between the human and the ship; and the human model which is subsequently described in this chapter.

The model used in the sagittal plane is an inverted pendulum, with a centre of mass located at a height of 0.93 metres above the ankles, which act as the point of rotation.

The model used in the frontal plane is a four-bar linkage, with rotational joints located at the ankles and hips. Both model are attached to the block through fixed joints at the base of both feet. This allows the block, with it four linear springs and joint, as derived in Chapter 2 to act as the method of interaction between the human biodynamical model and the deck. In the case of the biomechanical models, the block is reduced to a thickness < 1 mm in the z direction, and a width and length to match that of the stance width and foot length. Subsequent sections will describe the human characteristics, construction and control of the biomechanical models.

3.2 Human Characteristics

The articulated human models are based on the average North American male, with an approximate height of 174 cm, and an approximate mass of 78 kg as detailed by Diffrient et al. [30]. Individual body segments were considered, and the lengths were assigned as shown in Table 3.1. The head is represented by a sphere as seen in Figure 3.1, with a radius of 11.5 cm and each hand by a cylinder and sphere joined together. The cylinder is 7 cm in length, and the sphere has a radius of 5.75 cm. The radius of all cylinders used to represent the parts of the body, as seen in Figure 3.1 have a radius of 10 cm. The length given in the table for the foot is the length from the sole of the foot to the ankle. The width used for the foot is 11.2 cm and is the width if the individual were to be wearing a shoe.

The moments of inertia of the individual segments are based on regression equations calculated by Chandler et al. [31]. These equations base the moments of inertia of the individual segments on the total mass of the human being. The equations can be found in Appendix B. The location of the centres of mass of the segments are calculated as the average of the centres of mass found by Chandler et al. Although these equations are

Figure 3.1: Model of the human in ADAMS

Table 3.1: Body segment lengths defined by Diffrient et al. [30]

Body Segment	Length (cm)
torso	62.5
upper arm	28.2
forearm	25.4
thigh	42.4
calf	41.1
foot	8.9

based on only six individuals, they are used as approximations in this thesis. It is also assumed that the left and right sides of the body have the same moments of inertia, and locations of centres of mass. Chandler et al. showed that this is not strictly true, but this assumption allows for symmetry about the median plane and is appropriate in the context of this research.

After constructing the human, as illustrated in Figure 3.1, two methods for introducing the desired degrees of freedom can be used. The first method is to disable joint degrees of freedom where articulation should not occur, thereby causing the joints to remain stationary, or effectively “locked”. The second method is to unite the desired segments together

Table 3.2: Comparison of the joint constraints

Joint	X	Y	Z	none
neck				✓
shoulder				✓
elbow				✓
wrist				✓
hip	✓ (Four-bar)			
knee				✓
ankle	✓ (Four-bar)	✓ (Inverted Pendulum)		

Table 3.3: Angular joint limits (degrees)

	front	back	internal	external
ankle	35	38	23	23
hip	N / A	N / A	40	45

so they become one. Although both methods are valid, the second method involves less computation during a simulation than the first, and therefore was implemented for both models. The joints that were allowed rotations, and the applicable rotation axes are shown in Table 3.2, where allowable movements are indicated by a check mark.

Joint Limitations

Joint limits must be applied to the model to reflect actual limitations on the range of motion of human joints. The angular limits are applied at the values indicated in Table 3.3 [30].

The joint limits are applied in the model using a two-sided moment, which first slows the human as it approaches the set limit and then applies a rigid stop at the limit. The forcing function used is modeled after an over-damped rotational spring-damper system, with the values used indicated in Table 3.4.

Table 3.4: Values used in applying joint limitations

plane	spring constant N m/ rad	damping value N m s/ rad
sagittal	305	35220
frontal	3050	3549.1

Figure 3.2: Schematic illustration of the inverted pendulum model

3.3 Governing Dynamic Equations for the Articulated Models

Due to the method used to control the models, as derived in Section 3.4, it is necessary to first derive the equation of motion for use in determining the gains that will be applied. In this section the equations are derived for the human in the frontal and sagittal planes.

3.3.1 Inverted Pendulum

Figure 3.2 shows a schematic illustration of the inverted pendulum model. Derivation of the governing equations of motion begins with an expression of the position vector $\vec{r}_G$ from the origin of an assumed inertial coordinate system, attached to the deck, to the centre of mass

$$\vec{r}_G = \vec{r}_A + \vec{r}_{G/A} = x\vec{i} + z\vec{k} + L \sin \theta \vec{i} + L \cos \theta \vec{k} \quad (3.1)$$

where $\vec{r}_A$ is the position where the human is located on the ship deck, $\vec{r}_{G/A}$ is the position of G with respect to A, and θ is the angle between the pendulum and vertical, as illustrated in Figure 3.1. The velocity of G is obtained by differentiating the position vector, Equation 3.1, resulting in

$$\vec{v}_G = (\dot{x} + L\dot{\theta} \cos \theta)\vec{i} + (\dot{z} - L\dot{\theta} \sin \theta)\vec{k} \quad (3.2)$$

The acceleration of G is obtained by differentiating the velocity vector, Equation 3.2 resulting in

$$\vec{a}_G = (\ddot{x} + L\ddot{\theta} \cos \theta - L\dot{\theta}^2 \sin \theta)\vec{i} + (\ddot{z} - L\ddot{\theta} \sin \theta - L\dot{\theta}^2 \cos \theta)\vec{k} \quad (3.3)$$

The forces acting on the pendulum are then calculated using Newton's law in the $\vec{i}$ and $\vec{k}$ directions as

$$F_x = m(\ddot{x} + L\ddot{\theta} \cos \theta - L\dot{\theta}^2 \sin \theta) \quad (3.4)$$

$$F_z = m(\ddot{z} - L\ddot{\theta} \sin \theta - L\dot{\theta}^2 \cos \theta) + mg \quad (3.5)$$

The rotational motion of the pendulum about A is governed by

$$-M_A + F_z L \sin \theta - F_x L \cos \theta = I_G \ddot{\theta} \quad (3.6)$$

Combining Equations 3.4, 3.5, and 3.6 and linearizing for small inclination angles results in:

$$(I_G + mL^2)\ddot{\theta} + mgL\theta = -M_A + m\ddot{z}L\theta + m\ddot{x}L \quad (3.7)$$

This is the governing dynamic equation for the inverted pendulum on the moving base. However for the purpose of controller design, the equations is linearized for small departures from vertical (small θ) and the prescribed linear accelerations of the ship ($\dot{x}$ and $\dot{z}$) are deliberately omitted, resulting in

$$-M_a = (I_G + mL^2)\ddot{\theta} + mgL\theta \quad (3.8)$$

Equation 3.8 will be used in the derivation of the inverted pendulum model controller in Section 3.4. Although the values for φ may be large, the angle between the pendulum and vertical in Figure 3.2, θ , due to its vertical frame of reference should remain relatively small and therefore the linearization assumptions are reasonable.

3.3.2 Four-Bar Linkage

In order to assign the correct gains to the controllers developed in Section 3.4 the dynamics of the planar four-bar linkage model also must be derived. The model of the human in the frontal plane is a four-bar linkage, as illustrated in Figure 3.3, where the four links can be described as the:

torso that contains the entire upper body, including the trunk, arms, and head; the values for the moments of inertia and mass are calculated with the arms hanging at the side, as can be seen in Figure 3.1;

legs where each leg creates one link and the legs are considered to have no knees, that is they do not bend in the sagittal plane, and the link extends from the connection point at the hip joint to the ankle; both links are assumed to be the same length,

Figure 3.3: Schematic illustration of the four-bar linkage model

and have the same mass and mass moments of inertia; and

ground where the ground is considered as a solid link that connects the ankles together.

The generalized coordinates q_1, q_2, q_3 and q_4 displayed in Figure 3.3 are the angles between each link and the one that is next in the loop moving around the closed kinematic loop. All angles are measured positive in the counter-clockwise direction.

The dynamic equation used for designing the controller for this model was derived by hand using a multi-body dynamic formulation based on Kane's equation without multipliers. The reader is referred to [32] for details of the approach. In general, the method relies on developing expressions for the derivatives of the system momentum using a minimal number of generalized coordinates and eliminating unnecessary internal reaction forces using loop-closure velocity expressions. The process begins by evaluating the terms required to form a composite velocity vector that is a combination of the partial velocity matrix $[V]$ and the generalized speeds $\{u\}$. Generalized speeds are the time derivatives of the generalized coordinates. The partial velocity matrix reflects how each generalized speed affects the velocity of each key point in the system.

In this case the human forms a closed kinematic loop, and therefore introduces constraint equations that inter-relate the configuration coordinates and equivalently the generalized speeds. The constraint equations can be expressed as a single matrix equation of the form

$$[J]\{u\} = \{0\} \quad (3.9)$$

where $[J]$ is the constraint Jacobian matrix.

The generalized speed vector is then partitioned into independent (n) and dependent (m) generalized speeds. By partitioning Equation 3.9, an expression for the dependent generalized speeds in terms of independent generalized speeds can be obtained as

$$\{u_m\} = -[C]\{u_n\} \quad (3.10)$$

where $[C]$ is the matrix relating independent and dependent generalized speeds.

Kane's equation without multipliers is then used to formulate the dynamic equation in minimal coordinates. Kane's equation states

$$\{Q^*\} + \{Q^e\} = \{0\} \quad (3.11)$$

where $\{Q^*\}$ is a vector of generalized inertia forces; and $\{Q^e\}$ is a vector of generalized external forces.

The generalized inertial force contributed by individual bodies is calculated by

$$\{Q_i^*\} = \{Q_V^*\} + [Q_M^*]\{\dot{u}_n\} \quad (3.12)$$

where $\{Q_V^*\}$ contains centripetal and Coriolis force terms, and the second term depends on derivatives of the generalized speeds. The general matrix equation for $\{Q^*\}$ then becomes

$$\{Q^*\} = -[\underline{V}]^T ([I][\underline{V}]\{\dot{u}_n\} - [I][V_M][\dot{C}]\{\dot{u}_n\} + [I][\dot{V}]\{u\}) \quad (3.13)$$

where $[\underline{V}]$ is a reduced partial velocity matrix that relates the system velocities to the independent generalized speeds, $[I]$ is the inertial property matrix, and $[\dot{V}]$ is the time derivative of the partial velocity matrix.

The generalized external force vector $\{Q^e\}$ from Equation 3.11 is evaluated by considering the externally applied forces and moments acting on each degree of freedom

$$\{Q^e\} = [\underline{V}_{ik}^T]\{F_k\} \quad (3.14)$$

where $\{F_k\}$ represents the force k acting on body i as shown in Equation 3.15

$$\{F_k\} = \begin{Bmatrix} F_k \cdot \vec{i} \\ F_k \cdot \vec{j} \\ \vec{p}_k \times \vec{F}_k + \vec{M} \end{Bmatrix} \quad (3.15)$$

In the case of the human model, ankle torques are considered to be applied at one ankle and this moment is introduced into the formulation using Equation 3.14. Equations 3.13 and 3.14 are substituted into Equation 3.11 and simplified to form

$$[\underline{V}]^T [I][\underline{V}]\{\dot{u}_n\} = [\underline{V}]^T [I][V_M][\dot{C}]\{\dot{u}_n\} - [\underline{V}]^T [I][\dot{V}]\{u\} + [\underline{V}]^T \{F^e\} \quad (3.16)$$

The values required in Equation 3.16 in the case of the four-bar linkage model can be found in Appendix C. Equation 3.16 was then simplified into the form

$$A\ddot{q}_1 = B + CM \quad (3.17)$$

where the scalar coefficients are presented in Appendix C. This formula was obtained in order to develop the controller presented in Section 3.4.

3.4 Controller Design

The current configuration of the models allows for the use of single-input-single-output controllers. The controllers are designed such that the closed-loop poles have the desired damping ratios and undamped natural frequencies that match the experimental data. The initial derivation is based on the inverted pendulum. Values for the four-bar linkage are subsequently provided in Section 3.4.2.

3.4.1 Inverted Pendulum

The governing moment balance for the inverted pendulum model Equation 3.8 was found to be

$$-M_A = (I_G + mL^2)\ddot{\theta} + mgL\theta$$

This equation is used to develop a controller using the pole placement approach detailed by Ogata [33].

Defining the state variables x_1 and x_2 as

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

The output of the system becomes

$$y = \theta = x_1$$

In vector-matrix form the governing equations become

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgL}{I_G+mL^2} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_G+mL^2} \end{bmatrix} u \quad (3.18)$$

and

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (3.19)$$

The equations can be rewritten more compactly as

$$\{\dot{x}\} = [A]\{x\} + [B]u \quad (3.20)$$

and

$$[y] = [C]\{x\} \quad (3.21)$$

where

$$[A] = \begin{bmatrix} 0 & 1 \\ \frac{mgL}{I_G+mL^2} & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 \\ \frac{1}{I_G+mL^2} \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (3.22)$$

In order to find the value of the control signal u , a state-feedback control scheme is used such that

$$u = -[K]\{x\} \quad (3.23)$$

where $\{x\}$ is the state vector, u is the control signal, and $[K]$ is the state feedback gain matrix.

The characteristic equation of the system as indicated by Ogata [33] is

$$|s[I] - [A]| = \begin{vmatrix} s & -1 \\ \frac{mgL}{I_G + mL^2} & s \end{vmatrix} \quad (3.24)$$

$$s^2 + a_1s + a_2 = s^2 + \frac{mgL}{I_G + mL^2} \quad (3.25)$$

Therefore, based on Equation 3.25

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{mgL}{I_G + mL^2} \end{aligned}$$

The poles μ_1 and μ_2 are the desired poles of the system, and are chosen as discussed subsequently in Section 3.5. Once the poles are chosen, the desired characteristic equation becomes

$$(s - \mu_1)(s - \mu_2) = s^2 - (\mu_1 + \mu_2)s + \mu_1\mu_2 \quad (3.26)$$

alternatively written as

$$0 = s^2 + \alpha_1s + \alpha_2 \quad (3.27)$$

Therefore, solving for α_1 and α_2 results in

$$\alpha_1 = \mu_1 + \mu_2, \quad \alpha_2 = \mu_1\mu_2$$

The gain matrix $[K]$ is then given as

$$[K] = \begin{bmatrix} \alpha_2 - a_2 & \vdots & \alpha_1 - a_1 \end{bmatrix} [T]^{-1}$$

where

$$[T] = [M][W]$$

and $[M]$ and $[W]$ are given by

$$[M] = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{I_G + mL^2} \\ \frac{1}{I_G + mL^2} & 0 \end{bmatrix} \quad (3.28)$$

$$[W] = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore the matrix $[T]$ becomes

$$[T] = [M][W] = \begin{bmatrix} \frac{-1}{I_G + mL^2} & 0 \\ \frac{mgL}{I_G + mL^2} a_1 & \frac{mgL}{I_G + mL^2} \end{bmatrix}$$

and

$$[T]^{-1} = \begin{bmatrix} -I_G - mL^2 & 0 \\ a_1 I + a_1 mL^2 & \frac{I_G + mL^2}{mgL} \end{bmatrix}$$

The desired feedback gain matrix becomes

$$\begin{aligned} [K] &= \begin{bmatrix} \alpha_2 - a_2 & \vdots & \alpha_1 - a_1 \end{bmatrix} \begin{bmatrix} -I_G - mL^2 & 0 \\ a_1 I + a_1 mL^2 & \frac{I_G + mL^2}{mgL} \end{bmatrix} \\ &= \begin{bmatrix} (-\alpha_2 + a_2 + a_1 \alpha_1 - a_1^2)(I_G - mL^2) & (\alpha_1 - a_1) \frac{I_G + mL^2}{mgL} \end{bmatrix} \end{aligned}$$

The control signal u is then given by

$$u = -[K]\{x\} = -(-\alpha_2 + a_2 + a_1\alpha_1 - a_1^2)(I_G - mL^2)x_1 - (\alpha_1 - a_1)\frac{I_G + mL^2}{mgL}x_2$$

3.4.2 Four-Bar Linkage

In the case of the four-bar linkage model equation it is necessary to first rearrange and simplify the governing dynamic equation into the same form

$$A\ddot{q}_1 = Bq_1 + CM \quad (3.29)$$

where A, B and C are the values found by linearizing the equation about a value of 90° for q_1 . This is a reasonable assumption due to the small variations from vertical experienced by the model. The numerical values for A, B, and C are found by the following steps:

1. The value of B is found by holding all variables constant, with the exception of q_1 .
2. The equation is solved using the linearization point, for example for a q_1 of 90° , then the equation is solved again for a value of q_1 plus or minus a small increment. The values found by solving the equation are then subtracted from each other and divided by the increment used.
3. The chosen variable is successively modified by decreasing increments and step 2 is repeated until little to no change is found in the value calculated. This then becomes the desired number for that variable.
4. The process is then repeated until approximations have been found for all three variables in the equation.

Through this process the linearized equation for the four-bar linkage was found to be

Table 3.5: Comparison of MIIs in the NBDL experimental data

	Average MIIs (Observer)	Average MIIs (Subject)
Inverted pendulum tuning	20.2	17.8
Inverted pendulum validation	14.5	10.6
Four-bar tuning	6.7	9.6
Four-bar validation	2.3	2.5

$$-0.02476 \ddot{q}_1 = 0.0006133 q_1 - 0.00008913 M \quad (3.30)$$

The values were then substituted, and the matrixes from Section 3.4 become:

$$[A] = \begin{bmatrix} 0 & 1 \\ -0.02477 & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 \\ 0.00036 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

3.5 Pole Selection

The selection of the poles for the controller developed in Section 3.4 were obtained using experimental data.

Experiments were conducted at the US Naval Biodynamics Laboratory using volunteer naval personnel. During the experiment the navy personnel performed physical and cognitive tasks while experiencing simulated ship motions. The motions used representative motions for the flight deck of a United States Navy FFG7 class frigate in sea state five, with a ship speed of 10 knots operating in short-crested irregular seas with a significant wave height of 3.25 metres and wave modal period of 10 seconds. Two motion profiles were created, one representing bow seas, and the other representing stern quartering seas (designated in the experiment documentation as cases Lima and Hotel respectively). The total number of subjects used during the experiment was 15. They ranged in age from 17 to 24 years old. The subjects were asked to do the following five physical tasks during the

experiment:

1. stand facing astern to observe body lateral MIIs (2 minutes);
2. perform a weight positioning task which mimics the loading of a weapon (1 minute);
3. raise a weight, standing facing astern to observe the effect of a change in the centre of gravity location on body lateral MIIs (1 minute);
4. walking (1 minute); and
5. stand facing port in order to observe body longitudinal MIIs (1 minute).

This sequence of tasks was repeated 10 times during each experimental run. In this experiment incidences of MIIs were indicated by an independent observer as well as by the subject. As some of the data were not appropriate for the simulation of the MIIs done in this body of work, only the relevant time histories were used. The results from Task 1 were used in the tuning and validation of the four-bar linkage model, and the results from Task 5 were used in the tuning and validation of the inverted pendulum model.

Not all 15 of the subject results were used as some of the subjects did not complete the entire set of 10 cycles in the tests. Table 3.5 presents the average results for the MIIs experienced by those subjects that did finish the entire run, in this case 5.

3.5.1 Inverted Pendulum

The poles that were needed in order to determine the gains for the controllers developed in Section 3.4 were picked through an iterative process. For the inverted pendulum, the poles were chosen based on a damping ratio, ζ , of 0.9 as found by Corveil et al. [34] for quiet standing. Using this information and modifying the settling time in the range from 0 to 10 seconds, the real and imaginary poles were calculated from

$$\omega_n = \frac{4}{\zeta t_s} \quad (3.31)$$

$$\text{real pole} = \zeta \omega_n \quad (3.32)$$

$$\text{imaginary pole} = \omega_n(1 - \zeta^2) \quad (3.33)$$

where t_s is the settling time. The final gain matrix used was

$$[K] = \begin{bmatrix} 715.86 & 64.17 \end{bmatrix} \quad (3.34)$$

As well as the average number of occurrences, where possible, the major occurrences, which are occurrences that the majority (minimum of seven subjects) of the test subjects experienced, were also targeted for reproduction with the biomechanical model. For example, for the tuning of the inverted pendulum, the average number of occurrences was 18.5, there was an obvious common tip at 125 seconds. For the tuning of this model, the biomechanical model predicted 11 MIIs, but caught two major, which would indicate an acceptable result.

3.5.2 Four-Bar Linkage

The four-bar linkage poles were selected by varying both the real and imaginary components for combinations between 0 and 10 units. Once the poles were generated they were applied to the models and the resulting model performance evaluated. The models were tuned by running the model with the selected poles, and experimental motion, then graphing the results with the results from the NBDL experiments [4], as shown in Figures 3.4 and 3.5. Half of the data obtained were used to pick the poles and the other half were used to validate the poles as presented in Chapter 4. In Figures 3.4 and 3.5 the results

from Graham's model are indicated by a +, the results from the biomechanical model are indicated by a * and the results from the humans in the experiment are indicated by an x, and each row refers to a different subject. The models were tuned so that, where possible, the number of MIIs resulting from the articulated model was approximately the average number that the humans in the experiment experienced.

The final gain matrix used was

$$[K] = \begin{bmatrix} 593.49 & -446.90 \end{bmatrix} \quad (3.35)$$

As well as the average number of occurrences, where possible, the major occurrences, which are occurrences that the majority of the test subjects experienced, were also targeted for reproduction with the biomechanical model. For example, for the tuning of the four-bar linkage, the average number of occurrences was 8.5, there was one obvious common tip at 175 seconds, and a minor one around 420 seconds. For the tuning of that model, the biomechanical model predicted 9 MIIs, but caught all the major tips, which would indicate an acceptable result.

Figure 3.4: Graham's (+), experimental (x), and inverted pendulum models (*) plot of tipping events as a function of time for the tuning data set

Figure 3.5: Graham's (+), experimental (x), and four-bar linkage models (*) plot of tipping events as a function of time for the tuning data set

Chapter 4

Validation and Sample Results

The process of evaluating the models, and the selected poles was performed in two steps, first validating the dynamic models and selected controller poles against experimental data, and then by running the models with realistic ship motion as developed in Chapter 2.

4.1 Validation

The validation process used was to take half of the data obtained from the NBDL [4], that were not used for tuning the controllers and run the models with the poles obtained in Section 3.5. The results obtained from this process are shown in Figures 4.1 and 4.2.

If the following conditions were determined to be true for the validation, then the model can be considered valid within the range of conditions spanned by the validation process. The two conditions used are the following.

1. The number of MII occurrences, that is tips, would be the same for the biomechanical model as for the average number of tips experienced by the humans in the experiment.
2. The major MII occurrences felt by the humans, for example at time 417 in Figure 4.2, would also be detected by the biomechanical model.

Figure 4.1: Graham's model (+), experimental (x), and inverted pendulum (*) model results of tipping events as a function of time for the validation data set

Figure 4.2: Graham's model (+), experimental (x), and four-bar linkage (*) model results of tipping events as a function of time for the validation data set

Table 4.1: Comparison of MIIs in the NBDL experimental data

	Average NBDL MIIs	Graham's Model MIIs	Biomechanical Model MIIs
Inverted pendulum tuning	20.2	55	11
Inverted pendulum validation	14.5	66	8
Four-bar tuning	6.7	5	10
Four-bar validation	2.3	4	6

Table 4.2: Comparison of major incidents between the NBDL and the biomechanical models

	Average NBDL MIIs	Biomechanical Model MIIs
Inverted pendulum tuning	3	2
Inverted pendulum validation	4	3
Four-bar tuning	1	1
Four-bar validation	2	2

It can be seen in Figures 4.1 and 4.2 that for both models both conditions 1 and 2 are generally satisfied. This indicates that the poles picked through the process in Section 3.5 are valid for these conditions. In these plots the MIIs determined by Graham's model are plotted as a value of 12. By looking at the results for the inverted pendulum model, shown in Table 4.1, for both tuning and validation, Graham's model overpredicts the number of MIIs. In the case of the four-bar linkage model, for both tuning and validation, Graham's model generally underpredicts the values. This contrast of results for Graham's model indicates that it might have room for improvement in order to obtain results closer to those experienced by the subjects. It can be seen in Table 4.2, which compares the number of major tips for both models and trial data, that the biomechanical models predict a similar number of major tips to the experiments, and less than Graham's model.

Figure 4.3: Comparison of tipping results for Graham's and the combined biomechanical model for sea state 4

4.2 Sample Results

Once the poles were chosen and validated as in Section 4.1, ship motions, created as in Chapter 2, were applied to a combined model. The model was run as a combined model as Matjacic et al. [19] found that the two models could be treated independently, and subsequently combined together. The motions that were applied to the model were limited to the same three degrees of freedom as the models were tuned for, that is pitch, roll and heave. The results for these three motions for sea state 4 are shown in Figure 4.3 with the combined model results plotted from 0° to 180° and Graham's model results plotted from 180° to 360° . These results are for a human standing approximately at the centre of the flight deck, which for this simulation is at a point along the centreline of the ship, 50 metres behind, and 6.52 metres above the centre of rotation.

It is evident that the results predicted by Graham's model and the combined biomechanical model do not always agree. The models appear to agree for head seas in sea state 4, and for head seas at slower speeds for sea state 5 as is evident from Figure 4.4.

Figure 4.4: Comparison of tipping results for Graham's and the combined biomechanical model for sea state 5

This is not entirely unexpected as the ship motion that the experimental data were based on corresponded to a speed of 10 knots in irregular seas with a significant wave height of 3.2 metres and a wave modal period of 10 seconds, and therefore higher speeds are not guaranteed to agree, as the poles were not tuned for those conditions. It is also known that Graham's model is incorrect in regards to its prediction of MIIs, that is it generally overpredicts the number of MIIs, and the biomechanical model predicts fewer MIIs. In actuality there is no way to prove, without conducting further human trials, whether the model is accurate for higher speeds or sea states.

Chapter 5

Conclusion and Recommendations

This chapter provides a concluding discussion of the project as well as extensive recommendations for continuing research on articulated postural stability models.

5.1 Conclusion

This project has achieved the intended objective of developing and refining a dynamic postural stability model applicable in a moving environment. However, comprehensive validation of the model over a range of operating conditions was limited by available experimental data.

The sagittal plane model (the inverted pendulum) was found to be the simpler part of the overall combined model to construct, as it has been widely explored for quiet standing. The current model is new in the fact that it was tuned to be able to respond in a continuously moving environment and was validated in that moving environment. Once the model was tuned to the environment, it proved to be generally acceptable for the conditions for which it was tuned. This suggests that if a model was tuned in the manner described for the required environment, it would generally behave in the desired manner. If tuning over a range of conditions were possible, the model could evaluate the motion environment to

which it was subjected and use the appropriate controller gains in order to correctly react to the environment, thereby making it applicable over a wide range of conditions.

The frontal plane model (the four-bar linkage) was found to be the most complex aspect of the combined biomechanical model. In general, when this model has been used for quiet standing, controllers have been placed at all four joints. The model in this work combines these controllers into one torque at one ankle due the fact that by using a four-bar linkage the system degrees of freedom in the dynamic equations can be effectively reduced to one degree-of-freedom, in this case q_1 . By making the model into a one degree-of-freedom model, the method used to develop the inverted pendulum controller could also be used for four-bar linkage, with the appropriate state-space representation of the governing dynamics. From the tuning and validation results it can be seen that the method appears to be valid.

After validation of the individual models, they were combined and run for realistic, though three degree-of-freedom, ship motion. It was shown from the results that when the models are applied to conditions that differ from those for which they were tuned, the models may not behave in the expected manner, and can in fact overpredict MIIs. This tendency could be associated with the fact that the models are not able to adjust their gains to the severity of the ship motion; and the gains used for the models were tuned to less severe motions.

5.2 Recommendations

Due to the limited application of the developed model, it is recommended that effort be focussed in the following areas in order to improve the potential for applicability of future models to a wider range of environments.

5.2.1 Experimental Data

The current experimental data that were used in the development of this model were limited in their applicability. In the NBDL experiments the incidences of MIIs were obtained by observations made by either the test subject or an independent observer. The problem with this method is that the accuracy of the observations can come into question. For example, in one of the cases the observer indicated that the subject experienced 32 MIIs, whereas the subject only indicated 21 MIIs, this is a difference of 11, or over 50% based on the subject's perception. Another issue with the current data is that it only simulates a specific set of conditions, that is sea state 5 with a ship speed of 10 knots and bow and quartering seas. It is suggested that the following take place for any future experiments conducted.

- Future testing should be done in as many sea states as is physically possible with the motion simulator used. This would allow for testing and validation of any model in multiple conditions. As well, the testing should be done with all six ship motions included.
- Occurrences of MIIs should be captured using methods other than subjective observations. For example, detection could be accomplished using force plates under the subject's feet. This would also allow the occurrence of an MII to be detected the instant that it occurred, thus allowing better correlation between developed models and experimental results.
- Motion capture software and hardware could be used in order to obtain exact positions of the subject during the trial. Motion capture is the process of capturing the movement of a real object and mapping it onto a computer generated object. Garcia et al. [35] indicate that this process can be used to capture the position and orientation of several joints, in this case on the human, which are then recorded. These

recorded values then create a set of one-dimensional signals known as motion curves. These curves are then processed and mapped onto a skeleton hierarchy, producing virtual motion that mimics the captured motion. By producing these virtual motion drivers, simulation created within the computer could be tested in order to find out how closely it matches the actual subject motion.

- Another thing that could be done is measurement of muscle force, for example by using surface electromyography (sEMG). Surface electromyography measures the electrical current, or signal, produced by the muscles by placing a sensor on the skin directly above the muscle. This could be useful in determining control laws and activation of the required muscles and control laws, as well as for determining fatigue limits.

5.2.2 Dynamic Model

As well as modifications to future experiments, improvement of the existing models could be achieved by considering the following.

- The current model has joints only at the ankles for the sagittal plane, and at the hips and ankles for the frontal plane. In order to improve the model, it is possible to add more degrees of freedom. In the sagittal plane, joints at the knees and hips could be added, thus creating a multi-link inverted pendulum, as has been developed for quiet standing. In the sagittal plane, knees could be added in the form of sliders, as well as a joint at the neck, thus allowing the head to maintain its attitude, an important sensor as indicated by Van der Kooij et al. [20]. Further, an articulated spine or rotational joint that somewhat mimics the spine could be beneficial.
- The current model is based on linear control theory. In order to improve the response of this model to different input conditions, for example different sea states,

an adaptive controller could be used, as was explored for quiet standing by Soetanto [15]. In his work he indicated that although proportional-integral-derivative (PID) control works well for linearized skeletal models, it does not work well for models of muscles. He suggested other models, such as switching controllers, in which the controller changes states from one linear region to another. He also suggested that a neurocontroller might work well as it solves many simple operations all at once in a parallel fashion.

The research presented in this thesis introduces articulated biomechanical modelling for the computational estimation of MIIs in potentially-severe ship motion environments. As the subject area is somewhat novel, numerous opportunities for refinement exist. However, evidence included in this thesis clearly demonstrates that the potential for improved MII models exists by including articulation and appropriate control.

References

- [1] A.E. Baitis, D.A. Wollaver, and T.A. Beck. Rudder roll stablilzation for coast guard cutters and frigates. *Naval Engineers Journal*, 1983.
- [2] R. Graham, A.E. Baitis, and W.G. Meyers. On the development of seakeeping criteria. *Naval Engineers Journal*, pages 259–275, May 1992.
- [3] A.R.J.M. Lloyd. *SEAKEEPING: Ship behaviour in rough weather*. A R J M Lloyd, 26 Spithead Avenue, Gosport, Hampshire, United Kingdom, revised edition, 1998.
- [4] A.E. Baitis, F.D. Holcombe, S.L. Conwell, P. Crossland, J. Colwell, J.H. Pattison, and R. Strong. 1991-1992 Motion Induced Interruption (MII) and Motion Induced Fatigue (MIF) experiments at the Naval Biodynamics Laboratory. Technical Report CRDKNSWE-HD-1423-01, Carderock Division, Naval Surface Warfare Center, 1995.
- [5] C.H. Lewis and M.J. Griffin. Modelling the effects of deck motion on energy expenditure and motion-induced fatigue. Technical Report HFRU 98/05, Institue of Sound and Vibration Research, February 1998.
- [6] P. Crossland. Using the Large Motion Simulator (LMS) at DRA Bedford to investigate the effects of ship motions on motion induced interruptions. Technical Report DRA/SS/SSHE/CR96008, Defence Research Agency, February 1996.

- [7] P. Crossland, K.J.N.C. Rich, and D. Granshaw. Validating a model for predicting motion induced interruptions to task performance using simulated motions from the FFG-8 and Type 23 frigate. Technical Report DERA/SS/HE/CR971017, Defence Evaluation and Research Agency, October 1997.
- [8] S.L. Bales, W. T. Lee, and J.M. Voelker. Standardized wind and wave environments for NATO operational areas. Technical Report SPD-0919-01, DTNSRDC, 1980.
- [9] Y. Takaishi, T. Matsumoto, and Ohmatsu. S. Winds and waves of the North Pacific Ocean 1964-1973. Technical report, SRI, 1980.
- [10] K. A. McTaggart. *SHIPMO07: An updated strip theory program for predicting ship motions and sea loads in waves*. Defence Research Establishment Atlantic, Dartmouth, Nova Scotia, Canada, March 1996. DREA Technical Memorandum 96/243.
- [11] L. Vernooy and R. G. Langlois. *User's Guide for the ShipSim 4.2 Ship Motion Simulation Program*. Indal Technologies Inc., Mississauga, Ontario, Canada, August 1999.
- [12] A.E. Baitis, T.R. Applebee, and T.M. McNamara. Human factor considerations applied to operations of the FFG-8 and Lamps MK III. *Naval Engineers Journal*, 97(4), 1984.
- [13] R. Graham. Motion-induced interruptions on ship operability criteria. *Naval Engineers Journal*, 102(2), 1990.
- [14] Y.-C. Pai, B.E. Maki, K. Iqbal, W.E. McIlroy, and S.D. Perry. Thresholds for step initiation induced by support-surface translation: a dynamic center-of-mass model provides much better prediction than a static model. *Journal of Biomechanics*, 33:387–392, 2000.

- [15] D. Soetanto. *Stabilization of a Human Standing Posture Using Functional Neuromuscular Stimulation*. PhD thesis, Arizona State University, May 2000.
- [16] I.D. Loram, S.M. Kelly, and M. Lakie. Human balancing of an inverted pendulum: is sway size controlled by ankle impedance? *Journal of Physiology*, 532.3:879–891, 2001.
- [17] R. Kobetic, S.G. Carroll, and E.B. Marsolais. Functional electrical stimulation of hip extension and abduction affected by activity of the trunk. *Proceeding of the 10th Annual Conference on Rehabilitation Technology (RESNA), San Jose, CA*, pages 618–620, 1987.
- [18] H Hemami, M.J. Hines, R.E. Goddard, and B. Friedman. Biped sway in the frontal plane with locked knees. *IEEE Transactions on Systems, Man , and Cybernetics*, SMC-12(4):577–582, July/August 1982.
- [19] Z. Matjacic, M. Voigt, D. Popovic, and T. Stinkjær. Fonctional postural responses after perturbations in mulitple directions in a standing man:a principle of decoupled control. *Journal of Biomechanics*, 34:187–196, 2001.
- [20] H. van der Kooij, R. Jacobs, B. Koopman, and H. Grotenboer. A mulitsensory integration model of human stance control. *Biological Cybernetics*, 80(5):299–308, 1999.
- [21] van der Helm Fans C.T. Schouten Alfred C. Brouwn Guido G. de Vlugt, Erwin. Analysis of the reflexive feedback control loop during posture maintenance. *Biological Cybernetics*, 84:133–141, 2001.
- [22] Arthur D. Kuo. An optimal control model for analyzing human postural balance. *IEEE Transactions on Biomedical Engineering*, 42(01):87–101, January 1995.

- [23] H. van der Kooij, R. Jacobs, B. Kiipman, and F. van der Helm. An adaptive model of sensory integration in a dynamic environment applied to human stance control. *Biological Cybernetics*, 84:103–115, 2001.
- [24] P. Micheau, A. Kron, and Bourassa P. Analysis of human postural stability based on the inverted pendulum model with time-delay in feedback. *Proceedings of the American Control Conference*, pages 2297–2298, June 2001.
- [25] S. Onyshko and D.A. Winter. A mathematical model for the dynamics of human locomotion. *Journal of Biomechanics*, 13:361–368, 1980.
- [26] M.G. Pandy and N. Berme. A numerical method for simulating the dynamics of human walking. *Journal of Biomechanics*, 12:1043–1051, 1988.
- [27] C.E. Bauby and A.D. Kuo. Active control of lateral balance in human walking. *Journal of Biomechanics*, 33:1433–1440, 2000.
- [28] MSC.ADAMS Technical Staff. *Road Map to ADAMS/View Documentation*. Mechanical Dynamics, Inc., 2002.
- [29] Military Agency for Standardization. Standardized wave and wind environments and shipboard reporting of sea conditions. Standardization Agreement 4194, North Atlantic Treaty Organization, April 1983.
- [30] N. Diffrient, A. R. Tilley, and J. C. Bardagjy. *Humanscale 1/2/3*. The MIT Press, Massachusetts Institute of Technology, Cambridge, Mass., 1974.
- [31] R.F. Chandler, C.E. Clauser, J.T. McConville, M.H. Reynolds, and J.W. Young. Investigation of inertial properties of the human body. Technical Report DOT HS-801 430, U.S. Department of Transportation, 400 Seventh Street S.W. Washington D.C., March 1975.

- [32] R. G. Langlois and R. J. Anderson. Multibody dynamics of very flexible damped systems. *Multibody System Dynamics*, 3(2):109–136, May 1999.
- [33] K. Ogata. *Modern Control Engineering*. Prentice Hall Inc., Upper Saddle River, New Jersey 07458, third edition, 1997.
- [34] P. Corbeil, M. Simoneau, D. Rancourt, A. Tremblay, and N. Teasdale. Increased risk for falling associated with obesity: Mathematical modeling of postural control. *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, 9(2):126–136, June 2001.
- [35] L. Garcia, F. Da Silva, A. Oliveira, L. Velho, and J. Gohes. Embedding a motion-capture interface in a control structure for human-like agent behavior achievement. Four International Conference on AUTONOMOUS AGENTS, August 2000.

Appendix A

MII Results

The following are the complete set of results from Section 2.6.2, where the ADAMS model results are plotted from 0 to 180 degrees and Graham's model results are plotted from 180 to 360 degrees.

Figure A.1: Comparison of tipping from Graham's and ADAMS results for sea state 4

Figure A.2: Comparison of tipping from Graham's and ADAMS results for sea state 6

Figure A.3: Comparison of tipping from Graham's and ADAMS results for sea state 7

Appendix B

Inertial Properties of Biomechanical Model Segments

The calculations for the human are obtained through regression equations found by Chandler et al. [31]. In this case, the equations for the left portion of the body are shown, as the body was assumed to be symmetrical about the centreline.

Figure B.1 shows the locations of the corresponding segments defined below, and the orientation of the coordinate system used in the study done by Chandler.

The following equations are the regression equations presented for the head, which in this case does not include the neck.

$$Weight = 0.032m + 1906$$

$$I_x = 2.129m + 32030$$

$$I_y = 1.676m + 54818$$

$$I_z = 3.186m - 6846$$

where *Weight* is the segment mass measured in *grams*, I_i is the mass moment of inertia

Figure B.1: Location of individual human segments

about the ii axis with units of $gram - cm^2$, and m is the total mass of the human measured in *grams*. The same unit convention is also used for subsequent regression equations.

The following equations are the regression equations presented for the torso, which includes the neck.

$$Weight = 0.532m - 706$$

$$I_x = 296.900m - 3156034$$

$$I_y = 284.493m - 7664880$$

$$I_z = 102.507m - 2895524$$

The following equations are the regression equations presented for the upper arm.

$$Weight = 0.022m + 485$$

$$I_x = 2.096m + 15569$$

$$I_y = 1.352m + 49572$$

$$I_z = 0.567m - 14171$$

The following equations are the regression equations presented for the forearm.

$$Weight = 0.013m + 246$$

$$I_x = 0.659m + 21806$$

$$I_y = 0.727m + 15672$$

$$I_z = 0.230m - 6396$$

The following equations are the regression equations presented for the hand, which includes the wrist.

$$Weight = 0.005m + 76$$

$$I_x = 0.083m + 1437$$

$$I_y = 0.100m - 920$$

$$I_z = 0.028m - 6$$

The following equations are the regression equations presented for the thigh.

$$Weight = 0.127m - 1511$$

$$I_x = 21.310m - 172235$$

$$I_y = 23.633m - 319070$$

$$I_z = 5.404m - 139702$$

The following equations are the regression equations presented for the calf.

$$Weight = 0.044m - 178$$

$$I_x = 6.434m - 24410$$

$$I_y = 5.350m + 40974$$

$$I_z = 0.969m - 34567$$

The following equations are the regression equations presented for the foot, which includes the ankle.

$$Weight = 0.009m + 253$$

$$I_x = 0.371m + 8974$$

$$I_y = 0.391m + 4959$$

$$I_z = 0.130m - 956$$

Although these segments were created as individual bodies, where possible in the ADAMS biomechanical model they were combined using the built in merge function. Using this function ADAMS mathematically combines the properties and calculates the correct mass moments of inertia, and total weight for the new combined segments.

Appendix C

Equations of Motion for the Four-Bar linkage

The calculation of the dynamic equations used for the four-bar linkage are numerically developed here. Figure C.1 shows a schematic illustration of the inverted pendulum model. Derivation of the governing equations of motion begins with an expression of the position vector

$$\vec{r}_{j1} = 0$$

$$\vec{r}_{b1} = L_1 \cos(q_1) \vec{i} + L_1 \sin(q_1) \vec{j}$$

$$\vec{r}_{j2} = (L_1 + L_2) \cos(q_1) \vec{i} + (L_1 + L_2) \sin(q_1) \vec{j}$$

$$\begin{aligned} \vec{r}_{b2} &= \left((L_1 + L_2) \cos(q_1) + \sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) \right) \vec{i} \\ &= + \left((L_1 + L_2) \sin(q_1) + \sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) \right) \vec{j} \end{aligned}$$

$$\vec{r}_{j3} = ((L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2)) \vec{i}$$

$$= ((L_1 + L_2) \sin(q_1) + 2 L_3 \sin(q_1 + q_2)) \vec{j}$$

Figure C.1: Values for the four-bar linkage

$$\begin{aligned}
 \vec{r}_{b3} &= ((L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + L_2 \cos(q_1 + q_2 + q_3)) \vec{i} \\
 &= ((L_1 + L_2) \sin(q_1) + 2 L_3 \sin(q_1 + q_2) + L_2 \sin(q_1 + q_2 + q_3)) \vec{j} \\
 \vec{r}_{j4} &= ((L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) \cos(q_1 + q_2 + q_3)) \vec{i} \\
 &= ((L_1 + L_2) \sin(q_1) + 2 L_3 \sin(q_1 + q_2) + (L_1 + L_2) \sin(q_1 + q_2 + q_3)) \vec{j}
 \end{aligned}$$

Once the position is known the equations are differentiated in order to obtain the relevant velocities. The velocities of each link and joint are

$$\begin{aligned}
 \vec{v}_{j1} &= 0 \\
 \vec{v}_{b1} &= -L_1 (\dot{q}_1) \sin(q_1) \vec{i} + L_1 (\dot{q}_1) \cos(q_1) \vec{j} \\
 \vec{v}_{j2} &= (L_1 + L_2) (\dot{q}_1) \sin(q_1) \vec{i} + (L_1 + L_2) (\dot{q}_1) \cos(q_1) \vec{j} \\
 \vec{v}_{b2} &= \left((L_1 + L_2) (\dot{q}_1) \sin(q_1) + \sqrt{L_3^2 + h^2} (\dot{q}_1) \sin(q_1 + q_2 + \beta) \right) \vec{i} \\
 &= \left((L_1 + L_2) (\dot{q}_1) \cos(q_1) + \sqrt{L_3^2 + h^2} (\dot{q}_1) \cos(q_1 + q_2 + \beta) \right) \vec{j}
 \end{aligned}$$

$$\begin{aligned}
\vec{v}_{j3} &= ((L_1 + L_2) (\dot{q}_1) \sin(q_1) + 2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) \\
&\quad + (L_1 + L_2) (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3)) \vec{i} \\
&\quad ((L_1 + L_2) (\dot{q}_1) \cos(q_1) + 2 L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) \\
&\quad + (L_1 + L_2) (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3)) \vec{j} \\
\vec{v}_{b3} &= ((L_1 + L_2) (\dot{q}_1) \sin(q_1) + 2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) \\
&\quad + L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3)) \vec{i} \\
&\quad ((L_1 + L_2) (\dot{q}_1) \cos(q_1) + 2 L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) \\
&\quad + L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3)) \vec{j} \\
&= +L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3)) \vec{j}
\end{aligned}$$

These equations are then assembled into the partial velocity matrix such that

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_{11} & 0 & 0 \\ V_{12} & 0 & 0 \\ 0 & 0 & 0 \\ V_{14} & V_{24} & 0 \\ V_{15} & V_{25} & 0 \\ 0 & 0 & 0 \\ V_{17} & V_{27} & V_{37} \\ V_{18} & V_{28} & V_{38} \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{11} = -L_1 \cos(q_1) \dot{q}_1$$

$$V_{12} = -L_1 \sin(q_1) \dot{q}_1$$

$$\begin{aligned}
V_{14} &= -(\dot{q}_1 + \dot{q}_2) \sin(q_1) - (q_1 + q_2) \cos(q_1) \dot{q}_1 \\
&\quad - \sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2) \\
V_{24} &= -\sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2) \\
V_{15} &= (\dot{q}_1 + \dot{q}_2) \cos(q_1) - (q_1 + q_2) \sin(q_1) \dot{q}_1 \\
&\quad - \sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2) \\
V_{25} &= -\sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2) \\
V_{17} &= -(\dot{q}_1 + \dot{q}_2) \sin(q_1) - (q_1 + q_2) \cos(q_1) \dot{q}_1 \\
&\quad - 2 L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) - L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \\
V_{27} &= -2 L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) - L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \\
V_{37} &= -L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \\
V_{18} &= (\dot{q}_1 + \dot{q}_2) \cos(q_1) - (q_1 + q_2) \sin(q_1) \dot{q}_1 \\
&\quad - 2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) - L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3) \\
V_{28} &= -2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) - 2 \sin(q_1 + q_2 + q_3) (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\
V_{38} &= -L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3)
\end{aligned}$$

The partial velocity matrix is then differentiated in order to get $[\dot{V}]$

$$[\dot{V}] = \begin{bmatrix} \dot{V}_{11} & 0 & 0 \\ \dot{V}_{12} & 0 & 0 \\ 0 & 0 & 0 \\ \dot{V}_{14} & \dot{V}_{24} & 0 \\ \dot{V}_{15} & \dot{V}_{25} & 0 \\ 0 & 0 & 0 \\ \dot{V}_{17} & \dot{V}_{27} & \dot{V}_{37} \\ \dot{V}_{18} & \dot{V}_{28} & \dot{V}_{38} \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$\dot{V}_{11} = -L_1 (\dot{q}_1) \cos(q_1)$$

$$\dot{V}_{12} = -L_1 (\dot{q}_1) \sin(q_1)$$

$$\dot{V}_{14} = -(\dot{q}_1 + \dot{q}_2) \sin(q_1) - (q_1 + q_2) \cos(q_1) \dot{q}_1 - \sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2)$$

$$\dot{V}_{24} = -\sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2)$$

$$\dot{V}_{15} = (\dot{q}_1 + \dot{q}_2) \cos(q_1) - (q_1 + q_2) \sin(q_1) \dot{q}_1 - \sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2)$$

$$\dot{V}_{25} = -\sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) (\dot{q}_1 + \dot{q}_2)$$

$$\begin{aligned} \dot{V}_{17} = & -(\dot{q}_1 + \dot{q}_2) \sin(q_1) - (q_1 + q_2) \cos(q_1) \dot{q}_1 - 2L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) \\ & -L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \end{aligned}$$

$$\begin{aligned}
\dot{V}_{27} &= -2 L_3 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) - L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \\
\dot{V}_{37} &= -L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \cos(q_1 + q_2 + q_3) \\
\dot{V}_{18} &= (\dot{q}_1 + \dot{q}_2) \cos(q_1) - (q_1 + q_2) \sin(q_1) \dot{q}_1 - 2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) \\
&\quad - L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3) \\
\dot{V}_{28} &= -2 L_3 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) - 2 \sin(q_1 + q_2 + q_3) (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\
\dot{V}_{38} &= -L_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \sin(q_1 + q_2 + q_3)
\end{aligned}$$

Once $[\dot{V}]$ has been calculated the constraint equations must be evaluated. The positions of the other joints are then calculated in reference to the cotree joint. They are:

$$\begin{aligned}
\vec{r}_{j3j4} &= -(L_1 + L_2) \cos(q_1) \vec{i} - (L_1 + L_2) \sin(q_1) \vec{j} \\
\vec{r}_{j2j4} &= (-(L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2)) \vec{i} \\
&\quad + (-(L_1 + L_2) \sin(q_1) + 2 L_3 \sin(q_1 + q_2)) \vec{j} \\
\vec{r}_{j1j4} &= (-(L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) \cos(q_1 + q_2 + q_3)) \vec{i} \\
&\quad + (-(L_1 + L_2) \sin(q_1) + 2 L_3 \sin(q_1 + q_2) + (L_1 + L_2) \sin(q_1 + q_2 + q_3)) \vec{j}
\end{aligned}$$

In the case of this model, the cotree joint is a revolute joint and therefore imposes two orthogonal translational velocity constraints. Therefore, for a revolute joint the x and y equations are used such that the constraint Jacobian matrix becomes

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \end{bmatrix}$$

where

$$J_{11} = -(L_1 + L_2) \sin(q_1) - 2 L_3 \sin(q_1 + q_2) - (L_1 + L_2) \sin(q_1 + q_2 + q_3)$$

$$J_{12} = -2 L_3 \sin(q_1 + q_2) - (L_1 + L_2) \sin(q_1 + q_2 + q_3)$$

$$J_{13} = (L_1 + L_2) \sin(q_1 + q_2 + q_3)$$

$$J_{21} = (L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) \cos(q_1 + q_2 + q_3)$$

$$J_{22} = 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) L_3 \cos(q_1 + q_2 + q_3)$$

$$J_{23} = (L_1 + L_2) \cos(q_1 + q_2 + q_3)$$

The generalized speed vector is partitioned into n independent generalized speeds and m dependent generalized speeds. Using these two quantities, the matrix $[C]$ relating independent and dependent generalized speeds is

$$[C] = [J_m]^{-1}[J_n]$$

where

$$[J_m] = \begin{bmatrix} -2 L_3 \sin(q_1 + q_2) - (L_1 + L_2) \sin(q_1 + q_2 + q_3) & -(L_1 + L_2) \sin(q_1 + q_2 + q_3) \\ 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) \cos(q_1 + q_2 + q_3) & (L_1 + L_2) \cos(q_1 + q_2 + q_3) \end{bmatrix}$$

$$[J_n] = \begin{bmatrix} -(L_1 + L_2) \sin(q_1) - 2 L_3 \sin(q_1 + q_2) - (L_1 + L_2) \sin(q_1 + q_2 + q_3) \\ (L_1 + L_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + (L_1 + L_2) \cos(q_1 + q_2 + q_3) \end{bmatrix}$$

Solving this equation gives

$$[C] = \begin{bmatrix} 1/2 \frac{(L_1+L_2)(-\sin(q_2+q_3)-L_3 \sin(q_3))}{L_3 \sin(q_3)} \\ -1/2 \frac{(L_1+L_2)^2 \sin(q_2) \sin(q_2+q_3)}{\sin(q_3)} \end{bmatrix}$$

and the $[\dot{C}]$ of this is

$$[\dot{C}] = \begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \end{bmatrix}$$

where

$$\begin{aligned} \dot{C}_1 &= 1/2 \frac{(L_1+L_2)(\cos(q_2+q_3)(-\dot{q}_2-\dot{q}_3)-L_3 \cos(q_3)\dot{q}_3)}{L_3 \sin(q_3)} \\ &\quad -1/2 \frac{(L_1+L_2)(-\sin(q_2+q_3)-L_3 \sin(q_3)) \cos(q_3) \dot{q}_3}{L_3 (\sin(q_3))^2} \\ \dot{C}_2 &= -1/2 \frac{(L_1+L_2)^2 \cos(q_2)(\dot{q}_2) \sin(q_2+q_3)}{\sin(q_3)} \\ &\quad +1/2 \frac{(L_1+L_2)^2 \sin(q_2) \cos(q_2+q_3)(-\dot{q}_2-\dot{q}_3)}{\sin(q_3)} \\ &\quad +1/2 \frac{(L_1+L_2)^2 \sin(q_2) \sin(q_2+q_3) \cos(q_3) \dot{q}_3}{(\sin(q_3))^2} \end{aligned}$$

Once the $[C]$ is obtained, the reduced partial velocity matrix $[\underline{V}]$ can be obtained by

$$[\underline{V}] = [V_n] - [V_m][C]$$

where

$$[V] = \begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \\ \underline{V}_3 \\ \underline{V}_4 \\ \underline{V}_5 \\ \underline{V}_6 \\ \underline{V}_7 \\ \underline{V}_8 \\ \underline{V}_9 \end{bmatrix}$$

and

$$\underline{V}_1 = -L_1 \sin(q_1)$$

$$\underline{V}_2 = L_1 \cos(q_1)$$

$$\underline{V}_3 = 1$$

$$\begin{aligned} \underline{V}_4 &= -(q_1 + q_2) \sin(q_1) - \sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) \\ &= +1/2 \frac{\sqrt{L_3^2 + h^2} \sin(q_1 + q_2 + \beta) (L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)} \end{aligned}$$

$$\begin{aligned} \underline{V}_5 &= (q_1 + q_2) \cos(q_1) + \sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) \\ &= -1/2 \frac{\sqrt{L_3^2 + h^2} \cos(q_1 + q_2 + \beta) (L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)} \end{aligned}$$

$$\underline{V}_6 = 1 - 1/2 \frac{(L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)}$$

$$\begin{aligned} \underline{V}_7 &= -(q_1 + q_2) \sin(q_1) - 2 L_3 \sin(q_1 + q_2) - L_2 \sin(q_1 + q_2 + q_3) \\ &= -1/2 \frac{(-2 L_3 \sin(q_1 + q_2) - L_2 \sin(q_1 + q_2 + q_3)) (L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)} \\ &= -1/2 \frac{L_2 \sin(q_1 + q_2 + q_3) (L_1 + L_2)^2 \sin(q_2) \sin(q_2 + q_3)}{\sin(q_3)} \end{aligned}$$

$$\begin{aligned}
\underline{V}_8 &= (q_1 + q_2) \cos(q_1) + 2 L_3 \cos(q_1 + q_2) + L_2 \cos(q_1 + q_2 + q_3) \\
&= -1/2 \frac{(2 L_3 \cos(q_1 + q_2) + 2 \cos(q_1 + q_2 + q_3)) (L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)} \\
&= +1/2 \frac{L_2 \cos(q_1 + q_2 + q_3) (L_1 + L_2)^2 \sin(q_2) \sin(q_2 + q_3)}{\sin(q_3)} \\
\underline{V}_9 &= 1 - 1/2 \frac{(L_1 + L_2) (-\sin(q_2 + q_3) - L_3 \sin(q_3))}{L_3 \sin(q_3)} + 1/2 \frac{(L_1 + L_2)^2 \sin(q_2) \sin(q_2 + q_3)}{\sin(q_3)}
\end{aligned}$$

The value of the mass moments of inertia for the individual bodies are combined to form

$$[I] = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{bmatrix}$$

The above calculated parameters, and Equation 3.16, which is

$$[\underline{V}]^T [I] [\underline{V}] \{\dot{u}_n\} = [\underline{V}]^T [I] [\underline{V}_M] [\dot{C}] \{\dot{u}_n\} - [\underline{V}]^T [I] [\dot{\underline{V}}] \{u\} + [\underline{V}]^T \{F^e\}$$

developed in Section 3.3.2 is used to produce the governing dynamics for the four-bar linkage. The equation can be reduced in variables by calculating the values for $\dot{q}_2$ and $\dot{q}_3$, which are calculated by

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = -[C]\{\dot{q}_1\}$$

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 1/2 \frac{\dot{q}_1(L_1+L_2)(-\sin(q_2+q_3)-L_3 \sin(q_3))}{L_3 \sin(q_3)} \\ -1/2 \frac{\dot{q}_1(L_1+L_2)^2 \sin(q_2) \sin(q_2+q_3)}{\sin(q_3)} \end{bmatrix}$$

If the human stands with a stance width the same as the hip width, the values for q_2 and q_3 would be

$$\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 2\pi - q_1 \\ \pi + q_1 \end{bmatrix}$$

The equation is then placed in the form

$$A\ddot{q}_1 = B + CM \quad (\text{C.1})$$

In order to develop the controller Equation C.1 was linearized about a specific angle ($q_1 = 90^\circ$). Presented here are the values for A, B, and C before they were linearized.

$$\begin{aligned} A = & 8\dot{q}_1^2 L_3^2 m_3 L_1 (\cos(q_1))^3 L_2 - 3\dot{q}_1^2 L_1 L_2^2 m_2 L_3^2 \sin(q_1) \\ & + 32\dot{q}_1^2 L_1 L_2 m_3 L_3 \sin(q_1) \cos(q_1) - 48\dot{q}_1^2 L_3^2 m_3 \pi \sin(q_1) L_2 \\ & - 4\dot{q}_1^2 L_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 - 4\dot{q}_1^2 L_2^4 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 \\ & + 8\dot{q}_1^2 L_2^3 m_3 \cos(q_1) L_3 \sin(q_1) - 24\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2 \pi (\cos(q_1))^3 \\ & + 6\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_3^2 L_1 (\cos(q_1))^2 - 4\dot{q}_1^2 L_1^3 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) L_2 \\ & + 16\dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_1 L_2 + \dot{q}_1^2 L_1^3 m_3 L_2^2 \sin(q_1) (\cos(q_1))^2 \\ & + 8\dot{q}_1^2 L_3 m_3 L_2^2 (\cos(q_1))^2 + 4\dot{q}_1^2 L_3^2 m_3 L_1^2 (\cos(q_1))^3 \end{aligned}$$

$$\begin{aligned}
& +48\dot{q}_1^2 L_3^2 m_3 \pi \sin(q_1) L_1 (\cos(q_1))^2 + 8\dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_1^2 \\
& +8\dot{q}_1^2 L_3 m_3 \pi L_2 (\cos(q_1))^2 + 2\dot{q}_1^2 L_1^3 m_3 L_2^2 L_3 (\sin(q_1))^2 \\
& -16\dot{q}_1^2 L_1 L_2 m_3 L_3 \sin(q_1) (\cos(q_1))^3 - 8\dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_1^2 (\cos(q_1))^2 \\
& +8\dot{q}_1^2 L_1^3 m_3 L_2^2 L_3 (\sin(q_1))^2 (\cos(q_1))^2 + \dot{q}_1^2 L_1^4 m_3 L_2^2 L_3 (\sin(q_1))^2 \\
& +2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \sin(q_1) (\cos(q_1))^2 \\
& +24\dot{q}_1^2 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 (\cos(q_1))^2 \\
& -8\dot{q}_1^2 L_3 m_3 \pi L_2 + 3\dot{q}_1^2 L_1 L_2^4 m_3 \sin(q_1) (\cos(q_1))^2 \\
& -8\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \sin(q_1) (\cos(q_1))^2 - 8\dot{q}_1^2 L_3 m_3 L_2^2 (\cos(q_1))^4 \\
& -24\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi (\cos(q_1))^3 - 2\dot{q}_1^2 m_3 L_2 \sin(q_1) L_1^3 L_3^2 \\
& -2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 \cos(q_1) - 8\dot{q}_1^2 L_3 m_2 \pi L_1 \\
& -8\dot{q}_1^2 L_1^2 m_3 L_3 \sin(q_1) (\cos(q_1))^3 + 8\dot{q}_1^2 L_3 m_3 \pi L_1 (\cos(q_1))^2 \\
& -4\dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2 \cos(q_1) L_1^2 + 12\dot{q}_1^2 L_1 L_2^2 m_3 (\cos(q_1))^3 L_3 \sin(q_1) \\
& -8\dot{q}_1^2 L_3 m_3 L_2^2 \sin(q_1) (\cos(q_1))^3 - 2\dot{q}_1^2 L_2^2 I_3 \sin(q_1) - 2\dot{q}_1^2 L_1^2 I_3 \sin(q_1) \\
& -2\dot{q}_1^2 L_1^2 I_2 \sin(q_1) \dot{q}_1^2 L_2^3 I_3 \sin(q_1) - 2\dot{q}_1^2 L_2^2 I_2 \sin(q_1) \\
& -2\dot{q}_1^2 m_3 L_2^4 \sin(q_1) L_3^2 - 2\dot{q}_1^2 L_1^4 m_3 L_3^2 (\sin(q_1))^2 L_2 \cos(q_1) \\
& -2\dot{q}_1^2 L_2^3 m_3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 - \dot{q}_1^2 L_1^4 m_3 L_2^2 L_3 (\sin(q_1))^2 (\cos(q_1))^2 \\
& +16\dot{q}_1^2 m_3 L_3^2 \sin(q_1) L_1 (\cos(q_1))^2 + 20\dot{q}_1^2 L_1 L_2^2 m_3 \cos(q_1) L_3 \sin(q_1) \\
& +4\dot{q}_1^2 L_2^3 m_3 \pi (\sin(q_1))^3 + 4\dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 L_3 \pi (\cos(q_1))^3 \\
& -8\dot{q}_1^2 L_1 L_2^4 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) + 8\dot{q}_1^2 m_3 \pi \sin(q_1) L_1^2 (\cos(q_1))^2 \\
& -\dot{q}_1^2 L_1^3 m_2 L_3^2 \sin(q_1) - \dot{q}_1^2 m_3 L_2^5 \sin(q_1) L_3 (\cos(q_1))^3 - \dot{q}_1^2 L_2^3 m_2 \sin(q_1) h^2 \\
& -4\dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 L_3 \pi \cos(q_1) - 8\dot{q}_1^2 m_3 L_2^3 \sin(q_1) (\cos(q_1))^2 \\
& -3\dot{q}_1^2 L_2^2 I_3 L_1 \sin(q_1) - 4\dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 (\cos(q_1))^2 \\
& +2\dot{q}_1^2 L_1^4 m_3 (\cos(q_1))^2 L_3 (\sin(q_1))^2 L_2 + 16\dot{q}_1^2 m_3 \pi \sin(q_1) L_1 L_2 (\cos(q_1))^2
\end{aligned}$$

$$\begin{aligned}
& +3\dot{q}_1^2 L_1^2 m_3 L_2^3 \sin(q_1) (\cos(q_1))^2 + 4\dot{q}_1^2 L_1^2 m_3 \pi (\sin(q_1))^3 L_2 \\
& +2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 (\cos(q_1))^3 + 4\dot{q}_1^2 L_1^3 m_3 L_3 \sin(q_1) \cos(q_1) \\
& -8\dot{q}_1^2 L_2^2 m_3 \pi (\cos(q_1))^3 L_3 \sin(q_1) + 8\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \cos(q_1) \\
& +8 M L_3 (\cos(q_1))^2 - 2\dot{q}_1^2 L_3 m_3 L_2^4 \sin(q_1) \cos(q_1) + 8 L_1 (\cos(q_1))^3 m_1 g L_3 \\
& -2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \sin(q_1) + \dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1^3 L_3 \cos(q_1) \\
& -8 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) + 6\dot{q}_1^2 L_1^2 I_3 L_3 (\sin(q_1))^2 L_2 - 8 m_3 g L_3 (\cos(q_1))^3 L_1 \\
& -16\dot{q}_1^2 L_1^2 m_3 L_2 (\cos(q_1))^2 \sin(q_1) - 8\dot{q}_1^2 L_3 m_3 L_2 (\sin(q_1))^2 - 4\dot{q}_1^2 L_1 L_2 I_3 \sin(q_1) \\
& -\dot{q}_1^2 L_1^3 m_2 \sin(q_1) h^2 + 8\dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_2^2 - 2\dot{q}_1^2 m_3 L_2 (\sin(q_1))^2 L_3 L_1^2 (\cos(q_1))^2 \\
& -2\dot{q}_1^2 L_3 m_3 L_2^2 \sin(q_1) L_1^2 \cos(q_1) - 4\dot{q}_1^2 L_1^3 m_3 L_2^3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 \\
& -8\dot{q}_1^2 L_1^3 m_3 L_3^2 (\sin(q_1))^2 L_2^2 \cos(q_1) - 2\dot{q}_1^2 L_1^2 m_2 L_3^2 \sin(q_1) - \dot{q}_1^2 L_1^3 I_3 \sin(q_1) \\
& -16\dot{q}_1^2 L_3 m_3 \pi (\cos(q_1))^2 - \dot{q}_1^2 L_2^5 m_3 \sin(q_1) - 16 m_3 g L_3^2 - \dot{q}_1^2 L_1^3 m_3 L_2^2 \sin(q_1) \\
& -4\dot{q}_1^2 L_2^4 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) + 4\dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 L_3 \pi (\cos(q_1))^3 \\
& -4\dot{q}_1^2 L_1^3 m_3 (\cos(q_1))^2 \sin(q_1) - 8\dot{q}_1^2 L_3 m_3 \pi L_1 + 3\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_1^2 L_3 \cos(q_1) \\
& +16\dot{q}_1^2 L_1^2 m_3 L_3 \sin(q_1) \cos(q_1) - 3\dot{q}_1^2 L_1^2 m_3 L_2^3 \sin(q_1) \\
& -6\dot{q}_1^2 L_1^2 m_3 L_2^4 L_3 (\sin(q_1))^2 (\cos(q_1))^2 - 8\dot{q}_1^2 L_1 L_2 m_3 L_3^2 \sin(q_1) \\
& -4\dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 (\cos(q_1))^2 - 2\dot{q}_1^2 L_2^5 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) \\
& -16 \dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_1 L_2 (\cos(q_1))^2 + 2 (\cos(q_1))^2 \dot{q}_1^2 L_2^5 m_3 L_3 (\sin(q_1))^2 \\
& +12\dot{q}_1^2 L_3 m_3 L_2 (\cos(q_1))^2 L_1 + 2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 (\cos(q_1))^3 \\
& -8\dot{q}_1^2 L_1^2 m_3 L_3^2 \sin(q_1) - \dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1^3 L_3 (\cos(q_1))^3 \\
& -4\dot{q}_1^2 L_1 L_2 m_2 \sin(q_1) h^2 + 8\dot{q}_1^2 L_3 m_3 (\cos(q_1))^4 L_1 \\
& +16 m_3 g L_3 L_2 \cos(q_1) - 2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 \cos(q_1) \\
& +16 m_3 g L_3 \pi (\cos(q_1))^3 + 4\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \sin(q_1) L_2 (\cos(q_1))^2 \\
& -8\dot{q}_1^2 L_3 m_2 \pi L_2 + 24 \dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi \cos(q_1)
\end{aligned}$$

$$\begin{aligned}
& +3\dot{q}_1^2 m_3 L_2^4 \sin(q_1) L_3 \cos(q_1) L_1 - 3\dot{q}_1^2 L_1 L_2^2 m_2 \sin(q_1) h^2 \\
& +8\dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2^2 (\cos(q_1))^3 L_1 + 8 m_3 g L_3^2 L_2 (\cos(q_1))^2 \\
& -3\dot{q}_1^2 L_1 L_2^2 I_2 \sin(q_1) - \dot{q}_1^2 L_1^3 I_2 \sin(q_1) + 8\dot{q}_1^2 L_3 m_2 \pi L_2 (\cos(q_1))^2 \\
& +4\dot{q}_1^2 m_3 L_2^2 (\sin(q_1))^2 L_3 L_1 - 2\dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1^2 \\
& +8\dot{q}_1^2 L_3 m_3 L_2 (\sin(q_1))^2 (\cos(q_1))^2 - 4 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2 \\
& -8\dot{q}_1^2 L_1^2 m_3 \pi (\cos(q_1))^3 L_3 \sin(q_1) + 6\dot{q}_1^2 L_1^2 m_3 L_2^3 L_3 (\sin(q_1))^2 \\
& -8\dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 L_2 (\cos(q_1))^2 \\
& -16\dot{q}_1^2 L_3 m_3 L_2 (\cos(q_1))^2 - 12\dot{q}_1^2 L_1 L_2^3 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 \\
& -4\dot{q}_1^2 L_1 L_2^2 m_3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 + 4\dot{q}_1^2 L_3 m_3 L_2^3 \sin(q_1) L_1 (\cos(q_1))^3 \\
& +16 \dot{q}_1^2 L_1^2 m_3 L_2 \cos(q_1) L_3 \sin(q_1) + 4 \dot{q}_1^2 L_3^2 m_3 L_2^2 (\cos(q_1))^3 \\
& -24\dot{q}_1^2 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 + 6\dot{q}_1^2 L_1 L_2^2 I_3 L_3 (\sin(q_1))^2 - 8 M L_3 \\
& +2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \sin(q_1) (\cos(q_1))^2 + \dot{q}_1^2 L_2^4 I_3 L_3 (\sin(q_1))^2 \\
& +8\dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 L_3 \pi (\cos(q_1))^3 L_2 + 4\dot{q}_1^2 L_1^3 m_3 L_2^3 L_3 (\sin(q_1))^2 \\
& -4 \dot{q}_1^2 L_2^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 - \dot{q}_1^2 L_2^3 m_2 L_3^2 \sin(q_1) \\
& +2\dot{q}_1^2 m_3 L_2^2 (\cos(q_1))^2 \sin(q_1) L_1^2 + 8\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \sin(q_1) \\
& +4\dot{q}_1^2 L_3 m_3 L_2^3 \sin(q_1) (\cos(q_1))^3 - 3\dot{q}_1^2 L_1^2 m_2 L_2 \sin(q_1) h^2 \\
& -4\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_1 + 2\dot{q}_1^2 L_3 m_3 L_2^3 (\sin(q_1))^2 \\
& -2\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \sin(q_1) - 12\dot{q}_1^2 L_1^2 m_3 L_3^2 L_2^2 (\sin(q_1))^2 \cos(q_1) \\
& +2 \dot{q}_1^2 L_2^3 I_3 L_3 (\sin(q_1))^2 + 8 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) (\cos(q_1))^2 \\
& +8\dot{q}_1^2 L_3 m_2 \pi L_1 (\cos(q_1))^2 + 4\dot{q}_1^2 L_1 L_2^5 m_3 L_3 (\sin(q_1))^2 \\
& +4\dot{q}_1^2 L_1 L_2^3 I_3 L_3 (\sin(q_1))^2 - 4\dot{q}_1^2 L_3 m_3 (\cos(q_1))^4 L_1^2 \\
& -6\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_3^2 L_1 - 8 \dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 L_3 \pi \cos(q_1) L_2 \\
& +4\dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 + 4\dot{q}_1^2 L_1^3 I_3 L_3 (\sin(q_1))^2 L_2
\end{aligned}$$

$$\begin{aligned}
& +24\dot{q}_1^2 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 (\cos(q_1))^2 + \dot{q}_1^2 L_2^6 m_3 L_3 (\sin(q_1))^2 \\
& -3\dot{q}_1^2 L_1^2 I_2 L_2 \sin(q_1) + 6\dot{q}_1^2 L_3^2 m_3 L_2^2 (\cos(q_1))^2 \sin(q_1) L_1^2 \\
& -4\dot{q}_1^2 L_3 m_3 L_2^3 \sin(q_1) L_1 \cos(q_1) - 12\dot{q}_1^2 L_2^3 m_3 \pi (\sin(q_1))^4 L_3 L_1 \\
& -4\dot{q}_1^2 L_1 L_2 I_2 \sin(q_1) + 4\dot{q}_1^2 L_3 m_3 (\cos(q_1))^2 L_1^2 - 16\dot{q}_1^2 m_3 L_3^2 \sin(q_1) L_1 \\
& -12\dot{q}_1^2 L_1^2 m_3 L_3^2 L_2^3 (\sin(q_1))^2 \cos(q_1) + 12\dot{q}_1^2 L_3 m_3 (\cos(q_1))^3 L_1^2 L_2 \sin(q_1) \\
& +24\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2 \pi \cos(q_1) - 3\dot{q}_1^2 L_1 L_2^4 m_3 \sin(q_1) \\
& +4 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2 (\cos(q_1))^2 + 16\dot{q}_1^2 L_3 m_3 L_2 (\cos(q_1))^4 \\
& -2\dot{q}_1^2 L_2^2 m_2 L_3^2 \sin(q_1) + 4 (\cos(q_1))^3 \dot{q}_1^2 L_1^3 m_3 L_3 \sin(q_1) \\
& -16\dot{q}_1^2 L_3 m_3 \pi \sin(q_1) L_2 (\cos(q_1))^3 L_1 + 4 \dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2 (\cos(q_1))^3 L_1^2 \\
& -8 L_1 \cos(q_1) m_1 g L_3 + 6\dot{q}_1^2 L_1^2 m_3 L_2^4 L_3 (\sin(q_1))^2 \\
& +16 m_2 g L_3 \pi (\cos(q_1))^3 - 4\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \sin(q_1) L_2 \\
& -3\dot{q}_1^2 L_1^2 m_2 L_2 L_3^2 \sin(q_1) - 16 m_2 g L_3 \pi \cos(q_1) + 2\dot{q}_1^2 L_3 m_3 L_2^4 \sin(q_1) (\cos(q_1))^3 \\
& +\dot{q}_1^2 L_2^5 m_3 \sin(q_1) (\cos(q_1))^2 - 4\dot{q}_1^2 L_3^2 m_3 L_2^2 \cos(q_1) \\
& +2 \dot{q}_1^2 m_3 L_2^4 \sin(q_1) (\cos(q_1))^2 - 4\dot{q}_1^2 L_1^3 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 L_2 \\
& -8\dot{q}_1^2 L_3^2 m_3 L_1 \cos(q_1) L_2 + 4\dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 \\
& +\dot{q}_1^2 L_1^4 I_3 L_3 (\sin(q_1))^2 - 4\dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2^3 \cos(q_1) \\
& -8\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) (\cos(q_1))^3 - \dot{q}_1^2 L_2^3 I_2 \sin(q_1) \\
& +2 \dot{q}_1^2 m_3 L_2 \sin(q_1) L_1^3 L_3^2 (\cos(q_1))^2 + 16 m_3 g L_3^2 (\cos(q_1))^2 + 16\dot{q}_1^2 L_3 m_2 \pi \\
& +12\dot{q}_1^2 L_1^2 m_3 L_2^3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 - 12\dot{q}_1^2 L_3 m_3 L_2 (\cos(q_1))^4 L_1 \\
& +16\dot{q}_1^2 L_3^2 m_3 \cos(q_1) + 16\dot{q}_1^2 L_3 m_3 \pi - 12\dot{q}_1^2 L_1^2 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 L_2^2 \\
& +4\dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 (\cos(q_1))^3 L_2 - 4\dot{q}_1^2 L_3^2 m_3 L_1^2 \cos(q_1) \\
& +8 m_3 g L_3^2 L_1 (\cos(q_1))^2 + 6\dot{q}_1^2 L_1^2 I_3 L_2^2 L_3 (\sin(q_1))^2 \\
& -4\dot{q}_1^2 L_2^4 m_3 \pi (\sin(q_1))^4 L_3 - 8\dot{q}_1^2 m_3 \pi \sin(q_1) L_3^2 L_2^2 (\cos(q_1))^2
\end{aligned}$$

$$\begin{aligned}
& -4 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 - 4 \dot{q}_1^2 L_1^2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) \\
& + 48 \dot{q}_1^2 L_3^2 m_3 \pi \sin(q_1) L_2 (\cos(q_1))^2 - 3 \dot{q}_1^2 L_1^2 I_3 L_2 \sin(q_1) \\
& - 3 \dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_1^2 L_3 (\cos(q_1))^3 - 4 \dot{q}_1^2 L_1^3 m_3 \pi (\sin(q_1))^4 L_3 L_2 \\
& - 8 \dot{q}_1^2 L_1 L_2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 + \dot{q}_1^2 m_3 L_2^5 \sin(q_1) L_3 \cos(q_1) \\
& - 20 \dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1 (\cos(q_1))^2 + 2 \dot{q}_1^2 L_3 m_3 L_2 (\sin(q_1))^2 L_1^2 \\
& + 8 \dot{q}_1^2 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 L_2 - 6 \dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1^2 L_3^2 \\
& - 3 \dot{q}_1^2 m_3 L_2^4 \sin(q_1) L_3 (\cos(q_1))^3 L_1 - 12 \dot{q}_1^2 L_1 L_2^3 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) \\
& + 8 m_3 g L_3 \cos(q_1) L_1 - 4 \dot{q}_1^2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 \cos(q_1) L_2 \\
& + 6 \dot{q}_1^2 L_1 L_2^4 m_3 L_3 (\sin(q_1))^2 + 8 \dot{q}_1^2 m_3 \pi \sin(q_1) L_2^2 (\cos(q_1))^2 \\
& + 2 \dot{q}_1^2 L_3 m_3 L_2^2 \sin(q_1) L_1^2 (\cos(q_1))^3 + 2 \dot{q}_1^2 L_1^3 I_3 L_3 (\sin(q_1))^2 \\
& - 8 \dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2^2 \cos(q_1) L_1 - 4 \dot{q}_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 L_3 \pi \cos(q_1) \\
& - 8 \dot{q}_1^2 L_1 L_2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) - 2 \dot{q}_1^2 L_2^2 m_2 \sin(q_1) h^2 \\
& - 2 \dot{q}_1^2 m_3 L_2^4 \sin(q_1) - 16 \dot{q}_1^2 L_3 m_2 \pi (\cos(q_1))^2 \\
& - 4 \dot{q}_1^2 L_1 L_2 m_2 L_3^2 \sin(q_1) - 12 \dot{q}_1^2 L_1^2 m_3 \pi (\sin(q_1))^4 L_3 L_2^2 \\
& - 8 \dot{q}_1^2 L_3 m_3 (\cos(q_1))^2 L_1 - 4 \dot{q}_1^2 L_1 L_2^5 m_3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 \\
& + 4 m_2 g L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 (\cos(q_1))^2 - 16 m_3 g L_3 L_2 (\cos(q_1))^3 \\
& + 16 \dot{q}_1^2 L_3 m_3 L_2^2 \sin(q_1) \cos(q_1) + 8 \dot{q}_1^2 L_1 L_2^4 m_3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 \\
& - 8 \dot{q}_1^2 L_2^2 m_3 L_3^2 \sin(q_1) (\cos(q_1))^2 - 48 \dot{q}_1^2 L_3^2 m_3 \pi \sin(q_1) L_1 - 8 m_3 g L_3^2 L_1 \\
& - 8 m_3 g L_3^2 L_2 - 16 m_3 g L_3 \pi \cos(q_1) + 16 \dot{q}_1^2 m_3 L_2 \sin(q_1) L_3^2 (\cos(q_1))^2 \\
& - \dot{q}_1^2 L_2^6 m_3 L_3 (\sin(q_1))^2 (\cos(q_1))^2 - 16 \dot{q}_1^2 m_3 L_2 \sin(q_1) L_3^2 \\
& + 2 \dot{q}_1^2 L_2^5 m_3 L_3 (\sin(q_1))^2 - 12 \dot{q}_1^2 m_3 L_2^2 \sin(q_1) L_1 L_3^2 (\cos(q_1))^2 \\
& - 8 \dot{q}_1^2 m_3 \sin(q_1) L_3^2 L_1 L_2 (\cos(q_1))^2 - 12 \dot{q}_1^2 L_1^2 m_3 L_2 (\cos(q_1))^2 L_3^2 \sin(q_1) \\
& + 8 \dot{q}_1^2 L_1 L_2^2 m_3 \pi (\sin(q_1))^3 + 4 \dot{q}_1^2 m_3 \pi \sin(q_1) L_3 L_2^3 (\cos(q_1))^3
\end{aligned}$$

$$\begin{aligned}
& -4\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_3^2 (\cos(q_1))^2 - 16\dot{q}_1^2 L_3^2 m_3 (\cos(q_1))^3 \\
& -2\dot{q}_1^2 L_1^2 m_2 \sin(q_1) h^2 - 24\dot{q}_1^2 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 \\
& +2\dot{q}_1^2 m_3 L_2^4 \sin(q_1) L_3^2 (\cos(q_1))^2 + 4\dot{q}_1^2 m_3 L_2^3 \sin(q_1) L_1 (\cos(q_1))^2 \\
& -4\dot{q}_1^2 L_1^3 m_3 (\cos(q_1))^2 L_3^2 \sin(q_1) \\
& -4\dot{q}_1^2 L_2^2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) \\
& /2 \left(L_3 \left(-4 I_3 L_1 - 4 m_3 L_2^2 - 4 I_2 L_2 - 4 I_2 L_1 - 4 m_3 L_2^3 - 4 I_3 L_2 \right. \right. \\
& \left. -4 m_2 L_3^2 L_1 - m_3 L_2^2 L_1^2 + 16 m_3 \pi (\cos(q_1))^3 L_3 L_2 \right. \\
& \left. -m_2 h^2 L_1^2 + 4 m_3 (\cos(q_1))^2 L_3^2 L_1^2 - 4 I_1 - 4 I_2 - 4 I_3 + 2 m_3 L_2^2 L_1^2 (\cos(q_1))^2 \right. \\
& \left. -m_3 L_2^2 L_1^2 (\cos(q_1))^4 - 4 m_3 L_2^2 L_1 (\cos(q_1))^4 - 16 m_3 \pi (\cos(q_1))^4 L_1 \right. \\
& \left. +16 m_3 \pi (\cos(q_1))^2 L_1 + 16 m_3 L_2 (\cos(q_1))^4 L_1 + 4 m_3 L_2^3 L_1 (\cos(q_1))^2 \right. \\
& \left. -16 m_3 L_2 (\cos(q_1))^2 L_1 + 32 m_3 \pi (\cos(q_1))^2 L_2 - 4 m_2 L_3^2 L_2 - 16 m_3 \pi \cos(q_1) L_3 L_2 \right. \\
& \left. -16 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) + 4 L_1^2 m_1 (\cos(q_1))^2 + 8 m_3 L_2^3 (\cos(q_1))^2 \right. \\
& \left. +4 m_3 (\cos(q_1))^4 L_1^2 + 4 I_2 L_2 (\cos(q_1))^2 - 4 m_3 (\cos(q_1))^2 L_1^2 \right. \\
& \left. +8 m_3 L_3 \cos(q_1) L_1^2 + m_2 L_3^2 L_1^2 (\cos(q_1))^2 + m_2 L_3^2 L_2^2 (\cos(q_1))^2 \right. \\
& \left. +4 m_2 L_3^2 L_1 (\cos(q_1))^2 + 4 m_2 L_3^2 L_2 (\cos(q_1))^2 \right. \\
& \left. +16 m_3 L_3 \cos(q_1) L_2^2 - 32 m_3 L_3 L_2 (\cos(q_1))^3 + 16 m_3 \pi (\sin(q_1))^2 L_2 \right. \\
& \left. +2 I_3 L_1 L_2 (\cos(q_1))^2 + I_2 L_1^2 (\cos(q_1))^2 + I_3 L_1^2 (\cos(q_1))^2 - m_3 L_2^4 (\cos(q_1))^4 \right. \\
& \left. +4 I_2 L_1 (\cos(q_1))^2 - 8 m_3 L_2^2 (\cos(q_1))^2 + 4 I_3 L_1 (\cos(q_1))^2 \right. \\
& \left. +12 m_3 L_2^2 (\cos(q_1))^4 + 2 m_3 L_2^4 (\cos(q_1))^2 - 4 m_3 L_2^3 (\cos(q_1))^4 + 4 I_3 L_2 (\cos(q_1))^2 \right. \\
& \left. +16 m_2 \pi^2 (\cos(q_1))^2 + I_2 L_2^2 (\cos(q_1))^2 + I_3 L_2^2 (\cos(q_1))^2 \right. \\
& \left. +16 m_3 L_3^2 (\cos(q_1))^2 - 16 m_3 L_3^2 + 4 m_2 h^2 (\cos(q_1))^2 \right. \\
& \left. +4 m_2 L_3^2 (\cos(q_1))^2 - 8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 - 4 m_3 L_3^2 L_1^2 \right. \\
& \left. +2 m_2 L_3^2 L_1 L_2 (\cos(q_1))^2 + 16 m_3 \pi^2 (\cos(q_1))^2 - m_2 L_3^2 L_2^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -16 m_3 \pi (\sin(q_1))^2 L_2 (\cos(q_1))^2 - 16 m_3 \pi \cos(q_1) L_3 L_1 - 4 m_2 h^2 \\
& -16 m_3 L_3^2 L_1 - 24 m_3 L_3 (\cos(q_1))^3 L_1 L_2 - 4 m_2 L_3^2 - I_2 L_2^2 - 2 m_3 L_2^3 L_1 (\cos(q_1))^4 \\
& + 2 I_2 L_1 L_2 (\cos(q_1))^2 + 4 m_2 h^2 L_2 (\cos(q_1))^2 - 8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 \\
& + 8 m_3 L_2^2 L_1 (\cos(q_1))^2 + 4 m_2 h^2 L_1 (\cos(q_1))^2 + 8 m_3 \pi (\sin(q_1))^2 L_2^2 \\
& + m_2 h^2 L_1^2 (\cos(q_1))^2 + m_2 h^2 L_2^2 (\cos(q_1))^2 - 32 m_3 \pi (\cos(q_1))^4 L_2 \\
& + 24 m_3 L_3 \cos(q_1) L_1 L_2 - 4 L_1^2 m_1 - I_2 L_1^2 + 2 m_2 h^2 L_1 L_2 (\cos(q_1))^2 \\
& + 16 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) - 16 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) \\
& - I_3 L_1^2 + 16 m_3 L_3^2 L_1 (\cos(q_1))^2 + 16 m_3 L_3 \cos(q_1) L_1 - 32 m_3 L_3 \pi \cos(q_1) \\
& + 32 m_3 L_3 \pi (\cos(q_1))^3 + 16 m_3 L_3^2 L_2 (\cos(q_1))^2 - 2 m_2 L_3^2 L_1 L_2 \\
& + 32 m_3 L_3 L_2 \cos(q_1) - 2 m_3 L_2^3 L_1 - I_3 L_2^2 - 8 m_3 \pi (\sin(q_1))^2 L_2^2 (\cos(q_1))^2 \\
& + 8 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 + 8 m_3 \pi (\sin(q_1))^2 L_2 L_1 - 2 m_2 h^2 L_1 L_2 \\
& m_3 L_3^2 L_1 L_2 + 8 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2 \\
& - 4 m_3 L_2^2 L_3^2 - 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_2 + 4 I_2 (\cos(q_1))^2 + 4 I_1 (\cos(q_1))^2 \\
& - m_2 L_3^2 L_1^2 - m_3 L_2^4 - m_2 h^2 L_2^2 - 8 - 2 I_2 L_1 L_2 - 2 I_3 L_1 L_2 \\
& + 4 I_3 (\cos(q_1))^2 - 8 m_3 \pi (\sin(q_1))^2 L_2 L_1 (\cos(q_1))^2 \\
& - 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_1 + 16 m_3 \pi (\cos(q_1))^3 L_3 L_1 \\
& + 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_2 (\cos(q_1))^2 - 16 m_3 L_3^2 L_2 - 4 m_3 L_2^2 L_1 \\
& - 4 m_2 h^2 L_2 - 4 m_2 h^2 L_1 + 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_1 (\cos(q_1))^2 \\
& + 4 m_3 L_2^2 L_3^2 (\cos(q_1))^2 + 8 m_3 L_2 (\cos(q_1))^2 L_3^2 L_1 - 16 m_3 L_3 (\cos(q_1))^3 L_2^2 \\
& + 16 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) (\cos(q_1))^2 - 16 m_2 \pi^2 - 16 m_3 L_3 (\cos(q_1))^3 L_1 \\
& - 8 m_3 L_3 (\cos(q_1))^3 L_1^2 - 16 m_3 \pi^2)
\end{aligned}$$

$$\begin{aligned}
B = & \left(L_2^6 m_3 (\sin(q_1))^2 L_3 (\cos(q_1))^2 - 4 L_1^3 I_3 (\sin(q_1))^2 L_3 L_2 \right. \\
& - L_1^4 m_3 L_2^2 (\sin(q_1))^2 L_3 + 8 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi \cos(q_1) L_3 L_2 \\
& - L_1^4 I_3 (\sin(q_1))^2 L_3 + L_2^3 I_2 \sin(q_1) + L_2^3 I_3 \sin(q_1) \\
& + 8 L_1^3 m_3 L_2^2 L_3^2 (\sin(q_1))^2 \cos(q_1) - 2 L_3 m_3 L_2^3 (\sin(q_1))^2 - L_2^5 m_3 \sin(q_1) (\cos(q_1))^2 \\
& + 12 L_1^2 m_3 L_3^2 L_2^2 \cos(q_1) (\sin(q_1))^2 - 4 L_3 m_3 (\cos(q_1))^2 L_1^2 \\
& - 8 L_3 m_3 \pi L_1 (\cos(q_1))^2 + 16 L_3^2 m_3 (\cos(q_1))^3 + 2 L_2^2 I_2 \sin(q_1) \\
& + 2 L_1^2 I_2 \sin(q_1) + 2 L_2^2 I_3 \sin(q_1) + 4 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \sin(q_1) L_2 \\
& + 2 m_3 L_2^4 \sin(q_1) + 3 L_1 L_2^4 m_3 \sin(q_1) + 8 L_2^3 m_3 (\cos(q_1))^2 \sin(q_1) \\
& - 6 m_3 L_2^2 \sin(q_1) L_1^2 L_3^2 (\cos(q_1))^2 - 16 L_3^2 m_3 \cos(q_1) + 8 m_3 (\cos(q_1))^3 L_1^2 \sin(q_1) L_3 \\
& - 24 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 (\cos(q_1))^2 \\
& - 12 L_3 m_3 (\cos(q_1))^3 L_1 L_2^2 \sin(q_1) + 16 m_3 L_2 \sin(q_1) L_1 \pi (\cos(q_1))^3 L_3 \\
& - 16 m_3 L_2 \sin(q_1) L_3^2 (\cos(q_1))^2 + 8 L_2^2 m_3 L_3^2 \sin(q_1) (\cos(q_1))^2 \\
& + 4 m_3 \pi \sin(q_1) L_3 L_2^3 \cos(q_1) - 2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \sin(q_1) (\cos(q_1))^2 \\
& - 12 L_3 m_3 (\cos(q_1))^2 L_1 L_2 + 3 m_3 L_2^4 \sin(q_1) (\cos(q_1))^3 L_3 L_1 \\
& + 2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \sin(q_1) - 48 L_3^2 m_3 \pi \sin(q_1) L_1 (\cos(q_1))^2 \\
& + 8 m_3 L_2 \sin(q_1) L_3^2 L_1 + 4 L_2^4 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 \\
& - 6 L_1^2 I_3 L_2^2 (\sin(q_1))^2 L_3 + 12 L_1 L_2^3 m_3 \pi (\sin(q_1))^4 L_3 \\
& + 4 L_2^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 - 4 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \pi (\cos(q_1))^3 L_3 \\
& - 24 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi \cos(q_1) - 4 m_3 \pi \sin(q_1) L_3 L_2 (\cos(q_1))^3 L_1^2 \\
& + 4 L_3^2 m_3 L_1^2 \cos(q_1) + L_1^3 I_2 \sin(q_1) + 4 L_2^3 m_3 (\cos(q_1))^2 L_3^2 \sin(q_1) \\
& - 4 L_3 m_3 L_2^3 (\cos(q_1))^3 \sin(q_1) - 8 L_3 m_3 L_2^2 (\cos(q_1))^2 - 4 L_3^2 m_3 L_1^2 (\cos(q_1))^3 \\
& \left. - L_2^6 m_3 (\sin(q_1))^2 L_3 + 2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \sin(q_1) \right)
\end{aligned}$$

$$\begin{aligned}
& -4 L_3 m_3 L_2^2 (\sin(q_1))^2 L_1 - 2 L_1^4 m_3 (\cos(q_1))^2 L_2 L_3 (\sin(q_1))^2 \\
& -8 L_2^4 m_3 (\cos(q_1))^2 L_1 L_3 (\sin(q_1))^2 - 8 L_3 m_2 \pi L_2 (\cos(q_1))^2 \\
& -4 L_1^3 m_3 L_3 \cos(q_1) \sin(q_1) + 4 L_1^3 m_3 (\cos(q_1))^2 \sin(q_1) + 3 L_1 L_2^2 I_2 \sin(q_1) \\
& -24 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 (\cos(q_1))^2 - 8 L_3 m_3 \pi L_2 (\cos(q_1))^2 \\
& + L_2^3 m_2 \sin(q_1) h^2 - 4 m_3 (\cos(q_1))^3 L_1^3 \sin(q_1) L_3 - 8 L_3 m_2 \pi L_1 (\cos(q_1))^2 \\
& + L_1^3 m_3 L_2^2 \sin(q_1) + 4 L_1 L_2^5 m_3 (\sin(q_1))^2 L_3 (\cos(q_1))^2 \\
& + 24 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 - 6 L_1 L_2^2 I_3 L_3 (\sin(q_1))^2 \\
& -2 m_3 L_2^2 \sin(q_1) L_1^2 (\cos(q_1))^2 + 2 m_3 \cos(q_1) L_1^4 L_2 (\sin(q_1))^2 L_3^2 \\
& -20 L_3 m_3 \cos(q_1) L_1 L_2^2 \sin(q_1) + 4 L_1^3 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 L_2 \\
& -16 m_3 \pi \sin(q_1) L_1 L_2 (\cos(q_1))^2 + 3 L_1^2 m_3 L_2^3 \sin(q_1) - 4 L_2^3 I_3 L_1 (\sin(q_1))^2 L_3 \\
& -8 L_3 m_3 L_2 (\sin(q_1))^2 (\cos(q_1))^2 + 2 m_3 L_2 \sin(q_1) L_1^3 L_3^2 + 4 L_2^4 m_3 \pi (\sin(q_1))^4 L_3 \\
& -8 L_3^2 m_3 L_1 (\cos(q_1))^3 L_2 + 3 L_1^2 I_3 L_2 \sin(q_1) - 48 L_3^2 m_3 \pi \sin(q_1) L_2 (\cos(q_1))^2 \\
& -8 m_3 \pi \sin(q_1) L_1^2 (\cos(q_1))^2 - 16 L_3 m_2 \pi + 6 L_1^2 m_3 L_2^4 (\sin(q_1))^2 L_3 (\cos(q_1))^2 \\
& + 8 m_3 \pi \sin(q_1) L_3^2 L_2^2 (\cos(q_1))^2 - 2 L_3^2 m_3 L_2^4 (\cos(q_1))^2 \sin(q_1) \\
& + 12 L_1 L_2^3 m_3 L_3^2 \cos(q_1) (\sin(q_1))^2 + 16 L_1^2 m_3 L_2 (\cos(q_1))^2 \sin(q_1) \\
& -4 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 - 8 L_3 m_3 L_2^3 \cos(q_1) \sin(q_1) \\
& -4 L_1^2 m_3 \pi (\sin(q_1))^3 L_2 - 2 L_3 m_3 L_2 (\sin(q_1))^2 L_1^2 + m_3 L_2^5 \sin(q_1) (\cos(q_1))^3 L_3 \\
& -8 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi (\cos(q_1))^3 L_3 L_2 - 2 L_2^3 I_3 L_3 (\sin(q_1))^2 \\
& + 16 m_3 L_2 \sin(q_1) L_3^2 - 4 L_2^3 m_3 \pi (\sin(q_1))^3 + 2 m_3 L_2^2 \sin(q_1) L_1^2 \\
& + 2 L_2^2 m_2 \sin(q_1) h^2 + 16 L_3 m_2 \pi (\cos(q_1))^2 + 4 L_1 L_2 I_3 \sin(q_1) \\
& + 4 L_3 m_3 L_2^2 (\sin(q_1))^2 L_1 (\cos(q_1))^2 - 2 L_2^5 m_3 (\cos(q_1))^2 L_3 (\sin(q_1))^2 \\
& -4 m_3 \pi \sin(q_1) L_3 L_2^3 (\cos(q_1))^3 - L_1^3 m_3 L_2^2 \sin(q_1) (\cos(q_1))^2 \\
& + 4 L_3 m_3 (\cos(q_1))^4 L_1^2 - 2 L_3 m_3 L_2^4 \sin(q_1) (\cos(q_1))^3 + 48 m_3 \pi \sin(q_1) L_3^2 L_2
\end{aligned}$$

$$\begin{aligned}
& -8 m_3 \pi \sin(q_1) L_3^2 L_2^2 - 24 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2 \pi \cos(q_1) \\
& -16 m_3 \pi \sin(q_1) L_3^2 L_2 L_1 - 8 L_1^3 m_3 (\cos(q_1))^2 L_2^2 L_3 (\sin(q_1))^2 \\
& +8 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \sin(q_1) (\cos(q_1))^2 + 8 m_3 \pi (\cos(q_1))^3 L_3 \sin(q_1) L_1^2 \\
& +24 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2 \pi (\cos(q_1))^3 + 8 L_3 m_3 L_2^2 (\cos(q_1))^4 \\
& +4 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \pi \cos(q_1) L_3 + L_2^5 m_3 \sin(q_1) - 8 m_3 \pi \sin(q_1) L_2^2 (\cos(q_1))^2 \\
& -3 m_3 L_2^4 \sin(q_1) \cos(q_1) L_3 L_1 + L_2^3 m_2 \sin(q_1) L_3^2 + 2 L_3 m_3 L_2^2 \sin(q_1) L_1^2 \cos(q_1) \\
& -16 m_3 L_2^2 \cos(q_1) \sin(q_1) L_3 + 4 L_1^2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 \\
& +3 L_1^2 I_2 L_2 \sin(q_1) - L_2^4 I_3 (\sin(q_1))^2 L_3 + 4 L_1 L_2 m_2 \sin(q_1) h^2 \\
& +12 m_3 L_2^2 \sin(q_1) L_1 L_3^2 (\cos(q_1))^2 - 2 m_3 L_2 \sin(q_1) L_1^3 L_3^2 (\cos(q_1))^2 \\
& -8 m_3 \pi \sin(q_1) L_3 L_2^2 (\cos(q_1))^3 L_1 + 2 L_3 m_3 L_2^4 \sin(q_1) \cos(q_1) + 4 L_3^2 m_3 L_2^2 \cos(q_1) \\
& -m_3 L_2^2 \sin(q_1) L_1^3 \cos(q_1) L_3 - 4 L_3^2 m_3 L_2^2 (\cos(q_1))^3 - 2 L_1^3 I_3 L_3 (\sin(q_1))^2 \\
& -2 L_3 m_3 L_2^2 \sin(q_1) L_1^2 (\cos(q_1))^3 - 16 m_3 L_3^2 \sin(q_1) L_1 (\cos(q_1))^2 \\
& -16 L_3 m_3 \pi - 16 L_1^2 m_3 L_3 L_2 \cos(q_1) \sin(q_1) - m_3 L_2^5 \sin(q_1) \cos(q_1) L_3 \\
& +8 m_3 \pi \sin(q_1) L_3 L_2^2 \cos(q_1) L_1 - 12 L_3 m_3 (\cos(q_1))^3 L_1^2 L_2 \sin(q_1) + 2 L_1^2 I_3 \sin(q_1) \\
& +2 L_3 m_3 L_2 (\sin(q_1))^2 L_1^2 (\cos(q_1))^2 + 4 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1^2 \pi \cos(q_1) L_3 \\
& -2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 (\cos(q_1))^3 + m_3 L_2^2 \sin(q_1) L_1^3 (\cos(q_1))^3 L_3 \\
& -16 m_3 \cos(q_1) L_1^2 \sin(q_1) L_3 - 4 L_1 L_2^5 m_3 (\sin(q_1))^2 L_3 - 6 L_1 L_2^4 m_3 L_3 (\sin(q_1))^2 \\
& +12 L_1 L_2^3 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 + 8 L_1^2 m_3 L_3^2 \sin(q_1) - 16 L_3 m_3 L_2 (\cos(q_1))^4 \\
& +16 L_3 m_3 \pi (\cos(q_1))^2 + 4 L_1 L_2 I_2 \sin(q_1) + 4 m_3 L_2^3 \sin(q_1) L_1 \\
& +2 L_1^2 m_2 \sin(q_1) h^2 - 8 L_3 m_3 (\cos(q_1))^4 L_1 + 8 L_3 m_3 (\cos(q_1))^2 L_1 \\
& +8 L_1 L_2^4 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) - 4 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 (\cos(q_1))^3 L_2 \\
& +8 L_1 L_2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) + 8 m_3 \pi \sin(q_1) L_3^2 L_1^2 (\cos(q_1))^2 \\
& -2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 (\cos(q_1))^3 + 3 L_1^2 m_2 L_2 \sin(q_1) h^2
\end{aligned}$$

$$\begin{aligned}
& +4 L_2^2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) + 8 m_3 L_2^2 (\cos(q_1))^3 \sin(q_1) L_3 \\
& +4 m_3 \pi \sin(q_1) L_3 L_2 \cos(q_1) L_1^2 + 2 m_3 L_2^4 \sin(q_1) L_3^2 + 8 L_3 m_2 \pi L_2 + 8 L_3 m_3 \pi L_2 \\
& +2 L_1^2 m_2 L_3^2 \sin(q_1) + 8 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 L_2 (\cos(q_1))^2 \\
& +12 L_1^2 m_2 \pi (\sin(q_1))^4 L_3 L_2^2 + 8 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) (\cos(q_1))^3 \\
& -8 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \sin(q_1) + 24 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \pi (\cos(q_1))^3 \\
& +24 L_3 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 - 6 L_1^2 m_3 L_2^3 L_3 (\sin(q_1))^2 \\
& +4 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 \cos(q_1) L_2 - 4 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 \\
& -4 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \sin(q_1) L_2 (\cos(q_1))^2 + 8 L_3 m_2 \pi L_1 \\
& +16 m_3 \pi \sin(q_1) L_3^2 L_1 L_2 (\cos(q_1))^2 + 12 L_3^2 m_3 L_2^3 (\sin(q_1))^2 L_1^2 \cos(q_1) \\
& -6 m_3 L_2^3 \sin(q_1) L_1 L_3^2 (\cos(q_1))^2 + 16 L_3 m_3 L_2 (\cos(q_1))^2 + 2 L_2^2 m_2 L_3^2 \sin(q_1) \\
& -2 m_3 L_2^4 (\cos(q_1))^2 \sin(q_1) - 2 L_2^5 m_3 L_3 (\sin(q_1))^2 + 8 L_3 m_3 L_2 (\sin(q_1))^2 \\
& -4 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \pi (\cos(q_1))^3 L_3 + 4 L_2^4 m_3 L_3^2 \cos(q_1) (\sin(q_1))^2 \\
& -6 L_1^2 I_3 L_3 (\sin(q_1))^2 L_2 + 12 L_1^2 m_3 \pi (\cos(q_1))^2 L_3 (\sin(q_1))^2 L_2^2 \\
& +4 L_1^3 m_3 \pi (\sin(q_1))^4 L_3 L_2 - 4 L_1^3 m_3 L_2^3 (\sin(q_1))^2 L_3 + 16 m_3 L_3^2 \sin(q_1) L_1 \\
& +L_1^3 I_3 \sin(q_1) + 8 m_2 \sin(q_1) L_3^2 L_1 L_2 (\cos(q_1))^2 + 4 L_3 m_3 L_2^3 \sin(q_1) L_1 \cos(q_1) \\
& +16 m_3 (\cos(q_1))^3 L_1 L_2 \sin(q_1) L_3 + 4 L_1^2 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \pi \cos(q_1) \sin(q_1) \\
& -32 m_3 \cos(q_1) L_1 L_2 \sin(q_1) L_3 + 3 L_1^2 m_2 L_2 \sin(q_1) L_3^2 \\
& -3 L_1^2 m_3 L_2^3 \sin(q_1) (\cos(q_1))^2 + 8 L_1 L_2 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) \pi (\sin(q_1))^2 \\
& +6 m_3 L_2^3 \sin(q_1) L_1 L_3^2 + 8 m_3 L_2^2 \sin(q_1) \pi (\cos(q_1))^3 L_3 \\
& +L_1^4 m_3 L_2^2 (\sin(q_1))^2 L_3 (\cos(q_1))^2 + 3 L_2^2 I_3 L_1 \sin(q_1) \\
& -8 m_3 \pi \sin(q_1) L_3^2 L_1^2 - 8 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 L_2 \\
& +12 L_3 m_3 (\cos(q_1))^4 L_1 L_2 + 3 L_1 L_2^2 m_2 \sin(q_1) L_3^2 + 3 L_1 L_2^2 m_2 \sin(q_1) h^2 \\
& +4 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 (\cos(q_1))^2 + 4 L_1^3 m_3 L_3^2 \cos(q_1) (\sin(q_1))^2 L_2
\end{aligned}$$

$$\begin{aligned}
& -3 L_1 L_2^4 m_3 \sin(q_1) (\cos(q_1))^2 - 3 m_3 L_2^3 \sin(q_1) L_1^2 \cos(q_1) L_3 \\
& + 8 L_3 m_3 \pi L_1 - 8 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) \cos(q_1) - 8 L_1 L_2^2 m_3 \pi (\sin(q_1))^3 \\
& + 6 m_3 L_2^2 \sin(q_1) L_1^2 L_3^2 + 3 m_3 L_2^3 \sin(q_1) L_1^2 (\cos(q_1))^3 L_3 \\
& + 4 L_1^3 m_3 L_2^3 (\sin(q_1))^2 L_3 (\cos(q_1))^2 + 12 L_1^2 m_3 L_2 (\cos(q_1))^2 L_3^2 \sin(q_1) \\
& + L_1^3 m_2 \sin(q_1) L_3^2 - 6 L_1^2 m_3 L_2^4 (\sin(q_1))^2 L_3 - 4 m_3 L_2^3 \sin(q_1) L_1 (\cos(q_1))^2 \\
& + 2 L_2^5 m_3 L_3^2 (\sin(q_1))^2 \cos(q_1) + 48 L_3^2 m_3 \pi \sin(q_1) L_1 \\
& + 2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 \cos(q_1) - 12 L_1^2 m_3 L_2^3 (\cos(q_1))^2 L_3 (\sin(q_1))^2 \\
& + 4 L_1 L_2 m_2 L_3^2 \sin(q_1) - 2 L_1^3 m_3 L_2^2 L_3 (\sin(q_1))^2 + 20 L_2^2 m_3 (\cos(q_1))^2 L_1 \sin(q_1) \\
& + 2 L_3 m_3 L_2^3 (\sin(q_1))^2 (\cos(q_1))^2 - 2 L_3 m_2 \sqrt{L_3^2 + h^2} \sin(\beta) L_2^2 \sin(q_1) (\cos(q_1))^2 \\
& + 2 L_3 m_2 \sqrt{L_3^2 + h^2} \cos(\beta) L_1^2 \cos(q_1) + 8 L_3^2 m_3 L_1 \cos(q_1) L_2 \\
& + 4 m_3 L_3^2 L_1^3 \sin(q_1) (\cos(q_1))^2 - 4 L_3 m_3 L_2^3 \sin(q_1) L_1 (\cos(q_1))^3 \\
& + 4 m_2 \pi \sin(q_1) L_3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2^2 (\cos(q_1))^2 + L_1^3 m_2 \sin(q_1) h^2 (\dot{q}_1)^2 \\
& / - 2 L_3 \left(-4 I_3 L_1 - 4 m_3 L_2^2 - 4 I_2 L_2 - 4 I_2 L_1 - 4 m_3 L_2^3 - 4 I_3 L_2 - 4 m_2 L_3^2 L_1 \right. \\
& - m_3 L_2^2 L_1^2 + 16 m_3 \pi (\cos(q_1))^3 L_3 L_2 - m_2 h^2 L_1^2 + 4 m_3 (\cos(q_1))^2 L_3^2 L_1^2 \\
& - 4 I_1 - 4 I_2 - 4 I_3 + 2 m_3 L_2^2 L_1^2 (\cos(q_1))^2 - m_3 L_2^2 L_1^2 (\cos(q_1))^4 \\
& - 4 m_3 L_2^2 L_1 (\cos(q_1))^4 - 16 m_3 \pi (\cos(q_1))^4 L_1 + 16 m_3 \pi (\cos(q_1))^2 L_1 \\
& + 16 m_3 L_2 (\cos(q_1))^4 L_1 + 4 m_3 L_2^3 L_1 (\cos(q_1))^2 \\
& - 16 m_3 L_2 (\cos(q_1))^2 L_1 + 32 m_3 \pi (\cos(q_1))^2 L_2 - 4 m_2 L_3^2 L_2 \\
& - 16 m_3 \pi \cos(q_1) L_3 L_2 - 16 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) + 4 L_1^2 m_1 (\cos(q_1))^2 \\
& + 8 m_3 L_2^3 (\cos(q_1))^2 + 4 m_3 (\cos(q_1))^4 L_1^2 + 4 I_2 L_2 (\cos(q_1))^2 \\
& - 4 m_3 (\cos(q_1))^2 L_1^2 + 8 m_3 L_3 \cos(q_1) L_1^2 + m_2 L_3^2 L_1^2 (\cos(q_1))^2 \\
& + m_2 L_3^2 L_2^2 (\cos(q_1))^2 + 4 m_2 L_3^2 L_1 (\cos(q_1))^2 + 4 m_2 L_3^2 L_2 (\cos(q_1))^2 \\
& \left. + 16 m_3 L_3 \cos(q_1) L_2^2 - 32 m_3 L_3 L_2 (\cos(q_1))^3 + 16 m_3 \pi (\sin(q_1))^2 L_2 \right)
\end{aligned}$$

$$\begin{aligned}
& +2 I_3 L_1 L_2 (\cos(q_1))^2 + I_2 L_1^2 (\cos(q_1))^2 + I_3 L_1^2 (\cos(q_1))^2 \\
& -m_3 L_2^4 (\cos(q_1))^4 + 4 I_2 L_1 (\cos(q_1))^2 - 8 m_3 L_2^2 (\cos(q_1))^2 \\
& +4 I_3 L_1 (\cos(q_1))^2 + 12 m_3 L_2^2 (\cos(q_1))^4 + 2 m_3 L_2^4 (\cos(q_1))^2 \\
& -4 m_3 L_2^3 (\cos(q_1))^4 + 4 I_3 L_2 (\cos(q_1))^2 + 16 m_2 \pi^2 (\cos(q_1))^2 \\
& +I_2 L_2^2 (\cos(q_1))^2 + I_3 L_2^2 (\cos(q_1))^2 + 16 m_3 L_3^2 (\cos(q_1))^2 - 16 m_3 L_3^2 \\
& +4 m_2 h^2 (\cos(q_1))^2 + 4 m_2 L_3^2 (\cos(q_1))^2 - 8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 \\
& +16 m_3 L_3 \cos(q_1) L_1 - 24 m_3 L_3 (\cos(q_1))^3 L_1 L_2 + 8 m_3 L_2^2 L_1 (\cos(q_1))^2 \\
& -4 m_3 L_3^2 L_1^2 + 2 m_2 L_3^2 L_1 L_2 (\cos(q_1))^2 + 16 m_3 \pi^2 (\cos(q_1))^2 - m_2 L_3^2 L_2^2 \\
& -16 m_3 \pi (\sin(q_1))^2 L_2 (\cos(q_1))^2 - 16 m_3 \pi \cos(q_1) L_3 L_1 - 4 m_2 h^2 - 16 m_3 L_3^2 L_1 \\
& -2 m_3 L_2^3 L_1 (\cos(q_1))^4 + 2 I_2 L_1 L_2 (\cos(q_1))^2 + 4 m_2 h^2 L_2 (\cos(q_1))^2 \\
& +4 m_2 h^2 L_1 (\cos(q_1))^2 + 8 m_3 \pi (\sin(q_1))^2 L_2^2 + m_2 h^2 L_1^2 (\cos(q_1))^2 \\
& +m_2 h^2 L_2^2 (\cos(q_1))^2 - 32 m_3 \pi (\cos(q_1))^4 L_2 + 24 m_3 L_3 \cos(q_1) L_1 L_2 \\
& -4 L_1^2 m_1 - I_2 L_1^2 + 2 m_2 h^2 L_1 L_2 (\cos(q_1))^2 + 16 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) \\
& -16 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) - I_3 L_1^2 + 16 m_3 L_3^2 L_1 (\cos(q_1))^2 \\
& -32 m_3 L_3 \pi \cos(q_1) + 32 m_3 L_3 \pi (\cos(q_1))^3 + 16 m_3 L_3^2 L_2 (\cos(q_1))^2 \\
& +32 m_3 L_3 L_2 \cos(q_1) - 2 m_3 L_2^3 L_1 - I_3 L_2^2 - 8 m_3 \pi (\sin(q_1))^2 L_2^2 (\cos(q_1))^2 \\
& -8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 + 8 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) L_1 \\
& +8 m_3 \pi (\sin(q_1))^2 L_2 L_1 - 2 m_2 h^2 L_1 L_2 - 2 m_2 L_3^2 L_1 L_2 - m_2 L_3^2 L_1^2 - m_3 L_2^4 - m_2 h^2 L_2^2 \\
& -I_2 L_2^2 - 8 m_3 L_3^2 L_1 L_2 + 8 m_2 \pi (\cos(q_1))^3 \sqrt{L_3^2 + h^2} \cos(\beta) L_2 \\
& -2 I_2 L_1 L_2 - 2 I_3 L_1 L_2 - 4 m_3 L_2^2 L_3^2 - 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_2 \\
& +4 I_2 (\cos(q_1))^2 + 4 I_1 (\cos(q_1))^2 + 4 I_3 (\cos(q_1))^2 \\
& -8 m_3 \pi (\sin(q_1))^2 L_2 L_1 (\cos(q_1))^2 - 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_1 \\
& +8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_2 (\cos(q_1))^2 - 16 m_3 L_3^2 L_2 - 4 m_3 L_2^2 L_1
\end{aligned}$$

$$\begin{aligned}
& -4 m_2 h^2 L_2 - 4 m_2 h^2 L_1 + 8 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) L_1 (\cos(q_1))^2 \\
& + 4 m_3 L_2^2 L_3^2 (\cos(q_1))^2 + 8 m_3 L_2 (\cos(q_1))^2 L_3^2 L_1 \\
& - 4 m_2 L_3^2 + 16 m_3 \pi (\cos(q_1))^3 L_3 L_1 - 16 m_2 \pi^2 \\
& + 16 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) (\cos(q_1))^2 - 16 m_3 L_3 (\cos(q_1))^3 L_1 \\
& - 16 m_3 L_3 (\cos(q_1))^3 L_2^2 - 8 m_3 L_3 (\cos(q_1))^3 L_1^2 - 16 m_3 \pi^2
\end{aligned}$$

$$\begin{aligned}
C = & 1 / \left(I_3 L_1 + 4 m_3 L_2^2 + 4 I_2 L_2 + 4 I_2 L_1 + 4 m_3 L_2^3 + 4 I_3 L_2 + 4 m_2 L_3^2 L_1 \right. \\
& + m_3 L_2^2 L_1^2 + m_2 h^2 L_1^2 + 4 I_1 + 4 I_2 + 4 I_3 - m_3 L_2^2 L_1^2 (\cos(q_1))^2 \\
& - 16 m_3 \pi (\cos(q_1))^2 L_1 - 2 m_3 L_2^3 L_1 (\cos(q_1))^2 + 16 m_3 L_2 (\cos(q_1))^2 L_1 \\
& - 32 m_3 \pi (\cos(q_1))^2 L_2 + 4 m_2 L_3^2 L_2 + 16 m_3 \pi \cos(q_1) L_3 L_2 \\
& + 16 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) - 4 m_3 L_2^3 (\cos(q_1))^2 \\
& + 4 m_3 (\cos(q_1))^2 L_1^2 - 8 m_3 L_3 \cos(q_1) L_1^2 - 16 m_3 L_3 \cos(q_1) L_2^2 \\
& - 16 m_3 \pi (\sin(q_1))^2 L_2 + 12 m_3 L_2^2 (\cos(q_1))^2 - m_3 L_2^4 (\cos(q_1))^2 \\
& + 16 m_3 L_3^2 + 8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_2 + 4 m_3 L_3^2 L_1^2 \\
& + m_2 L_3^2 L_2^2 + 16 m_3 \pi \cos(q_1) L_3 L_1 + 4 m_2 h^2 + 16 m_3 L_3^2 L_1 \\
& + 4 m_2 L_3^2 + I_2 L_2^2 - 4 m_3 L_2^2 L_1 (\cos(q_1))^2 \\
& - 8 m_3 \pi (\sin(q_1))^2 L_2^2 - 24 m_3 L_3 \cos(q_1) L_1 L_2 + 4 L_1^2 m_1 + I_2 L_1^2 \\
& + 16 m_2 \pi \sin(q_1) \sqrt{L_3^2 + h^2} \sin(\beta) + I_3 L_1^2 - 16 m_3 L_3 \cos(q_1) L_1 \\
& + 32 m_3 L_3 \pi \cos(q_1) - 32 m_3 L_3 L_2 \cos(q_1) + 2 m_3 L_2^3 L_1 \\
& + I_3 L_2^2 + 8 m_2 \pi \cos(q_1) \sqrt{L_3^2 + h^2} \cos(\beta) L_1 - 8 m_3 \pi (\sin(q_1))^2 L_2 L_1 \\
& + 2 m_2 h^2 L_1 L_2 + 2 m_2 L_3^2 L_1 L_2 + m_2 L_3^2 L_1^2 + m_3 L_2^4 + m_2 h^2 L_2^2 \\
& \left. + 8 m_3 L_3^2 L_1 L_2 + 2 I_2 L_1 L_2 + 2 I_3 L_1 L_2 + 4 m_3 L_2^2 L_3^2 \right)
\end{aligned}$$

$$\begin{aligned}
& +8 m_2 \pi \sin (q_1) \sqrt{L_3^2+h^2} \sin (\beta) L_2+8 m_2 \pi \sin (q_1) \sqrt{L_3^2+h^2} \sin (\beta) L_1 \\
& +16 m_3 L_3^2 L_2+4 m_3 L_2^2 L_1+4 m_2 h^2 L_2+4 m_2 h^2 L_1+16 m_2 \pi^2+16 m_3 \pi^2)
\end{aligned}$$