LINEAR PROGRAMMING WITH INTERVAL COEFFICIENTS

by

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in partial fulfilment of
the requirements for the degree of
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Information and Systems Science
Department of Mathematics and Statistics

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ABSTRACT

Formulating and solving a Linear Programming model which incorporates imprecise coefficients (in the form of interval numbers), is a very interesting and challenging problem. The main difficulty is that if the constraints have interval coefficients, then the feasible region is not fixed. Instead, there is an infinite number of possible feasible regions, depending on the fixed values chosen for the interval coefficients. These different feasible regions may give different optimal values depending on the coefficient settings. In addition, if the objective function has interval coefficients then there is also an infinite choice of possible objective functions. For practical purposes, the problem is to determine the coefficient settings that produce the best optimum and the coefficient settings that produce the worst optimum. This gives an idea of the range of solutions spanned by the imprecise interval model. This thesis proposes a method of solving general linear programming models with interval numbers as coefficients (*LPIC*). The structure, properties and characteristics of LPIC are presented and illustrated with several examples.

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CHAPTER 1

Introduction

Since the introduction of the simplex method by George B. Dantzig in 1949, many people have contributed to the success and growth of linear programming in several ways; by introducing new efficient computational methods and new algorithms to improve its time complexity, by using linear programs as a base to solve more complex problems as in integer linear programming and nonlinear programming, etc. However one of the problems that has not been solved is the use of imprecise data, or more specifically interval numbers, as coefficients in a linear programming model.

In this chapter a brief description of linear programming problems with interval coefficients (*LPIC*) will be given. In addition, there will be an example to illustrate the nature of the problem, and an overview of the method and the main results will be outlined.

1.1 An Overview of Linear Programming with Interval Coefficients (LPIC)

Many optimization models in operations research such as linear programming assume that the data used to build the model is precisely known, that constraints delimit a crisp set of feasible decisions, and that criteria are well defined and easy to formalize (Dubois and Prade, 1980). However, in the real world most of these assumptions are only approximately true, and most of the data is known only within specific limits of the true value. In linear programming models, data uncertainty is associated with the coefficients of the model during the formulation stages. For instance during the early stages of design, the model designer may be unaware of the exact value of some coefficients. However, the designer may be able to use past experience to estimate the coefficient within certain limits. In this case, it is much safer and more appropriate to use a range of values to represent the coefficient in question instead of a fixed real number. Furthermore, in many real life models it is more realistic to represent some of the coefficients as interval numbers instead of fixed real numbers. By using interval numbers as input coefficients, the model becomes more flexible allowing for much richer output and better and more accurate decisions.

The goal of this thesis is to show how to solve the general interval linear programming problem in which some or all of the coefficients are interval numbers, a problem not fully addressed in the literature. A review of the literature (Chapter 3) shows

only a single method due to Shaocheng (1994) that can handle only one type of LPIC, i.e. LPIC with non-negative variables only.

1.2 Why LPIC is Difficult

Suppose we have a linear programming model where one of the coefficients is uncertain, as in the following example:

Example 1.1

$$\begin{aligned} \min \quad & z = x_2 \\ \text{s.t.} \quad & C_1: 2x_1 + x_2 \geq 4 \\ & C_2: 3x_1 - 2x_2 \leq \boxed{4 \pm 25\%} \\ & x_1, x_2 \geq 0 \end{aligned}$$

Note that the RHS of C_2 can take any value between 3 and 5.

$$\begin{aligned} \text{Let: } \quad & C_{2a}: 3x_1 - 2x_2 \leq 3 \\ & C_{2b}: 3x_1 - 2x_2 \leq 5 \end{aligned}$$

The uncertainty in the RHS of C_2 allows it to vary between the two constraints C_{2a} and C_{2b} depending on the value we choose, as in Figure 1.1. Furthermore, by choosing an arbitrary number $k \in [3, 5]$ as the RHS for constraint C_2 , we get a totally different optimal value depending on k . Our main objective is to determine the coefficient setting for all the interval coefficients that will give us the best optimal value and the worst optimal value, and to determine what the best and worst optimal values will be.

By analyzing Example 1.1 and Figure 1.1, one can easily see that the best optimum occurs at $(x_1, x_2) = (1.86, 0.29)$ with $z = 0.29$ and the RHS of constraint C_2 fixed at 5, and the worst optimum occurs at $(x_1, x_2) = (1.57, 0.855)$ with $z = 0.855$ and the RHS of constraint C_2 fixed at 3. This pictorial analysis is possible due to the small number of uncertain coefficients (in this case only one). However, if we have a model with 7 variables and 5 constraints with several uncertain coefficients and variables that are unrestricted in sign, it is impossible to determine the coefficient settings that give the best optimum and worst optimum solutions by inspection. Therefore, our goal is to develop a method which will allow us to determine these settings and solutions. In Chapter 6, this problem will be further described and analyzed, and a method which handles *all* cases of LPIC will be presented.

The main approach of the method proposed here, is to transform the non-deterministic LPIC model into a set of related deterministic LP models. The solution of

these deterministic LP models provides the best optimum solution and the worst optimum solution under the given uncertainties. The technique used here was found to be very effective in cases where the variables associated with interval coefficients are restricted to a specific orthant (i.e. $x_i \geq 0$ or $x_i \leq 0 \quad \forall i, x_i$ associated with interval coefficient). However, when some or all of the variables associated with interval coefficients are sign-unrestricted in any of the relationships, a combinatorial explosion in the number of deterministic LPs to be solved is unavoidable. To deal with the latter case, an enumerated approach is introduced. Several examples are presented.

1.3 Thesis Outline

In the next few chapters a complete description and analysis of LPIC, and a method to handle LPIC will be provided. An introduction to interval numbers and interval arithmetic, as well as an overview of linear programming will be covered in Chapter 2. In Chapter 3, the state of the art in LPIC will be reviewed. In Chapters 4, 5 and 6, which are the main chapters of this thesis, a complete and concise statement of the problem of linear programming with interval coefficients will be given, as well as a description, analysis and discussion of the main method used to solve LPIC. Throughout these chapters several examples are used to illustrate the concept of LPIC, and to uncover the different problems and scenarios that arise. In Chapter 7 a study case will be given to illustrate the use and effectiveness of this method in industry. A summary of the research and of the main results will be covered in the last chapter .

CHAPTER 2

Background Information

In this chapter a brief introduction to real interval numbers and linear programming problems is given. The basic properties and terminology of interval arithmetic are discussed and defined in Section 2.1. In Section 2.2, a review of the fundamental concepts of linear programming is outlined.

2.1 Introduction to Interval Numbers

Interval arithmetic is an elegant tool for practical work when representing approximate numbers, error bounds, imprecise data, etc. In mathematical models, coefficients can not always be represented in a floating-point form. In any computer application, numbers must be entered or read-in with a finite number of digits depending on the precision of the machine used. In general, this will lead to round-off error. For instance, $\frac{1}{3} \approx 0.33333\dots$, which will cause a small rounding error when entered in the computer. However, a more precise representation of $(\frac{1}{3})$ would be $0.3333 \leq \frac{1}{3} \leq 0.3334$, for example.

The basic definitions and properties of interval numbers and interval arithmetic summarized below follow text books written by Moore, (1966), Alefeld and Herzberger, (1983) and Neumaier, (1990).

Definition 2.1. A real *interval* number $A \in I(\mathbb{R})$ is a set of the form

$$A = [\underline{a}, \bar{a}] = \{t \in \mathbb{R} | \underline{a} \leq t \leq \bar{a}\}$$

where $\underline{a}, \bar{a} \in \mathbb{R}$ with $\underline{a} \leq \bar{a}$.

Definition 2.2. Two interval numbers $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$ are called equal iff $\underline{a} = \underline{b}$ and $\bar{a} = \bar{b}$.

Definition 2.3. An interval number, $A = [\underline{a}, \bar{a}]$ is called a point interval number, if $\underline{a} = \bar{a}$.

Definition 2.4. An interval is called unbounded, if the lower bound and/or the upper bound is infinity.

e.g. $[-\infty, 3]$, $[5, \infty]$, $[-\infty, \infty]$, ... etc.

2.1.1 Interval Arithmetic and Properties

Let $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$ with $0 \in B$, then:

- $A + B = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$ (addition)
- $A - B = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]$ (subtraction)
- $A * B = [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}]$ (multiplication)
- $A / B = [\underline{a}, \bar{a}] * [\frac{1}{\bar{b}}, \frac{1}{\underline{b}}]$ (division)

If $0 \in B$, then A / B is undefined.

Let A, B and $C \in I(\mathbb{R})$ then:

- $A + B = B + A$, $A * B = B * A$ *(commutativity)*
- $(A + B) + C = A + (B + C)$, $(A * B) * C = A * (B * C)$ *(associativity)*
- $[0,0]$ and $[1,1]$, are the unique neutral elements with respect to addition and multiplication.
- real interval number has no zero divisors.
- a real interval number $A = [\underline{a}, \bar{a}]$, $\underline{a} \neq \bar{a}$ has no inverse element with respect to addition and multiplication, however, $0 \in (A - A)$ and $1 \in (A / A)$.
e.g. $[2,4] - [2,4] = [-2,2]$.
- $A * (B + C) \subseteq A * B + A * C$ *(subdistributivity)*
- $a(B + C) = aB + aC$, $a \in \mathbb{R}$

The proofs of the above properties can be found in any of the above mentioned references, especially Alefeld and Herzberger, (1983). For the purpose of this thesis, there is no need to go through the proofs.

2.2 Introduction to Linear Programs

Linear programming is a mathematical tool developed to handle the optimization (minimization or maximization) of a linear function subject to a set of linear constraints.

The subject of linear programming is a very important area in applied mathematics and it has a large number of uses and applications in many industries. Some of the current

applications include; allocation of resources, transportation problems, scheduling of operations, management information systems, etc. ...

Mathematically a linear programming problem (LP), can be represented as follows:

$$\begin{aligned} \min (\max) \quad Z &= \sum_{j=1}^n c_j x_j \\ \text{subject to (s.t.)} \quad \sum_{j=1}^n a_{ij} x_j &\geq (=, \leq) b_i \quad (i = 1, \dots, m) \\ x_j &\in \mathbb{R}, \quad (j = 1, \dots, n) \end{aligned} \quad (1)$$

- $x_1, \dots, x_n$ are the problem variables,
- $c_1, \dots, c_n$ are constants called the cost coefficients,
- $b_1, \dots, b_m$ are the resource value coefficients,
- $a_{ij}, (i = 1, \dots, m), (j = 1, \dots, n)$ are the constraint coefficients,
- Z is the objective function.

Definition 2.5. A set of variable values $x_1, x_2, \dots, x_n$ satisfying all the constraints of an LP as in (1) is called a *feasible solution*. If the set fails to satisfy one or more of the constraints, then it is called an *infeasible solution*.

Definition 2.6. The set of all feasible solutions is called the *feasible region*.

Definition 2.7. A feasible solution that optimizes (minimizes/maximizes) the objective function is called an *optimal solution* with Z being the *optimal value*.

Theorem 2.1. The feasible set in $\mathbb{R}^n$ corresponding to any number of constraints of the types

$$a_1x_1 + a_2x_2 + \dots + a_nx_n (\leq, =, \text{ or } \geq) b$$

$$x_1, x_2, \dots, x_n \geq 0$$

is convex. (Calvert and Voxman, 1989)

Theorem 2.2. (Corner Point Theorem) Consider the LP problem

$$\begin{aligned} \min (\max) \quad Z &= \sum_{j=1}^n c_j x_j \\ \text{subject to (s.t.)} \quad &\sum_{j=1}^n a_{ij} x_j \geq (=, \leq) b_i \quad (i = 1, \dots, m) \\ &x_j \geq 0, \quad (j = 1, \dots, n) \end{aligned} \quad (2)$$

- a) If the feasible region of (2) is bounded, then an optimal solution is attained at a corner point of this feasible region.
- b) If the feasible region is unbounded, then an optimal solution may not exist; however, if an optimal solution exists, it is attained at a corner point of (2).

(Calvert and Voxman, 1989)

Non-negativity of variables:

Generally, most linear programming problems deal with variables that are non-negative in nature, and hence most of the methods developed to solve linear programming problems require the non-negativity of the variables involved.

Therefore, if a variable x_i is negative or unrestricted in sign, it can be replaced by positive variables as follows:

- if $x_i \leq 0$, let $x_i = -x'_i$ with $x'_i \geq 0$.
- if $x_i \geq l$, where $l \leq 0$, let $x_i = x'_i + l$ with $x'_i \geq 0$.
- if x_i is unrestricted in sign, then let $x_i = x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$.

For more information or proofs with regard to any of the theorems or the definitions given above, one can refer to any of these references (Calvert and Voxman, 1989), (Bazaraa et al, 1977), (Dantzig, 1963) or (Chvatal, 1983).

Example 2.1. Maximize: $z = -2x_1 + x_2$
 $C_1: -3x_1 + x_2 \leq 1$
 $C_2: -x_1 + x_2 \leq 3$
 Subject to: $C_3: -x_1 + 2x_2 \leq 2$
 $x_1, x_2 \geq 0$

Figure 2.1. An Example of LP problem.

This example shows that although the feasible region is unbounded, we still get an optimal solution to the problem at $(1,4)$ as in Figure 2.1 with $z = 2$.

CHAPTER 3

State of the Art in LPIC

A very small number of papers deal with linear programming models that involve interval coefficients. In fact only one paper was found that deals explicitly with general linear programming problems involving interval coefficients. Most papers deal with a specific case of LPIC (e.g. 0-1 LPIC, linear programming models with interval coefficients in the objective function only, etc.).

Note that there was extensive work done in the 1960s, 1970s and early 1980s in the area of interval linear programming. Refer to papers written by Zlobec and Ben-Israel (1970), Ben-Israel and P. Roberts (1977), and E. Gunn and G. Anders (1981). However this is not to be confused with LPIC; interval linear programming deals with models that have constraints of type $c_1 \leq a_1x_1 + a_2x_2 + \dots + a_nx_n \leq c_2$ and no interval coefficients anywhere.

In the following sections the relevant research will be outlined.

3.1 Boolean Linear Programming with Interval Coefficients

Levin (1994) deals with linear programming models that involve interval coefficients and boolean variables. The main approach is to transform the model from a non-deterministic problem into two deterministic problems as described below.

Suppose we have a non-deterministic maximization model as follows;

$$\begin{aligned} \max \quad & q = \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j \\ \text{s.t.} \quad & \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j \leq [\underline{b}_i, \bar{b}_i], \quad i = 1 \dots m \\ & x_j \in \{0,1\} \quad \forall j \end{aligned}$$

Then this model is transformed into two deterministic models as follows:

1)	$\begin{aligned} \max \quad & q_1 = \sum_{j=1}^n \underline{c}_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n \underline{a}_{ij} x_j \leq \underline{b}_i, \\ & \sum_{j=1}^n \bar{a}_{ij} x_j \leq \bar{b}_i \\ & x_j \in \{0,1\} \quad \forall j \end{aligned}$	2)	$\begin{aligned} \max \quad & q_2 = \sum_{j=1}^n \bar{c}_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n \underline{a}_{ij} x_j \leq \underline{b}_i, \\ & \sum_{j=1}^n \bar{a}_{ij} x_j \leq \bar{b}_i \\ & x_j \in \{0,1\} \quad \forall j \end{aligned}$
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The method used here is to create two objective functions by using the boundaries of each interval involved in the objective function, and similarly create two deterministic constraints for each non-deterministic constraint. The main drawback with this method resides in the approach used to create the two deterministic constraints. The author assumes that, given two intervals $A=[a_1, a_2]$ and $B=[b_1, b_2]$ then:

$$A \leq B \text{ iff } a_1 \leq b_1 \text{ and } a_2 \leq b_2$$

However this assumption is false. In interval numbers, A is less than or equal to B if and only if every element in A is less than or equal to every element in B i.e.

$$A \leq B \text{ iff } a_2 \leq b_1.$$

Example 3.1.

Let $A = [2, 6]$, and $B = [4, 7]$.

According to Levin, $A \leq B$ since $2 < 4$ and $6 < 7$. However, $5 \in A > 4 \in B$. Therefore, A is not less than or equal to B

Indeed the method used by Levin does not give the best and worst optimal solutions, instead it gives feasible solutions only, and fails in two aspects:

- 1) deals only with boolean variables.
- 2) does not give the best and worst *optimal* solutions.

However the basic idea is similar to the one used in this thesis.

3.2 Minimax Regret Solution

Imaiuchi and Sakawa (1995) deal only with linear programming models that involve interval coefficients in the objective function. The primary goal of their technique is to determine a single solution that is closest to all of the optimal solutions of the model under the given uncertainties in the objective function. They use the minimax regret approach to achieve one solution that minimizes the worst difference of the objective value between any two different solutions. There are many drawbacks to this approach; (1) only the objective function is allowed to take interval coefficients, (2) it does not give the optimal solution of the problem (best or worst) and hence provides no information for decision makers to analyze regarding the optimum value, and (3) as the authors indicate, the minimax solution is not always a basic solution i.e. it is not necessarily an extreme point of the feasible region in question.

3.3 Interval Number Linear Programming

Shaocheng (1994) provides the most important and relevant results regarding interval numbers and linear programming. He introduces the maximum value range inequalities and minimum value range inequalities (described in detail in Chapter 6) to reduce the LPIC from one non-deterministic model into two deterministic LP models.

These two models can then be solved by any of the available methods such as the Simplex method, to obtain an interval optimal solution to the original model. The technique is to determine the largest possible feasible region and the smallest possible feasible region under the given uncertainties, and these two regions then yield upper and lower bounds on the optimal solution of the original model in conjunction with appropriate versions of the objective function.

Shaocheng's basic idea underlies the work in this thesis. Unfortunately there were some drawbacks to his work, necessitating the research presented herein:

1. The paper deals only with non-negative variables.
2. No hints or explanations as to how to deal with non-positive variables or variables that are unrestricted in sign.
3. No hints or explanations as to how to deal with equality interval constraints.

CHAPTER 4

The Problem of Linear Programming With Interval Coefficients

In the following three sections, we will give a clear and concise statement of the problem of Linear Programming with Interval Coefficients (LPIC), further analyze the currently available approaches to this problem and close the chapter with a discussion of the importance of this type of mathematical program.

4.1 Thesis Problem Statement

The goal of this thesis is to develop a method capable of solving linear programming problems with interval numbers as coefficients.

The problem is stated as follows:

Given a linear programming problem with interval coefficients in either or both of the objective function and the constraints:

$$\min(\max) \quad Z = \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j$$

$$\text{s.t.} \quad \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j \geq (=, \leq) [\underline{b}_i, \bar{b}_i], \quad \text{for } i=1 \dots m$$

derive an algorithm which will compute:

a. the best optimum, and the coefficient settings that make this possible;

b. the worst optimum, and the coefficient settings that make this possible;

for both:

- x_j sign-restricted to a single orthant, $j = 1 \dots n$
(i.e. $x_j \geq 0$ or $x_j \leq 0$)
- some or all of x_j not sign-restricted to any specific orthant (i.e. x_j unrestricted in sign).

4.2 Analysis of the State of the Art In LPIC

As discussed earlier, this problem has not been dealt with completely in any of the previous literature. All previous papers design their methods and algorithms to handle only specific cases of LPIC. For instance Tong Shaocheng (1994) restricted his method to variables in the positive orthant only, and did not address how to deal with cases when some of the variables are negative or unrestricted in sign. Other work by V. I. Levin (1994) and Inuiguchi and Sakawa (1995) dealt only with cases in which the variables are Boolean and only the objective function involves interval coefficients, respectively. Furthermore, both of these approaches fail to provide the best optimum and the worst optimum even under these restricted conditions.

Because the current state of the art is not able to handle all cases of the problem outlined, we saw a need to develop a method that will provide a solution to an LPIC under the much broader range of conditions outlined in Section 4.1.

4.3 Importance of LPIC

In mathematical programming models, the coefficients of the objective function and constraints are assumed to be precisely known real numbers. However, in the real world most of the coefficients are only approximately known, and have a certain percentage of uncertainty and error. Solving an LP version of the model by fixing the coefficients can prove very misleading.

One way to deal with the problem of representing data with some uncertainty is to use interval numbers as coefficients instead of real numbers. This provides more realistic modelling, leading to better and more reliable decisions.

In an uncertain environment where the coefficients are known only approximately, decision makers would want to know:

- the best that can happen (i.e. best optimum);
- the worst that can happen (i.e. worst optimum);
- the setting of coefficients that leads to these optimum solutions.

By using LPIC as a modelling tool, decision makers will be able to get this information and hence make better decisions.

CHAPTER 5

Interval Linear Functions and Linear Programs

In this chapter the basic characteristics of interval linear functions and interval linear constraints will be defined, and the effect of interval coefficients in linear programs will be described and outlined by several examples.

5.1 Characteristics of Linear Equations with Interval Coefficients

Definition 5.1. Suppose we have a linear equation with interval coefficients such as:

$[a_1, a_2]x_1 + [b_1, b_2]x_2 = (\geq, \leq)[c_1, c_2]$, then an *extreme version* of this interval linear equation is obtained by fixing all coefficients at one of the boundary points. If one of the interval coefficients is fixed at an intermediate point, then it is called an *intermediate version*. e.g.

- $a_2x_1 + b_1x_2 = (\geq, \leq)c_1$ is an extreme version of the above interval linear equation.
- $a_2x_1 + (\frac{b_1 + b_2}{2})x_2 = (\geq, \leq)c_1$ is an intermediate version of the above interval linear equation.

Since the methods developed later in this thesis involve choosing various extreme or intermediate versions of the constraints and objective functions, it is important to understand how the different versions of a single interval linear equation are related. We examine below relationships such as shifts, tilts and reversals.

Definition 5.2. Let $[a_1, a_2]x_1 + [b_1, b_2]x_2 = (\geq, \leq)[c_1, c_2]$ be a given constraint in a linear program problem with interval coefficients. A *shift* of the constraint is a parallel movement of the constraint from one position to another without changing its slope. The shift is caused by changing only the RHS from one value in $[c_1, c_2]$ to another value in $[c_1, c_2]$.

Example 5.1. Let $2x_1 + 4x_2 \geq [4, 8]$ be a given linear constraint with interval RHS, then the constraint will shift from one extreme version ($2x_1 + 4x_2 \geq 4$), to the other ($2x_1 + 4x_2 \geq 8$) as illustrated in Figure 5.1.

Definition 5.3. Let $[a_1, a_2]x_1 + [b_1, b_2]x_2 = (\geq, \leq)[c_1, c_2]$ be a given constraint (or an objective function) in a linear programming problem with interval coefficients. A *tilt* is a change of the slope of the given constraint (or the objective function). The tilt is caused by changing at least one of the interval coefficients associated with any of the variables from one value in the interval to another value also in the interval.

Example 5.2. Let $2x_1 + [2, 4]x_2 \geq 4$ be a linear constraint with interval coefficient associated with x_2 , then the constraint will tilt from one extreme version ($2x_1 + 2x_2 \geq 4$), to the other ($2x_1 + 4x_2 \geq 4$) as illustrated in Figure 5.2.

Definition 5.4. A *reversal* of a linear constraint (or an objective function) of a linear programming problem is a special type of tilting that occurs when there is a simultaneous change of sign of all the coefficients on the LHS (i.e. all interval coefficients associated with variables span zero).

Example 5.3. Let $[-1,1]x_1 + [-1,2]x_2 \geq 2$ be a given linear constraint with interval LHS, then the constraint will flip from one version $(-x_1 - x_2 \geq 2)$, to the another version $(x_1 + 2x_2 \geq 2)$ as illustrated in Figure 5.3.

5.2 The Effect of Interval Coefficients in Linear Programs

In linear programming models where some or all of the parameters are interval coefficients, the simultaneous effect of tilting, shifting and reversal of the interval constraints causes the feasible region to move, shrink or expand depending on the fixed setting chosen for each interval coefficient of each constraint. Similarly, interval coefficients in the objective function cause its direction to change depending on the settings chosen for all the interval coefficients involved.

Let C be a set of constraints with interval coefficients, and let C_I and C_{II} be two different sets of constraints generated from C by using different extreme versions (different by at least one coefficient). Then, the feasible regions S_I and S_{II} generated by C_I and C_{II} are described by one of these possibilities:

- 1) $S_I \subseteq S_{II}$ or $S_{II} \subseteq S_I$, i.e. one feasible region is fully contained in the other.
- 2) $S_I \neq S_{II}$, and $S_I \cap S_{II} \neq \emptyset$, i.e. one feasible region partially intersects the other.
- 3) $S_I \cap S_{II} = \emptyset$, i.e. no overlap of the feasible regions.

Example 5.4. Let C be a set of constraints as follows:

$$C = \begin{cases} a: & 2x_1 - 2x_2 \leq [1, 2] \\ b: & x_1 + x_2 \geq [2, 2.5] \\ & x_1, x_2 \geq 0 \end{cases}$$

and let C_I and C_{II} be as follows:

$$C_I = \begin{cases} a_I: 2x_1 - 2x_2 \leq 2 \\ b_I: x_1 + x_2 \geq 2 \\ x_1, x_2 \geq 0 \end{cases} \quad \text{and} \quad C_{II} = \begin{cases} a_{II}: 2x_1 - 2x_2 \leq 1 \\ b_{II}: x_1 + x_2 \geq 2.5 \\ x_1, x_2 \geq 0 \end{cases} \quad \text{then}$$

S_{II} (feasible region bounded by C_{II}) $\subset S_I$ (feasible region bounded by C_I), as in Figure 5.4.

Example 5.5. Let C be a set of constraints as follows:

$$C = \begin{cases} a: [-1, 1]x_1 + [-1, 1]x_2 \geq 1 \\ b: x_1 \geq -2 \\ c: x_2 \geq -2 \end{cases}$$

and let C_I and C_{II} be as follows:

$$C_I = \begin{cases} a_I: x_1 + x_2 \geq 1 \\ b: x_1 \geq -2 \\ c: x_2 \geq -2 \end{cases}, \quad \text{and} \quad C_{II} = \begin{cases} a_{II}: -x_1 - x_2 \geq 1 \\ b: x_1 \geq -2 \\ c: x_2 \geq -2 \end{cases} \quad \text{then}$$

S_{II} (feasible region bounded by C_{II}) $\cap$ S_I (feasible region bounded by C_I) $= \emptyset$, as in

Figure 5.5.

The main difficulty is trying to determine which settings of the extreme versions of each constraint yield the best and worst optimal values of the problem. In the next chapter, this problem will be discussed and a method to determine these settings will be outlined.

CHAPTER 6

Solving LPIC Problems

The various methods developed to solve LPIC will be described in this chapter. The main objective of these methods is to determine the best optimum and its solution set, and the worst optimum and its solution set if it exists, as well as the coefficient settings that determine these solutions. At the end of each section a number of examples will be given to demonstrate how the method works, and to clarify the type of results obtained.

Definition 6.1. A variable x is called *sign-restricted* variable, if it is restricted to a single orthant (i.e. the variable does not change in sign), otherwise it is called sign-unrestricted.

Example 6.1.

a) $x \geq 0$, $y \leq 0$, $z \geq 6$, $-10 \leq w \leq -3$

b) $x \geq -4$, $y \leq 6$, $-3 \leq z \leq 3$, $w \in [-\infty, \infty]$

a) examples of sign-restricted variables

b) examples of sign-unrestricted variables.

Notation: Let,

$W_I = \{ \text{set of all sign-restricted variables associated with fixed or interval coefficients} \},$

$W_{II} = \{ \text{set of all sign-unrestricted variables associated with fixed coefficients} \},$

$W_{III} = \{ \text{set of all sign-unrestricted variables associated with interval coefficients} \},$

To deal with this problem more effectively, the general LPIC problem is classified into three different classes:

1. **Type I LPIC;** LPIC with variables that are all sign-restricted, i.e. all variables $\in W_I$.
2. **Type II LPIC;** LPIC with at least one variable that belongs to W_{II} , and all others $\in W_I$.
3. **Type III LPIC;** LPIC with at least one variable that belongs to W_{III} , and all others $\in W_I$ or W_{II} .

6.1 LPIC Models with Sign-Restricted Variables (Type I)

In this section a generalized version of the method used by Shaocheng (1994) will be described and several examples will be given to illustrate the use of this generalized version of the method.

As mentioned in Chapter 3 with regard to Shaocheng's approach, the main idea is to transform the LPIC from a non-deterministic problem into two analogous deterministic linear programming problems. This is achieved by the introduction of maximum value range and minimum value range inequalities. The idea of max/min value range inequalities

is based on the fact that, if you have interval coefficients associated with any constraint C , this constraint will yield at least two different extreme versions. This in turn will give at least two different feasible regions. Moreover, for any given objective function with interval coefficients there will be at least two different extreme versions, one of which will be the optimistic version (i.e. should give best optimum), and the other of which is the pessimistic version (i.e. should give the worst optimum).

Example 6.2.

Suppose we have the constraint $x_1 + [1,2]x_2 \leq 2$, with $x_1 \geq 0$ and $x_2 \geq 0$, then the extreme versions of this constraint are:

A: $x_1 + 2x_2 \leq 2$ and

B: $x_1 + x_2 \leq 2$; and the two resulting feasible regions are as in Figure 6.1.

From Figure 6.1. one can clearly see that the feasible region F_1 is completely contained in the other feasible region F_2 (i.e. F_1 is a subset of F_2). Furthermore, for any fixed objective function the optimal value with respect to the feasible region F_2 is always better than or equal to the optimal value with respect to the feasible region F_1 because F_2 adds more feasible points to F_1 . In general LPIC with all sign-restricted variables, the smaller feasible region (as in F_1) should yield the worst optimum solution and the bigger feasible region (as in F_2) should yield the best optimum solution. The question is how to determine which extreme versions of each constraint produce the smaller feasible region and the bigger feasible region? The remainder of this section will deal with this question.

Suppose we have:

$$\begin{aligned} \min Z &= \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j \\ \text{s.t.} \quad \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j &\geq [\underline{b}_i, \bar{b}_i], \quad \text{for } i=1 \dots m \end{aligned} \quad (1)$$

x_j is a sign-restricted variable $\forall j$ (i.e. $x_j \in W_j, \forall j$)

$[\underline{c}_j, \bar{c}_j], [\underline{a}_{ij}, \bar{a}_{ij}], [\underline{b}_i, \bar{b}_i] \in I(\mathbb{R})$, where $I(\mathbb{R})$ is the set of all

interval numbers in $\mathbb{R}$

According to the operations of interval numbers each inequality i in (1) can be transformed into 2^{n+1} different extreme inequalities, such as:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n \geq b \quad (2)$$

where $a_j \in \{\underline{a}_{ij}, \bar{a}_{ij}\}$ and $b \in \{\underline{b}_i, \bar{b}_i\}$, $i = 1 \dots m, j = 1 \dots n$.

Note: a_i and b each belong to a set of two elements.

Consider one inequality i of (1).

Let S_k = the set of solutions to the k^{th} extreme inequality (2) of the 2^{n-1} different extreme inequalities of i .

Then, let $\bar{S} = \bigcup_{k=1}^{2^{n-1}} S_k$, and $\underline{S} = \bigcap_{k=1}^{2^{n-1}} S_k$.

Suppose we have two inequalities e_1 and e_2 with their solution sets S_1 and S_2 respectively (in $\mathbb{R}^2$, with $x_i \geq 0$), then $\underline{S}$ and $\bar{S}$ may be as in Figure 6.2.

Definition. 6.2. For each constraint inequality i in (1), the inequality $\sum_{j=1}^n a_j x_j \geq b$, where

$a_j \in [\underline{a}_j, \bar{a}_j]$ and $b \in [\underline{b}, \bar{b}]$, is called the *characteristic formula* for the inequality i in

(1) (Shaocheng, 1994).

Note: Here, a_j and b each belong to an interval.

Definition 6.3. For each constraint inequality i in (1), if there exists one characteristic formula such that its solution set is the same as $\bar{S}$ or $\underline{S}$ then, this characteristic formula is called the *maximum value range* or *minimum value range* inequality respectively (Shaocheng, 1994).

In the following theorem, a method to determine the maximum value range inequalities and the minimum value range inequalities of a set of interval inequalities as in (1), will be outlined and proved. These maximum value range inequalities and minimum value range inequalities will be the main tool in solving the LPIC.

The theorem below was first stated without proof by Shaocheng (1994); a proof is given below.

Theorem 6.1. Suppose we have the interval inequality $\sum_{j=1}^n [a_j, \bar{a}_j] x_j \geq [\underline{b}, \bar{b}]$ where $x_j \geq 0 \forall j$ then:

$$\sum_{j=1}^n \bar{a}_j x_j \geq \underline{b} \quad \text{and} \quad \sum_{j=1}^n a_j x_j \geq \bar{b}$$

are the maximum value range and minimum value range for the given inequality (Shaocheng, 1994).

Proof:

Let $\bar{S}$ and $\underline{S}$ be as defined above.

Given the inequality $\sum_{j=1}^n [a_j, \bar{a}_j] x_j \geq [\underline{b}, \bar{b}]$ (3)

(3) can be written as: $\left[\sum_{j=1}^n a_j x_j, \sum_{j=1}^n \bar{a}_j x_j \right] \geq [\underline{b}, \bar{b}]$

This inequality holds iff

$$\sum_{j=1}^n a_j x_j \geq \bar{b} \quad (4)$$

by the operations of interval numbers (Moore, 1966).

i.e. any solution $\mathbf{x}$ for (4) is a solution for any other characteristic formula of (3)

Let S_1 be the set of solutions for (4) .

$$\therefore S_1 \subseteq \underline{S}.$$

Let $\mathbf{x}' \in \underline{S}$ be a particular solution.

$$\therefore \mathbf{x}' \text{ must be a solution for (4), i.e. } \mathbf{x}' \in S_1$$

$$\therefore \underline{S} \subseteq S_1.$$

$$\therefore \underline{S} = S_1$$

$$\therefore \sum_{j=1}^n a_j x_j \geq \bar{b} \text{ is the minimum value range inequality for (3) by Definition 6.3.}$$

Let $\sum_{j=1}^n a_j x_j \geq b$ be a characteristic formula of (3), where $\sum_{j=1}^n a_j x_j \in \left[\sum_{j=1}^n \underline{a}_j x_j, \sum_{j=1}^n \bar{a}_j x_j \right]$,

and $b \in [\underline{b}, \bar{b}]$ for any particular solution $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

$$\therefore \sum_{j=1}^n \bar{a}_j x_j \geq \sum_{j=1}^n a_j x_j \geq b \geq \underline{b}. \quad \text{Using the solution } \mathbf{x} = (x_1, x_2, \dots, x_n).$$

$$\therefore \sum_{j=1}^n \bar{a}_j x_j \geq \underline{b} \quad (5)$$

$\therefore$ The solution of any characteristic formula of (3) is also a solution of (5).

$\therefore$ The solution set of (5) is $\bar{S}$.

$$\therefore \sum_{j=1}^n \bar{a}_j x_j \geq \underline{b} \text{ is the maximum value range inequality for (3).}$$

□

By using this theorem one is able to determine the extreme version of each given interval constraint which will produce both the largest feasible region and the smallest feasible region.

Corollary 6.1. The previous theorem holds true for any sign-restricted variable x .
(i.e. $x \geq 0$ or $x \leq 0$).

Proof:

If $x \leq 0$ then use a simple variable substitution.

Let $x = -x'$ with $x' \geq 0$. Then substitute for x in the original problem, and the proof is identical to Theorem 6.1.

□

Theorem 6.2. Let $Z = \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j$ be a given objective function with $x_j \geq 0$, then;

$$\sum_{j=1}^n \bar{c}_j x_j \geq \sum_{j=1}^n \underline{c}_j x_j \quad \text{for any given vector } \mathbf{x} = (x_1, x_2, \dots, x_n), \quad x_j \geq 0.$$

Proof:

We have $\sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j$ with $x_j \geq 0$.

By using interval number properties for any interval number $[a, b]$, $b \geq a$.

Therefore;

$$\sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j = [\underline{c}_1, \bar{c}_1] x_1 + [\underline{c}_2, \bar{c}_2] x_2 + \dots + [\underline{c}_n, \bar{c}_n] x_n$$

$$= [\underline{c}_1x_1 + \underline{c}_2x_2 + \dots + \underline{c}_nx_n, \bar{c}_1x_1 + \bar{c}_2x_2 + \dots + \bar{c}_nx_n]$$

Therefore, $\bar{c}_1x_1 + \bar{c}_2x_2 + \dots + \bar{c}_nx_n \geq \underline{c}_1x_1 + \underline{c}_2x_2 + \dots + \underline{c}_nx_n,$

Therefore, $\sum_{j=1}^n \bar{c}_jx_j \geq \sum_{j=1}^n \underline{c}_jx_j$

□

Corollary 6.2. The previous theorem holds true for any sign-restricted variable x .

(i.e. $x \geq 0$ or $x \leq 0$).

Proof:

If $x \leq 0$ then use a simple variable substitution.

Let $x = -x'$ with $x' \geq 0$. Then substitute for x in the original problem, and the proof is

identical to Theorem 6.2 .

□

6.1.1 The General Method for Solving Type I LPIC

Without loss of generality, throughout this section we will only deal with minimization problems. For maximization problems we simply convert the problem into minimization by multiplying the objective function by (-1).

Definition 6.4. Assuming $x_i \geq 0$ and we have a minimization LPIC, then $\sum_{j=1}^n \underline{c}_j x_j$ is called the *most favorable objective function*, and $\sum_{j=1}^n \bar{c}_j x_j$ is the *least favorable objective function*. For a maximization problem we use the opposite, and for a non-positive variable we use a change of variable to convert it to a non-negative variable.

By using theorems 6.1. and 6.2. we will be able to compute both the best optimal solution and the worst optimal solution of the given LPIC problem. This is done by transforming the original LPIC problem into two classical linear programming problems. First, we use the most favorable version of the objective function (Theorem 6.2.) and the maximum value range inequalities (Theorem 6.1.) to determine the best optimal solution, and then we use the least favorable version of the objective function and the minimum value range inequalities to determine the worst optimal solution.

The General Method for solving Type I LPIC is described in Algorithm 1 assuming a minimization model and constraints of type ($\geq$) relationships. For constraints of type ($\leq$) multiply by (-1). Section 6.1.2 deals with equality interval constraints.

Suppose we are given the following LPIC problem:

$$\begin{aligned} \mathbb{P}: \min Z &= \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j \\ \text{s.t.} \quad & \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j \geq [\underline{b}_i, \bar{b}_i], \text{ for } i=1 \dots m \\ & x_j \text{ is a sign-restricted variable (i.e. } x_j \in W_j \forall j) \end{aligned}$$

Then the best optimum and worst optimum are computed as follows:

(I) For the Best Optimum:

$$\begin{aligned} \min \underline{z} &= \sum_{j=1}^n c'_j x_j, \quad \text{where } c'_j = \begin{cases} \underline{c}_j; & \text{if } x_j \geq 0 \\ \bar{c}_j; & \text{if } x_j \leq 0 \end{cases} \\ \text{s.t.} \quad & \sum_{j=1}^n a'_j x_j \geq \underline{b}_i \quad \forall i, \quad \text{where } a'_j = \begin{cases} \bar{a}_{ij}; & \text{if } x_j \geq 0 \\ \underline{a}_{ij}; & \text{if } x_j \leq 0 \end{cases} \\ & x_j \in W_j \quad \forall j. \end{aligned}$$

(II) For the Worst Optimum:

$$\begin{aligned} \min \bar{z} &= \sum_{j=1}^n c''_j x_j, \quad \text{where } c''_j = \begin{cases} \bar{c}_j; & \text{if } x_j \geq 0 \\ \underline{c}_j; & \text{if } x_j \leq 0 \end{cases} \\ \text{s.t.} \quad & \sum_{j=1}^n a''_j x_j \geq \bar{b}_i \quad \forall i, \quad \text{where } a''_j = \begin{cases} \underline{a}_{ij}; & \text{if } x_j \geq 0 \\ \bar{a}_{ij}; & \text{if } x_j \leq 0 \end{cases} \\ & x_j \in W_j \quad \forall j. \end{aligned}$$

Algorithm 1. The general method for solving Type I LPIC

Suppose $\underline{x} = (x'_1, x'_2, \dots, x'_n)$ is the optimal solution of (I) with $\underline{z}$ the optimal value, and

$\bar{x} = (x''_1, x''_2, \dots, x''_n)$ is the optimal solution of (II) with $\bar{z}$ the optimal value, then:

$\Rightarrow$ The best optimum solution will be $\underline{z}$, at $(x'_1, x'_2, \dots, x'_n)$ with the coefficient settings

c'_j, a'_q and $\underline{b}_i \forall i$ and j .

$\Rightarrow$ The worst optimum solution will be $\bar{z}$, at $(x''_1, x''_2, \dots, x''_n)$ with the coefficient settings

c''_j, a''_q and $\bar{b}_i \forall i$ and j .

The general solution of the LPIC problem $\mathbb{P}$ will be:

$Z = [\underline{z}, \bar{z}]$. i.e. the optimal value of the problem will be between $\underline{z}$ (the best case scenario) and $\bar{z}$ (the worst case scenario), depending on the coefficient setting of each interval coefficient.

Theorem 6.3. Given an LPIC problem, if for the best optimum we have an infeasible region (i.e. the best optimum does not exist), then the worst optimum has an infeasible region as well, and the original LPIC problem is infeasible.

Proof:

Since the region generated by the minimum value range inequalities is a subset of the region generated by the maximum value range inequalities, therefore, the set of minimum value range inequalities is infeasible (i.e. the worst optimum does not exist) if the set of maximum value range inequalities is infeasible.

□

Remarks and Notations: Assume a minimization model:

- For maximization problems, multiply the objective function by (-1) to get minimization.
- For constraints of type $(\leq)$, multiply the each constraint by (-1) to get $(\geq)$.
- For constraints of type $(=)$, see Section 6.1.2.
- If the best optimum solution is feasible with an optimum value $\underline{z}$, and the worst optimum is infeasible, then we set the worst optimum to be *inf*. (i.e. the optimum value for the original LPIC model is between $[\underline{z}, \text{inf}]$).
- If the worst optimum solution is feasible with an optimum value $\bar{z}$, and the best optimum is unbounded, then we set the best optimum to be $-\infty$. (i.e. the optimum value for the original model is between $[-\infty, \bar{z}]$).
- If both the best and worst optimum values are unbounded then the original model is unbounded.
- If both the best and worst optimum values are infeasible then the original model is infeasible.

6.1.2 Equality Constraints Involving Interval Coefficients

Although it is very unrealistic in a real life model to have equality constraints involving interval coefficients and interval RHS, we believe this case must be explored and studied from a mathematical point of view.

One of the major differences between fixed numbers and interval numbers is the fact that in fixed numbers we can express an equality relationship in terms of two inequalities. For instance if $a = b$ then we can say $a \geq b$ and $a \leq b$. However in interval analysis this property does not hold as shown below.

Assume $[a, b] = [c, d]$. This implies $a = c$ and $b = d$.

If we claim $[a, b] \geq [c, d]$, then by the properties of interval numbers every element in $[a, b] \geq$ every element in $[c, d]$, and hence $a \geq d$.

But $b = d$ is given, therefore $a \geq d \Rightarrow a \geq b$. However $[a, b]$ implies $a \leq b$ which is a contradiction unless $a = b$.

Therefore if $a \neq b$ and $[a, b] = [c, d]$ then $[a, b]$ is not $\geq [c, d]$, and similarly $[a, b]$ is not $\leq [c, d]$.

Theorem 6.4. If $[a, b] = [c, d]$ then $b \geq c$ and $a \leq d$.

Proof:

$$[a, b] = [c, d] \Rightarrow \begin{cases} a = c & (1) \\ b = d & (2) \end{cases}$$

Furthermore,

$$[a, b] \Rightarrow a \leq b \quad (3)$$

$$[c, d] \Rightarrow c \leq d \quad (4)$$

Therefore, from (2) & (3) $a \leq b = d \Rightarrow a \leq d$

from (1) & (4) $c \leq d = b \Rightarrow c \leq b \Rightarrow b \geq c$.

□

By applying Theorem 6.4 we can convert any equality constraint involving interval coefficients into two inequality constraints involving fixed coefficients only, as follows:

$$\text{Let } \sum_{j=1}^n [\underline{a}_j, \bar{a}_j] x_j = [\underline{b}, \bar{b}] \quad (5)$$

be a given equality constraint with interval coefficients, then by applying Theorem 6.4 we can substitute this constraint by the following two constraints;

$$\sum_{j=1}^n a'_j x_j \geq \underline{b} \quad , \quad \text{where } a'_j = \begin{cases} \bar{a}_j & \text{if } x_j \geq 0 \\ \underline{a}_j & \text{if } x_j \leq 0 \end{cases} \quad (6)$$

$$\sum_{j=1}^n a''_j x_j \leq \bar{b} \quad , \quad \text{where } a''_j = \begin{cases} \underline{a}_j & \text{if } x_j \geq 0 \\ \bar{a}_j & \text{if } x_j \leq 0 \end{cases} \quad (7)$$

Consequently, constraints (6) and (7) are in fact the extreme versions of the maximum value range inequality for the original equality constraint.

What makes equality constraints more difficult to handle is that they do not define a range of feasible points with full dimensionality, instead define a set of feasible points of lower dimensionality. Therefore by applying Theorem 6.4 as illustrated above, we can generate a fixed artificial feasible region "*region of possibility*" which contains all possible solutions satisfying constraints (6) and (7) simultaneously and hence satisfying (5) for any given settings. Consequently, we set up the maximum value range inequality of the remaining relationships (i.e. the regular inequality constraints with interval coefficients) and determine the best optimum solution of the problem.

Generally when solving for the best optimum using the two extreme constraints (6) and (7), the optimum point is unlikely to be exactly on one of these two constraints. Hence to recover the coefficient settings that correspond to the best optimum, we solve for the coefficient settings of each equality constraint (5) in terms of the optimum point as described in Algorithm 2 by using linear programming approach. The output of Algorithm 2 will be the coefficient settings needed.

For the worst optimum however, the solution must satisfy the equality restriction on the given equality constraint and hence we must determine which equality version that yield the worst optimum. The following theorem determines which version of the equality constraint we should examine for the worst optimum.

Suppose we have k equality constraints with interval coefficients, i.e.

$$\sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j = [\underline{b}_i, \bar{b}_i], \text{ for } i=1 \dots k$$

For each i let $a_{ij} \in [\underline{a}_{ij}, \bar{a}_{ij}]$ and $b_i \in [\underline{b}_i, \bar{b}_i]$.

We are given the optimum point $\mathbf{x} = (x_1, x_2, \dots, x_n)$

Use linear programming to determine a_{ij} and b_i for $i=1 \dots k$, and $j=1 \dots n$ as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^k \sum_{j=1}^n a_{ij} + \sum_{i=1}^k b_i \\ \text{s.t.} \quad & \sum_{j=1}^n x_j a_{ij} - b_i = 0, \text{ for } i=1 \dots k \\ & \left. \begin{aligned} \underline{a}_{ij} &\leq a_{ij} \leq \bar{a}_{ij} \\ \underline{b}_i &\leq b_i \leq \bar{b}_i \end{aligned} \right\}, \text{ for } i=1 \dots k, j=1 \dots n \\ & a_{ij}, b_i \text{ urs} \end{aligned}$$

Algorithm 2 Recovering coefficient settings

Theorem 6.5. Let P be a Type I LPIC minimization model with m regular inequality

interval constraints and k equality interval constraints ($k \geq 1$). Let $\sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j = [\underline{b}_i, \bar{b}_i]$

be a typical equality interval constraint i then the worst optimum solution will be at either,

$$(I) \quad \sum_{j=1}^n a'_{ij} x_j = \underline{b}_i, \quad \text{where } a'_{ij} = \begin{cases} \bar{a}_{ij} & \text{if } x_j \geq 0 \\ \underline{a}_{ij} & \text{if } x_j \leq 0 \end{cases}$$

or

$$(II) \quad \sum_{j=1}^n a''_{ij} x_j = \bar{b}_i, \quad \text{where } a''_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } x_j \geq 0 \\ \bar{a}_{ij} & \text{if } x_j \leq 0 \end{cases}$$

for each equality interval constraint i .

Proof:

Without loss of generality assume $k=1$ (i.e. we have only one equality interval constraint) and $x_j \geq 0 \forall j$.

$$\text{i.e.} \quad \sum_{j=1}^n [\underline{a}_j, \bar{a}_j] x_j = [\underline{b}, \bar{b}]. \quad (1)$$

$$\text{This can be rewritten as} \quad \left[\sum_{j=1}^n \underline{a}_j x_j, \sum_{j=1}^n \bar{a}_j x_j \right] = [\underline{b}, \bar{b}]. \quad (2)$$

This implies for any point $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $a_j \in [\underline{a}_j, \bar{a}_j]$ and $b \in [\underline{b}, \bar{b}]$ we have:

$$\sum_{j=1}^n a_j x_j = b \in \left[\sum_{j=1}^n \underline{a}_j x_j, \sum_{j=1}^n \bar{a}_j x_j \right]. \quad (3)$$

From Theorem 6.4 $\sum_{j=1}^n [\underline{a}_j, \bar{a}_j] x_j = [\underline{b}, \bar{b}]$ implies for any given point $\mathbf{x} = (x_1, x_2, \dots, x_n)$

$$\sum_{j=1}^n \bar{a}_j x_j \geq \underline{b} \text{ and } \sum_{j=1}^n \underline{a}_j x_j \leq \bar{b}. \quad (4)$$

Expression (4) creates a convex feasible region “region of possibility” where any expression as in (3) will be inside this region.

To determine the worst optimum we set up the objective function and the inequality interval constraints as described in Algorithm 1 in Section 6.1.1

We need to show that if the worst optimum exists then it must be on either one of the two extreme versions of (1);

$$(I) \sum_{j=1}^n \bar{a}_j x_j = \underline{b} \quad \text{or} \quad (II) \sum_{j=1}^n \underline{a}_j x_j = \bar{b}. \quad (5)$$

Let $\sum_{j=1}^n a_j x_j = b$ be an intermediate version of (1) (i.e. it satisfies (3)).

Assume $\sum_{j=1}^n a_j x_j = b$ is the constraint version at which we obtain the worst optimum $\bar{z}_0$.

Let $\bar{z}_I$ be the optimum obtained by using the constraint version (I).

Let $\bar{z}_H$ be the optimum obtained by using the constraint version (II).

Therefore from our assumption $\bar{z}_0 \geq \bar{z}_I$ and $\bar{z}_0 \geq \bar{z}_H$.

However from (3) $\sum_{j=1}^n a_j x_j = b \in \left[\sum_{j=1}^n \underline{a}_j x_j, \sum_{j=1}^n \bar{a}_j x_j \right]$ for $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $a_j \in [\underline{a}_j, \bar{a}_j]$

and $b \in [\underline{b}, \bar{b}]$, and all points linearly interpolated between the points defining $\bar{z}_I$ and $\bar{z}_H$ will be in the feasible region "region of possibility".

Therefore, all points on an interpolating line between the points defining $\bar{z}_I$ and $\bar{z}_H$ will have an objective function value between $\bar{z}_I$ and $\bar{z}_H$.

Let z_1 be the objective function value at the point defined by $\sum_{j=1}^n a_j x_j = b$ (an intermediate version of the equality interval constraint), and the interpolating line between the points defining $\bar{z}_I$ and $\bar{z}_H$. Then z_1 is between $\bar{z}_I$ and $\bar{z}_H$ (i.e. $z_1 \geq \min[\bar{z}_I, \bar{z}_H]$ and $z_1 \leq \max[\bar{z}_I, \bar{z}_H]$). Hence, due to the convexity of the feasible region, $\bar{z}_0 \leq z_1$.

Therefore $\bar{z}_0 \leq \max[\bar{z}_I, \bar{z}_H]$.

Therefore the assumption that $\bar{z}_0 \geq \max[\bar{z}_I, \bar{z}_H]$ is disproved.

Hence either version (I) or version (II) of the equality interval constraint will always yield the worst value.

□

By applying this theorem we would be able to determine if a worst optimum exists for the model. Unfortunately the process will lead to an exponential number of trials. See Example 6.8.

Note, if there are any equality constraints involving only fixed coefficients, then we treat them as in any regular LP.

6.1.3 Examples of Type I LPIC Models

In this section several examples will be given to illustrate the use of Algorithm 1 described in Section 6.1.1, as well as each example will show a different scenario such as both best and worst optimum are feasible, the best is feasible the worst is not feasible etc.

Example 6.3.

This is a minimization example that shows the case where both best and worst optimum are feasible.

$$\begin{aligned}
 \min \quad & Z = [0.5, 2]x_1 + [1, 3]x_2 \\
 \text{s.t.} \quad & C_1: [3, 4]x_1 + [-2, -15]x_2 \geq [-4, -3.5] \\
 & C_2: [4, 5]x_1 + [16, 17]x_2 \geq 16 \\
 & C_3: 6x_1 + 3x_2 \geq [8, 12] \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Using Algorithm 1 we get:

(I) <u>The Best Optimum:</u>	(II) <u>The Worst Optimum:</u>
$ \begin{aligned} \min \quad & \underline{z} = 0.5x_1 + x_2 \\ \text{s.t.} \quad & C_{1B}: 4x_1 - 15x_2 \geq -4 \\ & C_{2B}: 5x_1 + 17x_2 \geq 16 \\ & C_{3B}: 6x_1 + 3x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned} $	$ \begin{aligned} \min \quad & \bar{z} = 2x_1 + 3x_2 \\ \text{s.t.} \quad & C_{1W}: 3x_1 - 2x_2 \geq -3.5 \\ & C_{2W}: 4x_1 + 16x_2 \geq 16 \\ & C_{3W}: 6x_1 + 3x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{aligned} $
Using The Simplex method to solve these two standard LP models we get:	
$ \begin{aligned} x_1 &= 1.0115 \\ x_2 &= 0.6437 \\ \underline{z} &= 11.49425 \end{aligned} $	$ \begin{aligned} x_1 &= 1.7143 \\ x_2 &= 0.5714 \\ \bar{z} &= 5.142857 \end{aligned} $

Therefore, by using the best optimum solution and the worst optimum solution obtained above we would be able to determine an upper and lower bounds for the optimum value of the main problem. In this case the optimum value would be between 1.149425 and 5.142857.

i.e. $Z = [1.149425, 5.142857]$

The best and worst optimal solutions are clearly indicated in Figure 6.3.

This implies, for any selected setting of the interval coefficients involved, the optimum value will be between 1.149425 and 5.142857.

Example 6.4.

This example illustrate the use of Algorithm 1 for maximization models. Both optimum solutions are feasible for this model.

$$\begin{aligned}
 \max \quad & Z = 4x_1 + [8,12]x_2 \\
 \text{s.t.} \quad & C_1: 6x_1 + [4.25,5.75]x_2 \leq 30 \\
 & C_2: [0.95,1.05]x_1 \leq 3 \\
 & C_3: x_2 \leq [3.6,4.4] \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

First we have to set up the problem in the standard form by converting the problem into a minimization problem and change any $\leq$ inequalities into $\geq$ inequalities.

The new model is:

$$\begin{aligned}
 \min \quad & -Z = -4x_1 + [-12,-8]x_2 \\
 \text{s.t.} \quad & C_1: -6x_1 + [-5.75,-4.25]x_2 \geq -30 \\
 & C_2: [-1.05,-0.95]x_1 \geq -3 \\
 & C_3: -x_2 \geq [-4.4,-3.6] \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Using Algorithm 1 we get:

(I) <u>The Best Optimum:</u>	(II) <u>The Worst Optimum:</u>
$ \begin{aligned} \min \quad & -\underline{z} = -4x_1 - 12x_2 \\ \text{s.t.} \quad & C_{1U}: -6x_1 - 4.25x_2 \geq -30 \\ & C_{2U}: -0.95x_1 \geq -3 \\ & C_{3U}: -x_2 \geq -4.4 \\ & x_1, x_2 \geq 0 \end{aligned} $	$ \begin{aligned} \min \quad & -\bar{z} = -4x_1 - 8x_2 \\ \text{s.t.} \quad & C_{1W}: -6x_1 - 5.75x_2 \geq -30 \\ & C_{2W}: -1.05x_1 \geq -3 \\ & C_{3W}: -x_2 \geq -3.6 \\ & x_1, x_2 \geq 0 \end{aligned} $
Using The Simplex method to solve these two standard LP models we get:	
$ \begin{aligned} x_1 &= 1.8833 \\ x_2 &= 4.4 \\ -\underline{z} &= -60.3333 \end{aligned} $	$ \begin{aligned} x_1 &= 1.55 \\ x_2 &= 3.6 \\ -\bar{z} &= -35 \end{aligned} $

This implies that the best optimum solution occurs at $x_1=1.8833$ and $x_2=4.4$ with the optimal value of 60.3333, and the worst optimum solution occurs at $x_1=1.55$ and $x_2=3.6$ with the optimal value of 35.

Figure 6.4. clearly shows the best and worst optimum solutions. It also shows the maximum and minimum value range inequalities.

The optimum value of the main model lies between 35 and 60.3333 i.e. $Z = [35, 60.3333]$.

By using the best optimum value and the worst optimum value obtained above we were able to determine lower and upper bounds for the optimum value of the main problem. This implies, for any selected setting of the interval coefficients involved, the optimum value will be between 35 and 60.3333.

Example 6.5.

This is a minimization example where the best optimum is feasible, however the worst optimum is infeasible.

$$\begin{aligned}
 \min \quad & Z = x_1 - x_2 \\
 \text{s.t.} \quad & C_1: [-3, -15]x_1 + x_2 \geq 3 \\
 & C_2: x_1 + [-3, -15]x_2 \geq 3 \\
 & C_3: x_1 \geq -4 \\
 & C_4: x_2 \geq -4 \\
 & x_1, x_2 \leq 0
 \end{aligned}$$

Using Algorithm 1 we get:

(I) <u>The Best Optimum:</u>	(II) <u>The Worst Optimum:</u>
$ \begin{aligned} \min \quad & \underline{z} = x_1 - x_2 \\ \text{s.t.} \quad & C_1: -3x_1 + x_2 \geq 3 \\ & C_2: x_1 - 3x_2 \geq 3 \\ & C_3: x_1 \geq -4 \\ & C_4: x_2 \geq -4 \\ & x_1, x_2 \leq 0 \end{aligned} $	$ \begin{aligned} \min \quad & \bar{z} = x_1 - x_2 \\ \text{s.t.} \quad & C_1: -15x_1 + x_2 \geq 3 \\ & C_2: x_1 - 15x_2 \geq 3 \\ & C_3: x_1 \geq -4 \\ & C_4: x_2 \geq -4 \\ & x_1, x_2 \leq 0 \end{aligned} $
Using The Simplex method to solve these two standard LP models we get:	
$ \begin{aligned} x_1 &= -4 \\ x_2 &= -2.3333 \\ \underline{z} &= -1.6667 \end{aligned} $	Infeasible.

This implies that the best optimum solution occurs at $x_1 = -4$ and $x_2 = -2.3333$ with the optimal value of -1.6667 , however for the worst optimum the model becomes infeasible.

From Figure 6.5, we clearly see that the model becomes infeasible when using the minimum value range inequalities due to the fact that x_1 and x_2 must be greater than -4 and the feasible region generated by the first two constraints at the worst case have values of x_1 and x_2 less than -4 , thus infeasibility.

By using the best optimum value and the worst optimum value obtained above we would only be able to determine a lower bound of -1.6667 for the best optimum value of the main problem, and an infeasible worst optimum.

Example 6.6.

This is an example that illustrates a case where the best optimum is unbounded, however the worst optimum is bounded. Also this example involves non-positive variables.

$$\begin{aligned}
 \min \quad & Z = x_1 + x_2 \\
 \text{s.t.} \quad & C_1: [1,4]x_1 - 2x_2 \geq -2 \\
 & C_2: -x_1 + [1,2]x_2 \geq -3 \\
 & x_1, x_2 \leq 0
 \end{aligned}$$

Using Algorithm 1 we get:

(I) The Best Optimum:	(II) The Worst Optimum:
$ \begin{aligned} \min \quad & \underline{z} = x_1 + x_2 \\ \text{s.t.} \quad & C_1: x_1 - 2x_2 \geq -2 \\ & C_2: x_1 + x_2 \geq -3 \\ & x_1, x_2 \leq 0 \end{aligned} $	$ \begin{aligned} \min \quad & \bar{z} = x_1 + x_2 \\ \text{s.t.} \quad & C_1: 4x_1 - 2x_2 \geq -2 \\ & C_2: -x_1 + 2x_2 \geq -3 \\ & x_1, x_2 \leq 0 \end{aligned} $
Using The Simplex method to solve these two standard LP models we get:	
Unbounded	$ \begin{aligned} x_1 &= -1.667 \\ x_2 &= -2.333 \\ \bar{z} &= -4 \end{aligned} $

This implies that, the best optimum solution is unbounded, i.e. at the maximum value range inequalities the value of Z gets smaller and smaller ($Z \rightarrow -\infty$). However the worst optimum value exists at $x_1 = -1.667$ and $x_2 = -2.3333$ with the optimal value of -4 .

From Figure 6.6, we clearly see that the model has an unbounded feasible region using the maximum value range inequalities (i.e. C_U and C_L), however as both

constraints start to tilt inside toward the minimum value range inequalities a feasible region is attained, and hence we have a worst optimum value.

By using the best optimum value and the worst optimum value obtained above we would only be able to assign a lower bound of $-\infty$ for the best optimum value and an upper bound of -4 as the worst optimum value for the main LPIC model. Therefore, $Z = [-\infty, -4]$.

Example 6.7.

This is an example that illustrates the case where both optimum solutions are infeasible (i.e. the main model is infeasible).

$$\begin{aligned}
 \min \quad & Z = 2x_1 + 3x_2 \\
 \text{s.t.} \quad & C_1: [2,4]x_1 + x_2 \geq 4 \\
 & C_2: -3x_1 + 2x_2 \geq [7,8] \\
 & C_3: x_2 \leq 2 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Using Algorithm 1 we get:

(I) <u>The Best Optimum:</u>	(II) <u>The Worst Optimum:</u>
$ \begin{aligned} \min \quad & \underline{z} = 2x_1 + 3x_2 \\ \text{s.t.} \quad & C_{1L}: 4x_1 + x_2 \geq 4 \\ & C_{2L}: -3x_1 + 2x_2 \geq 7 \\ & C_3: x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned} $	$ \begin{aligned} \min \quad & \bar{z} = 2x_1 + 3x_2 \\ \text{s.t.} \quad & C_{1U}: 2x_1 + x_2 \geq 4 \\ & C_{2U}: -3x_1 + 2x_2 \geq 8 \\ & C_3: x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned} $
Using The Simplex method to solve these two standard LP models we get:	
Infeasible	Infeasible

Since both sub-problems are infeasible, i.e. the best optimum solution and the worst optimum solution do not exist, thus the main LPIC model is infeasible. From Figure 6.7. we clearly see that the model is infeasible for any given setting of the interval coefficients involved.

Example 6.8.

This example is a minimization model involving an equality constraint with interval coefficients. This example also illustrates the method described in Section 6.1.2 to handle this type of model.

$$\begin{aligned}
 \min \quad & Z = x_1 + x_2 \\
 \text{s.t.} \quad & C_1: -x_1 + x_2 \geq [-2, -1] \\
 & C_2: [2, 3]x_1 + x_2 = [3, 4] \\
 & C_3: x_2 \leq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

First we must convert C_2 into two inequality constraints, hence we have:

$$\begin{aligned}
 C_{2a}: 3x_1 + x_2 &\geq 3 \\
 C_{2b}: 2x_1 + x_2 &\leq 4
 \end{aligned}$$

Therefore to solve for the best optimum we proceed as follow:

For the objective function and constraints C_1 and C_3 we apply Algorithm 1, and for constraint C_2 we substitute with constraints C_{2a} and C_{2b} .

The best Optimum:

$$\begin{aligned}
 \min \quad & \underline{z} = x_1 + x_2 \\
 \text{s.t.} \quad & C_{1f}: -x_1 + x_2 \geq -2 \\
 & C_{2a}: 3x_1 + x_2 \geq 3 \\
 & C_{2b}: 2x_1 + x_2 \leq 4 \\
 & C_3: x_2 \leq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

The solution for the best optimum is:

$$x_1 = 1, x_2 = 0, \text{ and } \underline{z} = 1.$$

To determine the coefficient settings for the equality constraint at which we achieved this solution we apply Algorithm 2.

We have: $C_2: [2,3]x_1 + x_2 = [3,4]$.

Let $a \in [2,3]$ and $b \in [3,4]$.

$$\begin{aligned} \max \quad & a+b \\ \text{s.t.} \quad & a(1)+1(0)=b \\ & 2 \leq a \leq 3 \\ & 3 \leq b \leq 4. \end{aligned}$$

Therefore $a=3$ and $b=3$.

In this case the optimum point is exactly on C_{2a} and therefore we got the same coefficients, however in many models this is unlikely the case.

For the worst optimum we set up for the objective function and constraints C_1 and C_3 as in Algorithm 1, and for constraint C_2 we first solve it using C_{2a} (Model A) with an equality relationship and then we solve it using C_{2b} (Model B) with an equality relationship. Once we get the two solutions, we compare and select the worst optimum.

The worst Optimum:

Model A	Model B
$\min \quad \bar{z} = x_1 + x_2$ $\text{s.t.} \quad$ $C_{1a}: -x_1 + x_2 \geq -1$ $C_{2a}: 3x_1 + x_2 = 3$ $C_3: x_2 \leq 3$ $x_1, x_2 \geq 0$	$\min \quad \bar{z} = x_1 + x_2$ $\text{s.t.} \quad$ $C_{1b}: -x_1 + x_2 \geq -1$ $C_{2b}: 2x_1 + x_2 = 4$ $C_3: x_2 \leq 3$ $x_1, x_2 \geq 0$
$x_1 = 1$ $x_2 = 0$ $\bar{z} = 1$	$x_1 = 1.667$ $x_2 = 0.667$ $\bar{z} = 2.333$

From the solution we can clearly see that the worst optimum was obtained by using the extreme version as in C_{2b} as illustrated in Figure 6.8. Therefore the optimum solution for this model ranges between $Z=1$ and $Z=2.333$.

From this example we can clearly see that if we have more equality constraints involving interval coefficients this will lead to a large number of trials to determine the exact coefficient settings that will yield the worst optimum.

6.2 LPIC Models With Sign-Unrestricted Variables (Types II and III)

In this section the case of LPIC which involves unrestricted variables (i.e. LPIC with at least one variable which is unrestricted in sign) will be examined.

Due to the nature and complexity of this type of LPIC problem a simple change of variable for the sign-unrestricted variables before applying the general method outlined in Algorithm 1 generally will not work, as illustrated in Example 6.9.

Example 6.9.

Suppose we have an LPIC minimization problem with sign-unrestricted variables as follows:

$$\begin{aligned} \min \quad & Z = x_2 \\ \text{s.t.} \quad & C_1: [-4, -2]x_1 + 2x_2 \geq -2 \\ & C_2: 2x_1 + x_2 \geq [-4, 4] \\ & x_1, x_2 \text{ urs} \end{aligned}$$

From Figure 6.9 we clearly see that constraint C_1 can vary between the two extreme versions C_{1a} and C_{1b} and constraint C_2 can vary between its two extreme versions C_{2a} and C_{2b} . We also could see that the best optimum solution for this problem occurs at the point $(-0.75, -2.5)$ at the intersection of C_{1a} and C_{2a} , and the worst optimum solution occurs at $(125, 15)$ at the intersection of C_{1b} and C_{2b} .

Due to the unrestricted nature of both variables x_1 and x_2 , the general method outlined in Algorithm 1 can not be used directly. We can try to solve the problem by making a change of variable for both x_1 and x_2 and then using the general method.

Solving the problem by using change of variable for x_1 and x_2 :

Let $x_1 = x_1^+ - x_1^-$, and $x_2 = x_2^+ - x_2^-$; with $x_1^+, x_1^-, x_2^+, x_2^- \geq 0$

Substitute in the original problem, and apply the General Method to get:

$$\begin{array}{ll}
 \min & Z = x_2^+ - x_2^- \\
 \text{s.t.} & [-4, -2]x_1^- + [2, 4]x_1^+ + 2x_2^+ - 2x_2^- \geq -2 \\
 & 2x_1^+ - 2x_1^- + x_2^+ - x_2^- \geq [-4, 4] \\
 & x_1^+, x_1^-, x_2^+, x_2^- \geq 0
 \end{array}$$

At this point we can apply the General Method, we get:

For Best optimum:

$$\begin{array}{ll}
 \min \underline{z} = x_2^+ - x_2^- \\
 \text{s.t.} & -2x_1^+ + 4x_1^- + 2x_2^+ - 2x_2^- \geq -2 \\
 & 2x_1^+ - 2x_1^- + x_2^+ - x_2^- \geq -4 \\
 & x_1^+, x_1^-, x_2^+, x_2^- \geq 0
 \end{array}$$

Solution:

This version of the problem is
Unbounded.

For worst optimum:

$$\begin{array}{ll}
 \min \bar{z} = x_2^+ - x_2^- \\
 \text{s.t.} & -4x_1^+ + 2x_1^- + 2x_2^+ - 2x_2^- \geq -2 \\
 & 2x_1^+ - 2x_1^- + x_2^+ - x_2^- \geq 4 \\
 & x_1^+, x_1^-, x_2^+, x_2^- \geq 0
 \end{array}$$

Solution:

$$x_1^+ = 1.25, x_1^- = 0, x_2^+ = 1.5, x_2^- = 0$$

$$\bar{z} = 1.5$$

$$x_1 = 1.25 \text{ and } x_2 = 1.5$$

By applying a change of variable and using the General Method the conclusion is that the best optimum is unbounded. However, graphically we see that the best optimum is actually at the point $(x_1, x_2) = (-0.75, -2.5)$.

Clearly the direct use of the General Method described in Algorithm 1 in Section 6.1.1 accompanied by a change of variable for any sign-unrestricted variable is incorrect. The main problem encountered here is the fact that when there is an interval coefficient $A = [a_1, a_2]$ associated with a sign-unrestricted variable x , and the variable is substituted for by $(x^+ - x^-)$, then the intervals associated with x^+ and x^- (i.e. $[a_1, a_2]x^+ - [a_1, a_2]x^- \Rightarrow [a_1, a_2]x^+ + [-a_2, -a_1]x^-$) will be affected once we apply the General Method. By applying the General Method to obtain the best optimum and the worst optimum, the coefficient of x^+ is set separately from the coefficient of x^- when in fact the same interval is involved, so that the original interval associated with x is violated. For the best optimum case we have: $a_2x^+ - a_1x^-$ where $a_2 \neq a_1$, and for the worst optimum case we have: $a_1x^+ - a_2x^-$ where $a_1 \neq a_2$, and hence the substitution approach will not work. The cause of the problem is the pairing of an interval coefficient with a sign-unrestricted variable.

To deal with this type of LPIC more effectively, we first examine Type II LPIC (i.e. all sign-unrestricted variables have fixed coefficients in all relationships) then we examine Type III LPIC (i.e. at least one sign-unrestricted variable associated with interval coefficient in any one of the relationships).

Example 6.10.

(a) an example of TYPE II.

$$\begin{aligned} \min \quad & [1,2]x_1 + 5x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 3x_2 + [-1,1]x_3 \geq [2,3] \\ & [3,4]x_1 - x_2 + x_3 \geq 5 \\ & x_1 \geq 0, \quad x_2 \text{ urs}, \quad x_3 \leq 0 \end{aligned}$$

(b) an example of TYPE III.

$$\begin{aligned} \min \quad & [1,2]x_1 + 5x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 3x_2 + [-1,1]x_3 \geq [2,3] \\ & [3,4]x_1 - x_2 + x_3 \geq 5 \\ & x_1, x_2 \text{ urs}, \quad x_3 \leq 0. \end{aligned}$$

6.2.1 TYPE II LPIC

In this section we will be dealing with LPIC problems with sign-unrestricted variables of TYPE II. i.e. all sign-unrestricted variables have fixed coefficients in all relationships. With regards to this type of problems, a simple change of variable can be used to achieve the solution.

i.e. if x_i is of type sign-unrestricted then:

Let $x_i = x_i^+ - x_i^-$ where $x_i^+, x_i^- \geq 0$. Substitute $x_i^+ - x_i^-$ for x_i in the original problem, and use the General Method outlined in Algorithm 1. Once we get the solution, we substitute back to get the original variables.

Example 6.11.

$$\begin{aligned} \min \quad & Z = [0,1]x_1 + x_2 \\ \text{s.t.} \quad & C_1: [-4,-2]x_1 + 2x_2 \geq -2 \\ & C_2: 2x_1 + x_2 \geq [-4,4] \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

From Figure 6.10. we see that constraint C_1 can vary between (C_{1a}) and (C_{1b}) , and constraint C_2 can vary between (C_{2a}) and (C_{2b}) . The objective function can vary as indicated in the graph. Graphically, the best optimum occurs at $(0,-1)$ with $Z = -1$, and the worst optimum occurs at $(1.25,1.5)$ with $Z = 2.75$.

Now we solve the problem using change of variable and the general method outlined in Algorithm 1. Let $x_2 = x_2^+ - x_2^-$, and substitute in the original problem. Therefore,

$$\begin{aligned} \min Z &= [0,1]x_1 + x_2^+ - x_2^- \\ \text{s.t.} \quad &[-4,-2]x_1 + 2x_2^+ - 2x_2^- \geq -2 \\ &2x_1 + x_2^+ - x_2^- \geq [-4,4] \\ &x_1, x_2^+, x_2^- \geq 0 \end{aligned}$$

Therefore, the best optimum solution occurs at (0,-1) with the best optimum value of $z_1 = -1$ obtained by using the coefficients $\{0,1\}$ for the objective function and $\{-2, 2,-2\}$ for C_1 and $\{2, 1, -4\}$ for C_2 , and the worst optimum solution occurs at (1.25,1.5) with the worst optimum value of $z_2 = 2.75$ obtained by using the coefficients $\{1,1\}$ for the objective function and $\{-4, 2,-2\}$ for C_1 and $\{2, 1, 4\}$ for C_2 .

The final solution for the LPIC problem is:

The optimum value of this problem lies between $Z = [-1, 2.75]$, with the best optimum solution x_1 at $[0, -1]$ and the worst optimum solution x_2 at $[1.25, 1.5]$.

Using the change of variable approach and applying the General Method is successful in obtaining the correct answer for Type II LPIC. Since all the sign-unrestricted variables are associated with real number coefficients in all relationships, then substituting $x^+ - x^-$ for each of these variables and applying the General Method does not affect the coefficients of x^+ and x^- . Subsequently, in both cases (best optimum and worst optimum) the coefficients of x^+ and x^- are the same. Indeed, when dealing with TYPE II LPIC, we could simply apply the general method outlined in Algorithm 1 directly and relegate the change of variables to the step where we solve for the best optimum and the worst optimum.

6.2.2 TYPE III LPIC

In this section, LPIC with at least one sign-unrestricted variable that is associated with an interval coefficient in at least one of the relationships (i.e. the objective function or in any given constraint) will be examined.

As mentioned earlier, TYPE III LPIC involves two uncertain factors (interval coefficients and sign-unrestricted variables) which makes the problem unsolvable by any of the methods described so far. In the next few examples, some of the different types of difficulties which may arise from type III LPIC will be illustrated.

Referring back to Example 6.9 and Figure 6.9 we see that constraint C_1 can vary between its extreme versions C_{1a} and C_{1b} , and constraint C_2 can vary between C_{2a} and C_{2b} . We also note that the best optimum for this problem occurs in the negative orthant at $(-0.75, -2.5)$ from the intersection of C_{1a} and C_{2a} , and the worst optimum occurs in the positive orthant at $(1.25, 1.5)$ from the intersection of C_{1a} and C_{2b} .

As discussed earlier in Example 6.9., using a change of variable for x_1 and applying the general method did not get the correct answer. Graphically, we can see that the main problem of this example is the fact that one version of constraint C_1 (i.e. C_{1a}) is active at both the best optimum and the worst optimum solutions. C_{1a} is part of the maximum value range inequalities and the minimum value range inequalities. More specifically, C_{1a} is part of the maximum value range inequality when $x_1 \leq 0$, but is part of the minimum value

range inequality when $x_i \geq 0$. This phenomenon is in contradiction with the general method described in Algorithm 1, which requires the use of different versions of any constraints with interval coefficients in determining the maximum value range and the minimum value range inequalities. This occurs because the best and worst optimum solutions happen to be in different orthants with respect to x_i . Therefore, in order to avoid this problem, we solve the model twice: (1) by adding the artificial constraint $x_i \geq 0$, then (2) using $x_i \leq 0$, and comparing the results.

In this example there is only one sign-unrestricted variable associated with an interval coefficient. Consequently we need to solve two different LPIC models, as mentioned above. Therefore, if we had two sign-unrestricted variables x_i and x_j associated with interval coefficients then we would have to solve 4 different LPIC models. The first model will use $x_i \geq 0$ and $x_j \geq 0$, the second $x_i \geq 0$ and $x_j \leq 0$, the third $x_i \leq 0$ and $x_j \geq 0$, and the fourth will use $x_i \leq 0$ and $x_j \leq 0$. Clearly by using this process, there will be an exponential relationship between the number of sign-unrestricted variables associated with interval coefficients and the number of LPIC models to solve using the added artificial restriction on those variables. Hence this process leads to a combinatorial explosion in the number of LPs to solve.

Example 6.12.

This is an example where x_1 is sign-unrestricted and associated with interval coefficients in both constraints C_1 and C_2 .

$$\begin{array}{ll}
 \min & Z = 4x_1 + x_2 \\
 \text{s.t.} & C_1: [-2, 2]x_1 - x_2 \geq 3 \\
 & \quad \quad \quad \swarrow \quad \searrow \\
 & \quad \quad \quad C_{1a}: -2x_1 - x_2 \geq 3 \\
 & \quad \quad \quad C_{1b}: 2x_1 - x_2 \geq 3 \\
 & C_2: [-1, -0.5]x_1 + x_2 \geq -1 \\
 & \quad \quad \quad \swarrow \quad \searrow \\
 & \quad \quad \quad C_{2a}: -x_1 + x_2 \geq -1 \\
 & \quad \quad \quad C_{2b}: -0.5x_1 + x_2 \geq -1 \\
 & C_3: x_1 \geq -4 \\
 & x_1, x_2 \text{ urs}
 \end{array}$$

This is another example of TYPE III LPIC where we have one sign-unrestricted variable x_1 associated with an interval coefficient in C_1 and C_2 . From Figure 6.11, one can clearly see that the two *extreme versions* of C_1 and C_2 have created two disjoint feasible regions. At first, this may give the impression that the best optimum occurs at the point $(-4, -5)$ and the worst optimum occurs at the point $(2, 1)$. However if the coefficient of x_1 in C_1 is set to $1 \in [-2, 2]$, and the coefficient of x_1 in C_2 is set to -1 then the constraints C_1 and C_2 become parallel to each other in different directions and hence there is no solution for the worst optimum.

This problem can not be detected by using change of variable for x_1 and x_2 . Indeed using change of variable and the general method indicates an unbounded solution for the best optimum, which is not correct. The only option then is to artificially restrict x_1 and solve two LPIC models as in the previous example.

Similar to the previous example, there is an exponential relationship between the number of sign-unrestricted variables associated with interval coefficients and the number of LPIC models to solve using the added artificial restrictions.

Example 6.13.

This is an example where only the objective function has interval coefficients associated with sign-unrestricted variables.

$$\begin{array}{ll}
 \min & Z = [-1,1]x_1 + [-1,1]x_2 \\
 \text{s.t.} & C_1: 3x_1 - 5x_2 \geq -15 \\
 & C_2: 4x_1 - 2.5x_2 \geq -10 \\
 & C_3: -x_1 + 0.5x_2 \geq -1 \\
 & C_4: x_1 \leq 2 \\
 & C_5: x_2 \geq -3 \\
 & x_1, x_2 \text{ urs}
 \end{array}
 \quad
 \begin{cases}
 Z_a: -x_1 - x_2 \\
 Z_b: -x_1 + x_2 \\
 Z_c: x_1 - x_2 \\
 Z_d: x_1 + x_2
 \end{cases}$$

In this example both x_1 and x_2 are sign-unrestricted variables associated with interval coefficients. This example has a fixed feasible region as illustrated in Figure 6.12. and a variable objective function. All four extreme versions of the objective function are shown on the graph. By analyzing Figure 6.12. and the model parameters and relationships, we note the best optimum solution should occur at $(-4.375, -3)$ with $\underline{z} = -7.375$ using the extreme version Z_d of the objective function. The worst optimum should occur at any point with the coefficients of the objective function set to zero for both x_1 and x_2 and $\bar{z} = 0$, (i.e. intermediate version of the objective function).

This example is very different from the previous two. It involves a fixed feasible region and a variable objective function that flips and rotates, with respect to its direction, depending on the coefficients of its variables. By applying a change of variable for x_1 and x_2 and using the general method we get an unbounded solution as the best optimum, which

Figure 6.12. Example 3 of TYPE III LPIC

is completely wrong since we have a fixed bounded feasible region. Therefore, as in the previous two examples, we must add artificial restrictions on both x_1 and x_2 and solve the new LPIC models. In this case we would have four different new LPIC models; $x_1 \geq 0$ and $x_2 \geq 0$, $x_1 \geq 0$ and $x_2 \leq 0$, $x_1 \leq 0$ and $x_2 \geq 0$, and finally $x_1 \leq 0$ and $x_2 \leq 0$. Once we get the solution of these LPIC models we compare the results and choose the proper answer.

As in the previous two examples, this process will lead to a combinatorial explosion with regard to the number of new LPIC models to be solved.

In light of the previous three examples we conclude that, in general, the change of variable approach does not work when we try to use substitution for the sign-unrestricted variables in TYPE III LPIC. Hence, in order to determine the best optimum and worst optimum solutions for TYPE III LPIC, all different combinations of the orthants involved with the unrestricted variables must be examined. This is done by adding artificial restrictions on those variables involved. This enumeration procedure will ultimately lead to a combinatorial explosion of the number of new LPIC models to be examined, driven by the number of sign-unrestricted variables associated with interval coefficients.

In general, if we have:

- 1 sign-unrestricted variable with interval coefficients $\implies$ 2 different LPIC problems to be examined.
- 2 sign-unrestricted variables with interval coefficients $\implies$ 4 different LPIC problems to be examined.
- 3 sign-unrestricted variables with interval coefficients $\implies$ 8 different LPIC problems to be examined.
- $\vdots$
- n sign-unrestricted variables with interval coefficients $\implies$ 2^n different LPIC problems to be examined.

In the next section we will describe in detail the process of examining the best optimum and the worst optimum solutions using the new LPIC models introduced by adding the artificial restrictions.

6.2.3 Enumerated Solution for LPIC of TYPE III

In this section, a method which enumerates the different possible scenarios of TYPE III LPIC and selects the proper solution, will be outlined.

As we have seen in Section 6.2.2, a combinatorial explosion is unavoidable for the type III LPIC. However, since it is infrequent in real life problems to have a large number interval coefficients associated with sign-unrestricted variables, we believe that a method which enumerates the possible scenarios will be effective in most cases. Since our main objective is to avoid the combinatorial explosion, this method should be used only when the number of unrestricted variables associated with interval coefficients are within a certain limit (e.g. between 1-5 variables).

Consider Example 6.12. described earlier.

As we have seen earlier, this example could not be solved by any of the available methods. Since there is only one variable associated with an interval coefficient (i.e. x_1), we could consider solving two different problems; one where $x_1 \leq 0$ and the other $x_1 \geq 0$. Once we have the best and worst optimum solutions of both problems, we compare them and select the overall best and worst optimum solution.

The new LPIC sub-problems are:

problem (1):

$$\min \quad Z = 4x_1 + x_2$$

$$\text{s.t.} \quad C_1: \quad [-2,2]x_1 - x_2 \geq 3$$

$$C_2: \quad [-1,-0.5]x_1 + x_2 \geq -1$$

$$C_3: \quad x_1 \geq -4$$

$$\boxed{x_1 \leq 0}, x_2 \text{ urs}$$

problem (2):

$$\min \quad Z = 4x_1 + x_2$$

$$\text{s.t.} \quad C_1: \quad [-2,2]x_1 - x_2 \geq 3$$

$$C_2: \quad [-1,-0.5]x_1 + x_2 \geq -1$$

$$C_3: \quad x_1 \geq -4$$

$$\boxed{x_1 \geq 0}, x_2 \text{ urs}$$

Since x_2 in both sub-problems is not associated with any interval coefficient, both problem (1) and problem (2) are now of TYPE II.

The solutions for problems (1) and (2) are:

Problem (1):

Best Optimum:

$$x_1 = -4, x_2 = -5, \underline{z} = -21$$

Worst Optimum:

Infeasible.

Problem (2):

Best Optimum:

$$x_1 = 1.33, x_2 = -0.33, \underline{z} = 5$$

Worst Optimum:

Infeasible.

Therefore, the solution for the original problem is:

$$\text{Best Optimum:} \quad \implies \underline{z} = -21, \text{ at } x_1 = -4, x_2 = -5.$$

$$\text{Worst Optimum:} \quad \implies \text{Infeasible.}$$

This is the exact conclusion we achieved when we first analyzed this example in Section 6.2.2 using Figure 6.11.

Clearly this example illustrates that by examining each orthant for the best and worst optimum value using the artificial restriction on the urs-variables, we can determine the best and worst for the original problem. However, as we will see in the next example, this method generally does not work and further analysis of the solutions is required.

The initial steps required in solving type III LPIC problem are outlined in Algorithm 3.

Assume a minimization model and we have n variables, m of which are of type w_{III} (i.e. sign-unrestricted variable associated with an interval coefficient).

- I. enumerate the scenarios by adding artificial restrictions for each $x_j \in W_{III}$. x_j is artificially restricted to ≤ 0 or ≥ 0 . Since we have m of these variables, then there will be 2^m different scenarios.
- II. Let $\underline{z}^i$ and $\underline{x}^i$ be the best optimum and its solution set for scenario i .
 - If the best optimum is unbounded, then we let $\underline{z}^i = -\infty$.
(Minimization)
 - If the best optimum is infeasible, then we let $\underline{z}^i = \text{infeasible}$.
- III. Let $\bar{z}^i$ and $\bar{x}^i$ be the worst optimum and its solution set for scenario i .
 - If the worst optimum is infeasible, then we let $\bar{z}^i = \text{infeasible}$.

Algorithm 3. The initial steps of solving Type III LPIC problems

Theorem 6.6. If the worst (or best) optimum is infeasible for every orthant, then the worst (or best) optimum of the main model is infeasible.

Proof:

(By Contradiction)

The proof will be done for the worst optimum case only.

Given the worst optimum is infeasible for every orthant.

1. Assume there exists a worst optimum $\bar{z}^*$ at $\bar{x}^*$ for the main model. Then this optimum must occur in at least one of the orthants.
2. Assume it occurs in orthant 1. i.e. $\bar{x}^*$ belongs to orthant 1.

This implies that the worst model for orthant 1 should have $\bar{z}^*$ as the worst optimum.

However we are given that the worst optimum for orthant 1 is infeasible (our assumption).

Therefore, $\bar{z}^*$ can not be the worst optimum, since we found a setting at which the model is infeasible. Since our assumption (2) will hold for every orthant, therefore our assumption (1) regarding $\bar{z}^*$ is wrong.

Therefore, the worst optimum for the model is infeasible.

□

Definition 6.5. Given an LPIC model with at least one sign-unrestricted variable x_i . An *artificial orthant restriction* is an added restriction on x_i such that x_i is ≤ 0 or ≥ 0 .

Definition 6.6. Given a feasible region involving artificial orthant restrictions (e.g. $x_i \leq 0$ or $x_i \geq 0$ for some x_i which is an urs-variable associated with interval coefficient), an *artificial cornerpoint* is a cornerpoint in which at least one artificial orthant restriction is tight.

Example 6.14.

This is an LPIC minimization model of type III which illustrates the use of Algorithm 3 to obtain the desired solutions.

$$\begin{aligned} \min \quad & Z = x_2 \\ \text{s.t.} \quad & C_1: [-4, -2]x_1 + 2x_2 \geq -2 \\ & C_2: 2x_1 + x_2 \geq [-4, 4] \\ & x_1, x_2 \text{ urs} \end{aligned}$$

Since we have only one sign-unrestricted variable x_1 associated with interval coefficient in C_1 , then we will solve two different scenarios by adding artificial restrictions on x_1 (i.e. $x_1 \leq 0$ for scenario 1 and $x_1 \geq 0$ for scenario 2).

From table 6.1. one would conclude that $\underline{z}^1 = -2.5$ is the best optimum at $(-0.75, -2.5)$ and $\bar{z}^1 = 4$ is the worst optimum at $(0, 4)$. However, the point $(0, 4)$ is not a corner point of the original model, it is an *artificial corner point* created by the added *artificial orthant restriction* enforced on x_1 .

Table 6.1. Best and Worst optimums according to scenarios 1 and 2.

	Best Optimum	Worst Optimum
Scenario 1 $(x_1 \leq 0, x_2 \text{ urs})$	$\min \underline{z}^1 = x_2$ $s.t. \ C_{1a}: -4x_1 + 2x_2 \geq -2$ $\quad \quad C_{2a}: 2x_1 + x_2 \geq -4$ $\underline{z}^1 = -2.5$ $\underline{x}^1 = (x_1, x_2) = (-0.75, -2.5)$	$\min \bar{z}^1 = x_2$ $s.t. \ C_{1b}: -2x_1 + 2x_2 \geq -2$ $\quad \quad C_{2b}: 2x_1 + x_2 \geq 4$ $\bar{z}^1 = 4$ $\bar{x}^1 = (x_1, x_2) = (0, 4)$
Scenario 2 $(x_1 \geq 0, x_2 \text{ urs})$	$\min \underline{z}^2 = x_2$ $s.t. \ C_{1a}: -2x_1 + 2x_2 \geq -2$ $\quad \quad C_{2a}: 2x_1 + x_2 \geq -4$ $\underline{z}^2 = -1$ $\underline{x}^2 = (x_1, x_2) = (0, -1)$	$\min \bar{z}^2 = x_2$ $s.t. \ C_{1a}: -4x_1 + 2x_2 \geq -2$ $\quad \quad C_{2b}: 2x_1 + x_2 \geq 4$ $\bar{z}^2 = 1.5$ $\bar{x}^2 = (x_1, x_2) = (1.25, 1.5)$

From Figure 6.13. one can observe that the solution $\bar{z}^1 = 4$ was held at the point $(0, 4)$ mainly due to the added restriction on x_1 . However by removing the artificial restriction on x_1 and using the same setup as in scenario 1 worst optimum, the solution will move down to the point $(1.667, 0.667)$ with $z = 0.667$.

We clearly see that now we have a worst optimum at real corner points. However this new worst optimum ($z = 0.667$) is lower than the solution we obtained from scenario 2, $\bar{z}^2 = 1.5$. Hence the worst optimum $\bar{z}^2 = 1.5$ obtained at $(1.25, 1.5)$ using the coefficients in scenario 2 is indeed the worst optimum of this example.

Similarly the solution $\underline{z}^2 = -1$ at $(0, -1)$ for best optimum according to scenario 2, occurs at an artificial corner point. By removing the artificial restriction on x_1 and resolving for the best optimum using the same setup as in scenario 2 best optimum, we get ($z = -2$) at $(-1, -2)$. However this new best optimum ($z = -2$) is higher than the solution we obtained from scenario 1, ($z = -2.5$). Hence the best optimum $\underline{z}^1 = -2.5$ obtained at $(-0.75, -2.5)$ using the coefficients in scenario 1 is indeed the best optimum of this example.

We also observe that both new answers occurred in orthants different from the orthants used to set up for the best and worst optimums for each case. For instance in the first case we set up the model for the worst optimum using $x_1 \leq 0$, but the new solution $z = 0.667$ occurred in an orthant where $x_1 \geq 0$. Therefore, if we set up the model according to $x_1 \geq 0$ and search for the worst optimum in this orthant we would get a worst optimum that is not lower than 0.667. Indeed in this case as in scenario 2 the worst optimum we got $\bar{z}^2 = 1.5 \geq 0.667$. Similarly for the best optimum case.

In conclusion, when we have an optimum solution at an artificial corner point we must further investigate the result by resolving the same model without the use of the artificially added restrictions.

Let P be a Type III LPIC minimization model with n variables, m of which are sign-unrestricted variables associated with interval coefficients, and $m \leq n$. Therefore by adding artificial restrictions on the m unrestricted-in-sign variables, there will be 2^m scenarios (orthants) to test for the worst and best optimum values. Furthermore, the model setup of the coefficients is different according to each scenario.

Definition 6.7. Assume we are testing orthant k , and the worst optimum of P according to orthant k occurs at $\mathbf{x}$ with at least one of the artificial restrictions active (i.e. $\mathbf{x}$ is an *artificial corner point*). This implies that by removing the active artificial restrictions, there exists a better optimum in another orthant l (using the same model setup according to orthant (scenario) k). The two orthants k and l are called the *exiting* and *entering* orthants respectively.

Theorem 6.7. Assume we are testing for the best optimum in a minimization model. Then the best optimum obtained in the entering orthant using the model set up according to the exiting orthant without the artificial restrictions, is *no better* ($\geq$) than the best optimum obtained by using the model set up according to the entering orthant using the artificial restrictions.

Proof:

Let x_i be a sign-unrestricted variable. Assume we are testing for the best optimum in a given orthant (scenario) k using artificial restriction on x_i . Let $\underline{z}_1$ be the best optimum

obtained at an artificial corner point in the current orthant. Note, a solution at an artificial corner point means an attempt to “exit” the current orthant, which implies a “better” value exists in another orthant (the entering orthant).

Therefore, by removing the artificial restrictions on x_i and re-solving the problem using the same set up, we get a better optimum $\bar{z}_2$, ($\bar{z}_2 \leq \bar{z}_1$) in a different orthant (i.e. the entering orthant l). However, the optimum $\bar{z}_2$ obtained in orthant l is achieved by using the set up of the best model according to orthant k (i.e. the best model used to get $\bar{z}_2$ is not the maximum value range inequality according to orthant l). Therefore, according to Theorem 6.1. setting up the best model according to orthant l will yield a best optimum $\bar{z}_3$ such that ($\bar{z}_3 \leq \bar{z}_2$). Hence $\bar{z}_2$ is no better than $\bar{z}_3$. $\square$

The previous theorem also applies on the worst optimum case (from a worst optimum point of view and the minimum value range inequalities).

Example 6.15.

This is an LPIC minimization model of type III that illustrates the concept of entering and existing orthants.

$$\begin{aligned} \min \quad & Z = [0,4]x_1 + x_2 \\ \text{s.t.} \quad & C_1: \quad [-4,-2]x_1 + 2x_2 \geq -2 \\ & C_2: \quad 2x_1 + x_2 \geq [-4,4] \\ & C_3: \quad x_2 \leq 5 \\ & x_1, x_2 \quad \text{urs} \end{aligned}$$

Notation: Let $(\mathbb{N}, \mathbb{N})$ where $\mathbb{N} \in \{+, -\}$ indicates the orthant that we are working with. For example $(+, -)$ indicates the orthant where $x_1 \geq 0$ and $x_2 \leq 0$.

This example involves two sign-unrestricted variables but only x_1 associated with interval coefficients. Hence we have two scenarios to test for the best and worst optimum values. Note, the first scenario will deal with orthants where $x_1 \leq 0$ and x_2 unrestricted, and the second scenario will deal with orthants where $x_1 \geq 0$ and x_2 unrestricted.

From table 6.2, we observe that for the best optimum we have $\underline{z} = -13$ from scenario 1 however using scenario 2 we have an artificial corner point at $(0, -1)$. In this case the exiting orthant is $(+, -)$ and the entering orthant is $(-, -)$. However we have already tested the orthant $(-, -)$ in scenario 1 for the best optimum with $\underline{z} = -13$. Therefore by using Theorem 6.7, we conclude that $\underline{z} = -13$ is the best optimum for this example.

Table 6.2. Best and Worst optimums according to scenarios 1 and 2 using artificial restriction.

	Best Optimum	Worst Optimum
Scenario 1 $(x_1 \leq 0, x_2 \text{ urs})$	$\min \underline{z}^1 = 4x_1 + x_2$ <i>s.t.</i> $C_{1a}: -4x_1 + 2x_2 \geq -2$ $C_{2a}: 2x_1 + x_2 \geq -4$ $C_3: x_2 \leq 5$ $\underline{z}^1 = -13$ $\underline{x}^1 = (x_1, x_2) = (-4.5, 5)$	$\min \bar{z}^1 = x_2$ <i>s.t.</i> $C_{1b}: -2x_1 + 2x_2 \geq -2$ $C_{2b}: 2x_1 + x_2 \geq 4$ $C_3: x_2 \leq 5$ $\bar{z}^1 = 4$ $\bar{x}^1 = (x_1, x_2) = (0, 4)$
Scenario 2 $(x_1 \geq 0, x_2 \text{ urs})$	$\min \underline{z}^2 = x_2$ <i>s.t.</i> $C_{1b}: -2x_1 + 2x_2 \geq -2$ $C_{2b}: 2x_1 + x_2 \geq -4$ $C_3: x_2 \leq 5$ $\underline{z}^2 = -1$ $\underline{x}^2 = (x_1, x_2) = (0, -1)$	$\min \bar{z}^2 = 4x_1 + x_2$ <i>s.t.</i> $C_{1b}: -2x_1 + 2x_2 \geq -2$ $C_{2b}: 2x_1 + x_2 \geq 4$ $C_3: x_2 \leq 5$ $\bar{z}^2 = 4$ $\bar{x}^2 = (x_1, x_2) = (0, 4)$

With regard to the worst optimum we observe that from both scenarios we have an artificial corner point, namely the point $(0,4)$ with $\bar{z} = 4$ in both cases. Hence we must resolve both scenarios for the worst optimum without using artificial restriction on x_i to construct a cornerpoint.

Resolving for the worst optimum using the set up of scenarios 1 and 2 and allowing x_i to be sign-unrestricted we get $\bar{z}^1 = 0.667$ at $(1.667, 0.667)$, and $\bar{z}^2 = 3$ at $(-0.5, 5)$ respectively.

Figure 6.14. The extreme versions of Example 6.15.

In both cases we obtained a “lower” worst minimum than 4, but we are interested in a “higher” worst minimum. Furthermore, we observe that when testing scenario 1 (i.e. setting up the worst model according to $(x_1 \leq 0, x_2 \text{ urs})$) and allowing x_1 to be sign-unrestricted we had a worst optimum in orthant $(+,+)$, similarly when testing scenario 2 (i.e. setting up the worst model according to $(x_1 \geq 0, x_2 \text{ urs})$) and allowing x_1 to be sign-unrestricted we had a worst optimum in orthant $(-,+)$.

In summary, the extreme version of the model used in scenario 1 according to $(x_1 \leq 0, x_2 \text{ urs})$, yielded the optimum value $\bar{z}^1 = 4$ at $(0,4)$ in orthant $(-,+)$ with the artificial restriction on x_1 , and the optimum value $\bar{z}^1 = 0.667$ at $(1.667, 0.667)$ in the orthant $(+,+)$ without the artificial restriction on x_1 . Similarly, the extreme version of the model used in scenario 2 according to $(x_1 \geq 0, x_2 \text{ urs})$, yielded the optimum value $\bar{z}^2 = 4$ at $(0,4)$ in the orthant $(+,+)$ with the artificial restriction on x_1 , and the optimum value $\bar{z}^2 = 3$ at $(-0.5,5)$ in the orthant $(-,+)$ without the artificial restriction on x_1 . Hence both scenarios seem to balance at the point $(0,4)$ with $\bar{z} = 4$ as the worst optimum.

Let P be a Type III LPIC minimization model with n variables, m of which are sign-unrestricted variables associated with interval coefficients, and $m \leq n$.

Definition 6.8. Assume we are testing orthant k , and the worst optimum of P according to orthant k occurs at $\mathbf{x}$ ($\mathbf{x}$ is an artificial corner point). Let orthant k be the exiting orthant and orthant l be the entering orthant. Assume we are testing orthant l , and the worst optimum of P according to orthant l occurs at $\mathbf{y}$ ($\mathbf{y}$ is an artificial corner point and

l is the exiting orthant). If the entering orthant in this latter case is k , then k and l are called *balancing orthants*. Furthermore, if $\mathbf{x} = \mathbf{y}$ then $\mathbf{x}$ (and $\mathbf{y}$) is called the *balancing point*.

In Example 6.15, the point $(0,4)$ is a balancing point and the orthants $(-,+)$ and $(+,+)$ are the balancing orthants. The balance here occurs between $x_1 \leq 0$ and $x_1 \geq 0$.

Theorem 6.8. Given an LPIC model P of type III with two balancing orthants k and l , let $\bar{z}$ be the worst optimum for P according to orthants k and l using artificial restrictions on all unrestricted variables associated with interval coefficients. Let $\mathbf{x}$ be the point at which the worst optimum occurs (i.e. $\mathbf{x}$ is a balancing point). Then $\bar{z}$ is the worst optimum for the main model P .

Proof:

(Assume we have a minimization model.)

Given P , and the two balancing orthants k and l with the balancing point $\mathbf{x}$ and $\bar{z}$ the worst optimum of P according to orthants k and l using the artificial restrictions.

Let $\bar{z}'$ and $\bar{z}''$ be the worst optimum values of P at $\mathbf{x}'$ and $\mathbf{x}''$ using the worst model set up according to orthants k and l respectively *without* using artificial restrictions on the variables in question. Then,

$\bar{z}' \leq \bar{z}$, $\mathbf{x}' \in \text{orthant } l$, since k is the exiting orthant and l is the entering orthant.

$\bar{z}'' \leq \bar{z}$, $\mathbf{x}'' \in \text{orthant } k$, since l is the exiting orthant and k is the entering orthant.

See Figure 6.15 for a pictorial illustration using a model in $\mathbb{R}^2$.

$\bar{x}' \leq \bar{x}$ holds true since x is an artificial corner point. i.e. x happens to be an optimal solution only due to the artificial restriction added to the model when we are computing the worst optimum. Furthermore, x being an artificial corner point and an optimal solution, implies an attempt by the objective function to "exit" the current orthant k . This in turns means a better optimum for the objective function possibly exists in the entering orthant l . Hence by removing the added artificial restriction we may get a lower minimum

value $\bar{z}'$, i.e. $\bar{z}' \leq \bar{z}$ (similarly $\bar{z}'' \leq \bar{z}$). However we are interested in a “higher” worst minimum. Hence $\bar{z}$ is still a better candidate for the worst optimum of P . Therefore:

- given two different extreme versions of the relationships involved to set up for the worst optimum in each scenario (i.e. orthant k and orthant l) we get the same optimum solution $\bar{z}$ at x (the balancing point) using the artificial restrictions,
- given that the removal of the artificial restrictions for each scenario yields an optimal solution that is in the other orthant (i.e. the solution of the worst optimum according to the extreme versions corresponding to orthant k occurs in orthant l and vice versa)

then, there must exist an intermediate version of at least one relationship such that when setting up for the worst optimum according to the two balancing orthants we obtain $\bar{z}$ as the worst optimum for both scenarios without using any artificial restriction. Consequently, $\bar{z}$ is the worst optimum for the main LPIC model P .

□

Example 6.15. revisited:

We clearly observe from Figure 6.14 that if the coefficient of x_1 in the objective function is set to 2 (an intermediate value) then the worst optimum is indeed 4. Furthermore we observe that at this intermediate setting the objective function is parallel to constraint 2, and hence any feasible point on C_{2b} can be considered as a worst optimum solution including the point (0,4).

Refer to Example 6.13. and Figure 6.12.

As we have seen earlier this example could not be solved by using a change of variable for x_1 and x_2 . This example involves 2 sign-unrestricted variables associated with interval coefficients, therefore there will be 4 different scenarios to investigate. Also we observe (see Figure 6.12.) that the feasible region in this case is fixed and the objective function is the only relationship that varies. The solutions of the 4 scenarios are given in table 6.3.

Table 6.3. The best and worst optimum values for the 4 given scenarios.

	Best Optimum	Worst Optimum
Scenario 1 ($x_1 \leq 0, x_2 \leq 0$)	$\min \underline{z}^1 = x_1 + x_2$ s.t. $C_1 \dots C_5$ $\underline{z}^1 = -7.375$ $\underline{x}^1 = (x_1, x_2) = (-4.375, -3)$	$\min \bar{z}^1 = -x_1 - x_2$ s.t. $C_1 \dots C_5$ $\bar{z}^1 = 0.0$ $\bar{x}^1 = (x_1, x_2) = (0, 0)$
Scenario 2 ($x_1 \leq 0, x_2 \geq 0$)	$\min \underline{z}^2 = x_1 - x_2$ s.t. $C_1 \dots C_5$ $\underline{z}^2 = -3.4$ $\underline{x}^2 = (x_1, x_2) = (-1, 2.4)$	$\min \bar{z}^2 = -x_1 + x_2$ s.t. $C_1 \dots C_5$ $\bar{z}^2 = 0.0$ $\bar{x}^2 = (x_1, x_2) = (0, 0)$
Scenario 3 ($x_1 \geq 0, x_2 \leq 0$)	$\min \underline{z}^3 = -x_1 + x_2$ s.t. $C_1 \dots C_5$ $\underline{z}^3 = -2$ $\underline{x}^3 = (x_1, x_2) = (0, -2)$	$\min \bar{z}^3 = x_1 - x_2$ s.t. $C_1 \dots C_5$ $\bar{z}^3 = 0.0$ $\bar{x}^3 = (x_1, x_2) = (0, 0)$
Scenario 4 ($x_1 \geq 0, x_2 \geq 0$)	$\min \underline{z}^4 = -x_1 - x_2$ s.t. $C_1 \dots C_5$ $\underline{z}^4 = -6.2$ $\underline{x}^4 = (x_1, x_2) = (2, 2.4)$	$\min \bar{z}^4 = x_1 + x_2$ s.t. $C_1 \dots C_5$ $\bar{z}^4 = 0.0$ $\bar{x}^4 = (x_1, x_2) = (0, 0)$

From table 6.3. we see that the best optimum is $\bar{z}^1 = -7.375$ and it occurs in orthant $(-, -)$. Scenario 3 produces an optimum at artificial corner point $(0, -2)$. The entering orthant is $(-, -)$. Therefore using Theorem 6.5. the best optimum using scenario 3 without the artificial restrictions on x_1 and x_2 is no better than the best optimum obtained in $(-, -)$ (i.e. scenario 1). Thus the best optimum for this example is $\bar{z} = -7.375$.

With regard to the worst optimum, all 4 scenarios produced worst optima at the artificial corner point $(0, 0)$ with $\bar{z} = 0$. Since there is only one artificial corner point involved, then obviously it is the balancing point. However in more complex models we must first determine the balancing orthants to be able to determine the balancing point.

To determine the orthants that the worst optimum balances between, we must first resolve all the scenarios in question (i.e. scenarios 1 to 4) without using the artificial restrictions on x_1 and x_2 . Once we have the solutions we use a diagrammatic approach which will clearly demonstrate the balance between the different orthants in question and the balancing point involved. Table 6.4. shows the worst optimum of scenarios 1 to 4 without using the artificial restriction on x_1 and x_2 but using the same setup.

Table 6.4. The worst optimum values for the 4 given scenarios without the artificial restrictions on x_1 and x_2 .

	Worst Optimum
Scenario 1, Orthant $(-, -)$ x_1, x_2 <i>urs</i>	$\bar{z}^1 = -6.2$ $\bar{x}^1 = (x_1, x_2) = (2, 4.6)$
Scenario 2, Orthant $(-, +)$ x_1, x_2 <i>urs</i>	$\bar{z}^2 = -2.5$ $\bar{x}^2 = (x_1, x_2) = (-0.5, -3)$
Scenario 3, Orthant $(+, -)$ x_1, x_2 <i>urs</i>	$\bar{z}^3 = -3.4$ $\bar{x}^3 = (x_1, x_2) = (-1, 2.4)$
Scenario 4, Orthant $(+, +)$ x_1, x_2 <i>urs</i>	$\bar{z}^4 = -7.375$ $\bar{x}^4 = (x_1, x_2) = (-4.375, -3)$

The graph that shows the relationship between the different orthants is constructed as follows:

- Nodes represent the orthants to be tested, and are labeled with the z obtained by using the artificial restrictions.
- Arcs represent the movement from the exiting orthant to the entering orthant, and are labeled with the z obtained without using the artificial restrictions.

By using the diagrammatic approach as in Figure 6.16, we can easily determine the balancing orthants and the balancing point. In this case the balancing orthants are orthant $(-, -)$ and orthant $(+, +)$. The balancing point is $(0, 0)$. Therefore using Theorem 6.8, $\bar{z} = 0.0$ is the worst optimum for this example.

The main question still to be answered is how to determine the “intermediate” settings of the coefficients at which we get the worst optimum without using the artificial restrictions. With regard to Example 6.13, it is very easy to conclude that the intermediate version is $Z = 0x_1 + 0x_2$. However in more complex models it is very difficult to determine these settings.

6.2.4 Determining the Intermediate Setting for Type III LPIC

In this section, a method which determines the intermediate setting of a given TYPE III LPIC model to compute the worst optimum will be described.

As we have seen in Section 6.2.3. when dealing with Type III LPIC models the worst optimum, in some cases, can not be determined by using extreme versions of the model as in Type I and Type II LPIC's. Instead an intermediate version is required to determine the worst optimum.

In Section 6.2.3. we concluded that by using artificial restrictions on all variables belonging to W_{III} , in most cases we can determine the best and worst optimum. However, in a few cases we obtained a worst optimum $\bar{z}^*$ at an artificial corner point x^* due to the artificially added restrictions. Furthermore, we observed that this artificial corner point is a balancing point between two balancing orthants, and that there exists an intermediate setting of at least one interval coefficient in any relationship of the model which yields $\bar{z}^*$ as the worst optimum value and x^* as a worst optimum solution without using any artificial restrictions.

In general we are given a worst optimum value $\bar{z}^*$ and an optimum solution x^* , and we need to determine all the intermediate coefficient settings that give the same worst optimum $\bar{z}^*$ without using the artificial restrictions. In other words, we want to keep the given worst optimum value $\bar{z}^*$ but we want to change some (or all) of the coefficients. One way to handle this problem is by using sensitivity analysis.

Recall that *sensitivity analysis* is the study of the stability of LP models. It analyzes how changes to a solved LP model affect the optimal solution. Changes can include the addition or deletion of variables or constraints, or more importantly modification in the objective function, constraints or resource coefficients.

Let P be an LP minimization model with all variables ≥ 0 and the RHS ≥ 0 .

$$\begin{aligned} P: \quad \min z &= \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j &\geq b_i \quad i=1,2,\dots,m \\ x_j &\geq 0 \quad j=1,2,\dots,n \end{aligned}$$

then the sensitivity relationship at the optimal solution (Tableau) will be of the form:

$$\left[\begin{array}{c|cc} 1 & 0 & c_{BV} B^{-1} N - c_{NBV} \\ 0 & I & B^{-1} N \end{array} \right] \begin{bmatrix} Z \\ x_{BV} \\ x_{NBV} \end{bmatrix} = \begin{bmatrix} c_{BV} B^{-1} b \\ B^{-1} b \end{bmatrix}$$

where ;

- x_{BV} , $m \times 1$ vector of the basic variables at the optimal solution.
- x_{NBV} , $(n-m) \times 1$ vector of the non-basic variables at the optimal solution.
- B , $m \times m$ matrix whose j th column corresponds to the column of the j th basic variable in P .
- N , $m \times (n-m)$ matrix whose columns correspond to the columns of the non-basic variables in P .
- c_{BV} , $1 \times m$ row vector $[c_{BV_1} \ c_{BV_2} \ \dots \ c_{BV_m}]$ corresponding to the coefficients of the basic variables in the objective function.

- c_{NBV} , $1 \times (n-m)$ row vector $[c_{NBV_1} c_{NBV_2} \dots c_{NBV_{n-m}}]$ corresponding to the coefficients of the non-basic variables in the objective function.
- $c_{BV}B^{-1}N - c_{NBV}$, the entry corresponding to the non-basic variables in the objective function row of the final tableau.
- $c_{BV}B^{-1}b = Z$, the optimum value at the final tableau.
- $x_{BV} = B^{-1}b$, the value of the basic variables from the final tableau.
- $x_{NBV} = 0$, the value of non-basic variables in the final tableau.

(Bazzaraa et al, 1990) and (Winston, 1987).

Recall that for the sensitivity relationships the basis is fixed, but the coefficients are not fixed which is the point of sensitivity analysis. Therefore if some of the coefficients are modified and the model is to remain optimal, then the following sensitivity relationships must hold (for an LP minimization model as in P):

1. $c_{BV}B^{-1}N - c_{NBV} \leq 0$, (i.e. the model remains optimal).
2. $B^{-1}b \geq 0$, (i.e. the model remain feasible).

With regard to our case, we have a balancing point x^* that is an artificial corner point, and has been identified as a worst optimum solution (with an optimum value $\bar{z}^*$) according to at least two different orthants using artificial restrictions. Our main objective is to slightly change the coefficients within the given intervals to determine an intermediate setting of at least one coefficient such that we maintain the same optimum value $\bar{z}^*$ according to each balancing orthant without using any artificial restrictions.

This is to be done by representing some or all of the interval coefficients as variables in the sensitivity relationships (1) and (2). Once these relationships are set up, we solve for these variables to determine the intermediate settings needed.

The coefficients that may be treated as variables include all interval coefficients in the objective function, constraint rows and RHS. However since the variables of interest are always non-basic (they are the ones held at zero by the artificial orthant restrictions), therefore the interval coefficients that will be treated as variables will include *only* those ones associated with the non-basic variables that are original variables (i.e. no slack or artificial variables). Furthermore, since we are interested in keeping the same optimal solution, there is no need to change b (i.e. $B^{-1}b \geq 0$ will always be true). Similarly, the worst optimum value will not change (i.e. $c_{BV}B^{-1}b = \bar{z}^*$ will always be true). Consequently, the interval coefficients which will be treated as variable are in c_{NBV} and N .

Given a type III LPIC minimization model, assume we are testing for the worst optimum according to each given scenario using artificial restrictions as described earlier. Assume orthants A and B are two balancing orthants with x^* the balancing point (i.e. x^* is an artificial corner point), at which the worst optimum $\bar{z}^*$ occurs according to both orthants. Recall when the artificial restrictions are removed and resolving for the worst optimum according to the orthant A model, then the worst optimum solution x^A with $\bar{z}^A$ occurs in orthant B such that $\bar{z}^A \leq \bar{z}^*$. Similarly, when the artificial restrictions are removed and resolving for the worst optimum according to the orthant B model, then the worst optimum solution x^B with $\bar{z}^B$ occurs in orthant A such that $\bar{z}^B \leq \bar{z}^*$.

Furthermore, $\mathbf{x}^*$ being an artificial corner point implies that some of the components of $\mathbf{x}^*$ are set to zero due to the artificial restrictions.

Theorem 6.9. Let x_j be one of the components of $\mathbf{x}^*$ held at zero by the artificial restrictions (i.e. $\mathbf{x}^*$ is a balancing point and x_j is a non-basic variable). Let c_{NBj} and N_j be the unknown new settings associated with x_j which achieve the same worst optimum $\bar{z}^*$ without using artificial restrictions. The new settings c_{NBj} and N_j will yield the same worst optimum $\bar{z}^*$ without using artificial restrictions iff $c_{BV}B^{-1}N_j - c_{NBj} = 0$ for all j such that x_j is an original non-basic variable associated with interval coefficients.

Proof:

$\Rightarrow$ Given x_j , an original non-basic variable held at zero by the artificial restrictions. Also given the new settings c_{NBj} and N_j at which we get the same worst optimum $\bar{z}^*$ without using artificial restrictions.

Since the artificial restrictions are removed when we solve for the worst optimum using the new settings c_{NBj} and N_j associated with x_j , then the new optimum solution $\mathbf{x}^{**}$ will not be at an artificial corner point. Since $\mathbf{x}^*$ was at an artificial corner point, this implies that $\mathbf{x}^* \neq \mathbf{x}^{**}$ and $x_j \neq 0$. This implies x_j is in the basis. However we still have the same optimum value $\bar{z}^*$ (i.e. we have two optimum solutions $\mathbf{x}^*$ and $\mathbf{x}^{**}$ with the same optimum value $\bar{z}^*$). Therefore we have multiple optimum solutions. This implies that the entry for x_j in the objective function row (row 0) in the simplex tableau is zero. Therefore $c_{BV}B^{-1}N_j - c_{NBj} = 0$ for x_j which are original nonbasic variables associated with interval coefficients.

$$\Leftarrow \text{Let } c_{BV} B^{-1} N_j - c_{NBV_j} = 0.$$

Therefore, if we resolve the model with the new settings c_{NBV_j} and N_j without using the artificial restrictions, x_j will be a candidate to enter the basis. However since its value is zero in the objective function row (row 0), then the previous optimum value $\bar{z}^*$ will not be affected. Therefore we have a new optimal solution x^{**} with the same $\bar{z}^*$. $\square$

The general steps in determining the intermediate coefficient settings at which we achieve the worst optimum are outlined in Algorithm 4.

Consider a minimization model:

- I. Identify all balancing orthants, balancing points and their z -value.
 - If more than one balancing point, select the balancing point which has the highest worst optimum. (Select randomly to break ties).
- II. Set up the relation $c_{BV} B^{-1} N_j - c_{NBV_j} = 0$, for every variable x_j (x_j is a non-basic variable held at zero by the artificial restriction, and x_j is an original variable of type W_{III}). The entries in N_j and c_{NBV_j} which are interval coefficients in the original model will be represented by variables.
 - Since c_{BV} and B^{-1} are fixed (i.e. no variables), then the relationship $c_{BV} B^{-1} N_j - c_{NBV_j}$ will always be a linear relationship in terms of the variables in N_j and c_{NBV_j} .
- III. Set up all the limits for all the interval coefficients which are represented by variables in II.
- IV. Use II and III to solve for the new variables.
 - If there is more than one solution, then any solution will suffice.

Algorithm 4. The general steps in determining the intermediate coefficient settings

Once the solutions of these new variables are obtained (which represent the intermediate settings), we can resolve for the worst optimum by using the new intermediate setting and without using any artificial orthant restrictions.

Refer to Example 6.15., Figure 6.14. and Table 6.2.

As we have seen earlier in Example 6.15. the worst optimum of this problem is obtained at an artificial corner point (0,4) with $\bar{z} = 4$. Further analysis showed that this point is a balancing point and the two balancing orthants are (+,+) and (-,+). Therefore we need to determine the intermediate coefficient settings at which we obtain this worst optimum.

First we solve for the worst optimum according to one of the balancing orthants and note B^{-1} , x_{BV} , x_{NBV} , c_{BV} , c_{NBV} and N .

NOTE: We do not need to resolve the model for these orthants again, we simply take this information from the initial test (i.e. Table 6.2.).

Select the balancing orthant (-,+) (i.e. Scenario 1) for testing:

$$\begin{aligned} \min \bar{z}^1 &= x_2 \\ \text{s.t. } C_b: &-2x_1 + 2x_2 \geq -2 \\ C_{2b}: &2x_1 + x_2 \geq 4 \\ C_3: &x_2 \leq 5 \\ &(x_1 \leq 0, \ x_2 \text{ urs}) \end{aligned}$$

$$\begin{aligned} \bar{z}^1 &= 4 \\ \bar{x}^1 &= (x_1, x_2) = (0, 4) \end{aligned}$$

Now we use the sensitivity analysis information which can be obtained by using any LP solver.

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 2 & 0 \\ -2 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{x}_{BV} = \begin{bmatrix} s_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{x}_{NBV} = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\mathbf{c}_{BV} = (0, 1, 0), \mathbf{c}_{NBV} = (0, 0).$$

Note the coefficients in question are c_I and a_{II} . Furthermore, since we are only interested in x_I therefore we need to use only the first column of $\mathbf{N}$ corresponding to x_I .

$$\therefore \hat{\mathbf{N}}_1 = \begin{bmatrix} -a_{11} \\ -2 \\ 0 \end{bmatrix} \text{ and } \hat{\mathbf{c}}_{NBV_1} = (c_1).$$

$$\therefore \text{ we need } \mathbf{c}_{BV} \mathbf{B}^{-1} \hat{\mathbf{N}}_1 - \hat{\mathbf{c}}_{NBV_1} = 0;$$

$$\therefore \text{ we need } (0, 1, 0) \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -a_{11} \\ -2 \\ 0 \end{bmatrix} - c_1 = 0 \text{ for some } c_I \text{ and } a_{II}.$$

$$\therefore \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -a_{11} \\ -2 \\ 0 \end{bmatrix} - c_1 = 0 \quad \Rightarrow \quad 2 - c_1 = 0.$$

$$\therefore \quad c_1 = 2 \quad (1)$$

Therefore by using (1) and the interval limits on c_I and a_{II} from the original LPIC model we can solve for c_I and a_{II} .

The answer is: $c_I = 2$, a_{II} = any value between -4 and -2.

This implies if we set $c_1 = 2$ for both scenarios (i.e. the two identified balancing orthants) and we set a_{11} to any value between -4 and -2 , when we solve for the worst optimum we should be able to get the optimum value $z = 4$ both using the artificial restrictions and without using the artificial restrictions.

Table 6.5. clearly shows that by setting $c_1 = 2$ and using the artificial restriction on x_1 we obtained $\bar{z} = 4$ in both scenarios at $(0,4)$, however when both scenarios were solved without the artificial restriction on x_1 we obtained $\bar{z} = 4$ in both scenarios but at different points. This clearly shows that when solving for the worst optimum and using this intermediate setting of c_1 the objective function becomes parallel to constraint C_{2b} and consequently we have an infinite number of worst optimum solutions with the same worst optimum value $\bar{z} = 4$.

Table 6.5. The Worst optimum according to scenarios 1 and 2 using the pre-determined intermediate setting, with and without the artificial restriction on x_1 .

	Worst Optimum
Scenario 1 using the artificial restriction on x_1 ($x_1 \leq 0$, x_2 urs)	$\min \bar{z}^1 = 2x_1 + x_2$ s.t. $C_{1b}: -2x_1 + 2x_2 \geq -2$ $C_{2b}: 2x_1 + x_2 \geq 4$ $C_3: x_2 \leq 5$ $\bar{z}^1 = 4$ $\bar{x}^1 = (x_1, x_2) = (0, 4)$
without the artificial restriction (x_1 , x_2 urs)	$\bar{z}^1 = 4$ $\bar{x}^1 = (x_1, x_2) = (1.667, 0.667)$
Scenario 2 using the artificial restriction on x_1 ($x_1 \geq 0$, x_2 urs)	$\min \bar{z}^2 = 2x_1 + x_2$ s.t. $C_{1b}: -4x_1 + 2x_2 \geq -2$ $C_{2b}: 2x_1 + x_2 \geq 4$ $C_3: x_2 \leq 5$ $\bar{z}^2 = 4$ $\bar{x}^2 = (x_1, x_2) = (0, 4)$
without the artificial restriction (x_1 , x_2 urs)	$\bar{z}^2 = 4$ $\bar{x}^2 = (x_1, x_2) = (1.25, 1.5)$

Example 6.16.

Determine the best and the worst optimum of the following LPIC problem.

$$\begin{aligned}
 \min Z &= [7,8]x_1 + [6,9]x_2 + x_3 \\
 \text{s.t. } C_1: & [6,10]x_1 + [2,8]x_2 + [0.2,1]x_3 \geq [-4,4] \\
 C_2: & [-2,-1]x_1 + [-1,-0.5]x_2 + [1,2]x_3 \geq -2 \\
 C_3: & 4x_1 - 4x_2 + x_3 \geq -4 \\
 C_4: & -2x_1 + 3x_2 + x_3 \geq -6 \\
 C_5: & x_3 \leq 5 \\
 & x_1, x_2, x_3 \text{ urs}
 \end{aligned}$$

In this example there are 3 variables, all of type W_{III} (i.e. sign-unrestricted and associated with interval coefficients). Therefore we will have a total of $2^3 = 8$ different scenarios to examine. Table 6.6. shows the best optimum solution for every scenario, as well as the worst optimum solution according to each scenario for both cases, with and without the artificial restrictions (i.e. *restricted* and *relaxed* versions respectively).

With regard to the best optimum, Table 6.6. clearly shows the best optimum solution obtained at $(-0.23, -3.82, 5)$ with $\underline{z} = -31.18$. The coefficient settings used to achieve this optimum are as indicated in scenario 2. However with regard to the worst optimum, all solutions obtained by using the artificial restrictions are either at an artificial corner point or infeasible. Therefore further analysis of the solutions obtained using the relaxed version is required to determine the balancing orthants and the balancing point. Once the balancing orthants and the balancing point are identified we can use sensitivity analysis, as described earlier, to determine the intermediate setting at which we achieve the worst optimum.

By using the diagrammatic approach as in Figure 6.17, we can easily conclude that orthant (+,+) (i.e. scenario 6) and orthant (-,+) (i.e. scenario 3) are the balancing orthants, and the point $x = (x_1, x_2, x_3) = (0.667, 0, 0)$ is the balancing point with $\bar{z} = 533$.

Table 6.6. (Part 1) Best and Worst optimums according to scenarios 1-8. The worst optimum was computed for both cases, with and without the artificial restrictions.

Scenario 1: $x_1 \leq 0, x_2 \leq 0, x_3 \leq 0$ Orthant (—)		Scenario 2: $x_1 \leq 0, x_2 \leq 0, x_3 \geq 0$ Orthant (—+)	
Best Opt.	Worst Opt.	Best Opt.	Worst Opt.
$\max 8x_1 + 9x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + 0.2x_3 \geq 4$ $C_2: -2x_1 - x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 2x_1 + 6x_2 + x_3$ s.t. $C_1: 10x_1 + 8x_2 + x_3 \geq 4$ $C_2: -x_1 - 0.5x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\max 8x_1 + 9x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + x_3 \geq 4$ $C_2: -2x_1 - x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 2x_1 + 6x_2 + x_3$ s.t. $C_1: 10x_1 + 8x_2 + 0.2x_3 \geq 4$ $C_2: -x_1 - 0.5x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$
Restricted	Restricted Relaxed	Restricted	Restricted Relaxed
$x_1 = 0$ $x_2 = -2$ $x_3 = 0$ $\bar{z}^1 = -18$	(Infeasible) $\begin{cases} x_1 = 1.24 \\ x_2 = -0.97 \\ x_3 = -0.62 \\ \bar{z}^1 = 223 \end{cases}$	$x_1 = -0.23$ $x_2 = -3.82$ $x_3 = 5$ $\bar{z}^1 = -3.118$ *** Best Opt ***	(infeasible) $\begin{cases} x_1 = 1.09 \\ x_2 = -0.83 \\ x_3 = -1.32 \\ \bar{z}^1 = 1.33 \end{cases}$

Scenario 3: $x_1 \leq 0, x_2 \geq 0, x_3 \leq 0$ Orthant (-,+)		Scenario 4: $x_1 \leq 0, x_2 \geq 0, x_3 \geq 0$ Orthant (-++)	
Best Opt.	Worst Opt.	Best Opt.	Worst Opt.
$\max 8x_1 + 9x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + 0.2x_3 \geq 4$ $C_2: -2x_1 - x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 2x_1 + 6x_2 + x_3$ s.t. $C_1: 10x_1 + 8x_2 + x_3 \geq 4$ $C_2: -x_1 - 0.5x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\max 8x_1 + 9x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + x_3 \geq 4$ $C_2: -2x_1 - x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 2x_1 + 6x_2 + x_3$ s.t. $C_1: 10x_1 + 8x_2 + 0.2x_3 \geq 4$ $C_2: -x_1 - 0.5x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$
Restricted	Restricted Relaxed	Restricted	Restricted Relaxed
$x_1 = -0.62$ $x_2 = 0$ $x_3 = -1.54$ $\bar{z}^1 = -6.46$	(Infeasible) $\begin{cases} x_1 = 0.56 \\ x_2 = -3.29 \\ x_3 = 5 \\ \bar{z}^1 = -20.74 \end{cases}$	$x_1 = -1.93$ $x_2 = 0.32$ $x_3 = 5$ $\bar{z}^1 = -8.5$	$x_1 = 0$ $x_2 = 1.71$ $x_3 = 2.86$ $\bar{z}^1 = 18.29$

Table 6.6. (Part 2)

Scenario 5: $x_1 \geq 0, x_2 \leq 0, x_3 \leq 0$ Orthant (+--)		Scenario 6: $x_1 \geq 0, x_2 \leq 0, x_3 \geq 0$ Orthant (+-+)	
Best Opt.	Worst Opt.	Best Opt.	Worst Opt.
$\min 7x_1 + 9x_2 + x_3$ s.t. $C_1: 10x_1 + 2x_2 + 0.2x_3 \geq -4$ $C_2: -x_1 - x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 8x_1 - 6x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + x_3 \geq 4$ $C_2: -2x_1 - 0.5x_2 - 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 7x_1 + 9x_2 + x_3$ s.t. $C_1: 10x_1 + 2x_2 + x_3 \geq -4$ $C_2: -x_1 - x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 8x_1 - 6x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + 0.2x_3 \geq 4$ $C_2: -2x_1 - 0.5x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$
Restricted	Restricted Relaxed	Restricted	Restricted Relaxed
$x_1 = 0$ $x_2 = -2$ $x_3 = 0$ $\bar{z}^* = -18$	$x_1 = 0.667$ $x_2 = 0$ $x_3 = 0$ $\bar{z}^* = 5.33$	$x_1 = 0$ $x_2 = -3.67$ $x_3 = 5$ $\bar{z}^* = -28$	$x_1 = 0.667$ $x_2 = 0$ $x_3 = 0$ $\bar{z}^* = 5.33$

Scenario 7: $x_1 \geq 0, x_2 \geq 0, x_3 \leq 0$ Orthant (++-)		Scenario 8: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ Orthant (+++)	
Best Opt.	Worst Opt.	Best Opt.	Worst Opt.
$\min 7x_1 + 9x_2 + x_3$ s.t. $C_1: 10x_1 + 2x_2 + 0.2x_3 \geq -4$ $C_2: -x_1 - x_2 + 0.5x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 8x_1 - 6x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + x_3 \geq 4$ $C_2: -2x_1 - x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 7x_1 + 9x_2 + x_3$ s.t. $C_1: 10x_1 + 2x_2 + x_3 \geq -4$ $C_2: -x_1 - 0.5x_2 + 2x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$	$\min 8x_1 - 6x_2 + x_3$ s.t. $C_1: 6x_1 + 2x_2 + 0.2x_3 \geq 4$ $C_2: -2x_1 - x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_1 \leq 5$
Restricted	Restricted Relaxed	Restricted	Restricted Relaxed
$x_1 = 0$ $x_2 = 0$ $x_3 = -2$ $\bar{z}^* = -2$	$x_1 = 0.667$ $x_2 = 0$ $x_3 = 0$ $\bar{z}^* = 5.33$	$x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $\bar{z}^* = 0$	$x_1 = 0.667$ $x_2 = 0$ $x_3 = 0$ $\bar{z}^* = 5.33$

Since orthant (- + -) (i.e. scenario 3) is infeasible when using artificial restriction, hence we select orthant (+ - +) (i.e. scenario 6) to determine the intermediate coefficient settings at which we achieve the worst optimum $\bar{z} = 5.33$.

Orthant (+ - +) scenario 6:

$$\begin{aligned}
 &\min 8x_1 + 6x_2 + x_3 \\
 &\text{s.t. } C_1: 6x_1 + 8x_2 + 0.2x_3 \geq 4 \\
 &\quad C_2: -2x_1 - 0.5x_2 + x_3 \geq -2 \\
 &\quad C_3: 4x_1 - 4x_2 + x_3 \geq -4 \\
 &\quad C_4: -2x_1 + 3x_2 + x_3 \geq -6 \\
 &\quad C_5: \quad \quad \quad x_3 \leq 5 \\
 &\quad x_1 \geq 0, x_2 \leq 0, x_3 \geq 0
 \end{aligned}$$

The optimal solution using the artificial restrictions is:

(0.667, 0, 0) with $\bar{z} = 533$

From the optimal tableau we get:

$$\mathbf{x}_{BV} = \begin{bmatrix} x_1 = 0.667 \\ s_2 = 0.667 \\ s_3 = 6.667 \\ s_4 = 4.667 \\ s_5 = 5 \end{bmatrix}, \mathbf{x}_{NBV} = \begin{bmatrix} x_2 = 0 \\ x_3 = 0 \\ s_1 = 0 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} -0.1667 & 0 & 0 & 0 & 0 \\ 0.3333 & 1 & 0 & 0 & 0 \\ -0.6667 & 0 & 1 & 0 & 0 \\ 0.3333 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} -8 & -0.2 & -1 \\ 0.5 & -1 & 0 \\ 4 & -1 & 0 \\ -3 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{c}_{BV} = [8, 0, 0, 0, 0], \mathbf{c}_{NBV} = [6, 1, 0].$$

Note: The solver used to solve this LP model, converts all constraints to $\leq$ type. This is reflected in both $\mathbf{B}^{-1}$ and $\mathbf{N}$.

Since we are only interested in the original variables held at zero by the artificial restrictions, therefore we will only work with x_2 and x_3 . Also we can delete the third column of $\mathbf{N}$ since it corresponds to the slack variable s_1 . Furthermore, the interval coefficients which will be represented by variables are associated with x_2 and x_3 in $\mathbf{c}_{NBV}$ and $\mathbf{N}$.

$$\text{Therefore we have: } \hat{\mathbf{N}} = \begin{bmatrix} -a_{12} & -a_{13} \\ -a_{22} & -a_{23} \\ 4 & -1 \\ -3 & -1 \\ 0 & 1 \end{bmatrix}, \hat{\mathbf{c}}_{NBV} = [c_2, 1].$$

$$\text{Set } \mathbf{c}_{BV} \mathbf{B}^{-1} \hat{\mathbf{N}} - \hat{\mathbf{c}}_{NBV} = \mathbf{0}.$$

$$\text{For } x_2 \text{ we get: } \mathbf{c}_{BV} \mathbf{B}^{-1} \hat{\mathbf{N}}_1 - \hat{c}_{NBV_1} = 0 \Rightarrow 1.333a_{12} - c_2 = 0 \quad (1)$$

$$\text{For } x_3 \text{ we get: } \mathbf{c}_{BV} \mathbf{B}^{-1} \hat{\mathbf{N}}_2 - \hat{c}_{NBV_2} = 0 \Rightarrow 1.333a_{13} - 1 = 0 \quad (2)$$

$$2 \leq a_{12} \leq 8, 0.2 \leq a_{13} \leq 1, -1 \leq a_{22} \leq -0.5, 1 \leq a_{23} \leq 2, 6 \leq c_2 \leq 9.$$

$$\text{From (2) } a_{13} = \chi_{133} = 0.75$$

Let $c_2 = 6$, then from (1) $a_{12} = \chi_{133} = 4.5011$. Note a_{12} is dependent on c_2 .

The variables a_{22} and a_{23} can take any value within their interval limits (i.e. $a_{22} \in [-1, -0.5]$ and $a_{23} \in [1, 2]$). Therefore these values will be the intermediate coefficient settings at which we achieve the worst optimum solution.

Finally, if we substitute for these intermediate coefficient settings in (scenario 6), we should get the same worst optimum value $\bar{z} = 5.33$ without using artificial restrictions as in table 6.7.

Table 6.7. The Worst optimum according to scenarios 6 using the pre-determined intermediate settings, with and without the artificial restrictions.

	Worst Optimum
Orthant (+ - +) (Scenario 6) Using the artificial restriction on x_1, x_2, x_3 $(x_1 \geq 0, x_2 \leq 0, x_3 \geq 0)$	$\min 8x_1 + 6x_2 + x_3$ s.t. $C_1: 6x_1 + 4.5011x_2 + 0.75x_3 \geq 4$ $C_2: -2x_1 - 0.5x_2 + x_3 \geq -2$ $C_3: 4x_1 - 4x_2 + x_3 \geq -4$ $C_4: -2x_1 + 3x_2 + x_3 \geq -6$ $C_5: x_3 \leq 5$ $\bar{z}^6 = 5.33$ $\bar{x}^6 = (x_1, x_2, x_3) = (0.667, 0, 0)$
Without the artificial restriction $(x_1, x_2, x_3 \text{ urs})$	$\bar{z}^6 = 5.33$ $\bar{x}^6 = (x_1, x_2, x_3) = (-0.94, 1.31, 5.0)$

Therefore, the final solution for this example is:

- The best optimum is $\underline{z} = -31.18$ at $(-0.23, -3.82, 5)$ with the coefficient settings as indicated in table 6.6. scenario 2.
- The worst optimum is $\bar{z} = 5.33$ at $(-0.94, 1.31, 5.0)$ with the coefficient settings as in table 6.7.

The general procedure to solve any type III LPIC is outlined in Algorithm 5.

Consider a minimization model:

START:

1. Identify all scenarios as described in Algorithm 3, then add the required artificial restrictions.
 2. For each scenario, determine the best and worst optimums and their respective solutions.
 - Let $\underline{z}^i$ be the best optimum according to scenario i with its solution $\underline{x}^i$.
 - Let $\bar{z}^i$ be the worst optimum according to scenario i with its solution $\bar{x}^i$.
 3. Let $BO = \{\underline{z}^i \mid \forall i \mid \underline{z}^i \neq \text{inf}\}$, then the best optimum $\underline{z} = \min \{BO\}$ with its optimal solution $\underline{x}$. If $BO = \emptyset$ then the best optimum $\underline{z}$ is infeasible, which implies the entire model is infeasible, hence STOP.
 4. Let $WO = \{\bar{z}^i \mid \forall i \mid \bar{z}^i \neq \text{inf}\}$
 - while $WO \neq \emptyset$
 - $\bar{z}' := \max \{WO\}$ with its optimal solution $\bar{x}'$.
 - if $\bar{x}'$ is an original corner point then $\bar{z}'$ is the worst optimum.
Goto 5.
 - else ($\bar{x}'$ is an artificial corner point)
 - if $\bar{x}'$ is a balancing point between two balancing orthants then
Determine the intermediate settings as described in
Algorithm 4 and set $\bar{z}'$ as the worst optimum. Set $\bar{x}'$ to
be the solution obtained by resolving the specific scenario
using the new intermediate settings.
Goto 5.
 - else ($\bar{x}'$ is not a balancing point)
 $WO = WO - \bar{z}'$
 5. if $WO = \emptyset$ then the worst optimum is infeasible.
else $\bar{z}'$ is the worst optimum with its optimal solution $\bar{x}'$.
- END.

Algorithm 5. The general procedure for solving Type III LPIC models.

CHAPTER 7

Practical Examples

In this chapter practical examples will be used to demonstrate the use and the practicality of interval coefficients in linear programming models.

7.1 Quarrying over Periods of time

This is an example adapted from Stark and Mayer, *Quantitative Construction Management* (1983). The original example incorporates real numbers only. In this thesis we modified some of the coefficients to be represented by interval numbers.

Rollin Along, a major road builder operates two quarries to produce three different types of stones; Softa, Harda and Common. Suppose that Rollin Along's construction operations are to begin in a week, and a three-week forecast for quantities (in tons per week) of needed stone is given in table 7.1. For example the forecast for stone Softa in week 1 is between 390 and 420 tons.

Table 7.1. A three-week forecast for quantities of needed stone.

Stone	Construction Week		
	1	2	3
Softa	[390,420]	[580,620]	[640,670]
Harda	[480,515]	[570,600]	[590,610]
Common	[160,180]	[300,340]	[310,340]

Assume that Rollin Along has a single crew available for up to 5 days per week at the two quarries. The respective daily operating costs at the two quarries are:

- Quarry 1: between \$350 and \$410.
- Quarry 2: between \$1350 and \$1450.

The estimated daily yields of stones at the two quarries are :

- Quarry 1: between 70 and 80 tons of Softa, between 50 and 65 tons of Harda, and between 20 and 25 tons of Common.
- Quarry 2: between 35 and 45 tons of Softa, between 60 and 80 tons of Harda, and between 50 and 60 tons of Common.

Suppose that an outside commercial source of stone can provide stone of any type at a flat rate of \$12 per ton. This source has an ample supply of Softa stone, but its current supplies of Harda and Common stones are limited, between 550 and 600, and 150 and 200 tons, respectively. However, effectively *unlimited* supplies of Harda and Common stones are expected in time for the third week of construction (but not earlier).

Site storage of stone poses no problem for Rollin Along. However the needed weekly quantities must be on hand at the beginning of each week to avoid interference with site operations.

Rollin Along must decide where to assign the quarry crew and how much stone to purchase in order to obtain the needed stones at least cost.

Let:

X_{ij} = {the number of days to work at quarry i during the week preceding week j . $i = 1, 2$
and $j = 1, 2, 3$ }

Y_k = {the quantities of stone k to be purchased immediately preceding week j of
production. $k = S, H, C$ and $j = 1, 2, 3$ }

Now Rollin Along's problem of minimizing the total cost of stone can be formulated beginning with the objective function as follows:

$$\begin{aligned} \text{minimize: } Z = & [350,410]X_{11} + [350,410]X_{12} + [350,410]X_{13} \\ & + [1350,1450]X_{21} + [1350,1450]X_{22} + [1350,1450]X_{23} \\ & + 12Y_{S1} + 12Y_{S2} + 12Y_{S3} + 12Y_{H1} + 12Y_{H2} + 12Y_{H3} + 12Y_{C1} + 12Y_{C2} + 12Y_{C3} \end{aligned}$$

The first three terms represent the costs at the first quarry, the next three terms those at the second quarry, and the remaining terms express the cost of purchased stone, if any.

For the constraints, at least between 390 and 420 tons of Softa stone is needed for work in the first week, that is:

$$\text{SoftaW1: } [70,80]X_{11} + [35,45]X_{21} + Y_{S1} \geq [390,420]$$

Clearly, any excess stone that is not used during the first week will be available to meet the requirements in the second week, and so on. For Softa stone requirements in the second week:

$$([70,80]X_{11} + [35,45]X_{21} + Y_{s1} - [390,420]) + [70,80]X_{12} + [35,45]X_{22} + Y_{s2} \geq [580,620]$$

The quantity $[70,80]X_{11} + [35,45]X_{21} + Y_{s1} - [390,420]$ is the excess, if any, of Softa stone not used in the first week. This last expression can be rewritten as:

$$\text{SoftaW2: } [70,80]X_{11} + [70,80]X_{12} + [35,45]X_{21} + [35,45]X_{22} + Y_{s1} + Y_{s2} \geq [970,1040]$$

Similarly, for the third week:

$$\begin{aligned} \text{SoftaW3: } & [70,80]X_{11} + [70,80]X_{12} + [70,80]X_{13} \\ & + [35,45]X_{21} + [35,45]X_{22} + [35,45]X_{23} + Y_{s1} + Y_{s2} + Y_{s3} \geq [1610,1710] \end{aligned}$$

In the same way, constraints are written for Harda and Common stones. Therefore, there are nine constraints (one for each type of stone for each of three weeks) to ensure the needed quantities as follows:

For Harda:

$$\text{HardaW1: } [50,65]X_{11} + [60,80]X_{21} + Y_{H1} \geq [480,515]$$

$$\text{HardaW2: } [50,65]X_{11} + [50,65]X_{12} + [60,80]X_{21} + [60,80]X_{22} + Y_{H1} + Y_{H2} \geq [1050,1115]$$

$$\begin{aligned} \text{HardaW3: } & [50,65]X_{11} + [50,65]X_{12} + [50,65]X_{13} \\ & + [60,80]X_{21} + [60,80]X_{22} + [60,80]X_{23} + Y_{H1} + Y_{H2} + Y_{H3} \geq [1640,1725] \end{aligned}$$

For Common:

$$\text{CommonW1: } [20,25]X_{11} + [50,60]X_{21} + Y_{C1} \geq [160,180]$$

$$\text{CommonW2: } [20,25]X_{11} + [20,25]X_{12} + [50,60]X_{21} + [50,60]X_{22} + Y_{C1} + Y_{C2} \geq [400,520]$$

$$\text{CommonW3: } [20,25]X_{11} + [20,25]X_{12} + [20,25]X_{13} \\ + [50,60]X_{21} + [50,60]X_{22} + [50,60]X_{23} + Y_{C1} + Y_{C2} + Y_{C3} \geq [710,860]$$

To limit the work week to five days at each quarry to produce Softa, Harda and Common stones, these additional constraints are necessary:

$$\text{WorkW1: } X_{11} + X_{21} \leq 5 \text{ days, i.e. } -X_{11} - X_{21} \geq -5$$

$$\text{WorkW2: } X_{12} + X_{22} \leq 5 \text{ days, i.e. } -X_{12} - X_{22} \geq -5$$

$$\text{WorkW3: } X_{13} + X_{23} \leq 5 \text{ days, i.e. } -X_{13} - X_{23} \geq -5$$

Finally, constraints are needed because of the limited availability of Harda and Common stones (recall that the external supply of Softa stone is unlimited). These are written as:

$$\text{ExtraH: } Y_{H1} + Y_{H2} \leq [550,600] \text{ tons, i.e. } -Y_{H1} - Y_{H2} \geq [-600,-550]$$

$$\text{ExtraC: } Y_{C1} + Y_{C2} \leq [150,200] \text{ tons, i.e. } -Y_{C1} - Y_{C2} \geq [-200,-150]$$

Note that these constraints only involve variables for the first and second weeks. The reason, of course, is that the supply available for the third week (Y_{H3} and Y_{C3}) is effectively limited.

Summarizing, Rollin Along's problem has been formulated as an LPIC model. The objective is to minimize the total cost under the given uncertainties such that 14 linear constraints (nine quantity constraints, three work week constraints, and the two external

supply constraints) are satisfied. Of course, none of the variables can be permitted to have negative values.

Since this LPIC model involves only non-negative variables therefore it is of type I LPIC and we use Algorithm 1 to solve it. First we transform the LPIC model into two deterministic LP models and then we solve them separately to determine the best and worst optimum solutions under the given uncertainties. The set up and the solution for the first LP model (The Best optimum model) is shown in Display 7.1, and the set up and the solution for the second LP model (The Worst Optimum model) is shown in Display 7.2.

```

MIN    350 X11 + 350 X12 + 350 X13 + 1350 X21 + 1350 X22 + 1350 X23
      + 12 YS1 + 12 YS2 + 12 YS3 + 12 YH1 + 12 YH2 + 12 YH3 + 12 YC1
      + 12 YC2 + 12 YC3

SUBJECT TO

SoftaW1) 80 X11 + 45 X21 + YS1 >= 390
SoftaW2) 80 X11 + 80 X12 + 45 X21 + 45 X22 + YS1 + YS2 >= 970
SoftaW3) 80 X11 + 80 X12 + 80 X13 + 45 X21 + 45 X22
      + 45 X23 + YS1 + YS2 + YS3 >= 1610
HardaW1) 65 X11 + 80 X21 + YH1 >= 480
HardaW2) 65 X11 + 65 X12 + 80 X21 + 80 X22 + YH1 + YH2 >= 1050
HardaW3) 65 X11 + 65 X12 + 65 X13 + 80 X21 + 80 X22
      + 80 X23 + YH1 + YH2 + YH3 >= 1640
CommonW1) 25 X11 + 60 X21 + YC1 >= 160
CommonW2) 25 X11 + 25 X12 + 60 X21 + 60 X22 + YC1 + YC2 >= 400
CommonW3) 25 X11 + 25 X12 + 25 X13 + 60 X21 + 60 X22
      + 60 X23 + YC1 + YC2 + YC3 >= 710
WorkW1) - X11 - X21 >= - 5
WorkW2) - X12 - X22 >= - 5
WorkW3) - X13 - X23 >= - 5
ExtraH) - YH1 - YH2 >= - 600
ExtraC) - YC1 - YC2 >= - 200

END

LP OPTIMUM FOUND AT STEP 10

OBJECTIVE FUNCTION VALUE

1) 22170.000

VARIABLE      VALUE      REDUCED COST
X11           5.000000      .000000
X12           5.000000      .000000
X13           5.000000      .000000
X21           .000000      820.000000
X22           .000000      820.000000
X23           .000000      820.000000
YS1          410.000000      .000000
YS2           .000000      .000000
YS3           .000000      .000000
YH1          600.000000      .000000
YH2           .000000      .000000
YH3           65.000000      .000000
YC1          200.000000      .000000
YC2           .000000      .000000
YC3          135.000000      .000000

```

Display 7.1. Quarrying over time (Best Optimum)

ROW	SLACK OR SURPLUS	DUAL PRICES
SoftaW1	420.000000	.000000
SoftaW2	240.000000	.000000
SoftaW3	.000000	-12.000000
HardaW1	445.000000	.000000
HardaW2	200.000000	.000000
HardaW3	.000000	-12.000000
CommonW1	165.000000	.000000
CommonW2	50.000000	.000000
CommonW3	.000000	-12.000000
WorkW1	.000000	-1690.000000
WorkW2	.000000	-1690.000000
WorkW3	.000000	-1690.000000
ExtraH	.000000	.000000
ExtraC	.000000	.000000

NO. ITERATIONS= 10

RANGES IN WHICH THE BASIS IS UNCHANGED:

VARIABLE	CURRENT COEF	OBJ COEFFICIENT RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
X11	350.000000	820.000000	INFINITY
X12	350.000000	820.000000	INFINITY
X13	350.000000	820.000000	INFINITY
X21	1350.000000	INFINITY	820.000000
X22	1350.000000	INFINITY	820.000000
X23	1350.000000	INFINITY	820.000000
Y11	12.000000	.000000	12.000000
Y12	12.000000	INFINITY	.000000
Y13	12.000000	INFINITY	.000000
YH1	12.000000	.000000	INFINITY
YH2	12.000000	INFINITY	.000000
YH3	12.000000	54.666670	.000000
YC1	12.000000	.000000	INFINITY
YC2	12.000000	INFINITY	.000000
YC3	12.000000	23.428570	.000000

ROW	CURRENT RHS	RIGHTHAND SIDE RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
SoftaW1	390.000000	420.000000	INFINITY
SoftaW2	970.000000	240.000000	INFINITY
SoftaW3	1610.000000	INFINITY	240.000000
HardaW1	480.000000	445.000000	INFINITY
HardaW2	1050.000000	200.000000	INFINITY
HardaW3	1640.000000	INFINITY	65.000000
CommonW1	160.000000	165.000000	INFINITY
CommonW2	400.000000	50.000000	INFINITY
CommonW3	710.000000	INFINITY	135.000000
WorkW1	-5.000000	2.000000	1.000000
WorkW2	-5.000000	2.000000	1.000000
WorkW3	-5.000000	5.000000	1.000000
ExtraH	-600.000000	200.000000	65.000000
ExtraC	-200.000000	50.000000	135.000000

Display 7.1. Continued.

For both models (The best and the worst), LINDO (A linear programming solver) was used to obtain the solutions. From the output in Display 7.1 one can observe that the best optimum is $z = \$22,170.00$, and the coefficients used to get this solution are as indicated in Display 7.1.

Similarly for the worst model, Display 7.2 shows that the worst optimum for this LPIC problem is $\bar{z} = \$38,860.00$, and the fixed coefficients used to get this solution are as indicated in Display 7.2.

MIN			410 X11 + 410 X12 + 410 X13 + 1450 X21 + 1450 X22 + 1450 X23
			+ 12 YS1 + 12 YS2 + 12 YS3 + 12 YH1 + 12 YH2 + 12 YH3 + 12 YC1
			+ 12 YC2 + 12 YC3
SUBJECT TO			
SoftaW1)	70 X11 + 35 X21 + YS1	>=	420
SoftaW2)	70 X11 + 70 X12 + 35 X21 + 35 X22 + YS1 + YS2	>=	1040
SoftaW3)	70 X11 + 70 X12 + 70 X13 + 35 X21 + 35 X22 +		
	35 X23 + YS1 + YS2 + YS3	>=	1710
HardaW1)	50 X11 + 60 X21 + YH1	>=	515
HardaW2)	50 X11 + 50 X12 + 60 X21 + 60 X22 + YH1 + YH2	>=	1115
HardaW3)	50 X11 + 50 X12 + 50 X13 + 60 X21 + 60 X22 +		
	60 X23 + YH1 + YH2 + YH3	>=	1725
CommonW1)	20 X11 + 50 X21 + YC1	>=	280
CommonW2)	20 X11 + 20 X12 + 50 X21 + 50 X22 + YC1 + YC2	>=	520
CommonW3)	20 X11 + 20 X12 + 20 X13 + 50 X21 + 50 X22 +		
	50 X23 + YC1 + YC2 + YC3	>=	860
WorkW1)	- X11 - X21	>=	- 5
WorkW2)	- X12 - X22	>=	- 5
WorkW3)	- X13 - X23	>=	- 5
ExtraH)	- YH1 - YH2	>=	- 550
ExtraC)	- YC1 - YC2	>=	- 150
END			
LP OPTIMUM FOUND AT STEP			10
OBJECTIVE FUNCTION VALUE			
1)	38860.000		
VARIABLE	VALUE	REDUCED COST	
X11	3.500000	.000000	
X12	.000000	.000000	
X13	5.000000	.000000	
X21	1.500000	.000000	
X22	5.000000	.000000	
X23	.000000	980.000000	
YS1	887.500000	.000000	
YS2	.000000	.000000	
YS3	.000000	.000000	
YH1	550.000000	.000000	
YH2	.000000	.000000	
YH3	360.000000	.000000	
YC1	150.000000	.000000	
YC2	.000000	.000000	
YC3	215.000000	.000000	
ROW	SLACK OR SURPLUS	DUAL PRICES	
SoftaW1)	765.000000	.000000	
SoftaW2)	320.000000	.000000	
SoftaW3)	.000000	-12.000000	
HardaW1)	300.000000	.000000	
HardaW2)	.000000	-98.000000	
HardaW3)	.000000	-12.000000	
CommonW1)	115.000000	.000000	
CommonW2)	25.000000	.000000	
CommonW3)	.000000	-12.000000	
WorkW1)	.000000	-6170.000000	
WorkW2)	.000000	-6170.000000	
WorkW3)	.000000	-1270.000000	
ExtraH)	.000000	-98.000000	
ExtraC)	.000000	.000000	

Display 7.2. Quarrying over time (Worst Optimum)

RANGES IN WHICH THE BASIS IS UNCHANGED:			
VARIABLE	CURRENT COEF	OBJ COEFFICIENT RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
X11	410.000000	.000000	INFINITY
X12	410.000000	INFINITY	.000000
X13	410.000000	980.000000	INFINITY
X21	1450.000000	INFINITY	.000000
X22	1450.000000	.000000	INFINITY
X23	1450.000000	INFINITY	980.000000
Y21	12.000000	.000000	12.000000
Y22	12.000000	INFINITY	.000000
Y23	12.000000	INFINITY	.000000
YH1	12.000000	.000000	INFINITY
YH2	12.000000	INFINITY	.000000
YH3	12.000000	98.000000	12.000000
YU1	12.000000	.000000	INFINITY
YU2	12.000000	INFINITY	.000000
YU3	12.000000	32.666670	.000000
RHS	CURRENT RHS	RIGHTHAND SIDE RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
SoftaM1	420.000000	765.000000	INFINITY
SoftaM2	1040.000000	320.000000	INFINITY
SoftaM3	1710.000000	INFINITY	320.000000
HardaM1	515.000000	300.000000	INFINITY
HardaM2	1115.000000	35.000000	8.333333
HardaM3	1725.000000	INFINITY	360.000000
CommonM1	180.000000	115.000000	INFINITY
CommonM2	520.000000	25.000000	INFINITY
CommonM3	860.000000	INFINITY	215.000000
WorkM1	-5.000000	.563333	.192308
WorkM2	-5.000000	.563333	.192308
WorkM3	-5.000000	5.000000	4.571429
ExtraM	-550.000000	35.000000	8.333333
ExtraC	-150.000000	25.000000	215.000000

Display 7.2. Continued.

In summary, by using this method to solve this LPIC model one can conclude that the optimum solution Z of this problem (i.e. the minimum total cost of stone for Rollin Along) lies between the best optimum $\underline{z} = \$22,170.00$ and the worst optimum $\bar{z} = \$38,860.00$ under any given settings of the interval coefficients involved. Furthermore we could analyze each solution by using the sensitivity analysis obtained from each solution independently and consequently could predict whether another setting within the given interval coefficients would lead to a different optimal solution or not.

7.2 Fuel Allocation Model

This example is adapted from Roy D. Shapiro, *Optimization Models for Planning and Allocation* (1984). The original example incorporates real numbers only. We modified some of the coefficients to be represented by interval numbers.

Shortly after the beginning of the oil embargo, the Eastern District Director of the Government Office of Fuel Allocation was concerned that he would soon have to start issuing monthly allocations specifying the amounts of heating oil that each refinery in the district would send to each city to ensure that every city would receive its quota.

Since different refineries used a variety of alternative sources of crude oil, both foreign and domestic, the cost of products at each refinery varied considerably. Consequently, under current government regulations, the prices that could be charged for heating oil would vary from one refinery to another. To avoid political criticism, it was felt that, in making allocations of supplies from the refineries to the cities, it was best to maintain a reasonably uniform average refinery price for each city. In fact, the Director felt that the average refinery price paid by any city should not be more than 3% above the overall average.

Another complication had arisen in recent months, since some of the emergency allocations of supplies to certain cities could not be delivered due to a shortage in tanker capacity. If delivery of the allocated supplies were to be feasible, the limited transportation facilities on some of the shipping routes would have to be recognized. Finally, it would be

most desirable to maintain distribution costs as low as possible under any allocation plan, if charges of government inefficiency were to be avoided (Roy D. Shapiro, 1984).

Data for a simplified problem with three refineries and four cities are given below. Since some of the data are only approximately known, interval numbers are used here to represent these data.

Table 7.2 Refinery Prices and over all average Refinery Price.

Refinery	December Availability (thousands of barrels)	Price at Refinery (\$ per barrel)	Average Refinery Price
A. Norton	[90,100]	$10.53 \pm 5\% = [10.01, 11.06]$	10.53
B. Chatham	[60,70]	$9.39 \pm 6\% = [8.83, 9.95]$	9.39
C. Eastport	[100,120]	$12.43 \pm 5\% = [11.81, 13.05]$	12.43
Overall Average Refinery price			10.78

For example, from Table 7.2 the December availability from Norton refinery is between 90 and 100 barrels with the price per barrel $\$10.53 \pm 5\%$ (i.e. varying between \$10.01 and \$11.06).

Table 7.3 December Quotas (barrels) for each city.

City	December Quotas (barrels)
1. Westville	55,000
2. Bridgeton	73,000
3. Brookfield	105,000
4. Droval	38,000

Tables 7.4a and 7.4b show the shipping costs per barrel. Where table 7.4a shows the cost of shipping within a certain percentage of error, table 7.4b shows the costs in interval form, which will be used in this model.

Table 7.4a Shipping Costs in percentage form.

Refinery	Shipping Costs $\pm 10\%$ (\$ per barrel) City			
	1	2	3	4
A	0.10	0.16	0.32	0.28
B	0.20	0.34	0.30	0.12
C	0.34	0.38	0.22	0.18

Table 7.4b Shipping Costs in interval number form.

Refinery	Shipping Costs (\$ per barrel) City			
	1	2	3	4
A	[0.09,0.11]	[0.14,0.18]	[0.29,0.35]	[0.25,0.31]
B	[0.18,0.22]	[0.31,0.37]	[0.27,0.33]	[0.11,0.13]
C	[0.31,0.37]	[0.34,0.42]	[0.20,0.24]	[0.16,0.20]

Table 7.5 Capacities (thousands of barrels per month)

Refinery	Capacities (thousands of barrels per month) City			
	1	2	3	4
A	*	25	*	*
B	*	20	20	*
C	25	*	*	*

* No effective limit.

The decision variables are:

A_n = Barrels of heating oil (in thousands) to be supplied from Refinery A to City n ,
where $n=1,2,3$, or 4.

B_n = Barrels of heating oil (in thousands) to be supplied from Refinery B to City n ,
where $n=1,2,3$, or 4.

C_n = Barrels of heating oil (in thousands) to be supplied from Refinery C to City n ,
where $n=1,2,3$, or 4.

PMAX = A value higher than the over all average refinery price paid by any of the four cities.

There will be 4 different types of constraints:

1. Supply constraints at each refinery.
2. Quota restrictions for each city.
3. Average price restrictions which impose an upper limit to the average refinery price that will be paid by the city.
4. Shipping capacity limits on some of the routes.

Constraints type 1:

$$\text{REF-A: } A_1 + A_2 + A_3 + A_4 \leq [90, 100]$$

$$\text{REF-B: } B_1 + B_2 + B_3 + B_4 \leq [60, 70]$$

$$\text{REF-C: } C_1 + C_2 + C_3 + C_4 \leq [100, 120]$$

Constraints type 2:

$$\text{CIT-1: } A_1 + B_1 + C_1 = 55.00$$

$$\text{CIT-2: } A_2 + B_2 + C_2 = 73.00$$

$$\text{CIT-3: } A_3 + B_3 + C_3 = 105.00$$

$$\text{CIT-4: } A_4 + B_4 + C_4 = 38.00$$

Constraints type 3:

For these constraints we use the following expression:

$$\text{AP-CIT-1} = \frac{[0.01, 1.106]A_1 + [8.83, 9.95]B_1 + [1.181, 13.05]C_1}{55} \leq \text{PMAX} \quad \text{or}$$

$$\text{AP-CIT-1} = [0.182, 0.201]A_1 + [0.161, 0.181]B_1 + [0.215, 0.237]C_1 - \text{PMAX} \leq 0$$

$$\text{AP-CTI-2} = [0.137, 0.151]A_2 + [0.121, 0.136]B_2 + [0.162, 0.179]C_2 - \text{PMAX} \leq 0$$

$$\text{AP-CTI-3} = [0.095, 0.105]A_3 + [0.084, 0.095]B_3 + [0.113, 0.124]C_3 - \text{PMAX} \leq 0$$

$$AP-CTI-4 = [0.263, 0.291]A_4 + [0.232, 0.262]B_4 + [0.311, 0.343]C_4 - PMAX \leq 0$$

The value of PMAX should not be more than 3% greater than the overall average price.

Therefore :

$$PRICELM = PMAX \leq (10.78 + 0.03 * 10.78) = 11.10$$

Constraints type 4:

$$ROUTE-A2: A_2 \leq 25.00$$

$$ROUTE-B2: B_2 \leq 20.00$$

$$ROUTE-B3: B_3 \leq 20.00$$

$$ROUTE-C1: C_1 \leq 25.00$$

Finally the objective is to minimize the total shipping costs for the month of December under the given constraints. Furthermore, all variables must be non-negative.

$$\begin{aligned} \text{MINIMIZE } Z = & [0.09, 0.11]A_1 + [0.14, 0.18]A_2 + [0.29, 0.35]A_3 + [0.25, 0.31]A_4 + \\ & [0.18, 0.22]B_1 + [0.31, 0.37]B_2 + [0.27, 0.33]B_3 + [0.11, 0.13]B_4 + \\ & [0.31, 0.37]C_1 + [0.34, 0.42]C_2 + [0.20, 0.24]C_3 + [0.16, 0.20]C_4 \end{aligned}$$

Since all variables are restricted, we apply the method described in Algorithm 1, provided that we convert all constraints of type $\leq$ (that involve interval coefficients) to type $\geq$ by multiplying the constraints by -1. Once this step is done, we set up and solve for the best and worst optimums as shown in Display 7.3 and Display 7.4 respectively.

```

MIN      0.09 A1 + 0.14 A2 + 0.29 A3 + 0.25 A4 + 0.18 B1 + 0.31 B2
        + 0.27 B3 + 0.11 B4 + 0.31 C1 + 0.34 C2 + 0.2 C3 + 0.16 C4
SUBJECT TO
REF(A) - A1 - A2 - A3 - A4 >= -100
REF(B) - B1 - B2 - B3 - B4 >= -70
REF(C) - C1 - C2 - C3 - C4 >= -120
CITY1) A1 + B1 - C1 = 55
CITY2) A2 + B2 - C2 = 73
CITY3) A3 + B3 - C3 = 105
CITY4) A4 + B4 - C4 = 38
ARCT1) - 0.182 A1 - 0.161 B1 - 0.215 C1 + PMAX >= 0
ARCT2) - 0.137 A2 - 0.121 B2 - 0.162 C2 + PMAX >= 0
ARCT3) - 0.095 A3 - 0.084 B3 - 0.113 C3 + PMAX >= 0
ARCT4) - 0.263 A4 - 0.232 B4 - 0.311 C4 + PMAX >= 0
PRICELM PMAX <= 11.1
ROUTEA2) A2 <= 25
ROUTEB2) B2 <= 20
ROUTEB3) B3 <= 20
ROUTECL) C1 <= 25
END

```

LP OPTIMUM FOUND AT STEP 12

OBJECTIVE FUNCTION VALUE

1) 51.915000

VARIABLE	VALUE	REDUCED COST
A1	55.000000	.000000
A2	25.000000	.000000
A3	10.277750	.000000
A4	.000000	.110000
B1	.000000	.120000
B2	12.000000	.000000
B3	20.000000	.000000
B4	38.000000	.000000
C1	.000000	.220000
C2	36.000000	.000000
C3	74.722250	.000000
C4	.000000	.020000
PMAX	11.100000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
REF(A)	9.722254	.000000
REF(B)	.000000	-.030000
REF(C)	9.277746	.000000
CITY1)	.000000	-.090000
CITY2)	.000000	-.340000
CITY3)	.000000	-.765000
CITY4)	.000000	-.140000
ARCT1)	1.090001	.000000
ARCT2)	.391001	.000000
ARCT3)	.000000	-5.000000
ARCT4)	2.284001	.000000
PRICELM	.000000	5.000000
ROUTEA2)	.000000	.200000
ROUTEB2)	8.000000	.000000
ROUTEB3)	.000000	.045000
ROUTECL)	25.000000	.000000

NO. ITERATIONS= 12

Display 7.3. The Fuel Allocation Model (Best Optimum)

RANGES IN WHICH THE BASIS IS UNCHANGED:			
VARIABLE	CURRENT COEF	OBJ COEFFICIENT RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
A1	.090000	.120000	INFINITY
A2	.140000	.200000	INFINITY
A3	.290000	INFINITY	.027931
A4	.250000	INFINITY	.110000
B1	.180000	INFINITY	.120000
B2	.310000	.030000	.020000
B3	.270000	.045000	INFINITY
B4	.110000	.020000	INFINITY
C1	.310000	INFINITY	.220000
C2	.340000	.020000	.030000
C3	.200000	.073636	INFINITY
C4	.160000	INFINITY	.020000
PMAX	.000000	5.000000	INFINITY
ROW	CURRENT RHS	RIGHTHAND SIDE RANGES	
		ALLOWABLE INCREASE	ALLOWABLE DECREASE
REFA	-100.000000	9.722254	INFINITY
REFB	-70.000000	9.277746	8.000000
REFC	-120.000000	9.277746	INFINITY
CIT1	55.000000	5.989015	55.000000
CIT2	73.000000	2.413583	36.000000
CIT3	105.000000	1.548678	1.637163
CIT4	38.000000	9.277746	8.000000
APCIT1	.000000	1.090001	INFINITY
APCIT2	.000000	.391001	INFINITY
APCIT3	.000000	.175001	.166999
APCIT4	.000000	2.284001	INFINITY
PRICELM	11.100000	.166999	.175001
ROUTEA2	25.000000	9.722254	9.277746
ROUTEB2	20.000000	INFINITY	8.000000
ROUTEB3	20.000000	5.758602	6.034502
ROUTECL	25.000000	INFINITY	25.000000

Display 7.3. Continued.

Clearly we see from the output obtained from LINDO, that the best optimum (minimum) for this LPIC model is $\underline{z} = 51.915$ (i.e. the minimum shipping cost under the given uncertainties is \$51,915.00). The solution and the settings of the coefficients are as shown in Display 7.3. However, as we observe from Display 7.4 the worst optimum (minimum) is infeasible (i.e. as some of the coefficients are being set at their worst setting, the problem becomes unsolvable and consequently further analysis of the coefficients is required).

```

MIN      0.17 A1 + 0.19 A2 + 0.35 A3 + 0.31 A4 + 0.22 B1 + 0.37 B2
        + 0.11 B3 + 0.13 B4 + 0.37 C1 + 0.42 C2 + 0.24 C3 + 0.2 C4
SUBJECT TO
REFA) - A1 - A2 - A3 - A4 Y = 90
REFB) - B1 - B2 - B3 - B4 Y = 60
REFC) - C1 - C2 - C3 - C4 Y = 100
CIT1) A1 + B1 + C1 = 55
CIT2) A2 + B2 + C2 = 73
CIT3) A3 + B3 + C3 = 105
CIT4) A4 + B4 + C4 = 38
APCIT1) - 0.201 A1 - 0.181 B1 - 0.237 C1 + PMAX Y = 0
APCIT2) - 0.151 A2 - 0.136 B2 - 0.179 C2 + PMAX Y = 0
APCIT3) - 0.105 A3 - 0.095 B3 - 0.124 C3 + PMAX Y = 0
APCIT4) - 0.219 A4 - 0.262 B4 - 0.343 C4 + PMAX Y = 0
PRICE10) PMAX <= 11.1
ROUTEA2) A2 <= 25
ROUTEB2) B2 <= 20
ROUTEB3) B3 <= 20
ROUTECL) C1 <= 25
END

```

NO FEASIBLE SOLUTION AT STEP 13
 SUM OF INFEASIBILITIES= 21.7973

VIOLATED ROWS HAVE NEGATIVE SLACK.
 OR (EQUALITY ROWS) NONZERO SLACKS.
 ROWS CONTRIBUTING TO INFEASIBILITY
 HAVE NONZERO DUAL PRICE.

OBJECTIVE FUNCTION VALUE

1) 80.748150

VARIABLE	VALUE	REDUCED COST
A1	.000000	.009710
A2	11.062430	.000000
A3	70.526280	.000000
A4	8.411285	.000000
B1	9.000000	.000000
B2	20.000000	.000000
B3	20.000000	.000000
B4	11.000000	.000000
C1	25.000000	.000000
C2	41.937560	.000000
C3	14.473720	.000000
C4	18.588710	.000000
PMAX	11.100000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICE
REFA)	.000000	-1.009710
REFB)	.000000	-1.000000
REFC)	.000000	-.981710
CIT1)	21.000000	-1.000000
CIT2)	.000000	-1.160710
CIT3)	.000000	-1.164447
CIT4)	.000000	-1.059161
APCIT1)	3.546000	.000000
APCIT2)	-.797252	-1.000000
APCIT3)	.000000	-1.473685
APCIT4)	.000000	-.225807
PRICE10)	.000000	2.699491
ROUTEA2)	13.937570	.000000
ROUTEB2)	.000000	.024710
ROUTEB3)	.000000	.024447
ROUTECL)	.000000	.018290

NO. ITERATIONS= 13

Display 7.4. The Fuel Allocation Model (Worst Optimum)

Technically the optimum value for this model lies between 51.915 (i.e. \$51,915.00) and infeasibility, however an infeasible solution is unacceptable in any industry. Further analysis of the coefficients to determine the worst feasible solution is desirable. In many instances a study of the infeasible model and consequently determining a minimal infeasible set of constraints might be sufficient to the analysts to localize the coefficients causing the infeasibility of the worst model. J. Chinneck and E. Dravnieks (1991), developed an algorithm that locates the minimal infeasible set of constraints that causes infeasibility.

CHAPTER 8

Conclusions and Summary of Results

The main objective of this thesis has been to develop a method to solve linear programming models that have interval numbers as coefficients. The main difference between our approach and previous work on this problem is the generality and the practicality of our method. We developed a method that deals with every type of LPIC (i.e. interval coefficients in the objective function, in the constraint coefficients, in the RHS, and with variables that are sign-restricted and sign-unrestricted) which made the problem more interesting and difficult to work with.

8.1 Conclusions

- The problem stated in Chapter 4 has been solved as shown in Chapter 6.
- The main problem has been divided into 3 sub-problems:
 1. Type I LPIC (i.e. LPIC with sign-restricted variables only).

2. Type II LPIC (i.e. LPIC with at least one sign-unrestricted variable associated with fixed coefficient).

3. Type III LPIC (i.e. LPIC with at least one sign-unrestricted variable associated with interval coefficient).

- For type I and type II LPIC, Algorithm 1 (described in Section 6.2) is used as the main solution. For LPIC models involving equality constraints with interval coefficients, Theorems 6.4 and 6.5 in Section 6.1.2 are used as main tool to this type of LPIC models.
- For type III LPIC a different approach is applied, as outlined in Algorithms 3 and 4 in Section 6.2.4. The main difficulty in solving type III LPIC is the fact that a simple change of variable for all the sign-unrestricted variables associated with interval coefficients, coupled with Algorithm 1, does not work. Therefore, a new enumerated approach is employed to identify all the different orthants in which the solutions (best and worst optimum) might occur. However, this enumeration leads to a combinatorial explosion in the number of LPIC's (of type I and type II) that must be solved and analyzed before we can determine a solution for the main model. Nevertheless, since type III LPIC models are rare, and are likely to have few variables of type W_{III} the enumerative method ought to suffice in practice.

8.2 Summary of Contributions

The main contributions of the thesis are summarized below:

- Proved the minimum and maximum value range inequality theorem, which is the main tool in solving any LPIC model.
- Outlined and proved several theorems related to type III LPIC problems.
- Enhanced an existing method to solve LPIC models with restricted variables. The existing method was generalized by allowing it to handle any sign-restricted variable (non-negative or non-positive) without the use of new variables. The method was further generalized to handle sign-unrestricted variables associated with fixed coefficients (Type II LPIC).
- Developed an algorithm to handle type III LPIC models.
- Demonstrated the advantages and the practicality of using interval coefficients in linear programming models by using the solution methods developed in this thesis to solve a case study example.

8.3 Future Work

Although the main objective of this thesis was accomplished (solving linear programming models with interval coefficients), future research is needed to explore the following two points:

- Given an LPIC model of any type. Given the best optimum is feasible, and the worst optimum is infeasible, at what intermediate setting do we obtain a worst “feasible” optimum (i.e. $\underline{z}$ is a real number)? In Example 6.6. we had an infeasible worst optimum due mainly to the infeasibility of the feasible region at the worst optimum setup. However, there exists an intermediate setting at which we have a feasible region that yields a worst “feasible” optimum.
- Future research is also needed to improve the performance of the algorithms presented in Section 6.2.4. The research might concentrate on reducing the number of orthants examined, which in turn reduces the number of LPIC’s to be solved and may consequently eliminate the combinatorial explosion.

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