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Boundary Element Analysis of Interface Cracks Between Orthotropic Layers

by

Jon Erik Torrance

*A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Engineering*

**Ottawa-Carleton Institute of Mechanical and Aerospace Engineering
Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario, Canada K1S 5B6
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**Boundary Element Analysis of Interface Cracks
Between Orthotropic Layers**

submitted by

Jon Erik Torrance, B.Eng.

in partial fulfilment of the requirements
for the degree of Master of Engineering

Chair, Department of Mechanical and Aerospace Engineering

Thesis Supervisor

Carleton University

June 13, 1994

Abstract

The theory of linear elastic fracture mechanics of bimaterial interface cracks and the formulation of the boundary integral equation (BIE) method with quarter-point crack-tip elements are reviewed in this study. The concepts of the complex stress intensity factor and the phase angle are introduced. These are required for the full characterisation of the oscillatory singular near-tip stress fields of bimaterial interface cracks. A review of the formulae which may be used in evaluating the stress intensity factors directly from the BIE computed values of the tractions on the crack tip element is also presented here. The veracity and efficiency of the BIE approach undertaken are demonstrated by several test problems.

The multi-region BIE method with quarter-point crack-tip elements is then applied to the two-dimensional linear elastic fracture mechanics analysis of cracks occurring between layers in orthotropic bilayer and trilayer beams, subjected to mechanical loading. The aspect ratio of the beams is varied in the analysis, and for each beam geometry and loading condition, a range of relative crack sizes is treated. The effects of varying the mechanical properties of the orthotropic layers is also investigated. The numerical results obtained will be useful in the interfacial fracture toughness testing of orthotropic laminates using the specimens treated in this study.

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Nomenclature

- a** - Crack length
- a_{ij}** - Compliance constants from the generalized Hooke's law
- a_{ij}** - Compliance constants
- $C_{jk}(P)$** - Limiting value of the surface integral of $T_{jk}(P,Q)$
- C_{klmn}** - Stiffness constants
- c_{ij}** - Stiffness constants
- E** - Young's modulus
- G** - Strain energy release rate
- G_0** - Normalising parameter for strain energy release rates
- G_{IC}** - Critical strain energy release rate for mode I crack deformation
- H** - Layer thickness
- i** - Square root of -1, i.e. $i=\sqrt{-1}$
- K** - Complex stress intensity factor
- K_0** - Magnitude of K
- K_0** - Normalising parameter for stress intensity factors
- K_1** - Real part of K
- K_2** - Imaginary part of K
- K_I** - Mode I stress intensity factor
- K_{II}** - Mode II stress intensity factor
- K_{IC}** - Critical stress intensity factor for mode I crack deformation

- h - Characteristic length for the phase angle ψ
- $J(\zeta)$ - Jacobian of transformation
- L - Characteristic length
- l - Crack-tip element length
- M - Total number of boundary elements
- N - Total number of boundary nodes
- $N^e(\zeta)$ - Shape functions
- n_j - Components of the unit outward normal
- P - A point on the boundary
- P - Mechanical load
- P^a - a-th node of the boundary element mesh
- $P^{a(b,c)}$ - c-th node of the b-th element of the boundary element mesh
- Q - General boundary point
- r - Distance away from the crack-tip; a polar coordinate.
- r - Distance between P and Q
- S - Boundary of domain
- S_b - Portion of S represented by the b-th element
- S_k - Roots of the characteristic equation for a material
- $T_{jk}(P,Q)$ - The fundamental solution for tractions
- t_i - Traction
- $t_{i(e)}$ - Nodal tractions
- $U_{jk}(P,Q)$ - The fundamental solution for displacements

u_i	- Displacements
W	- Length of a specimen
x_i	- Cartesian coordinate directions
α, β	- Dundur's parameters
β	- Parameter characterising an anisotropic bimaterial interface
γ	- Bimaterial constant for anisotropic bimaterials
δ_{jk}	- Kronecker delta
e	- Bimaterial constant for isotropic bimaterials
e_{ij}	- Strain components
ζ	- A local or intrinsic coordinate
η_1, η_2	- Orthotropic parameters
η_{ijk}	- Coefficient of mutual influence of the first kind
η_{ljk}	- Coefficient of mutual influence of the second kind
θ	- A polar coordinate
μ, μ_{ij}	- Shear moduli
μ_{ijk}	- Chentsov coefficients
ν, ν_{ij}	- Poisson's ratios
$\sigma_{ij}, \sigma_{\alpha\beta}$	- Stress components
ϕ	- Angle between two coordinate systems
ψ	- Phase angle of K for bimaterial interface crack

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Chapter 1

Introduction

Advanced composite materials are increasingly being used in structural applications because of their higher strength-to-weight ratios compared to homogeneous materials. Composite laminates, produced by stacking and bonding together layers of orthotropic fibre-reinforced laminae, are a common form in which they are used. These laminates offer the possibility of tailoring the directional properties of the material to suit the particular engineering application. They are, however, susceptible to failure through the propagation of delamination cracks between layers.

The mechanical behaviour of composite materials may be studied from a micromechanics viewpoint or a macromechanics viewpoint. In the former approach, the details of the matrix and the reinforcing inclusions must be considered; whereas in the latter approach, the material is considered to be homogeneous and continuous, with material properties related to those of both the matrix and the inclusions. From an applied engineering standpoint, the macromechanics approach is the more appropriate one to adopt. This is particularly so if the global displacements and stresses are to be determined under external loads.

The fracture analysis of problems involving cracks situated at layer interfaces in composite laminates is of practical importance because cracks in these laminates often propagate just along the interface between the layers, due to the relative weakness of the interface compared to the layer materials. The fracture analysis of these interface cracks therefore helps to provide an understanding of the factors that affect the structural integrity of composite laminate structural members.

The mechanics of cracks with crack-tips located in homogeneous materials are well understood, with the near-tip stress and displacement fields under the assumptions of linear elastic fracture mechanics being well known. The opening mode and shear mode stress intensity factors, and the strain energy release rate are well established as fracture characterising parameters for cracks in homogeneous bodies. For bimaterial interface cracks, however, there is an oscillatory singularity in the stresses as the crack-tip is approached, and an inherent coupling of tensile and shear effects. It is therefore no longer possible to define classical stress intensity factors for the opening and shear modes of fracture separately. In addition, the oscillation of the stress and displacement fields imply interpenetration of the crack surfaces. This physically inadmissible result is due to the contact conditions of the crack surfaces not being considered in the analysis. However, the contact region has been demonstrated to be sufficiently small, for a wide range of loading, that the complex stress intensity factor can still be used as a fracture characterising parameter, see Rice (1988) and Wang and Suo (1990). The strain energy release rate, however, is not oscillatory in form, and can therefore still be used as a

fracture characterising parameter without modifying its interpretation. The foregoing discussion applies equally to isotropic and anisotropic bimaterial interface cracks, although the stress and displacement fields are more complex in form for the latter case.

Analytical solutions for cracks between dissimilar materials are available for only a few cases with relatively simple geometries and load conditions. For dissimilar anisotropic materials, such solutions are even more scarce. It is therefore normally necessary to resort to numerical methods, or to experimental techniques, when dealing with problems involving interface cracks between dissimilar materials. The finite element method (FEM) and the boundary integral equation (BIE) method (also called the boundary element method (BEM)) are the numerical methods most commonly employed for treating crack problems. The BIE method, in particular, has been found to be an efficient tool for the fracture analysis of cracks, whether located in homogeneous bodies or at the interface between dissimilar isotropic materials. The procedures for determining the fracture characterising parameters from BIE computed results for displacements and tractions in the homogeneous case are well known. The treatment of bimaterial interface crack problems using the BIE method is more difficult due to the oscillatory singularity in stress near the crack tip. Contributions in this area include the works of Lee and Choi (1988), Yuuki et al. (1987), Tan and Gao (1990a,b, 1991) and Tan, Gao and Afagh (1992). Of these, only Tan, Gao and Afagh (1992) dealt with dissimilar anisotropic materials. The authors derived analytical relations for the evaluation of the fracture characterising parameters directly from the BIE computed results, and demonstrated that

accurate numerical results could be obtained using this approach, even with relatively coarse mesh discretization.

In the current study, the conventional multi-region BIE method with quarter-point crack-tip elements, following Tan, Gao and Afagh (1992), is used for the two-dimensional analysis of cracks occurring along the interface between layers in orthotropic bilayer and trilayer beams. These configurations are frequently used in the fracture toughness testing of composite materials (see, e.g. Hashemi et al. (1990), Bao et al. (1992), or Ozdil and Carlsson (1992)). For each of these cases, two different mechanical load cases are treated for a range of overall beam dimensions and crack lengths. The effects of varying the mechanical properties of the orthotropic composite are also investigated. The primary objective here is to obtain the fracture characterising parameters for these geometries, load conditions and material property combinations. Before the BIE analysis of these problems is discussed, in Chapters 4 and 5, respectively, however, the theory of linear elastic fracture mechanics of bimaterial interface cracks will first be reviewed in the next Chapter. This is followed by a review of the formulation of the BIE method for fracture mechanics of anisotropic materials, including the treatment of interface cracks between dissimilar anisotropic regions.

Chapter 2

Fracture Mechanics of Bimaterial Interface Cracks

2.0 Introduction

The study of interface cracks between dissimilar elastic media is important in obtaining an understanding of the integrity of bonded interfaces between dissimilar materials. It is, for example, important in determining the factors which influence the mechanical properties of composites and composite laminates. The near-tip stress and displacement fields for cracks in homogeneous materials are well known and their characterization in terms of the stress intensity factors, K_I and K_{II} , has been well established. However, the near-tip fields for bimaterial cracks are less well understood, although there has been much recent work done in this area, spurred by the increasing use of composite materials in structural applications. Differences from the homogeneous case include an inherent coupling between tensile and shear effects which prevent the stress intensity factors from being interpreted separately. There is also the difficulty that for bimaterials, the stress singularity near the interface crack is oscillatory in nature, which gives rise to some numerical difficulties. This chapter will present a review of anisotropic elasticity, and of the mechanics of cracks in homogeneous materials and at bimaterial interfaces.

2.1 Review of Anisotropic Elasticity

An anisotropic body is one in which the elastic properties are different for different directions passing through the same point. In a Cartesian co-ordinate system the stresses acting on each plane normal to a co-ordinate axis can be resolved into three components; one normal stress and two shear stresses as shown in Figure 2.1. There are therefore nine components required to express the state of stress at a point in an elastic body. These are

$$\sigma_{ij}, \quad i,j=1,3$$

Considerations of equilibrium of an infinitesimal element require that

$$\sigma_{ij} = \sigma_{ji}$$

so that there are six independent stress components, namely three direct stresses and three shear stresses.

Corresponding to these are six strain components

$$\epsilon_{ij}, \quad i,j=1,3$$

$$\text{with} \quad \epsilon_{ij} = \epsilon_{ji}$$

Assuming small strains which vary linearly with the applied stresses, the relation between them is expressed in the generalized Hooke's law. This is often written in the form

$$\begin{aligned}
 \epsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} + a_{14}\sigma_{23} + a_{15}\sigma_{13} + a_{16}\sigma_{12} \\
 \epsilon_{22} &= a_{21}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} + a_{24}\sigma_{23} + a_{25}\sigma_{13} + a_{26}\sigma_{12} \\
 \epsilon_{33} &= a_{31}\sigma_{11} + a_{32}\sigma_{22} + a_{33}\sigma_{33} + a_{34}\sigma_{23} + a_{35}\sigma_{13} + a_{36}\sigma_{12} \\
 \epsilon_{23} &= a_{41}\sigma_{11} + a_{42}\sigma_{22} + a_{43}\sigma_{33} + a_{44}\sigma_{23} + a_{45}\sigma_{13} + a_{46}\sigma_{12} \\
 \epsilon_{13} &= a_{51}\sigma_{11} + a_{52}\sigma_{22} + a_{53}\sigma_{33} + a_{54}\sigma_{23} + a_{55}\sigma_{13} + a_{56}\sigma_{12} \\
 \epsilon_{12} &= a_{61}\sigma_{11} + a_{62}\sigma_{22} + a_{63}\sigma_{33} + a_{64}\sigma_{23} + a_{65}\sigma_{13} + a_{66}\sigma_{12}
 \end{aligned}$$

(2.1)

where the coefficients a_{ij} are referred to as compliance constants.

It is also possible to write the generalized Hooke's law in an equivalent form

$$\begin{aligned}
 \sigma_{11} &= c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33} + c_{14}\epsilon_{23} + c_{15}\epsilon_{13} + c_{16}\epsilon_{12} \\
 \sigma_{22} &= c_{21}\epsilon_{11} + c_{22}\epsilon_{22} + c_{23}\epsilon_{33} + c_{24}\epsilon_{23} + c_{25}\epsilon_{13} + c_{26}\epsilon_{12} \\
 \sigma_{33} &= c_{31}\epsilon_{11} + c_{32}\epsilon_{22} + c_{33}\epsilon_{33} + c_{34}\epsilon_{23} + c_{35}\epsilon_{13} + c_{36}\epsilon_{12} \\
 \sigma_{23} &= c_{41}\epsilon_{11} + c_{42}\epsilon_{22} + c_{43}\epsilon_{33} + c_{44}\epsilon_{23} + c_{45}\epsilon_{13} + c_{46}\epsilon_{12} \\
 \sigma_{13} &= c_{51}\epsilon_{11} + c_{52}\epsilon_{22} + c_{53}\epsilon_{33} + c_{54}\epsilon_{23} + c_{55}\epsilon_{13} + c_{56}\epsilon_{12} \\
 \sigma_{12} &= c_{61}\epsilon_{11} + c_{62}\epsilon_{22} + c_{63}\epsilon_{33} + c_{64}\epsilon_{23} + c_{65}\epsilon_{13} + c_{66}\epsilon_{12}
 \end{aligned}$$

(2.2)

where the coefficients c_{ij} are referred to as stiffness constants.

Both versions of the law contain 36 constants but it can be demonstrated that these matrices are symmetric i.e. $a_{ij} = a_{ji}$ and $c_{ij} = c_{ji}$. Thus the number of independent constants is reduced to 21.

The compliances can be expressed in terms of the more commonly used engineering material constants as follows, see Lekhnitskii (1963).

$$\begin{aligned}
 a_{11} &= \frac{1}{E_1} & a_{12} &= a_{21} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \\
 a_{22} &= \frac{1}{E_2} & a_{23} &= a_{32} = -\frac{\nu_{23}}{E_2} = -\frac{\nu_{32}}{E_3} \\
 a_{33} &= \frac{1}{E_3} & a_{13} &= a_{31} = -\frac{\nu_{13}}{E_1} = -\frac{\nu_{31}}{E_3} \\
 a_{44} &= \frac{1}{\mu_{23}} & a_{45} &= a_{54} = \frac{\mu_{31,23}}{\mu_{23}} = \frac{\mu_{23,31}}{\mu_{13}} \\
 a_{55} &= \frac{1}{\mu_{13}} & a_{56} &= a_{65} = \frac{\mu_{12,31}}{\mu_{13}} = \frac{\mu_{31,12}}{\mu_{12}} \\
 a_{66} &= \frac{1}{\mu_{12}} & a_{46} &= a_{64} = \frac{\mu_{12,23}}{\mu_{23}} = \frac{\mu_{23,12}}{\mu_{12}} \\
 & & a_{14} &= a_{41} = \frac{\eta_{23,1}}{E_1} = \frac{\eta_{1,23}}{\mu_{12}} \\
 & & & \text{etc.}
 \end{aligned}$$

(2.3)

where

E_1 is the Young's modulus in the direction parallel to the x_1 -axis (similarly for E_2 and E_3);

μ_{23} is the shear modulus in the plane perpendicular to the x_1 -axis (similarly for μ_{13} and μ_{12});

ν_{12} is the Poisson's ratio in the x_1 - x_2 -plane (similarly for ν_{23} and ν_{13});

$\mu_{12,23}$ is the Chentsov coefficient expressing the coupling between shear stresses in the x_2 - x_3 -plane and shear strains in the x_1 - x_2 -plane (Other Chentsov coefficients exist for the other pairings of planes); and

$\eta_{12,3}$ is the coefficient of mutual influence of the first kind and $\eta_{1,23}$ is the coefficient of mutual influence of the second kind. They express the coupling between shear stresses and direct strains and between direct stresses and shear strains respectively. The subscripts indicate the plane of the shear stress or strain and the direction of the direct stress or strain as applicable.

Note that the a_{ij} values not explicitly given have the same form as those of a_{14} and a_{41} with the appropriate changes in the subscripts of the coefficients of mutual influence, Young's moduli and shear moduli which correspond to the proper axis directions.

For many materials, some of the compliances will be zero depending on the type of elastic symmetry the material possesses. Some materials have one plane of elastic symmetry i.e. any two directions passing through a point in the body that are symmetrical with respect to this plane have equivalent elastic properties. For these materials it can be shown that the number of non-zero coefficients in the Hooke's law will be 20, 13 which will be independent as shown below, see Vinson and Sierakowski (1986).

$$\begin{aligned}
 \epsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} + 0 + 0 + a_{16}\sigma_{12} \\
 \epsilon_{22} &= a_{21}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} + 0 + 0 + a_{26}\sigma_{12} \\
 \epsilon_{33} &= a_{31}\sigma_{11} + a_{32}\sigma_{22} + a_{33}\sigma_{33} + 0 + 0 + a_{36}\sigma_{12} \\
 \epsilon_{23} &= 0 + 0 + 0 + a_{44}\sigma_{23} + a_{45}\sigma_{13} + 0 \\
 \epsilon_{13} &= 0 + 0 + 0 + a_{54}\sigma_{23} + a_{55}\sigma_{13} + 0 \\
 \epsilon_{12} &= a_{61}\sigma_{11} + a_{62}\sigma_{22} + a_{63}\sigma_{33} + 0 + 0 + a_{66}\sigma_{12}
 \end{aligned}$$

(2.4)

An orthotropic material possesses three mutually orthogonal planes of elastic symmetry at any point. If the co-ordinate axes are normal to these three planes, Hooke's law can then be expressed as

$$\begin{aligned}
 \epsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} + 0 + 0 + 0 \\
 \epsilon_{22} &= a_{21}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} + 0 + 0 + 0 \\
 \epsilon_{33} &= a_{31}\sigma_{11} + a_{32}\sigma_{22} + a_{33}\sigma_{33} + 0 + 0 + 0 \\
 \epsilon_{23} &= 0 + 0 + 0 + a_{44}\sigma_{23} + 0 + 0 \\
 \epsilon_{13} &= 0 + 0 + 0 + 0 + a_{55}\sigma_{13} + 0 \\
 \epsilon_{12} &= 0 + 0 + 0 + 0 + 0 + a_{66}\sigma_{12}
 \end{aligned}$$

(2.5)

Therefore, only 12 elastic constants, of which 9 are independent, are required for orthotropic materials.

Transversely isotropic materials possess an axis of rotational symmetry i.e. all directions perpendicular to this axis are elastically equivalent. Hooke's law has the same form as for orthotropic materials but $a_{11} = a_{22}$, $a_{13} = a_{23}$, $a_{44} = a_{55}$ and $a_{66} = 2(a_{11} - a_{12})$ so that only 5 of the 12 elastic constants are independent. In isotropic materials all directions through a point are elastically equivalent and only two independent elastic constants are necessary to describe the material. The generalized Hooke's law will have the familiar form.

$$\begin{aligned}
 \epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] & \epsilon_{12} &= \frac{1}{\mu}\sigma_{12} \\
 \epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] & \epsilon_{23} &= \frac{1}{\mu}\sigma_{23} \\
 \epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] & \epsilon_{13} &= \frac{1}{\mu}\sigma_{13}
 \end{aligned}$$

(2.6)

Only two of E , μ and ν are therefore required to fully describe the properties of an isotropic material, since $\mu = E/2(1 + \nu)$.

The three-dimensional form of the generalized Hooke's law is not needed for many problems of interest. In two dimensional or plane problems it can be expressed in a 3×3 matrix. For a homogeneous, fully anisotropic, elastic material under conditions of plane stress, this matrix can be written

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

(2.7)

where the elastic compliances can be expressed in terms of engineering material constants as follows

$$\begin{aligned} a_{11} &= \frac{1}{E_1} & ; & & a_{16} &= \frac{\eta_{12,1}}{E_1} = \frac{\eta_{1,12}}{\mu_{12}} \\ a_{22} &= \frac{1}{E_2} & ; & & a_{26} &= \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{\mu_{12}} \\ a_{12} &= -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} & ; & & a_{66} &= \frac{1}{\mu_{12}} \end{aligned}$$

(2.8)

Equation (2.7) is also applicable to conditions of plane strain if the elastic compliances a_{ij} are replaced with b_{ij} (Sih and Liebowitz (1968)) where

$$b_{ij} = a_{ij} - a_{i3}a_{j3}/a_{33} \quad ; \quad i, j = 1, 2 \quad (2.9)$$

and where

$$\begin{aligned} a_{i3} &= -\nu_{i3}/E_i = -\nu_{3i}/E_3 \quad ; \\ a_{33} &= 1/E_3 \quad ; \\ a_{63} &= \eta_{12,3}/E_3 = \tau_{i3,12}/\mu_{12} \end{aligned} \quad (2.10)$$

The index 3 above refers to the out-of-plane direction.

2.2 Cracks in Homogeneous Bodies

In a homogeneous body, the material properties are identical at all points inside the body. Composite materials do not conform to this definition when scales comparable to the size of the constituent materials are considered, but they can be considered to be homogeneous when the scale of a problem is such as to reduce the constituents to insignificance by comparison.

The mechanics of cracks in homogeneous isotropic bodies are well developed and are described in a wide body of literature, see e.g. Sih and Liebowitz (1972) or Irwin (1958). The near-tip stress fields of such a crack have the form

$$\sigma_{ij} = K_I(2\pi r)^{-1/2}\sigma'_{ij}(\theta) + K_{II}(2\pi r)^{-1/2}\sigma''_{ij}(\theta) \quad (2.11)$$

in both plane stress and plane strain. r and θ are the polar coordinates centred at the crack tip, as shown in Figure 2.2.; and K_I and K_{II} are the Mode I and Mode II stress intensity factors. The functions, $\sigma'_{ij}(\theta)$ and $\sigma''_{ij}(\theta)$, expressing the θ variance are

$$\begin{aligned} \sigma'_{11} &= \cos\frac{\theta}{2}(1 - \sin\frac{\theta}{2}\sin\frac{3}{2}\theta) ; & \sigma''_{11} &= \sin\frac{\theta}{2}(2 + \cos\frac{\theta}{2}\cos\frac{3}{2}\theta) ; \\ \sigma'_{22} &= \cos\frac{\theta}{2}(1 + \sin\frac{\theta}{2}\sin\frac{3}{2}\theta) ; & \sigma''_{22} &= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3}{2}\theta ; \\ \sigma'_{12} &= \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3}{2}\theta ; & \sigma''_{12} &= \cos\frac{\theta}{2}(1 - \sin\frac{\theta}{2}\sin\frac{3}{2}\theta) \end{aligned} \quad (2.12)$$

Thus the singular tractions ahead of the crack tip are

$$\sigma_{22} = K_I(2\pi r)^{-1/2} ; \quad \sigma_{12} = K_{II}(2\pi r)^{-1/2} \quad (2.13)$$

and the displacements, u_i , of the crack faces relative to the crack tip in the area dominated by the singular crack tip fields are given by

$$\begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} K_I \\ K_{II} \end{bmatrix} \times (8/\bar{E})[r/(2\pi)]^{1/2} \quad (2.14)$$

where

$$\begin{aligned} \bar{E} &= E/(1 - \nu^2) & (\text{plane strain}) \\ \bar{E} &= E & (\text{plane stress}) \end{aligned} \quad (2.15)$$

and E is Young's modulus.

Also, the energy release rate, G , for stable crack advance along $\theta=0$ is

$$G = (K_I^2 + K_{II}^2)/\bar{E} \quad (2.16)$$

It is well documented that in isotropic materials a crack will advance at an angle such that it will experience only Mode I loading along the $\theta=0$ line at the crack tip. It is therefore not usually necessary to consider the effect of mixed mode loading in isotropic materials except when establishing the path which the crack growth will follow.

Crack advance will initiate when the strain energy release rate reaches a critical value. This value is related to the fracture toughness of the material. For mode I loading, this means

$$G = G_{IC} = \frac{K_{IC}^2}{E} \quad (2.17)$$

where K_{IC} is the mode I fracture toughness and is assumed invariant with the direction for isotropic materials.

Fracture mechanics methodology is equally applicable to homogeneous anisotropic materials but certain complications arise. These are due to the fact that crack propagation is often confined to specific weak directions in composites. It is therefore necessary to consider the effect of mixed mode loading at the crack tip and of material orientation.

Many composites fall into the category of orthotropic materials. Two additional non-dimensional elastic parameters are required to characterise the stresses in an orthotropic material in which the crack is aligned with one of the axes of orthotropy, Suo (1990a). For plane stress, these are

$$\lambda = a_{11}/a_{22} \quad ; \quad \rho = (a_{12} + a_{66}/2)/(a_{11}a_{22})^{1/2} \quad (2.18)$$

where the a_{ij} 's are the elastic compliances for plane stress. For plane strain conditions the

coefficients b_{ij} in Equation (2.9) may be substituted.

The stress fields near the crack tip will have the same form as previously but the functions σ^I_{ij} and σ^II_{ij} now depend on λ and ρ as well as on r and θ . These stresses have been given explicitly by Sih et al. (1965), and they will be shown later in Equation (2.36).

The crack face displacements will now be given by

$$\begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} \lambda^{-3/4} K_I \\ \lambda^{-1/4} K_{II} \end{bmatrix} \times 8na_{11}[r/(2\pi)]^{1/2} \quad (2.19)$$

where

$$n = [(1 + \rho)/2]^{1/2} \quad (2.20)$$

and the energy release rate for straight ahead crack advance in the plane of the original crack will be, see, e.g. Sih et al (1965),

$$G = a_{11}n(\lambda^{-3/4}K_I^2 + \lambda^{-1/4}K_{II}^2) \quad (2.21)$$

2.3 Interface Cracks between Dissimilar Isotropic Media

The analysis of a crack lying at a bimaterial interface presents the added complication of an inherent mode mixity. Pure mode I loading will cause both mode I and mode II to exist at the crack tip. Pure mode II loading will result in both mode II and mode I stress components at the crack tip. In addition, the fracture toughness of the interface will not generally be a constant but will depend on the mode mixity.

Consider a case where both materials are homogeneous and isotropic with material 1 above the x_1 -axis, material 2 below it and the crack lying along the negative x_1 -axis so that the origin is at the crack tip (See Figure 2.3). Let

$$\begin{aligned}\kappa_j &= 3 - 4\nu_j && \text{for plane strain} \\ \kappa_j &= (3 - \nu_j)/(1 + \nu_j) && \text{for plane stress} \\ j &= 1, 2\end{aligned}$$

(2.22)

Dundurs' elastic mismatch parameters α and β can now be defined as, see Dundurs (1969)

$$\alpha = \frac{\mu_1(\kappa_2 + 1) - \mu_2(\kappa_1 + 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}$$

(2.23)

$$\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \quad (2.24)$$

The expression for α can also be written as follows

$$\alpha = (E_1 - E_2)/(E_1 + E_2) \quad (2.25)$$

It is evident that α quantifies the mismatch in tensile properties across the interface. Similarly, β measures the mismatch in the in-plane bulk modulus across the interface. Note that both α and β are zero when no mismatch of the material properties exists. α approaches 1 if material 1 is much stiffer than material 2 and -1 if it is much more compliant.

The dominant stress singularity for any plane problem having traction free crack surfaces is of the form, see Rice (1988),

$$\sigma_{\alpha\beta} = \text{Re}[Kr^{i\epsilon}](2\pi r)^{-1/2}\sigma'_{\alpha\beta}(\theta, \epsilon) + \text{Im}[Kr^{i\epsilon}](2\pi r)^{-1/2}\sigma''_{\alpha\beta}(\theta, \epsilon) \quad (2.26)$$

where $i = \sqrt{-1}$, r and θ are polar co-ordinates centred at the crack tip; Re and Im represent the real and imaginary parts, respectively; and

$$\epsilon = \frac{1}{2\pi} \ln \left(\frac{1 - \beta}{1 + \beta} \right) \quad (2.27)$$

The explicit forms of $\sigma_{\alpha\beta}^I$ and $\sigma_{\alpha\beta}^{II}$ are relatively complicated and have been given by Sun and Jih (1987) or Rice et al. (1990).

Note that the near-tip stress field of the bimaterial interface crack is now characterized by a complex stress intensity factor K . The complex stress intensity factor $K = K_1 + iK_2$ has real and imaginary parts K_1 and K_2 which are similar to the mode I and mode II stress intensity factors of conventional fracture mechanics. Ahead of the crack tip the tractions are given by Hutchinson et al. (1987) as

$$\sigma_{22} + i\sigma_{12} = (K_1 + iK_2)(2\pi r)^{-1/2} r^{ie} \quad (2.28)$$

where $r^{ie} = \cos(e \ln r) + i \sin(e \ln r)$. The stresses near the crack tip therefore have an oscillatory singularity as r approaches zero. It is also evident from Equation (2.28) that the tensile and shear effects in the vicinity of the crack-tip are coupled and that K_1 and K_2 do not represent the opening and shear modes of fracture as do K_I and K_{II} in conventional fracture mechanics.

The crack face displacements can be shown to be given by, see e.g Rice (1988)

$$u_2 + iu_1 = \frac{8}{(1 + 2ie) \cosh(\pi e)} \frac{(K_1 + iK_2)}{E_*} \left(\frac{r}{2\pi}\right)^{1/2} r^{ie} \quad (2.29)$$

where

$$\frac{i}{E_*} = \frac{1}{2} \left(\frac{i}{E_1} + \frac{1}{E_2} \right) \quad (2.30)$$

and the energy release rate for crack advance in the interface is

$$G = \frac{(1 - \beta^2)}{E_*} (K_1^2 + K_2^2) \quad (2.31)$$

It should be noted that when $\beta=0$, K_1 and K_2 become decoupled and they govern the amplitude of the mode I and mode II stress singularities respectively. Also, when $\beta \neq 0$ it is necessary to define a measure of the mode mixity, ψ , such that

$$\psi = \tan^{-1} \left[\frac{\text{Im}(Kh^{1/2})}{\text{Re}(Kh^{1/2})} \right] \quad (2.32)$$

K is the complex stress intensity factor as before and h is a reference length which can be arbitrarily chosen within certain limits. It is necessary to specify h because the ratio of normal traction to shear traction will vary slowly with distance to the crack tip when $\beta \neq 0$.

The choice of h is somewhat arbitrary due to the simplicity of the rule for transformation between ψ_1 based on h_1 and ψ_2 based on h_2 , given below

$$\psi_2 = \psi_1 + \epsilon \ln(h_2/h_1) \quad (2.33)$$

The magnitude or modulus of K , namely K_σ , which equals $(K_1^2 + K_2^2)^{1/2}$, remains constant during the transformation. Note that for small values of ϵ the change in ψ will be quite small. A choice of h based on a material length scale is useful for presentation of fracture data while a choice based on some in-plane length of the specimen geometry is better for discussion of the mixed mode character of a bimaterial crack solution (See, e.g. Suo and Hutchinson (1992)). A similar transformation for fracture toughness data is possible with

$$\Gamma(\psi_2, h_2) = \Gamma(\psi_1 + \epsilon \ln(h_2/h_1), h_1) \quad (2.34)$$

The criterion for advance of an interface crack is

$$G = \Gamma(\psi, h) \quad (2.35)$$

An interesting feature of bimaterial interface fracture mechanics presented above is that crack face interpenetration is predicted by Equation (2.29). This is clearly a physically inadmissible result. Fortunately, the size of the predicted contact zone has been shown, see Rice (1988) or Wang and Suo (1990), to be small enough compared to other relevant length scales for most loading situations that this does not significantly compromise the accuracy of the solutions obtained.

2.4 Interface Cracks between Dissimilar Anisotropic Media

The foregoing discussion concerns only interfaces between dissimilar isotropic materials. There has, however, also been some work done to extend it to anisotropic and orthotropic materials. Much of this work consisted of experimental investigations of the fracture toughness of various interfaces under mixed mode loading, and calibration of specimens for carrying out such investigations, see e.g. Suo and Hutchinson (1989), Charalambides et al. (1989), Wang and Suo (1990), O.Dowd et al. (1991). Numerical methods have also been applied by several researchers, see e.g. Sun and Manoharan (1989), Lin and Hartmann (1989), Tan, Gao and Afagh (1992), Cho et al. (1992), Carlsson and Prasad (1993), with the finite element method (FEM) predominantly employed. Analytical work in this area has also been undertaken by a few researchers, see e.g. Bassani and Qu (1989a,b), Nakagawa et al. (1990), Wu (1990), Suo (1990), Ni and Nemat-Nasser (1991), Wu (1991), Suo et al. (1991), Bao et al. (1991) and Erdogan and Wu (1993). The results of some of this analytical work are exploited by the boundary element analysis used in the present study.

In the present work, the characterisation of the near tip fields of an interface crack between dissimilar anisotropic materials is based on the use of a positive definite Hermitian matrix. The entries in this matrix are functions of the elastic constants of the two materials. This approach incorporates the proper $r^{-1/2+i\gamma}$ order singularity of stress (γ is a bimaterial constant defined in Equation (3.18) in Chapter 3). Some aspects of this

approach are described in Chapter 3, in the discussion of how the stress intensity factors are obtained. Of importance to note at this point is that the fracture characterising parameters obtained are the same as for the case of isotropic bimaterial interfaces and they have essentially the same physical meanings. The complex stress intensity factor and phase angle are still used and they have the same interpretations as presented thus far. The need for a characteristic dimension of the problem in defining K still applies as does the rule for transforming values based on one characterising dimension to being based on another. The difference lies in the relationship between the displacement and stress fields and K being more complicated due to the material anisotropy.

For a crack in a generally anisotropic plane with a Cartesian (x_1 - x_2) and a polar (r - θ) coordinate system both having their origin at the crack tip (see Figure 2.4) the near-tip stress and displacement fields have been presented by Sih and Liebowitz (1968). They have the form

Stresses:

$$\begin{aligned} \sigma_{11} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left\{ \frac{s_2}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_1}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right] \\ & + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_2^2}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_1^2}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right], \end{aligned}$$

$$\begin{aligned}
\sigma_{22} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_2}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_1}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right] \\
& + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{1}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{1}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right] \\
\sigma_{12} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left\{ \frac{1}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{1}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right] \\
& + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_1}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_2}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right\} \right].
\end{aligned}
\tag{2.36}$$

Displacements:

$$\begin{aligned}
u_1 = & K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \{ s_1 p_2 (\cos\theta + s_2 \sin\theta)^{1/2} - s_2 p_1 (\cos\theta + s_1 \sin\theta)^{1/2} \} \right] \\
& + K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \{ p_2 (\cos\theta + s_2 \sin\theta)^{1/2} - p_1 (\cos\theta + s_1 \sin\theta)^{1/2} \} \right].
\end{aligned}$$

$$\begin{aligned}
u_2 = & K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \{ s_1 q_2 (\cos \theta + s_2 \sin \theta)^{1/2} - s_2 q_1 (\cos \theta + s_1 \sin \theta)^{1/2} \} \right] \\
& + K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \{ q_2 (\cos \theta + s_2 \sin \theta)^{1/2} - q_1 (\cos \theta + s_1 \sin \theta)^{1/2} \} \right],
\end{aligned}
\tag{2.37}$$

where

$$p_i = a_{11}^* s_i^2 + a_{12}^* - a_{16}^* s_i ; \quad q_i = \frac{a_{12}^* s_i^2 + a_{22}^* - a_{26}^*}{s_i} \quad i = 1, 2
\tag{2.38}$$

The a_{ij}^* 's are the compliance constants in the local co-ordinate system. If the principal material axes and crack are oriented such that these do not correspond to the (x_1-x_2) Cartesian coordinate system, a further transformation of the compliance constants is necessary, see Lekhnitskii (1963).

Also, in Equations (2.36)-(2.38), s_1 and s_2 are material property parameters with respect to the local (x_1-x_2) co-ordinate system. They are related to the roots S_1 and S_2 of the characteristic equation for the material, namely

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0
\tag{2.39}$$

These roots have been shown by Lekhnitskii to be either complex or purely imaginary.

The relationship between s_k and S_k can be shown to be

$$s_k = \frac{S_k \cos \phi + \sin \phi}{\cos \phi - S_k \sin \phi}; \quad k = 1, 2 \quad (2.40)$$

where ϕ is the angle between the co-ordinate system in which the compliance constants, a_{ij}^* , apply and the (x_1-x_2) coordinate system as shown in Figure 2.4.

2.5 Closing remarks

In this chapter, the basic equations for anisotropic elasticity and for the near-tip stress and displacement fields for cracks in homogeneous bodies and between dissimilar elastic materials have been reviewed. It has also been noted that the near-tip stress fields in the latter case are oscillatory singular and the complex stress intensity factor K was introduced as the fracture characterising parameter. It is also of importance to note that for cracks lying between dissimilar materials, the stress intensity factors K_I and K_{II} do not represent the opening and sliding modes of crack deformation as when they lie within a homogeneous medium (for this reason, the symbols K_1 and K_2 are often used instead for interfacial cracks). From dimensional considerations, these stress intensity factors alone are inadequate to properly characterise the near-tip fields and a reference length which is characteristic of the bimaterial interface crack problem must now also be included in the presentation of the stress intensity factor solutions.

Chapter 3

The Boundary Integral Equation Method in Fracture Mechanics of Anisotropic Materials

3.0 Introduction

The increased use of advanced composite materials has recently led to greater interest in solutions to problems in anisotropic elasticity. Problems involving the presence of stress concentrations and cracks are of particular interest since these can seriously affect the behaviour of composite structural members. With the exception of some very simple cases, there are few analytical solutions available for cracks or cutouts in anisotropic elastic bodies. Solutions to problems involving interface cracks between dissimilar anisotropic materials are even more scarce. It is therefore normally necessary to resort to numerical methods in treating these problems.

The finite element method (FEM) has hitherto been used almost exclusively in numerical studies of interface cracks. However, the boundary integral equation (BIE) method, also called the boundary element method (BEM), is an efficient alternative numerical method. It is particularly well suited to problems involving rapidly varying stress gradients such as those which occur near cracks. However, only very few numerical studies of problems

in anisotropic elasticity using the BIE method have been reported in the literature.

This chapter will first review the theoretical basis for the BIE method in anisotropic elasticity and its numerical solution. Its application to fracture mechanics will then be presented. Some features of the approach used in this study will be described and the results of a number of example problems are shown and discussed.

3.1 The BIE for Anisotropic Elasticity

Application of Airy's stress functions shows, see Lekhnitskii (1963), that the characteristic equation for an anisotropic material in stable equilibrium in two dimensions is

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0 \quad (3.1)$$

The four roots of this equation will be distinct and complex provided that the material involved is not isotropic. The roots may be written

$$\begin{aligned} S_j &= \alpha_j + i\beta_j \\ \bar{S}_j &= \alpha_j - i\beta_j ; \quad j = 1, 2 \end{aligned} \quad (3.2)$$

The characteristic directions of the material may be denoted by

$$z_j = x_1 + S_j x_2 ; \quad j = 1, 2 \quad (3.3)$$

and their complex conjugates.

The development of the BIE for plane anisotropy may now be discussed. The steps are the same as for isotropic elasticity, see Cruse (1977), Banerjee and Butterfield (1981). The derivation of the BIE is based on unit load solutions in an infinite body, known as the fundamental solutions. These will be different from those for isotropic elasticity. Application of the Betti-Rayleigh reciprocal work theorem with these fundamental solutions and carrying out appropriate mathematical limiting operations results in the BIE. The differential equations which are applicable throughout the solution domain are transformed into integral equations which are written for just its boundary.

In essence, the BIE relates the displacements u_i and tractions t_i at the boundary S of a homogeneous elastic domain, see Figure 3.1. Using indicial notation, it can be written as, see, for example, Jia et al. (1988) , Martinez and Dominguez (1984) or Tan and Gao (1992),

$$C_{jk}(P)u_k(P) + \int_S T_{jk}(P, Q)u_k(Q)dS(Q) = \int_S U_{jk}(P, Q)t_k(Q)dS(Q) ;$$

$$j, k = 1, 2 \quad (3.4)$$

The value of the function $C_{jk}(P)$ depends on the local geometry of the boundary, S , at P , a point on that boundary. Also, $U_{jk}(P,Q)$ and $T_{jk}(P,Q)$ are the fundamental solutions for the displacements and tractions respectively. They represent the value of the displacement or traction in the x_j -direction at Q due to a line load of unit intensity in the x_k -direction at P in a plane homogeneous infinite body. The derivation of the fundamental solutions for anisotropic elasticity has been described in detail by Cruse (1988) and as such it will not be presented here. Their explicit forms are

$$U_{jk} = 2\text{Re}[r_{kj}A_{j1}\ln(z_1) + r_{k2}A_{j2}\ln(z_2)] \quad (3.5a)$$

$$T_{j1} = 2n_1\text{Re}[S_1^2A_{j1}/z_1 + S_2^2A_{j2}/z_2] - 2n_2\text{Re}[S_1A_{j1}/z_1 + S_2A_{j2}/z_2] \quad (3.5b)$$

$$T_{j2} = -2n_1\text{Re}[S_1A_{j1}/z_1 + S_2A_{j2}/z_2] + 2n_2\text{Re}[A_{j1}/z_1 + A_{j2}/z_2] \quad j=1,2 \quad (3.5c)$$

In Equation (3.5),

$$\begin{aligned} r_{1j} &= a_{11}S_j^2 + a_{12} - a_{16}S_j \\ r_{2j} &= a_{12}S_j + a_{22}/S_j - a_{26} \end{aligned} \quad (3.6)$$

and n_j are the components of the unit outward normal at Q in the x_1 - x_2 coordinate system. Also A_{jk} are complex constants, which may be obtained from the following equations, shown below in matrix form

$$[Im(B_1) \ Re(B_1) \ Im(B_2) \ Re(B_2)] \begin{bmatrix} Re(A_{j1}) \\ Im(A_{j1}) \\ Re(A_{j2}) \\ Im(A_{j2}) \end{bmatrix} = \begin{bmatrix} -1/4\pi & \delta_{j2} \\ 1/4\pi & \delta_{j1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad j=1,2 \quad (3.7)$$

where δ_{jk} is the Kronecker delta and

$$\{B_k\} = [iS_k \ r_{1k} \ r_{2k}]^T ; \quad k=1,2 \quad (3.8)$$

3.2 Numerical Solution of the BIE

To solve the BIE numerically it is necessary to discretize the boundary S of the two-dimensional solution domain into a mesh of M line elements containing N distinct nodes. Quadratic isoparametric elements are used in this study. With these three-noded elements, the geometry and the primary solution variables are assumed to vary in a quadratic manner over each of the elements. These variables are defined in terms of the nodal values and shape functions $N^e(\zeta)$ of a coordinate, ζ , intrinsic to the element in question, as follows:

$$\begin{aligned}
x_j(\zeta) &= N^c(\zeta)x_j^c \quad c=1,3 \\
u_j(\zeta) &= N^c(\zeta)u_j^c \\
t_j(\zeta) &= N^c(\zeta)t_j^c
\end{aligned}
\tag{3.9}$$

where

$$\begin{aligned}
N^1(\zeta) &= -\frac{1}{2}\zeta(1-\zeta) \\
N^2(\zeta) &= (1-\zeta^2) \\
N^3(\zeta) &= \frac{1}{2}\zeta(1+\zeta)
\end{aligned}
\tag{3.10}$$

These may be substituted into Equation (3.4) to obtain the discretized form, as follows:

$$\begin{aligned}
& C_{jk}(P^a)u_k(P^a) + \sum_{b=1}^M \sum_{c=1}^3 u_k(P^{d(b,c)}) \int_{S_b} T_{jk}(P^a, Q) N^c(\zeta) J(\zeta) dS \\
& = \sum_{b=1}^M \sum_{c=1}^3 t_k(P^{d(b,c)}) \int_{S_b} U_{jk}(P^a, Q) N^c(\zeta) J(\zeta) dS \\
& P^a = 1, N
\end{aligned}
\tag{3.11}$$

where P^a is the a -th node of the mesh, $P^{d(b,c)}$ is the c -th node of the b -th element and $J(\zeta)$ is the Jacobian of transformation. The integrals over each of the elements S_b can be evaluated by Gaussian quadrature although some care is required for cases where $P^{d(b,c)}$ is a node of the element over which the integration is performed. Details of this may be found in e.g. Tan (1987). Equation (3.11) represents $2N$ linear algebraic equations for

the unknown displacements and tractions at the nodes on the boundary of the solution domain. These may be solved numerically by, for example, direct Gaussian elimination.

3.3 Application of the BIE Method to Linear Elastic Fracture Mechanics

As mentioned previously, the FEM, because of its earlier development, is more commonly used for fracture mechanics analysis than the BIE method in spite of certain advantages the latter possesses. It is better suited, for example, for problems with sharp stress gradients such as those found near crack tips. Mesh refinement there can be carried out more easily in the BIE method than in FEM since it is not necessary for elements to form a two-dimensional grid as in FEM.

The approach used for obtaining fracture characterizing parameters is in principle, similar for both numerical methods. The numerical analysis yields tractions or displacements at selected points in the cracked body. Theoretical relationships between these and the desired fracture characterizing parameters are then exploited to calculate the latter.

3.3.1 Quarter-Point Singular Crack-Tip Elements

It is well known that near a crack tip the displacements are of an $r^{1/2}$ order while the tractions exhibit an $r^{-1/2}$ order singularity. Use of the BIE method with the quadratic, isoparametric element formulation and the shape functions already mentioned does not accurately model these variations. However, by shifting the mid point nodes of the crack tip elements to the quarter points, see Figure 3.2, the following form for the variations of the displacements and tractions along these elements can be obtained; see Henshall and Shaw (1975), Barsoum (1976):

$$\begin{Bmatrix} u_i \\ t_i \end{Bmatrix} = \Phi_i + \Phi_i' \sqrt{\frac{r}{l}} + \Phi_i'' \left(\frac{r}{l} \right); \quad i = 1, 2 \quad (3.12)$$

where Φ_i , Φ_i' and Φ_i'' are constants for the elements, r is the distance from the crack tip and l is the length of the crack tip element.

Clearly, by shifting the mid-side nodes to quarter points along the element, the correct $r^{1/2}$ variation for the displacements is present but the traction is not of the $r^{-1/2}$ order indicated by fracture mechanics theory. This is because in the BIE method, both the displacements and the tractions are primary solution variables and are represented independently. This is in contrast to the FEM, where the tractions/stresses are obtained via differentiating the displacements and are therefore secondary variables. A simple way to remedy this is to multiply the right hand side of the expression for the tractions by $\sqrt{(l/r)}$ to obtain the

proper singularity of tractions, see Cruse and Wilson (1977). The new computed nodal values of the tractions are now related to the physical values by the following equations, see Martinez and Dominguez (1984):

$$\bar{t}_i^{(3)} = t_i^{(3)}$$

(3.13a)

$$\bar{t}_i^{(2)} = \frac{1}{2} t_i^{(2)}$$

(3.13b)

$$\bar{t}_i^{(1)} = \lim_{r \rightarrow 0} t_i^{(1)} \sqrt{\frac{r}{l}}$$

(3.13c)

where the superscripts denote the nodes shown in Figure 3.2 and the overbars denote the computed tractions.

By equating the singular term of the classical near-tip traction field to Equation (3.13c), Martinez and Dominguez have shown that the stress intensity factors may be obtained from the BIE computed nodal tractions on the traction-singular crack-tip element as follows:

(3.14)

$$K_I = \sqrt{2\pi l} \quad \tilde{t}_2^{(1)}$$

$$K_{II} = \sqrt{2\pi l} \quad \tilde{t}_1^{(1)}$$

3.3.2 Determination of Characterising Parameters for Interface Cracks

The main objective of any two-dimensional fracture mechanics analysis of cracks in homogeneous materials is to determine the values of K_I and K_{II} ; the mode I and mode II stress intensity factors. These characterize the amplitude of the direct and shear stress fields near the crack tip. However for interface cracks, as mentioned previously, they do not have the same physical interpretations because of the coupled nature of the shear and tensile effects even for simple tensile loading. For clarity, the symbols K_1 and K_2 will be used in the ensuing discussion.

Consider a crack lying at the interface between two semi-infinite solids composed of linear elastic anisotropic materials with r and θ being polar co-ordinates as shown in Figure 2.4. The stresses along the interface in the vicinity of the crack tip are written as

$$t = (\sigma_{12} \quad \sigma_{22} \quad \sigma_{32})^T \quad (3.15)$$

The stress intensity factors will be written as

$$K(L) = (K_2 \ K_1 \ K_3)^T \quad (3.16)$$

and will bear the following relation to the stresses

$$K(L) = \sqrt{2\pi r} \cdot R \left[\left(\frac{r}{L} \right)^{-\gamma} \right] t \quad (3.17)$$

where γ is defined as, see Ting (1986)

$$\gamma = \frac{1}{2\pi} \ln \left(\frac{1+\beta}{1-\beta} \right) \quad (3.18)$$

where β is given by

$$\beta = \left\{ -\frac{1}{2} \text{tr}[(WD^{-1})^2] \right\}^{1/2} \quad (3.19)$$

In Equation (3.16), W and D are the negative real and imaginary parts of a complex matrix M , that is,

$$M = -(W + iD) \quad (3.20)$$

and M is related to a Hermitian matrix A which is a function of the elastic constants of the two materials.

$$M = -i(2\pi)^{1/2} [A^T]^{-1} \quad (3.21)$$

$$A = (B^1 + B^2)^{-1} \quad (3.22)$$

$$B^\alpha = i(2\pi)^{-1/2} \sum_{N=1}^3 Adj L^\alpha(p_N^\alpha) \left[\sum_{M=1}^3 C^\alpha(p_M^\alpha) Adj L^\alpha(p_M^\alpha) \right]^{-1} \quad (3.23)$$

$$L^\alpha(p_N^\alpha) = Q^\alpha + p_N^\alpha [S^\alpha + (S^\alpha)^T] + p_N^2 V^\alpha \quad (3.24)$$

$$C^\alpha(p_N^\alpha) = (S^\alpha)^T + p_N^\alpha V^\alpha \quad (3.25)$$

where the superscript $\alpha = 1$ or 2 refers to material 1 and 2 respectively and $p_{N\alpha}$ are three of the roots of a sixth-order polynomial derived, using Fourier transforms, from Navier's equation of equilibrium for plane deformation. Q , S and V are matrices of stiffness constants. They can be expressed in indicial notation as

$$Q = (Q_{jk}) = (C_{jlk}) \quad (3.26a)$$

$$S = (S_{jk}) = (C_{jlk2}) \quad (3.26b)$$

$$V = (V_{jk}) = (C_{j2k2}) \quad (3.26c)$$

where the stiffness constants C_{klmn} are as given in the stress-strain relation for a generally anisotropic body in Cartesian co-ordinate space, shown below

$$\sigma_{kl} = C_{klmn} \epsilon_{mn} \quad (3.27)$$

This statement is equivalent to Equation (2.2).

Two other matrices, P and R , may be introduced and defined as

$$P = \frac{1}{\beta} W D^{-1} \quad (3.28)$$

$$M^{-1} = W + iD \quad (3.29)$$

$$R[c] = I + Im[c]P + (1 - Re[c])P^2 \quad (3.30)$$

c must be a dimensionless quantity. R has the following properties

$$\begin{aligned} R[1] &= I \\ R[c] \cdot R[e] &= R[ce] \end{aligned} \quad (3.31)$$

For materials in which the x_3 -axis is a two fold symmetry axis, the elastic constants $C_{ij23} = C_{ij31} = C_{2312} = C_{3112} = 0$. There is thus no coupling between the displacement in the x_3 direction and the in-plane displacement components. Under these conditions, the matrix P^2 in Equation (3.30) can be shown to be

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.32)$$

For plane problems σ_{23} and K_3 therefore disappear. Using Equations (3.29) and (3.32), Equation (3.17) can be rewritten for plane problems as

$$K(L) = \begin{bmatrix} K_2 \\ K_1 \end{bmatrix} = \sqrt{2\pi r} \cdot R' \left[\left(\frac{r}{L} \right)^{-i\gamma} \right] \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix} \quad (3.33)$$

where R' is a two dimensional version of R , given by

$$\begin{aligned} R' \left[\left(\frac{r}{L} \right)^{-i\gamma} \right] &= Re \left[\left(\frac{r}{L} \right)^{-i\gamma} \right] I + Im \left[\left(\frac{r}{L} \right)^{-i\gamma} \right] P \\ &= \cos \left[\gamma \ln \left(\frac{r}{L} \right) \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sin \left[\gamma \ln \left(\frac{r}{L} \right) \right] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{aligned} \quad (3.34)$$

It is evident from Equations (3.33) and (3.34) that when $r \rightarrow 0$ the stresses exhibit an oscillatory singularity, as fracture mechanics theory predicts.

R is used to modify the shape functions of the quarter-point crack tip element so that they incorporate the oscillatory nature of the traction singularity which is predicted by fracture mechanics theory. The tractions over this element are now taken to be, see Tan, Gao and Afagh (1992)

$$t_i = N^1(\zeta) \cdot \sqrt{\frac{l}{r}} \cdot R' \left[\left(\frac{r}{l} \right)^{\gamma} \right] \cdot \bar{t}_i^{(1)} + N^2(\zeta) \cdot \left(\frac{l}{r} \right) \cdot R' \left[\left(\frac{r}{l} \right)^{\gamma} \right] \cdot \bar{t}_i^{(2)} + N^3(\zeta) \cdot R' \left[\left(\frac{r}{l} \right)^{\gamma} \right] \cdot \bar{t}_i^{(3)} \quad (3.35)$$

where $\bar{t}_i^c$ is the traction in the i -direction computed for the c -th node of the crack tip element with the modified shape functions. The relationship between the computed tractions and the physical tractions is now given by

$$\bar{t}_i^{(3)} = t_i^{(3)}$$

$$\bar{t}_i^{(2)} = \frac{1}{2} R' \left[\left(\frac{1}{2} \right)^{-\gamma} \right] \cdot t_i^{(2)}$$

$$\bar{t}_i^{(1)} = \lim_{r \rightarrow 0} \sqrt{\frac{r}{l}} \cdot R' \left[\left(\frac{r}{l} \right)^{-\gamma} \right] \cdot t_i^{(1)}$$

(3 36)

For two different characteristic dimensions L_1 and L_2 it has been shown (Wu, 1990) that

$$K(L_1) = R' \left[\left(\frac{L_1}{L_2} \right)^{\gamma} \right] \cdot K(L_2) \quad (3.37)$$

Thus it can be written, using the crack-tip element length l as one of the characteristic lengths, and another characteristic dimension L as the other, that

$$\begin{aligned} K(L) &= R' \left[\left(\frac{L}{l} \right)^{\gamma} \right] \cdot K(l) \\ &= R' \left[\left(\frac{L}{l} \right)^{\gamma} \right] \cdot \sqrt{2\pi r} \cdot R' \left[\left(\frac{r}{l} \right)^{-\gamma} \right] \cdot \sqrt{\frac{l}{r}} \cdot R' \left[\left(\frac{r}{l} \right)^{\gamma} \right] \cdot \tilde{t}^{(1)} \\ &= R' \left[\left(\frac{L}{l} \right)^{\gamma} \right] \cdot \sqrt{2\pi l} \cdot \tilde{t}^{(1)} \end{aligned} \quad (3.38)$$

By equating the singular term of the near-tip traction field to that shown in Equation (3.38), the complex stress intensity factor, $K(L)$, may be related to the BIE computed traction components at the crack tip node by the following equation

$$\begin{bmatrix} K_2(L) \\ K_1(L) \end{bmatrix} = \sqrt{2\pi l} \cdot R' \left[\left(\frac{L}{l} \right)^{\gamma} \right] \cdot \begin{bmatrix} \tilde{t}_1^{(1)} \\ \tilde{t}_2^{(1)} \end{bmatrix} \quad (3.39)$$

K_1 and K_2 can therefore be calculated directly from the BIE computed nodal values of the tractions using Equation (3.39).

The strain energy release rate can also be found from the equation, see Wu (1990)

$$G = \frac{1}{4} K^T \bar{D}^{-1} K \quad (3.40)$$

where $\bar{D}$ is the imaginary part of the matrix M , see Equation (3.20).

3.4 Examples

Five problems were treated using the BIE method to demonstrate the general accuracy of the technique and the veracity of the computer program used in this study. The problems include stress intensity factor computations for cracks in both isotropic and anisotropic homogeneous materials. The results for interfacial cracks between isotropic materials and between anisotropic materials are also presented. Solutions for all of the problems treated are available in the literature for comparison.

3.4.1 Cracks Emanating from a Circular Hole in an Isotropic Plate in Tension

The problem is that of a finite rectangular panel containing a circular hole with two cracks of length $(a-r)$ emanating from opposite sides. The upper and lower edges of the plate are subjected to a tensile stress σ , as shown in Figure 3.3. The aspect ratio (c/b)

of the plate is 2 while the diameter of the circular hole is one quarter the width of the plate. Three relative crack lengths, $a/b = 0.4, 0.6, 0.8$, were treated in the BIE analysis. A typical boundary element mesh for this problem is shown in Figure 3.4.

This problem has been treated using the FEM, see Newman (1971). These solutions are compared with the results of the BIE analysis in Table 3.1. Both sets of results are normalised with respect to $\sigma\sqrt{\pi a}$. The discrepancies between the BIE computed results and those by Newman were less than 2% for all three relative crack lengths treated.

3.4.2 Edge Crack in an Isotropic Plate in Tension

An edge crack of length a is located at the edge of a plate of width b and height $2h$. The plate is subject to a uniform normal tensile stress σ at its ends as shown in Figure 3.5. The normalised stress intensity factor for this problem has been obtained by Gross and Srawley (1964) using a modified mapping collocation technique, as follows:

$$\frac{K_1}{K_0} = 1.12 - 0.23\left(\frac{a}{b}\right) + 10.6\left(\frac{a}{b}\right)^2 - 21.7\left(\frac{a}{b}\right)^3 + 30.4\left(\frac{a}{b}\right)^4 \quad (3.41)$$

where $K_0 = \sigma\sqrt{\pi a}$. For the BIE analysis b and h were taken to be $4a$. The predicted normalized stress intensity factor according to Equation (3.41) is then 1.5047. The BIE mesh used is shown in Figure 3.6. Note that symmetry made it possible to treat only half of the plate. The computed normalized stress intensity factor from the BIE analysis was

1.5226 giving a discrepancy of 1.2% for the relatively simple mesh used.

3.4.3 Bimaterial Interface Crack between Isotropic Materials

The next problem treated was that of a crack of length $2a$ between two dissimilar isotropic half planes subjected to biaxial normal stresses at infinity as shown in Figure 3.7. Plane stress conditions were assumed. To ensure continuity of strain across the interface the normal stresses parallel to the interface applied to materials 1 and 2 were related by the following equation

$$\sigma_1^{(2)} = \left(\frac{E_2}{E_1} \right) \sigma_1^{(1)} + \left[\nu_2 - \left(\frac{E_2}{E_1} \right) \nu_1 \right] \sigma_2 \quad (3.42)$$

The analytical solution for the stress intensity factors for this problem has been obtained by Rice and Sih (1965), viz:

$$K_1 = \frac{\sqrt{a}}{\cosh(\pi e)} \sigma_2 [\cos(e \ln 2a) + 2e \sin(e \ln 2a)] \quad (3.43a)$$

$$K_2 = -\frac{\sqrt{a}}{\cosh(\pi e)} \sigma_2 [\sin(e \ln 2a) + 2e \cos(e \ln 2a)] \quad (3.43b)$$

where e is defined in Equation (2.27). In the BIE analysis, the width and height of the

plate were taken to be $80a$. The mesh used is shown in Figure 3.8 with the crack-tip element length equal to $0.1a$. Values of the ratio of the shear moduli of the two materials, μ_2/μ_1 , of 1, 10 and 100 were considered. Table 3.2 shows the exact and BIE solutions for the stress intensity factors at one of the crack tips for the different values of μ_2/μ_1 . The error is within 6% for all the K_1 values obtained and within 5% for the K_2 values, with the error increasing with the mismatch in elastic properties. This error trend has also been observed by Tan and Gao (1990a), who showed that smaller crack-tip elements are required as μ_2/μ_1 increased, if the same percentage errors as for smaller μ_2/μ_1 values are to be maintained.

3.4.4 Cracks Emanating from a Circular Hole in an Orthotropic Plate in Tension

The problem considered here is that of two symmetric cracks, each of length $(L-R)$, emanating from the edge of a circular hole, of radius R , in a thin, infinitely large, generally orthotropic plate, under remote tension σ_0 , as shown in Figure 3.9. The stress intensity factors for this problem have been determined by Cheong and Hong (1988) using a modified mapping collocation technique. The material they considered was E-glass/Epoxy with the following properties

$$E_1 = 53.74 \text{ GPa}, \quad E_2 = 17.91 \text{ GPa}$$

$$\mu_{12} = 8.96 \text{ GPa}, \quad \nu_{12} = 0.24$$

In the BIE analysis, the plate was taken to be finite with width $2W$ twenty times the radius of the hole, and height $2H$ forty times R . Relative crack sizes $(L - R)/L = 0.2$ and 0.4 were analysed. The material principal directions were varied by using values of ϕ of 0° , 30° , 60° and 90° , see Figure 3.9. The BIE meshes used for the two crack sizes are shown in Figure 3.10a and Figure 3.10b. Note that the whole geometry was modelled, as the material properties and the geometry do not generally share the same axes of symmetry.

The stress intensity factors that were obtained are displayed in Figures 3.11 and 3.12 together with the results of Cheong and Hong (1988); they have been normalised with respect to $\sigma_0 \sqrt{\pi(L-R)}$. The $\phi=0^\circ$ and $\phi=90^\circ$ cases and the $\phi=30^\circ$ and $\phi=60^\circ$ cases are shown separately for the sake of clarity. It can be seen that there is generally very good agreement between the two sets of results.

3.4.5 Bimaterial Interface Crack between Anisotropic Materials

The last problem treated here is that of a crack of length $2a$ lying at the interface between two dissimilar semi-infinite bodies, subjected to internal pressure σ_0 , see Figure 3.13. Material 1 was treated as anisotropic with its orthogonal material axes rotated by an angle θ with respect to the global axes. Material 2 was treated as isotropic for simplicity. The

material properties of single crystal alumina and of fully stabilized polycrystal (FSP) zirconia were used for materials 1 and 2 respectively. FSP zirconia has $E=192$ GPa and $\nu=0.3$. Single crystal alumina has engineering constants with respect to the orthogonal material axes as follows

$$\begin{aligned} E_{11} &= 345 \text{ GPa}, & E_{22} &= 516 \text{ GPa}, & E_{33} &= 345 \text{ GPa} \\ \nu_{12} &= 0.131, & \nu_{13} &= 0.362, & \nu_{23} &= 0.196 \\ \eta_{12,1} &= 59 \text{ GPa}, & \eta_{12,2} &= 0, & \eta_{12,3} &= -59 \text{ GPa} \\ \mu_{12} &= 173 \text{ GPa}, \end{aligned}$$

In the BIE analysis, a finite body was treated, with width and height equal to $80a$. Plane strain conditions were assumed. Values of θ of 0° , 30° , 60° and 90° were considered. The mesh used was the same as that for the isotropic bimaterial interface, shown in Figure 3.8, except that the crack tip element length l was taken to be $0.02a$ and $0.1a$ in two separate analyses. The characterizing dimension L used in the computations was $20a$.

The exact closed form solution for the stress intensity factors of this problem has been given by Wu (1990). The BIE results obtained are compared to these in Table 3.3. The accuracy of the BIE results can be seen to be good when $l/a=0.02$; the errors were all less than 2% for the K_1 values, while the errors of the K_2 values, which are about an order of magnitude smaller, ran as high as 4.7%. The error in K_1 was also within 2% for $l/a=0.1$, though the BIE results were consistently higher than the exact ones, whereas for $l/a=0.02$, they were consistently lower. The percentage errors of K_2 were somewhat higher when

$l/a=0.1$, but then the absolute values were relatively small indeed.

3.5 Closing Remarks

In this chapter, the basic theory of the boundary integral equation method for anisotropic elasticity has been reviewed. Some features of its application to linear elastic fracture mechanics, including the use of quarter-point singular crack tip elements and the method for obtaining fracture characterizing parameters from BIE computed tractions, have been discussed. Finally, the solutions obtained using the BIE method for a few two-dimensional fracture mechanics problems have also been presented to demonstrate the validity of the technique and of the particular BIE method computer code used in this study.

Chapter 4

Fracture Mechanics Analysis of an Interface Crack in Double Cantilever Beam (DCB) Specimens

4.0 Introduction

Fibre reinforced composites are increasingly being used for structural applications and composite laminates are a common form in which they are used. A composite laminate typically consists of bonded layers of fibre reinforced composite material each with their material principal axes oriented in specified directions. Delamination, resulting from the propagation of cracks along the interface between layers, is a common failure mode for these structural materials. The macromechanics of these cracks is, as yet, not fully understood. As with isotropic bimaterial interface cracks, the mismatch in mechanical properties across the interface will lead to an inherent coupling of the mode I and mode II stress intensity factors at the crack tip. However, the problem is even more complex when the materials are anisotropic, due to the variation of mechanical properties with orientation on each side of the interface.

The BIE method has been shown to be an efficient numerical tool for the analysis of interface crack problems between anisotropic materials, see, e.g. Tan, Gao and Afagh (1992). In this chapter, two-dimensional interface crack problems in bilayer beams subjected to two different mechanical load conditions are considered using the BIE method discussed in Chapter 3. The geometry considered is that of a double cantilever beam divided along the midplane by the interface between the two materials, with the crack introduced along the interface from the free end of the beam. The double cantilever beam (DCB) specimen is one commonly used in fracture mechanics studies and fracture toughness testing (see, e.g. Whitney et al. (1982), Hashemi et al. (1990) and Ozdil and Carlsson (1992)). Attention is restricted here to the case where the material in each layer is the same. In the analysis, the fibre direction in one layer was set parallel to the longitudinal axis of the beam, while that in the other layer was set perpendicular to this axis, i.e. in the out-of-plane direction.

4.1 Interface Crack in an Orthotropic Bilayer

Figure 4.1 shows an interface crack of length a between the two layers of an orthotropic, bilayer beam. The beam has length W and height $2H$; and relative crack lengths, $a/W = 0.1, 0.2, 0.3, 0.4$ and 0.5 were treated for each of the W/H ratios $5, 7.5$ and 10 .

Thermal effects and body forces were not considered in the BIE analysis. Two mechanical load cases were analyzed throughout and they are referred to herein as Problems (I) and (II), as shown in Figure 4.2. In Problem (I), the load consisted of a tensile force P , perpendicular to the plane of the interface crack, applied to the free end of each layer. In Problem (II), however, the force P was applied only to the free end of the upper layer of the beam. The upper layer in both these problem was taken to be the one with the reinforcing fibres aligned in the out-of-plane direction.

The mechanical properties of each orthotropic layer were characterized by the quantities η_1 and η_2 , known as the orthotropic parameters. These are defined as follows:

$$\eta_1 \eta_2 = \left(\frac{E_1}{E_2} \right)^{1/2} \quad (4.1a)$$

$$\eta_1 + \eta_2 = \sqrt{2} \left[\left(\frac{E_1}{E_2} \right)^{1/2} + \frac{E_1}{2\mu_{12}} - \nu_{12} \right]^{1/2} \quad (4.1b)$$

Poisson's ratio was taken to be 0.3 throughout, and conditions of plane stress were assumed. For the different layups of the orthotropic layers here the effects of whether plane strain or plane stress is assumed in the analysis on the fracture characterising parameters are not expected to be very significant, see Davidson and Schapery (1988). Setting values for η_1 and η_2 and assuming E_1 allows for all the mechanical properties

required in the BIE analysis to be properly defined. For each geometry and load case, the material properties represented by the following combinations of η_1 and η_2 values, namely $\eta_1 = 1$, $\eta_2 = 1, 2, 3, 4, 6$, and $\eta_1 = 0.8$, $\eta_2 = 2, 4, 6$ were considered. A number of composite materials fall within the range of properties encompassed by these values, see Table 4.1.

In the numerical analysis, it was necessary to model the entire solution domain, as the material properties did not display any symmetry to match that of the geometry. The boundary of each layer was represented by a mesh of boundary elements, with continuity of displacements being enforced for those elements on the bonded interface ahead of the crack tip. Figure 4.3 shows a typical BIE mesh used. The quadratic isoparametric element formulation with quarter-point crack tip elements was employed throughout. The length of the crack tip element was generally one-tenth the length of the crack. To simulate the end force P in both Problems (I) and (II), a uniform tensile stress was applied over a small boundary element. Stress intensity factors were computed using the traction formula discussed in Section 3.3.2 and energy release rates were computed from these using Equation (3.39). The characteristic dimension used in computing stress intensity factors was taken to be equal to W . In the results presented below, the computed stress intensity factors are normalized by the quantity K_o where

$$K_o = \frac{P}{H} \sqrt{\pi a} \quad (4.2)$$

while the strain energy release rates are normalised with respect to the quantity G_o where

$$G_o = \frac{12P^2a^2}{E_1H^3} \quad (4.3)$$

In Equations (4.2) and (4.3), the width of the beam was taken as unity.

4.2 Results and Discussion

(a) Problem (I)

Figures 4.4-4.27 show the results of the normalized stress intensity factors, K_1/K_o and K_2/K_o , obtained from the BIE analysis. It can be seen in Figures 4.4-4.9 that the magnitudes of K_1/K_o are highest for the isotropic case when $\eta_1=\eta_2=1$ and they decrease asymptotically with increasing η_2 . The K_2/K_o results, shown in Figures 4.10-4.15, display a similar asymptotic trend with increasing value of η_2 , although in this case the magnitude increases with increasing η_2 . The variation of K_1/K_o with relative crack length, for the different η_2 values treated when $\eta_1=1$, are shown in Figures 4.16-4.18. The corresponding results for K_2/K_o are shown in Figures 4.19-4.21. Of interest to note is that K_1/K_o increases in an approximately linear manner with a/W for all the η_2 values considered. The magnitudes of K_2/K_o are all significantly smaller than the corresponding values of K_1/K_o , although the approximately linear trend is not evident. Also noteworthy is that both K_1/K_o and K_2/K_o increase quite markedly, for the same relative crack size a/W and material property combination of the two layers, when the aspect ratio of the beam, W/H ,

is increased. This increase is, however, not proportional to W/H . The variations of K_1/K_0 and K_2/K_0 with a/W when $\eta_1=0.8$ are shown in Figures 4.22-4.27 for the various values of η_2 and W/H ratios considered. The qualitative trends are very similar to those seen in the $\eta_1=1$ case, so are the orders of magnitude of the values of the normalised stress intensity factors.

The computed results of the normalised strain energy release rates, G/G_0 , are shown plotted in Figures 4.28-4.39. It can be seen in Figures 4.28-4.33 that G/G_0 increases in a quadratic manner as η_2 increases. This not surprising, since one half of the beam becomes increasingly more compliant as η_2 increases due to the associated decrease in the value of E_2 . Of interest to note is that the curves for the different W/H values become closer together as a/W increases. The significance of this is perhaps more apparent in Figures 4.34-4.39, where it is evident that for any given combination of W/H , η_1 and η_2 , G/G_0 decreases asymptotically as a/W increases. This has implications in fracture toughness testing in that it is not critical to specify a certain relative crack size, as long as it exceeds that value at which the constant value of G/G_0 has been reached. For all the cases examined, with the exception of some cases when $\eta_2=6$, this constant value of G/G_0 is reached at $a/W \leq 0.5$. The numerical values of G/G_0 at $a/W=0.5$ for all the cases treated are listed in Table 4.2. Also listed are the corresponding values of the phase angle $\psi^* = \tan^{-1}(K_2/K_1)$ which denote the degree of mode mixity. It is noteworthy that the value of ψ^* is relatively more sensitive to the change in the material parameters, η_1 and η_2 , than to the slenderness of the beam as measured by W/H .

(b) Problem (II)

The computed normalised stress intensity factors for Problem (II) are shown plotted in Figures 4.40-4.63 for the different geometric and material properties cases considered. The various curves of K_1/K_0 versus η_2 , see Figures 4.40-4.45, all show K_1/K_0 increasing in magnitude with η_2 , but none of them reach an asymptotic value within the range of η_2 considered. As would be expected, the value for the isotropic case is one half of the corresponding value for Problem (I); however, the trends are significantly different from those seen in Problem (I). From linear superposition, it can be deduced that the portion of K_1/K_0 due to the load on the lower half of the beam in Problem (I) must decrease towards a horizontal asymptote as η_2 increases and that the portion of K_1/K_0 caused by the load on the upper half of the beam becomes very small for high values of η_2 . This difference in the effects of the upper and lower loads in Problem (I) demonstrates that the asymmetry of material properties changes the behaviour from that in isotropic fracture mechanics. The variation of K_1/K_0 with a/W , shown in Figures 4.52-4.54 and 4.58-4.60, is approximately linear for the range of a/W treated, as for Problem (I). Also, K_1/K_0 increases with increasing value of W/H ratio, but the increase is again not directly proportional to W/H , as in Problem (I).

The K_2/K_0 results, shown in Figures 4.46-4.51, 4.55-4.57 and 4.61-4.63 are also significantly different in features from the corresponding results of Problem (I). In contrast to the previous problem K_2 is no longer zero for the isotropic case. This is due to the Poisson's ratio effect and the lack of symmetry of the loading. From Figures 4.55-

4.57 and 4.61-4.63, it is evident that the magnitude of K_2/K_0 increases with the relative crack length a/W for a given material property combination (η_1, η_2) and the beam aspect ratio W/H . This trend, however, is similar to that seen in Problem (I).

Unlike the results for the normalised stress intensity factors, the results of the normalised strain energy release rate G/G_0 , as shown in Figures 4.64-4.75, are identical in form to those obtained for Problem (I). As before, G/G_0 increases in a quadratic manner with η_2 , and, more significantly, as shown in Figures 4.70-4.75, the normalised strain energy release rate decreases towards a constant value as a/W increases. Furthermore, for a given value of η_1 , this constant value of G/G_0 is reached at smaller relative crack lengths. This feature makes this specimen, with loading just on the upper surface, suitable for fracture toughness testing as well, since the relative crack length is one less critical dimension to consider. Table 4.3 lists the computed values of G/G_0 at $a/W=0.5$ for all the cases analyzed. Beyond this relative crack length, changes in the value of G/G_0 with a/W are expected to be very minimal indeed. Finally, as in Problem (I), it is observed that the magnitudes of G/G_0 decrease with increasing slenderness of the specimen.

Chapter 5

Fracture Mechanics Analysis of an Interface Crack in Trilayers

5.0 Introduction

The geometry investigated in the previous Chapter, with only two orthotropic layers, is the simplest possible composite laminate. Practical composite laminates, in general, are made of more than two layers. The macromechanics of cracks between laminae in multilayer laminates are therefore also of practical interest. In this chapter, two-dimensional interface crack problems in orthotropic three-layer beams subjected to end loads are considered. The geometry considered is similar to the bilayer cantilever beam considered in Chapter 4, except for an additional orthotropic layer. A crack exists along the interface between the upper and middle layers of the beam, from the free end of the beam. As before, the material in each layer was treated as being the same, but the first material principal direction in each layer was treated to be always perpendicular to that in the adjoining layer. Figures 5.1a and 5.1b show the two possible combinations of the lay-up. The objective of the investigation here is, again, to determine the stress intensity factors and strain energy release rates for interfacial cracks in such multi-layer beam specimens used in interfacial fracture mechanics studies.

5.1 Interface crack in a three-layer beam

Figure 5.2 shows the problem considered in which an interface crack exists between two adjoining layers of an orthotropic, trilayer cantilever beam. The length of the beam is W and the height is $3H$; and relative crack lengths, $a/W = 0.1, 0.2, 0.3, 0.4, 0.5$ were treated for each of the following values of W/H : 7.5, 11.25 and 15. Thermal and body force effects were again ignored in the BIE analysis, and, as in Chapter 4, two load cases were considered, as shown in Figure 5.2(a) and 5.2(b), respectively. In Problem I, shown in Figure 5.2(a), a tensile force P was applied to the free end of the upper and lower layers of the beam, perpendicular to the plane of the crack. In Problem II, the force P was only applied to the free end of the upper layer of the cantilever beam. There were therefore four cases studied, distinguished by loading and layup, which will be referred herein to as Problems I(A), I(B), II(A) and II(B), with the (A) or (B) distinguishing the two layups in Figures 5.1(A) and 5.1(B), respectively.

As in Chapter 4, the quantities η_1 and η_2 defined in Equation (4.1) are used to characterise the mechanical properties in each layer in the x_1 - x_2 coordinate system. Poisson's ratio was also taken to be 0.3 throughout, and conditions of plane stress were assumed. Setting values for η_1 and η_2 and assuming E_1 allows for all the mechanical properties required in the BIE analysis to be properly defined. Also, as before, for each geometry and load case, the material properties represented by the set of η_1 and η_2 values, namely $\eta_1 = 1, \eta_2 = 1, 2, 3, 4, 6$, and $\eta_1 = 0.8, \eta_2 = 2, 4, 6$ were considered.

In the BIE analysis, it was again necessary to model the entire solution domain, with the boundary of each layer represented by an assemblage of boundary elements. A typical BIE mesh employed is shown in Figure 5.3; it contains a total of 113 elements. In the computation of the stress intensity factors, the length of the beam W was again taken to be the characteristic dimension. The results of the stress intensity factor obtained are all normalised here by K_0 , as defined in Equation (4.2). Similarly, the strain energy release rates obtained from the BIE analysis are normalised by G_0 , as defined in Equation (4.3) previously. It should perhaps be reminded here that H is one-third of the beam height as opposed to one-half in Chapter 4. K_0 and G_0 are therefore, 1.5 and 3.375 times, respectively, the values for the beam with the same aspect ratio from Chapter 4. These results will now be presented and discussed below.

5.2 Results and Discussion

(a) *Problem I*

Figures 5.4-5.27 show the results of the normalised stress intensity factors K_1/K_0 and K_2/K_0 obtained from the BIE analysis of Problem I(A). It can be seen in Figures 5.4-5.9 that the behaviour of K_1/K_0 with changes in η_2 is similar to that observed in Problem (I), treated in the previous Chapter; it decreases in magnitude as η_2 increases. However, unlike in Problem (I) before, the corresponding K_2/K_0 results, shown in Figures 5.10-5.15, also decrease in magnitude with increasing η_2 . Examining Figures 5.16-5.18 and 5.22-5.24, it is evident that, for given values of η_1 and η_2 , K_1/K_0 varies in an approximately linear manner with a/W , with the slope of the plot decreasing with increasing η_2 . Indeed,

in a number of cases when $\eta_2=6$, K_1/K_0 is approximately constant for the range of a/W treated. The corresponding variation of K_2/K_0 with a/W , shown in Figures 5.19-5.21 and 5.25-5.27, is also approximately linear for $a/W \geq 0.2$; the trend in slope with increasing η_2 being also similar to that of K_1/K_0 .

The normalised strain energy release rate results, G/G_0 , obtained from the BIE analysis are shown in Figures 5.28-5.39. They are similar in form to the corresponding results of Problem (I) in Chapter 4, but the magnitudes are significantly lower here. This is a consequence of the reduced compliance of the top layer of the beam compared to the DCB and of the higher value of the normalising parameter G_0 compared to that in Chapter 4. The value of G/G_0 increases with increasing η_2 as shown in Figures 5.28-5.33, and it decreases asymptotically with increasing a/W , Figures 5.34-5.39. As before, the results appear to reach their asymptotic values by $a/W=0.5$, for the lower values of η_1 and η_2 treated. The numerical values of G/G_0 at $a/W=0.5$ for Problem I(A) are tabulated in Table 5.1. Also listed in Table 5.1 are the corresponding values of $\psi^* = \tan^{-1}(K_2/K_0)$ at $a/W=0.5$. Of interest to note is that, as in the DCB specimen treated in the previous chapter, ψ^* is more sensitive to the changes in η_1 and η_2 (geometry fixed) than to the W/H ratio (material parameters η_1 and η_2 fixed). Again, the magnitude of G/G_0 is higher for less slender beams. One feature of interest is that the increase in G/G_0 with η_2 is evidently non-linear only when $a/W=0.1$. For higher relative crack lengths, the plots are, for all practical purposes, straight lines.

The results of the normalised stress intensity factors, K_1/K_0 and K_2/K_0 , for Problem I(B) are shown plotted in Figures 5.40-5.75. The K_1/K_0 results are similar to those for Problem (II) of Chapter 4 even though the load case is different. As seen in Figures 5.40-5.45, the value of K_1/K_0 increases slowly with increasing value of η_2 , as previously, except that the rate of change of K_1/K_0 is not as rapid with η_2 . The variation of K_2/K_0 with η_2 , shown in Figures 5.46-5.51, is also very similar in trend to that of Problem (II) in the previous Chapter. As is evident in Figures 5.51-5.54 and 5.58-5.60, the K_1/K_0 results follow the now familiar pattern of approximately linear increase with increasing a/W . The variation of K_2/K_0 with a/W , shown in Figures 5.55-5.57 and 5.61-5.63, follows a similar trend of increasing magnitude of K_2/K_0 with relative crack size, but the approximately linear variation seen above for K_1/K_0 is not apparent.

The plots of the results for the normalized strain energy release rates G/G_0 for Problem I(B) are shown in Figures 5.64-5.75. They share the same qualitative features as in the previous problems, although they resemble the corresponding results of Problem (II) in Chapter 4 more than those of Problem I(A) above. The magnitudes of G/G_0 are generally higher than those for Problem I(A) but are still lower than those for Problems (I) and (II) in the previous Chapter. These values and ψ^* at $a/W=0.5$ for the cases analyzed are listed in Table 5.2. Unlike in Problem I(A), the value of ψ^* does not appear to change very significantly with either the W/H ratio of the material parameters, η_1 and η_2 .

(b) Problem II

Figures 5.76-5.81 show the variations of the computed values of K_1/K_0 with η_2 when the load is applied to just the top layer. The qualitative features are similar to those seen in Problem I(A) above; the magnitudes are lower, however, due to the absence of the load on the lower surface of the beam. The magnitude of K_1/K_0 increases with increasing W/H , as has been the case for all of the problems investigated. The variation of K_2/K_0 with η_2 , shown in Figures 5.82-5.87, is similar in trend to that for K_1/K_0 and to the corresponding variation for Problem I(A) above. The variation of K_1/K_0 with a/W , shown in Figures 5.88-5.90 and 5.94-5.96 for the different geometric and material property cases, is also very similar to that in Problem I(A), being approximately linear for $a/W \geq 0.2$, with the slopes of the curves in general decreasing with increasing η_2 . This feature is repeated for the variation of K_2/K_0 with a/W , as shown in Figures 5.91-5.93 and 5.97-5.99; the magnitude of K_2/K_0 being greater for higher W/H , as before.

The results of the computed normalized strain energy release rates G/G_0 , shown plotted in Figures 5.100-5.111, are again very similar in trends to those of Problem I(A). G/G_0 increases monotonically with increasing η_2 for a given cracked geometry, and it decreases asymptotically with increasing a/W , for a given combination of η_1 and η_2 . The numerical values of G/G_0 and $\psi^* = \tan^{-1}(K_2/K_1)$ at $a/W = 0.5$ for all the cases of this problem are listed in Table 5.3. As before, G/G_0 is not expected to change significantly for a/W higher than 0.5, for a given set of values of η_1 , η_2 and W/H .

The variations of K_1/K_0 with η_2 for the different crack sizes analyzed. Problem II(B), are shown in Figures 5.112-5.117. The corresponding variations of K_2/K_0 with η_2 are presented in Figures 5.118-5.123. It is evident that the qualitative features of these plots are very similar to those seen previously for Problem (II) in Chapter 4. The similarity in the trends to those of Problem (II) also extend to the variation of K_1/K_0 and K_2/K_0 with the relative crack length. These are shown in Figures 5.124-5.135. The similarity of the results here with those of Problem (II) in the previous Chapter is not very surprising in view of the identical loading condition and the layup of the composite beam, save for the extra layer at the bottom of the beam where no loads are applied.

The computed results of the normalized strain energy release rate G/G_0 for Problem II(B) are shown in Figures 5.136-5.147. They are identical in trends to those of Problem (II) in the previous Chapter; the magnitudes are lower here, however, although they are still significantly higher than in Problem II(A) above. The numerical values of G/G_0 and ψ^* at $a/W=0.5$ for this Problem are listed in Table 5.4. As for Problem (II) analyzed in Chapter 4, G/G_0 appears to reach the asymptote at a slightly lower relative crack length for this loading condition than for Problem I(B), where the load was applied to the upper and lower surfaces of the beam.

5.3 Concluding Remarks

In this chapter, the BIE method has been used to investigate two problems in interfacial fracture mechanics. The problems investigated consisted of orthotropic trilayer beams with a crack between the middle and upper layers. Stress intensity factors and strain energy release rates have been obtained for two load cases involving tensile forces on the parts of the beam on each side of the crack. The effects of varying the beam geometry, the crack length and the properties of the orthotropic materials of which the beam is composed have also been investigated. The addition of another layer to the DCB specimen treated in the previous Chapter evidently produced significant changes in the magnitudes of the normalized stress intensity factors and strain energy release rates. For beams having the same overall aspect ratio, the normalized stress intensity factors generally increased in magnitude with the addition of the third layer while the normalised strain energy release rates decreased in magnitude.

Chapter 6

Summary and Conclusions

In this study, the theory of linear elastic fracture mechanics (LEFM) has been reviewed. This review included a review of anisotropic elasticity, the presentation of the near-tip stress and displacement fields for cracks in homogeneous isotropic and orthotropic materials, and a review of the mechanics of bimaterial interface cracks. It is noteworthy that, for the bimaterial interface crack, the stresses in the vicinity of the crack-tip are, according to LEFM theory, oscillatory singular, and the tensile and shear effects at the crack tip are inherently coupled. The full characterisation of the crack tip fields therefore requires that the complex stress intensity factor be introduced. However, the strain energy release rate does not exhibit the oscillatory singularity, and thus it is generally now being used as a measure of fracture toughness of cracked bonded layers.

A review of the formulation of the multi-region boundary integral equation (BIE) method has also been presented. The quadratic isoparametric element formulation, with quarter point crack-tip elements to accurately represent the variation of displacement and the singularity of stress in the vicinity of the crack-tip, has been introduced. The procedures followed to obtain the stress intensity factors for an interface crack, directly from the BIE computed nodal tractions on the crack tip elements, have been described. The accuracy

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and efficiency of the BIE method have been illustrated by five test problems.

The multi-region BIE method with quarter-point crack-tip elements was then used to treat cracks in bilayer and trilayer orthotropic beams. These problems are of importance in understanding the delamination and consequent failure of composite laminate structural members. The geometries examined have relevance to interfacial fracture toughness testing of bonded layers. The specimens considered were subjected to mechanical loads only. For each crack problem, different beam aspect ratios and crack sizes relative to the length of the beam were treated and the effects of varying the properties of the orthotropic composite, and, for the trilayer beam, of varying the layup, were investigated. Stress intensity factor and strain energy release rate results have been presented and discussed for all of the cases analysed.

An interesting feature of the results obtained is that the magnitudes and qualitative trends of the fracture characterising parameters were not necessarily similar for cases having similar loading conditions. The layup of the orthotropic layers was found to strongly influence the results. Another particularly significant feature of the results is that, for both the bilayer and trilayer beams, the normalised strain energy release rate has been found to reach a constant value as the crack length increases, for a given beam aspect ratio and a set of material properties of the orthotropic layers. This implies that the relative crack length is not a critical dimension in fracture toughness testing using these specimens, provided that the crack length is greater than the minimum value at which the

constant value of the normalised strain energy release rate is achieved.

This study has further demonstrated that the multi-region BIE method with quarter-point crack tip elements is an efficient numerical tool for fracture analysis of cracks between dissimilar anisotropic materials. The results obtained will be useful in the interfacial fracture toughness testing of orthotropic laminates using the specimens above.

References

- Banerjee, P.K. and Butterfield, R. (1981), *Boundary Element Methods in Engineering Science*, McGraw-Hill, Berkshire, UK
- Bao, G., Ho, S., Suo, Z. and Fan, B. (1992), *The role of material orthotropy in fracture specimens for composites*, Int. J. Solids Structures **29**, 9, 105-116
- Barsoum, R.S. (1976), *On the use of isoparametric finite elements in linear fracture mechanics*, Int. J. Numer. Methods. Engng. **10**, 25-37
- Bassani, J.L. and Qu, J. (1989a), *Finite crack on bimaterial and bicrystal interfaces*, J. Mech. Phys. Solids **37**, 435-453
- Bassani, J.L. and Qu, J. (1989b), *On elasticity solutions for cracks on bimaterial and bicrystal interfaces*, Mater. Sci. Engng. **A107**, 177-184
- Carlsson, L.A. and Prasad, S. (1993), *Interfacial fracture of sandwich beams*, Engng Fracture Mech. **44**, 4, 581-590
- Charlambides, P.G., Lund, J., Evans, A.E. and McMeeking, R.M. (1989), *A test specimen for determining the fracture resistance of bimaterial interfaces*, J. Appl. Mech. **56**, 77-82

Cheong, S.K. and Hong, C.S. (1988), *Analysis of cracks emanating from a circular hole in an orthotropic plate under mixed mode deformation*, Engng. Fracture Mech. **31**, 237-248

Cho, S.B., Lee, K.R., Choy, Y.S. (1992), *Determination of stress intensity factors and boundary element analysis for interface cracks in dissimilar anisotropic materials*, Engng. Fracture Mech. **43**, 4, 603-614

Cruse, T.A. (1977), *Mathematical foundation of the boundary integral equation method in solid mechanics*, AFOSR-TR-77-1022 Report

Cruse, T.A. (1988), *Boundary Element Analysis in Computational Fracture Mechanics*, Kluwer Academic Publisher, Dordrecht, The Netherlands

Cruse, T.A. and Wilson, R.B. (1977), *Boundary integral equation method for elastic fracture mechanics*, AFOSR-TR-78-0355 Report

Davidson, B.D. and Schapery, R.A. (1988), *Effect of finite width on deflection and energy release rate of an orthotropic double cantilever specimen*, J. Comp. Mater. **22**, 640-656

Dundurs, J. (1969), *Edge-bonded dissimilar orthogonal elastic wedges*, J. Appl. Mech. **32**, 650-652

Erdogan, F. and Wu, B. (1993), *Interface crack problems in layered orthotropic materials*, J. Mech. Phys. Solids **41**, 5, 889-917

Gross, B. and Srawley, J.E. (1964), *Stress intensity factors for a single edge notch tension specimen by boundary collocation of a stress function*, NASA TN D-2395

Hashemi, S., Kinloch, A.J. and Williams, J.G. (1990), *The effects of geometry, rate and temperature on the mode I , mode II and mixed-mode III interlaminar fracture of carbon-fibre/poly(ether-ether ketone) composites*, J. Comp. Mater. **24**, 918-956

Henshell, R.D. and Shaw, K.G. (1975), *Crack-tip elements are unnecessary*, Int. J. Numer. Methods Engng **9**, 495-509

Hutchinson, J.W., Mear, M.E. and Rice, J.R. (1987), *Crack paralleling and interface between dissimilar materials*, J. Appl. Mech. **54**, 828-832

Irwin, G.R. (1958), *Fracture*, In *Handbuch der Physik*, Vol.VI, Springer

Jia, Z.H., Shippy, D.J. and Rizzo, F.J. (1988), *On the computation of two-dimensional stress intensity factors using the boundary element method*, Int. J. Numer. Methods Engng. **26**, 2739-53

Lee, K.Y. and Choi, H.J. (1988), *Boundary element analysis of stress intensity factors for bimaterial interface cracks*, Engng. Fract. Mech. **29**, 461-472

Lekhnitskii, S.G. (1963), *Theory of Elasticity of an Anisotropic Body*, Holden-Day, San Francisco

Lin, K.Y. and Hartmann (1989), *Numerical analysis of stress singularities at a bonded anisotropic wedge*, Engng. Fracture Mech. **32**, 211-224

Martinez, J. and Dominguez, J. (1984), *On the use of quarter-point boundary elements for stress intensity factor computations*, Int. J. Numer. Methods Engng. **20**, 1941-50

Nakagawa, K., Anma, T. and Duan, S.J. (1990), *A mathematical approach of the interface crack between dissimilar anisotropic composite materials*, Engng. Fracture Mech. **36**, 439-449

Newman, Jr., J.C. (1971), *An improved method of collocation for the stress analysis of cracked plates with various shaped boundaries*, NASA TN D-6376

Ni, L. and Nemat-Nasser, S. (1991), *Interface cracks in anisotropic dissimilar materials: an analytic solution*, J. Mech. Phys. Solids **39**, 113-114

- O'Dowd, N.P., Shih, C.F. and Stout, M.G. (1992), *Test geometries for measuring interfacial fracture toughness*, Int. J. Solids Structures 29, 5, 571-589
- Ozdil, F. and Carlsson, L.A. (1992), *Mode I interlaminar fracture of interleaved graphite/epoxy*, J. Comp. Mater. 26, 433-459
- Rice, J.R. (1988), *Elastic fracture mechanics concepts for interfacial cracks*, J. Appl. Mech. 55, 98-103
- Rice, J.R. and Sih, G.C. (1965), *Plane problems of cracks in dissimilar media*, J. Appl. Mech. 32, 418-423
- Rice, J.R., Suo, Z. and Wang, J.S. (1990), *Mechanics and thermodynamic of brittle interfacial failure in bimaterial systems*. In *Composite Material: Testing and Design*, ASTM STP 972 (J.D. Whitcomb ed.), 161-179
- Sih, G.C., Paris, P.C. and Irwin, G.R. (1965), *On cracks in rectilinearly anisotropic bodies*, Int. J. Fracture Mech. 1, 189-203
- Sih, G.C. and Liebowitz, H. (1968), *Mathematical theories of brittle fracture*. In *Treatise on Fracture* (Ed. H. Liebowitz), Vol II, p. 69, Academic Press, New York

Sih, G.C. and Liebowitz, H. (1972), *Mathematical theories of brittle fracture*. In *Fracture, An Advanced Treatise* ,(Ed. H. Liebowitz), Vol II, Academic Press, New York

Suo, Z. (1990a), *Delamination specimens for orthotropic materials*, J. Appl. Mech. **57**, 627-634

Suo, Z. (1990b), *Singularities, interfaces and cracks in dissimilar anisotropic media*, Proc. R. Soc. Lond. A **427**, 331-358

Suo, Z., Bao, G., Fan, B. and Wang, T.C. (1990), *Orthotropy rescaling and implications for fracture in composites*, Int. J. Solids Structures **28**, 235-248

Suo, Z. and Hutchinson, J.W. (1989), *Sandwich test specimens for measuring interface crack toughness*, Mater. Sci. Engng. A **107**, 135-143

Suo, Z. and Hutchinson, J.W. (1992), *Mixed mode cracking in layered materials*, In *Advances in Applied Mechanics* **29**, 63-191

Sun, C.T. and Jih, C.J. (1987), *On strain energy release rates for interfacial cracks in bi-material media*, Engng. Fracture Mech. **28**, 1, 13-20

Sun, C.T. and Manoharan, M.G. (1989), *Strain energy release rates of an interfacial*

crack between two orthotropic solids, J. Comp. Mater. **23**, 460-478

Tan, C.L. (1987), *The Boundary Element Method: A Short Course*, Monograph, Carleton University

Tan, C.L. and Gao, Y.L. (1990a), *Treatment of bimaterial interface crack problems using the boundary element method*, Engng. Fracture. Mech. **36**, 6, 919-932

Tan, C.L. and Gao, Y.L. (1990b), *Stress intensity factors for cracks at spherical inclusions by the boundary integral equation method*, J. Strain Analysis **25**, 4, 197-206

Tan, C.L. and Gao, Y.L. (1991), *Axisymmetric boundary integral equation analysis of interface cracks between dissimilar materials*, Computational Mech. **7**, 381-396

Tan, C.L. and Gao, Y.L. (1992), *Boundary element analysis of plane anisotropic bodies with stress concentrations and cracks*, Composite Structures **20**, 17-28

Tan, C.L., Gao, Y.L. and Afagh, F.F. (1992), *Boundary element analysis of interface cracks between dissimilar anisotropic materials*, Int. J. Solids Structures **29**, 24, 3201-3220

Ting, T.C.T. (1986), *Explicit solution and invariance of the singularities at an interface*

crack in anisotropic composites, Int. J. Solids Structures **22**, 965-983

Vinson, J.R. and Sierakowski, R.L. (1986), *The behaviour of structures composed of composite materials*, p. 60, Kluwer Academic Publishers

Wang, J.S. and Suo, Z. (1990), *Experimental determination of interfacial toughness using Brazil-nut sandwich*, Aca Met. **38**, 1279-1290

Whitney, J.M., Browning, C.E. and Hoogsteden, W. (1982), *A double cantilever beam test for characterising mode I delamination of composite materials*, J. Reinforced Plastics and Composites, **1**:297

Wu, K.C. (1990), *Stress intensity factor and energy release rate for interfacial cracks between dissimilar anisotropic materials*, J. Appl. Mech. **57**, 882-886

Wu, K.C. (1991), *Explicit crack-tip fields of an extending interface crack in an anisotropic bimaterial*, Int. J. Solids Structures **27**, 455-466

Yuuki, R., Cho, S., Matsumoto, T. and Kisu, H. (1987), *Usefulness of Hetenyi's solution for boundary element analysis of crack problems in dissimilar materials*, In *Role of Fracture Mechanics in Modern Technology* (Eds. G.C. Sih, H. Nisitani and T. Isihara), 823-834

Table 3.1 Normalised stress intensity factors, K_I/K_0 , for cracks emanating from a circular hole in an isotropic plate (Example 3.4.1). $K_0 = \sigma\sqrt{\pi a}$.

a/b	K_I/K_0 theoretical	K_I/K_0 BIE	% error
0.4	1.2156	1.2390	1.92
0.6	1.3965	1.4192	1.63
0.8	1.9044	1.8962	0.43

Table 3.2 Normalised stress intensity factors for an interface crack between two semi-infinite dissimilar isotropic media (Example 3.4.3). $K_0 = \sigma^2\sqrt{\pi a}$.

μ_2/μ_1	K_I/K_0 (exact)	K_I/K_0 (BIE)	% error	K_2/K_0 (exact)	K_2/K_0 (BIE)	% error
1	1.000	1.003	0.3	0	.0001	-----
10	0.9677	1.006	4.0	0.117	0.121	3.7
100	0.9532	1.003	5.2	0.139	0.146	4.6

Table 3.3 Comparison of exact and BIE computed stress intensity factors for
Example 3.4.5 - Pressurized interface crack between alumina and FSP
 zirconia. $K_0 = \sigma\sqrt{\pi a}$. Superscripts A and B refer to the left and right hand
 crack tips respectively.

θ (°)		Exact	BIE ($l/a=0.02$)	% Error	BIE ($l/a=0.1$)	% Error
0	K^A_1/K_0	0.992	0.980	-1.2	1.003	+1.2
	K^B_1/K_0	0.999	0.986	-1.3	1.010	+1.1
	K^A_2/K_0	0.107	0.102	-4.7	0.100	-6.5
	K^B_2/K_0	-0.107	-0.102	-4.7	-0.100	-6.5
30	K^A_1/K_0	0.979	0.974	-0.5	0.998	+1.9
	K^B_1/K_0	0.992	0.986	-0.6	1.010	+1.8
	K^A_2/K_0	0.193	0.186	-3.6	0.184	-4.7
	K^B_2/K_0	-0.193	-0.186	-3.6	-0.184	-4.7
60	K^A_1/K_0	1.000	0.985	-1.5	1.008	+0.8
	K^B_1/K_0	1.000	0.985	-1.5	1.008	+0.8
	K^A_2/K_0	0.035	0.034	-2.9	0.032	-8.6
	K^B_2/K_0	-0.035	-0.034	-2.9	-0.032	-8.6
90	K^A_1/K_0	0.999	0.986	-1.3	1.010	+1.1
	K^B_1/K_0	0.992	0.980	-1.2	1.004	+1.2
	K^A_2/K_0	0.111	0.106	-4.5	0.103	-7.2
	K^B_2/K_0	-0.111	-0.106	-4.5	-0.103	-7.2

Table 4.1 Values of the orthotropic parameters for some typical composite materials.

Material properties from Vinson and Sierakowski (1986).

Material	η_1	η_2
High Strength GR/Epoxy	0.8835	5.4305
High Modulus GR/Epoxy	0.8490	6.6629
Ultra High Modulus GR/Epoxy	0.9918	6.6882
Kevlar49/Epoxy	0.6190	6.3862
S Glass/Epoxy	0.7337	3.8851
Boron/Epoxy (50% vol fraction)	0.5850	5.4054

Table 4.2 Computed values of G/G_0 at $a/W=0.5$ for Problem (I). $\psi^*=\tan^{-1}(K_2/K_1)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
5	1	1.453	0
	2	3.744	-13.73
	3	7.390	-18.91
	4	12.437	-21.09
	6	26.672	-22.24
7.5	1	1.221	-1.18
	2	3.156	-14.95
	3	6.334	-20.17
	4	10.785	-22.12
	6	23.391	-22.87
10.0	1	1.106	-1.30
	2	2.866	-15.43
	3	5.810	-20.62
	4	9.942	-22.44
	6	21.680	-22.95

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
5	2	2.714	-9.80
	4	8.434	-19.68
	6	17.712	-22.43
7.5	2	2.265	-11.40
	4	7.230	-21.17
	6	15.426	-23.46
10.0	2	2.048	-11.88
	4	6.632	-21.69
	6	14.242	-23.78

Table 4.3 Computed values of G/G_0 at $a/W=0.5$ for Problem (II). $\psi^* = \tan^{-1}(K_2/K_1)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
5	1	0.579	-38.61
	2	2.429	-37.74
	3	5.667	-34.90
	4	10.363	-32.37
	6	24.000	-28.82
7.5	1	0.512	-40.66
	2	2.159	-39.26
	3	5.039	-36.04
	4	9.235	-33.12
	6	21.476	-28.96
10.0	1	0.482	-41.95
	2	2.036	-39.93
	3	4.744	-36.45
	4	8.670	-33.34
	6	20.120	-28.91

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
5	2	1.552	-38.52
	4	6.558	-34.73
	6	15.242	-31.62
7.5	2	1.370	-40.83
	4	5.822	-36.07
	6	13.615	-32.20
10.0	2	1.290	-41.75
	4	5.470	-36.50
	6	12.768	-32.34

Table 5.1 Computed values of G/G_0 at $a/W=0.5$ for Problem I(A). $\psi^* = \tan^{-1}(K_2/K_1)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
7.5	1	0.731	-20.52
	2	1.004	-14.69
	3	1.302	-10.88
	4	1.586	-8.58
	6	2.052	-6.29
11.25	1	0.638	-22.20
	2	0.821	-16.38
	3	1.039	-12.23
	4	1.257	-9.87
	6	1.652	-8.06
15.0	1	0.591	-23.07
	2	0.728	-17.68
	3	0.906	-13.40
	4	1.083	-10.95
	6	1.403	-9.27

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
7.5	2	0.913	-16.36
	4	1.436	-9.66
	6	1.894	-6.20
11.25	2	0.761	-18.15
	4	1.143	-11.01
	6	1.518	-7.74
15.0	2	0.687	-19.39
	4	0.994	-12.18
	6	1.298	-8.91

Table 5.2 Computed values of G/G_0 at $a/W=0.5$ for Problem I(B). $\psi^*=\tan^{-1}(K_2/K_1)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
7.5	1	0.731	-20.52
	2	2.692	-24.73
	3	5.921	-25.00
	4	10.420	-24.72
	6	23.102	-23.84
11.25	1	0.638	-22.20
	2	2.395	-25.93
	3	5.309	-25.67
	4	9.377	-25.05
	6	20.886	-23.76
15.0	1	0.591	-23.07
	2	2.251	-26.17
	3	5.011	-25.44
	4	8.871	-24.62
	6	19.757	-22.89

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
7.5	2	1.774	-24.12
	4	6.789	-25.48
	6	15.047	-25.27
11.25	2	1.568	-25.56
	4	6.086	-26.17
	6	13.554	-25.53
15.0	2	1.467	-26.06
	4	5.743	-25.94
	6	12.806	-24.96

Table 5.3 Computed values of G/G_0 at $a/W=0.5$ for Problem II(A). $\psi^*=\tan^{-1}(K_2/K_1)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
7.5	1	0.587	-36.67
	2	0.711	-39.49
	3	0.836	-41.04
	4	0.942	-42.12
	6	1.079	-43.50
11.25	1	0.530	-38.65
	2	0.603	-42.24
	3	0.690	-43.94
	4	0.776	-45.14
	6	0.928	-47.26
15.0	1	0.502	-39.73
	2	0.548	-44.24
	3	0.615	-46.07
	4	0.681	-47.33
	6	0.811	-49.33

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
7.5	2	0.674	-38.13
	4	0.898	-40.59
	6	1.059	-41.90
11.25	2	0.583	-40.64
	4	0.743	-42.39
	6	0.887	-44.87
15.0	2	0.539	-42.33
	4	0.658	-45.51
	6	0.775	-47.07

Table 5.4 Computed values of G/G_0 at $a/W=0.5$ for Problem II(B). $\psi^*=\tan^{-1}(K_y/K_x)$

$\eta_1=1$

W/H	η_2	G/G_0	ψ^* (°)
7.5	1	0.587	-36.67
	2	2.298	-36.52
	3	5.224	-34.24
	4	9.425	-32.06
	6	21.618	-28.64
11.25	1	0.530	-38.65
	2	2.101	-37.63
	3	4.787	-34.80
	4	8.627	-32.27
	6	19.763	-28.42
15.0	1	0.502	-39.73
	2	2.007	-37.87
	3	4.570	-34.56
	4	8.231	-31.73
	6	18.803	-27.55

$\eta_1=0.8$

W/H	η_2	G/G_0	ψ^* (°)
7.5	2	1.485	-37.27
	4	6.007	-34.12
	6	13.786	-31.57
11.25	2	1.352	-38.69
	4	5.496	-34.99
	6	12.606	-31.66
15.0	2	1.287	-39.24
	4	5.240	-34.73
	6	11.994	-31.06

Figure 2.1 Stress Components in a Three-Dimensional Cartesian Coordinate System

Figure 2.2 Local Polar Coordinate System for Crack in Homogeneous Material

Figure 2.3 Bimaterial Interface Crack

Figure 2.4 Crack in an Infinite Anisotropic Plate

Figure 3.1 **Boundary Value Problem in Elastostatics**

Figure 3.2 Quarter Point Crack Tip Elements

Figure 3.3 Cracks Emanating from a Circular Hole in an Isotropic Plate in Tension

Figure 3.4 Typical Boundary Element Mesh for Example 3.4.1

Figure 3.5 **Edge Crack in an Isotropic Plate in Tension**

Figure 3.6 Boundary Element Mesh for Example 3.4.2

Figure 3.7 Isotropic Bimaterial Interface Crack Subjected to Biaxial Tension

Figure 3.8 Boundary Element Mesh for Example 3.4.3

Figure 3.9 Cracks Emanating from a Circular Hole in an Orthotropic Plate Under Remote Tension

Figure 3.10a Boundary Element Mesh for Example 3.4.4; $(L-R)/L=0.4$

Figure 3.10b Boundary Element Mesh for Example 3.4.4; $(L-R)/L=0.6$

Figure 3.11 Variation of normalised stress intensity factor, K_I/K_0 , with relative crack length, $(L-R)/R$, for cracks emanating from a circular hole in an orthotropic plate (Example 3.4.4). $K_0 = \sigma\sqrt{\pi L}$.

Figure 3.12 Variation of normalised stress intensity factors, K_I/K_0 and K_{II}/K_0 , with relative crack length, $(L-R)/R$, for cracks emanating from a circular hole in an orthotropic plate (Example 3.4.4). $K_0 = \sigma\sqrt{\pi L}$.

Figure 3.13 Bimaterial Interface Crack Between Anisotropic Materials ((1) Single Crystal Alumina and (2) FSP Zirconia) Subjected to Internal Pressure

Figure 4.1 **A Double Cantilever Beam Specimen**

(a) Problem (I)

(b) Problem (II)

Figure 4.2 Load Cases for the Double Cantilever Beam Specimen

Figure 4.3 Typical BIE Mesh for Problems (I) and (II)

Figure 4.4 Variation of K_1/K_0 with η_2 for Problem (I):
 $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.5 Variation of K_1/K_0 with η_2 for Problem (I):
 $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.6 Variation of K_1/K_0 with η_2 for Problem (I):
 $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.7 Variation of K_1/K_0 with η_2 for Problem (I):
 $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.8 Variation of K_1/K_0 with η_2 for Problem (I): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.9 Variation of K_1/K_0 with η_2 for Problem (I): $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.10 Variation of K_2/K_0 with η_2 for Problem (I): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.11 Variation of K_2/K_0 with η_2 for Problem (I): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.12 Variation of K_2/K_0 with η_2 for Problem (I):
 $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.13 Variation of K_2/K_0 with η_2 for Problem (I):
 $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.14 Variation of K_2/K_0 with η_2 for Problem (I):
 $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.15 Variation of K_2/K_0 with η_2 for Problem (I):
 $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.16 Variation of K_1/K_0 with a/W for Problem (D):
 $\eta_1=1, W/H=5, K_0=(P/H)\sqrt{\pi a}$

Figure 4.17 Variation of K_1/K_0 with a/W for Problem (I):
 $\eta_1=1, W/H=7.5, K_0=(P/H)\sqrt{\pi a}$

Figure 4.18 Variation of K_1/K_0 with a/W for Problem (I): $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.19 Variation of K_2/K_0 with a/W for Problem (I): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.20 Variation of K_2/K_0 with a/W for Problem (I): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.21 Variation of K_2/K_0 with a/W for Problem (I): $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.22 Variation of K_1/K_0 with a/W for Problem (I): $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.23 Variation of K_1/K_0 with a/W for Problem (I): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.24 Variation of K_I/K_0 with a/W for Problem (I): $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.25 Variation of K_{II}/K_0 with a/W for Problem (I): $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.26 Variation of K_2/K_0 with a/W for Problem (I):
 $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.27 Variation of K_2/K_0 with a/W for Problem (I):
 $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.28 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=1$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.29 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.30 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=1$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.31 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=0.8$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.32 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^3/E_1H^3$

Figure 4.33 Variation of G/G_0 with η_2 for Problem (I): $\eta_1=0.8$, $W/H=10$, $G_0=12P^2a^3/E_1H^3$

Figure 4.34 Variation of G/G_0 with a/W for Problem (I): $\eta_1=1$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.35 Variation of G/G_0 with a/W for Problem (I): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.36 Variation of G/G_0 with a/W for Problem (I): $\eta_1=1$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.37 Variation of G/G_0 with a/W for Problem (I): $\eta_1=0.8$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.38 Variation of G/G_0 with a/W for Problem (I):
 $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.39 Variation of G/G_0 with a/W for Problem (I):
 $\eta_1=0.8$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.40 Variation of K_1/K_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.41 Variation of K_1/K_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.42 Variation of K_i/K_o with η_2 for Problem (II): $\eta_1=1$, $W/H=10$, $K_o=(P/H)\sqrt{\pi a}$

Figure 4.43 Variation of K_i/K_o with η_2 for Problem (II): $\eta_1=0.8$, $W/H=5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 4.44 Variation of K_i/K_o with η_2 for Problem (II):
 $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 4.45 Variation of K_i/K_o with η_2 for Problem (II):
 $\eta_1=0.8$, $W/H=10$, $K_o=(P/H)\sqrt{\pi a}$

Figure 4.46 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.47 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.48 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.49 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.50 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.51 Variation of K_2/K_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.52 Variation of K_I/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.53 Variation of K_I/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.54 Variation of K_1/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.55 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.56 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.57 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=1$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.58 Variation of K_1/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.59 Variation of K_1/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.60 Variation of K_1/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.61 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.62 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.63 Variation of K_2/K_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=10$, $K_0=(P/H)\sqrt{\pi a}$

Figure 4.64 Variation of G/G_0 with η_2 for Problem (II):
 $\eta_1 = 1$, $W/H = 5$, $G_0 = 12P^2a^2/E_1H^3$

Figure 4.65 Variation of G/G_0 with η_2 for Problem (II):
 $\eta_1 = 1$, $W/H = 7.5$, $G_0 = 12P^2a^2/E_1H^3$

Figure 4.66 Variation of G/G_0 with η_2 for Problem (II): $\eta_1=1$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.67 Variation of G/G_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.68 Variation of G/G_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.69 Variation of G/G_0 with η_2 for Problem (II): $\eta_1=0.8$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.70 Variation of G/G_0 with a/W for Problem (II): $\eta_1=1$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.71 Variation of G/G_0 with a/W for Problem (II): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.72 Variation of G/G_0 with a/W for Problem (II): $\eta_1=1$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

Figure 4.73 Variation of G/G_0 with a/W for Problem (II): $\eta_1=0.8$, $W/H=5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.74 Variation of G/G_0 with a/W for Problem (II):
 $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 4.75 Variation of G/G_0 with a/W for Problem (II):
 $\eta_1=0.8$, $W/H=10$, $G_0=12P^2a^2/E_1H^3$

(A) Fibres in the middle layer are in the out- of- plane direction

(B) Fibres in the middle layer are parallel to the long axis of the beam

Figure 5.1 Layups for the Composite Beam with an Interface Crack

Problem I

Problem II

Figure 5.2 Load Cases for Trilayer Cantilever Beam

Figure 5.3 Typical BIE Mesh for Problems I and II

Figure 5.4 Variation of K_i/K_o with η_2 for Problem I(A): $\eta_1=1$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.5 Variation of K_i/K_o with η_2 for Problem I(A): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.6 Variation of K_1/K_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.7 Variation of K_1/K_0 with η_2 for Problem I(A): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.8 Variation of K_i/K_o with η_2 for Problem I(A):
 $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.9 Variation of K_i/K_o with η_2 for Problem I(A):
 $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.10 Variation of K_2/K_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.11 Variation of K_2/K_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.12 Variation of K_2/K_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.13 Variation of K_2/K_0 with η_2 for Problem I(A): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.14 Variation of K_2/K_0 with η_2 for Problem I(A):
 $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.15 Variation of K_2/K_0 with η_2 for Problem I(A):
 $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.16 Variation of K_I/K_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.17 Variation of K_I/K_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.18 Variation of K_1/K_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.19 Variation of K_2/K_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.20 Variation of K_j/K_o with a/W for Problem I(A): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.21 Variation of K_j/K_o with a/W for Problem I(A): $\eta_1=1$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.22 Variation of K_1/K_0 with a/W for Problem I(A):
 $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.23 Variation of K_1/K_0 with a/W for Problem I(A):
 $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.24 Variation of K_i/K_o with a/W for Problem I(A): $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.25 Variation of K_j/K_o with a/W for Problem I(A): $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.26 Variation of K_2/K_0 with a/W for Problem I(A):
 $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.27 Variation of K_2/K_0 with a/W for Problem I(A):
 $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.28 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.29 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.30 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=1$, $W/H=15$, $G_0=12P^2a^3/E_1H^3$

Figure 5.31 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^3/E_1H^3$

Figure 5.32 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=0.8$, $W/H=11.25$, $G_0=12P_0^2a^2/E_1I^3$

Figure 5.33 Variation of G/G_0 with η_2 for Problem I(A): $\eta_1=0.8$, $W/H=15$, $G_0=12P_0^2a^2/E_1I^3$

Figure 5.34 Variation of G/G_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.35 Variation of G/G_0 with a/W for Problem I(A): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.36 Variation of G/G_0 with a/W for Problem I(A): $\eta_1 = 1$, $W/H = 15$, $G_0 = 12P^2a^2/E_1H^3$

Figure 5.37 Variation of G/G_0 with a/W for Problem I(A): $\eta_1 = 0.8$, $W/H = 0.8$, $G_0 = 12P^2a^2/E_1H^3$

Figure 5.38 Variation of G/G_0 with a/W for Problem I(A):
 $\eta_1 = 0.8$, $W/H = 11.25$, $G_0 = 12P^2a^2/E_1H^3$

Figure 5.39 Variation of G/G_0 with a/W for Problem I(A):
 $\eta_1 = 0.8$, $W/H = 15$, $G_0 = 12P^2a^2/E_1H^3$

Erdogan, F. and Wu, B. (1993), *Interface crack problems in layered orthotropic materials*, J. Mech. Phys. Solids **41**, 5, 889-917

Gross, B. and Srawley, J.E. (1964), *Stress intensity factors for a single edge notch tension specimen by boundary collocation of a stress function*, NASA TN D-2395

Hashemi, S., Kinloch, A.J. and Williams, J.G. (1990), *The effects of geometry, rate and temperature on the mode I , mode II and mixed-mode I/II interlaminar fracture of carbon-fibre/poly(ether-ether ketone) composites*, J. Comp. Mater. **24**, 918-956

Henshell, R.D. and Shaw, K.G. (1975), *Crack-tip elements are unnecessary*, Int. J. Numer. Methods Engng **9**, 495-509

Hutchinson, J.W., Mear, M.E. and Rice, J.R. (1987), *Crack paralleling and interface between dissimilar materials*, J. Appl. Mech. **54**, 828-832

Irwin, G.R. (1958), *Fracture*, In *Handbuch der Physik*, Vol.VI, Springer

Jia, Z.H., Shippy, D.J. and Rizzo, F.J. (1988), *On the computation of two-dimensional stress intensity factors using the boundary element method*, Int. J. Numer. Methods Engng. **26**, 2739-53

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Figure 5.40 Variation of K_i/K_o with η_2 for Problem I(B):
 $\eta_1 = i$, $W/H = 7.5$, $K_o = (P/H)\sqrt{\pi a}$

Figure 5.41 Variation of K_i/K_o with η_2 for Problem I(B):
 $\eta_1 = 1$, $W/H = 11.25$, $K_o = (P/H)\sqrt{\pi a}$

Figure 5.42 Variation of K_1/K_0 with η_2 for Problem I(B):
 $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\tau a}$

Figure 5.43 Variation of K_1/K_0 with η_2 for Problem I(B):
 $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\tau a}$

Figure 5.44 Variation of K_1/K_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.45 Variation of K_1/K_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.46 Variation of K_2/K_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.47 Variation of K_2/K_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.48 Variation of K_j/K_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.49 Variation of K_j/K_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.50 Variation of K_2/K_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.51 Variation of K_2/K_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.52 Variation of K_i/K_o with a/W for Problem I(B): $\eta_1=1$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.53 Variation of K_i/K_o with a/W for Problem I(B): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.54 Variation of K_i/K_o with a/W for Problem I(B):
 $\eta_1=1$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.55 Variation of K_j/K_o with a/W for Problem I(B):
 $\eta_1=1$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.56 Variation of K_y/K_o with a/W for Problem I(B): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.57 Variation of K_y/K_o with a/W for Problem I(B): $\eta_1=1$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.58 Variation of K_i/K_o with a/W for Problem I(B): $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.59 Variation of K_i/K_o with a/W for Problem I(B): $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.60 Variation of K_I/K_0 with a/W for Problem I(B): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.61 Variation of K_2/K_0 with a/W for Problem I(B): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.62 Variation of K_2/K_0 with a/W for Problem I(B):
 $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.63 Variation of K_2/K_0 with a/W for Problem I(B):
 $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.64 Variation of G/G_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.65 Variation of G/G_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.66 Variation of G/G_0 with η_2 for Problem I(B): $\eta_1=1$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.67 Variation of G/G_0 with η_2 for Problem I(B): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.68 Variation of G/G_0 with η_2 for Problem I(B):
 $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.69 Variation of G/G_0 with η_2 for Problem I(B):
 $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.70 Variation of G/G_0 with a/W for Problem I(B):
 $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.71 Variation of G/G_0 with a/W for Problem I(B):
 $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.72 Variation of G/G_0 with a/W for Problem I(B):
 $\eta_1 = 1$, $W/H = 15$, $G_0 = 12P^2a^2/E_1H^3$

Figure 5.73 Variation of G/G_0 with a/W for Problem I(B):
 $\eta_1 = 0.8$, $W/H = 7.5$, $G_0 = 12P^2a^2/E_1H^3$

Figure 5.74 Variation of G/G_0 with a/W for Problem I(B): $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.75 Variation of G/G_0 with a/W for Problem I(B): $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.76 Variation of K_i/K_o with η_2 for Problem II(A): $\eta_1=1$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.77 Variation of K_i/K_o with η_2 for Problem II(A): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.78 Variation of K_1/K_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.79 Variation of K_1/K_0 with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.80 Variation of K_i/K_o with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.81 Variation of K_i/K_o with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.82 Variation of K_j/K_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.83 Variation of K_j/K_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.84 Variation of K_2/K_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.85 Variation of K_2/K_0 with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.86 Variation of K_2/K_0 with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.87 Variation of K_2/K_0 with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.88 Variation of K_I/K_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.89 Variation of K_I/K_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.90

Variation of K_1/K_0 with a/W for Problem II(A):
 $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.91

Variation of K_2/K_0 with a/W for Problem II(A):
 $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.92 Variation of K_2/K_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.93 Variation of K_2/K_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.94 Variation of K_1/K_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.95 Variation of K_1/K_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.96 Variation of K_I/K_0 with a/W for Problem II(A):
 $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.97 Variation of K_f/K_0 with a/W for Problem II(A):
 $\eta_1=0.8$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.98 Variation of K_2/K_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.99 Variation of K_2/K_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.100 Variation of G/G_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.101 Variation of G/G_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.102 Variation of G/G_0 with η_2 for Problem II(A): $\eta_1=1$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.103 Variation of G/G_0 with η_2 for Problem II(A): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.104 Variation of G/G_0 with η_2 for Problem II(A):
 $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.105 Variation of G/G_0 with η_2 for Problem II(A):
 $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.106 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.107 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.108 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=1$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.109 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.110 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.111 Variation of G/G_0 with a/W for Problem II(A): $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.112 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=1$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.113 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=1$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.114 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=1$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.115 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.116 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.117 Variation of K_i/K_o with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.118 Variation of K_2/K_0 with η_2 for Problem II(B): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.119 Variation of K_2/K_0 with η_2 for Problem II(B): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.120 Variation of K_z/K_o with η_2 for Problem II(B):
 $\eta_1=1$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.121 Variation of K_z/K_o with η_2 for Problem II(B):
 $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.122 Variation of K_2/K_0 with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.123 Variation of K_2/K_0 with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.124 Variation of K_I/K_0 with a/W for Problem II(B):
 $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.125 Variation of K_I/K_0 with a/W for Problem II(B):
 $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.126 Variation of K_1/K_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.127 Variation of K_2/K_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=7.5$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.128 Variation of K_2/K_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.129 Variation of K_2/K_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.130 Variation of K_i/K_o with a/W for Problem II(B): $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.131 Variation of K_i/K_o with a/W for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.132 Variation of K_i/K_o with a/W for Problem II(B):
 $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.133 Variation of K_2/K_o with a/W for Problem II(B):
 $\eta_1=0.8$, $W/H=7.5$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.134 Variation of K_I/K_0 with a/W for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.135 Variation of K_I/K_0 with a/W for Problem II(B): $\eta_1=0.8$, $W/H=15$, $K_0=(P/H)\sqrt{\pi a}$

Figure 5.136 Variation of G/G_0 with η_2 for Problem II(B): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.137 Variation of G/G_0 with η_2 for Problem II(B): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.138 Variation of G/G_0 with η_2 for Problem II(B):
 $\eta_1=1$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.139 Variation of G/G_0 with η_2 for Problem II(B):
 $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.140 Variation of G/G_0 with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.141 Variation of G/G_0 with η_2 for Problem II(B): $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.142 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.143 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.144 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=1$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

Figure 5.145 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=0.8$, $W/H=7.5$, $G_0=12P^2a^2/E_1H^3$

Figure 5.146 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=0.8$, $W/H=11.25$, $G_0=12P^2a^2/E_1H^3$

Figure 5.147 Variation of G/G_0 with a/W for Problem II(B): $\eta_1=0.8$, $W/H=15$, $G_0=12P^2a^2/E_1H^3$

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Figure 5.44 Variation of K_i/K_o with η_2 for Problem I(B):
 $\eta_1=0.8$, $W/H=11.25$, $K_o=(P/H)\sqrt{\pi a}$

Figure 5.45 Variation of K_i/K_o with η_2 for Problem I(B):
 $\eta_1=0.8$, $W/H=15$, $K_o=(P/H)\sqrt{\pi a}$