An investigation of weak amenability of hyperbolic groups

by

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Abstract

The purpose of this thesis is to prove that hyperbolic groups are weakly amenable. Precisely, the definition of weak amenability is as follows: Let Γ be a countable discrete group. From [CH89] we say **Γ is weakly amenable with constant C** if there exist a sequence of finitely supported functions $\phi_n : \Gamma \to \mathbb{C}$, such that $\phi_n \to 1$ pointwise, and $\sup\{\|\phi_n\|_{CB} : n \in \mathbb{N}\} \leq C$.

In the definition of weak amenability, we interpret the sequence of functions, $(\phi_n)_{n=1}^{\infty}$, *to be an approximate identity whose CB-norm is uniformly bounded by C*. The proof of the statement actually draws on many areas of mathematics. In this thesis, we give a quick treatment of the necessary background information before moving onto the associated propositions, lemmas, and theorems used in the proof.

In chapter 1 we begin with an introduction to Hilbert spaces and tensor products. While the tensor product of Hilbert spaces remains an inner product space, what might be surprising is that it is no longer complete (i.e. it is a pre-Hilbert space). The main purpose of chapter 1 is to take the (Hilbert) completion of this inner product space, and to explore the associated properties.

In chapter 2 we turn our attention towards the linear maps between operator spaces. By the work of Ruan [WBF+01], we define an operator space to be a closed linear subspace of a $\mathcal{C}^*$ algebra. We are interested in studying the class of linear maps between operator spaces, and not the operator spaces themselves. One such class of linear maps are the Schur multipliers. Next, we introduce the completely-bounded (CB) norm, which is a norm on the linear maps. In general, it is difficult to compute the CB norm directly, but fortunately, when we restrict our attention to Schur multipliers, there are some theorems which help us in that regard.

In chapter 3 the focus shifts to geometric group theory. Here we introduce the concepts of graphs, word metrics, geodesicity, and hyperbolicity. In particular, hyperbolicity provides a rich source of propositions and theorems which can be far reaching. Nearing the end of chapter 3, we introduce a theorem about hyperbolic graphs which is connected to the tensor product of Hilbert spaces.

In chapter 4 we connect hyperbolic graphs with kernels and Schur multipliers, and prove some theorems about the CB norms. The main result is to prove that hyperbolic groups are weakly amenable, and we show that this comes as a corollary to the work from chapter 4.

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I would also like to thank my family and friends because without them, I wouldn't have made it very far in life.

I have learned a lot, and I still have a lot learn.

Table of Contents

Chapter 1

Preliminary Material

The focus of this chapter is to provide the preliminary background material used in this thesis. Chapter 1 is not intended to be a comprehensive treatment of its topics.

1.1 Banach Spaces, Hilbert Spaces, the tensor product

In section 1.1 we cover the basic definitions, which will be familiar if the reader has taken a course in functional analysis and abstract algebra.

Definition 1.1.1 *A* ***Banach space*** $(X, \|\cdot\|)$ *is a normed vector space which is complete with respect to its induced metric. We define* $X_1 = \{x \in X \mid \|x\| \leq 1\}$ *to be the closed unit ball of* X *centered at 0.*

We usually will write X is a Banach space; it's understood that there is an associated norm. Unless stated otherwise, we exclusively work with infinite dimensional vector spaces over the complex field.

Definition 1.1.2 *Let* $f : X \to Y$ *be a linear map between normed vector spaces. The* ***operator norm*** *of* f *is defined as*

$$\|f\|_{op} = \sup\{\|f(x)\|_Y : x \in X_1\}$$

which is equal to

$$\|f\|_{op} = \inf\{k : \|f(x)\|_Y \leq k\|x\|_X \textit{ for all } x \in X\}$$

We can view the first definition as the supremum of $\|f(x)\|_Y$ over the closed unit ball of X, and the second definition as the infimum of the Lipschitz constants. We say $f : X \to Y$ is a **contractive linear map** when $\|f\|_{op} \leq 1$.

Definition 1.1.3 *Let X and Y be normed vector spaces. We define the **set of bounded linear maps from X to Y** to be $\mathbb{B}(X,Y) = \{f : X \to Y \mid f \text{ is linear}, \|f\|_{op} < \infty\}$ and we note $\mathbb{B}(X,Y)$ equipped with $\|\cdot\|_{op}$ is a normed vector space.*

In the setting where $X = Y$, we simply write $\mathbb{B}(X)$ in place of $\mathbb{B}(X,Y)$. Also, we usually will not distinguish the norm when it is obvious. For example, if $f \in \mathbb{B}(X)$, then we use $\|f\|$ to denote $\|f\|_{op}$.

Lemma 1.1.4 *Let X and Y be metric spaces, with metric completions X' and Y' respectively. If $f : X \to Y$ a uniformly continuous function, then there is a unique continuous extension of f, denoted $f' : X' \to Y'$.*

Lemma 1.1.5 *Let X and Y be normed vector spaces, with norm-metric completions X' and Y' respectively (i.e. they are Banach spaces). If $f : X \to Y$ is a bounded linear map, there is an unique continuous extension of f, denoted $f' : X' \to Y'$ which will remain bounded and linear. Furthermore, $\|f'\| = \|f\|$.*

The main difference between the two lemmas above is with the object in question. In the setting of linear maps between normed vector spaces, we know that the concepts of boundedness, Lipschitz continuity, uniformly continuity, continuity, and continuity at a point are all equivalent.

Definition 1.1.6 *A **Hilbert space H** is an inner product vector space which is complete with respect to its induced metric. Let $\mathcal{E} \subseteq H$ be an orthonormal set. We say **$\mathcal{E}$ is an orthonormal basis** when $\mathcal{E}$ is a maximal orthonormal set with respect to containment.*

*Hilbert spaces H and K are **unitarily isomorphic** when there exists an isomorphism $\phi : H \to K$ which preserves the inner product; that is $\langle \phi(h), \phi(h') \rangle_K = \langle h, h' \rangle$ for all $h, h' \in H$.*

Examples.

- The real line, $\mathbb{R}$, is a Hilbert space with respect to the dot inner product.
- By a standard theorem in real analysis, all finite dimensional normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$ are complete, and hence are Banach spaces.
- As a corollary, all finite dimensional inner product spaces over $\mathbb{R}$ or $\mathbb{C}$ are complete, and hence are Hilbert spaces.
- $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$ are both Banach spaces with respects to $\|\cdot\|_\infty$, the supremum norm.
- By a standard theorem in measure theory, all $L^p(X,\mu)$ spaces are Banach spaces for $p \in [0,\infty]$, and is a Hilbert space if and only if $p = 2$ or $|X| = 1$.
- Trivially, all Hilbert spaces are already Banach spaces.

Lemma 1.1.7 *By Zorn's lemma, all Hilbert Spaces H have an orthonormal basis.*

Lemma 1.1.8 *Let H be a Hilbert space, and $\mathcal{E} \subseteq H$ be an orthonormal set. $\mathcal{E} \subseteq H$ is an orthonormal basis if and only if $H = \overline{\text{Span}(\mathcal{E})}$.*

Lemma 1.1.9 *If Hilbert space H has an orthonormal basis $\mathcal{E}$, then $H \simeq \ell_2(\mathcal{E})$ unitarily.*

Lemma 1.1.10 *If H and K are Hilbert spaces then $H \oplus K$ is also a Hilbert space with an inner product which satisfies*

$$\langle (h,k),(h',k') \rangle_{H\oplus K} = \langle h,h' \rangle_H + \langle k,k' \rangle_K$$

Definition 1.1.11 *If $f : H \to K$ is a bounded linear map between Hilbert spaces, then **the adjoint of f** is a bounded linear map, which is denoted by $f^* : K \to H$, and it satisfies*

$$\langle f(h), k \rangle_K = \langle h, f^*(k) \rangle_H$$

where $h \in H$ and $k \in K$.

That is if $f \in \mathbb{B}(H,K)$, then $f^* \in \mathbb{B}(K,H)$, and not surprisingly, $f^{**} = f$.

Definition 1.1.12 *Borrowing a processes from [AM94], given complex vector spaces V and W, the tensor product is constructed as follows:*

Let $\mathbb{C}^{(V\times W)}$ be the set of functions of finite support from $V \times W$ to $\mathbb{C}$. When $v \in V$ and $w \in W$, let $\chi_{(v,w)} : V \times W \to \mathbb{C}$ be the indicator function on (v,w). Let $S \subseteq \mathbb{C}^{(V\times W)}$ be the subspace spanned by elements of the following four types:

1. $\chi_{v+v',w} - \chi_{v,w} - \chi_{v',w}$
2. $\chi_{v,w+w'} - \chi_{v,w} - \chi_{v,w'}$
3. $s \cdot \chi_{v,w} - \chi_{sv,w}$
4. $s \cdot \chi_{v,w} - \chi_{v,sw}$

where $v, v' \in V, w, w' \in W, s \in \mathbb{C}$.

*The **(algebraic) tensor product** of V and W is defined as $V \otimes W = \mathbb{C}^{(V\times W)}/S$. An **elementary tensor** is denoted $v \otimes w$ where $v \in V$ and $w \in W$. We can think of $v \otimes w$ as the corresponding element of $\chi_{v,w}$ in the quotient space. We note that elementary tensors are objects which are bilinear. That is if $v, v' \in V$, $w, w' \in W$ and $s \in \mathbb{C}$, then*

$$(sv + v') \otimes w = s \cdot v \otimes w + v' \otimes w$$

and

$$v \otimes (sw + w') = s \cdot v \otimes w + v \otimes w'$$

Therefore $V \otimes W = \text{Span}\{v \otimes w : v \in V, w \in W\}$ remains a complex vector space.

Proposition 1.1.13 *Let V and W be vector spaces. Then there exists a vector space $V \otimes W$ and a bilinear map $\phi : V \times W \to V \otimes W$ with the "factor through" property:*

> *If $f : V \times W \to P$ is a bilinear map and P is a vector space, then there is a linear map $\overline{f} : V \otimes W \to P$ such that $f = \overline{f} \circ \phi$.*
>
> *Furthermore, if the bilinear map, $\phi' : V \times W \to T$, where T is a vector space, also has the "factor through" property, then there exists an isomorphism $\psi : T \to V \otimes W$ such that $\phi = \psi \circ \phi'$.*

This is known as the **universal property of tensors**, and it will be frequently used in section 1.2. In the case of conjugate bilinear functions, an analogous process exists.

Proposition 1.1.14 *Let V and W be vector spaces with bases $\mathcal{B}_V$ and $\mathcal{B}_W$ respectively. Then $\{x \otimes y : x \in \mathcal{B}_V, y \in \mathcal{B}_W\}$ is a basis of $V \otimes W$.*

1.2 The tensor product of Hilbert Spaces

Unlike section 1.1, when working with Hilbert spaces H and K, there are a few differences. To begin, we refer to the algebraic tensor product as $H \odot K$, which happens to be an inner product space.

However, it turns out that $H \odot K$ is not complete because both H and K are infinite dimensional. Thus, the Hilbert space completion of $H \odot K$ will be denoted as $H \otimes K$. The purpose of section 1.2 will be to fill in these details.

Proposition 1.2.1 *Let H and K be Hilbert spaces. If $v = \sum_{i=1}^{n} h_i \otimes k_i$ in $H \odot K$, then there exists a finite orthonormal set in $\{e_1, \ldots, e_n\} \subseteq H$ such that $v = \sum_{p=1}^{n} e_p \otimes y_p$ where $y_p \in K$.*

Proof. Without a loss of generality, $\{h_1, \ldots, h_n\}$ is linearly independent, so by Gram-Schmidt's procedure, there exists a finite orthonormal set in H such that $\text{Span}\{e_1, \ldots, e_n\} = \text{Span}\{h_1, \ldots h_n\}$. Then

$$\begin{aligned}
v &= h_1 \otimes k_1 + \cdots + h_n \otimes k_n \\
&= \left(\sum_{p=1}^{n} \langle h_1, e_p \rangle e_p \right) \otimes k_1 + \cdots + \left(\sum_{p=1}^{n} \langle h_n, e_p \rangle e_p \right) \otimes k_n \\
&= \sum_{p=1}^{n} (\langle h_1, e_p \rangle e_p \otimes k_1 + \cdots + \langle h_n, e_p \rangle e_p \otimes k_n) \\
&= \sum_{p=1}^{n} e_p \otimes (\langle h_1, e_p \rangle k_1 + \cdots + \langle h_n, e_p \rangle k_n) \\
&= \sum_{p=1}^{n} e_p \otimes y_p
\end{aligned}$$

where $y_p = \langle h_1, e_p\rangle k_1 + \cdots + \langle h_n, e_p\rangle k_n \in K$.

$\square$

Proposition 1.2.2 *If H and K are Hilbert spaces, then their (algebraic) tensor product, $H \odot K$, is an inner product space which satisfies*

$$\left\langle \sum_{i=1}^{m} h_i \otimes k_i, \sum_{j=1}^{n} x_j \otimes y_j \right\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} \langle h_i, x_j\rangle_H \langle k_i, y_j\rangle_K$$

Note that this proposition is listed as proposition 3.2.1 in [BO08]. Since H and K are infinite dimensional, then $H \odot K$ is not complete.

Proof. **For the existence**, we let $(h, k) \in H \times K$ and define $f_{h,k} : H \times K \to \mathbb{C}$ such that

$$f_{h,k}(x, y) = \langle h, x\rangle_H \langle k, y\rangle_K$$

Immediately, $f_{h,k}$ is conjugate-bilinear, so by the universal tensor property, there exists a well-defined mapping, $\overline{f_{h,k}} : H \odot K \to \mathbb{C}$ such that

$$\overline{f_{h,k}}(x \otimes y) = \langle h, x\rangle_H \langle k, y\rangle_K$$

and extend it by conjugate-linearity.

Let $(H \odot K)^{*,\mathcal{C}}$ be the set of conjugate-bilinear functionals on $H \odot K$. It's not hard to see that $(H \odot K)^{*,\mathcal{C}}$ is a complex vector space.

We define $\Phi : H \times K \to (H \odot K)^{*,\mathcal{C}}$ where $\Phi(h, k) = \overline{f_{h,k}}$. Similar to the work above, Φ is bilinear. Then, by the universal tensor property, we have a well-defined mapping, $\overline{\Phi} : H \odot K \to (H \odot K)^{*,\mathcal{C}}$, where

$$\overline{\Phi}(h \otimes k) = \overline{f_{h,k}}$$

and extend it by linearity.

Finally, we define $\langle \cdot, \cdot\rangle : H \odot K \times H \odot K \to \mathbb{C}$ as follows:

$$\langle v, w\rangle = \overline{\Phi}(v)(w)$$

where $v = \sum_{i=1}^{m} h_i \otimes k_i$ and $w = \sum_{j=1}^{n} x_j \otimes y_j$ are elements of $H \odot K$.

By simplifying,

$$\begin{aligned}
\langle v, w\rangle &= \overline{\Phi}(v)(w) \\
&= \sum_{i=1}^{m} \overline{\Phi}(h_i \otimes k_i)(w) \\
&= \sum_{i=1}^{m} \overline{f_{h_i,k_i}}(w) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{f_{h_i,k_i}}(x_j \otimes y_j) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h_i, x_j\rangle \cdot \langle k_i, y_j\rangle
\end{aligned}$$

and this expression is well-defined.

Next, we verify the inner product axioms. We remark that the axioms of conjugate symmetry and left linearity are trivially true by definition. It remains to check $\langle\cdot,\cdot\rangle$ is positive-definite.

Let $v = \sum_{i=1}^{n} h_i \otimes k_i$ in $H \odot K$. By proposition 1.2.1, there exists an orthonormal set $\{e_1, \ldots, e_n\} \subseteq H$ such that

$$v = \sum_{p=1}^{n} e_p \otimes k'_p$$

where $k'_p \in K$. Then

$$\begin{aligned}
\langle v, v\rangle &= \left\langle \sum_{p=1}^{n} e_p \otimes k'_p, \sum_{q=1}^{n} e_q \otimes k'_q \right\rangle \\
&= \sum_{p,q} \langle e_p, e_q\rangle \cdot \langle k'_p, k'_q\rangle \\
&= \sum_{p=1}^{n} \|k'_p\|^2 \\
&\geq 0
\end{aligned}$$

If $\langle v, v\rangle = 0$, then $\|k'_p\|^2 = 0$, which implies $k'_p = 0$. Then

$$v = \sum_{p=1}^{n} e_p \otimes k'_p = 0$$

We conclude $(H \odot K, \langle\cdot,\cdot\rangle)$ is indeed a pre-Hilbert space with the desired inner-product.

□

Remark. The next step is to take the completion. From metric space theory, there are no issues with the existence of a complete $H \otimes K$, but the question then becomes, is this new metric still induced by an inner product on $H \otimes K$? **That is, we've extended the metric, but have we extended the inner product?** Fortunately, by use of the proposition 1.2.3, the answer here is yes. We use a familiar process, which borrows heavily from the completion of a metric space.

Let H be a pre-Hilbert space, and consider the set of Cauchy sequences in H. We can form an equivalence relation on the Cauchy sequences as follows:

$$(x_n)_n \sim (y_n)_n \text{ if and only if } \lim_n \|x_n - y_n\|_H = 0$$

Next, we define H' to be the set of equivalence classes of Cauchy sequences. It is not hard to see that H' is a complex vector space when endowed with the usual candidates for addition and scalar multiplication, which are well-defined. Define $\langle \cdot, \cdot \rangle : H' \times H' \to \mathbb{C}$ as

$$\langle [(x_n)_n], [(y_n)_n] \rangle_{H'} = \lim_n \langle x_n, y_n \rangle_H$$

where $[(x_n)_n]$ and $[(y_n)_n]$ are in H'.

Proposition 1.2.3 *We will verify the following:*

i. $\lim_n \langle x_n, y_n \rangle_H$ *does exist*

ii. $\langle \cdot, \cdot \rangle_{H'}$ *is well-defined*

iii. $\langle \cdot, \cdot \rangle_{H'}$ *is an inner product*

iv. H *embeds into* H' *as an inner product space*

v. H' *is complete and* H *is dense* H'

Proof. **For (i)**, we know that $(x_n)_n$ and $(y_n)_n$ are Cauchy in H, which implies that their norms are bounded above by B. Then

$$\begin{aligned} |\langle x_m, y_m \rangle_H - \langle x_n, y_n \rangle_H| &\leq |\langle x_m, y_m \rangle_H - \langle x_m, y_n \rangle_H| + |\langle x_m, y_n \rangle_H - \langle x_n, y_n \rangle_H| \\ &= |\langle x_m, y_m - y_n \rangle_H| + |\langle x_m - x_n, y_n \rangle_H| \\ &\leq \|x_m\| \cdot \|y_m - y_n\| + \|x_m - x_n\| \cdot \|y_n\| \\ &\leq B \cdot (\|y_m - y_n\| + \|x_m - x_n\|) \end{aligned}$$

Thus, $(\langle x_n, y_n \rangle_H)_n$ is a Cauchy sequence in $\mathbb{C}$, and so the limit will exist.

For (ii), we consider $(x_n)_n \sim (x'_n)_n$ and $(y_n)_n \sim (y'_n)_n$. Again, these are Cauchy sequences, which are bounded above. As $n \to \infty$, we have

$$\begin{aligned}|\langle x_n, y_n\rangle_H - \langle x'_n, y'_n\rangle_H| &\le |\langle x_n, y_n\rangle_H - \langle x_n, y'_n\rangle_H| + |\langle x_n, y'_n\rangle_H - \langle x'_n, y'_n\rangle_H| \\ &= |\langle x_n, y_n - y'_n\rangle_H| + |\langle x_n - x'_n, y_n\rangle_H| \\ &\le \|x_n\| \cdot \|y_n - y'_n\| + \|x_n - x'_n\| \cdot \|y'_n\| \\ &\le B(\|y_n - y'_n\| + \|x_n - x'_n\|) \\ &\to 0\end{aligned}$$

which implies

$$\langle [(x_n)_n], [(y_n)_n]\rangle_{H'} = \lim_n \langle x_n, y_n\rangle_H = \lim_n \langle x'_n, y'_n\rangle_H = \langle [(x'_n)_n], [(y'_n)_n]\rangle_{H'}$$

and so $\langle \cdot, \cdot\rangle_{H'}$ is well-defined.

For (iii), we will only verify that $\langle \cdot, \cdot\rangle_{H'}$ is positive definite because the other inner-product axioms are obvious. Then

$$\langle [(x_n)_n], [(x_n)_n]\rangle_{H'} = \lim_n \langle x_n, x_n\rangle_H = \lim_n \|x_n\|_H^2 \ge 0$$

If $\langle [(x_n)_n], [(x_n)_n]\rangle_{H'} = 0$, then $\lim_n \|x_n - 0\|_H = 0$. Thus $(x_n)_n \sim (0_n)_n$ represent the same equivalence class. Thus, $[(x_n)_n]$ is the zero-element of H'.

For (iv), we let $\phi : H \to H'$ be defined as $\phi(h) = [(h, h, h, \ldots)]$. We will verify that ϕ embeds H into H' as an inner product space.

The fact that ϕ is linear is obvious. For injectivity, if $h_1 \neq h_2$, then $\lim_n \|h_1 - h_2\|_H \neq 0$ which implies $[(h_1, h_1, h_1, \ldots)] \neq [(h_2, h_2, h_2, \ldots)]$. Furthermore,

$$\langle \phi(h_1), \phi(h_2)\rangle_{H'} = \langle [(h_1)_n], [(h_2)_n]\rangle_{H'} = \lim_n \langle h_1, h_2\rangle_H = \langle h_1, h_2\rangle_H$$

Thus $\phi : H \to H'$ is indeed an embedding of inner product spaces.

For (v). Since H embeds into H' as an inner product space, then it also embeds as a metric space. To check that H' is complete and $H \subseteq H'$ is dense, the proof is exactly the same as it is in metric spaces.

□

Corollary 1.2.4 *By the propositions above, the pre-Hilbert space, $H \odot K$, embeds into the Hilbert Space, $H \otimes K$ with the desired inner product.*

As with the convention from metric space theory, we will unabashedly assume that $H \odot K \subseteq H \otimes K$, and that the inner product of $H \otimes K$ extends the inner product of $H \odot K$. For this reason, we will not distinguish between $\langle \cdot, \cdot\rangle_{H\odot K}$ and $\langle \cdot, \cdot\rangle_{H\otimes K}$.

1.3 Properties of the tensor product of Hilbert spaces

In the previous section we carefully constructed the tensor product of Hilbert spaces and its completion. In this section, we focus on various properties involved in the tensor product of functions.

Corollary 1.3.1 *If H and K are Hilbert spaces with $h \in H$ and $k \in K$, then immediate from the definitions, $\|h \otimes k\| = \|h\| \cdot \|k\|$.*

Lemma 1.3.2 *Given Hilbert spaces H and K, we pick $h \in H$ and $k \in K$. Then the functions $f_k : H \to H \otimes K$ and $g_h : K \to H \otimes K$ defined by*

i. $f_k(\cdot) = \cdot \otimes k$
ii. $g_h(\cdot) = h \otimes \cdot$

are bounded and linear.

In other words, the processes of left-tensoring and right-tensoring are bounded and linear, hence continuous.

Proof. By the tensor product properties, the linearity of f_k and g_h is obvious. By corollary 1.3.1, for all $x \in H$ and $y \in K$, we have

$$\|f_k(x)\| = \|x\|\|k\| \text{ and } \|g_h(y)\| = \|h\|\|y\|$$

Thus,

$$\|f_k\| \leq \|k\| \text{ and } \|g_h\| \leq \|h\|$$

□

Corollary 1.3.3 *Let H and K be Hilbert spaces. If $\mathcal{E} \subseteq H$ and $\mathcal{F} \subseteq K$ are orthonormal bases of their respective spaces, then $B = \{e \otimes f : e \in \mathcal{E}, f \in \mathcal{F}\}$ is an orthonormal basis of $H \otimes K$.*

Proof. It's easy to see that B is an orthonormal set. Our goal will be to show that $H \otimes K = \overline{\text{Span}(B)}$.

If $h \in H$ and $k \in K$, then by the properties of the orthonormal bases and lemma 1.3.2 we have

$$h \otimes k = \sum_{e \in \mathcal{E}} \sum_{f \in \mathcal{F}} \langle h, e \rangle \langle k, f \rangle e \otimes f \in \overline{\text{Span}(B)}$$

Note that the closure of a normed vector space is still a vector space. Thus, $H \odot K \subseteq \overline{\text{Span}(B)} \subseteq H \otimes K$. Since $H \odot K \subseteq H \otimes K$ is dense, then $H \otimes K = \overline{\text{Span}(B)}$ follows.

□

Proposition 1.3.4 *Let H, K be Hilbert spaces. If $S \in \mathbb{B}(H)$ and $T \in \mathbb{B}(K)$ then there exists a unique linear operator $f_{S\otimes T} \in \mathbb{B}(H \otimes K)$ such that $f_{S\otimes T}(v \otimes w) = Sv \otimes Tw$ for all $v \in H$ and $w \in K$. Furthermore,*

i. $\|f_{S\otimes T}\| = \|S\|\|T\|$

ii. $(f_{S\otimes T})^* = f_{S^*\otimes T^*}$

iii. $(f_{S\otimes T})(f_{S'\otimes T'}) = f_{SS'\otimes TT'}$

iv. the function $\mathbb{B}(H) \times \mathbb{B}(K) \to \mathbb{B}(H \otimes K)$ where $(S, T) \mapsto f_{S\otimes T}$ is bilinear

v. if $S \otimes T \neq 0$ in $\mathbb{B}(H) \odot \mathbb{B}(K)$, then $f_{S\otimes T} \neq 0$ in $\mathbb{B}(H \otimes K)$

In other words, there is a natural *-embedding $\mathbb{B}(H) \odot \mathbb{B}(K) \hookrightarrow \mathbb{B}(H \otimes K)$ such that $S \otimes T \mapsto f_{S\otimes T}$. For convenience we identify $S \otimes T = f_{S\otimes T}$, which is a convention that is also used in [BO08]. More generally, $\mathbb{B}(H) \odot \mathbb{B}(K) \subseteq \mathbb{B}(H \otimes K)$.

Note. This proposition is listed as proposition 3.2.3 in [BO08].

Proof. **For existence** we consider the following maps, $f_{S,id_K} : H \times K \to H \odot K$ and $f_{id_H,T} : H \times K \to H \odot K$ defined as

$$f_{S,id_K}(v, w) = Sv \otimes id_K(w) = Sv \otimes w$$

and

$$f_{id_H,T}(v, w) = id_H(v) \otimes Tw = v \otimes Tw$$

Immediately, both f_{S,id_K} and $f_{id_H,T}$ are bilinear. Then by the universal tensor property, we have the corresponding linear maps, $f_{S\odot id_K}$, $f_{id_H\odot T} : H \odot K \to H \odot K$.

Observe $f_{S\odot id_K}$ and $f_{id_H\odot T}$ are also bounded. Indeed, consider $x \in H \odot K$. By proposition 1.2.1, there exists an orthonormal set $\{e_i : 1 \leq i \leq n\} \subseteq H$ such that $x = \sum_{i=1}^n e_i \otimes k_i$ where $k_i \in K$. Observe that both $\{e_i \otimes k_i : 1 \leq i \leq n\}$ and $\{e_i \otimes T(k_i) : 1 \leq i \leq n\}$ are orthogonal sets. Then

$$\|f_{id_H\odot T}(x)\|^2 = \sum_{i=1}^n \|e_i \otimes T(k_i)\|^2 = \sum_{i=1}^n \|e_i\|^2\|T(k_i)\|^2 \leq \sum_{i=1}^n \|e_i\|^2\|T\|^2\|k_i\|^2$$

and also,

$$\sum_{i=1}^n \|e_i\|^2\|T\|^2\|k_i\|^2 = \|T\|^2 \cdot \sum_{i=1}^n \|e_i\|^2\|k_i\|^2 = \|T\|^2 \cdot \sum_{i=1}^n \|e_i \otimes k_i\|^2 = \|T\|^2 \cdot \|x\|^2$$

which implies $\|f_{id_H\odot T}\| \leq \|T\|$.

Likewise, the same reasoning shows $\|f_{S\odot id_K}\| \leq \|S\|$. Thus, both $f_{S\odot id_K}$ and $f_{id_H\odot T}$ are in $\mathbb{B}(H \odot K)$.

By lemma 1.1.5 we denote $f_{id_H\otimes T}$ and $f_{S\otimes id_K}$ in $\mathbb{B}(H\otimes K)$ to be the (unique) continuous extensions of $f_{id_H\odot T}$ and $f_{S\odot id_K}$. Define $f_{S\otimes T} = f_{id_H\otimes T}\circ f_{S\otimes id_K} \in \mathbb{B}(H\otimes K)$. Observe that

$$f_{S\otimes T}(v\otimes w) = f_{id_H\otimes T}(f_{S\otimes id_K}(v\otimes w)) = f_{id_H\otimes T}(Sv\otimes w) = Sv\otimes Tw$$

and thus existence is done.

For uniqueness, if $f' \in \mathbb{B}(H\otimes K)$ also satisfies $f'(v\otimes w) = Sv\otimes Tw$, then $f_{S\otimes T}$ and f' match on all of $H\odot K$, which is a dense subset of $H\otimes K$. Then by continuity, $f_{S\otimes T} = f'$ on the entire domain.

For (i), we recall $\|S\| = \sup\{\|Sh\| : h \in H_1\}$. Then, there exists a sequence $(h_n)_{n=1}^{\infty}$ in H_1 such that $\lim_{n\to\infty}\|Sh_n\| = \|S\|$. Likewise, there exists a sequence $(k_n)_{n=1}^{\infty}$ in K_1 such that $\lim_{n\to\infty}\|Tk_n\| = \|T\|$. Then

$$\lim_{n\to\infty}\|f_{S\otimes T}(h_n\otimes k_n)\| = \lim_{n\to\infty}\|Sh_n\otimes Tk_n\| = \lim_{n\to\infty}\|Sh_n\|\|Tk_n\| = \|S\|\|T\|$$

By lemma 1.1.5 the norms are unchanged; that is to say, $\|f_{S\odot id_K}\| = \|f_{S\otimes id_K}\|$ and $\|f_{id_H\odot T}\| = \|f_{id_H\otimes T}\|$. Since $\|h_n\otimes k_n\| \leq 1$, then

$$\|f_{S\otimes T}(h_n\otimes k_n)\| \leq \|f_{S\otimes T}\| \leq \|f_{S\otimes id_K}\|\|f_{id_H\otimes T}\| \leq \|S\|\|T\|$$

Thus $\|f_{S\otimes T}\| = \|S\|\cdot\|T\|$ is verified.

For (ii). Let $v\otimes w$ and $x\otimes y \in H\odot K$. By properties of the adjoint of a Hilbert space operator we observe

$$\begin{aligned}
\langle f^*_{S\otimes T}(v\otimes w), x\otimes y\rangle &= \langle v\otimes w, f_{S\otimes T}(x\otimes y)\rangle \\
&= \langle v\otimes w, Sx\otimes Ty\rangle \\
&= \langle v, Sx\rangle\cdot\langle w, Ty\rangle \\
&= \langle S^*v, x\rangle\cdot\langle T^*w, y\rangle \\
&= \langle S^*v\otimes T^*w, x\otimes y\rangle \\
&= \langle f_{S^*\otimes T^*}(v\otimes w), x\otimes y\rangle
\end{aligned}$$

which implies that $f^*_{S\otimes T}\big|_{H\odot K} = f_{S^*\otimes T^*}|_{H\odot K}$. In other words, the continuous functions, $f^*_{S\otimes T}$ and $f_{S^*\otimes T^*}$, are equal on a dense subset of $H\otimes K$. It follows that $f^*_{S\otimes T} = f_{S^*\otimes T^*}$.

For (iii). If $v\in H$ and $w \in K$, then

$$f_{S\otimes T}\circ f_{S'\otimes T'}(v\otimes w) = f_{S\otimes T}(S'v\otimes T'w) = SS'v\otimes TT'w = f_{SS'\otimes TT'}(v\otimes w)$$

By using the same argument from (ii), we have $f_{S\otimes T} \circ f_{S'\otimes T'}$ and $f_{SS'\otimes TT'}$ are equal when restricted to $H \odot K$. Thus $f_{S\otimes T} \circ f_{S'\otimes T'}$ and $f_{SS'\otimes TT'}$ are equal on $H \otimes K$ by continuity.

For (iv). It is trivial that $(S, T) \mapsto f_{S\otimes T}$ is bilinear.

For (v). Suppose $S \otimes T \neq 0$. Then $S \neq 0_H$ in $\mathbb{B}(H)$ and $T \neq 0_K$ in $\mathbb{B}(K)$. Thus, $Sv \in H$ and $Tw \in K$ are non-zero vectors for some suitable $v \in H$ and $w \in K$. Since

$$\|f_{S\otimes T}(v \otimes w)\| = \|Sv \otimes Tw\| = \|Sv\| \cdot \|Tw\| \neq 0$$

then $f_{S\otimes T} \neq 0$ in $\mathbb{B}(H \otimes K)$.

□

Remark. By [BO08], $\mathbb{B}(H) \odot \mathbb{B}(K) \subseteq \mathbb{B}(H \otimes K)$ is weakly dense.

Proposition 1.3.5 *Let H and K be Hilbert spaces and $\Gamma \subseteq H$ be an orthonormal basis. Suppose for each $x \in \Gamma$, there exist $k_x \in K$ in the closed unit ball. Then there exists a contractive linear map, $f : H \to H \otimes K$, such that $f(x) = x \otimes k_x$ for all $x \in \Gamma$.*

Proof. We begin by defining $f : \text{Span}(\Gamma) \to H \odot K$. For $y = \sum_{i=1}^{n} c_i x_i \in \text{Span}(\Gamma)$, then define

$$f(y) = \sum_{i=1}^{n} c_i x_i \otimes k_{x_i}$$

Recall Γ is orthonormal, so $\{x \otimes k_x : x \in \Gamma\}$ is orthogonal. Then

$$\|f(y)\|^2 = \sum_{i=1}^{n} |c_i|^2 \|x_i\|^2 \|k_{x_i}\|^2 \leq \sum_{i=1}^{n} |c_i|^2 = \|y\|^2$$

which implies $f : \text{Span}(\Gamma) \to H \odot K$ is a linear contraction.

By lemma 1.1.5, we can continuously extend f without affecting its norm. With a small abuse of notation, we shall refer to this extended function as $f : H \to H \otimes K$.

□

Chapter 2

Operator spaces, the completely bounded norm, kernels and Schur multipliers

We begin 2.1 and 2.2 with definitions. In 2.3 we focus on the technical, but necessary propositions and lemmas. Finally, in 2.4 we prove the main results of chapter 2, which are theorem D.4 of [BO08], theorem 3 of [Oza08], and lemma 4 of [Oza08].

2.1 Operator spaces and the completely bounded norm

Definition 2.1.1 *We say $\mathcal{A}$ is a Banach algebra when $\mathcal{A}$ is a Banach space from 1.1.1, which is also endowed with ring multiplication (which is separate from, but compatible with scalar multiplication) such that the norm is submultiplicative, which is to say $\|ab\| \leq \|a\|\|b\|$ for all a, $b \in \mathcal{A}$.*

Definition 2.1.2 *We say $\mathcal{A}$ is a Banach involution algebra or a Banach *-algebra when it is a Banach algebra, and there exists an involution function $^* : \mathcal{A} \to \mathcal{A}$ which satisfies the following properties:*

1. $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$
2. $(ab)^* = b^* a^*$
3. $(a^*)^* = a$

where $\lambda, \mu \in \mathbb{C}$ and $a, b \in X$.

Definition 2.1.3 *If $\mathcal{A}$ and $\mathcal{B}$ are both Banach *-algebras, we say $\phi : \mathcal{A} \to \mathcal{B}$ is a *-algebra homomorphism when*

1. $\phi(\lambda a + \mu a') = \lambda\phi(a) + \mu\phi(a')$

2. $\phi(aa') = \phi(a)\phi(a')$
3. $\phi(a)^* = \phi(a^*)$

where $\lambda, \mu \in \mathbb{C}$ and $a, a' \in \mathcal{A}$.

Definition 2.1.4 *We say $\mathcal{A}$ is a $\mathcal{C}^*$ algebra when it is a Banach involution algebra which also satisfies the $\mathcal{C}^*$ equation, that is $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.*

Definition 2.1.5 *Let $\mathcal{A}$ be a (unital) Banach algebra with unit e. The spectrum of $a \in \mathcal{A}$ is*

$$\sigma(a) = \{\lambda \in \mathbb{C} \ : \ a - \lambda e \text{ is not invertible (under ring multiplication)}\}$$

Definition 2.1.6 *Let $\mathcal{A}$ be a Banach *-algebra. We say **$a \in \mathcal{A}$ is normal** when it $a^*a = aa^*$. We say **$a \in \mathcal{A}$ is self-adjoint** when $a = a^*$.*

Examples.

- For any Banach space X, $\mathbb{B}(X)$ is a Banach algebra with respect to composition.
- For any $S \neq \emptyset$, $\ell^\infty(S, \mathbb{C})$ is a $\mathcal{C}^*$ algebra with respects to pointwise multiplication, and the involution is the complex conjugate of the functions (taken pointwise).
- For a locally compact and Hausdorff topological space X, $C_0(X, \mathbb{C})$ is a $\mathcal{C}^*$ algebra, and the involution is the conjugate of the functions taken pointwise.
- For a Hilbert space H, $\mathbb{B}(H)$ is a $\mathcal{C}^*$ algebra, and by viewing $A \in \mathbb{B}(H)$ as a matrix, the involution is interpreted as being *the conjugate transpose* of A.
- For a Hilbert space H and $n \in \mathbb{N}$, we note H^n is a Hilbert space by 1.1.10, and $M_n(\mathbb{B}(H)) = \mathbb{B}(H^n)$ is a $\mathcal{C}^*$ algebra.

Definition 2.1.7 *By the work of Ruan [WBF+01], we say X is an **operator space** when it is a closed (linear) subspace of a $\mathcal{C}^*$ algebra $\mathcal{A}$. If $\phi : X \to Y$ is a linear map between operator spaces, where $X \subseteq \mathcal{A}$ and $Y \subseteq \mathcal{B}$, we define $\phi^n : M_n(X) \to M_n(Y)$ as follows:*

$$\phi^n([x_{ij}]) = [\phi(x_{ij})]_{1 \le i,j \le n}$$

where each x_{ij} is an element of X.

*We say ϕ^n is the **n^{th} amplification of ϕ**. Also, $M_n(X) = M_n(\mathbb{C}) \otimes X$ inherits a norm from $M_n(\mathcal{A})$ because by the GNS theorem $\mathcal{A}$ embeds into $\mathbb{B}(H)$ as a $\mathcal{C}^*$ subalgebra for some Hilbert space H [WBF+01].*

*The **completely bounded norm** of ϕ is defined as $\|\phi\|_{CB} = \sup\{\|\phi^n\|_{op} : n \in \mathbb{N}\}$. We say ϕ is **completely bounded** when $\|\phi\|_{CB} < \infty$, and we say ϕ is **completely contractive** when $\|\phi\|_{CB} \le 1$. Note that $\| \cdot \|_{CB}$ is indeed a norm [BO08].*

Remark. While the definition of the CB-norm is not too complicated, trying to compute it directly can be inconvenient. Fortunately, when working with kernels and Schur multipliers, we have theorem 2.4.1 which mitigates this problem.

2.2 Kernels and Schur multipliers

Given a non-empty set Γ, we know from chapter 1 that $\ell_2(\Gamma)$ is a Hilbert space and $\{\delta_x : x \in \Gamma\}$ is the canonical orthonormal basis. Then each $A = [A_{x,y}]_{x,y\in\Gamma} \in \mathbb{B}(\ell_2\Gamma)$ can be represented as a matrix, and by using the properties of inner products, the $(x, y)^{th}$ entry of A is

$$A_{x,y} = \langle A\delta_y, \delta_x\rangle \in \mathbb{C}$$

Definition 2.2.1 *A* ***kernel of*** $\boldsymbol{\Gamma}$ *is a function* $k : \Gamma \times \Gamma \to \mathbb{C}$*. A* ***Schur multiplier*** *associated to a kernel* k *is a bounded function* $m_k : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma)$ *such that*

$$m_k(A) = [k(x, y)A_{x,y}]_{x,y\in\Gamma}$$

where $A = [A_{x,y}]_{x,y\in\Gamma} \in \mathbb{B}(\ell_2\Gamma)$*.*

We think of m_k as taking matrix $A = [A_{x,y}]_{x,y\in\Gamma}$ and multiplying each entry by $k(x, y)$. Depending on the choice of kernel k, it is possible that $m_k(A)$ remains linear, but not bounded. To remedy this problem, we follow a convention from [Pis01], which is that we only consider the kernels k such that m_k is well-defined.

Remark. As noted before, $\mathbb{B}(\ell_2\Gamma)$ is a $\mathcal{C}^*$ algebra because $\ell_2(\Gamma)$ is a Hilbert space. Since $\mathbb{B}(\ell_2\Gamma)$ is closed (relative to itself), then it is already an operator space. Given a kernel $k : \Gamma \times \Gamma \to \mathbb{C}$, the corresponding Schur multiplier $m_k : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma)$ is a linear function between operator spaces. **Then** $\mathbb{B}(\boldsymbol{\ell_2\Gamma})$ **is our operator space of interest, and** $\boldsymbol{m_k}$ **is our linear map of interest, and we will further study these objects**.

In this context, $H = \ell_2(\Gamma)$ is a Hilbert space, and $m_k^n : M_n(\mathbb{B}(H)) \to M_n(\mathbb{B}(H))$ is defined as

$$m_k^n([A^{ij}]_{1\le i,j\le n}) = [m_k(A^{ij})]_{1\le i,j\le n}$$

which is the n^{th} amplification of m_k.

Definition 2.2.2 *Let* $S \subseteq \Gamma \times \Gamma$*. We define the indicator function* $\chi_S : \Gamma \times \Gamma \to \{0, 1\}$ *as follows:*

$$\chi_S(x, y) = \begin{cases} 1 & \textit{if } (x, y) \in S \\ 0 & \textit{otherwise} \end{cases}$$

We note χ_S is a kernel, and it corresponds to the Schur multiplier, m_{χ_S}. In general, the indicator functions are a rich source of kernels and Schur multipliers when it is well-defined.

Example 2.2. The constant function 1 is a kernel. Find the CB-norm of m_k.

$$\begin{aligned} k : \Gamma \times \Gamma &\to \mathbb{C} \\ (x, y) &\mapsto 1 \end{aligned}$$

Consider $m_k : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma)$, and let $A = [A_{x,y}]_{x,y\in\gamma} \in \mathbb{B}(\ell_2\Gamma))$. Then

$$m_k(A) = [k(x, y)A_{x,y}]_{x,y\in\Gamma} = [A_{x,y}]_{x,y\in\Gamma}$$

and so m_k is the identity of $\mathbb{B}(\ell_2\Gamma)$.

Consider $m_k^n : M_n(\mathbb{B}(\ell_2\Gamma)) \to M_n(\mathbb{B}(\ell_2\Gamma))$, and let $[A^{ij}]_{1\le i,j\le n} \in M_n(\mathbb{B}(\ell_2\Gamma))$. Then

$$m_k^n([A^{ij}]_{1\le i,j\le n}) = [m_k(A^{ij})]_{1\le i,j\le n} = [A^{ij}]_{1\le i,j\le n}$$

and so m_k^n is the identity of $M_n(\mathbb{B}(\ell_2\Gamma))$.

In this particular case, it's easy to see that $\|m_k^n\|_{op} = 1$ for all $n \in \mathbb{N}$, and so $\|m_k\|_{CB} = 1$.

A non-example. Let $\Gamma = \mathbb{N}$ and define $k : \Gamma \times \Gamma \to \mathbb{C}$ as

$$k(x, y) = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Then $m_k : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma)$ is false because the codomain is not $\mathbb{B}(\ell_2\Gamma)$. Indeed, let id be the identity of $\mathbb{B}(\ell_2\Gamma)$, which is equivalent to the identity matrix. Then

$$m_k(id) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $m_k(id)$ is linear, but is not bounded. In this case, $\|m_k\|_{CB} = \infty$, and we say that m_K is not well-defined.

2.3 Miscellaneous propositions and lemmas

Proposition 2.3.1 *If A, B are unital $\mathcal{C}^*$-algebras, and $\phi : A \to B$ is a *-algebra homomorphism, then ϕ is a contraction.*

Proof. Let $\text{rad}(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$ be the spectral radius of $a \in \mathcal{A}$. We claim that $\sigma(\phi(a)) \cup \{0\} \subseteq \sigma(a) \cup \{0\}$. Let $\lambda \in \sigma(\phi(a))$, which implies $\phi(a) - \lambda e_{\mathcal{B}} \in \mathcal{B}$ is non-invertible. By way of contrapositive, suppose $a - \lambda e_{\mathcal{A}}$ is invertible with inverse $x \in \mathcal{A}$. Since $\phi(e_{\mathcal{A}}) = e_{\mathcal{B}}$, then

$$(\phi(a) - \lambda e_{\mathcal{B}}) \cdot \phi(x) = \phi(a - \lambda e_{\mathcal{A}}) \cdot \phi(x) = \phi((a - \lambda e_{\mathcal{A}})x) = \phi(e_{\mathcal{A}}) = e_{\mathcal{B}}$$

because ϕ is a *-algebra homomorphism. By a similar argument, $\phi(x) \cdot (\phi(a) - \lambda e_{\mathcal{B}}) = e_{\mathcal{B}}$. This contradicts that $\phi(a) - \lambda e_{\mathcal{B}} \in \mathcal{B}$ is not invertible. Then we conclude $a - \lambda e_{\mathcal{A}}$ is non-invertible, and so $\lambda \in \sigma(a)$.

From [Arv76], if $a \in \mathcal{A}$ is normal, then $\text{rad}(a) = \|a\|$. We observe that $a^*a \in \mathcal{A}$ is (trivially) normal. By the $\mathcal{C}^*$ equation and the work above we have

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \text{rad}(\phi(a)^*\phi(a)) = \text{rad}(\phi(a^*a)) \leq \text{rad}(a^*a) = \|a^*a\| = \|a\|^2$$

□

Proposition 2.3.2 *If $\phi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism between $\mathcal{C}^*$ algebras, then the n^{th}-amplification of ϕ is still a *-homomorphism.*

Proof. Let $[a_{x,y}]$ and $[a'_{x,y}] \in M_n(A)$ and $s \in \mathbb{C}$. Then the goal is to verify the following statements for each $n \in \mathbb{N}$:

i. $\phi^{(n)}([a_{x,y}])^* = \phi^{(n)}([a_{x,y}]^*)$
ii. $\phi^{(n)}([a_{x,y}] + s[a'_{x,y}]) = \phi^{(n)}([a_{x,y}]) + s \cdot \phi^{(n)}([a'_{x,y}])$
iii. $\phi^{(n)}([a_{x,y}] \cdot [a'_{x,y}]) = \phi^{(n)}([a_{x,y}]) \cdot \phi^{(n)}([a'_{x,y}])$

The verification is intuitive and straight forwards, but is too tedius to present here. The proof is omitted.

□

Theorem 2.3.3 *Let $X \subseteq A$ be an operator space and $\phi : X \to \mathbb{B}(H)$ be a completely contractive map. Then there exists a Hilbert space $\hat{H}$, and a *-algebra homomorphism $\pi : A \to \mathbb{B}(\hat{H})$ and isometries $V, W : H \to \hat{H}$ such that*

$$\phi(x) = V^*\pi(x)W$$

for all $x \in X$.

This theorem is also known as **Wittstock's factorization theorem**, and is listed as theorem B.7 in [BO08].

2.4 The main theorem of chapter 2

Theorem 2.4.1 *Consider kernel $k : \Gamma \times \Gamma \to \mathbb{C}$. Then the following are equivalent:*

i. The Schur multiplier is completely contractive. i.e. $\|m_k\|_{CB} \leq 1$

ii. There exists a Hilbert space H and vectors $\xi_s, \eta_t \in H$ with norms at most 1 such that $\langle \eta_t, \xi_s \rangle = k(s,t)$ for every $s,t \in \Gamma$.

Note that this proposition is listed as theorem D.4 in [BO08].

Proof. **For the forward direction**, we suppose $m_k : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma)$ is completely contractive. By theorem 2.3.3 (i.e. Wittstock), there is a Hilbert space H and a *-(algebra) homomorphism, $\pi : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(H)$ and isometries $V, W : \ell_2(\Gamma) \to H$ such that $m_k(x) = V^*\pi(x)W$ for every $x \in \mathbb{B}(\ell_2\Gamma)$.

We recall that the elements of $\mathbb{B}(\ell_2\Gamma)$ can be represented as $\Gamma \times \Gamma$ matrices. Then we define $e_{x,y} \in \mathbb{B}(\ell_2\Gamma)$ to be an elementary matrix with an entry of 1 at $(x,y) \in \Gamma \times \Gamma$ and 0 everywhere else.

For each $r \in \Gamma$ we define $\xi_s = \pi(e_{r,s})V(\delta_s)$ and $\eta_t = \pi(e_{r,t})W(\delta_t) \in H$ where $s,t \in \Gamma$. By proposition 2.3.1, *-homomorphism are contractions, and the operator norm is submultiplicative, so both ξ_s and η_t have norm at most 1.

Since $\pi(e_{r,s})^*\pi(e_{r,t}) = \pi(e_{s,r})\pi(e_{r,t}) = \pi(e_{s,t})$, then

$$\begin{aligned}
\langle \eta_t, \xi_s \rangle &= \langle \pi(e_{r,t})W(\delta_t), \pi(e_{r,s})V(\delta_s) \rangle \\
&= \langle \pi(e_{s,t})W(\delta_t), V(\delta_s) \rangle \\
&= \langle V^*\pi(e_{s,t})W(\delta_t), \delta_s \rangle \\
&= \langle m_k(e_{s,t})\delta_t, \delta_s \rangle \\
&= (s,t)^{th} \text{ entry of } m_k(e_{s,t}) \\
&= k(s,t)
\end{aligned}$$

and the forward direction is done.

For the reverse direction, we assume H is a Hilbert space such that $k(s,t) = \langle \eta_t, \xi_s \rangle$ where $\eta_t, \xi_s \in H_1$ (i.e. $\|\eta_t\|, \|\xi_s\| \leq 1$). Recall from chapter 1 that $\ell_2\Gamma \otimes H$ is a Hilbert space, and we distinguish it from the algebraic tensors $\ell_2\Gamma \odot H$ which is a pre-Hilbert space.

Since $\{\delta_s : s \in \Gamma\} \subseteq \ell_2(\Gamma)$ is an orthonormal basis, then by proposition 1.3.5 we have contractive linear maps, $V, W : \ell_2(\Gamma) \to \ell_2\Gamma \otimes H$ which satisfy $V(\delta_s) = \delta_s \otimes \xi_s$ and $W(\delta_t) = \delta_t \otimes \eta_t$. Also consider $\pi : \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2\Gamma \otimes H)$ where $\pi(A) = A \otimes id_H$. Recall that the notation, $A \otimes id_H$, was made rigorous by proposition 1.3.4 (i.e. proposition 3.2.3 of [BO08]).

Next we will verify $m_k(A) = V^*\pi(A)W$ for all $A \in \mathbb{B}(\ell_2\Gamma)$. Since the inner product is bilinear and continuous in each component, then it is sufficient to check $\langle V^*\pi(A)W(\delta_t), \delta_s\rangle = \langle m_k(A)\delta_t, \delta_s\rangle$ for all δ_s, δ_t in the standard orthonormal basis of $\ell_2(\Gamma)$.

In other words, when the objects are viewed as $\Gamma \times \Gamma$ matrices, for all $s, t \in \Gamma$, our goal is to verify that

$$\text{the } (s,t)^{th} \text{ entry of } m_k(A) = \text{the } (s,t)^{th} \text{ entry of } V^*\pi(A)W$$

To that end, we observe

$$\begin{aligned}
\langle V^*\pi(A)W(\delta_t), \delta_s\rangle &= \langle \pi(A)W(\delta_t), V(\delta_s)\rangle \\
&= \langle A \otimes id_H(\delta_t \otimes \eta_t), \delta_s \otimes \xi_s\rangle \\
&= \langle A\delta_t \otimes id_H(\eta_t), \delta_s \otimes \xi_s\rangle \\
&= \langle A\delta_t, \delta_s\rangle \cdot \langle \eta_t, \xi_s\rangle \\
&= (\text{the } (s,t)^{th} \text{ entry of } A) \cdot k(s,t) \\
&= \text{the } (s,t)^{th} \text{ entry of } m_k(A) \\
&= \langle m_k(A)\delta_t, \delta_s\rangle
\end{aligned}$$

Lastly, we will verify that $\|m_k\|_{CB} \leq 1$. If $A = [A_{ij}]_{1 \leq i,j \leq n} \in M_n(\mathbb{B}(\ell_2\Gamma))$, then

$$\begin{aligned}
m_k^n(A) &= [m_k(A_{ij})]_{1 \leq i,j \leq n} \\
&= [V^*\pi(A_{ij})W]_{1 \leq i,j \leq n} \\
&= (V^* \otimes id_n)[\pi(A_{ij})](W \otimes id_n) \\
&= (V^* \otimes id_n) \cdot \pi^n(A) \cdot (W \otimes id_n)
\end{aligned}$$

We note that π is a *-algebra homomorphism. As a consequence of propositions 2.3.1 and 2.3.2, the amplifications of π are contractions. Also, V, W are contractions and the involution is an isometry. Then,

$$\begin{aligned}
\|m_k^n(A)\| &\leq \|V^* \otimes id_n\| \cdot \|\pi^n(A)\| \cdot \|W \otimes id_n\| \\
&\leq \|V^*\|\|id_n\| \cdot \|\pi^n\|\|A\| \cdot \|W\|\|id_n\| \\
&\leq \|A\|
\end{aligned}$$

Then $\|m_k^n\| \leq 1$ for all $n \in \mathbb{N}$, and we conclude $\|m_k\|_{CB} \leq 1$.

□

In light of theorem D.4 of [BO08] we have theorem 2.4.2 as a corollary.

Theorem 2.4.2 *Let kernel $k : \Gamma \times \Gamma \to \mathbb{C}$ and $C \geq 0$ be given. Then the following are equivalent:*

i. The Schur multiplier is completely bounded by C; i.e. $\|m_k\|_{CB} \leq C$

ii. There exists a Hilbert space H and vectors $\zeta^+(x), \zeta^-(y) \in H$ with norms at most $\sqrt{C}$ such that $\langle \zeta^-(y), \zeta^+(x) \rangle = k(x, y)$ for every $x, y \in \Gamma$.

Note. This theorem is listed as theorem 3 in [Oza08].

Proof. **For the forward direction** we suppose $\|m_k\|_{CB} \leq C$. Consider a new kernel, $k' = \frac{1}{C}k$. Note that $\|m_{k'}\|_{CB} = \frac{1}{C}\|m_k\|_{CB} \leq 1$. By theorem 2.4.1 (i.e. theorem D.4 of [BO08]), we have a Hilbert space H and vectors $\beta^+(x), \beta^-(y) \in H$ which have norms at most 1 and that $\langle \beta^-(y), \beta^+(x) \rangle = k'(x, y)$ for every $x, y \in \Gamma$.

Now define $\zeta^+(x) = \sqrt{C}\beta^+(x)$ and $\zeta^-(y) = \sqrt{C}\beta^-(y) \in H$ and the forward direction is complete.

For the reverse direction, we define $\xi_x = \frac{1}{\sqrt{C}}\zeta^+(x)$ and $\eta_y = \frac{1}{\sqrt{C}}\zeta^-(y) \in H$, which have norms bounded by 1. We also define $k' = \frac{1}{C}k$. Quite obviously, $\langle \eta_y, \xi_x \rangle = k'(x, y)$. Then, by theorem 2.4.1 we have $\|m_{k'}\|_{CB} \leq 1$, and by the scaling properties, $\|m_k\|_{CB} \leq C$.

□

Lemma 2.4.3 *Define $\mathcal{P}_f(\Gamma)$ to be the set of finite subsets of Γ. Note that $\emptyset \in \mathcal{P}_f(\Gamma)$. For $\ell_2(\mathcal{P}_f\Gamma)$ we define the inner product as*

$$\langle g, h \rangle = \sum_{\omega \in \mathcal{P}_f(\Gamma)} g(\omega)\overline{h(\omega)}$$

where $g, h \in \ell_2(\mathcal{P}_f\Gamma)$.

For $S \in \mathcal{P}_f(\Gamma)$, we define $\widetilde{\xi}_S^+$ and $\widetilde{\xi}_S^- : \mathcal{P}_f(\Gamma) \to \mathbb{C}$ by

$$\widetilde{\xi}_S^+(\omega) = \begin{cases} 1 & \text{if } \omega \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

and

$$\widetilde{\xi}_S^-(\omega) = \begin{cases} (-1)^{|\omega|} & \text{if } \omega \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

Also define $\xi_S^+ = \widetilde{\xi}_S^+ - \delta_\emptyset$ and $\xi_S^- = -(\widetilde{\xi}_S^- - \delta_\emptyset)$. If $S, T \in \mathcal{P}_f(\Gamma)$, then

i. $\xi_S^\pm \perp \xi_T^\pm$ when $S \cap T = \emptyset$

ii. $\|\xi_S^{\pm}\|^2 + 1 = \|\tilde{\xi}_S^{\pm}\|^2 = 2^{|S|}$

iii. $\langle \xi_T^-, \xi_S^+ \rangle = 1 - \langle \tilde{\xi}_T^-, \tilde{\xi}_S^+ \rangle = \begin{cases} 1 & \textit{if } S \cap T \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$

and as a consequence of part (ii), we have $\widetilde{\xi}_S^{\pm}$ *and* $\xi_S^{\pm} \in \ell_2(\mathcal{P}_f\Gamma)$.

Note: This lemma is listed as lemma 4 in [Oza08].

Proof. **For (i)**, we observe that

$$|\langle \xi_S^{\pm}, \xi_T^{\pm} \rangle| \leq \sum_{\omega \in \mathcal{P}_f(\Gamma)} |A_\omega|$$

where

$$A_\omega = \xi_S^{\pm}(\omega) \cdot \xi_T^{\pm}(\omega) = \left(\widetilde{\xi}_S^{\pm}(\omega) - \delta_\emptyset(\omega)\right) \cdot \left(\widetilde{\xi}_T^{\pm}(\omega) - \delta_\emptyset(\omega)\right)$$

Since $S \cap T = \emptyset$, then

$$A_\omega = \begin{cases} (1-1)(1-1) = 0 & \text{if } \omega = \emptyset \\ (\pm 1 - 0)(0-0) = 0 & \text{if } \omega \subseteq S \text{ and } \omega \neq \emptyset \\ (0-0)(\pm 1 - 0) = 0 & \text{if } \omega \subseteq T \text{ and } \omega \neq \emptyset \\ (0-0)(0-0) = 0 & \text{if } \omega \not\subseteq S, \omega \not\subseteq T, \omega \neq \emptyset \end{cases}$$

which implies $\langle \xi_S^{\pm}, \xi_T^{\pm} \rangle = 0$.

For (ii). We note that $\widetilde{\xi}_S^{\pm}(\emptyset) - \delta_\emptyset(\emptyset) = 0$. Then

$$\begin{aligned} \|\xi_S^{\pm}\|^2 + 1 &= \sum_{\omega \in \mathcal{P}_f(\Gamma)} \left(\xi_S^{\pm}(\omega)\right)^2 + 1 \\ &= \sum_{\omega \in \mathcal{P}_f(\Gamma)} \left(\widetilde{\xi}_S^{\pm}(\omega) - \delta_\emptyset(\omega)\right)^2 + 1 \\ &= \sum_{\omega \in \mathcal{P}_f(\Gamma) \setminus \{\emptyset\}} \left(\widetilde{\xi}_S^{\pm}(\omega) - \delta_\emptyset(\omega)\right)^2 + 1 \\ &= \sum_{\omega \in \mathcal{P}_f(\Gamma) \setminus \{\emptyset\}} \left(\widetilde{\xi}_S^{\pm}(\omega)\right)^2 + 1 \\ &= \sum_{\omega \in \mathcal{P}_f(\Gamma)} \left(\widetilde{\xi}_S^{\pm}(\omega)\right)^2 \\ &= \|\widetilde{\xi}_S^{\pm}\|^2 \end{aligned}$$

Furthermore,

$$\begin{aligned}\|\widetilde{\xi}_S^{\pm}\|^2 &= \sum_{\omega\in\mathcal{P}_f(\Gamma)} \left(\widetilde{\xi}_S^{\pm}(\omega)\right)^2 \\ &= \sum_{\omega\subseteq S} ({}^{\pm}1)^2 \\ &= 2^{|S|}\end{aligned}$$

with the last equality coming from the binomial theorem. We can think of it as counting the number of subsets of $S \in \mathcal{P}_f(\Gamma)$.

For (iii). Observe

$$\begin{aligned}\langle \xi_T^-, \xi_S^+\rangle &= \langle -\widetilde{\xi}_T^- + \delta_\emptyset, \widetilde{\xi}_S^+ - \delta_\emptyset\rangle \\ &= -\langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+\rangle + \langle \widetilde{\xi}_T^-, \delta_\emptyset\rangle + \langle \delta_\emptyset, \widetilde{\xi}_S^+\rangle - \langle \delta_\emptyset, \delta_\emptyset\rangle \\ &= -\langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+\rangle + 1 + 1 - 1 \\ &= 1 - \langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+\rangle\end{aligned}$$

If $S \cap T \neq \emptyset$, then $n = |S \cap T| \geq 1$. By the binomial theorem we have

$$\langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+\rangle = \sum_{\omega\subseteq S\cap T} (-1)^{|\omega|} \cdot 1 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1 + (-1))^n \ = 0$$

which implies

$$\langle \xi_T^-, \xi_S^+\rangle = 1 - \langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+\rangle = 1$$

On the other hand, if $S \cap T = \emptyset$, then we already have $\langle \xi_T^-, \xi_S^+\rangle = 0$ from part (i). In summary,

$$\langle \xi_T^-, \xi_S^+\rangle = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{if } S \cap T = \emptyset \end{cases}$$

□

Chapter 3

An introduction to hyperbolic graphs and hyperbolic groups

We begin section 3.1 by defining our notion of graphs, and thereafter we explain the concept of geodesic paths and the hyperbolic property. Later in the section, we introduce the concept of the Cayley graph, which allows us to reconsider a group as a graph. We finish section 3.1 by introducing the notion of hyperbolic groups.

Section 3.2 is somewhat mundane, as it is mostly a bunch of technical lemmas and propositions, but will be useful when we move onto chapter 4.

3.1 Hyperbolic graphs and hyperbolic groups

Definition 3.1.1 *A* ***(directed) graph***, $\boldsymbol{\Gamma = (V_\Gamma, E_\Gamma)}$, *is comprised of a* ***set of vertices***, $\boldsymbol{V_\Gamma}$, *and a* ***set of edges***, $\boldsymbol{E_\Gamma}$. *We also have functions* ∂_0, $\partial_1 : E_\Gamma \to V_\Gamma$. *For* $e \in E_\Gamma$, *let* $x = \partial_0(e)$, $y = \partial_1(e) \in V_\Gamma$.

We say ***x and y are the endpoints of e***. *We say the* ***directed edge e joins the adjacent vertices x and y***.

For this thesis we will adopt the following conventions on graph Γ:

1. Γ does not allow for loops. That is for edge $e \in E_\Gamma$, we require $\partial_0(e) \neq \partial_1(e)$.
2. Γ is connected, which means that for any two distinct vertices x, y we have a (finite) sequence of vertices, $x = x_0, x_1, \ldots, x_n = y$ such that x_i and x_{i+1} are adjacent.
3. Γ cannot have multiple edges in the same direction that join vertices x and y. If edges e and e' satisfy $x = \partial_0(e) = \partial_0(e')$ and $y = \partial_1(e) = \partial_1(e')$, then $e = e'$. We may use the notation $e = (x, y)$.

Remark. We work with directed graphs because the Cayley graphs introduced in 3.1.15 are directed. However, we do not actually make use of the directed structure.

Next, we iron out some details. For instance, the expression $x \in \Gamma$ refers to a point on Γ, but is x a vertex of Γ, or is it a point inside an edge of Γ that is not an endpoint? The details are presented below.

We borrow a technique from [BH99], and we define an equivalence relation on $E_\Gamma \times [0,1]$. For $e, e' \in E_\Gamma$ and $t, t' \in [0,1]$ we have $(e,t) \sim (e',t')$ when

$$\partial_0(e) = \partial_1(e') \text{ and } \partial_1(e) = \partial_0(e') \qquad \text{OR} \qquad \partial_0(e) = \partial_0(e') \text{ and } \partial_1(e) = \partial_1(e')$$

Let $X_\Gamma = {}^{E_\Gamma \times [0,1]}/_{\sim}$ and $q : E_\Gamma \times [0,1] \to X_\Gamma$ be the projection map. This equivalence relation on $E_\Gamma \times [0,1]$ allows one to identify an edge with its inverse as geometric segments.

Now consider $\partial : E_\Gamma \times \{0,1\} \to V_\Gamma$ which is defined as $\partial(e,t) = \partial_t(e)$. ∂_t is the function which returns an endpoint of edge e relative to the choice of t. We note that ∂ is surjective because Γ is connected. Thus, the quotient map $\overline{\partial} : {}^{E_\Gamma \times \{0,1\}}/_{\sim} \to V_\Gamma$ is a bijection, and we identify $V_\Gamma = q(E_\Gamma \times \{0,1\})$.

Definition 3.1.2 *Using the work from above, we defined $X_\Gamma = {}^{E_\Gamma \times [0,1]}/_{\sim}$ and we saw $V_\Gamma \subseteq X_\Gamma$. We say that* ***$a, b \in X_\Gamma$ belong to a common edge*** *when there exists $e \in E_\Gamma$ such that $a = q(e,t_1)$ and $b = q(e,t_2)$ for some $t_1, t_2 \in [0,1]$.*

Our next goal is to build a metric, d, but first, we define $d^* : X_\Gamma \times X_\Gamma \to [0,\infty]$. If a, b belong to a common edge $e \in E_\Gamma$, then there exists $t_1, t_2 \in [0,1]$ such that $a = q(e,t_1)$ and $b = q(e,t_2)$. We define

$$d^*(a,b) = \begin{cases} |t_1 - t_2| & \text{if } a, b \text{ belong to a common edge} \\ \infty & \text{if } a, b \text{ don't belong to a common edge} \end{cases}$$

We note d^* is well-defined. Indeed if a, b belong to a common edge, we have a few cases:

• At-least either a or b is not an endpoint, then both the choice of common edge and the choice of $t_1, t_2 \in [0,1]$ are unique, so that $a = q(e,t_1)$ and $b = q(e,t_2)$

• Both a and b are endpoints (i.e. vertices) and $a = b$, then $t_1 = t_2$ for all possible choices of common edge, and $|t_1 - t_2| = 0$

• Both a and b are endpoints and $a \neq b$, then for all possible choices of common edge, $|t_1 - t_2| = 1$. i.e. a and b represent the two opposing endpoints of whichever common edge they lie on.

Definition 3.1.3 *The graph metric, $d_\Gamma : X_\Gamma \times X_\Gamma \to \mathbb{R}$, is defined as follows:*

$$d_\Gamma(a,b) = \inf \left\{ \sum_{i=0}^{n-1} d^*(u_{i+1}, u_i) \;\middle|\; \exists a = u_0, u_1, \ldots, u_n = b \text{ in } X_\Gamma \text{ such that } u_i, u_{i+1} \text{ belong to a common edge for all } 0 \le i \le n-1 \right\}$$

We note that d_Γ is well-defined because Γ is connected. Furthermore, $d_\Gamma(a,b) = d^*(a,b)$ when a and b belong to a common edge of Γ.

Remark. In summary, the purpose of X_Γ is to include the vertices of Γ and the points within the edges. For instance, let $e \in E_\Gamma$. If $a = q(e, 0.5) \in X_\Gamma$, then a is the midpoint of e. By construction, each edge e is an isometric copy of $[0,1]$ via the isometry $q(e,t)$ where $0 \le t \le 1$. Thus (X_Γ, d_Γ) is our metric space of interest.

From now on, we will use the notation Γ and X_Γ interchangeably, and it should be clear in the context whether we are working with Γ or X_Γ. For example, the expression $x \in \Gamma$ is really saying $x \in X_\Gamma$.

Definition 3.1.4 *Given a metric space (M,d), let $E \subseteq M$ and $R > 0$. We define the **R-neighbourhood** (or R-nhood) of E by*

$$\mathcal{N}_R(E) = \{y \in M : d(y,E) < R\}$$

where

$$d(y,E) = \inf\{d(x,y) : x \in E\}$$

If $E = \{x\}$, then we denote $\mathcal{B}_R(x) = \mathcal{N}_R(\{x\})$, which is the usual ball around x of radius R. Our next objective is to establish our notion of paths and geodesics.

Definition 3.1.5 *Let (M,d) be a metric space. For $a, b \in M$, a **path from a to b** is a continuous function $\mathfrak{p} : [\alpha, \beta] \to M$ such that $\alpha \le \beta$, $\mathfrak{p}(\alpha) = a$, and $\mathfrak{p}(\beta) = b$. We say that **path $\mathfrak{p}$ joins a to b**.*

*A **subpath of $\mathfrak{p}$** is $\mathfrak{p}|_{[t_1,t_2]}$ where $\alpha \le t_1 \le t_2 \le \beta$.*

*The **length of path $\mathfrak{p}$** is*

$$\ell(\mathfrak{p}) = \sup \left\{ \sum_{i=0}^{n-1} d(\mathfrak{p}(t_{i+1}), \mathfrak{p}(t_i)) \;\middle|\; \alpha = t_0 \le t_1 \le \cdots \le t_n = \beta \text{ where } n \ge 1 \right\} \le \infty$$

If $\mathfrak{p}_1 : [\alpha_1, \beta_1] \to M$ *and* $\mathfrak{p}_2 : [\alpha_2, \beta_2] \to M$ *are paths and* $p_1(\beta_1) = p_2(\alpha_2)$*, then the* ***concatenation of*** $\boldsymbol{\mathfrak{p}_1}$ ***and*** $\boldsymbol{\mathfrak{p}_2}$ *is* $\mathfrak{p} : [\alpha_1, \beta_1 + \beta_2 - \alpha_2] \to M$ *where*

$$\mathfrak{p}(t) = \begin{cases} \mathfrak{p}_1(t) & \text{if } \alpha_1 \leq t \leq \beta_1 \\ \mathfrak{p}_2(t - \beta_1 + \alpha_2) & \text{if } \beta_1 \leq t \leq \beta_1 + \beta_2 - \alpha_2 \end{cases}$$

Remark. As seen in figure 3.1, the red lines approximate the length of $\mathfrak{p}$. The length of $\mathfrak{p}$ is the supremum of such expressions. Furthermore, $\ell(\mathfrak{p}) = \ell(\mathfrak{p}_1) + \ell(\mathfrak{p}_2)$, which is a consequence of definition 3.1.5. It is possible to have paths of infinite length as seen in example 3.1.

Example 3.1. Consider a partition $0 = t_0 < t_1 < \cdots < t_n < \cdots < 1$ where $t_n = 1 - \frac{1}{10^n}$. We construct a continuous path $\mathfrak{p} : [0, 1] \to (\mathbb{R}, d_{Euclidean})$ which satisfies

$$\mathfrak{p}(t_n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 1 - \frac{1}{2} & \text{if } n = 2 \\ 1 - \frac{1}{2} + \frac{1}{3} & \text{if } n = 3 \\ \vdots & \vdots \end{cases}$$

In general, $\mathfrak{p}(t_n) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$ for $n \geq 1$. For $x \in (t_n, t_{n+1})$ we think of $\mathfrak{p}$ as traveling from $\mathfrak{p}(t_n)$ to $\mathfrak{p}(t_{n+1})$ linearly.

From calculus, the alternating series $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = \ln(2)$, so we define $\mathfrak{p}(1) = \ln(2)$. Conversely, the harmonic series

$$\ell(\mathfrak{p}) \geq \lim_{n \to \infty} \left(\sum_{i=0}^{n} |\mathfrak{p}(t_{i+1}) - \mathfrak{p}(t_i)| \right) \geq \sum_{i=1}^{\infty} \left| \frac{(-1)^{i+1}}{i} \right| = \infty$$

Definition 3.1.6 *Let* $\mathfrak{p} : [\alpha, \beta] \to M$ *be a path joining* $a = \mathfrak{p}(\alpha)$ *to* $b = \mathfrak{p}(\beta)$*. We say* $\boldsymbol{\mathfrak{p}}$ ***is a geodesic path*** *when*

$$d(\mathfrak{p}(t_1), \mathfrak{p}(t_2)) = |t_1 - t_2|$$

for all $t_1, t_2 \in [\alpha, \beta]$*.*

A ***metric space*** $\boldsymbol{(M, d)}$ ***is geodesic*** *when for all* $a, b \in M$*, there is a geodesic path which joins* a *to* b*. For brevity, we say* M *is a geodesic space.*

A ***geodesic triangle*** $\boldsymbol{\Delta xyz}$ *inside a geodesic space* (M, d) *is a set of three (distinct) points* $x, y, z \in M$ *which are joined by geodesic paths* $[x, y], [x, z]$*, and* $[y, z]$*.*

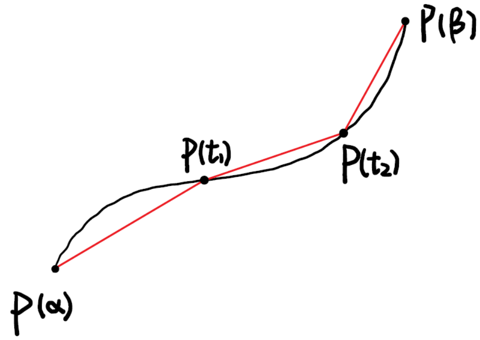

Figure 3.1: $\alpha = t_0 < t_1 < t_2 < t_3 = \beta$.

Examples. $(\mathbb{R}^2, d_{Euclidean})$ is a geodesic space, and furthermore, it is uniquely geodesic in the sense that a geodesic path from x to $y \in \mathbb{R}^2$ is unique.

On the other hand, let $X = \{(x, y) \in \mathbb{R}^2 : x \text{ or } y \in \mathbb{Z}\}$ represent a grid within $\mathbb{R}^2$. Then (X, d_{ℓ_1}) is a geodesic space where d_{ℓ_1} is the taxi-cab metric. Consider:

- path $\mathfrak{p}_1$ starts at $(0, 0)$, travels to $(1, 0)$, and then to $(1, 1)$ at unit speed.
- path $\mathfrak{p}_2$ starts at $(0, 0)$, travels to $(0, 1)$, and then to $(1, 1)$ at unit speed.

and we note both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are geodesic paths from $(0, 0)$ to $(1, 1)$ in (X, d_{ℓ_1}). Thus (X, d_{ℓ_1}) is a geodesic space, but is not uniquely geodesic.

Now consider a **non-example**: $(\mathbb{R}^2 \backslash \{0\}, d_{Euclidean})$ is not a geodesic space because there does not exist a geodesic path from $(-1, -1)$ to $(1, 1)$.

Remark. For convenience we use $[a, b]$ to represent a geodesic path from a to b. If the geodesics are not unique, we can interpret $[a, b]$ to mean *a geodesic path* as opposed to *the geodesic path*. As a consequence of the the definitions, a subpath of a geodesic path will remain geodesic. However, the concatenation of two geodesic paths need not be geodesic.

Lemma 3.1.7 *Let (M, d) be a metric space and $\mathfrak{p} : [\alpha, \beta] \to M$ be a path from $a = \mathfrak{p}(\alpha)$ to $b = \mathfrak{p}(\beta)$. Then*

i. $d(a, b) \leq \ell(\mathfrak{p})$
ii. *if $\mathfrak{p}$ is geodesic, then* $d(a, b) = \beta - \alpha$

Proof. **For (i)**. The length of a path is defined in 3.1.5. Take $\alpha = t_0 \leq t_1 = \beta$ and so $d(a, b) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)) \leq \ell(\mathfrak{p})$.

For (ii). We suppose $\mathfrak{p}$ is geodesic. Then $d(a, b) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)) = \beta - \alpha$.

□

Lemma 3.1.8 *Let (M, d) be a geodesic space and let $\mathfrak{p} : [\alpha, \beta] \to M$ be a path joining $a = \mathfrak{p}(\alpha)$ and $b = \mathfrak{p}(\beta)$. Consider*

i. $\mathfrak{p}$ *is geodesic*

ii. $\ell(\mathfrak{p}) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))$

iii. $\mathfrak{p}$ *is a path of shortest length that joins* $\mathfrak{p}(\alpha)$ *to* $\mathfrak{p}(\beta)$

iv. $d((\mathfrak{p}(t), \mathfrak{p}(t')) = d(\mathfrak{p}(t), \mathfrak{p}(u)) + d(\mathfrak{p}(u), \mathfrak{p}(t'))$ *for all* $\alpha \leq t \leq u \leq t' \leq \beta$

Then $(i) \to (ii) \leftrightarrow (iii) \leftrightarrow (iv)$.

Proof. **For (i) implies (ii)**. We suppose $\mathfrak{p}$ is geodesic. We recall the definition of the length of a path from 3.1.5 and apply 3.1.7. Then

$$\sum_{i=0}^{n-1} d(\mathfrak{p}(t_{i+1}), \mathfrak{p}(t_i)) = \sum_{i=0}^{n-1} |t_{i+1} - t_i| = t_n - t_0 = \beta - \alpha = d(a, b)$$

where $\alpha = t_0 \leq t_1 \leq \cdots \leq t_n = \beta$, and so $\ell(\mathfrak{p}) = d(a, b)$.

For (ii) implies (iii). We suppose $\ell(\mathfrak{p}) = d(a, b)$. Let $\mathfrak{p}' : [\alpha', \beta'] \to M$ be another path that joins a to b. Consider $\alpha' = t_0 \leq t_1 = \beta'$. Then

$$\ell(\mathfrak{p}') \geq d(\mathfrak{p}'(t_1), \mathfrak{p}'(t_0)) = d(\mathfrak{p}'(\beta'), \mathfrak{p}'(\alpha')) = d(b, a) = d(\mathfrak{p}(\beta), \mathfrak{p}(\alpha)) = \ell(\mathfrak{p})$$

For (iii) implies (ii). We suppose $\mathfrak{p}$ is the shortest path joining a to b. Since M is a geodesic space, then there exists a geodesic path $p' : [\alpha', \beta'] \to M$ from a to b. Then $\ell(\mathfrak{p}') = d(a, b)$ as proven in (i) $\to$ (ii). Then

$$d(a, b) \leq \ell(\mathfrak{p}) \leq \ell(\mathfrak{p}') = d(a, b)$$

For (iii) implies (iv). We suppose $\mathfrak{p}$ is a path of shortest length from $\mathfrak{p}(\alpha)$ to $\mathfrak{p}(\beta)$. Let $\alpha \leq t \leq u \leq t' \leq \beta$. Define $\mathfrak{p}_0 = \mathfrak{p}|_{[t,t']}$, $\mathfrak{p}_1 = \mathfrak{p}|_{[t,u]}$, and $\mathfrak{p}_2 = \mathfrak{p}|_{[u,t']}$. We observe that $\mathfrak{p}_0$, $\mathfrak{p}_1$, and $\mathfrak{p}_2$ are shortest paths on their respective endpoints. By applying (iii) $\to$ (ii) for $\mathfrak{p}_0$, $\mathfrak{p}_1$, and $\mathfrak{p}_2$, we have

$$d(\mathfrak{p}(t), \mathfrak{p}(t')) = \ell(\mathfrak{p}_0) = \ell(\mathfrak{p}_1) + \ell(\mathfrak{p}_2) = d(\mathfrak{p}(t), \mathfrak{p}(u)) + d(\mathfrak{p}(u), \mathfrak{p}(t'))$$

For (iv) implies (ii). We recall the length of a path $\mathfrak{p}$ at 3.1.5. Consider $\alpha = t_0 \leq t_1 \leq \cdots \leq t_n = \beta$, and by repeatedly applying (iv), we have

$$d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)) = \sum_{i=0}^{n-1} d(\mathfrak{p}(t_{i+1}), \mathfrak{p}(t_i))$$

Thus $d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)) = \ell(\mathfrak{p})$.

□

Lemma 3.1.9 *Let (M, d) be a path-connected metric space and $\mathfrak{p} : [\alpha, \beta] \to M$ be a path which satisfies $\ell(\mathfrak{p}) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))$. If $\mathfrak{p}$ is injective, then up to reparamterization, $\mathfrak{p}$ is geodesic.*

Proof. Define $s : [\alpha, \beta] \to [0, d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))]$ as follows

$$s(t) = d(\mathfrak{p}(\alpha), \mathfrak{p}(t))$$

Our goals are to show that s is a homeomorphism and that $\mathfrak{p} \circ s^{-1}$ is geodesic.

The function, $d(\mathfrak{p}(\alpha), \ \cdot\ ) : M \to \mathbb{R}$ is continuous. Indeed, if $\epsilon > 0$, then take $\delta = \epsilon$. For $m_0 \in M$, then $|d(\mathfrak{p}(\alpha), m_0) - d(\mathfrak{p}(\alpha), m)| \le d(m_0, m) < \delta = \epsilon$. Since path $\mathfrak{p}$ is continuous, then $s(\cdot) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\cdot))$ is continuous.

Observe that $s(\alpha) = 0$ and $s(\beta) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))$. By the intermediate value theorem, s is a surjection.

To demonstrate that s is an injection we suppose $s(t) = s(t')$ where $\alpha \le t \le t' \le \beta$. Define $\mathfrak{p}_1 = \mathfrak{p}|_{[\alpha,t]}$, $\mathfrak{p}_2 = \mathfrak{p}|_{[t,t']}$, and $\mathfrak{p}_3 = \mathfrak{p}|_{[t',\beta]}$. Observe that $\mathfrak{p}$ is the concatenation of $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$. By lemma 3.1.7 we have

- $\ell(\mathfrak{p}_1) \ge d(\mathfrak{p}(\alpha), \mathfrak{p}(t)) = s(t) = s(t') = d(\mathfrak{p}(\alpha), \mathfrak{p}(t'))$
- $\ell(\mathfrak{p}_2) \ge d(\mathfrak{p}(t), \mathfrak{p}(t'))$
- $\ell(\mathfrak{p}_3) \ge d(\mathfrak{p}(t'), \mathfrak{p}(\beta))$

By assumption, $\ell(\mathfrak{p}) = d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))$. Applying the first and third equations we have

$$\ell(\mathfrak{p}_1) + \ell(\mathfrak{p}_3) \ge d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)) = \ell(\mathfrak{p}) = \ell(\mathfrak{p}_1) + \ell(\mathfrak{p}_2) + \ell(\mathfrak{p}_3)$$

Thus $d(\mathfrak{p}(t), \mathfrak{p}(t')) \le \ell(\mathfrak{p}_2) \le 0$. By metric properties, $\mathfrak{p}(t) = \mathfrak{p}(t')$, and since $\mathfrak{p}$ is injective by assumption, then $t = t'$.

Overall $s : [\alpha, \beta] \to [0, d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))]$ is a continuous bijection from a compact space to a Hausdorff space. To show that s^{-1} is continuous, it suffices to show the image of a closed set under $(s^{-1})^{-1} = s$ remains a closed set. This is true because a a closed subset of a compact set is compact, and the continuous image of a compact set remains compact. Since the codomain is a Hausdorff space, then the compact set is also closed. Thus s is a homeomorhpism.

So far, $\mathfrak{p} \circ s^{-1} : [0, d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta))] \to M$ is a path. To check that it is geodesic, we consider $0 \le u_1 \le u_2 \le d(\mathfrak{p}(\alpha), \mathfrak{p}(\beta)$.

Since s is a continuous bijection from an interval to an interval, then by the intermediate value theorem, it must be (strictly) monotone. In this case, s is a strictly increasing bijection. Thus there exists $\alpha \le t_1 \le t_2 \le \beta$ such that $u_1 = d(\mathfrak{p}(\alpha), \mathfrak{p}(t_1)) = s(t_1)$ and $u_2 = d(\mathfrak{p}(\alpha), \mathfrak{p}(t_2)) = s(t_2)$. Then by applying lemma 3.1.8,

$$d(\mathfrak{p} \circ s^{-1}(u_1), \mathfrak{p} \circ s^{-1}(u_2)) = d(\mathfrak{p}(t_1), \mathfrak{p}(t_2)) = d(\mathfrak{p}(\alpha), \mathfrak{p}(t_2)) - d(\mathfrak{p}(\alpha), \mathfrak{p}(t_1)) = |u_1 - u_2|$$

□

In summary, by definition 3.1.6, a geodesic path from a to b is a function that is an isometry, and hence it travels at *unit speed uniformly*. Lemmas 3.1.7, 3.1.8, and 3.1.9 come from [Bow06], and their purpose is to provide the (necessary and sufficient) conditions to determine when a path is geodesic.

Remark. As stated before, given a graph Γ, each edge e can be viewed as an isometric copy of $[0, 1]$ via the isometry $q(e, t)$ where $0 \le t \le 1$. This is equivalent to saying that edges are geodesic paths of length 1. Recall that if the concatenation of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ is $\mathfrak{p}$, then $\ell(\mathfrak{p}_1) + \ell(\mathfrak{p}_2) = \ell(\mathfrak{p})$.

Now let us revisit the graph metric, d_Γ, from definition 3.1.3. **We claim that $d_\Gamma(a, b)$ is actually the minimum of the length of paths from a to b**. Indeed, for case 1, suppose $a, b \in \Gamma$ belong to a common edge e. Since edge e is viewed as an isometric copy of [0,1], then there exists a (geodesic) path $\mathfrak{p}$ from a to b. Then $d^*(a, b) = d_\Gamma(a, b) = \ell(\mathfrak{p})$.

For case 2, if $a, b \in \Gamma$ belong to different edges, then the expression $\sum_{i=0}^{n-1} d^*(u_i, u_{i+1})$ can be interpreted as the length of a path which starts at $a = u_0$, moves from u_i to u_{i+1} at unit speed, and stops at $u_n = b$. Keep in mind we still require u_i, u_{i+1} to belong to a common edge, and so $n \ge 2$. Since we are interested in taking the infimum of such expressions, then we further assume $u_i \in V_\Gamma$ for all $1 \le i \le n - 1$. Since the length of an edge is 1, then the following

$$\left\{ \sum_{i=0}^{n-1} d^*(u_{i+1}, u_i) \;\middle|\; \begin{array}{l} \exists a = u_0, \dots, u_n = b \text{ such that } u_i \in V_\Gamma \text{ for all } 1 \le i \le n-1 \\ \qquad \text{and } u_i, u_{i+1} \text{ belong to a common edge for all } 0 \le i \le n-1 \end{array} \right\}$$

is actually a discrete set.

Proposition 3.1.10 *If Γ is a connected graph, then (Γ, d_Γ) is geodesic.*

Proof. By the discussion above, if $a, b \in \Gamma$, then there exists $\mathfrak{p} : [\alpha, \beta] \to \Gamma$ such that $d_\Gamma(a, b) = \ell(\mathfrak{p})$. We may further assume that there exists $\alpha = t_0 < t_1 < \dots < t_n = \beta$ such that $\mathfrak{p}|_{[t_i, t_{i+1}]}$ is actually a geodesic path from $\mathfrak{p}(t_i)$ to $\mathfrak{p}(t_{i+1})$. In other words $\mathfrak{p}|_{[t_i, t_{i+1}]}$ is an isometry and is injective on its domain.

Observe $\mathfrak{p}$ is injective on $[\alpha, \beta]$. By way of contradiction, suppose $\mathfrak{p}$ is not injective. Then there exists $\alpha \leq u_1 < u_2 \leq \beta$ such that $\mathfrak{p}(u_1) = \mathfrak{p}(u_2)$. Take $\mathfrak{p}_1 = \mathfrak{p}|_{[\alpha,u_1]}$, $\mathfrak{p}_2 = \mathfrak{p}|_{[u_1,u_2]}$, and $\mathfrak{p}_3 = \mathfrak{p}|_{[u_2,\beta]}$. Define $\mathfrak{p}'$ to be the concatenation of $\mathfrak{p}_1$ and $\mathfrak{p}_3$. Then

$$d_\Gamma(a, b) \leq \ell(\mathfrak{p}') \leq \ell(\mathfrak{p}) = d_\Gamma(a, b)$$

and that implies $\ell(\mathfrak{p}_2) = 0$. In other words, $\mathfrak{p}$ is a constant function when restricted to $[u_1, u_2]$.

Since $u_1 < u_2$, this implies there exists $u_1 \leq u_1' < u_2' \leq u_2$ such that $[u_1', u_2'] \subseteq [t_i, t_{i+1}]$ for some $i \in \{0, \ldots, n-1\}$. In other words, $\mathfrak{p}|_{[t_i,t_{i+1}]}$ is a constant function on the (non-degenerate) subinterval of $[u_1', u_2']$. This contradicts that $\mathfrak{p}|_{[t_i,t_{i+1}]}$ is an injection. Thus $\mathfrak{p}$ is actually an injection on all of $[\alpha, \beta]$.

By applying lemma 3.1.9 we see, up to reparameterization, $\mathfrak{p}$ is a geodesic path. We conclude (Γ, d_Γ) is a geodesic space.

□

Definition 3.1.11 *A* ***geodesic space*** (X, d) ***is hyperbolic*** *when there exists a constant* $\delta \geq 0$ *such that for every geodesic triangle, each edge is contained in the* δ*-neighbourhood of the union of the other two edges (i.e. geodesic triangles are* δ*-slim). For brevity, we say* X *is a hyperbolic space (as opposed to* X *is a hyperbolic geodesic metric space).*

We provide an illustration in figure 3.2. The idea of δ-slim triangles is that we could pick w in $[x, y]$, $[x, z]$ or $[y.z]$ and it would be contained in the δ neighbourhood of the other two edges.

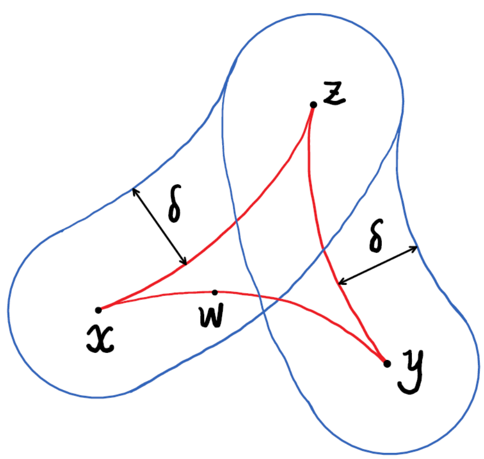

Figure 3.2: w in $[x, y]$ has distance less than δ to $[x, z] \cup [y, z]$

Definition 3.1.12 *Given a geodesic space (X, d), we pick $w \in X$ to be a base point. For $x, y \in X$, we define the **Gromov product** (with respect to base point w) as*

$$\langle x, y \rangle_w = \frac{1}{2} \left(d(x, w) + d(y, w) - d(x, y) \right)$$

*We say the **Gromov product is hyperbolic** (with respects to base point w) when there exists $\delta \geq 0$ such that for all $x, y, z \in X$ we have*

$$\langle x, y \rangle_w \geq \min\{\langle x, z \rangle_w, \langle y, z \rangle_w\} - \delta$$

Note that $\langle x, y \rangle_w \geq 0$ by the triangle inequality.

Theorem 3.1.13 *Let (X, d) be a geodesic space. The following are equivalent:*

i. X is a hyperbolic space.
ii. the Gromov product on X is hyperbolic with respect to any base point.

This theorem is listed as theorem 2.1 in [ABC+91]. In fact, the theorem actually lists five equivalent criteria in determining a hyperbolic space, but for this thesis we only look at two criteria.

Remark. Before we move onto the definition of internal points of geodesic triangles, we first look at the motivation. Given a geodesic triangle Δxyz in geodesic space (X, d), we can construct a comparison triangle in the $\mathbb{R}^2$ with Euclidean metric, $d_{Euclidean} = d_E$, with the same lengths. With a small abuse of notation, we shall still use Δxyz to denote the triangle in $\mathbb{R}^2$. In other words, $d_E(x, y) = d(x, y)$, $d_E(x, z) = d(x, z)$, and $d_E(y, z) = d(y, z)$.

Then we inscribe the maximal circle in triangle Δxyz in $\mathbb{R}^2$, which meets each of the edges precisely once. As seen in figure 3.3, we label these points c_x, c_y, c_z and refer to them as the **internal points** of Δxyz. By Euclidean geometry, the internal points have the following property: $d_E(x, c_y) = d_E(x, c_z)$, $d_E(y, c_x) = d_E(y, c_z)$, and $d_E(z, c_x) = d_E(z, c_y)$.

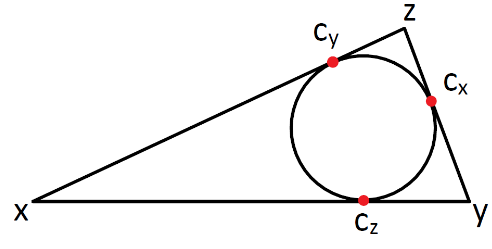

Figure 3.3: The internal points of a triangle in $\mathbb{R}^2$

Definition 3.1.14 *Using the same geodesic space (X, d) and geodesic triangle Δxyz as described above, the **internal points** of Δxyz are a set of three points $c_x, c_y, c_z \in X$ which satisfy the following properties:*

$$d(x, c_y) = d(x, c_z),\ d(y, c_x) = d(y, c_z),\ \text{and}\ d(z, c_x) = d(z, c_y)$$

such that $c_x \in [y, z]$, $c_y \in [x, z]$, and $c_z \in [y, z]$.

Next, we introduce the concept of a Cayley graph, which allows us to reconsider a group as a graph. This idea will be made precise in the following definition.

Definition 3.1.15 *Given a group G and a finite generating set S such that the identity $e \notin S$, the **Cayley graph of G** (with respect to S) is denoted by $\Gamma = \text{Cay}(G, S)$, and is constructed as follows:*

Let $V_\Gamma = G$ be the vertex set, and $E_\Gamma = \{(g, ga) : g \in G, a \in S\}$ be the edge set. While the Cayley graph is a directed graph, in this thesis we won't make use of the directed structure.

Remark.

i. We require identity $e \notin S$ so that vertices will not be adjacent to themselves; i.e. Γ won't have any loops.

ii. We require that S be a finite generating set of G to ensure that $\Gamma = \text{Cay}(G, S)$ is a connected graph of bounded degree. By 3.1.10, (Γ, d_Γ) is a geodesic space.

iii. The Cayley graph is dependent on the choice of generating set; that is to say, if we change the generating set, then the Cayley graph will also change.

iv. By construction, if $x, y \in V_\Gamma$, then there is at-most one (directed) edge from x to y and at-most one (directed) edge from y to x. Thus, it is valid to use the notation $e = (x, y)$ or $e = (y, x)$ when appropriate.

Definition 3.1.16 *A **group G is hyperbolic** if there exists a finite generating set S such that $\Gamma = \text{Cay}(G, S)$ is the Cayley graph and (Γ, d_Γ) is hyperbolic.*

Examples. For figures 3.4 and 3.5, we see that the Cayley graph of $\mathbb{Z}$ depends on the choice of generating set. The same phenomenon applies to figures 3.6 and 3.7. For figures 3.8 and 3.9 we see that both F_2 and $\mathbb{Z}^2$ are generated by two elements, but the relation for $\mathbb{Z}^2$ changes the graph.

Figures 3.4 and 3.9 are trees and hence are both 0-hyperbolic. Figure 3.5 is hyperbolic due to theorem 3.1.21, and figures 3.6 and 3.7 are hyperbolic because they are bounded as metric spaces.

For figure 3.8 we see $\mathbb{Z}^2$ is quasi-isometric (defined at 3.1.17) to $\mathbb{R}^2$. $\mathbb{R}^2$ is not hyperbolic because for any choice of $\delta > 0$, the equilateral triangle of side lengths 4δ would not be δ-slim. The midpoint of an edge would not be in the δ neighbourhood of either of the other two edges. Then by theorem 3.1.20, $\mathbb{Z}^2$ will also not be hyperbolic.

Figure 3.4: $\mathbb{Z} = \langle a \rangle$

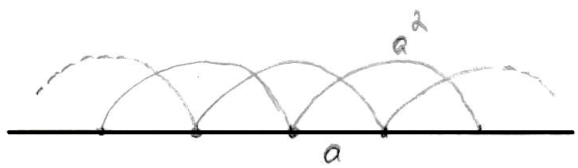

Figure 3.5: $\mathbb{Z} = \langle a, a^2 \rangle$

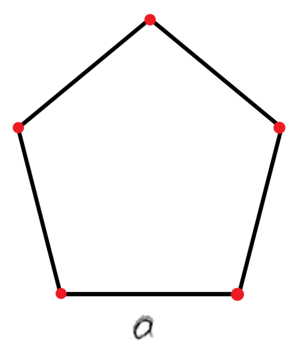

Figure 3.6: $\mathbb{Z}_5 = \langle a \mid a^5 \rangle$

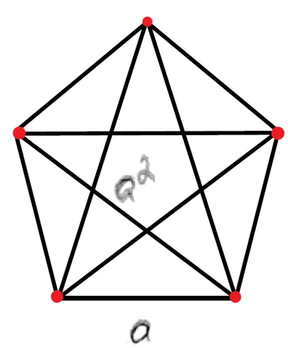

Figure 3.7: $\mathbb{Z}_5 = \langle a, a^2 \mid a^5 \rangle$

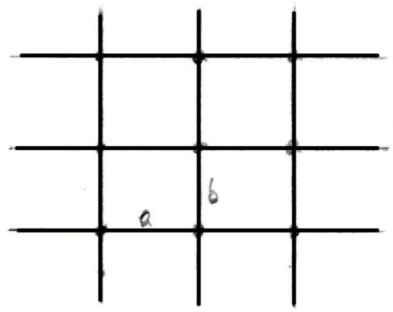

Figure 3.8: $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$

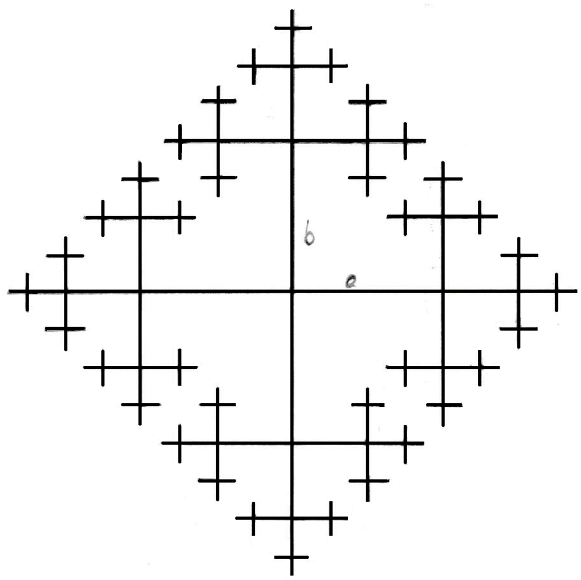

Figure 3.9: $\mathbb{F}_2 = \langle a, b \rangle$

Remark. What's interesting is that the hyperbolicity of a group is independent of the choice of finite generating subset by 3.1.21. While we won't prove the theorem we will outline the main definitions and theorems. For full details, the reader is directed to [Bow06].

Definition 3.1.17 *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\phi : X \to Y$ is a **quasi-isometry** when it satisfies the following two properties:*

i. there exist constants $k_1 > 0$ and $k_2, k_3, k_4 \geq 0$ such that

$$k_1 d_X(x, y) - k_2 \leq d_Y(\phi(x_1), \phi(x_2)) \leq k_3 d_X(x_1, x_2) + k_4$$

for all $x_1, x_2 \in X$

ii. there exists $k_5 \geq 0$ such that for all $y \in Y$, there exists $x \in X$ where $d(y, \phi(x)) \leq k_5$

Lemma 3.1.18 *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces.*

i. If $\phi : X \to Y$ and $\psi : Y \to Z$ are both quasi-isometries, then so is their composition, $\psi \circ \phi$.

ii. If $\phi : X \to Y$ is a quasi-isometry, then there exists a quasi-isometry $\psi : Y \to X$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are a bounded distance away from the identity maps.

Note: This lemma is listed as proposition 3.2 in [Bow06].

As a consequence of lemma 3.1.18 we see that quasi-isometries form an equivalence relation; that is $X \sim Y$ if and only if there exists a quasi-isometry between them.

Theorem 3.1.19 *If S and S_0 are both finite generating subsets of group Γ, then $\mathrm{Cay}(\Gamma, S) \sim \mathrm{Cay}(\Gamma, S_0)$.*

Note: This theorem is listed as theorem 3.3 in [Bow06]

Theorem 3.1.20 *If X and Y are both geodesic spaces and $X \sim Y$, then X is hyperbolic if and only if Y is hyperbolic.*

Note: This theorem is listed as theorem 6.19 in [Bow06]

Corollary 3.1.21 *The hyperbolicity of a group is independent of its finite generating subset, and it comes as a consequence of theorems 3.1.19 and 3.1.20.*

3.2 Technical lemmas and propositions

In section 3.1 we established our notion of graphs Γ. Then we carefully constructed the metric space (Γ, d_Γ), and we also looked at paths. We begin section 3.2 by recreating the same definitions, but in the context of V_Γ.

Definition 3.2.1 *Given a connected graph Γ, **the word metric**, $d_w : V_\Gamma \times V_\Gamma \to \mathbb{R}$, is defined as follows:*

$$d_w(x, y) = \min\{n : \exists x = x_0, \dots, x_n = y \text{ in } \Gamma \text{ such that } x_i \text{ and } x_{i+1} \text{ are adjacent}\}$$

*A **(finite) path of length n in (V_Γ, d_w)** is a function $\mathfrak{p} : \{0, 1, \dots, n\} \to V_\Gamma$ such that $p(i)$ and $p(i+1)$ are adjacent.*

*An **infinite path in (V_Γ, d_w)** is a function $\mathfrak{p} : \mathbb{N}_0 \to V_\Gamma$ such that $\mathfrak{p}(i)$ and $\mathfrak{p}(i+1)$ are adjacent.*

We note d_w is well-defined because Γ is connected. As a convention, a path $\mathfrak{p}$ in V_Γ joining x to y is denoted by its vertices; that is, $\mathfrak{p} = x_0 x_1 \dots x_n$ where $x_0 = x$ and $x_n = y$.

Motivation. We want to visualize paths in (V_Γ, d_w) as *coming from* paths in (X_Γ, d_Γ). Furthermore, we want to view (V_Γ, d_w) as a submetric space of (X_Γ, d_Γ). The details are presented below.

From definition 3.1.5, a path $\mathfrak{p} : [\alpha, \beta] \to (\Gamma, d_\Gamma)$ from $a = \mathfrak{p}(\alpha)$ to $b = \mathfrak{p}(\beta)$ is a continuous function, and $\ell(\mathfrak{p})$ is the length. We now consider the additional restrictions:

- $\mathfrak{p}$ is geodesic in the sense of definition 3.1.6
- $\ell(p) = n \in \mathbb{N}_0$
- $\alpha = 0$
- $\beta = n$
- $\mathfrak{p}(i) \in V_\Gamma$ for all $i \in \{0, 1, \dots, n\}$
- $\mathfrak{p}(i),\ \mathfrak{p}(i+1)$ are adjacent for all $i \in \{0, \dots, n-1\}$

and we will see that paths in V_Γ (as defined in 3.2.1) can be expressed as $\mathfrak{p}|_{[\alpha,\beta]\cap\mathbb{N}_0}$.

Remark. Observe that $d_w(x, y) = d_\Gamma(x, y)$ when $x, y \in V_\Gamma$. Broadly speaking, in both definitions we are looking for paths of shortest length from x to y, relative to the ambient spaces. For d_w, this is by the definition 3.2.1, and for d_Γ, it was discussed in the remarks preceding proposition 3.1.10.

Suppose $\mathfrak{p} : \{0, 1, \dots, n\} \to V_\Gamma$ is a path of shortest length from x to y in (V_Γ, d_w). We can lift $\mathfrak{p}$ to $\mathfrak{p}' : [0, n] \to \Gamma$ by *travelling* from vertex to vertex *at unit speed*. By our construction, $\mathfrak{p}'|_{[0,n]\cap\mathbb{N}_0} = \mathfrak{p}$ and $\ell(\mathfrak{p}') = \ell(\mathfrak{p})$. Also, $\mathfrak{p}'$ is a path of shortest length in (Γ, d_Γ) because it essentially travels directly from x to y. Then,

$$d_w(x, y) = \ell(\mathfrak{p}) = \ell(\mathfrak{p}') = d_\Gamma(x, y)$$

and so d_Γ extends d_w.

Based on the work above, *paths in* (V_Γ, d_w) *come from paths in* (X_Γ, d_Γ) *when they satisfy the additional restrictions above*. Furthermore, (V_Γ, d_w) *can be viewed as a submetric space of* (X_Γ, d_Γ). Since (V_Γ, d_w) is a discrete space, we now provide a more reasonable definition of a geodesic path.

Definition 3.2.2 *A (finite or infinite) path* $\mathfrak{p}$ *in* (V_Γ, d_w) *is* ***geodesic*** *when*

$$d_w(\mathfrak{p}(m), \mathfrak{p}(n)) = |m - n|$$

for all m, n *in the domain of* $\mathfrak{p}$.

Thus the definitions and lemmas from section 3.1 will also hold for (V_Γ, d_w). Based on definition 3.2.1, it's obvious that (V_Γ, d_w) remains a geodesic space because Γ is connected. This is analogous to proposition 3.1.10.

Unless indicated otherwise, from now on, we will use the notations $(\boldsymbol{V_\Gamma, d_w})$ and $(\boldsymbol{\Gamma, d_w})$ interchangably. For example, in the expression $d_w(x, y)$, it should be clear that $x, y \in V_\Gamma$.

Lemma 3.2.3 *Let (Γ, d_w) be a geodesic graph and $\mathfrak{p} : \mathbb{N}_0 \to \Gamma$ be an infinite geodesic path. If $y \in \Gamma$, then there exists an infinite geodesic path $\mathfrak{p}_y$ which starts at y and eventually flows into $\mathfrak{p}$. i.e. the symmetric difference, $\mathfrak{p}\Delta\mathfrak{p}_y$, is finite.*

Note: This lemma is listed as lemma 5 in [Oza08].

Proof. Let $\mathfrak{p} = x_0x_1x_2\dots$ be an infinite geodesic path and $y \in \Gamma$. If $m \leq n$, define $h_y(m, n) = d_w(y, x_m) + d_w(x_m, x_n) - d_w(y, x_n)$.

Our initial goal is to show there exists $m_0 \in \mathbb{N}_0$ such that $h_y(m_0, n) = 0$ for all $n \geq m_0$. The idea is that if $h_y(m_0, n) = 0$ and $[y, x_{m_0}]$ and $[x_{m_0}, x_n]$ are geodesic, then their concatenation, $[y, x_{m_0}] \cup [x_{m_0}, x_n]$, will remain geodesic. The details are presented below.

Immediately $h_y(m, n) \geq 0$ by triangle inequality. Also, if $m \leq n \leq k$, then $h_y(m, k) = h_y(m, n) + h_y(n, k) \geq h_y(m, n)$, which is easily verified by expanding the terms. Thus, if m is fixed, then the sequence, $(h_y(m, n))_{n\geq m}^{\infty}$ is increasing.

Observe $h_y(m, n) \leq 2d_w(y, x_0)$. Indeed, $d_w(x_n, x_0) - d_w(x_n, y) \leq d_w(y, x_0)$ by the triangle inequality. Also $d_w(y, x_m) - d_w(x_0, x_m) = d_w(y, x_0) - h_y(0, m)$ follows by the definitions. Recall $\mathfrak{p} = x_0x_1x_2\dots$ is geodesic. Since $m \leq n$, then

$$\begin{aligned}
h_y(m, n) &= d_w(y, x_m) + d_w(x_m, x_n) - d_w(y, x_n) \\
&= d_w(y, x_m) - d_w(x_0, x_m) + d_w(x_0, x_n) - d_w(y, x_n) \\
&= d_w(y, x_0) - h_y(0, m) + d_w(x_0, x_n) - d_w(y, x_n) \\
&\leq d_w(y, x_0) + d_w(x_0, x_n) - d_w(y, x_n) \\
&\leq d_w(y, x_0) + d_w(y, x_0) \\
&= 2d_w(y, x_0)
\end{aligned}$$

We can view $(h_y(m, k))_{k\geq m}^{\infty}$ as a sequence indexed by k. By the work above, we know this sequence is increasing and bounded, so we define $h(m) = \lim_{k\to\infty} h_y(m, k)$. If $m \leq n$, then

$$h(m) = \lim_{k\to\infty} (h_y(m, n) + h_y(n, k)) = h_y(m, n) + h(n)$$

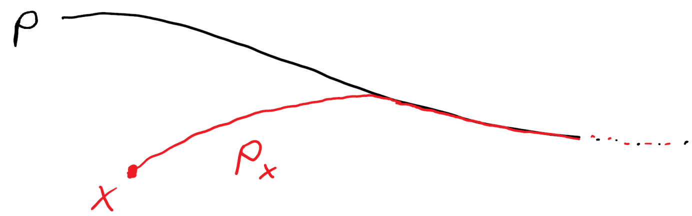

Figure 3.10: geodesic path $\mathfrak{p}_x$ starts at $x \in V_\Gamma$ and flows into $\mathfrak{p}$

Pick $m_0 \in \mathbb{N}$ such that $h(m_0) = \min\{h(m) : m \in \mathbb{N}_0\}$. Observe that $h_y(m_0, n) = 0$ when $m_0 \leq n$. Indeed, $h_y(m_0, n) = h(m_0) - h(n) \leq 0$ by choice of m_0. Then by the triangle inequality,

$$h_y(m_0, n) = d_w(y, x_{m_0}) + d_w(x_{m_0}, x_n) - d_w(y, x_n) \geq 0$$

and so, we have verified that $h_y(m_0, n) = 0$ for all $m_0 \leq n$.

Pick a (finite) geodesic path $\mathfrak{p}_1 = [y, x_{m_0}]$ and let $\mathfrak{p}_2 = x_{m_0} x_{m_0+1} x_{m_0+2} \ldots$ be a subpath of $\mathfrak{p}$. Define $\mathfrak{p}_y$ as a concatenation of $\mathfrak{p}_1$ and $\mathfrak{p}_2$. To complete the proof, we will show $\mathfrak{p}_y$ is geodesic and that $\mathfrak{p} \Delta \mathfrak{p}_y$ is finite. Let ℓ be the length of $\mathfrak{p}_1$. Without harm, we can rename $y = y_0$ and $x_{m_0} = y_\ell$. In general, $x_{m_0+i} = y_{\ell+i}$ for all $i \in \mathbb{N}_0$. Hence,

$$\mathfrak{p}_1 = [y_0, y_\ell], \;\; \mathfrak{p}_2 = y_\ell y_{\ell+1} y_{\ell+2} \ldots, \;\; \text{and } \mathfrak{p}_y = y_0 y_1 y_2 \ldots$$

Next, we will verify that all finite subpaths of $\mathfrak{p}_y$ are geodesic. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are geodesic, it is only necessary to check $y_m y_{m+1} \ldots y_n$ is geodesic for $m \leq \ell \leq n$. Recall $h_y(m_0, k) = 0$ when $m_0 \leq k$. For $k = m_0 + n - \ell$ we have

$$d_w(y_0, y_n) = d_w(y, x_k) = d_w(y, x_{m_0}) + d_w(x_{m_0}, x_k) = \ell + (k - m_0) = n$$

Thus $y_0 y_1 \ldots y_n$ and the subpath $y_m y_{m+1} \ldots y_n$ are both geodesic, because they represent paths of shortest length. Overall, p_y is geodesic because all finite subpaths are geodesic. Lastly, the symmetric difference $\mathfrak{p} \Delta \mathfrak{p}_y$ is at-most $m_0 + \ell$.

□

Lemma 3.2.4 *Let (Γ, d_w) is an infinite δ-geodesic graph with a bounded degree. Then there exists an infinite geodesic path.*

Proof. Let $x_0 \in \Gamma$ be fixed. Since Γ is of bounded degree, for each $n \in \mathbb{N}$, there exists $x_n \in \Gamma$ such that $d_w(x_0, x_n) \geq n$. Let $\mathfrak{p}_n = [x_0, x_n]$ be a geodesic path in Γ.

Let $S = \{\mathfrak{p}_n : n \in \mathbb{N}\}$. For each $r \in \mathbb{N}$, each $B(x_0, r)$ is a finite set because Γ is of bounded degree. Thus there exists an infinite set, $S_r \subseteq S$, such that all $p_n \in S_r$ have the same initial segment of length r.

Likewise, there exists an infinite set $S_{r+1} \subseteq S_r$ such that the paths in S_{r+1} have the same initial segment of length $r+1$. More importantly, they extend the initial segment of length r from before. Since $r \in \mathbb{N}$, we can repeatedly apply process to find an infinitely long geodesic path.

□

Remark. For the remainder of this thesis, we assume that (Γ, d_w) is an infinite δ-hyperbolic graph with a bounded degree, that is $\sup\{|B_R(x)| : x \in \Gamma\} < \infty$ for all $R > 0$. We also assume $\delta \geq 1$ is fixed.

As a consequence of lemmas 3.2.3 (i.e. lemma 5 of [Oza08]) and 3.2.4 we now fix an infinite geodesic path $\mathfrak{p}$, and for every $x \in \Gamma$, we pick and fix an infinite geodesic path $\mathfrak{p}_x$ which starts at x and eventually flows into $\mathfrak{p}$. We refer to figure 3.10.

In lemma 3.2.5, we will work with hyperbolic graphs. The triangles in figures 3.3, 3.11, and 3.12 will appear to be Euclidean, but that is merely to simplify the nature of the drawings. The point is that in (Γ, d_w), our geodesic triangles are δ-slim.

Lemma 3.2.5 *Let (V_Γ, d_w) be a δ-hyperbolic graph. As assumed above, $\delta \geq 1$. If x to $y \in V_\Gamma$ is joined by geodesic path $[x, y]$, and $z \in V_\Gamma$, then*

$$d_w(z, [x, y]) \leq \langle x, y \rangle_z + 10\delta$$

Note: This lemma is listed as lemma 6 in [Oza08], and proposition 2.17 in [GdlH90]. To avoid confusion, we will be careful to distinguish V_Γ from X_Γ and d_w from d_Γ.

Proof. Consider $x, y, z \in V_\Gamma$ and a corresponding geodesic triangle, Δxyz in (X_Γ, d_Γ). We label the internal points c_x, c_y, $c_z \in X_\Gamma$ in $[y, z]$, $[x, z]$, and $[x, y]$ respectively. Define $k = \lfloor d_\Gamma(x, c_y) \rfloor = \lfloor d_\Gamma(x, c_z) \rfloor$. Pick $a_z \in V_\Gamma$ in $[x, y]$ and $a_y \in V_\Gamma$ in $[x, z]$ such that $k = d_w(x, a_z) = d_w(x, a_y)$, as seen in figure 3.11.

We introduce b_x and $b_z \in V_\Gamma$ which have a similar purpose. Specifically, $b_x \in [y, z]$ and $b_z \in [x, y]$ are chosen such that $d_w(y, b_x) = d_w(y, b_z) = \lfloor d_w(y, c_x) \rfloor = \lfloor d_w(y, c_z) \rfloor$.

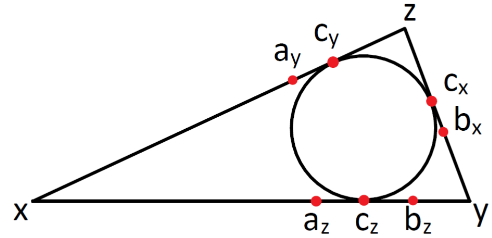

Figure 3.11: $d_\Gamma(a_z, c_z)$, $d_\Gamma(a_y, c_y)$, $d_\Gamma(b_z, c_z)$, $d_\Gamma(b_x, c_x)$ are all less than 1.

To prove this lemma we create three subgoals:

- $d_\Gamma(z, c_x) = d_\Gamma(z, c_y) = \langle x, y\rangle_z$
- $d_w(a_y, a_z) \leq 2\delta$ or $d_w(b_x, b_z) \leq 6\delta$
- $d_w(z, a_y) \leq \langle x, y\rangle_z + \delta$ and $d_w(z, b_x) \leq \langle x, y\rangle_z + \delta$

For the **first claim**, we recall d_w is extended by d_Γ. By property of internal points and geodesic paths we see $d_\Gamma(x, c_y) + d_\Gamma(y, c_x) = d_\Gamma(x, y)$. Then,

$$\begin{aligned}
\langle x, y\rangle_z &= \frac{1}{2}\left(d_w(x, z) + d_w(y, z) - d_w(x, y)\right) \\
&= \frac{1}{2}\left(d_\Gamma(x, c_y) + d_\Gamma(z, c_y) + d_\Gamma(y, c_x) + d_\Gamma(z, c_x) - d_\Gamma(x, y)\right) \\
&= \frac{1}{2}\left(d_\Gamma(z, c_y) + d_\Gamma(z, c_x) + d_\Gamma(x, c_y) + d_\Gamma(y, c_x) - d_\Gamma(x, y)\right) \\
&= \frac{1}{2}\left(d_\Gamma(z, c_y) + d_\Gamma(z, c_x)\right) \\
&= d_\Gamma(z, c_y)
\end{aligned}$$

and we note that $d_\Gamma(z, c_y) = d_\Gamma(z, c_x)$ by definition of the internal points.

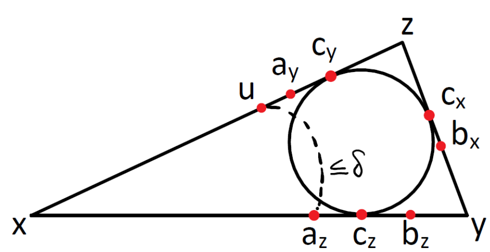

Figure 3.12: u lies on $[xz]$ and $d_\Gamma(a_z, u) \leq \delta$

For the **second claim**, we will verify $d_w(a_y, a_z) \leq 2\delta$ or $d_w(b_x, b_z) \leq 6\delta$. Since (V_Γ, d_w) is δ-hyperbolic, then by the δ-slim property, there exists $u \in [x, z] \cup [y, z]$ such that $d_w(a_z, u) \leq \delta$. As a side remark, $u \in V_\Gamma$ because (V_Γ, d_w) is assumed to be δ-hyperbolic, not (X_Γ, d_Γ).

For the first case, we consider $u \in [x, z]$. As seen in figure 3.12, we have $d_\Gamma(a_z, u) \leq \delta$ and we recall $d_w(x, a_z) = d_w(x, a_y)$. Then,

$$d_w(x, a_y) = d_w(x, a_z) \leq d_w(x, u) + d_w(a_z, u) \leq d_w(x, u) + \delta$$

and

$$d_w(x, u) \leq d_w(x, a_z) + d_w(a_z, u) \leq d_w(x, a_y) + \delta$$

Together, they imply $d_w(a_y, u) = |d_w(x, u) - d_w(x, a_y)| \leq \delta$. Since $d_w(a_z, u) \leq \delta$ was assumed, then $d_w(a_y, a_z) \leq 2\delta$ follows by the triangle inequality.

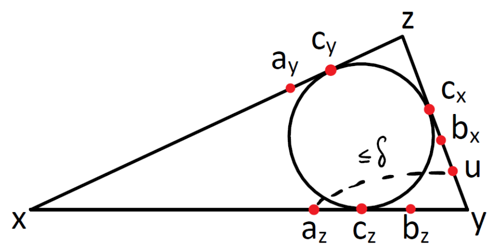

Figure 3.13: u lies on $[yz]$ and $d_\Gamma(a_z, u) \leq \delta$

For the second case, we consider $u \in [y, z]$. As seen in figure 3.13, we have $d_\Gamma(a_z, u) \leq \delta$ and we recall $d_w(y, b_x) = d_w(y, b_z)$. Note that $d_w(a_z, b_z) \leq d_\Gamma(a_z, c_z) + d_\Gamma(c_z, b_z) \leq 2 \leq 2\delta$ because $\delta \geq 1$. Then,

$$d_w(y, u) \leq d_w(y, a_z) + d_w(a_z, u) \leq d_w(y, b_z) + d_w(b_z, a_z) + \delta \leq d_w(y, b_x) + 3\delta$$

and

$$d_w(y, b_x) = d_w(y, b_z) \leq d_w(y, u) + d_w(u, a_z) + d_w(a_z, b_z) \leq d_w(y, u) + 3\delta$$

Together, they imply $d_w(u, b_x) = |d_w(u, y) - d_w(b_x, y)| \leq 3\delta$. Since $d_w(a_z, u) \leq \delta$ was assumed, then

$$d_w(b_z, b_x) \leq d_w(b_z, a_z) + d_w(a_z, u) + d_w(u, b_x) \leq 2 + \delta + 3\delta \leq 6\delta$$

For the **third claim**, we will verify $d_w(z, a_y) \leq \langle x, y \rangle_z + \delta$ and $d_w(z, b) \leq \langle x, y \rangle_z + \delta$. Recall $\delta \geq 1$. Then, by applying claim 1 we see

$$d_w(z, a_y) \leq d_\Gamma(z, c_y) + d_\Gamma(a_y, c_y) \leq \langle x, y \rangle_z + 1 \leq \langle x, y \rangle_z + \delta$$

and

$$d_w(z, b_x) \leq d_\Gamma(z, c_x) + d_\Gamma(c_x, b_x) \leq \langle x, y \rangle_z + 1 \leq \langle x, y \rangle_z + \delta$$

By combining the results from the second and third claims, we have

$$d_w(z, a_z) \leq d_w(z, a_y) + d_w(a_y, a_z) \leq d_w\langle x, y \rangle_z + 3\delta$$

or

$$d_w(z, b_z) \leq d_w(z, b_x) + d_w(b_x, b_y) \leq d_w\langle x, y \rangle_z + 7\delta$$

Whether $u \in [x, z]$ or $u \in [y, z]$ we conclude $d_\Gamma(z, [x, y]) \leq \langle x, y \rangle_z + 10\delta$

□

Disclaimer: **Now that lemma 3.2.5 is proven, we can safely forget about the graph metric from 3.1.3. From here on we will exclusively work with (V_Γ, d_w).**

Remark. We provide an illustration of $\mathcal{N}_{100\delta}(\mathfrak{p}_x)$ in figure 3.14. $\mathcal{N}_{100\delta}(\mathfrak{p}_x)$ is an expression of interest and it will reappear many times throughout section 3.2.

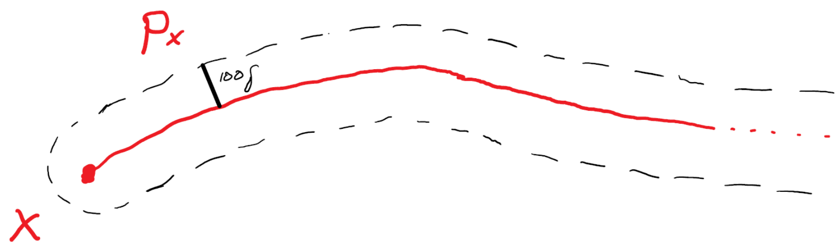

Figure 3.14: We think of it as a *tube* of radius 100δ surrounding $\mathfrak{p}_x$.

Lemma 3.2.6 *For $x \in \Gamma$ and $k \in \mathbb{Z}$ we define*

$$T(x,k) = \{w \in \mathcal{N}_{100\delta}(\mathfrak{p}_x) : d_w(w,x) \in \{k-1,k\}\}$$

There exists a constant R_0 such that if $x \in \Gamma$, $k \in \mathbb{N}_0$ and v_x denotes the unique point on $\mathfrak{p}_x$ such that $d_w(v_x,x) = k$, then

$$T(x,k) \subseteq B_{R_0}(v_x)$$

Note: This lemma is listed as lemma 7 in [Oza08]. Immediately, $T(x,k) = \emptyset$ whenever $k < 0$. We provide an illustration in figure 3.15. Also, the δ-hyperbolic assumption is not needed here; a bounded infinite geodesic graph is enough.

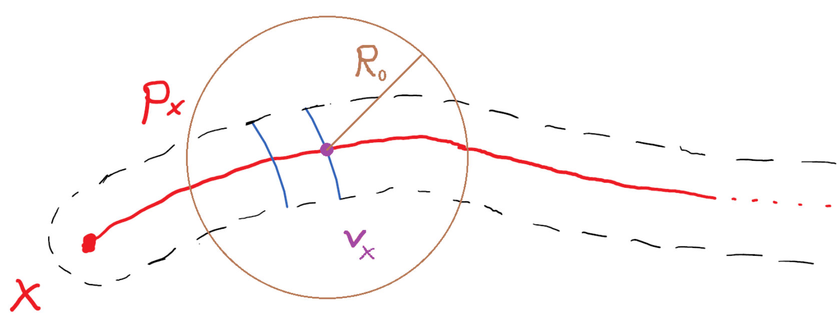

Figure 3.15: The two blue lines are of distance $k-1$ and k away from x respectively. $T(x,k)$ is the union of the two blue lines.

Proof. Let $w \in T(x,k)$. Since $w \in \mathcal{N}_{100\delta}(\mathfrak{p}_x)$, then choose $w' \in \mathfrak{p}_x$ such that $d_w(w,w') < 100\delta$.

Our goal is to show that $T(x,k) \subseteq B_{R_0}(v_x)$ for $R_0 = 200\delta + 1$. Indeed, by the triangle inequality $d_w(w,v_x) \leq d_w(w,w') + d_w(w',v_x)$. Recall the path $\mathfrak{p}_x$ is geodesic which contains x, w', v_x. If the points of the paths are ordered x, w', v_x, then

$$d_w(w',v_x) = -d_w(x,w') + d_w(x,v_x)$$

or if the points are ordered x, v_x, w', then

$$d_w(w',v_x) = d_w(x,w') - d_w(x,v_x)$$

Thus $d_w(w', v_x) = |d_w(x, w') - d_w(x, v_x)|$ is true in both cases. Recall that $d_w(v_x, x) = k$. Then,

$$\begin{aligned} d_w(w, v_x) &\le d_w(w, w') + d_w(w', v_x) \\ &= d_w(w, w') + |d_w(x, w') - d_w(x, v_x)| \\ &= d_w(w, w') + |d_w(x, w') - k)| \\ &\le d_w(w, w') + |d_w(x, w') - d_w(w, x)| + |d_w(w, x) - k| \\ &\le d_w(w, w') + d_w(w, w') + |d_w(w, x) - k| \end{aligned}$$

To complete the proof, recall $w \in T(x, k)$ implies $d_w(w, x) = k - 1$ or k. Also $w' \in \mathfrak{p}_x$ was picked such that $d_w(w, w') < 100\delta$. Then

$$d_w(w, v_x) < 100\delta + 100\delta + 1 = R_0$$

and we conclude $T(x, k) \subseteq B_{R_0}(v_x)$.

□

Lemma 3.2.7 *For $n \in \mathbb{N}_0$, $k, l \in \mathbb{Z}$, and borrowing $T(x, k)$ from lemma 3.2.6, we define*

$$E(n) = \{(x, y) \in \Gamma \times \Gamma : d_w(x, y) \le n\}$$

and

$$W(k, l) = \{(x, y) \in \Gamma \times \Gamma : T(x, k) \cap T(y, l) \ne \emptyset\}$$

i. Then, for every $n \in \mathbb{N}_0$

$$E(n) = \bigcup_{k=0}^{n} W(k, n - k)$$

ii. Also, there exists a constant $R_1 \ge 0$ such that if $j > R_1$, then

$$W(k, l) \cap W(k + j, l - j) = \emptyset$$

Note: This lemma is listed as lemma 8 in [Oza08]. We think of $W(k, l)$ as all pairs of points (x, y) such that there exists a point $z \in \Gamma$ which lies in both tubes surrounding $\mathfrak{p}_x$ and $\mathfrak{p}_y$ with a distance of k or $k-1$ from x and a distance of l or $l-1$ from y. Also, if $k < 0$ or $l < 0$, then immediate from the definitions, $W(k, l) = \emptyset$. We provide an illustration of $W(k, l)$ in figure 3.16.

Proof. **For (i)**, we consider $(x, y) \in W(k, n-k)$, so there exists $w \in T(x, k) \cap T(y, n-k)$. Then $d_w(x, y) \le d_w(x, w) + d_w(w, y) \le k + (n - k) = n$, which implies $(x, y) \in E(n)$. Thus $\supseteq$ is done.

For the reverse containment we suppose $(x, y) \in E(n)$. Immediately, $d_w(x, y) \le n$. Both $\mathfrak{p}_x$ and $\mathfrak{p}_y$ eventually flow into $\mathfrak{p}$ by lemma 3.2.3 (i.e. lemma 5 of [Oza08]), so without

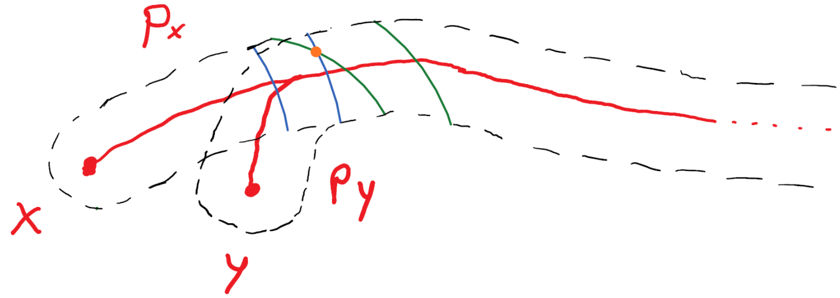

Figure 3.16: $z \in \Gamma$ is the orange dot.

a loss of generality, pick $p \in \mathfrak{p}_x \cap \mathfrak{p}_y$ such that $d_w(p,x) + d_w(p,y) \geq n$. Let $[x,y]$ represent a geodesic path connecting x and y. Then $d_w(p,[x,y]) \leq \langle x,y \rangle_p + 10\delta$ by lemma 3.2.5 (i.e. lemma 6 of [Oza08]). Pick point a in $[x,y]$ such that $d_w(p,a) \leq \langle x,y \rangle_p + 10\delta$.

Let $[a,p]$ be a geodesic path and let $0 \leq m \leq d_w(a,p)$ be a integer. We denote $w(m)$ to be the point on $[a,p]$ such that $d_w(w(m),a) = m$. By a property of geodesic paths, there is precisely one point on $[a,p]$ whose distance to a is m. Thus $w(m)$ is indeed a well-defined function. Note that $w(0) = a$ and $w(d_w(a,p)) = p$.

Define $f : \{0,1,2,\ldots,d_w(a,p)\} \to \mathbb{N}$ as $f(m) = d_w(w(m),x) + d_w(w(m),y)$. By a rather meticulous verification we have $f(0) \leq n \leq f(d_w(a,p))$. Indeed, since $(x,y) \in E(n)$, then

$$f(0) = d_w(w(0),x) + d_w(w(0),y) = d_w(a,x) + d_w(a,y) = d_w(x,y) \leq n$$

because $a \in [x,y]$. Also, by choice of $p \in \mathfrak{p}_x \cap \mathfrak{p}_y$ we have

$$n \leq d_w(p,x) + d_w(p,y) = d_w(w(d_w(a,p)),x) + d_w(w(d_w(a,p)),y) = f(d_w(a,p))$$

Also $d_w(w(m+1),w(m)) = 1$ for all $m \in \mathbb{N}_0$ because the path $[a,p]$ is geodesic. Then by the triangle inequality, we have

$$\begin{aligned} f(m+1) &= d_w(w(m+1),x) + d_w(w(m+1),y) \\ &\leq 1 + d_w(w(m),x) + 1 + d_w(w(m),y) \\ &= 2 + f(m) \end{aligned}$$

In summary, $f(0) \leq n \leq f(d_w(a,p))$ and $f(m+1) - f(m) \leq 2$ for all $m \in \mathbb{N}_0$. Thus, there exists $m_0 \in \{0,1,\ldots,d_w(a,p)\}$ such that $f(m_0) \in \{n-1,n\}$, where $w(m_0)$ is in $[a,p]$.

To complete part (i), we will prove $(x,y) \in W(k,n-k)$ where $k = d_w(w(m_0),x) \leq n$ by showing $w(m_0) \in T(x,k) \cap T(y,n-k)$. Recall $\langle y,p \rangle_a \geq 0$ by triangle inequality. By lemma 3.2.5, we have

$$\langle x,p \rangle_a \leq \langle x,p \rangle_a + \langle y,p \rangle_a = d_w(a,p) - \langle x,y \rangle_p \leq 10\delta$$

Recall $w(m_0) \in [a,p]$. Then

$$\begin{aligned}\langle x,p\rangle_{w(m_0)} &= \frac{1}{2}\big(d_w(x,w(m_0)) + d_w(p,w(m_0)) - d_w(x,p)\big)\\ &\le \frac{1}{2}\big(d_w(x,a) + d_w(a,w(m_0)) + d_w(p,w(m_0)) - d_w(x,p)\big)\\ &= \frac{1}{2}\big(d_w(x,a) + d_w(a,p) - d_w(x,p)\big)\\ &= \langle x,p\rangle_a\\ &\le 10\delta\end{aligned}$$

Since $[x,p] \subseteq \mathfrak{p}_x$, then

$$d_w(w(m_0),\mathfrak{p}_x) \le d_w(w(m_0),[x,p]) \le \langle x,p\rangle_{w(m_0)} + 10\delta \le 20\delta < 100\delta$$

by lemma 3.2.5. Then $w(m_0) \in \mathcal{N}_{100\delta}(\mathfrak{p}_x)$ is indeed true, and $k = d_w(w(m_0),x)$ by definition. Thus, $w(m_0) \in T(x,k)$ is verified.

Next, we will demonstrate that $w(m_0) \in T(y,n-k)$ where $k = d_w(w(m_0),x)$. Recall that $f(m_0) \in \{n-1,n\}$, which implies $d_w(w(m_0),y) = f(m_0) - k \in \{n-k-1,n-k\}$. By a similar process as above, $d_w(w(m_0),\mathfrak{p}_y) \le 20\delta < 100\delta$. Thus $w(m_0) \in T(y,n-k)$.

Overall, $w(m_0) \in T(x,k) \cap T(y,n-k) \neq \emptyset$ which implies $(x,y) \in W(k,n-k)$. Also, $k = d_w(w(m_0),x) \le f(m_0) \le n$. Hence $\subseteq$ is verified, and part (i) is complete.

For (ii), we prove the contrapositive statement. Suppose $(x,y) \in W(k,l) \cap W(k+j,l-j)$ where $k,l \in \mathbb{Z}$, $j \in \mathbb{N}_0$. By the definitions, then there exists $v \in T(x,k) \cap T(y,l)$ and $w \in T(x,k+j) \cap T(y,l-j)$.

Define v_x, w_x in $\mathfrak{p}_x$ such that $d_w(v_x,x) = k$, $d_w(w_x,x) = k+j$. Likewise, define v_y, w_y in $\mathfrak{p}_y$ such that $d_w(v_y,y) = l$ and $d_w(w_y,y) = l-j$. By lemma 3.2.6 (i.e. lemma 7 of [Oza08]), for a large $R_0 \ge 0$ we have

$$v \in T(x,k) \subseteq B_{R_0}(v_x) \text{ and } w \in T(x,k+j) \subseteq B_{R_0}(w_x)$$

$$v \in T(y,l) \subseteq B_{R_0}(v_y) \text{ and } w \in T(y,l-j) \subseteq B_{R_0}(w_y)$$

Also, by lemma 3.2.3, both $\mathfrak{p}_x$ and $\mathfrak{p}_y$ will flow into $\mathfrak{p}$, so there exists a point p in both $\mathfrak{p}_x$ and $\mathfrak{p}_y$. By the (reverse) triangle inequality,

$$|d_w(v_x,p) - d_w(v_y,p)| \le d_w(v_x,v_y) \le d_w(v_x,v) + d_w(v_y,v) < 2R_0$$

and

$$|d_w(w_x,p) - d_w(w_y,p)| \le d_w(w_x,w_y) \le d_w(w_x,w) + d_w(w_y,w) < 2R_0$$

Recall $\mathfrak{p}_x$ and $\mathfrak{p}_y$ are geodesic paths and the points are ordered as x, v_x, w_x, p on $\mathfrak{p}_x$ and y, w_y, v_y, p on $\mathfrak{p}_y$. Note that without a loss of generality, we can assume p occurs "very far" into $\mathfrak{p}_x$ and $\mathfrak{p}_y$, which justifies the ordering of the points. Then

$$\begin{aligned}
2j &= (k+j) - k + l - (l-j) \\
&= d_w(w_x,x) - d_w(v_x,x) + d_w(v_y,y) - d_w(w_y,y) \\
&\le d_w(w_x,v_x) + d_w(w_y,v_y) \\
&= d_w(v_x,p) - d_w(w_x,p) + d_w(w_y,p) - d_w(v_y,p) \\
&\le |d_w(v_x,p) - d_w(v_y,p)| + |d_w(w_y,p) - d_w(w_x,p)| \\
&\le 4R_0
\end{aligned}$$

To complete the proof, we define $R_1 = 2R_0$.

□

Lemma 3.2.8 *Borrowing the notation from lemma 3.2.7, we define*

$$Z(k,l) = W(k,l) \cap \bigcap_{j=1}^{R_1} W(k+j, l-j)^c$$

and for every $n \in \mathbb{N}_0$ we have

$$\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}$$

Note: This lemma is listed as lemma 9 in [Oza08].

Proof. The idea is to show $E(n) = \bigsqcup_{k=0}^{n} Z(k, n-k)$ is a disjoint union. Immediately, $Z(k,l) \subseteq W(k,l)$, so then $E(n) = \bigcup_{k=0}^{n} W(k,n-k) \supseteq \bigcup_{k=0}^{n} Z(k,n-k)$, by part (i) of lemma 3.2.7 (i.e. lemma 8 of [Oza08]).

For the reverse inclusion, suppose $(x,y) \in E(n) = \bigcup_{k=0}^{n} W(k,n-k)$. By part (ii) of lemma 3.2.7, we have $Z(k,l) = W(k,l) \cap \bigcap_{j=1}^{\infty} W(k+j,l-j)^c$ for all $k,l \in \mathbb{Z}$.

Pick $k_0 \in \{0,1,\ldots,n\}$ to be the largest index where $(x,y) \in W(k_0, n-k_0)$. Thus $(x,y) \notin W(k_0+j, n-k_0-j)$ for all $j \ge 1$. Recall $W(k,l) = \emptyset$ whenever $k < 0$ or $l < 0$. Then

$$(x,y) \in W(k_0, n-k_0) \cap \bigcap_{j=1}^{\infty} W(k_0+j, n-k_0-j)^c = Z(k_0, n-k_0)$$

and thus $E(n) \subseteq \bigcup_{k=0}^{n} Z(k, n-k)$.

To complete the proof, we will demonstrate that $E(n) = \cup_{k=0}^{n} Z(k, n-k)$ is a disjoint union. Let $(x, y) \in Z(k, n-k) \cap Z(k', n-k')$. Suppose $k \neq k'$, and without a loss of generality, $k < k'$. Then

$$(x, y) \in W(k, n-k) \cap \bigcap_{j=1}^{\infty} W(k+j, n-k-j)^c \subseteq W(k', n-k')^c$$

However, $(x, y) \in Z(k', n-k') \subseteq W(k', n-k')$ yields an obvious contradiction. Thus, $E(n) = \bigsqcup_{k=0}^{n} Z(k, n-k)$ is indeed a disjoint union, and as a immediate consequence, we have $\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}$.

□

Proposition 3.2.9 *Let Γ be a hyperbolic graph with bounded degree, and we borrow $E(n)$ and $Z(k, l)$ from the previous lemmas. Then, there exist a constant $C_0 > 0$, a Hilbert space H and functions $\eta_k^+, \eta_l^- : \Gamma \to H$ which satisfy the following properties:*

i. $\eta_m^{\pm}(\omega) \perp \eta_{m'}^{\pm}(\omega)$ *for every* $\omega \in \Gamma$ *and* $m, m' \in \mathbb{N}_0$ *with* $|m - m'| \geq 2$.

ii. $\|\eta_m^{\pm}(\omega)\| \leq \sqrt{C_0}$ *for every* $\omega \in \Gamma$ *and* $m \in \mathbb{N}_0$

iii. $\langle \eta_l^-(y), \eta_k^+(x) \rangle = \chi_{Z(k,l)}(x, y)$ *for every* $x, y \in \Gamma$ *and* $k, l \in \mathbb{N}_0$

iv. $\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}$ *for every* $n \in \mathbb{N}_0$.

Note: This proposition is listed as proposition 10 in [Oza08].

Proof. We begin by recalling the notation used in lemma 2.4.3 (i.e. lemma 4 of [Oza08]). As a word of caution, we must distinguish between $\xi_S^{\pm}$ and $\widetilde{\xi}_S^{\pm} \in \ell_2(\mathcal{P}_f\Gamma)$. We know $H = \ell_2(\mathcal{P}_f\Gamma)^{\otimes(1+R_1)}$ is a Hilbert space by the work in chapter 1.

Next, we define η_k^+ and $\eta_k^- : \Gamma \to H$ as follows:

$$\eta_k^+(x) = \xi_{T(x,k)}^+ \otimes \widetilde{\xi}_{T(x,k+1)}^+ \otimes \cdots \otimes \widetilde{\xi}_{T(x,k+R_1)}^+$$

and

$$\eta_l^-(y) = \xi_{T(y,l)}^- \otimes \widetilde{\xi}_{T(y,l-1)}^- \otimes \cdots \otimes \widetilde{\xi}_{T(y,l-R_1)}^-$$

For (i), if $|m - m'| \geq 2$, then $T(\omega, m) \cap T(\omega, m') = \emptyset$. Thus, $\xi_{T(\omega,m)}^{\pm} \perp \xi_{T(\omega,m')}^{\pm}$ follows from part (i) of lemma 2.4.3. Then

$$\langle \eta_m^{\pm}(\omega), \eta_{m'}^{\pm}(\omega) \rangle = \left\langle \xi_{T(\omega,m)}^{\pm}, \xi_{T(\omega,m')}^{\pm} \right\rangle \cdot \prod_{i=1}^{R_1} \left\langle \widetilde{\xi}_{T(\omega,m\pm i)}^{\pm}, \widetilde{\xi}_{T(\omega,m'\pm i)}^{\pm} \right\rangle = 0$$

For (ii), we note that Γ has bounded degree. Then

$$C_1 := \sup_{\omega,m} |T(\omega,m)| \le \sup_v |B_{R_0}(v)| < \infty$$

by lemma 3.2.6 (i.e. lemma 7 of [Oza08]). Then by part (ii) of lemma 2.4.3 we have

$$\begin{aligned}
\|\eta_m^{\pm}(\omega)\|^2 &= \|\xi^{\pm}_{T(\omega,n)}\|^2 \cdot \prod_{i=1}^{R_1} \left\|\widetilde{\xi}^{\pm}_{T(w,m\pm i)}\right\|^2 \\
&\le 2^{|T(\omega,m)|} \cdot \prod_{i=1}^{R_1} 2^{|T(\omega,m\pm i)|} \\
&\le 2^{C_1(1+R_1)}
\end{aligned}$$

Now take $C_0 = 2^{C_1(1+R_1)}$, and part (ii) is complete.

For (iii), we recall $W(k,l) = \{(x,y) \in \Gamma \times \Gamma : T(x,k) \cap T(y,l) \neq \emptyset\}$. Then by part (iii) of lemma 2.4.3 we have

$$\langle \xi_T^-, \xi_S^+ \rangle = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle \widetilde{\xi}_T^-, \widetilde{\xi}_S^+ \rangle = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned}
\left\langle \eta_l^-(y), \eta_k^+(x) \right\rangle &= \left\langle \xi^-_{T(y,l)}, \xi^+_{T(x,k)} \right\rangle \cdot \prod_{i=1}^{R_1} \left\langle \widetilde{\xi}^-_{T(y,l-i)}, \widetilde{\xi}^+_{T(x,k+i)} \right\rangle \\
&= \chi_{W(k,l)}(x,y) \cdot \prod_{i=1}^{R_1} \chi_{W(k+i,l-i)^C}(x,y) \\
&= \chi_{Z(k,l)}(x,y)
\end{aligned}$$

and the last equality is justified by definition of $Z(k,l)$.

For (iv), it's purely a restatement of lemma 3.2.8 (i.e. lemma 9 of [Oza08]).

□

Chapter 4

Hyperbolic groups and weak amenability

Theorem 1 of [Oza08] is the final prerequisite theorem. Theorem 1 is special in that it is the only theorem used in the proof of the main theorem.

4.1 The last prerequisite theorem

As a convention, if $k : \Gamma \times \Gamma \to \mathbb{C}$ is a kernel, we denote $\|k\|_{CB} = \|m_k\|_{CB}$. That is,

$$\|k\|_{CB} \text{ is the } CB\text{-norm of the Schur multiplier } m_k$$

Theorem 4.1.1 *Let Γ be a hyperbolic graph with bounded degree and d_w be the word metric. We take $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ to be the open unit disk centered around the origin in the complex plane. Then there exists a constant C such that the following are true:*

i. $\left\|z^{d_w}\right\|_{CB} \leq \frac{C|1-z|}{1-|z|}$ *where $z \in \mathbb{D}$ and*

$$\begin{aligned} z^{d_w(\cdot,\cdot)} &: \Gamma \times \Gamma \to \mathbb{C} \\ (x, y) &\mapsto z^{d_w(x,y)} \end{aligned}$$

ii. $\left\|\chi_{F(n)}\right\|_{CB} \leq C(n+1)$ *where $F(n) = \{(x, y) \in \Gamma \times \Gamma : d_w(x, y) = n\}$ and*

$$\chi_{F(n)}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in F(n) \\ 0 & \text{otherwise} \end{cases}$$

iii. For $n \in \mathbb{N}$, if $0 < r_n < 1$, then there exists $K_n \in \mathbb{N}$ such that

$$\psi_n(x, y) = r_n^{d(x,y)} \cdot \chi_{E(K_n)}(x, y)$$

where $x, y \in \Gamma$, and $\|\psi_n\|_{CB} \leq 2C$ uniformly.

Note: This theorem is also known as theorem 1 of [Oza08].

Proof. **For (i)**, we take $\eta_m^\pm : \Gamma \to \mathcal{H}$ as defined in proposition 3.2.9 (i.e. proposition 10 of [Oza08]). We note that by part (ii) of proposition 3.2.9, we have $\eta_m^\pm \in \ell_\infty(\Gamma, \mathcal{H})$. For every $z \in \mathbb{D}$ in the unit disk (i.e. $|z| < 1$), we define $\zeta_z^\pm : \Gamma \to \mathcal{H}$ as follows:

$$\zeta_z^+(x) = \overline{\sqrt{1-z}} \sum_{k=0}^{\infty} \overline{z}^k \eta_k^+(x)$$

and

$$\zeta_z^-(y) = \sqrt{1-z} \sum_{l=0}^{\infty} z^l \eta_l^-(y)$$

such that $\sqrt{1-z}$ is the principal branch of the square root.

If $x, y \in \Gamma$, then by parts (iii) and (iv) of proposition 3.2.9, we have

$$\begin{aligned}
\langle \zeta_z^-(y), \zeta_z^+(x) \rangle &= (1-z) \sum_{k,l} z^{k+l} \cdot \langle \eta_l^-(y), \eta_k^+(x) \rangle \\
&= (1-z) \sum_{k,l} z^{k+l} \cdot \chi_{Z(k,l)}(x,y) \\
&= (1-z) \sum_{n=0}^{\infty} z^n \cdot \chi_{E(n)}(x,y) \\
&= (1-z) \sum_{n=d_w(x,y)}^{\infty} z^n \\
&= (1-z) \cdot (z^{d_w(x,y)} + z^{d_w(x,y)+1} + z^{d_w(x,y)+2} + \cdots) \\
&= (1-z) \cdot z^{d_w(x,y)} \cdot (1 + z + z^2 + \cdots) \\
&= z^{d_w(x,y)}
\end{aligned}$$

and the last equality makes use of $\frac{1}{1-z} = 1 + z + z^2 + \cdots$ whenever $|z| < 1$.

For $\omega \in \Gamma$,

$$\|\zeta_z^\pm(\omega)\|^2 = |1-z| \cdot \left\| \sum_{k=0}^{\infty} z^k \cdot \eta_k^\pm(\omega) \right\|^2 \leq |1-z| \cdot \left(\sum_{k=0}^{\infty} |z|^k \cdot \left\| \eta_k^\pm(\omega) \right\| \right)^2$$

and this summation expression converges absolutely because of part (ii) of 3.2.9. Then to simplify the expression, we separate the even powers from the odd powers. We define $A = \sum_{k=0}^{\infty} z^{2k} \cdot \eta_{2k}^\pm(\omega)$ and $B = \sum_{k=0}^{\infty} z^{2k+1} \cdot \eta_{2k+1}^\pm(\omega)$.

Now apply parts (i) and (ii) of proposition 3.2.9, and we have

$$\|A\|^2 = \sum_{k=0}^{\infty} |z|^{4k} \|\eta_{2k}^{\pm}(\omega)\|^2 \leq C_0 \sum_{k=0}^{\infty} |z|^{4k}$$

and

$$\|B\|^2 = \sum_{k=0}^{\infty} |z|^{4k+2} \|\eta_{2k+1}^{\pm}(\omega)\|^2 \leq C_0 \sum_{k=0}^{\infty} |z|^{4k+2}$$

Note that $(\|A\| + \|B\|)^2 \leq 2\|A\|^2 + 2\|B\|^2$. Since $|z| < 1$, then

$$\begin{aligned}
\|\zeta_z^{\pm}(\omega)\|^2 &= |1-z| \left\| \sum_{k=0}^{\infty} z^k \cdot \eta_k^{\pm}(\omega) \right\|^2 \\
&\leq |1-z|(\|A\| + \|B\|)^2 \\
&\leq 2|1-z|(\|A\|^2 + \|B\|^2) \\
&\leq 2|1-z| \cdot C_0 \sum_{k=0}^{\infty} |z|^{2k} \\
&= 2|1-z| \cdot C_0 \frac{1}{1-|z|^2} \\
&\leq C \frac{|1-z|}{1-|z|}
\end{aligned}$$

such that $C = 2C_0$.

Therefore, by theorem 2.4.2 (i.e. theorem 3 of [Oza08]), the Schur multiplier, θ_z, associated to the kernel, z^d, has completely bounded norm at most $C\frac{|1-z|}{1-|z|}$.

For (ii), by combining parts (ii) and (iii) of proposition 3.2.9 with theorem 2.4.2, then we have $\|\chi_{Z(k,l)}\|_{CB} \leq C_0$ for all $k, l \in \mathbb{N}_0$. Then by part (iv) of proposition 3.2.9 we have

$$\|\chi_{E(n)}\|_{CB} \leq \sum_{k=0}^{n} \|\chi_{Z(k,n-k)}\|_{CB} \leq C_0(n+1)$$

for all $n \in \mathbb{N}_0$.

Since $F(n) = E(n) \backslash E(n-1)$, then

$$\|\chi_{F(n)}\|_{CB} = \|\chi_{E(n)} - \chi_{E(n-1)}\|_{CB} \leq 2C_0(n+1) = C(n+1)$$

For (iii). We borrow an idea from [BP93]. Let $0 < r_n < 1$, $K_n \in \mathbb{N}$, and $x, y \in \Gamma$, and define $\psi_n : \Gamma \times \Gamma \to \mathbb{C}$ as

$$\psi_n(x,y) = r_n^{d_w(x,y)} \cdot \chi_{E(K_n)}(x,y) = \begin{cases} r_n^{d_w(x,y)} & \text{if } d_w(x,y) \leq K_n \\ 0 & \text{otherwise} \end{cases}$$

Equivalently,

$$r_n^{d_w(x,y)} - \sum_{i \geq K_n+1} r_n^i \cdot \chi_{F(i)}(x,y) = \begin{cases} r_n^{d_w(x,y)} & \text{if } d_w(x,y) \leq K_n \\ 0 & \text{otherwise} \end{cases}$$

By using parts (i) and (ii) of theorem 4.1.1 (i.e. theorem 1 [Oza08]) we have

$$\begin{aligned} \|\psi_n\|_{CB} &\leq \|r_n^d\|_{CB} + \sum_{i \geq K_n+1} |r_n|^i \cdot \|\chi_{F(i)}\|_{CB} \\ &\leq C\frac{|1-r_n|}{1-|r_n|} + \sum_{i \geq K_n+1} r_n^i \cdot C(i+1) \\ &= C + C\sum_{i \geq K_n+1} (i+1) r_n^i \end{aligned}$$

because $0 < r_n < 1$.

From calculus, if $0 < r < 1$, then the expression, $\sum_{i=0}^{\infty}(i+1)r^i = \left(\frac{1}{1-r}\right)^2$ is convergent. Thus, if $0 < r_n < 1$, there exists $K_n \in \mathbb{N}$ such that

$$\sum_{i \geq K_n+1} (i+1) r_n^i \leq 1$$

Quite obviously, if $r_n \to 1$, then $K_n \to \infty$. Overall, by a careful selection of $0 < r_n < 1$ and $K_n \in \mathbb{N}_0$ we see $\|\psi_n\|_{CB} \leq 2C$.

□

4.2 The main theorem

Definition 4.2.1 *Let Γ be a countable discrete group. If $\phi : \Gamma \to \mathbb{C}$, then define the kernel $k_\phi : \Gamma \times \Gamma \to \mathbb{C}$ by $k_\phi(x,y) = \phi(x^{-1}y)$.*

We note $\|\phi\|_{CB}$ is actually the CB-norm of the Schur multiplier associated to k_ϕ.

*We say Γ is **weakly amenable with constant C** if there exist a sequence of finitely supported functions, $\phi_n : \Gamma \to \mathbb{C}$, such that $\phi_n \to 1$ pointwise, and $\sup\{\|\phi_n\|_{CB} : n \in \mathbb{N}\} \leq C$.*

Remark. This definition of weak amenability is inspired by [CH89], and in that paper, the notion of weak amenability is defined for locally compact discrete groups G, and it involves nets in Fourier $A(G)$ that converge to 1 uniformly on compact sets, and is uniformly bounded above.

Theorem 4.2.2 *Every hyperbolic group is weakly amenable.*

Note: This theorem is listed as theorem 2 in [Oza08], and it is the main theorem of this thesis.

Proof. Let Γ be a hyperbolic group. By corollary 3.1.21, (Γ, d_Γ) is hyperbolic for any choice of finite generating set S. We note (Γ, d_w) is hyperbolic because if Δxyz is δ-slim relative to (Γ, d_Γ), then Δxyz is $(\delta+1)$-slim relative to (Γ, d_w). Note $x, y, z \in V_\Gamma$.

Case 1 is when Γ is finite. For $n \in \mathbb{N}$, define $\phi_n(x) = 1$ for all $x \in \Gamma$. The corresponding kernel is the constant function 1 which comes from example 2.2, so $\|\phi_n\|_{CB} = 1$. Also ϕ_n are finitely supported because Γ is a finite set. Thus finite sets (not just groups) are already weakly amenable with constant 1.

Case 2 is when Γ is infinite. We note that it's countably infinite because hyperbolic groups are finitely generated.

We make use of $\psi_n : \Gamma \times \Gamma \to \mathbb{C}$ from theorem 4.1.1 (i.e. theorem 1 of [Oza08]). Let $e \in \Gamma$ be the identity, and pick $0 < r_n < 1$ such that $r_n \to 1$ as $n \to \infty$. We choose $K_n \in \mathbb{N}_0$ as seen in the proof of part (iii) of theorem 4.1.1. Now define $\phi_n : \Gamma \to \mathbb{C}$ as

$$\phi_n(x) = \psi_n(e, x) = r_n^{d_w(e,x)} \cdot \chi_{E(K_n)}(e, x)$$

We see ϕ_n is finitely supported because Γ is finitely generated. As discussed in the proof of part (iii) of theorem 4.1.1, we know $r_n \to 1$ implies $K_n \to \infty$. Thus $\phi_n \to 1$ pointwise as $n \to \infty$.

Finally, $k_{\phi_n}(x, y) = \phi_n(x^{-1}y) = \psi_n(e, x^{-1}y) = \psi_n(x, y)$. Then by part (iii) of theorem 4.1.1,

$$\sup\{\|\phi_n\|_{CB} : n \in \mathbb{N}_0\} \leq 2C$$

We conclude hyperbolic groups are weakly amenable with constant $2C$.

□

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