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APPLICATION OF BOUNDARY ELEMENT METHOD
TO
CERTAIN PROBLEMS IN STRUCTURAL DYNAMICS

by

Y.G. HANNA

A thesis

submitted to the Faculty of Graduate Studies in
partial fulfillment of the requirements for the degree of

MASTER OF ENGINEERING

Department of Civil Engineering

Carleton University

Ottawa, Canada

April 24, 1980

TO
MY WIFE
AND
CHILD

The undersigned recommend to the Faculty of Graduate Studies
acceptance of the thesis

"APPLICATION OF BOUNDARY ELEMENT METHOD
TO
CERTAIN PROBLEMS IN STRUCTURAL DYNAMICS"

submitted by Y.G. Hanna, in partial fulfillment of the requirements
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MASTER OF ENGINEERING

Thesis Supervisor

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April 24, 1980

ABSTRACT

The mathematical basis for the formulation of the boundary element method is reviewed. The method is then applied to certain problems in structural dynamics. The solution of the problem of forced vibrations of stretched membranes by boundary element method is presented first as an example of the application of the method to a simple two-dimensional problem.

The method is then applied to the problem of hydrodynamic pressure distribution on a rigid dam at one end of a finite water reservoir when the system is subjected to a horizontal ground motion. The effect of certain parameters on the pressure distribution is studied.

To examine the efficiency of the method in dealing with infinite domains, the method is applied to the problem of hydrodynamic pressure generated on the face of a rigid dam at the downstream end of an infinite water reservoir by a ground motion.

A combined finite element boundary element method is then used to study the effect of the flexibility of dam on its response and on the hydrodynamic pressure.

It is observed that in the application of the boundary element method, solutions for the exterior domain may be intermixed with those for the domain under consideration.

The results obtained by using the boundary element method are compared with known classical solutions.

The advantages and disadvantages of the method as applied to the above-mentioned problems are discussed.

It is concluded that the boundary element method can be a powerful tool in certain problems of engineering analysis.

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CHAPTER 1

INTRODUCTION

1.1 GENERAL

The analysis of many engineering problems can be reduced to the solution of one or a set of differential equations. It is possible to solve these equations exactly in only a limited number of cases and that too under very restricted conditions. Therefore, in the solution of many engineering problems, approximate numerical solutions play an important role. Several numerical methods have been developed to obtain such approximate solutions.

The finite element method and the finite differences method have been applied successfully and extensively to many engineering problems. In the finite element method, a continuum with infinite degrees of freedom is approximated by an assemblage of elements so that the problem reduces to one having a finite number of unknowns. Each component element interconnects with others at common nodal points. A detailed discussion of the finite element method will not be presented here because the method has received extensive treatment in the literature [3,35].

The basic concept of the finite differences method of discretization is to replace the continuous variables of the governing differential equation by discrete variables. The relationship between these

discrete variables is expressed in the form of finite differences equations. More details about the finite differences method have been extensively used in the solution of engineering problems and are well described in the literature [5].

The finite element and finite differences methods are called 'domain' methods since they deal with the domain under consideration. In certain class of problems, the domain methods are either inefficient or fail to give an accurate solution. For example, problems involving infinite domains cannot be efficiently dealt with by using the domain methods.

The finite element method requires special care in data preparation and errors are frequent. Domains with complicated geometry pose special problems when the finite differences method is used. These problems have resulted in a renewal of interest in other numerical methods, including the boundary solution techniques such as "Boundary Element Method".

1.2 BOUNDARY ELEMENT METHOD

The term 'boundary element method' is used to indicate the method whereby the external surface of a domain is divided into a series of elements, called boundary elements, over which the functions under consideration can vary in different ways in much the same manner as the domain is divided into elements in the finite element method. The points on the boundary where the unknown values of the variable or variables are sought are called 'nodes'. For the constant element method,

a node is taken to be in the middle of each segment. One can also choose linear elements (i.e. elements for which the nodes are at the intersection of elements) and curved elements such as the quadratic elements (Fig. 1.1). The boundary elements need not be of equal lengths. They can be chosen in any convenient and suitable way.

In certain class of problems, the boundary element method offers important advantages over the 'domain' type methods, such as the finite element and the finite difference methods. One of the important features of the boundary element method is the much smaller resulting system of equations and the considerable reduction in the data input required in the solution of a problem.

Also, the boundary element method is well suited to problems with infinite domain such as those occurring in soil mechanics, hydraulics, and hydrodynamics, for which the classical 'domain' methods are unsuitable. Another important feature of the boundary element method is that the values of the functions under consideration are calculated for internal points of the domain only where they are needed.

The boundary element method may be coupled with other methods such as the finite element method to deal with situations where neither the boundary element method nor the other methods are appropriate by themselves. This implies that the weaknesses of one approach may be offset by the strengths of the other to form a combination more powerful than either approach taken individually. This is indeed the case for a class of problems which involves large or infinite domains while simultaneously containing a subregion with non-homogeneous material properties.

A - CONSTANT ELEMENTS

B - LINEAR ELEMENTS

C - QUADRATIC ELEMENTS

FIG. 1.1. DIFFERENT TYPES OF BOUNDARY ELEMENTS.

The boundary element method has some drawbacks which are worthy of further research effort. Following are some of them:

- a) The boundary element method sometimes leads to unwanted solutions for a domain exterior or interior to the one being considered. The solutions are intermixed and it is sometimes difficult to distinguish one from the other.
- b) Boundary element matrices are usually fully populated which means that the method may be less efficient computationally than other methods in certain applications.
- c) The boundary element method is not well suited to problems involving regions with inhomogeneous material properties.

1.3 REVIEW OF THE LITERATURE

A brief review of the literature related to the problems discussed in the present study is presented in this section. Problems involving time dependent variables have been solved by the boundary element method either by numerically integrating over time, Chang, Kang and Chen [6]⁺ or by removing the time variation by the use of Laplace transform, Cruse and Rizzo [7], Rizzo and Shippy [8,9] and Shippy [10].

James R. Hutchinson [11] has applied the boundary element method to find the eigenvalues and eigenvectors of stretched membranes. In his work, the time dependence is removed by the assumption of a sinu-

+ Numbers in square brackets refer to references given at the end of this study.

soidal motion. The process of finding the eigenvalues and eigenvectors of the considered membranes was confused because of the false frequencies which correspond to solutions for a membrane external to the boundary of the membrane under consideration.

To the knowledge of the author, the boundary element method has not been applied to the problem of hydrodynamic pressure distribution on gravity dams subject to ground motion. The hydrodynamic pressure on dams subjected to an earthquake type motion is generally obtained by using Westergaard's results [13]. Westergaard's work on hydrodynamic pressure dates back to 1933 and was the first known application of hydrodynamics to earthquakes. Although it was an excellent piece of work of its time, it has several limitations. These include the assumptions that the dam is rigid, straight and its upstream face is vertical; the reservoir bottom is rigid, horizontal and extends to infinity; the foundation is rigid; the ground motion is horizontal, simple harmonic and in the direction perpendicular to the dam. In order to use Westergaard's results in design, water is assumed to be incompressible, for which case the hydrodynamic pressure does not depend upon the history of excitation.

Several authors have extended the work of Westergaard to lift some of the assumptions made by him.

Zangar [14] obtained hydrodynamic pressures on inclined faces by using an electric analogue and constructing flow nets. In this experimental study, water had to be treated as incompressible. Bustamante et al [15] studied the effects of the length of reservoir. Chopra [16] found that typical strong motion earthquakes cause hydrodynamic

pressures of appreciable magnitude, that vertical components of earthquakes have a very significant effect on the response, and that neglecting the compressibility of water may lead to significant errors. Hydrodynamic pressure due to vertical excitation for both vertical and inclined faces of dams and for finite lengths of reservoir were obtained by Nath [17]. Effects of flexibility of the dam on the hydrodynamic pressure have also been reported by Brahtz and Heilbron [18]. They assumed linear deformation of the dam face and used an iterative procedure to obtain the response of the coupled system. Bustamante et al [15] assumed a parabolic deformation of the dam face and concluded that the effect of the flexibility of the dam on the hydrodynamic pressure was very significant. The effects of dam-reservoir interaction were further investigated by Chopra [19,20,21] assuming that the dam vibrated only in its fundamental mode. Later, Chakrabarti and Chopra [22,23,24,25] developed a more general method in which any number of modes could be considered. In their analysis, they assumed that the deformation of the interface of the dam may be represented with sufficient accuracy by a linear combination of the first few modes of the dam without the reservoir. The major conclusion of their analysis was that consideration of vertical component of excitation and compressibility of water could lead to an increase in the stress in the dam by more than 100%. The dam reservoir problem was also studied by Finn and Varoglu [2,26] who first solved the case of a rectangular dam analytically and later extended their work to a finite element model of the dam. All investigations of the dam-reservoir systems have treated the reservoir as a continuum for which the governing equation of motion

was solved directly by the classical solution methods or indirectly through the finite element method. The classical solution methods cannot handle the case of sloping dam face or an inclined reservoir bed. In the finite element method, it is difficult to model the infinite boundary, and the proposed radiation damping terms often fail to simulate accurately the energy loss in the waves moving away from the reservoir.

1.4 SCOPE OF THE PRESENT STUDY

The main objectives of the present study are as follows:

- i) To review the mathematical basis and formulation of the boundary element method.
- ii) To study the feasibility of applying the boundary element method to structural dynamic problems. As a first step, the method is applied to the problem of forced vibrations of membranes because it is the simplest two-dimensional vibrations problem that can be investigated.
- iii) To study the feasibility of applying the boundary element method to the problem of hydrodynamic pressure distribution on a rigid dam storing a finite water reservoir when the system is subjected to a ground motion.
- iv) To investigate the possibility of representing infinite domains by applying the method to the problem of hydrodynamic pressure distribution on a rigid dam storing an infinite water reservoir.
- v) To investigate the possibility of combining the boundary element method with other numerical methods such as the finite element method, in investigating the problem of hydrodynamic pressure

distribution on a flexible gravity dam storing an infinite water reservoir. The dam is represented by a finite element model and the reservoir by boundary elements.

- vi) To present the advantages and disadvantages of the boundary element method in its application to the field of structural dynamics.

1.5 FORMAT OF THESIS

The thesis has been divided into seven chapters. Chapter 1 introduces the boundary element method as a numerical technique and presents its advantages over the domain type methods such as the finite element and the finite difference methods. This chapter also surveys the literature associated with the problems studied in this thesis and presents the scope of this study and the format of the thesis.

Chapter 2 presents the mathematical basis of the boundary element method and identifies the method as an application of the Green's theorem. Since the governing differential equations of all the problems discussed in the present work can be reduced to the Helmholtz equation, the application of the boundary element method to the solution of this equation is presented in Chapter 2. Solution techniques are described for both the constant and the linear boundary elements.

In Chapter 3, the boundary element method is applied to the problem of forced vibration of stretched membranes. The results obtained for both the constant and the linear elements are presented and compared with the known classical solutions. It is shown that the agreement of results is satisfactory.

In Chapter 4, the boundary element method is applied to the problem of hydrodynamic pressure distribution on a rigid dam storing a finite water reservoir and subjected to a horizontal ground motion. The classical solutions for the problem are discussed for both the finite and infinite water reservoirs. Results are obtained for both vertical and inclined upstream face of the dam. The effects of sloping reservoir bed and the compressibility of water are investigated. The results are compared with known classical solutions and it is shown that the agreement is excellent.

In Chapter 5, the application of the boundary element method to the solution of the vibration problem of an infinite reservoir is presented. The choice of an appropriate fundamental solution is discussed. Different approaches that may be taken in representing the boundary at infinity are presented. The results are compared with known classical solutions and the agreement is shown to be satisfactory.

In Chapter 6, a combination of the boundary element method and the finite element method is applied to the vibration problem of a flexible dam storing an infinite water reservoir. The effects of interaction between the water reservoir and the flexible dam are investigated. The results obtained are compared with known solutions and the agreement is shown to be excellent.

A summary of the work done in the present study and the conclusions arrived at are contained in Chapter 7.

Several computer programs have been developed to carry out the computation work required for this study.

CHAPTER 2

• FORMULATION OF THE BOUNDARY ELEMENT METHOD2.1 INTRODUCTION

The mathematical formulation of the boundary element method is developed in this chapter. In many problems of structural dynamics, the time variable can be eliminated from the governing differential equation provided the exciting function is assumed to be harmonic so that the resulting response is also harmonic. In such cases, the governing equation very often reduces to the well known Helmholtz equation. The development presented here takes the Helmholtz equation as the starting point. This equation is first multiplied by an arbitrary function of the plane coordinates and then converted to an integral equation by integrating over the domain. It is shown that by using the Green's theorem and by an appropriate choice of the arbitrary function, the integration over the domain can be converted to an integral over the boundary. The appropriately chosen multiplying function must itself satisfy a certain differential equation. The solution of this equation is called the fundamental solution and plays an important role in the development of the method.

This chapter then describes how the resulting integral equations are discretized so that the problem reduces to the solution of a set of simultaneous equations.

2.2 GREEN'S THEOREM

The boundary element method is based on the application of Green's Theorem which converts an integral over an entire domain to one over the boundary of that domain. It follows from Green's Theorem that if both u and u^* are functions of x and y then:

$$\iint_A u^* \frac{\partial u}{\partial y} dx dy = - \iint_A u \frac{\partial u^*}{\partial y} dx dy - \int_c uu^* dx \quad (2.1)$$

and

$$\iint_A u^* \frac{\partial u}{\partial x} dx dy = - \iint_A u \frac{\partial u^*}{\partial x} dx dy + \int_c uu^* dy \quad (2.2)$$

where

A represents the domain under consideration and

c represents the boundary of the domain A .

Also, it is known that:

$$\int_c u \frac{\partial u^*}{\partial x} dy - \int_c u \frac{\partial u^*}{\partial y} dx = \int_c u \frac{\partial u^*}{\partial n} dc \quad (2.3)$$

$$\int_c u^* \frac{\partial u}{\partial x} dy - \int_c \frac{\partial u}{\partial y} u^* dx = \int_c u^* \frac{\partial u}{\partial n} dc \quad (2.4)$$

where $\frac{\partial u}{\partial n}$ represents differentiation with respect to the outward normal to the boundary.

Eqns. (2.1), (2.2), (2.3) and (2.4) form the starting point in the formulation of the boundary element method.

2.3 HELMHOLTZ EQUATION

The reduced Helmholtz equation is a partial differential equation which governs many practical problems in structural dynamics. Its general form is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.0 \quad (2.5)$$

where

u is a function of x and y , and

k is a constant.

Since the governing differential equations for all problems discussed in the present study can be reduced to a Helmholtz equation, the formulation of the boundary element method is explained by treating Eqn. (2.5) as the governing differential equation.

Multiplying Eqn. (2.5) by u^* , a function of x and y , the properties of which will be discussed later, and integrating over the domain under consideration, A , one gets:

$$\iint_A \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u \right) u^* dx dy = 0.0 \quad (2.6)$$

By the application of Green's Theorem, Eqn. (2.6) can be reduced as follows:

$$\begin{aligned} \iint_A \frac{\partial^2 u}{\partial x^2} u^* dx dy &= - \iint_A \frac{\partial u}{\partial x} \frac{\partial u^*}{\partial x} dx dy + \int_C \frac{\partial u}{\partial x} u^* dy \\ &= + \iint_A u \frac{\partial^2 u^*}{\partial x^2} dx dy - \int_C u \frac{\partial u^*}{\partial x} dy + \int_C \frac{\partial u}{\partial x} u^* dy \quad (2.7) \end{aligned}$$

and

$$\iint_A \frac{\partial^2 u}{\partial y^2} u^* dx dy = + \iint_A u \frac{\partial^2 u^*}{\partial y^2} dx dy + \int_c u \frac{\partial u^*}{\partial y} dx - \int_c \frac{\partial u}{\partial y} u^* dx \quad (2.8)$$

Substitution of Eqns. (2.7) and (2.8) into Eqn. (2.6) gives:

$$\begin{aligned} \iint_A \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u \right) u^* dx dy &= + \iint_A \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + k^2 u^* \right) u dx dy \\ &- \left[\int_c u \frac{\partial u^*}{\partial x} dy - \int_c u \frac{\partial u^*}{\partial y} dx \right] \\ &+ \left[\int_c u^* \frac{\partial u}{\partial x} dy - \int_c \frac{\partial u}{\partial y} u^* dx \right] \end{aligned} \quad (2.9)$$

Using Eqns. (2.3) and (2.4), Eqn. (2.9) becomes:

$$\begin{aligned} \iint_A \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u \right) u^* dx dy &= \iint_A \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + k^2 u^* \right) u dx dy \\ &- \int u \frac{\partial u^*}{\partial n} dc + \int u^* \frac{\partial u}{\partial n} dc \\ &= 0.0 \end{aligned}$$

or

$$\iint_A \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + k^2 u^* \right) u dx dy = \int_c u \frac{\partial u^*}{\partial n} dc - \int_c u^* \frac{\partial u}{\partial n} dc \quad (2.10)$$

It can be noticed that the right-hand side of Eqn. (2.10) contains integration over only the boundary c of the domain. By an appropriate

choice of u^* , the integration over the domain A appearing on the left-hand side of Eqn. (2.10) can be avoided. This is discussed in some detail in the following section.

2.4 FUNDAMENTAL SOLUTION

In order to reduce the problem being considered to a boundary problem, the function u^* is chosen so as to satisfy the following equation:

$$\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + k^2 u^* = \Delta^i \quad (2.11)$$

where

Δ^i is a Dirac Delta function centred at point 'i' having coordinates (x_i, y_i) .

The solution of Eqn. (2.11) is called the Fundamental Solution.

Substitution of Eqn. (2.11) into Eqn. (2.10) gives:

$$\iint_A \Delta^i u \, dx dy = \int_C u \frac{\partial u^*}{\partial n} \, dc - \int_C u^* \frac{\partial u}{\partial n} \, dc \quad (2.12)$$

Using the properties of the Dirac Delta function [12], one can write:

$$\iint_A \Delta^i u \, dx dy = u_i \quad \text{for a point inside the domain A,}$$

$$\iint_A \Delta^i u \, dx dy = \frac{1}{2} u_i \quad \text{for a point on the smooth boundary of the domain A}$$

$$\text{and} \quad \iint_A \Delta^i u \, dx dy = 0.0 \quad \text{for a point outside the domain A.}$$

Hence, Eqn. (2.12), in its general form, becomes:

$$c_i u_i = \int_C u \frac{\partial u^*}{\partial n} dc - \int_C u^* \frac{\partial u}{\partial n} dc \quad (2.13)$$

where

$$c_i = 1 \quad \text{for a point inside the domain A,}$$

$$c_i = \frac{1}{2} \quad \text{for a point on the smooth boundary of the domain A}$$

$$c_i = 0 \quad \text{for a point outside the domain A.}$$

Putting

$$\frac{\partial u^*}{\partial n} = q^*$$

and

$$\frac{\partial u}{\partial n} = q$$

Eqn. (2.13) becomes:

$$c_i u_i = \int_C u q^* dc - \int_C u^* q dc \quad (2.14)$$

Eqn. (2.14) relates the value of the function under consideration, u , at any point i to its values on the boundary of the domain. It can be seen that the integrations contained in Eqn. (2.14) are to be carried out only on the boundary of the domain.

2.5 BOUNDARY ELEMENT DISCRETIZATION

In order to evaluate the integrals contained in Eqn. (2.14), it is assumed that the boundary of the domain is divided into N boundary elements as shown in Fig. 1.1. The points where the unknown values of

the variable are sought are called 'nodes'. The position of the nodes differs according to the type of boundary elements used. For constant elements, the nodes are taken to be at the middle of each element; for linear elements, they are taken to be at the intersection of the adjacent elements. The following two types of elements will be considered in the present work:

1. Constant Boundary Elements
2. Linear Boundary Elements

2.5.1 Constant Boundary Elements

For the constant boundary elements, the values of u and q are assumed to be constant over each element and equal to their values at the mid-node of the element. Eqn. (2.14) for a given point, i , on the boundary becomes in its discretized form:

$$-\frac{1}{2} u_i + \sum_{j=1}^N u_j \int_{c_j} q^* dc = \sum_{j=1}^N q_j \int_{c_j} u^* dc \quad (2.15)$$

where u_j and q_j are the values of these variables over the j^{th} element and c_j represents the boundary of that element. Eqn. (2.15) applies for a particular node ' i '. The terms $\int_{c_j} q^* dc$ relate the ' i ' node with

the segment ' j ' over which the integral is carried out. These integral values will be denoted by h_{ij} . The integral values $\int_{c_j} u^* dc$ will be denoted by G_{ij} . Eqn. (2.15) then becomes:

$$-\frac{1}{2} u_i + \sum_{j=1}^N u_j h_{ij} = \sum_{j=1}^N q_j G_{ij} \quad (2.16)$$

Putting

$$H_{ij} = h_{ij} \text{ when } i \neq j$$

$$H_{ij} = h_{ij} - \frac{1}{2} \text{ when } i=j$$

Eqn. (2.16) becomes:

$$\sum_{j=1}^N H_{ij} u_j = \sum_{j=1}^N G_{ij} q_j \quad (2.17)$$

The whole set of equations for the N nodes can now be expressed in matrix form as:

$$[H]\{u\} = [G]\{q\} \quad (2.18)$$

where

$[H]$ is a matrix containing the elements H_{ij}

$[G]$ is a matrix containing the elements G_{ij}

$\{u\}$ is a vector containing the known and unknown values of the considered function u , and

$\{q\}$ is a vector containing the known and unknown values of the derivative of the considered function u .

Now, if it is assumed that n_1 values of u and n_2 values of q are known so that $n_1 + n_2 = N$, there are N unknowns in Eqn. (2.18). Re-ordering the equations in such a way that all the unknowns are transferred to the left-hand side, one can write:

$$[B]\{X\} = \{F\} \quad (2.19)$$

where $\{X\}$ is the vector of unknowns u_s and q_s . Once the values of u_s

and q_s on the entire boundary are known one can calculate the value of u at any interior point using Eqn. (2.14), which after discretization, becomes:

$$u_i = \sum_{j=1}^N u_j \int_{c_j} q^* dc - \sum_{j=1}^N q_j \int_{c_j} u^* dc \quad (2.20)$$

2.5.2 Linear Boundary Elements

For linear boundary elements, the variation of u and q is assumed to be linear within each element. The nodes are considered to be at the intersections of the adjacent elements as shown in Fig. 2.1.

Eqn. (2.14) can be written for the N elements as:

$$-c_i u_i + \sum_{j=1}^N \int_{c_j} u q^* dc = \sum_{j=1}^N \int_{c_j} u^* q dc \quad (2.21)$$

In Eqn. (2.21), u and q cannot be taken out of the integral as they vary over the element.

Consider an arbitrary segment such as the one shown in Fig. 2.1.

The values of u and q at any point of the element can be defined in terms of their nodal values as follows:

$$u = \frac{u_j + u_{j+1}}{2} + \frac{u_{j+1} - u_j}{2} \xi \quad (2.22)$$

and

$$q = \frac{q_j + q_{j+1}}{2} + \frac{q_{j+1} - q_j}{2} \xi \quad (2.23)$$

FIG. 2.1. LINEAR BOUNDARY ELEMENT.

or

$$u = \phi_1 u_j + \phi_2 u_{j+1} \quad (2.24)$$

and

$$q = \phi_1 q_j + \phi_2 q_{j+1} \quad (2.25)$$

where ξ is the dimensionless coordinate ($\xi = 2x/\ell$) and ϕ_1, ϕ_2 are given by:

$$\phi_1 = \frac{1}{2} (1 - \xi) \quad (2.26)$$

$$\phi_2 = \frac{1}{2} (1 + \xi) \quad (2.27)$$

Substitution of Eqns. (2.24), (2.25), (2.26) and (2.27) into Eqn. (2.21) gives:

$$-c_i u_i + \sum_{j=1}^N \int_{c_j} [\phi_1 u_j + \phi_2 u_{j+1}] q^* dc = \sum_{j=1}^N \int_{c_j} [\phi_1 q_j + \phi_2 q_{j+1}] u^* dc \quad (2.28)$$

or

$$\begin{aligned} & -c_i u_i + \sum_{j=1}^N [u_j \int_{c_j} \phi_1 q^* dc + u_{j+1} \int_{c_j} \phi_2 q^* dc] \\ & = \sum_{j=1}^N [q_j \int_{c_j} \phi_1 u^* dc + q_{j+1} \int_{c_j} \phi_2 u^* dc] \end{aligned} \quad (2.29)$$

Let the integral

$$\int_{c_j} \phi_1 q^* dc \quad \text{be called } h_{i1},$$

$$\int_{c_j} \phi_2 q^* dc \quad \text{be called } h_{i2},$$

$$\int_{c_j} \phi_1 u^* dc \quad \text{be called } g_{i1},$$

$$\int_{c_j} \phi_2 u^* dc \quad \text{be called } g_{i2}.$$

Then, Eqn. (2.29) becomes:

$$-c_i u_i + \sum_{j=1}^N [u_j \cdot h_{i1} + u_{j+1} \cdot h_{i2}] = \sum_{j=1}^N [q_j \cdot g_{i1} + q_{j+1} \cdot g_{i2}] \quad (2.30)$$

Eqn. (2.30) can be written as:

$$-c_i u_i + [H'_{i1} \ H'_{i2} \ \dots \ H'_{iN}] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = [G_{i1} \ G_{i2} \ \dots \ G_{iN}] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} \quad (2.31)$$

where H'_{ij} is equal to the h_{i1} term of element 'j' plus h_{i2} term of element 'j-1' and similarly for G_{ij} .

Eqn. (2.31) can be written as:

$$-c_i u_i + \sum_{j=1}^N H'_{ij} u_j = \sum_{j=1}^N G_{ij} q_j \quad (2.32)$$

Putting

$$H_{ij} = H'_{ij} \quad \text{for } i \neq j$$

and

$$H_{ij} = H'_{ij} - c_i \quad \text{for } i=j$$

Eqn. (2.32) becomes:

$$\sum_{j=1}^N H_{ij} u_j = \sum_{j=1}^N G_{ij} q_j \quad (2.33)$$

The whole system of equations for the N nodes becomes:

$$[H]\{U\} = [G]\{q\} \quad (2.34)$$

which is the same as for constant elements. It should be noticed that unless the boundary is smooth at point 'i', the value of c_i will be different from $\frac{1}{2}$. However, one can always calculate the diagonal terms of the matrix $[H]$ by noting the fact that when u is constant over the whole boundary, the normal derivatives (q values) must be zero. In such a case, Eqn. (2.34) becomes:

$$[H][U] = 0 \quad (2.35)$$

When all u_j 's are equal, Eqn. (2.35) will have non-trivial solution only when the sum of all the elements of $[H]$ in any one row is zero. The value of the coefficient on the diagonal can therefore be easily determined once the off-diagonal coefficients have been calculated. Thus,

$$H_{ii} = - \sum_{j=1}^N H_{ij} \quad i \neq j \quad (2.36)$$

Once the matrices in Eqn. (2.34) have been determined, the solution follows steps similar to those for constant boundary elements. Unfortunately, with linear elements, problems appear at corner points such as

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those shown in Fig. 2.2a. At such points, one can define two different sets of values for u and q depending on which of the two adjacent segments is considered. A simple way to avoid the corner problem is to assume that there are two points very near to each other but which belong to different sides (see Fig. 2.2b). This approach has been used in the present work and gives reasonable solutions to the problem of corner points of linear boundary elements.

3

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a. One Corner Point

b. Two Corner Points

FIG. 2.2. CORNER POINTS IN LINEAR BOUNDARY ELEMENTS.

CHAPTER 3

FORCED VIBRATIONS OF STRETCHED MEMBRANES

3.1 INTRODUCTION

This chapter describes the application of the boundary element method to the problem of forced vibrations of stretched membranes. The stretched membrane represents a simple two-dimensional vibration problem that can be investigated by the boundary element method. The problem serves as a test case in exploring the effectiveness of the method in the solution of problems in structural dynamics.

The boundary element method has been applied in the past to the solution of time-dependent problems either by numerically integrating over time, Chang, Kang and Chen [6] or by removing the time variation by use of the Laplace transform, Cruse and Rizzo (1968), Cruse [7], Rizzo and Shippy [8,9] and Shippy [10]. In the present work, the classical vibration approach is used whereby the time dependence is removed by the assumption of sinusoidal motion.

3.2 CLASSICAL SOLUTION OF THE MEMBRANE PROBLEM

Consider a perfectly flexible homogeneous membrane bounded by a plane curve c in the x - y plane (see Fig. 3.1), and under a uniform tension S per unit length along the edge. The membrane is assumed to undergo small transverse vibrations and the following assumptions are made in obtaining a solution to the problem:

FIG. 3.1. FORCES ACTING ON A STRETCHED MEMBRANE.

- i) The equilibrium position of the membrane is in the x - y plane, and a point P of the membrane is identified by its coordinates x and y at the equilibrium position. Each point $P(x,y)$ moves only in the z -direction and its displacement is denoted by $w(x,y,t)$.
- ii) The values of w are very small compared with the dimensions of the membrane.
- iii) The tension S is sufficiently great so that the plane, tangent to the membrane at P , makes a very small angle with the x and y axes.
- iv) The surface density ρ is constant and damping is negligible.
- v) The displacement w on the boundary is taken as zero.

To obtain the equation of motion of the membrane, consider a small element of area $dxdy$. Forces Sdy and Sdx act on the sides of this area. The sum of the projections of the two Sdy forces on the Z -axis is:

$$Sdy \left[\frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) dx \right] - Sdy \frac{\partial w}{\partial x} = S \frac{\partial^2 w}{\partial x^2} dxdy \quad (3.1)$$

Similarly, the sum of projections of the two Sdx forces on the Z -axis is $S \frac{\partial^2 w}{\partial y^2} dxdy$. The acceleration of the element $dxdy$ is $\frac{\partial^2 w}{\partial t^2}$. Then the equation of free vibration of the membrane is:

$$\rho dxdy \frac{\partial^2 w}{\partial t^2} = S \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) dxdy$$

or

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (3.2)$$

where

$$c = \sqrt{\frac{S}{\rho}}$$

If the membrane is subjected to an external force $p(x,y,t)$ per unit area acting in the z -direction, the equation of motion becomes:

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{p(x,y,t)}{\rho} \quad (3.3)$$

For a rectangular membrane of dimensions a and b as shown in Fig. 3.2, if the forcing function $p(x,y,t)$ is of the form

$$p(x,y,t) = p_0(x,y)e^{i\omega t} \quad (3.4)$$

it can be assumed that the steady state response is of

the form:

$$w(x,y,t) = e^{i\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b} \quad (3.5)$$

where A_{mn} is a constant.

Substitution of Eqn. (3.5) into Eqn. (3.3) gives:

$$\begin{aligned} & -\omega^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b} \\ & + c^2 \pi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b} \cdot \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} \right] = \frac{p_0}{\rho} \quad (3.6) \end{aligned}$$

Multiplying both sides of Eqn. (3.6) by $\sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b}$ and integrating with respect to x from zero to a and with respect to y from zero to b , Eqn. (3.6) becomes:

THE POINT UNDER CONSIDERATION

FIG. 3.3. DEFINITION OF THE ANGLE θ .

FIG. 3.2. RECTANGULAR MEMBRANE.

$$-\frac{ab}{4} A_{mn}^2 \omega^2 + c^2 \pi^2 A_{mn} \frac{ab}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = \int_0^a \int_0^b \frac{p_0}{\rho} \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b} dx dy$$

which gives for a constant value of p_0

$$A_{mn} \left[-\omega^2 \frac{ab}{4} + c^2 \pi^2 \frac{ab}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right] = \frac{4p_0 ab}{\rho \pi^2 mn} \text{ for } m \text{ and } n \text{ both odd}$$

$$= 0 \text{ for either } m \text{ or } n \text{ even}$$

and hence

$$A_{mn} = \frac{16p_0}{\rho \pi^2 mn [\pi^2 c^2 (\frac{m^2}{a^2} + \frac{n^2}{b^2}) - \omega^2]} \quad (3.7)$$

Substitution of Eqn. (3.7) into (3.5) gives:

$$w(x,y,t) = \frac{16p_0}{\rho \pi^2 mn [\pi^2 c^2 (\frac{m^2}{a^2} + \frac{n^2}{b^2}) - \omega^2]} e^{i\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{b} \quad (3.8)$$

which is the classical solution for the forced vibration of rectangular membranes (m and n are odd).

3.3 BOUNDARY ELEMENT FORMULATION

The governing differential equation of the forced vibration of a stretched membrane is (Eqn. (3.3)):

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{p(x,y,t)}{\rho}$$

Assuming $p(x,y,t)$ to be of the form:

$$p(x,y,t) = p_0(x,y) e^{i\omega t} \quad (3.9)$$

it is reasonable to seek a solution $w(x,y,t)$ of the form:

$$w(x,y,t) = w_0(x,y)e^{i\omega t} \quad (3.10)$$

Substitution of Eqns. (3.9) and (3.10) into the governing differential equation gives:

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} + w_0 \frac{\omega^2}{c^2} = \frac{-p_0}{\rho c^2} \quad (3.11)$$

Putting

$$\frac{\omega^2}{c^2} = k^2$$

and

$$\frac{-p_0(x,y)}{\rho c^2} = \phi(x,y)$$

Eqn. (3.11) becomes:

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} + k^2 w_0 = \phi(x,y) \quad (3.12)$$

Multiplying both sides of Eqn. (3.12) by $w_0^*(x,y)$ and applying Green's theorem as described in Chapter 2, Eqn. (3.12) becomes:

$$\iint_A \left(\frac{\partial^2 w_0^*}{\partial x^2} + \frac{\partial^2 w_0^*}{\partial y^2} + k^2 w_0^* \right) w_0 \, dA = \iint_A \phi w_0^* \, dA + \int_C w_0^* q \, dc - \int_C w_0^* q^* \, dc \quad (3.13)$$

where A is the region covering the membrane,

$$q = \frac{\partial w_0}{\partial n} \quad \text{and}$$

$$q^* = \frac{\partial w_0^*}{\partial n}$$

Following the steps outlined in Chapter 2, Eqn. (3.13) can be converted into the boundary problem:

$$c_i(w_o)_i = \int_C w_o q^* dc - \int_C w_o^* q dc + \iint_A \phi w_o^* dA \quad (3.14)$$

where w_o^* is the fundamental solution of the equation

$$\frac{\partial^2 w_o^*}{\partial x^2} + \frac{\partial^2 w_o^*}{\partial y^2} + k^2 w_o^* = \Delta^i \quad (3.15)$$

The fundamental solution of Eqn. (3.15), w_o^* , may be regarded as the potential induced at any point (x,y) in the domain 'A' by a point mass of unit strength at point 'i'. It should therefore be symmetric about the point 'i' and will depend only on the radial variable, r , where

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

and x_i and y_i are the coordinates of point 'i'.

In polar coordinates, the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ can be written as:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Noting that

$$\frac{\partial^2}{\partial \theta^2} w_o^* = 0$$

(since w_o^* depends only on r , Eqn. (3.15) becomes in polar coordinates:

$$\frac{d^2 w_0^*}{dr^2} + \frac{1}{r} \frac{dw_0^*}{dr} + k^2 w_0^* = \Delta^i$$

For r greater than zero, this will give

$$r \frac{d^2 w_0^*}{dr^2} + \frac{dw_0^*}{dr} + k^2 r w_0^* = 0, \quad r > 0 \quad (3.16)$$

The substitution $z=kr$ transforms Eqn. (3.16) into:

$$zk \frac{d^2 w_0^*}{dz^2} + k \frac{dw_0^*}{dz} + kz w_0^* = 0 \quad \text{for } r > 0$$

or

$$\frac{d^2 w_0^*}{dz^2} + \frac{1}{z} \frac{dw_0^*}{dz} + w_0^* = 0 \quad \text{for } r > 0 \quad (3.17)$$

Eqn. (3.17) is a Bessel equation of order zero, with linearly independent solutions $J_0(z)$ and $Y_0(z)$ or $J_0(kr)$ and $Y_0(kr)$ given by the following equations [27]:

$$Y_0(z) = \frac{2}{\pi} \left[\ln \frac{z}{2} + c + \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{(k!)^2} \left(\ln \frac{z}{2} + c - \sum_{m=1}^k \frac{1}{m} \right) \right]$$

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{(k!)^2}$$

where

$$c = \text{Euler's constant} = .5772$$

Of these two solutions, the Bessel function of first kind and order zero, $J_0(kr)$, will not be considered since it is regular (has a value of 1.0) at $r=0.0$ and hence cannot lead to the desired delta function

singularity at $r=0.0$. Hence, the fundamental solution of Eqn. (3.15) will be taken as:

$$w_o^* = D Y_o(kr) \quad (3.18)$$

In the above, $Y_o(kr)$ is the Bessel function of the second kind of zero order which is singular at $r=0$ and D is a constant. The value of D can be found by integrating Eqn. (3.15) over a disk of arbitrary small radius ϵ , centred at point 'i', as follows:

$$\iint (\nabla^2 w_o^* + k^2 w_o^*) d\sigma = \iint \Delta^i d\sigma \quad (a)$$

where σ is the region over which the integration is carried out. For small values of (kr) , the fundamental solution w_o^* can be approximated as follows:

$$\begin{aligned} w_o^* &= D Y_o(kr) \\ &= \frac{2D}{\pi} \ln(kr) \end{aligned} \quad (b)$$

Also,

$$\frac{\partial w_o^*}{\partial r} = \frac{2D}{\pi r} \quad (c)$$

Applying the Gauss divergence theorem [28] and using Eqns. (b) and (c), the integral $\iint \nabla^2 w_o^* d\sigma$ becomes:

$$\int_S \left. \frac{\partial w_o^*}{\partial r} \right|_{r=\epsilon} ds = \frac{2D}{\pi \epsilon} \int_S dS = 4D$$

where S is the boundary of the small disk.

The integral $\iint k^2 w_0^* d\sigma$ is equal to:

$$\begin{aligned} & \iint k^2 \frac{2}{\pi} D \ln(kr) d\sigma \\ &= \iint \frac{2k^2 D}{\pi} \ln(kr) r dr d\theta \\ &= \frac{2k^2 D}{\pi} \int_0^E \left[\frac{r^2}{2} \ln kr - \frac{r^2}{4} \right] d\theta \end{aligned}$$

As ε tends to zero, the term within the integral tends to zero. Hence, the left-hand side of Eqn. (a) becomes:

$$\iint (V^2 w_0^* + k^2 w_0^*) d\sigma = 4D$$

Since the right-hand side of Eqn. (a) is unity, it follows that:

$$D = \frac{1}{4}.$$

The fundamental solution thus takes the form:

$$w_0^* = \frac{1}{4} Y_0(kr) \quad (3.19)$$

where r is the distance from the considered point 'i'.

Once w_0^* is known, q^* can be found as follows:

$$\begin{aligned} w_0^* &= \frac{1}{4} Y_0(kr) \\ q^* &= \frac{dw_0^*}{dn} \\ &= \frac{dw_0^*}{dr} \frac{dr}{dn} \\ &= \frac{k}{4} Y_0'(kr) \cos \theta \end{aligned}$$

FIG. 3.4. BOUNDARY ELEMENTS AND INTERNAL CELLS.

FIG. 3.5. COORDINATES SYSTEM.

$$[H]\{w_o\} = [G]\{q\} + \{b\} \quad (3.21)$$

On applying the appropriate boundary conditions and re-ordering the system of Eqn. (3.21) in such a way that all the unknowns are transferred to the left-hand side, Eqn. (3.21) becomes:

$$[B]\{X\} = \{F\}$$

where $\{X\}$ is the vector of unknowns w_o and q .

After the values of w_o and q have been calculated for the entire boundary by solving the above set of simultaneous equations, the response at time t can be found for any point in the membrane using the following equation:

$$\begin{aligned} w_i &= (w_o)_i e^{i\omega t} \\ &= \left[\sum_{j=1}^N (w_o)_j H_{ij} - \sum_{j=1}^N (q)_j G_{ij} + b_i \right] e^{i\omega t} \end{aligned}$$

It must be noted that the description of the formulation of boundary element method presented above is quite general. For the special case of a membrane, the value of w_o on the entire boundary is zero and therefore the elements of the $[H]$ matrix need not be calculated and the system of Eqns. (3.23) will be simply:

$$[G]\{q\} = -\{b\}$$

Reference will be made in Chapters 4 and 5 to the method of evaluation of the $[H]$ matrix described above.

FIG. 3.4. BOUNDARY ELEMENTS AND INTERNAL CELLS.

FIG. 3.5. COORDINATES SYSTEM.

The first term of H_{11} will vanish due to the orthogonality of r and n ($\theta = \pi/2$, hence, $\cos \theta = 0.0$). Thus,

$$H_{11} = -\frac{1}{2}$$

The term G_{1j} is given by:

$$\begin{aligned} G_{1j} &= \int_{c_j} w_o^* dc \\ &= \int_{c_j} \frac{1}{4} Y_o(kr) dc \end{aligned}$$

Transforming the coordinates into the ξ system

$$\begin{aligned} G_{1j} &= \int_{-1}^{+1} \frac{1}{4} Y_o(kr) \frac{\ell}{2} d\xi \\ &= \frac{\ell}{8} \int_{-1}^{+1} Y_o(kr) d\xi \\ &= \frac{\ell}{8} \sum_{i=1}^4 Y_o(kr) \cdot w_i \end{aligned}$$

The term G_{ii} is calculated analytically using the approximation:

$$Y_o(kr) \approx \frac{2}{\pi} \ln(kr) \text{ as } r \rightarrow 0.0$$

Thus,

$$\begin{aligned} G_{ii} &= \int_{c_i} w_o^* dc \\ &= \int_{c_i} \frac{1}{4} Y_o(kr) dc \\ &= \int_{c_i} \frac{1}{4} \frac{2}{\pi} \ln(kr) dc \\ &= \frac{1}{2\pi} \int_{c_i} \ln(kr) dc \end{aligned}$$

and on transforming into ξ coordinates

$$\begin{aligned} G_{ii} &= \frac{\ell}{2\pi} \int_0^1 \ln(k \xi \frac{\ell}{2}) d\xi \\ &= \frac{\ell}{2\pi} [\ln(k \frac{\ell}{2}) - 1] \end{aligned}$$

where ℓ is the length of element 'i'.

The terms b_{is} are calculated using Gaussian quadrature for rectangular domains [30] as follows:

$$b_{is} = \iint_{A_s} \phi w_o^* dA_s$$

Setting $\phi w_o^* = f(x,y)$ and transferring into ξ and η coordinates, the above reduces to:

$$\begin{aligned} b_{is} &= \frac{A_s}{4} \int_{-1}^{+1} \int_{-1}^{+1} f d\xi d\eta \\ &= \frac{A_s}{4} \sum_{j=1}^L \sum_{i=1}^L w_i \cdot w_j f(\xi_i, \eta_j) \end{aligned}$$

where w_i and w_j are the weighting functions used in the Gaussian quadrature and L is the number of points used in the Gaussian quadrature.

Having obtained values of all b_{is} terms corresponding to the point i , the value of b_i can be obtained by using the equation

$$b_i = \sum_{s=1}^M b_{is}$$

Once the elements of $[G]$, $[H]$ and $\{b\}$ have been calculated, the boundary conditions are applied to the system of equations:

$$[H]\{w_o\} = [G]\{q\} + \{b\} \quad (3.21)$$

On applying the appropriate boundary conditions and re-ordering the system of Eqn. (3.21) in such a way that all the unknowns are transferred to the left-hand side, Eqn. (3.21) becomes:

$$[B]\{X\} = \{F\}$$

where $\{X\}$ is the vector of unknowns w_o and q .

After the values of w_o and q have been calculated for the entire boundary by solving the above set of simultaneous equations, the response at time t can be found for any point in the membrane using the following equation:

$$\begin{aligned} w_i &= (w_o)_i e^{i\omega t} \\ &= \left[\sum_{j=1}^N (w_o)_j H_{ij} - \sum_{j=1}^N (q)_j G_{ij} + b_i \right] e^{i\omega t} \end{aligned}$$

It must be noted that the description of the formulation of boundary element method presented above is quite general. For the special case of a membrane, the value of w_o on the entire boundary is zero and therefore the elements of the $[H]$ matrix need not be calculated and the system of Eqns. (3.23) will be simply:

$$[G]\{q\} = -\{b\}$$

Reference will be made in Chapters 4 and 5 to the method of evaluation of the $[H]$ matrix described above.

ii. Evaluation of the integrals for linear boundary elements

As mentioned in Chapter 2, the terms required to fill in the two matrices $[G]$ and $[H]$ are, in this case, h_{i1} , h_{i2} , g_{i1} and g_{i2} . They are calculated as follows:

$$\begin{aligned} h_{i1} &= \int_{c_j} \phi_1 q^* dc \\ &= \int_{c_j} \frac{1}{2} (1-\xi) q^* dc \\ &= \int_{c_j} \frac{1}{2} (1-\xi) \frac{k}{4} Y'_0(kr) \cos\theta dc \end{aligned}$$

Transforming the coordinates into the ξ system as before, the above reduces to:

$$\begin{aligned} h_{i1} &= \int_{-1}^{+1} \frac{1}{2} (1-\xi) \frac{k}{4} \frac{\ell}{2} Y'_0(kr) \cos\theta d\xi \\ &= \frac{k\ell}{16} \sum_{i=1}^4 (1-\xi_i) Y'_0(kr)_i (\cos\theta)_i w_i \end{aligned}$$

where w_i is a weighting factor used in the Gaussian quadrature.

The terms h_{i2} , g_{i1} and g_{i2} are obtained in a similar manner. The final expressions for these terms are:

$$\begin{aligned} h_{i2} &= \frac{k\ell}{16} \sum_{i=1}^4 (1+\xi_i) Y'_0(kr) (\cos\theta)_i w_i, \\ g_{i1} &= \frac{\ell}{16} \sum_{i=1}^4 (1-\xi_i) Y_0(kr) w_i \end{aligned}$$

and

$$g_{i2} = \frac{\ell}{16} \sum_{i=1}^4 (1+\xi_i) Y_0(kr) \cdot w_i$$

when $i=j$.

The term H_{ii} is given by:

$$H_{ii} = - \sum_{j=1}^N H_{ij} \quad i \neq j$$

The term G_{ii} is obtained as follows:

$$\begin{aligned} g_{i1} &= \int_0^{\ell} \phi_1 w_0^* dr \\ &= \int_0^{\ell} \frac{1}{2} (1-\xi) \frac{1}{4} Y_0(kr) dr \\ &= \int_0^{\ell} \frac{1}{4\pi} (1-\xi) \ln(kr) dr \end{aligned}$$

On transforming into ξ coordinates, the above becomes:

$$g_{i1} = \frac{\ell}{8\pi} \int_{-1}^{+1} (1-\xi) [\ln k + \ln \frac{\ell}{2} (1+\xi)] d\xi$$

Evaluation of the above integral gives:

$$g_{i1} = \frac{\ell}{4\pi} [\ln \ell + \ln k - 1.5]$$

In a similar manner,

$$\begin{aligned}
 g_{12} &= \int_0^{\ell} \phi_2 w_0^* dr \\
 &= \int_0^{\ell} \frac{1}{2} (1+\xi) \frac{1}{4} Y_0(kr) dr \\
 &= \frac{\ell}{4\pi} [\ln k + \ln \ell - 0.5]
 \end{aligned}$$

Having obtained the matrices $[H]$ and $[G]$, the remaining steps in the solution are the same as for the case of constant elements.

3.5 RESULTS OF THE INVESTIGATION

Computer programs were developed to determine the forced vibration response of a stretched membrane. The results of an example problem solved to compare the results obtained from the boundary element method with those obtained from the classical methods of solution are presented in this section. The membrane selected for analysis has dimensions $a=b=5$ ms. The forcing function is assumed to be of the form:

$$p(x,y,t) = p_0 \cos \omega t$$

The amplitude p_0 is taken as equal to 0.02 t/m^2 and the exciting frequency as equal to 2π . Assuming that the tension per unit length of the edge, S , is 5 t/m and that the mass per unit area of the membrane is 1 kg/m^2 , the response of the membrane at points a, b, and c shown in Fig. 3.6a is determined for $t=1.0 \text{ s}$. The boundary of the membrane is divided into 12 elements and the membrane area itself is divided

into 9 internal cells as shown in Fig. 3.6b. Both the constant element and the linear element programs were used. The results obtained from the classical solution method and the boundary element method are compared in Table 1. The agreement of the results is satisfactory for both the linear and the constant elements.

To test the convergence of the solution, the same problem was solved using 16 boundary elements and 16 internal cells as shown in Fig. 3.6c. The results obtained with 16 boundary elements are also shown in Table 1. In this particular problem, the use of linear elements does not increase the accuracy of the solution.

A - SQUARE MEMBRANE

B - 12 BOUNDARY ELEMENTS AND 9 INTERNAL CELLS

C - 16 BOUNDARY ELEMENTS AND 16 INTERNAL CELLS

FIG. 3.6. BOUNDARY ELEMENT DISCRETIZATION OF A SQUARE MEMBRANE.

TABLE I

DISPLACEMENTS IN METERS AT TIME $t=1.0s$			
POINT	a	b	c
Classical Solution	0.00737534	0.0034677	0.00548964
<u>Constant Element:</u>			
12 Elements	0.00722010	0.0033880	0.00541393
16 Elements	0.00729138	0.0033777	0.00538482
20 Elements	0.00726402	0.0034030	0.00538134
<u>Linear Element:</u>			
12 Elements	0.00701586	0.00323036	0.00521447
16 Elements	0.00714169	0.00323956	0.00523682

CHAPTER 4

HYDRODYNAMIC PRESSURE DISTRIBUTION
IN
A FINITE WATER RESERVOIR4.1 INTRODUCTION

The safety of dams during earthquakes is extremely important because the failure of such a structure may have disastrous consequences on life and property. It is therefore extremely important that dams be designed to be adequately safe against earthquake loads. Recently, the importance of earthquake analysis of dams has been realized because of damages caused to some dams by earthquakes [32]. It has been found that the dynamic behaviour of a dam is far different from that obtained by the traditional static method which treats earthquake loads as equivalent static horizontal forces equal to the product of mass of the system and a design ground acceleration [33,34]. The dynamics of a dam differs from that of any other ground structure because of the interaction of the structure with the reservoir in addition to the interaction with the foundation. The deformations of the structure and the foundation and the hydrodynamic pressure on the structure depend upon each other. Therefore, an accurate analytical procedure must consider these triple interactions, i.e. interactions between dam-foundation, dam-reservoir and reservoir-foundation. Since the water reservoir and the soil under foundation are mathematically represented as infinite domains which can not be suitably represented by the domain methods,

there is promise for the use of the boundary element method in the analysis of the complete dam-reservoir-foundation system.

In this chapter, the boundary element method is applied to find the distribution of the hydrodynamic pressure acting on the upstream face of a dam of height H , storing a finite reservoir of depth H_0 and length L , (see Fig. 4.1). Two different conditions are considered for the far-end of the reservoir. The first condition specifies that the far end is stationary and the second that it is moving with the same acceleration as the upstream face of the dam. The cases of sloping upstream face of the dam and sloping base are investigated. Two types of boundary elements are used, the constant element and the linear element. The results for the case of vertical upstream face of the dam and horizontal reservoir bed were compared with the known classical solutions.

4.2 CLASSICAL SOLUTIONS

The following assumptions are made in determining the distribution of the hydrodynamic pressure on the upstream face of the dam considered rigid:

1. The motion of the dam-reservoir system is considered as two dimensional, i.e. it is the same for any vertical plane perpendicular to the axis of the dam.
2. Water is assumed to be compressible but its internal viscosity is neglected.

FIG. 4.1. RIGID DAM STORING FINITE WATER RESERVOIR.

3. The effects of waves at the free surface of water are ignored because the associated errors were found to be small by other investigators [1].

4. The dam is assumed to be rigid.

5. The motion of water is limited to small amplitudes.

With the assumptions listed above, the motion of water in the reservoir is governed by the following equations:

$$\frac{\partial^2 u}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (4.1)$$

$$\frac{\partial^2 v}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

where

u is the x component of displacement of a particle of water,

v is the y component of displacement of a particle of water,

ρ is the mass of water per unit volume and

p is the hydrodynamic pressure.

In addition, the following equation of continuity must be satisfied:

$$p = -k \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (4.2)$$

where k is the bulk modulus of water.

On introducing a velocity potential $\phi(x,y,t)$ defined as:

$$- \frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial t}, \quad (4.3)$$

$$- \frac{\partial \phi}{\partial y} = \frac{\partial v}{\partial t}$$

Eqs. (4.1), (4.2) and (4.3) yield:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \quad (4.4)$$

and

$$p = \rho \frac{\partial \phi}{\partial t} \quad (4.5)$$

where c denotes the **velocity** of sound in water given by:

$$c^2 = k/\rho \quad (4.6)$$

The solution of Eqn. (4.4) is discussed below for the two cases of an infinite and a finite water reservoir.

4.2.1 Infinite Water Reservoir

For the case of infinite water reservoir, the following boundary conditions should be satisfied:

1. The vertical component of velocity of the particles of water on the bed of the reservoir should vanish for the case of horizontal ground acceleration, i.e.

$$\frac{\partial \phi}{\partial y} (x, 0, t) = 0.0 \quad (4.7)$$

2. The hydrodynamic pressure, p , should vanish at the free surface of the reservoir, i.e.

$$\frac{\partial \phi}{\partial t} (x, H, t) = 0.0 \quad (4.8)$$

3. The hydrodynamic pressure, as well as the horizontal and vertical components of the velocity, should vanish at

infinity, i.e.

$$\phi(\infty, y, t) = 0.0 \quad (4.9)$$

4. The horizontal acceleration of the particles of water at the upstream face of the dam should be equal to the horizontal ground acceleration, i.e.

$$-\frac{\partial}{\partial t} \frac{\partial \phi}{\partial x} (0, y, t) = \frac{\partial^2 u_g}{\partial t^2} \quad (4.10)$$

where

$$\frac{\partial^2 u_g}{\partial t^2}$$

is the horizontal ground acceleration.

Assuming that the horizontal ground acceleration is of the form

$$\frac{\partial^2 u_g}{\partial t^2} = e^{i\omega t}$$

Eqn. (4.10) becomes:

$$-\frac{\partial}{\partial t} \frac{\partial \phi}{\partial x} (0, y, t) = e^{i\omega t}$$

In addition to the above boundary conditions, there exist the following initial conditions:

$$1. \quad \phi(x, y, 0) = 0$$

$$2. \quad \frac{\partial \phi}{\partial t} (x, y, 0) = 0$$

Assuming that $\phi(x, y, t)$ can be expressed as:

$$\phi(x, y, t) = X(x) \cdot Y(y) \cdot T(t) \quad (4.11)$$

Eqn. (4.4) becomes:

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = \frac{1}{c^2} + \frac{1}{T} \frac{d^2 T}{dt^2} = -k^2 \quad (4.12)$$

where k^2 is a constant.

From Eqn. (4.12),

$$\frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad (4.13)$$

which gives

$$T = A_1 e^{ickt} + B_1 e^{-ickt} \quad (4.14)$$

where A_1 and B_1 are constants.

Again, from Eqn. (4.12),

$$\frac{1}{X} \frac{d^2 X}{dx^2} + k^2 = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = m^2 \quad (4.15)$$

where m^2 is a constant.

Eqn. (4.15) can be written as:

$$\frac{d^2 Y}{dy^2} + m^2 Y = 0 \quad (4.16)$$

and

$$\frac{d^2 X}{dx^2} - (m^2 - k^2) X = 0 \quad (4.17)$$

The solution of Eqn. (4.16) gives:

$$Y = A_2 \cos my + B_2 \sin my \quad (4.18)$$

where A_2 and B_2 are constants.

Introduction of boundary condition (1), Eqn. (4.7), gives:

$$B_2 = 0.0$$

and

$$Y = A_2 \cos my \quad (4.19)$$

Boundary condition (2), Eqn. (4.8), will require that

$$\cos mh = 0.0$$

or that

$$m = \frac{(2n-1)\pi}{2H} \quad (4.20)$$

where $n=1,2,3,4,\dots,\infty$.

Assuming that $m > k$ and putting $m^2 - k^2 = q_n^2$ in Eqn. (4.17), the solution for function X becomes:

$$X = A_3 e^{q_n x} + B_3 e^{-q_n x} \quad (4.21)$$

Boundary condition (3), Eqn. (4.9), requires that the solution $e^{q_n x}$ be discarded, so that

$$X = B_3 e^{-q_n x} \quad (4.22)$$

Combining Eqns. (4.14), (4.19), (4.20) and (4.22)

$$\phi = \sum_{n=1}^{\infty} [A_n e^{ickt} + B_n e^{-ickt}] \cos my \cdot e^{-q_n x} \quad (4.23)$$

Introduction of boundary condition (4), Eqn. (4.10), gives the relationship:

$$e^{i\omega t} = \sum_{n=1}^{\infty} q_n \cos my [A_n(ick) e^{ickt} + B_n(-ick) e^{-ickt}] \quad (4.24)$$

The above relationship will be satisfied if:

$$B_n = 0.0 \quad (4.25)$$

and

$$\omega = ck \text{ or } k = \frac{\omega}{c} \quad (4.26)$$

and

$$1.0 = \sum_{n=1}^{\infty} q_n A_n(i\omega) \cos my \quad (4.27)$$

The value of constant A_n is obtained by multiplying both sides of Eqn.

(4.27) by $\cos my$ and integrating from zero to H :

$$A_n = \frac{(-1)^{n+1}}{m \cdot q_n} * \frac{2}{i\omega H} \quad (4.28)$$

Substitution of Eqns. (4.25) and (4.28) into Eqn. (4.23) gives:

$$\phi = \sum_{n=1}^{\infty} \frac{2}{i\omega H} \cdot \frac{(-1)^{n+1}}{m \cdot q_n} \cdot \cos my \cdot e^{-q_n x} e^{i\omega t} \quad (4.29)$$

An expression for the hydrodynamic pressure p is now obtained by using

Eqn. (4.5):

$$p = \frac{2\rho}{H} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{m \cdot q_n} \cdot \cos my \cdot e^{-q_n x} e^{i\omega t} \quad (4.30)$$

The natural frequencies of the reservoir are obtained from the condition

that $q_n = 0.0$, which leads to:

$$m = k \quad (4.31)$$

or

$$\frac{\pi(2n-1)}{2H} = \frac{\omega}{c}, \quad n=1,2,3,\dots,$$

The fundamental natural frequency, ω_1 , is given by:

$$\omega_1 = \frac{\pi c}{2H} \quad (4.32)$$

Depending on the value of the exciting frequency ω the following two different cases may arise.

a) The exciting frequency ω is less than the fundamental natural frequency ω_1 . For this case,

$$\frac{\omega}{c} < \frac{\omega_1}{c} < \frac{\pi}{2H} < \frac{3\pi}{2H} < \dots$$

or

$$k < m_1 < m_2 < m_3 < \dots$$

Hence,

$$q_n = \sqrt{m^2 - k^2}$$

is real and the assumption made in deriving Eqn. (4.22) is correct.

In this case, the hydrodynamic pressure p is given by:

$$p = \frac{2\rho}{H} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{m \cdot q_n} e^{-q_n x} \cos m y e^{i\omega t} \quad (4.33)$$

b) The exciting frequency ω is greater than the fundamental natural frequency ω_1 . For this case, we have:

$$k = \frac{\omega}{c} > \frac{(2n-1)\pi}{2H}, \quad n=1,2,\dots,\bar{n}$$

and

$$k = \frac{\omega}{c} < \frac{(2n-1)\pi}{2H}, \quad n=\bar{n}+1,\dots,\infty$$

Therefore,

$$q_n = \sqrt{m^2 - k^2} \text{ is imaginary for } n=1, 2, \dots, \bar{n}$$

and

$$q_n = \sqrt{m^2 - k^2} \text{ is real for } n=\bar{n}+1, \bar{n}+2, \dots, \infty.$$

For $n=1, 2, \dots, \bar{n}$, q_n is imaginary and one can write

$$q_n = i \cdot \bar{q}_n$$

where

$$\bar{q}_n = \sqrt{k^2 - m^2}$$

The rejection of the solution $e^{q_n x}$ when q_n is not real requires some explanation. In this case, the value of $e^{q_n x}$ is not bounded at infinity. However, from the classical wave theory, it is known that the solution $e^{q_n x} e^{i\omega t} = e^{i(q_n x + \omega t)}$ represents an incoming wave reflected from the far boundary. In an infinite reservoir, such a wave cannot exist and the solution $e^{q_n x}$ must therefore be discarded.

The hydrodynamic pressure p is then given by:

$$p = \frac{2\rho}{H} \sum_{n=1}^{\bar{n}} \frac{(-1)^{n+1}}{i \cdot m \cdot \bar{q}_n} \cos my e^{-i\bar{q}_n x} e^{i\omega t} + \frac{2\rho}{H} \sum_{n=\bar{n}+1}^{\infty} \frac{(-1)^{n+1}}{m \cdot q_n} e^{-q_n x} \cos my e^{i\omega t} \quad (4.34)$$

4.2.2 Finite Water Reservoir

As for the case of infinite reservoir, the velocity potential $\phi(x,y,t)$ is assumed to be of the form:

$$\phi(x,y,t) = X(x) \cdot Y(y) \cdot T(t).$$

By a procedure similar to that for the case of infinite reservoir, the following expressions are obtained for the functions T , Y and X :

$$T = A_1 e^{i\omega t} \quad (4.35)$$

$$Y = A_2 \cos my \quad (4.36)$$

and

$$X = A_3 e^{-q_n x} + B_3 e^{q_n x} \quad (4.37)$$

where

$$q_n = \sqrt{m^2 - k^2}$$

The velocity potential ϕ is thus given by:

$$\phi(x,y,t) = \sum_{n=1}^{\infty} [A_n e^{-q_n x} + B_n e^{q_n x}] \cos my e^{i\omega t} \quad (4.38)$$

The boundary condition at the far end is now different from that for the infinite reservoir.

The following two boundary conditions are studied.

a) Stationary far end

The far end of the reservoir is assumed to be stationary. In this case, the boundary condition can be expressed as:

$$\frac{\partial \phi}{\partial x} (L,y,t) = 0.0 \quad (4.39)$$

where L is the length of the reservoir.

Introduction of the above boundary condition gives the following relationship between coefficients B_n and A_n .

$$B_n = A_n e^{-2q_n L} \quad (4.40)$$

giving the following value for the derivative of the velocity potential:

$$\frac{\partial \phi}{\partial x} = \sum_{n=1}^{\infty} \left[-A_n q_n e^{-q_n x} + A_n q_n e^{-2q_n L} e^{q_n x} \right] \cos m y e^{i\omega t} \quad (4.41)$$

From the boundary condition given by Eqn. (4.10),

$$1.0 = \sum_{n=1}^{\infty} A_n q_n i\omega (1 - e^{-2q_n L}) \cos m y \quad (4.42)$$

The value of A_n is obtained by multiplying both sides of Eqn. (4.42) by $\cos m y$ and integrating from zero to H .

$$A_n = \frac{(-1)^{n+1}}{m q_n} \cdot \frac{2}{i\omega H (1 - e^{-2q_n L})} \quad (4.43)$$

Substitution of Eqns. (4.40) and (4.43) into Eqn. (4.38) yields:

$$\phi = \sum_{n=1}^{\infty} \frac{2}{i\omega H} \frac{(-1)^{n+1} \left[e^{q_n(L-x)} + e^{-q_n(L-x)} \right]}{m q_n \left[e^{q_n L} - e^{-q_n L} \right]} \cos m y e^{i\omega t} \quad (4.44)$$

The hydrodynamic pressure p can be expressed as:

$$p = \frac{2\rho}{H} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{m q_n} \frac{e^{q_n(L-x)} + e^{-q_n(L-x)}}{e^{q_n L} - e^{-q_n L}} \cos m y e^{i\omega t} \quad (4.45)$$

As for the case of infinite reservoir, the following two cases will be considered.

1. The exciting frequency ω is less than the fundamental natural frequency ω_1 , so that

$$\frac{\omega}{c} < \frac{\pi}{2H} < \frac{3\pi}{2H} < \dots$$

Hence,

$$q_n = \sqrt{\left(\frac{(2n-1)\pi}{2H}\right)^2 - \frac{\omega^2}{c^2}}, \quad n=1,2,\dots,\infty$$

It is clear that q_n will be real for all values of n , hence

$$p = \frac{2\rho}{H} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{m q_n} \cdot \frac{\cosh[q_n(L-x)]}{\sinh(q_n L)} \cos m y e^{i\omega t} \quad (4.46)$$

2. The exciting frequency ω is greater than the fundamental natural frequency ω_1 , so that

$$\frac{\omega}{c} > \frac{(2n-1)\pi}{2H}, \quad n=1,2,\dots,\bar{n}$$

which means that q_n is imaginary for $n=1,2,\dots,\bar{n}$ and is real for $n=\bar{n}+1, \bar{n}+2, \dots, \infty$.

In this case, the hydrodynamic pressure p is given by:

$$p = \frac{-2\rho}{H} \sum_{n=1}^{\bar{n}} \frac{(-1)^{n+1}}{m \bar{q}_n} \cdot \frac{\cos \bar{q}_n(L-x)}{\sin \bar{q}_n(L)} \cdot \cos m y \cdot e^{i\omega t} \\ + \frac{2\rho}{H} \sum_{n=\bar{n}+1}^{\infty} \frac{(-1)^{n+1}}{m q_n} \cdot \frac{\cosh q_n(L-x)}{\sinh q_n L} \cdot \cos m y \cdot e^{i\omega t} \quad (4.47)$$

b) Moving Far End

An alternative boundary condition for the far end is to assume that the end is moving at the same acceleration as the upstream face of the dam. The following expressions are obtained for the hydrodynamic pressure p in this case.

$$1. \omega < \omega_1$$

$$p = \frac{2\rho}{H} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{m q_n} \frac{\cosh q_n(L-x) - \cosh q_n x}{\sinh q_n L} \right] \cos my e^{i\omega t} \quad (4.48)$$

$$2. \omega > \omega_1$$

$$p = \frac{-2\rho}{H} \sum_{n=1}^{\bar{n}} \frac{(-1)^{n+1}}{m \bar{q}_n} \frac{\cos \bar{q}_n(L-x) - \cos \bar{q}_n x}{\sin \bar{q}_n L} \cos my e^{i\omega t} \\ + \frac{2\rho}{H} \sum_{n=\bar{n}+1}^{\infty} \frac{(-1)^{n+1}}{m q_n} \frac{\cosh q_n(L-x) - \cosh q_n x}{\sinh q_n L} \cos my e^{i\omega t} \quad (4.49)$$

4.3 BOUNDARY ELEMENT FORMULATION

Equations (4.4) and (4.5) show that the small amplitude motion of water in the reservoir is governed by the equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (4.50)$$

The time dependence is removed by assuming that p is of the form:

$$p = p_0(x, y) e^{i\omega t} \quad (4.51)$$

Substitution of Eqn. (4.51) into Eqn. (4.50) gives:

$$\frac{\partial^2 p_o}{\partial x^2} + \frac{\partial^2 p_o}{\partial y^2} + \frac{\omega^2}{c^2} p_o = 0 \quad (4.52)$$

Putting $\frac{\omega^2}{c^2} = k^2$, Eqn. (4.52) becomes:

$$\frac{\partial^2 p_o}{\partial x^2} + \frac{\partial^2 p_o}{\partial y^2} + k^2 p_o = 0 \quad (4.53)$$

which is the familiar Helmholtz equation discussed in Chapter 2. Following the steps outlined in Chapter 2, the Helmholtz equation is converted into the following boundary problem:

$$c_i(p_o)_i = \int_C p_o q^* dc - \int_C p_o^* q dc \quad (4.54)$$

where p_o^* is the fundamental solution of the equation

$$\frac{\partial^2 p_o^*}{\partial x^2} + \frac{\partial^2 p_o^*}{\partial y^2} + k^2 p_o^* = \Delta^i \quad (4.55)$$

As mentioned in Chapter 3; the fundamental solution p_o^* is given by:

$$p_o^* = \frac{1}{4} Y_o(kr) \quad (4.56)$$

and

$$q^* = \frac{k}{4} Y_o'(kr) \cdot \cos\theta \quad (4.57)$$

The system of boundary equations governing the problem can be written as:

$$[H]\{p_o\} = [G]\{q\}$$

The method of calculating the elements of [H] and [G] for both the constant element and linear element was described in Chapter 3. After the elements of [G] and [H] matrices have been obtained, the boundary conditions are applied as follows.

4.3.1 Boundary Conditions

For the case of finite reservoir, the following boundary conditions should be satisfied:

1. The hydrodynamic pressure p should vanish at the free surface of the reservoir, i.e.

$$p(x, H, t) = 0$$

which gives

$$p_0(x, H) = 0.0$$

2. For a horizontal ground acceleration, the vertical acceleration of the particles of water on the bed of the reservoir should vanish, i.e.

$$\left(\frac{\partial^2 v}{\partial t^2} \right) (x, 0, t) = 0.0$$

which gives

$$\frac{\partial p_0}{\partial n} = 0.0$$

3. The horizontal acceleration of the particles of water at the upstream face of the dam should be equal to the horizontal ground acceleration, i.e.

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{(0,y,t)} = \frac{\partial^2 u}{\partial t^2} \quad (4.58)$$

Assuming that

$$\frac{\partial^2 u}{\partial t^2} = e^{i\omega t}$$

and noting that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{\partial p}{\partial n}$$

Eqn. (4.58) becomes:

$$\frac{1}{\rho} \frac{\partial p}{\partial n} = e^{i\omega t}$$

or

$$\left(\frac{\partial p}{\partial n}\right)_{(0,y)} = \rho$$

4. For the boundary condition at the far end of the reservoir, the following two cases arise:

- i) Assuming that the horizontal acceleration of the particles of water at the far end of the reservoir is zero, the following boundary condition should be satisfied:

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{(L,y,t)} = 0.0$$

which gives:

$$\left(\frac{\partial p}{\partial n}\right)_{(L,y,t)} = 0.0$$

- ii) Assuming that the horizontal acceleration of the particles of water at the far end is equal to the horizontal ground acceleration, the following boundary condition should be satisfied:

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{(L,y,t)} = \frac{\partial^2 u_g}{\partial t^2}$$

which gives

$$\frac{\partial p_o}{\partial n} = -\rho$$

On applying the appropriate boundary conditions and re-ordering the equations in such a way that all the unknowns are transferred to the left-hand side, the system of equations becomes:

$$[B]\{X\} = \{F\}$$

where $\{X\}$ is the vector of unknowns p_o and q s.

After obtaining the values of p_o and q on the entire boundary, the hydrodynamic pressure at any point can be found from the following equation:

$$\begin{aligned} (p)_i &= (p_o)_i e^{i\omega t} \\ &= \left[\sum_{j=1}^N (p_o)_j H_{ij} - \sum_{j=1}^N (q)_j G_{ij} \right] e^{i\omega t} \end{aligned}$$

A computer program was written to determine the hydrodynamic pressure distribution for the finite reservoir case.

A number of example problems were solved. The results are presented in the following sections.

4.4 RESULTS OF INVESTIGATION

The following cases have been investigated using the computer program mentioned earlier.

4.4.1 Finite Water Reservoir with Stationary Far End

a) The exciting frequency less than the fundamental natural frequency:

The response of finite water reservoir shown in Fig. 4.2 to ground motion is calculated using the boundary element method. To test the convergence of the method, the problem is solved twice, first with 30 constant boundary elements (C.B.E.) and then with 35 constant boundary elements (C.B.E.). Two different values of exciting frequency were considered, $\frac{1}{4}$ and $\frac{1}{2}$ of the fundamental natural frequency, ω_1 . Solutions were obtained for four different values of L ($L=H$, $2H$, $3H$ and $4H$). The results are presented in Figs. 4.3, 4.4, 4.5 and 4.6. For purpose of comparison, the results obtained from the known classical solutions are also presented. The agreement of results is excellent. It is found that if the length of the reservoir is equal to or greater than twice its depth and the far end of the reservoir is stationary, the hydrodynamic pressure distribution will be the same as for the case of infinite reservoir of the same depth. It is also observed that the hydrodynamic pressure on the far end of the reservoir decreases as the length of the reservoir increases as shown in Figs. 4.7 and 4.8.

b) Exciting frequency greater than the fundamental natural frequency:

Two different values for L were considered, i.e. $L=H$ and $L=2H$. The results are compared with the known classical solutions and the

A - 30 CONSTANT BOUNDARY ELEMENTS

B - 35 CONSTANT BOUNDARY ELEMENTS

FIG. 4.2. BOUNDARY ELEMENT DISCRETIZATION OF FINITE WATER RESERVOIR.

FIG. 4.3., HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.4. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.5. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.6. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.7. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.8. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

agreement of the results is excellent. It is found that for this case the hydrodynamic pressure distribution for an infinite reservoir cannot be deduced from that for a finite reservoir as can be seen from Figs. 4.9 and 4.10.

4.4.2 Finite Reservoir with All Boundary Points Moving with the Same Horizontal Acceleration

a) Exciting frequency less than the fundamental natural frequency:

The problem is solved for several different values for the length of the reservoir. For each value of the length, the hydrodynamic pressure distribution on the upstream face of the dam is plotted and compared with the known classical solutions. The agreement of the results obtained from the boundary element method with known classical solutions is excellent. It is found that the length of the reservoir that should be considered to obtain the same hydrodynamic pressure distribution as for an infinite reservoir of the same depth depends on the ratio of the exciting frequency to the fundamental natural frequency. For example, the length required for a ratio of $\frac{1}{4}$ is $4H$, and for a ratio of $\frac{1}{2}$ is $5H$. This conclusion is drawn from results presented in Figs. 4.11, 4.12, 4.13, 4.14, 4.15 and 4.16.

b) Exciting frequency greater than the fundamental natural frequency:

Again for this case, the hydrodynamic pressure distribution for the infinite reservoir cannot be deduced from that for the finite reservoir. Solutions are, however, obtained for two different values of L : $L=H$ and $L=2H$. The results are shown in Figs. 4.17 and 4.18 and the agreement with classical solutions is excellent.

FIG. 4.9. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.10. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.11. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.12. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.13. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.14. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.15. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.16. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.17. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

FIG. 4.18. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR MOVING WITH THE GROUND ACCELERATION.

4.4.3 Effect of Sloping Reservoir Bed on Response

To study the case of sloping reservoir bed, solutions are obtained for different bed-slope angles, β , and exciting frequencies (see Fig. 4.19). It is assumed that all boundary points of the reservoir except those at the surface are moving with the same acceleration. It is found that the effect of the reservoir bed slope depends on the following factors:

- i) the slope angle β
- ii) the value of the exciting frequency.

The results are presented in Figs. 4.20 and 4.21. In general, the hydrodynamic pressure decreases as the bed slope increases.

4.4.4 Effect of Sloping Upstream Face of the Dam on Response

To study the effect of the slope of the upstream face of the dam, the hydrodynamic pressure on dams inclined at various angles, θ , varying from 0° to 75° are obtained (see Fig. 4.22). Two different cases are investigated, one with the water considered incompressible and the other with the water considered compressible. The results are presented in Figs. 4.23 and 4.24 and are compared with the exact solution obtained by Chwang [4] for incompressible water. The agreement is excellent. It is found that the hydrodynamic pressure decreases as the angle θ increases. It is also found that the effect of the compressibility of water depends on the value of the exciting frequency ω as shown in Fig. 4.25.

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FIG. 4.19. SLOPING-RESERVOIR BED.

FIG. 4.20. EFFECTS OF RESERVOIR BED SLOPE ON HYDRODYNAMIC PRESSURE.

FIG. 4.21. EFFECTS OF RESERVOIR BED SLOPE ON HYDRODYNAMIC PRESSURE.

FIG. 4.22. SLOPING UPSTREAM FACE OF DAM.

FIG. 4.23. HYDRODYNAMIC PRESSURE AGAINST INCLINED UPSTREAM FACE (INCOMPRESSIBLE WATER).

FIG. 4.24. HYDRODYNAMIC PRESSURE AGAINST INCLINED UPSTREAM FACE (COMPRESSIBLE WATER).

FIG. 4.25. EFFECT OF COMPRESSIBILITY OF WATER.

4.5 COMMENTS

4.5.1 Effect of the Frequencies of Exterior Domain

As would be expected, when the exciting frequency is close to one of the natural frequencies of the reservoir, the response becomes very large. A problem associated with the use of the boundary element method is that the frequencies of the domain external to the one being considered may be intermixed with those of the latter.

When the exciting frequency is close to these spurious frequencies, the response values obtained by the boundary element method become unreliable.

At a natural frequency whether of the domain under consideration or the one external to it, the determinant of the matrix $[B]$ in the equation $[B]\{X\} = \{F\}$ will become zero. Thus, by locating the zeros of the determinant of $[B]$ all natural frequencies of the system can be determined. To study the effect of the spurious frequencies, the finite reservoir shown in Fig. 4.26 was analyzed for several different values of ω/ω_1 . For each value of the exciting frequency, the determinant of the matrix $[B]$ was determined. By plotting the values of the determinant of the matrix $[B]$ vs the values of the exciting frequencies, the curve shown in Fig. 4.27 is obtained. This curve indicates that the value of the determinant is zero for a value of the exciting frequency which is not one of the natural frequencies of the reservoir. To study the effect of such a frequency, the total hydrodynamic pressure force has been plotted against the exciting frequency for both the classical and

FIG. 4.26. FINITE WATER RESERVOIR WITH STATIONARY FAR END.

FIG. 4.27. NATURAL FREQUENCIES OF FINITE WATER RESERVOIR.

the boundary element solutions. The results are presented in Fig. 4.28. It will be noticed that the results obtained from the boundary element method are not reliable only for those values of the exciting frequency which are close to a natural frequency of the exterior domain.

4.5.2 Comparison of the Constant and Linear Boundary Element Methods

A comparison of the hydrodynamic pressure obtained from the constant and the linear boundary element methods is presented in Fig. 4.29. It is seen that the constant boundary element method gives more accurate results. The linear boundary element method thus offers no advantage in this case.

— CLASSICAL SOLUTION
 - - - B.E.M.

FIG. 4.28. TOTAL HYDRODYNAMIC PRESSURE FORCE.

FIG. 4.29. HYDRODYNAMIC PRESSURE IN FINITE WATER RESERVOIR WITH STATIONARY FAR END.

CHAPTER 5

HYDRODYNAMIC PRESSURE DISTRIBUTION
IN
AN INFINITE WATER RESERVOIR5.1 INTRODUCTION

In many cases, the domain methods such as the finite element method and finite difference methods are not appropriate for representing infinite domains. One of the main advantages of the boundary element method over the domain method is the capability of the former in representing infinite domains.

In the present chapter, the boundary element method is applied to the problem of finding the hydrodynamic pressure distribution on rigid dams storing infinite water reservoirs and subjected to a ground motion. The choice of an appropriate fundamental solution for the problem is discussed. Methods of representing the boundary conditions at infinity are presented. The solution technique for the resulting system of equations is discussed and the results obtained by using the boundary element method are compared with known classical solutions. It is observed that the boundary element method may prove to be an efficient and powerful tool for the solution of many engineering problems containing infinite domains.

5.2 BOUNDARY ELEMENT FORMULATION

As for the case of finite water reservoir, the governing differential equation for the motion of water is given by:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (5.1)$$

Assuming the pressure p to be of the form

$$p = p_0(x, y) e^{i\omega t}$$

Eqn. (5.1) becomes:

$$\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2} + k^2 p_0 = 0 \quad (5.2)$$

where

$$k^2 = \omega^2 / c^2.$$

Multiplying both sides of Eqn. (5.2) by p_0^* , integrating over the domain and applying Green's theorem, the following integral equation is obtained. The detailed steps are the same as those shown in Chapter 4 for the finite water reservoir.

$$\iint_A \left(\frac{\partial^2 p_0^*}{\partial x^2} + \frac{\partial^2 p_0^*}{\partial y^2} + k^2 p_0^* \right) p_0 \, dA = \int_c \frac{\partial p_0^*}{\partial n} p_0 \, dc - \int_c p_0^* \frac{\partial p_0}{\partial n} \, dc \quad (5.3)$$

where c is the boundary of the domain A .

If function p_0^* is chosen so as to satisfy the equation

$$\frac{\partial^2 p_0^*}{\partial x^2} + \frac{\partial^2 p_0^*}{\partial y^2} + k^2 p_0^* = \Delta i \quad (5.4)$$

where Δ_i is a Delta Dirac function at point 'i' with coordinates (x_i, y_i)

Eqn. (5.3) becomes:

$$(p_o)_i = \int_c \frac{\partial p_o^*}{\partial n} p_o \, dc - \int_c p_o^* \frac{\partial p_o}{\partial n} \, dc \quad (5.5)$$

Eqn. (5.5) is valid for any point in the domain A.

5.3 FUNDAMENTAL SOLUTION (p_o^*)

For the case of finite water reservoir, the fundamental solution was given by:

$$p_o^* = \frac{1}{4} Y_o(kr) \quad (5.6)$$

From the known asymptotic behaviour of $Y_o(kr)$ [27], it is noted that

$$p_o^* = \frac{1}{4} Y_o(kr) \approx \frac{1}{4} \sqrt{\frac{2}{\pi kr}} \sin(kr - \frac{\pi}{4}) \text{ as } r \rightarrow \infty$$

and

$$\begin{aligned} p_o^* e^{i\omega t} &\approx \frac{1}{4} \sqrt{\frac{2}{\pi kr}} \sin(kr - \frac{\pi}{4}) e^{i\omega t} \\ &\approx \frac{1}{4} \sqrt{\frac{2}{\pi kr}} \frac{1}{2i} [e^{ik(r+ct-\pi/4k)} - e^{-ik(r-ct-\pi/4k)}] \end{aligned}$$

Now, the general form $F(r-ct)$ represents a wave moving radially outwards at a constant speed c and the general form $F(r+ct)$ represents a wave moving inwards at a constant speed c [28]. The first term in the last expression for $p_o^* e^{i\omega t}$ therefore represents an inward moving wave and the second term an outward moving wave. The solution given by Eqn. (5.6) is not therefore suitable for the case of infinite water reservoir because the presence of waves coming in from infinity is unacceptable from a physical point of view.

It is known that there are two other fundamental solutions for Eqn. (5.4) [27] given by Hankel functions of zero order, first and second kind. These solutions are:

$$-\frac{i}{4} H_0^{(1)}(kr)$$

and

$$\frac{i}{4} H_0^{(2)}(kr)$$

where $H_0^{(1)}(kr)$ is Hankel function of zero order and first kind and $H_0^{(2)}(kr)$ is Hankel function of zero order and second kind. From the known asymptotic behaviour of these two solutions, it can be shown that:

$$-\frac{i}{4} H_0^{(1)}(kr) e^{i\omega t} \approx -\frac{i}{4} \sqrt{\frac{2}{\pi kr}} e^{-ik(-r+ct+\pi/4k)} \quad \text{as } r \rightarrow \infty \quad (5.7)$$

and

$$\frac{i}{4} H_0^{(2)}(kr) e^{i\omega t} \approx \frac{i}{4} \sqrt{\frac{2}{\pi kr}} e^{-ik(r-ct-\pi/4k)} \quad \text{as } r \rightarrow \infty \quad (5.8)$$

It can be seen that the fundamental solutions given by Eqns. (5.7) and (5.8) both contain only outgoing waves. The waves have a diminishing amplitude because of the $r^{-1/2}$ factor. The fundamental solution $\frac{i}{4} H_0^{(2)}(kr)$ which satisfies the physical requirements is adopted for the case of infinite water reservoir.

5.4 SOLUTION TECHNIQUE

As for the case of finite water reservoir, the equation for a point on the boundary can be written as:

$$\sum_{j=1}^N H_{ij}(p_o)_j = \sum_{j=1}^N G_{ij} \left(\frac{\partial p_o}{\partial n} \right)_j \quad (5.9)$$

where

$$\begin{aligned}
 H_{ij} &= \int_{c_j} \frac{\partial p_o^*}{\partial n} dc && \text{when } i \neq j \\
 H_{ij} &= -\frac{1}{2} + \int_{c_j} \frac{\partial p_o^*}{\partial n} dc && \text{when } i=j, \\
 G_{ij} &= \int_{c_j} p_o^* dc
 \end{aligned} \tag{5.10}$$

For the case of infinite water reservoir

$$\begin{aligned}
 p_o^* &= \frac{i}{4} H_o^{(2)}(kr) \\
 &= \frac{i}{4} J_o(kr) + \frac{1}{4} Y_o(kr)
 \end{aligned} \tag{5.11}$$

and since

$$\frac{\partial p_o^*}{\partial n} = \frac{\partial p_o^*}{\partial r} \cdot \cos \theta$$

where θ is the angle between $\underline{r}$ and the outward normal $\underline{n}$.

$$\frac{\partial p_o^*}{\partial n} = \left[\frac{i}{4} k \cos \theta J_o'(kr) + \frac{k}{4} \cos \theta Y_o'(kr) \right] \tag{5.12}$$

Substitution of Eqns. (5.11) and (5.12) into Eqn. (5.10) gives:

$$H_{ij} = \int_{c_j} \frac{i}{4} k \cos \theta J_o'(kr) dc + \int_{c_j} \frac{k}{4} \cos \theta Y_o'(kr) dc$$

where $J_o'(kr)$ and $Y_o'(kr)$ are the first derivatives of $J_o(kr)$ and $Y_o(kr)$ respectively. Transforming into ξ coordinates (Section 3.4):

$$H_{ij} = \frac{ik\ell}{8} \int_{-1}^{+1} J'_0(kr) \cos \theta \, d\xi + \frac{k\ell}{8} \int_{-1}^{+1} Y'_0(kr) \cos \theta \, d\xi$$

If a 4-point Gauss quadrature is used for calculating the integrals, the following expressions result:

$$H_{ij} = i(H_{ij})_I + (H_{ij})_R \quad (5.13)$$

where

$$(H_{ij})_I = \frac{k\ell}{8} \sum_{i=1}^4 J'_0(kr)_i \cdot (\cos \theta)_i \cdot w_i$$

$$(H_{ij})_R = \frac{k\ell}{8} \sum_{i=1}^4 Y'_0(kr)_i \cdot (\cos \theta)_i \cdot w_i$$

In a similar manner, the term G_{ij} is obtained as follows:

$$\begin{aligned} G_{ij} &= \int_{c_j} p_o^* \, dc \\ &= \int_{c_j} \left[\frac{i}{4} J_0(kr) + \frac{1}{4} Y_0(kr) \right] dc \\ &= \frac{i\ell}{8} \int_{-1}^{+1} J_0(kr) d\xi + \frac{\ell}{8} \int_{-1}^{+1} Y_0(kr) d\xi \end{aligned}$$

Use of a 4-point Gauss quadrature gives the following expressions:

$$G_{ij} = i(G_{ij})_I + (G_{ij})_R$$

where

$$(G_{ij})_I = \frac{\ell}{8} \sum_{i=1}^4 J_0(kr)_i \cdot w_i$$

$$(G_{ij})_R = \frac{\ell}{8} \sum_{i=1}^4 Y_0(kr)_i \cdot w_i$$

For the special case $i=j$

$$\begin{aligned}
 H_{ij} = H_{ii} &= -\frac{1}{2} + \int_{c_j} \frac{\partial p_o^*}{\partial n} dc \\
 &= -\frac{1}{2} + \int_{c_j} \frac{\partial p_o^*}{\partial r} \cos \theta dc \\
 &= -\frac{1}{2}
 \end{aligned} \tag{5.14}$$

because $\underline{r}$ and $\underline{n}$ are perpendicular to each other.

Eqn. (5.14) can be written in the same form as Eqn. (5.13), i.e.

$$H_{ii} = i(H_{ii})_I + (H_{ii})_R \tag{5.15}$$

where

$$(H_{ii})_I = 0.0.$$

$$(H_{ii})_R = -\frac{1}{2}$$

The term G_{ii} is obtained as follows:

$$\begin{aligned}
 G_{ii} &= \int_{c_i} p_o^* dc = \int_{c_i} \frac{i}{4} H_o^{(2)}(kr) dc \\
 &= \int_{c_i} \left[\frac{i}{4} J_o(kr) + \frac{1}{4} Y_o(kr) \right] dc
 \end{aligned}$$

For small values of (kr) , the Bessel functions J_o and Y_o can be approximated as follows [27]:

$$J_0(kr) \approx 1.0$$

$$Y_0(kr) \approx \frac{2}{\pi} \ln(kr)$$

Thus the term G_{ii} becomes:

$$G_{ii} = \frac{i}{4} \int_{c_i} 1.0 \, dc + \int_{c_i} \frac{1}{2\pi} \ln(kr) \, dc$$

The second integral on the right was evaluated earlier (Section 3.4)

and was found to be equal to

$$\int_{c_i} \frac{1}{2\pi} \ln(kr) \, dc = \frac{\ell}{2\pi} \left[\ln \left(\frac{k\ell}{2} \right) - 1 \right]$$

Hence

$$G_{ii} = i \frac{\ell}{4} + \frac{\ell}{2\pi} \left[\ln \left(\frac{k\ell}{2} \right) - 1 \right] \quad (5.16)$$

Eqn. (5.16) can be written as:

$$G_{ii} = i(G_{ii})_I + (G_{ii})_R \quad (5.17)$$

where

$$(G_{ii})_I = \frac{\ell}{4}$$

$$(G_{ii})_R = \frac{\ell}{2\pi} \left[\ln \left(\frac{k\ell}{2} \right) - 1 \right]$$

As for the case of finite water reservoir, the complete system of equations can be written in matrix form as:

$$[H][p_o] = [G][q_o] \quad (5.18)$$

where p_o and q_o are now complex and can be written in the form:

$$p_o = (p_o)_R + i(p_o)_I$$

and

$$q_o = (q_o)_R + i(q_o)_I$$

Eqn. (5.18) can be written as:

$$[[H]_R + i[H]_I][[p_o]_R + i[p_o]_I] = [[G]_R + i[G]_I][[q_o]_R + i[q_o]_I]$$

where:

(5.19)

$[H]_R$: A matrix containing the elements $(H_{ij})_R$

$[H]_I$: A matrix containing the elements $(H_{ij})_I$

$[G]_R$: A matrix containing the elements $(G_{ij})_R$

$[G]_I$: A matrix containing the elements $(G_{ij})_I$

$[p_o]_R$: A vector containing the elements $(p_o)_R$

$[p_o]_I$: A vector containing the elements $(p_o)_I$

$[q_o]_R$: A vector containing the elements $(q_o)_R$

$[q_o]_I$: A vector containing the elements $(q_o)_I$

On carrying out the matrix multiplication contained in Eqn. (5.19) and

separating the real and the imaginary parts, the following matrix equations are obtained:

$$[H]_R [p_o]_R - [H]_I [p_o]_I = [G]_R [q_o]_R - [G]_I [q_o]_I \quad (5.20)$$

and

$$[H]_I [p_o]_R + [H]_R [p_o]_I = [G]_I [q_o]_R + [G]_R [q_o]_I \quad (5.21)$$

The two systems of Eqns. (5.20) and (5.21) can be written as:

$$\begin{bmatrix} [H]_R & -[H]_I \\ [H]_I & [H]_R \end{bmatrix} \begin{bmatrix} [p_o]_R \\ [p_o]_I \end{bmatrix} = \begin{bmatrix} [G]_R & -[G]_I \\ [G]_I & [G]_R \end{bmatrix} \begin{bmatrix} [q_o]_R \\ [q_o]_I \end{bmatrix} \quad (5.22)$$

The system of Eqns. (5.22) can be written in a more compact form as:

$$[H'] \{p_o'\} = [G'] \{q_o'\} \quad (5.23)$$

where

$$[H'] = \begin{bmatrix} [H]_R & -[H]_I \\ [H]_I & [H]_R \end{bmatrix}, \quad [G'] = \begin{bmatrix} [G]_R & -[G]_I \\ [G]_I & [G]_R \end{bmatrix}$$

$$\{p_o'\} = \begin{bmatrix} [p_o]_R \\ [p_o]_I \end{bmatrix}, \quad \{q_o'\} = \begin{bmatrix} [q_o]_R \\ [q_o]_I \end{bmatrix}$$

On solving the system of Eqns. (5.23), the unknown values of $(p_o)_R$ and $(p_o)_I$ at points on the upstream face of the dam can be obtained. The hydrodynamic pressure (p) at any point on the upstream face of the dam can thus be expressed as follows:

$$\begin{aligned} p &= p_o e^{i\omega t} \\ &= [(p_o)_R + i(p_o)_I][\cos \omega t + i \sin \omega t] \\ &= [(p_o)_R \cos \omega t - (p_o)_I \sin \omega t] + i[(p_o)_I \cos \omega t + (p_o)_R \sin \omega t] \end{aligned}$$

Therefore, if the ground acceleration is a $\cos \omega t$ function the hydrodynamic pressure p is given by:

$$p = (p_o)_R \cos \omega t - (p_o)_I \sin \omega t \quad (5.24)$$

which can be expressed as

$$p = \sqrt{(p_o)_R^2 + (p_o)_I^2} \cos(\omega t + \phi)$$

where

$$\tan \phi = \frac{(p_o)_I}{(p_o)_R}$$

Similarly, if the ground acceleration is a $\sin \omega t$ function, the hydrodynamic pressure p is given by

$$p = (p_o)_I \cos \omega t + (p_o)_R \sin \omega t$$

which can be expressed as

$$p = \sqrt{(p_o)_R^2 + (p_o)_I^2} \sin(\omega t + \phi)$$

where again

$$\tan \phi = \frac{(p_o)_I}{(p_o)_R}$$

It is clear from these expressions that the maximum pressure which is equal to $\sqrt{(p_o)_R^2 + (p_o)_I^2}$ will not occur at the same time for all points on the upstream face of the dam.

5.5 BOUNDARY CONDITIONS

For the case of infinite water reservoir, the following boundary conditions should be satisfied (Fig. 5.1).

1. The hydrodynamic pressure p should vanish at the upper surface of water, i.e.

$$p(x, H, t) = p_o e^{i\omega t} = 0.0$$

or

$$p_o = 0.0$$

2. For the case of horizontal ground acceleration, the vertical acceleration of the particles of water at the base of the reservoir should vanish, i.e.

$$\left(\frac{\partial^2 v}{\partial t^2}\right)(x, 0, t) = + \frac{1}{\rho} \frac{\partial p}{\partial n} = 0.0$$

FIG. 5.1. INFINITE WATER RESERVOIR.

or

$$\frac{\partial p_o}{\partial n} = 0.0$$

3. The horizontal acceleration of the particles of water at the upstream face of the dam should be equal to the horizontal ground acceleration, i.e.

$$\left(\frac{1}{\rho} \frac{\partial p}{\partial n}\right)_{(0,y,t)} = \left(\frac{\partial^2 u_g}{\partial t^2}\right)$$

If

$$\left(\frac{\partial^2 u_g}{\partial t^2}\right) = e^{i\omega t}$$

this condition gives

$$\frac{\partial p_o}{\partial n} = \rho$$

4. The hydrodynamic pressure p should vanish at infinity, i.e.

$$p(\infty, y, t) = p_o e^{i\omega t} = 0.0$$

or

$$p_o(\infty, y, t) = 0.0$$

To satisfy this last condition, the following solutions were tried:

i) Omitting of the Boundary at Infinity

Referring to Fig. 5.2, for any point on the boundary:

$$\frac{1}{2} (p_o) = \int_c p_o \frac{\partial p_o^*}{\partial n} dc - \int_c p_o^* \frac{\partial p_o}{\partial n} dc \quad (5.25)$$

Since the fundamental solution p_o^* given by Eqn. (5.8) tends to zero as (kr) tends to infinity, the integrations on the right-hand side of Eqn. (5.25) need not cover the boundary at infinity. The integration may be restricted to a finite region such as ABCD shown in Fig. 5.2.

ii) Application of Radiation Condition to the Boundary at Infinity

Let the domain be divided into two parts, A which is finite and A' which extends to infinity, as shown in Fig. 5.3. The two regions are connected at the interface L. Now, for any point in domain A',

$$(p_o)_i = \int_c \frac{\partial p_o^*}{\partial n} p_o dc - \int_c p_o^* \frac{\partial p_o}{\partial n} dc \quad (5.26)$$

If we take the point 'i' to be far from our problem region and neglect the integration over the boundaries MP, PO and ON as in case i, we may make the following simplification using the asymptotic form for $H_o^{(2)}(kr)$ [12]. Hence,

$$H_o^{(2)'}(kr) = -k H_1^{(2)}(kr) \approx -ik \sqrt{\frac{2}{\pi kr}} e^{-i(kr-\pi/4)}$$

where $H_1^{(2)}(kr)$ is the Hankel function of order 1 and of second kind.

Then, Eqn. (5.26) becomes:

FIG. 5.2. INTEGRATION OVER FINITE REGION. SOLUTION "I".

FIG. 5.3. DIVIDING THE INFINITE DOMAIN. SOLUTION "II".

$$0.0 = \int_L \frac{k}{4} \sqrt{\frac{2}{\pi k r}} e^{-i(kr-\pi/4)} \cdot p_o \, dL$$

$$- \frac{i}{4} \int_C \sqrt{\frac{2}{\pi k r}} e^{-i(kr-\pi/4)} \frac{\partial p_o}{\partial n} \, dL$$

or

$$p_o k = \frac{\partial p_o}{\partial n} i$$

or

$$\frac{\partial p_o}{\partial n} = - i k p_o \quad (5.27)$$

Eqn. (5.27) is known as Sommerfeld or radiation condition. To apply the radiation condition to the system of Eqns. (5.23), the boundary of domain A is divided into two parts, δA and δB , as shown in Fig. 5.4. Taking into account the direction of the outward normal to δB when considered as a part of the boundary of domain A, Eqn. (5.27) becomes:

$$\frac{\partial p_o}{\partial n} = i k p_o \quad (5.28)$$

The system of Eqns. (5.23) will take the form:

$$\begin{bmatrix} H_{AA} & H_{AB} \\ H_{BA} & H_{BB} \end{bmatrix} \begin{bmatrix} (p_o)_A \\ (p_o)_B \end{bmatrix} = \begin{bmatrix} G_{AA} & G_{AB} \\ G_{BA} & G_{BB} \end{bmatrix} \begin{bmatrix} (q_o)_A \\ (q_o)_B \end{bmatrix} \quad (5.29)$$

where H_{AA} is a matrix, the elements of which are obtained when the boundary point 'i' is on δA and the integration is carried out on the

FIG. 5.4.

boundary elements of δA .

H_{AB} is a matrix, the elements of which are obtained when the boundary point 'i' is on the boundary δA and the integration is carried out on the boundary elements of δB .

H_{BA} is a matrix, the elements of which are obtained when the boundary point 'i' is on the boundary δB and the integration is carried out on the boundary elements of δA .

H_{BB} is a matrix, the elements of which are obtained when the boundary point 'i' is on the boundary δB and the integration is carried out on the boundary elements of δB .

$(p_o)_A$ and $(p_o)_B$ are the values of (p_o) for the boundary elements of δA and δB respectively.

Submatrices G_{AA} , G_{AB} , G_{BA} , G_{BB} , $(q_o)_A$ and $(q_o)_B$ are defined in a similar manner. On carrying out the matrix multiplication contained in Eqns. (5.29) and making use of Eqn. (5.28), the following equations are obtained.

$$H_{AA} \cdot (p_o)_A = -H_{AB} \cdot (q_o)_B \cdot \frac{1}{ik} + G_{AA} \cdot (q_o)_A + G_{AB} \cdot (q_o)_B$$

$$H_{BA} \cdot (p_o)_B = -H_{BB} \cdot (q_o)_B \cdot \frac{1}{ik} + G_{BA} \cdot (q_o)_A + G_{BB} \cdot (q_o)_B$$

In matrix form, these equations become:

$$\begin{bmatrix} H_{AA} & H_{AB} \\ H_{BA} & H_{BB} \end{bmatrix} \begin{bmatrix} (p_o)_A \\ 0 \end{bmatrix} = \begin{bmatrix} G_{AA} & G_{AB} - \frac{1}{ik} H_{AB} \\ G_{BA} & G_{BB} - \frac{1}{ik} H_{BB} \end{bmatrix} \begin{bmatrix} (q_o)_A \\ (q_o)_B \end{bmatrix} \quad (5.30)$$

7

On setting

$$[H_{AA}] = [H_{AA}]_R + i[H_{AA}]_I$$

$$[H_{AB}] = [H_{AB}]_R + i[H_{AB}]_I$$

$$[H_{BA}] = [H_{BA}]_R + i[H_{BA}]_I$$

$$[H_{BB}] = [H_{BB}]_R + i[H_{BB}]_I$$

$$[G_{AA}] = [G_{AA}]_R + i[G_{AA}]_I$$

$$[G_{AB}] = [G_{AB}]_R + i[G_{AB}]_I$$

$$[G_{BA}] = [G_{BA}]_R + i[G_{BA}]_I$$

$$[G_{BB}] = [G_{BB}]_R + i[G_{BB}]_I$$

$$[(p_o)_A] = [(p_o)_A]_R + i[(p_o)_A]_I$$

and separating the real and imaginary parts, Eqn. (5.30) can be written in their expanded form as follows.

$$\begin{bmatrix} [H_{AA}]_R & [H_{AB}]_R & -[H_{AA}]_I & -[H_{AB}]_I \\ [H_{BA}]_R & [H_{BB}]_R & -[H_{BA}]_I & -[H_{BB}]_I \\ [H_{AA}]_I & [H_{AB}]_I & [H_{AA}]_R & [H_{AB}]_R \\ [H_{BA}]_I & [H_{BB}]_I & [H_{BA}]_R & [H_{BB}]_R \end{bmatrix} \cdot \begin{bmatrix} [(p_o)_A]_R \\ 0 \\ [(p_o)_A]_I \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix}
 [G_{AB}]_R & -\frac{1}{k}[H_{AB}]_I & -[G_{AA}]_I & -[G_{AB}]_I & [(q_o)_A]_R \\
 [G_{BA}]_R & [G_{BB}]_R & -[G_{BA}]_I & -\frac{1}{k}[H_{BB}]_R & [(q_o)_B]_R \\
 [G_{AA}]_I & \frac{1}{k}[H_{AB}]_R & [G_{AA}]_R & [G_{AB}]_R & [(q_o)_A]_I \\
 [G_{BA}]_I & \frac{1}{k}[H_{BB}]_R & [G_{BA}]_R & [G_{BB}]_R & [(q_o)_B]_I
 \end{bmatrix}
 \begin{bmatrix}
 [G_{AB}]_R \\
 -\frac{1}{k}[H_{AB}]_I \\
 [G_{AA}]_I \\
 -[G_{AB}]_I \\
 [G_{BA}]_R \\
 [G_{BB}]_R \\
 -\frac{1}{k}[H_{BB}]_I \\
 [G_{AA}]_R \\
 [G_{AB}]_R \\
 -\frac{1}{k}[H_{AB}]_I \\
 [G_{BA}]_R \\
 [G_{BB}]_R \\
 -\frac{1}{k}[H_{BB}]_I
 \end{bmatrix}
 \begin{bmatrix}
 [(q_o)_A]_R \\
 [(q_o)_B]_R \\
 [(q_o)_A]_I \\
 [(q_o)_B]_I
 \end{bmatrix}
 \quad (5.31)$$

So, terms in Eqn. (5.31) are now real. Solving the system of Eqns. (5.31) the hydrodynamic pressure on the whole boundary can be determined.

5.6 RESULTS OF INVESTIGATION AND COMMENTS

A rigid dam of height 100ms was analyzed by the two procedures outlined in Sections 5.5 (i) and (ii) respectively. Several different values for the ratio ω/ω_1 were considered. The results obtained by the boundary element method are presented along with those obtained from the known classical solutions in Figs. 5.5 through 5.12. The agreement between the two sets of results is found to be quite good.

FIG. 5.5. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR SOLUTION ("I").

FIG. 5.6. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "I").

FIG. 5.7. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "I").

FIG. 5.8. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "I").

FIG. 5.9. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "II").

FIG. 5.10: HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "I").

FIG. 5.11. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "II").

FIG. 5.12. HYDRODYNAMIC PRESSURE IN INFINITE WATER RESERVOIR (SOLUTION "II").

In order to examine the problem of exterior domains discussed earlier in Chapter 4, the total hydrodynamic pressure force, F_{dynamic} , was obtained for different values of the ratio ω/ω_1 and compared with the values obtained from classical solutions. The results are presented in Fig. 5.13. For the case of infinite water reservoir, the problem of exterior domains does not appear for the range of the ratio ω/ω_1 for which the analysis was carried out.

The following conclusions are drawn from the study presented in this chapter:

- i) The application of the boundary element method to infinite domains reveals that it is a powerful tool for the solution of problems containing infinite domains.
 - ii) For the problem discussed here, it is found that it is easier to use the solution omitting the infinite boundary because this procedure gives satisfactory results with an easier formulation.
 - iii) It is found that as the value of the ratio ω/ω_1 increases the convergence to the exact classical solution becomes better.
 - iv) Classical solutions are not available for the cases of complicated geometry of the water reservoir. However, the boundary element method can be successfully applied in such situations.
- It is, in fact, only in such situations that there is a real advantage in applying the boundary element method. The reason the results presented here are all for simple geometry of

FIG. 5.13. TOTAL HYDRODYNAMIC PRESSURE FORCE.

reservoir is that only for these cases the boundary solutions can be compared with exact solutions.

CHAPTER 6

DYNAMICS OF GRAVITY DAM-RESERVOIR SYSTEM

6.1 INTRODUCTION

As mentioned earlier, the boundary element method can be combined with other numerical methods such as the finite element method or the finite difference method, to deal with problems where neither the boundary element method nor other methods are appropriate by themselves.

It is evident from the results presented in Chapter 5 that the boundary element method is appropriate for dealing with infinite domains such as, for example, an infinite water reservoir. On the other hand, a finite domain such as a dam can be quite efficiently analyzed by the finite element method which can also take into account any non-homogeneity in the material properties.

In the present chapter, the problem of the complete dam-reservoir system is solved using finite element idealization for the dam and boundary element idealization for the reservoir.

It is known that the ground motion and the deformation of the upstream face of a dam will cause hydrodynamic pressures in the reservoir, and the structural deformation in turn will be affected by the hydrodynamic pressures on the upstream face of the dam. To account for this closed cycle of cause and effect, the problem formulation must recognize the dynamic interaction between the dam and the reservoir.

The fundamental idea behind the method of analysis presented in this chapter is to treat the dam and the water in the reservoir as separate sub-structures. The analysis procedure then couples the sub-structures by taking account of the forces interacting between the reservoir and the face of the dam. The analysis presented here deals with the calculation of hydrodynamic pressure on the dam face due to a horizontal harmonic excitation of dam and reservoir base.

6.2 ASSUMPTIONS

The following assumptions, many of which have been discussed earlier in Chapters 4 and 5, are made in the analysis.

1. The motion of the dam-reservoir system is considered as two-dimensional, i.e. assumed as being the same for any vertical plane perpendicular to the axis of the dam.
2. The reservoir is assumed to extend to infinity in the upstream direction.
3. Water is assumed to be compressible but its internal viscosity is neglected.
4. The effect of waves at the free surface of the reservoir is ignored.
5. The material of the dam is assumed to be linearly elastic. This assumption is appropriate for structures such as gravity dams which should be designed to withstand major earthquakes with little or no damage.

6. The dam cross-section is idealized as a plane-stress system.

In the present study, no restriction is imposed on the shape of both the upstream face of the dam and the bed of the reservoir.

6.3 FINITE ELEMENT DISCRETIZATION OF THE DAM CROSS-SECTION

The basic concept of the finite element method of discretization is the idealization of an elastic continuum as an assemblage of discrete elements interconnected at their nodal points. A cross-section of a concrete gravity dam is thus represented by a number of finite elements. Within an element, the displacements are assumed to vary according to appropriately chosen displacement functions. Material properties are defined individually for each element in the assemblage. Consequently, dam cross-section of arbitrary shape with arbitrary variation of material properties can be represented as an assemblage of finite elements.

The element stiffness matrix which is a function of the geometry and the constitutive properties of the element may be calculated by standard procedures once the displacement field has been specified [3,35]. The stiffness matrix $[k]$ of the complete structural assemblage is obtained from the individual elements stiffness matrices. The structural stiffness matrix can be easily arranged in banded form so as to improve the efficiency of the solution procedure.

The mass matrix, required for the dynamic analysis, may be approximated by 'lumping' the total mass of an element at its nodes. A lumped mass approximation leads to a diagonal mass matrix $[M]$ for the

total structure and provides results having a desired degree of accuracy with less computational effort and computer storage. In the present work, constant strain triangles are selected as the finite elements.

For the triangular element shown in Fig. 6.1, the lumped mass matrix is given by:

$$[M]^e = \frac{\rho t \Delta}{3} \begin{bmatrix} I & 0 & 0 \\ \text{Sym.} & I & 0 \\ & & I \end{bmatrix}$$

and the stiffness matrix is given by:

$$[k]^e = \Delta t [A^{-1}]^T [B]^T [D] [B] [A^{-1}]$$

where

ρ is the mass density of the material of the dam,

t is the thickness of the element

Δ is the area of the element

I is a 2x2 identity matrix

$[M]^e$ is the lumped mass matrix of the element

$[k]^e$ is the stiffness matrix of the element

$[A]$, $[B]$ and $[D]$ are matrices given by:

$$[A] = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}$$

FIG. 6.1. TRIANGULAR FINITE ELEMENT.

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

where E and ν are Young's modulus and Poisson's ratio of the material of the dam, respectively.

6.4 EQUATIONS OF MOTION FOR THE DAM-RESERVOIR SYSTEM

A gravity dam of height H_s storing a reservoir of depth H is shown in Fig. 6.2.

The equations of motion for a dam, idealized as a two-dimensional finite element system, subjected to transverse and vertical components of earthquake ground motion, and including hydrodynamic effects, are:

$$[M]\{\ddot{r}\} + [C]\{\dot{r}\} + [k]\{r\} = -\{E\}^h a_g^h(t) - \{E\}^v a_g^v(t) + \{R(t)\} \quad (6.1)$$

In the above equation, $\{r\}$ is the vector of nodal displacements relative to the ground, and

$$\{r\}^T = \{r_1^x, r_1^y, r_2^x, r_2^y, \dots, r_n^x, r_n^y, \dots, r_N^x, r_N^y\}$$

where r_n^x and r_n^y are respectively the x and y components of the displacement of nodal point ' n ' and N is the number of nodal points above the base in the finite element idealization.

a) Horizontal Ground Motion.

b) Vertical Ground Motion.

FIG. 6.2. DAM RESERVOIR SYSTEM.

$\{\dot{r}\}$ and $\{\ddot{r}\}$ are respectively the nodal velocity and acceleration vectors and $[M]$, $[C]$ and $[k]$ are respectively the mass, damping and stiffness matrices for the finite element system. The vectors $\{E^h\}$ and $\{E^v\}$ are defined by:

$$\{E^h\}^T = \{M_1 \ 0 \ M_2 \ 0 \dots M_n \ 0 \dots M_N \ 0\}$$

$$\{E^v\}^T = \{0 \ M_1 \ 0 \ M_2 \ \dots \ 0 \ M_n \ \dots \ 0 \ M_N\}$$

where M_n is the mass lumped at nodal point 'n'. $a_g^h(t)$ and $a_g^v(t)$ are respectively the horizontal and vertical components of earthquake ground acceleration. $\{R(t)\}$ in Eqn. (6.1) is the vector of nodal loads associated with the hydrodynamic pressures. Because these pressures act only on the upstream face, all elements in $\{R(t)\}$ except those corresponding to displacements of the nodal points on the upstream face are zero.

6.5 MODE SHAPES AND FREQUENCIES OF THE DAM

The mode shapes and frequencies of a structural system are obtained by solving the undamped free vibration equations of motion given by:

$$[M]\{\ddot{r}\} + [k]\{r\} = 0$$

These equations lead to the familiar eigenvalue problem:

$$(-\omega^2[M] + [k])\{\phi\} = 0 \quad (6.2)$$

in which ω represents a natural frequency or an eigenvalue, and $\{\phi\}$ represents a mode shape or an eigenvector. For a system with N effective

degrees of freedom, there are N independent real and positive values of ω and N corresponding mode shapes. In the dynamic analysis of structures, one is usually interested only in the first few modes, starting with the one which has the lowest value of frequency, referred to as the fundamental mode. The natural frequencies $\{\omega\}$ and mode shapes $\{\phi\}$ of vibration may be computed by various techniques [36,38]. The particular method to be followed depends on the size and properties of the matrices involved.

In the present work, the mode shapes and frequencies of the dam are obtained by first transforming the banded stiffness matrix into a tridiagonal matrix using the Lanczos method [39] and then using a standard program for finding the eigenvalues and eigenvectors of a tridiagonal matrix [36].

In the generalized Lanczos method, the form of the eigenvalue problem is assumed to be:

$$[k]\{\phi\} = \lambda[M]\{\phi\} \quad (6.3)$$

in which $[k]$ and $[M]$ are real, symmetric matrices, and $[M]$ is positive-definite. The vector $\{\phi\}$ represents an eigenvector, and λ is the corresponding eigenvalue. The method is based on the assumption that a transformation matrix $[R]$ can be found such that the following relationship holds:

$$[R]^T[k][R] = [T] \quad (6.4)$$

and

$$[R]^T[M][R] = [I] \quad (6.5)$$

where $[T]$ is a symmetric tridiagonal matrix and $[I]$ is the identity matrix. Eqn. (6.5) states that the matrix $[R]$ is orthogonal with respect to $[M]$.

Introducing a new eigenvector $\{\phi'\}$ defined by:

$$[R]\{\phi'\} = \{\phi\} \quad (6.6)$$

and using Eqns. (6.4) and (6.5), Eqn. (6.3) is reduced to:

$$[R]^T[k][R]\{\phi'\} = \lambda[R]^T[M][R]\{\phi'\} \quad (6.7)$$

or

$$[T]\{\phi'\} = \{\phi'\}\lambda \quad (6.8)$$

Pre-multiplying Eqns. (6.4) and (6.5) by $[R]^T$ and combining:

$$[k][R] = [M][R][T] \quad (6.9)$$

Let $[R]$ be defined in terms of its component vectors as:

$$[R] = [r_1 \quad r_2 \quad \dots \quad r_n] \quad (6.10)$$

and let the tridiagonal matrix $[T]$ be of the form:

$$[T] = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ & \alpha_2 & \beta_2 & & \\ \beta_1 & & & & \\ & \beta_2 & & & \\ & & \beta_{n-1} & & \\ & & & \alpha_n & \end{bmatrix} \quad (6.11)$$

Substitution of Eqns. (6.10) and (6.11) into Eqn. (6.9) gives the following expression for the i -th vector of $[R]$:

$$[k]\{r_i\} = \beta_{i-1}[M]\{r_{i-1}\} + \alpha_i[M]\{r_i\} + \beta_i[M]\{r_{i+1}\} \dots \quad (6.12)$$

Pre-multiplication of Eqn. (6.12) by $\{r_i\}^T$ gives:

$$\alpha_i = \{r_i\}^T [k] \{r_i\} \quad (6.13)$$

because the vectors of $[R]$ are orthogonal with respect to $[M]$. At this stage α_i can be computed if $\{r_i\}$ is known. Re-arrangement of terms in Eqn. (6.12) gives:

$$-\beta_i[M]\{r_{i+1}\} = \{v_i\} \quad (6.14)$$

where

$$\{v_i\} = [k]\{r_i\} - \beta_{i-1}[M]\{r_{i-1}\} - \alpha_i[M]\{r_i\} \quad (6.15)$$

Eqn. (6.14) represents a recursive process where the unknown on the left-hand side is defined in terms of its values at the previous step.

Recursion is commenced by taking the following value for r_1 :

$$[r_1]^T = \left[\frac{1}{\sqrt{M_{11}}} \quad 0 \quad 0 \quad 0 \dots\dots\dots 0 \right] \quad (6.16)$$

The value of β_0 is obviously zero.

Now, since $[M]$ is positive-definite and symmetric, it can be factored into the product of a lower and an upper triangular matrix

where the two component matrices are the transpose of each other. Thus,

$$[M] = [u^T][u] \quad (6.17)$$

Substitution of Eqn. (6.17) into Eqn. (6.14) gives:

$$\beta_i [u^T][u] \{r_{i+1}\} = \{v_i\} \quad (6.18)$$

Setting

$$\{Z_i\} = \beta_i [u] \{r_{i+1}\} \quad (6.19)$$

a solution for $\{Z_i\}$ is obtained from Eqn. (6.18) through a forward-elimination process represented by:

$$[u^T]\{Z_i\} = \{v_i\} \quad (6.20)$$

For the purpose of computing β_i , consider the result of the inner product of Z_i with its transpose,

$$\{Z_i\}^T \{Z_i\} = \beta_i^2 \{r_{i+1}\}^T [u^T][u] \{r_{i+1}\} = \beta_i^2 \quad (6.21)$$

Thus, β_i is computed as:

$$\beta_i = ([\{Z_i\}^T \{Z_i\}])^{1/2} \quad (6.22)$$

Division of Eqn. (6.19) by β_i yields an expression from which $\{r_{i+1}\}$ may be obtained through a backward-substitution process. That is,

$$[u] \{r_{i+1}\} = \frac{1}{\beta_i} \{Z_i\}$$

The process is repeated until all of the elements of $[T]$ and the vectors

of $[R]$ are computed.

Having obtained the tridiagonal matrix $[T]$ and the matrix $[R]$, the eigenvalues and eigenvectors of the problem are evaluated by the well-known Q-R algorithm [40].

6.6 RESPONSE TO GROUND MOTION

The earthquake response of the dam and the reservoir is obtained by solving the equations of motion (Eqn. (6.1)). Two different methods of analysis can be used:

- (i) Time Domain Analysis
- (ii) Frequency Domain Analysis.

Application of the boundary element method to determine the hydrodynamic pressure caused by the reservoir requires that the excitation be harmonic. This implies that the analysis of the reservoir-dam system can be carried out only in the frequency domain. In this method of analysis, the ground motion is resolved into its harmonic components by using the Fourier transform theory. The response to ~~each~~ harmonic component is determined next. Superposition of the harmonic responses then gives the total response. The superposition is carried out by using the inverse Fourier transform. Thus, if $x(t)$ is an arbitrary non-periodic exciting function it can be represented as an infinite sum of harmonic components. The amplitude of the component with frequency ω is given by the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \quad (6.23)$$

Multiplying this by the complex frequency response function $H(\omega)$ therefore yields the amplitude of the response component at frequency ω . The total response is obtained by summing these response components over the entire range of frequency. Mathematically, the response $y(t)$ is given by the inverse Fourier transform:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) X(\omega) e^{i\omega t} d\omega \quad (6.24)$$

In general, the integrals represented by Eqns. (6.23) and (6.24) must be evaluated by numerical techniques. Development of the Fast Fourier techniques [41] has made the numerical evaluation of the above integrals highly efficient increasing the speed of computation by several orders of magnitude. The analysis in the frequency domain is, in many cases, equally or more efficient than an analysis in the time domain. Of course, for the frequency domain analysis to be valid, the system must be linear.

The present work is limited to the evaluation of the complex frequency response functions of the reservoir-dam system subjected to a harmonic excitation. The details of the procedure used are described in the following sections.

6.7 RESPONSE TO HARMONIC EXCITATION

Although Eqns. (6.1) can be solved in their original form, it is computationally more efficient to transform them using the coordinates represented by the mode shapes of the dam without the reservoir. The transformation is given by:

$$\{r(t)\} = \sum_{m=1}^{2N} \phi_m Y_m(t) \quad (6.25)$$

where Y_m is the m th generalized coordinate. It should be noted that Eqn. (6.25) is exact if all $2N$ mode shapes are used because they span a complete space of dimension $2N$. In general, however, reasonable accuracy can be obtained by using the first few modes.

Provided the damping matrix c is such that

$$\phi_r^T c \phi_s = 0 \quad \text{for } r \neq s,$$

the transformation gives $2N$ equations of the form:

$$M_m^* \ddot{Y}_m(t) + c_m^* \dot{Y}_m(t) + k_m^* Y_m(t) = -\{\phi_m\}^T \{E^h\} a_g^h(t) + \{\phi_m\}^T \{R(t)\} \quad (6.26)$$

where it is assumed that the vertical ground acceleration is zero and

$$M_m^* = \{\phi_m\}^T [m] \{\phi_m\}$$

$$c_m^* = \{\phi_m\}^T [c] \{\phi_m\} = 2 \xi_m \omega_m M_m^*$$

$$k_m^* = \{\phi_m\}^T [k] \{\phi_m\} = \omega_m^2 M_m^*$$

ω_m = natural circular frequency in the m th mode

ϕ_m = m th mode shape, and

ξ_m = damping ratio in the m th mode.

If the ground acceleration $a_g^h(t)$ is equal to $e^{i\omega t}$, the generalized displacements and accelerations can be expressed as follows:

$$Y_m(t) = \bar{Y}_m(\omega) e^{i\omega t} \quad (6.27)$$

$$\ddot{Y}_m(t) = \ddot{\bar{Y}}_m(\omega) e^{i\omega t} = -\omega^2 \bar{Y}_m(\omega) e^{i\omega t}$$

where $\bar{Y}_m(\omega)$ and $\ddot{\bar{Y}}_m(\omega)$ are the complex frequency response functions for displacement and acceleration, respectively.

The horizontal and vertical accelerations of the nodes on the upstream face of the dam are given by:

$$\begin{aligned} \{\ddot{u}(t)\} &= \{\hat{1} + \sum_{k=1}^L \phi_k^{fh} \ddot{\bar{Y}}_k(\omega)\} e^{i\omega t} \\ \{\ddot{v}(t)\} &= \left\{ \sum_{k=1}^L \phi_k^{fv} \ddot{\bar{Y}}_k(\omega) \right\} e^{i\omega t} \end{aligned} \quad (6.28)$$

in which $\hat{1}$ is a vector with all elements equal to unity, ϕ_k^{fh} represents those components of the k th mode which lie on the upstream face of the dam and are oriented in a horizontal direction, and ϕ_k^{fv} represents those components of the k th mode which lie on the upstream face of the dam and are oriented in a vertical direction.

The hydrodynamic pressure due to the accelerations given by Eqn. (6.28) can be obtained by using the Boundary Element Method. Because the governing differential equation for the motion of water as well as the boundary conditions are linear, the principle of superposition applies and the hydrodynamic pressures can be expressed as:

$$\{p(t)\} = \{p_o(t)\} + \sum_{k=1}^L \ddot{\bar{Y}}_k \{p_k(t)\} \quad (6.29)$$

where $\{p(t)\}$ is a vector of the nodal pressures on the upstream face.

In Eqn. (6.29), the vector $\{p_0(t)\}$ is obtained from a solution of the wave equation for the following boundary conditions:

$$\frac{\partial p}{\partial n}(x, 0, t) = 0$$

$$p(x, H, t) = 0$$

and along the ~~face~~ of the dam

$$\frac{\partial p}{\partial n} = \frac{w}{g} e^{i\omega t} \cos \theta$$

where θ is the angle that the upstream face of the dam makes with the vertical.

Vector $\{p_k(t)\}$ is obtained in a similar manner except that the boundary conditions along the face of the dam are now:

$$\frac{\partial p}{\partial n} = \frac{w}{g} [\phi_k^{fh}(y) \cos \theta + \phi_k^{fv} \sin \theta] e^{i\omega t}$$

Vectors $\{p_0(t)\}$ and $\{p_k(t)\}$ can be expressed in terms of their complex frequency response functions as follows:

$$\{p_0(t)\} = \{\bar{p}_0(\omega)\} e^{i\omega t}$$

and

$$\{p_k(t)\} = \{\bar{p}_k(\omega)\} e^{i\omega t}$$

Thus, $\bar{p}_0(\omega)$ is the complex frequency response function for the hydrodynamic pressures when the excitation is the acceleration of the ground and the dam is rigid and $\bar{p}_k$ is the corresponding function when the excitation is the acceleration of the dam in the k th mode of vibration

and there is no motion of the base. Both $\bar{p}_0(\omega)$ and $\bar{p}_k(\omega)$ are easily computed by using the Boundary Element Method.

The nodal force vector $\{R(t)\}$ can be expressed as follows:

$$\begin{aligned}\{R(t)\} &= \{R_0(t)\} + \sum_{k=1}^L \{R_k(t)\} \bar{Y}_k \\ &= \{\bar{R}_0(\omega)\} e^{i\omega t} + \sum_{k=1}^L \bar{Y}_k \{\bar{R}_k(\omega)\} e^{i\omega t}\end{aligned}\quad (6.30)$$

The vectors $\{\bar{R}_0(\omega)\}$ and $\{\bar{R}_k(\omega)\}$ are the static equivalents of the corresponding pressures $\{\bar{p}_0(\omega)\}$ and $\{\bar{p}_k(\omega)\}$ respectively. They may be computed directly by using the principle of virtual work assuming linear variation of pressure between the nodal points. Thus, in Fig. 6.3,

let $i-1$, i and $i+1$ represent three consecutive points on the face of the dam and the pressure at these points be p^{i-1} , p^i and p^{i+1} respectively.

Then it can be easily proved that

$$R_x^i = - \left[\left(\frac{1}{3} p_o^i + \frac{1}{6} p_o^{i-1} \right) \ell_i + \left(\frac{1}{3} p_o^i + \frac{1}{6} p_o^{i+1} \right) \ell_{i+1} \right] \cos \theta \quad (6.31)$$

$$R_y^i = - \left[\left(\frac{1}{3} p_o^i + \frac{1}{6} p_o^{i-1} \right) \ell_i + \left(\frac{1}{3} p_o^i + \frac{1}{6} p_o^{i+1} \right) \ell_{i+1} \right] \sin \theta$$

Thus, when the excitation $a_g^h(t)$ is equal to $e^{i\omega t}$, Eqn. (6.26) becomes:

$$\begin{aligned}& [-\omega^2 M_m^* + i\omega c_m^* + k_m^*] \bar{Y}_m(\omega) \\ &= -\{\phi_m\}^T \{E^h\} + \{\phi_m\}^T (\{\bar{R}_0(\omega)\} - \omega^2 \sum_{k=1}^L \bar{Y}_k \bar{R}_k(\omega))\end{aligned}\quad (6.32)$$

These are L equations of the form 6.32. They can be re-arranged and expressed in matrix form as:

FIG. 6.3. NODAL PRESSURE FORCES.

$$\begin{bmatrix} S_{11}(\omega) & S_{12}(\omega) & S_{1L}(\omega) \\ S_{21}(\omega) & S_{22}(\omega) & S_{2L}(\omega) \\ \hline S_{L1}(\omega) & S_{L2}(\omega) & S_{LL}(\omega) \end{bmatrix} \begin{bmatrix} \bar{Y}_1(\omega) \\ \bar{Y}_2(\omega) \\ \hline \bar{Y}_L(\omega) \end{bmatrix} = \begin{bmatrix} B_1(\omega) \\ B_2(\omega) \\ \hline B_L(\omega) \end{bmatrix} \quad (6.33)$$

or

$$[S]\{\bar{Y}\} = \{B\} \quad (6.34)$$

in which

$$S_{mk}(\omega) = \omega^2 \{\phi_m\}^T \{\bar{R}_k(\omega)\}; \quad m \neq k \quad (m=1, 2, \dots, L)$$

$$S_{mm}(\omega) = [-\omega^2 M_m^* + i\omega c_m^* + k_m^*] + \omega^2 \{\phi_m\}^T \{\bar{R}_m(\omega)\} \quad (k=1, 2, \dots, L)$$

$$B_m(\omega) = -\{\phi_m\}^T \{E^h\} + \{\phi_m\}^T \{\bar{R}_0(\omega)\}$$

The coefficient matrix $S(\omega)$ in Eqn. (6.34) is a frequency-dependent matrix which relates the generalized displacement vector $\bar{Y}(\omega)$ with the corresponding generalized loads $B(\omega)$. Unlike in the classical modal matrix, the matrix $S(\omega)$ is not diagonal because the displacement vector ϕ_m is not the normal mode of the dam reservoir system; it is simply the mode of the dam with the reservoir empty.

6.8 SOLUTION TECHNIQUE

A technique for solving Eqn. (6.34) by separating the real and the imaginary parts of the elements of $[S]$, $\{\bar{Y}\}$ and $\{B\}$ is described in this section. Let these matrices be expressed as follows:

$$[S] = [S_R] + i[S_I]$$

$$\{\bar{Y}\} = \{\bar{Y}_R\} + i\{\bar{Y}_I\}$$

and

$$\{B\} = \{B_R\} + i\{B_I\}$$

Then Eqn. (6.34) becomes:

$$[[S_R] + i[S_I]] \{\bar{Y}_R\} + i\{\bar{Y}_I\} = \{\{B_R\} + i\{B_I\}\}$$

or

$$[S_R]\{\bar{Y}_R\} - [S_I]\{\bar{Y}_I\} = \{B_R\} \quad (6.35)$$

and

$$[S_I]\{\bar{Y}_R\} + [S_R]\{\bar{Y}_I\} = \{B_I\} \quad (6.36)$$

Eqns. (6.35) and (6.36) can be written in matrix form as:

$$\begin{bmatrix} [S_R] & -[S_I] \\ [S_I] & [S_R] \end{bmatrix} \begin{bmatrix} \{\bar{Y}_R\} \\ \{\bar{Y}_I\} \end{bmatrix} = \begin{bmatrix} \{B_R\} \\ \{B_I\} \end{bmatrix} \quad (6.37)$$

The elements of the matrices in Eqn. (6.37) are given by the following expressions:

$$(S_{mk})_R = \omega^2 \{\phi_m\}^T \{\bar{R}_k\}_R$$

$$(S_{mk})_I = \omega^2 \{\phi_m\}^T \{\bar{R}_k\}_I$$

$$(S_{mm})_R = -\omega^2 M_m^* + k_m^* + \omega^2 \{\phi_m\}^T \{\bar{R}_m\}_R$$

$$(S_{mm})_I = \omega c_m^* + \omega^2 \{\phi_m\}^T \{\bar{R}_m\}_I$$

$$(B_m)_R = -\{\phi_m\}^T \{E^h\} + \{\phi_m\}^T \{\bar{R}_o\}_R$$

and

$$(B_m)_I = \{\phi_m\}^T \{\bar{R}_o\}_I$$

A solution of Eqn. (6.37) will give the complete frequency response functions for all the generalized coordinate displacements, $\bar{Y}_m$, $m=1,2, \dots, L$. The frequency responses for generalized accelerations may be obtained from those for the generalized displacements by the relationship:

$$\bar{\ddot{Y}}(\omega) = -\omega^2 \bar{Y}_m(\omega)$$

Having obtained the generalized acceleration $\{\bar{\ddot{Y}}_m(\omega)\}$, the corresponding hydrodynamic pressure distribution taking the reservoir effect into consideration can be obtained from Eqn. (6.29) in which $\{p_o(t)\}$ and $\{p_k(t)\}$ have already been computed using the Boundary Element Method.

6.9 RESULTS OF INVESTIGATION AND COMMENTS

As an example, a concrete dam-reservoir system of the dimensions shown in Fig. 6.4 was analyzed. The following data was used:

Young's Modulus of the concrete = 0.72×10^9 lb/ft²

Poisson's Ratio = 0.17

Mass density of concrete = 4.814 lb mass/ft³

FIG. 6.4. DAM-RESERVOIR SYSTEM.

The distribution of the hydrodynamic pressure taking the effect of the interaction between the dam and the reservoir into account was found for zero structural damping. The dam cross-section was approximated by 87 constant stress triangular elements as shown in Fig. 6.5. The reservoir was approximated by 30 constant boundary elements as shown in Fig. 6.6, omitting the boundary at infinity as explained in Chapter 5. The results obtained were compared with those obtained from a closed form solution by Liam Finn and Varoglu [2].

Table 1 presents a comparison of the first three frequencies of the dam. The first and second mode shapes of vibration of the dam when the reservoir is empty are shown in Figs. 6.7 and 6.8. The hydrodynamic pressure distribution was plotted for several different values of the non-dimensional excitation frequency ω/ω_1 where ω_1 is the fundamental natural frequency of reservoir alone. The results shown in Figs. 6.9 and 6.10 show that for small values of the ratio ω/ω_1 (0.19 and 0.55), the effect of taking the flexibility into account is small while the results shown in Figs. 6.11, 6.12 and 6.13 show that the hydrodynamic pressure distribution is greatly affected by taking the flexibility of the dam into consideration. It can be stated that the hydrodynamic pressure force on a gravity dam can be underestimated or overestimated under the assumption that the dam is rigid.

To determine the number of mode shapes to be considered, solutions were obtained using 2, 3 and 4 modes, respectively. It was found

that the results obtained with 3 modes were the same as those obtained with 4 modes (see Fig. 6.12). Therefore, the remaining results presented here are those obtained with 3 modes only. The results are summarized in Table 2. The effect of sloping upstream face was investigated and the results are shown in Fig. 6.14.

FIG. 6.5. FINITE ELEMENT DISCRETIZATION OF DAM.

FIG. 6.6. BOUNDARY ELEMENT DISCRETIZATION OF WATER RESERVOIR.

TABLE 1

SOLUTION	NO. OF FINITE ELEMENTS	ω_1	ω_2	ω_3
by FINN	335	32.98	73.96	89.04
Present Work	87	33.56	73.90	88.43

FREQUENCIES OF THE DAM.

FIG. 6.7. FIRST MODE SHAPE.

FIG. 6.8. SECOND MODE SHAPE.

FIG. 6.9. HYDRODYNAMIC PRESSURE ON FLEXIBLE DAM.

FIG. 6.10. HYDRODYNAMIC PRESSURE ON FLEXIBLE DAM.

FIG. 6.11. HYDRODYNAMIC PRESSURE ON FLEXIBLE DAM.

FIG. 6.12. HYDRODYNAMIC PRESSURE ON FLEXIBLE DAM.

FIG. 6.13. HYDRODYNAMIC PRESSURE ON FLEXIBLE DAM.

EXCITING FREQUENCY FIRST NATURAL FREQUENCY OF THE RESERVOIR ω/ω_1	F DYNAMIC F STATIC % (PRESENT WORK)		F DYNAMIC F STATIC % LIAM FINN	
	RIGID DAM	FLEXIBLE DAM	RIGID DAM	FLEXIBLE DAM
.19	3.300	3.300	3.400	3.400
.55	4.050	4.450	4.000	4.600
2.5	1.500	2.850	1.550	3.000
4.0	1.016	0.395	1.000	0.400

TABLE 2. COMPARISON OF TOTAL HYDRODYNAMIC PRESSURE FORCE FOR RIGID AND FLEXIBLE DAMS.

FIG. 6.14. HYDRODYNAMIC PRESSURE AGAINST INCLINED UPSTREAM FACE (FLEXIBLE DAM).

CHAPTER 7

SUMMARY AND CONCLUSION

7.1 SUMMARY

The boundary element method is introduced as a numerical technique. The mathematical basis and the formulation of the method are discussed.

To establish the feasibility of applying the method to the field of structural dynamics, it is applied to several problems in structural dynamics. As a first step, the method is applied to the problem of forced vibrations of stretched membranes since it represents one of the simplest two-dimensional vibration problems. The reduction of the governing differential equation to the well known Helmholtz equation and the choice of an appropriate fundamental solution to convert the domain problem to a boundary problem are discussed. The classical solution of the problem is presented and the results obtained from the boundary element method are compared with those obtained from the classical method. The agreement of results is satisfactory and encouraging.

Having established the feasibility of applying the boundary element method to structural vibration problems, the method is applied to an important practical problem, i.e. the problem of the hydrodynamic pressure distribution on rigid dams storing finite water reservoirs and

subjected to ground motion. The classical solution of the problem is discussed. Solutions for vertical and inclined upstream face of the dam are obtained and compared with known classical solutions. The solutions obtained converge to the classical solutions as more points are chosen on the boundary. The effect of sloping reservoir bed is also studied.

The problem of spurious solutions pertaining to the exterior domain instead of to the domain under consideration is examined and is found to be one of the drawbacks of the method.

To investigate the efficiency of the boundary element method when dealing with problems containing infinite domains, the problem of hydrodynamic pressure distribution on rigid dams storing infinite water reservoirs is investigated. The choice of a fundamental solution satisfying the physical requirements of the problem is discussed. Appropriate methods of representing the boundary at infinity are presented. The results obtained from the boundary element method are compared with known classical solutions. A close agreement of the results reveals that the method is a powerful and efficient tool when applied to problems containing infinite domains. Finally, the boundary element method is coupled with the finite element method to solve the problem of the dynamics of gravity dam-reservoir systems. The complete system is considered as composed of two substructures. The dam, represented as a finite element system, and the reservoir as a boundary element system of infinite length in the upstream direction. The analysis procedure.

couples the substructures by taking account of the forces interacting between the reservoir and the face of the dam. The hydrodynamic pressure distribution on the upstream face of the dam is obtained and compared with available known solutions. The effect of neglecting the flexibility of the dam is also investigated. It is found that the total hydrodynamic pressure force can be underestimated or over-estimated if the dam is assumed to be rigid.

As a result of the present study, it can be stated that the boundary element method presents a powerful tool that can be efficiently applied for the numerical solution of certain class of vibration problems.

7.2 CONCLUSION

As a result of the present study, the following conclusions are drawn:

1. The boundary element method offers several important advantages over the domain methods. These advantages are:
 - i) a reduction in size of the problem because instead of the domain only the surface enclosing the domain needs to be defined
 - ii) facility in representing the boundaries at infinity.
2. One of the more interesting features of the boundary element method is the simplicity of the input data required to run a problem. This contrasts with the large amount of data required to run a finite element program. This may be a

very important point in practice as many man-hours are lost in the preparation and checking of finite element data.

3. The boundary element method yields accurate solutions for the problems discussed in the present study. Further, the solutions converge to the known solutions as more points are chosen on the boundary. The feasibility of the application of this method to the field of structural dynamics has, therefore, been clearly established.
4. Although the boundary element method leads to a reduction in the size of the problem when compared to the domain methods, the coefficient matrices in the resulting simultaneous equations are unsymmetrical and fully populated. This may, at times, offset the advantage obtained from a reduction in the size of the problem.
5. One of the major drawbacks of the boundary element method as applied to the problems of the present study is that spurious solutions for an exterior domain may be intermixed with the solutions for the domain under consideration.
6. The present study confirms the following conclusions obtained by previous researchers in respect of the problem of hydrodynamic pressure distribution on gravity dams:
 - a) For gravity dams having inclined upstream face, the assumption that the face is vertical could lead to significant errors. However, when the inclination is small, the face can be assumed to be vertical without significant error.

- b) The effect of a small slope in the reservoir bed on the hydrodynamic pressure on gravity dams is not found to be very significant for horizontal excitation.
 - c) The hydrodynamic pressure force can be underestimated or overestimated if the dam is assumed to be rigid.
7. In conclusion, it can be asserted that the boundary element method presents a powerful tool for the analysis of certain problems in structural dynamics, particularly those dealing with infinite domains.

7.3 RECOMMENDATIONS FOR FUTURE RESEARCH

The following recommendations are made for future research:

- 1) In the present study, it has been found that the solutions obtained from the boundary element method are sometimes related to the exterior domain and not to the considered domain. When the exciting frequency coincides with the frequency of external domain, the results are not reliable. Further research is needed to weed out the spurious solutions pertaining to the exterior domain.
- 2) The solution methods for the symmetrical banded system of equations which occur in the finite element method are well developed. Similar development is required in relation to the boundary element method, where large fully populated non-symmetrical matrices are encountered.

- 3) In calculating the hydrodynamic pressure on dams, the effect of viscosity of water has always been neglected in the past because the problem becomes untractable if the viscosity is taken into account. It would be of great interest to know the extent of error resulting from neglecting the viscosity.
- 4) In the present work, the boundary element method was applied to study the effect of the interaction between the dam and the water reservoir using finite element discretization for the dam. It would be of great interest to apply the method to the problem of the complete dam-reservoir-foundation system, considering both the water reservoir and the soil under the foundation as infinite domains and discretizing the dam using the boundary element method.
- 5) The application of the method could be extended to include three-dimensional analysis of arch dam-reservoir-foundation systems. The effects of curvatures of the dam and different shapes of the valley may be investigated.

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