

Cooperative Linear-Quadratic Mean Field Control and Hamiltonian Matrix Analysis

by

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Abstract

In this thesis, we present a new methodology to investigate the existence and uniqueness of the solution of the Social Certainty Equivalence (SCE) equation system related to social optima in mean field linear-quadratic-Gaussian (LQG) control problems. The methodology involves Hamiltonian matrices and continuous-time algebraic Riccati equations (CARE), and it avoids the restrictive contractive assumption typically used in a fixed point approach. For computing the stabilizing solution of CARE, we also develop a computational method related to generalized eigenvectors and Schur vectors. We further extend our method to solve a linear-quadratic mean field game.

Keywords: SCE Equation System; Hamiltonian Matrices; Continuous-time Algebraic Riccati Equations

Dedication

I dedicate this thesis to my parents, Tianxiong Chen and Hui Li, who always encourage me and support me.

I would like to thank my supervisor, Minyi Huang, for his patient guidance and advice throughout my time as his student.

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Chapter 1

Introduction

Mean field game (MFG) theory studies stochastic decision making problems involving a large number of noncooperative agents. In a mean field game, the individual behavior has insignificant influence on others, while the mass behavior has significant impacts on individuals. Traditional approaches to mean field optimization problems involve high complexity for analysis and computation because of the large size of the population of players. MFG theory provides a powerful methodology to reduce the complexity by applying mean field approximations, which can construct a decentralized strategy and then show an ϵ -Nash equilibrium property [6]. Due to the advantages of MFG theory, it has been widely used in many domains related to large-scale dynamical systems, such as social science [2, 3], economics [5] and engineering [14].

There is a parallel development in social optima. Different from MFG, mean field socially optimal control problems involve a large number of cooperative agents. The aim of this model is seeking socially optimal decisions. Similar to MFG, one may consider centralized and decentralized strategies in social optima. For centralized strategies, each agent has access to information of all agents. Centralized strategies can be found by solving a standard LQG control problem [8]. For decentralized strategies, each agent only uses local information, and this is a highly desirable feature for reducing complexity. We describe a uniform agent example of social optima in mean field LQG control to show how to find decentralized strategies by mean approximations. This example is based on [8]. Throughout this thesis, we will use the same notations as in the following example.

Consider an LQG control system of N agents. The dynamics of agent i are given by the stochastic differential equation (SDE):

$$dx_i = Ax_i dt + Bu_i dt + DdW_i, \quad t \geq 0. \quad (1.1)$$

We use $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ to denote an underlying filtered probability space. The state x_i and the control u_i are n and n_1 dimensional vectors respectively. The initial states $\{x_i(0), 1 \leq i \leq N\}$ are independent. The noise processes $\{W_i, 1 \leq i \leq N\}$ are n_2 dimensional independent standard Brownian motions adapted to \mathcal{F}_t , which are also independent of $\{x_j(0), 1 \leq j \leq N\}$. The constant matrices A, B and D have compatible dimensions. Given a matrix Q , the quadratic form $z^T Q z$ may be denoted as $|z|_Q^2$.

The individual cost for agent i is given by

$$J_i(u(\cdot)) = E \int_0^\infty e^{-\rho t} \left[|x_i - \Phi(x^{(N)})|_Q^2 + u_i^T R u_i \right] dt, \quad (1.2)$$

where $\Phi(x^{(N)}) = \Gamma x^{(N)} + \eta$ and $x^{(N)} := (1/N) \sum x_i$ is called the mean field term. The constant matrices or vectors $\Gamma, Q \geq 0, R > 0$ and η have compatible dimensions, and $\rho > 0$ is a discount factor.

The objective of the social optimization problem is to minimize the social cost, which is defined as

$$J_{soc}^{(N)}(u(\cdot)) = \sum_{i=1}^N J_i(u(\cdot)). \quad (1.3)$$

Denote $u_{-i} := (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$, and $\hat{u}^c = (\hat{u}_1^c, \dots, \hat{u}_N^c)$ is the solution with centralized strategies.

Firstly, considering a new social cost $J_i^0(u_i) = J_{soc}^{(N)}(\hat{u}_1^c, \dots, \hat{u}_{i-1}^c, u_i, \hat{u}_{i+1}^c, \dots, \hat{u}_N^c)$, it is known that for minimizing J_i^0 , the best strategy is $\hat{u}_i = \hat{u}_i^c$. This property is called person-by-person optimality [8].

Secondly, since J_i^0 is only optimized by u_i , terms within J_i^0 not affected by u_i can be dropped. Thus, minimizing J_i^0 is equivalent to minimizing:

$$J_i^1(u_i) = E \int_0^\infty e^{-\rho t} L(x_i, \hat{x}_{-i}^{(N)}, u_i)(t) dt, \quad (1.4)$$

where $\hat{x}_{-i}^{(N)} = (1/N) \sum_{j \neq i} \hat{x}_j^c$; \hat{x}_j^c corresponds to \hat{u}_j^c and

$$\begin{aligned}
L = & x_i^T \left[\left(I - \frac{\Gamma}{N} \right)^T Q \left(I - \frac{\Gamma}{N} \right) + \frac{(N-1)\Gamma^T Q \Gamma}{N^2} \right] x_i \\
& - 2(\Gamma \hat{x}_{-i}^{(N)} + \eta)^T Q \left(I - \frac{\Gamma}{N} \right) x_i \\
& - 2 \left\{ \left[1 - \left(1 - \frac{1}{N} \right) \Gamma \right] \hat{x}_{-i}^{(N)} - \left(1 - \frac{1}{N} \right) \eta \right\}^T Q \Gamma x_i \\
& + u_i^T R u_i.
\end{aligned} \tag{1.5}$$

The detail of proof is in [8].

Finally, for large N , it is reasonable to use a deterministic function \bar{x} to approximate $\hat{x}_{-i}^{(N)}$ in (1.5). Thus, the limiting optimal control problem is:

$$\begin{aligned}
dx_i &= Ax_i dt + Bu_i dt + DdW_i, \quad t \geq 0. \\
\min J_i^*(u_i) &= E \int_0^\infty e^{-\rho t} L^*(x_i, \bar{x}, u_i)(t) dt,
\end{aligned} \tag{1.6}$$

where $L^*(x_i, \bar{x}, u_i)(t)$ is the approximation of (1.5) and takes the form:

$$L^*(x_i, \bar{x}, u_i)(t) = x_i^T Q x_i - 2(\Gamma \bar{x} + \eta)^T Q x_i - 2[(I - \Gamma)\bar{x} - \eta]^T Q \Gamma x_i + u_i^T R u_i.$$

By this mean field approximation, the social cost in (1.6) only depends on x_i and u_i . Thus, the resulting solution \hat{u}_i only uses local information, which means this \hat{u}_i is a decentralized strategy. On the other hand, according to LQG theory, the optimal control is:

$$\hat{u}_i = -R^{-1} B^T (\Pi x_i + s), \tag{1.7}$$

where Π is the solution of the following algebraic Riccati equation (ARE):

$$\rho \Pi = \Pi A + A^T \Pi - \Pi B R^{-1} B^T \Pi + Q, \tag{1.8}$$

and \bar{x} and s satisfy the Social Certainty Equivalence(SCE) equation system:

$$\frac{d\bar{x}}{dt} = (A - B R^{-1} B^T \Pi) \bar{x} - B R^{-1} B^T s, \tag{1.9}$$

$$\frac{ds}{dt} = Q \Gamma \bar{x} + (\rho I - A^T + \Pi B R^{-1} B^T) s + \eta \Gamma, \tag{1.10}$$

where $\bar{x}(0) = x_0$, $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$ (see Section 2.1 for the definition), $Q_\Gamma = \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma$ and $\eta_\Gamma = (I - \Gamma^T)Q\eta$.

Now, the social optimization problem has been reduced to solving a SCE equation system [8]. Thus, if there exists a solution (\bar{x}, s) , a decentralized control \hat{u}_i can be found by (1.7). To summarize, by person-by-person optimality and mean field approximations, the solution of the social optimization problem is equivalent to solving the ODE system (1.9)-(1.10). Thus, the key issue here is whether there exists a unique solution of (\bar{x}, s) . When Q_Γ is negative semidefinite, existence and uniqueness of the solution has been proved [8, Theorem 4.3].

There are some similar social optimization problems presented in [1], [10] and [19]. In [10], the authors introduce a mean field LQG game model involving a major player. Here the major player has a strong impact on other players. Another related work is [1], which considers nonlinear mean field optimal control to minimize a social cost.

In this thesis, we develop a new approach for the social optimization problem to analyze the solution of SCE equation system (1.9)-(1.10) when Q_Γ is only symmetric. Our method involves Hamiltonian matrices, frequency domain conditions and continuous-time algebraic Riccati equations (CARE).

The organization of this thesis is as follow. Chapter 2 develops a new methodology to solve the SCE equation system and proves the existence and uniqueness of a solution (\bar{x}, s) in $C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$ by identifying a unique initial condition s_0 . The mean field game solution is presented in Chapter 3. Chapter 4 concludes the thesis.

1.1 Claim of Original Contributions

We prove existence and uniqueness of a solution to SCE equation systems by using a decomposition technique based on Hamiltonian matrix analysis and develop the computational method to solve a CARE when a related state weight matrix is indefinite. We further extend this approach to solve a mean field game.

Chapter 2

Social Optimization Problem

2.1 Description of Problem

In Chapter 1, we have described a social optimum problem in mean field LQG control to find decentralized strategies by person-by-person optimality and mean field approximations.

Definition 2.1.1. For N is an integer and r is a positive real number, define $C_r([0, \infty), \mathbb{R}^N)$ as a set including all function $f \in C([0, \infty), \mathbb{R}^N)$ such that $\sup_{t \geq 0} |f(t)|e^{-r't} < +\infty$, for some positive real number $r' < r$. Here, r' may depend on $f(t)$.

Now our interest is to analyze the solution of the following ODE system:

$$\frac{d\bar{x}}{dt} = (A - BR^{-1}B^T\Pi)\bar{x} - BR^{-1}B^T s \quad (2.1)$$

$$\frac{ds}{dt} = Q_\Gamma \bar{x} + (\rho I - A^T + \Pi BR^{-1}B^T)s + \eta_\Gamma, \quad (2.2)$$

where $\bar{x}(0) = x_0$, both \bar{x} and s are in $C_{\rho/2}([0, \infty), \mathbb{R}^n)$, $Q_\Gamma = \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma$ and $\eta_\Gamma = (I - \Gamma^T)Q\eta$. Since the initial condition s_0 is unspecified, a key part of finding a solution is to determine the initial condition s_0 .

We can transform the coefficient matrix in the ODE system (2.1)-(2.2) to a Hamiltonian matrix by the following transformation:

$$\tilde{x} = e^{-\rho t/2} \bar{x}, \quad \tilde{s} = e^{-\rho t/2} s,$$

and then we obtain new differential equations for (\tilde{x}, \tilde{s}) :

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix}}_H \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\eta}_\Gamma \end{pmatrix}, \quad (2.3)$$

where $\tilde{x}(0) = x_0$ and $\tilde{\eta}_\Gamma = e^{-\frac{\rho}{2}t}\eta_\Gamma$.

Denote $\mathcal{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ and

$$H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathcal{A}^T \end{pmatrix}. \quad (2.4)$$

Then

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{pmatrix} = H \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\eta}_\Gamma \end{pmatrix}. \quad (2.5)$$

We should notice that in ODE (2.2), η_Γ is a constant vector. However, in system (2.3), $\tilde{\eta}_\Gamma$ is a function of t . Indeed, since $\rho > 0$, $\lim_{t \rightarrow +\infty} \tilde{\eta}_\Gamma = 0$, which means that it does not result in much difficulty in finding a bounded solution of ODE (2.3).

2.2 Eigenvalues and Eigenvectors of Hamiltonian Matrices

Before analyzing the solution of ODE system (2.3), we introduce a special class of matrices - Hamiltonian matrices.

Definition 2.2.1. A matrix $K \in \mathbb{R}^{2n \times 2n}$ is called a Hamiltonian matrix if JK is symmetric, where

$$J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}. \quad (2.6)$$

Since $JH = \begin{pmatrix} Q_\Gamma & -\mathcal{A}^T \\ -\mathcal{A} & -BR^{-1}B^T \end{pmatrix}$ is symmetric, H is a Hamiltonian matrix and satisfies the following properties.

Proposition 2.2.2. [13] If matrix K is a Hamiltonian matrix, $p_K(x)$ is the characteristic polynomial of matrix K and $x_0 \in \mathbb{C}$ is a root of $p_K(x) = 0$, then we have

$$p_K(-x_0) = p_K(\bar{x}_0) = p_K(-\bar{x}_0) = 0.$$

Theorem 2.2.3. (The real Schur-Hamiltonian decomposition) [16] Suppose

$$M = \begin{pmatrix} A & N \\ K & -A^T \end{pmatrix},$$

where $A \in \mathbb{R}^{n \times n}$, $N^T = N \in \mathbb{R}^{n \times n}$ and $K^T = K \in \mathbb{R}^{n \times n}$. If M has no purely imaginary eigenvalues, then there exists an orthogonal matrix

$$U = \begin{pmatrix} U_{11} & U_{12} \\ -U_{12} & U_{11} \end{pmatrix}, \quad U_{11}, U_{12} \in \mathbb{R}^{n \times n},$$

such that

$$U^T M U = \begin{pmatrix} T & R \\ 0 & -T^T \end{pmatrix}, \quad T, R \in \mathbb{R}^{n \times n},$$

where T is upper quasi-triangular and $R^T = R$. U can be chosen such that the eigenvalues of T are in the open left half plane.

A block upper triangular matrix with 1×1 and 2×2 blocks on the diagonal is called upper quasi-triangular. Theorem 2.2.3 is presented in [16, p. 26] and we briefly overview the proof. This result is a real valued matrix situation of Theorem 3.1 in [16, p. 19]. The authors apply an induction method in their proof. The first step is to prove the existence of unitary transformation matrix when M is a 2×2 matrix. There is an error in their proof. By the condition of no purely imaginary eigenvalues, we can obtain $k\eta + \alpha^2 > 0$, not $k\eta > 0$ as in [16, p. 20]. The second step is to assume any Hamiltonian matrices of order $2(n-1)$ without purely imaginary eigenvalues have the decomposition. The final step is to prove under the above assumption the decomposition exists for any Hamiltonian matrices of order $2n$ without purely imaginary eigenvalues. Here the key point is to use an algorithm that is introduced in [21] to prove the eigenvalues of Hamiltonian matrix M of order $2n$ are λ and $-\bar{\lambda}$ together with the eigenvalues of a Hamiltonian matrix M_1 of order $2(n-1)$.

Here the condition that there is no purely imaginary eigenvalues means no eigenvalues with zero real parts. Thus, zero is interpreted as a purely imaginary eigenvalue.

2.3 Continuous-time Algebraic Riccati Equations

Our analysis of the SCE equation system in this thesis depends on a matrix decomposition method which involves a class of algebraic Riccati equations. In this part, we introduce a set of continuous-time algebraic Riccati equation (CARE) of the form:

$$XDX - XA - A^T X - C = 0, \quad (2.7)$$

where A, C, D are given matrices in $\mathbb{R}^{n \times n}$, with $C^T = C$ and $D^T = D$.

For the reader's convenience, we summarize some standard definitions on Riccati equations and linear control systems [11, 23].

Definition 2.3.1. For a matrix $M \in \mathbb{R}^{n \times n}$, M is stable if all the eigenvalues of M are in the open left half-plane.

Definition 2.3.2. For a real symmetric solution X_+ of the CARE (2.7), X_+ is the maximal real symmetric solution if for any real symmetric solution X , $(X_+ - X)$ is nonnegative definite.

Definition 2.3.3. For a real symmetric solution X of the CARE (2.7), X is an almost stabilizing solution if the eigenvalues of $(A - DX)$ are in the closed left half-plane. In addition, X is called a stabilizing solution if the eigenvalues of $(A - DX)$ are in the open left half-plane.

Before analyzing the solution of the CARE, we first introduce some concept related to a system of the form

$$\dot{x} = Ax(t) + Bu(t), \quad x(0) = x_0. \quad (2.8)$$

Definition 2.3.4. For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, a pair of matrices (A, B) is called a controllable pair if

$$\text{rank}[B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] = n.$$

We say the system (2.8) is controllable if, for any fixed x_0 and x_1 , there exists a continuous control $u(t)$ such that $x(t)$ satisfies (2.8) and the terminal state $x(T) = x_1$, for terminal time $T > 0$. In addition, it can be proved that the system (2.8) is controllable is equivalent to that the pair (A, B) is controllable [11].

Definition 2.3.5. For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, a pair of matrices (A, B) is called a stabilizable pair if there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable (i.e. all eigenvalues of $A + BK$ are in the open left half-plane).

The system (2.8) is stabilizable means for any fixed initial state x_0 , there exists a continuous control $u(t)$ such that $x(t)$ satisfies (2.8) and $\lim_{t \rightarrow +\infty} x(t) = 0$. In parallel to controllability, the system is stabilizable if and only if the pair (A, B) is stabilizable.

Theorem 2.3.6. [11, p. 90] *If the pair (A, B) is controllable, then the pair (A, B) is stabilizable.*

Example 2.3.7. Let $x \in \mathbb{R}$, $\dot{x} = -1.1x + 0u$.

This system is not controllable but stabilizable. Indeed, since for fixed initial state x_0 , the solution $x(t)$ is always a fixed trajectory no matter what the control $u(t)$ is. Thus, for a given T , $x(T)$ is fixed, which means uncontrollability. However, since $A = -1.1$ is a stable matrix, the solution $x(t)$ is stable (i.e. $\lim_{t \rightarrow +\infty} x(t) = 0$), which means this system is stabilizable.

Theorem 2.3.6 shows that controllable systems must be stabilizable. Furthermore, Example 2.3.7 shows that stabilizable systems may be uncontrollable. Thus, controllability is a stronger condition. Actually, all the following theorems are proved to be correct under the controllability condition firstly, and then we can relax the condition to the stabilizability condition by selecting a special orthogonal matrix to transform. The details are as follow.

Considering a space $\mathcal{C} := \text{Im}[D \quad \mathcal{A}D \quad \mathcal{A}^2D \quad \dots \quad \mathcal{A}^{n-1}D]$, where Im means the subspace spanned by the columns. By [11, p. 85], \mathcal{C} is \mathcal{A} -invariant. By the discussion in [11, p. 193], there exists a set of orthogonal basis $T = (T_1 \quad T_2)$, where $T_1 \in \mathcal{C}$ and $T_2 \in \mathcal{C}^\perp$, such that

$$T^T AT = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_{12} \\ 0 & \mathcal{A}_2 \end{pmatrix}, T^T DT = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, T^T CT = \begin{pmatrix} C_1 & C_{12} \\ C_{12}^T & C_2 \end{pmatrix},$$

where $D_1 \geq 0$; C_1, C_2 is symmetric; (\mathcal{A}_1, D_1) is controllable and \mathcal{A}_2 is stable. If we set $Y = T^T XT$ and X is real symmetric, (2.7) is equivalent to

$$Y(T^T DT)Y - Y(T^T AT) - (T^T AT)^T Y - T^T CT = 0.$$

Thus, if our interest is the real symmetric solution of (2.7), we can assume:

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_{12} \\ 0 & \mathcal{A}_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_{12} \\ C_{12}^T & C_2 \end{pmatrix}. \quad (2.9)$$

Thus, letting X be a block matrix

$$\begin{pmatrix} X_1 & X_{12} \\ X_{12}^T & X_2 \end{pmatrix}, \quad (2.10)$$

where X_1 and \mathcal{A}_1 have the same dimension, (2.7) is equivalent to the following equations:

$$X_1 D_1 X_1 - X_1 \mathcal{A}_1 - \mathcal{A}_1^T X_1 - C_1 = 0, \quad (2.11)$$

$$(\mathcal{A}_1^T - X_1 D_1) X_{12} + X_{12} \mathcal{A}_2 + (C_{12} + X_1 \mathcal{A}_{12}) = 0, \quad (2.12)$$

$$-\mathcal{A}_2^T X_2 - X_2 \mathcal{A}_2 + X_{12}^T D_1 X_{12} - X_{12}^T \mathcal{A}_{12} - \mathcal{A}_{12}^T X_{12} - C_{12} = 0. \quad (2.13)$$

Theorem 2.3.8. [11, p. 195] *If $D \geq 0$, (\mathcal{A}, D) is stabilizable and there is a real symmetric solution of (2.7), then there is a maximal real symmetric solution X_+ .*

Theorem 2.3.9. [11, p. 196] *Suppose that $D \geq 0$, (\mathcal{A}, D) is stabilizable and there is a real symmetric solution of (2.7). Then for the maximal real symmetric solution X_+ of (2.7), $\mathcal{A} - DX_+$ is stable if and only if the matrix $M := \begin{pmatrix} \mathcal{A} & -D \\ -C & -\mathcal{A}^T \end{pmatrix}$ has no eigenvalues on the imaginary axis.*

In [11, p. 196], there is a proof when the coefficient matrices \mathcal{A} , C , D are complex. Here, for the reader's convenience, we provide a proof of Theorem 2.3.9 when the coefficient matrices are real.

Proof. Without loss of generality, we can assume the coefficient matrices \mathcal{A} , C , D have the form of (2.9).

1) The "if" part. Suppose M has no eigenvalues on the imaginary axis, and M has the form:

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_{12} & -D_1 & 0 \\ 0 & \mathcal{A}_2 & 0 & 0 \\ -C_1 & -C_{12} & -\mathcal{A}_1^T & 0 \\ -C_{12}^T & -C_2 & -\mathcal{A}_{12} & -\mathcal{A}_2^T \end{pmatrix}.$$

By orthogonal transformation matrix $T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$, we have

$$T^T M T = \begin{pmatrix} \mathcal{A}_1 & -D_1 & \mathcal{A}_{12} & 0 \\ -C_1 & -\mathcal{A}_1^T & -C_{12} & 0 \\ 0 & 0 & \mathcal{A}_2 & 0 \\ -C_{12}^T & -\mathcal{A}_{12} & -C_2 & -\mathcal{A}_2^T \end{pmatrix}. \quad (2.14)$$

From (2.14), we can obtain the eigenvalues of M consisting of the eigenvalues of \mathcal{A}_2 , $-\mathcal{A}_2^T$ and M_1 , where

$$M_1 = \begin{pmatrix} \mathcal{A}_1 & -D_1 \\ -C_1 & -\mathcal{A}_1^T \end{pmatrix}.$$

Thus, M_1 has no eigenvalues on the imaginary axis. According to [11, Theorem 7.5.1, p. 168], since (\mathcal{A}_1, D_1) is controllable and M_1 has no eigenvalues on the imaginary, there exists a solution X_{1+} of (2.11) such that $\mathcal{A} - D_1 X_{1+}$ is stable. Let $X_+ = \begin{pmatrix} X_{1+} & X_{12+} \\ X_{12+}^T & X_{2+} \end{pmatrix}$, where X_{12+} and X_{2+} are the corresponding solution of (2.12) and (2.13).

From (2.9) and (2.10), we have

$$\mathcal{A} - D X_+ = \begin{pmatrix} \mathcal{A}_1 - D_1 X_{1+} & \mathcal{A}_{12} - D_1 X_{12+} \\ 0 & \mathcal{A}_2 \end{pmatrix}. \quad (2.15)$$

The eigenvalues of $\mathcal{A} - D X_+$ consist of the eigenvalue of $\mathcal{A}_1 - D_1 X_{1+}$ and \mathcal{A}_2 , which are both stable. Thus, $\mathcal{A} - D X_+$ is stable.

2) The "only if" part. Suppose $\mathcal{A} - DX_+$ are stable. Since \mathcal{A}_2 is stable, then the eigenvalues of $-\mathcal{A}_2^T$ are all in the open right half-plane. We just need to prove M_1 has no eigenvalues on the imaginary axis. Let X_{1+} be the corresponding block matrix of X_+ , and it is obvious that X_{1+} is the maximal solution of (2.11). From (2.15), since $\mathcal{A} - DX_+$ is stable and \mathcal{A}_2 is stable, $\mathcal{A}_1 - D_1X_{1+}$ must be stable.

Consider an invertible transformation

$$T = \begin{pmatrix} I & 0 \\ X_{1+} & I \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} I & 0 \\ -X_{1+} & I \end{pmatrix}.$$

Then $T^{-1}M_1T = \begin{pmatrix} \mathcal{A}_1 - D_1X_{1+} & -D_1 \\ 0 & -(\mathcal{A}_1 - D_1X_{1+})^T \end{pmatrix}$. Since $\mathcal{A}_1 - D_1X_{1+}$ is stable, then the eigenvalues of $-(\mathcal{A}_1 - D_1X_{1+})^T$ are all in the open right half-plane, so $T^{-1}M_1T$ has no eigenvalues on the imaginary axis. In addition, M_1 shares the same eigenvalues with $T^{-1}M_1T$. Thus M_1 has no eigenvalues on the imaginary axis. \square

Definition 2.3.10. For a matrix $M \in \mathbb{R}^{n \times n}$ and its eigenvalue λ_0 , the sizes of its Jordan canonical block in the Jordan canonical form of M is called partial multiplicities of λ_0 .

Example 2.3.11. Consider

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

This matrix M has two eigenvalues 1 and -1 . For the eigenvalue 1, its partial multiplicities are 3, 1 and 1. For the eigenvalue -1 , its partial multiplicities are 5.

Theorem 2.3.12. [11, p. 239] *There exists a unique almost stabilizing solution of the CARE (2.7) if and only if (\mathcal{A}, D) is stabilizable and the partial multiplicities of purely imaginary (or zero) eigenvalues of the matrix $M := \begin{pmatrix} \mathcal{A} & -D \\ -C & -\mathcal{A}^T \end{pmatrix}$ are all even.*

2.4 Frequency Domain Condition

In this part, we consider a special form of the CARE:

$$XBR^{-1}B^T X - \mathcal{A}^T X - X\mathcal{A} - C = 0, \quad (2.16)$$

where $\mathcal{A}, Q \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $R \in \mathbb{R}^{m \times m}$; C is symmetric and R is positive definite.

Firstly, we introduce the Frequency-Domain Inequality (FDI) [22]:

$$\hat{H}(\bar{s}, s) = R + B^T(I\bar{s} - \mathcal{A}^T)^{-1}C(Is - \mathcal{A})^{-1}B \geq 0. \quad (2.17)$$

The following theorem is related to FDI.

Theorem 2.4.1. [22] *Assume that the pair (\mathcal{A}, B) is controllable. Then the following are true.*

1) *Existence: CARE (2.16) has a real symmetric solution if and only if $\hat{H}(-j\omega, j\omega) \geq 0$ for all real ω . There always exists exactly one such solution, denoted by X_+ , which has the additional property that $\text{Re } \lambda(\mathcal{A} - BR^{-1}B^T X_+) \leq 0$, i.e. X_+ is the almost stabilizing solution of CARE (2.16).*

2) *Strict Inequality: the strict inequality*

$$\hat{H}(-j\omega, j\omega) \geq \epsilon B^T(-Ij\omega - \mathcal{A}^T)^{-1}(Ij\omega - \mathcal{A})^{-1}B \quad (2.18)$$

holds, for some $\epsilon > 0$ if and only if the CARE (2.16) has a real symmetric solution X_+ , such that $\text{Re } \lambda(\mathcal{A} - BR^{-1}B^T X_+) < 0$, i.e. X_+ is the stabilizing solution of CARE (2.16).

Example 2.4.2. Consider

$$H = \begin{pmatrix} a - b^2\Pi/r - \rho/2 & -b^2/r \\ (2\gamma - \gamma^2)q & -(a - b^2\Pi/r - \rho/2) \end{pmatrix},$$

where

$$\rho\Pi = 2a\Pi - b^2\Pi/r + q. \quad (2.19)$$

Actually this matrix is the size 2×2 matrix of (2.4), where $\mathcal{A} = a - b^2\Pi/r - \rho/2$, $B = b$, $R = r \geq 0$ and $Q_\Gamma = (2\gamma - \gamma^2)q$, where $q > 0$ and $0 \leq \gamma \leq 1$.

From (2.19), denote $(a - \rho/2)$ as a_ρ and b^2/r as b_r ; then we can obtain that $\Pi = a_\rho/b_r \pm \sqrt{a_\rho^2/b_r^2 + q/b_r}$ and choose the greater one.

Firstly, consider the following CARE:

$$XBR^{-1}B^T X - \mathcal{A}^T X - X\mathcal{A} + Q_\Gamma = 0,$$

and then the corresponding FDI is $\hat{H}(-j\omega, j\omega) = r - qb^2(2\gamma - \gamma^2)/(w^2 + a_\rho^2 + qb/r^2) \geq 0$.

Then we can obtain the strict FDI:

$$1 - qb_r^2(2\gamma - \gamma^2)/(w^2 + a_\rho^2 + qb_r) \geq \epsilon b_r^2/(w^2 + a_\rho^2 + qb_r),$$

which is equivalent to

$$\omega^2 + a_\rho^2 + qb_r(\gamma - 1)^2 \geq \epsilon b_r. \quad (2.20)$$

If for all $\omega \in R$ and some $\epsilon > 0$, (2.20) holds, it means $a_\rho \neq 0$ or $\{\gamma \neq 1 \text{ and } b_r \neq 0\}$.

On the other hand, letting

$$\det(\lambda I - H) = \det \begin{pmatrix} \lambda + \sqrt{a_\rho^2 + qb_r} & b_r \\ (\gamma^2 - 2\gamma)q & \lambda - \sqrt{a_\rho^2 + qb_r} \end{pmatrix} = 0,$$

we can obtain $\lambda^2 = a_\rho^2 + qb_r(\gamma - 1)^2$. If the matrix H has no purely imaginary eigenvalues, it is equivalent to $a_\rho^2 + qb_r(\gamma - 1)^2 > 0$, i.e. $a_\rho \neq 0$ or $\{\gamma \neq 1 \text{ and } b_r \neq 0\}$.

Here, we find that H has no purely imaginary eigenvalues is equivalent to that the corresponding strict FDI holds when H is a 2×2 matrix.

2.5 Existence and Uniqueness of A Solution

In Section 2.1, we transform the SCE equation system (2.1)-(2.2) to the following Hamiltonian system:

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{pmatrix} = H \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\eta}_\Gamma \end{pmatrix}, \quad (2.21)$$

where $\mathcal{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ and

$$H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathcal{A}^T \end{pmatrix}.$$

Now we develop a new method to analyze the solution of the SCE equation system (2.1)-(2.2) by using the properties of Hamiltonian matrices and the continuous-time algebraic Riccati equation.

Lemma 2.5.1. *If there exists an invertible matrix $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, where U_{11} is invertible and*

$$U^{-1}HU = \begin{pmatrix} \mathcal{A}_C & F \\ 0 & -\mathcal{A}_C^T \end{pmatrix},$$

where \mathcal{A}_C is stable, we can obtain the following assertions:

1) For any fixed initial state x_0 of SCE equation system (2.1)-(2.2), there exists an initial condition s_0 , such that SCE equation system (2.1)-(2.2) has a unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.

2) Such an initial condition s_0 is unique and is equal to

$$U_{21}U_{11}^{-1}\bar{x}_0 + (U_{21}U_{11}^{-1}U_{12} - U_{22}) \int_0^{+\infty} e^{A_C^T\tau} V_{22}\eta_\Gamma e^{-\rho\tau/2} d\tau,$$

where $V = U^{-1} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$.

Proof. 1) Proof of existence.

Let $\begin{pmatrix} \hat{x} \\ \hat{s} \end{pmatrix} = V \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix}$. The ODE systems of (\hat{x}, \hat{s}) is:

$$\frac{d\hat{x}}{dt} = \mathcal{A}_C \hat{x} + F \hat{s} + V_{12} \tilde{\eta}_\Gamma, \quad (2.22)$$

$$\frac{d\hat{s}}{dt} = -\mathcal{A}_C^T \hat{s} + V_{22} \tilde{\eta}_\Gamma. \quad (2.23)$$

Firstly, since $\tilde{\eta}_\Gamma = e^{-\rho t/2} \eta_\Gamma$, from (2.23), we can obtain that

$$\frac{d\hat{s}}{dt} = -\mathcal{A}_C^T \hat{s} + V_{22} \eta_\Gamma e^{-\rho t/2}. \quad (2.24)$$

Thus, the solution of ODE (2.23) has the form:

$$\hat{s}(t) = e^{-\mathcal{A}_C^T t} \hat{s}_0 + \int_0^t e^{-\mathcal{A}_C^T (t-\tau)} V_{22} \eta_\Gamma e^{-\rho \tau/2} d\tau.$$

Since \mathcal{A}_C^T is stable, the integral $\int_0^{+\infty} e^{\mathcal{A}_C^T \tau} V_{22} \eta_\Gamma e^{-\rho \tau/2} d\tau$ is finite. Let

$$\hat{s}_0 = - \int_0^{+\infty} e^{\mathcal{A}_C^T \tau} V_{22} \eta_\Gamma e^{-\rho \tau/2} d\tau.$$

Then we obtain that

$$\begin{aligned} \hat{s}(t) &= - \int_t^\infty e^{-\mathcal{A}_C^T (t-\tau)} V_{22} \eta_\Gamma e^{-\rho \tau/2} d\tau \\ &= - \int_0^\infty e^{\mathcal{A}_C^T \tau} V_{22} \eta_\Gamma e^{-\rho(\tau+t)/2} d\tau \\ &= -e^{-\rho t/2} \int_0^\infty e^{(\mathcal{A}_C^T - \frac{\rho}{2} I) \tau} V_{22} \eta_\Gamma d\tau. \end{aligned} \quad (2.25)$$

Denote the maximal real part of eigenvalues of \mathcal{A}_C as σ_s . Since \mathcal{A}_C is stable, σ_s is negative. Consider a real positive number ε such that $\varepsilon < \min(-\sigma_s, \frac{\rho}{2})$. From equation (2.25), we can obtain

$$e^{\varepsilon t} \hat{s}(t) = -e^{-(\frac{\rho}{2} - \varepsilon)t} \int_0^\infty e^{(\mathcal{A}_C^T - \frac{\rho}{2} I) \tau} V_{22} \eta_\Gamma d\tau. \quad (2.26)$$

Since $\mathcal{A}_C^T - \frac{\rho}{2} I$ is stable and $\frac{\rho}{2} - \varepsilon > 0$, we have $e^{\varepsilon t} \hat{s}(t)$ is bounded. Thus, $e^{\frac{\rho}{2} t} \hat{s}(t)$ is within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$.

Secondly, from the ODE (2.22), we can obtain that

$$\frac{d\hat{x}}{dt} = \mathcal{A}_C \hat{x} + F \hat{s}(t) + V_{12} \eta_\Gamma e^{-\rho t/2}, \quad (2.27)$$

where $\hat{s}(t)$ is the bounded solution (2.25).

Now consider $e^{\varepsilon t}\hat{x}(t)$, where $\hat{x}(t)$ is the solution of ODE (2.27). We can obtain that

$$\frac{d[e^{\varepsilon t}\hat{x}(t)]}{dt} = \varepsilon e^{\varepsilon t}\hat{x}(t) + e^{\varepsilon t}\frac{d\hat{x}}{dt} \quad (2.28)$$

$$= \varepsilon e^{\varepsilon t}\hat{x}(t) + e^{\varepsilon t}\left[\mathcal{A}_C\hat{x}(t) + F\hat{s}(t) + V_{12}\eta_{\Gamma}e^{-\rho t/2}\right] \quad (2.29)$$

$$= (\mathcal{A}_C + \varepsilon I)\left[e^{\varepsilon t}\hat{x}(t)\right] + F\left[e^{\varepsilon t}\hat{s}(t)\right] + V_{12}\eta_{\Gamma}e^{(\varepsilon - \frac{\rho}{2})t}. \quad (2.30)$$

From (2.26), we can obtain that $\lim_{t \rightarrow +\infty} e^{\varepsilon t}\hat{s}(t) = 0$. In addition, since the $(\mathcal{A}_C + \varepsilon I)$ is stable, for any initial condition \hat{x}_0 , we conclude that $e^{\varepsilon t}\hat{x}(t)$ is bounded, which means that $e^{\frac{\rho}{2}t}\hat{x}(t)$ is within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$.

Now, we obtain $e^{\frac{\rho}{2}t}\hat{x}(t)$ and $e^{\frac{\rho}{2}t}\hat{s}(t)$ are both within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$. Since

$$\begin{pmatrix} \bar{x}(t) \\ s(t) \end{pmatrix} = U \begin{pmatrix} e^{\frac{\rho}{2}t}\hat{x}(t) \\ e^{\frac{\rho}{2}t}\hat{s}(t) \end{pmatrix},$$

we obtain that $\bar{x}(t) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ and $s(t) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$, i.e. $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.

Finally, since

$$\begin{aligned} \bar{x}_0 &= U_{11}\hat{x}_0 + U_{12}\hat{s}_0, \\ s_0 &= U_{21}\hat{x}_0 + U_{22}\hat{s}_0, \end{aligned}$$

and U_{11} is invertible, for any fixed \bar{x}_0 and fixed \hat{s}_0 , there always exists an initial condition

$$s_0 = U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}_0) + U_{22}\hat{s}_0.$$

Thus, for obtaining $\bar{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ and $s \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$, we can choose the initial condition

$$\begin{aligned} s_0 &= U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}_0) + U_{22}\hat{s}_0 \\ &= U_{21}U_{11}^{-1}\bar{x}_0 - (U_{21}U_{11}^{-1}U_{12} - U_{22})\hat{s}_0 \\ &= U_{21}U_{11}^{-1}\bar{x}_0 + (U_{21}U_{11}^{-1}U_{12} - U_{22})\int_0^{+\infty} e^{\mathcal{A}_C^T\tau}V_{22}\eta_{\Gamma}e^{-\rho\tau/2}d\tau \end{aligned}$$

2) Proof of uniqueness.

Assume there exists another initial condition s'_0 such that the corresponding solution $(\bar{x}'(t), s'(t)) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$. Since

$$\bar{x}_0 = U_{11}\hat{x}'_0 + U_{12}\hat{s}'_0, \quad (2.31)$$

$$s'_0 = U_{21}\hat{x}'_0 + U_{22}\hat{s}'_0, \quad (2.32)$$

and U is invertible, we can obtain the corresponding \hat{s}'_0 .

Now we denote the corresponding solution of ODE (2.24) as $\hat{s}'(t)$, where the initial condition of ODE (2.24) is \hat{s}'_0 . Thus, $\hat{s}'(t)$ has the following form:

$$\hat{s}'(t) = e^{-\mathcal{A}_C^T t} \hat{s}'_0 + \int_0^t e^{-\mathcal{A}_C^T (t-\tau)} V_{22} \eta_{\Gamma} e^{-\rho\tau/2} d\tau. \quad (2.33)$$

Since $s'(t) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$, from the first part of proof, we can obtain that $\hat{s}'(t)$ must be bounded. In addition, since $\hat{s}(t)$ is also bounded, we can obtain $\hat{s}(t) - \hat{s}'(t)$ is bounded.

From (2.25) and (2.33), we have

$$\hat{s}(t) - \hat{s}'(t) = e^{-\mathcal{A}_C^T t} (\hat{s}_0 - \hat{s}'_0). \quad (2.34)$$

Since the eigenvalues of $-\mathcal{A}_C^T$ are all in the open right half-plane, $\hat{s}_0 - \hat{s}'_0$ must be 0 and otherwise the RHS of (2.34) is not bounded. Thus, $\hat{s}'_0 = \hat{s}_0$.

According to equations (2.31)-(2.32) and the invertibility of U_{11} , we can obtain that that

$$\begin{aligned} s'_0 &= U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}'_0) + U_{22}\hat{s}'_0 \\ &= U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}_0) + U_{22}\hat{s}_0 \\ &= s_0. \end{aligned}$$

Thus, the initial condition s_0 of SCE equation system (2.1)-(2.2) such that $(\bar{x}(t), s(t)) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$ is unique. It is obvious that the solution (\bar{x}, s) within $C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$ is unique. □

Theorem 2.5.2. *Assume the pair (A, B) is controllable and this inequality*

$$R + B^T(-Ij\omega - \mathcal{A}^T)^{-1}(-Q_{\Gamma})(Ij\omega - \mathcal{A})^{-1}B \geq \epsilon B^T(-Ij\omega - \mathcal{A}^T)^{-1}(Ij\omega - \mathcal{A})^{-1}B \quad (2.35)$$

holds, for some $\epsilon > 0$. Then there exists a unique initial condition s_0 , such that the SCE equation system (2.1)-(2.2) has a unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.

Proof. Consider a CARE:

$$XBR^{-1}B^T X - XA - A^T X + Q_\Gamma = 0. \quad (2.36)$$

Since (A, B) is controllable, (\mathcal{A}, B) is controllable [23]. According to Theorem 2.4.1, there exists a solution X_+ of CARE (2.36) such that $(\mathcal{A} - BR^{-1}B^T X_+)$ is stable.

$$\text{Setting } U = \begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}, U \text{ is an invertible matrix and } U^{-1} = \begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix}.$$

$$U^{-1}HU = \begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix} \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -A^T \end{pmatrix} \begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix} \quad (2.37)$$

$$= \begin{pmatrix} \mathcal{A} - BR^{-1}B^T X_+ & -BR^{-1}B^T \\ X_+ BR^{-1}B^T X_+ - X_+ A - A^T X_+ + Q_\Gamma & -(\mathcal{A} - BR^{-1}B^T X_+)^T \end{pmatrix} \quad (2.38)$$

$$= \begin{pmatrix} \mathcal{A}_C & -BR^{-1}B^T \\ 0 & -\mathcal{A}_C^T \end{pmatrix}, \quad (2.39)$$

where $\mathcal{A}_C = \mathcal{A} - BR^{-1}B^T X_+$ is stable.

According to Lemma 2.5.1, by selecting the initial condition

$$s_0 = X_+ \bar{x}_0 - \int_0^{+\infty} e^{\mathcal{A}_C^T \tau} \eta_\Gamma e^{-\rho\tau/2} d\tau,$$

there exists a solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$. In addition, such an initial condition and such a solution are both unique. \square

Theorem 2.5.2 guarantees that for a controllable system, if the corresponding coefficient matrices satisfy the strict frequency domain inequality, we can choose an explicit initial condition s_0 , such that the corresponding solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$. Notice that Q_Γ in Theorem 2.5.2 corresponds to $-C$ in Theorem 2.4.1.

Recall Example 2.4.2:

$$H = \begin{pmatrix} a - b^2\Pi/r - \rho/2 & -b^2/r \\ (2\gamma - \gamma^2)q & -(a - b^2\Pi/r - \rho/2) \end{pmatrix}.$$

For matrix H , it has no purely imaginary eigenvalues if and only if the corresponding strict FDI holds. Since matrix H in Example 2.4.2 is the 2×2 dimension matrix H in Theorem 2.5.1, it is reasonable for us to try to replace the frequency domain inequality condition (2.35) by an eigenvalue condition on H , as an alternative requirement on the model.

On the other hand, for $b \neq 0$, the pair (a, b) is always controllable. However, for the matrix pair (A, B) , controllability is restrictive. Thus, it is desirable to relax the condition on the pair (A, B) .

Theorem 2.5.3. *Assume that the pair (A, B) is stabilizable and matrix H has no purely imaginary eigenvalues, then there exists a unique initial condition s_0 , such that the SCE equation system (2.1)-(2.2) has a unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.*

Proof. Since \mathcal{A} is stable, $(\mathcal{A}, BR^{-1}B^T)$ is stabilizable.

For matrix $H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathcal{A}^T \end{pmatrix}$, it has no purely imaginary eigenvalues, which means the partial multiplicities of pure imaginary eigenvalue are all 0. Consider the following CARE

$$XBR^{-1}B^TX - X\mathcal{A} - \mathcal{A}^TX + Q_\Gamma = 0 \quad (2.40)$$

According to Theorem 2.3.12, there exists a unique almost stabilizing solution of CARE (2.40). Furthermore, CARE (2.40) satisfies the condition of Theorem 2.3.9. Thus, according to Theorem 2.3.9, there exists a maximal real symmetric solution X_+ , such that $\mathcal{A} - BR^{-1}B^TX_+$ is stable. Actually since X_+ is also the almost stabilizing solution, by Theorem 2.3.12, X_+ is unique.

Setting $U = \begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}$, U is an invertible matrix and $U^{-1} = \begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix}$. Then

$$\begin{aligned} U^{-1}HU &= \begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix} \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathcal{A}^T \end{pmatrix} \begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A} - BR^{-1}B^TX_+ & -BR^{-1}B^T \\ X_+BR^{-1}B^TX_+ - X_+\mathcal{A} - \mathcal{A}^TX_+ + Q_\Gamma & -(\mathcal{A} - BR^{-1}B^TX_+)^T \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A}_C & -BR^{-1}B^T \\ 0 & -\mathcal{A}_C^T \end{pmatrix}, \end{aligned}$$

where $\mathcal{A}_C = \mathcal{A} - BR^{-1}B^T X_+$ is stable. According to Lemma 2.5.1, by selecting the initial condition $s_0 = X_+ \bar{x}_0 - \int_0^{+\infty} e^{\mathcal{A}_C^T \tau} \eta_\Gamma e^{-\rho\tau/2} d\tau$, there exists a solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$. In addition, such an initial condition and such a solution are both unique. \square

Theorem 2.5.3 relaxes the condition of pair (A, B) from controllability to stabilizability and establishes the connection between SCE equation system and the eigenvalues of the corresponding matrix. Actually similar to Theorem 2.5.2, it also involves the CARE. We use the stabilizing solution of the corresponding CARE to construct an invertible transformation U satisfying the condition in Lemma 2.5.1. Notice that in the proof of Theorem 2.5.3, $\mathcal{A} = A - BR^{-1}B^T \Pi - \frac{\rho}{2}I$ corresponds to A in Theorem 2.3.9; Q_Γ corresponds to $-C$ and $BR^{-1}B^T$ corresponds to D . In addition, D in Theorem 2.3.9 should be nonnegative definite. Since R is positive definite, we can obtain that R^{-1} is also positive and then $BR^{-1}B^T$ is nonnegative definite.

2.6 Computational Methods

2.6.1 The Stabilizing Solution

From Theorem 2.5.2, Theorem 2.5.3 and their proofs, we can find that it is quite important to find a stabilizing solution of CARE of the following form:

$$XA + \mathcal{A}^T X - XBR^{-1}B^T X - Q = 0, \quad (2.41)$$

where Q is symmetric and R is positive definite. We will use eigenvectors and generalized eigenvectors of H to compute the stabilizing solution, where

$$H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q & -\mathcal{A}^T \end{pmatrix},$$

and assume that H has no purely imaginary eigenvalues. Notices that Q is allowed to be indefinite.

Definition 2.6.1. Consider an eigenvalue problem

$$Hz = \lambda z. \quad (2.42)$$

λ is an eigenvalue when $\det(H - \lambda I) = 0$ and eigenvectors of λ is all nonzero vectors which satisfy equation (2.42). The generalized eigenvectors are all the nonzero vectors z such that $(H - \lambda I)^r z = 0$ and $(H - \lambda I)^{r-1} z \neq 0$, where r is a positive integer.

From the theory related to Jordan canonical form [20], there exists an invertible matrix T , such that $T^{-1}HT = S$, where T consists of $2n$ independent generalized eigenvectors and S is the Jordan canonical form of H . According to our assumption that H has no purely imaginary eigenvalues, H has n stable eigenvalues and n unstable eigenvalues (see Proposition 2.2.2). Thus, by reordering the generalized eigenvectors, we can find an invertible $T = \begin{pmatrix} U & V \end{pmatrix}$, such that

$$T^{-1}HT = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$$

where U consists of generalized eigenvectors corresponding to stable eigenvalues which are all in Jordan block S_{11} , and V consists of generalized eigenvectors corresponding to unstable eigenvalues which are all in Jordan block S_{22} .

Now by partitioning U into two $n \times n$ matrices

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

we obtain that

$$H \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} S_{11} = \begin{pmatrix} U_1 S_{11} \\ U_2 S_{11} \end{pmatrix},$$

which implies that

$$\mathcal{A}U_1 - BR^{-1}B^T U_2 = U_1 S_{11}, \tag{2.43}$$

$$QU_1 - A^T U_2 = U_2 S_{11}. \tag{2.44}$$

Here, if we assume that U_1 is invertible (actually we will prove it), from (2.43), we can obtain

$$U_1^{-1}\mathcal{A}U_1 - U_1^{-1}BR^{-1}B^T U_2 = S_{11}. \tag{2.45}$$

Then combining (2.45) with (2.44), we have

$$QU_1 - \mathcal{A}^T U_2 = U_2 U_1^{-1} \mathcal{A} U_1 - U_2 U_1^{-1} B R^{-1} B^T U_2, \quad (2.46)$$

i.e.

$$U_2 U_1^{-1} \mathcal{A} + \mathcal{A}^T U_2 U_1^{-1} - U_2 U_1^{-1} B R^{-1} B^T U_2 U_1^{-1} - Q = 0,$$

which means $U_2 U_1^{-1}$ is the solution of CARE (2.41). Actually, we will prove a conclusion that since S_{11} is stable, $U_2 U_1^{-1}$ is the stabilizing solution of CARE (2.41).

Pappas, Laub and Sandell [17] presented a discrete-time version of Lemma 2.6.2 and Lemma 2.6.3 below for the case where the state weight matrix in their Riccati equation is nonnegative definite.

Lemma 2.6.2. *If there exists a solution X of CARE (2.41), X must have the form $X = T_2 T_1^{-1}$, where $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ consists of n generalized eigenvectors of H .*

Proof. Let $K = \mathcal{A} - B R^{-1} B^T X$ and then there exists an invertible T_1 , such that $T_1^{-1} K T_1 = \hat{S}$, where \hat{S} is the Jordan canonical form of K . Define $T_2 = X T_1$. Then we have $X = T_2 T_1^{-1}$.

Since $K = \mathcal{A} - B R^{-1} B^T X$ and $T_1^{-1} K T_1 = \hat{S}$, we have

$$\mathcal{A} - B R^{-1} B^T X = T_1 \hat{S} T_1^{-1},$$

and then

$$\mathcal{A} T_1 - B R^{-1} B^T X T_1 = T_1 \hat{S}.$$

Finally we obtain

$$\mathcal{A} T_1 - B R^{-1} B^T T_2 = T_1 \hat{S}. \quad (2.47)$$

From CARE (2.41), we have

$$\mathcal{A}^T X + X K - Q = 0.$$

Thus, we have

$$\mathcal{A}^T X + X T_1 \hat{S} T_1^{-1} - Q = 0,$$

and then

$$Q - \mathcal{A}^T T_2 = T_2 \hat{S}. \quad (2.48)$$

From (2.47) and (2.48), we have

$$\begin{aligned} \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q & -\mathcal{A}^T \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} &= \begin{pmatrix} \mathcal{A}T_1 - BR^{-1}B^T T_2 \\ Q - \mathcal{A}^T T_2 \end{pmatrix} = \begin{pmatrix} T_1 \hat{S} \\ T_2 \hat{S} \end{pmatrix} \\ &= \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \hat{S}. \end{aligned} \quad (2.49)$$

Thus, $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ is n generalized eigenvectors of H . □

Lemma 2.6.3. Consider $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ consisting of n generalized eigenvectors of H and corresponding to the Jordan canonical block \hat{S} . Assume that V_1 is invertible. Then

- 1) $X = V_2 V_1^{-1}$ is a real solution of CARE (2.41).
- 2) $X = V_2 V_1^{-1}$ is the stabilizing solution CARE (2.41) if and only if \hat{S} is stable.

Proof. Our method follows [18].

- 1) Similar to (2.49), we have

$$H \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \hat{S} = \begin{pmatrix} V_1 \hat{S} \\ V_2 \hat{S} \end{pmatrix}, \quad (2.50)$$

which implies that:

$$\mathcal{A}V_1 - BR^{-1}B^T V_2 = V_1 \hat{S}, \quad (2.51)$$

$$QV_1 - \mathcal{A}^T V_2 = V_2 \hat{S}. \quad (2.52)$$

Since V_1 is invertible, we have

$$V_1^{-1} \mathcal{A}V_1 - V_1^{-1} BR^{-1} B^T V_2 = \hat{S}. \quad (2.53)$$

Then combining (2.53) with (2.52), we have

$$QV_1 - \mathcal{A}^T V_2 = V_2 V_1^{-1} \mathcal{A} V_1 - V_2 V_1^{-1} B R^{-1} B^T V_2,$$

i.e.

$$V_2 V_1^{-1} \mathcal{A} + \mathcal{A}^T V_2 V_1^{-1} - V_2 V_1^{-1} B R^{-1} B^T V_2 V_1^{-1} - Q = 0.$$

Thus, $X = V_2 V_1^{-1}$ is the solution of CARE (2.41).

2) From (2.50), we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} H \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \hat{S}. \quad (2.54)$$

Since H is a Hamiltonian matrix, the left-hand side of (2.54) is symmetric. Thus, the RHS of (2.54) is also symmetric and then we obtain

$$\hat{S}^T \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \hat{S},$$

i.e.

$$(V_1^T V_2 - V_2^T V_1) \hat{S} + \hat{S}^T (V_1^T V_2 - V_2^T V_1) = 0 \quad (2.55)$$

Denote $V_1^T V_2 - V_2^T V_1$ as Y . Equation (2.55) is

$$Y \hat{S} + \hat{S}^T Y = 0,$$

which is a Lyapunov equation of Y . If \hat{S} is stable, this Lyapunov equation has a unique solution $Y = 0$. Thus, $V_1^T V_2 = V_2^T V_1$, and then $V_2 V_1^{-1} = (V_1^T)^{-1} V_2^T = (V_2 V_1^{-1})^T$, i.e. $X = V_2 V_1^{-1}$ is symmetric. Since we have

$$\mathcal{A} V_1 - B R^{-1} B^T V_2 = V_1 \hat{S},$$

so

$$\mathcal{A} - B R^{-1} B^T X = V_1 \hat{S} V_1^{-1}.$$

We have $V_1 \hat{S} V_1^{-1}$ is stable. Thus $\mathcal{A} - BR^{-1}B^T X$ is stable, which means X is the stabilizing solution.

If X is the stabilizing solution, then $\mathcal{A} - BR^{-1}B^T X$ is stable. Thus, $V_1 \hat{S} V_1^{-1}$ is stable, which implies \hat{S} is stable. □

Theorem 2.6.4. *Assume (\mathcal{A}, B) is stabilizable and the Hamiltonian matrix H has no purely imaginary eigenvalues. Then U_1 is invertible and $X = U_2 U_1^{-1}$ is the stabilizing solution of CARE (2.41), where $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ consists of n generalized eigenvectors of H and corresponds to the stable Jordan canonical block S_{11} .*

Proof. According to Theorem 2.3.9 and Theorem 2.3.12, there exists a unique stabilizing solution of CARE (2.41), denoted by X_+ . By Lemma 2.6.2, X_+ must have the form $X_+ = V_2 V_1^{-1}$, where $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ consists of n generalized eigenvectors of H , and the corresponding Jordan canonical block is \hat{S} . By Lemma 2.6.3, \hat{S} is stable. Since S_{11} and \hat{S} are both stable, they both includes all the stable eigenvalues of H . Thus, there exists a orthogonal matrix T , such that $T^{-1} S_{11} T = \hat{S}$. In addition, we have $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$. Thus, $U_1 = TV_1$ is invertible and then by Lemma 2.6.3, $X = U_2 U_1^{-1}$ is the stabilizing solution of CARE (2.41). □

Theorem 2.6.4 guarantees that when we find a set of generalized eigenvectors corresponding to the stable Jordan canonical block of H , the stabilizing solution can be found by this set of generalized eigenvectors. In addition, we just require two assumptions that (\mathcal{A}, B) is stabilizing and H has no purely imaginary eigenvalues. Indeed, these two assumptions are to guarantee the existence and uniqueness of the stabilizing solution. On the other hand, a similar idea is presented in [17]. When Q is nonnegative definite, Pappas, Laub and Sandell [17] used generalized eigenvectors to find the positive definite solution of a discrete-time algebraic Riccati equation. In our assumption, Q only needs to be symmetric.

However, computing generalized eigenvectors is still complicated. Now we introduce a more convenient method to compute the stabilizing solution of CARE (2.41). Here, we still assume (\mathcal{A}, B) is stabilizable and H has no purely imaginary eigenvalues.

Theorem 2.6.5. [12] *There exists an orthogonal transformation $V \in \mathbb{R}^{n \times n}$, such that*

$$V^T H V = \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix},$$

where all eigenvalues of H_{11} are the stable eigenvalues of H .

We refer to $\begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix}$ as the real Schur form and V consists of $2n$ independent vectors which are called Schur vectors. If we partition V into four $n \times n$ block $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, $\begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$ consists of n Schur vectors corresponding to stable Schur block H_{11} .

Lemma 2.6.6. [12, p. 915] *Denote $\begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$ as the Schur vectors corresponding to the stable Schur block of H and $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ as the generalized vectors corresponding to the stable Jordan canonical block of H , there exists an invertible $T \in \mathbb{R}^{n \times n}$ such that*

$$\begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} T.$$

The proof of Lemma 2.6.6 is presented in [12], which corresponds to [12, Remark 1, p915]. This lemma establishes the connection between Schur vectors and generalized eigenvectors, so we can extend the conclusion about generalized eigenvectors to the Schur vectors.

Theorem 2.6.7. *Assume (A, B) is stabilizable and the Hamiltonian matrix H has no purely imaginary eigenvalues. Then V_{11} is invertible and $X = V_{21}V_{11}^{-1}$ is the stabilizing solution of CARE (2.41), where $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ consists of n Schur vectors of H and corresponds to the stable Schur block H_{11} .*

Proof. By Theorem 2.6.4, there exists a set of generalized eigenvectors corresponding to stable Jordan block of H , such that $U_2U_1^{-1}$ is the stabilizing solution of CARE (2.41).

By Lemma 2.6.6, we have $\begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} T$, so $V_{11} = U_1T$ is invertible. Furthermore, $X = V_{21}V_{11}^{-1} = (U_2T)(U_1T)^{-1} = U_2TT^{-1}U_1^{-1} = U_2U_1^{-1}$ is a stabilizing solution. \square

Now we show some examples to compute the stabilizing solution of the following CARE (2.56) and the results will be used in further discussions. All the computation results are based on Matlab.

$$XA + A^T X - XBR^{-1}B^T X - Q_\Gamma = 0, \quad (2.56)$$

where $\mathcal{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I$, $Q_\Gamma = \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma$ and the corresponding Hamiltonian matrix is

$$H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & \mathcal{A}^T \end{pmatrix}.$$

In the examples below, we specify the parameters, including the matrix A , which further determines \mathcal{A} .

Example 2.6.8. Consider a scalar example where $A = 2, B = 1, Q = 2, R = 1, \rho = 1, \Gamma = 1$. We have

$$H = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & \mathcal{A}^T \end{pmatrix} = \begin{pmatrix} -2.0616 & -1.0000 \\ 2.0000 & 2.0616 \end{pmatrix},$$

and the eigenvalues of H is -1.5 and 1.5 , so H has no purely imaginary eigenvalues. The Schur decomposition is as follow:

$$H = \begin{pmatrix} -0.8719 & -0.4896 \\ 0.4896 & -0.8719 \end{pmatrix} \begin{pmatrix} -1.5000 & -3.0000 \\ 0 & 1.5000 \end{pmatrix} \begin{pmatrix} -0.8719 & -0.4896 \\ 0.4896 & -0.8719 \end{pmatrix}^{-1},$$

so $V_{11} = -0.8719, V_{21} = 0.4896$ and then $X = V_{21}V_{11}^{-1} = -0.5616$.

Example 2.6.8 shows that a stabilizing solution X may be negative (definite).

Example 2.6.9. Consider $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Gamma = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}$ and $\rho = 1, R = 1$.

We have

$$H = \begin{pmatrix} 2.7370 & -8.3699 & -1.0000 & -1.0000 \\ 2.2370 & -5.8699 & -1.0000 & -1.0000 \\ -1.0000 & -1.0000 & -2.7370 & -2.2370 \\ -1.0000 & 0 & 8.3699 & 5.8699 \end{pmatrix},$$

and the eigenvalues of H are $\pm 1.1901 \pm 0.2034i$, so H has no purely imaginary eigenvalues. The Schur decomposition is as follow:

$$H = V \begin{pmatrix} -1.1901 & 2.7201 & -0.0793 & -8.4223 \\ -0.0152 & -1.1901 & -12.1723 & 0.0793 \\ 0 & 0 & 1.1901 & 2.7201 \\ 0 & 0 & -0.0152 & 1.1901 \end{pmatrix} V^{-1},$$

where

$$V = \begin{pmatrix} 0.6136 & -0.5467 & 0.2808 & -0.4958 \\ 0.2808 & -0.4957 & -0.6137 & 0.5466 \\ -0.4958 & -0.2808 & -0.5467 & -0.6136 \\ 0.5466 & 0.6137 & -0.4957 & -0.2808 \end{pmatrix}.$$

Thus, $V_{11} = \begin{pmatrix} 0.6136 & -0.5467 \\ 0.2808 & -0.4957 \end{pmatrix}$, $V_{21} = \begin{pmatrix} -0.4958 & -0.2808 \\ 0.5466 & 0.6137 \end{pmatrix}$ and $X = V_{21}V_{11}^{-1} = \begin{pmatrix} -2.1548 & 2.9428 \\ 2.9428 & -4.4836 \end{pmatrix}$. Actually, we can check that $A - BR^{-1}B^T X = \begin{pmatrix} 1.9489 & -6.8292 \\ 1.4489 & -4.3292 \end{pmatrix}$, so its eigenvalues are $-1.1901 \pm 0.2034i$. Thus, X is a stabilizing solution, which means that our computational method works.

2.6.2 Computational Results

Now recall our methods of finding the unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$. Our original ODE system has the following form:

$$\begin{pmatrix} \frac{d\bar{x}}{dt} \\ \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^T\Pi & -BR^{-1}B^T \\ Q_\Gamma & \rho I - A^T + \Pi BR^{-1}B^T \end{pmatrix} \begin{pmatrix} \bar{x} \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_\Gamma \end{pmatrix}, \quad (2.57)$$

By the transformation matrix $T = \begin{pmatrix} e^{\frac{\rho}{2}t} & 0 \\ 0 & e^{\frac{\rho}{2}t} \end{pmatrix}$, we obtain a new ODE system of (\tilde{x}, \tilde{s}) :

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-\frac{\rho}{2}t}\eta_\Gamma \end{pmatrix}.$$

By selecting a transformation matrix $U = \begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}$, we obtain the ODE system of (\hat{x}, \hat{s}) :

$$\begin{pmatrix} \frac{d\hat{x}}{dt} \\ \frac{d\hat{s}}{dt} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_C^T & -BR^{-1}B^T \\ 0 & -\mathcal{A}_C^T \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{s} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-\frac{\rho}{2}t}\eta_\Gamma \end{pmatrix}, \quad (2.58)$$

where $\mathcal{A}_C = A - BR^{-1}B^T X_+$. Since

$$\begin{pmatrix} \hat{x} \\ \hat{s} \end{pmatrix} = U^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} \quad (2.59)$$

and

$$\begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} = T^{-1} \begin{pmatrix} \bar{x} \\ s \end{pmatrix} \quad (2.60)$$

we obtain that

$$\begin{pmatrix} \bar{x}(t) \\ s(t) \end{pmatrix} = TU \begin{pmatrix} \hat{x}(t) \\ \hat{s}(t) \end{pmatrix} = \begin{pmatrix} e^{\frac{\rho}{2}t}I & 0 \\ e^{\frac{\rho}{2}t}X_+ & e^{\frac{\rho}{2}t}I \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ \hat{s}(t) \end{pmatrix} = \begin{pmatrix} e^{\frac{\rho}{2}t}\hat{x}(t) \\ X_+e^{\frac{\rho}{2}t}\hat{x}(t) + e^{\frac{\rho}{2}t}\hat{s}(t) \end{pmatrix}. \quad (2.61)$$

Here X_+ is the stabilizing solution of the following CARE:

$$XBR^{-1}B^T X - XA - A^T X + Q_\Gamma = 0, \quad (2.62)$$

where $\mathcal{A} = A - BR^{-1}B^T \Pi - \frac{\rho}{2}I$.

According to Theorem 2.5.3, for obtaining the solution (\bar{x}, s) of SCE equation system (2.57), such that $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$, we (must) select the initial condition

$$s_0 = X_+ \bar{x}_0 - \int_0^{+\infty} e^{(\mathcal{A}_C^T - \frac{\rho}{2}I)\tau} \eta_\Gamma d\tau,$$

where $\mathcal{A}_C^T = (A - BR^{-1}B^T X_+)^T$. We can firstly find the solution (\tilde{x}, \tilde{s}) of ODE system (2.58) under the initial condition $\hat{s}_0 = -\int_0^{+\infty} e^{\mathcal{A}_C^T \tau} \eta_\Gamma e^{-\rho\tau/2} d\tau$ and then use (2.61) to obtain the corresponding solution of SCE equation system (2.57). Indeed, we can also solve the SCE equations system (2.57) directly under the initial condition s_0 . Here is a summary of our computational method:

| | |
|--------|--|
| Step 1 | Check the eigenvalues of the Hamiltonian matrix H |
| Step 2 | Obtain the Schur vectors by Schur decomposition |
| Step 3 | Compute the stabilizing solution to the corresponding CARE |
| Step 4 | Compute the initial condition s_0 |
| Step 5 | Compute the solution to the SCE equation system |

Example 2.6.10. Consider a scalar example: $A = 2, B = 1, Q = 2, R = 1, \eta = 1, \rho = 1, \Gamma = \gamma$ and the initial state $x_0 = 1$. The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}}{dt} \\ \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} -1.5616 & -1.0000 \\ 2(2\gamma - \gamma^2) & 2.5616 \end{pmatrix} \begin{pmatrix} \bar{x} \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ 2(1 - \gamma) \end{pmatrix}. \quad (2.63)$$

Now, we consider some special cases.

1) For $\gamma = 1$, we have $Q_\Gamma = Q > 0$. The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}}{dt} \\ \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} -1.5616 & -1.0000 \\ 2 & 2.5616 \end{pmatrix} \begin{pmatrix} \bar{x} \\ s \end{pmatrix},$$

and

$$H = \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} = \begin{pmatrix} -2.0616 & -1.0000 \\ 2.0000 & 2.0616 \end{pmatrix}.$$

The eigenvalues of H is -1.5 and 1.5 , so H has no purely imaginary eigenvalues. By solving the CARE (2.62), $X_+ = -0.5615$ and then $\mathcal{A}_C = -1.5$.

Thus, we select the initial condition $s_0 = X_+\bar{x}_0 - \int_0^{+\infty} e^{(\mathcal{A}_C^T - \frac{\rho}{2}I)\tau} \eta_\Gamma d\tau = -0.5615$. By solving the SCE equation system under the initial condition $(1, -0.5615)$, we obtain the solution $(\bar{x}(t), s(t)) = (1.0 * e^{-1.0t}, -0.561553 * e^{-1.0*t})$ and obviously $(\bar{x}(t), s(t)) \in C_{1/2}([0, \infty), \mathbb{R}^2)$.

2) For $\gamma = 2$, we have $Q_\Gamma = 0$. The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}}{dt} \\ \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} -1.5616 & -1.0000 \\ 0 & 2.5616 \end{pmatrix} \begin{pmatrix} \bar{x} \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

and

$$H = \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} = \begin{pmatrix} -2.0616 & -1.0000 \\ 0 & 2.0616 \end{pmatrix}.$$

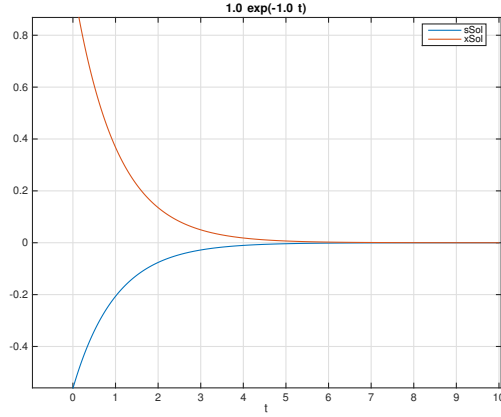


Figure 2.1: Solution when $\gamma = 1$

The eigenvalues of H are -2.0616 and 2.0616 , so H has no purely imaginary eigenvalues. Obviously the stabilizing solution $X_+ = 0$ and the initial condition we choose is $s_0 = -\int_0^{+\infty} e^{(A_C^T - \frac{\rho}{2}I)\tau} \eta_\Gamma d\tau = -0.780776$. Then the corresponding solution $(\bar{x}(t), s(t)) = (1.5 * e^{-1.56155*t} - 0.5, 0.780776)$, where $s(t)$ is a constant.

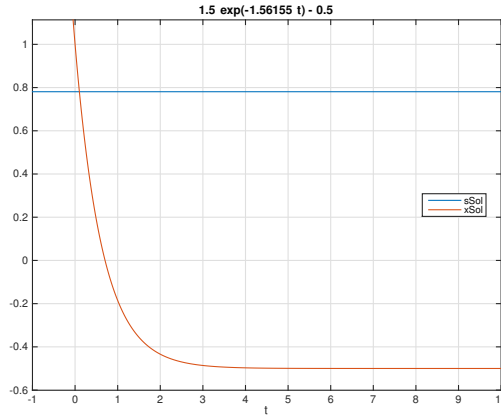


Figure 2.2: Solution when $\gamma = 2$

3) For $\gamma = 3$, we have $Q_\Gamma = -3Q < 0$.

The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}}{dt} \\ \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} -1.5616 & -1.0000 \\ -6.0000 & 2.5616 \end{pmatrix} \begin{pmatrix} \bar{x} \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix},$$

and

$$H = \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} = \begin{pmatrix} -2.0616 & -1.0000 \\ -6 & 2.0616 \end{pmatrix}.$$

The eigenvalues of H are -3.2016 and 3.2016 , so H has no purely imaginary eigenvalues. By solving the CARE (2.62), $X_+ = 1.1400$ and then $\mathcal{A}_C = -3.2016$.

Thus, we select the initial condition $s_0 = X_+\bar{x}_0 - \int_0^{+\infty} e^{(\mathcal{A}_C - \frac{\rho}{2}I)\tau} \eta_\Gamma d\tau = 2.22063$. By solving the SCE equation system under the initial condition $(1, 2.22063)$, we obtain the solution $(\bar{x}(t), s(t)) = (1.4e^{-2.70156t} - 0.4, 1.59601e^{-2.70156t} + 0.624621)$ and obviously $(\bar{x}(t), s(t)) \in C_{1/2}([0, \infty), \mathbb{R}^2)$.

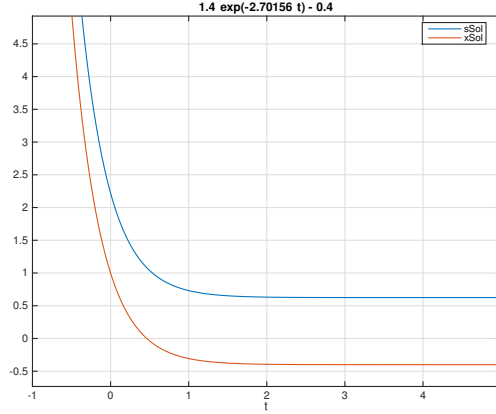


Figure 2.3: Solution when $\gamma = 3$

Example 2.6.10 shows that our method works well when Q_Γ is positive, negative and zero. Thus, we can handle the situation that $Q_\Gamma > 0$. This situation is not covered by [8, Theorem 4.3]. However, if Q_Γ is not a scalar, we should consider a situation that Q_Γ is an indefinite matrix.

Example 2.6.11. Consider $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Gamma = \gamma \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}$, $\rho = 1$, $R = 1$, and the initial state is $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

1) For $\gamma = 2$, we have $Q_\Gamma = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$. Since the eigenvalues of Q_Γ are -1.618 and 0.618 , Q_Γ is an indefinite matrix.

The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}_1}{dt} \\ \frac{d\bar{x}_2}{dt} \\ \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} 3.2370 & -8.3699 & -1.0000 & -1.0000 \\ 2.2370 & -5.3699 & -1.0000 & -1.0000 \\ -1.0000 & -1.0000 & -2.2370 & -2.2370 \\ -1.0000 & 0 & 8.3699 & 6.3699 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} H &= \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} \\ &= \begin{pmatrix} 2.7370 & -8.3699 & -1.0000 & -1.0000 \\ 2.2370 & -5.8699 & -1.0000 & -1.0000 \\ -1.0000 & -1.0000 & -2.7370 & -2.2370 \\ -1.0000 & 0 & 8.3699 & 5.8699 \end{pmatrix}. \end{aligned}$$

The eigenvalues of H are $-1.9655 \pm 0.3365i$ and $1.9655 \pm 0.3365i$, so H has no purely imaginary eigenvalues. By solving the CARE (2.62), $X_+ = \begin{pmatrix} 3.5811 & -4.4401 \\ -4.4401 & 6.0972 \end{pmatrix}$ and then

$$\mathcal{A}_C = \begin{pmatrix} 3.5960 & -10.0270 \\ 3.0960 & -7.5270 \end{pmatrix}.$$

Thus, we select the initial condition $s_0 = X_+\bar{x}_0 - \int_0^{+\infty} e^{(\mathcal{A}_C^T - \frac{\rho}{2}I)\tau} \eta_\Gamma d\tau = \begin{pmatrix} 0.437344 \\ 0.0377865 \end{pmatrix}$.

By solving the SCE equation system under the initial condition $(1, 1, 0.437344, 0.0377865)$, we obtain the corresponding solution. From Figure 2.4, obviously $(\bar{x}_1(t), \bar{x}_2(t), s_1(t), s_2(t)) \in C_{1/2}([0, \infty), \mathbb{R}^4)$.

2) For $\gamma = 1$, we have $Q_\Gamma = \begin{pmatrix} 0.7500 & 0 \\ 0 & 1.0000 \end{pmatrix}$. Since the eigenvalues of Q_Γ are 0.7500 and 1, Q_Γ is a positive definite matrix.

The SCE equation system is

$$\begin{pmatrix} \frac{d\bar{x}_1}{dt} \\ \frac{d\bar{x}_2}{dt} \\ \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} 3.2370 & -8.3699 & -1.0000 & -1.0000 \\ 2.2370 & -5.3699 & -1.0000 & -1.0000 \\ 0.7500 & 0 & -2.2370 & -2.2370 \\ 0 & 1 & 8.3699 & 6.3699 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ s_1 \\ s_2 \end{pmatrix},$$

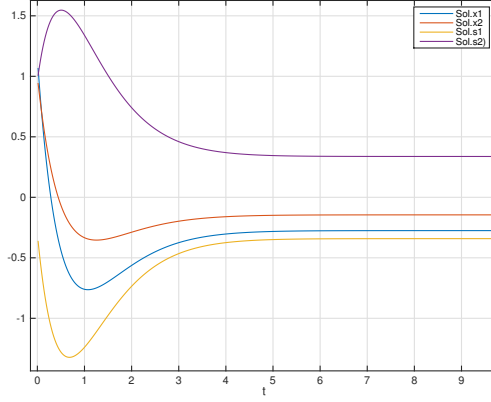


Figure 2.4: Solution when $\gamma = 2$

and

$$\begin{aligned}
 H &= \begin{pmatrix} A - BR^{-1}B^T\Pi - \frac{\rho}{2}I & -BR^{-1}B^T \\ Q_\Gamma & \frac{\rho}{2}I - A^T + \Pi BR^{-1}B^T \end{pmatrix} \\
 &= \begin{pmatrix} 2.7370 & -8.3699 & -1.0000 & -1.0000 \\ 2.2370 & -5.8699 & -1.0000 & -1.0000 \\ -1.0000 & -1.0000 & -2.7370 & -2.2370 \\ -1.0000 & 0 & 8.3699 & 5.8699 \end{pmatrix}.
 \end{aligned}$$

The eigenvalues of H are $-1.1901 \pm 0.2034i$ and $1.1901 \pm 0.2034i$, so H has no purely imaginary eigenvalues. By solving the CARE (2.62), $X_+ = \begin{pmatrix} -2.1548 & 2.9428 \\ 2.9428 & -4.4836 \end{pmatrix}$.

Thus, we select the initial condition $s_0 = X_+\bar{x}_0 = \begin{pmatrix} 0.788068 \\ -1.54074 \end{pmatrix}$. By solving the SCE equation system under the initial condition $(1, 1, 0.788068, -1.54074)$, we obtain the corresponding solution. From Figure 2.5, obviously $(\bar{x}_1(t), \bar{x}_2(t), s_1(t), s_2(t)) \in C_{1/2}([0, \infty), \mathbb{R}^4)$.

2.6.3 Comparison

In this section, we introduce a fixed point method, which is used to solve the SCE equation system and the Hamiltonian ODE system in [7] and [15]. Since our purpose is to compare the fixed point method with our method, we will omit some assumptions and details related to the fixed point method.

Consider this Hamiltonian ODE:

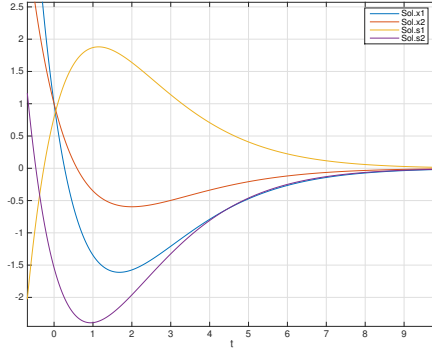


Figure 2.5: Solution when $\gamma = 1$

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathcal{A}^T \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\eta}_\Gamma \end{pmatrix}, \quad (2.64)$$

where $x(0) = x_0$ and the value of the initial condition \tilde{s}_0 is to be determined according to Section 2.1. Under some reasonable assumptions [7, 15], the solution $\tilde{x}(t)$ can be solved by fixed point equation $x(t) = [\Psi x](t)$, where

$$\Psi(x(t)) = e^{\mathcal{A}t}x_0 + \int_0^t e^{\mathcal{A}(t-s)}BR^{-1}B^T \left[\int_s^\infty e^{-\mathcal{A}^T(s-\tau)} \left(Q_\Gamma x(\tau) + e^{-\frac{\rho}{2}\tau} \eta_\Gamma \right) d\tau \right] ds.$$

And the solution of this equation is called a fixed point. According to the theory related to the fixed point method [7, 15], the fixed point exists and is unique if there exists a real number $\beta \in (0, 1)$, such that for each $t \geq 0$

$$\|\Psi(x(t)) - \Psi(y(t))\| \leq \beta \|x - y\|,$$

which is called the contraction mapping condition. Here, $\|\cdot\|$ is the Frobenius norm [4]. Indeed, we have

$$\begin{aligned}
\|\Psi(x(t)) - \Psi(y(t))\| &= \left\| \int_0^t e^{\mathcal{A}(t-s)} B R^{-1} B^T \left\{ \int_s^\infty e^{-\mathcal{A}^T(s-\tau)} [Q_\Gamma (x(\tau) - y(\tau))] d\tau \right\} ds \right\| \\
&\leq \int_0^t \left\| e^{\mathcal{A}(t-s)} B R^{-1} B^T \right\| \left(\int_s^\infty \left\| e^{-\mathcal{A}^T(s-\tau)} Q_\Gamma \right\| \|x - y\| d\tau \right) ds \\
&\leq \|x - y\| \times \int_0^t \left\| e^{\mathcal{A}(t-s)} B R^{-1} B^T \right\| \left(\int_s^\infty \left\| e^{-\mathcal{A}^T(s-\tau)} Q_\Gamma \right\| d\tau \right) ds \\
&\leq \|x - y\| \times \int_0^t \left\| e^{\mathcal{A}s} B R^{-1} B^T \right\| \left(\int_0^\infty \left\| e^{\mathcal{A}^T\tau} Q_\Gamma \right\| d\tau \right) ds \\
&\leq \|x - y\| \times \int_0^\infty \left\| e^{\mathcal{A}s} B R^{-1} B^T \right\| \left(\int_0^\infty \left\| e^{\mathcal{A}^T\tau} Q_\Gamma \right\| d\tau \right) ds
\end{aligned}$$

Let $\beta = \int_0^\infty \left\| e^{\mathcal{A}s} B R^{-1} B^T \right\| \left(\int_0^\infty \left\| e^{\mathcal{A}^T\tau} Q_\Gamma \right\| d\tau \right) ds$, we consider Example 2.6.11 and the computational results are based on Matlab.

In Example 2.6.11, we have $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Gamma = \gamma \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}$, $\rho = 1$, $R = 1$. When we set $\gamma = 2$, $\beta = 12.4298 > 1$. When we set $\gamma = 1$, $\beta = 6.24287 > 1$. Thus, the fixed point method cannot work in these two situations, while we solve these two situations successfully by our computational method. Actually, if we set $\gamma = 0.05$, $\beta = 0.894025 < 1$, which means that we can apply the fixed point method to solve Hamiltonian ODE system (2.64). However, the fixed point method has high computational load since it involves the iteration of function curves.

Our method avoids the contraction mapping condition. It solves the SCE equation system and the only assumption involved is that the corresponding Hamiltonian matrix has no purely imaginary eigenvalues. Also it avoids the iteration of function curves as in the fixed point method.

Chapter 3

Mean Field Game Problem

3.1 Description of Problem

Now, we consider a continuous time linear quadratic mean field game (MFG):

$$\begin{cases} dx_i = (Ax_i + Bu_i)dt + DdW_i, & (3.1) \\ \min J_i(u_i) = E \int_0^\infty e^{-\rho t} (|x_i - (\Gamma x^{(N)} + \eta)|_Q^2 + u_i^T R u_i) dt. & (3.2) \end{cases}$$

The initial states $\{x_i(0), 1 \leq i \leq N\}$ are independent. The noise processes $\{W_i, 1 \leq i \leq N\}$ are n_2 dimensional independent standard Brownian motions adapted to \mathcal{F}_t , which are also independent of $\{x_j(0), 1 \leq j \leq N\}$. The constant matrices A , B and D have compatible dimensions. Given a matrix Q , the quadratic form $z^T Q z$ may be denoted as $|z|_Q^2$.

According to mean field game theory [9], the decentralized strategies for the above problem can be designed by using the following ODE system:

$$\begin{cases} \frac{d\bar{x}}{dt} = (A - BR^{-1}B^T\Pi)\bar{x} - BR^{-1}B^T s, & (3.3) \\ \frac{ds}{dt} = Q\Gamma\bar{x} + (\rho I - A^T + \Pi BR^{-1}B^T)s + Q\eta. & (3.4) \end{cases}$$

Letting $\tilde{x} = e^{-\rho t/2}\bar{x}$ and $\tilde{s} = e^{-\rho t/2}s$, we have an ODE system of (\tilde{x}, \tilde{s}) :

$$\begin{cases} \frac{d\tilde{x}}{dt} = \mathcal{A}\tilde{x} - BR^{-1}B^T\tilde{s}, & (3.5) \\ \frac{d\tilde{s}}{dt} = Q\Gamma\tilde{x} - \mathcal{A}^T\tilde{s} + \tilde{\eta}_\Gamma, & (3.6) \end{cases}$$

where $\mathcal{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ and $\tilde{\eta}_\Gamma = e^{-\rho t/2}Q\eta$.

Notice that $Q\Gamma$ is generally not symmetric, so ODE system (3.3)-(3.4) is not a Hamiltonian ODE system, which means our method cannot be applied directly. However, we can follow the similar idea in Section 2.5 to find a solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.

3.2 Existence of A Solution

Let

$$M = \begin{pmatrix} \mathcal{A} & -BR^{-1}B^T \\ Q\Gamma & -\mathcal{A}^T \end{pmatrix}.$$

Theorem 3.2.1. *If there exists an invertible matrix $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, where U_{11} is invertible and*

$$U^{-1}MU = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix},$$

where M_{11} has n stable eigenvalues and M_{22} has n eigenvalues with positive real part. Then we can obtain the following conclusion:

- 1) For any fixed initial state x_0 of ODE system (3.3)-(3.4), there exists an initial condition s_0 , such that ODE system (3.3)-(3.4) has a unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.
- 2) Such an initial condition s_0 is unique and is equal to

$$U_{21}U_{11}^{-1}\bar{x}_0 + (U_{21}U_{11}^{-1}U_{12} - U_{22}) \int_0^{+\infty} e^{-M_{22}\tau} V_{22}Q\eta e^{-\rho\tau/2} d\tau,$$

where $V = U^{-1} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$.

Proof. 1) Proof of existence.

Letting $\begin{pmatrix} \hat{x} \\ \hat{s} \end{pmatrix} = V \begin{pmatrix} \tilde{x} \\ \tilde{s} \end{pmatrix}$, the ODE systems of (\hat{x}, \hat{s}) is:

$$\frac{d\hat{x}}{dt} = M_{11}\hat{x} + M_{12}\hat{s} + V_{12}\tilde{\eta}_\Gamma, \quad (3.7)$$

$$\frac{d\hat{s}}{dt} = M_{22}\hat{s} + V_{22}\tilde{\eta}_\Gamma. \quad (3.8)$$

Firstly, since $\tilde{\eta}_\Gamma = e^{-\rho t/2}Q\eta$, from (3.8), we can obtain that

$$\frac{d\hat{s}}{dt} = M_{22}\hat{s} + V_{22}Q\eta e^{-\rho t/2}. \quad (3.9)$$

Thus, the solution of ODE (3.8) has the form:

$$\hat{s}(t) = e^{M_{22}t} \hat{s}_0 + \int_0^t e^{M_{22}(t-\tau)} V_{22} Q \eta e^{-\rho\tau/2} d\tau.$$

Since M_{22} is stable, the integral $\int_0^{+\infty} e^{-M_{22}\tau} V_{22} Q \eta e^{-\rho\tau/2} d\tau$ exists and finite. Letting

$$\hat{s}_0 = - \int_0^{+\infty} e^{-M_{22}\tau} V_{22} Q \eta e^{-\rho\tau/2} d\tau,$$

we obtain that

$$\begin{aligned} \hat{s}(t) &= - \int_t^\infty e^{M_{22}(t-\tau)} V_{22} Q \eta e^{-\rho\tau/2} d\tau \\ &= - \int_0^\infty e^{-M_{22}\tau} V_{22} Q \eta e^{-\rho(\tau+t)/2} d\tau \\ &= -e^{-\rho t/2} \int_0^\infty e^{(-M_{22}-\frac{\rho}{2}I)\tau} V_{22} Q \eta d\tau. \end{aligned} \tag{3.10}$$

Denote the maximal real part of eigenvalues of M_{11} as σ_s . Since M_{11} is stable, σ_s is negative. Consider a real positive number ε such that $\varepsilon < \min(-\sigma_s, \frac{\rho}{2})$. From equation (3.10), we can obtain

$$e^{\varepsilon t} \hat{s}(t) = -e^{-(\frac{\rho}{2}-\varepsilon)t} \int_0^\infty e^{(-M_{22}-\frac{\rho}{2}I)\tau} V_{22} Q \eta d\tau. \tag{3.11}$$

Since $(-M_{22} - \frac{\rho}{2}I)$ is stable and $\frac{\rho}{2} - \varepsilon > 0$, we have that $e^{\varepsilon t} \hat{s}(t)$ is bounded. Thus, $e^{\frac{\rho}{2}t} \hat{s}(t)$ is within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$.

Secondly, from the ODE (3.7), we can obtain that

$$\frac{d\hat{x}}{dt} = M_{11}\hat{x} + M_{12}\hat{s}(t) + V_{12}Q\eta e^{-\rho t/2}, \tag{3.12}$$

where $\hat{s}(t)$ is the solution in (3.10).

Now consider $e^{\varepsilon t} \hat{x}(t)$, where $\hat{x}(t)$ is the solution of ODE (2.27). We can obtain that

$$\begin{aligned} \frac{d[e^{\varepsilon t} \hat{x}(t)]}{dt} &= \varepsilon e^{\varepsilon t} \hat{x}(t) + e^{\varepsilon t} \frac{d\hat{x}}{dt} \\ &= \varepsilon e^{\varepsilon t} \hat{x}(t) + e^{\varepsilon t} [M_{11}\hat{x}(t) + M_{12}\hat{s}(t) + V_{12}Q\eta e^{-\rho t/2}] \\ &= (M_{11} + \varepsilon I) [e^{\varepsilon t} \hat{x}(t)] + M_{12} [e^{\varepsilon t} \hat{s}(t)] + V_{12}Q\eta e^{(\varepsilon-\frac{\rho}{2})t}. \end{aligned}$$

From (3.11), we can obtain that $\lim_{t \rightarrow +\infty} e^{\varepsilon t} \hat{s}(t) = 0$. In addition, since $(M_{11} + \varepsilon I)$ is stable, for any initial state \hat{x}_0 , we conclude that $e^{\varepsilon t} \hat{x}(t)$ is bounded, which means that $e^{\frac{\rho}{2}t} \hat{x}(t)$ is within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$

Now, we obtain $e^{\frac{\rho}{2}t}\hat{x}(t)$ and $e^{\frac{\rho}{2}t}\hat{s}(t)$ are both within $C_{\rho/2}([0, \infty), \mathbb{R}^n)$. Since

$$\begin{pmatrix} \bar{x}(t) \\ s(t) \end{pmatrix} = U \begin{pmatrix} e^{\frac{\rho}{2}t}\hat{x}(t) \\ e^{\frac{\rho}{2}t}\hat{s}(t) \end{pmatrix},$$

we obtain that $\bar{x}(t) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ and $s(t) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$, i.e. $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$.

Finally, since

$$\bar{x}_0 = U_{11}\hat{x}_0 + U_{12}\hat{s}_0,$$

$$s_0 = U_{21}\hat{x}_0 + U_{22}\hat{s}_0,$$

and U_{11} is invertible, for any fixed \bar{x}_0 and fixed \hat{s}_0 , there always exists an initial condition

$$s_0 = U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}_0) + U_{22}\hat{s}_0.$$

Thus, for obtaining $\bar{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ and $s \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$, we can choose the initial condition

$$\begin{aligned} s_0 &= U_{21}U_{11}^{-1}(\bar{x}_0 - U_{12}\hat{s}_0) + U_{22}\hat{s}_0 \\ &= U_{21}U_{11}^{-1}\bar{x}_0 - (U_{21}U_{11}^{-1}U_{12} - U_{22})\hat{s}_0 \\ &= U_{21}U_{11}^{-1}\bar{x}_0 + (U_{21}U_{11}^{-1}U_{12} - U_{22}) \int_0^{+\infty} e^{-M_{22}\tau} V_{22} Q \eta e^{-\rho\tau/2} d\tau \end{aligned}$$

2) The proof of uniqueness is the same as in the corresponding part of Lemma 2.5.1. \square

Theorem 3.2.1 provides a method to find a unique solution $(\bar{x}, s) \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ when we can find such an invertible matrix U . Actually, from the condition of Theorem 2.5.1, this method can be only applied when M has n stable eigenvalues and n eigenvalues with positive real parts. Here we provide an example:

Example 3.2.2. Consider $A = \begin{pmatrix} 5 & -5 \\ 0 & 10 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 5 & 0 \\ 2.5 & 5 \end{pmatrix}$, $\rho = 2$, $R = 1$, and the initial state is $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

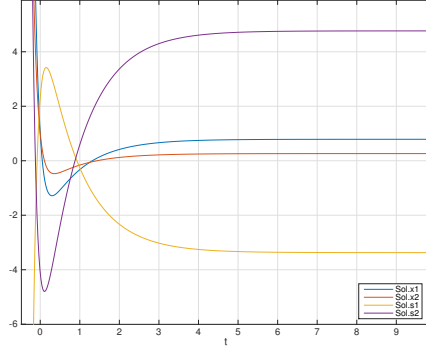


Figure 3.1: Solution of Example 3.2.2

Since

$$M = \begin{pmatrix} 14.7999 & -42.0915 & -1.0000 & -1.0000 \\ 10.7999 & -28.0915 & -1.0000 & -1.0000 \\ 5.0000 & 0 & -14.7999 & -10.7999 \\ 2.5000 & 5.0000 & 42.0915 & 28.0915 \end{pmatrix},$$

the eigenvalues of M are $9.2522, 1.7783, -2.0950$ and -8.9356 . The Schur decomposition is

$$U \begin{pmatrix} -8.9356 & -13.4806 & 27.2557 & -35.7593 \\ 0 & -2.0950 & -46.5898 & -32.5048 \\ 0 & 0 & 9.2522 & -15.8037 \\ 0 & 0 & 0 & 1.7783 \end{pmatrix} U^{-1},$$

where

$$U = \begin{pmatrix} -0.5425 & -0.6081 & -0.0928 & 0.5721 \\ -0.3060 & -0.4284 & -0.3027 & -0.7945 \\ 0.5543 & -0.1947 & -0.7863 & 0.1911 \\ -0.5521 & 0.6394 & -0.5305 & 0.0700 \end{pmatrix}.$$

Since $U_{11} = \begin{pmatrix} -0.5425 & -0.6081 \\ -0.3060 & -0.4284 \end{pmatrix}$ and then $\det(U_{11}) = 0.0464 > 0$, so U_{11} is invertible.

Thus, the initial condition $s_0 = U_{21}U_{11}^{-1}\bar{x}_0 + (U_{21}U_{11}^{-1}U_{12} - U_{22}) \int_0^{+\infty} e^{-M_{22}\tau} V_{22} Q \eta e^{-\rho\tau/2} d\tau = \begin{pmatrix} 2.31075 \\ -4.11538 \end{pmatrix}$. From Figure 3.1, it is obvious that the corresponding solution is within $C_{1/2}([0, \infty), \mathbb{R}^4)$.

Chapter 4

Conclusion

This thesis develops a methodology to prove the existence and uniqueness of the solutions of the social certainty equivalence equation systems, which is related to social optima in mean field linear-quadratic-Gaussian (LQG) control problems. When the corresponding Hamiltonian matrix has no purely imaginary eigenvalues, a solution within $C_{\rho/2}([0, \infty), \mathbb{R}^{2n})$ can be found by choosing the initial condition s_0 . On the other hand, this thesis also develops a computational method for the stabilizing solution of continuous-time algebraic Riccati equation by using generalized eigenvectors or Schur vectors. We further extend the stable/unstable subspace decomposition method to solve a linear quadratic mean field game where the ODE system does not have a Hamiltonian structure.

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Appendix A

Matlab Code

Here is the matlab code of my example in the section 2.6.2 and 2.6.3.

```
clear ;
clc ;
A=[1 -1; 0 2];
B=[1;1];
Q=[1 ,0;0 ,1];
R=1;
eta = [1;0];
gam=1;
Gamma=[1 ,0;0.5 ,1]*gam;
rho=1;
E=[1 ,0;0 ,1];
O2=[0;0];
O4=[0 ,0;0 ,0];
x0=[1;1];

%Calculate coefficient
Pi=Sol_Pi(A,B,R,Q,rho);
Pi=double(Pi);
B_r=B*R^-1 *B';
A_n=A-B_r*Pi-rho/2*E;
```

```

Q_g=Gamma'*Q+Q*Gamma-Gamma'*Q*Gamma;
eta_g=(E-Gamma')*Q*eta;

%SCE coefficient

SCE_m=[A-B_r*Pi,-B_r;Q_g,rho*E-A'+Pi*B_r];
eta_l=[O2;eta_g];

%Hamiltonian matrix
H=[A_n,-B_r;Q_g,-A_n'];

%Calculate stabilizing solution X_p=care(A_n,B,-Q_g,R,O2,E)
[V,T]=schur(H);
[V_r,T_r]=ordschur(V,T,'lhp');
V_b=mat2cell(V_r,[2 2],[2 2]);
V_1=V_b{1,1};
V_2=V_b{2,1};
X_p=V_2*V_1^-1;

U=[E,O4;X_p,E];
H_c=U^-1*H*U;
H_b=mat2cell(H_c,[2 2],[2 2]);
A_c=H_b{1,1};
A_c0=H_b{2,2};
eta_c=U^-1*eta_l;

%Calculate the initial condition s0
syms z; %z is tau
s0=X_p*x0-int(expm((A_c'-rho/2*E)*z)*eta_g,z,0,inf);
s0=vpa(s0,6);

syms x1(t) x2(t) s1(t) s2(t)

```

```

Y=[x1;x2;s1;s2];
G=eta_1;
C=Y(0)=[x0;s0];
Sol=dsolve('diff(Y)=-SCE_m*Y+G,C');

clf
ezplot(Sol.x1,[-1,10])
hold on
ezplot(Sol.x2,[-1,10])
hold on
ezplot(Sol.s1,[-1,10])
hold on
ezplot(Sol.s2,[-1,10])
grid on
axis([-1 12 -3 3]);
legend('Sol.x1','Sol.x2','Sol.s1','Sol.s2','Location','best')

```

```

clear;
clc;

A=[1 -1; 0 2];
B=[1;1];
Q=[1,0;0,1];
R=1;
eta=[1;0];
rho=1;
E=[1,0;0,1];
O2=[0;0];
O4=[0,0;0,0];
x0=[1;1];
B_r=B*R^-1*B';
Pi=Sol_Pi(A,B,R,Q,rho);
Pi=double(Pi);
A_n=A-B_r*Pi-rho/2*E;

```

```

gam = 0.05
Gamma=[1,0;0.5,1]*gam;
Q_g=Gamma'*Q+Q*Gamma-Gamma'*Q*Gamma;

syms t s z positive

f=expm(A_n*s)*B_r;
g=expm(A_n'*z)*Q_g;
fn=norm(f, 'fro');
gn=norm(g, 'fro');
k1=int(fn, s, 0, inf);
k2=int(gn, z, 0, inf);
beta=vpa(k1*k2, 6)

```