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**DYNAMIC ANALYSIS
OF FLEXIBLE DISC - ROTOR SYSTEMS
USING COMPONENT MODES**

by

**NIKOLAOS PERDIKOLOGOS
B.ENG. (MECH.), UNIVERSITY OF PATRAS,
PATRA, GREECE, 1983**

**A Thesis submitted to the Faculty of Graduate
Studies and Research in partial fulfillment
of the requirements for the degree of:
Master of Engineering**

**DEPARTMENT OF MECHANICAL
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The undersigned hereby recommend to the Faculty of Graduate Studies and
Research acceptance of the thesis:

**Dynamic Analysis
of Flexible Disc - Rotor Systems
Using Component Modes**

submitted by Nikolaos Perdikologos, B.Eng., in partial fulfillment of the re-
quirements for the degree of Master of Engineering.

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ABSTRACT

In this investigation a flexible whirling shaft - disc system is studied and modeled using Finite Element techniques. By using disc lateral deflections relative to the undeformed body (rigid disc) two component problems are formed

Flexible shaft carrying rigid discs

Flexible disc on rigid shaft

Comparison of whirling modes with values available for exact and approximate analyses demonstrates the superiority of the 'relative deflections' method to the more traditional 'absolute deflections' method, since it allows conventional analysis of the above independent problems while the effect of coupling of the two flexible systems can be accurately and efficiently obtained with the greatly reduced set of modal equations. It is shown that a few lower normal modes of each component in the global modal matrix is sufficient to give accurate numerical results for forward and backward whirl and for non - rotating cases. This method is particularly useful for forced and transient analysis of the rotor system.

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Nik. Perdikologos

July 1985

The mind is like a mirror.

It reflects but does not hold.

It grasps nothing.

It expects nothing.

It reflects but does not hold.

Therefore the Perfect Man

can act without effort.

Chuang Tzu

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List of principal symbols

A	- Shaft cross - section
E	- modulus of elasticity for shaft, disc; $207 \cdot 10^9 N/m^2$, unless otherwise stated
I	- shaft section moment of inertia
I_D	- diametral moment of inertia of the disc
I_P	- polar moment of inertia of the disc
L	- shaft element length
M_e	- disc mass
N_S, N_D	- normal modes for shaft, disc
NMS, NMD	- number of modes used in modal matrix
p, q	- shaft lateral deflections
R	- outer radius of disc
S	- shifting constant
T_k	- kinetic energy
U	- potential energy
u, v	- disc lateral deflections
$w(r)$	- disc deflection function
α	- precession factor
θ, ϕ	- shaft slope at a node
ν	- Poisson's ratio
ρ	- material density for shaft and disc $7744 kg/m^3$, unless otherwise stated
ω	- whirling speed
ω_c	- critical speed
Ω	- rotational speed
ω_n	- natural frequency
$[B^{-1}]$	- disc element boundary values matrix
$[BG]$	- global modal matrix

$[B_S], [B_D]$	- modal matrices
$[C_e]$	- disc annular element coupling vector
$[G_E]$	- gyroscopic matrix for disc annular element
$[D]$	- dynamic stiffness matrix
$[E_K], [E_M]$	- shaft element stiffness and mass matrices
$[G_M], [G_K]$	- global stiffness and mass matrices
$[K_E], [M_E]$	- disc annular element stiffness and mass matrices
$\{p_e\}, \{p_e\}$	- shaft element coordinate vectors
$\{w_d\}$	- disc coordinate vector
$\{Z\}, \{Z_1\}$	- global coordinate vectors

Chapter 1

INTRODUCTION

1.1 Intro ductory problem statement

In the design of turbine rotors the dynamic response of the shafts was normally treated with the assumption of rigid discs. This was valid as long as the discs were thick and heavy and the turbines were operating in a low speed giving low power.

In modern gas turbines, particularly for aircraft use, the trend is towards lighter more highly stressed structures. Thus the discs are reduced in thickness and exhibit more flexibility, and instead of experiencing rigid body rotational motion when attached to the shaft, are now forced to vibrate while spinning. Furthermore, the flexibility of the attached blades is expected to additionally complicate this increased disc flexibility so that the bladed - disc system should be considered when assessing the effect of disc dynamics on the complete blade - disc - shaft system.

We expect the presence of the disc flexibility to influence the system's response. However, the general concepts and definitions regarding the system's motion (shape of orbit, precession, whirl, etc.) are not changed by the shaft - disc dynamic interaction and will be presented next, for the purpose of the introduction to the problem.

1.2 Operation and response of a rotor system

In practice, some part of the rotor will always have small unbalanced masses, that is, in some of the discs the mass center (G) will not coincide with the geometric center (O). Even if perfect manufacturing eliminates originally such defects, the performance conditions will soon develop them. An unbalanced mass could be negligibly small

but at high speed the resulting dynamic forces can be large, since their amplitude is proportional to the square of this speed.

The direction of the unbalanced force is always fixed relative to the disc (along the OG line). So the rotor will assume a bent shape which will rotate with some angular velocity around the straight line determined by the bearings (z -axis). This doesn't generally mean that the orbit will be circular. It can be an ellipse, circle, or even straight line (ref. 7¹) depending on the initial conditions of the motion and the axial symmetry of the structure (both in mass and stiffness distribution).

The rotation of the bent shaft shape is called 'whirl' and the rotational speed 'whirling speed'.

A stationary observer located at a plane perpendicular to z -axis will see a flexural shaft vibration with period equal to the period of rotation, while an observer on the z -axis will notice the 'whirl' orbit of the shaft.

Another important point - not fully understood yet - is the relation between the amplitudes and directions of the spinning and whirling speeds. When the whirling speed is equal to the rotational speed both in magnitude and sense we are talking about 'synchronous whirl' or 'synchronous precession'. Every other situation, every combination of amplitudes and directions is theoretically possible, [ref. 41], and then we have the 'non-synchronous whirl'. When the direction of the spin is the same as the whirl direction we have 'forward whirl' or 'forward precession', otherwise we have 'backward whirl' or 'backward precession'.

Two situations usually appear in practice:

- The synchronous precession

The backward precession with whirling speed equal in magnitude to the rotational speed. This can occur in the case of concentric counter-rotating shafts such as used in some jet-engines.

In the general solution of the system's differential equations both forward and

¹ The references are listed in alphabetical order.

backward whirl motions are superimposed [ref. 6-7]

If we define B' the trace of the point 'O' on the z -axis (permanently fixed in space), the vectors $\vec{B'O}$ and $\vec{OG}$ can describe the resulting motion for ranges of the rotational speed below and beyond the critical, with presence or not of external or internal friction

Neglecting friction (as in the present reports) the lines of the vectors $\vec{B'O}$ and $\vec{OG}$ coincide (ref. 7) (figure 1.1). For a simple one degree of freedom system it can be shown that when the rotational speed is less than the critical (subcritical operation) $\vec{B'O}$ and $\vec{OG}$ have the same direction, in other words the mass center G rotates with radius $|\vec{B'O}| + |\vec{OG}|$, (figure 1.1 a), while in 'supercritical operation' $\vec{B'O}$ and $\vec{OG}$ have opposite directions [Ref. 7], so G rotates with radius $|\vec{B'O}| - |\vec{OG}|$ (figure 1.1 b). For very high speeds the points B and G tend to coincide so the disc rotates around its mass center G with radius equal to the eccentricity²[ref. 41]

1.3 Literature review

Corresponding to the design concepts and assumptions for the disc flexibility, the published works can be divided into two groups. Prior to 1970 the literature was concerned with whirling problems assuming rigid discs, or with isolated disc analyses. Results for rotor systems accounting for disc's flexibility began to appear after 1972. Around 1980 analyses accounting also for the blades flexibility start to be reported and the 'flexible disc' aspect is sometimes replaced by the 'flexible blade - disc assembly'.

The development of the calculation methods begins when Prohl [ref. 35] gives in a classic paper in 1945 the first numerical method for calculating synchronous whirling of complex shafts comprising variable shaft sections and multiple discs. The method is

² In the literature, two different definitions of the 'eccentricity' exist. In [ref. 7] eccentricity is the $\vec{OG}$ vector. In the rest of the thesis eccentricity is the distance 'e' between the geometric center 'O' and the concentrated unbalanced mass m . Their relation is $|\vec{OG}|(M + m) = em$, where M is the disc's mass. Since $M \gg m \Rightarrow e \gg |\vec{OG}|$

sometimes referred to as the 'Myklestad - Prohl' method, and is perhaps best classified as one of the general class of 'Transfer matrix' methods. Variations of Prohl's method still remain the most widely used calculation schemes for complex practical rotors.

The earliest extensive analytical work was presented in 1948 by R B Green, [ref. 16] for the analysis of the problem of non-synchronous whirl of a flexible rotor with a rigid disc including gyroscopic forces. The shaft mass was neglected and supports were considered to be rigid. Using the influence number method put forward by Den Hartog in 1927, the analysis could handle overhung discs, discs between supports, even multiple disc rotors. It could be easily expanded to include bearing flexibility. This method though could not include details for the shaft such as mass distribution and variable sections.

In 1961 Dimentberg introduces in [ref. 6] the very useful concept of 'dynamic stiffness'. It is used for the dynamic coupling of several members into one structure. With this technique, the entire dynamic effect of a disc, at a specified rotational and precession speed, can be expressed as two simple terms. One gives moment versus slope and the other gives transverse force versus deflection. With this method he solves in simple elegant manner a wide range of rotor problems, including forced response. His work is limited to rotors of simple geometry for which 'dynamic stiffness' terms could be described in closed form by elementary mechanics. For practical rotor geometries however, application of finite element methods, undeveloped in 1961, could now be used in a very similar manner.

Another approach in developing the analysis was the problem of a single overhung rigid disc on a shaft with two stiff bearings solved by T C Huang and F C C Huang and presented in 1967 in [ref. 20]. It handled the shaft as a 'Timoshenko beam' that is, shear deflection was included in the classical beam deflection equation, which was integrated to give the solution. The disc was assumed thin for the purpose of the diametral inertia being half the polar. The precession speed was dealt with as an unknown in the integration. To evaluate it from the boundary conditions an iteration was used. This model was later extended to include several discs using Laplace transformation.

methods

At the same period Eshleman and Eubanks appear with a similar analysis [ref 11] to Huang and Huang, but it could also handle a disc between bearings. Very good agreement of experimental results - included in the work - with the calculations was shown. In a later paper [ref 12] they include the shaft torque effect and long bearings were considered which provide zero slope of the shaft - in other words, rotationally stiff bearings.

All the analytical methods discussed above (until 1969) rely on a uniform section shaft which can be described by a differential equation with constant coefficients; therefore they are inefficient for a practical rotor study. Going back to the numerical methods, a modified Prohl's method, known as 'Holzer - Myklestad - Prohl' method was used by Ekong, Eshleman and Bonthron in [ref. 13], to describe a shaft of variable cross-section on multiple rigid supports, and carrying several discs. Synchronous and non-synchronous precession frequencies were calculated and compared to the results of Prohl [ref. 35] and Urban [ref. 43], to end up with the conclusion that it is probably the most accurate analysis, because of the fewer approximations, and because it includes shear deformation and distributed mass rather than lumped mass models for the shaft.

To apply the Transfer Matrix method the shaft is divided into a number of constant diameter segments with discs or bearings located at the junctures. A numerical integration is then performed step by step from one shaft end to the other, during which an assumed value for the precession frequency is used. This frequency is then incremented by iteration until finally the actual set of boundary conditions are satisfied. Numerical methods under the names of Holzer, Myklestad or Prohl are basically the same 'Transfer Matrix Method' merely using a different technique to carry out the arithmetic.

The energy methods should be also mentioned here, after all, they are the first step for the Finite Element development. Jacobson and Ayre [ref. 21], Hurty and Rubinstein [ref. 19] and many others present energy methods known as Rayleigh

Rayleigh - Ritz, Stodola, Galerkin. In all these methods the solution of the differential equations (usually the deflection of the shaft) is approximated by a known function that satisfies the geometrical (deflection and slope) boundary conditions. The results can be improved by using a series of functions to approximate the deflected shape. Although for simple rotors and first bending modes these methods give excellent results, when multiple bearings and large disc gyroscopic loads are present it is difficult to adequately approximate the deflection shape. These difficulties are now overcome by the use of finite element techniques. Morton in [ref. 31] uses classical Rayleigh - Ritz procedures, while Ruhl [ref. 36] compares the finite element method with the transfer matrix (Prohl) methods. The 'lumped mass' discrete transfer matrix model is shown to be computationally inefficient compared to the finite element model.

In 1972 two publications, one by Shapiro and Rumbarger [ref. 40] and one by Eshleman [ref. 14], give a comprehensive review of the problems faced in rotor dynamics up to date as well as of the methods and the analytical approaches adopted. In the meantime, the analysis of rotor dynamics assuming disc flexibility, has already started.

In 1970 Sann in [ref. 39] was the first to study a flexible disc gyroscope supported by a shaft which has angular and axial flexibility but no mass. The boundary conditions of the disc to the shaft and to a heavy rigid ring on the outer edge were taken into account when the differential equations were integrated numerically over the area of the circular plate. Results were obtained for non - synchronous precession natural frequencies as well as for forced vibration response.

Four years later Red [ref. 37] pointed out that Sann did not handle properly the boundary conditions in his analysis, which was correct from every other point of view. According to Red, Sann's results did not approximate the true state of stress in the plate to the desired accuracy. Red examined the case in more depth and presented a more complete investigation of the free vibration of the rotor plate in the spin mode. He also obtained closed form solution of the governing differential equation for the natural vibration of the rotor when free of imposed gyroscopic rates.

Dopkin [ref. 8, 9] developed in 1972 and 1974 respectively a general analysis to

predict the natural frequencies of vibration of a rotating shaft on linear undamped bearings. Any number of flexible discs was included. To solve the system he used the transfer matrix techniques (previously discussed), after selection from several other applicable analytical methods. Constant thickness segments and lumped masses were used to describe the shaft and discs. The method could be expected to analyse any practical rotor configurations commonly appearing in turbomachinery, and many other types of rotors. Results were compared with data given in the literature and covering a wide range of assumptions were found to give satisfactory agreement.

Dopkin concluded that the disc flexibility always reduces the rotor natural frequency, sometimes to only a few percent of the rotor with rigid disc value. In his analysis only a circular orbit is considered for synchronous or non-synchronous precession. His model is linear, described by small deformations. The wide use of the method in industry shows its success.

Numerically however it was a poor model, round-off errors are common. Masses are assumed concentrated at discrete points while flexibility is distributed according to classical beam and plate theory. It is also a disadvantage the fact that stress fields in the plane of the disc due to thermal, residual stress, or centrifugal effects are not included.

An analytical investigation of the influence of disc flexibility on the transverse bending natural frequencies and critical speeds of a rotating shaft-disc system is reported in 1973 by Chivers [ref. 4, 5]. The shaft was treated as a Bernoulli-Euler beam, and standard differential thin plate bending equations described the disc. The in-plane stresses were included and the attachment of the disc to shaft was through a rigid core. Assuming a given precession factor, the linear differential equations were solved exactly after non-dimensionalization. Results were presented for a wide range of parameters. The model could be described as follows:

- The differential equations were linearized under small deflections assumption.

- The effect of shear and rotary inertia were neglected.

- The rigid core and flexible plate were thin enough so that the interaction with

the shaft occurred at a point

The plate in-plane stresses were assumed to be generated only by the spin rate. Structural damping was neglected.

The shaft and plate were taken as isotropic.

The shaft was not allowed to take any torsional deformation.

The analysis was not applicable to practical rotor geometries.

A finite element approach to the rotor analysis is presented by Kuo in 1975 (ref. 25). It was a three-dimensional model, (using the computer program NASTRAN), that could describe any practical shaft, and normal mode analysis was employed giving reasonable accuracy for the dynamic stress distribution. Good agreement with measurements was shown. Results are given for non-rotating systems only.

In 1977 Wilgen [ref. 46, 47] approaches the rotor dynamics problem from the point of view of the whirling stability influenced by the appendage flexibility. The stability criteria were obtained by employing a Liapunov stability analysis (energy method), using 'admissible' functions for the approximation of the deflected shapes of shaft and appendage. Different approximations were also used for the formulation with respect to Euler-Bernoulli theory. This method is not suitable for practical geometry, however one significant advantage is that no assumption regarding the precession factor was made.

In [ref. 32] Meirovitch discussed the types of functions to be used in the Rayleigh-Ritz method for rotating structures. His 'stationary principle' implies that R-R approach can be used for rotating structures in a similar way to that for non-rotating structures.

As long as a complete set of admissible functions is used, one needs only calculate the eigenfunctions of the non-rotating appendage. The results will be satisfactory as if the rotating appendage eigenfunctions were used. This will help develop efficient computational procedures and will improve the understanding of the behaviour of linear gyroscopic systems.

Klompas [ref. 26] in 1978 described a new method of calculating the critical speeds of multishaft jet engines. He defines new coordinates for elliptical orbits, and the resulting scheme requires solution of eight simultaneous equations for each bearing, after the employment of the well - known transfer matrix technique. Analysis of rotors on asymmetric supports as well as unbalance response can be obtained. However, Klompas work is very difficult to follow.

A Finite Element approach for the study of the effect of disc flexibility on the rotor whirling frequencies is reported by Masoom in 1980 [ref. 29]. Beam finite elements, not including shear deflection and rotary inertia, are used to model the shaft, and annular elements developed by Kirkhope and Wilson [ref. 23, 24], Kaldas [ref. 22] and Wilson [ref. 44] are used for the flexible disc. The latter includes rotational stiffening effects. The shaft - disc interaction occurs at a point and only circular whirling orbits are considered.

Overhanging disc configurations as well as disc between bearings and a combination of the above are presented for different conditions of rotation and a range of disc flexibility. The results are compared with analytical or alternative numerical data and show excellent agreement, with very efficient modelling.

Finally, an attempt is presented to approximate the flexible rotor with a simple 4 - degree of freedom model. Data are used from the partial problems 'shaft with rigid disc' and 'flexible disc on rigid shaft' to conclude whether a shaft - disc interaction is likely to occur. It is also a useful first approximation to the actual whirling frequencies of the system, giving satisfactory results, but cannot represent more than one flexible disc at a time.

However, this simple analysis revealed the potential benefits of using deflections of the disc relative to the rigid disc configuration. By using relative deflections the complete system could be analysed as two component problems

- a shaft with rigid discs,
- a disc mounted on a rigid shaft

These two component problems are recognized as those which would normally be treated in practical analysis. The interaction of these two component systems can then be most efficiently treated by modal analysis, using only those component modes which significantly affect the combined system response.

Using the Lagrangian of the rotor system, Takahashi, Suzuki and Yamaguchi [ref. 42] studied the flexural vibrations of a bar with circular plates, considering the lateral vibration of the plates. They produced some numerical and experimental results, but these were not elaborated. The method is not suitable for practical problems and rotational effects are not included.

From this point, the vibration of the blades is also considered to affect the system's dynamics, that is, the blades form part of the flexible disc system.

In 1980, Palladino in [ref. 34] and in 1982 in [ref. 33] (with Rossettos) reports the development of a special shaft finite element which incorporates analytical solutions for disc dynamics and approximations for blade effects. It thus has the advantage that the size of the system depends only on the number of the shaft finite elements used and not on the disc and blade assemblies.

The disc is described as an isotropic circular plate, while the blades are handled as orthotropic plates. In general conclusions he gives descriptions of cases where blade and disc flexibility are affecting the system's critical speeds. He also notes that the trends for multiple disc rotors are much the same as those for the single disc, and that backward whirl and second modes are most sensitive to disc and blade flexibility.

Due to the versatility of the finite element method, complex configurations and boundary conditions can be considered, and good and reliable analysis of practical rotors can be obtained.

One of the most recent works on the field of rotor dynamics was published in 1984 by Lowey and Khader [ref. 28]. Initially a finite element analysis is used to give the non-rotating modes of the disc-blade assembly which are then coupled to the rotating flexible shaft. The resulting eigenvalue problem is of relatively small size, with the

number of degrees of freedom limited to those of the shaft plus two times the number of the non-rotating modes for each disc. They conclude that in the range of design practice for stiffness values some of the one - diametral mode frequencies are shown to be affected significantly by shaft degrees of freedom. Accounting for Coriolis forces introduces small changes in the resulting frequencies for the most part, but in regions where rigid and flexible modes are highly coupled these terms can have a substantial effect.

A summary can be made here regarding the literature about the analysis of flexible rotor systems.

1. The lumped-mass transfer matrix method is so far the most extensively used for the analysis of practical rotor systems. Compared to the finite element method it is relatively inefficient due to the round-off errors, and to the many concentrated masses needed for good representation. It cannot be applied easily to response calculations.
2. Undamped systems have been mostly examined, especially for the calculation of whirling frequencies and normal modes. Internal (structural) and external damping is present in practice and forced response is not only due to imbalance but also due to aerodynamic forces. Thus, much work is still needed, for forced response, instability, asymmetric supports, etc.
3. Studies of the effect of disc flexibility have thus far largely focussed on whirling frequencies, and under artificially imposed constraints of precession. There is a need to study the forced vibration and stability of such rotors where the precession rate is obtained as part of the solution. To efficiently carry out such studies on practical rotor geometries a form of modal analysis is desirable which retains the detailed rotor behaviour accurately, but with a manageable number of degrees of freedom.

4. The lack of extensive experimental data should be noted, and the need of such results is always necessary for the verification of the analysis.

1.4 Goals and overview of the thesis

In the previously completed literature review a comparison made concerning the approximate methods of rotor analysis showed the advantages of the finite element method. Due to the efficiency, accuracy, ability to describe any physical distribution of properties, ability to include forced response or stability analysis, it was decided that the finite element model should be used for the needs of the present investigation.

The advantages of using disc relative deflections have been noted above, where the solution equations evolve naturally from the two component problems of 'rigid discs' and 'rigid shaft' which would be given detailed analysis in practice. Thus the thesis develops this method of analysis with a view to efficiently modelling the interacting disc - shaft system.

In Chapter II the model used is presented, the matrix equations that describe it are derived with respect to the relative deflections of the flexible disc to a hypothetical rigid disc. Normal mode analysis is employed twice, first to the partial problems 'shaft with rigid discs' and 'flexible disc on rigid shaft' and second to the reduced combined system.

In Chapter III cases from the literature are analysed and solved and numerical results are presented to validate the accuracy of the computer solutions. The rotor of a practical gas turbine engine is finally modeled and studied in some detail.

In Chapter IV the conclusions of this investigation are described, achievements are discussed and some recommendations and orientation for future studies is given.

Chapter 2

ANALYTICAL APPROACH

2.1 General assumptions - definitions

Certain assumptions are made for the model used in the analysis shown in figure (2.1)

- 1 The shaft can be considered to behave as a thin beam, i.e. transverse shear and rotatory inertia are neglected
- 2 The rotor and bearing supports are axisymmetric, in other words we have a 'circumferentially symmetric' structure
- 3 The only significant gyroscopic effects are due to the discs attached.
- 4 Assumptions for small deformations lead to linearization and simplification of the system's differential equations of motion.
- 5 The discs are thin and therefore the disc polar inertia is twice its diametral moment of inertia

$$I_P = 2I_D$$

- 6 The definition of the precession factor ' α ' should be made here

$$\alpha = \frac{\Omega}{\omega} = \frac{\text{rotational speed}}{\text{precession speed}}$$

When $\alpha = 0$ the non-rotating case is analysed, when $\alpha > 0$ and $\alpha < 0$ we get forward and backward whirl respectively.

Of these, only assumption (2) is significantly restrictive. Inclusion of transverse

shear and rotatory inertia is not difficult and has been done elsewhere (ref 38) and requires only the use of suitable shaft (beam) finite elements. Likewise including the small contribution of shaft element gyroscopic effects is readily carried out (ref 38) since the shaft element gyroscopic matrix is simply twice the shaft element mass matrix (of the disc gyroscopic effect treated in this thesis). For assumption (5) the correct polar inertia I_p and diametral inertia I_d for the disc can be used where known. None of these assumptions substantially detracts from the method of analysis presented above.

To describe the general behaviour case that results in the whirling orbit we have to employ orthogonal deflections p and q for the shaft (figure 2.2) and orthogonal lateral deflections u and v for the disc (figure 2.3). Thus, any type of elliptical orbit can be expressed¹

2.2 Shaft model

The shaft is divided into elements by a series of nodes. Nodes must be placed at bearing supports and at the points where discs are attached or, generally speaking, at points of application of concentrated loads. Nodes should be also placed where the cross-section of the shaft changes.

Conventional 4-degree of freedom elements are used, that is, in each node i both the lateral deflection p_i and the slope of the axial line $\theta_i = \left. \frac{dp}{dx} \right|_i$ are described. The element thickness is taken uniform. A more complicated thickness variation could be imposed, but is of limited value for modelling practical systems.

Under our assumptions, the whirling problem for the shaft is nothing but lateral vibration on a rotating plane. Therefore, for the shaft element located between the nodes i and $i+1$ (figure 2.4), the matrix equation for the whirling frequencies is the same as the matrix equation for the lateral deflection eigenvalues, given by

¹ A circular orbit is regarded here as a special case of the general elliptical orbit having the two axes equal.

$$[EK - \omega^2 EM]\{q_e\} = 0 \quad (2.1)$$

where

EK = the stiffness matrix for the element

EM = the mass matrix and

$$\omega = \text{angular velocity of the shaft} = \text{disc angular velocity}.$$

The matrices, the development of which can be found in any text on Finite Elements, are given in Appendix A. At the assembly of the shaft stiffness matrix, the two bearing constants (linear stiffness and rotational stiffness) have to be added to the diagonal positions (corresponding to the nodal coordinates) for each bearing.

Note that the above does not contain the degrees of freedom of the deflection q (orthogonal to p). An equation similar to (2.1) can be written to contain the additional degrees of freedom. Thus

$$\{[EKT] - \omega^2[EMT]\}\{q_{et}\} = 0 \quad (2.2)$$

where

$$[EKT] = \begin{bmatrix} [EK] & 0 \\ 0 & [EK'] \end{bmatrix} \quad [EMT] = \begin{bmatrix} [EM] & 0 \\ 0 & [EM'] \end{bmatrix}$$

$$\{q_{et}\} = \{p_i, \theta_i, p_{i+1}, \theta_{i+1}, q_i, \phi_i, q_{i+1}, \phi_{i+1}\}$$

$$\text{with } \phi_i = \left(\frac{dq}{dx} \right)_i$$

The description of the shaft matrices is not yet complete. The contribution to the mass matrix of the disc(s) attached will be fully analysed in the next paragraph.

2.3 Disc model

The disc is divided into annular elements according to the disc model for isotropic thin discs given by Kirkhope and Wilson in [ref 23]. Linear ($\eta = a + b\xi$) and parabolic

($\eta = a + b\xi^2$) thickness variations may be used for the disc profile. In this, η is the non-dimensional thickness, and ξ the non-dimensional radial variable

For the purpose of this thesis $a=1$, $b=0$ is suitable (uniform thickness)

Another point that requires consideration is that the inner perimeter of the disc is not clearly determined, since the mass distribution for the shaft is not considered radially, in other words the mass of the shaft is assumed to be concentrated along the axial line. In [ref 23] the exact size of the inner hole appears to be of little importance so, a full disc simply supported at the perimeter was modelled for convenience with a small central hole $R_a/R_b = 0.001$, where R_a was the inner and R_b the outer radius and divided into equal² elements

The mass distribution for the system would be still correct if the disc was modelled with a very small central hole ($R_a/R_b = 0.001$). However, the stiffness around the inner clamped edge more drastically depends on the diameter of the hole, altering the natural frequencies accordingly. Therefore the disc inner diameter will be taken equal to the shaft's diameter at the point of attachment throughout this investigation

2.4 Energy expressions for the disc annular element

a. Kinetic energy

To obtain the kinetic energy function for a whirling disc annular element consider a general rotational deformation around the principal axes which are shown in figure (2.5)

For a differential element located between radii r and $r+dr$, the kinetic energy is:

$$T_k = T_{k1} + T_{k2} \quad (3)$$

² For an element located between the circular nodes i and $i+1$, $R_{i+1} - R_i = c$ (constant), with $c = (R_b - R_a)/e$, where e is the number of elements the disc is divided into

where from [ref 29]

$$\begin{aligned} T_{k1} &= \frac{1}{2} dI_D [\dot{\phi}^2 + \dot{\beta}^2] \\ T_{k2} &= \frac{1}{2} dI_P [2\dot{\phi}\dot{\beta}\Omega] \end{aligned} \quad (4)$$

$\dot{\phi}$ and $\dot{\beta}$ are the rotational velocities around two of the principal axes coinciding with the shaft's deflection vectors (at the point of disc's attachment to the shaft) q_S and p_S respectively

Another term should be added to (3) regarding the vertical translation of the element

$$T_{kV} = \frac{1}{2} dm [\dot{p}_S^2 + \dot{q}_S^2] \quad (5)$$

so that (3) can be completed as:

$$T_k = T_{kV} + T_{k1} + T_{k2} \quad (6)$$

If ρ is the density of the disc, and t the uniform thickness, the dm , dI_D , dI_P of (4), (5), literally the mass, diametral moment of inertia and polar moment of inertia for the differential element, can be expressed as:

$$\begin{aligned} dm &= 2\pi r \rho t dr \\ dI_D &= \frac{r^2}{2} dm \\ dI_P &= 2dI_D \end{aligned} \quad (7)$$

The next step is to express the rotations $\dot{\phi}$, $\dot{\beta}$ in terms of the lateral disc deflections u and v (fig. 2.3).

$$\dot{\phi} = \frac{\dot{u}}{r} \quad \text{and} \quad \dot{\beta} = \frac{\dot{v}}{r} \quad (8)$$

Substituting (7) and (8) in (5) and (4) we obtain the general form of the kinetic energy terms

$$\begin{aligned}
T_{kV} &= \frac{1}{2} 2\pi r \rho t (p_3^2 + q_3^2) dr \\
T_{k1} &= \frac{1}{2} 2\pi r \rho t \frac{r^2}{2} \left[\left(\frac{u}{r} \right)^2 + \left(\frac{v}{r} \right)^2 \right] dr \\
T_{k2} &= \frac{1}{2} 2\pi r \rho t r^2 \left[2 \left(\frac{\dot{u}}{r} \right) \left(\frac{v}{r} \right) \Omega \right] dr
\end{aligned} \tag{9}$$

The energy terms for the finite element between R_1 and R_2 can be found by integrating (9)

$$\begin{aligned}
T_{EV} &= \frac{1}{2} 2\pi \rho t \int_{R_1}^{R_2} r (p_3^2 + q_3^2) dr \\
T_{E1} &= \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r (u\dot{u} + v\dot{v}) dr \\
T_{E2} &= \frac{1}{2} 4\pi \rho t \Omega \int_{R_1}^{R_2} r (\dot{u}v) dr
\end{aligned} \tag{10}$$

The T_{EV} term can be immediately calculated giving:

$$T_{EV} = \frac{1}{2} M_e (p_3^2 + q_3^2) \tag{11}$$

where

$$M_e = \pi \rho (R_2^2 - R_1^2) t$$

is the total mass of the finite element.

For the other terms of (10) we have to choose an expression for $u(r)$ and $v(r)$. In [Ref. 23] the polynomial for the deflection approximation along the radius is

$$w(r) = A + Br + Cr^2 + Dr^3$$

a third order polynomial that allows both the deflection $w(r)$ and the slope $\frac{dw}{dr}$ to be described at each node. The above can also be written as

$$w(r) = [S][a] \quad (12)$$

where

$[S] = [1 \ r \ r^2 \ r^3]$ is the vector of functions

$[a] = \{A \ B \ C \ D\}$ is the vector of constants

Furthermore, our purpose is to use deflections for the flexible disc relative to a hypothetical rigid disc being attached to the shaft at the same point. So,

$$\begin{aligned} u(r) &= r\theta_s + [S][a] \\ v(r) &= r\psi_s + [S][a] \end{aligned} \quad (13)$$

where θ_s , ψ_s are the slopes of the shaft at the attachment point corresponding to deflections p_s and q_s respectively, (see Fig. 2.6).

The coordinate vector $\{\bar{w}\}$ which is in [ref. 29]:

$$\{\bar{w}\} = \left\{ u, \left. \frac{du}{dr} \right|_i, u_{i+1}, \left. \frac{du}{dr} \right|_{i+1} \right\}$$

has to be slightly modified to contain the shaft deflection and slope:

$$\{\bar{w}_s\} = \left\{ p_s, \theta_s, u, \left. \frac{du}{dr} \right|_i, u_{i+1}, \left. \frac{du}{dr} \right|_{i+1} \right\} \quad (14)$$

As with the shaft, the imposition of the v - deflection doubles the length of the coordinate vector:

$$\{\bar{w}\} = \{\bar{w}_p, \bar{w}_q\}$$

where

$$\{\bar{w}_q\} = \left\{ q_s \phi_s v, \frac{dv}{dr} \Big|_i, v_{i+1} \frac{dv}{dr} \Big|_{i+1} \right\}$$

if we define

$$\begin{aligned} \{\bar{u}\} &= \left\{ u_i \frac{du}{dr} \Big|_i, u_{i+1} \frac{du}{dr} \Big|_{i+1} \right\} \\ \{\bar{v}\} &= \left\{ v_i \frac{dv}{dr} \Big|_i, v_{i+1} \frac{dv}{dr} \Big|_{i+1} \right\} \end{aligned} \quad (15)$$

we can express more conveniently the above:

$$\begin{aligned} \{\bar{w}_p\} &= \{p_s \theta_s \bar{u}\} \\ \{\bar{w}_q\} &= \{q_s \phi_s \bar{v}\} \end{aligned}$$

Noting that:

$$[S][a] = [S][B^{-1}][\bar{u}] \quad [ref.23]$$

where $[B^{-1}]$ is the inverse of the shape matrix, and substituting the formulas (13) in T_{E1} and T_{E2} of (10) we get:

$$\begin{aligned} T_{E1} &= \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r \left[\left(r \theta_s + [S][B^{-1}][\bar{u}] \right)^2 + \left(r \phi_s + [S][B^{-1}][\bar{v}] \right)^2 \right] dr \\ T_{E2} &= \frac{1}{2} 4\pi \rho t \Omega \int_{R_1}^{R_2} r \left[r \theta_s + [S][B^{-1}][\bar{u}] \right] \left[r \phi_s + [S][B^{-1}][\bar{v}] \right] dr \end{aligned}$$

taking the above separately we have for T_{E1} :

$$\begin{aligned}
T_{E1} &= \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r^3 \dot{\theta}_S^2 dr + \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r [\ddot{u}]^T [B^{-1}]^T [S]^T [S] [B^{-1}] [\ddot{u}] dr \\
&\quad + \frac{1}{2} 2 \pi \rho t \int_{R_1}^{R_2} r^2 \dot{\theta}_S [S] [B^{-1}] [\ddot{u}] dr + \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r^3 \dot{\psi}_S^2 dr \\
&\quad + \frac{1}{2} \pi \rho t \int_{R_1}^{R_2} r [\ddot{v}]^T [B^{-1}]^T [S]^T [S] [B^{-1}] [\ddot{v}] dr + \frac{1}{2} 2 \pi \rho t \int_{R_1}^{R_2} r^2 \dot{\psi}_S [S] [B^{-1}] [\ddot{v}] dr \\
\Rightarrow T_{E1} &= \frac{1}{2} I_{DE} \dot{\theta}_S^2 + \frac{1}{2} [\ddot{u}]^T [ME] [\ddot{u}] + \frac{1}{2} 2 \pi \rho t \int_{R_1}^{R_2} r^2 [S] dr \dot{\theta}_S [B^{-1}] [\ddot{u}] \\
&\quad + \frac{1}{2} I_{DE} \dot{\psi}_S^2 + \frac{1}{2} [\ddot{v}]^T [ME] [\ddot{v}] + \frac{1}{2} 2 \pi \rho t \int_{R_1}^{R_2} r^2 [S] dr \dot{\psi}_S [B^{-1}] [\ddot{v}]
\end{aligned}$$

where

$$[ME] = [B^{-1}]^T \int_{R_1}^{R_2} [S]^T (\pi \rho t r) [S] dr [B^{-1}] \quad (16)$$

is the mass matrix for the annular element, given in Appendix B. Furthermore,

$$\begin{aligned}
T_{E1} &= \frac{1}{2} I_{DE} \dot{\theta}_S^2 + \frac{1}{2} [\ddot{u}]^T [ME] [\ddot{u}] + \frac{1}{2} \dot{\theta}_S [C_*] [\ddot{u}] \\
&\quad + \frac{1}{2} I_{DE} \dot{\psi}_S^2 + \frac{1}{2} [\ddot{v}]^T [ME] [\ddot{v}] + \frac{1}{2} \dot{\psi}_S [C_*] [\ddot{v}]
\end{aligned} \quad (17)$$

where

$$[C_*] = 2 \int_{R_1}^{R_2} \pi \rho t r^2 [S] dr [B^{-1}] \quad (18)$$

can be called the element coupling matrix, and is also given in Appendix B

The kinetic energy term T_{E2} gives

$$\begin{aligned}
T_{E2} &= \frac{1}{2} 4\pi \rho l \Omega \int_{R_1}^{R_2} r \left[r^2 \theta_S \psi_S + r \psi_S [S][B^{-1}][\bar{u}] \right. \\
&\quad \left. + r \theta_S ([S][B^{-1}][\bar{v}])^T + [\bar{v}]^T [B^{-1}]^T [S]^T [S][B^{-1}][\bar{u}] \right] dr \\
&\Rightarrow T_{E2} = \Omega I_{PE} \theta_S \psi_S + \Omega [C_s][\bar{u}]\psi_S + \Omega [C_s][\bar{v}]\theta_S \\
&\quad + 2\Omega [\bar{v}]^T [ME][\bar{u}]
\end{aligned} \tag{19}$$

b. Potential energy

Since the strain energy is due to relative deflections, the potential energy and, therefore, the stiffness matrix is identical to that of [ref. 29]. Centrifugal stiffening effects resulting from rotational in-plane stresses are accounted for, and a 'rotational' stiffness matrix is added to the element stiffness matrix $[KE]$

$$[K] = [KE] + \Omega^2 [KE_R]$$

The $[KE_R]$ matrix is developed by Wilson in [ref. 44], who uses the same coordinate vector as in the present thesis. A description of the $[KE_R]$ matrix is also given in Appendix B.

The submatrices of the global stiffness matrix corresponding to the shaft and to the disc are uncoupled, and so we use an (8x1) coordinate vector for the stiffness matrix assembly

2.5 Lagrange equations

The element's inertia and gyroscopic matrices will be derived from the kinetic energy expressions applying the Lagrange equations:

$$F_i = \frac{d}{dt} \left(\frac{\partial T_k}{\partial \dot{q}_i} \right) - \frac{\partial T_k}{\partial q_i} \tag{20}$$

where the q_i variables are

$$p_s, q_s, \theta_s, \psi_s, [\bar{u}], [\bar{v}]$$

Combining (11), (17), (19), as described in (6), the full expression for the kinetic energy is

$$\begin{aligned} T_k = & \frac{1}{2} M_s p_s^2 + \frac{1}{2} I_{DE} \theta_s^2 + \frac{1}{2} [\bar{u}]^T [ME][\bar{u}] + \frac{1}{2} \theta_s [C_s][\bar{u}] \\ & + \frac{1}{2} M_s q_s^2 + \frac{1}{2} I_{DE} \psi_s^2 + \frac{1}{2} [\bar{v}]^T [ME][\bar{v}] + \frac{1}{2} \psi_s [C_s][\bar{v}] \\ & + \Omega I_{PE} \theta_s \psi_s + \Omega [C_s][\bar{u}] \psi_s + \Omega [C_s][\bar{v}] \theta_s + 2\Omega [\bar{v}]^T [ME][\bar{u}] \end{aligned}$$

We now apply (20) on the above for each variable q_i

$$1. \quad q_i = p_s \quad \Rightarrow$$

$$\Rightarrow F_{p_s} = M_s \ddot{p}_s$$

$$2. \quad q_i = \theta_s \quad \Rightarrow$$

$$\Rightarrow F_{\theta_s} = I_{DE} \ddot{\theta}_s + \frac{1}{2} [C_s][\bar{u}] + \Omega I_{PE} \dot{\psi}_s + \Omega [C_s][\bar{v}]$$

$$3. \quad q_i = [\bar{u}] \quad \Rightarrow$$

$$\Rightarrow F_{[\bar{u}]} = [ME][\bar{u}] + \frac{1}{2} [C_s]^T \ddot{\theta}_s + \Omega [C_s]^T \dot{\psi}_s + 2\Omega [ME][\bar{v}] \quad (21)$$

$$4. \quad q_i = q_s \quad \Rightarrow$$

$$\Rightarrow F_{q_s} = M_s \ddot{q}_s$$

$$5. \quad q_i = \psi_s \quad \Rightarrow$$

$$\Rightarrow F_{\psi_s} = I_{DE} \ddot{\psi}_s + \frac{1}{2} [C_s][\bar{v}] - \Omega I_{PE} \dot{\theta}_s - \Omega [C_s][\bar{u}]$$

$$6 \quad q_1 = [v] \Rightarrow$$

$$\Rightarrow F_{[v]} = [ME][\dot{v}] + \frac{1}{2}[C_s]^T \dot{\psi}_s - \Omega[C_s]^T \theta_s - 2\Omega[ME][u]$$

Combining all the above in one system, we have

$$[EM_T] \{ \bar{W} \} + \Omega [EC_T] \{ \bar{W} \} = \{ F_L \} \quad (22)$$

where $[EM_T]$ and $[EC_T]$ are the element's full (for the two perpendicular planes) matrices for mass and gyroscopic effect respectively, can be further divided using submatrices $[ME]$ and $[CE]$

$$[EM_T] = \begin{bmatrix} [ME] & 0 \\ 0 & [ME] \end{bmatrix} \quad [EC_T] = \begin{bmatrix} 0 & [CE] \\ -[CE] & 0 \end{bmatrix}$$

shown in figure 2.7

From the above figure, as well as from the thin disc assumption that the polar moment of inertia (I_{PE}) is twice the diametral (I_{DE}) it is obvious that the elements of the $[CE]$ submatrix can be obtained from the $[ME]$, using the relation.

$$CE(i, j) = 2ME(i, j), \quad i, j = 1, 2, \dots, 6 \quad (23)$$

except the element $CE(1,1)$, which is zero, actually being a direct consequence of the assumption that the concentrated disc's mass does not contribute to the gyroscopic forces.

In the mass matrix $[GM]$ the element's mass and diametral inertia are placed in the diagonal positions corresponding to the shaft deflection and slope at the node of the disc attachment. The contribution of all the disc elements requires that in the appropriate positions of the shaft's diagonal, the total mass and total diametral inertia of the disc are added. Therefore, the first part of the global matrix (Fig. 2.8) represents the shaft carrying rigid discs. In this same figure the remainder of the global matrix includes

A band symmetric diagonal matrix. It is assembled from the $[ME]$ matrices, and the first two rows and corresponding columns have been deleted, imposing the boundary conditions for disc (disc's horizontal translation is zero, disc's rotation is same with shaft's slope, therefore the disc's relative rotation is zero). This matrix represents a flexible disc clamped on a rigid shaft.

A single row matrix (and its transpose) that corresponds to the shaft's slope θ_s . It is the coupling matrix, since it couples the disc to the shaft.

For the whole system the matrix equation for the free whirling vibration is,

$$[GM][\bar{Z}] + \Omega[GC][\bar{Z}] + \{[GK] + \Omega^2[GKR]\}[Z] = 0 \quad (24)$$

Imposing the general solution,

$$\{Z\} = \left\{ \{\bar{Z}_p\} e^{i\omega t} \quad \{\bar{Z}_q\} e^{i\omega t} \right\} \quad (25)$$

and dividing the system's matrices into p and q submatrices we have:

$$\begin{aligned} & -\omega^2 \begin{bmatrix} [GM]_p & 0 \\ 0 & [GM]_q \end{bmatrix} \begin{bmatrix} \bar{Z}_p \\ \bar{Z}_q \end{bmatrix} + i\omega\Omega \begin{bmatrix} 0 & [GC]_p \\ -[GC]_q & 0 \end{bmatrix} \begin{bmatrix} \bar{Z}_p \\ \bar{Z}_q \end{bmatrix} \\ & + \left(\begin{bmatrix} [GK]_p & 0 \\ 0 & [GK]_q \end{bmatrix} + \Omega^2 \begin{bmatrix} [GKR]_p & 0 \\ 0 & [GKR]_q \end{bmatrix} \right) \begin{bmatrix} \bar{Z}_p \\ \bar{Z}_q \end{bmatrix} = 0 \end{aligned} \quad (26)$$

Considering axial symmetry being one of our assumptions, the corresponding submatrices with p and q subscripts are identical, and so replacing both subscripts by s we have the system

$$\begin{aligned} & \left\{ -\omega^2[GK]_s + [GK]_s + \Omega^2[GKR]_s \right\} \{\bar{Z}_p\} + i\omega\Omega[GC]_s \{\bar{Z}_q\} = 0 \\ & \left\{ -\omega^2[GK]_s + [GK]_s + \Omega^2[GKR]_s \right\} \{\bar{Z}_q\} - i\omega\Omega[GC]_s \{\bar{Z}_p\} = 0 \end{aligned} \quad (27)$$

The system of equations (27) describe the linear vibrations of any rotating, rotationally symmetric structure [Ref. 45]. It consists of a symmetric mass matrix, a skew

- symmetric gyroscopic matrix and a symmetric stiffness matrix (including rotational stiffening)

In the case of steady state whirling with imbalance being the only excitation, we can use the relation:

$$\{\bar{Z}_p\} = i\{\bar{Z}_q\} \quad (28)$$

which denotes a circular whirling orbit [Ref. 29]. Using also the precession factor $\alpha = \Omega/\omega$, and substituting (28) into (27 b) we get

$$\begin{aligned} -\omega^2[GM]_S\{\bar{Z}_q\} + \omega\Omega[GC]_S\{\bar{Z}_q\} + [GK]_S\{\bar{Z}_q\} + \Omega^2[GKR]_S\{\bar{Z}_q\} &= 0 \\ \Rightarrow \left\{[GK]_S - \omega^2\{[GM]_S - \alpha[GC]_S - \alpha^2[GKR]_S\}\right\}\{\bar{Z}_p\} &= 0 \end{aligned} \quad (29.a)$$

substituting (28) into (27 a) we get a similar expression for the p - subsystem.

$$\left\{[GK]_S - \omega^2\{[GM]_S - \alpha[GC]_S - \alpha^2[GKR]_S\}\right\}\{\bar{Z}_p\} = 0 \quad (29.b)$$

Either of equations (29) is the eigenvalue problem of the rotating system. It can be also written in the form:

$$\{[AK] - \omega^2[AM]\}\{z\} = 0 \quad (30)$$

with

$$\begin{aligned} [AK] &= [GK]_S \\ [AM] &= [GM]_S - \alpha[GC]_S - \alpha^2[GKR]_S \end{aligned} \quad (31)$$

being the generalized stiffness and mass matrices respectively

2.6 Reduction of the eigenvalue problem

Though the eigenvalue problem (30) can be solved as it is, the advantage of this

scheme is that the size of the system can be reduced.

As described earlier, the problem setup consists of the following parts:

- a) Flexible shaft carrying rigid disc(s)
- b) Flexible disc(s) on rigid shaft

The introduction of the disc relative deflection (with respect to a hypothetical rigid disc attached at the same point) has also created a:

- c) Coupling matrix,

which couples the disc to the shaft slope θ_S and, therefore, it is a one - dimensional vector corresponding to the θ_S degree of freedom.

Note that the coupling vector exists only in the mass and gyroscopic matrices, while the stiffness matrices are uncoupled.

The following step is to perform the modal analysis of the independent problems a) and b) above. We can arrange a modal matrix $[B_S]$ to consist of the N_S lower modes of problem a), and, similarly a modal matrix $[B_D]$ to consist of the N_D lower modes of problem b). Using them as submatrices, the global modal matrix $[BG]$ will be.

$$[BG] = \begin{bmatrix} [B_S] & 0 \\ 0 & [B_D] \end{bmatrix}$$

and if N is the dimension of the global problem (described by matrices $N \times N$), the dimensions of $[BG]$ are $(N \times [N_S + N_D])$ as can be seen in fig. 2.9.

Introducing the transformation:

$$\{Z_S\} = [BG]\{Z_1\} \quad (32)$$

in (30) and premultiplying with $[BG]^T$, we have:

$$\left\{ [BG]^T [AK]_S [BG] - \omega^2 [BG]^T [AM]_S [BG] \right\} \{Z_1\} = 0 \Rightarrow$$

$$\Rightarrow \{ [GK1] - \omega^2 [GM1] \} \{ Z_1 \} = 0 \quad (33)$$

where, in (33),

$$[GK1] = [BG]^T [AK]_s [BG]$$

and

$$[GM1] = [BG]^T [AM]_s [BG]$$

The eigenvalue problem (33) can be seen to be reduced to $([N_S + N_D] \cdot [N_S^* + N_D])$

The $[GK1]$ matrix is diagonal because the modal submatrices $[B_S]$ and $[B_D]$ have diagonalized both submatrices of problems a) and b) respectively. The $[GM1]$ matrix is full; the non - diagonal terms coming from the existence of the coupling matrix:

$$[BG]^T [GM]_s [BG] = \begin{bmatrix} [B_S]^T & 0 \\ 0 & [B_D]^T \end{bmatrix} \begin{bmatrix} [M_S] & [M_C] \\ [M_C]^T & [M_D] \end{bmatrix} \begin{bmatrix} [B_S] & 0 \\ 0 & [B_D] \end{bmatrix}$$

$$\Rightarrow [GM1] = \begin{bmatrix} \text{diag } M_{SS}^* & M_{SD} \\ M_{DS} & \text{diag } M_{DD}^* \end{bmatrix}$$

where,

$$\text{diag } M_{SS}^* = [B_S]^T [M_S] [B_S]$$

$$M_{SD} = [B_S]^T [M_C] [B_D]$$

$$\text{diag } M_{DD}^* = [B_D]^T [M_D] [B_D]$$

$$M_{DS} = [B_D]^T [M_C] [B_S]$$

The individual terms of M_{SD} say corresponding to $[b_S']$ (mode 1 on shaft) and $[b_D']$ (mode 1 on disc) are

$$M_{SD}'' = [b_S']^T [M_E] [b_D']$$

Another modal analysis for the reduced problem (33) is required to obtain the eigenvalues and normal modes of the coupled shaft - flexible disc system. Finally to go back to the original global coordinates, we only have to consider that the modal analysis of (33) is equivalent to a transformation.

$$[Z_1] = [B_1][Z_2] \quad (34)$$

substituting (34) in (32) we get the eigenvectors of the full problem in the original global coordinates:

$$[Z_S] = [B_G][B_1][Z_2] \quad (35)$$

2.7 Determination of the modal submatrices

A The 'shaft with rigid disc(s)' problem from (30) can be described as

$$\left\{ [AK]_S - \omega^2 [AM]_S \right\} \left\{ Z_{PS} \right\} = 0$$

The modal analysis of the above eigenvalue problem provides the shaft's modal matrix $[B_S]$

Note 1 s subscript for shaft's matrices

Note 2 For the rotating case ($\alpha = 1$) not all eigenvalues generally exist $[B_S]$ will then consist of the lower eigenvectors

Note 3 The precession factor α is contained in the appropriate diagonal position of $[AM]_D$.

B The 'disc on rigid support' problem from (30) is

$$\left\{ [AK]_D - \omega^2 [AM]_D \right\} \{ Z_{PD} \} = 0 \quad (32)$$

where

$$\begin{aligned} [AM]_D &= [DM] - \alpha [DG] - \alpha^2 [DKR] \\ &= (1 - 2\alpha) [DM] - \alpha^2 [DKR] \end{aligned}$$

It can be seen that for $\alpha = 1$ the generalized mass matrix will have it's diagonal elements negative and no eigenvalues or eigenvectors can be obtained

This physically means that in the case of a spinning disc, lateral resonance is possible only with external excitation. But even when we have resonance, the rotational velocity is never equal to the frequency of the excitation (it is always smaller). Ideally, (with negligible rotational stiffening effects), the highest possible value for the rotational speed in resonance is to be half the forcing frequency. When rotational stiffening is present, this value is even more decreased. (Look at Appendix C for the disc analysis)

In our case, the shaft (carrying the flexible disc) will be at resonance (for the sake of the critical speeds) where the whirling speed will be equal to the rotational speed of the shaft (or the rotational speed of the disc). Consequently, the flexible disc will be under forced vibration with a frequency well below (more than 50%) the critical!

Finally, the value $\alpha = 1$ has meaning only for the system, not for the disc alone. In other words, the problem (32) is NOT an eigenvalue problem.

But we still need the normal modes

To override this difficulty, we can work without correspondence to a physical model. We are searching for the matrix $[B_D]$ that diagonalizes the matrices $[AK]_D$ and $[AM]_D$ above. Observing that the $[B_D]$ transformation should also diagonalize the $-[AM]_D$ matrix, we form the eigenvalue problem

$$\left\{ [AK]_D - \omega^2 (-1) [AM]_D \right\} \{ Z_{PD} \} = 0$$

This problem gives eigenvalues - meaningless - and the eigenvectors that form the required modal matrix $[B_D]$

2.8 Numerical solution

To solve numerically the a) and b) problems of the last paragraph, the power iteration method was employed. The method uses the 'Dynamic stiffness matrix' concept which allows the determination of the frequencies starting from the lowest

$$\begin{aligned} [K]\{Z\} &= \omega^2 [M]\{Z\} \Rightarrow \\ \Rightarrow \left(\frac{1}{\omega^2} \right) \{Z\} &= [K]^{-1} [M]\{Z\} \end{aligned}$$

where $[K]^{-1} [M] = [D]$ is the dynamic matrix [Ref. 6]

Every eigenvalue problem of order $(n \times n)$ has n roots ω^2 . In our case, the roots are real because the mass and stiffness matrices are real and symmetric. They are also discrete

$$0 < |\omega_1^2| < |\omega_2^2| < \dots < |\omega_n^2|$$

a negative root $|\omega_i^2|$ is meaningless for the system. However, if ω_i^2 (the i^{th} root) is negative, the system will not converge to the i^{th} mode (because the value of the eigenvector elements will change sign in every iteration step). After that the calculation of the $i+1, i+2, \dots$ modes is impossible.

In some problems of the present thesis, the first root (ω_1^2) is negative, so in order to calculate the first existing modes and frequencies, a 'shift' technique is used.

In [Ref. 2] Bathe describes the 'eigenvalue shifting' procedure where, by performing the shift S on the stiffness matrix $[K]$

$$[\dot{K}] = [K] - S[M] \quad S > 0$$

and solving the eigenvalue problem

$$\{[\dot{K}] - \omega^2[M]\}\{Z\} = 0$$

he calculates the eigenvectors and eigenvalues, and:

- the eigenvectors $\{Z\}$ of the 'shifted' problem are identical to the eigenvectors of the original problem $\{[K] - \omega^2[M]\}\{Z\} = 0$

- the eigenvalues ω_i^2 of the 'shifted' problem are the eigenvalues of the original problem reduced by the value S

While the purpose of this technique is to accelerate the calculations, in our case the shift

$$[\dot{K}] = [K] + S[M] \quad S > 0$$

will give eigenvalues increased by S. Therefore, if S is chosen to be

$$S > |\omega_1^2|$$

all the roots $\omega_1^2, \omega_2^2, \dots, \omega_n^2$ will be calculated and the positive lower frequencies will be obtained.

Chapter 3

NUMERICAL RESULTS

3.1 Introduction

To verify the validity of the previous analysis a number of test cases is used, chosen from the existing literature.

Results are obtained both for the non-reduced rotor system and for the reduced system, using a variable number of modes for each flexible component. Comparisons are made with exact or alternative numerical data, where possible, to indicate the minimum number of normal modes required for accuracy, for several precession factors covering the range from non-rotating ($\alpha = 0$), to forward precession ($\alpha = 1$).

Finally, a practical rotor is modelled - which is actually a part of an aircraft turbine engine - and again the system's reduction using component modes is studied.

3.2 Test case I

Overhanging disc configuration

A simply supported rotor with an overhanging disc is chosen from Dopkin's studies [ref 8] shown in figure 3.1 (Rotor No 1). The bearing supports are flexible, and the disc is thin, unrealistically large and impractically flexible, so that the effects of the disc flexibility on the system's natural frequencies will be exaggerated.

The results provided in [ref 8] have been calculated without considering the shaft rotary inertia and shear deflection. Dopkin actually showed, after including the above in the analysis, that their effects are negligible compared to the disc effects.

The shaft is divided in two equal elements by the two bearings and the disc. Both the shaft and the disc are of uniform cross-section and thickness respectively.

The bearings have a stiffness of $18 \cdot 10^9 \text{ N/m}$ to deflection. They have no stiffness to rotation (free to rotate).

The shaft length is 1.22m, with an outer radius of 0.0381m.

The disc outer radius is 1.22m, thickness is 0.006m. Modulus of Elasticity $200 \cdot 10^9 \text{ N/m}^2$, density 7744 kg/m^3 , Poisson's ratio 0.33.

Though Dopkin used 20 segments to model the shaft, according to Masoom [ref 29] division into two finite elements only is sufficient to give excellent results. The disc, again according to [ref 29] is divided into 4 annular elements (the minimum number of elements for good accuracy).

Table 3.1 contains the natural frequencies of both the partial problems and the combined system and compares these values with the values obtained in [ref 8] and [ref 29].

This simple case of a non-rotating system was chosen to compare the relative deflections method to the traditional finite element model (Masoom, ref.29). It appears that the results for the fundamental frequency and first mode, the 'deflections' mode, are exactly the same. It is the higher frequencies (and modes related to higher derivatives of the deflection line) that are approximated slightly better using the relative deflections method.

However the real advantage of this method is that the size of the global system can be reduced, solving first the partial problems. The study of the reduction using a variable number of shaft and disc modes in the modal matrix comes next. Table 3.2 gives the natural frequencies of the reduced system. The number of modes used for the shaft and the flexible disc is shown.

As expected, the first disc mode dominates and the fundamental frequency is a little less than the frequency of the flexible disc alone. It is evident though, that higher disc modes do not practically influence the system's natural frequencies, while more

shaft modes have to be used for better accuracy. With 2 - shaft and 1 - disc modes we observe that the first frequency converges and is equal to the value of the non - reduced system. The approximation for the second frequency is already only 0.28% higher than the non - reduced value. The reduced system is of order $(2+1) \cdot (2+1)$ or $(3 \cdot 3)$, while the global structure with 7 nodes in total is of order $14 \cdot 14$. With 2 - shaft and 2 - disc modes (order $4 \cdot 4$) even the second frequency converges to the value for the non - reduced system.

3.3 Test case II

Disc between bearings

Rotor No 2 is used in this case (shown in figure 3.2), to investigate the natural frequencies and critical speeds of a flexible rotor with a disc located between the two flexible bearings, dividing the shaft length into 2/1 ratio. The disc is the same as that of rotor No 1, and the shaft and bearings have the same properties.

The disc is divided into 4 annular elements and the shaft into 3 beam finite elements.

Again the results for the partial problems and for the global non - reduced system are compared to the values given by Dopkin and Masoom. The partial problem 'flexible disc on rigid support' will have the same eigenvalues as before, and is not presented.

Table 3.3 contains the whirling frequencies of the rotor with rigid and with flexible disc for several precession factors.

The fundamental frequency (for $\alpha = 0$) is not different from the traditional finite element model value. The approximation on the second eigenvalue is 0.2% lower compared to the traditional finite element model, the same as in the overhanging disc configuration. However, as the rotor is spinning this slight improvement in accuracy tends to disappear, and for the precession factor $\alpha = 0.5$ the two methods give the same value. That is because the term $I_D - \alpha I_P = I_D(1 - 2\alpha)$ tends to zero, and the

flexible disc tends to behave as a rigid body. For $\alpha > 0.5$, higher eigenvalues do not exist, so the two methods give identical results.

The system reduction is studied next for the non-rotating disc, ($\alpha = 0$), as well for the synchronous precession ($\alpha = 1$).

Table 3.4 shows the number of disc and shaft modes used and the corresponding eigenvalues for the reduced system. In this case it is the disc modes that significantly affect the system's frequencies. Two disc and one shaft mode are needed for satisfactory results, while the addition of a second shaft mode will give exactly the non-reduced rotor eigenvalues.

The system's reduction for the rotating case is presented in Table 3.5. With two disc modes to interact with the fundamental shaft mode the critical speed converges for the synchronous whirl ($\alpha = 1$). That means that the effects of the flexible components practically remain the same for the whole range of forward precession. The above is another verification of the statement by Lowey and Khader [ref. 44] that the non-rotating modes for the disc can be efficiently used to analyze the rotating case as well, as long as a complete set of functions is used to approximate the disc deflection.

3.4 Test case III

Study of a Model of a Practical Rotor System

In order to display the efficiency of the 'relative deflections' method in modelling practical systems, a complicated rotor, representing multiple disc configuration, was chosen to be examined. Multiple bearings with radial flexibility only, (flexible to deflection, free to rotation) are used to support the system, all the previous cases, i.e., discs between bearings as well as overhanging disc are combined and contribute to the system's response.

The rotor chosen is a low speed or fan rotor of a two-shaft aircraft gas turbine engine, also studied in [ref. 29].

This engine comprises two coaxial shaft systems - a high speed shaft operating at a maximum speed of 30,000 RPM approximately, and a low speed rotor operating at a maximum speed of approximately 15,000 RPM

In figure 3.3 the low speed rotor is presented, isolated from the rest of the system. This isolation is a simplification, since the centre bearing is an intershaft bearing which will couple whirling motions of the two rotors. Therefore for the purposes of the present study, the expected whirling interaction was disregarded.

At the front of the shaft, a single fan compressor stage is mounted, and at the rear we have two turbine stages. The fan - disc can be considered flexible, compared to the turbine discs, and comprising blade flexibility mainly, so this study examines the effect of fan - disc flexibility on the whirling speeds of the rotor.

Originally assuming all discs were rigid, the design calculations revealed that, in the synchronous precession mode, $\alpha = 1$, the whirling of the rotor should be avoided at speeds close to design speed, 15,000 RPM, or approximately 1500 rad/sec. Because intershaft coupling was possible, the range of whirling speed also excluded values close to 30,000 RPM, or 3000 rad/sec approximately.

Uniform thickness annular finite elements are used to model the disc. The blade flexibility is approximated arranging the disc geometry and physical properties [ref 29]. A range of values for the Modulus of Elasticity for the disc is used, allowing for an indication as to when the flexible rotor modes are likely to vary significantly from the shaft with rigid disc modes.

The data for the disc (from [ref 29]) are

- Outer radius 0.164m
- Inner radius 0.0287m

Using a material density of 4432.4 kg/m^3 , the disc thickness must be 0.023m in order to match the actual mass and diametral inertia.

The Poisson's ratio is taken 0.33 and the Modulus of Elasticity E ranges from $17 \cdot 10^9 \text{ N/m}^2$ to $55 \cdot 10^9 \text{ N/m}^2$

The data for the shaft are

- Outer radius 0.0287 m

- Inner radius 0.025 m

The density is 7744 kg/m^3 , and the Modulus of Elasticity is 207 GPa (207 N/m^2)

The first step is to check whether the flexible disc - shaft interaction is likely to occur. The fundamental frequency of the flexible disc rotating at 15,000 RPM is found [ref. 29] to be less than the above operational speed. An interaction therefore is expected.

In table 3.6 the whirling frequencies of the rotor calculated assuming a rigid disc, and assuming a range of disc flexibilities are compared to the non-rotating ($\alpha = 0$), forward precession ($\alpha = 1$), and backward precession ($\alpha = -1$) eigenvalues.

It is interesting to notice for the forward precession case ($\alpha = 1$) the small effect of the disc flexibility on the system's eigenvalues (not more than 3% reduction from the rigid disc values for the first mode). The other existing normal modes are likewise essentially unchanged.

For the non-rotating case, and for the backward precession, however, large reductions in the frequencies of all modes are quite evident.

Tables 3.7 to 3.9 contain the system's reduction for precession factors $\alpha = 1$, $\alpha = 0$ and $\alpha = -1$. The flexible disc with Elastic Modulus $E = 1.725 \cdot 10^9 \text{ N/m}^2$ is used, for better representation of the eigenvalues reduction. As in the overhanging disc configuration of test case I, it is the shaft modes that play the most important role in improving the calculated values for the natural frequencies, for the whole range of the precession factor.

For $\alpha = 0$ and $\alpha = 1$ only 1 - disc mode is sufficient. Using also 4 shaft modes the results converge, while the reduced system is of order $(5 \cdot 5)$ ¹.

¹ It should be noted that the global system has 22 D O F, or the non-reduced system is of order $(22 \cdot 22)$.

For $\alpha = -1$ only one eigenvector exists for the disc alone partial problem. Again 1 - disc and 4 - shaft modes are sufficient to give exactly the results of the non - reduced flexible rotor.

Chapter 4

SUMMARY , CONCLUSIONS AND RECOMMENDATIONS

4.1 Summary

In this thesis, a new scheme for the dynamic analysis of a whirling rotor carrying flexible discs has been developed, the 'relative deflections' method. By assuming the flexible disc lateral deflections to be taken relatively to a hypothetical rigid disc attached to the shaft at the same point, the mathematical formulation comprises two component problems, and their coupling. The two component problems, namely 'whirling flexible shaft carrying rigid discs' and 'flexible disc clamped on a whirling and spinning rigid shaft', can be solved independently modeled (here) using Finite Element techniques. The Finite Element models can provide refinements such as distributed inertia loading and gyroscopic loading, and variation in taper of shaft and disc segments, so that few elements are required to obtain accurate solutions to the equations.

From the kinetic energy expressions and the application of Lagrange equations the interconnection of the above partial problems has also been formulated as a vector which couples the disc to the shaft slope at the point of attachment.

The independent problems are solved numerically, a modal analysis is performed employing the power iteration method. At this point we can study the shaft - flexible disc interaction by arranging a variable number of shaft and disc modes in the global modal matrix. Furthermore, we can consequently reduce the size of the rotor system considerably, without significant loss in accuracy. This has been verified for a variety of configurations by comparison with alternative numerical solutions.

A relatively complicated rotor representative of a practical geometry has been studied to assess the practical importance of shaft - disc whirl interaction study and

global system reduction.

The Finite Element models used are well established, however the description of the shaft - disc coupling in a vector is new. The important feature of this work is the simple and direct analysis of the component modes, and the use of these component modes to reduce the global system equations while retaining high accuracy. This reduction of the system is extremely efficient for calculation of the whirling speeds when interaction between shaft and flexible discs occurs. More important advantages are evident when attempting to calculate forced response or transient behaviour of the shaft - flexible disc system.

4.2 Discussion

1. The description of the disc deflections relative to a rigid body at the same location has developed the independent problem 'flexible disc on a rigid whirling shaft'. This problem can again be divided into two, 'flexible spinning disc centrally clamped' and 'flexible whirling disc'. The first has eigenvalues for the whole positive range of the spin speed. Rotational stiffening effects have to be accounted for, while the energy expressions do not include gyroscopic effects or rigid body translation. The second does not always have eigenvalues. In any case - for $0 < \alpha < .5$ - the eigenvalues are quite different from these of the first problem. Furthermore, the energy formulation includes gyroscopic effects.

2. The 'relative deflections' method does not generally improve the results given by the traditional Finite Element method, which is already quite accurate. However, it reduces the system greatly, and shows directly which (lower) normal modes from each component play the most important role in altering the 'shaft with rigid disc' eigenvalues. Therefore, it has been found that in the case of discs between bearings, it is the disc modes that drastically improve the approximation to the system's eigenvectors, while in the case of overhanging disc the shaft modes are more significant. This conclusion is also valid for every combined case with more discs, as can be seen

from the study of the practical rotor. Of course, the larger the system is, the more modes from each component - depended on the configuration - are needed for satisfactory approximation. However, the rotor system will still be greatly reduced, e.g., for the practical rotor the system of equations was reduced to (5×5) from (22×22) . This is a 77% reduction for the coordinate vector length, or a 95.5% reduction for the dynamic stiffness matrix area! Similar reduction has been obtained for the other test cases.

4.3 Recommendations

1. The equations of motion have been formulated for a circular orbit whirling which corresponds to a circumferentially symmetric rotor, both in geometry and stiffness, performing steady state vibration excited by unbalance. It has been discussed earlier though, that the actual orbit shape depends on the initial conditions and on the axial symmetry in the flexible bearings (different stiffness along x and y axes). Therefore, an extension of the scheme requires the feature of describing any possible elliptical orbit, accounting for all types of initial conditions and excitations.

2. Due to the assumption that the shaft - disc interaction occurs at a point, the coupling matrix is reduced to a vector corresponding to the shaft slope at the disc location. Thus, a further improvement and simplification could possibly be to get the shaft - disc coupling terms in closed form.

3. For the previous analysis a precession factor is always assumed, in other words, the relation between the rotational speed and the whirling speed cannot be calculated here (cannot be treated as an unknown). However, this ability would provide a more complete analysis and a general understanding as to what external performance conditions contribute to arranging the value of the precession factor.

4. An important case where induce rearward precession is the transient response to sudden imbalance, e.g., sudden blade loss. This is because in the solution of the differential equation of motion for the transient analysis, terms representing both forward and rearward whirl are present. The above should be the subject of further

study since disc flexibility may significantly modify the response

5 Finally, the most important requirement, and the most difficult, is the verification of the analysis by experimental studies. A variety of cases should be examined: frequency calculations of non-rotating and rotating cases; forced response produced by unbalance or aerodynamic forces; bearing support stiffness asymmetry where shaft-disc interaction effects may be large.

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Figure 1.1.a.:
Subcritical operation.

Figure 1.1.b.:
Supercritical operation

1

Figure 2.1.: general model for the rotor system analysis

Figure 2.2.: orthogonal deflections 'p' and 'q' for the shaft

Figure 2.3.: orthogonal deflections 'u' and 'v' for the disc, on a stationary axial system XYZ

Figure 2.4.: Deflections described on a shaft finite element

Figure 2.5.: Principal axes for the description of the disc rotational deformation. Z is the polar axis.

Figure 2.6.: Disc deflection u relative to rigid disc, small deformations.

$$\begin{array}{c}
 \begin{array}{|c|c|c|}
 \hline
 M_s & 0 & p_s \\
 \hline
 I_{DE} & C_s & \theta_s \\
 \hline
 \end{array} \\
 \\
 \begin{array}{c}
 MD = \\
 \begin{array}{|c|c|c|}
 \hline
 C_s^T & EM & \\
 \hline
 \end{array}
 \end{array}
 \begin{array}{l}
 u_s \\
 \frac{du}{dr} \\
 u_{s+1} \\
 \frac{du}{dr} \rightarrow 1
 \end{array}
 \end{array}$$

Figure 2.7.a.: Complete disc element mass matrix.

$$\begin{array}{c}
 \begin{array}{|c|c|c|}
 \hline
 & & \\
 \hline
 & I_{PE} & 2[C] \\
 \hline
 & & \\
 \hline
 & 2[C]^T & 2[EM] \\
 \hline
 \end{array}
 \end{array}
 \begin{array}{l}
 p_s \\
 \theta_s \\
 u_s \\
 \frac{du}{dr}_s \\
 \frac{du}{dr}_{s+1}
 \end{array}$$

2

Figure 2.7.β.: Complete disc element gyroscopic matrix.

Figure 2.8.: Global submatrix $[GM]_p$ for the $p + u$ domain.

Figure 2.9.: Modal matrix $[B]$ containing N_s shaft modes and N_D disc modes. The dimension is $N \times (N_s + N_D)$.

Figure 3.1.: Test case I rotor

Figure 3.2.: Test case II rotor. The shaft, disc and bearings have the same properties as the rotor of test case I.

Figure 3.3.: Model of a gas turbine fan engine rotor (schematic representation)

Figure C.1.: The whirling disc critical speeds as a function of the precession factor α for the case of forward precession ($\alpha > 0$).

TABLE 3.1

Test case I results

Natural frequency in rad/sec for the problems:

- shaft with rigid disc (Rotor No. 1)
- flexible disc on rigid support (disc alone)
- combined, non - reduced system (flexible rotor)

Precession factor $\alpha = 0$ (non - rotating)

Problem type	n	Relative deflections	Dopkin [8]	Masoom [29]
shaft & rigid disc	1 2	60.66 257.15	60.41 -	60.66 257.15
disc alone	1 2	16.6 152.4	- -	16.6 152.4
flexible rotor	1 2	16.2 102.3	15.13 -	16.2 102.5

TABLE 3.2**Test case I results**

Natural frequency in rad/sec for the reduced flexible rotor No. 1 system

NMS	n	NMD		
		1	2	3
1	1	16.34	16.34	16.34
	2	108.83	108.29	108.29
2	1	16.20	16.20	16.20
	2	102.59	102.31	102.31
3	1	16.20	16.20	16.20
	2	102.59	102.31	102.31

NMS: number of modes used from the 'shaft with rigid disc' problem

NMD: number of modes used from the 'flexible disc on rigid shaft' problem

TABLE 3.3**Test case II results**

Whirling frequency in rad/sec for rotor No. 2, non - reduced system.

α	n	shaft with	shaft with	flexible	disc
		rigid disc	R.D.	M.[29]	D. [8]
0.0	1	153.7	16.53	16.53	15.41
	2	312.0	151.9	152.20	-
0.25	1	184.5	23.3	23.3	21.78
	2	366.7	213.4	213.6	-
0.5	1	218.5	218.5	218.5	217.3
	2	1944.1	1947.0	1947.1	-
0.75	1	239.0	218.6	218.6	217.7
	2	*	*	*	*
1.0	1	248.9	218.7	218.7	-
	2	*	*	*	*

R.D.:Relative deflections M. Masoom D. Dopkin

TABLE 3.4

Test case II results

Whirling frequency in rad/sec for the reduced flexible rotor No 2 system, $\alpha = 0$

NMS	n	NMD		
		1	2	3
1	1	16.54	16.54	16.54
	2	262.30	151.91	151.91
2	1	16.53	16.53	16.53
	2	215.83	151.90	151.90
3	1	16.53	16.53	16.53
	2	215.83	151.90	151.90

TABLE 3.5

Test case II results

Whirling frequency in rad/sec for the reduced flexible rotor No. 2 system, $\alpha = 1$

n	NMD		
	1	2	3
1	219.23	218.73	218.72
2	*	*	*

TABLE 3.6

Test case III results

Whirling frequency in rad/sec for the combined shaft - disc system representing the effect of disc flexibility and the rotational speed

α	n	E	N/m^2	Shaft with
		$1.725 \cdot 10^9$	$27.6 \cdot 10^9$	rigid disc
1.0	1	1103.7	1127.8	1134.0
	2	1308.5	1311.5	1312.6
	3	5584.0	5756.7	6052.9
	4	11830.0	11893.0	11967.0
0.0	1	669.6	981.1	991.2
	2	1115.0	1293.6	1294.7
	3	1302.8	2026.4	2445.7
	4	2562.7	2572.7	2624.4
-1.0	1	420.0	786.9	836.5
	2	1144.1	1276.4	1281.4
	3	1298.9	1498.9	1560.7
	4	1578.0	1609.2	1824.5

TABLE 3.7

Test case III results

Whirling frequency in rad/sec for the reduced rotor system in forward precession
 ($\alpha = 1$)

NMD	n	NMS		
		1	2	3
1	1	1107.3	1101.5	1094.9
	2	-	1315.4	1307.2
	3	-	-	-
2	1	1107.2	1101.5	1094.9
	2	-	1314.7	1307.2
	3	-	-	-

TABLE 3.8

Test case III results

Whirling frequency in rad/sec for the reduced rotor system, non-rotating case
 ($\alpha = 0$)

NMS	n	NMD	
		1	2
1	1	682.5	682.5
	2	1155.7	1155.6
	3	-	5361.5
2	1	681.9	681.9
	2	1148.8	1148.7
	3	1309.0	1309.0
	4	-	5362.2
3	1	673.2	673.2
	2	1123.2	1123.2
	3	1303.9	1303.9
	4	3731.1	3731.1
4	1	669.9	669.9
	2	1115.6	1115.5
	3	1302.8	1302.8
	4	2563.0	2562.9

TABLE 3.9

Test case III results

Whirling frequency in rad/sec for the reduced rotor system, backward precession
 ($\alpha = -1$)

NMS	n	NMD
		l_n
1	1	424.6
	2	1415.4
	3	-
	4	-
2	1	424.4
	2	1243.1
	3	1482.1
	4	-
3	1	424.0
	2	1233.8
	3	1380.4
	4	1764.5
4	1	420.0
	2	1144.8
	3	1299.0
	4	1578.1

APPENDIX (A)

Shaft Element Matrices and Assembly

The stiffness matrix $[EK]$ for the flexural shaft element can be obtained in a conventional way assuming a complete cubic displacement function and is given as

$$[EK] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

where

E is the Modulus of Elasticity

I is the Moment of Inertia.

L is the length of the element

The matrix equation for the static analysis (stiffness matrix) is

$$\{P_e\} = [EK]\{p_e\}$$

where

$\{P_e\}^T = \{V_i, M_i, V_{i+1}, M_{i+1}\}$ is the force vector

$\{p_e\}^T = \{p_i, \theta_i, p_{i+1}, \theta_{i+1}\}$ is the deflection vector

For the assembly of the system's stiffness matrix, the nodal connectivity of the element has to be taken under consideration, or, in other words, the coordinate vector of the element has to match the system's coordinate vector. If, for example, the element is between nodes $N1$ and $N2$, the $EK(1,1)$ value has to be added to the $(2N1-1, 2N1-1)$ position of the system's matrix, the $EK(4,1)$ to the $(2N2, 2N1-1)$ position, etc.

The consistent mass matrix can be conventionally derived using the same cubic deflection function. Considering only the translational inertia, the element matrix is as

follows

$$EM = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

where

ρ is the density and

A is the shaft cross section

The same coordinate vector and therefore the same assembly line is used to form the shaft mass matrix. Concentrated masses and flexible bearings can be treated as described in section 2.2

APPENDIX (B)

Element Matrices for a Disc Annular Element

1. Flexural Stiffness and Inertia Matrices

The stiffness and inertia matrices of the thin disc annular element are obtained by substituting the assumed deflection function into the Strain energy and Kinetic energy expressions

From ref. 23, 24 the $[KE]$ matrix (element stiffness matrix) is given by

$$[KE] = [B^{-1}]^T [K_e] [B^{-1}] \quad (B.1)$$

where $[B^{-1}]$ is the inverse of boundary values matrix given in table B.1 and $[K_e]$ is a matrix used in forming the element bending stiffness matrix given in table B.2

Similarly the element inertia matrix $[ME]$ can be obtained from

$$[ME] = [B^{-1}]^T [M_e] [B^{-1}] \quad (B.2)$$

where $[M_e]$ is a matrix used in forming the element inertia matrix given in table B.3

The above tables (originating from [ref. 23, 24]), have been simplified due to the assumptions for uniform thickness isotropic disc and one nodal diameter

2. Vector coupling the disc to the shaft slope

The coupling matrix $[C_e]$, already described in eq. (18) of Chapter 2, can be expressed as:

$$[C_e] = [C][B^{-1}] \quad 9(B.3)$$

where

$$C_1 = 2 \int_{R_1}^{R_2} \pi \rho t r^2 [S] dr \quad (B 4)$$

is given in table B 4

3. Effect of in - plane stresses on the disc flexural vibration

When in - plane radial stress σ_r and tangential stresses σ_θ are present at the middle plane of the annular thin disc, the strain energy expressions for transverse bending will contain additional terms accounting for the effect of the stresses

These terms are described with an additional matrix $[KE_R]$, discussed in chapter 2. This can be obtained from

$$[KE_R] = [B^{-1}]^T [K_R] [B^{-1}] \quad (B 5)$$

where, $[K_R]$ is the matrix used in forming the additional stiffness matrix given in table B 5. The two integrals involved in this matrix are

$$\begin{aligned} S_1 &= \pi t \int_{R_1}^{R_2} r^3 \cdot \sigma_\theta(r) dr \\ C_1 &= \pi t \int_{R_1}^{R_2} r^3 \cdot \sigma_r(r) dr \end{aligned} \quad (B 6)$$

The stresses $\sigma_r(r)$ and $\sigma_\theta(r)$ are assumed to vary linearly within the element which requires that these stresses need only be known at the nodal points.

For a uniform thickness disc the expressions used to calculate these stresses, found in any standard text on Theory of Elasticity, are as follows:

$$\begin{aligned}\sigma_r(r) &= \frac{3+\nu}{8} \rho \Omega^2 R^2 \left[1 - \left(\frac{r}{R} \right)^2 \right] \\ \sigma_\theta(r) &= \frac{3+\nu}{8} \rho \Omega^2 R^2 - \frac{1+3\nu}{8} \rho \Omega^2 r^2\end{aligned}\tag{B 7}$$

where

Ω is the rotational speed in *rpm*

r is the radial distance of the node

R is the outer radius of the disc

ρ is the disc density

Table B.1

Matrix $[B^{-1}]$

$\frac{R_2^2 (R_2 - 3R_1)}{(R_2 - R_1)^3}$	$\frac{R_1 R_2^2}{(R_2 - R_1)^2}$	$\frac{R_1^2 (3R_2 - R_1)}{(R_2 - R_1)^2}$	$\frac{R_1^2 R_2}{(R_2 - R_1)^2}$
$\frac{6R_1 R_2}{(R_2 - R_1)^3}$	$\frac{R_2 (2R_1 + R_2)}{(R_2 - R_1)^2}$	$\frac{6R_1 R_2}{(R_2 - R_1)^3}$	$\frac{R_1 (R_1 + 2R_2)}{(R_2 - R_1)^2}$
$\frac{3(R_1 + R_2)}{(R_2 - R_1)^3}$	$\frac{(R_1 + 2R_2)}{R_2 - R_1^2}$	$\frac{3(R_1 + R_2)}{R_2 - R_1^3}$	$\frac{(2R_1 + R_2)}{(R_2 - R_1)^2}$
$\frac{2}{(R_2 - R_1)^3}$		$\frac{2}{(R_2 - R_1)^3}$	$\frac{1}{(R_2 - R_1)^2}$

Table B.2

Matrix $[K_e]$ of the thin disc bending annular element.¹

$$\begin{bmatrix} (3 - 2\nu)P_{-3} & 0 & -3P_{-1} & -2(3 + \nu)P_0 \\ & 0 & 0 & 0 \\ & & (7 + 2\nu)P_1 & (17 + 6\nu)P_2 \\ \text{symm} & & & (48 + 16\nu)P_3 \end{bmatrix}$$

where

ν is the Poisson's ratio

$$P_i = \frac{\pi E t}{12(1 - \nu^2)} \int_{R_1}^{R_2} r^i dr$$

¹ The above matrix is simplified from [ref.23] for isotropic, uniform thin disc vibrating in 1 - nodal diameter.

Table B.3

Matrix $[M_e]$ of the thin plate bending annular element

$$\begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ & Q_3 & Q_4 & Q_5 \\ & & Q_5 & Q_6 \\ \text{symm} & & & Q_7 \end{bmatrix}$$

where

$$Q_i = \pi \rho t \int_{R_1}^{R_2} r^i dr$$

Table B.4

Coupling vector $[C]$ of the thin disc bending annular element

$$[C_0 \ C_1 \ C_2 \ C_3]$$

where

$$C_i = 2\pi \rho t \int_{R_1}^{R_2} r^{2+i} dr$$

Table B.5

Rotational stiffness matrix $[K_R]$ of a thin disc annular element

$$\begin{bmatrix}
 S_{-1} & S_0 & S_1 & S_2 \\
 & C_1 + S_1 & 2C_2 + S_2 & 3C_3 + S_3 \\
 & & 4C_3 + S_3 & 6C_4 + S_4 \\
 \text{symm} & & & 9C_5 + S_5
 \end{bmatrix}$$

where

$$S_i = \pi t \int_{R_1}^{R_2} r^i \cdot \sigma_{\xi}(r) dr$$

$$C_i = \pi t \int_{R_1}^{R_2} r^i \cdot \sigma_r(r) dr$$

$$\sigma_r(r) = \frac{3+\nu}{8} \rho \Omega^2 R^2 \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

$$\sigma_{\xi}(r) = \frac{3+\nu}{8} \rho \Omega^2 R^2 - \frac{1+3\nu}{8} \rho \Omega^2 r^2$$

APPENDIX (C)

The partial problem 'flexible disc on a rigid shaft'

According to the analysis of Chapter 2, due to the introduction of lateral disc deflections relative to the undeformed body (rigid disc), the component problem 'flexible disc on rigid shaft' has been developed and the matrix equations that describe it can be easily isolated out of the system's global matrices. These submatrices describe a circumferentially symmetric structure²

For the non-rotating case ($\alpha = 0$) the analysis of this partial problem is simple. A modal analysis is performed and the natural frequencies and eigenvectors (forming the $[B_D]$ modal matrix) are directly calculated.

The analysis of the rotating system ($\alpha > 0$) also introduces the above partial problem. However, a more precise description of the flexible disc problem should be 'flexible disc mounted on a whirling rigid shaft', represented by the matrix equation.

$$\left\{ [K] - \omega^2 (1 - 2\alpha) [M] \right\} \{ Z \} = 0 \quad (C.1)$$

The above includes gyroscopic effects but not rotational stiffening, and is a superposition of two motions, spin and whirl.

Pre- and - post multiplying (C.1) with the $[B_D]$ matrix we obtain:

$$\left[K_i - \omega_{Ci}^2 (1 - 2\alpha) M_i \right] Z_i = 0 \quad , \quad i = 1, 2, \dots, n \quad (C.2)$$

where K_i and M_i are the i^{th} row and column diagonal elements of the matrices $[B_D]^T [K] [B_D]$ and $[B_D]^T [M] [B_D]$ respectively.

From (C.2) the critical speed ω_{Ci} is obtained:

² The gyroscopic matrix exists and is exactly twice the mass matrix.

$$\omega_{C_i} = \frac{1}{\sqrt{1 - 2\alpha}} \sqrt{K_i/M_i} \quad (C.3)$$

where $(K_i/M_i)^{1/2}$ is the natural frequency ω_i .

In figure (C.1) there is the plot of ω_{C_i} versus the precession factor $\alpha = \Omega/\omega$ as described in (C.3)

For $\alpha = 0$ the plot gives the natural (non-rotating) frequency ω_i . In the right side boundary we have:

$$\lim_{\alpha \rightarrow .5} \omega_{C_i} = +\infty$$

In the same figure the rotational speed $\Omega = \alpha\omega_{C_i}$ is presented. Notice that Ω is always much smaller than ω_{C_i} for the whole range of the precession factor. In other words, for any given value of α , the disc is forced to spin and whirl, and performs forced lateral vibration with frequency well below the critical.

It is obvious that:

1. We cannot have resonance without external³ excitation.
2. The rotational speed $\Omega = \alpha\omega$ is much smaller than the forcing frequency (since $0 < \alpha < .5$).

The inclusion of the rotational stiffening matrix is not expected to significantly affect the above conclusions. Since this matrix is always positive, it will further reduce the limit of α where ω_{C_i} tends to infinity:

$$\lim_{\alpha \rightarrow \alpha_C} \omega_{C_i} = +\infty, \quad \alpha_C < .5$$

This will happen because the problem is now described by the equation:

$$\left\{ [K] - \omega^2 ((1 - 2\alpha)[M] - \alpha^2[K_R]) \right\} \{Z\} = 0 \quad (C.4)$$

³ External force perpendicular to the plane of the disc

However, although the whirling flexible disc is not excited (e.g., for $\alpha = 1$, critical speeds do not even exist), the disc flexibility does affect (reduce) the system's eigenvalues. This is because, as it was mentioned before, equation (C.1) represents two superimposed problems. The eigenvalues for the spinning disc do exist, as is shown in [ref. 24], for the whole range⁴ of the positive rotational speed Ω ; and these natural modes actually influence the system's response.

Therefore, to check whether a flexible disc - shaft interaction will appear at a certain operational speed, the first step is to calculate the eigenvalues of the rotating disc at this speed, including the rotational stiffening effects. If the fundamental disc frequency is below the rotational speed Ω , ($0 < \omega_{C1} < \Omega$), or in the neighbourhood of Ω , a shaft - disc interaction should be expected.

⁴ The plot ω_{C1} versus Ω (presented in [ref. 24]) does not seem to have a right limit.

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