Nonlinear Interactions between Atmospheric Rossby Waves and Internal Gravity Waves

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Abstract

In this thesis, we use analytical and numerical methods to study a mathematical problem describing two types of waves in an atmospheric fluid flow, namely planetary Rossby waves resulting from the rotation of the earth and internal gravity waves resulting from the gravitational and buoyancy forces. We consider a three-dimensional or baroclinic configuration and make the quasi-geostrophic approximation in the special case where the background flow velocity depends on a linear combination of the meridional (south-north) variable and the vertical variable. Both types of waves are forced at the same location and propagate upwards and southwards to a critical line where they interact with and modify the background mean flow and background mean temperature.

We carry out a weakly-nonlinear analysis to derive equations describing the rate of change of the wave-induced mean velocity and temperature with time due to the nonlinear interactions in the vicinity of the critical line. We also carry out numerical simulations using finite difference approximations and spectral methods.

We compare the results obtained with combined gravity waves and Rossby waves with those obtained with Rossby waves only. We find that with the additional effects of the gravity waves, the background mean flow becomes more eastward and the background mean temperature is higher around the critical line. In addition, in the presence of gravity waves, the amplitude of the Rossby wave zonal velocity is reduced and the Rossby wave temperature is increased.

These conclusions are consistent with observations of planetary wave dissipation and enhanced temperatures in the mesospheric inversion layer that occurs in the middle atmosphere.

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Dedication

I lovingly dedicate this thesis to:

*My dad, **Musaad** [1938-2016] (May Allah have mercy on his soul) and my mom, **Khadra** [1945-2021] (May Allah have mercy on her soul), who both have always loved me and whose good examples have taught me to work hard for the things that I aspire to achieve.*

*My wife, **Samirah**, for all of her love and support.*

*My sons, **Saood** and **Musaad**, who have been a source of motivation and strength during moments of despair and discouragement.*

List of Symbols

- $(\dots)^*$ dimensional quantity of $(\dots)$
- $(\dots)^*$ complex conjugate of $(\dots)$
- c.c. complex conjugate of preceding term
- $(\hat{\dots})$ Fourier transform of $(\dots)$
- $\frac{D}{Dt}$ Lagrangian derivative
- ∇ gradient vector
- ∇^2 Laplacian operator
- x zonal space variable (west to east)
- y meridional space variable (south to north) or latitude
- z vertical space variable (ground to upward) or altitude
- t time variable
- $\mathbf{u}$ total velocity vector of the fluid
- ρ density of the fluid
- ρ_0 constant reference density.
- p pressure of the fluid

- p_s standard reference pressure
- T temperature of the fluid
- T_s global average temperature
- g the acceleration due to gravity
- e the internal energy of the fluid
- J the rate of heating per unit mass
- Ω uniform angular velocity
- R_{earth} the radius of the earth
- R the gas constant for dry air
- $f = 2\Omega \sin \theta$ Coriolis parameter varying with latitude θ
- β the latitudinal gradient representing the effects of the earth's rotation
- Φ scalar potential function
- c_p the specific heat at constant pressure
- u^* x -component of dimensional total velocity of the fluid
- v^* y -component of dimensional total velocity of the fluid
- w^* z -component of dimensional total velocity of the fluid
- U the typical velocity scale in the x -direction
- V the typical velocity scale in the y -direction
- W the typical velocity scale in the z -direction
- L_x the typical length scale in the x -direction

- L_y the typical length scale in the y -direction
- L_z the typical length scale in the z -direction
- t_{scale} the time scale, i.e. $t_{\text{scale}} = \frac{L_x}{U} = \frac{L_y}{V} = \frac{L_z}{W}$
- $T_{\text{scale}} = 288.15$ Kelvin the typical temperature scale
- $\mathcal{U}$ x -component of the nondimensional total velocity of the fluid
- $\mathcal{V}$ y -component of the nondimensional total velocity of the fluid
- $\mathcal{W}$ z -component of the nondimensional total velocity of the fluid
- $\mathcal{T}$ the nondimensional temperature of the fluid
- $\delta = \frac{L_y^2}{L_x^2}$ the square of the meridional to zonal aspect ratio
- $\alpha = \frac{L_z^2}{L_y^2}$ the square of the vertical to meridional aspect ratio.
- H standard scale height
- $\bar{u}(y, z)$ mean flow velocity
- $\bar{T}(y, z)$ mean flow temperature
- u_R zonal Rossby wave velocity perturbation
- v_R meridional Rossby wave velocity perturbation
- w_R vertical Rossby wave velocity perturbation
- T_R Rossby wave temperature perturbation
- ψ_R Rossby wave streamfunction perturbation
- ζ_R Rossby wave vorticity perturbation
- u_G zonal gravity wave velocity perturbation

- v_G meridional gravity wave velocity perturbation
- w_G vertical gravity wave velocity perturbation
- T_G gravity wave temperature perturbation
- ψ_G gravity wave streamfunction perturbation
- ζ_G gravity wave vorticity perturbation
- $N^2 = \frac{g}{T_s} \left(-\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right)$ the buoyancy frequency squared
- k x -component of the wavevector
- l y -component of the wavevector
- m z -component of the wavevector
- $\overline{(\dots)}$ x -average over a zonal wavelength $\frac{2\pi}{k}$ of $(\dots)$
- $\psi_R^{(0)}$ leading-order term in the perturbation series of the Rossby wave streamfunction
- $\psi_R^{(1)}$ first-nonlinear correction term in the perturbation series of the Rossby wave streamfunction
- $\psi_{0R}^{(1)}$ zero-wavenumber component of the Rossby wave streamfunction perturbation
- $\psi_{2R}^{(1)}$ second-wavenumber component (second harmonic) of the Rossby wave streamfunction perturbation
- $\bar{u}_{0R}$ the wave-induced mean flow velocity resulting from the nonlinear interactions of Rossby waves
- $\bar{T}_{0R}$ the wave-induced mean flow temperature resulting from the nonlinear interactions of Rossby waves

- F_R the leading-order meridional component of the zonal mean of the zonal momentum flux (Rossby Reynolds stress)
- $X_R = \frac{\partial F_R}{\partial y}$ the Rossby wave drag force
- H_R the leading-order meridional component of the zonal mean of the Rossby heat flux
- $\psi_G^{(0)}$ leading-order term in the perturbation series of the gravity wave streamfunction
- $\psi_G^{(1)}$ first-nonlinear correction term in the perturbation series of the gravity wave streamfunction
- $\psi_{0G}^{(1)}$ zero-wavenumber component of the gravity wave streamfunction perturbation
- $\psi_{2G}^{(1)}$ second-wavenumber component (second harmonic) of the gravity wave streamfunction perturbation
- $\bar{u}_{0G}$ the wave-induced mean flow velocity resulting from the nonlinear interactions of gravity waves
- $\bar{T}_{0G}$ the wave-induced mean flow temperature resulting from the nonlinear interactions of gravity waves
- F_G the leading-order vertical component of the zonal mean of the zonal momentum flux (gravity Reynolds stress)
- $X_G = \frac{\partial F_G}{\partial z}$ the gravity wave drag force
- H_G the leading-order vertical component of the zonal mean of the gravity heat flux

- X_{RG} the drag force resulting from the nonlinear interactions between Rossby waves and gravity waves
- H_{RG1} the leading-order component of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the Rossby wave velocity v_R
- H_{RG2} the leading-order component the zonal mean of the flux of Rossby wave heat being transported to the vertical direction by the gravity wave velocity w_G

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Chapter 1

Introduction

1.1 Waves in geophysical fluid flows

This thesis describes an investigation using mathematical methods to study interactions between two types of waves in a geophysical fluid flow. The topic of geophysical fluid dynamics deals with the dynamics of the air and water in the atmosphere and the ocean. The two features that differentiate geophysical fluid dynamics from other types of fluid dynamics are the rotation of the earth and the vertical density stratification of the medium [Kundu, Cohen and Dowling (2012)].

The earth's rotation affects the direction of the flow of fluids in the earth's atmosphere and oceans. For instance, a fluid in the Northern Hemisphere that moves toward the pole will deviate toward the east, whereas a fluid moving toward the equator will deviate toward the west. On the other hand, a fluid in the Southern Hemisphere that moves toward the pole will deviate toward the west, whereas a fluid moving toward the equator will deviate toward the east. These effects give rise to a force called the Coriolis force that acts on the fluid in the atmosphere and the oceans [Cushman-Roisin and Becker, (2010)].

The density of a geophysical fluid varies in the vertical direction due to variations in fluid properties such as temperature or salinity. In regions where the lower layers

are denser than the upper layers, there is stable stratification. In the atmosphere, the temperature varies with altitude as shown in Figure 1.1. This figure shows the temperature profile based on a model called the U.S. Standard Atmosphere [U.S. Standard Atmosphere, (1976)].

Variations in the way temperature changes with height indicate that the atmosphere is composed of several different layers. These variations are due to changes in the chemical and physical characteristics of the atmosphere with altitude [Pidwirny, (2006)].

The atmosphere consists of four distinct primary layers based on the variation of temperature. The four layers of the atmosphere are the troposphere, the stratosphere, the mesosphere and the thermosphere, and there are interfaces between these layers. These interfaces are tropopause, stratopause and mesopause, as shown in Figure 1.1.

The temperature decreases in the lower atmosphere or troposphere, the region from the ground (sea level) to the tropopause level. The troposphere's depth varies from about 8 to 16 kilometres, and the average depth of this layer is approximately 10 kilometres [Pidwirny, (2006)].

Above the tropopause is the region known as the stratosphere, extending from an average altitude of 10 to 50 kilometres above the ground. As shown in Figure 1.1, in the first 9 kilometres of the stratosphere, the temperature nearly remains constant with height. A region with constant temperature in the atmosphere is called an isothermal layer [Pidwirny, (2006)]. From an altitude of 20 to 50 kilometres, the temperature increases with altitude.

At an altitude of about 50 kilometres, the stratosphere and the mesosphere layers are separated by a transition region called the stratopause. In the mesosphere, the temperature decreases rapidly with altitude, and the atmosphere reaches its coldest temperature that is 190 Kelvin (about -90° Celsius) at an altitude of approximately 90 kilometres. At the top of the mesosphere is another transition region known as the mesopause.

Figure 1.1: Vertical change in average global atmospheric temperature based on the U.S. Standard Atmosphere. Source: [U.S. Standard Atmosphere, (1976)], [<https://commons.wikimedia.org/wiki/File:%D9%86%D9%85%D9%88%D8%AF%D8%A7%D8%B1.gif>]

The uppermost layer of the atmosphere is called the thermosphere and has an altitude greater than 90 kilometres. In the lowest 5 kilometres of the thermosphere, the temperature almost remains constant with height. From an altitude greater than 95 kilometres, the temperature increases rapidly with altitude, and the temperatures in this layer can be greater than 1200° Celsius [Pidwirny, (2006)].

The atmosphere contains waves that are influenced by the spherical shape of the earth and its rotation and waves that are forced by buoyancy and gravity [Kundu,

Cohen and Dowling (2012)].

The waves that result from the Coriolis force, which is caused by the rotation of the earth, are called planetary waves or Rossby waves. They are called planetary waves because their wavelengths are on the scale of the magnitude of the planet earth. They are called Rossby waves after the Swedish meteorologist Carl-Gustaf Arvid Rossby (1898-1957), who first identified such waves in the earth's atmosphere in 1939 and continued to explain their motion [Cushman-Roisin and Becker, (2010)]. Figure 1.2 shows how Rossby waves develop.

Figure 1.2: Jet streams (Rossby waves). Source:

https://commons.wikimedia.org/wiki/File:Jetstream_-_Rossby_Waves_-_N_hemisphere.svg

Within the moving air, streams of air move very fast like an atmospheric highway, and many weather systems are travelling with it. These streams are called jet streams separating warm air from the tropics and cold air from the pole (as presented in pink in Figure 1.2). These jet streams travel inside more significant air movements known as Rossby waves.

When the shape and position of the jet streams change, cold polar air may interfere southward. The primary influence on the jet streams is the temperature difference between the pole and the tropics. When the temperature difference is large, the jet streams are faster and less wavy (Figure 1.2(b)). Conversely, the jet

streams are weaker and meander over the globe when the temperature difference is slight (Figure 1.2(c)).

The jet streams, which are weaker and wavier (Figure 1.2(c)), facilitate cold air travelling further south. Therefore, some areas may experience record-breaking cold weather, and at the same time, unusual warming may occur at higher latitudes.

Rossby waves naturally occur in rotating fluids. Within the earth's atmosphere and ocean, these planetary waves play an essential role in shaping weather. In their most general form, Rossby waves have an important bearing on the weather and the structure of the general circulation of the atmosphere and ocean. Rossby waves occur widely in fluid flows of many kinds (for example, in hurricanes). Rossby waves oscillations and propagation are primarily horizontal motion rather than vertical [Cushman-Roisin and Becker, (2010)]

Rossby waves are large-scale waves with wavelengths thousands of kilometres long. They are the most important type of wave for large-scale meteorological processes. In the atmosphere, Rossby waves form primarily as a result of the earth's rotation. They facilitate the transfer of heat from the tropics toward the poles and cold air toward the tropics to return the atmosphere to balance. They also facilitate the jet stream's location and mark out the track of low-pressure surface systems [Kundu, Cohen and Dowling (2012)].

The competing effects of the upward buoyancy force from the density stratification and the downward gravitational force give rise to gravity waves. When these waves occur on the interface between two fluids, such as water and air, they are called surface gravity waves, and when they occur in the interior of a fluid, e.g. within the atmosphere or within a body of water like the ocean, they are called internal gravity waves.

In the atmosphere, internal gravity waves are generated by obstacles such as topography (mountains) or by convective circulation in the lower atmosphere. Internal gravity waves often propagate upward or downward in the atmosphere, and

they may break down, leading to a turbulent flow. They influence the weather and climate. Such influence should be considered, especially in simulating wind speed and temperature in the atmosphere [Sutherland (2010)].

Atmospheric gravity waves are medium-scale waves with a horizontal wavelength ranging between several tens and several thousands of kilometres and a vertical wavelength of several kilometres. Atmospheric gravity waves have a shorter wavelength than planetary Rossby waves mentioned above.

Collectively, the atmosphere and oceans have an extensive and significant effect on humanity. The combined dynamics of the atmosphere and oceans are essential contributors to global climate and weather and wave interactions have profound effects on the global circulation. All of us live within the atmosphere and are influenced by the weather and its behaviour. Therefore, understanding and reliably estimating geophysical fluid dynamic activities and trends is a scientific, economic and humanitarian issue [Cushman-Roisin and Becker, (2010)].

Rossby waves and gravity waves both exist in the atmosphere. There are also other large-scale waves, e.g. Kelvin waves and Rossby-gravity waves in the tropics. Convection in the tropical troposphere, ranging from the scale of mesoscale convective complexes (spanning more than 100 km) to planetary-scale phenomena, produces a broad spectrum of waves, including gravity, inertia-gravity, Kelvin, and Rossby-gravity waves, which propagate into the stratosphere (see, for example, Plate 2 in [Baldwin *et al.*, (2001)]).

Planetary-scale waves (Rossby waves) occur between the tropopause and the stratopause at middle to high latitudes. There is a pronounced annual cycle at high latitudes, with strong westerly (from the west) winds during the winter season. To the north of the equator in the lower stratosphere, tropical winds alter the waveguide for upward and equatorward propagating planetary-scale waves (see, for example, Plate 2 in [Baldwin *et al.*, (2001)]).

In a theoretical formulation, Rossby waves (similarly, gravity waves) can be rep-

resented by a discrete or continuous spectrum of Fourier modes in space; e.g. a sum or integral over k_R of waves of the following form

$$A e^{ik_R x} + A^* e^{-ik_R x},$$

where x is a horizontal variable, and the wave amplitude A depends on the Rossby wavenumber k_R and the other spatial variables and time. The amplitude A is complex, and its complex conjugate is A^* .

Many theoretical studies of Rossby waves and gravity waves have been based on linear theory. If the wave amplitude is sufficiently small relative to the background flow, the governing equations can be linearized. The linearization process expresses each of the total quantities in the governing equations as the sum of a background steady and perturbation. The waves are considered to be perturbations to a basic background state. Linearity simplifies the fundamental equations of fluid dynamics but means there can be no interaction between waves (different Fourier modes) or between the waves and the background flow.

In a nonlinear problem, there are interactions between waves and between the waves and the mean flow. The mean flow changes as a result of these interactions. For example, the nonlinear terms in Rossby wave equations involve products of the Fourier modes. This means that, as the waves propagate, Fourier modes with Rossby wavenumbers $\pm 2k_R, \pm 3k_R, \dots$, develop in addition to forced Rossby wavenumbers $k_R + (-k_R) = 0$. These are proportional to $e^0, e^{\pm 2ik_R x}, e^{\pm 3ik_R x}$ and so on, as we shall see in chapters 3 and 4. The zero wavenumber component corresponds to a change in the mean flow, and this change is referred to as the wave-induced mean flow.

Planetary Rossby waves are generated at high latitudes and propagate towards the equator. Gravity waves occur at all latitudes in the lower atmosphere and propagate upwards in the middle atmosphere. In regions where both Rossby waves and gravity waves are present, they both interact with the general background circulation and may also interact with each other.

Most theoretical studies model Rossby wave and gravity wave effects separately and do not consider interactions between the two types of waves.

In this thesis, we aim to study these interactions. Some mathematical models to study these interactions will be presented and analyzed using asymptotic methods and numerical solutions. In the next section, some previous work is described.

1.2 Previous work and motivation

A large number of theoretical investigations have been carried out in the past 50 years to improve our understanding of Rossby wave and gravity wave effects on the general circulation. For both types of waves, interactions with the background flow occur when there is a critical line where the wave phase speed is equal to the background wind speed. For Rossby waves travelling on a horizontal plane, this is a critical latitude, and for gravity waves travelling on a vertical plane, it is a critical level. The region around the critical line is called the critical layer.

The interactions between the waves and the mean flow at critical layers have been proposed as the basis of theories for many phenomena observed in the atmosphere and ocean, such as stratospheric sudden warmings [Matsuno, (1971)] and the quasi-biennial oscillation (QBO) [Lindzen and Holton, (1968)].

There have been linear and nonlinear analytical and numerical studies of critical layers for horizontally propagating Rossby wave critical layers and for vertically propagating internal gravity waves.

A first study of forced Rossby wave critical layers was formulated by Dickinson (1970), which led to closer examination by Warn and Warn (1976). Both of these papers assumed that the disturbance's amplitude develops with time and derived asymptotic solutions to the linear barotropic vorticity equation to examine a forced Rossby wave propagating horizontally toward a critical layer on a β -plane (i.e. a horizontal plane on which the Coriolis force varies linearly with latitude). Their

results revealed that the disturbance is absorbed in the critical layer. As a result, the Reynolds stress (the momentum flux of the waves) decreases significantly across the critical layer. Besides, Warn and Warn observed that the neglected nonlinear terms would become non-negligible for large time.

Warn and Warn (1978) examined the full nonlinear equation without the long-wave assumption. They derived an analytic solution valid for large time outside the critical layer and an equation describing the long-time evolution of the vorticity in the inner region, which they solved numerically. Stewartson (1978) independently examined waves with long zonal wavelength and derived the equation for the vorticity in the critical layer. He revealed that it was possible to derive an approximate solution to this equation for a particular flow domain's unique configuration. Both papers concluded that the Reynolds stress jump was modified for large time, ultimately oscillating around zero, indicating reflection of the incident waves at the critical layer. The numerical investigation of Beland (1978) confirmed all the results mentioned above for Rossby waves on a β -plane.

Analytical and numerical investigations of the linear equations of Rossby waves forced by a horizontally localized wave packet were studied by Campbell and Maslowe (1998). They found that the waves are absorbed at the critical level, and the wave-induced mean flow is proportional to the wave packet length. Campbell (2004) carried out a nonlinear numerical and analytical analysis of Rossby waves forced by a zonally localized wave packet. She observed that the waves are absorbed at the critical level at early time and reflected at later time when the nonlinearity becomes non-negligible. However, it was found that the absorption is dominant compared to the reflection.

The above-mentioned studies describe Rossby wave mean flow interactions on a horizontal plane. This is a reasonable first approximation for Rossby waves in situations where their propagation is mostly horizontal. However, internal gravity waves propagate vertically in most circumstances. Consequently, representing both types of waves in the same configuration requires a three-dimensional model that

allows vertical propagation of Rossby waves and gravity waves.

The vertical propagation of planetary scale disturbances from lower to upper atmosphere was described by Charney and Drazin (1961). The energy in the form of heat and momentum flux carried by the small scale disturbances modifies the zonal mean flow when they propagate from lower to upper atmosphere. The authors also proposed that the planetary scale disturbances could propagate vertically only when the background winds are westerly (blowing from the west) relative to the phase speed. Moreover, they suggested that transient waves can interact with background flow and the interaction can bring the background winds toward the phase speed of the wave.

Booker and Bretherton (1967) studied the related critical layer problem for forced internal gravity waves propagating vertically in a stratified density flow. They studied the linear equations of internal gravity waves with a background shear flow. They showed that small-amplitude gravity waves are absorbed at their critical level, and the mean flow was enhanced at the critical level.

Brown and Stewartson (1980, 1982a,b) continued the work of Booker and Bretherton (1967) by including the nonlinear terms in the internal gravity wave equations to investigate the behaviour at the critical level. They found that the waves at the critical level are absorbed at early time and reflected at later time.

Campbell and Maslowe (2003) carried out a full numerical investigation of the nonlinear critical level evolution for internal gravity waves with a horizontally localized wave packet forcing as well as monochromatic forcing. They noticed that the waves are absorbed at early time, as suggested by linear critical theory, and reflected at later time, as the nonlinear terms become essential at the critical level. However, they noticed that there is more absorption in the case of wave packet forcing compared to that with monochromatic forcing.

In addition to wave effects on the mean flow, there are nonlinear interactions between wave modes that give rise to higher wavenumbers. Wave-wave interactions

have been discussed for the case of Rossby waves [e.g., Longuet-Higgins *et al.*, (1967); Connaughton *et al.*, (2015)] and for the case of gravity waves [e.g., Olbers, (1976); Nazarenko, (2011)] based on the weak-interaction assumption (i.e., assuming that the wave amplitudes are changing only slowly) formulated for geophysical applications first by Hasselmann (1962).

All the above studies modeled Rossby wave and gravity wave effects separately and did not take into account interactions between the two types of waves.

Although commonly separated by immensely different time scales, it is known that both wave modes can interact with each other [e.g., Müller, (1977); Lighthill, (1978); Ford *et al.*, (2000)]. A general framework for the interaction between long and short waves was first introduced by Benney (1977), investigating nonlinear differential equations that allow, in the linearized case, for long and short wave solutions. The theory can be applied to the resonant interaction of two short internal gravity waves with a long Rossby wave. The resonance condition of such a triad of waves is an approximate equality of the long wave phase velocity and the short wave group velocity [Eden *et al.*, (2019a)].

Eden *et al.*, (2019a) analytically derived mixed triad wave-wave interactions between Rossby and gravity waves using the kinetic equations. They used the weak-interaction assumption, which is a necessary assumption to derive the kinetic equations. They applied the kinetic equations to a simple two-dimensional layered model and allowed for the interaction of all waves present in the models, that is, internal gravity waves and Rossby waves. Since the interactions that occur only within the gravity wave field or only within the Rossby wave field conserve energy by themselves, Eden *et al.*, (2019a) found that the same is true for the mixed interactions between gravity and Rossby waves in the kinetic equation.

It has become manifest that gravity waves can affect the propagation of the Rossby waves, particularly in the mesosphere [Miyahara, (1984)]. Miyahara (1984) examined the interaction between stationary Rossby waves and gravity waves using

a simplified quasi-geostrophic model on a midlatitude β -plane. *Steady* waves mean waves whose amplitudes are independent of time while *stationary* waves are steady waves with zero horizontal phase speed [Holton, (2004)]. Miyahara (1984) found that the amplitudes of stationary planetary waves are immensely reduced by the drag forces due to small scale internal gravity wave breaking in the mesosphere. The wave drag forces and the effect of induced eddy diffusion are the suggested reasons for planetary wave suppression in the mesosphere [Miyahara, (1984)]. Teitelbaum and Vial (1991) also suggested that by the fact that tides, which share many properties with the gravity waves, are known to interact with the Rossby waves.

Vanneste and Vial, (1994) developed an analytical model of the barotropic divergent equations on a sphere to study interactions of the Rossby waves and gravity waves using the potential-streamfunction formulation and the normal modes as basic functions. They exhibited how gravity waves can exchange energy through interaction with a Rossby wave. Moreover, a particular interaction between a Rossby wave and a gravity wave whose zonal wavenumber is twice as large as the wavenumber of the Rossby wave was discussed. They showed that this interaction will lead to a frequency-shift in the Rossby wave. Vanneste and Vial, (1994) suggested that using a plane geometry and constant stratification would simplify the problem of the interaction between Rossby and gravity waves.

A number of investigations have studied interactions between gravity waves and the large-scale "balanced" flow, which includes Rossby waves. Spontaneous emission or loss of balance [e.g., Vanneste, (2013)] has attracted much attention and indicates the process of gravity wave generation by a large-scale flow spontaneously in the absence of any external forcing, which usually happens during baroclinic instability [Plougonven and Snyder, (2007)].

In the laboratory, the spontaneous generation of internal gravity waves from the large-scale flow has been studied by, for example, Williams *et al.*, (2005), (2008). Numerical studies of this process have been discussed by, for example, Molemaker

et al., (2005); Sugimoto and Plougonven, (2016); Chouksey *et al.*, (2018). It was shown that this process can transfer energy from the balanced flow to gravity waves. It was considered that this process may represent a significant energy sink for the balanced flow in the ocean [e.g., Williams *et al.*, (2008); Brüggemann and Eden, (2015)].

It is well known that gravity waves in the stratosphere and mesosphere affect the mean flow by the mechanisms of the critical layer interaction and gravity wave saturation. According to linear saturation theory, the amplitude of the gravity wave increases exponentially as it propagates vertically upward in response to the decreasing atmospheric density and breaks at a particular altitude where it becomes convectively unstable [Ramesh *et al.*, (2013)]. When the wave breaks, it deposits a net momentum in the mean flow and accelerates the background wind in the direction of propagation of the wave and causes a downward heat flux, which results in cooling of the atmosphere at the breaking height [Walterscheid, (1981); Gardner and Yang, (1998); Ramesh and Sridharan, (2012)].

Lindzen (1981) offered a simple theory suggesting that the drag forces due to the gravity wave breaking decelerates the mean zonal winds to balance the momentum budget in the mesosphere. He additionally suggested that the drag force peaks near the lowest level at which the gravity wave breaks.

The effect of gravity wave breaking on the general circulation of middle atmosphere was discussed by Holton (1983). He suggested that the vertical eddy diffusion generated by gravity wave breaking modifies the zonal mean flow and causes the planetary wave dissipation in the mesosphere. The momentum transferred by vertically propagating internal gravity waves could reduce the amplitudes of planetary waves when they interact with each other [Lindzen, (1984); Schoeberl and Strobel, (1984)]. This would suggest that gravity-wave-critical-layer interactions may also have an effect of planetary Rossby waves that occur in the same vicinity.

Mesospheric inversion layers are a context in which gravity waves and planetary

Rossby waves are known to occur together. A mesospheric inversion layer (MIL) is known as the layer of enhanced temperature with several kilometers thickness showing a positive lapse rate at the bottom and is superposed upon the characteristically decreasing temperature in the mesosphere [Ramesh *et al.*, (2013)]. There are several causative mechanisms proposed for the formation of these layers [Meriwether and Gardner, (2000); Meriwether and Gerrard, (2004)], which include gravity wave breaking and planetary wave critical level interaction.

Salby *et al.* (2002) described the relationship between mesospheric inversion layers (MIL) and the planetary wave activity from UARS/Microwave Limb Sounder. They suggested that the strong absorption of the planetary wave in the mesospheric inversion region is due to the turbulent mixing associated with gravity wave breaking.

The contribution of planetary waves in causing MILs using Whole Atmosphere Community Climate Model (WACCM) was discussed by Sassi *et al.* (2002). They suggested that the existence of the critical level for the planetary waves in the mesosphere is due to the deposition of easterly momentum by small scale gravity waves [Matsuno, (1982); Holton, (1983); Garcia and Solomon, (1985)], so that the planetary waves are dissipated at this level.

Ramesh *et al.*, (2013) discussed planetary wave-gravity wave interactions during mesospheric inversion layer events using Rayleigh Lidar temperature observations over Gadanki (13.5°N, 79.2°E). In this study, it was shown that mesospheric temperature inversion is caused fundamentally due to gravity wave breaking and that the inversion amplitudes got changed by the interaction of gravity waves and planetary waves.

The studies described in this section suggest that there may be important interactions between internal gravity waves and planetary Rossby waves in regions in the stratosphere and mesosphere where both types of waves interact with and modify the background mean wind and background mean temperature.

Consequently, understanding the interactions between Rossby waves, gravity waves, and the mean flow can contribute to a better understanding of the dynamics of

the atmosphere. Some information may be obtained by carrying out a study involving both types of waves in a simplified configuration where approximate analytical solutions and numerical solutions can be obtained.

A better understanding of atmospheric behaviour can be obtained by modelling the atmospheric circulation using global circulation models (large-scale atmospheric models). Global circulation models (GCMs) are complex computer programs based on the governing equations of motion, which are solved using numerical methods to accurately predict weather (short term) and climate (long term). For these predictions to be reliable, the modellers need to carefully simulate phenomena such as internal gravity waves and other atmospheric waves. However, gravity waves are relatively small-scale phenomena with wavelengths smaller than a few hundred kilometers, so a very high spatial resolution is required for GCMs to represent them accurately. Typically, the order of magnitude of the GCM grid spacing is 100 km, and hence they are not always able to resolve small-scale gravity waves.

Atmospheric scientists and modellers parameterize the effects of the gravity waves in GCMs by adding to the equations in the model a term called the gravity wave drag to represent the drag force that would have resulted from the gravity waves if the model had correctly resolved them. Some examples of gravity wave drag parameterization schemes are those developed by Lindzen (1981), Hines (1997) and Alexander and Dunkerton (1999). In order to develop accurate parameterizations of the gravity wave drag, it is important to understand how gravity waves are affected by Rossby waves.

It is understood that the quasi-biennial oscillation (QBO) is driven by large-scale atmospheric waves as well as small-scale gravity waves [Baldwin *et al.*, (2001)], although there has been considerable debate in the past few decades on the relative contributions of the two types of waves. A realistic representation of the QBO in a climate model is one of the significant issues in global climate modelling. To correctly model the QBO and other phenomena in the middle atmosphere, it is essential to

accurately represent the generation of the gravity waves in the lower atmosphere. The QBO is now simulated in a growing number of climate models [e.g., Takahashi (1996) and Giorgetta, Manzini and Roeckner (2006)] either by using very high resolution or by using a gravity wave drag parameterization.

In brief, a fundamental goal of studying gravity wave interactions with Rossby waves and the mean flow is to develop reasonable physical and theoretically based gravity wave parameterizations for use in GCMs. Most studies describe Rossby wave-mean flow interactions in a two-dimensional (2D) barotropic model on a horizontal plane and internal gravity wave-mean flow interactions in a 2D model on a vertical plane. Therefore, in order to represent both waves in the same configuration, we need a three-dimensional (3D) baroclinic model allowing vertical propagation of Rossby waves and internal gravity waves.

In this thesis, we consider a 3D configuration where Rossby waves and gravity waves are forced at the same location and propagate upwards and southwards to a critical line where both waves interact with the background mean wind and background mean temperature to examine the additional effects of the gravity wave interactions with Rossby waves compared with cases with Rossby waves only (without the gravity wave effects). We shall see that our results are consistent with the studies of Charney and Drazin (1961), Lindzen (1981), Walterscheid, (1981), Holton (1983), Lindzen (1984), Schoeberl and Strobel (1984), Miyahara (1984), Gardner and Yang (1998), Meriwether and Gardner (2000), Meriwether and Gerrard (2004), Ramesh and Sridharan, (2012) and Ramesh *et al.*, (2013).

1.3 Overview of the thesis

An overview of the thesis is as follows. Chapter 2 is devoted to deriving the fundamental equations to study a configuration where Rossby waves and gravity waves are both present and interact with the atmosphere's general circulation. We begin, in

section 2.1, by describing the fundamental equations of fluid dynamics: the continuity, momentum and energy equations written in Cartesian coordinates. The complete derivation of the fundamental equations used in our study in a three-dimensional configuration under the hydrostatic approximation and a quasi-geostrophic approximation written in terms of log-pressure coordinates is given in Appendix A. In sections 2.2 and 2.3, we start by writing these equations in terms of dimensionless quantities. Afterwards, we use the perturbation method to derive the perturbation equations representing the continuity equation, the zonal momentum equation, the meridional momentum equation, the hydrostatic balance equation, and the thermodynamic energy equation for Rossby waves and gravity waves. These perturbation equations comprise Rossby wave terms together, gravity wave terms together and Rossby-gravity wave terms together. In section 2.4, we describe the configuration and domain of our model. In section 2.5, we conclude this chapter by obtaining the nondimensional values of the parameters used in our numerical computations in chapter 5.

Our dimensional analysis yields two small parameters $\varepsilon_R \ll 1$ and $\varepsilon_G \ll 1$. In chapter 3, we carry out a weakly-nonlinear analysis of the perturbation equations with these two small parameters and write the nonlinear perturbation equations in streamfunction-vorticity formulation. We reveal, in section 3.1, that our derivation of the perturbation equations allows us to consider different possibilities for ε_R and ε_G where the nature of the solution depends on the relative magnitude of these two small parameters. In section 3.2, we combine the Rossby wave terms and Rossby-gravity wave terms of the nonlinear perturbation equations into a single nonlinear equation. In addition, we combine the gravity wave terms of the nonlinear perturbation equations into two nonlinear equations. Lastly, we state our initial-boundary value problem.

In chapter 4, we carry out a weakly nonlinear analysis of the initial-boundary value problem. The first step of the weakly-nonlinear analysis involves solving the linearized equations which appear at the leading order. The leading order terms in the

weakly nonlinear formulation give uncoupled linear equations for the Rossby wave and gravity wave perturbations, which can be solved separately. In sections 4.1 and 4.2, we review the linear Rossby wave solutions [Dickinson, (1970), Warn and Warn, (1976); Campbell and Maslowe (1998)] and gravity wave solutions [Booker and Bretherton, (1967)]. Furthermore, we discuss some properties of these solutions, including the dispersion relations and definitions of group velocity. This information is required for the weakly-nonlinear analysis (sections 4.3-4.5) and the numerical solutions (chapter 5).

The nonlinear terms in the initial-boundary value problem give rise to zero wavenumber components representing a wave-induced mean flow velocity and a wave-induced mean flow temperature. The effects of the Rossby wave self-interactions and the gravity wave self-interactions on the mean flow are well known. We review these effects in sections 4.3 and 4.4. Besides, we discuss the wave-induced mean temperature resulting from the Rossby wave self-interactions and the gravity wave self-interactions.

This chapter is terminated in section 4.5, where we consider a case where both Rossby waves and gravity waves are present and of the same order of magnitude. In order to examine the behaviours of the mean flow and the mean temperature in a critical line, we derive two equations describing the rate of change of the wave-induced mean flow velocity and the rate of change of the wave-induced mean flow temperature with time, respectively, resulting from the nonlinear interactions of Rossby waves, the nonlinear interactions of gravity waves and the nonlinear interactions between Rossby waves and gravity waves.

In chapter 5, we carry out numerical simulations for our nonlinear initial-boundary value problem to examine the interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line. We describe, in section 5.1, the numerical implementation, including the configuration, numerical methods and restrictions on the choice of the nondimensional parameters which must be satisfied in order for the solutions to be stable. In section 5.2, we show and discuss the results

consisting of cases with barotropic (pressure gradient force does not vary with height) and baroclinic (pressure gradient force varies with height) Rossby wave effects only (without gravity wave effects), cases with dominant baroclinic Rossby wave effects (with minimal gravity wave effects) and cases with dominant gravity wave effects (with minimal baroclinic Rossby wave effects).

In section 5.3, we show and discuss the results of a case where both Rossby waves and gravity waves are present and of the same order of magnitude. We examine this case where Rossby waves and gravity waves have the same forced wavenumbers, and where they have different forced wavenumbers. We conclude this chapter by making a comparison between different behaviours of the mean flow and the mean temperature around a critical line as a consequence of the nonlinear interactions of Rossby waves, the nonlinear interactions of gravity waves and the nonlinear interactions between Rossby waves and gravity waves as illustrated in section 5.4.

Specifically, we conclude that the gravity wave interaction with Rossby waves causes less momentum flux and heat flux depositions in the critical line indicating the considerable effects of drag force and outward heat flux. Therefore, the mean flow accelerates more in the eastward direction relative to that in the baroclinic Rossby waves only, and the mean temperature is warmer than that in baroclinic Rossby waves only. Furthermore, we observe that the effects of drag force and outward heat flux in the interaction between Rossby waves and gravity waves whose zonal forced wavenumber is twice the zonal forced Rossby wavenumber are more considerable than those in the case where Rossby waves and gravity waves have the same forced wavenumber. In addition, we conclude that the gravity wave interaction with Rossby waves reduces the amplitude of zonal Rossby wave velocity perturbation compared to that in baroclinic Rossby wave only. These conclusions are consistent with the conclusions from Charney and Drazin (1961), Lindzen (1981), Walterscheid, (1981), Holton (1983), Lindzen (1984), Schoeberl and Strobel (1984), Miyahara (1984), Gardner and Yang (1998), Meriwether and Gardner (2000), Meriwether and Gerrard (2004), Ramesh

and Sridharan, (2012) and Ramesh *et al.*, (2013).

Chapter 6 contains the conclusions of our study of the nonlinear interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line comprising of a summary of the thesis, a discussion of our results relative to the conclusions of some associated studies and possible future work.

Afterwards, we finish the thesis with three appendices that contain the derivation of the fundamental equations of fluid dynamics, some basic concepts and background material required in various parts of the thesis. Precisely, in Appendix A, we exhibit the complete derivation of fundamental equations used in our study in a three-dimensional configuration under the hydrostatic approximation and a quasi-geostrophic approximation written in terms of log-pressure coordinates. In Appendix B, time-dependent linear solution for baroclinic Rossby waves and time-dependent linear solution for hydrostatic gravity waves are described. In Appendix C, we present stability criteria for disturbances in barotropic and baroclinic flows and the Miles-Howard theorem for the stability of gravity waves in a stratified shear flow.

1.4 Original contributions

My analyses and numerical computations are for the particular configuration where the background mean flow speed is a function of a linear combination of the meridional variable y and the vertical variable z . This particular choice of the background mean flow speed enables me to introduce a single variable Y for the linear combination of y and z so that I can do analyses and obtain numerical solutions in terms of two variables x and Y .

My original contributions in the study of the nonlinear interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line are:

- In chapter 2, I formulate the perturbation equations representing the continuity,

the momentum and the thermodynamic energy equations in a form comprising of Rossby wave terms together, gravity wave terms together and Rossby-gravity wave terms together (section 2.3).

■ In chapter 3:

1. I carry out a weakly-nonlinear analysis of these perturbation equations with the parameters $\varepsilon_R \ll 1$ and $\varepsilon_G \ll 1$ and consider different possibilities for ε_R and ε_G . Besides, I present the magnitude of the zero wavenumber components of the solutions representing the wave-induced mean flow for different magnitudes of Rossby wave and gravity wave amplitudes and various choices of the Rossby wave and gravity wave wavenumbers (sections 3.1 and 3.2).
2. I formulate the momentum and the thermodynamic energy equations that model the interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line in terms of the Rossby wave streamfunction, Rossby wave vorticity, gravity wave streamfunction and gravity wave vorticity; then, I state the nonlinear initial-boundary value problem (section 3.3).

■ In chapter 4:

1. I derive an equation describing the total wave-induced mean flow acceleration as a consequence of the nonlinear interactions of Rossby waves, the nonlinear interactions of gravity waves and the nonlinear interactions between Rossby waves and gravity waves (equation (4.5.16) in section 4.5). This equation includes the well-known Rossby wave drag force, well-known gravity wave drag force and a new drag force resulting from the nonlinear interactions of the Rossby-gravity waves. For large t , I additionally determine which of these drag forces is dominant and consequently determine

the order of the wave-induced mean flow acceleration and the order of the wave-induced mean flow velocity.

2. I derive an equation describing the rate of change of the total wave-induced mean temperature resulting from the Rossby wave self-interactions, the gravity wave self-interactions and the interactions between Rossby waves and gravity waves (equation (4.5.30) in section 4.5). This equation is proportional to the meridional and vertical divergences of four heat fluxes plus three cooling terms.

■ In chapter 5:

1. I carry out numerical simulations of the interactions of Rossby waves (without gravity wave effects) and the background mean flow (subsection 5.2.1).
2. I carry out numerical simulations of the interactions of dominant Rossby waves (with very small different magnitudes of gravity wave amplitudes) and the background mean flow (subsections 5.2.2 and 5.2.3).
3. I carry out numerical simulations of the interactions of Rossby waves, gravity waves (both have the same magnitudes of amplitudes), and the background mean flow where Rossby waves and gravity waves have the same forced wavenumbers (subsection 5.3.1).
4. I carry out numerical simulations of the interactions of Rossby waves, gravity waves (both have the same magnitudes of amplitudes), and the background mean flow where the forced Rossby wavenumber is less than the forced gravity wavenumber (subsection 5.3.2).

Chapter 2

A mathematical model for Rossby waves and gravity waves

2.1 Equations for fluid flow influenced by the Coriolis force

In this chapter, we present nonlinear partial differential equations that describe a fluid flow in the earth's atmosphere under the influence of the Coriolis force. We use these equations to study a configuration where Rossby waves and gravity waves are both present and interact with the general circulation of the atmosphere.

2.1.1 The fundamental equations of fluid dynamics

The fundamental equations of fluid dynamics are based on the laws of conservation of mass, momentum and energy. The full derivation of these equations can be found in any fluid dynamics textbook, for example, [Gill, (1982)], [Anderson, (1985)], [Holton, (2004)] or [Kundu, Cohen and Dowling (2012)]. In this section, we describe these equations following Kundu, Cohen and Dowling (2012).

The physical principle of the law of conservation of mass states that the rate

of increase mass within a fixed volume must equal the rate of inflow through the boundaries. The principle of conservation of mass can be expressed by the following differential equation

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \quad (2.1.1)$$

where the scalar function ρ is the fluid density, the vector function $\mathbf{u}$ is the velocity of the fluid motion and both are functions of space represented by the position vector $\mathbf{x}$ (defined by $\mathbf{x} = (x, y, z) = x \hat{i} + y \hat{j} + z \hat{k}$, where $(\hat{i}, \hat{j}, \hat{k})$ are the unit vectors in the x (going from west to east), y (going from south to north) and z (going from the ground to the upward) directions, respectively) and the time variable t . The differential operator $\frac{D}{Dt}$, known as Lagrangian derivative, is given by $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$, and the divergence of the vector function $\mathbf{u}$ is $\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. Equation (2.1.1) is called the *continuity equation*

A fluid is said to be incompressible if its density does not change with pressure. Liquids are almost incompressible whereas gases are compressible. However, for flow speeds less than or equal to 100 m sec^{-1} , changes in pressure are small and the density changes in the flow are also small [Kundu, Cohen and Dowling (2012)]. In the case of incompressible fluid flows, we can assume that the first term of the continuity equation (2.1.1) is small and consequently we can neglect it and rewrite the continuity equation (2.1.1) as

$$\nabla \cdot \mathbf{u} = 0. \quad (2.1.2)$$

The physical principle of conservation of momentum (known as Newton's second law) states that the net forces that act on a fluid element equal the time rate of change of momentum. In other words, the net forces acting on a fluid element equal its mass times its acceleration. Applied to incompressible fluid flows this leads to the following *Navier-Stokes equation* in vector form

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}, \quad (2.1.3)$$

where the scalar function p is the fluid pressure, the vector $\mathbf{g}$ is the acceleration due

to gravity, the term $\mu \nabla^2 \mathbf{u}$ represents the effects of viscosity or friction in the fluid and the gradient of the scalar function p is $\nabla p = \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k}$.

If we divide (2.1.3) by ρ , we get

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (2.1.4)$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient. If viscous effects are negligible ($\mu = 0$), which is usually true away from the boundaries of the flow field, equation (2.1.3) simplifies to the *Euler's equation*

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}. \quad (2.1.5)$$

If the fluid is at rest with $\mathbf{u} = 0$, the first and the second terms on the right-hand side of the Euler's equation (2.1.5) are balanced and hence we get

$$\nabla p = \rho \mathbf{g}. \quad (2.1.6)$$

Equation (2.1.6) is called the *hydrostatic balance*. In Cartesian coordinates (x, y, z) , the vector $\mathbf{g}$ acts only in the vertical direction, so $\mathbf{g}$ can be written as $(g_1, g_2, g_3) = (0, 0, -g)$ where g is a scalar ($g = 9.8 \text{ m/s}^2$), thus; equation (2.1.6) can be alternatively written as follows

$$\frac{\partial p}{\partial z} = -\rho g. \quad (2.1.7)$$

The physical principle of conservation of energy (known as the first law of thermodynamics) states that the rate of change of total energy of the fluid as it flows through the control volume is equal to the rate of work done by body forces and surface forces on the fluid plus the rate of heat added to the fluid.

The thermal energy equation based on the principle of conservation of energy is given by

$$\rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{u} + \rho J, \quad (2.1.8)$$

where e is the internal energy of the fluid and J here is the rate of heating per unit mass due to radiation, conduction, and latent heat release.

2.1.2 The Coriolis force

The equations governing geophysical fluid dynamics can be stated with respect to an inertial framework of reference, one that is stationary or that is moving at a constant speed with respect to a stationary frame of reference [Cushman-Roisin and Becker, (2010)]. The motion of the atmosphere and the ocean is naturally studied in an inertial coordinate frame rotating with the earth. The rotation of the earth gives rise to the Coriolis force [Kundu, Cohen and Dowling (2012)].

The continuity equation (2.1.2), in a non-inertial frame of reference, is unchanged but the momentum equation (2.1.4) must be modified. A frame of reference $\mathbf{1}'\mathbf{2}'\mathbf{3}'$ rotating at a uniform angular velocity $\boldsymbol{\Omega}$ (does not change with time) with respect to a fixed (inertial) frame of reference $\mathbf{123}$ is shown in Figure 2.1.

Figure 2.1: Geometry showing the relationship between a fixed coordinate system $\mathbf{123}$ and a noninertial coordinate system $\mathbf{1}'\mathbf{2}'\mathbf{3}'$ that is moving with respect to $\mathbf{123}$.

The fluid particle acceleration in a fixed frame is given by [Kundu, Cohen and Dowling, (2012)]

$$\left(\frac{D\mathbf{u}}{Dt}\right)_{fixed} = \left(\frac{D\mathbf{u}}{Dt}\right)_{rot.} + 2\boldsymbol{\Omega} \times \mathbf{u}_{rot.} - |\boldsymbol{\Omega}|^2 \mathbf{R}, \quad (2.1.9)$$

where $\left(\frac{D\mathbf{u}}{Dt}\right)_{fixed}$ is the derivative of the fixed velocity in the fixed frame and $\left(\frac{D\mathbf{u}}{Dt}\right)_{rot.}$ is the derivative of the rotating velocity in the rotating frame and $\mathbf{u}_{rot.}$ is the fluid velocity in the rotating frame and $\mathbf{R}$ is the radius of rotation. On the right-hand side, the second term is called the Coriolis acceleration and the third term is the particle's centripetal acceleration. The Coriolis term depends on the fluid particle's velocity, not on its position. The centripetal acceleration depends strongly on the rotation rate and the distance of the fluid particle from the axis of rotation [Kundu, Cohen and Dowling (2012)].

Now, the momentum equation in the rotating frame of reference is written as follows. We substitute the right-hand side of (2.1.9) into the left hand side of the momentum equation (2.1.4) $\left(\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}\right)$ and attain

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} + 2 \boldsymbol{\Omega} \times \mathbf{u} - |\boldsymbol{\Omega}|^2 \mathbf{R} &= -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \\ \frac{D\mathbf{u}}{Dt} &= -\frac{1}{\rho}\nabla p + \underbrace{(\mathbf{g}_{Newtonian} + |\boldsymbol{\Omega}|^2 \mathbf{R})}_{\mathbf{g}_{total}} - 2 \boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} \\ \frac{D\mathbf{u}}{Dt} &= -\frac{1}{\rho}\nabla p + \mathbf{g}_{total} - 2 \boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u}. \end{aligned} \quad (2.1.10)$$

Equation (2.1.10) is the momentum equation in the rotating frame of reference. This equation describes conservation of momentum for a fluid under the influence of a pressure gradient, a body force (gravitational force), the Coriolis force and the viscosity in the fluid.

2.1.3 Approximate equations for a thin layer on a rotating sphere

In this section, we present the derivation of approximate equations for the fluid motion on a rectangular domain on a tangent plane. The atmosphere and the ocean are very thin layers in which the depth scale of flow is a few kilometers, whereas the horizontal scale is of the order of hundreds, or even thousands, of kilometers. The trajectories

of fluid elements are nearly horizontal while vertical velocities are much smaller than horizontal velocities [Kundu, Cohen and Dowling, (2012)].

In order to study large-scale geophysical flow problems, we need to use spherical coordinates. If, however, the horizontal length scales are much smaller than the radius of the earth ($= 6371$ km), then the curvature of the earth can be ignored, and the motion can be studied by adopting a local Cartesian system on a plane tangent to the surface of the earth (see Figure 2.2, following Figure 13.3 in [Kundu, Cohen and Dowling, (2012)]).

Figure 2.2: Local Cartesian coordinates. The x -axis points into the plane of the paper. The y -axis is tangent to the earth's surface and points toward the north pole. The z -axis is vertical, opposing gravity. The earth's angular rotation vector has positive y and z components in the northern hemisphere.

On this plane we take an xyz coordinate system, with x increasing eastward, y northward, and z upward. The corresponding velocity components are u (eastward), v (northward), and w (upward). The thin-sheet approximation requires that $w \ll v$.

The earth rotates at the rate

$$\Omega = \frac{2\pi \text{ rad}}{\text{day}} = 0.73 \times 10^{-4} \text{ s}^{-1}$$

around the polar axis in a counterclockwise sense when looking from above the north pole [Kundu, Cohen and Dowling (2012)]. From Figure 2.2, the components of angular velocity of the earth in the local Cartesian system are $\boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z) = (0, \Omega \cos \theta, \Omega \sin \theta)$, where θ is the latitude which is the angle defined such as $\theta = 0$ corresponds to the equator, $0 < \theta \leq \pi/2$ corresponds to the northern hemisphere and $-\pi/2 \leq \theta < 0$ corresponds to the southern hemisphere. Thus, the Coriolis acceleration is

$$2\boldsymbol{\Omega} \times \mathbf{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2\Omega \cos \theta & 2\Omega \sin \theta \\ u & v & w \end{vmatrix} = 2\Omega \left[(w \cos \theta - v \sin \theta) \hat{i} + u \sin \theta \hat{j} - u \cos \theta \hat{k} \right].$$

In the term multiplied by $\hat{i}$ we can use the condition $w \cos \theta \ll v \sin \theta$ since $w \ll v$. Thus, the three components of the Coriolis acceleration are

$$2\boldsymbol{\Omega} \times \mathbf{u} \approx (-2\Omega v \sin \theta, 2\Omega u \sin \theta, -2\Omega u \cos \theta) = (-fv, fu, -2\Omega u \cos \theta),$$

where we define $f = 2\Omega \sin \theta$ (twice the vertical component of $\boldsymbol{\Omega}$). Typically, f is referred to as the *Coriolis parameter*, or the *Coriolis frequency*. The Coriolis parameter f is positive in the northern hemisphere ($\theta > 0$) and negative in the southern hemisphere ($\theta < 0$).

The f -plane approximation The Coriolis parameter $f = 2\Omega \sin \theta$ varies with latitude θ . This variation, however, is important only for phenomena having very long time scales (several weeks) or very long length scales (thousands of kilometers). For many purposes we can assume f to be a constant, say $f_0 = 2\Omega \sin \theta_0$, where θ_0 is the central latitude of the region under study. An approximation that uses a constant Coriolis parameter is called an *f -plane approximation*.

The β -plane approximation A better approximation for the Coriolis parameter f , which takes into account the variation of f with latitude, can be obtained by

linearizing f about some fixed latitude θ_0 . Since dealing with spherical geometry is more complicated than with planar geometry, it is common to represent a strip of the sphere that is limited in latitude but going all the way around the world in longitude as a plane as in Figure 2.3.

Figure 2.3: The β plane.

Assuming that the width is small enough, the variation of f with latitude can be approximately represented by expanding f in a Taylor series about the central latitude $\theta = \theta_0$ and approximating by the first two terms:

$$f(\theta) \approx f_0 + \left(\frac{df}{d\theta} \right)_{\theta=\theta_0} (\theta - \theta_0),$$

where

$$f_0 = f(\theta_0) = 2\Omega \sin \theta_0 \quad \text{and} \quad \left(\frac{df}{d\theta} \right)_{\theta=\theta_0} = 2\Omega \cos \theta_0.$$

Also, we use the approximation for the arclength $y \approx R_{earth} (\theta - \theta_0)$ and consequently the Coriolis parameter can be approximated as

$$f(\theta) \approx f_0 + \left(\frac{df}{d\theta} \right)_{\theta=\theta_0} (\theta - \theta_0) \approx f_0 + \left(\frac{df}{d\theta} \right)_{\theta=\theta_0} \frac{y}{R_{earth}} = f_0 + \left(\frac{df}{dy} \right)_{y=0} y = f_0 + \beta y = f(y),$$

where

$$\beta \equiv \left(\frac{df}{dy} \right)_{y=0} = \frac{1}{R_{earth}} \left(\frac{df}{d\theta} \right)_{\theta=\theta_0} = \frac{2\Omega \cos \theta_0}{R_{earth}}$$

is the gradient of planetary vorticity in y -direction. This constant β is always positive since $-\frac{\pi}{2} \leq \theta_0 \leq \frac{\pi}{2}$. Here, we have used $f = 2\Omega \sin \theta$. The radius of the earth is

approximately $R_{earth} = 6371$ km. An approximation that takes into account the variation of the Coriolis parameter in the simplified form $f = f_0 + \beta y$, with β a constant, is called a **β -plane approximation**. It approximates the curved surface of the earth by a horizontal plane.

An important definition which we need in our model is the *geostrophic balance*. The following horizontal momentum equations, without the viscous effects, (2.1.10) are used to define the geostrophic balance

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u &= -\frac{1}{\rho} \frac{\partial p}{\partial y}. \end{aligned} \quad (2.1.11)$$

If we consider quasi-steady, large-scale motions within the atmosphere or the ocean, away from boundaries, an excellent approximation, for these flows, for the horizontal equilibrium is a balance between the Coriolis force and the pressure gradient:

$$-f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad (2.1.12)$$

$$f u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \quad (2.1.13)$$

where ρ_0 is a constant reference density. The balance of forces shown in (2.1.12) and (2.1.13), in which the horizontal pressure gradients are balanced by Coriolis forces, is called a *geostrophic balance* [Holton, (2004)].

This thesis deals with a set of equations in a three-dimensional configuration under the hydrostatic approximation and a quasi-geostrophic approximation written in terms of log-pressure coordinates [Holton, (2004)]. The derivation of equations is given in Appendix A.

In section A.1 we transform equations (2.1.1), (2.1.11), (2.1.6) and (2.1.8) which assume hydrostatic balance from the height coordinate system to an isobaric coordinate system in which pressure is the vertical coordinate [Holton, (2004)]. Then in section A.2, we make the quasi-geostrophic approximation in which the horizontal flow is assumed to be approximately geostrophic and that the ratio of the vertical

velocity component to the horizontal velocity components is taken to be small. Under these conditions, the horizontal velocity is considered as the sum of a geostrophic part and a small ageostrophic part [Holton, (2004)].

In section A.3, we re-write the quasi-geostrophic equations in terms of log-pressure vertical coordinates defined as

$$z = -H \ln\left(\frac{p}{p_s}\right),$$

where p_s is a standard reference pressure and H is a standard scale height, $H = \frac{RT_s}{g}$ with R is the gas constant for dry air and T_s is a global average temperature [Holton, (2004)]. This gives the continuity equation (A.3.5) (with ρ_0 is constant), the horizontal momentum equations (A.3.6), the hydrostatic balance (A.3.7) and the thermodynamic energy equation (A.3.8).

2.2 Nondimensional equations for Rossby waves and gravity waves

The equations for the fluid flow are the hydrostatic and geostrophic equations (A.3.5), (A.3.6), (A.3.7) and (A.3.8) in log-pressure coordinates which were derived in Appendix A.3 [Holton, (2004)]. These equations are repeated here for reference. The continuity equation (A.3.5) (where ρ_0 is constant), the horizontal momentum equations (A.3.6), the hydrostatic balance (A.3.7) and the thermodynamic energy equation (A.3.8) (with $J = 0$ for simplicity) are, respectively,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0, \quad (2.2.1)$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} - f^* v^* = -\frac{\partial \Phi^*}{\partial x^*}, \quad (2.2.2)$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} + f^* u^* = -\frac{\partial \Phi^*}{\partial y^*}, \quad (2.2.3)$$

$$\frac{\partial \Phi^*}{\partial z^*} = \frac{g^* T^*}{T_s^*}, \quad (2.2.4)$$

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \left(\frac{g^*}{c_p^* T_s^*} T^* + \frac{\partial T^*}{\partial z^*} \right) = 0, \quad (2.2.5)$$

where Φ is a scalar potential function, T is a temperature and c_p is the specific heat at constant pressure. All the variables and quantities in (2.2.1)-(2.2.5) are dimensional and we denote this by a star ($\star$).

Our aim now is to nondimensionalize the equations (2.2.1)-(2.2.5). We define the nondimensional variables and parameters (without stars) by:

$$\begin{aligned} x &= \frac{x^*}{L_x}, \quad y = \frac{y^*}{L_y}, \quad z = \frac{z^*}{L_z}, \quad t = \frac{t^*}{t_{\text{scale}}} = \frac{U t^*}{L_x}, \quad \mathcal{U} = \frac{u^*}{U}, \\ \mathcal{V} &= \frac{v^*}{V}, \quad \mathcal{W} = \frac{w^*}{W}, \quad f = \frac{f^*}{U/L_y}, \quad g = \frac{g^*}{U^2/L_z}, \quad \Phi = \frac{\Phi^*}{U^2} \\ \mathcal{T} &= \frac{T^*}{T_{\text{scale}}}, \quad T_s = \frac{T_s^*}{T_{\text{scale}}}, \quad c_p = \frac{c_p^*}{U^2/T_{\text{scale}}} \end{aligned} \quad (2.2.6)$$

where

- L_x , L_y and L_z are the typical length scales in the x , y and z directions respectively.
- U , V and W are the typical velocity scales in the x , y and z directions respectively.
- t_{scale} is the time scale, i.e. $t_{\text{scale}} = \frac{L_x}{U} = \frac{L_y}{V} = \frac{L_z}{W}$.
- $\frac{U^2}{L_z}$ is the gravity acceleration scale.
- T_{scale} is the typical temperature scale, i.e. $T_{\text{scale}} = 288.15$ Kelvin.
- U^2/T_{scale} is the typical specific heat scale at constant pressure.

We rewrite the equations (2.2.1)-(2.2.5) in terms of the nondimensional quantities as follows. The continuity equation (2.2.1) becomes

$$\frac{U}{L_x} \frac{\partial \mathcal{U}}{\partial x} + \frac{V}{L_y} \frac{\partial \mathcal{V}}{\partial y} + \frac{W}{L_z} \frac{\partial \mathcal{W}}{\partial z} = 0, \quad (2.2.7)$$

and since $\frac{1}{t_{\text{scale}}} = \frac{U}{L_x} = \frac{V}{L_y} = \frac{W}{L_z}$, then (2.2.7) becomes

$$\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{V}}{\partial y} + \frac{\partial \mathcal{W}}{\partial z} = 0. \quad (2.2.8)$$

Equation (2.2.8) is the *nondimensional* continuity equation.

The momentum equation in the x direction (2.2.2) becomes

$$\frac{U^2}{L_x} \frac{\partial \mathcal{U}}{\partial t} + \frac{U^2}{L_x} \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \frac{UV}{L_y} \mathcal{V} \frac{\partial \mathcal{U}}{\partial y} + \frac{UW}{L_z} \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} - \frac{UV}{L_y} f \mathcal{V} = -\frac{U^2}{L_x} \frac{\partial \Phi}{\partial x}. \quad (2.2.9)$$

Multiplying (2.2.9) by $\frac{L_x}{U^2}$ leads to the *nondimensional* momentum equation in the x -direction

$$\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{V} \frac{\partial \mathcal{U}}{\partial y} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} - f \mathcal{V} = -\frac{\partial \Phi}{\partial x}. \quad (2.2.10)$$

The momentum equation in the y -direction (2.2.3) becomes

$$\frac{\partial \mathcal{V}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{V}}{\partial x} + \mathcal{V} \frac{\partial \mathcal{V}}{\partial y} + \mathcal{W} \frac{\partial \mathcal{V}}{\partial z} + \frac{L_x^2}{L_y^2} f \mathcal{U} = -\frac{L_x^2}{L_y^2} \frac{\partial \Phi}{\partial y} \quad (2.2.11)$$

Multiplying (2.2.11) by the square of the aspect ratio $\delta = \frac{L_y^2}{L_x^2}$ leads to the following *nondimensional* momentum equation in the y -direction

$$\delta \left[\frac{\partial \mathcal{V}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{V}}{\partial x} + \mathcal{V} \frac{\partial \mathcal{V}}{\partial y} + \mathcal{W} \frac{\partial \mathcal{V}}{\partial z} \right] + f \mathcal{U} = -\frac{\partial \Phi}{\partial y}. \quad (2.2.12)$$

The hydrostatic balance equation (2.2.4) becomes

$$\frac{U^2}{L_z} \frac{\partial \Phi}{\partial z} = \frac{U^2}{L_z} \frac{g}{T_s} \mathcal{T}. \quad (2.2.13)$$

Multiplying (2.2.13) by $\frac{L_z}{U^2}$ leads to the following *nondimensional* hydrostatic balance equation

$$\frac{\partial \Phi}{\partial z} = \frac{g}{T_s} \mathcal{T}. \quad (2.2.14)$$

The thermodynamic energy equation (2.2.5) becomes

$$\frac{UT_{\text{scale}}}{L_x} \frac{\partial \mathcal{T}}{\partial t} + \frac{UT_{\text{scale}}}{L_x} \mathcal{U} \frac{\partial \mathcal{T}}{\partial x} + \frac{VT_{\text{scale}}}{L_y} \mathcal{V} \frac{\partial \mathcal{T}}{\partial y} + W \mathcal{W} \left(\frac{\frac{U^2}{L_z}}{\frac{U^2}{T_{\text{scale}}}} \frac{g}{c_p T_{\text{scale}} T_s} T_{\text{scale}} \mathcal{T} + \frac{T_{\text{scale}}}{L_z} \frac{\partial \mathcal{T}}{\partial z} \right) = 0. \quad (2.2.15)$$

Multiplying (2.2.15) by L_x/UT_{scale} leads to the following *nondimensional* thermodynamic energy equation

$$\frac{\partial \mathcal{T}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{T}}{\partial x} + \mathcal{V} \frac{\partial \mathcal{T}}{\partial y} + \mathcal{W} \left(\frac{g}{c_p T_s} \mathcal{T} + \frac{\partial \mathcal{T}}{\partial z} \right) = 0. \quad (2.2.16)$$

2.3 The perturbed equations

We are interested in a situation where Rossby waves and gravity waves are generated at a fixed location in xyz -space in a fluid whose properties are governed by equations (2.2.8), (2.2.10), (2.2.12) and (2.2.16). The waves are considered to be oscillations in the fluid quantities $\mathcal{U}$, $\mathcal{V}$, Φ and $\mathcal{T}$. In order to represent the waves, we make use of the perturbation method in which each field variable is written as the sum of a basic steady part and two time dependent perturbations that represent the local deviations of the field from the basic part.

Accordingly, we write

$$\begin{aligned}
 \mathcal{U}(x, y, z, t) &= \bar{u}(y, z) + \varepsilon_R [u_a(x, y, z, t) + u_R(x, y, z, t)] + \varepsilon_G u_G(x, y, z, t), \\
 \mathcal{V}(x, y, z, t) &= \varepsilon_R [v_a(x, y, z, t) + v_R(x, y, z, t)] + \varepsilon_G v_G(x, y, z, t), \\
 \mathcal{W}(x, y, z, t) &= \varepsilon_R w_R(x, y, z, t) + \varepsilon_G w_G(x, y, z, t) \\
 \Phi(x, y, z, t) &= \bar{\Phi}(y, z) + \varepsilon_R \Phi_R(x, y, z, t) + \varepsilon_G \Phi_G(x, y, z, t), \\
 \mathcal{T}(x, y, z, t) &= \bar{T}(y, z) + \varepsilon_R T_R(x, y, z, t) + \varepsilon_G T_G(x, y, z, t),
 \end{aligned} \tag{2.3.1}$$

where the overbar indicates the zonal average of the basic steady parts, the subscripts R and G denote Rossby waves and gravity waves, respectively, and the parameters ε_R and ε_G characterize the extent of nonlinearity of the equations. As discussed in section 2.1, the horizontal velocity is considered as the sum of a geostrophic part and a small ageostrophic part. In our notation, u_a and v_a are the ageostrophic horizontal velocities and u_R and v_R are the geostrophic horizontal velocities.

With the terms representing Rossby waves, we use the definitions and assumptions presented and explained in Appendix A. These definitions and assumptions are repeated here for reference. Firstly, the expression for the background geostrophic balance and the expression for perturbation of the geostrophic balance in log-pressure coordinates are

$$f_0 \bar{u} = -\frac{\partial \bar{\Phi}}{\partial y}, \quad f_0 u_R = -\frac{\partial \Phi_R}{\partial y} \quad \text{and} \quad f_0 v_R = \frac{\partial \Phi_R}{\partial x} \tag{2.3.2}$$

so, the divergence of the geostrophic wind vanishes

$$\frac{\partial u_R}{\partial x} + \frac{\partial v_R}{\partial y} = 0. \quad (2.3.3)$$

Secondly, as point out in Appendix A, the rate of change of momentum following the total motion is approximately equal to the rate of change of the geostrophic momentum following the geostrophic wind, i.e. $\frac{D}{Dt} \approx \frac{\partial}{\partial t} + u_R \frac{\partial}{\partial x} + v_R \frac{\partial}{\partial y}$.

Substituting from (2.3.1) into (2.2.8) yields to the following continuity equation for Rossby waves and gravity waves

$$\varepsilon_R \left[\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_R}{\partial z} \right] + \varepsilon_G \left[\frac{\partial u_G}{\partial x} + \frac{\partial v_G}{\partial y} + \frac{\partial w_G}{\partial z} \right] = 0, \quad (2.3.4)$$

We substitute (2.3.1) into (2.2.10) and group terms of $\mathcal{O}(\varepsilon_R)$, $\mathcal{O}(\varepsilon_R^2)$, $\mathcal{O}(\varepsilon_G)$, $\mathcal{O}(\varepsilon_G^2)$ and $\mathcal{O}(\varepsilon_R \varepsilon_G)$, respectively, to obtain the horizontal (zonal) momentum equation for Rossby waves and gravity waves

$$\begin{aligned} & \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u_R + \frac{\partial \bar{u}}{\partial y} v_R - f_0 v_a - \beta y v_R \right] \\ & + \varepsilon_R^2 \left(u_R \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_R}{\partial y} \right) \\ & + \varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u_G + \frac{\partial \bar{u}}{\partial y} v_G + \frac{\partial \bar{u}}{\partial z} w_G - f v_G + \frac{\partial \Phi_G}{\partial x} \right] \\ & + \varepsilon_G^2 \left(u_G \frac{\partial u_G}{\partial x} + v_G \frac{\partial u_G}{\partial y} + w_G \frac{\partial u_G}{\partial z} \right) \\ & + \varepsilon_R \varepsilon_G \left[u_R \frac{\partial u_G}{\partial x} + u_G \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_G}{\partial y} + v_G \frac{\partial u_R}{\partial y} \right] = 0. \end{aligned} \quad (2.3.5)$$

We should remark here that the last two terms in the first line of (2.3.5) are written after using (2.3.2), so the terms $-f_0 v_R$ and $-\frac{\partial \Phi}{\partial x}$ are cancelled. Following Holton (2004), the ageostrophic part of the horizontal velocity $-\beta y v_a$ is neglected compared to the geostrophic part of the horizontal velocity $-\beta y v_R$, as discussed in section A.2.

Similarly, we substitute (2.3.1) into (2.2.12) to obtain the horizontal (meridional)

momentum equation for Rossby waves and gravity waves

$$\begin{aligned}
& \varepsilon_R \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) v_R + f_0 u_a + \beta y u_R \right] \\
& + \delta \varepsilon_R^2 \left(u_R \frac{\partial v_R}{\partial x} + v_R \frac{\partial v_R}{\partial y} \right) \\
& + \varepsilon_G \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) v_G + f u_G + \frac{\partial \Phi_G}{\partial y} \right] \\
& + \delta \varepsilon_G^2 \left(u_G \frac{\partial v_G}{\partial x} + v_G \frac{\partial v_G}{\partial y} + w_G \frac{\partial v_G}{\partial z} \right) \\
& + \delta \varepsilon_R \varepsilon_G \left[u_R \frac{\partial v_G}{\partial x} + u_G \frac{\partial v_R}{\partial x} + v_R \frac{\partial v_G}{\partial y} + v_G \frac{\partial v_R}{\partial y} + w_G \frac{\partial v_R}{\partial z} \right] = 0,
\end{aligned} \tag{2.3.6}$$

where the last two terms in the first line of (2.3.6) are written after using the same argument presented below (2.3.5).

Substituting (2.3.1) into (2.2.14) leads to the hydrostatic balance equation for Rossby waves and gravity waves

$$\left[\frac{\partial \bar{\Phi}}{\partial z} - \frac{g \bar{T}}{T_s} \right] + \varepsilon_R \left[\frac{\partial \Phi_R}{\partial z} - \frac{g T_R}{T_s} \right] + \varepsilon_G \left[\frac{\partial \Phi_G}{\partial z} - \frac{g T_G}{T_s} \right] = 0 \tag{2.3.7}$$

Finally, we substitute (2.3.1) into (2.2.16) and attain the thermodynamic energy equation for Rossby waves and gravity waves

$$\begin{aligned}
& \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_R + \frac{\partial \bar{T}}{\partial y} v_R + w_R \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \right] \\
& + \varepsilon_R^2 \left[u_R \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_R}{\partial y} + w_R \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] \\
& + \varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G + \frac{\partial \bar{T}}{\partial y} v_G + w_G \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \right] \\
& + \varepsilon_G^2 \left[u_G \frac{\partial T_G}{\partial x} + v_G \frac{\partial T_G}{\partial y} + w_G \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) \right] \\
& + \varepsilon_R \varepsilon_G \left[u_R \frac{\partial T_G}{\partial x} + u_G \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_G}{\partial y} + v_G \frac{\partial T_R}{\partial y} + w_R \left(\frac{g T_G}{c_p T_s} + \frac{\partial T_G}{\partial z} \right) + w_G \left(\frac{g T_R}{c_p T_s} + \frac{\partial T_R}{\partial z} \right) \right] = 0.
\end{aligned} \tag{2.3.8}$$

2.4 The domain

In this section, we present the domain where we will solve the Rossby wave and the gravity wave problems. The domain throughout the thesis is considered to be a rectangular prism domain defined in terms of Cartesian coordinates x, y and z .

We consider a Rossby wave perturbation sinusoidal and periodic in the x direction with a wavelength of $2\pi/k_R$ (k_R a positive integer) and a gravity wave perturbation sinusoidal and periodic in the x direction with a wavelength of $2\pi/k_G$ (k_G a positive integer) both generated on a plane surface in xyz space. Thus, the x domain is specified as $0 \leq x < 2\pi$.

The waves are forced on a plane defined by the line $\mu y + \Lambda z + b = Y_1, z \geq 0$ for $0 \leq x < 2\pi$. We choose the parameters so that $\mu \geq 0, \Lambda \leq 0, Y_1 \geq 1$ and $Y_1 > b$, as shown in Figure 2.4 so that the waves propagate southwards (in the direction of negative y) and upwards (in the direction of positive z).

Figure 2.4: The domain in the yz plane (in grey shade)

This is representative of a situation where waves generated in the troposphere in the mid-latitude region of the northern hemisphere propagate towards the equator and up in the stratosphere.

The wave forcing line in the yz plane is

$$\mu y + \Lambda z + b = Y_1 \implies z = \left(\frac{-\mu}{\Lambda} \right) y - \left(\frac{1}{\Lambda} \right) (b - Y_1), \quad (2.4.1)$$

as shown in Figure 2.4 (blue line). For the reason that we must have $z \geq 0$, the wave forcing line (2.4.1) stops at $z = 0$ at the point with coordinates $(y, z) = (\frac{-1}{\mu} (b - Y_1), 0)$.

The equation of the line that is perpendicular to the wave forcing line (2.4.1), passes through the point $(y, z) = (\frac{-1}{\mu} (b - Y_1), 0)$ and forms the side boundary of the domain in the yz plane is

$$z = \left(\frac{\Lambda}{\mu} \right) y + \left(\frac{\Lambda}{\mu^2} \right) (b - Y_1), \quad (2.4.2)$$

as shown in Figure 2.4 (red line). The waves propagate into the shaded region bounded by the line (2.4.1) and the perpendicular line (2.4.2).

We consider Rossby waves and gravity waves as perturbations to a background flow with zonal mean speed $\bar{u}(y, z)$. In order to allow asymptotic analysis (chapters 3 and 4) and to facilitate the numerical discretization described in chapter 5, we consider the special case where $\bar{u}(y, z)$ is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$.

With this choice of $\bar{u}$, equations (2.3.4)-(2.3.8) can be written in terms of a new variable Y defined as $Y = \mu y + \Lambda z + b$. In terms of this variable, the domain is $0 \leq x < 2\pi$ and $-\infty < Y \leq Y_1$ and the waves are forced at the boundary $\mu y + \Lambda z + b = Y = Y_1$. As $Y \rightarrow -\infty$, the waves in the yz plane propagate in the direction of negative Y (i.e. southwards in the y -direction and upwards in the z -direction).

We choose $\bar{u}(y, z)$ such that there is a critical line in the yz plane where $\bar{u}$ is equal to the phase speed of both waves. We consider a situation where both the Rossby waves and gravity waves have the same zero phase speed, and thus their critical lines

coincide at a value of $Y = Y_c$. This critical line is parallel to the forcing line $Y = Y_1$ in the yz plane and corresponds to a plane (critical surface) in xyz space. This is exhibited in Figure 2.4.

2.5 Dimensional analysis

In chapters 3-5, we examine the nondimensional perturbation equations (2.3.4)-(2.3.8) written in terms of x, Y and t with background flow given by $\bar{u}(Y)$ on the domain $0 \leq x < 2\pi$, $-\infty < Y \leq Y_1$ and for $0 \leq t < \infty$.

The nondimensional parameters are chosen to correspond to reasonable values of the corresponding dimensional quantities in the atmosphere according to (2.2.6). The reference quantities $L_x, L_y, L_z, t_{\text{scale}}$ and T_{scale} used in (2.2.6) are chosen so that the nondimensional parameters $g, T_s, c_p, f_0, N^2, \kappa$ and β are $\mathcal{O}(1)$ while the wave amplitude parameters $\varepsilon_R, \varepsilon_G$ and the square of the meridional to zonal aspect ratio $\delta = (L_y/L_x)^2$ are $\ll 1$. An explanation of these choices is given below.

We consider waves generated at mid-latitudes around 55° N, for example, propagating towards the equator and reaching a critical line around 25° N. This means that the waves travel a distance of about 30° in the southward direction from the forcing latitude to the critical line. We call this difference in latitude $\Delta\theta^*$. We take $\theta_{\text{ref}}^* = 25^\circ$ to be a reference latitude corresponding to the location y_{ref}^* in Figure 2.4.

We consider a reference altitude in the stratosphere $z_{\text{ref}}^* = 8.5$ km corresponding to the level z_{ref}^* in Figure 2.4.

We define our dimensional domain on the zonal, meridional and vertical directions, respectively, to be as follows

$$0 \leq x^* \leq 16,859 \text{ km}, \quad -4800 \leq y^* \leq 9600 \text{ km}, \quad 0 \leq z^* \leq 36 \text{ km}.$$

In the earth's atmosphere, the Coriolis parameter is $f^* = 2\Omega \sin(\theta_{\text{ref}}^*)$ and its latitudinal gradient representing the effects of the earth's rotation is $\beta^* = \frac{2\Omega \cos(\theta_{\text{ref}}^*)}{R_{\text{earth}}}$, where $\Omega = 7.292 \times 10^{-5} \text{ rad sec}^{-1}$ is the earth's angular speed of rotation and R_{earth}

is the radius of the earth that is approximately 6.378×10^6 m [Holton, (2004)]. At our reference latitude $\theta_{ref}^* = 25^\circ$, we have $f_0^* = 6.1 \times 10^{-5} \text{ sec}^{-1}$ and $\beta^* = 2.07 \times 10^{-11} \text{ m}^{-1} \text{ sec}^{-1}$.

We consider a location where the dimensional sea level temperature and the background mean temperature are $T_s^* = \bar{T}^* = 288.15$ Kelvin [Holton, (2004)]. In the atmosphere, the specific heat of dry air at constant pressure is approximately $c_p^* = 1004 \text{ Joules Kelvin}^{-1} \text{ kg}^{-1}$, where Joules = $\text{kg m}^2 \text{ sec}^{-2}$ [Holton, (2004)].

In the earth's atmosphere, the acceleration due to gravity is $g^* = 9.8 \text{ m sec}^{-2}$. Thus, the dimensional Brunt-Väisälä frequency is approximately $N^* = \sqrt{\frac{g^*}{T_s^*} \left(\frac{g^*}{c_p^*} \frac{\bar{T}^*}{T_s^*} \right)} = 0.0182 \text{ sec}^{-1}$. The zonal, meridional and vertical length scales are, respectively, $L_x = 2683 \text{ km}$, $L_y = 1200 \text{ km}$ and $L_z = 3 \text{ km}$. The reference zonal velocity is $U = 38.34 \text{ m sec}^{-1}$. Thus, the time scale is $t_{\text{scale}} = \frac{L_x}{U} = 6.9 \times 10^4 \text{ sec} \approx 19.4 \text{ hours}$.

With these references values, the aforementioned dimensional values of the parameters and (2.2.6), we attain all the nondimensional values of the parameters that are used in our numerical computations in chapter 5. We summarize all the dimensional values and the nondimensional values in Table 2.1.

Table 2.1: The values of the dimensional and the nondimensional parameters

Dimensional values	Nondimensional value
$\theta_{ref.}^* = 25^\circ$	$y_{ref.} = 2.3$
$z_{ref.}^* = 8.5 \text{ km}$	$z_{ref.} = 2.8$
— — — — —	$\delta = 0.2$
$f_0^* = 6.1 \times 10^{-5} \text{ sec}^{-1}$	$f_0 = 1.9291$
$\beta^* = 2.07 \times 10^{-11} \text{ m}^{-1} \text{ sec}^{-1}$	$\beta = 0.7783$
$T_s^* = \bar{T}^* = 288.15 \text{ Kelvin}$	$T_s = \bar{T} = 1$
$c_p^* = 1004 \text{ Joules Kelvin}^{-1} \text{ kg}^{-1}$	$c_p = 196.8$
$g^* = 9.8 \text{ m sec}^{-2}$	$g = 20$
$N^* = 0.0182 \text{ sec}^{-1}$	$N = \sqrt{2.035}$

Chapter 3

Weakly-nonlinear formulation

3.1 Perturbation equations

In this chapter, we carry out a weakly-nonlinear analysis of equations (2.3.4)-(2.3.8) with the parameters $\varepsilon_R \ll 1$, $\varepsilon_G \ll 1$ and $f_0, \beta, g, N \sim \mathcal{O}(1)$. Our derivation of equations (2.3.4)-(2.3.8) allows us to consider different possibilities for ε_R and ε_G . The solutions of (2.3.4)-(2.3.8) can be written as perturbation series in powers of the parameters ε_R and ε_G as:

$$\left. \begin{aligned} u_R &\sim u_R^{(0)} + \varepsilon_R u_R^{(1)} + \mathcal{O}(\varepsilon_R^2) \\ v_R &\sim v_R^{(0)} + \varepsilon_R v_R^{(1)} + \mathcal{O}(\varepsilon_R^2) \\ w_R &\sim w_R^{(0)} + \varepsilon_R w_R^{(1)} + \mathcal{O}(\varepsilon_R^2) \\ \Phi_R &\sim \Phi_R^{(0)} + \varepsilon_R \Phi_R^{(1)} + \mathcal{O}(\varepsilon_R^2) \\ T_R &\sim T_R^{(0)} + \varepsilon_R T_R^{(1)} + \mathcal{O}(\varepsilon_R^2). \end{aligned} \right\} \quad (3.1.1)$$

and

$$\left. \begin{aligned} u_G &\sim u_G^{(0)} + \varepsilon_G u_G^{(1)} + \mathcal{O}(\varepsilon_G^2) \\ v_G &\sim v_G^{(0)} + \varepsilon_G v_G^{(1)} + \mathcal{O}(\varepsilon_G^2) \\ w_G &\sim w_G^{(0)} + \varepsilon_G w_G^{(1)} + \mathcal{O}(\varepsilon_G^2) \\ \Phi_G &\sim \Phi_G^{(0)} + \varepsilon_G \Phi_G^{(1)} + \mathcal{O}(\varepsilon_G^2) \\ T_G &\sim T_G^{(0)} + \varepsilon_G T_G^{(1)} + \mathcal{O}(\varepsilon_G^2). \end{aligned} \right\} \quad (3.1.2)$$

In order to simplify the notation, we define a vector $\vec{F}$ whose components are $\vec{F} = (F_1^{(i)}, F_2^{(i)}, F_3^{(i)}, F_4^{(i)}, F_5^{(i)}) = (u^{(i)}, v^{(i)}, w^{(i)}, \Phi^{(i)}, T^{(i)})$. We then can have $F_{Rj}^{(i)}$, with $j = 1, 2, 3, 4, 5$, which correspond to the Rossby wave functions, $F_{Gj}^{(i)}$, with $j = 1, 2, 3, 4, 5$, that correspond to the gravity wave functions, where the superscript (i) , $i = 0, 1, 2, 3, \dots$, represents the order of each function in terms of ε_R or ε_G .

For simplicity, each of equations (2.3.5), (2.3.6) and (2.3.8) can then be written in the form

$$\begin{aligned} &\varepsilon_R \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] \\ &+ \varepsilon_R^2 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] \\ &+ \varepsilon_G \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] \\ &+ \varepsilon_G^2 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] \\ &+ \varepsilon_R \varepsilon_G \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] = 0, \end{aligned} \quad (3.1.3)$$

with $jj = 1, 2, 3$, j and k can be $1, 2, 3, 4, 5$, $\mathcal{L}_{Rjj}$ are the linear operators that operate on each of the Rossby wave functions $F_{Rj}^{(i)}$, $\mathcal{L}_{Gjj}$ are the linear operators that operate on each of the gravity wave functions $F_{Gj}^{(i)}$, $\mathcal{N}_{Rjj}$ are the nonlinear operators that operate on the Rossby wave functions $F_{Rj}^{(i)}$ and $F_{Rk}^{(i)}$, $\mathcal{N}_{Gjj}$ are the nonlinear operators that operate on the gravity wave functions $F_{Gj}^{(i)}$ and $F_{Gk}^{(i)}$ and $\mathcal{N}_{RGj}$ are the nonlinear operators that operate on pairs of Rossby wave and gravity wave functions.

The nature of the solution depends on the relative magnitude of the two small parameters ε_R and ε_G . We write $\varepsilon_G = \varepsilon_R^p$, where $p > 0$. We have cases where the

Rossby wave functions are dominant

$$\begin{aligned}
 \underline{p > 2}: \quad & \text{Case R1 : } \varepsilon_R \gg \varepsilon_R^2 \gg \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2 \\
 \underline{p = 2}: \quad & \text{Case R2 : } \varepsilon_R \gg \varepsilon_R^2 = \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2 \\
 \underline{1 < p < 2}: \quad & \text{Case R3 : } \varepsilon_R \gg \varepsilon_G \gg \varepsilon_R^2 \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2,
 \end{aligned}$$

cases where the gravity wave functions are dominant

$$\begin{aligned}
 \underline{p < \frac{1}{2}}: \quad & \text{Case G1 : } \varepsilon_G \gg \varepsilon_G^2 \gg \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2 \\
 \underline{p = \frac{1}{2}}: \quad & \text{Case G2 : } \varepsilon_G \gg \varepsilon_G^2 = \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2 \\
 \underline{\frac{1}{2} < p < 1}: \quad & \text{Case G3: } \varepsilon_G \gg \varepsilon_R \gg \varepsilon_G^2 \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2,
 \end{aligned}$$

and the case where the Rossby wave and gravity wave functions are the same order of magnitude

$$\underline{p = 1}: \quad \text{Case RG: } \varepsilon_R = \varepsilon_G \gg \varepsilon_R^2 = \varepsilon_R \varepsilon_G = \varepsilon_G^2.$$

In Case R1, for instance, we consider a configuration described by equation (3.2.5) in a domain such as that given in Figure 2.4 with a boundary condition generating Rossby waves of the form $e^{\pm i k_R}$ and gravity waves of the form $e^{\pm i k_G}$ where the wavenumbers k_R and k_G are positive integers.

The products of these functions with themselves and with each other generate combinations of wave modes $\pm(nk_R \pm mk_G)$, where n and m are positive integers. If $nk_R = mk_G$, the combinations $\pm nm(k_R - k_G)$ give zero wavenumber components. This corresponds to a wave-induced mean flow. The magnitude of the wave-induced mean flow depends on the order at which these combinations occur.

We provide more discussion of these cases in section 3.2.

3.2 Orders of magnitude

In this section, we discuss the perturbation equations and the magnitude of the zero wavenumber components of the solutions for the different magnitudes of Rossby wave and gravity wave amplitudes listed in section 3.1 and for different choices of the Rossby wave and gravity wave wavenumbers.

3.2.1 Case R1 : $\varepsilon_R \gg \varepsilon_R^2 \gg \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2$

In Case R1, we set $\varepsilon_G = \varepsilon_R^p$, where $p > 2$, in (3.1.3) to get

$$\begin{aligned} \varepsilon_R \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_R^2 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \varepsilon_R^p \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_R^{p+1} \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_R^{2p} \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] = 0. \end{aligned} \quad (3.2.1)$$

We show that p has to be an integer. If we consider p is not an integer, for example $p = 5/2$, we then rewrite (3.2.1) as

$$\begin{aligned} \varepsilon_R \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_R^2 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \varepsilon_R^{5/2} \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_R^{7/2} \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_R^5 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] = 0. \end{aligned} \quad (3.2.2)$$

We use (3.2.2), (3.1.1) and (3.1.2) to have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_R) : \quad \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] &= 0, \\ \mathcal{O}(\varepsilon_R^2) : \quad \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] &= -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^{5/2}) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] &= 0 \\ \mathcal{O}(\varepsilon_R^{7/2}) : \quad \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] &= 0. \end{aligned} \quad (3.2.3)$$

As we can see that at $\mathcal{O}(\varepsilon_R^{7/2})$, we do not have linear terms to match with at this order. Thus, p in (3.2.1) should be an integer and the smallest choice of p to have a good match of terms is $p = 3$. Setting $p = 3$ in (3.2.1) leads to

$$\begin{aligned} \varepsilon_R \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_R^2 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \varepsilon_R^3 \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_R^4 \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_R^6 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] = 0. \end{aligned} \quad (3.2.4)$$

We use (3.2.4), (3.1.1) and (3.1.2) to have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_R) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] = 0, \\ \mathcal{O}(\varepsilon_R^2) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^3) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(2)}] + \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(1)}] \\ \mathcal{O}(\varepsilon_R^4) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(3)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(2)}] - \mathcal{N}_{Rjj} [F_{Rj}^{(1)}, F_{Rk}^{(1)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^5) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(4)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(3)}] - \mathcal{N}_{Rjj} [F_{Rj}^{(1)}, F_{Rk}^{(2)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(1)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^6) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(5)}] + \mathcal{L}_{Gjj} [F_{Gj}^{(1)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(4)}] - \mathcal{N}_{Rjj} [F_{Rj}^{(2)}, F_{Rk}^{(2)}] \\ & - \mathcal{N}_{Rjj} [F_{Rj}^{(1)}, F_{Rk}^{(3)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(2)}, F_{Gk}^{(0)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(0)}] \end{aligned} \quad (3.2.5)$$

At the leading order $\mathcal{O}(\varepsilon_R)$ in Case R1, we can study the linear Rossby problem only, and we have $\mathcal{L}_{Rjj} [F_{Rj}^{(0)}] = 0$, where $F_{Rj}^{(0)}$ is a linear solution as in chapter 4 corresponding to the modes $\pm k_R$ specified at the forced boundary. If we only look at $\mathcal{O}(\varepsilon_R)$ and $\mathcal{O}(\varepsilon_R^2)$, we can study the nonlinear Rossby problem and do not have to deal with gravity waves as we shall see in chapter 4. At $\mathcal{O}(\varepsilon_R^2)$, the terms $\mathcal{L}_{Rjj} [F_{Rj}^{(1)}]$ and $\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}]$ correspond to wavenumbers 0 and $\pm 2k_R$ as explained in chapter 4. Since the $\mathcal{O}(\varepsilon_R^2)$ terms include a zero wavenumber component, it means that the nonlinear interactions of the Rossby waves affect the mean flow at this order.

At $\mathcal{O}(\varepsilon_R^3)$, the Rossby wave terms and the gravity wave terms are uncoupled, so we can solve the nonlinear Rossby problem and the linear gravity problem separately. At this order, we have $\mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = 0$, where $F_{Gj}^{(0)}$ is a linear solution as in chapter 4 corresponding to the modes $\pm k_G$ specified at the forced boundary. The Rossby terms $\mathcal{L}_{Rjj} [F_{Rj}^{(2)}]$ and $\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(1)}]$ correspond to wavenumbers $\pm k_R$ and $\pm 3k_R$.

At $\mathcal{O}(\varepsilon_R^4)$, the terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(2)}]$ and $\mathcal{N}_{Rjj}[F_{Rj}^{(1)}, F_{Rk}^{(1)}]$ correspond to wavenumbers $0, \pm 2k_R, \pm 4k_R$ and the terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$. It can be noted that the zero wavenumber component resulting from the nonlinear interactions of the Rossby waves affects the mean flow. If $k_R = k_G$, then we have also a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity waves that affects the mean flow.

At $\mathcal{O}(\varepsilon_R^5)$, the terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(3)}]$ correspond to wavenumbers $\pm k_R, \pm 3k_R, \pm 5k_R, \pm(2k_R \pm k_G), \pm k_G$ and the terms $\mathcal{N}_{Rjj}[F_{Rj}^{(1)}, F_{Rk}^{(2)}]$ correspond to wavenumbers $\pm k_R, \pm 3k_R, \pm 5k_R$. The terms $\mathcal{N}_{RGjj}[F_{Rj}^{(1)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(2k_R \pm k_G)$ and $\pm k_G$.

It can be noted that in Case R1 with $p = 3$, the nonlinear gravity wave effects appear at $\mathcal{O}(\varepsilon_R^6)$. The Rossby wave terms and the gravity wave terms are uncoupled at this order, so we can solve the nonlinear Rossby problem and the nonlinear gravity problem separately. The Rossby wave terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(4)}]$ and $\mathcal{N}_{Rjj}[F_{Rj}^{(1)}, F_{Rk}^{(3)}]$ correspond to wavenumbers $0, \pm 2k_R, \pm 4k_R, \pm 6k_R, \pm(k_R \pm k_G), \pm(3k_R \pm k_G)$ and the Rossby term $\mathcal{N}_{Rjj}[F_{Rj}^{(2)}, F_{Rk}^{(2)}]$ correspond to wavenumbers $0, \pm 2k_R, \pm 4k_R, \pm 6k_R$. The Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(2)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G), \pm(3k_R \pm k_G)$. The gravity wave terms $\mathcal{L}_{Gjj}[F_{Gj}^{(1)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $0 \pm 2k_G$. If $k_R = k_G$, then we have also a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity waves affecting the mean flow. The wavenumbers at each order in Case R1 with $p = 3$ are summarized in Table 3.1.

Table 3.1: Wavenumbers corresponding to Case R1 ($\varepsilon_G = \varepsilon_R^3$)

Order	Wavenumbers
ε_R	$\pm k_R$
ε_R^2	$0, \pm 2k_R$
ε_R^3	$\pm k_R, \pm 3k_R, \pm k_G$
ε_R^4	$0, \pm 2k_R, \pm 4k_R, \pm(k_R \pm k_G)$
ε_R^5	$\pm k_R, \pm 3k_R, \pm 5k_R, \pm(2k_R \pm k_G) \pm k_G$
ε_R^6	$0, \pm 2k_R, \pm 4k_R, \pm 6k_R, \pm 2k_G, \pm(k_R \pm k_G), \pm(3k_R \pm k_G)$

3.2.2 Case G1: $\varepsilon_G \gg \varepsilon_G^2 \gg \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2$

In Case G1, we have $\varepsilon_G = \varepsilon_R^p$, where $p = \frac{1}{3}$, in (3.1.3) to obtain

$$\begin{aligned} \varepsilon_G \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_G^2 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] + \varepsilon_G^3 \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_G^4 \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_G^6 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] = 0. \end{aligned} \quad (3.2.6)$$

We use (3.2.6), (3.1.1) and (3.1.2) to have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_G) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = 0, \\ \mathcal{O}(\varepsilon_G^2) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(1)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_G^3) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(2)}] + \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(1)}] \\ \mathcal{O}(\varepsilon_G^4) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(3)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(2)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(1)}, F_{Gk}^{(1)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_G^5) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(4)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(3)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(1)}, F_{Gk}^{(2)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(1)}] \\ \mathcal{O}(\varepsilon_G^6) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(5)}] + \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(4)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(2)}, F_{Gk}^{(2)}] \\ & - \mathcal{N}_{Gjj} [F_{Gj}^{(1)}, F_{Gk}^{(3)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(2)}] - \mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}]. \end{aligned} \quad (3.2.7)$$

At the leading order $\mathcal{O}(\varepsilon_G)$, we can study the linear gravity problem only, and we have $\mathcal{L}_{Gjj}[F_{Gj}^{(0)}] = 0$, where $F_{Gj}^{(0)}$ is a linear solution as in chapter 4 corresponding to the modes $\pm k_G$ specified at the forced boundary. If we only look at $\mathcal{O}(\varepsilon_G)$ and $\mathcal{O}(\varepsilon_G^2)$, we can study the nonlinear gravity problem and do not have to deal with Rossby waves. At $\mathcal{O}(\varepsilon_G^2)$, the terms $\mathcal{L}_{Gjj}[F_{Gj}^{(1)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers 0 and $\pm 2k_G$ as discussed in chapter 4, and the zero wavenumber component resulting from the nonlinear interactions of gravity waves affects the mean flow as presented in chapter 4.

At $\mathcal{O}(\varepsilon_G^3)$, we have $\mathcal{L}_{Rjj}[F_{Rj}^{(0)}] = 0$, where $F_{Rj}^{(0)}$ is a linear Rossby solution as in chapter 4 corresponding to the modes $\pm k_R$ specified at the forced boundary. The gravity terms $\mathcal{L}_{Gjj}[F_{Gj}^{(2)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(1)}]$ correspond to wavenumbers $\pm k_G$ and $\pm 3k_G$.

At $\mathcal{O}(\varepsilon_G^4)$, the terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(2)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(1)}, F_{Gk}^{(1)}]$ correspond to wavenumbers 0, $\pm 2k_G$, $\pm 4k_G$ and the terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$. It can be noted that the zero wavenumber component resulting from the nonlinear interactions of the gravity waves affects the mean flow and if $k_R = k_G$, then we have also another zero wavenumber component resulting from the nonlinear interactions of Rossby-gravity waves affects the mean flow as well.

At $\mathcal{O}(\varepsilon_G^5)$, the terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(3)}]$ correspond to wavenumbers $\pm k_G$, $\pm 3k_G$, $\pm 5k_G$, $\pm(k_R \pm 2k_G)$, $\pm k_R$, the terms $\mathcal{N}_{Gjj}[F_{Gj}^{(1)}, F_{Gk}^{(2)}]$ correspond to wavenumbers $\pm k_G$, $\pm 3k_G$, $\pm 5k_G$ and the terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(1)}]$ correspond to wavenumbers $\pm(k_R \pm 2k_G)$ and $\pm k_R$.

It can be noted that in Case G1 with $p = \frac{1}{3}$, the nonlinear Rossby wave effects appear at $\mathcal{O}(\varepsilon_G^6)$. At this order, the gravity wave terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(4)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(1)}, F_{Gk}^{(3)}]$ correspond to wavenumbers 0, $\pm 2k_G$, $\pm 4k_G$, $\pm 6k_G$, $\pm(k_R \pm k_G)$, $\pm(k_R \pm 3k_G)$ and the gravity terms $\mathcal{N}_{Gjj}[F_{Gj}^{(2)}, F_{Gk}^{(2)}]$ correspond to wavenumbers 0, $\pm 2k_G$, $\pm 4k_G$, $\pm 6k_G$. The Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(2)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$, $\pm(k_R \pm 3k_G)$ and the Rossby wave terms $\mathcal{L}_{Rjj}[F_{Rj}^{(1)}]$ and

$\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(0)}]$ correspond to wavenumbers $0 \pm 2k_R$.

At $\mathcal{O}(\varepsilon_G^6)$, the zero wavenumber components resulting from the nonlinear interactions of the Rossby waves and the nonlinear interactions of the gravity waves affect the mean flow. If $k_R = k_G$, then we have also a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity waves affecting the mean flow. The wavenumbers at each order in Case G1 with $p = \frac{1}{3}$ are summarized in Table 3.2.

Table 3.2: Wavenumbers corresponding to Case G1 ($\varepsilon_G = \varepsilon_R^{\frac{1}{3}}$)

Order	Wavenumbers
ε_G	$\pm k_G$
ε_G^2	$0, \pm 2k_G$
ε_G^3	$\pm k_G, \pm 3k_G, \pm k_R$
ε_G^4	$0, \pm 2k_G, \pm 4k_G, \pm(k_R \pm k_G)$
ε_G^5	$\pm k_G, \pm 3k_G, \pm 5k_G, \pm(k_R \pm 2k_G) \pm k_R$
ε_G^6	$0, \pm 2k_G, \pm 4k_G, \pm 6k_G, \pm 2k_R, \pm(k_R \pm k_G), \pm(k_R \pm 3k_G)$

3.2.3 Case R2: $\varepsilon_R \gg \varepsilon_R^2 = \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2$

In Case R2, we have $\varepsilon_G = \varepsilon_R^2$, and use (3.2.1) to obtain

$$\begin{aligned} \varepsilon_R \mathcal{L}_{Rjj}[F_{Rj}^{(i)}] + \varepsilon_R^2 \mathcal{N}_{Rjj}[F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \varepsilon_R^2 \mathcal{L}_{Gjj}[F_{Gj}^{(i)}] + \varepsilon_R^3 \mathcal{N}_{RGjj}[F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_R^4 \mathcal{N}_{Gjj}[F_{Gj}^{(i)}, F_{Gk}^{(i)}] = 0. \end{aligned} \quad (3.2.8)$$

Thus, we have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_R) : \quad \mathcal{L}_{Rjj}[F_{Rj}^{(0)}] &= 0, \\ \mathcal{O}(\varepsilon_R^2) : \quad \mathcal{L}_{Rjj}[F_{Rj}^{(1)}] + \mathcal{L}_{Gjj}[F_{Gj}^{(0)}] &= -\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^3) : \quad \mathcal{L}_{Rjj}[F_{Rj}^{(2)}] &= -\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(1)}] - \mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}] \end{aligned} \quad (3.2.9)$$

$$\begin{aligned} \mathcal{O}(\varepsilon_R^4) : \quad \mathcal{L}_{Rjj}[F_{Rj}^{(3)}] + \mathcal{L}_{Gjj}[F_{Gj}^{(1)}] = & -\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(2)}] - \mathcal{N}_{Rjj}[F_{Rj}^{(1)}, F_{Rk}^{(1)}] \\ & - \mathcal{N}_{RGjj}[F_{Rj}^{(1)}, F_{Gk}^{(0)}] - \mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}] \end{aligned}$$

At the leading order $\mathcal{O}(\varepsilon_R)$, we can study the linear Rossby problem only and we have $\mathcal{L}_{Rjj}[F_{Rj}^{(0)}] = 0$, where $F_{Rj}^{(0)}$ is a linear solution corresponding to the modes $\pm k_R$ specified at the forced boundary as in Case R1.

At $\mathcal{O}(\varepsilon_R^2)$, we can study the nonlinear Rossby wave problem and the linear gravity wave problem separately, whereas in Case R1 we can study the nonlinear Rossby wave problem only at $\mathcal{O}(\varepsilon_R^2)$. At this order, the Rossby terms $\mathcal{L}_{Rjj}[F_{Rj}^{(1)}]$ and $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(0)}]$ correspond to wavenumbers 0 and $\pm 2k_R$, and the gravity terms $\mathcal{L}_{Gjj}[F_{Gj}^{(0)}] = 0$ correspond to the modes $\pm k_G$ specified at the forced boundary. The nonlinear interactions of the Rossby waves generate a zero wavenumber component affecting the mean flow at this order.

At $\mathcal{O}(\varepsilon_R^3)$, the nonlinear Rossby terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(1)}]$ correspond to wavenumbers $\pm k_R, \pm 3k_R$ and the nonlinear Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$. If $k_R = k_G$, then we have a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity wave interactions affecting the mean flow whilst at $\mathcal{O}(\varepsilon_R^3)$ in Case R1, there are no interactions between the waves to affect the mean flow.

At $\mathcal{O}(\varepsilon_R^4)$, the nonlinear gravity wave effects appear whereas they appear at $\mathcal{O}(\varepsilon_R^6)$ in Case R1. The Rossby terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(2)}]$ correspond to wavenumbers 0, $\pm 2k_R, \pm 4k_R, \pm(2k_R \pm k_G), \pm k_G$ and the Rossby terms $\mathcal{N}_{Rjj}[F_{Rj}^{(1)}, F_{Rk}^{(1)}]$ correspond to wavenumbers 0, $\pm 2k_R, \pm 4k_R$.

The Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(1)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm k_G, \pm(2k_R \pm k_G)$. The gravity wave terms $\mathcal{L}_{Gjj}[F_{Gj}^{(1)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers 0 $\pm 2k_G$. The zero wavenumber components resulting from the nonlinear interactions of the Rossby waves and the nonlinear interactions of the gravity waves affect the mean flow. The wavenumbers at each order in Case R2 are summarized in

Table 3.3.

Table 3.3: Wavenumbers corresponding to Case R2 ($\varepsilon_G = \varepsilon_R^2$)

Order	Wavenumbers
ε_R	$\pm k_R$
ε_R^2	$0, \pm 2k_R, \pm k_G$
ε_R^3	$\pm k_R, \pm 3k_R, \pm(k_R \pm k_G)$
ε_R^4	$0, \pm 2k_R, \pm 4k_R, \pm k_G, \pm 2k_G \pm(2k_R \pm k_G)$

3.2.4 Case G2: $\varepsilon_G \gg \varepsilon_G^2 = \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2$

In Case G2, we have $\varepsilon_G = \varepsilon_R^{\frac{1}{2}}$, and use (3.1.3) to get

$$\begin{aligned} \varepsilon_G \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_G^2 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] + \varepsilon_G^2 \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_G^3 \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_G^4 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] = 0. \end{aligned} \quad (3.2.10)$$

We have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_G) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] &= 0, \\ \mathcal{O}(\varepsilon_G^2) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(1)}] + \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] &= -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_G^3) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(2)}] &= -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(1)}] - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_G^4) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(3)}] + \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] &= -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(2)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(1)}, F_{Gk}^{(1)}] \\ &\quad - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(1)}] - \mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}] \end{aligned} \quad (3.2.11)$$

At the leading order $\mathcal{O}(\varepsilon_G)$, we can study the linear gravity wave problem only, and we have $\mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = 0$, where $F_{Gj}^{(0)}$ is a linear solution as in chapter 4 corresponding to the modes $\pm k_G$ specified at the forced boundary as in Case G1.

At $\mathcal{O}(\varepsilon_G^2)$, we can study the nonlinear gravity wave problem and the linear Rossby wave problem while in Case G1 we can only study the nonlinear gravity wave problem at $\mathcal{O}(\varepsilon_G^2)$. At $\mathcal{O}(\varepsilon_G^2)$, the gravity terms $\mathcal{L}_{Gjj}[F_{Gj}^{(1)}]$ and $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers 0 and $\pm 2k_G$, and the Rossby terms $\mathcal{L}_{Rjj}[F_{Rj}^{(0)}] = 0$ correspond to the modes $\pm k_R$ specified at the forced boundary. The nonlinear interactions of the gravity waves generate a zero wavenumber component affecting the mean flow at this order.

At $\mathcal{O}(\varepsilon_G^3)$, the nonlinear gravity terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(1)}]$ correspond to wavenumbers $\pm k_G$, $\pm 3k_G$ and the nonlinear Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$. If $k_R = k_G$, then we have a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity wave interactions affecting the mean flow whilst at $\mathcal{O}(\varepsilon_G^3)$ in Case G1, there are no interactions between the waves to affect the mean flow.

At $\mathcal{O}(\varepsilon_G^4)$, the nonlinear Rossby wave effects appear whereas they appear at $\mathcal{O}(\varepsilon_G^6)$ in Case G1. The nonlinear gravity terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(2)}]$ correspond to wavenumbers 0, $\pm 2k_G$, $\pm 4k_G$, $\pm(k_R \pm 2k_G)$, $\pm k_R$ and the nonlinear gravity terms $\mathcal{N}_{Gjj}[F_{Gj}^{(1)}, F_{Gk}^{(1)}]$ correspond to wavenumbers 0, $\pm 2k_G$, $\pm 4k_G$.

The nonlinear Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(1)}]$ correspond to wavenumbers $\pm k_R$, $\pm(k_R \pm 2k_G)$, and the Rossby terms $\mathcal{L}_{Rjj}[F_{Rj}^{(1)}]$ and $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(0)}]$ correspond to wavenumbers 0 $\pm 2k_R$. The zero wavenumber components resulting from the nonlinear interactions of the Rossby waves and the nonlinear interactions of the gravity waves affect the mean flow. The wavenumbers at each order in Case G2 are summarized in Table 3.4.

Table 3.4: Wavenumbers corresponding to Case G2 ($\varepsilon_G = \varepsilon_R^{\frac{1}{2}}$)

Order	Wavenumbers
ε_G	$\pm k_G$
ε_G^2	$0, \pm 2k_G, \pm k_R$
ε_G^3	$\pm k_G, \pm 3k_G, \pm(k_R \pm k_G)$
ε_G^4	$0, \pm 2k_G, \pm 4k_G, \pm k_R, \pm 2k_R \pm(k_R \pm 2k_G)$

3.2.5 Case R3: $\varepsilon_R \gg \varepsilon_G \gg \varepsilon_R^2 \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2$

In Case R3, we have $\varepsilon_G = \varepsilon_R^p$, where $1 < p < 2$, and use (3.1.3) to obtain

$$\begin{aligned} \varepsilon_R \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_R^p \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_R^2 \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \varepsilon_R^{p+1} \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_R^{2p} \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] = 0. \end{aligned} \quad (3.2.12)$$

Therefore, in Case R3, we have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_R) : \quad \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] &= 0, \\ \mathcal{O}(\varepsilon_R^p) : \quad \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] &= 0 \\ \mathcal{O}(\varepsilon_R^2) : \quad \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] &= -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}] \\ \mathcal{O}(\varepsilon_R^{p+1}) : \quad \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] &= 0. \end{aligned} \quad (3.2.13)$$

As we can see that at $\mathcal{O}(\varepsilon_R^{p+1})$ in Case R3, where $1 < p < 2$, we do not have linear terms to match with at this order. As a result, Case R3 does not work for asymptotic analysis since there is not enough separations of scales.

3.2.6 Case G3: $\varepsilon_G \gg \varepsilon_R \gg \varepsilon_G^2 \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2$

In Case G3, we have $\varepsilon_G = \varepsilon_R^p$, where $\frac{1}{2} < p < 1$ and use (3.1.3) to obtain

$$\begin{aligned} \varepsilon_G \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] + \varepsilon_G^{\frac{1}{p}} \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \varepsilon_G^2 \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] + \varepsilon_G^{\frac{p+1}{p}} \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \\ + \varepsilon_G^{\frac{2}{p}} \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] = 0. \end{aligned} \quad (3.2.14)$$

We have at the following orders

$$\begin{aligned} \mathcal{O}(\varepsilon_G) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = 0, \\ \mathcal{O}(\varepsilon_G^{\frac{1}{p}}) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] = 0 \\ \mathcal{O}(\varepsilon_G^2) : \quad & \mathcal{L}_{Gjj} [F_{Gj}^{(1)}] = -\mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(0)}] \\ \mathcal{O}(\varepsilon_G^{\frac{p+1}{p}}) : \quad & \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] = 0. \end{aligned} \quad (3.2.15)$$

At $\mathcal{O}(\varepsilon_G^{\frac{p+1}{p}})$ in Case G3, where $\frac{1}{2} < p < 1$, we do not have linear terms to match with at this order. Thus, Case G3 does not work for asymptotic analysis since there is not enough separations of scales.

3.2.7 Case RG: $\varepsilon_R = \varepsilon_G \gg \varepsilon_R^2 = \varepsilon_R \varepsilon_G = \varepsilon_G^2$

In Case RG, we have $\varepsilon_G = \varepsilon_R$, so we let $\varepsilon_G = \varepsilon_R = \varepsilon$ in (3.1.3) to obtain

$$\varepsilon \left\{ \mathcal{L}_{Rjj} [F_{Rj}^{(i)}] + \mathcal{L}_{Gjj} [F_{Gj}^{(i)}] \right\} + \varepsilon^2 \left\{ \mathcal{N}_{Rjj} [F_{Rj}^{(i)}, F_{Rk}^{(i)}] + \mathcal{N}_{Gjj} [F_{Gj}^{(i)}, F_{Gk}^{(i)}] + \mathcal{N}_{RGjj} [F_{Rj}^{(i)}, F_{Gk}^{(i)}] \right\} = 0. \quad (3.2.16)$$

According to (3.2.16), we have at

$$\begin{aligned} \mathcal{O}(\varepsilon) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(0)}] + \mathcal{L}_{Gjj} [F_{Gj}^{(0)}] = 0, \\ \mathcal{O}(\varepsilon^2) : \quad & \mathcal{L}_{Rjj} [F_{Rj}^{(1)}] + \mathcal{L}_{Gjj} [F_{Gj}^{(1)}] = -\mathcal{N}_{Rjj} [F_{Rj}^{(0)}, F_{Rk}^{(0)}] - \mathcal{N}_{Gjj} [F_{Gj}^{(0)}, F_{Gk}^{(0)}] \\ & - \mathcal{N}_{RGjj} [F_{Rj}^{(0)}, F_{Gk}^{(0)}] \end{aligned} \quad (3.2.17)$$

At the leading order $\mathcal{O}(\varepsilon)$, we have $\mathcal{L}_{Rjj}[F_{Rj}^{(0)}] = 0$, where $F_{Rj}^{(0)}$ is a linear solution corresponding to the modes $\pm k_R$ and $\mathcal{L}_{Gjj}[F_{Gj}^{(0)}] = 0$, where $F_{Gj}^{(0)}$ is a linear solution corresponding to the modes $\pm k_G$ specified at the forced boundary.

At $\mathcal{O}(\varepsilon^2)$, the nonlinear Rossby terms $\mathcal{N}_{Rjj}[F_{Rj}^{(0)}, F_{Rk}^{(0)}]$ correspond to wavenumbers $0, \pm 2k_R$. The nonlinear gravity terms $\mathcal{N}_{Gjj}[F_{Gj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $0, \pm 2k_G$. The nonlinear Rossby-gravity terms $\mathcal{N}_{RGjj}[F_{Rj}^{(0)}, F_{Gk}^{(0)}]$ correspond to wavenumbers $\pm(k_R \pm k_G)$.

At $\mathcal{O}(\varepsilon^2)$, the zero wavenumber components resulting from the nonlinear interactions of the Rossby waves and the nonlinear interactions of the gravity waves affect the mean flow. If $k_R = k_G$, then we have also a zero wavenumber component resulting from the nonlinear interactions of the Rossby-gravity waves affects the mean flow as well. The wavenumbers at each order in Case RG are summarized in Table 3.5.

Table 3.5: Wavenumbers corresponding to Case RG ($\varepsilon_G = \varepsilon_R = \varepsilon$)

Order	Wavenumbers
ε	$\pm k_R, \pm k_G$
ε^2	$0, \pm 2k_R, \pm 2k_G, \pm(k_R \pm k_G)$

3.3 Nonlinear equations in streamfunction-vorticity formulation

In the numerical solutions in chapter 5, we solve equations (2.3.4)-(2.3.8) to study Rossby-gravity wave interactions by separating each equation into a part comprising the $\mathcal{O}(\varepsilon_R)$, $\mathcal{O}(\varepsilon_R^2)$ and $\mathcal{O}(\varepsilon_R \varepsilon_G)$ terms and a part comprising the $\mathcal{O}(\varepsilon_G)$, $\mathcal{O}(\varepsilon_G^2)$ terms. We consider $\mathcal{O}(\varepsilon_R \varepsilon_G)$ terms as a part of the $\mathcal{O}(\varepsilon_R)$, $\mathcal{O}(\varepsilon_R^2)$ terms for the reason that we are setting $v_G = 0$ in order to be consistent with the gravity wave problem in the literature.

In subsection 3.3.2, we combine the $\mathcal{O}(\varepsilon_G)$, $\mathcal{O}(\varepsilon_G^2)$ terms in equations (2.3.4)-(2.3.8) into a single equation describing the gravity waves.

3.3.1 Rossby wave and Rossby-gravity wave terms

In this subsection, we consider the $\mathcal{O}(\varepsilon_R)$, $\mathcal{O}(\varepsilon_R^2)$ and $\mathcal{O}(\varepsilon_R \varepsilon_G)$ terms in equations (2.3.4)-(2.3.8) and combine them into a single equation as follows. We consider

$$\varepsilon_R \left[\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_R}{\partial z} \right] = 0, \quad (3.3.1)$$

$$\begin{aligned} \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u_R + \frac{\partial \bar{u}}{\partial y} v_R - f_0 v_a - \beta y v_R \right] + \varepsilon_R^2 \left(u_R \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_R}{\partial y} \right) \\ + \varepsilon_R \varepsilon_G \left(u_R \frac{\partial u_G}{\partial x} + u_G \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_G}{\partial y} \right) = 0, \end{aligned} \quad (3.3.2)$$

$$\begin{aligned} \varepsilon_R \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) v_R + f_0 u_a + \beta y u_R \right] + \delta \varepsilon_R^2 \left(u_R \frac{\partial v_R}{\partial x} + v_R \frac{\partial v_R}{\partial y} \right) \\ + \delta \varepsilon_R \varepsilon_G \left(u_G \frac{\partial v_R}{\partial x} + w_G \frac{\partial v_R}{\partial z} \right) = 0, \end{aligned} \quad (3.3.3)$$

$$\begin{aligned} \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_R + \frac{\partial \bar{T}}{\partial y} v_R + w_R \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \right] \\ + \varepsilon_R^2 \left[u_R \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_R}{\partial y} + w_R \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] \\ + \varepsilon_R \varepsilon_G \left[u_R \frac{\partial T_G}{\partial x} + u_G \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_G}{\partial y} + w_R \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) + w_G \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] = 0 \end{aligned} \quad (3.3.4)$$

We differentiate (3.3.3) with respect to x , differentiate (3.3.2) with respect to y and subtract the former from the latter to get

$$\varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\delta \frac{\partial v_R}{\partial x} - \frac{\partial u_R}{\partial y} \right) + f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} \right) v_R \right]$$

$$\begin{aligned}
& + \varepsilon_R^2 \left[v_R \frac{\partial}{\partial y} \left(\delta \frac{\partial v_R}{\partial x} - \frac{\partial u_R}{\partial y} \right) + u_R \frac{\partial}{\partial x} \left(\delta \frac{\partial v_R}{\partial x} - \frac{\partial u_R}{\partial y} \right) \right] \\
& + \varepsilon_R \varepsilon_G \left[u_G \frac{\partial}{\partial x} \left(\delta \frac{\partial v_R}{\partial x} - \frac{\partial u_R}{\partial y} \right) + \frac{\partial u_G}{\partial x} \left(\delta \frac{\partial v_R}{\partial x} - \frac{\partial u_R}{\partial y} \right) + \delta \frac{\partial}{\partial x} \left(w_G \frac{\partial v_R}{\partial z} \right) \right. \\
& \quad \left. - u_R \frac{\partial^2 u_G}{\partial x \partial y} - v_R \frac{\partial^2 u_G}{\partial y^2} \right] = 0. \quad (3.3.5)
\end{aligned}$$

We make use of the ageostrophic Rossby wave continuity equation (3.3.1) to write the ageostrophic terms as $-f_0 \frac{\partial w_R}{\partial z}$. Since the perturbation of the geostrophic wind is nondivergent (2.3.3), we can define a geostrophic streamfunction perturbation ψ_R by $u_R = -\frac{\partial \psi_R}{\partial y}$ and $v_R = \frac{\partial \psi_R}{\partial x}$. With this definition, ψ_R is related to the potential function Φ_R by $\psi_R = f_0^{-1} \Phi_R$. The gravity wave continuity equation (3.3.14) allows us to define a gravity wave streamfunction perturbation by $u_G = -\frac{\partial \psi_G}{\partial z}$ and $w_G = \frac{\partial \psi_G}{\partial x}$.

Equation (3.3.5) can then be written as

$$\begin{aligned}
& \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi_R - f_0 \frac{\partial w_R}{\partial z} + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} \right) \frac{\partial \psi_R}{\partial x} \right] \\
& + \varepsilon_R^2 \left[\frac{\partial \psi_R}{\partial x} \nabla^2 \frac{\partial \psi_R}{\partial y} - \frac{\partial \psi_R}{\partial y} \nabla^2 \frac{\partial \psi_R}{\partial x} \right] \\
& + \varepsilon_R \varepsilon_G \left[-\frac{\partial \psi_G}{\partial z} \nabla^2 \frac{\partial \psi_R}{\partial x} - \frac{\partial^2 \psi_G}{\partial x \partial z} \nabla^2 \psi_R + \delta \frac{\partial}{\partial x} \left(\frac{\partial \psi_G}{\partial x} \frac{\partial^2 \psi_R}{\partial x \partial z} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial^3 \psi_G}{\partial x \partial y \partial z} \right. \\
& \quad \left. + \frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_G}{\partial y^2 \partial z} \right] = 0, \quad (3.3.6)
\end{aligned}$$

where $\nabla^2 = (\delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the nondimensional Laplacian operator in terms of the horizontal coordinates.

Our goal now is to write the term $-f_0 \frac{\partial w_R}{\partial z}$ in (3.3.6) in terms of ψ_R , ψ_G and T_G as follows. We multiply (3.3.4) by $\frac{g}{T_s}$ to get

$$\begin{aligned}
& \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{g}{T_s} T_R + \frac{g}{T_s} \frac{\partial \bar{T}}{\partial y} v_R + w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \right] \\
& + \varepsilon_R^2 \left[u_R \frac{\partial}{\partial x} \left(\frac{g}{T_s} T_R \right) + v_R \frac{\partial}{\partial y} \left(\frac{g}{T_s} T_R \right) + w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon_R \varepsilon_G \left[u_R \frac{\partial}{\partial x} \left(\frac{g}{T_s} T_G \right) + u_G \frac{\partial}{\partial x} \left(\frac{g}{T_s} T_R \right) + v_R \frac{\partial}{\partial y} \left(\frac{g}{T_s} T_G \right) \right. \\
& \quad \left. + w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) + w_G \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] = 0. \quad (3.3.7)
\end{aligned}$$

The terms $w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right)$ and $w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right)$ in (3.3.7) can be omitted for the reason that the horizontal flow is assumed to be approximately geostrophic and the ratio of the vertical velocity component to the horizontal velocity components is taken to be small as discussed in Appendix A.2. Also, we use the hydrostatic balance of Rossby waves (2.3.7), i.e. $\frac{\partial \Phi_R}{\partial z} = \frac{g}{T_s} T_R$, and the buoyancy frequency squared (A.3.11), i.e. $N^2 \equiv \frac{g}{T_s} \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right)$. Therefore, equation (3.3.7) can be written as

$$\begin{aligned}
& \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \Phi_R}{\partial z} + \frac{\partial}{\partial y} \left(\frac{\partial \bar{\Phi}}{\partial z} \right) v_R + N^2 w_R \right] \\
& + \varepsilon_R^2 \left[u_R \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right) + v_R \frac{\partial}{\partial y} \left(\frac{\partial \Phi_R}{\partial z} \right) \right] \\
& + \varepsilon_R \varepsilon_G \left[u_R \left(\frac{g}{T_s} \frac{\partial T_G}{\partial x} \right) + u_G \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right) + v_R \left(\frac{g}{T_s} \frac{\partial T_G}{\partial y} \right) + w_G \left(\frac{g}{c_p T_s} \frac{\partial \Phi_R}{\partial z} + \frac{\partial^2 \Phi_R}{\partial z^2} \right) \right] = 0. \quad (3.3.8)
\end{aligned}$$

We now differentiate (3.3.8) with respect to z to get

$$\begin{aligned}
& \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \Phi_R}{\partial z^2} + \frac{\partial}{\partial y} \left(\frac{\partial^2 \bar{\Phi}}{\partial z^2} \right) v_R + N^2 \frac{\partial w_R}{\partial z} + \frac{\partial \bar{u}}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \bar{\Phi}}{\partial z} \right) \frac{\partial v_R}{\partial z} \right] \\
& + \varepsilon_R^2 \left[u_R \frac{\partial}{\partial x} \left(\frac{\partial^2 \Phi_R}{\partial z^2} \right) + v_R \frac{\partial}{\partial y} \left(\frac{\partial^2 \Phi_R}{\partial z^2} \right) + \frac{\partial u_R}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right) + \frac{\partial v_R}{\partial z} \frac{\partial}{\partial y} \left(\frac{\partial \Phi_R}{\partial z} \right) \right] \\
& + \varepsilon_R \varepsilon_G \left[u_R \left(\frac{g}{T_s} \frac{\partial^2 T_G}{\partial x \partial z} \right) + \frac{\partial u_R}{\partial z} \left(\frac{g}{T_s} \frac{\partial T_G}{\partial x} \right) + u_G \frac{\partial}{\partial x} \left(\frac{\partial^2 \Phi_R}{\partial z^2} \right) + \frac{\partial u_G}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right) \right. \\
& \quad \left. + v_R \left(\frac{g}{T_s} \frac{\partial^2 T_G}{\partial y \partial z} \right) + \frac{\partial v_R}{\partial z} \left(\frac{g}{T_s} \frac{\partial T_G}{\partial y} \right) + w_G \left(\frac{g}{c_p T_s} \frac{\partial^2 \Phi_R}{\partial z^2} + \frac{\partial^3 \Phi_R}{\partial z^3} \right) \right. \\
& \quad \left. + \frac{\partial w_G}{\partial z} \left(\frac{g}{c_p T_s} \frac{\partial \Phi_R}{\partial z} + \frac{\partial^2 \Phi_R}{\partial z^2} \right) \right] = 0. \quad (3.3.9)
\end{aligned}$$

If we differentiate the perturbation of the geostrophic balance in log-pressure coordinates (2.3.2) with respect to z , then we find that $\frac{\partial}{\partial y} \left(\frac{\partial \bar{\Phi}}{\partial z} \right) \frac{\partial v_R}{\partial z} = -\frac{\partial \bar{u}}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right)$ and

$\frac{\partial}{\partial y} \left(\frac{\partial \Phi_R}{\partial z} \right) \frac{\partial v_R}{\partial z} = -\frac{\partial u_R}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_R}{\partial z} \right)$. Thus, the last two terms in the first line of (3.3.9) and the last two terms in the second line of (3.3.9) can be omitted.

We then substitute $\Phi_R = f_0 \psi_R$ into (3.3.9) and multiply the equation by f_0/N^2 to write the term $-\varepsilon_R f_0 \frac{\partial w_R}{\partial z}$ in terms of ψ_R , ψ_G and T_G as

$$\begin{aligned} -\varepsilon_R f_0 \frac{\partial w_R}{\partial z} &= \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \kappa \frac{\partial^2 \psi_R}{\partial z^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \frac{\partial \psi_R}{\partial x} \right] \\ &+ \varepsilon_R^2 \left[-\frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] \\ &+ \varepsilon_R \varepsilon_G \left[-\frac{f_0}{N^2} \frac{\partial \psi_R}{\partial y} \left(\frac{g}{T_s} \frac{\partial^2 T_G}{\partial x \partial z} \right) - \frac{f_0}{N^2} \frac{\partial^2 \psi_R}{\partial y \partial z} \left(\frac{g}{T_s} \frac{\partial T_G}{\partial x} \right) - \frac{\partial \psi_G}{\partial z} \frac{\partial}{\partial x} \left(\kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right. \\ &\quad - \frac{\partial^2 \psi_G}{\partial z^2} \frac{\partial}{\partial x} \left(\kappa \frac{\partial \psi_R}{\partial z} \right) + \frac{f_0}{N^2} \frac{\partial \psi_R}{\partial x} \left(\frac{g}{T_s} \frac{\partial^2 T_G}{\partial y \partial z} \right) + \frac{f_0}{N^2} \frac{\partial^2 \psi_R}{\partial x \partial z} \left(\frac{g}{T_s} \frac{\partial T_G}{\partial y} \right) \\ &\quad \left. + \frac{\partial \psi_G}{\partial x} \left(\frac{g}{c_p T_s} \frac{\partial^2 \psi_R}{\partial z^2} + \kappa \frac{\partial^3 \psi_R}{\partial z^3} \right) + \frac{\partial^2 \psi_G}{\partial x \partial z} \left(\frac{g}{c_p T_s} \frac{\partial \psi_R}{\partial z} + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right]. \quad (3.3.10) \end{aligned}$$

where $\kappa = \frac{f_0^2}{N^2}$. As defined in section 2.5, $\kappa = \frac{\kappa^*}{\alpha} = (\frac{f_0^*}{N^*})^2/\alpha$, where $\alpha = \frac{L_z^2}{L_y^2}$ is the square of the vertical to meridional aspect ratio. Based on the values given in section 2.5, we consider κ to be an $\mathcal{O}(1)$ parameter in the analysis and numerical computations.

Substituting from (3.3.10) into the second term of (3.3.6) and rearranging the terms leads to

$$\begin{aligned} &\varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} \right] \\ &+ \varepsilon_R^2 \left[\frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] \\ &+ \varepsilon_R \varepsilon_G \left[-\frac{\partial \psi_G}{\partial z} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \frac{\partial^2 \psi_G}{\partial x \partial z} \left(\frac{g}{c_p T_s} \frac{\partial \psi_R}{\partial z} + \kappa \frac{\partial^2 \psi_R}{\partial z^2} - \nabla^2 \psi_R \right) \right. \\ &\quad \left. + \frac{\partial \psi_G}{\partial x} \left(\delta \frac{\partial^3 \psi_R}{\partial x^2 \partial z} + \frac{g}{c_p T_s} \frac{\partial^2 \psi_R}{\partial z^2} + \kappa \frac{\partial^3 \psi_R}{\partial z^3} \right) + \frac{\partial^2 \psi_R}{\partial x \partial z} \left(\delta \frac{\partial^2 \psi_G}{\partial x^2} + \frac{f_0}{N^2} \frac{g}{T_s} \frac{\partial T_G}{\partial y} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial \psi_R}{\partial x} \left(\frac{\partial^3 \psi_G}{\partial y^2 \partial z} + \frac{f_0}{N^2} \frac{g}{T_s} \frac{\partial^2 T_G}{\partial y \partial z} \right) - \frac{\partial \psi_R}{\partial y} \left(\frac{\partial^3 \psi_G}{\partial x \partial y \partial z} + \frac{f_0}{N^2} \frac{g}{T_s} \frac{\partial^2 T_G}{\partial x \partial z} \right) \\
& - \frac{f_0}{N^2} \frac{g}{T_s} \frac{\partial^2 \psi_R}{\partial y \partial z} \frac{\partial T_G}{\partial x} - \kappa \frac{\partial^2 \psi_R}{\partial x \partial z} \frac{\partial^2 \psi_G}{\partial z^2} \Big] = 0.
\end{aligned} \tag{3.3.11}$$

If we set $\varepsilon_G = 0$ in (3.3.11), we obtain

$$\begin{aligned}
\varepsilon_R \Big[& \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} \Big] \\
& + \varepsilon_R^2 \left[\frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] = 0.
\end{aligned} \tag{3.3.12}$$

Equation (3.3.12) is the nonlinear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves. The solution of ψ_R can be used to find u_R and v_R since they are related by $u_R = -\frac{\partial \psi_R}{\partial y}$ and $v_R = \frac{\partial \psi_R}{\partial x}$.

Moreover, we can find T_R from the solution of ψ_R as follows. As presented in the beginning of this subsection, ψ_R is related to the potential function Φ_R by $\psi_R = f_0^{-1} \Phi_R$. Hence, using the hydrostatic balance of Rossby waves (2.3.7), i.e. $\frac{\partial \Phi_R}{\partial z} = \frac{g}{T_s} T_R$, leads to $\Lambda \frac{\partial \psi_R}{\partial Y} = \frac{\partial \psi_R}{\partial z} = f_0^{-1} \frac{\partial \Phi_R}{\partial z} = \frac{g}{f_0 T_s} T_R$. Thus, the Rossby wave temperature perturbation is given by

$$T_R = \frac{f_0 T_s}{g} \frac{\partial \psi_R}{\partial z}. \tag{3.3.13}$$

3.3.2 Gravity wave terms

In this subsection, we consider the $\mathcal{O}(\varepsilon_G)$ and $\mathcal{O}(\varepsilon_G^2)$ terms in equations (2.3.4)-(2.3.8) with $v_G = 0$ to get

$$\varepsilon_G \left[\frac{\partial u_G}{\partial x} + \frac{\partial w_G}{\partial z} \right] = 0, \tag{3.3.14}$$

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u_G + \frac{\partial \bar{u}}{\partial z} w_G + \frac{\partial \Phi_G}{\partial x} \right] + \varepsilon_G^2 \left(u_G \frac{\partial u_G}{\partial x} + w_G \frac{\partial u_G}{\partial z} \right) = 0, \tag{3.3.15}$$

$$\varepsilon_G \left[\frac{\partial \Phi_G}{\partial z} - \frac{g}{T_s} T_G \right] = 0, \tag{3.3.16}$$

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G + w_G \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \right] + \varepsilon_G^2 \left[u_G \frac{\partial T_G}{\partial x} + w_G \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) \right] = 0. \quad (3.3.17)$$

We first differentiate equation (3.3.15) with respect to z , and by making use of the continuity equation (3.3.14), we get

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial u_G}{\partial z} + \frac{\partial^2 \bar{u}}{\partial z^2} w_G + \frac{\partial^2 \Phi_G}{\partial x \partial z} \right] + \varepsilon_G^2 \left(u_G \frac{\partial^2 u_G}{\partial x \partial z} + w_G \frac{\partial^2 u_G}{\partial z^2} \right) = 0.$$

We then differentiate equation (3.3.16) with respect to x to get

$$\frac{\partial^2 \Phi_G}{\partial x \partial z} = \frac{g}{T_s} \frac{\partial T_G}{\partial x}.$$

Now by subtraction and making use of the definition of the gravity wave streamfunction perturbation ψ_G , i.e. $u_G = -\frac{\partial \psi_G}{\partial z}$ and $w_G = \frac{\partial \psi_G}{\partial x}$, we get

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_G}{\partial z^2} - \frac{\partial^2 \bar{u}}{\partial z^2} \frac{\partial \psi_G}{\partial x} - \frac{g}{T_s} \frac{\partial T_G}{\partial x} \right] + \varepsilon_G^2 \left(\frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_G}{\partial z^3} - \frac{\partial \psi_G}{\partial z} \frac{\partial^3 \psi_G}{\partial x \partial z^2} \right) = 0 \quad (3.3.18)$$

and

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G + \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \frac{\partial \psi_G}{\partial x} \right] + \varepsilon_G^2 \left[\frac{\partial \psi_G}{\partial x} \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) - \frac{\partial \psi_G}{\partial z} \frac{\partial T_G}{\partial x} \right] = 0. \quad (3.3.19)$$

Equations (3.3.18) and (3.3.19) are the nonlinear Boussinesq equations for the gravity wave streamfunction ψ_G and T_G .

3.3.3 The nonlinear initial-boundary value problem

In this subsection, we state the nonlinear initial-boundary value problem. As discussed in section 2.4, we consider Rossby waves and gravity waves as perturbations to a background flow with zonal mean speed $\bar{u}(y, z)$. In order to allow asymptotic analysis and to facilitate the numerical discretization described in chapter 5, we consider the special case where $\bar{u}(y, z)$ is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$.

With this choice of $\bar{u}$, equations (3.3.11), (3.3.18) and (3.3.19) can then be written in terms of a new variable Y defined as $Y = \mu y + \Lambda z + b$. In terms of this variable, the

domain is $0 \leq x < 2\pi$ and $-\infty < Y \leq Y_1$ and the waves are forced at the boundary $\mu y + \Lambda z + b = Y = Y_1$.

Therefore, the Rossby-gravity wave equations (3.3.11), (3.3.18) and (3.3.19) can be written in terms of the independent variables x , Y and t as follows

$$\begin{aligned} \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u}(Y) \frac{\partial}{\partial x} \right) \zeta_R + \left(\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right) \frac{\partial \psi_R}{\partial x} \right] + \varepsilon_R^2 \left(\mu \frac{\partial \psi_R}{\partial x} \frac{\partial \zeta_R}{\partial Y} - \mu \frac{\partial \psi_R}{\partial Y} \frac{\partial \zeta_R}{\partial x} \right) \\ + \varepsilon_R \varepsilon_G \left[-\Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial \zeta_R}{\partial x} + \left(\Lambda \frac{\partial^2 \psi_G}{\partial x \partial Y} \right) \left(\frac{\Lambda g \kappa}{c_p T_s} \frac{\partial \psi_R}{\partial Y} + (\Lambda^2 \kappa - \mu^2) \frac{\partial^2 \psi_R}{\partial Y^2} - \delta \frac{\partial^2 \psi_R}{\partial x^2} \right) \right. \\ + \frac{\partial \psi_G}{\partial x} \left(\Lambda \delta \frac{\partial^3 \psi_R}{\partial x^2 \partial Y} + \frac{\Lambda^2 g \kappa}{c_p T_s} \frac{\partial^2 \psi_R}{\partial Y^2} + \Lambda^3 \kappa \frac{\partial^3 \psi_R}{\partial Y^3} \right) + \Lambda \frac{\partial^2 \psi_R}{\partial x \partial Y} \left(\delta \frac{\partial^2 \psi_G}{\partial x^2} + \frac{\mu f_0 g}{N^2 T_s} \frac{\partial T_G}{\partial Y} \right) \\ + \frac{\partial \psi_R}{\partial x} \left(\mu^2 \Lambda \frac{\partial^3 \psi_G}{\partial Y^3} + \frac{\mu \Lambda f_0 g}{N^2 T_s} \frac{\partial^2 T_G}{\partial Y^2} \right) - \mu \frac{\partial \psi_R}{\partial Y} \left(\mu \Lambda \frac{\partial^3 \psi_G}{\partial x \partial Y^2} + \frac{\Lambda f_0 g}{N^2 T_s} \frac{\partial^2 T_G}{\partial x \partial Y} \right) \\ \left. - \frac{\mu \Lambda f_0 g}{N^2 T_s} \frac{\partial^2 \psi_R}{\partial Y^2} \frac{\partial T_G}{\partial x} - \Lambda^3 \kappa \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_G}{\partial Y^2} \right] = 0, \end{aligned} \quad (3.3.20)$$

$$\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} = \zeta_R, \quad (3.3.21)$$

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \zeta_G - \Lambda^2 \frac{\partial^2 \bar{u}}{\partial Y^2} \frac{\partial \psi_G}{\partial x} - \frac{g}{T_s} \frac{\partial T_G}{\partial x} \right] + \varepsilon_G^2 \left(\Lambda \frac{\partial \psi_G}{\partial x} \frac{\partial \zeta_G}{\partial Y} - \Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial \zeta_G}{\partial x} \right) = 0, \quad (3.3.22)$$

$$\Lambda^2 \frac{\partial^2 \psi_G}{\partial Y^2} = \zeta_G, \quad (3.3.23)$$

$$\varepsilon_G \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G + \left(\frac{g \bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) \frac{\partial \psi_G}{\partial x} \right] + \varepsilon_G^2 \left[\frac{\partial \psi_G}{\partial x} \left(\frac{g T_G}{c_p T_s} + \Lambda \frac{\partial T_G}{\partial Y} \right) - \Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial T_G}{\partial x} \right] = 0. \quad (3.3.24)$$

Equations (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) are the numerical model for the Rossby-gravity wave problem. In this thesis, we study these equations in the domain described in section 2.4 and with the parameters given in section 2.5.

The initial conditions are

$$\begin{aligned}
\psi_R(x, Y, 0) &= \psi_G(x, Y, 0) = 0 & \text{for } -\infty < Y < Y_1, \\
\zeta_R(x, Y, 0) &= \zeta_G(x, Y, 0) = 0 & \text{for } -\infty < Y \leq Y_1, \\
T_R(x, Y, 0) &= T_G(x, Y, 0) = 0 & \text{for } -\infty < Y \leq Y_1.
\end{aligned} \tag{3.3.25}$$

On the boundary $\mu y + \Lambda z + b = Y = Y_1$, we impose the boundary conditions

$$\begin{aligned}
\psi_R(x, Y_1, t) &= e^{ik_R x} + \text{c.c.} \\
\psi_G(x, Y_1, t) &= e^{ik_G x} + \text{c.c.} \\
\zeta_R(x, Y_1, t) &= \zeta_G(x, Y_1, t) = T_G(x, Y_1, t) = 0,
\end{aligned} \tag{3.3.26}$$

where c.c. denotes the complex conjugate of the preceding term. The first two conditions in (3.3.26) force Rossby waves and gravity waves that propagate into the domain. Furthermore, we impose the conditions

$$\begin{aligned}
\frac{\partial \zeta_R(x, Y_1, t)}{\partial Y} &= T_R(x, Y_1, t) = \frac{\partial T_R(x, Y_1, t)}{\partial Y} = \frac{\partial^2 T_R(x, Y_1, t)}{\partial Y^2} = 0 \\
\frac{\partial \zeta_G(x, Y_1, t)}{\partial Y} &= \frac{\partial T_G(x, Y_1, t)}{\partial Y} = \frac{\partial^2 T_G(x, Y_1, t)}{\partial Y^2} = 0.
\end{aligned} \tag{3.3.27}$$

The conditions in (3.3.27) ensure that the nonlinear terms in (3.3.20), (3.3.22) and (3.3.24) are zero at the inflow boundary $Y = Y_1$. We impose the boundary conditions on T_R to ensure that some of the nonlinear Rossby-gravity terms in (3.3.20) are zero since T_R is related to ψ_R by (3.3.13).

As $Y \rightarrow -\infty$, we have $y \rightarrow -\infty$ and $z \rightarrow \infty$. At the out flow boundary $Y \rightarrow -\infty$, we consider that there is no incoming wave energy. This means that in deriving the solutions, we only consider solutions corresponding to Rossby waves and gravity waves where the y -component of the group velocity of Rossby waves is negative (southward) (see subsection 4.1.1) and the z -component of the group velocity of gravity waves is positive (upward) (see subsection 4.2.1).

The numerical solutions described in chapter 5 are carried out on a finite domain $Y_2 \leq Y \leq Y_1$ with the waves forced at $Y = Y_1$. The conditions on the wave group

velocity y and z components mean that in the numerical implementation, we require a nonreflecting boundary condition or a radiation condition to prevent wave reflection and instabilities [Nijimbere and Campbell, (2016)].

A time-dependent radiation condition was first produced by Béland and Warn (1975) for a configuration involving the propagation of Rossby waves on a two-dimensional horizontal β -plane. They linearized the governing equations and used a Laplace transform to obtain a boundary condition in terms of a Laplace convolution integral of the time variable. Campbell (2004) generalized it for Rossby wave packets, Campbell and Maslowe (2003) developed a time-dependent nonreflecting boundary condition for gravity waves and gravity wave packets, and Nijimbere and Campbell (2016) extended this to include nonlinear effects for cases where there is no critical level and the waves propagate to the outflow boundary.

With the choice of $\bar{u}(y, z)$ used in our numerical solutions, the Rossby waves and gravity waves both encounter a critical line in the yz plane in the interior of the domain, where $\bar{u}$ is equal to the phase speed of both the waves. At the critical line the waves are absorbed or reflected and their amplitudes are greatly reduced, and hence the wave amplitude is close to zero at the outflow boundary $Y = Y_2$. Consequently, it is sufficient to set zero outflow boundary conditions. This conclusion is consistent with the observations of Campbell (2004) and Campbell and Maslowe (2003).

In the numerical computations described in chapter 5, we set

$$\begin{aligned}\zeta_R(x, Y_2, t) &= \psi_R(x, Y_2, t) = 0 \\ \zeta_G(x, Y_2, t) &= \psi_G(x, Y_2, t) = T_G(x, Y_2, t) = 0.\end{aligned}\tag{3.3.28}$$

Moreover, we impose the conditions

$$T_R(x, Y_2, t) = \frac{\partial T_R(x, Y_2, t)}{\partial Y} = \frac{\partial T_G(x, Y_2, t)}{\partial Y} = 0\tag{3.3.29}$$

The conditions in (3.3.29) ensure that the nonlinear terms in (3.3.20), (3.3.22) and (3.3.24) are zero at the outflow boundary $Y = Y_2$. The form of the wave forcing functions (3.3.26) allows us to consider periodic boundary conditions at $x = 0$ and

$x = 2\pi$. All the boundary conditions in (3.3.26), (3.3.27), (3.3.28) and (3.3.29) are for $0 \leq t < \infty$.

Equations (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) with the initial and boundary conditions (3.3.25), (3.3.26), (3.3.28), (3.3.27) and (3.3.29) give the nonlinear initial boundary-value problem.

In chapter 4, we carry out a weakly nonlinear analysis of this problem to examine the behaviour of the wave-induced mean flow and the wave-induced mean temperature arising from Rossby wave nonlinear interactions, gravity wave nonlinear interactions and the nonlinear interactions between Rossby waves and gravity waves, which all contribute to the wave-induced mean flow and the wave-induced mean temperature.

The leading-order terms in the weakly nonlinear formulation give uncoupled linear equations for the Rossby wave and gravity wave perturbations. These linear equations are obtained by first dividing (3.3.20) by ε_R and dividing (3.3.22) and (3.3.24) by ε_G , then setting ε_R and ε_G both to zero in these equations.

The linear Rossby wave problem is given by the linear terms in (3.3.20) and (3.3.21) with the initial and boundary conditions (3.3.25), (3.3.26) and (3.3.28). The linear gravity wave problem is given by the linear terms in (3.3.22), (3.3.23) and (3.3.24) with the initial and boundary conditions (3.3.25), (3.3.26) and (3.3.28).

When we linearize the equations and look for steady-amplitude normal mode solutions, as we shall see in sections 4.1 and 4.2, we end up with those singular ordinary differential equations and solutions that are discontinuous across singular points. Those singularities are the consequences of having simplified the equations by linearizing and considering steady-amplitude normal mode solutions. However, the original nonlinear equations (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) are not singular. Therefore, when we solve them numerically (or if we could solve them exactly), we would obtain solutions that were continuous with continuous derivatives, up to the third derivative appearing in the nonlinear terms. Thus, the solutions are at least C^3 . In chapter 5, the nonlinear problem is solved numerically.

Chapter 4

Linear solutions and weakly-nonlinear analysis

4.1 Linear baroclinic Rossby wave solutions

The nonlinear terms in equations (3.3.11), (3.3.18) and (3.3.19) give rise to zero wavenumber components that represent a wave-induced mean flow. We can represent these nonlinear effects using weakly-nonlinear analysis. The effects of nonlinear Rossby wave interactions and nonlinear gravity wave interactions are well known. We describe these in sections 4.3 and 4.4. In our formulation, both Rossby waves and gravity waves are present, so there are additional effects from the terms involving products of Rossby wave and gravity wave functions. We derive these in section 4.5.

The first step of the weakly-nonlinear analysis involves solving the linearized equations which appear at the leading order. In the leading-order linear problem, the Rossby wave and gravity wave equations are uncoupled and can be solved separately. In sections 4.1 and 4.2, we review the linear Rossby wave and gravity wave solutions. With appropriate boundary and initial conditions, we can obtain steady-amplitude solutions (subsections 4.1.1 and 4.2.1) or time-dependent solutions (subsections 4.1.2 and 4.2.2). For certain special cases, exact analytical solutions can be obtained; in

general approximate asymptotic solutions can be derived. We discuss some properties of these solutions, including the dispersion relations and definitions of group velocity. This information is required for the weakly-nonlinear analysis (sections 4.3-4.5) and numerical solutions (chapter 5).

4.1.1 Steady-amplitude solutions for baroclinic Rossby waves

The nonlinear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (3.3.12) can be linearized by dividing through first by ε_R and then setting $\varepsilon_R = 0$ to get

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} = 0. \quad (4.1.1)$$

Equation (4.1.1) is the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves.

We are considering the case where the background flow velocity is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$, where a new variable Y defined as $Y = \mu y + \Lambda z + b$ is introduced (as presented in section 3.3) and the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (4.1.1) is written in terms of Y after dividing the first line of (3.3.20) by ε_R and then setting $\varepsilon_R = 0$.

Solution for the case where $\bar{u}(Y) = Y$ and $\delta = 0$

We first consider the special case where $\bar{u}(Y) = Y$ and $\delta = 0$. If the dimensional wavelength of the forced wave L_x is very large relative to the length scale in the meridional and vertical directions, L_y and L_z , then the square of the ratio $\delta = \frac{L_y^2}{L_x^2}$ (defined earlier in section 2.2) will be small and the x -derivative term in the nondimensional Laplacian operator $\nabla^2 = \left(\delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ can be neglected by setting $\delta = 0$.

This is known as the long-wave limit and gives the linear quasi-geostrophic potential vorticity equation under the long-wave approximation

$$(\mu^2 + \kappa\Lambda^2) \left(\frac{\partial}{\partial t} + \bar{u}(Y) \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 \psi_R}{\partial Y^2} \right) + \beta \frac{\partial \psi_R}{\partial x} = 0. \quad (4.1.2)$$

We solve this equation in the domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ with the boundary conditions (3.3.26) and (3.3.28). This equation has the same general form as the corresponding barotropic equation. Replacing Y with y and setting $\mu = 1$ and $\kappa = 0$ gives the linear barotropic vorticity equation under the long-wave approximation. The solutions of the barotropic problem for forced waves are well-known [Dickinson, (1970) and Warn and Warn, (1976)]. We review these solutions here and plot graphs to illustrate them.

Steady-amplitude solutions can be written as

$$\psi_R(x, Y, t) = \hat{\psi}_R(Y) e^{ik_R(x-ct)} + \text{c.c.} \quad (4.1.3)$$

Substituting (4.1.3) into (4.1.2) leads to

$$\frac{\partial^2 \hat{\psi}_R}{\partial Y^2} + \left(\frac{\beta}{(\mu^2 + \kappa\Lambda^2)(\bar{u} - c)} \right) \hat{\psi}_R = 0. \quad (4.1.4)$$

Equation (4.1.4) has a singularity when there is a surface $Y = Y_c$ where the mean velocity is equal to the phase speed of the perturbation, i.e. $\bar{u}(Y_c) = c$. The solution of (4.1.4) with the following boundary conditions

$$\hat{\psi}_R(-\infty) = 0 \quad \text{and} \quad \hat{\psi}_R(Y_1) = 1, \quad (4.1.5)$$

can be expressed in terms of the modified Bessel functions [Abramowitz and Stegun (1964)], as exhibited in the succeeding paragraphs.

The independent variable Y and the dependent variable $\hat{\psi}_R$ can be transformed as follows

$$\tilde{Y} = 2(Y_c - Y)^{\frac{1}{2}} \implies \frac{\tilde{Y}^2}{4} = (Y_c - Y) \quad (4.1.6)$$

and

$$F = (Y_c - Y)^{-\frac{1}{2}} \hat{\psi}_R \implies \hat{\psi}_R = \underbrace{(Y_c - Y)^{\frac{1}{2}}}_{\frac{\tilde{Y}}{2}} F. \quad (4.1.7)$$

In order to obtain first $\frac{\partial \hat{\psi}_R}{\partial Y}$ and then $\frac{\partial^2 \hat{\psi}_R}{\partial Y^2}$, we use the chain rule to get

$$\begin{aligned} \frac{\partial \hat{\psi}_R}{\partial Y} &= \frac{\partial \hat{\psi}_R}{\partial \tilde{Y}} \frac{\partial \tilde{Y}}{\partial Y} = \underbrace{\frac{-2}{\tilde{Y}}}_{\frac{\partial \tilde{Y}}{\partial Y}} \underbrace{\frac{\partial}{\partial \tilde{Y}} \left[\frac{\tilde{Y}}{2} F \right]}_{\hat{\psi}_R} = -F_{\tilde{Y}} - \frac{1}{\tilde{Y}} F, \\ \frac{\partial^2 \hat{\psi}_R}{\partial Y^2} &= \frac{\partial}{\partial \tilde{Y}} \frac{\partial \hat{\psi}_R}{\partial Y} = \frac{-2}{\tilde{Y}} \frac{\partial}{\partial \tilde{Y}} \left[\underbrace{-F_{\tilde{Y}} - \frac{1}{\tilde{Y}} F}_{\frac{\partial \hat{\psi}_R}{\partial Y}} \right] = \frac{2}{\tilde{Y}} F_{\tilde{Y}\tilde{Y}} + \frac{2}{\tilde{Y}^2} F_{\tilde{Y}} - \frac{2}{\tilde{Y}^3} F. \end{aligned} \quad (4.1.8)$$

We substitute from (4.1.7) and (4.1.8) into (4.1.4) to obtain

$$\begin{aligned} -\frac{\tilde{Y}^2}{4} \left[\frac{2}{\tilde{Y}} F_{\tilde{Y}\tilde{Y}} + \frac{2}{\tilde{Y}^2} F_{\tilde{Y}} - \frac{2}{\tilde{Y}^3} F \right] + \frac{\beta}{(\mu^2 + \kappa \Lambda^2)} \frac{\tilde{Y}}{2} F &= 0 \\ \frac{-\tilde{Y}}{2} F_{\tilde{Y}\tilde{Y}} - \frac{1}{2} F_{\tilde{Y}} + \frac{1}{2\tilde{Y}} F + \frac{\beta}{(\mu^2 + \kappa \Lambda^2)} \frac{\tilde{Y}}{2} F &= 0. \end{aligned} \quad (4.1.9)$$

Multiplying (4.1.9) by $-2\tilde{Y}$ leads to

$$\tilde{Y}^2 F_{\tilde{Y}\tilde{Y}} + \tilde{Y} F_{\tilde{Y}} - \left(\frac{\beta}{(\mu^2 + \kappa \Lambda^2)} \tilde{Y}^2 + 1 \right) F = 0. \quad (4.1.10)$$

The solutions of (4.1.10) are the modified Bessel functions of the first kind $\mathcal{I}_1$, and second kind $\mathcal{K}_1$, of order 1 [Abramowitz and Stegun (1964), page 374], i.e.,

$$\hat{\psi}_R(Y) = A (Y_c - Y)^{\frac{1}{2}} \mathcal{I}_1 \left[2 \left(\frac{\beta (Y_c - Y)}{\mu^2 + \kappa \Lambda^2} \right)^{\frac{1}{2}} \right] + B (Y_c - Y)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\beta (Y_c - Y)}{\mu^2 + \kappa \Lambda^2} \right)^{\frac{1}{2}} \right], \quad (4.1.11)$$

where A and B will be determined from the boundary conditions (4.1.5). Applying the boundary conditions (4.1.5) gives A=0 and allows us to determine B. We obtain the solution

$$\psi_R(x, Y, t) = e^{ik_R(x-ct)} \frac{(Y_c - Y)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\beta (Y_c - Y)}{\mu^2 + \kappa \Lambda^2} \right)^{\frac{1}{2}} \right]}{(Y_c - Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\beta (Y_c - Y_1)}{\mu^2 + \kappa \Lambda^2} \right)^{\frac{1}{2}} \right]}. \quad (4.1.12)$$

The solution (4.1.12) is plotted in Figure 4.1 on the domain $0 \leq x \leq 2\pi$ and $-4 \leq Y \leq Y_1 = 4$. The nondimensional parameters in Table 2.1, $\mu = \frac{1}{2}$, $\Lambda = \frac{-1}{2}$ and $b = 2$ are used in all figures with the only difference being that $\delta = 0$ here. The figures are given in yz -plane where $-4 \leq y \leq 8$ and $0 \leq z \leq 12$.

Since $Y = \mu y + \Lambda z + b$, the domain in yz -plane is given by $\left(\frac{\Lambda}{\mu}\right) z - \left(\frac{\Lambda^2}{\mu^3}\right) (b - Y_1) \leq y \leq \left(\frac{-\Lambda}{\mu}\right) z - \left(\frac{1}{\mu}\right) (b - Y_1)$. With these values the domain in yz -plane is given by $-z + 4 \leq y \leq z + 4$ as shown in Figure 4.1(c). Also, we set the phase velocity of the of the perturbation to be zero (i.e. $c = 0$) so that according to (4.1.4) we have a critical surface at $Y_c = 0$.

Figure 4.1(a) shows a surface plot of the perturbation of geostrophic streamfunction $\psi_R(x, Y, t)$, according to (4.1.12). It can be seen that the wave amplitude is decreasing swiftly at $Y = 0$, and for $-4 \leq Y < 0$, the wave amplitude is greatly reduced.

A contour plot of the perturbation of the geostrophic streamfunction is shown in Figure 4.1(b), which is similar to Figure 2 in [Campbell and Maslowe, (1998)]. Wave absorption can be seen at the critical layer, the region around $Y = 0$. In Figure 4.1(c), we can see wave absorption, in terms of y and z , at the critical surface, i.e. $z = y + 4$, corresponding to $Y = 0$.

The amplitude of the perturbation of geostrophic streamfunction $\hat{\psi}_R(Y)$ is displayed in Figure 4.1(d). We see from this figure that the wave amplitude decreases rapidly at $Y = 0$, and south of the critical layer, the wave amplitude is small and decreases in the direction of negative Y . It is so small at the lower boundary that it appears to be zero in the plots.

Figure 4.1: (a) and (b) Surface and contour plots of the geostrophic streamfunction perturbation $\psi_R(x, Y, t)$, respectively. (c) Contour plot of the geostrophic streamfunction perturbation $\psi_R(x, y, z, t)$ in yz -plane with $x = 0$. (d) The amplitude of the perturbation of geostrophic streamfunction.

Solution for the case where $\bar{u}$ is constant

Next we consider the special case where $\bar{u}$ is constant with general δ . In that case we look for solutions for (4.1.1) of the form

$$\psi_R(x, y, z, t) = \phi e^{i(k_R x + l y + m z - \omega t)} + \text{c.c.}, \quad (4.1.13)$$

where ϕ is the wave amplitude. We obtain the following dispersion relation for the wave frequency ω in terms of the wavenumbers k_R , l and m in the x (zonal), y (meridional) and z (vertical) directions

$$\omega = \bar{u} k_R - \frac{\beta k_R}{\delta k_R^2 + l^2 + \kappa m^2}. \quad (4.1.14)$$

The group velocity is the speed and direction of propagation of the wave group and it describes the direction of propagation of energy [Sutherland, (2010)]. It is given by

$$\mathbf{c}_g = \nabla_{\mathbf{k}} \omega,$$

where $\nabla_{\mathbf{k}}$ is the gradient operator that takes derivatives with respect to each components of the wavenumber vector $\mathbf{k}$ with the following components in the x , y and z directions

$$c_{g_x} = \frac{\partial \omega}{\partial k_R}, \quad c_{g_y} = \frac{\partial \omega}{\partial l}, \quad \text{and} \quad c_{g_z} = \frac{\partial \omega}{\partial m}. \quad (4.1.15)$$

The phase velocity gives the speed and direction in which the wave crests move. The one-dimensional phase speed in the x -direction is

$$c_{p_x} = \frac{\omega}{k_R} = \bar{u} - \frac{\beta}{\delta k_R^2 + l^2 + \kappa m^2}. \quad (4.1.16)$$

Since β (defined in section 2.1), $\delta = \frac{L_y^2}{L_x^2}$ and $\kappa = \frac{f_0^2}{N^2}$ are positive numbers, the phase velocity of baroclinic Rossby wave in the x -direction is always westward (travel from east to west) relative to the background flow $\bar{u}$. The zonal phase speed of *baroclinic* Rossby waves (4.1.16) is *slower* than that of the *barotropic* Rossby waves (i.e. $\bar{u} - \frac{\beta}{\delta k_R^2 + l^2}$).

The zonal group velocity of baroclinic Rossby waves is

$$c_{gx} = \frac{\partial \omega}{\partial k_R} = \bar{u} - \frac{\beta (l^2 + \kappa m^2 - \delta k_R^2)}{(\delta k_R^2 + l^2 + \kappa m^2)^2}. \quad (4.1.17)$$

Rossby waves are said to be dispersive since the group velocity and phase velocity are not equal.

The meridional group velocity of baroclinic Rossby waves is

$$c_{gy} = \frac{\partial \omega}{\partial l} = \frac{2\beta l k_R}{(\delta k_R^2 + l^2 + \kappa m^2)^2}. \quad (4.1.18)$$

The sign of c_{gy} depends on the signs of k_R and l since β (defined in section 2.1) is always positive. Thus, if $k_R > 0$, then ($l > 0$) means that the meridional group velocity of baroclinic Rossby waves travel northward (travel from south to north) whereas ($l < 0$) means that the meridional group velocity of baroclinic Rossby waves travel southward (travel from north to south). On the contrary, if $k_R < 0$, then ($l > 0$) and ($l < 0$) means that the meridional group velocity of baroclinic Rossby waves travel southward and northward, respectively.

The vertical group velocity of baroclinic Rossby waves is

$$c_{gz} = \frac{\partial \omega}{\partial m} = \frac{2\beta m k_R}{(\delta k_R^2 + l^2 + \kappa m^2)^2}. \quad (4.1.19)$$

The sign of c_{gz} depends on the signs of k_R and m since β is always positive. Thus, if $k_R > 0$, then ($m > 0$) and ($m < 0$) means that the vertical group velocity of baroclinic Rossby waves travel upward and downward, respectively, and vice versa if $k_R < 0$.

Solution for general $\bar{u}(Y)$ (analytic function) with general δ

We now consider the case where $\bar{u}(Y)$ is not necessarily linear and with general δ . Thus, the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (4.1.1) is written in terms of Y as follows

$$\left[\frac{\partial}{\partial t} + \bar{u}(Y) \frac{\partial}{\partial x} \right] \left[\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} \right] + \left[\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right] \frac{\partial \psi_R}{\partial x} = 0. \quad (4.1.20)$$

We solve this equation in the same domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ with the same boundary conditions (3.3.26) and (3.3.28).

Substituting (4.1.3) into (4.1.20) leads to

$$\frac{\partial^2 \hat{\psi}_R}{\partial Y^2} + \frac{1}{(\mu^2 + \kappa \Lambda^2)} \left(\frac{\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2}}{\bar{u} - c} - \delta k_R^2 \right) \hat{\psi}_R = 0. \quad (4.1.21)$$

Equation (4.1.21) is of the same form as the Rayleigh-Kuo equation (C.1.1) which is obtained in the barotropic case [equation (2) in Kuo, (1949)]. The Rayleigh-Kuo equation could be obtained from (4.1.21) by setting $\mu = 1$, $\kappa = 0$, and replacing Y with y . Equation (4.1.21) has a singularity when there is a surface $Y = Y_c$ where the mean velocity is equal to the phase speed of the perturbation, i.e. $\bar{u}(Y_c) = c$.

The solutions of (4.1.21), valid near the singular surface $Y = Y_c$ can be obtained by the method of Frobenius and take the same form as the solutions obtained in the barotropic case [Lin, (1955)], are

$$\begin{aligned} \hat{\psi}_{R_1}(Y) &= \frac{1}{(\mu^2 + \kappa \Lambda^2)} \left[(Y - Y_c) - \left(\frac{\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}_c}{\partial Y^2}}{2 \frac{\partial \bar{u}_c}{\partial Y}} \right) (Y - Y_c)^2 \right] + \dots, \\ \hat{\psi}_{R_2}(Y) &= \frac{1}{(\mu^2 + \kappa \Lambda^2)} \left[1 - \left(\frac{\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}_c}{\partial Y^2}}{\frac{\partial \bar{u}_c}{\partial Y}} \right) \hat{\psi}_{R_1} \log(Y - Y_c) \right] + \dots, \end{aligned} \quad (4.1.22)$$

where the subscript c denotes evaluation at Y_c . The general solution can be written as a linear combination $\hat{\psi}_R = a \hat{\psi}_{R_1} + b \hat{\psi}_{R_2}$ and the constants a and b determined from the boundary conditions. The solution (4.1.22) is not valid at the critical line $Y = Y_c$ since at that line the zonal component of the perturbation velocity, i.e. $u_R = -\frac{\partial \psi_R}{\partial y} = -\mu \frac{\partial \psi_R}{\partial Y}$, has a logarithmic singularity.

The logarithmic term $\log(Y - Y_c)$ in (4.1.22) can be written as

$$\log(Y - Y_c) = \log|Y - Y_c| + i \arg(Y - Y_c), \quad (4.1.23)$$

where $\arg(Y - Y_c) = 0$ for $Y > Y_c$ and for $Y < Y_c$, $\arg(Y - Y_c) = \pm\pi$. The correct sign of the argument of $(Y - Y_c)$ when $Y < Y_c$ is determined by considering the time-dependent problem where the wave amplitude $\hat{\psi}_R(Y)$ is varying with time (see the next subsection 4.1.2).

The general solution $\hat{\psi}_R(Y)$ is given by

$$\hat{\psi}_R(Y) = \begin{cases} a^+ \hat{\psi}_{R_1}(Y) + b^+ \hat{\psi}_{R_2}(Y) & \text{for } Y > Y_c \\ a^- \hat{\psi}_{R_1}(Y) + b^- \hat{\psi}_{R_2}(Y) & \text{for } Y < Y_c \end{cases} \quad (4.1.24)$$

The singularity at $Y = Y_c$ can be overcome by two well-known approaches. One approach, taken by Lin (1955), includes viscosity in the critical layer region to obtain a non-singular critical layer solution. In the viscous theory, the viscous critical layer solution is matched with the inviscid outer solutions (4.1.24) and the relationship between a^+, b^+ and a^-, b^- is found to be

$$b^+ = b^- = b, \quad \text{and} \quad a^+ - a^- = i \frac{\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}_c}{\partial Y^2}}{|\frac{\partial \bar{u}_c}{\partial Y}|} b \theta,$$

where $\theta = \pm\pi$ is the logarithmic phase shift. Lin (1955) found that the correct branch is with the negative sign so the phase change is $\theta = -\pi$. Consequently, the outer solutions (4.1.24) can be written as

$$\hat{\psi}_R(Y) = a^\pm \hat{\psi}_{R_1}(Y) + b \hat{\psi}_{R_2}(Y),$$

where

$$\hat{\psi}_{R_2}(Y) = \frac{1}{(\mu^2 + \kappa \Lambda^2)} \left[1 - \left(\frac{\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}_c}{\partial Y^2}}{\frac{\partial \bar{u}_c}{\partial Y}} \right) \hat{\psi}_{R_1} \log(Y - Y_c) \right] + \dots,$$

with the logarithmic term now defined to be

$$\log(Y - Y_c) = \log|Y - Y_c| + i \theta \operatorname{sgn}\left(\frac{\partial \bar{u}_c}{\partial Y}\right), \quad \text{when } Y < Y_c.$$

The alternative approach, taken by Dickinson (1970) and Warn and Warn (1976), is to introduce time-dependence, i.e., assuming that the solution of equation (4.1.21) takes the form

$$\psi_R(x, Y, t) = \operatorname{Re}\{\hat{\psi}_R(Y, t) e^{ik_R x}\}.$$

They solved the initial-value problem of forced Rossby waves propagating towards a critical layer region by taking a Laplace transform of $\hat{\psi}_R$ and found that most of the

wave energy was absorbed at the critical layer and that, for large time t , the solution outside the critical layer reached a quasi-steady state exhibiting similar characteristics to the viscous solutions found earlier by Lin (1955).

In the inviscid time-dependent analysis and numerical solutions presented in section 4.5 and chapter 5, our linear solutions take this form. The nonlinear solutions behave the same as the linear solutions at early time, but additional features develop at late time, including a wave-induced mean flow component and higher wavenumber components, consistent with what occurs in the barotropic case in Warn and Warn (1978) and Béland (1978).

In subsection 4.1.2, we discuss the linear time-dependent solution of (4.1.1) and in section 4.3, we present a weakly-nonlinear analysis.

4.1.2 Time-dependent linear solution for baroclinic Rossby waves

In this subsection, we consider the initial-boundary value problem comprising the linear equation (4.1.1) in the same domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ for $t \geq 0$ with the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28). We discuss first the case where $\bar{u}(Y) = Y$, $\delta = \frac{L_y^2}{L_x^2} = 0$ and $\kappa = \beta = 1$.

In this case, we can find a solution of the form

$$\psi_R(x, Y, t) = \hat{\psi}_R(Y, t) e^{ik_R x} + \text{c.c.}, \quad (4.1.25)$$

This gives an equation for the amplitude function $\hat{\psi}_R(Y, t)$. We solve for $\hat{\psi}_R(Y, t)$ by using a Laplace transform. This is described in section B.1. We obtain the solution

$$\psi_R(x, Y, t) \sim e^{ik_R x} \left\{ h_1(Y) + h_2(Y) \frac{e^{-ik_R Y t}}{k_R^2 Y t^2} + h_3(Y) \frac{e^{-ik_R Y_1 t}}{k_R^2 Y_1 t^2} \right\}, \quad (4.1.26)$$

where

$$h_1(Y) = \frac{(-Y)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{(-Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}, \quad (4.1.27)$$

$$h_2(Y) = \frac{-1}{2(Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y - Y_1)^{\frac{1}{2}} \right]}, \quad (4.1.28)$$

and

$$h_3(Y) = \frac{2(Y_1 - Y)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y_1 - Y)^{\frac{1}{2}} \right]}{(\mu^2 + \Lambda^2)}, \quad (4.1.29)$$

The time dependent terms are both of order $1/t^2$ for $t \gg 1$, so as $t \rightarrow \infty$ these two terms go to zero. Setting $\mu = 1$, $\Lambda = 0$ and $b = 0$ in $Y = \mu y + \Lambda z + b$ and letting $Y_1 = y_1$, then (4.1.26) leads exactly to Warn and Warn (1976) and Campbell and Maslowe (1998) solutions.

In the case where $\bar{u}(Y)$ is not necessarily linear (analytic function) and for general $\delta > 0, \delta \ll 1$, the linear time-dependent amplitude solution of (4.1.1) is

$$\begin{aligned} \psi_R(x, Y, t) &= \hat{\psi}_R(Y, t) e^{ik_R x} + \text{c.c.} \\ &\sim e^{ik_R x} \left\{ h_1(Y) e^{-ik_R c t} + h_2(Y) \frac{e^{-ik_R(\bar{u}(Y)-c)t}}{k_R^2 (\bar{u}(Y) - c) t^2} + h_3(Y) \frac{e^{-ik_R(\bar{u}(Y_1)-c)t}}{k_R^2 (\bar{u}(Y_1) - c) t^2} \right\} + \text{c.c.} \end{aligned} \quad (4.1.30)$$

It can be noted that for general $\bar{u}(Y)$ (analytic function),

$\bar{u}(Y) \sim c + \frac{\partial \bar{u}(Y_c)}{\partial Y} (Y - Y_c) + \mathcal{O}(Y - Y_c)^2$, so there is a logarithmic singularity in the $h_1(Y)$ term arising from the modified Bessel function $\mathcal{K}_1$ after expanding in powers of $(Y - Y_c)$; therefore, $h_1(Y) \sim (Y - Y_c) \log(Y - Y_c)$ for $|Y - Y_c|t \ll 1$. Thus, there are singularities in the log term in $h_1(Y)$ and in the $(Y - Y_c)^{-1}$ factor in the $h_2(Y)$ term indicating a discontinuity from $Y > Y_c$ to $Y < Y_c$ and a phase change as

$$(Y - Y_c) \log(Y - Y_c) = \begin{cases} |Y - Y_c| \log |Y - Y_c| & \text{for } Y > Y_c \\ |Y - Y_c| \log |Y - Y_c| + i\theta & \text{for } Y < Y_c \end{cases} \quad (4.1.31)$$

where $\theta = \pm\pi$ is the logarithmic phase shift. We can determine the phase shift of the steady solution or in other words, the correct sign of the argument of $(Y - Y_c)$ when $Y < Y_c$ as follows.

It can be observed that the steady solution $h_1(Y)$ corresponds to the pole $s = 0$. Considering the limit $s \rightarrow 0$ leads to $(Y - \frac{is}{k_R})\log(Y - \frac{is}{k_R}) \rightarrow (Y - Y_c)\log(Y - Y_c)$ which can be written as follows

$$\left|Y - \frac{is}{k_R}\right| \log\left|Y - \frac{is}{k_R}\right| + i \arg\left(Y - \frac{is}{k_R}\right) \rightarrow |Y - Y_c| \log|Y - Y_c| + i \arg(Y - Y_c).$$

It can be noted that $\text{Re}(s) > 0$ along the contour of integration $(\iota - i\infty, \iota + i\infty)$ since $\iota > 0$ as shown in Figure B.1. Consequently, the imaginary part of $(Y - \frac{is}{k_R})$ is less than zero meaning that

$$-\pi < \arg\left(Y - \frac{is}{k_R}\right) < 0.$$

Thus, we can conclude that as $s \rightarrow 0$, $\arg(Y - Y_c) \rightarrow -\pi$ for $Y < Y_c$ and the phase shift of the steady solution is $\theta = -\pi$.

In the simple linear barotropic (2D) Rossby wave problem where the initial mean flow $\bar{u}(y)$ is a function of y only, the linear time-dependent amplitude solution is

$$\begin{aligned} \psi_R(x, y, t) &= \hat{\psi}_R(y, t) e^{ik_R x} + \text{c.c.} \\ &\sim e^{ik_R x} \left\{ h_1(y) e^{-ik_R c t} + h_2(y) \frac{e^{-ik_R(\bar{u}(y)-c)t}}{k_R^2 (\bar{u}(y) - c) t^2} + h_3(y) \frac{e^{-ik_R(\bar{u}(y_1)-c)t}}{k_R^2 (\bar{u}(y_1) - c) t^2} \right\} + \text{c.c.} \end{aligned} \quad (4.1.32)$$

which is similar to the solution (4.1.30) with the differences being that we set $\mu = 1$, $\Lambda = 0$ and $b = 0$ in $Y = \mu y + \Lambda z + b$ so that the variable $Y = \mu y + \Lambda z + b$ is reduced to the variable y and the barotropic Rossby wave propagation takes place on a horizontal plane. This gives the linear barotropic problem originally studied analytically by Dickinson (1970) and Warn and Warn (1976). The steady part term with $h_1(y)$ has a term $(y - y_c) \log(y - y_c)$, so there is a discontinuity from $y > y_c$ to $y < y_c$ and a phase change as follows

$$(y - y_c) \log(y - y_c) = \begin{cases} |y - y_c| \log|y - y_c| & \text{for } y > y_c \\ |y - y_c| \log|y - y_c| + i\theta & \text{for } y < y_c \end{cases} \quad (4.1.33)$$

where $\theta = \pm\pi$ is the logarithmic phase shift. Lin (1955) found that the correct branch is with the negative sign so the phase change is $\theta = -\pi$.

4.2 Linear gravity wave solutions

4.2.1 Steady-amplitude solution for hydrostatic gravity waves

The nonlinear model for gravity waves (3.3.18) and (3.3.19) can be linearized by dividing through first by ε_G and then setting $\varepsilon_G = 0$ to get

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_G}{\partial z^2} - \frac{\partial^2\bar{u}}{\partial z^2}\frac{\partial\psi_G}{\partial x} - \frac{g}{T_s}\frac{\partial T_G}{\partial x} = 0, \quad (4.2.1)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)T_G + \left(\frac{g}{c_p T_s}\bar{T} + \frac{\partial\bar{T}}{\partial z}\right)\frac{\partial\psi_G}{\partial x} = 0. \quad (4.2.2)$$

Equations (4.2.1) and (4.2.2) are the linear equations describing the propagation of gravity waves.

For mathematical analysis, it is convenient to combine these two equations in one equation. In order to do so, we apply the operator $\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)$ to (4.2.1) and $\frac{g}{T_s}\frac{\partial}{\partial x}$ to (4.2.2) to obtain respectively

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2\psi_G}{\partial z^2} - \frac{\partial^2\bar{u}}{\partial z^2}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial\psi_G}{\partial x} - \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{g}{T_s}\frac{\partial T_G}{\partial x} = 0, \quad (4.2.3)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{g}{T_s}\frac{\partial T_G}{\partial x} + N^2\frac{\partial^2\psi_G}{\partial x^2} = 0, \quad (4.2.4)$$

where $N^2 = \frac{g}{T_s}\left(\frac{g}{c_p T_s}\bar{T} + \frac{\partial\bar{T}}{\partial z}\right)$ is the buoyancy frequency squared as presented in (A.3.11).

Substituting from (4.2.4) into the last term of (4.2.3) leads to the linear stream-function perturbation equation that describes the propagation of gravity waves

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2\psi_G}{\partial z^2} - \frac{\partial^2\bar{u}}{\partial z^2}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial\psi_G}{\partial x} + N^2\frac{\partial^2\psi_G}{\partial x^2} = 0. \quad (4.2.5)$$

Solution for the case where $\bar{u}(Y) = Y$

We first consider the special case where $\bar{u}(Y) = Y$. As we did in subsection 4.1.1, we are considering the case where the background flow velocity is a function of the

linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$, where a new variable Y defined as $Y = \mu y + \Lambda z + b$ is introduced.

Equation (4.2.5) is written in terms of Y as follows

$$\Lambda^2 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \frac{\partial^2 \psi_G}{\partial Y^2} + N^2 \frac{\partial^2 \psi_G}{\partial x^2} = 0. \quad (4.2.6)$$

We solve this equation in the domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ with the boundary conditions (3.3.26) and (3.3.28). In this section, we derive steady-amplitude solutions of the form

$$\psi_G(x, Y, t) = \hat{\psi}_G(Y) e^{ik_G(x-ct)} + \text{c.c.} \quad (4.2.7)$$

Substituting (4.2.7) into (4.2.6) leads to

$$\frac{\partial^2 \hat{\psi}_G}{\partial Y^2} + \frac{N^2}{\Lambda^2 (\bar{u} - c)^2} \hat{\psi}_G = 0. \quad (4.2.8)$$

with the boundary conditions

$$\hat{\psi}_G(-\infty) = 0 \quad \text{and} \quad \hat{\psi}_G(Y_1) = 1. \quad (4.2.9)$$

Equation (4.2.8) is of the same form as the Taylor-Goldstein equation [Miles, (1961); Howard (1961)] with z replaces by Y and for the special case of hydrostatic background flow. Equation (4.2.8) has a singularity when there is a surface $Y = Y_c$ where the mean velocity is equal to the phase speed of the perturbation, i.e. $\bar{u}(Y_c) = c$. For the reason that $\bar{u}(Y) = Y$ is a linear function, we can expand $\bar{u}(Y)$ in powers of $(Y - Y_c)$ and find that $(\bar{u} - c) = (Y - Y_c)$. Consequently, (4.2.8), under this assumption, simplifies to

$$\frac{\partial^2 \hat{\psi}_G}{\partial Y^2} + \frac{N^2}{\Lambda^2 (Y - Y_c)^2} \hat{\psi}_G = 0. \quad (4.2.10)$$

Equation (4.2.10) known as an Euler-Cauchy equation. We now look for solutions to (4.2.10) in the form

$$\hat{\psi}_G(Y) = (Y - Y_c)^\alpha. \quad (4.2.11)$$

Substituting (4.2.11) into (4.2.10) leads to

$$\Lambda^2 (\alpha(\alpha - 1)) + N^2 = 0,$$

and solving for α gives the distinct roots

$$\alpha_{1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4N^2}{\Lambda^2}} = \frac{1}{2} \pm i\tilde{\alpha},$$

where $\tilde{\alpha} = \sqrt{R_i - \frac{1}{4}} = \sqrt{\frac{N^2}{\Lambda^2} - \frac{1}{4}}$ and $R_i = \frac{N^2}{(\Lambda \frac{\partial \bar{u}}{\partial Y})^2}$ is the Richardson number that gives a measure of the relative importance of the stratification measured by

$N^2 = \frac{g}{T_s} \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right)$ and the vertical shear measured by $\Lambda \frac{\partial \bar{u}}{\partial Y}$. A condition on stability is that $R_i \geq \frac{1}{4}$ [Miles, (1961); Howard (1961)]. This criterion is known as the Miles-Howard theorem (see section C.3).

The solution to the Euler-Cauchy equation (4.2.10) is given by

$$\hat{\psi}_G(Y) = c_1 (Y - Y_c)^{\frac{1}{2}+i\sqrt{R_i-\frac{1}{4}}} + c_2 (Y - Y_c)^{\frac{1}{2}-i\sqrt{R_i-\frac{1}{4}}}. \quad (4.2.12)$$

where c_1 and c_2 will be determined from the boundary conditions (4.2.9). Applying the boundary conditions (4.2.9) gives $c_2 = 0$ and allows us to determine c_1 . We obtain the solution

$$\psi_G(x, Y, t) = e^{ik_G(x-ct)} \frac{(Y - Y_c)^{\frac{1}{2}+i\sqrt{R_i-\frac{1}{4}}}}{(Y_1 - Y_c)^{\frac{1}{2}+i\sqrt{R_i-\frac{1}{4}}}}. \quad (4.2.13)$$

The solution (4.2.13) is plotted in Figure 4.2 on the domain $0 \leq x \leq 2\pi$ and $-4 \leq Y \leq Y_1 = 4$. As in section 4.1, the nondimensional parameters in Table 2.1, $\mu = \frac{1}{2}$, $\Lambda = \frac{-1}{2}$ and $b = 2$ are used in all figures. With these values the Richardson number $R_i = \frac{N^2}{(\Lambda \frac{\partial \bar{u}}{\partial Y})^2} = 8.14$.

As in section 4.1, the domain in yz -plane is given by $-z + 4 \leq y \leq z + 4$ as shown in Figure 4.2(c). Also, we set the phase velocity of the of the perturbation to be zero (i.e. $c = 0$) so that according to (4.2.8) we have a critical surface at $Y_c = 0$.

Figure 4.2(a) shows a surface plot of the streamfunction steady perturbation $\psi_G(x, Y, t)$, according to (4.2.13). It can be seen that the wave amplitude is decreasing swiftly at $Y = 0$, and for $-4 \leq Y < 0$, the wave amplitude is greatly reduced.

A contour plot of the streamfunction steady perturbation $\psi_G(x, Y, t)$ is shown in Figure 4.2(b). Wave absorption can be seen at the critical layer, the region around $Y = 0$.

In Figure 4.2(c), we can see wave absorption, in terms of y and z , at the critical surface, i.e. $z = y + 4$, corresponding to $Y = 0$.

The amplitude of the streamfunction steady perturbation $\hat{\psi}_G(Y)$ is displayed in Figure 4.2(d). We see from this figure that the wave amplitude is small and decreases in the direction of negative Y . It is so small at the lower boundary that it appears to be zero in the plots.

Figure 4.2: (a) and (b) Surface and contour plots of the geostrophic streamfunction perturbation $\psi_G(x, Y, t)$, respectively. (c) Contour plot of the geostrophic streamfunction perturbation $\psi_G(x, y, z, t)$ in yz -plane with $x = \pi$. (d) The amplitude of the perturbation of geostrophic streamfunction.

Solution for the case where $\bar{u}$ is constant

Next we consider the special case where $\bar{u}$ is constant. In that case, equation (4.2.5) can be written as

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \frac{\partial^2 \psi_G}{\partial z^2} + N^2 \frac{\partial^2 \psi_G}{\partial x^2} = 0, \quad (4.2.14)$$

and we look for solutions for (4.2.14) of the form

$$\psi_G(x, y, z, t) = \phi e^{i(k_G x + l y + m z - \omega t)} + \text{c.c.} \quad (4.2.15)$$

where ϕ is the wave amplitude. We obtain the following dispersion relation for the gravity wave frequency ω in terms of the wavenumbers k_G and m in the x (zonal) and z (vertical) directions

$$\omega = \bar{u} k_G \pm \frac{N k_G}{m}. \quad (4.2.16)$$

The zonal and vertical group velocities of gravity waves are, respectively,

$$c_{g_x} = \frac{\partial \omega}{\partial k_G} = \bar{u} \pm \frac{N}{m}, \quad (4.2.17)$$

$$c_{g_z} = \frac{\partial \omega}{\partial m} = \mp \frac{N k_G}{m^2}. \quad (4.2.18)$$

The phase velocity in x -direction is defined as

$$c_{p_x} = \frac{\omega}{k_G} = \bar{u} \pm \frac{N}{m}. \quad (4.2.19)$$

The z -component of the group velocity c_{g_z} (4.2.18) and the phase speed in x -direction $c_{p_x} = c$ (4.2.19) can be combined as

$$(\bar{u} - c)^3 = \mp \frac{N^3}{m^3} = \frac{N^2 c_{g_z}}{k_G m}.$$

Thus, we have

$$c_{g_z} = \frac{k_G m (\bar{u} - c)^3}{N^2}.$$

Now, we define m so that $\text{sgn}(m) = \text{sgn}(\bar{u} - c)$. If we consider the case where the zonal wavenumber $k_G > 0$, then c_{g_z} is positive regardless of the sign of m and $(\bar{u} - c)$ and

the solution with the wavenumber $(+m)$ means that the vertical group velocity of gravity waves travel towards the upward.

However, the group velocity corresponding to the wavenumber $(-m)$ is given by

$$c_{gz} = \frac{\partial \omega}{\partial(-m)} = -\frac{k_G m (\bar{u} - c)^3}{N^2},$$

so c_{gz} is negative which means that the solution with the wavenumber $(-m)$ travel towards the downward.

4.2.2 Time-dependent linear solution for hydrostatic gravity waves

In this section, we consider the initial-boundary value problem comprising the linear equation (4.2.5) in the same domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ for $t \geq 0$ with the boundary conditions (3.3.26) and (3.3.28) and initial conditions (3.3.25). We discuss first the case where $\bar{u}(Y) = Y$.

In this case, we can find a solution of the form

$$\psi_G(x, Y, t) = \hat{\psi}_G(Y, t) e^{ik_G x} + \text{c.c.}, \quad (4.2.20)$$

This gives an equation for the amplitude function $\hat{\psi}_G(Y, t)$. We solve for $\hat{\psi}_G(Y, t)$ by using a Laplace transform. This is described in section B.2. Following Booker and Bretherton (1967), we obtain the solution

$$\begin{aligned} \psi_G(x, Y, t) = e^{ik_G x} \left\{ g_1(Y) - e^{-ik_G Y_1 t} \left[\frac{g_3(Y)}{t^{\frac{1}{2}-i\tilde{\alpha}}} + \frac{g_5(Y)}{t^{\frac{3}{2}-i\tilde{\alpha}}} + \mathcal{O}(t^{-5/2}) \right] \right. \\ \left. - e^{-ik_G Y t} \left[\frac{g_2(Y)}{t^{\frac{3}{2}+i\tilde{\alpha}}} Y + \frac{g_4(Y)}{t^{\frac{5}{2}+i\tilde{\alpha}}} Y^2 + \mathcal{O}(t^{-7/2}) \right] \right\} + \text{c.c.} \end{aligned} \quad (4.2.21)$$

where

$$g_1(Y) = \left(\frac{Y}{Y_1} \right)^{1/2+i\tilde{\alpha}} \quad (4.2.22)$$

$$g_2(Y) = -\frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[3/2 + i\tilde{\alpha}]}{2i\pi(i(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}} k_G^{\frac{3}{2}+i\tilde{\alpha}}} \quad (4.2.23)$$

$$\begin{aligned}
g_3(Y) &= \frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[1/2 - i\tilde{\alpha}] (i(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}}}{2i\pi Y_1 k_G^{\frac{1}{2}-i\tilde{\alpha}}} \\
g_4(Y) &= -\frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[5/2 + i\tilde{\alpha}] [(-1/2 + i\tilde{\alpha})Y + Y_1]}{2i\pi(i(Y - Y_1))^{\frac{3}{2}+i\tilde{\alpha}} k_G^{\frac{5}{2}+i\tilde{\alpha}}} \\
g_5(Y) &= \frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[3/2 - i\tilde{\alpha}] (i(Y_1 - Y))^{-\frac{1}{2}+i\tilde{\alpha}} [(1/2 - i\tilde{\alpha})Y_1 - Y]}{2i\pi Y_1 k_G^{\frac{3}{2}-i\tilde{\alpha}}}.
\end{aligned} \tag{4.2.24}$$

This solution consists of a steady term that arises from the residue at the pole $s = 0$ and time-dependent terms which arise from the branch points at $s = -ik_G Y$ and $s = -ik_G Y_1$. The residue at the pole $s = 0$ gives the steady part of the solution which corresponds to the steady solution that was obtained in subsection 4.2.1.

As $t \rightarrow \infty$, the time-dependent terms approach zero and only the steady-state solution remains. The time-dependent terms on the second line of equation (4.2.21) are singular at $Y = 0$. Thus, this solution is valid only in an outer region away from the critical level $Y = 0$.

Based on the time-dependent solution for ψ_G (4.2.21), the time-dependent solution for $T_G(x, Y, t)$ can be found. The solution (4.2.21) can be written as

$$\psi_G(x, Y, t) = e^{ik_G x} \left[g_1(Y) + \psi_{G1}(Y, t) e^{-ik_G Y_1 t} + \psi_{G2}(Y, t) e^{-ik_G Y t} \right] + \text{c.c.}, \tag{4.2.25}$$

where

$$\psi_{G1}(Y, t) = - \left[\frac{g_3(Y)}{t^{\frac{1}{2}-i\tilde{\alpha}}} + \frac{g_5(Y)}{t^{\frac{3}{2}-i\tilde{\alpha}}} + \mathcal{O}(t^{-5/2}) \right] \tag{4.2.26}$$

$$\psi_{G2}(Y, t) = - \left[\frac{g_2(Y)}{t^{\frac{3}{2}+i\tilde{\alpha}} Y} + \frac{g_4(Y)}{t^{\frac{5}{2}+i\tilde{\alpha}} Y^2} + \mathcal{O}(t^{-7/2}) \right] \tag{4.2.27}$$

This implies that we can find a solution of the form

$$T_G(x, Y, t) = e^{ik_G x} \left[T_{G\text{steady}}(Y) + T_{G1}(Y, t) e^{-ik_G Y_1 t} + T_{G2}(Y, t) e^{-ik_G Y t} \right] + \text{c.c.}, \tag{4.2.28}$$

where $T_{G\text{steady}}(Y)$ is the steady term, $T_{G1}(Y, t)$ and $T_{G2}(Y, t)$ can be found individually using the integrating factor method. In terms of Y , equation (4.2.2) can be written

as

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right) T_G + \left(\frac{g\bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y}\right) \frac{\partial \psi_G}{\partial x} = 0. \quad (4.2.29)$$

If we substitute (4.2.25) and (4.2.28) into (4.2.29), we can find at leading order

$$\begin{aligned} T_G(x, Y, t) \sim e^{ik_G x} & \left\{ T_{Gsteady}(Y) - \frac{1}{Y} \left(\frac{g\bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) e^{-ik_G Y t} \left[\frac{g_3(Y)}{t^{\frac{1}{2}-i\tilde{\alpha}}} + \frac{g_5(Y)}{t^{\frac{3}{2}-i\tilde{\alpha}}} \right] \right. \\ & \left. + ik_G \left(\frac{g\bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) e^{-ik_G Y t} \left[\frac{g_2(Y)}{(\frac{1}{2} + i\tilde{\alpha}) t^{\frac{1}{2}+i\tilde{\alpha}} Y} + \frac{g_4(Y)}{(\frac{3}{2} + i\tilde{\alpha}) t^{\frac{3}{2}+i\tilde{\alpha}} Y^2} \right] \right\} + \text{c.c.}, \end{aligned} \quad (4.2.30)$$

where

$$T_{Gsteady}(Y) = \frac{1}{Y} \left(\frac{g\bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) g_1(Y). \quad (4.2.31)$$

In the case where $\bar{u}(Y)$ is not necessarily linear (analytic function), the linear time-dependent amplitude solution of (4.2.5) is

$$\psi_G(x, Y, t) \sim e^{ik_G x} \left\{ g_1(Y) e^{-ik_G c t} + g_2(Y) \frac{e^{-ik_G(\bar{u}(Y)-c)t}}{(\bar{u}(Y)-c) t^{\frac{3}{2}+i\tilde{\alpha}}} + g_3(Y) \frac{e^{-ik_G(\bar{u}(Y_1)-c)t}}{(\bar{u}(Y_1)-c) t^{\frac{1}{2}-i\tilde{\alpha}}} \right\} + \text{c.c.} \quad (4.2.32)$$

For general $\bar{u}(Y)$ (analytic function), $\bar{u}(Y) \sim c + \frac{\partial \bar{u}(Y_c)}{\partial Y} (Y - Y_c) + \mathcal{O}(Y - Y_c)^2$, so $\frac{\partial \bar{u}}{\partial Y}$ is an asymptotic approximation and the first term in a Frobenius series solution is $g_1(Y) \sim (Y - Y_c)^{1/2+i\tilde{\alpha}} \sim e^{(1/2+i\tilde{\alpha}) \log(Y-Y_c)}$. Thus, there are singularities in the log factor in the exponent of $g_1(Y)$ and in the $(Y - Y_c)^{-1}$ factor in the $g_2(Y)$ term.

4.3 Weakly-nonlinear analysis of baroclinic Rossby waves

In this section, we carry out a weakly-nonlinear analysis of the nonlinear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (3.3.12), i.e.

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} \\ & + \varepsilon_R \left[\frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] = 0, \end{aligned}$$

As before, we consider an initial zonal mean background flow with velocity $\bar{u}(\mu y + \Lambda z + b) = \bar{u}(Y)$ where $\bar{u}$ and ψ_R are analytic (infinitely differentiable) functions of Y . We can then rewrite (3.3.12) in terms of x, Y and t as

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} \right] + \left[\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right] \frac{\partial \psi_R}{\partial x} \\ & + \varepsilon_R \left\{ \mu \frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial Y} \left[\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} \right] \right. \\ & \quad \left. - \mu \frac{\partial \psi_R}{\partial Y} \frac{\partial}{\partial x} \left[\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} \right] \right\} = 0. \quad (4.3.1) \end{aligned}$$

We can solve this equation in the domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ for $t \geq 0$ with the initial conditions (3.3.25) and the boundary conditions (3.3.26), (3.3.28), (3.3.27) and (3.3.29).

Solutions of the linearized equation obtained by setting $\varepsilon_R = 0$, as discussed in section 4.1, with the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28), the linear solutions are proportional to $e^{\pm i k_R x}$.

The nonlinear terms describe the interaction between waves with positive wavenumbers $(+k_R)$ and negative wavenumbers $(-k_R)$. These products generate waves with higher wavenumbers $(\pm 2k_R, \pm 3k_R, \dots)$ and a zero wavenumber component which represents changes in the mean flow. This change is referred to as the wave-induced mean flow.

Consequently, the mean flow varies with time due to the nonlinear interaction. In this section, we discuss the wave-induced mean flow. This gives us insight into the numerical results in chapter 5.

The zonal mean is defined as the x -average over a zonal wavelength $\lambda_x = 2\pi/k_R$

and is denoted with an overbar as follows:

$$\overline{(\dots)} = \frac{k_R}{2\pi} \int_0^{\frac{2\pi}{k_R}} (\dots) dx. \quad (4.3.2)$$

Solutions of (4.3.1) can be written as perturbation series in powers of the non-linear parameter ε_R as

$$\psi_R(x, Y, t) \sim \psi_R^{(0)}(x, Y, t) + \varepsilon_R \psi_R^{(1)}(x, Y, t) + \mathcal{O}(\varepsilon_R^2). \quad (4.3.3)$$

We substitute (4.3.3) into the nonlinear quasi-geostrophic potential vorticity equation (4.3.1) and group terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon_R)$.

At $\mathcal{O}(1)$, we obtain the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (4.1.1) which can be written in terms of x, Y and t as

$$\left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\delta \frac{\partial^2 \psi_R^{(0)}}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} \right] + \left[\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right] \frac{\partial \psi_R^{(0)}}{\partial x} = 0. \quad (4.3.4)$$

The solutions satisfying the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28) is

$$\psi_R^{(0)}(x, Y, t) = \hat{\psi}_R(Y, t) e^{ik_R x} + \hat{\psi}_R^*(Y, t) e^{-ik_R x}, \quad (4.3.5)$$

as discussed in section 4.1.

At $\mathcal{O}(\varepsilon_R)$, we find that $\psi_R^{(1)}$ satisfies the equation

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\delta \frac{\partial^2 \psi_R^{(1)}}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R^{(1)}}{\partial Y^2} \right] + \left[\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right] \frac{\partial \psi_R^{(1)}}{\partial x} \\ &= - \left\{ \mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial}{\partial Y} \left[\delta \frac{\partial^2 \psi_R^{(0)}}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} \right] - \mu \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial}{\partial x} \left[\delta \frac{\partial^2 \psi_R^{(0)}}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} \right] \right\}, \end{aligned} \quad (4.3.6)$$

with the initial and boundary conditions (3.3.25), (3.3.26), (3.3.28), (3.3.27) and (3.3.29). If we substitute (4.3.5) into the right hand side of (4.3.6), then we have

$$\left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\delta \frac{\partial^2 \psi_R^{(1)}}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R^{(1)}}{\partial Y^2} \right] + \left[\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right] \frac{\partial \psi_R^{(1)}}{\partial x}$$

$$\begin{aligned}
&= ik_R \left\{ -\mu \delta k_R^2 \left[\hat{\psi}_R \left(\frac{\partial \hat{\psi}_R^*}{\partial Y} - \frac{\partial \hat{\psi}_R^*}{\partial Y} \right) + \hat{\psi}_R^* \left(\frac{\partial \hat{\psi}_R}{\partial Y} - \frac{\partial \hat{\psi}_R}{\partial Y} \right) \right] \right. \\
&\quad \left. + \left[\mu(\mu^2 + \kappa \Lambda^2) \left(\hat{\psi}_R^* \frac{\partial^3 \hat{\psi}_R}{\partial Y^3} - \hat{\psi}_R \frac{\partial^3 \hat{\psi}_R^*}{\partial Y^3} - \frac{\partial^2 \hat{\psi}_R}{\partial Y^2} \frac{\partial \hat{\psi}_R^*}{\partial Y} + \frac{\partial \hat{\psi}_R}{\partial Y} \frac{\partial^2 \hat{\psi}_R^*}{\partial Y^2} \right) \right] \right\} \\
&+ ik_R \left[-\mu \delta k_R^2 \hat{\psi}_R \left(\frac{\partial \hat{\psi}_R}{\partial Y} - \frac{\partial \hat{\psi}_R}{\partial Y} \right) - \mu(\mu^2 + \kappa \Lambda^2) \left(\hat{\psi}_R \frac{\partial^3 \hat{\psi}_R}{\partial Y^3} + \frac{\partial^2 \hat{\psi}_R}{\partial Y^2} \frac{\partial \hat{\psi}_R}{\partial Y} \right) \right] e^{+2ik_R x} \\
&- ik_R \left[-\mu \delta k_R^2 \hat{\psi}_R^* \left(\frac{\partial \hat{\psi}_R^*}{\partial Y} - \frac{\partial \hat{\psi}_R^*}{\partial Y} \right) - \mu(\mu^2 + \kappa \Lambda^2) \left(\hat{\psi}_R^* \frac{\partial^3 \hat{\psi}_R^*}{\partial Y^3} + \frac{\partial^2 \hat{\psi}_R^*}{\partial Y^2} \frac{\partial \hat{\psi}_R^*}{\partial Y} \right) \right] e^{-2ik_R x}.
\end{aligned} \tag{4.3.7}$$

It can be noted that the terms with δ on the right-hand side of (4.3.7) are zero. Equation (4.3.7) is a nonhomogenous equation. The solutions of the corresponding homogeneous equation are zero because of zero initial and boundary conditions (3.3.25), (3.3.26), (3.3.28), (3.3.27) and (3.3.29). Thus, the particular solutions of (4.3.7) can be written as

$$\psi_R^{(1)} = \psi_{0R}^{(1)}(Y, t) + \psi_{2R}^{(1)}(Y, t) e^{2ik_R x} + \text{c.c.} \tag{4.3.8}$$

The wave-induced mean flow velocity is then defined as

$$\bar{u}_{0R}(Y, t) = -\frac{\overline{\partial \psi_R(x, y, z, t)}}{\partial y} \approx -\varepsilon_R \frac{\partial \bar{\psi}_{0R}^{(1)}}{\partial y}. \tag{4.3.9}$$

We substitute (4.3.5) and (4.3.8) into (4.3.6) and apply (4.3.2) to get

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0R}^{(1)}}{\partial Y^2} \right) &= -\mu \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^3}} - \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x} \right) \\
\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0R}^{(1)}}{\partial Y^2} \right) &= -\mu \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^3}} - 2 \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x} + \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x} \right).
\end{aligned} \tag{4.3.10}$$

In order to integrate the second term on the right-hand side of (4.3.10) with respect to x , we perform integration by parts by differentiating $\frac{\partial \psi_R^{(0)}}{\partial Y}$ and integrating $\frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x}$ to get

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0R}^{(1)}}{\partial Y^2} \right) = -\mu \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^3}} + 2 \frac{\partial^2 \psi_R^{(0)}}{\partial x \partial Y} \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} + \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x} \right) \tag{4.3.11}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0R}^{(1)}}{\partial Y^2} \right) = -\mu \frac{\partial^2}{\partial Y^2} \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y}} \right). \quad (4.3.12)$$

Integrating first both sides of (4.3.12) twice with respect to Y , then differentiating with respect to y and using the definition of the wave-induced mean flow velocity (4.3.9), i.e., $\bar{u}_{0R} = -\varepsilon_R \frac{\partial \bar{\psi}_{0R}^{(1)}}{\partial y}$ leads to

$$\frac{\partial \bar{u}_{0R}}{\partial t} = \varepsilon_R \frac{\partial}{\partial y} \left(\overline{\mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y}} \right) \quad (4.3.13)$$

Equation (4.3.13) describes the rate of change of the mean flow velocity with time resulting from the nonlinear interactions of the Rossby waves. We define

$$F_R(Y, t) = \left(\overline{\mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y}} \right) = \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial y}} \right) = -\overline{(u_R^{(0)} v_R^{(0)})} \quad (4.3.14)$$

where $u_R^{(0)}$ and $v_R^{(0)}$ are the leading order components of the Rossby wave velocity. F_R is the leading order meridional component of the zonal mean zonal momentum flux and is also called the Rossby Reynolds stress. In terms of F_R equation (4.3.13) can be written as

$$\frac{\partial \bar{u}_{0R}}{\partial t} = \varepsilon_R \frac{\partial F_R}{\partial y}, \quad (4.3.15)$$

so the change in the mean flow results from the meridional divergence of the momentum flux.

If the initial mean flow velocity $\bar{u}(Y)$ is such that there is a critical line $Y = Y_c$ in the Y domain ($-\infty < Y \leq Y_1$) and the wave amplitude is reduced across the critical line, then there is also a jump in F_R across the critical line. This means that the meridional divergence of the momentum flux $\frac{\partial F_R}{\partial y} = \mu \frac{\partial F_R}{\partial Y}$ is nonzero in the critical layer and there is a (positive or negative) acceleration of the mean flow in the critical layer given by $\frac{\partial \bar{u}_{0R}}{\partial t}$ which results in the wave-induced mean flow $\bar{u}_{0R}$.

Therefore, in the linear baroclinic Rossby wave problem where the forcing is at the upper boundary $Y = Y_1$ and the baroclinic Rossby waves propagate downward, we expect, in the numerical simulations in chapter 5 that if the function F_R increases

across the critical line in the direction of negative Y , then the mean flow acceleration $\frac{\partial \bar{u}_{0R}}{\partial t}$ is negative in the critical layer. On the other hand, if the function F_R decreases across the critical line in the direction of negative Y , then the mean flow acceleration $\frac{\partial \bar{u}_{0R}}{\partial t}$ is positive in the critical layer.

Using the linear time-dependent amplitude solution (4.1.30), the leading order term of F_R is

$$F_R = \left(\mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \right) = ik_R \mu \left[h_1(Y) \frac{\partial h_1^*(Y)}{\partial Y} - h_1^*(Y) \frac{\partial h_1(Y)}{\partial Y} \right],$$

where $h_1^*(Y)$ is the complex conjugate of the function $h_1(Y)$. As discussed in subsection 4.1.2, there is a logarithmic singularity in the function $h_1(Y)$ that is $(Y - Y_c) \log(Y - Y_c)$, so F_R is discontinuous from $Y > Y_c$ to $Y < Y_c$ and a phase change as in (4.1.31).

In the simple linear barotropic (2D) Rossby wave problem where the initial mean flow $\bar{u}$ is constant, the leading order linear study amplitude solution of the leading order equation is

$$\psi_R^{(0)}(x, y, t) = A e^{i(k_R x + l y - \omega t)} + \text{c.c.},$$

where A is a complex constant. In that case, we have

$$F_R = \left(\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial y} \right) = 2 k_R l |A|^2$$

which is constant. More generally, if $\bar{u}$ is locally constant, independent of y on same y interval, then F_R is independent of y on that interval. This is the Rossby wave analogue of Eliassen-Palm theorem for gravity waves [Eliassen and Palm, (1961)] (see section 4.4).

If $\bar{u}$ is a function of y only and there is a critical line in the y domain and the wave amplitude is reduced across the critical line, then there is also a jump in F_R across the critical line. This means that the $\frac{\partial F_R}{\partial y}$ is nonzero in the critical layer and there is a (positive or negative) acceleration of the mean flow in the critical layer given by $\frac{\partial \bar{u}_{0R}}{\partial t}$ which results in the wave-induced mean flow $\bar{u}_{0R}$.

Using the linear time-dependent amplitude solution (4.1.32), the leading order term of F_R is the steady part term with $h_1(y)$ which has a term $(y - y_c) \log(y - y_c)$, so F_R is discontinuous from $y > y_c$ to $y < y_c$ and a phase change as (4.1.33).

In the linear barotropic Rossby wave configuration where the forcing is at the upper boundary of the rectangular domain and the barotropic Rossby waves propagate southward, F_R is negative for $y > y_c$ (y_c critical latitude) and close to zero for $y < y_c$ as shown in Figure 5.3(a), so the mean flow acceleration $\frac{\partial \bar{u}_{0R}}{\partial t}$ is negative in the critical layer as shown in Figure 5.5(a).

We now use (3.3.4) to derive an equation for the wave-induced mean temperature arising from the Rossby effects. The terms $w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right)$ and $w_R \frac{g}{T_s} \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right)$ in (3.3.4) can be omitted since the horizontal flow is assumed to be approximately geostrophic and the ratio of the vertical velocity component to the horizontal velocity components is taken to be small as discussed in section A.2. Thus, equation (3.3.4) can then be written after dividing through by ε_R as

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_R + \frac{\partial \bar{T}}{\partial y} v_R + w_R \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) + \varepsilon_R \left[u_R \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_R}{\partial y} \right] \\ & + \varepsilon_G \left[u_R \frac{\partial T_G}{\partial x} + u_G \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_G}{\partial y} + w_G \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] = 0. \end{aligned} \quad (4.3.16)$$

Setting $\varepsilon_G = 0$, using the technique of writing the solutions as perturbation series in powers of the nonlinear parameter ε_R , taking the zonal mean and group terms using integration by parts leads to the following equation at $\mathcal{O}(\varepsilon_R)$

$$\frac{\partial \bar{T}_{0R}}{\partial t} = -\frac{N^2 T_s}{g} \bar{w}_{0R} - \varepsilon_R \frac{\partial}{\partial y} \left(\overline{v_R^{(0)} T_R^{(0)}} \right). \quad (4.3.17)$$

Equation (4.3.17) describes the rate of change of the wave-induced mean temperature with time resulting from the nonlinear interactions of the Rossby waves. The term $\frac{N^2 T_s}{g} \bar{w}_{0R}$ is the adiabatic cooling term [Holton, (2004)]. We can use (4.3.12) to obtain the left-hand side of (4.3.17) as follows. We first integrate both sides of (4.3.12) with respect to Y , then multiply through by $\frac{f_0 T_s \Lambda}{g}$ and use (3.3.13), i.e. $T_R = \frac{f_0 T_s}{g} \frac{\partial \psi_R}{\partial z}$ which

can be written in terms of Y as $T_R = \frac{f_0 T_s \Lambda}{g} \frac{\partial \psi_R}{\partial Y}$. Thus, $\frac{\partial \bar{T}_{0R}}{\partial t} = -\varepsilon_R \frac{\partial}{\partial y} (\overline{v_R^{(0)} T_R^{(0)}})$. As a result, the adiabatic cooling term $\frac{N^2 T_s}{g} \bar{w}_{0R}$ in (4.3.17) is zero. We have a special case where the divergence of the heat flux gives a change in mean $\bar{T}_{0R}$ with no cooling term since y derivatives are proportional to z derivatives because we have a linear relation between y and z , so y and z derivatives all become Y derivatives. We define

$$H_R(Y, t) = -(\overline{v_R^{(0)} T_R^{(0)}}) \quad (4.3.18)$$

where $T_R^{(0)}$ is the leading order component of the Rossby wave temperature. H_R is the leading order meridional component of the zonal mean Rossby heat flux. The heat flux vector is directed towards regions of lower temperature. In terms of H_R equation (4.3.17) can be written as

$$\frac{\partial \bar{T}_{0R}}{\partial t} = \varepsilon_R \frac{\partial H_R}{\partial y}, \quad (4.3.19)$$

so the change in the wave-induced mean Rossby wave temperature results from the meridional divergence of the Rossby heat flux.

4.4 Weakly-nonlinear analysis for gravity waves

In this section, we carry out a weakly-nonlinear analysis of the nonlinear equations describing the propagation of gravity waves (3.3.18) and (3.3.19), i.e.

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_G}{\partial z^2} - \frac{\partial^2 \bar{u}}{\partial z^2} \frac{\partial \psi_G}{\partial x} - \frac{g}{T_s} \frac{\partial T_G}{\partial x} = \varepsilon_G \left(\frac{\partial \psi_G}{\partial z} \frac{\partial^3 \psi_G}{\partial x \partial z^2} - \frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_G}{\partial z^3} \right)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G + \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \frac{\partial \psi_G}{\partial x} = \varepsilon_G \left[\frac{\partial \psi_G}{\partial z} \frac{\partial T_G}{\partial x} - \frac{\partial \psi_G}{\partial x} \left(\frac{g}{c_p T_s} T_G + \frac{\partial T_G}{\partial z} \right) \right],$$

As before, we consider an initial zonal mean background flow with velocity

$$\bar{u}(\mu y + \Lambda z + b) = \bar{u}(Y) \text{ where } \bar{u}, \psi_G \text{ and } T_G \text{ are analytic (infinitely differentiable) functions of } Y. \text{ We can then rewrite (3.3.18) and (3.3.19) in terms of } x, Y \text{ and } t \text{ as}$$

$$\Lambda^2 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_G}{\partial Y^2} - \Lambda^2 \frac{\partial^2 \bar{u}}{\partial Y^2} \frac{\partial \psi_G}{\partial x} - \frac{g}{T_s} \frac{\partial T_G}{\partial x} = \Lambda^3 \varepsilon_G \left(\frac{\partial \psi_G}{\partial Y} \frac{\partial^3 \psi_G}{\partial x \partial Y^2} - \frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_G}{\partial Y^3} \right) \quad (4.4.1)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right) T_G + \left(\frac{g}{c_p T_s} \bar{T} + \Lambda \frac{\partial \bar{T}}{\partial Y}\right) \frac{\partial \psi_G}{\partial x} = \varepsilon_G \left[\Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial T_G}{\partial x} - \frac{\partial \psi_G}{\partial x} \left(\frac{g}{c_p T_s} T_G + \Lambda \frac{\partial T_G}{\partial Y}\right) \right]. \quad (4.4.2)$$

We can solve this equation in the domain $0 \leq x \leq 2\pi$ and $-\infty < Y \leq Y_1$ for $t \geq 0$ with the initial conditions (3.3.25) and the boundary conditions (3.3.26), (3.3.28), (3.3.27) and (3.3.29).

Solutions of the linearized equations obtained by setting $\varepsilon_G = 0$, as discussed in section 4.2, with the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28), the linear solutions are proportional to $e^{\pm i k_G x}$.

The nonlinear terms describe the interaction between waves with positive wavenumbers ($+k_G$) and negative wavenumbers ($-k_G$). These products generate waves with higher wavenumbers ($\pm 2k_G, \pm 3k_G, \dots$) and a zero wavenumber component which represents changes in the mean flow.

Therefore, the mean flow varies with time due to the nonlinear interaction. In this section, we discuss the wave-induced mean flow. This gives us insight into the numerical results in chapter 5.

The zonal mean is defined as the x -average over a zonal wavelength $\lambda_x = 2\pi/k_G$ and is denoted with an overbar as follows:

$$\overline{(\dots)} = \frac{k_G}{2\pi} \int_0^{\frac{2\pi}{k_G}} (\dots) dx. \quad (4.4.3)$$

Solutions of (4.4.1) and (4.4.2) can be written as perturbation series in powers of the nonlinear parameter ε_G :

$$\psi_G(x, Y, t) \sim \psi_G^{(0)}(x, Y, t) + \varepsilon_G \psi_G^{(1)}(x, Y, t) + \mathcal{O}(\varepsilon_G^2) \quad (4.4.4)$$

and

$$T_G(x, Y, t) \sim T_G^{(0)}(x, Y, t) + \varepsilon_G T_G^{(1)}(x, Y, t) + \mathcal{O}(\varepsilon_G^2). \quad (4.4.5)$$

We substitute (4.4.4) and (4.4.5) into the nonlinear gravity wave equations (4.4.1) and (4.4.2) and group terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon_G)$.

At $\mathcal{O}(1)$, we obtain the linear gravity wave equations (4.2.1) and (4.2.2) which can be written in terms of x, Y and t as

$$\Lambda^2 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_G^{(0)}}{\partial Y^2} - \Lambda^2 \frac{\partial^2 \bar{u}}{\partial Y^2} \frac{\partial \psi_G^{(0)}}{\partial x} - \frac{g}{T_s} \frac{\partial T_G^{(0)}}{\partial x} = 0 \quad (4.4.6)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G^{(0)} + \left(\frac{g}{c_p T_s} \bar{T} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) \frac{\partial \psi_G^{(0)}}{\partial x} = 0. \quad (4.4.7)$$

The solutions satisfying the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28) are

$$\psi_G^{(0)}(x, Y, t) = \hat{\psi}_G(Y, t) e^{ik_G x} + \hat{\psi}_G^*(Y, t) e^{-ik_G x}, \quad (4.4.8)$$

and

$$T_G^{(0)}(x, Y, t) = \hat{T}_G(Y, t) e^{ik_G x} + \hat{T}_G^*(Y, t) e^{-ik_G x}, \quad (4.4.9)$$

as discussed in section 4.2.

At $\mathcal{O}(\varepsilon_G)$, we find that $\psi_G^{(1)}$ and $T_G^{(1)}$ satisfy the equations

$$\Lambda^2 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_G^{(1)}}{\partial Y^2} - \Lambda^2 \frac{\partial^2 \bar{u}}{\partial Y^2} \frac{\partial \psi_G^{(1)}}{\partial x} - \frac{g}{T_s} \frac{\partial T_G^{(1)}}{\partial x} = -\Lambda^3 \left(\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial^3 \psi_G^{(0)}}{\partial Y^3} - \frac{\partial \psi_G^{(0)}}{\partial Y} \frac{\partial^3 \psi_G^{(0)}}{\partial x \partial Y^2} \right) \quad (4.4.10)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_G^{(1)} + \left(\frac{g}{c_p T_s} \bar{T} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) \frac{\partial \psi_G^{(1)}}{\partial x} = \left[\Lambda \frac{\partial \psi_G^{(0)}}{\partial Y} \frac{\partial T_G^{(0)}}{\partial x} - \frac{\partial \psi_G^{(0)}}{\partial x} \left(\frac{g}{c_p T_s} T_G^{(0)} + \Lambda \frac{\partial T_G^{(0)}}{\partial Y} \right) \right]. \quad (4.4.11)$$

As we did in section 4.3, if we substitute (4.4.8) and (4.4.9) into the right-hand side of (4.4.10) and (4.4.11), then the particular solutions of (4.4.10) and (4.4.11) can be written as

$$\psi_G^{(1)} = \psi_{0G}^{(1)}(Y, t) + \psi_{2G}^{(1)}(Y, t) e^{2ik_G x} + \text{c.c.} \quad (4.4.12)$$

and

$$T_G^{(1)} = T_{0G}^{(1)}(Y, t) + T_{2G}^{(1)}(Y, t) e^{2ik_G x} + \text{c.c.}, \quad (4.4.13)$$

and the solutions of the corresponding homogeneous equations are zero because of zero initial and boundary conditions (3.3.25), (3.3.26), (3.3.28), (3.3.27) and (3.3.29).

The wave-induced mean flow velocity resulting from the nonlinear interactions of the gravity waves is then defined as

$$\bar{u}_{0G}(Y, t) = -\overline{\frac{\partial \psi_G(x, y, z, t)}{\partial z}} \approx -\varepsilon_G \frac{\partial \bar{\psi}_{0G}^{(1)}}{\partial z}. \quad (4.4.14)$$

Substituting (4.4.12) and (4.4.13) into (4.4.10) and applying (4.4.3) leads to

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0G}^{(1)}}{\partial Y^2} \right) = -\Lambda \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial^3 \psi_G^{(0)}}{\partial Y^3}} - \frac{\partial \psi_G^{(0)}}{\partial Y} \frac{\partial^3 \psi_G^{(0)}}{\partial Y^2 \partial x} \right). \quad (4.4.15)$$

If we repeat the simplifications and steps we did in equations (4.3.10) and (4.3.11), we then obtain

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_{0G}^{(1)}}{\partial Y^2} \right) = -\Lambda \frac{\partial^2}{\partial Y^2} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial Y}} \right). \quad (4.4.16)$$

Integrating first both sides of (4.4.16) twice with respect to Y , then differentiating with respect to z and using the definition of the wave-induced mean flow velocity resulting from the nonlinear interactions of the gravity waves (4.4.14), i.e.,

$\bar{u}_{0G}(Y, t) = -\varepsilon_G \frac{\partial \bar{\psi}_{0G}^{(1)}}{\partial z}$ leads to

$$\frac{\partial \bar{u}_{0G}}{\partial t} = \varepsilon_G \frac{\partial}{\partial z} \left(\overline{\Lambda \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial Y}} \right). \quad (4.4.17)$$

The left hand side of (4.4.17) is the rate of change of the mean flow with time resulting from the nonlinear interactions of the gravity waves. We define

$$F_G(Y, t) = \left(\overline{\Lambda \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial Y}} \right) = \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial z}} \right) = -\overline{(u_G^{(0)} w_G^{(0)})} \quad (4.4.18)$$

where $u_G^{(0)}$ and $w_G^{(0)}$ are the leading order components of the gravity wave velocity. F_G is the leading order vertical component of the zonal mean zonal momentum flux and is also called the gravity Reynolds stress. In terms of F_G equation (4.4.17) can be written as

$$\frac{\partial \bar{u}_{0G}}{\partial t} = \varepsilon_G \frac{\partial F_G}{\partial z}, \quad (4.4.19)$$

so the change in the mean flow results from the vertical divergence of the momentum flux.

As discussed in section 4.3, the presence of a critical line $Y = Y_c$ means that the function F_G (decreases or increases) across the critical line. The change in F_G means that the vertical divergence of the momentum flux $\frac{\partial F_G}{\partial Y}$ is nonzero around $Y = Y_c$ and there is a (positive or negative) acceleration of the mean flow in the critical layer according to (4.4.19) which results in the wave-induced mean flow $\bar{u}_{0G}$.

In a 2D linear gravity wave problem where the initial mean flow $\bar{u}$ and the buoyancy frequency N^2 are constants, the leading order linear steady-amplitude solution of the leading order equation is

$$\psi_G^{(0)}(x, z, t) = A e^{i(k_G x + m z - \omega t)} + \text{c.c.},$$

where A is a complex constant. In that case, we have

$$F_G = \left(\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \overline{\psi_G^{(0)}}}{\partial z} \right) = 2 k_G m |A|^2$$

which is constant. More generally, if $\bar{u}$ is locally constant, independent of z on same z interval, then F_G is independent of z on that interval. This result is known as the Eliassen-Palm theorem.

The Eliassen-Palm theorem [Eliassen and Palm, (1961)]:

For linear waves with a constant background velocity ($\bar{u}=\text{constant}$), the momentum flux is independent of height.

If $\bar{u}$ is a function of z only and there is a critical line in the z domain and the wave amplitude is reduced across the critical line, then there is also a jump in F_G across the critical line. This means that the $\frac{\partial F_G}{\partial z}$ is nonzero in the critical layer and there is a (positive or negative) acceleration of the mean flow in the critical layer given by $\frac{\partial \bar{u}_{0G}}{\partial t}$ which results in the wave-induced mean flow $\bar{u}_{0G}$.

In the 2D linear gravity wave configuration where the forcing is at the lower boundary of the rectangular domain and the gravity waves propagate upward, the

function F_G is almost constant (negative) for $z < z_c$ (z_c critical altitude) and close to zero for $z > z_c$, so the mean flow acceleration $\frac{\partial \bar{u}_{0G}}{\partial t}$ is positive in the critical layer.

The linear time-dependent amplitude solution of the leading order equation with $\bar{u}$ is a function of z only is

$$\begin{aligned}\psi_G^{(0)}(x, z, t) &= \hat{\psi}_G(z, t) e^{ik_G x} + \text{c.c.} \\ &\sim e^{ik_G(x-ct)} (z - z_c)^{1/2 \pm i\tilde{\alpha}} \sim e^{ik_G(x-ct)} e^{(1/2 \pm i\tilde{\alpha}) \log(z-z_c)}\end{aligned}$$

where $\tilde{\alpha} = (R_i - 1/4)^{1/2}$ as presented in section 4.2. In this case, the leading order term of $F_G = \left(\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial z} \right)$ is discontinuous from $z < z_c$ to $z > z_c$ and a phase change as follows

$$e^{(1/2 \pm i\tilde{\alpha}) \log(z-z_c)} = \begin{cases} e^{(1/2 + i\tilde{\alpha}) \log|z-z_c|} & \text{for } z < z_c \\ e^{(1/2 + i\tilde{\alpha})(\log|z-z_c| + i\theta)} & \text{for } z > z_c \end{cases}$$

Miles (1961) showed that the logarithmic phase shift is $\theta = -\pi$.

Now, we derive the equation for the wave-induced mean temperature arising from the gravity effects as follows. We substitute (4.4.8), (4.4.9), (4.4.12) and (4.4.13) into (4.4.11) and take the x -average of (4.4.11) to get

$$\begin{aligned}\frac{\partial \bar{T}_{0G}}{\partial t} &= \Lambda \varepsilon_G \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial Y} \frac{\partial T_G^{(0)}}{\partial x}} - \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial T_G^{(0)}}{\partial Y} \right) - \varepsilon_G \frac{g}{c_p T_s} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} T_G^{(0)}} \right) \\ \frac{\partial \bar{T}_{0G}}{\partial t} &= \Lambda \varepsilon_G \left(\frac{\partial}{\partial x} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial Y} T_G^{(0)}} \right) - \frac{\partial^2 \psi_G^{(0)}}{\partial Y \partial x} T_G^{(0)} - \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial T_G^{(0)}}{\partial Y} \right) - \varepsilon_G \frac{g}{c_p T_s} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} T_G^{(0)}} \right) \\ \frac{\partial \bar{T}_{0G}}{\partial t} &= -\Lambda \varepsilon_G \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} T_G^{(0)}} \right) - \varepsilon_G \frac{g}{c_p T_s} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} T_G^{(0)}} \right) \tag{4.4.20}\end{aligned}$$

Equation (4.4.20) describes the rate of change of the mean gravity wave temperature with time resulting from the nonlinear interactions of the gravity waves. We define

$$H_G(Y, t) = -\left(\overline{w_G^{(0)} T_G^{(0)}} \right) \tag{4.4.21}$$

where $T_G^{(0)}$ is the leading order component of the gravity wave temperature. H_G is the leading order vertical component of the zonal mean gravity heat flux. In terms

of H_G equation (4.4.20) can be written as

$$\frac{\partial \bar{T}_{0G}}{\partial t} = \varepsilon_G \frac{\partial H_G}{\partial z} - \varepsilon_G \frac{g}{c_p T_s} \overline{(w_G^{(0)} T_G^{(0)})}, \quad (4.4.22)$$

so the change in the mean gravity wave temperature results from the vertical divergence of the gravity heat flux plus the adiabatic lapse rate (cooling term) [Holton, (2004)].

4.5 Combined effects of Rossby waves and gravity waves (Case RG)

In this section, we consider the case where both Rossby waves and gravity waves are present and of the same order of magnitude, e.g. Case RG where $\varepsilon_R = \varepsilon_G = \varepsilon$. In this case with $k_R = k_G$, the momentum flux divergence comprises terms arising from Rossby wave nonlinear interactions, gravity wave nonlinear interactions and Rossby-gravity wave nonlinear interactions which all contribute to the wave-induced mean flow at $\mathcal{O}(\varepsilon)$.

In order to derive an equation for this, we consider equation (3.3.2), i.e.

$$\begin{aligned} \varepsilon_R \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u_R + \frac{\partial \bar{u}}{\partial y} v_R - f_0 v_a - \beta y v_R \right] + \varepsilon_R^2 \left(u_R \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_R}{\partial y} \right) \\ + \varepsilon_R \varepsilon_G \left(u_R \frac{\partial u_G}{\partial x} + u_G \frac{\partial u_R}{\partial x} + v_R \frac{\partial u_G}{\partial y} \right) = 0, \end{aligned}$$

divide by ε_R , rewrite it in terms of the streamfunctions ψ_R, ψ_G and take the zonal mean. We simplify and group terms using integration by parts to get

$$\frac{\partial}{\partial t} \left(-\frac{\partial \bar{\psi}_R}{\partial y} \right) - f_0 \bar{v}_a = \varepsilon_R \frac{\partial}{\partial y} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial y}} \right) + \varepsilon_G \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^2 \psi_G}{\partial y \partial z}} \right). \quad (4.5.1)$$

Differentiating (4.5.1) with respect to y leads to

$$\frac{\partial}{\partial t} \left(-\frac{\partial^2 \bar{\psi}_R}{\partial y^2} \right) - f_0 \frac{\partial \bar{v}_a}{\partial y} = \varepsilon_R \frac{\partial^2}{\partial y^2} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial y}} \right) + \varepsilon_G \frac{\partial}{\partial y} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^2 \psi_G}{\partial y \partial z}} \right). \quad (4.5.2)$$

Taking the zonal mean of the Rossby wave continuity equation, i.e. $\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_R}{\partial z} = 0$, the second term of (4.5.2) can be written as $-f_0 \frac{\partial \bar{v}_a}{\partial y} = f_0 \frac{\partial \bar{w}_R}{\partial z}$. Thus, equation (4.5.2) can be rewritten in terms of x, Y and t as

$$\frac{\partial}{\partial t} \left(-\mu^2 \frac{\partial^2 \bar{\psi}_R}{\partial Y^2} \right) + \Lambda f_0 \frac{\partial \bar{w}_R}{\partial Y} = \mu^3 \varepsilon_R \frac{\partial^2}{\partial Y^2} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial Y}} \right) + \mu^2 \Lambda \varepsilon_G \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^2 \psi_G}{\partial Y^2}} \right) \quad (4.5.3)$$

The term $\Lambda f_0 \frac{\partial \bar{w}_R}{\partial Y}$ can be obtained by taking the zonal mean of equation (3.3.10) to get

$$\begin{aligned} -\Lambda f_0 \frac{\partial \bar{w}_R}{\partial Y} = & \kappa \Lambda^2 \frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_R}{\partial Y^2} \right) + \kappa \mu \Lambda^2 \varepsilon_R \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} - \frac{\partial \psi_R}{\partial Y} \frac{\partial^3 \psi_R}{\partial x \partial Y^2} \right) \\ & + \frac{g \kappa \Lambda^2}{c_p T_s} \varepsilon_G \left(\overline{\frac{\partial \psi_G}{\partial x} \frac{\partial^2 \psi_R}{\partial Y^2}} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial \psi_R}{\partial Y} \right) \\ & + \frac{f_0 g \Lambda \mu}{N^2 T_s} \varepsilon_G \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^2 T_G}{\partial Y^2}} + \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial T_G}{\partial Y} - \frac{\partial \psi_R}{\partial Y} \frac{\partial^2 T_G}{\partial x \partial Y} - \frac{\partial^2 \psi_R}{\partial Y^2} \frac{\partial T_G}{\partial x} \right) \\ & + \kappa \Lambda^3 \varepsilon_G \left(\overline{\frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} - \frac{\partial \psi_G}{\partial Y} \frac{\partial^3 \psi_R}{\partial x \partial Y^2} - \frac{\partial^2 \psi_G}{\partial Y^2} \frac{\partial^2 \psi_R}{\partial x \partial Y} \right). \end{aligned} \quad (4.5.4)$$

We simplify and group the terms on the right-hand side of (4.5.4) using integration by parts. The second term on the right-hand side of (4.5.4) can be written as

$$\begin{aligned} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} - \frac{\partial \psi_R}{\partial Y} \frac{\partial^3 \psi_R}{\partial x \partial Y^2} \right) &= \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} + \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} \right) \\ &= \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} + 2 \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} - \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} \right) \\ &= \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3}} + 2 \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} + \frac{\partial \psi_R}{\partial Y} \frac{\partial^3 \psi_R}{\partial x \partial Y^2} \right) \\ &= \frac{\partial^2}{\partial Y^2} \left(\overline{\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial Y}} \right). \end{aligned} \quad (4.5.5)$$

The third term on the right-hand side of (4.5.4) can be written as

$$\left(\overline{\frac{\partial \psi_G}{\partial x} \frac{\partial^2 \psi_R}{\partial Y^2}} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial \psi_R}{\partial Y} \right) = \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y}} \right). \quad (4.5.6)$$

The fourth term on the right-hand side of (4.5.4) can be written as

$$\begin{aligned}
& \left(\frac{\partial \psi_R}{\partial x} \frac{\partial^2 T_G}{\partial Y^2} + \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial T_G}{\partial Y} - \frac{\partial \psi_R}{\partial Y} \frac{\partial^2 T_G}{\partial x \partial Y} - \frac{\partial^2 \psi_R}{\partial Y^2} \frac{\partial T_G}{\partial x} \right) \\
&= \left(\frac{\partial \psi_R}{\partial x} \frac{\partial^2 T_G}{\partial Y^2} + \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial T_G}{\partial Y} + \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial T_G}{\partial Y} + \frac{\partial^3 \psi_R}{\partial x \partial Y^2} T_G - \frac{\partial}{\partial x} \left(\frac{\partial^2 \psi_R}{\partial Y^2} T_G \right) \right) \\
&= \frac{\partial^2}{\partial Y^2} \left(\frac{\partial \psi_R}{\partial x} T_G \right). \tag{4.5.7}
\end{aligned}$$

The last term on the right-hand side of (4.5.4) can be written as

$$\begin{aligned}
& \left(\frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} - \frac{\partial \psi_G}{\partial Y} \frac{\partial^3 \psi_R}{\partial x \partial Y^2} - \frac{\partial^2 \psi_G}{\partial Y^2} \frac{\partial^2 \psi_R}{\partial x \partial Y} \right) \\
&= \left(\frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} + \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} + \frac{\partial^3 \psi_G}{\partial x \partial Y^2} \frac{\partial \psi_R}{\partial Y} \right) \\
&= \left(\frac{\partial \psi_G}{\partial x} \frac{\partial^3 \psi_R}{\partial Y^3} + 2 \frac{\partial^2 \psi_G}{\partial x \partial Y} \frac{\partial^2 \psi_R}{\partial Y^2} + \frac{\partial^3 \psi_G}{\partial x \partial Y^2} \frac{\partial \psi_R}{\partial Y} \right) \\
&= \frac{\partial^2}{\partial Y^2} \left(\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y} \right). \tag{4.5.8}
\end{aligned}$$

Substituting (4.5.5), (4.5.6), (4.5.7) and (4.5.8) into (4.5.4) and multiplying through by (-1) leads to

$$\begin{aligned}
\Lambda f_0 \frac{\partial \bar{w}_R}{\partial Y} &= -\kappa \Lambda^2 \frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\psi}_R}{\partial Y^2} \right) - \kappa \mu \Lambda^2 \varepsilon_R \frac{\partial^2}{\partial Y^2} \left(\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial Y} \right) - \frac{g \kappa \Lambda^2}{c_p T_s} \varepsilon_G \frac{\partial}{\partial Y} \left(\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y} \right) \\
&\quad - \frac{f_0 g \Lambda \mu}{N^2 T_s} \varepsilon_G \frac{\partial^2}{\partial Y^2} \left(\frac{\partial \psi_R}{\partial x} T_G \right) - \kappa \Lambda^3 \varepsilon_G \frac{\partial^2}{\partial Y^2} \left(\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y} \right). \tag{4.5.9}
\end{aligned}$$

We substitute (4.5.9) into (4.5.3), integrate with respect to Y and multiply through by μ to obtain

$$\begin{aligned}
(\mu^2 + \kappa \Lambda^2) \frac{\partial}{\partial t} \left(-\mu \frac{\partial \bar{\psi}_R}{\partial Y} \right) &= \mu^4 \varepsilon_R \frac{\partial}{\partial Y} \left(\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial Y} \right) + \kappa \mu^2 \Lambda^2 \varepsilon_R \frac{\partial}{\partial Y} \left(\frac{\partial \psi_R}{\partial x} \frac{\partial \psi_R}{\partial Y} \right) \\
&\quad + \mu^3 \Lambda \varepsilon_G \left(\frac{\partial \psi_R}{\partial x} \frac{\partial^2 \psi_G}{\partial Y^2} \right) + \frac{g \mu \kappa \Lambda^2}{c_p T_s} \varepsilon_G \left(\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y} \right)
\end{aligned}$$

$$+ \frac{f_0 g \Lambda \mu^2}{N^2 T_s} \varepsilon_G \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_R}{\partial x} T_G} \right) + \kappa \mu \Lambda^3 \varepsilon_G \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_G}{\partial x} \frac{\partial \psi_R}{\partial Y}} \right). \quad (4.5.10)$$

Since $\bar{u}_{0R} = -\mu \varepsilon_R \frac{\partial \bar{\psi}_R}{\partial Y}$, then equation (4.5.10) can be written in terms of x, y and z as

$$\begin{aligned} \frac{\partial \bar{u}_{0R}}{\partial t} = & -\varepsilon_R^2 \frac{\partial}{\partial y} (\overline{u_R v_R}) - \frac{\mu^2}{(\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \left(\overline{\frac{\partial u_G}{\partial y} v_R} \right) \\ & - \frac{g \kappa \Lambda^2}{c_p T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G (\overline{u_R w_G}) + \frac{f_0 g \mu \Lambda}{N^2 T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \frac{\partial}{\partial y} (\overline{v_R T_G}) \\ & - \frac{\kappa \Lambda^2}{(\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \frac{\partial}{\partial z} (\overline{u_R w_G}). \end{aligned} \quad (4.5.11)$$

As a result, the equation describing the rate of change of the mean flow with time resulting from the nonlinear interactions of the Rossby waves, the nonlinear interactions of the gravity waves and the nonlinear interactions of the Rossby-gravity waves is

$$\begin{aligned} \frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t} = & -\varepsilon_R^2 \frac{\partial}{\partial y} (\overline{u_R v_R}) - \frac{\mu^2}{(\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \left(\overline{\frac{\partial u_G}{\partial y} v_R} \right) \\ & - \frac{g \kappa \Lambda^2}{c_p T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G (\overline{u_R w_G}) + \frac{f_0 g \mu \Lambda}{N^2 T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \frac{\partial}{\partial y} (\overline{v_R T_G}) \\ & - \frac{\kappa \Lambda^2}{(\mu^2 + \kappa \Lambda^2)} \varepsilon_R \varepsilon_G \frac{\partial}{\partial z} (\overline{u_R w_G}) - \varepsilon_G^2 \frac{\partial}{\partial z} (\overline{u_G w_G}). \end{aligned} \quad (4.5.12)$$

In this configuration where both Rossby waves and gravity waves are present, we can represent the solution as a perturbation series

$$\begin{aligned} \varepsilon_R \psi_R + \varepsilon_G \psi_G \sim & \varepsilon_R \psi_R^{(0)}(x, Y, t) + \varepsilon_G \psi_G^{(0)}(x, Y, t) \\ & + \varepsilon_R^2 \psi_R^{(1)}(x, Y, t) + \varepsilon_R \varepsilon_G \psi_{RG}^{(1)}(x, Y, t) + \varepsilon_G^2 \psi_G^{(1)}(x, Y, t). \end{aligned} \quad (4.5.13)$$

We consider Case RG where $\varepsilon_R = \varepsilon_G = \varepsilon$ so that we can rewrite (4.5.13) as

$$\psi_R + \psi_G \sim \psi_R^{(0)} + \psi_G^{(0)} + \varepsilon \psi_R^{(1)} + \varepsilon \psi_{RG}^{(1)} + \varepsilon \psi_G^{(1)}. \quad (4.5.14)$$

Substituting (4.5.14) into (3.3.11) and (4.4.1) gives the linear baroclinic Rossby wave equation (4.3.4) and the linear gravity wave equation (4.4.6) at the leading order. With the initial conditions (3.3.25) and the boundary conditions (3.3.26) and (3.3.28), the linear baroclinic Rossby wave streamfunction perturbation solution is (4.3.5), i.e. $\psi_R^{(0)} = \hat{\psi}_R(Y, t) e^{ik_R x} + \hat{\psi}_R^*(Y, t) e^{-ik_R x}$, the linear gravity wave streamfunction perturbation solution is (4.4.8), i.e., $\psi_G^{(0)} = \hat{\psi}_G(Y, t) e^{ik_G x} + \hat{\psi}_G^*(Y, t) e^{-ik_G x}$, and the linear gravity wave temperature perturbation solution is (4.4.9), i.e.,

$$T_G^{(0)}(x, Y, t) = \hat{T}_G(Y, t) e^{ik_G x} + \hat{T}_G^*(Y, t) e^{-ik_G x}.$$

At $\mathcal{O}(\varepsilon)$, we have equation (4.5.12) describing the rate of change of the mean flow with time resulting from the nonlinear interactions of the Rossby waves, the gravity waves and the Rossby-gravity waves. We are interested in determining the effects of the nonlinear interactions between Rossby waves and gravity waves on the mean flow.

Therefore, we first rewrite (4.5.12) as

$$\begin{aligned} \frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t} = & \varepsilon \mu \frac{\partial}{\partial Y} \left(\mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \right) + \frac{\mu^2}{(\mu^2 + \kappa \Lambda^2)} \varepsilon \left(\mu \Lambda \frac{\partial^2 \psi_G^{(0)}}{\partial Y^2} \frac{\partial \psi_R^{(0)}}{\partial x} \right) \\ & + \frac{g \kappa \Lambda^2}{c_p T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon \left(\mu \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \right) + \frac{f_0 g \mu^2 \Lambda}{N^2 T_s (\mu^2 + \kappa \Lambda^2)} \varepsilon \frac{\partial}{\partial Y} \left(\frac{\partial \psi_R^{(0)}}{\partial x} T_G^{(0)} \right) \\ & + \frac{\kappa \Lambda^3}{(\mu^2 + \kappa \Lambda^2)} \varepsilon \frac{\partial}{\partial Y} \left(\mu \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \right) + \varepsilon \Lambda \frac{\partial}{\partial Y} \left(\Lambda \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial Y} \right). \end{aligned} \quad (4.5.15)$$

This equation describes the rate of change of the total wave-induced mean flow due to the Rossby wave self-interactions, the gravity wave self-interactions and the interactions between the Rossby waves and the gravity waves in Case RG where $k_R = k_G$. As defined before that $F_R = -\overline{(u_R^{(0)} v_R^{(0)})}$ is the meridional component of the zonal mean zonal momentum flux, $X_R = \frac{\partial F_R}{\partial y}$ is the Rossby wave drag force which is given by the first term on the right-hand side. Also, as defined before that $F_G = -\overline{(u_G^{(0)} w_G^{(0)})}$ is the vertical component of the zonal mean zonal momentum flux, $X_G = \frac{\partial F_G}{\partial z}$ is the gravity wave drag force which is given by the last term on the right-hand side.

The second, third, fourth and fifth terms on the right-hand side of (4.5.15) represent the drag force X_{RG} resulting from the nonlinear interactions of the Rossby-gravity waves. Consequently, equation (4.5.15) can be written as

$$\frac{\partial \bar{u}_0}{\partial t} = X_R + X_G + X_{RG}. \quad (4.5.16)$$

If the sum of the drag forces on the right-hand side of (4.5.16) is positive, then the total wave-induced mean flow acceleration becomes more positive or eastward and if the sum is negative, then the total wave-induced mean flow acceleration becomes more negative or westward. If the sum of the drag forces has the same sign as the mean flow, then there is a positive acceleration either in the westward or eastward direction; otherwise, there is a deceleration (negative acceleration).

For large t , we can determine which of these terms is dominant and consequently determine the order of the wave-induced mean flow acceleration. Integrating (4.5.15) with respect to t gives us the order of the wave-induced mean flow velocity. We substitute the linear the solutions of $\psi_R^{(0)}$, $\psi_G^{(0)}$ and $T_G^{(0)}$ into (4.5.15) to find the order in t of each of the parts on the right-hand side as follows.

The linear time-dependent amplitude solution $\psi_R^{(0)}(x, Y, t)$ is given in (4.1.30). Thus, the steady term, the largest time-dependent term of $\psi_R^{(0)}(x, Y, t)$ in (4.1.30) and its partial derivatives with respect to Y are

$$\begin{aligned} \frac{\partial \psi_R^{(0)}}{\partial Y} &\sim \frac{\partial h_1(Y)}{\partial Y} e^{ik_R(x-ct)} - \frac{i h_2(Y) e^{-ik_R(Y-Y_c)t} e^{ik_R x}}{k_R (Y - Y_c) t} \\ \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} &\sim \frac{\partial^2 h_1(Y)}{\partial Y^2} e^{ik_R(x-ct)} + \frac{h_2(Y) e^{-ik_R(Y-Y_c)t} e^{ik_R x}}{(Y - Y_c)} \\ \frac{\partial^3 \psi_R^{(0)}}{\partial Y^3} &\sim \frac{\partial^3 h_1(Y)}{\partial Y^3} e^{ik_R(x-ct)} - \frac{ik_R t h_2(Y) e^{-ik_R(Y-Y_c)t} e^{ik_R x}}{(Y - Y_c)} \end{aligned} \quad (4.5.17)$$

Thus, the Rossby wave drag force is

$$X_R = \frac{\partial F_R}{\partial y} = \varepsilon \mu \frac{\partial}{\partial Y} \left(\mu \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \right)$$

$$\begin{aligned}
& \sim ik_R \mu^2 \varepsilon \frac{\partial}{\partial Y} \left[h_1(Y) \frac{\partial h_1^*(Y)}{\partial Y} - h_1^*(Y) \frac{\partial h_1(Y)}{\partial Y} \right] + \mathcal{O}(t^{-2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{-2}).
\end{aligned} \tag{4.5.18}$$

The linear time-dependent amplitude solution $\psi_G^{(0)}(x, Y, t)$ is given in (4.2.32). Therefore, the steady term, the largest time-dependent term of $\psi_G^{(0)}(x, Y, t)$ and its partial derivatives with respect to Y are

$$\begin{aligned}
\frac{\partial \psi_G^{(0)}}{\partial Y} & \sim \frac{\partial g_1(Y)}{\partial Y} e^{ik_G(x-ct)} - \frac{ik_G g_2(Y) e^{-ik_G(Y-Y_c)t} e^{ik_G x}}{(Y-Y_c) t^{\frac{1}{2}+i\tilde{\alpha}}} \\
\frac{\partial^2 \psi_G^{(0)}}{\partial Y^2} & \sim \frac{\partial^2 g_1(Y)}{\partial Y^2} e^{ik_G(x-ct)} + \frac{t^{\frac{1}{2}-i\tilde{\alpha}} k_G^2 g_2(Y) e^{-ik_G(Y-Y_c)t} e^{ik_G x}}{(Y-Y_c)} \\
\frac{\partial^3 \psi_G^{(0)}}{\partial Y^3} & \sim \frac{\partial^3 g_1(Y)}{\partial Y^3} e^{ik_G(x-ct)} - \frac{i t^{\frac{3}{2}-i\tilde{\alpha}} k_G^3 g_2(Y) e^{-ik_G(Y-Y_c)t} e^{ik_G x}}{(Y-Y_c)}
\end{aligned} \tag{4.5.19}$$

Similarly, the steady term, the largest time-dependent term of $T_G^{(0)}(x, Y, t)$ and its a first partial derivative with respect to Y are

$$\begin{aligned}
\frac{\partial T_G^{(0)}}{\partial Y} & \sim \frac{\partial T_{G\text{steady}}}{\partial Y} e^{ik_G(x-ct)} \\
& + \frac{t^{\frac{1}{2}-i\tilde{\alpha}} \left(\frac{g\bar{T}}{c_p T_s} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) k_G^2 g_2(Y) e^{-ik_G(Y-Y_c)t} e^{ik_G x}}{(\frac{1}{2} + i\tilde{\alpha}) (Y-Y_c)}
\end{aligned} \tag{4.5.20}$$

Therefore, the gravity wave drag force is

$$\begin{aligned}
X_G = \frac{\partial F_G}{\partial z} & = \varepsilon \Lambda \frac{\partial}{\partial Y} \left(\overline{\Lambda \frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_G^{(0)}}{\partial Y}} \right) \\
& \sim ik_G \Lambda^2 \varepsilon \frac{\partial}{\partial Y} \left[g_1(Y) \frac{\partial g_1^*(Y)}{\partial Y} - g_1^*(Y) \frac{\partial g_1(Y)}{\partial Y} \right] + \mathcal{O}(t^{1/2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{1/2})
\end{aligned} \tag{4.5.21}$$

In order to find the order of the drag force X_{RG} resulting from the nonlinear interactions of the Rossby-gravity waves in t , we need to find the order in t of each

of the second, third, fourth and fifth terms on the right-hand side of (4.5.15). The order of the second term on the right-hand side of (4.5.15) in t is

$$\begin{aligned}
& \frac{\mu^3 \Lambda \varepsilon}{(\mu^2 + \kappa \Lambda^2)} \left(\overline{\frac{\partial^2 \psi_G^{(0)}}{\partial Y^2} \frac{\partial \psi_R^{(0)}}{\partial x}} \right) \\
& \sim \frac{ik_R \mu^3 \Lambda \varepsilon}{(\mu^2 + \kappa \Lambda^2)} \left[h_1(Y) \frac{\partial^2 g_1^*(Y)}{\partial Y^2} - h_1^*(Y) \frac{\partial^2 g_1(Y)}{\partial Y^2} \right] + \mathcal{O}(t^{1/2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{1/2})
\end{aligned} \tag{4.5.22}$$

The order of the third term on the right-hand side of (4.5.15) in t is

$$\begin{aligned}
& \frac{g \mu \kappa \Lambda^2 \varepsilon}{c_p T_s (\mu^2 + \kappa \Lambda^2)} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y}} \right) \\
& \sim \frac{ik_G g \mu \kappa \Lambda^2 \varepsilon}{c_p T_s (\mu^2 + \kappa \Lambda^2)} \left[g_1(Y) \frac{\partial h_1^*(Y)}{\partial Y} - g_1^*(Y) \frac{\partial h_1(Y)}{\partial Y} \right] + \mathcal{O}(t^{-1/2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{-1/2})
\end{aligned} \tag{4.5.23}$$

The order of the fourth term on the right-hand side of (4.5.15) in t is

$$\begin{aligned}
& \frac{f_0 g \mu^2 \Lambda \varepsilon}{N^2 T_s (\mu^2 + \kappa \Lambda^2)} \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_R^{(0)}}{\partial x} T_G^{(0)}} \right) \\
& \sim \frac{ik_R f_0 g \mu^2 \Lambda \varepsilon}{N^2 T_s (\mu^2 + \kappa \Lambda^2)} \frac{\partial}{\partial Y} \left[h_1(Y) T_{Gsteady}^*(Y) - h_1^*(Y) T_{Gsteady}(Y) \right] + \mathcal{O}(t^{1/2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{1/2}).
\end{aligned} \tag{4.5.24}$$

The order of the fifth term on the right-hand side of (4.5.15) in t is

$$\begin{aligned}
& \frac{\mu \kappa \Lambda^3 \varepsilon}{(\mu^2 + \kappa \Lambda^2)} \frac{\partial}{\partial Y} \left(\overline{\frac{\partial \psi_G^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y}} \right) \\
& \sim \frac{ik_G \mu \kappa \Lambda^3 \varepsilon}{(\mu^2 + \kappa \Lambda^2)} \frac{\partial}{\partial Y} \left[g_1(Y) \frac{\partial h_1^*(Y)}{\partial Y} - g_1^*(Y) \frac{\partial h_1(Y)}{\partial Y} \right] + \mathcal{O}(t^{-1/2}) \\
& \sim \mathcal{O}(1) + \mathcal{O}(t^{-1/2})
\end{aligned} \tag{4.5.25}$$

Using (4.5.22), (4.5.23), (4.5.24) and (4.5.25), the drag force X_{RG} , due to the interactions between Rossby waves and gravity waves, is

$$X_{RG} \sim \mathcal{O}(1) + \mathcal{O}(t^{1/2}) + \mathcal{O}(t^{-1/2}). \quad (4.5.26)$$

We can see from (4.5.18), (4.5.21) and (4.5.26) that the order of the sum of the drag forces is $\mathcal{O}(t^{1/2})$ coming from the X_G and X_{RG} drag forces. Thus, the order of the wave induced mean acceleration is $\mathcal{O}(t^{1/2})$. To find the order of the wave induced mean velocity, we integrate both sides of (4.5.16) with respect to t . For the reason that the order of the sum of the drag forces is $\mathcal{O}(t^{1/2})$, then the order of the wave-induced mean velocity is $\mathcal{O}(t^{3/2})$.

In the case where both Rossby waves and gravity waves are present and of the same order of magnitude $\varepsilon_R = \varepsilon_G = \varepsilon$ and $k_R = k_G$, the heat flux divergence comprises terms arising from Rossby wave nonlinear interactions, gravity wave nonlinear interactions and Rossby-gravity wave nonlinear interactions which all contribute to the wave-induced mean temperature at $\mathcal{O}(\varepsilon)$.

In order to derive an equation for this, we consider equation (4.3.16) where $\varepsilon_R = \varepsilon_G = \varepsilon$, i.e.

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T_R + \frac{\partial \bar{T}}{\partial y} v_R + w_R \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) + \varepsilon \left[u_R \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_R}{\partial y} \right] \\ & + \varepsilon \left[u_R \frac{\partial T_G}{\partial x} + u_G \frac{\partial T_R}{\partial x} + v_R \frac{\partial T_G}{\partial y} + w_G \left(\frac{g}{c_p T_s} T_R + \frac{\partial T_R}{\partial z} \right) \right] = 0. \end{aligned}$$

Writing the solutions as perturbation series in powers of the nonlinear parameter ε as before, taking the zonal mean and group terms using integration by parts leads to the following equation at $\mathcal{O}(\varepsilon)$

$$\begin{aligned} \frac{\partial \bar{T}_{0R}}{\partial t} = & -\frac{N^2 T_s}{g} \bar{w}_{0R} - \varepsilon \frac{\partial}{\partial y} (\overline{v_R^{(0)} T_R^{(0)}}) - \varepsilon \frac{\partial}{\partial y} (\overline{v_R^{(0)} T_G^{(0)}}) - \varepsilon \frac{\partial}{\partial z} (\overline{w_G^{(0)} T_R^{(0)}}) \\ & - \frac{g}{c_p T_s} \varepsilon (\overline{w_G^{(0)} T_R^{(0)}}) \end{aligned} \quad (4.5.27)$$

where the first and last terms in (4.5.27) are the cooling terms. We define

$$H_{RG1}(Y, t) = -\overline{(v_R^{(0)} T_G^{(0)})} \quad \text{and} \quad H_{RG2}(Y, t) = -\overline{(w_G^{(0)} T_R^{(0)})}. \quad (4.5.28)$$

where H_{RG1} is leading order component of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the Rossby wave velocity v_R and H_{RG2} is leading order component the zonal mean of the flux of Rossby wave heat being transported to the vertical direction by the gravity wave velocity w_G . Equation (4.5.27) can be written as

$$\frac{\partial \bar{T}_{0R}}{\partial t} = \varepsilon \frac{\partial H_R}{\partial y} + \varepsilon \frac{\partial H_{RG1}}{\partial y} + \varepsilon \frac{\partial H_{RG2}}{\partial z} - \frac{N^2 T_s}{g} \bar{w}_{0R} - \frac{g}{c_p T_s} \varepsilon \overline{(w_G^{(0)} T_R^{(0)})}, \quad (4.5.29)$$

so the change in the wave-induced mean Rossby temperature results from the meridional divergence of the meridional component of the zonal mean Rossby heat flux, the meridional divergence of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the meridional Rossby wave velocity, the vertical divergence of the zonal mean of the flux of Rossby wave heat being transported to the vertical direction by the vertical gravity wave velocity plus two cooling terms.

The wave-induced mean temperature arising from the gravity wave effects is given by equation (4.4.22). Thus, using equations (4.5.29) and (4.4.22) leads to

$$\begin{aligned} \frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t} = & \varepsilon \frac{\partial H_R}{\partial y} + \varepsilon \frac{\partial H_{RG1}}{\partial y} + \varepsilon \frac{\partial H_{RG2}}{\partial z} - \frac{N^2 T_s}{g} \bar{w}_{0R} - \varepsilon \frac{g}{c_p T_s} \overline{(w_G^{(0)} T_R^{(0)})} \\ & + \varepsilon \frac{\partial H_G}{\partial z} - \varepsilon \frac{g}{c_p T_s} \overline{(w_G^{(0)} T_G^{(0)})}. \end{aligned} \quad (4.5.30)$$

Equation (4.5.30) describes the rate of change of the wave-induced mean temperature with time in Case RG where $k_R = k_G$ resulting from the Rossby wave self-interactions, the gravity wave self-interactions and the interactions between Rossby waves and gravity waves. The rate of change of the wave-induced mean temperature with time t is proportional to the meridional and vertical divergences of four heat fluxes plus three cooling terms.

It can be noted that at $\mathcal{O}(\varepsilon)$ in Case RG where $k_R \neq k_G$, specifically where $k_R < k_G$, the Rossby wave self-interactions and the gravity wave self-interactions give zero wave number terms. On the other hand, the drag force from the interactions between Rossby waves and gravity waves X_{RG} does not appear at $\mathcal{O}(\varepsilon)$, it appears at high order as discussed in subsection 3.2.7 (see Table 3.5). Therefore, the rate of change of the total wave-induced mean flow is affected by the Rossby wave self-interactions and the gravity wave self-interactions at $\mathcal{O}(\varepsilon)$ where $k_R < k_G$.

Similarly, at $\mathcal{O}(\varepsilon)$ in Case RG where $k_R \neq k_G$, specifically where $k_R < k_G$, the rate of change of the wave-induced mean temperature is affected by the Rossby wave self-interactions and the gravity wave self-interactions. The interactions between Rossby waves and gravity waves appear at high order.

Chapter 5

Results of the numerical simulations

5.1 Numerical implementation

5.1.1 Configuration

Equations (3.3.20)-(3.3.24) with the initial and boundary conditions (3.3.25)-(3.3.29) are the numerical model for the Rossby-gravity wave problem which we solve numerically in this chapter.

The goal is to examine the interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line, as described in section 3.3. For this reason, we choose the background flow velocity to be $\bar{u}(Y) = \tanh(Y)$ as shown in Figure 5.1. With this choice of function, the Rossby and gravity waves propagate through a region where the mean flow varies slowly with Y , so that they have the approximate form given by the linear Rossby and gravity wave solutions (4.1.13) and (4.2.14).

Figure 5.1: The background shear flow $\bar{u} = \tanh(Y)$.

Near the centre of the domain, the hyperbolic tangent profile $\bar{u}(Y) = \tanh(Y)$ varies approximately linearly with Y , so we expect the Rossby wave and gravity wave profiles to be qualitatively similar to the linear solutions for the case with $\bar{u}(Y) = Y$ described in section 4.1 for the case of Rossby waves (Figure 4.1) and in section 4.2 for the case of gravity waves (Figure 4.2).

At the critical line, $Y = 0$, the Rossby and gravity wave amplitudes are greatly reduced. Below the critical line, $\bar{u}$ is approximately constant, and the Rossby and gravity waves again take the approximate form given by the linear Rossby and gravity wave solutions (4.1.13) and (4.2.14), and propagate towards the lower boundary with the negative group velocity. As noted in section 3.3, we find that the wave amplitudes are so small below the critical line; thus, it is sufficient to apply zero boundary conditions at the lower boundary.

As discussed in section 3.3, the numerical simulations are carried out on a finite domain $Y_2 \leq Y \leq Y_1$ with the waves forced at $Y = Y_1$. Thus, we solve our initial-boundary-value problem on the two-dimensional rectangular domain $0 \leq x \leq 2\pi$ and $-4 \leq Y \leq 4$ with the parameters given in section 2.5. On this domain, we are

interested in the situation where the waves are generated by the oscillatory boundary conditions (3.3.26) given at $Y = 4$, and they propagate downward as $Y \rightarrow -4$. The dimensional analysis in section 2.5 shows that the parameter values are reasonable values for representing the atmosphere in a region in one hemisphere and within a vertical range of $0 \leq z \leq 36$ km altitude. These parameter values satisfy the criteria for stability of both Rossby wave and gravity wave solutions. The stability criteria will be discussed in subsection 5.1.3.

5.1.2 Numerical methods

The numerical implementation is performed using a finite difference approximation and a spectral method. The spectral method is used since the boundary conditions are periodic in x and the Fast Fourier Transform (FFT) algorithm is utilized in the numerical implementation.

If one wants to solve an ordinary differential equation or a partial differential equation to high accuracy on a simple domain and if the data defining the problem are smooth, then the spectral methods are usually the most beneficial tool [Trefethen, (2000)].

The discrete Fourier transform $\mathcal{F}$ of a function χ and its inverse $\mathcal{F}^{-1}$ of $\hat{\chi}$ are defined, respectively, by

$$\begin{aligned}\hat{\chi}(k, Y, t) &= \mathcal{F}[\chi(x_j, Y, t)] = \frac{1}{J} \sum_{j=0}^{J-1} \chi(x_j, Y, t) e^{-ikx_j}, \\ \chi(x_j, Y, t) &= \mathcal{F}^{-1}[\hat{\chi}(k, Y, t)] = \sum_{k=-J/2}^{J/2-1} \hat{\chi}(k, Y, t) e^{ikx_j},\end{aligned}$$

where the function $\chi \in L^2$, $x_j = \frac{2\pi j}{J}$, $j = 0, 1, \dots, J$ with J even and $-\frac{J}{2} \leq k \leq \frac{J}{2} - 1$. The symbol L^2 denotes the set of all functions for which the following integral is finite

$$L^2 = \{\chi, \|\chi\| = \left[\int_{-\infty}^{\infty} |\chi|^2 dx \right]^{1/2} < \infty\}.$$

[Trefethen, (1994), page 82]. We apply these transforms for the functions

$\chi = \zeta_R, \psi_R, \zeta_G, \psi_G, T_G$ and $\hat{\chi} = \hat{\zeta}_R, \hat{\psi}_R, \hat{\zeta}_G, \hat{\psi}_G, \hat{T}_G$ with wavenumbers $k = k_R = k_G$.

Dividing (3.3.20) by ε_R and dividing (3.3.22) and (3.3.24) by ε_G , then applying the Fourier transform in x to (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) leads to

$$\begin{aligned}
& \frac{\partial \hat{\zeta}_R}{\partial t} + ik_R \left[\bar{u} \hat{\zeta}_R + \left(\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \right) \frac{\partial \hat{\psi}_R}{\partial x} \right] + \varepsilon_R \mathcal{F} \left[\mu \frac{\partial \psi_R}{\partial x} \frac{\partial \zeta_R}{\partial Y} - \mu \frac{\partial \psi_R}{\partial Y} \frac{\partial \zeta_R}{\partial x} \right] \\
& + \varepsilon_G \left\{ \mathcal{F} \left[-\Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial \zeta_R}{\partial x} \right] + \mathcal{F} \left[\left(\Lambda \frac{\partial^2 \psi_G}{\partial x \partial Y} \right) \left(\frac{\Lambda g \kappa}{c_p T_s} \frac{\partial \psi_R}{\partial Y} \right) \right] \right. \\
& + \mathcal{F} \left[\left(\Lambda \frac{\partial^2 \psi_G}{\partial x \partial Y} \right) \left((\Lambda^2 \kappa - \mu^2) \frac{\partial^2 \psi_R}{\partial Y^2} \right) \right] - \mathcal{F} \left[\left(\Lambda \frac{\partial^2 \psi_G}{\partial x \partial Y} \right) \left(\delta \frac{\partial^2 \psi_R}{\partial x^2} \right) \right] \\
& + \mathcal{F} \left[\frac{\partial \psi_G}{\partial x} \left(\Lambda \delta \frac{\partial^3 \psi_R}{\partial x^2 \partial Y} \right) \right] + \mathcal{F} \left[\frac{\partial \psi_G}{\partial x} \left(\frac{\Lambda^2 g \kappa}{c_p T_s} \frac{\partial^2 \psi_R}{\partial Y^2} \right) \right] + \mathcal{F} \left[\frac{\partial \psi_G}{\partial x} \left(\Lambda^3 \kappa \frac{\partial^3 \psi_R}{\partial Y^3} \right) \right] \\
& + \mathcal{F} \left[\left(\Lambda \frac{\partial^2 \psi_R}{\partial x \partial Y} \right) \left(\delta \frac{\partial^2 \psi_G}{\partial x^2} \right) \right] + \mathcal{F} \left[\left(\Lambda \frac{\partial^2 \psi_R}{\partial x \partial Y} \right) \left(\frac{\mu f_0 g}{N^2 T_s} \frac{\partial T_G}{\partial Y} \right) \right] \\
& + \mathcal{F} \left[\frac{\partial \psi_R}{\partial x} \left(\mu^2 \Lambda \frac{\partial^3 \psi_G}{\partial Y^3} \right) \right] + \mathcal{F} \left[\frac{\partial \psi_R}{\partial x} \left(\frac{\mu \Lambda f_0 g}{N^2 T_s} \frac{\partial^2 T_G}{\partial Y^2} \right) \right] - \mathcal{F} \left[\left(\mu \frac{\partial \psi_R}{\partial Y} \right) \left(\mu \Lambda \frac{\partial^3 \psi_G}{\partial x \partial Y^2} \right) \right] \\
& - \mathcal{F} \left[\left(\mu \frac{\partial \psi_R}{\partial Y} \right) \left(\frac{\Lambda f_0 g}{N^2 T_s} \frac{\partial^2 T_G}{\partial x \partial Y} \right) \right] - \mathcal{F} \left[\left(\frac{\mu \Lambda f_0 g}{N^2 T_s} \frac{\partial^2 \psi_R}{\partial Y^2} \frac{\partial T_G}{\partial x} \right) \right] \\
& \left. - \mathcal{F} \left[\left(\Lambda^3 \kappa \frac{\partial^2 \psi_R}{\partial x \partial Y} \frac{\partial^2 \psi_G}{\partial Y^2} \right) \right] \right\} = 0, \tag{5.1.1}
\end{aligned}$$

$$\frac{\partial^2 \hat{\psi}_R}{\partial Y^2} - \frac{\delta k_R^2}{(\mu^2 + \kappa \Lambda^2)} \hat{\psi}_R = \frac{\hat{\zeta}_R}{(\mu^2 + \kappa \Lambda^2)}, \tag{5.1.2}$$

$$\frac{\partial \hat{\zeta}_G}{\partial t} + ik_G \left[\bar{u} \hat{\zeta}_G - \Lambda^2 \frac{\partial^2 \bar{u}}{\partial Y^2} \hat{\psi}_G - \frac{g}{T_s} \hat{T}_G \right] + \varepsilon_G \mathcal{F} \left[\Lambda \frac{\partial \psi_G}{\partial x} \frac{\partial \zeta_G}{\partial Y} - \Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial \zeta_G}{\partial x} \right] = 0, \tag{5.1.3}$$

$$\frac{\partial^2 \hat{\psi}_G}{\partial Y^2} = \frac{\hat{\zeta}_G}{\Lambda^2}, \tag{5.1.4}$$

$$\frac{\partial \hat{T}_G}{\partial t} + ik_G \left[\bar{u} \hat{T}_G + \left(\frac{g}{c_p T_s} \bar{T} + \Lambda \frac{\partial \bar{T}}{\partial Y} \right) \hat{\psi}_G \right] + \varepsilon_G \left\{ \mathcal{F} \left[\frac{\partial \psi_G}{\partial x} \left(\frac{g}{c_p T_s} T_G + \Lambda \frac{\partial T_G}{\partial Y} \right) - \Lambda \frac{\partial \psi_G}{\partial Y} \frac{\partial T_G}{\partial x} \right] \right\} = 0. \quad (5.1.5)$$

The initial conditions are

$$\begin{aligned} \hat{\zeta}_R(k_R, Y, 0) = \hat{T}_R(k_R, Y, 0) = 0 & \quad \text{for } -4 \leq Y \leq 4 \\ \hat{\zeta}_G(k_G, Y, 0) = \hat{T}_G(k_G, Y, 0) = 0 & \quad \text{for } -4 \leq Y \leq 4 \\ \hat{\psi}_R(k_R, Y, 0) = \hat{\psi}_G(k_G, Y, 0) = 0 & \quad \text{for } -4 \leq Y < 4. \end{aligned} \quad (5.1.6)$$

The boundary conditions at the inflow boundary $Y = 4$ are

$$\begin{aligned} \hat{\zeta}_R(k_R, 4, t) = \hat{T}_R(k_R, 4, t) = \frac{\partial \hat{\zeta}_R(k_R, 4, t)}{\partial Y} = \frac{\partial \hat{T}_R(k_R, 4, t)}{\partial Y} = \frac{\partial^2 \hat{T}_R(k_R, 4, t)}{\partial Y^2} = 0, \\ \hat{\zeta}_G(k_G, 4, t) = \hat{T}_G(k_G, 4, t) = \frac{\partial \hat{\zeta}_G(k_G, 4, t)}{\partial Y} = \frac{\partial \hat{T}_G(k_G, 4, t)}{\partial Y} = \frac{\partial^2 \hat{T}_G(k_G, 4, t)}{\partial Y^2} = 0, \end{aligned} \quad (5.1.7)$$

$$\hat{\psi}_R(k, 4, t) = \begin{cases} 1 & \text{if } k = \pm k_R, \\ 0 & \text{if } k \neq \pm k_R, \end{cases} \quad \text{and} \quad \hat{\psi}_G(k, 4, t) = \begin{cases} 1 & \text{if } k = \pm k_G, \\ 0 & \text{if } k \neq \pm k_G. \end{cases} \quad (5.1.8)$$

The boundary conditions at the outflow boundary $Y = -4$ are

$$\begin{aligned} \hat{\zeta}_R(k_R, -4, t) = \hat{\psi}_R(k_R, -4, t) = \hat{T}_R(k_R, -4, t) = \frac{\partial \hat{T}_R(k_R, -4, t)}{\partial Y} = 0, \\ \hat{\zeta}_G(k_G, -4, t) = \hat{\psi}_G(k_G, -4, t) = \hat{T}_G(k_G, -4, t) = \frac{\partial \hat{T}_G(k_G, -4, t)}{\partial Y} = 0. \end{aligned} \quad (5.1.9)$$

Discretization

We carry out the numerical simulations on the two dimensional rectangular domain $0 \leq x \leq 2\pi$ and $-4 \leq Y \leq 4$ with the following spatial discretization of the independent variables

$$x_j = j\Delta x, \quad j = 0, 1, 2, 3, \dots, J,$$

and

$$Y_i = Y_{\text{outflow}} + (i - 1)\Delta Y, \quad i = 1, 2, 3, \dots, I,$$

where the increment in x is $\Delta x = 2\pi/J$ and the increment in Y is $\Delta Y = (Y_I - Y_{\text{outflow}})/(I - 1)$. The time is discretized as $t_n = n\Delta t$, $n = 0, 1, 2, \dots, N$, with the time increment $\Delta t = t_N/N$.

The approximations for the functions $\hat{\zeta}_R(k_R, Y, t)$, $\hat{\psi}_R(k_R, Y, t)$, $\hat{\zeta}_G(k_G, Y, t)$, $\hat{\psi}_G(k_G, Y, t)$ and $\hat{T}_G(k_G, Y, t)$ are denoted, respectively, by

$$\begin{aligned} \hat{\zeta}_R(k_R, Y_i, t_n) &= \hat{\zeta}_{k_R, i}^n, & \hat{\psi}_R(k_R, Y_i, t_n) &= \hat{\psi}_{k_R, i}^n \\ \hat{\zeta}_G(k_G, Y_i, t_n) &= \hat{\zeta}_{k_G, i}^n, & \hat{\psi}_G(k_G, Y_i, t_n) &= \hat{\psi}_{k_G, i}^n, \quad \text{and} \quad \hat{T}_G(k_G, Y_i, t_n) = \hat{T}_{k_G, i}^n, \end{aligned}$$

The simulations are done with following spectrum of wavenumbers in the x -direction

$$\begin{aligned} &\dots, -3k_R, -2k_R, -k_R, 0, +k_R, +2k_R, +3k_R, \dots \\ &\dots, -3k_G, -2k_G, -k_G, 0, +k_G, +2k_G, +3k_G, \dots, \end{aligned}$$

where k_R and k_G are the wavenumbers at the forcing (at the inflow $Y = 4$).

In order to calculate the discrete Fourier transform of the nonlinear terms in (5.1.1), (5.1.3) and (5.1.5), we apply the pseudo-spectral method. At each time step, we compute the x and Y derivatives of ζ_R , ψ_R , ζ_G , ψ_G and T_G for each x_j , $j = 0, 1, 2, \dots, J$ by evaluating the inverse Fourier transforms of $\frac{\partial \hat{\psi}_R}{\partial Y}$, $\frac{\partial^2 \hat{\psi}_R}{\partial Y^2}$, $\frac{\partial^3 \hat{\psi}_R}{\partial Y^3}$, $\frac{\partial \hat{\zeta}_R}{\partial Y}$, $\frac{\partial \hat{\psi}_G}{\partial Y}$, $\frac{\partial^2 \hat{\psi}_G}{\partial Y^2}$, $\frac{\partial \hat{\zeta}_G}{\partial Y}$, $\frac{\partial \hat{T}_G}{\partial Y}$, $\frac{\partial^2 \hat{T}_G}{\partial Y^2}$, $ik_R \hat{\psi}_R$, $(ik_R)^2 \hat{\psi}_R$, $ik_R \hat{\zeta}_R$, $ik_G \hat{\psi}_G$, $(ik_G)^2 \hat{\psi}_G$, $ik_G \hat{\zeta}_G$ and $ik_G \hat{T}_G$. We then compute the nonlinear functions and take their Fourier transforms to add to the numerical computations at the next time step.

The time derivative is approximated by using the Adams-Bashforth method of order four. The second-derivative with respect to Y in the equations (5.1.2) and (5.1.4) is approximated using the second order central finite difference approximation.

Grid size

The choice of the grid size is based on grid sensitivity analysis. Generally, in the process of mesh refinement, we first specify a tolerance. We keep reducing the grid size by half and comparing the results until we obtain that the maximum absolute error of the whole spatial domain is less than our previously specified tolerance. Then we stop the mesh refinement and plot the results.

In our numerical simulations, we specified tolerance of 1×10^{-2} . For the choice of the grid size in the Y -direction, the maximum absolute error of solutions of ψ_R and ψ_G , in Case RG where $k_R = k_G = \pm 1$ and $\varepsilon = 0.05$ at $t = 50$, with $\Delta Y = 0.03$ compared with $\frac{1}{2}\Delta Y = 0.015$ was found to be 9×10^{-3} which is less than our specified tolerance of 1×10^{-2} , and the plotted results seemed identical. Therefore, we stopped the mesh refinement and plotted the results for $\Delta Y = 0.03$.

For the reason that we are applying the Fourier transform in x and for efficiency reasons, it is preferable for the FFT algorithm that the number of points in x -direction I is a power of 2 which has to be the same number as k Fourier modes. We increased the number of Fourier modes and x grid points until the maximum absolute error of solutions of ψ_R and ψ_G with $I = 2^6 = 64$ Fourier modes compared with $I = 2^7 = 128$ Fourier modes was found to be 3.5×10^{-3} which lies within the tolerance we specified, and we plotted the results with 64 Fourier modes.

As for the time step, we found that the time step depends on the number of wavenumbers (Fourier modes). In particular, we found that with 64 wavenumbers or less, the time step $\Delta t = 2.5 \times 10^{-3}$ was enough in order to preserve stability. With 128 wavenumbers or more, the time step was set to $\Delta t = 1 \times 10^{-3}$ and this value of Δt was found to be sufficiently small to maintain stability.

Although the time step $\Delta t = 2.5 \times 10^{-3}$ was enough to maintain stability with 64 wavenumbers, we chose $\Delta t = 1 \times 10^{-3}$ in our numerical simulations with 64 wavenumbers.

Tools

All computations were performed in MATLAB Version 9.9.0 (R2020b) on a MacBook Pro (16-inch, 2019) with the following characteristics:

- ◇ Processor: 2.4 GHz 8-Core Intel Core i9.
- ◇ Memory: 64 GB 2667 MHz DDR4.
- ◇ Operating system: Mac OS Catalina Version 10.15.7.

5.1.3 Choice of parameters

The nondimensional domain and parameters used in the numerical computations are chosen to give a configuration representative of a region in the atmosphere with reasonable values of the dimensional quantities, as discussed in section 2.5.

In the numerical computations, there are a number of restrictions on the choice of the nondimensional parameters which must be satisfied in order for the solutions to be stable. Specifically, we have to choose all the nondimensional parameters in such a way that the perturbation would be able to propagate towards the critical region without attenuation or amplification. The values given in chapter 2 satisfy these restrictions.

It can be seen that equation (4.1.21) with $\bar{u}$ constant, i.e.,

$\frac{\partial^2 \hat{\psi}_R}{\partial Y^2} + \frac{1}{(\mu^2 + \kappa \Lambda^2)} \left(\frac{\beta}{\bar{u} - c} - \delta k_R^2 \right) \hat{\psi}_R = 0$, can have periodic solutions near the forced boundary, $Y = 4$, only if

$$\frac{1}{(\mu^2 + \kappa \Lambda^2)} \left(\frac{\beta}{\bar{u} - c} - \delta k_R^2 \right) > 0,$$

otherwise the solutions grow or decay exponentially with Y and there is no Rossby wave propagation. In the region near the forced boundary, $\bar{u}(Y) = \tanh(Y)$ is approximately constant, as discussed in section 5.1.1. In this chapter, we have $\bar{u} = \tanh(Y) \approx 1$, and since $\beta > 0$, $\delta = \frac{L_y^2}{L_x^2} > 0$ and $\kappa = \frac{f_0^2}{N^2} > 0$, then we must

choose the parameters to satisfy the inequality $\beta > \delta k_R^2$ in order to have Rossby wave propagation. Moreover, we have to choose $\beta > (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2}$ everywhere in the domain to satisfy the converse of the Rayleigh-Kuo criterion that guarantees stability of the Rossby wave solutions (see sections C.1 and C.2).

In a similar manner, equation (4.2.8), i.e. $\frac{\partial^2 \hat{\psi}_G}{\partial Y^2} + \frac{N^2}{\Lambda^2 (\bar{u} - c)^2} \hat{\psi}_G = 0$, can have periodic solutions near the forced boundary, $Y = 4$, only if

$$\frac{N^2}{\Lambda^2 (\bar{u} - c)^2} > 0.$$

Furthermore, we must choose the Richardson number, $R_i = N^2 / (\Lambda \frac{\partial \bar{u}(Y_c)}{\partial Y})^2$, to be bigger than $\frac{1}{4}$ in order to ensure that the mean shear flow is stable with respect to small disturbances [Miles, (1961); Howard, (1961)] (see section C.3).

Taking into account these considerations and as discussed in section 2.5, we choose the following values of the nondimensional parameters given in Table 5.1. These conditions are satisfied with the values of the parameters given in Table 5.1. It can be seen that by using these values given in Table 5.1 the maximum of $\frac{\partial^2 \bar{u}}{\partial Y^2} = \frac{\partial^2 \tanh(Y)}{\partial Y^2} \approx 0.76$, so we satisfy the converse of the Rayleigh-Kuo criterion $\beta - (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2} \neq 0$ everywhere in the domain. We additionally note that by using these values given in Table 5.1, $\frac{\partial \bar{u}(Y_c)}{\partial Y} = \frac{\partial \tanh(Y_c)}{\partial Y} = 1 - \tanh^2(Y_c)$ at the critical line $Y_c = 0$, so the Richardson number is $R_i = 4 \times N^2$ which is bigger than $\frac{1}{4}$, so we satisfy the Miles-Howard theorem.

As noted in section 1.1, Rossby waves have horizontal wavelengths thousands of kilometers long, while gravity waves have wavelengths ranging from several tens to thousands of kilometres and are typically shorter than Rossby waves. In all the numerical cases, we set $k_R = 1$, corresponding to a dimensional wavelength of about 16800 km since $L_x = 2683$ km, as presented in section 2.5. We set $k_G = 1$ or $k_G = 2$, corresponding to wavelengths 8400 – 16800 km.

The simulations were carried out over the nondimensional time interval from $t = 0$ to $t = 50$ with the nondimensional parameters summarized in Tables 5.1 and the different values of ε_R , ε_G and the forced wavenumbers k_R , k_G summarized in Table

5.2. We found that the Rossby wavenumber domain $-32 \leq k_R \leq 32$ and the gravity wavenumber domain $-32 \leq k_G \leq 32$ used in the simulations are sufficient. However, we only show the intervals of $|k_R| \leq 15$ and $|k_G| \leq 15$ in all figures presented in this chapter.

Table 5.1: The values of the nondimensional parameters used in all figures

The nondimensional parameters	The value
The square of the aspect ratio: $\delta = \left(\frac{L_y}{L_x}\right)^2$	$\delta = 0.2$
The Coriolis parameter	$f_0 = 1.9291$
The parameter β representing the effects of the earth's rotation	$\beta = 0.7783$
The sea level temperature and the background mean temperature	$T_s = \bar{T} = 1$
The specific heat of dry air at constant pressure	$c_p = 196.8$
The acceleration due to gravity	$g = 20$
The Brunt-Väisälä frequency	$N = \sqrt{2.035}$
The parameter κ	$\kappa = 1.83$
The parameter μ	$\mu = \frac{1}{2}$
The parameter Λ	$\Lambda = -\frac{1}{2}$

Table 5.2: The values of the parameters ε_R and ε_G , and the forced wavenumbers k_R and k_G used for the different cases

Cases	ε_R	ε_G	k_R	k_G
Barotropic Rossby wave ($Y = \mu y$) [Figures 5.2-5.7]	0.1	0	± 1	--
Baroclinic Rossby wave [Figures 5.2-5.7]	0.1	0	± 1	--
Case R1 [Figures 5.8-5.14]	0.1	$(0.1)^3$	± 1	± 2
Case R2 [Figures 5.15-5.19]	0.04	$(0.04)^2$	± 1	± 2
Case G1 [Figures 5.20-5.26]	$(0.07)^3$	0.07	± 1	± 1
Case G1 [Figures 5.20-5.26]	$(0.008)^3$	0.008	± 1	± 2
Case G2 [Figures 5.27-5.30]	$(0.07)^2$	0.07	± 1	± 1
Case RG [Figures 5.31-5.39]	0.05	0.05	± 1	± 1
Case RG [Figures 5.40-5.48]	0.008	0.008	± 1	± 2
Comparison between some cases [Figures 5.49-5.56]				

5.2 Cases with dominant Rossby wave effects or dominant gravity wave effects

5.2.1 Barotropic and baroclinic Rossby wave numerical solutions

In this subsection, we describe the results of preliminary computations of baroclinic Rossby waves in the absence of gravity waves. For comparison, we also describe the results of computations of barotropic Rossby waves. We solve equations (3.3.20) and (3.3.21) with $\varepsilon_G = 0$ in (3.3.20), i.e.

$$\left(\frac{\partial}{\partial t} + \bar{u}(Y)\frac{\partial}{\partial x}\right)\zeta_R + \left(\beta - (\mu^2 + \kappa \Lambda^2)\frac{\partial^2 \bar{u}}{\partial Y^2}\right)\frac{\partial \psi_R}{\partial x} + \varepsilon_R \left(\mu \frac{\partial \psi_R}{\partial x} \frac{\partial \zeta_R}{\partial Y} - \mu \frac{\partial \psi_R}{\partial Y} \frac{\partial \zeta_R}{\partial x}\right) = 0 \quad (5.2.1)$$

$$\delta \frac{\partial^2 \psi_R}{\partial x^2} + (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \psi_R}{\partial Y^2} = \zeta_R.$$

The simulations are carried out on the same domain with the same initial and boundary conditions presented at the beginning of this chapter. The imposed boundary condition (3.3.26), i.e. $\psi_R(x, 4, t) = e^{ik_R x} + \text{c.c.}$ is used with $k_R = 1$.

In the case of the baroclinic Rossby waves, we set the parameters $\delta, \beta, \kappa, \mu$ and Λ to the values given in Table 5.1. In the case of the barotropic Rossby waves, we set the parameters δ, β, μ and Λ to the values given in Table 5.1, but we set $\kappa = 0$ which means that the variable $Y = \mu y + \Lambda z + b$ is reduced to the variable $\mu y + b$ and the barotropic Rossby wave propagation takes place on a horizontal plane. This gives the barotropic problem originally studied numerically by Béland(1978). Campbell (2004) studied a variant of this problem with a spatially localized wave packet forcing comprising a continuous spectrum of Fourier modes and compared the wave packet solutions with wave solutions obtained in the case where the wave forcing is monochromatic as in (3.3.26). Hence, we can compare our barotropic results with

the results of Beland(1978) and with the monochromatic wave results of Campbell (2004).

The simulations are carried out over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters given in Table 5.1. The value of ε_R determines the time rate at which the nonlinear behaviour of the solution becomes evident. With small ε_R , the numerical computations are more stable, but the nonlinear effects develop more slowly, and hence it is necessary to continue the computation to later time in order to examine the nonlinear behaviour of the solution. Large values of ε_R may result in instability, but with larger ε_R , the nonlinear effects develop faster and hence a shorter time interval is needed to observe the nonlinear behaviour. The nonlinear results shown here are for $\varepsilon_R = 0.1$.

Figures 5.2(a) and 5.2(b) present the linear ($\varepsilon_R = 0$) behaviours of the zonal barotropic and baroclinic Rossby wave velocity perturbations $u_R(x, Y, t)$ as a function of x and Y at $t = 50$ whilst Figures 5.2(c) and 5.2(d) present the nonlinear ($\varepsilon_R = 0.1$) behaviours. These barotropic results are similar to those obtained by Campbell (2004) (precisely, Figure 2(a) and Figure 2(b) in [Campbell, (2004)]) with the differences being that we have used $\beta = 0.7783$ and $\varepsilon_R = 0.1$ instead of $\beta = 1$ and $\varepsilon_R = 0.02$.

In all figures there is a little or no transmission of wave activity beyond the critical level around $Y = 0$. The linear numerical solutions shown in Figures 5.2(a) and 5.2(b) can be compared with the approximate analytical solutions that were obtained in section 4.1 for the configuration with $\bar{u}(Y) = Y$ as shown in Figure 4.1(b). In the numerical solutions, we have $\bar{u}(Y) = \tanh(Y) \sim Y$ and we observe similar qualitative behaviour of the solutions around the critical line $Y = 0$ where the wave is almost entirely absorbed.

In Figures 5.2(c) and 5.2(d), on the other hand, when the effects of the nonlinear interactions of Rossby waves become evident, we note that the solutions experience a different behaviour around the critical level. The distortion of the contours above the critical line suggests that the wave solution may include a reflected component in

addition to the downward propagating incident component. This can be confirmed by examining the behaviour of the momentum flux.

In order to compare the effects of the nonlinear interactions of barotropic and baroclinic Rossby waves on the mean flow and to observe wave absorption and reflection, we examine the behaviour of the meridional component of the zonal mean zonal momentum flux presented in section 4.3, $F_R = -(\overline{u_R v_R})$ in the vicinity of the critical line. Also, at each time step, the jump in the momentum flux $[F_R]$ across the critical line is computed as follows $[F_R] = F_R(Y = -1) - F_R(Y = 1)$. As discussed in section 4.3, the change in the mean flow results from the meridional divergence of the momentum flux according to (4.3.15), i.e. $\frac{\partial \bar{u}_{0R}}{\partial t} = \varepsilon_R \frac{\partial F_R}{\partial y} = \mu \varepsilon_R \frac{\partial F_R}{\partial Y}$.

In the linear simulation, the momentum fluxes F_R go from negative values above the critical line to zero below as shown in Figure 5.3(a) meaning that F_R is increasing across the critical line, so the jump $[F_R]$ is positive as shown in Figure 5.4(a). However, $\frac{\partial F_R}{\partial Y}$ and $\frac{\partial F_R}{\partial y}$ are negative because the waves are travelling in the direction of negative Y and y . Consequently, the mean flow accelerations in both cases are negative (westward acceleration) in the critical line as exhibited in Figure 5.5(a). We also observe that the mean flow in the barotropic case accelerates more in the westward direction compared to that in the baroclinic case indicating that the absorption of the linear barotropic Rossby wave by the mean flow is relatively more considerable than the absorption of the linear baroclinic Rossby wave by the mean flow.

In the nonlinear simulation, the behaviours of the momentum fluxes F_R in the barotropic and baroclinic cases are different in the vicinity of the critical line as given in Figure 5.3(b) which can also be noted in the jumps in Figure 5.4(b). From Figure 5.4(b), we observe that the jumps, at early time, are positive as in the linear case where the waves are absorbed by the mean flow; on the other hand, when the nonlinear interactions of Rossby waves became manifest, the jumps begin to oscillate indicating wave reflection (eastward acceleration) and wave absorption (westward acceleration) and at the same time decay as t becomes large. In Figure 5.5(b), we see that in

both cases, the mean flow accelerations are unnoticeable up to $t = 10$, indicating that the effects of the nonlinearity have not been evident to affect the mean flow. When $t \gtrsim 10$, the mean flow has a westward acceleration indicating wave absorption and then begins to oscillate between an eastward acceleration (positive acceleration) and a westward acceleration (negative acceleration), indicating wave reflection and wave absorption, respectively.

The wave-induced mean flow velocities resulting from the nonlinear interactions of the barotropic and baroclinic Rossby waves (4.3.9), i.e. $\bar{u}_{0R}(Y, t) \approx -\varepsilon_R \frac{\partial \bar{\psi}_{0R}^{(1)}}{\partial y}$ are shown in Figure 5.6. For both, we observe that the wave-induced mean flow velocities are changing only in the critical line vicinity and almost constant away from the critical level revealing the important effect of the nonlinearity around the critical line.

The Fourier spectrum of the zonal barotropic and baroclinic Rossby wave velocity perturbations $|\hat{u}_R(k_R, Y, t)|$ at the critical level $Y = 0$ at $t = 50$ are shown in Figures 5.7(a) and 5.7(b). In the linear plot in Figure 5.7(a), there is only the forced wavenumbers and the baroclinic value is smaller. In the nonlinear plot in Figure 5.7(b) showing $\varepsilon_R |\hat{u}_R(k_R, Y, t)|$, we see that the zero wavenumber component $k_R = 0$ in both cases grow and exceed the forced wavenumbers $k_R = \pm 1$ as a result of the nonlinear interactions of Rossby waves. Moreover, we note that the zero wavenumber component in the barotropic case is slightly bigger than the zero wavenumber component in the baroclinic case. In the nonlinear plot, the higher wavenumbers develop. The $\pm 2k_R$ values correspond to the $e^{\pm 2ik_R x}$ terms that appear in the $\mathcal{O}(\varepsilon_R)$ solutions in section 4.3. It can be seen that the $e^{\pm 2ik_R x}$ terms in the barotropic case are bigger than the $e^{\pm 2ik_R x}$ terms in the baroclinic case.

Figure 5.2: The zonal Rossby wave velocity perturbation $u_R(x, Y, t)$ as a function of x and Y at $t = 50$: Linear ($\varepsilon_R = 0$) (a) barotropic and (b) baroclinic. Nonlinear ($\varepsilon_R = 0.1$) (c) barotropic and (d) baroclinic. The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

Figure 5.3: Variation of the zonal mean momentum flux F_R with Y at $t = 50$. (a) Comparison of linear barotropic (dashed curve) and baroclinic (solid curve). (b) Comparison of nonlinear barotropic (dashed curve) and baroclinic (solid curve). The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

Figure 5.4: Variation of the jump in the zonal mean momentum flux F_R with t . (a) Comparison of linear barotropic (dashed curve) and baroclinic (solid curve). (b) Comparison of nonlinear barotropic (dashed curve) and baroclinic (solid curve). The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

Figure 5.5: (a) Comparison of linear barotropic (dashed curve) and baroclinic (solid curve) of the wave-induced mean flow acceleration at the critical line $Y = 0$. (b) Comparison of nonlinear barotropic (dashed curve) and baroclinic (solid curve) of the wave-induced mean flow acceleration at the critical line $Y = 0$. The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

Figure 5.6: Comparison of nonlinear barotropic (dashed curve) and baroclinic (solid curve) of variation of the wave-induced mean flow velocity with Y at $t = 50$. The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

Figure 5.7: Fourier spectrum of the zonal Rossby wave velocity perturbation $|\hat{u}_R(k_R, Y, t)|$ at the critical level $Y = 0$ at $t = 50$. (a) Comparison of linear barotropic (dashed curve) and baroclinic (solid curve). (b) Comparison of nonlinear barotropic (dashed curve) and baroclinic (solid curve). The periodic forcing $e^{ik_R x}$, $k_R = 1$, has been applied at the upper boundary of the rectangular domain $Y = 4$.

5.2.2 Case R1

In this subsection, we present and discuss the results of Case R1 where

$\varepsilon_R \gg \varepsilon_R^2 \gg \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2$. As discussed in chapter 3, that we should choose $\varepsilon_G = \varepsilon_R^3$ in order to discuss this case.

As we discussed in chapter 3 that in Case R1, the first two orders are the nonlinear Rossby wave problem only and the nonlinear gravity wave effects appears at $\mathcal{O}(\varepsilon_R^6)$. Thus, we expect to see the nonlinear Rossby wave effects and the linear gravity wave effects in all figures in this subsection.

We carry out the simulations on the same domain with the same initial and boundary conditions presented at the beginning of this chapter. At the inflow boundary $Y = 4$, the boundary conditions (3.3.26) are used with $k_R = 1$ and $k_G = 2$. We carry out the simulations over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters in Table 5.1 and with $\varepsilon_R = 0.1$, so $\varepsilon_G = (0.1)^3$.

The zonal Rossby wave velocity perturbation $\varepsilon_R u_R(x, Y, t)$, the zonal gravity

wave velocity perturbation $\varepsilon_G u_G(x, Y, t)$ and the sum of these velocities $\varepsilon_R u_R + \varepsilon_G u_G$ as a function of x and Y at $t = 50$ are shown in Figures 5.8(a) and 5.8(b). The $\varepsilon_R u_R$ and $\varepsilon_R u_R + \varepsilon_G u_G$ plots are both shown in Figure 5.8(b), but they are identical so we only see one curve indicating that the Rossby wave effects are dominant in Case R1, as expected in section 3.2. Rossby wave reflection can be seen at the critical line $Y = 0$ in Figure 5.8(b), while gravity wave absorption at the critical line can be seen in Figure 5.8(a).

The wave-induced mean flow velocity resulting from the nonlinear interactions of Rossby waves, i.e., $\bar{u}_{0R}(Y, t) = -\mu\varepsilon_R \frac{\partial}{\partial Y}[\hat{\psi}_R(k_R = 0, Y, t)]$, the wave-induced mean flow velocity resulting from the nonlinear interactions of gravity waves, i.e., $\bar{u}_{0G}(Y, t) = -\Lambda\varepsilon_G \frac{\partial}{\partial Y}[\hat{\psi}_G(k_G = 0, Y, t)]$ and the sum of the wave-induced mean flow velocities $\bar{u}_{0R} + \bar{u}_{0G}$ are exhibited in Figures 5.9(a) and 5.9(b). The $\bar{u}_{0R}$ and $\bar{u}_{0R} + \bar{u}_{0G}$ plots are both shown in Figure 5.9(b), but they are identical so we only see one curve. We observe that the wave-induced mean flow velocities are changing only in the critical line vicinity and almost constant away from the critical line. This reveals the important effect of the nonlinearity around the critical level. In order to compare $\bar{u}_{0R}(Y, t)$ and $\bar{u}_{0G}(Y, t)$, we present them in Figure 5.10 with same horizontal scale. We note that $\bar{u}_{0G}(Y, t)$ is almost not changing with Y at $t = 50$ compared to $\bar{u}_{0R}(Y, t)$ which is changing considerably in the critical line vicinity due to the nonlinear interactions of Rossby waves.

Figure 5.11(a) presents the momentum flux of Rossby waves F_R according to (4.3.14), i.e. $F_R = -(\overline{u_R v_R})$ whereas Figure 5.11(b) presents the momentum flux of gravity waves F_G according to (4.4.18), i.e. $F_G = -(\overline{u_G w_G})$. In both figures, it is zero almost everywhere below the critical level, indicating nearly no transmission of wave activity below the critical level. This confirms that it is sufficient to apply zero outflow boundary conditions if a flow has a critical level as discussed in section 3.3.

We also examine the behaviours of the momentum flux F_R and the momentum flux F_G in the vicinity of the critical line. Moreover, we compute, at each time

step, the jump in the $[F_R]$ and the jump in $[F_G]$ across the critical line as follows, $[F_R] = F_R(Y = -1) - F_R(Y = 1)$ and $[F_G] = F_G(Y = -1) - F_G(Y = 1)$. As discussed in chapter 4, the change in the mean flow resulting from the nonlinear interactions of the Rossby waves according to (4.3.15), i.e. $\frac{\partial \bar{u}_{0R}}{\partial t} = \varepsilon_R \frac{\partial F_R}{\partial y} = \mu \varepsilon_R \frac{\partial F_R}{\partial Y}$ and the change in the mean flow resulting from the nonlinear interactions of the gravity waves according to (4.4.19) $\frac{\partial \bar{u}_{0G}}{\partial t} = \varepsilon_G \frac{\partial F_G}{\partial z} = \Lambda \varepsilon_G \frac{\partial F_G}{\partial Y}$.

The momentum flux of gravity waves F_G goes from positive values above the critical line to zero below as shown in Figure 5.11(b) meaning that F_G is decreasing across the critical line, so the jump $[F_G]$ is negative as shown in Figure 5.12(b). For the reason that the waves are travelling in the direction of negative Y , then $\frac{\partial F_G}{\partial Y}$ is positive; however, $\frac{\partial F_G}{\partial z}$ is negative since Y and z have opposed signs in our configuration. Therefore, we note in Figure 5.12(b) that $[F_G]$ is negative (westward acceleration), and at about $t = 20$, the jump is an approximately negative constant indicating that there is continuous absorption of the gravity wave by the mean flow. In contrast, the jump $[F_R]$ in Figure 5.12(a), at early time, behaves as the linear case where the mean flow absorbs the baroclinic Rossby wave. However, when the nonlinear interactions of Rossby waves became evident, the jump begins to oscillate, indicating wave reflection (eastward acceleration) and wave absorption (westward acceleration).

To compare the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$, we present them in Figure 5.13 with same vertical scale. We observe that $\frac{\partial \bar{u}_{0G}}{\partial t}$ is almost not varying from $t = 0$ to $t = 50$, denoting that the nonlinear gravity wave effects are absent compared to the nonlinear Rossby wave effects. Therefore, the mean flow is modified by the nonlinear interactions of Rossby waves only, as expected by our analysis of Case R1 in section 3.2.

The Fourier spectrum of the zonal wave velocity perturbations $\varepsilon_R |\hat{u}_R(k_R, Y, t)|$, $\varepsilon_G |\hat{u}_G(k_G, Y, t)|$ and the sum of them $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$ at the critical line $Y = 0$ at $t = 50$ are shown in Figures 5.14(a) and 5.14(b). The $\varepsilon_R |\hat{u}_R|$ and $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$

plots are both shown in Figure 5.14(b), but they are identical so we only see one curve indicating that the nonlinear Rossby wave effects are dominant in the Case R1 which is consistent with the observations presented in Table 3.1 exhibiting that the Rossby wavenumbers are dominant at the first two leading orders.

We see in Figure 5.14(b) that the zero Rossby wavenumber component $k_R = 0$ grows and exceeds the forced wavenumbers $k_R = \pm 1$ as a result of the nonlinear interactions Rossby waves. In Figure 5.14(a), on the other hand, we note that the peaks are only at the forced gravity wavenumbers $k_G = \pm 2$ and there are no higher Fourier modes of k_G indicating that we have just the linear gravity effects in Case R1, as explained in section 3.2. We also see in Figure 5.14(b) that the higher Rossby wavenumbers develop, and the $\pm 2k_R$ values correspond to the $e^{\pm 2ik_R x}$ terms appearing in the $\mathcal{O}(\varepsilon_R)$ solutions in section 4.3.

Figure 5.8: Case R1: the zonal velocity perturbations at $t = 50$ (a) $\varepsilon_G u_G(x, Y, t)$. (b) $\varepsilon_R u_R + \varepsilon_G u_G$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.9: Case R1: the wave-induced mean flow velocities with Y at $t = 50$: (a) $\bar{u}_{0G}(Y, t)$. (b) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.10: Case R1: Comparison of variation of the wave-induced mean flow velocities $\bar{u}_{0R}(Y, t)$ and $\bar{u}_{0G}(Y, t)$ with Y at $t = 50$. The periodic forcing $e^{ik_R x}$, $k_R = 1$, and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.11: Case R1: variation of the zonal mean momentum fluxes with Y at $t = 50$: (a) momentum flux of Rossby waves F_R . (b) momentum flux of gravity waves F_G . The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.12: Case R1: variation of the jump in the zonal mean momentum fluxes with t . (a) $[F_R]$. (b) $[F_G]$. The periodic forcing $e^{ik_R x}$, $k_R = 1$, and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.13: Case R1: comparison of the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ at the critical line $Y = 0$ with t . The periodic forcing $e^{ik_R x}$, $k_R = 1$, and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

Figure 5.14: Case R1: the Fourier spectrum of the zonal velocity perturbations at the critical line $Y = 0$ at $t = 50$ (a) $\varepsilon_G|\hat{u}_G|$. (b) $\varepsilon_R|\hat{u}_R| + \varepsilon_G|\hat{u}_G|$. The periodic forcing $e^{ik_R x}, k_R = 1$ and the periodic forcing $e^{ik_G x}, k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.1$ and $\varepsilon_G = (0.1)^3$ are used.

5.2.3 Case R2

In this subsection, we present and discuss the results of Case R2 where $\varepsilon_R \gg \varepsilon_R^2 = \varepsilon_G \gg \varepsilon_R \varepsilon_G \gg \varepsilon_G^2$. As discussed in section 3.2, in Case R2, the first two orders are the nonlinear Rossby wave problem and the linear gravity wave problem. The nonlinear interactions of the gravity waves develop at $\mathcal{O}(\varepsilon_R^4)$. Thus, we expect to see the nonlinear Rossby wave effects and the linear gravity wave effects in all figures in this section. However, the linear gravity wave effects are relatively more considerable than that in Case R1, as we shall see.

At the inflow boundary $Y = 4$, the boundary conditions (3.3.26) are used with $k_R = 1$ and $k_G = 2$. We carry out the simulations over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters in Table 5.1 and with $\varepsilon_R = 0.04$, so $\varepsilon_G = (0.04)^2$.

The zonal Rossby wave velocity perturbation $\varepsilon_R u_R(x, Y, t)$, the zonal gravity wave velocity perturbation $\varepsilon_G u_G(x, Y, t)$ and the sum of these velocities $\varepsilon_R u_R + \varepsilon_G u_G$ as a function of x and Y at $t = 50$ are shown in Figures 5.15(a) and 5.15(b). The $\varepsilon_R u_R$ and $\varepsilon_R u_R + \varepsilon_G u_G$ plots are both shown in Figure 5.15(b), but they are identical; accordingly, we only see one curve indicating that the Rossby wave effects are dominant in Case R2, as expected in section 3.2. Rossby wave reflection can be seen at the critical line $Y = 0$ in Figure 5.15(b), while gravity wave absorption at the critical line can be seen in Figure 5.15(a).

Figure 5.16(b) manifests the sum of the wave-induced mean flow velocities $\bar{u}_{0R} + \bar{u}_{0G}$ and $\bar{u}_{0R}$ which are identical plots. We observe that the wave-induced mean flow velocities change only in the critical line vicinity and almost constant away from the critical line. This figure also confirms that the modifications of the mean flow in Case R2 are a consequence of the Rossby wave nonlinear interactions. We also plotted the wave-induced mean flow velocities $\bar{u}_{0R}$ and $\bar{u}_{0G}$ in Figure 5.17 with the same horizontal scale to show that $\bar{u}_{0G}$ is relatively not changing compared to $\bar{u}_{0R}$.

A comparison between the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$

in Figure 5.18 indicates that the nonlinear gravity wave effects are almost absent contrary to the nonlinear Rossby wave effects.

A transfer of wave activity to higher wavenumbers can be seen in Figure 5.19(b), but the amplitude of $\varepsilon_R|\hat{u}_R| + \varepsilon_G|\hat{u}_G|$ is still small at the ends of the wavenumber interval shown in the graph. The zero Rossby wavenumber component $k_R = 0$ has dramatically increased and is slightly bigger than the Rossby forced wavenumbers $k_R = \pm 1$ due to the nonlinear interactions of Rossby waves. It can be noted that the peaks at the forced gravity wavenumbers $k_G = \pm 2$ are bigger than that in Case R1 (Figure 5.14(a)), indicating that the linear gravity wave effects are more notable than that in Case R1.

We can conclude from the preceding figures that the mean flow is modified only by the nonlinear interactions of Rossby waves in Case R2, as discussed in section 3.2.

Figure 5.15: Case R2: the zonal velocity perturbations at $t = 50$ (a) $\varepsilon_G u_G$.

(b) $\varepsilon_R u_R + \varepsilon_G u_G$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.04$ and $\varepsilon_G = (0.04)^2$.

Figure 5.16: Case R2: the wave-induced mean flow velocities with Y at $t = 50$:

(a) $\bar{u}_{0G}(Y, t)$. (b) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.04$ and $\varepsilon_G = (0.04)^2$ are used.

Figure 5.17: Case R2: comparison of variation of the wave-induced mean flow velocities $\bar{u}_{0R}(Y, t)$ and $\bar{u}_{0G}(Y, t)$ with Y at $t = 50$. The periodic forcing $e^{ik_R x}$, $k_R = 1$, and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.04$ and $\varepsilon_G = (0.04)^2$ are used.

Figure 5.18: Case R2: comparison of the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ at the critical line $Y = 0$ with t . The periodic forcing $e^{ik_R x}$, $k_R = 1$, and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.04$ and $\varepsilon_G = (0.04)^2$ are used.

Figure 5.19: Case R2: the Fourier spectrum of the zonal velocity perturbations at the critical line $Y = 0$ at $t = 50$ (a) $\varepsilon_G |\hat{u}_G|$. (b) $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$. The periodic forcing $e^{ik_R x}, k_R = 1$ and the periodic forcing $e^{ik_G x}, k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_R = 0.04$ and $\varepsilon_G = (0.04)^2$ are used.

5.2.4 Case G1

In this subsection, we present and discuss the results of Case G1 where $\varepsilon_G \gg \varepsilon_G^2 \gg \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2$. As discussed in section 3.2, in Case G1, the first two orders are the nonlinear gravity wave problem only, and the nonlinear Rossby wave effects appear at $\mathcal{O}(\varepsilon_G^6)$. Thus, we expect to see the nonlinear gravity wave effects and the linear Rossby wave effects in all figures in this subsection.

Because the nonlinear Rossby-gravity terms in (3.3.20) are multiplied by ε_G , so if the forced gravity wavenumbers $k_G = \pm 1$, we can choose ε_G as large as $\varepsilon_G = 0.07$. If ε_G is greater than 0.07, we get instabilities. With largely forced gravity wavenumbers $|k_G| > 1$, we need smaller ε_G to preserve the stability. Specifically, if the forced gravity wavenumbers $k_G = \pm 2$, we can choose ε_G as large as $\varepsilon_G = 0.008$. If $\varepsilon_G > 0.008$, we get instabilities. As before, the forced Rossby wavenumbers are $k_R = \pm 1$.

At the inflow boundary $Y = 4$, the boundary conditions (3.3.26) are used with $k_R = 1$, $k_G = 1$, and $\varepsilon_G = 0.07$, so $\varepsilon_R = (0.07)^3$. Also, At the inflow boundary $Y = 4$, the boundary conditions (3.3.26) are used with $k_R = 1$, $k_G = 2$, and $\varepsilon_G = 0.008$, so $\varepsilon_R = (0.008)^3$.

The $\varepsilon_G u_G$ and $\varepsilon_R u_R + \varepsilon_G u_G$ plots are both shown in Figures 5.20(b) and 5.21(b), but they are identical so we only see one curve. The contour plots of the sum of the zonal Rossby and gravity wave velocity perturbations show that the gravity wave effects are dominant in Case G1, as expected in section 3.2. Gravity wave reflection at the critical line $Y = 0$ can be seen in these figures.

The wave-induced mean flow velocities $\bar{u}_{0R}$ and $\bar{u}_{0G}$ in Figures 5.22(a) and 5.22(b) with same horizontal scale show that $\bar{u}_{0G}$ changes only in the critical line vicinity due to the gravity wave nonlinear interactions and $\bar{u}_{0R}$ does not change compared to $\bar{u}_{0G}$.

At each time step, the jump in the momentum flux $[F_R]$ and the jump in the momentum flux $[F_G]$ are computed across the critical line as before (see subsection 5.2.2) and the corresponding results are presented in Figures 5.23(a) and 5.24(a). As discussed in chapter 4, the change in the mean flow resulting from the nonlinear

interactions of the Rossby waves according to (4.3.15), i.e. $\frac{\partial \bar{u}_{0R}}{\partial t} = \varepsilon_R \frac{\partial F_R}{\partial y} = \mu \varepsilon_R \frac{\partial F_R}{\partial Y}$ and the change in the mean flow resulting from the nonlinear interactions of the gravity waves according to (4.4.19) $\frac{\partial \bar{u}_{0G}}{\partial t} = \varepsilon_G \frac{\partial F_G}{\partial z} = \Lambda \varepsilon_G \frac{\partial F_G}{\partial Y}$.

Therefore, we note that $[F_R]$ is positive and at about $t = 20$ the jumps are approximately positive constants, indicating that there is continuous absorption of the Rossby wave by the mean flow (westward acceleration of $\bar{u}_{0R}$). In contrast, the jumps $[F_G]$, at early time, behave as the linear case where the mean flow absorbs the gravity wave, but when the nonlinear interactions of gravity waves became evident, the jumps begin to oscillate, indicating gravity wave reflection (eastward acceleration of $\bar{u}_{0G}$) and gravity wave absorption (westward acceleration of $\bar{u}_{0G}$).

The rate of change of the wave-induced mean flow velocities $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ with t are shown in Figures 5.23(b) and 5.24(b) with same vertical scale. We observe that $\frac{\partial \bar{u}_{0R}}{\partial t}$ is not changing from $t = 0$ to $t = 50$, meaning that the nonlinear Rossby wave effects are unnoticeable compared to the nonlinear gravity wave effects. Consequently, the mean flow is affected by the nonlinear interactions of gravity waves only, as expected by our analysis of Case G1 in section 3.2.

The Fourier spectrum of the zonal wave velocity perturbations $\varepsilon_R |\hat{u}_R(k_R, Y, t)|$, $\varepsilon_G |\hat{u}_G(k_G, Y, t)|$ and the sum of them $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$ at the critical line $Y = 0$ at $t = 50$ are shown in Figures 5.25(a), 5.25(b), 5.26(a) and 5.26(b). The $\varepsilon_G |\hat{u}_G|$ and $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$ plots are both shown in Figures 5.25(b) and 5.26(b), but they are identical so we only see one curve indicating that the nonlinear gravity wave effects are dominant in the Case G1 which is consistent with the observations presented in Table 3.2 exhibiting that the gravity wavenumbers are dominant at the first two leading orders. The zero gravity wavenumber component $k_G = 0$ is hugely bigger than the zero Rossby wavenumber component $k_R = 0$ indicating that the mean flow is affected by the gravity wave nonlinear interactions. It is evident that the mean flow is more affected by the gravity wave nonlinear interactions with $\varepsilon_G = 0.07$ compared to $\varepsilon_G = 0.008$. We also see in Figures 5.25(b) and 5.26(b) that the higher gravity

wavenumbers develop and the $\pm 2k_G$ values correspond to the $e^{\pm 2ik_G x}$ terms appearing in the $\mathcal{O}(\varepsilon_G)$ solutions in section 4.4.

Figure 5.20: Case G1: the zonal velocity perturbations at $t = 50$: (a) $\varepsilon_R u_R$.

(b) $\varepsilon_R u_R + \varepsilon_G u_G$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^3$ are used.

Figure 5.21: Case G1: the zonal velocity perturbations at $t = 50$: (a) $\varepsilon_R u_R$.

(b) $\varepsilon_R u_R + \varepsilon_G u_G$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.008$ and $\varepsilon_R = (0.008)^3$ are used.

Figure 5.22: Case G1: comparison of variation of the wave-induced mean flow velocities $\bar{u}_{0R}(Y, t)$ and $\bar{u}_{0G}(Y, t)$ with Y at $t = 50$: (a) The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$ with $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^3$. (b) The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$ with $\varepsilon_G = 0.008$ and $\varepsilon_R = (0.008)^3$.

Figure 5.23: Case G1: (a) Variation of the jump in the momentum flux of Rossby waves F_R and the momentum flux of gravity waves F_G with t . (b) Comparison of the rate of change of the wave-induced mean flow velocities $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ at the critical line $Y=0$ with t . The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y=4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^3$ are used.

Figure 5.24: Case G1: (a) Variation of the jump in the momentum flux of Rossby waves F_R and the momentum flux of gravity waves F_G with t . (b) Comparison of the rate of change of the wave-induced mean flow velocities $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ at the critical line $Y=0$ with t . The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y=4$. The parameters in Table 5.1, $\varepsilon_G = 0.008$ and $\varepsilon_R = (0.008)^3$ are used.

Figure 5.25: Case G1: the Fourier spectrum of the zonal velocity perturbations at the critical line $Y = 0$ at $t = 50$: (a) $\varepsilon_R |\hat{u}_R|$. (b) $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^3$ are used.

Figure 5.26: Case G1: the Fourier spectrum of the zonal velocity perturbations at the critical line $Y = 0$ at $t = 50$: (a) $\varepsilon_R |\hat{u}_R|$. (b) $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$. The periodic forcing $e^{ik_R x}$, $k_R = 2$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.008$ and $\varepsilon_R = (0.008)^3$ are used.

5.2.5 Case G2

In this subsection, we present the results of Case G2 where

$\varepsilon_G \gg \varepsilon_G^2 = \varepsilon_R \gg \varepsilon_R \varepsilon_G \gg \varepsilon_R^2$. As discussed in section 3.2, in Case G2, the first two orders are the nonlinear gravity wave problem and the linear Rossby wave problem. The nonlinear interactions of the Rossby waves develop at $\mathcal{O}(\varepsilon_G^4)$. Thus, we expect to see the nonlinear gravity wave effects and the linear Rossby wave effects in all figures in this subsection. However, the linear Rossby wave effects are relatively more considerable than that in Case G1, as we shall see.

At the inflow boundary $Y = 4$, the boundary conditions (3.3.26) are used with $k_R = 1$ and $k_G = 1$. We carry out the simulations over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters in Table 5.1 and with $\varepsilon_G = 0.07$, so $\varepsilon_R = (0.07)^2$.

The contour plots of the zonal Rossby and gravity wave velocity perturbations are shown in Figures 5.27(a) and 5.27(b) where the $\varepsilon_G u_G$ and $\varepsilon_R u_R + \varepsilon_G u_G$ plots are both shown in Figure 5.27(b), but they are identical so we only see one curve. This confirms that the gravity wave effects are dominant in Case G2, as expected in section 3.2. Gravity wave reflection can be observed at the critical line $Y = 0$ in Figure 5.27(b).

The wave-induced mean flow velocities $\bar{u}_{0R}$ and $\bar{u}_{0G}$ in Figure 5.28 with the same horizontal scale show that $\bar{u}_{0G}$ changes only in the critical line vicinity due to the nonlinear interactions of gravity waves and $\bar{u}_{0R}$ changes insignificantly.

The wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ are shown in Figures 5.29 with the same vertical scale. We observe that $\frac{\partial \bar{u}_{0R}}{\partial t}$ is slightly negative (westward acceleration of $\bar{u}_{0R}$) at early time, indicating the absorption of the Rossby wave by the mean flow, then it is very small indicating that the nonlinear Rossby wave effects are not noticeable compared to the nonlinear gravity wave effects. Contrarily, $\frac{\partial \bar{u}_{0G}}{\partial t}$ at early time behaves as the linear case where the mean flow absorbs the gravity wave. However, when the nonlinear interactions of gravity waves became apparent,

we see oscillations, indicating gravity wave reflection (eastward acceleration of $\bar{u}_{0G}$) and gravity wave absorption (westward acceleration of $\bar{u}_{0G}$). Hence, the mean flow is modified by the nonlinear interactions of gravity waves only, as expected by our analysis of Case G2 in section 3.2.

The Fourier spectrum of the zonal wave velocity perturbations in Figures 5.30(a) and 5.30(b) indicate that the mean flow is affected by the gravity wave nonlinear interactions since the zero gravity wavenumber component $k_G = 0$ is bigger than the zero Rossby wavenumber component $k_R = 0$.

Figure 5.27: Case G2: the zonal velocity perturbations at $t = 50$: (a) $\varepsilon_R u_R$.

(b) $\varepsilon_R u_R + \varepsilon_G u_G$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^2$ are used.

Figure 5.28: Case G2: Comparison of variation of the wave-induced mean flow velocities $\bar{u}_{0R}(Y, t)$ and $\bar{u}_{0G}(Y, t)$ with Y at $t = 50$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^2$ are used.

Figure 5.29: Case G2: Comparison of the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ and $\frac{\partial \bar{u}_{0G}}{\partial t}$ at the critical line $Y = 0$ with t . The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^2$ are used.

Figure 5.30: Case G2: the Fourier spectrum of the zonal velocity perturbations at the critical line $Y = 0$ at $t = 50$: (a) $\varepsilon_R |\hat{u}_R|$. (b) $\varepsilon_R |\hat{u}_R| + \varepsilon_G |\hat{u}_G|$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1, $\varepsilon_G = 0.07$ and $\varepsilon_R = (0.07)^2$ are used.

5.3 Combined effects of Rossby waves and gravity waves (Case RG)

In this section, we exhibit and discuss the results of Case RG where

$\varepsilon_R = \varepsilon_G \gg \varepsilon_R^2 = \varepsilon_R \varepsilon_G = \varepsilon_G^2$. In this case, we let $\varepsilon_R = \varepsilon_G = \varepsilon$. The nonlinear interactions of the Rossby, gravity and Rossby-gravity waves in Case RG can be observed at the first two orders, as discussed in section 3.2. Therefore, we expect to see these effects in the figures shown in this section. We solve the same initial-boundary value problem as before.

In subsection 5.3.1, we present and discuss the results of Case RG where Rossby waves and gravity waves have the same forced wavenumbers, $k_R = k_G = \pm 1$, as discussed in section 4.5. In subsection 5.3.2, we present and discuss the results of Case RG where the Rossby waves and gravity waves have different forced wavenumbers, $k_R = \pm 1$ and $k_G = \pm 2$, as discussed in section 4.5.

5.3.1 The case with $k_R = k_G = \pm 1$

In this subsection, the simulations are carried out over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters in Table 5.1 and $\varepsilon = 0.05$.

The zonal Rossby wave and gravity wave velocity perturbations $\varepsilon u_R(x, Y, t)$ and $\varepsilon u_G(x, Y, t)$ and the sum of these velocities are shown, respectively, in Figures 5.31(a), 5.31(b) and 5.31(c) at $t = 50$ as a function of y and z for fixed $x = 0$ in the yz -domain which is of the form shown in Figure 2.4.

The zonal Rossby wave and gravity wave velocity perturbations $\varepsilon u_R(x, Y, t)$ and $\varepsilon u_G(x, Y, t)$ and the sum of these velocities are shown, respectively, in Figures 5.32(a), 5.32(b) and 5.32(c) at $t = 10$ and in Figures 5.32(d), 5.32(e) and 5.32(f) at $t = 50$. At early time $t = 10$, Rossby, gravity and Rossby-gravity wave absorptions can be seen at the critical line $Y = 0$, while at late time $t = 50$, wave reflections due to the nonlinear interactions of Rossby, gravity and Rossby-gravity waves can be seen at the

critical line $Y = 0$.

The wave-induced mean flow velocity $\bar{u}_{0R}$ resulting from the nonlinear interactions of Rossby and Rossby-gravity waves with Y at $t = 50$ are shown in Figure 5.33(a). In Figure 5.33(b), we present the wave-induced mean flow velocity $\bar{u}_{0G}$ resulting from the nonlinear interactions of gravity waves with Y at $t = 50$. In Figure 5.33(c), we show the sum of these wave-induced mean flow velocities $\bar{u}_{0R} + \bar{u}_{0G}$ corresponding to the nonlinear interactions of Rossby, gravity and Rossby-gravity waves. We note from Figure 5.33(c) that below the critical line, $\bar{u}_{0R} + \bar{u}_{0G}$ is so small and approaches zero at the lower boundary. On the other hand, $\bar{u}_{0R} + \bar{u}_{0G}$ is remarkably changing in the region above the critical line as a consequence of the reflections of Rossby waves, gravity waves and Rossby-gravity waves in that region.

We also have plotted the evolutions of the wave-induced mean flow velocities with time t at the critical line $Y = 0$ (Figure 5.34). The evolution of $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$ starts to develop at around $t = 10$, then it becomes negative (westward) up to approximately $t = 25$ then begins to oscillate between less negative (eastward) and more negative (westward) when the effects of the nonlinearity become manifest.

In Figures 5.35(a) and 5.35(b), we show with time t at the critical line $Y = 0$, the wave-induced mean flow accelerations $\frac{\partial \bar{u}_{0R}}{\partial t}$ resulting from the sum of the Rossby and Rossby-gravity drag forces X_R and X_{RG} according to (4.5.11), and $\frac{\partial \bar{u}_{0G}}{\partial t}$ resulting from the gravity drag force X_G . The sum of $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$ corresponding to the sum of the drag forces X_R, X_G and X_{RG} according to (4.5.15) is shown in Figure 5.35(c). In Figure 5.35(c), we see that $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$ becomes negative (westward) at early time indicating that the mean flow absorbs the waves. Next, it becomes less negative (eastward) up to nearly $t = 33$, indicating that the waves are reflected at the critical line $Y = 0$, then it oscillates, indicating wave absorption and wave reflection. The considerable effects of the nonlinear interactions of Rossby, gravity and Rossby-gravity waves on the mean flow appear at late time as the acceleration oscillates more rapidly with t .

The evolutions of the wave-induced mean temperatures $\bar{T}_{0R}$, $\bar{T}_{0G}$ with time t at the critical line $Y = 0$ and the sum $\bar{T}_{0R} + \bar{T}_{0G}$ are shown in Figure 5.36. It can be seen that $\bar{T}_{0R} + \bar{T}_{0G}$ begins to develop at approximately $t = 10$ and decreases swiftly with respect to t during the wave absorption stage up to $t = 25$. When the nonlinear interactions between the waves become apparent, the wave-induced mean temperature increases and decreases during the wave reflection stage and wave absorption stage, respectively.

In Figure 5.37(a), we show the rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ according to (4.5.27) with t at the critical line $Y = 0$. As exhibited in (4.5.27), the rate of change of $\bar{T}_{0R}$ with t results from the Rossby wave self-interactions and the interactions between Rossby waves and gravity waves. This change is proportional to the meridional divergence of the meridional component of the zonal mean Rossby heat flux $\frac{\partial}{\partial y}(\overline{v_R T_R})$, the meridional divergence of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the meridional Rossby wave velocity $\frac{\partial}{\partial y}(\overline{v_R T_G})$, the vertical divergence of the zonal mean of the flux of Rossby wave heat being transported to the vertical direction by the vertical gravity wave velocity $\frac{\partial}{\partial z}(\overline{w_G T_R})$ plus two cooling terms $\frac{N^2 T_s}{g} \bar{w}_{0R}$ and $\frac{g}{c_p T_s}(\overline{w_G T_R})$. We see in Figure 5.37(a) that $\frac{\partial \bar{T}_{0R}}{\partial t}$ drops considerably with respect to t at early time, then rises during the wave reflection stage. At late time, $\frac{\partial \bar{T}_{0R}}{\partial t}$ oscillates between dropping and rising.

The vertical divergence of the vertical component of the zonal mean gravity heat flux $\frac{\partial}{\partial z}(\overline{w_G T_G})$ plus the cooling term $\frac{g}{c_p T_s}(\overline{w_G T_G})$ according to (4.4.22) cause a change in the wave-induced mean gravity wave temperature $\frac{\partial \bar{T}_{0G}}{\partial t}$ with t at $Y = 0$ as shown in Figure 5.37(b). This change results from the gravity wave self-interactions. It can be observed that $\frac{\partial \bar{T}_{0G}}{\partial t}$ oscillates slightly with t between decreasing and increasing at early time, then it oscillates rapidly between decreasing and increasing with t at late time.

The sum $\frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t}$ with t is shown in Figure 5.37(c). During the wave absorption

at early time, $\frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t}$ drops rapidly with t , then it rises when the effects of the nonlinearity become evident. It can be noticed that the substantial effects of the meridional and vertical divergences of the four heat fluxes plus the three cooling terms according to (4.5.30) cause rapid dropping and rising in the wave-induced mean temperature at late time. This is expected, as we discussed in section 4.5, that at $\mathcal{O}(\varepsilon)$ in Case RG where $k_R = k_G$, the rate of change of the wave-induced mean temperature is affected by the Rossby wave self-interactions, the gravity wave self-interactions and the interactions between Rossby waves and gravity waves.

The Fourier spectrum of both velocities $|\hat{u}_R(k_R, Y, t)|$, $|\hat{u}_G(k_G, Y, t)|$ and the sum of them at the critical line $Y = 0$ at early time $t = 10$ are displayed in Figures 5.38(a), 5.38(b) and 5.38(c) and at late time $t = 50$ are displayed in Figures 5.39(a), 5.39(b) and 5.39(c). The value of $\hat{u}_R$ at $k_R = 0$ is the wave-induced mean flow velocity $\bar{u}_{0R}$ and the value of $\hat{u}_G$ at $k_G = 0$ is the wave-induced mean flow velocity $\bar{u}_{0G}$.

It can be seen that the effects on the mean flow begin to develop very early, by $t = 10$, there are peaks at $k_R = 0$, $k_G = 0$ and $k_R + k_G = 0$. These peaks have increased noticeably at $t = 50$. This is expected since that in Case RG with same forced wavenumbers, the mean flow is affected by the nonlinear interactions of the Rossby, gravity and Rossby-gravity waves at $\mathcal{O}(\varepsilon)$ as discussed in section 4.5. Also, it can be noted that the dominant wavenumbers are $0, \pm 1$ and ± 2 at the first two leading orders as discussed in Table 3.5.

In order to check the values of the wave amplitude at the high end of the spectrum, we plot $\log(|\hat{u}_R| + |\hat{u}_G|)$ as a function of k at $t = 50$ in Figure 5.40. We see from the plot that $\log(|\hat{u}_R| + |\hat{u}_G|)$ is decreasing with k and approximately -11 at $k = 32$, indicating that the value of the amplitude function is e^{-11} which is much less than the maximum value of the amplitude function e , which is attained at $k = 1$.

Figure 5.31: Case RG: the zonal velocity perturbations with y and z at $t = 50$ and $x = 0$:

(a) $\varepsilon u_R(x, y, z, t)$. (b) $\varepsilon u_G(x, y, z, t)$. (c) $\varepsilon(u_R + u_G)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

At $t = 10$:

At $t = 50$:

Figure 5.32: Case RG: the zonal velocity perturbations: At $t = 10$: (a) $\varepsilon u_R(x, Y, t)$.

(b) $\varepsilon u_G(x, Y, t)$. (c) $\varepsilon(u_R + u_G)$. At $t = 50$: (d) $\varepsilon u_R(x, Y, t)$. (e) $\varepsilon u_G(x, Y, t)$. (f) $\varepsilon(u_R + u_G)$.

The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.33: Case RG: the wave-induced mean flow velocities with Y at $t = 50$:

(a) $\bar{u}_{0R}(Y, t)$. (b) $\bar{u}_{0G}(Y, t)$. (c) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.34: Case RG: the wave-induced mean flow velocities with t at the critical line $Y = 0$: (a) $\bar{u}_{0R}(Y, t)$. (b) $\bar{u}_{0G}(Y, t)$. (c) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}, k_R = 1$ and the periodic forcing $e^{ik_G x}, k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.35: Case RG: The wave-induced mean flow accelerations with t at the critical line $Y = 0$: (a) $\frac{\partial \bar{u}_{0R}}{\partial t}$. (b) $\frac{\partial \bar{u}_{0G}}{\partial t}$. (c) $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.36: Case RG: the wave-induced mean temperatures with t at the critical line $Y = 0$: (a) $\bar{T}_{0R}(Y, t)$. (b) $\bar{T}_{0G}(Y, t)$. (c) $\bar{T}_{0R}(Y, t) + \bar{T}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}, k_R = 1$ and the periodic forcing $e^{ik_G x}, k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.37: Case RG: the rate of change of the wave-induced mean temperatures with t at the critical line $Y = 0$: (a) $\frac{\partial \bar{T}_{0R}}{\partial t}$. (b) $\frac{\partial \bar{T}_{0G}}{\partial t}$. (c) $\frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t}$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.38: Case RG: the Fourier spectrum of the perturbations of the zonal velocities at the critical level $Y = 0$ at $t = 10$: (a) $\varepsilon|\hat{u}_R(k_R, Y, t)|$. (b) $\varepsilon|\hat{u}_G(k_G, Y, t)|$. (c) $\varepsilon(|\hat{u}_R| + |\hat{u}_G|)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.39: Case RG: the Fourier spectrum of the perturbations of the zonal velocities at the critical level $Y = 0$ at $t = 50$: (a) $\varepsilon|\hat{u}_R(k_R, Y, t)|$. (b) $\varepsilon|\hat{u}_G(k_G, Y, t)|$. (c) $\varepsilon(|\hat{u}_R| + |\hat{u}_G|)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

Figure 5.40: Case RG: $\log(|\hat{u}_R| + \hat{u}_G)$ as a function of k at $t = 50$. The periodic forcing $e^{ik_R x}, k_R = 1$ and the periodic forcing $e^{ik_G x}, k_G = 1$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.05$ are used.

5.3.2 The case with $k_R < k_G$

In this subsection, we set $k_G = \pm 2$, with k_R still set to $k_R = \pm 1$. This gives a situation where the gravity wave horizontal wavelength is shorter than that of the Rossby wave, which is typical in the atmosphere. As previously explained in subsection 5.2.4, if the forced gravity wavenumber is $|k_G| > 1$, we need to use a smaller ε_G to maintain the stability because, in our model, the nonlinear Rossby-gravity terms in (3.3.20) are multiplied by ε_G . Thus, in this subsection we set $\varepsilon_R = \varepsilon_G = \varepsilon = 0.008$. The simulations are carried out over the nondimensional time interval from $t = 0$ to $t = 50$ with the parameters in Table 5.1 and $\varepsilon = 0.008$.

The zonal Rossby wave and gravity wave velocity perturbations $\varepsilon u_R(x, Y, t)$ and $\varepsilon u_G(x, Y, t)$ and sum of these velocities are shown, respectively, in Figures 5.41(a), 5.41(b) and 5.41(c) at $t = 50$ as a function of y and z for fixed $x = 0$ in the yz -domain

which is of the form shown in Figure 2.4.

At early time $t = 10$, Rossby, gravity and Rossby-gravity waves are absorbed at the critical line $Y = 0$ as it can be seen in Figures 5.42(a), 5.42(b) and 5.42(c). At late time $t = 50$, Rossby, gravity and Rossby-gravity waves are reflected at the critical line $Y = 0$ when the effects of the nonlinearity become observable as exhibited in Figures 5.42(d), 5.42(e) and 5.42(f).

In Figure 5.43(c), the sum of the wave-induced mean flow velocities $\bar{u}_{0R} + \bar{u}_{0G}$ with Y at $t = 50$ is shown. We observe that $\bar{u}_{0R} + \bar{u}_{0G}$ is changing only in the critical line vicinity and almost constant away from the critical line. In contrast, $\bar{u}_{0R} + \bar{u}_{0G}$ in Case RG with the same forced wavenumbers is changing not only in the critical line vicinity but in the whole region above the critical line (Figure 5.33(c)).

Also, we have plotted the evolutions of the wave-induced mean flow velocities $\bar{u}_{0R}$ and $\bar{u}_{0G}$ with time t at $Y = 0$ (Figure 5.44). The evolution of $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$ begins to develop at about $t = 10$ and becomes negative (westward) during the wave absorption stage, then it becomes less negative (eastward) during the wave reflection stage.

In Figures 5.45(a) and 5.45(b), we show the rate of change of the wave-induced mean flow velocities $\frac{\partial \bar{u}_{0R}}{\partial t}$ resulting from the sum of the Rossby and Rossby-gravity drag forces X_R and X_{RG} according to (4.5.11), and $\frac{\partial \bar{u}_{0G}}{\partial t}$ resulting from the gravity drag force X_G with time t at the critical line $Y = 0$. The sum of $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$ corresponding to the sum of the drag forces X_R, X_G and X_{RG} according to (4.5.15) is shown in Figure 5.45(c). In Figure 5.45(c), the acceleration $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$ becomes negative (westward) at early time. However, the effects of the drag forces can be noted at early time since the acceleration $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$ becomes slightly less negative (eastward). At late time, the acceleration changes rapidly between the westward and eastward directions as a consequence of the considerable effects of the drag forces.

The evolutions of the wave-induced mean temperatures $\bar{T}_{0R}$, $\bar{T}_{0G}$ and the sum $\bar{T}_{0R} + \bar{T}_{0G}$ with time t at the critical line $Y = 0$ are shown in Figure 5.46. It can

be observed that $\bar{T}_{0R} + \bar{T}_{0G}$ begins to develop when approximately $t > 10$ and drops swiftly with respect to t during the wave absorption stage. When the nonlinear interactions between the waves become apparent, the wave-induced mean temperature rises during the wave reflection stage.

In Figure 5.47(a), the rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ according to (4.5.27) with t at the critical line $Y = 0$ drops rapidly before the effects of the meridional and vertical divergences of the heat fluxes become noticeable. At the wave reflection stage, $\frac{\partial \bar{T}_{0R}}{\partial t}$ starts to rise as a consequence of the effects of the meridional and vertical divergences of the heat fluxes, then drops and rises at late time. The rate of change of the wave-induced mean gravity wave temperature $\frac{\partial \bar{T}_{0G}}{\partial t}$ with t at $Y = 0$ decreases and increases swiftly as a result of the rapid changes in the vertical divergence of the vertical component of the zonal mean gravity heat flux $\frac{\partial H_G}{\partial z}$ (Figure 5.47(b)). In Figure 5.47(c), the sum $\frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t}$ decreases at early time with slight oscillations indicating increasing and decreasing. Then, it increases with significant oscillations during the wave reflection stage. Nevertheless, these significant oscillations at late time are not considerable compared to that in Case RG where $k_R = k_G$ (Figure 5.37(c)) for the reason that the wave-induced mean temperature in Case RG where $k_R < k_G$, is affected by the Rossby wave self-interactions and the gravity wave self-interactions only at $\mathcal{O}(\varepsilon)$. The effects of the interactions between Rossby waves and gravity waves appear at high order, as discussed in section 4.5.

The Fourier spectrum of both velocities $|\hat{u}_R(k_R, Y, t)|$, $|\hat{u}_G(k_G, Y, t)|$ and the sum of them at the critical line $Y = 0$ at early time $t = 10$ are shown in Figures 5.48(a), 5.49(b) and 5.48(c). As before, the value of $\hat{u}_R$ at $k_R = 0$ is the wave-induced mean flow velocity $\bar{u}_{0R}$ and the value of $\hat{u}_G$ at $k_G = 0$ is the wave-induced mean flow velocity $\bar{u}_{0G}$. We observe from Figure 5.48(c) that there is almost no peak at the zero wavenumber showing that there are no effects on the mean flow at early time. In contrast, the effects on the mean flow start to evolve very early in Case RG with same forced wavenumbers (Figure 5.39(c)). Moreover, it can be noted that the magnitude

of the zero wavenumber at $t = 10$ in Case RG with same forced wavenumbers (Figure 5.39(c)) is approximately equal to the magnitude of the forced wavenumbers at $t = 10$ in Case RG with different forced wavenumbers (Figure 5.48(c)).

At $t = 50$, we note from Figure 5.49(c) that there is a transfer of wave activity to higher wavenumbers, but the amplitude of $|\hat{u}_R| + |\hat{u}_G|$ is still very small at the ends of the wavenumber interval shown in the graph. Although the zero wavenumber in Figure 5.49(c) has increased but it is dominated by the higher wavenumbers $\pm 1, \pm 2, \pm 3$ and ± 4 . This is expected as we discussed in chapter 4 that the effects on the mean flow in Case RG with different forced wavenumbers would appear at high order. Moreover, it can be observed from Figures 5.38(c) and 5.49(c) that the effects on the mean flow in Case RG with same forced wavenumbers, $k_R = k_G = \pm 1$, at early time are approximately equal to the effects on the mean flow in Case RG with different forced wavenumbers, $k_R = \pm 1, k_G = \pm 2$, at late time.

In order to check the values of the wave amplitude at the high end of the spectrum, we plot $\log(|\hat{u}_R| + |\hat{u}_G|)$ as a function of k at $t = 50$ in Figure 5.50. We see from the plot that $\log(|\hat{u}_R| + |\hat{u}_G|)$ is decreasing with k and approximately -11 at $k = 32$, indicating that the value of the amplitude function is e^{-11} which is much less than the maximum value of the amplitude function $e^0 = 1$, which is attained at $k = 2$.

Figure 5.41: Case RG: the zonal velocity perturbations with y and z at $t = 50$ and $x = 0$:

(a) $\varepsilon u_R(x, y, z, t)$. (b) $\varepsilon u_G(x, y, z, t)$. (c) $\varepsilon(u_R + u_G)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

At $t = 10$:

At $t = 50$:

Figure 5.42: Case RG: the zonal velocity perturbations: At $t = 10$: (a) $\varepsilon u_R(x, Y, t)$.

(b) $\varepsilon u_G(x, Y, t)$. (c) $\varepsilon(u_R + u_G)$. At $t = 50$: (d) $\varepsilon u_R(x, Y, t)$. (e) $\varepsilon u_G(x, Y, t)$. (f) $\varepsilon(u_R + u_G)$.

The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.43: Case RG: the wave-induced mean flow velocities with Y at $t = 50$:

(a) $\bar{u}_{0R}(Y, t)$. (b) $\bar{u}_{0G}(Y, t)$. (c) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.44: Case RG: the wave-induced mean flow velocities with t at the critical line $Y = 0$: (a) $\bar{u}_{0R}(Y, t)$. (b) $\bar{u}_{0G}(Y, t)$. (c) $\bar{u}_{0R}(Y, t) + \bar{u}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.45: Case RG: the wave-induced mean flow accelerations with t at the critical line $Y = 0$: (a) $\frac{\partial \bar{u}_{0R}}{\partial t}$. (b) $\frac{\partial \bar{u}_{0G}}{\partial t}$. (c) $\frac{\partial \bar{u}_{0R}}{\partial t} + \frac{\partial \bar{u}_{0G}}{\partial t}$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.46: Case RG: the wave-induced mean temperatures with t at the critical line $Y = 0$: (a) $\bar{T}_{0R}(Y, t)$. (b) $\bar{T}_{0G}(Y, t)$. (c) $\bar{T}_{0R}(Y, t) + \bar{T}_{0G}(Y, t)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.47: Case RG: the rate of change of the wave-induced mean temperatures with t at the critical line $Y = 0$: (a) $\frac{\partial \bar{T}_{0R}}{\partial t}$. (b) $\frac{\partial \bar{T}_{0G}}{\partial t}$. (c) $\frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t}$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.48: Case RG: the Fourier spectrum of the perturbations of the zonal velocities at the critical level $Y = 0$ at $t = 10$: (a) $\varepsilon|\hat{u}_R(k_R, Y, t)|$. (b) $\varepsilon|\hat{u}_G(k_G, Y, t)|$. (c) $\varepsilon(|\hat{u}_R| + |\hat{u}_G|)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.49: Case RG: the Fourier spectrum of the perturbations of the zonal velocities at the critical level $Y = 0$ at $t = 50$: (a) $\varepsilon|\hat{u}_R(k_R, Y, t)|$. (b) $\varepsilon|\hat{u}_G(k_G, Y, t)|$. (c) $\varepsilon(|\hat{u}_R| + |\hat{u}_G|)$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

Figure 5.50: Case RG: $\log(|\hat{u}_R| + \hat{u}_G)$ as a function of k at $t = 50$. The periodic forcing $e^{ik_R x}$, $k_R = 1$ and the periodic forcing $e^{ik_G x}$, $k_G = 2$, have been applied at the upper boundary of the rectangular domain $Y = 4$. The parameters in Table 5.1 and $\varepsilon = 0.008$ are used.

5.4 Comparison between baroclinic Rossby waves and Case RG

In this section, our new results formerly shown in subsections 5.3.1 and 5.3.2 are compared with the well-known results of baroclinic Rossby waves in the absence of gravity waves manifested in subsection 5.2.1.

It can be seen in Figure 5.51 that the wave-induced mean Rossby wave acceleration $\frac{\partial \bar{u}_{0R}}{\partial t}$ at the critical line $Y = 0$ with time t in baroclinic Rossby waves only (without the effects of gravity waves) becomes more negative (more westward) in comparison to that in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$), so the wave-induced mean Rossby wave velocity $\bar{u}_{0R}$ in baroclinic Rossby waves only is more negative (more westward) than that in Case RG ($k_R = k_G = 1$) and Case RG

($k_R = 1 < k_G = 2$). Furthermore, it can be noted that $\frac{\partial \bar{u}_{0R}}{\partial t}$ at the critical line $Y = 0$ with t in Case RG ($k_R = k_G = 1$) is relatively more negative (more westward) compared to that in Case RG ($k_R = 1 < k_G = 2$), so $\bar{u}_{0R}$ in Case RG ($k_R = k_G = 1$) is relatively more westward (more negative) than in Case RG ($k_R = 1 < k_G = 2$). This can also be observed in the zero Rossby wavenumber component of the Fourier spectrum of the perturbation of the zonal Rossby wave velocity $\varepsilon |\hat{u}_R(k_R = 0, Y = 0, t = 50)|$ representing the wave-induced mean Rossby wave velocity $\bar{u}_{0R}$ at the critical line as shown in Figure 5.52. We can conclude from this comparison that the momentum deposition in the background mean flow around the critical line in the case of baroclinic Rossby waves in the absence of gravity waves is more considerable than that in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$). Additionally, the momentum deposition in the background mean flow in the vicinity of the critical line in Case RG ($k_R = k_G = 1$) is relatively larger than in Case RG ($k_R = 1 < k_G = 2$).

In Figure 5.53, we compare the amplitude of zonal Rossby wave velocity perturbation u_R in our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) with that in baroclinic Rossby wave case only with Y at $t = 50$. We observe that the effects of the nonlinear interactions between Rossby waves and gravity waves significantly modify the zonal Rossby wave velocity perturbation profile, and the gravity wave effects reduce the amplitude of u_R . Moreover, the amplitude of u_R in Case RG ($k_R < k_G$) is immensely reduced in comparison to that in Case RG ($k_R = k_G$).

In Figure 5.54, the rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ at the critical line $Y = 0$ with time t in our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) is compared with that in baroclinic Rossby wave case only. As discussed in section 4.3, the rate of change of the wave-induced mean Rossby wave temperature with time t in baroclinic Rossby wave case only results from the meridional divergence of the meridional component of the zonal mean Rossby heat flux according to (4.3.19), i.e. $\frac{\partial \bar{T}_{0R}}{\partial t} = \varepsilon_R \frac{\partial H_R}{\partial y}$. On the other hand, the rate of change of the wave-induced mean Rossby temperature in Case RG results from the merid-

ional divergence of the meridional component of the zonal mean Rossby heat flux, the meridional divergence of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the meridional Rossby wave velocity, the vertical divergence of the zonal mean of the flux of Rossby wave heat being transported to the vertical direction by the vertical gravity wave velocity plus two cooling terms according to (4.5.29), i.e.,

$$\frac{\partial \bar{T}_{0R}}{\partial t} = \varepsilon \frac{\partial H_R}{\partial y} + \varepsilon \frac{\partial H_{RG1}}{\partial y} + \varepsilon \frac{\partial H_{RG2}}{\partial z} - \frac{N^2 T_s}{g} \bar{w}_{0R} - \frac{g}{c_p T_s} \varepsilon (\overline{w_G^{(0)} T_R^{(0)}}).$$

In Figure 5.54, we see a considerable decrease of $\frac{\partial \bar{T}_{0R}}{\partial t}$ with t at the critical line $Y = 0$ in baroclinic Rossby wave case (without the effects of gravity waves) in comparison to that in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$), so the wave-induced mean Rossby wave temperature $\bar{T}_{0R}$ in baroclinic Rossby wave case greatly decreases in comparison to that in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$). Besides, the decrease of $\frac{\partial \bar{T}_{0R}}{\partial t}$ in Case RG ($k_R = k_G = 1$) is relatively more than that in Case RG ($k_R = 1 < k_G = 2$); hence, the decrease of $\bar{T}_{0R}$ in Case RG ($k_R = k_G = 1$) is relatively more than that in Case RG ($k_R = 1 < k_G = 2$). This can also be seen in the zero Rossby wavenumber component of the Fourier spectrum of the perturbation of the Rossby wave temperature $\varepsilon |\hat{T}_R(k_R = 0, Y = 0, t = 50)|$ representing the wave-induced mean Rossby temperature $\bar{T}_{0R}$ at the critical line as shown in Figure 5.55.

In Figure 5.56, the amplitude of the Rossby wave temperature perturbation T_R with Y at $t = 50$ in our results in Case RG ($k_R = k_G = 1$) is compared to that in baroclinic Rossby wave case only. It can be seen that the nonlinear interactions between Rossby waves and gravity waves modify the amplitude of the Rossby wave temperature perturbation T_R profile. The gravity wave effects cause an increase in the amplitude of T_R . In addition, it can be seen that the amplitude of T_R in Case RG ($k_R = k_G = 1$) increases after adding the amplitude of the gravity wave temperature perturbation T_G .

A comparison between the amplitude of the Rossby wave temperature perturbation T_R with Y at $t = 50$ in our results in Case RG ($k_R = 1 < k_G = 2$) with that in baroclinic Rossby wave case only is shown in Figure 5.57. It can be seen that the Rossby-gravity wave nonlinear interactions modify the Rossby wave temperature profile. The gravity wave effects in Case RG ($k_R = 1 < k_G = 2$) cause a decrease in the amplitude of T_R . However, after adding the amplitude of the gravity wave temperature perturbation T_G , the amplitude of T_R is increased.

In closing, we compare, in Figure 5.58, the amplitude of Rossby wave temperature T_R with Y at $t = 50$ in baroclinic Rossby wave case only and the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$). The gravity wave effects in both cases cause an increase in the amplitude of T_R . Moreover, it can be observed that the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R = k_G = 1$) is relatively greater than that in Case RG ($k_R = 1 < k_G = 2$).

Figure 5.51: Comparison between the wave-induced mean Rossby wave acceleration $\frac{\partial \bar{u}_{0R}}{\partial t}$ with t at the critical level $Y = 0$ in baroclinic Rossby wave case only, Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$).

Figure 5.52: Comparison of the Fourier spectrum of the perturbation of the zonal Rossby wave velocity at the critical level $Y = 0$ at $t = 50$: (a) baroclinic Rossby wave only, Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$). (b) as in (a), but with smaller vertical scale.

Figure 5.53: Comparison between the amplitude of the zonal Rossby wave velocity perturbation u_R with Y at $t = 50$ in baroclinic Rossby wave case only, Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$).

Figure 5.54: Comparison between the rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ with time t at the critical line $Y = 0$ in baroclinic Rossby wave case only, Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$).

Figure 5.55: Comparison between the Fourier spectrum of the perturbation of the Rossby wave temperature at the critical level $Y = 0$ at $t = 50$ for the baroclinic Rossby wave case only, Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$).

Figure 5.56: Comparison between the amplitude of the Rossby wave temperature perturbation T_R with Y at $t = 50$ for the baroclinic Rossby wave case only, Case RG ($k_R = k_G = 1$) and the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R = k_G = 1$).

Figure 5.57: Comparison between the amplitude of the Rossby wave temperature perturbation T_R with Y at $t = 50$ for the baroclinic Rossby wave case only, Case RG ($k_R = 1 < k_G = 2$) and the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R = 1 < k_G = 2$).

Figure 5.58: Comparison between the amplitude of Rossby wave temperature T_R with Y at $t = 50$ in baroclinic Rossby wave case only with the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R = k_G = 1$) and Case RG ($k_R = 1 < k_G = 2$).

Chapter 6

Conclusion

6.1 Summary

In this thesis, we examined a three-dimensional configuration where Rossby waves and gravity waves are forced at the same location and propagate upwards and southwards to a critical line where both types of waves interact with the background flow.

The thesis began by presenting the nonlinear partial differential governing equations describing a fluid flow in the earth's atmosphere under the influence of the Coriolis force in terms of Cartesian coordinates x, y and z . In the derivation of the governing equations, we considered the general three-dimensional inviscid equations of fluid dynamics written in terms of the velocity components, the density and the pressure.

Then we rewrote these equations under the hydrostatic approximation and the geostrophic approximation. After that, we transformed the equations which assume the hydrostatic approximation from the height coordinate system x, y and z to an isobaric coordinate system in which pressure is the vertical coordinate x, y and p since the use of the isobaric coordinate system simplifies the development of these approximate equations.

These governing equations are quite complicated even when the hydrostatic ap-

proximation is applied. Therefore, we made the quasi-geostrophic approximation in which the horizontal flow is assumed to be approximately geostrophic and that the ratio of the vertical velocity component to the horizontal velocity components is taken to be small. Under these conditions, we considered the horizontal velocity as the sum of a geostrophic part and a small ageostrophic part.

Afterward, we derived these governing equations in a three-dimensional configuration under the hydrostatic approximation and a quasi-geostrophic approximation in terms of log-pressure coordinates defined as $z = -H \ln\left(\frac{p}{p_s}\right)$, and wrote them in terms of dimensionless quantities to compare the magnitude of the terms of any equations regardless of their units.

In order to represent the Rossby waves and gravity waves, we used the perturbation method in which each field variable is written as the sum of a basic steady part and two time dependent perturbations representing the local deviations of the field from the basic part. Thus, we derived the perturbation equations (2.3.4), (2.3.5), (2.3.6), (2.3.7) and (2.3.8) representing the continuity equation, the zonal momentum equation, the meridional momentum equation, the hydrostatic balance equation and the thermodynamic energy equation, respectively, for Rossby waves and gravity waves.

We considered a Rossby wave perturbation sinusoidal and periodic in the x direction with a wavelength of $2\pi/k_R$ and a gravity wave perturbation sinusoidal and periodic in the x direction with a wavelength of $2\pi/k_G$ both generated on a plane surface in xyz space. Thus, the x domain is specified as $0 \leq x < 2\pi$.

The waves are forced on a plane defined by the line $\mu y + \Lambda z + b = Y_1, z \geq 0$ for $0 \leq x < 2\pi$. We chose the parameters so that $\mu \geq 0$, $\Lambda \leq 0$, $Y_1 \geq 1$ and $Y_1 > b$, as shown in Figure 2.4 so that the waves propagate southwards (in the direction of negative y) and upwards (in the direction of positive z). This is representative of a situation where waves generated in the troposphere in the mid-latitude region of the northern hemisphere propagate towards the equator and up in the stratosphere.

We considered Rossby waves and gravity waves as perturbations to a background flow with zonal mean speed $\bar{u}(y, z)$. In order to allow asymptotic analysis (as in chapters 3 and 4) and to facilitate the numerical discretization described in chapter 5, we considered the special case where $\bar{u}(y, z)$ is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$.

We carried out a weakly-nonlinear analysis of the perturbation equations (2.3.4)-(2.3.8) where the solutions can be written as perturbation series in powers of the parameters ε_R as in (3.1.1) and ε_G as in (3.1.2). Our derivation of the perturbation equations (2.3.4)-(2.3.8) allowed us to consider different possibilities for ε_R and ε_G . The nature of the solution depended on the relative magnitude of the two small parameters ε_R and ε_G . We wrote $\varepsilon_G = \varepsilon_R^p$, where $p > 0$. Therefore, we had cases where the Rossby wave functions are dominant (Case R1, Case R2 and Case R3), cases where the gravity wave functions are dominant (Case G1, Case G2 and Case G3), and the case where the Rossby wave and gravity wave functions have the same order of magnitude (Case RG).

In each case, we considered a configuration described by, for example, equations (3.2.5) in a domain such as that given in Figure 2.4 with a boundary condition generating Rossby waves of the form $e^{\pm i k_R}$ and gravity waves of the form $e^{\pm i k_G}$ where the wavenumbers k_R and k_G are positive integers.

The products of these functions with themselves and with each other generated combinations of wave modes $\pm(nk_R \pm mk_G)$, where n and m are positive integers. If $nk_R = mk_G$, the combinations $\pm nm(k_R - k_G)$ gave zero wavenumber components corresponding to a wave-induced mean flow. The magnitude of the wave-induced mean flow depended on the order at which these combinations occur.

With our choice of $\bar{u}$ where $\bar{u}(y, z)$ is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$, we rewrote the perturbation equations (2.3.4)-(2.3.8) in terms of a new variable Y defined as $Y = \mu y + \Lambda z + b$. In terms of this variable, the domain is $0 \leq x < 2\pi$ and $-\infty < Y \leq Y_1$ and the waves are forced at the boundary

$\mu y + \Lambda z + b = Y = Y_1$. As $Y \rightarrow -\infty$, the waves in the yz plane propagate in the direction of negative Y (i.e. southwards in the y -direction and upwards in the z -direction).

We chose $\bar{u}(y, z)$ such that there is a critical line in the yz plane where $\bar{u}$ is equal to the phase speed of both the waves. We considered a situation where both Rossby waves and gravity waves have the same zero phase speed, and thus their critical lines coincide at a value of $Y = Y_c$. This critical line is parallel to the forcing line $Y = Y_1$ in the yz plane and corresponds to a plane (critical surface) in xyz space as exhibited in Figure 2.4.

In order to study the interactions of Rossby waves, gravity waves and the background mean flow, it was convenient to combine the perturbation equations (2.3.4)-(2.3.8) into the equation (3.3.11) comprising the $\mathcal{O}(\varepsilon_R)$, $\mathcal{O}(\varepsilon_R^2)$ and $\mathcal{O}(\varepsilon_R \varepsilon_G)$ terms, and the equations (3.3.18) and (3.3.19) comprising the $\mathcal{O}(\varepsilon_G)$ and $\mathcal{O}(\varepsilon_G^2)$ terms. Setting $\varepsilon_G = 0$ in (3.3.11) allowed us to obtain the nonlinear quasi-geostrophic potential vorticity equation (3.3.12) describing the propagation of baroclinic Rossby waves only (without the gravity wave effects). Afterwards, we stated our nonlinear initial-boundary value problem that model the interactions of Rossby waves and gravity waves with the background shear flow in the vicinity of a critical line.

We then carried out a weakly nonlinear analysis of our initial-boundary value problem. The first step of the weakly-nonlinear analysis involved solving the linearized equations which appear at the leading order. The leading order terms in the weakly nonlinear formulation gave uncoupled linear equations for the Rossby wave and gravity wave perturbations which can be solved separately. In sections 4.1 and 4.2, we reviewed the linear Rossby wave solutions [Dickinson, (1970), Warn and Warn, (1976); Campbell and Maslowe (1998)] and gravity wave solutions [Booker and Bretherton, (1967)]. We discussed some properties of these solutions, including the dispersion relations and definitions of group velocity. This information was required for the weakly-nonlinear analysis (sections 4.3-4.5) and the numerical solutions

(chapter 5).

For Rossby waves and gravity waves, interactions with the background flow occur when there is a critical line where the wave phase speed is equal to the background wind speed. In the nonlinear time-dependent problem, there are interactions between waves and between the waves and the mean flow. The mean flow changes as a result of these interactions. For instance, the nonlinear terms in Rossby wave equations involve products of the Fourier modes. This means that, as the waves propagate, Fourier modes with Rossby wavenumbers $\pm 2k_R, \pm 3k_R, \dots$, develop in addition to the Rossby wavenumbers $k_R + (-k_R) = 0$. These are proportional to $e^0, e^{\pm 2ik_R x}, e^{\pm 3ik_R x}$ and so on. The zero wavenumber component corresponds to a change in the mean flow called the wave-induced mean flow.

The effects of the nonlinear Rossby wave interactions and nonlinear gravity wave interactions on the mean flow are well known. We reviewed these effects in sections 4.3 and 4.4. In addition, we discussed the effects of the nonlinear Rossby wave interactions and nonlinear gravity wave interactions on the mean temperature.

In section 4.5 where both Rossby waves and gravity waves are present and of the same order of magnitude ($\varepsilon_R = \varepsilon_G = \varepsilon$) and have the same forced wavenumbers ($k_R = k_G = 1$), the momentum flux divergence and the heat flux divergence comprised terms arising from Rossby wave nonlinear interactions, gravity wave nonlinear interactions and Rossby-gravity wave nonlinear interactions which all contribute to the wave-induced mean flow and the wave-induced mean temperature at $\mathcal{O}(\varepsilon)$.

In order to examine the behaviours of the mean flow and the mean temperature around the critical line, we derived the equations (4.5.16) and (4.5.30) describing the wave-induced mean flow acceleration with time t and the rate of change of the wave-induced mean temperature with time t , respectively, resulting from the nonlinear Rossby wave interactions, the nonlinear gravity wave interactions and the nonlinear interactions between Rossby waves and gravity waves.

Equation (4.5.16) which describes the wave-induced mean flow acceleration with

time t around the critical line is, i.e.,

$$\frac{\partial \bar{u}_0}{\partial t} = X_R + X_G + X_{RG},$$

where X_R is the Rossby wave drag force, X_G is the gravity wave drag force and X_{RG} is the drag force resulting from the nonlinear interactions of the Rossby-gravity waves. If the sum of the drag forces is positive, then the wave-induced mean flow acceleration becomes more positive or eastward and if the sum is negative, then the wave-induced mean flow acceleration becomes more negative or westward. If the sum of the drag forces has the same sign as the mean flow, then there is a positive acceleration either in the westward or eastward direction; otherwise, there is a deceleration (negative acceleration).

For large t , we determined the order of the drag force X_{RG} resulting from the nonlinear interactions between Rossby waves and gravity waves which is $\mathcal{O}(t^{1/2})$. The order of the Rossby wave drag force X_R and the gravity wave drag force X_G are well known to be $\mathcal{O}(1)$ and $\mathcal{O}(t^{1/2})$, respectively. For the reason that the wave-induced mean flow acceleration is proportional to the sum of the drag forces in (4.5.16), we found that the order of the wave-induced mean flow acceleration is $\mathcal{O}(t^{1/2})$, and consequently the order of the wave-induced mean flow velocity is $\mathcal{O}(t^{3/2})$.

Equation (4.5.30) which describes the rate of change of the wave-induced mean temperature with time around the critical line is, i.e.,

$$\begin{aligned} \frac{\partial \bar{T}_{0R}}{\partial t} + \frac{\partial \bar{T}_{0G}}{\partial t} = & \varepsilon \frac{\partial H_R}{\partial y} + \varepsilon \frac{\partial H_{RG1}}{\partial y} + \varepsilon \frac{\partial H_{RG2}}{\partial z} - \frac{N^2 T_s}{g} \bar{w}_{0R} - \varepsilon \frac{g}{c_p T_s} (\overline{w_G^{(0)} T_R^{(0)}}) \\ & + \varepsilon \frac{\partial H_G}{\partial z} - \varepsilon \frac{g}{c_p T_s} (\overline{w_G^{(0)} T_G^{(0)}}), \end{aligned}$$

so the rate of change of the wave-induced mean temperature with time t results from the meridional divergence of the meridional component of the zonal mean Rossby heat flux $\frac{\partial H_R}{\partial y}$, the meridional divergence of the zonal mean of the flux of gravity wave heat being transported to the meridional direction by the meridional Rossby wave velocity $\frac{\partial H_{RG1}}{\partial y}$, the vertical divergence of the zonal mean of the flux of Rossby wave

heat being transported to the vertical direction by the vertical gravity wave velocity $\frac{\partial H_{RG2}}{\partial z}$, the vertical divergence of the vertical component of the zonal mean gravity heat flux $\frac{\partial H_{G2}}{\partial z}$ plus three cooling terms.

Also, we noted that in Case RG, where both Rossby waves and gravity waves are present and of the same order of magnitude ($\varepsilon_R = \varepsilon_G = \varepsilon$), but have different forced wavenumbers $k_R = 1$ and $k_G = 2$, the wave-induced mean flow acceleration and the rate of change of the wave-induced mean temperature were affected by the nonlinear interactions of Rossby waves and the nonlinear interactions of gravity waves only at $\mathcal{O}(\varepsilon)$; and the nonlinear interactions between Rossby waves and gravity waves appear at high order.

In a numerical comparison, in chapter 5, between barotropic (pressure gradient force does not vary with z) and baroclinic (pressure gradient force varies with z) Rossby waves, we found that barotropic Rossby waves deposited more significant momentum in the background mean flow near the critical line compared to that in baroclinic Rossby waves. Thus, the background mean flow around the critical line in the barotropic case became more negative (westward) in comparison to that in the baroclinic case.

In Case R1 and Case R2, which are cases with dominant baroclinic Rossby wave effects (with minimal gravity wave effects), the wave-induced mean flow acceleration with time t around the critical line became more positive (eastward) compared to that in baroclinic Rossby waves (without the gravity wave effects), so the wave-induced mean flow velocity in Case R1 and Case R2 was more positive (eastward) compared to that in baroclinic Rossby wave case only. Moreover, in Case R2, where the magnitude of the gravity wave amplitude was larger than in Case R1, the wave-induced mean flow acceleration was more positive (eastward) than that in Case R1; consequently, the wave-induced mean flow velocity was more eastward in comparison to that in Case R1.

Furthermore, we investigated Case G1, which is a case with dominant gravity

wave effects (with minimal Rossby wave effects) with two different forced gravity wavenumbers $k_G = 1$ and $k_G = 2$. It was observed that the effects of the gravity drag force with the forced gravity wavenumber $k_G = 2$ was dominant; therefore, the wave-induced mean flow acceleration with time t in the vicinity of the critical line was more eastward (positive) compared to that with the forced gravity wavenumber $k_G = 1$. Thus, the wave-induced mean flow velocity in Case G1 ($k_G = 2$) was more eastward than that in Case G1 ($k_G = 1$).

In our Case RG, where the Rossby waves and gravity waves are present, of the same order of magnitude ($\varepsilon_R = \varepsilon_G = \varepsilon$) and have the same forced wavenumbers $k_R = k_G = 1$, the mean flow absorbed the waves at early time, so the wave-induced mean flow acceleration with time t at the critical line $Y = 0$ became negative (westward). Then, when the effects of the nonlinearity became evident, the wave-induced mean flow acceleration became less negative (eastward) during the wave reflection stage. At late time, the wave-induced mean flow acceleration was changing rapidly between westward and eastward directions (negative and positive values) as a consequence of the considerable effects of the drag forces X_R , X_G and X_{RG} .

In order to observe the effects of adding the gravity waves in our Case RG ($k_R = k_G$), we compared our results in Case RG ($k_R = k_G$) with the well-known results of baroclinic Rossby waves in the absence of gravity waves, as presented in section 5.4. The addition of the gravity waves in Case RG ($k_R = k_G$) had the effect of making the wave-induced mean flow acceleration with t more positive (more eastward) and hence making the wave-induced mean flow velocity $\bar{u}_{0R}$ more positive (more eastward) compared to that in baroclinic Rossby waves (without the gravity wave effects). The effects of the gravity waves caused a reduction in the amplitude of the zonal Rossby wave velocity perturbation u_R contrasted to u_R in baroclinic Rossby waves only. The rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ significantly increased with time t in the vicinity of the critical line in Case RG ($k_R = k_G$) compared to $\frac{\partial \bar{T}_{0R}}{\partial t}$ in baroclinic Rossby waves only and hence the wave-induced mean Rossby

wave temperature $\bar{T}_{0R}$ in Case RG ($k_R = k_G$) raised in comparison to that in baroclinic Rossby waves only as a consequence of adding the gravity wave effects. The last observation in this comparison was that the addition of the gravity waves increased the amplitude of the Rossby wave temperature perturbation $T_R + T_G$ in contrast to T_R without the gravity wave effects.

In our Case RG, where the Rossby waves and gravity waves are present and of the same order of magnitude ($\varepsilon_R = \varepsilon_G = \varepsilon$) but have different forced wavenumbers $k_R = 1 < k_G = 2$, the wave-induced mean flow accelerated with t in the westward direction at early time during the wave absorption stage, then accelerated in the eastward and westward directions at late time during the wave reflection. However, as noted in section 4.5, that at $\mathcal{O}(\varepsilon)$ in Case RG ($k_R < k_G$), the wave-induced mean flow acceleration is influenced by the nonlinear interactions of Rossby wave and the nonlinear interactions of gravity waves only; and the nonlinear interactions between Rossby waves and gravity waves develop at high order.

The effects of adding the gravity waves in our Case RG ($k_R < k_G$) were observed by comparing our results in Case RG ($k_R < k_G$) with the well-known results of baroclinic Rossby waves without gravity waves, as discussed in section 5.4. The addition of the gravity waves in Case RG ($k_R < k_G$) had the effect of making the wave-induced mean flow acceleration with t considerably more positive (more eastward) and hence making the wave-induced mean flow velocity $\bar{u}_{0R}$ more positive (more eastward) unlike $\bar{u}_{0R}$ in baroclinic Rossby waves only which became more westward. The effects of the gravity waves considerably reduced the amplitude of the zonal Rossby wave velocity perturbation u_R Case RG ($k_R < k_G$) compared to u_R in baroclinic Rossby waves only. The rate of change of the wave-induced mean Rossby wave temperature $\frac{\partial \bar{T}_{0R}}{\partial t}$ considerably increased with time t in the vicinity of the critical line in Case RG ($k_R < k_G$) compared to $\frac{\partial \bar{T}_{0R}}{\partial t}$ in baroclinic Rossby waves only and hence the wave-induced mean Rossby wave temperature $\bar{T}_{0R}$ in Case RG ($k_R < k_G$) significantly raised due to the gravity wave effects unlike $\bar{T}_{0R}$ in baroclinic Rossby waves only.

The last observation in this comparison was that the addition of the gravity waves insignificantly increased the amplitude of the Rossby wave temperature perturbation $T_R + T_G$ opposed to T_R without the gravity wave effects.

Finally, a comparison between our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) was made. Although the effect of the drag force resulting from the nonlinear interactions between Rossby waves and gravity waves X_{RG} does not appear at $\mathcal{O}(\varepsilon)$ in Case RG ($k_R < k_G$), the wave-induced mean flow acceleration with t in the vicinity of the critical line was more positive (eastward) compared to that in Case RG ($k_R = k_G$) for the reason that the effects of the gravity waves in Case RG ($k_R < k_G$) was relatively more considerable than that in Case RG ($k_R = k_G$). Therefore, the wave-induced mean velocity in Case RG ($k_R < k_G$) with time t around the critical line became more eastward compared to that in Case RG ($k_R = k_G$). The rate of change of the wave-induced mean temperature in Case RG ($k_R < k_G$) was approximately greater than that in Case ($k_R = k_G$), hence the wave-induced mean temperature in Case RG ($k_R < k_G$) was relatively warmer than that in Case RG ($k_R = k_G$). Finally, the sum of the amplitude of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ in Case RG ($k_R < k_G$) was relatively smaller than that in Case RG ($k_R = k_G$).

In section 1.2, we discussed previous observational and modelling studies that showed the effects of gravity wave interactions in configurations where Rossby waves were present. In the next section, we discuss the relevance of our results to these studies and observe that our conclusions agree with these studies.

6.2 Discussion

A number of previous observational, analytical and numerical studies by other researchers have investigated interactions between Rossby waves and gravity waves in regions in the stratosphere and mesosphere where both types of waves interact with

and modify the background mean wind and the background mean temperature. This section discusses our results relative to the conclusions of some associated studies, which were previously discussed in section 1.2.

As mentioned in section 1.2, Charney and Drazin (1961) proposed that the planetary scale disturbances could propagate vertically when the background winds are eastward relative to the phase speed and found that the energy in the form of heat and momentum flux carried by the small scale disturbances modifies the zonal mean flow. They also suggested that transient waves can interact with background flow and this interaction can bring the background winds toward the phase speed of the wave. As noted in section 1.2, Lindzen (1981) and Holton (1983) proposed that the drag forces due to the gravity wave breaking modifies the mean zonal winds to balance the momentum budget in the mesosphere. According to the linear saturation theory, the amplitude of the gravity wave increases exponentially as it propagates vertically upward in response to the decreasing atmospheric density and breaks at a particular altitude where it becomes convectively unstable [Ramesh *et al.*, (2013)], as mentioned in section 1.2. When the wave breaks, it deposits a net momentum in the mean flow and accelerates the background wind in the direction of the propagation of the wave as observed by Walterscheid, (1981), Gardner and Yang, (1998), Ramesh and Sridharan, (2012), as we discussed in section 1.2.

Our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) show that at early time the mean flow absorbed the waves, so the wave-induced mean flow acceleration around the critical line became negative (westward) with time t . Then, when the effects of the nonlinearity became evident, the wave-induced mean flow acceleration became less negative (eastward) during the wave reflection stage. At late time, the wave-induced mean flow acceleration was changing rapidly between westward and eastward directions (negative and positive values) due to the effects of the drag forces X_R , X_G and X_{RG} . We also observed that although the effect of the drag force coming from the nonlinear interactions between Rossby waves and gravity waves X_{RG} does

not appear at $\mathcal{O}(\varepsilon)$ in Case RG ($k_R < k_G$), the wave-induced mean flow acceleration with t in the vicinity of the critical line was more positive (eastward) compared to that in Case RG ($k_R = k_G$) because the effects of the drag force due to the gravity wave in Case RG ($k_R < k_G$) was relatively more considerable than that in Case RG ($k_R = k_G$), indicating that the deposition of momentum by gravity waves in the background mean flow around the critical line in Case RG ($k_R < k_G$) was less than that in Case RG ($k_R = k_G$). Consequently, the wave-induced mean velocity in Case RG ($k_R < k_G$) with time t around the critical line became more eastward compared to that in Case RG ($k_R = k_G$). Our results agree qualitatively with the results of Charney and Drazin (1961), Lindzen (1981), Holton (1983), Walterscheid, (1981), Gardner and Yang, (1998), Ramesh and Sridharan, (2012).

As presented in section 1.2, Holton (1983) suggested that the vertical eddy diffusion generated by gravity wave breaking modifies the zonal mean flow and causes the planetary wave dissipation in the mesosphere. As noted in section 1.2, Lindzen (1984) and Schoeberl and Strobel (1984) suggested that the amplitudes of *stationary* planetary waves may be reduced when they interact with internal gravity waves due to the momentum transferred by vertically propagating internal gravity waves. *Stationary* waves are waves whose amplitudes are independent of time with zero horizontal phase speed (as mentioned in chapter 1), so they can be presented as the form (4.1.3), i.e., $\psi_R(x, Y, t) = \hat{\psi}_R(Y) e^{ik_R(x-ct)} + \text{c.c.}$, with $c = 0$.

The foregoing studies motivated Miyahara (1984) to examine the interactions between *stationary* planetary waves and internal gravity waves using a simplified quasi-geostrophic model on a midlatitude β -plane. He assumed that the basic mean zonal wind varies only with height. In log-pressure coordinates, the linearized quasi-geostrophic potential vorticity equation in his model (equation (9)) is similar to our linear quasi-geostrophic potential vorticity equation (4.1.1), i.e.,

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} = 0,$$

but with $\frac{\partial}{\partial t} = 0$ and $\frac{\partial^2 \bar{u}}{\partial y^2} = 0$ since he considered *stationary* planetary waves and the

basic mean zonal wind is independent of y in his model. On the other hand, in our model, we considered time-dependent Rossby waves and the basic zonal mean velocity varies with both y and z . He revealed that the amplitudes of stationary planetary waves in the mesosphere are considerably reduced by the effects of drag forces due to internal gravity wave breaking as suggested by Holton (1983), Lindzen (1984) and Schoeberl and Strobel (1984).

Our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) illustrate that the amplitude of zonal Rossby wave velocity perturbation u_R was reduced due to the gravity wave effects. Furthermore, the amplitude of u_R was considerably reduced in Case RG ($k_R < k_G$) compared to that in Case RG ($k_R = k_G$) as a result of the considerable effects of the gravity waves in Case RG ($k_R < k_G$) in comparison to that in Case RG ($k_R = k_G$). Our results are consistent with the results of Holton (1983), Lindzen (1984), Schoeberl and Strobel (1984) and Miyahara (1984).

As presented in section 1.2, Ramesh *et al.*, (2013) discussed planetary wave-gravity wave interactions during mesospheric inversion layer events. Mesospheric inversion layers (MILs) are a context in which gravity waves and planetary Rossby waves are known to occur together. A mesospheric inversion layer is recognized as the layer of enhanced temperature with several kilometres showing the positive lapse rate at the bottom and is superposed upon the characteristically decreasing temperature in the mesosphere [Ramesh *et al.*, (2013)]. Meriwether and Gardner (2000) and Meriwether and Gerrard (2004) proposed several causative mechanisms for forming these layers. One of them includes the gravity wave breaking and the planetary wave critical level interaction. Ramesh *et al.*, (2013) revealed that the gravity wave breaking causes the mesospheric temperature inversion, and the mesospheric temperature inversion amplitudes got modified by the interactions of gravity waves and planetary waves.

Our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) show that the sum of the Rossby wave and gravity wave temperature perturbations $T_R + T_G$ increased with

altitude z near the critical line before it decreased swiftly with altitude z . Not only that, but we also noticed that the increase in $T_R + T_G$ with altitude z near the critical line in Case RG ($k_R < k_G$) was greater than that in Case RG ($k_R = k_G$). Our results agree qualitatively with the results of Meriwether and Gardner (2000), Meriwether and Gerrard (2004) and Ramesh *et al.*, (2013).

As mentioned in section 1.2 that according to the linear saturation theory, Walterscheid (1981), Gardner and Yang (1998), Ramesh and Sridharan (2012) noted that when a gravity wave breaks at a particular altitude, it caused a downward heat flux resulting in cooling of the atmosphere at the breaking height.

Our results in Case RG ($k_R = k_G$) and Case RG ($k_R < k_G$) show that the rate of change of the wave-induced mean temperature in the vicinity of the critical line considerably decreased with time t during the wave absorption stage at early time. During the wave reflection, when the effects of the nonlinearity became apparent, the rate of change of the wave-induced mean temperature in the vicinity of the critical line increased with time t due to the effects of the meridional and vertical divergence of the heat fluxes resulting from the Rossby wave self-interactions, the gravity wave self-interactions and the nonlinear interactions between Rossby waves and gravity waves. Additionally, we observed that the rate of change of the wave-induced mean temperature with t around the critical line in Case RG ($k_R < k_G$) was approximately more significant than that in Case ($k_R = k_G$); thus, the wave-induced mean temperature in Case RG ($k_R < k_G$) was relatively warmer than that in Case RG ($k_R = k_G$). Our results are in agreement with the results of Walterscheid (1981), Gardner and Yang (1998), Ramesh and Sridharan (2012).

In climate modelling, researchers and modellers need to include the gravity wave effects from wave saturation and critical layers using wave drag parameterization schemes. This is because the gravity waves are relatively small-scale, and when their scale is less than or of the same order of length as the grid-scale, they are not adequately resolved by the numerical scheme [Campbell and Shepherd, (2005a,b)]. The

gravity wave momentum flux, drag and heat flux are represented by expressions in sections 4.4 and 4.5 based on weakly-nonlinear theory and added to the numerical model to forcing terms. The expressions used in the parameterization schemes do not take into account gravity wave interactions with Rossby waves. Our investigation includes these interactions and hence can be used to improve the representation of gravity wave effects in parameterization schemes.

In this thesis, we considered the particular case where $\bar{u}(y, z)$ is a function of the linear combination $\bar{u}(y, z) = \mu y + \Lambda z + b$ to allow asymptotic analysis and facilitate the numerical discretization. Our choice of $\bar{u}(y, z)$ enables us to introduce the variable $Y = \mu y + \Lambda z + b$ and solve the 2D problem in terms of x and Y . More general $\bar{u}(y, z)$ profiles with no linear combination of y and z could be considered to try to solve the 3D problem asymptotically and numerically. If we could not be able to do asymptotic analysis, we could then carry out 2D numerical simulations similar to what we did in chapter 5, but they would be more expensive. Furthermore, we could solve the 3D problem numerically, which would be more costly. The case studied in this thesis is a good starting point that can be used for comparison with the other more complicated cases.

6.3 Future work

In this section, we propose some of the future projects that can be considered in relation to what we have done in the thesis. These include, for instance,:

Evaluate the second harmonic terms in the weakly-nonlinear analysis:

The weakly-nonlinear analysis in section 4.5 can be continued to give us information about the second harmonic terms that arise from the nonlinear interactions between Rossby waves and gravity waves. The particular solutions of the nonhomogenous equation (4.3.7) in Rossby wave case without the gravity wave effects (section 4.3)

can be written as in (4.3.8), i.e.,

$$\psi_R^{(1)} = \psi_{0R}^{(1)}(Y, t) + \psi_{2R}^{(1)}(Y, t) e^{2ik_R x} + \text{c.c.},$$

the $\pm 2k_R$ terms were discussed by Warn and Warn (1978). The particular solutions of the nonhomogenous equations (4.4.10) and (4.4.11) in gravity wave case (section 4.4) can be written as in (4.4.12) and (4.4.13), i.e.,

$$\psi_G^{(1)} = \psi_{0G}^{(1)}(Y, t) + \psi_{2G}^{(1)}(Y, t) e^{2ik_G x} + \text{c.c.},$$

$$T_G^{(1)} = T_{0G}^{(1)}(Y, t) + T_{2G}^{(1)}(Y, t) e^{2ik_G x} + \text{c.c.},$$

the $\pm 2k_G$ terms were discussed by Brown and Stewartson (1980, 1982a,b).

In our configuration where both Rossby waves and gravity waves are present, the solution as a perturbation series can be represented as in (4.5.14), i.e.,

$$\psi_R + \psi_G \sim \psi_R^{(0)} + \psi_G^{(0)} + \varepsilon \psi_R^{(1)} + \varepsilon \psi_{RG}^{(1)} + \varepsilon \psi_G^{(1)}.$$

Substituting (4.5.14) into (3.3.11) and (4.4.1) leads to the second harmonic terms $\pm 2k_R, \pm 2k_G$ and $\pm(k_R \pm k_G)$ in our Case RG. As discussed in subsection 3.2.7 that at $\mathcal{O}(\varepsilon^2)$, the nonlinear Rossby terms correspond to wavenumbers 0, $\pm 2k_R$, the nonlinear gravity terms correspond to wavenumbers 0, $\pm 2k_G$ and the nonlinear Rossby-gravity terms correspond to wavenumbers $\pm(k_R \pm k_G)$. These wavenumbers at each order in Case RG are shown in Table 3.5. We could derive asymptotic expressions for these terms. However, it can be noted that the results may be different depending on whether $k_R = k_G$ or $k_R < k_G$. Although we did not evaluate these terms in section 4.5, these terms were all included in the numerical simulations and they are shown in Figures 5.39 and 5.48 showing the Fourier spectrum. The second harmonics correspond to the peaks in the curve seen at $\pm 2k_R, \pm 2k_G$ and $\pm(k_R \pm k_G)$.

Other $\bar{u}(Y)$ profiles in the numerical solution:

We could consider different $\bar{u}(Y)$ profiles in the numerical simulation as long as the conditions for the stability of the Rossby wave and gravity wave solutions are satisfied.

As discussed in section 5.1, we had to choose $\beta > (\mu^2 + \kappa \Lambda^2) \frac{\partial^2 \bar{u}}{\partial Y^2}$ everywhere in the domain to satisfy the converse of the Rayleigh-Kuo criterion that guarantees stability of the Rossby wave solutions. In addition, we had to choose the Richardson number, $R_i = N^2 / (\Lambda \frac{\partial \bar{u}(Y_c)}{\partial Y})^2$, to be bigger than $\frac{1}{4}$ in order to ensure that the mean shear flow is stable with respect to small disturbances [Miles, (1961); Howard, (1961)].

In this thesis, we chose the background flow velocity to be $\bar{u}(Y) = \tanh(Y)$ as shown in Figure 5.1 for the reason that $\tanh(Y) \sim Y$ for Y close to zero, so we could compare the numerical solution with $\bar{u}(Y) = \tanh(Y)$ with the analytical solution for $\bar{u}(Y) = Y$ discussed in sections 4.1 and 4.2. Also, we wanted $\bar{u}(Y)$ to be approximately constant near the upper and lower boundaries. We could consider different $\bar{u}(Y)$ profiles in the numerical simulation that also satisfy this, for example, a jet like $\bar{u}(Y) = \text{sech}^2(Y)$ used in some previous studies. This profile is called a Bickley jet and its stability is discussed by, for instance, Maslowe (1991) and Swaters (1999). In our case we would have a jet profile in both the y and z directions since Y depends on y and z .

More general $\bar{u}(y, z)$ profile:

We could consider general $\bar{u}(y, z)$ profiles with no linear combination of y and z . For instance, we could carry out a weakly-nonlinear analysis of the nonlinear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (3.3.12), i.e.

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) + \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \kappa \frac{\partial^2 \bar{u}}{\partial z^2} \right) \frac{\partial \psi_R}{\partial x} \\ + \varepsilon_R \left[\frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] = 0. \end{aligned}$$

We could consider an initial zonal mean background flow with velocity

$\bar{u}(y, z) = \bar{u}(Y)$, where Y is a function of y and z with continuous first and second partial derivatives with respect to y and z , and $\bar{u}$ and ψ_R are analytic (infinitely

differentiable) functions of Y . Thus, (3.3.12) can be written as

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right] \\ & + \left\{ \beta - \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial y} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial y^2} \right] - \kappa \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial z} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial z^2} \right] \right\} \frac{\partial \psi_R}{\partial x} \\ & + \varepsilon_R \left[\frac{\partial \psi_R}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) - \frac{\partial \psi_R}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R + \kappa \frac{\partial^2 \psi_R}{\partial z^2} \right) \right] = 0. \end{aligned} \quad (6.3.1)$$

Therefore, the solutions can be written as perturbation series in powers of the non-linear parameter ε_R as in (4.3.3). We substitute (4.3.3) into (6.3.1) and group terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon_R)$. At $\mathcal{O}(1)$, we get

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \bar{u}(Y) \frac{\partial}{\partial x} \right] \left[\nabla^2 \psi_R^{(0)} + \kappa \frac{\partial^2 \psi_R^{(0)}}{\partial z^2} \right] \\ & + \left\{ \beta - \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial y} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial y^2} \right] - \kappa \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial z} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial z^2} \right] \right\} \frac{\partial \psi_R^{(0)}}{\partial x} = 0. \end{aligned} \quad (6.3.2)$$

Thus, $\psi_R^{(0)}$ can be written as in (4.3.5), i.e. $\psi_R^{(0)} = \hat{\psi}_R(Y, t) e^{ik_R x} + \text{c.c.}$

At $\mathcal{O}(\varepsilon_R)$, we find that $\psi_R^{(1)}$ satisfies the equation

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right] \left[\nabla^2 \psi_R^{(1)} + \kappa \frac{\partial^2 \psi_R^{(1)}}{\partial z^2} \right] \\ & + \left\{ \beta - \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial y} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial y^2} \right] - \kappa \left[\frac{\partial^2 \bar{u}}{\partial Y^2} \left(\frac{\partial Y}{\partial z} \right)^2 + \frac{\partial \bar{u}}{\partial Y} \frac{\partial^2 Y}{\partial z^2} \right] \right\} \frac{\partial \psi_R^{(1)}}{\partial x} \\ & = - \left[\frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi_R^{(0)} + \kappa \frac{\partial^2 \psi_R^{(0)}}{\partial z^2} \right) - \frac{\partial \psi_R^{(0)}}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi_R^{(0)} + \kappa \frac{\partial^2 \psi_R^{(0)}}{\partial z^2} \right) \right]. \end{aligned} \quad (6.3.3)$$

If we substitute from (4.3.5) into the right hand side of (6.3.3), then we should write $\psi_R^{(1)}$ as in (4.3.8), i.e. $\psi_R^{(1)} = \psi_{0R}^{(1)}(Y, t) + \psi_{2R}^{(1)}(Y, t) e^{2ik_R x} + \text{c.c.}$

In order to compute the wave-induced mean flow, we take the x -average of (6.3.3) over the zonal wavelength $\lambda_x = \frac{2\pi}{k_R}$ as we did in section 4.3. Substituting (4.3.8) into the left-hand side of (6.3.3) and applying (4.3.2) to (6.3.3) leads to

$$\frac{\partial}{\partial t} \left\{ \left[\left(\frac{\partial Y}{\partial y} \right)^2 + \kappa \left(\frac{\partial Y}{\partial z} \right)^2 \right] \frac{\partial^2 \bar{\psi}_{0R}^{(1)}}{\partial Y^2} + \left[\frac{\partial^2 Y}{\partial y^2} + \kappa \frac{\partial^2 Y}{\partial z^2} \right] \frac{\partial \bar{\psi}_{0R}^{(1)}}{\partial Y} \right\}$$

$$\begin{aligned}
&= - \left\{ \frac{k_R}{2\pi} \int_0^{\frac{2\pi}{k_R}} \left\{ \frac{\partial Y}{\partial y} \left[\left(\frac{\partial Y}{\partial y} \right)^2 + \kappa \left(\frac{\partial Y}{\partial z} \right)^2 \right] \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^3} \right. \right. \\
&\quad + \left[3 \frac{\partial Y}{\partial y} \frac{\partial^2 Y}{\partial y^2} + \kappa \left(\frac{\partial Y}{\partial y} \frac{\partial^2 Y}{\partial z^2} + 2 \frac{\partial Y}{\partial z} \frac{\partial^2 Y}{\partial y \partial z} \right) \right] \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial^2 \psi_R^{(0)}}{\partial Y^2} \\
&\quad + \left[\frac{\partial^3 Y}{\partial y^3} + \kappa \frac{\partial^3 Y}{\partial y \partial z^2} \right] \frac{\partial \psi_R^{(0)}}{\partial x} \frac{\partial \psi_R^{(0)}}{\partial Y} \\
&\quad \left. \left. - \frac{\partial Y}{\partial y} \left[\left(\frac{\partial Y}{\partial y} \right)^2 + \frac{\partial^2 Y}{\partial y^2} \right] + \left[\kappa \left(\frac{\partial Y}{\partial z} \right)^2 + \kappa \frac{\partial^2 Y}{\partial z^2} \right] \frac{\partial \psi_R^{(0)}}{\partial Y} \frac{\partial^3 \psi_R^{(0)}}{\partial Y^2 \partial x} \right\} dx. \right. \tag{6.3.4}
\end{aligned}$$

Equation (6.3.4) includes additional factors involving derivatives with respect to y and z . Therefore, we may not be able to do asymptotic analysis. However, we could then carry out 2D numerical simulations similar to what we did in chapter 5, but they would be more expensive.

More general $\bar{u}(y, z)$ profile (3D problem):

We could consider general $\bar{u}(y, z)$ profiles with no linear combination of y and z . We could solve the 3D problem numerically which would be more expensive. The case studied in this thesis is a good starting point that can be used for comparison with the other more complicated cases listed in this section.

Parameterization:

Because gravity waves are relatively small scale and their scale is less than or of the same order of length as the grid-scale, they are not properly resolved by the numerical scheme. Therefore, researchers and modellers in climate modelling need to include the gravity wave effects from wave saturation and critical layers using wave drag parameterization schemes. We could use the expressions for the momentum flux, drag, heat flux and the cooling terms for our Case RG (section 4.5) to develop a parameterization scheme for gravity wave interactions with Rossby waves and the mean flow. These expressions could then be added to numerical models where the

gravity waves are not adequately resolved to represent the missing effects of the unresolved gravity waves.

Appendix A

Fluid equations under the influence of the Coriolis force

A.1 The fundamental equations of fluid dynamics in isobaric coordinates

In this section, we transform the fundamental equations of fluid dynamics presented in section 2.1 from height coordinates to isobaric coordinates. We make use of scale analysis and the geostrophic balance (2.1.12) and (2.1.13). Scale analysis is used to determine whether some terms in the fundamental equations are negligible for motions of meteorological concern.

There are approximate expressions for the relationships among velocity, pressure, and temperature fields, which are useful in the analysis of weather systems. These are discussed most conveniently using a coordinate system in which pressure is the vertical coordinate. A coordinate system in which pressure is the vertical coordinate is called *an isobaric coordinate* [Holton,(2004)].

From the hydrostatic equation (2.1.7), $\frac{\partial p}{\partial z} = -\rho g$, we note that a single-valued monotonic relationship exists between pressure and height in each vertical column of

the atmosphere. Therefore, we may use pressure as the independent vertical coordinate and height as a dependent variable.

The fundamental equations of fluid dynamics, written in section 2.1, are expressed in height coordinates (where x, y, z, t are the independent variables). They can be expressed in isobaric coordinates (where x, y, p, t are the independent variables).

Some texts cover the derivation of the fundamental equations of fluid dynamics in isobaric coordinates from first principles. Procedures of this can be found in [Gill, (1982)] (pp. 180-182), [Holton, (2004)] (pp. 57-59), or [Wallace and Hobbs, (2006)] (pp. 291-294). In the following paragraphs, transforming the fundamental equations of fluid dynamics from height coordinates to isobaric coordinates is primarily based on that in Holton (2004).

The continuity equation in isobaric coordinates

Before we begin to express the continuity equation in isobaric coordinates, we need to represent gravity in terms of the gradient of a scalar potential function which we want in the derivation of the horizontal momentum equations in isobaric coordinates. Thus, the gradient of a scalar potential function Φ is defined as

$$\nabla\Phi = -\mathbf{g}.$$

However, since $\mathbf{g} = -g\hat{k}$ where $g \equiv |\mathbf{g}|$, we note that $\Phi = \Phi(z)$ and $d\Phi/dz = g$.

In order to transform equations from a system in which height z is an independent variable to one in which pressure p is the vertical coordinate, we have the following coordinate transformations for an arbitrary variable quantity A :

$$\left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_p - \left(\frac{\partial A}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p,$$

where the partial derivative of $A(x, y, z(x, y, p), t)$ with respect to x (keeping y, p, t fixed) is given by the chain rule in terms of $\frac{\partial A}{\partial x}$ and $\frac{\partial A}{\partial z} \frac{\partial z}{\partial x}$. The subscripts mean variables that remain fixed in evaluating the differentials.

The transformation of the derivative of p with respect to x from height coordinates to isobaric coordinates is given by

$$\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_z = \left(\frac{\partial \Phi}{\partial x} \right)_p. \quad (\text{A.1.1})$$

Similarly, the derivative of p with respect to y is transformed from height coordinates to isobaric coordinates as follows

$$\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)_z = \left(\frac{\partial \Phi}{\partial y} \right)_p. \quad (\text{A.1.2})$$

The continuity equation (2.1.1), $\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$, which is expressed in height coordinates, can be expressed in isobaric coordinates. First, we transform the first term on the left hand side of (2.1.1), $\frac{1}{\rho} \frac{D\rho}{Dt}$, as follows. We note that

$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right).$$

Now, the partial derivative of $\rho(x, y, z(x, y, p), t)$ with respect to t , x , y and z are respectively

$$\left(\frac{\partial \rho}{\partial t} \right)_z = \left(\frac{\partial \rho}{\partial t} \right)_p - \left(\frac{\partial \rho}{\partial z} \right)_t \left(\frac{\partial z}{\partial t} \right)_p, \quad (\text{A.1.3})$$

$$\left(\frac{\partial \rho}{\partial x} \right)_z = \left(\frac{\partial \rho}{\partial x} \right)_p - \left(\frac{\partial \rho}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_p, \quad (\text{A.1.4})$$

$$\left(\frac{\partial \rho}{\partial y} \right)_z = \left(\frac{\partial \rho}{\partial y} \right)_p - \left(\frac{\partial \rho}{\partial z} \right)_y \left(\frac{\partial z}{\partial y} \right)_p, \quad (\text{A.1.5})$$

$$\frac{\partial \rho}{\partial z} = -\rho g \frac{\partial \rho}{\partial p}. \quad (\text{A.1.6})$$

The first term on the right hand side of (A.1.3) is equal to zero because

$$\left(\frac{\partial \rho}{\partial t} \right)_p = \left(\frac{\partial}{\partial t} \right)_p \left(\frac{-1}{g} \frac{\partial p}{\partial z} \right) = \frac{-1}{g} \frac{\partial}{\partial z} \left(\frac{\partial p}{\partial t} \right)_p = 0.$$

Thus, we use the same argument to show that

$$\left(\frac{\partial \rho}{\partial x} \right)_p = 0 \quad \text{and} \quad \left(\frac{\partial \rho}{\partial y} \right)_p = 0.$$

Now, since $\left(\frac{\partial \rho}{\partial z}\right)_t = -\rho g \frac{\partial \rho}{\partial p}$ and $\left(\frac{\partial z}{\partial t}\right)_p = \frac{1}{g} \left(\frac{\partial \Phi}{\partial t}\right)_p$, equation (A.1.3) can be written as

$$\left(\frac{\partial \rho}{\partial t}\right)_z = \rho \left(\frac{\partial \Phi}{\partial t}\right)_p \frac{\partial \rho}{\partial p}.$$

Also, by using the same argument, equations (A.1.4) and (A.1.5) can be written as

$$\begin{aligned} \left(\frac{\partial \rho}{\partial x}\right)_z &= \rho \left(\frac{\partial \Phi}{\partial x}\right)_p \frac{\partial \rho}{\partial p}, \\ \left(\frac{\partial \rho}{\partial y}\right)_z &= \rho \left(\frac{\partial \Phi}{\partial y}\right)_p \frac{\partial \rho}{\partial p}. \end{aligned}$$

We note that

$$\frac{\partial \rho}{\partial p} = \frac{\partial}{\partial p} \left(\frac{-1}{g} \frac{\partial p}{\partial z} \right) = \frac{-1}{g} \frac{\partial}{\partial z} \left(\frac{\partial p}{\partial p} \right) = 0.$$

Therefore, the first term on the left hand side of (2.1.1), $\frac{1}{\rho} \frac{D\rho}{Dt}$, is equal to zero.

Now, we just need to transform the second term on the left hand side of (2.1.1) from height coordinates to isobaric coordinates as listed below.

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \tag{A.1.7}$$

The partial derivative of $u(x, y, z(x, y, p), t)$ with respect to x (keeping y, p, t fixed) and the partial derivative of $v(x, y, z(x, y, p), t)$ with respect to y (keeping x, p, t fixed) are respectively

$$\left(\frac{\partial u}{\partial x}\right)_z = \left(\frac{\partial u}{\partial x}\right)_p, \tag{A.1.8}$$

$$\left(\frac{\partial v}{\partial y}\right)_z = \left(\frac{\partial v}{\partial y}\right)_p \tag{A.1.9}$$

As for the partial derivative of $w(x, y, z(x, y, p), t)$ with respect to z , we use the hydrostatic balance (2.1.7), $\frac{\partial p}{\partial z} = -\rho g$, to get

$$\frac{\partial w}{\partial z} = -\rho g \frac{\partial w}{\partial p}. \tag{A.1.10}$$

Also, since $w = \frac{Dz}{Dt}$ and $\omega = \frac{Dp}{Dt}$, we have

$$w = \frac{\omega}{-\rho g}. \quad (\text{A.1.11})$$

Now, if we substitute (A.1.8), (A.1.9), (A.1.10) and (A.1.11) in (A.1.7), we attain

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0. \quad (\text{A.1.12})$$

Equation (A.1.12) is the continuity equation in isobaric coordinates. We can see that equation (A.1.12) contains no reference to the density field and does not include time derivatives. The simplicity of (A.1.12) is one of the principal advantages of the isobaric coordinate system.

The horizontal momentum equations in isobaric coordinates

The horizontal momentum equations, without the viscous effects, (2.1.10) can be written using (A.1.10), (A.1.11), (A.1.1) and (A.1.2) as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x}, \quad (\text{A.1.13})$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} + fu = -\frac{\partial \Phi}{\partial y}. \quad (\text{A.1.14})$$

Equations (A.1.13) and (A.1.14) can be written in vectorial form as

$$\frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla_p \Phi, \quad (\text{A.1.15})$$

where $\mathbf{V} = u\hat{i} + v\hat{j}$ is the horizontal velocity vector and ∇_p is the horizontal gradient operator applied with pressure held constant. Equation (A.1.15) is the horizontal momentum equations expressed in isobaric coordinates.

Because p is the independent vertical coordinate, the total derivative in (A.1.15) must be expanded as

$$\begin{aligned} \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt} \frac{\partial}{\partial x} + \frac{Dy}{Dt} \frac{\partial}{\partial y} + \frac{Dp}{Dt} \frac{\partial}{\partial p} \\ &= \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p}, \end{aligned} \quad (\text{A.1.16})$$

where $\frac{Dp}{Dt} \equiv \omega$ (normally called the omega vertical motion) is the pressure change following the motion, which plays the same role in the isobaric coordinate system that $w = \frac{Dz}{Dt}$ plays in height coordinates.

In order to simplify (A.1.15) for synoptic scale (synoptic scale in the atmosphere is a horizontal length scale of the order of thousands of kilometres) motions, the following characteristic scales of the field variables based on observed values for midlatitude synoptic systems are defined in Table A.1 [Holton, (2004)].

Table A.1: The characteristic scales of the field variables

Variable	Characteristic Value	Description
u, v	$U \sim 10 \text{ m s}^{-1}$	Horizontal velocity scale
w	$W \sim 1 \text{ cm s}^{-1}$	Vertical velocity scale
x, y	$L \sim 10^6 \text{ m}$	Length scale
z	$H \sim 10^4 \text{ m}$	Height scale
$\frac{1}{\rho} p$	$\frac{\delta p}{\rho} \sim 10^3 \text{ m}^2 \text{ s}^{-2}$	Horizontal pressure fluctuation scale
t	$\frac{L}{U} \sim 10^5 \text{ s}$	Time scale
f	$f_0 \sim 10^{-4} \text{ s}^{-1}$	Coriolis parameter

Here, the time scale is an advective time scale, which is appropriate for pressure systems that move at approximately the speed of the horizontal wind, as is observed for synoptic scale motions. Hence, L/U is the time required to travel a distance L at speed U , and the differential operator scales as $D/Dt \sim U/L$ for such motions.

The characteristic magnitudes of each term in (A.1.15) based on the scaling considerations given in Table A.1 are respectively

$$\begin{aligned}
 \frac{D\mathbf{V}}{Dt} &\sim \frac{U^2}{L} = 10^{-4} \text{ m s}^{-2}, \\
 f\hat{k} \times \mathbf{V} &\sim f_0 U = 10^{-3} \text{ m s}^{-2}, \\
 -\nabla_p \Phi &\sim \frac{\delta p}{\rho L} = 10^{-3} \text{ m s}^{-2}.
 \end{aligned}$$

According to these values, it is seen that for midlatitude synoptic-scale disturbances the Coriolis force and the geopotential gradient are in approximate balance. Keeping only these two terms in (A.1.15) yields to an approximation which is called the geostrophic balance. The geostrophic balance in height coordinates, which is shown in (2.1.12) and (2.1.13), can be written in vectorial form as

$$f\mathbf{V}_g \equiv \hat{k} \times \frac{1}{\rho} \nabla p, \quad (\text{A.1.17})$$

where $\mathbf{V}_g = u_g \hat{i} + v_g \hat{j}$, is called the geostrophic wind. The geostrophic balance in isobaric coordinates, which can be noted from (A.1.15), is

$$f\mathbf{V}_g \equiv \hat{k} \times \nabla_p \Phi. \quad (\text{A.1.18})$$

If we compare (A.1.17) and (A.1.18), we can see that density ρ does not appear in the geostrophic balance in isobaric coordinates. Consequently, a given geopotential gradient implies the same geostrophic wind at any height, whereas a given horizontal pressure gradient implies different values of the geostrophic wind depending on the density. This is another advantage of using the isobaric coordinates.

It can be noted that the geostrophic balances (A.1.17) and (A.1.18) have no reference to time and consequently cannot be used to predict the evolution of the velocity field. Therefore, it is necessary to retain the acceleration term in (A.1.15) in order to predict the evolution of the velocity field.

A useful measure of the magnitude of the acceleration compared to the Coriolis force can be obtained by forming the ratio of the characteristic scales for the acceleration and the Coriolis force terms: $(U^2/L)/(f_0 U)$. This ratio is a nondimensional number called the Rossby number after the Swedish meteorologist Carl-Gustaf Arvid Rossby (1898-1957) and is designated by

$$R_0 = \frac{U}{f_0 L} = 10^{-1}. \quad (\text{A.1.19})$$

Therefore, the smallness of the Rossby number is a measure of the validity of the geostrophic approximation.

The hydrostatic balance in isobaric coordinates

The hydrostatic balance (2.1.7), $\frac{\partial p}{\partial z} = -\rho g$, can also be transformed from height coordinates to isobaric coordinates. We need first to present the following equation of state for dry air which is used for the derivation of the hydrostatic balance in isobaric coordinates:

$$p = \rho RT \quad \text{or} \quad p\alpha = RT, \quad (\text{A.1.20})$$

where $\alpha \equiv \rho^{-1}$ is the specific volume and R is the gas constant for dry air.

Now, from the hydrostatic balance (2.1.7), we have

$$\begin{aligned} \frac{\partial p}{\partial z} = -\rho g &\implies \frac{\partial p}{-\rho} = g dz = \partial\Phi, \quad \text{since } d\Phi/dz = g, \\ \frac{\partial\Phi}{\partial p} &= \frac{1}{-\rho} = -\alpha, \\ \frac{\partial\Phi}{\partial p} &= \frac{-RT}{p}. \end{aligned} \quad (\text{A.1.21})$$

Equation (A.1.21) is the hydrostatic balance in isobaric coordinates.

The thermodynamic energy equation in isobaric coordinates

The thermal energy equation (2.1.8) can be transformed from height coordinates to isobaric coordinates as follows. Equation (2.1.8) can be written in more familiar form as follows. Firstly, from (2.1.1), we note that

$$\begin{aligned} \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} &= 0, \\ \frac{1}{\rho} \nabla \cdot \mathbf{u} &= \frac{-1}{\rho^2} \frac{D\rho}{Dt} = \frac{D\alpha}{Dt}, \quad \text{since } \alpha = \rho^{-1}. \end{aligned} \quad (\text{A.1.22})$$

Secondly, the internal energy per unit mass, for dry air, is given by

$$e = c_v T, \quad (\text{A.1.23})$$

where c_v is the specific heat at constant volume. If we now divide (2.1.8) by ρ and substitute (A.1.22) and (A.1.23) in (2.1.8), we obtain

$$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J, \quad (\text{A.1.24})$$

which is the customary form of the thermodynamic energy equation. Therefore, the first law of thermodynamics indeed is applicable to a fluid in motion.

The total derivative of the equation of state for dry air (A.1.20) is

$$\begin{aligned}\frac{D(p\alpha)}{Dt} &= R \frac{DT}{Dt}, \\ p \frac{D\alpha}{Dt} + \alpha \frac{Dp}{Dt} &= R \frac{DT}{Dt}.\end{aligned}\tag{A.1.25}$$

If we substitute for $p \frac{D\alpha}{Dt}$ in (A.1.24) and use $c_p = c_v + R$, where c_p is the specific heat at constant pressure, we can rewrite the first law of thermodynamics (A.1.24) as

$$\begin{aligned}c_v \frac{DT}{Dt} + \left(R \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} \right) &= J, \\ (c_v + R) \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} &= J, \\ c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} &= J\end{aligned}\tag{A.1.26}$$

Equation (A.1.26) is the thermodynamic energy equation which we want to transform from height coordinates to isobaric coordinates. If we expand the total derivative $\frac{DT}{Dt}$ in (A.1.26) by applying (A.1.16) and note that $\frac{Dp}{Dt} = \omega$, we obtain

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha \omega = J.\tag{A.1.27}$$

If we divide equation (A.1.27) by c_p , equation (A.1.27) can be rewritten as follows

$$\left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - S_p \omega = \frac{J}{c_p},\tag{A.1.28}$$

where $S_p = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p}$. Equation (A.1.28) is the thermodynamic energy equation in isobaric coordinates. The expression of S_p can be written with the aid of the equation of state for dry air (A.1.20) as

$$S_p = \frac{RT}{c_p p} - \frac{\partial T}{\partial p}.\tag{A.1.29}$$

A.2 Quasi-geostrophic analysis

Synoptic scale in the atmosphere is a horizontal length scale of the order of thousands of kilometres. Rossby waves occur on this scale and effects the weather system.

Gravity waves occur on a smaller scale called mesoscale which is of the order of hundreds of kilometres.

The governing equations for the atmospheric fluid flows are quite complicated even when the hydrostatic approximation (which is valid for all large-scale meteorological systems) is applied. However, for midlatitude synoptic-scale motions, the horizontal velocities are approximately geostrophic. Such motions are commonly referred to as quasi-geostrophic [Holton, (2004)].

In this section, we show that for motions that are hydrostatic and nearly geostrophic, the three-dimensional flow field is determined approximately by the isobaric distribution of geopotential $\Phi(x, y, p, t)$. For this analysis, it is convenient to use the isobaric coordinate system which simplifies the development of approximate equations.

The fundamental equations of fluid dynamics in isobaric coordinates under the hydrostatic approximation developed in section A.1 are repeated here for reference. The continuity equation, the horizontal momentum equation, the hydrostatic equation, and the thermodynamic energy equation are expressed, respectively, as

$$\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \quad (\text{A.2.1})$$

$$\frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla\Phi, \quad (\text{A.2.2})$$

$$\frac{\partial \Phi}{\partial p} = \frac{-RT}{p}, \quad (\text{A.2.3})$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) T - S_p \omega = \frac{J}{c_p}. \quad (\text{A.2.4})$$

The total derivative in (A.2.2) is defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)_p + \omega \frac{\partial}{\partial p}, \quad (\text{A.2.5})$$

where $\omega \equiv \frac{Dp}{Dt}$ is the pressure change following the motion (defined earlier).

These equations include several terms that are of secondary significance for midlatitude synoptic-scale systems, so they can be simplified further by making a geostrophic approximation as follows. This approximation is justified by con-

sidering that the magnitude of the ratio of vertical velocity to horizontal velocity based on the scaling considerations given in Table A.1 is of order 10^{-3} (i.e. $W/U = 1 \text{ cm s}^{-1}/10^3 \text{ cm s}^{-1}$).

Firstly, the horizontal velocity is separated into geostrophic and ageostrophic parts by letting

$$\mathbf{V} = \mathbf{V}_g + \mathbf{V}_a, \quad (\text{A.2.6})$$

where the geostrophic wind is defined as

$$\mathbf{V}_g \equiv f_0^{-1} \hat{k} \times \nabla \Phi, \quad (\text{A.2.7})$$

and the ageostrophic wind, $\mathbf{V}_a$, is the difference between the total horizontal wind and the geostrophic wind.

On the synoptic scale, the length scale, L , is small compared to the radius of the earth, so the geostrophic wind (A.2.7) can be defined using a constant reference latitude value of the Coriolis parameter f_0 . For the synoptic scale system, $|\mathbf{V}_a|/|\mathbf{V}_g| \sim O(R_0)$, that is the ratio of the magnitudes of the ageostrophic and geostrophic winds is the same order of magnitude as the Rossby number (A.1.19).

The horizontal momentum can be approximated to $O(R_0)$ by its geostrophic value, and the rate of change of momentum or temperature following the horizontal motion can be approximated to the same order by the rate of change following the geostrophic wind.

Therefore, $\mathbf{V}$, in (A.2.5), can be replaced by $\mathbf{V}_g$ and the vertical advection ($\omega \frac{\partial}{\partial p}$), which arises only from the ageostrophic flow, can be neglected. The rate of change of momentum following the total motion is approximately equal to the rate of change of the geostrophic momentum following the geostrophic wind:

$$\frac{D\mathbf{V}}{Dt} \approx \frac{D_g \mathbf{V}_g}{Dt},$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}. \quad (\text{A.2.8})$$

Although a constant f_0 can be used in defining $\mathbf{V}_g$, it is still indispensable to keep the dynamical effect of the variation of the Coriolis parameter with latitude in the Coriolis force term in the momentum equation.

As presented in subsection 2.1.3, the variation of f with latitude can be approximately represented by expanding f in a Taylor series about the central latitude $\theta = \theta_0$ and approximating by the first two terms:

$$f = f_0 + \beta y, \quad (\text{A.2.9})$$

where $\beta \equiv (df/dy)_{\theta=\theta_0} = 2\Omega \cos \theta_0 / R_{\text{earth}}$ is the gradient of planetary vorticity in y -direction and R_{earth} is the radius of the earth. For synoptic-scale motions, the ratio of the first two terms in the expansion of f has an order of magnitude:

$$\frac{\beta L}{f_0} \sim \frac{\cos \theta_0}{\sin \theta_0} \frac{L}{R_{\text{earth}}} \sim O(R_0) \ll 1,$$

[Holton, (2004)]. This justifies letting the Coriolis parameter have a constant value f_0 in the geostrophic approximation and approximating its variation in the Coriolis force term by (A.2.9).

From (A.2.2), we see that the acceleration following the motion is equal to the difference between the Coriolis force and the pressure gradient force. This difference depends on the departure of the actual wind from the geostrophic wind. Thus, it is not allowable to just replace the horizontal velocity by its geostrophic value in the Coriolis term [Holton, (2004)].

Rather, we use (A.2.6), (A.2.7) and (A.2.9) to write

$$\begin{aligned} f \hat{k} \times \mathbf{V} + \nabla \Phi &= (f_0 + \beta y) \hat{k} \times (\mathbf{V}_g + \mathbf{V}_a) - f_0 \hat{k} \times \mathbf{V}_g \\ &\approx f_0 \hat{k} \times \mathbf{V}_a + \beta y \hat{k} \times \mathbf{V}_g, \end{aligned}$$

where the geostrophic relation (A.2.7) is used to eliminate the pressure gradient force and the ageostrophic wind is neglected compared to the geostrophic wind in the term proportional to βy . Consequently, the form of the approximate horizontal momentum

equation is

$$\frac{D_g \mathbf{V}_g}{Dt} + f_0 \hat{k} \times \mathbf{V}_a + \beta y \hat{k} \times \mathbf{V}_g = 0. \quad (\text{A.2.10})$$

The continuity equation (A.2.1) can be rewritten as follows.

$$\begin{aligned} \nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} &= 0, \\ \nabla \cdot (\mathbf{V}_g + \mathbf{V}_a) + \frac{\partial \omega}{\partial p} &= 0, \\ \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} &= 0, \end{aligned} \quad (\text{A.2.11})$$

since the divergence of the geostrophic wind defined in (A.2.7) vanishes. Equation (A.2.11) shows that if the geostrophic wind is defined by (A.2.7), ω is determined only by the ageostrophic part of the wind field.

In the thermodynamic energy equation (A.2.4), the horizontal advection can be approximated by its geostrophic value. However, the vertical advection is not neglected but forms part of the adiabatic heating and cooling term. This term must be retained because the static stability is ordinarily large enough on the synoptic scale so that the adiabatic heating or cooling due to vertical motion is of the same order as the horizontal temperature advection, notwithstanding the smallness of the vertical velocity [Holton, (2004)].

The adiabatic heating and cooling term can be slightly simplified by dividing the total temperature field, $T_{\text{tot.}}$, into a basic state (standard atmosphere) part that depends only on pressure, $T_0(p)$, plus a deviation from the basic state, $T(x, y, p, t)$ as follows

$$T_{\text{tot.}}(x, y, p, t) = T_0(p) + T(x, y, p, t),$$

[Holton, (2004)].

Now because $|dT_0/dp| \gg |\partial T/\partial p|$, only the basic state part of the temperature field needs to be included in the static stability term and the quasi-geostrophic thermodynamic energy equation (A.2.4) may be expressed in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla \right) T - \left(\frac{\sigma p}{R} \right) \omega = \frac{J}{c_p}, \quad (\text{A.2.12})$$

where $\sigma \equiv -\frac{RT_0}{p} \frac{d \ln \theta_0}{dp}$ and θ_0 is the potential temperature corresponding to the basic state temperature T_0 . If we use (A.2.3), we can write the thermodynamic energy equation in terms of the geopotential field as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla \right) \left(-\frac{\partial \Phi}{\partial p} \right) - \sigma \omega = \frac{R}{p} \frac{J}{c_p}. \quad (\text{A.2.13})$$

Equations (A.2.7), (A.2.10), (A.2.11), and (A.2.13) constitute the quasi-geostrophic equations. These form a complete set in the dependent variables Φ , $\mathbf{V}_g$, $\mathbf{V}_a$, and ω (provided that the diabatic heating rate is known).

A.3 The fundamental equations of fluid dynamics in the log-pressure coordinates

The vertical coordinate, in the *log-pressure coordinates*, is defined as

$$z = -H \ln \left(\frac{p}{p_s} \right), \quad (\text{A.3.1})$$

where p_s is a standard reference pressure and H is a standard scale height, $H = \frac{RT_s}{g}$ with T_s a global average temperature [Holton, (2004)].

For the special case of an isothermal atmosphere (an atmosphere in hydrostatic equilibrium, in which the temperature is constant with height and the pressure decreases exponentially upward) at temperature T_s and the density profile is given by the reference density

$$\rho_0(z) = \rho_s e^{\frac{-z}{H}}, \quad (\text{A.3.2})$$

where ρ_s is the density at $z = 0$ [Holton, (2004)].

The vertical velocity in the log-pressure coordinates is

$$w = \frac{Dz}{Dt}. \quad (\text{A.3.3})$$

Now, we can transform the continuity equation (A.1.12), the horizontal momentum equations (A.1.15), the hydrostatic balance (A.1.21) and the thermodynamic

energy equation (A.2.4) from the isobaric coordinates to the log-pressure coordinate as listed below.

The continuity equation in log-pressure coordinates

The continuity equation (A.1.12), $(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})_p + \frac{\partial \omega}{\partial p} = 0$, can be transformed from isobaric coordinates to the log-pressure coordinates as follows. We first note that $w = \frac{Dz}{Dt}$ and by using (A.3.1), we obtain

$$w = -\frac{H}{p} \frac{Dp}{Dt} = -\frac{H}{p} \omega.$$

We can now find

$$\frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p} \left(\frac{pw}{H} \right) = -\frac{p}{H} \frac{\partial w}{\partial p} - \frac{w}{H} = \frac{\partial w}{\partial z} - \frac{w}{H}, \quad \text{since } \partial z = -\frac{H}{p} \partial p.$$

We can use (A.3.2) to write $\frac{\partial \omega}{\partial p}$ as follows:

$$\frac{\partial \omega}{\partial p} = \frac{\partial w}{\partial z} - \frac{w}{H} = \frac{1}{\rho_0} \frac{\partial(\rho_0 w)}{\partial z}. \quad (\text{A.3.4})$$

Therefore, the continuity equation in log-pressure coordinates becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0} \frac{\partial(\rho_0 w)}{\partial z} = 0. \quad (\text{A.3.5})$$

The horizontal momentum equations in log-pressure coordinates

The horizontal momentum equations (A.1.15) in the log-pressure system is the same as that in the isobaric system:

$$\frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla\Phi. \quad (\text{A.3.6})$$

However, the total derivative $\frac{D}{Dt}$ is defined in the log-pressure coordinate as

$$\begin{aligned} \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla + w \frac{\partial}{\partial z}. \end{aligned}$$

The hydrostatic equation in log-pressure coordinates

The log-pressure form of the hydrostatic equation, $\frac{\partial \Phi}{\partial p} = \frac{-RT}{p}$, can be obtained by transforming from the isobaric coordinate form (A.1.21). If we multiply (A.1.21) by p , we obtain

$$\frac{\partial \Phi}{\partial \ln p} = -RT.$$

If we divide this equation, $\frac{\partial \Phi}{\partial \ln p} = -RT$, by $-H$ and use (A.3.1), we get

$$\frac{\partial \Phi}{\partial z} = \frac{RT}{H} = \frac{g}{T_s} T \quad (\text{A.3.7})$$

Equation (A.3.7) is the hydrostatic balance in the log-pressure coordinates.

The thermodynamic energy equation in log-pressure coordinates

The thermodynamic energy equation (A.2.4) can be expressed in log-pressure coordinates in the following way. We first use the static stability parameter for the isobaric system $S_p = \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$. Then, we use $(w = -\frac{H}{p} \omega)$ and $(\partial z = -\frac{H}{p} \partial p)$ to write (A.2.4) as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \left(\frac{g}{c_p T_s} T + \frac{\partial T}{\partial z} \right) = \frac{J}{c_p}. \quad (\text{A.3.8})$$

The fourth term in the left-hand side of (A.3.8) can be slightly simplified by dividing the total temperature field into a basic state (standard atmosphere) part that depends only on y and z , $\bar{T}(y, z)$, plus a deviation from the basic state, $T_R(x, y, z, t)$, such that $T_R \ll \bar{T}$. For the reason that $|\partial \bar{T} / \partial z| \gg |\partial T_R / \partial z|$, only the basic state part of the temperature field needs to be included in the static stability term and (A.3.8) can be written as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) = \frac{J}{c_p}. \quad (\text{A.3.9})$$

Now, if we multiply (A.3.9) by $\frac{g}{T_s}$ and use (A.3.7), we can write (A.3.9) as

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\partial \Phi}{\partial z} + N^2 w = \frac{g}{T_s} \frac{J}{c_p}, \quad (\text{A.3.10})$$

where

$$N^2 = \frac{g}{T_s} \left(\frac{g}{c_p T_s} \bar{T} + \frac{\partial \bar{T}}{\partial z} \right) \quad (\text{A.3.11})$$

is the buoyancy frequency squared.

Appendix B

Time-dependent solutions

B.1 Time-dependent linear solution for baroclinic Rossby waves

In this section, we present the asymptotic solution (4.1.26) to the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves (4.1.1). As presented in subsection 4.1.2, we have $\bar{u}(Y) = Y$, $\delta = 0$ and $\kappa = \beta = 1$, so equation (4.1.1) is written in terms of Y as

$$(\mu^2 + \Lambda^2) \left(\frac{\partial}{\partial t} + Y \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 \psi_R}{\partial Y^2} \right) + \frac{\partial \psi_R}{\partial x} = 0. \quad (\text{B.1.1})$$

Substituting (4.1.25), i.e. $\psi_R(x, Y, t) = \hat{\psi}_R(Y, t) e^{ik_R x} + \text{c.c.}$, into (B.1.1) leads to

$$(\mu^2 + \Lambda^2) \left[\frac{\partial}{\partial t} + ik_R Y \right] \frac{\partial^2 \hat{\psi}_R}{\partial Y^2} + ik_R \hat{\psi}_R = 0, \quad (\text{B.1.2})$$

with the initial condition $\hat{\psi}_R(Y, 0) = 0$ when $Y < Y_1$ and the boundary condition $\hat{\psi}_R(Y_1, t) = 1$. Equation (B.1.2) can be solved by taking a Laplace transform in time t . The Laplace transform of $\hat{\psi}_R(Y, t)$ is defined by

$$\tilde{\psi}_R(Y, s) = \int_0^\infty \hat{\psi}_R(Y, t) e^{-st} dt.$$

Taking the Laplace transform of (B.1.2) gives

$$\left(\mu^2 + \Lambda^2\right) \left[Y - \frac{is}{k_R}\right] \frac{\partial^2 \tilde{\psi}_R}{\partial Y^2} + \tilde{\psi}_R = 0, \quad (\text{B.1.3})$$

with the following initial and boundary conditions

$$\begin{aligned} \tilde{\psi}_R(Y, 0) &= 0, \\ \tilde{\psi}_R(Y_1, s) &= \frac{1}{s}. \end{aligned} \quad (\text{B.1.4})$$

The solution of (B.1.3) with the initial and boundary conditions (B.1.4) can be expressed in terms of the modified Bessel functions of first kind $\mathcal{I}_1$ and second kind $\mathcal{K}_1$ of order 1 as we did in subsection 4.1.1. Consequently, the solution of (B.1.3) with the initial and boundary conditions (B.1.4) is given by

$$\tilde{\psi}_R(Y, s) = \frac{\left(\frac{is}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]}{s \left(\frac{is}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y_1}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]}. \quad (\text{B.1.5})$$

We can find $\hat{\psi}_R(Y, t)$ by inverting the Laplace transform as follows

$$\hat{\psi}_R(Y, t) = \mathcal{L}^{-1}[\tilde{\psi}_R(Y, s)] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \tilde{\psi}_R(Y, s) e^{st} ds = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{G(k, s)}{s} e^{st} ds, \quad (\text{B.1.6})$$

where

$$G(k, s) = \frac{\left(\frac{is}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]}{\left(\frac{is}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y_1}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]}.$$

Our goal now is to calculate $\mathcal{L}^{-1}\left[\frac{G(k, s)}{s}\right]$ as follows

$$\mathcal{L}^{-1}\left[\frac{G(k, s)}{s}\right] = \frac{1}{2\pi i} \int_{\iota-i\infty}^{\iota+i\infty} \frac{\left(\frac{is}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]}{s \left(\frac{is}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y_1}{\mu^2 + \Lambda^2}\right)^{\frac{1}{2}}\right]} e^{st} ds, \quad (\text{B.1.7})$$

where $\iota > 0$ is a real number chosen so that the contour of integration will lie to the right of all the singularities of the integrand. The integral in (B.1.7) has three

singularities which are at $s = 0$, $s = -ik_R Y$ and $s = -ik_R Y_1$. The integral in (B.1.7) is evaluated along the contour in the imaginary plane shown in Figure B.1.

Figure B.1: Contour of integration in the complex plane for (B.1.7)

The integral in (B.1.7) is the sum of the contributions from the three singularities:

- the pole at $s = 0$,
- the following branch points:
 1. at the point $\mathcal{P}$ where $s = -ik_R Y$,
 2. at the point $\mathcal{P}_1$ where $s = -ik_R Y_1$.

Consequently,

$$\mathcal{L}^{-1} \left[\frac{G(k, s)}{s} \right] = \{\text{residue at the pole}\} - \frac{1}{2\pi i} \{\text{contributions from the branch points}\}.$$

The residue at the pole $s = 0$ is given by

$$\lim_{s \rightarrow 0} \left[s \frac{G(k, s)}{s} \right] = \frac{(-Y)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{(-Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}. \quad (\text{B.1.8})$$

It can be noted that (B.1.8) is the steady-amplitude solution (4.1.12), with $\kappa = \beta = 1$ and $Y_c = 0$ that was obtained in subsection 4.1.4., to the linear quasi-geostrophic potential vorticity equation (4.1.1).

In the paragraphs that follow, we calculate the integral (B.1.7) along the contours C_1 , C_2 , C_3 , C_4 , C_5 , C_6 and C_7 as shown in Figure B.1. The contour C_1 can be described in polar coordinates (ρ, θ) as

$$s = \iota + R e^{i\theta}, \quad \text{where} \quad \frac{\pi}{2} < \theta < \frac{-\pi}{2},$$

hence,

$$\begin{aligned} & \int_{C_1} \frac{\left(\frac{\iota s}{k_R} - Y \right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{\iota s}{k_R} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{s \left(\frac{\iota s}{k_R} - Y_1 \right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{\iota s}{k_R} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} e^{st} ds \\ &= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\left(\frac{\iota(\iota + R e^{i\theta})}{k} - Y \right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{\iota(\iota + R e^{i\theta})}{k} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{(\iota + R e^{i\theta}) \left(\frac{\iota(\iota + R e^{i\theta})}{k} - Y_1 \right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{\iota(\iota + R e^{i\theta})}{k} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} e^{(\iota + R e^{i\theta})t} i R e^{i\theta} d\theta, \end{aligned} \quad (\text{B.1.9})$$

as $R \rightarrow \infty$, the integral (B.1.9) has the order of $e^{R e^{i\theta}} = e^{R(\cos \theta + i \sin \theta)}$. For the reason that $\cos \theta$ is negative when $\theta \in (\frac{\pi}{2}, \frac{-\pi}{2})$, then the integral (B.1.9) along the contour C_1 converges to zero.

The contour C_2 can be described in polar coordinates (ρ, θ) as

$$s = r e^{i\theta} - i k_R Y, \quad \text{where} \quad \pi < \theta < -\pi,$$

hence,

$$\begin{aligned}
& \int_{C_2} \frac{\left(\frac{is}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{s \left(\frac{is}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} e^{st} ds \\
&= \int_{\pi}^{-\pi} \frac{\left(\frac{i(re^{i\theta} - ik_R Y)}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{i(re^{i\theta} - ik_R Y)}{k_R} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right] e^{(re^{i\theta} - ik_R Y)t} ir e^{i\theta}}{(re^{i\theta} - ik_R Y) \left(\frac{i(re^{i\theta} - ik_R Y)}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{i(re^{i\theta} - ik_R Y)}{k_R} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} d\theta,
\end{aligned} \tag{B.1.10}$$

as $r \rightarrow 0$, the integral (B.1.10) has the order of r . Thus, the integral (B.1.10) along the contour C_2 converges to zero as $r \rightarrow 0$. In a similar way, it can be proved that the integral (B.1.7) along the contour C_3 (described in polar coordinates (ϱ, θ) as $s = re^{i\theta} - ik_R Y_1$, where $\pi < \theta < -\pi$) converges to zero as $r \rightarrow 0$.

The contours C_4 and C_5 are lines that can be described as

$$C_4 : s = \varrho e^{i\pi} - ik_R Y, \quad \text{where } \varrho : -\infty \rightarrow 0,$$

$$C_5 : s = \varrho e^{-i\pi} - ik_R Y, \quad \text{where } \varrho : 0 \rightarrow -\infty.$$

Hence,

$$\begin{aligned}
& \int_{C_4} \frac{\left(\frac{is}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{s \left(\frac{is}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{is}{k_R} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} e^{st} ds \\
&= \int_{-\infty}^0 \frac{\left(\frac{i(\varrho e^{i\pi} - ik_R Y)}{k_R} - Y\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{i(\varrho e^{i\pi} - ik_R Y)}{k_R} - Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right] e^{(\varrho e^{i\pi} - ik_R Y)t} e^{i\pi}}{(\varrho e^{i\pi} - ik_R Y) \left(\frac{i(\varrho e^{i\pi} - ik_R Y)}{k_R} - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{\frac{i(\varrho e^{i\pi} - ik_R Y)}{k_R} - Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} d\varrho \\
&= - \int_{-\infty}^0 \frac{e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} \right] e^{-ik_R Y t} e^{-\varrho t}}{(-\varrho - ik_R Y) \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}} \right]} d\varrho.
\end{aligned} \tag{B.1.11}$$

In a similar manner, we obtain the following expression along the contour C_5

$$-\int_0^{-\infty} \frac{-e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} \mathcal{K}_1\left[-\frac{2}{\sqrt{\mu^2+\Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}}\right] e^{-ik_R Y t} e^{-\varrho t}}{(-\varrho - ik_R Y) \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}}\right]} d\varrho. \quad (\text{B.1.12})$$

The two expressions (B.1.11) and (B.1.12) give the only non-zero contributions from the branch point $\mathcal{P}$ where $s = -ik_R Y$. Consequently, $I_{\mathcal{P}}$ is equal to their sum,

$$\begin{aligned} I_{\mathcal{P}} = & -\int_0^{\infty} \frac{e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} e^{-ik_R Y t} e^{-\varrho t}}{(-\varrho - ik_R Y) \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}}\right]} \\ & \times \left[\mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}}\right] + \mathcal{K}_1\left[-\frac{2}{\sqrt{\mu^2+\Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}}\right] \right] d\varrho, \\ I_{\mathcal{P}} = & \int_0^{\infty} \frac{e^{-ik_R Y t} e^{-\varrho t} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} i\pi \mathcal{I}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}}\right]}{(-\varrho - ik_R Y) \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} \left(\frac{-i\varrho}{k_R} + Y - Y_1\right)^{\frac{1}{2}}\right]} d\varrho. \end{aligned} \quad (\text{B.1.13})$$

The integral $I_{\mathcal{P}}$ (B.1.13) is quite difficult to allow the integral to be evaluated exactly. However, if t is sufficiently large, we can obtain an asymptotic approximation for $I_{\mathcal{P}}$.

It may be observed that for large t , most of the contribution to the integral $I_{\mathcal{P}}$ will be produced when ϱ is small. Therefore, we can expand the integrand in powers of ϱ and approximate it by the first term in the series in the outer region far away from the critical surface where $|Y|t \gg 1$. The integrand is simplified to obtain

$$\begin{aligned} I_{\mathcal{P}} & \sim i\pi e^{-ik_R Y t} \int_0^{\infty} \frac{e^{-\varrho t} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R}\right)^{\frac{1}{2}}}{(-ik_R Y) (Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} (Y - Y_1)^{\frac{1}{2}}\right]} d\varrho, \\ I_{\mathcal{P}} & \sim i\pi e^{-ik_R Y t} \int_0^{\infty} \frac{e^{-\varrho t} e^{\frac{3\pi i}{2}} \left(\frac{\varrho}{k_R}\right)}{(-ik_R Y) (Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} (Y - Y_1)^{\frac{1}{2}}\right]} d\varrho, \\ I_{\mathcal{P}} & \sim \frac{i\pi e^{-ik_R Y t}}{k_R^2 Y (Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1\left[\frac{2}{\sqrt{\mu^2+\Lambda^2}} (Y - Y_1)^{\frac{1}{2}}\right]} \int_0^{\infty} e^{-\varrho t} \varrho d\varrho. \end{aligned}$$

In order to integrate $e^{-\varrho t}$ ϱ with respect to ϱ , we perform integration by parts by differentiating ϱ and integrating $e^{-\varrho t}$ to obtain

$$I_{\mathcal{P}} \sim \frac{i\pi e^{-ik_R Y t}}{k_R^2 Y (Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y - Y_1)^{\frac{1}{2}} \right] t^2}. \quad (\text{B.1.14})$$

Similarly, we can find the contribution to the integral (B.1.7) from the branch point $\mathcal{P}1$ where $s = -ik_R Y_1$. The contribution is obtained by integrating along C_6 and C_7 :

$$I_{\mathcal{P}1} = - \int_0^\infty \frac{e^{-ik_R Y_1 t} e^{-\varrho t} \left(\frac{-i\varrho}{k_R} + Y_1 - Y \right)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} \left(\frac{-i\varrho}{k_R} + Y_1 - Y \right)^{\frac{1}{2}} \right]}{(-\varrho - ik_R Y_1) e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R} \right)^{\frac{1}{2}}} \times \left\{ \frac{1}{\mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R} \right)^{\frac{1}{2}} \right]} + \frac{1}{\mathcal{K}_1 \left[-\frac{2}{\sqrt{\mu^2 + \Lambda^2}} e^{\frac{3\pi i}{4}} \left(\frac{\varrho}{k_R} \right)^{\frac{1}{2}} \right]} \right\} d\varrho. \quad (\text{B.1.15})$$

Again, the integral $I_{\mathcal{P}1}$ (B.1.15) is quite difficult to allow the integral to be evaluated exactly. But, if t is sufficiently large, we can obtain an asymptotic approximation for $I_{\mathcal{P}1}$. For large t , the integrand can be approximated by the first term in the series expansion, and after simplifying and integrating, it is found that

$$I_{\mathcal{P}1} \sim \frac{-4i\pi e^{-ik_R Y_1 t} (Y_1 - Y)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y_1 - Y)^{\frac{1}{2}} \right]}{(\mu^2 + \Lambda^2) k_R^2 Y_1 t^2}. \quad (\text{B.1.16})$$

Therefore, the time-dependent solution of equation (B.1.1) is

$$\psi_R(x, Y, t) = \mathcal{L}^{-1} \left[\frac{G(k, s)}{s} \right] e^{ik_R x},$$

where the inverse Laplace transform is the sum of the residue at the pole and the

quantity $(-2\pi i)^{-1} (I_{\mathcal{P}} + I_{\mathcal{P}_1})$, i.e.

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{G(k, s)}{s} \right] &= \frac{(-Y)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]}{(-Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[2 \left(\frac{-Y_1}{\mu^2 + \Lambda^2} \right)^{\frac{1}{2}} \right]} \\ &\quad - \frac{e^{-ik_R Y t}}{2 k_R^2 Y (Y - Y_1)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y - Y_1)^{\frac{1}{2}} \right] t^2} \\ &\quad + \frac{2 e^{-ik_R Y_1 t} (Y_1 - Y)^{\frac{1}{2}} \mathcal{K}_1 \left[\frac{2}{\sqrt{\mu^2 + \Lambda^2}} (Y_1 - Y)^{\frac{1}{2}} \right]}{(\mu^2 + \Lambda^2) k_R^2 Y_1 t^2}. \end{aligned} \quad (\text{B.1.17})$$

B.2 Time-dependent linear solution for hydrostatic gravity waves

In this section, we follow Booker and Bretherton (1967) to present the asymptotic solution (4.2.21) to the linear streamfunction perturbation equation that describes the propagation of gravity waves (4.2.5). As presented in subsection 4.2.2, we have $\bar{u}(Y) = Y$, so equation (4.2.5) is written in terms of Y as in (4.2.6), i.e.

$$\Lambda^2 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \frac{\partial^2 \psi_G}{\partial Y^2} + N^2 \frac{\partial^2 \psi_G}{\partial x^2} = 0.$$

Substituting (4.2.20), i.e. $\psi_G(x, Y, t) = \hat{\psi}_G(Y, t) e^{ik_G x} + \text{c.c.}$, in (4.2.6) leads to

$$\Lambda^2 \left(\frac{\partial}{\partial t} + ik_G Y \right)^2 \frac{\partial^2 \hat{\psi}_G}{\partial Y^2} + (ik_G)^2 N^2 \hat{\psi}_G = 0, \quad (\text{B.2.1})$$

with the initial condition $\hat{\psi}_G(Y, 0) = 0$ when $Y < Y_1$ and the boundary condition $\hat{\psi}_G(Y_1, t) = 1$. Equation (B.2.1) can be solved by taking a Laplace transform in time t . The Laplace transform of $\hat{\psi}_G(Y, t)$ is defined by

$$\tilde{\psi}_G(Y, s) = \int_0^\infty \hat{\psi}_G(Y, t) e^{-st} dt,$$

where s is a complex variable. Taking the Laplace transform of (B.2.1) gives

$$\frac{\partial^2 \tilde{\psi}_G}{\partial Y^2} + \frac{N^2}{\Lambda^2 \left(Y - \frac{is}{k_G} \right)^2} \tilde{\psi}_G = 0, \quad (\text{B.2.2})$$

with the initial and boundary conditions, respectively,

$$\tilde{\psi}_G(Y, 0) = 0, \quad \text{and} \quad \tilde{\psi}_G(Y_1, s) = \frac{1}{s}. \quad (\text{B.2.3})$$

Equation (B.2.2) is similar to (4.2.10) with the only difference being that in (B.2.2) we have $\frac{is}{k_G}$ instead of Y_c in (4.2.10). Therefore, the solution of (B.2.2) corresponding to a downward-propagating wave since the forcing is at the upper boundary $Y = Y_1$ is

$$\tilde{\psi}_G(Y, s) \sim A(s) \left(Y - \frac{is}{k_G} \right)^{\frac{1}{2} + i\tilde{\alpha}} \quad (\text{B.2.4})$$

where $\tilde{\alpha} = \sqrt{R_i - \frac{1}{4}} = \sqrt{\frac{N^2}{\Lambda^2} - \frac{1}{4}}$ and $R_i = \frac{N^2}{(\Lambda \frac{\partial u}{\partial Y})^2}$ is the Richardson number as presented earlier in subsection 4.2.1. After applying the boundary condition $\tilde{\psi}_G(Y_1, s) = \frac{1}{s}$, we attain

$$\tilde{\psi}_G(Y, s) \sim \frac{\left(Y - \frac{is}{k_G} \right)^{\frac{1}{2} + i\tilde{\alpha}}}{s \left(Y_1 - \frac{is}{k_G} \right)^{\frac{1}{2} + i\tilde{\alpha}}}. \quad (\text{B.2.5})$$

The solution $\hat{\psi}_G(Y, t)$ of (B.2.1) is obtained by taking the inverse Laplace transform of (B.2.5) as follows

$$\hat{\psi}_G(Y, t) = \mathcal{L}^{-1}[\tilde{\psi}_G(Y, s)] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \tilde{\psi}_G(Y, s) e^{st} ds. \quad (\text{B.2.6})$$

The integral (B.2.6) is evaluated by integrating along a straight vertical line from $s = \iota - iR$ to $s = \iota + iR$ in the complex plane and then taking the limit as $R \rightarrow \infty$. The real constant ι is chosen such that the contour of integration will lie to the right of all the singularities of the integrand. The integral (B.2.6) has three singularities which are at $s = 0$, $s = -ik_G Y$ and $s = -ik_G Y_1$. Thus, we must choose $\iota > 0$ and the integral (B.2.6) can be written as

$$\hat{\psi}_G(Y, t) = \mathcal{L}^{-1}[\tilde{\psi}_G(Y, s)] = \frac{1}{2\pi i} \int_{\iota - iR}^{\iota + iR} \tilde{\psi}_G(Y, s) e^{st} ds. \quad (\text{B.2.7})$$

The integral (B.2.7) is evaluated along the contour in the imaginary plane shown in Figure B.1. The integral (B.2.7) is found to be equal to the sum of the contributions from the three singularities:

- the pole at $s = 0$,
- the following branch points:
 1. at the point $\mathcal{P}$ where $s = -ik_G Y$,
 2. at the point $\mathcal{P}_1$ where $s = -ik_G Y_1$.

Consequently,

$$\mathcal{L}^{-1}[\tilde{\psi}_G(Y, s)] = \{\text{residue at the pole}\} - \frac{1}{2\pi i} \{\text{contributions from the branch points}\}.$$

The residue at the pole $s = 0$ is given by

$$\lim_{s \rightarrow 0} \left[s \tilde{\psi}_G(Y, s) \right] = \frac{(Y)^{\frac{1}{2} + i\tilde{\alpha}}}{(Y_1)^{\frac{1}{2} + i\tilde{\alpha}}}, \quad (\text{B.2.8})$$

where $\tilde{\psi}_G(Y, s)$ as in (B.2.5). It can be noted that (B.2.8) is the steady amplitude solution (4.2.13) with $Y_c = 0$, that was obtained in subsection 4.2.1.

In the paragraphs that follow, we calculate the integral (B.2.7) along the contours $C_1, C_2, C_3, C_4, C_5, C_6$ and C_7 as shown in Figure B.1. The contour C_1 can be described in polar coordinates (ϱ, θ) as

$$s = \iota + R e^{i\theta}, \quad \text{where} \quad \frac{\pi}{2} < \theta < \frac{-\pi}{2},$$

hence,

$$\int_{C_1} \frac{\left(Y - \frac{\iota s}{k_G}\right)^{\frac{1}{2} + i\tilde{\alpha}}}{s \left(Y_1 - \frac{\iota s}{k_G}\right)^{\frac{1}{2} + i\tilde{\alpha}}} e^{st} ds = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\left(Y - \frac{\iota(\iota + R e^{i\theta})}{k_G}\right)^{\frac{1}{2} + i\tilde{\alpha}} e^{(\iota + R e^{i\theta})t} i R e^{i\theta}}{(\iota + R e^{i\theta}) \left(Y_1 - \frac{\iota(\iota + R e^{i\theta})}{k_G}\right)^{\frac{1}{2} + i\tilde{\alpha}}} d\theta.$$

As $R \rightarrow \infty$, the integral along the contour C_1 has the order of $e^{Re^{i\theta}} = e^{R(\cos\theta + i\sin\theta)}$. For the reason that $\cos\theta$ is negative when $\theta \in (\frac{\pi}{2}, \frac{-\pi}{2})$, then the integral along the contour C_1 converges to zero.

The contour C_2 can be described in polar coordinates (ϱ, θ) as

$$s = r e^{i\theta} - i k_G Y, \quad \text{where} \quad \pi < \theta < -\pi,$$

hence,

$$\int_{C_2} \frac{\left(Y - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}}{s \left(Y_1 - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}} e^{st} ds = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\left(Y - \frac{i(re^{i\theta} - ik_G Y)}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}} e^{(re^{i\theta} - ik_G Y)t} i r e^{i\theta}}{(re^{i\theta} - ik_G Y) \left(Y_1 - \frac{i(re^{i\theta} - ik_G Y)}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}} d\theta.$$

As $r \rightarrow 0$, the integral along the contour C_2 has the order of $r^{\frac{3}{2}+i\tilde{\alpha}}$. Thus, the integral along the contour C_2 converges to zero as $r \rightarrow 0$. In a similar way, it can be proved that the integral along the contour C_3 converges to zero as $r \rightarrow 0$.

The contours C_4 and C_5 are lines that can be described as

$$C_4 : s = \varrho e^{i\pi} - ik_G Y, \quad \text{where } \varrho : -\infty \rightarrow 0,$$

$$C_5 : s = \varrho e^{-i\pi} - ik_G Y, \quad \text{where } \varrho : 0 \rightarrow -\infty.$$

Hence,

$$\int_{C_4} \frac{\left(Y - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}}{s \left(Y_1 - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}} e^{st} ds = - \int_{-\infty}^0 \frac{(\varrho e^{i\pi})^{\frac{1}{2}+i\tilde{\alpha}} e^{(\varrho e^{i\pi} - ik_G Y)t}}{(\varrho e^{i\pi} - ik_G Y) (ik_G(Y_1 - Y) - \varrho)^{\frac{1}{2}+i\tilde{\alpha}}} d\varrho.$$

In a similar manner, we obtain the following expression along the contour C_5

$$\int_{C_5} \frac{\left(Y - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}}{s \left(Y_1 - \frac{is}{k_G}\right)^{\frac{1}{2}+i\tilde{\alpha}}} e^{st} ds = - \int_0^{-\infty} \frac{(\varrho e^{-i\pi})^{\frac{1}{2}+i\tilde{\alpha}} e^{(\varrho e^{-i\pi} - ik_G Y)t}}{(\varrho e^{-i\pi} - ik_G Y) (ik_G(Y_1 - Y) - \varrho)^{\frac{1}{2}+i\tilde{\alpha}}} d\varrho.$$

The sum of the integral along the contours C_4 and C_5 can be written as

$$\int_{C_4} + \int_{C_5} = i \int_0^{\infty} \frac{e^{-ik_G Y t} e^{-\varrho t} (e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) (\varrho)^{\frac{1}{2}+i\tilde{\alpha}}}{(-\varrho - ik_G Y) (ik_G(Y_1 - Y) - \varrho)^{\frac{1}{2}+i\tilde{\alpha}}} d\varrho.$$

We note that because of the exponential factor $e^{-\rho t}$ in the integrand, the dominant contribution to the integral for $t \gg 1$ comes from an interval of values of ϱ that are close to $\rho = 0$. Thus, we can obtain an asymptotic approximation for this integral valued for $t \gg 1$ by considering ϱ small and expanding the denominator in powers of ϱ and then taking the first term of the expansion,

$$\int_{C_4} + \int_{C_5} = - \frac{e^{-ik_G Y t} (e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}})}{k_G Y (ik_G(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}}} \int_0^{\infty} \frac{e^{-\varrho t} \varrho^{\frac{1}{2}+i\tilde{\alpha}}}{\left(1 + \frac{\varrho}{ik_G Y}\right) \left(1 - \frac{\varrho}{ik_G(Y_1 - Y)}\right)^{\frac{1}{2}+i\tilde{\alpha}}} d\varrho.$$

We obtain

$$\int_{C_4} + \int_{C_5} = (e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \left[\frac{h_{G1}(Y, t)}{k_G^{\frac{3}{2}+i\tilde{\alpha}}} \int_0^\infty e^{-\varrho t} \varrho^{\frac{1}{2}+i\tilde{\alpha}} d\varrho + \frac{h_{G2}(Y, t)}{k_G^{\frac{5}{2}+i\tilde{\alpha}}} \int_0^\infty e^{-\varrho t} \varrho^{\frac{3}{2}+i\tilde{\alpha}} + \dots \right],$$

where

$$h_{G1}(Y, t) = -\frac{e^{-ik_G Y t}}{Y (i(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}}},$$

and

$$h_{G2}(Y, t) = -\frac{(-\frac{1}{2} + i\tilde{\alpha})Y + Y_1}{Y^2 (i(Y_1 - Y))^{\frac{3}{2}+i\tilde{\alpha}}} e^{-ik_G Y t}.$$

By using the definition of the gamma function (Abramowitz and Stegun, 1964) we write

$$\int_0^\infty e^{-\varrho t} \varrho^{d-1} d\varrho = \Gamma(d) t^{-d},$$

where d is complex. We obtain

$$\int_{C_4} + \int_{C_5} = 2i\pi e^{-ik_G Y t} \left[\frac{g_2(Y)}{t^{\frac{3}{2}+i\tilde{\alpha}}} + \frac{g_4(Y)}{t^{\frac{5}{2}+i\tilde{\alpha}}} + \mathcal{O}(t^{-7/2}) \right],$$

where

$$g_2(Y) = -\frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[3/2 + i\tilde{\alpha}]}{2i\pi (i(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}} k_G^{\frac{3}{2}+i\tilde{\alpha}}}$$

and

$$g_4(Y) = -\frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[5/2 + i\tilde{\alpha}] [(-1/2 + i\tilde{\alpha})Y + Y_1]}{2i\pi (i(Y - Y_1))^{\frac{3}{2}+i\tilde{\alpha}} k_G^{\frac{5}{2}+i\tilde{\alpha}}}.$$

A similar procedure can be used to obtain the integrals along the contours C_6 and C_7

$$\int_{C_6} + \int_{C_7} = 2i\pi e^{-ik_G Y_1 t} \left[\frac{g_3(Y)}{t^{\frac{1}{2}-i\tilde{\alpha}}} + \frac{g_5(Y)}{t^{\frac{3}{2}-i\tilde{\alpha}}} + \mathcal{O}(t^{-5/2}) \right],$$

where

$$g_3(Y) = \frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[1/2 - i\tilde{\alpha}] (i(Y_1 - Y))^{\frac{1}{2}+i\tilde{\alpha}}}{2i\pi Y_1 k_G^{\frac{1}{2}-i\tilde{\alpha}}}$$

and

$$g_5(Y) = \frac{(e^{-\pi\tilde{\alpha}} + e^{\pi\tilde{\alpha}}) \Gamma[3/2 - i\tilde{\alpha}] (i(Y_1 - Y))^{-\frac{1}{2}+i\tilde{\alpha}} [(1/2 - i\tilde{\alpha})Y_1 - Y]}{2i\pi Y_1 k_G^{\frac{3}{2}-i\tilde{\alpha}}}.$$

Appendix C

Stability Analysis

In this appendix, we present stability criteria for disturbances in barotropic and baroclinic flows in sections C.1 and C.2, respectively. In section C.3, we present the Miles-Howard theorem for stability of gravity waves in a stratified shear flow.

C.1 Barotropic instability

In this section, we present a necessary condition for barotropic instability of a shear flow $\bar{u}(y)$ in a coordinate frame rotating with the earth. This condition was found by Rayleigh (1879) and extended by Kuo (1949). The following discussion is based on that in Kundu, Cohen and Dowling (2012).

The inviscid linear barotropic vorticity equation, i.e.

$$\left(\frac{\partial}{\partial t} + \bar{u}(y) \frac{\partial}{\partial x} \right) \nabla^2 \psi_R(x, y, t) + \left(\beta - \frac{d^2 \bar{u}(y)}{dy^2} \right) \frac{\partial \psi_R(x, y, t)}{\partial x} = 0,$$

is used to present the necessary condition for the inviscid instability of a barotropic current $\bar{u}(y)$, in a channel $y_1 < y < y_2$ in the two dimensional xy -space. Since the coefficients of the inviscid linear barotropic vorticity equation are independent of x and t , there can be solutions of the form:

$\psi_R(x, y, t) = \hat{\psi}_R(y) e^{ik_R(x-ct)}$, where $\hat{\psi}_R(y)$ is the amplitude. The phase speed

$c = c_r + ic_i$ is complex, so $\psi_R(x, y, t) = \hat{\psi}_R(y) e^{ik_R(x - c_r t)} e^{k_R c_i t}$, and solutions are unstable if $c_i > 0$.

The amplitude of the disturbance is given by the Rayleigh-Kuo equation

$$(\bar{u}(y) - c) \left(\frac{d^2 \hat{\psi}_R(y)}{dy^2} - \delta k_R^2 \hat{\psi}_R(y) \right) + \left(\beta - \frac{d^2 \bar{u}(y)}{dy^2} \right) \hat{\psi}_R(y) = 0. \quad (\text{C.1.1})$$

The y -component of the velocity is equal to zero, i.e. $v = 0$, so the boundary conditions are

$$\hat{\psi}_R(y_1) = \hat{\psi}_R(y_2) = 0. \quad (\text{C.1.2})$$

Equation (C.1.1) with boundary conditions (C.1.2) defines an eigenvalue problem with $c(k_R)$ as the eigenvalue and $\hat{\psi}_R$ as the eigenfunction. Since these equations do not include i , we can take the complex conjugate. The complex conjugate shows that if $\hat{\psi}_R$ is an eigenfunction with eigenvalue c for some k_R , then $\hat{\psi}_R^*$ is also an eigenfunction with eigenvalue c^* for the same k_R . Consequently, to each eigenvalue with a positive c_i there is a corresponding eigenvalue with a negative c_i . Namely, to each growing mode there is a corresponding decaying mode. Consequently, stable solutions can have only a real c . This is true only if the flows are inviscid.

It can be shown that certain velocity distributions $\bar{u}(y)$ are unstable according to the equation (C.1.1).

Criterion for barotropic instability [Kuo, (1949)]: *A necessary condition for the inviscid instability of a barotropic current $\bar{u}(y)$ is that the gradient of the absolute vorticity,*

$$\frac{d}{dy}(\bar{\zeta} + f) = \left(\beta - \frac{d^2 \bar{u}(y)}{dy^2} \right), \quad (\text{C.1.3})$$

must change sign somewhere in the flow.

The criterion for barotropic instability can be proven by rewriting equation (C.1.1) as follows

$$\frac{d^2 \hat{\psi}_R}{dy^2} - \delta k_R^2 \hat{\psi}_R + \frac{1}{\bar{u} - c} \left(\beta - \frac{d^2 \bar{u}}{dy^2} \right) \hat{\psi}_R = 0,$$

and considering the unstable mode for which $c_i > 0$ and hence $\bar{u} - c \neq 0$. We multiply this equation by the complex conjugate $\hat{\psi}_R^*$, integrate from y_1 to y_2 and apply the boundary conditions (C.1.2).

The first term is thereby transformed to

$$\int_{y_1}^{y_2} \hat{\psi}_R^* \left(\frac{d^2 \hat{\psi}_R}{dy^2} \right) dy = \left[\hat{\psi}_R^* \left(\frac{d \hat{\psi}_R}{dy} \right) \right]_{y_1}^{y_2} - \int_{y_1}^{y_2} \left(\frac{d \hat{\psi}_R^*}{dy} \right) \left(\frac{d \hat{\psi}_R}{dy} \right) dy = - \int_{y_1}^{y_2} \left| \frac{d \hat{\psi}_R}{dy} \right|^2 dy. \quad (\text{C.1.4})$$

Therefore, the equation (C.1.1) gives

$$\int_{y_1}^{y_2} \left(\left| \frac{d \hat{\psi}_R}{dy} \right|^2 + \delta k_R^2 |\hat{\psi}_R|^2 \right) dy - \int_{y_1}^{y_2} \frac{1}{\bar{u} - c} \left(\beta - \frac{d^2 \bar{u}}{dy^2} \right) |\hat{\psi}_R|^2 dy = 0. \quad (\text{C.1.5})$$

The first term in (C.1.5) is real. The second term in (C.1.5) is complex and its imaginary part can be obtained by multiplying the numerator and denominator by $(\bar{u} - c^*)$ where c^* is the complex conjugate of c . Hence, the imaginary part of (C.1.5) is

$$c_i \int_{y_1}^{y_2} \frac{1}{|\bar{u} - c|^2} \left(\beta - \frac{d^2 \bar{u}}{dy^2} \right) |\hat{\psi}_R|^2 dy = 0. \quad (\text{C.1.6})$$

Equation (C.1.6), for the unstable case ($c_i > 0$), can be satisfied only if $(\beta - \frac{d^2 \bar{u}}{dy^2})$ changes sign at least once in the open interval $y_1 < y < y_2$.

Namely, for barotropic instability, the gradient of the absolute vorticity $(\beta - \frac{d^2 \bar{u}}{dy^2})$ must have at least one point of inflection within the flow, i.e. where $(\beta - \frac{d^2 \bar{u}}{dy^2}) = 0$. We note here that the existence of a point of inflection does not guarantee a nonzero c_i . As a result, the inflection point is a necessary but not sufficient condition for barotropic instability. If there is no inflection point in the interval y_1 and y_2 , then stability is guaranteed. Since the criterion includes β and not f , the instability has no preference for any latitude [Kundu, Cohen and Dowling (2012)].

Setting $\delta = 1$ and $\beta = 0$ in (C.1.1), gives the Rayleigh equation.

Rayleigh's inflection point criterion [Rayleigh, (1879)]: *A necessary (but not sufficient) criterion for instability of an inviscid parallel flow is that the basic velocity profile $\bar{u}(y)$ has a point of inflection.*

C.2 Baroclinic instability: Charney-Stern theorem

Charney (1947) and Eady (1949) first formulated the theoretical problem of the stability of large-scale zonal flow with vertical shear. This theory is called baroclinic instability. Baroclinic instability is the most important mechanism for generating weather in the middle latitudes. It plays a significant role in the meridional transport of properties in the atmosphere. Being a large-scale phenomenon, baroclinic instability can be investigated within the quasi geostrophic approximation.

In this section, we present the stability problem for an adiabatic fluid flow in a continuously stratified atmosphere on the midlatitude β -plane [Charney and Stern, (1962)]. The following discussion is based on that in Holton (2004).

We consider the linear quasi-geostrophic potential vorticity equation describing the propagation of baroclinic Rossby waves as in (4.1.1), but for the more general density profile $\rho_0(z)$ rather than for constant ρ_0 . In that case, (4.1.1) can be rewritten as

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) q_R + \frac{\partial \bar{q}}{\partial y} \frac{\partial \psi_R}{\partial x} = 0, \quad (\text{C.2.1})$$

where

$$\begin{aligned} \bar{q}(y, z) &= \nabla^2 \bar{\psi}(y, z) + f + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \bar{\psi}}{\partial z} \right), \\ q_R(x, y, z, t) &= \nabla^2 \psi_R(x, y, z, t) + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \psi_R}{\partial z} \right), \\ \frac{\partial \bar{q}}{\partial y} &= \beta - \frac{\partial^2 \bar{u}(y, z)}{\partial y^2} - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \bar{u}}{\partial z} \right). \end{aligned} \quad (\text{C.2.2})$$

Equation (C.2.1) is examined in the domain $0 \leq x < 2\pi$, $-L < y < L$ and $0 < z < \infty$.

The flow is assumed to be adiabatic (an adiabatic process occurs without transfer of heat or mass of substances between a thermodynamic system and its surroundings) and the vertical motion w vanishes at the boundaries, so $J = w = 0$ in the thermodynamic energy equation (A.3.10). Then, we substitute the perturbations (2.3.1) into (A.3.10), use the expression for perturbation of the geostrophic balance in log-pressure coordinate (2.3.2) and use the relation $\psi_R = f_0^{-1} \Phi_R$ to write the linearized

form of the thermodynamic energy equation (A.3.10) valid at horizontal boundary surfaces as

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \psi_R}{\partial z} - \frac{\partial \bar{u}}{\partial z} \frac{\partial \psi_R}{\partial x} = 0. \quad (\text{C.2.3})$$

The sidewall boundary conditions are

$$v_R = \frac{\partial \psi_R}{\partial x} = 0, \quad \text{thus, } \psi_R = 0 \quad \text{at } y = \pm L. \quad (\text{C.2.4})$$

Now, we consider a perturbation geostrophic streamfunction ψ_R to the basic flow consisting of a single zonal Fourier component propagating in the x direction as follows

$$\psi_R(x, y, z, t) = \text{Re} \{ \Psi(y, z) e^{ik_R(x-ct)} \} \quad (\text{C.2.5})$$

where $\Psi(y, z) = \Psi_r + i\Psi_i$ is a complex amplitude. Alternatively, (C.2.5) can be written as

$$\psi_R(x, y, z, t) = e^{k_R c t} [\Psi_r \cos k_R(x - c t) - \Psi_i \sin k_R(x - c t)].$$

Therefore, the relative magnitudes of Ψ_r and Ψ_i determine the phase of the wave for any y, z .

If we substitute from (C.2.5) into (C.2.1) and (C.2.3), we get

$$(\bar{u} - c) \left[\frac{\partial^2 \Psi}{\partial y^2} - \delta k_R^2 \Psi + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \Psi}{\partial z} \right) \right] + \frac{\partial \bar{q}}{\partial y} \Psi = 0, \quad (\text{C.2.6})$$

and

$$(\bar{u} - c) \frac{\partial \Psi}{\partial z} - \frac{\partial \bar{u}}{\partial z} \Psi = 0, \quad \text{at } z = 0. \quad (\text{C.2.7})$$

The upper boundary condition can be specified by requiring $\Psi \rightarrow \text{constant}$ as $z \rightarrow \infty$. Periodic boundary conditions are specified at $x = 0$ and $x = 2\pi$. Equation (C.2.6), together with its boundary conditions, constitutes a linear boundary value problem for $\Psi(y, z)$.

If (C.2.6) is divided by $(\bar{u} - c)$ and the resulting equation is respectively separated into real and imaginary parts, we get

$$\frac{\partial^2 \Psi_r}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \Psi_r}{\partial z} \right) - \left[\delta k_R^2 - \chi_r \left(\frac{\partial \bar{q}}{\partial y} \right) \right] \Psi_r - \chi_i \frac{\partial \bar{q}}{\partial y} \Psi_i = 0, \quad (\text{C.2.8})$$

$$\frac{\partial^2 \Psi_i}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \Psi_i}{\partial z} \right) - \left[\delta k_R^2 - \chi_r \left(\frac{\partial \bar{q}}{\partial y} \right) \right] \Psi_i + \chi_i \frac{\partial \bar{q}}{\partial y} \Psi_r = 0, \quad (\text{C.2.9})$$

where

$$\chi_r = \frac{\bar{u} - c_r}{(\bar{u} - c_r)^2 + c_i^2}, \quad \text{and} \quad \chi_i = \frac{c_i}{(\bar{u} - c_r)^2 + c_i^2}.$$

In a similar way, (C.2.7) is divided by $(\bar{u} - c)$ and the resulting equation is respectively separated into real and imaginary parts to give for the boundary condition at $z = 0$:

$$\begin{aligned} \frac{\partial \Psi_r}{\partial z} + \frac{\partial \bar{u}}{\partial z} \left[\chi_i \Psi_i - \chi_r \Psi_r \right] &= 0, \\ \frac{\partial \Psi_i}{\partial z} - \frac{\partial \bar{u}}{\partial z} \left[\chi_r \Psi_i + \chi_i \Psi_r \right] &= 0. \end{aligned} \quad (\text{C.2.10})$$

Multiplying (C.2.8) by $\rho_0 \Psi_i$, (C.2.9) by $\rho_0 \Psi_r$, and subtracting the former from the latter yields

$$\begin{aligned} \rho_0 \left[\Psi_i \frac{\partial^2 \Psi_r}{\partial y^2} - \Psi_r \frac{\partial^2 \Psi_i}{\partial y^2} \right] + \left[\Psi_i \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \Psi_r}{\partial z} \right) - \Psi_r \frac{\partial}{\partial z} \left(\kappa \rho_0 \frac{\partial \Psi_i}{\partial z} \right) \right] \\ - \rho_0 \chi_i \left(\frac{\partial \bar{q}}{\partial y} \right) (\Psi_i^2 + \Psi_r^2) = 0. \end{aligned} \quad (\text{C.2.11})$$

If the chain rule of differentiation is used, then (C.2.11) can be expressed in the following form

$$\begin{aligned} \rho_0 \frac{\partial}{\partial y} \left[\Psi_i \frac{\partial \Psi_r}{\partial y} - \Psi_r \frac{\partial \Psi_i}{\partial y} \right] + \frac{\partial}{\partial z} \left[\kappa \rho_0 \left(\Psi_i \frac{\partial \Psi_r}{\partial z} - \Psi_r \frac{\partial \Psi_i}{\partial z} \right) \right] \\ - \rho_0 \chi_i \left(\frac{\partial \bar{q}}{\partial y} \right) (\Psi_i^2 + \Psi_r^2) = 0. \end{aligned} \quad (\text{C.2.12})$$

The first term in brackets in (C.2.12) is a perfect differential in y and the second term in brackets in (C.2.12) is a perfect differential in z . Consequently, if (C.2.12) is integrated over the y, z domain, the result can be expressed as

$$\begin{aligned} \int_0^\infty \rho_0 \left[\Psi_i \frac{\partial \Psi_r}{\partial y} - \Psi_r \frac{\partial \Psi_i}{\partial y} \right]_{y=-L}^{y=+L} dz + \int_{-L}^{+L} \left[\kappa \rho_0 \left(\Psi_i \frac{\partial \Psi_r}{\partial z} - \Psi_r \frac{\partial \Psi_i}{\partial z} \right) \right]_{z=0}^{z=\infty} dy \\ - \int_{-L}^{+L} \int_0^\infty \rho_0 \chi_i \left(\frac{\partial \bar{q}}{\partial y} \right) (\Psi_i^2 + \Psi_r^2) dz dy = 0. \end{aligned} \quad (\text{C.2.13})$$

The first integral in (C.2.13) vanishes for the reason that from (C.2.4) $\Psi_r = \Psi_i = 0$ at $y = \pm L$. Moreover, the contribution to the second integral of (C.2.13) at the upper boundary vanishes if $\Psi \rightarrow \text{constant}$ as $z \rightarrow \infty$.

If (C.2.10) is used to eliminate the vertical derivatives in this term at the lower boundary ($z = 0$), (C.2.13) can be written as

$$c_i \left[\int_{-L}^{+L} \int_0^\infty \frac{\partial \bar{q}}{\partial y} \frac{\rho_0 |\Psi|^2}{|\bar{u} - c|^2} dz dy - \int_{-L}^{+L} \kappa \frac{\partial \bar{u}}{\partial z} \frac{\rho_0 |\Psi|^2}{|\bar{u} - c|^2} \Big|_{z=0} dy \right] = 0, \quad (\text{C.2.14})$$

where $|\Psi|^2 = \Psi_r^2 + \Psi_i^2$ is the disturbance amplitude squared.

Equation (C.2.14) has important implications for the stability of quasi-geostrophic perturbations. Unstable modes require that c_i must be nonzero. Hence, the quantity in square brackets in (C.2.14) must vanish. Because $|\Psi|^2/|\bar{u} - c|^2$ and $|\Psi|^2/\kappa|\bar{u} - c|^2$ are nonnegative, instability is possible only when $(\partial \bar{q}/\partial y)$ in the whole domain and $(\partial \bar{u}/\partial z)$ at the lower boundary satisfy particular constraints as follows

- (i) If $\partial \bar{u}/\partial z$ at $z = 0$ (which by thermal wind balance implies that the meridional temperature gradient vanishes at the boundary), then the second integral in (C.2.14) vanishes. Therefore, for instability to occur, the first integral in (C.2.14) must also vanish. The first integral vanishes only if $\partial \bar{q}/\partial y$ changes sign within the domain (i.e. $\partial \bar{q}/\partial y = 0$ somewhere). This is referred to as the *Rayleigh necessary condition*.
- (ii) If $\partial \bar{q}/\partial y \geq 0$ everywhere, then it is necessary that $\partial \bar{u}/\partial z > 0$ somewhere at the lower boundary for instability to occur.
- (iii) If $\partial \bar{u}/\partial z < 0$ everywhere at the lower boundary, then it is necessary that $\partial \bar{q}/\partial y < 0$ somewhere for instability to occur.

If none of these conditions hold, then stability is guaranteed. In the numerical computations described in chapter 5, it is important to choose $\bar{u}$ and the relevant parameters such that none of these conditions hold.

C.3 Miles-Howard theorem

The Richardson number R_i is named after the English mathematician Lewis Fry Richardson (1881-1953). The Richardson number is a measure of stability of the fluid flow. In atmospheric and ocean science the Richardson number R_i has a more general form which takes into account the stratification of the fluid flow. It is a nondimensional number that expresses the ratio of the buoyancy term to the flow shear term

$$R_i = \frac{\text{buoyancy term}}{\text{flow shear term}} = \frac{-\frac{g}{\rho} \frac{\partial \rho}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2},$$

where, as presented before, g is the gravity, ρ is the density, u is the flow velocity, and z is the height.

The term $-\frac{g}{\rho} \frac{\partial \rho}{\partial z}$ measures the extent of stratification while $\frac{\partial u}{\partial z}$ measures the strength of the shear. A condition on stability is that $R_i \geq \frac{1}{4}$. This criterion is known as the Miles-Howard theorem.

Miles-Howard theorem [Miles, (1961)], [Howard, (1961)]:

For stationary solutions of the incompressible Euler equations in the form of parallel flows, a sufficient condition for stability is that the Richardson number must be greater than or equal to $\frac{1}{4}$ everywhere in the fluid flow. (Ref. to [Warren, (1968)] for the proof).

The Richardson number is always considered positive. The Richardson number is negative if $\frac{\partial \rho}{\partial z} > 0$ which means that the density ρ increases with height. This is known as unstable stratification. In an incompressible, in a static state, the criterion for determining the stability of the medium is that:

- ☐ the medium is **unstable** if the density **increases** with height.
- ☐ the medium is **neutral** if the density is **uniform** with height.
- ☐ the medium is **stable** if the density **decreases** with height.

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