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***T*-Stress Solutions for Three-Dimensional Cracked Components**

by

JIE QU, B. Sc., M. Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Applied Science (Engineering)

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
Carleton University
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submitted by
Jie Qu, B. Sc., M. Eng.
in partial fulfillment of the requirements for
the degree of Master of Applied Science (Engineering)

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ABSTRACT

The elastic T -stress has been found to be an important parameter in characterizing the elastic-plastic stress state near crack tip for two-dimensional and three-dimensional problems. In addition to the stress intensity factor K in elastic analysis, or J -integral in elastic-plastic analysis, T -stress provides an effective two-parameter characterization of the constraint levels in the vicinity of crack. For three-dimensional problems, it is suggested that S -stress should be also considered in addition to T -stress.

Embedded elliptical crack in an infinite solid subject to uniaxial or biaxial loadings represents the excellent model for many engineering components with three-dimensional cracks. Although extensive analyses have been conducted for stress intensity factor K , no solutions for elastic T -stress are available so far. This thesis presents the analytical solutions for elastic T -stress and S -stress of an embedded elliptical crack in an infinite body under remote uniaxial and biaxial tension or bending. Three-dimensional finite element analysis is conducted to verify these solutions.

Quarter-elliptical corner cracks in a finite thickness plate are representative models for corner cracks found in welding toes, rivet holes or the nozzle areas of pressure vessels. Stress intensity factor K solutions for those cracks have been published in literature. But comprehensive T -stress solutions are not available. In this thesis, three-dimensional finite element analyses are conducted for a quarter-elliptical corner crack in a finite thickness plate with different aspect ratios and relative depths under remote uniaxial tension and bending. The T -stress and S -stress solutions are obtained. For the convenience of engineering applications, the empirical equations of T -stress and S -stress at two surface points are obtained.

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NOTATION

α	rotation angle of coordinate transformation
φ	parametric angle
λ	biaxial ratio
σ_{ij}	normal stress components
σ_0	uniform tension stress
τ_{ij}	shear stress components
ν	Poisson's ratio
Φ	suitable harmonic stress function
(ξ, ζ, η)	ellipsoidal coordinate system
(r, θ, ω)	spherical coordinate system
G	shear modulus of elasticity
K	stress intensity factor
J	J -integral
K_{IC}	material toughness of opening mode
E	Yung's modulus of elasticity
$V(s)$	volume which encloses the crack front segment
$V^{(n)}$	potential function
$\omega(s)$	$\omega(s) = 1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{s}$
x_i	local Cartesian coordinates
x, y, z	global Cartesian coordinates

k	modulus of Jacobian elliptical functions
k'	complementary modulus of Jacobian elliptical functions
$u_1, E(u_1)$	Jacobian elliptical function
$K(k)$	complete elliptical integral of the first kind
$E(k)$	complete elliptical integral of the second kind
f	point force
$q_k(s)$	spatial weighting function
u_i	components of displacement vector
u, v, w	displacement components
a	half length of the major axis of elliptical cracks or half length of the minor axis of quarter-elliptical cracks
b	half length of the major axis of elliptical cracks
c	half length of the minor axis of quarter-elliptical cracks
W	half width of the plate for elliptical cracks or the width for corner cracks
h	the distance from the crack plane to the remote end
T	T -stress
t	normalized T -stress or the coordinate in the tangential direction
S	S -stress
s	normalized S -stress
A, B	surface crack points for corner cracks
c_{ij}, A_{ij}, C_i	constants
$I(s)$	interaction integral
(n, t, z)	local principal normal coordinate system
F	boundary correction factor

Q	Q -stress
$Q(s)$	$Q(s) = s(a^2 + s)(b^2 + s)$
$p, p(x, y)$	nominal stress at remote ends
p_0	uniform stress magnitude
f_i	terms in T -stress calculation for embedded elliptical cracks
h_i	terms in S -stress calculation for embedded elliptical cracks
B_i, B_i'	terms in stress calculation
D_i, D_i'	terms in stress calculation
F_i	terms in T -stress calculation for embedded elliptical cracks
L_i, L_i'	terms in stress calculation
M_i, M_i'	terms in stress calculation
g_i	terms in T -stress calculation for embedded elliptical cracks
V_{Ai}, V_{Bi}	referential T -stress for corner cracks
W_{Ai}, W_{Bi}	referential S -stress for corner cracks

Chapter 1

Introduction

Failure of engineering structures could cause substantial property and human losses. How to predict and prevent the failure has been the interest of engineers and scientists. The traditional way of failure analysis is to calculate the stress distribution in the structure under a number of assumptions of materials, for example, continuum, isotropic and uniform. The failure is predicted when the calculated maximum stress is equal to or larger than the material strengths such as yield strength or ultimate strength. However, Griffith (1921) found that failure happened when stress level is far below the material strength when he conducted tests on glass. Fracture mechanics is needed to explain these phenomena, which is based on the assumption that no material is perfect and cracks exist in all engineering materials. However, fracture mechanics did not catch the eyes of many engineers and scientists until all-welded structures were widely used. The U.S.A. built approximately 5,000 Liberty ships during the World War II using all-welded structures. More than one-fifth of these ships developed cracks before April 1946, and most of them were younger than three years old, as reported by Gdoutos (1993).

In the view of fracture mechanics, fracture failure is controlled by fracture parameter such as stress intensity factor or J -integral. For brittle materials where linear elasticity applies, the failure criteria are established by equating the stress intensity factors (SIF) to the critical value of the material toughness (K_{Ic}), Gdoutos (1993). But many metals have

good ductility and fracture failure is accompanied by plastic deformation. For cracks in ductile materials, elastic-plastic analyses are often conducted. The commonly used fracture parameter is J -integral or crack tip opening displacement (CTOD). These methodologies are generally regarded to as the “one-parameter approach”.

Although the one-parameter fracture analysis provides the means for real engineering materials that consists cracks, it is based on the fact that the crack has to be well constrained so that the minimum value of the material toughness can be achieved. This methodology leads to the following limitations: it is over conservative for the structures that are not well constrained, and thus raises the cost of product and maintenance.

Therefore, constraint-based failure analysis has been developed in order to overcome the above drawbacks. T -stress, suggested by Rice (1974), together with stress intensity factor K for elastic analysis, or Q -stress, suggested by Shih *et al* (1993), with J -integral for elastic-plastic analysis, has been considered as the constraint parameter. Under small scale yielding, Shih *et al* (1993) indicated that Q -stress could be related to T -stress directly.

T -stress is defined as the second term of Williams (1957) series expansion of the stress component parallel to the crack plane. It relates to the constraint in crack tip. It has been shown that T -stress not only changes the size but also the shape of the plastic zone near the crack tip. It has been recognized that the T -stress, in addition to the fracture parameter K or J , provides an effective two-parameter characterization of the stress fields in the vicinity of crack tip in a variety of crack configuration and loading conditions for two-dimensional problems, Du and Hancock (1991).

Based on this two-parameter characterization, constraint-based failure assessment method has been established, (Ainsworth and O'Dowd, 1995). Recently, the effects of the constraint on fatigue crack propagation have been investigated by several researchers (Tong, 2002).

For three-dimensional elastic fracture problems, Rice (1974) has suggested that both T -stress and S -stress should be considered as the constraint parameters in order to characterize the stress field in the vicinity of crack front. S -stress is the second term of Williams (1957) series expansion of the stress component tangential along the crack front. It is therefore very important to calculate the T -stress and S -stress solutions for three-dimensional engineering components with defects.

Extensive studies have been conducted and methodologies have been developed for determining the solutions of T -stress for two-dimensional problems. T -stress solutions for two-dimensional problems can be found in the work of Larsson and Carlsson (1973), Leivers and Radon (1983), Kfoury (1986), Sham (1991) and Sherry *et al* (1995). However, T -stress solutions are very limited for three-dimensional problems.

Embedded elliptical crack is the common model (Figure 1.1) for three-dimensional analysis and the solutions of stress intensity factors are available. The solutions of T -stress and S -stress have to be derived. Quarter-elliptical corner cracks, shown in Figure 1.2, are often found in weld toes, rivet holes or the nozzle areas of pressure vessels. Stress intensity factors for this case are published in handbooks or in research papers, i.e., Shiratori and Miyoshi (1992), Zheng *et al* (1996) and Kiciak *et al* (1998). But T -stress and S -stress have not been calculated.

The most commonly used methods of determining T -stress and S -stress are the analytical methods using three-dimensional elasticity theory, and the numerical methods such as finite element analysis. The objectives of this thesis are summarized as follows. The first is to derive the analytical solutions of T -stress and S -stress for an elliptical crack embedded in an infinite solid under remote uniaxial and biaxial tension and bending. The second objective is to calculate the T -stress and S -stress for a quarter-elliptical corner crack in a finite thickness plate under remote tension and bending conditions.

There are five chapters in this thesis. Chapter 1 is an introduction to the current research. In Chapter 2, the background and literature review are presented. The solutions of T -stress and S -stress are derived in Chapter 3 for an elliptical crack embedded in an infinite solid under remote tension and bending conditions. These solutions are also verified by finite element method. In Chapter 4, T -stress and S -stress of quarter-elliptical corner cracks in finite thickness plates are calculated using finite element method. And empirical equations of T -stress and S -stress at two surface points are obtained. Chapter 5 is the conclusion and recommendations of the thesis.

Figure 1.1 An embedded elliptical crack in a solid and coordinates

Figure 1.2 A quarter-elliptical crack in a finite thickness plate and coordinates

Chapter 2

Background and Literature Review

2.1 Fracture Mechanics

Fracture mechanics is based on the assumption that cracks are inevitable in all engineering materials. There are three basic modes characterized by the displacements of upper and lower crack surfaces, and are shown in Figure 2.1. Mode I is called the opening mode, where the two crack surfaces are pulled apart. For mode II, the two crack surfaces slide over each other in plane; it is called the sliding mode. In mode III, the tearing mode, the two crack surfaces tear apart out of plane. In this thesis, only mode I crack problems will be discussed.

2.1.1 One-Parameter Fracture Mechanics

One-parameter fracture mechanics had been well accepted for more than three decades since fracture mechanics emerged based on brittle materials. For linear elastic fracture mechanics (LEFM) or small scale yielding where cracking is dominated by extensively elastic field enclosing the elastic-plastic zone in the vicinity of the crack tip, instable crack propagation is predicted when stress intensity factor K exceeds the material toughness K_{IC} , expressed by

$$K \geq K_{IC} \quad (2.1)$$

The K at left hand side of eq. (2.1) characterizes the magnitude of the stress singularity in the vicinity of the crack tip, while the K_{IC} at the right hand side is the fracture toughness.

However, many engineering materials have good ductility and cracking is often accompanied by larger elastic-plastic deformation. The stress intensity factor (SIF), which is based on LEFM, has to be replaced by other parameters that are suitable for elastic-plastic analysis. In the scope of elastic-plastic fracture mechanics, the fracture parameter used is either J -integral or crack opening displacement (COD). Because of the different definitions of COD and the difficulties of measurement, COD is less used in theoretical analysis than J -integral, which physically represents the decreasing rate of potential energy with respect to the crack length. For elastic-plastic fracture mechanics, the one-parameter criteria becomes

$$J \geq J_{IC} \quad (2.2)$$

For a elastic problem, the relation of the stress intensity factor K and J -integral can be expressed by

$$J = \frac{K^2}{E} \quad (\text{Plane Stress}) \quad (2.3)$$

$$J = \frac{(1-\nu^2)K^2}{E} \quad (\text{Plane Strain}) \quad (2.4)$$

2.1.2 Constraint-Based Fracture Mechanics

Since Larsson and Carlsson (1973) noticed that the range of validity of the “small scale yielding” approximation for crack tip plastic zone is substantially more limited than the analyses had suggested by Rice (1967), the stress fields near the crack tip have been further investigated by Rice (1974) and Parks (1991) for two-dimensional and three-

dimensional problems, respectively. In an isotropic linear elastic body containing a crack subject to symmetric (Mode I) loading, the Williams (1957) series expansion of stress components near the crack tip can be written as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix} f_{11}(\theta) & f_{12}(\theta) & 0 \\ f_{21}(\theta) & f_{22}(\theta) & 0 \\ 0 & 0 & f_{33}(\theta) \end{bmatrix} + \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \quad (2.5)$$

where r and θ are polar coordinates centered at the crack tip with $\theta = 0$ corresponding to a line ahead of the crack and normal to the crack front, shown in Figure 2.2. K_I is the Mode I stress intensity factor (SIF) of the crack.

The first term on the right-hand side describes the square root singularity of the stress field near the crack tip. It is obvious that the value goes to infinity when $r \rightarrow 0$. The second term on the right-hand side is non-singular and this term is related the constraints in the plane of the crack. The term T_{11} in eq. (2.5) is defined as T -stress, while T_{33} is called S -stress, Rice (1974). They represent the stresses in crack surface plane normal to and tangential to the crack front, respectively.

A significant amount of studies have been conducted, trying to describe the important effect of T -stress on the fracture behavior experimentally and numerically. It has been observed by a number of researchers that for two-dimensional cracks, the fracture toughness K_{IC} and crack growth direction is dependent on the magnitude and the sign of T -stress.

Larsson and Carlsson (1973) noticed the differences of fracture toughness K_{IC} values getting from two groups of specimens of the same material, in which one group is of compact tension specimens and three-point bending specimens while another group is of center cracked specimens and double edge-cracked specimens. Similar phenomena were

also observed by Hall (1971) in his experimental studies and by Shih and German (1981) through finite element analysis on three-point bending specimen and center-cracked specimen.

The discrepancies of fracture toughness corresponding to various specimen geometries reflect the constraint effects on the stress field. One-parameter criteria can fully describe the stress field and fracture properties for high constraint geometries, such as compact tension and three-point bending specimens. However, the fracture toughness estimated under high constraint conditions would result in the lower value than the actual ones if low constraint geometries were applied, such as center-cracked and double edge-cracked specimens. The lower values of fracture toughness will lead to over-conservative results and unnecessary cost in the assessment of engineering structures.

The constraint effect can be accounted for by using elastic T -stress. The sign and magnitude of T -stress substantially changed the size and shape of the plane strain crack tip plastic zone at finite load. Positive T -stress strengthens the level of stress triaxiality and leads to high crack-tip constraint; while negative T -stress reduces the level of crack-tip stress triaxiality and leads to the loss to the crack tip constraint. Betegon and Hancock (1991), Du and Hancock (1991) and O'Dowd and Shih (1991) indicated that the T -stress, in addition to the J -integral, provides an effective two-parameter characterization of plane strain elastic-plastic crack tip fields in a variety of crack configurations and loading conditions.

In addition, Cotterell and Rice (1980) demonstrated that the T -stress determines the stability of a straight-ahead model I cracking; cracks with $T < 0$ are stable in this mode

while those having $T > 0$ are prone to curve off the initial crack plane. The work of Hancock *et al* (1993) shows that T -stress affects crack tip deformation as well.

Based on the two-parameter characterization, constraint-based failure assessment methodology has been developed, (Ainsworth and O'Dowd, 1995). Recently, the effect of the constraint on fatigue crack propagation have been analysed by several researchers (Tong, 2002).

For three-dimensional cracks, in addition to the T -stress, Rice (1974) suggested that S -stress should be required in order to characterize the stress field in the vicinity of the crack tip. Therefore, it is very important to obtain the T -stress and S -stress solutions for three-dimensional engineering components.

2.2 The Calculation of the Constraint Parameters

Various methodologies have been developed in finding out the solutions of T -stress and S -stress. These methods include finite element method (FEM), weight function method, boundary element method (BEM) and analytical method. The analytical method and finite element method will be reviewed in this section.

2.2.1 Analytical Method for T -stress Determination

The analytical method for three-dimensional problems involves three-dimensional elasticity theory. Some of the fundamental knowledge employed in the thesis is reviewed in the following.

1) Fundamental Equations of Three-Dimensional Potential Function Method

In the absence of body force and shear stress ($\tau_{xz} = 0$, $\tau_{yz} = 0$ on $z = 0$ plane), the boundary value problem can be solved by finding a harmonic function Φ defined as

$$\nabla^2 \Phi = 0 \quad (2.6)$$

where Φ is a suitable harmonic function and ∇^2 is the Laplacian operator in three dimensions. The displacements in x , y and z directions in terms of Φ are given by

$$u = z \frac{\partial^2 \Phi}{\partial x \partial z} + (1 - 2\nu) \frac{\partial \Phi}{\partial x} \quad (2.7a)$$

$$v = z \frac{\partial^2 \Phi}{\partial y \partial z} + (1 - 2\nu) \frac{\partial \Phi}{\partial y} \quad (2.7b)$$

$$w = z \frac{\partial^2 \Phi}{\partial z^2} - (1 - \nu) \frac{\partial \Phi}{\partial z} \quad (2.7c)$$

where ν is Poisson's ratio. Stress components in Cartesian coordinate system (x, y, z) can be represented by

$$\sigma_{xx} = 2G \left(z \frac{\partial^3 \Phi}{\partial x^2 \partial z} + \frac{\partial^2 \Phi}{\partial x^2} + 2\nu \frac{\partial^2 \Phi}{\partial y^2} \right) \quad (2.8a)$$

$$\sigma_{yy} = 2G \left(z \frac{\partial^3 \Phi}{\partial y^2 \partial z} + \frac{\partial^2 \Phi}{\partial y^2} + 2\nu \frac{\partial^2 \Phi}{\partial x^2} \right) \quad (2.8b)$$

$$\sigma_{zz} = 2G \left(z \frac{\partial^3 \Phi}{\partial z^3} - \frac{\partial^2 \Phi}{\partial z^2} \right) \quad (2.8c)$$

$$\tau_{xy} = 2G \left[z \frac{\partial^3 \Phi}{\partial x \partial y \partial z} + (1 - 2\nu) \frac{\partial^2 \Phi}{\partial x \partial y} \right] \quad (2.8d)$$

$$\tau_{yz} = 2Gz \frac{\partial^3 \Phi}{\partial y \partial z^2} \quad (2.8e)$$

$$\tau_{xz} = 2Gz \frac{\partial^3 \Phi}{\partial x \partial z^2} \quad (2.8f)$$

where G is shear modulus.

2) Extraction of the T -stress

Nakamura and Parks (1992) give the series expansion of eq. (2.9) (the detail form of Eq. (2.5)) for the stress field very close to the crack front in $z = 0$ plane.

$$\sigma_{11} = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + T \quad (2.9a)$$

$$\sigma_{22} = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \quad (2.9b)$$

$$\sigma_{33} = \frac{K_1}{\sqrt{2\pi r}} 2\nu \cos \frac{\theta}{2} + S \quad (2.9c)$$

$$\tau_{12} = \frac{K_1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \quad (2.9d)$$

$$\tau_{13} = \tau_{23} = 0 \quad (2.9e)$$

where r and θ are the polar coordinates in $z = 0$ plane. $\theta = 0$ along the principal normal direction x_1 , as shown in Figure 2.3. Then T -stress and S -stress can be obtained from the following equations

$$T = \lim_{r \rightarrow 0} \left(\sigma_{11} \Big|_{\theta=0} - \sigma_{22} \Big|_{\theta=0} \right) \Big|_{z=0} \quad (2.10)$$

$$S = \lim_{r \rightarrow 0} \left(\sigma_{33} \Big|_{\theta=0} - 2\nu \sigma_{22} \Big|_{\theta=0} \right) \Big|_{z=0} \quad (2.11)$$

For a general point P along the crack front, shown in Figure 2.3, the stress components σ_{11} , σ_{22} and σ_{33} in eq. (2.10) and eq. (2.11) can be obtained from the stress components in Cartesian system (x, y, z) through a rotation around z axis to the Cartesian

system (x_1, x_2, x_3) . The stress components in these two systems have the following relationship in $z = 0$ plane:

$$\sigma_{11} = \sigma_{xx} \cos^2 \alpha + 2\tau_{xy} \sin \alpha \cos \alpha + \sigma_{yy} \sin^2 \alpha \quad (2.12)$$

$$\sigma_{22} = \sigma_{zz} \quad (2.13)$$

$$\sigma_{33} = \sigma_{xx} \sin^2 \alpha - 2\tau_{xy} \sin \alpha \cos \alpha + \sigma_{yy} \cos^2 \alpha \quad (2.14)$$

If the Cartesian system (x_1, x_2, x_3) is also named as (n, t, z) system, where n and t are in normal and tangential directions at point P , respectively, then the stress components σ_{11} , σ_{22} , σ_{33} , τ_{12} , τ_{13} and τ_{23} can also be denoted by σ_{nn} , σ_{zz} , σ_{tt} , τ_{nz} , τ_{nt} and τ_{zt} , respectively, then the eq. (2.12), (2.13) and (2.14) become

$$\sigma_{nn} = \sigma_{xx} \cos^2 \alpha + 2\tau_{xy} \sin \alpha \cos \alpha + \sigma_{yy} \sin^2 \alpha \quad (2.15)$$

$$\sigma_{zz} = \sigma_{zz} \quad (2.16)$$

$$\sigma_{tt} = \sigma_{xx} \sin^2 \alpha - 2\tau_{xy} \sin \alpha \cos \alpha + \sigma_{yy} \cos^2 \alpha \quad (2.17)$$

And the T in Eq. (2.10) also can be obtained by

$$T = \lim_{r \rightarrow 0} \left(\sigma_{nn} \Big|_{\theta=0} - \sigma_{zz} \Big|_{\theta=0} \right) \Big|_{z=0} \quad (2.18)$$

The S in eq. (2.11) can be calculated by

$$S = \lim_{r \rightarrow 0} \left(\sigma_{tt} \Big|_{\theta=0} - 2\nu \sigma_{zz} \Big|_{\theta=0} \right) \Big|_{z=0} \quad (2.19)$$

where σ_{nn} and σ_{tt} in eq. (2.18) and (2.19) are given by eq. (2.15) and eq. (2.17), respectively. Eq. (2.18) and (2.19) will be used in Chapter 3 of the present thesis to derive T -stress and S -stress solutions for embedded elliptical cracks.

2.2.2 Finite Element Method

Finite element method is a very effective tool in fracture mechanics. It has been being used for more than three decades in solving the conventional elastic and elastic-plastic problems. When finite element method is applied to the analysis, the structure considered is divided into small elements. In each of these small elements, the displacement components are interpolated by the nodal values using shape functions. Because quadratic or cubic shape functions are usually used, therefore the stress field is in linear or quadratic, Zienkiewicz and Taylor (1991). However, a problem occurs for fracture analysis. The conventional elements are not suitable for modeling the singular property because the stress field near a crack tip is singular. In order to solve this problem, Barsoum (1977) demonstrated that when three-dimensional prism elements with four mid-side nodes at the quarter points (a degenerated cube with one face collapsed), shown in Figure 2.4, were used and the separate crack tip nodal points were constrained to have the same displacement, the stress field with the $r^{-1/2}$ singularity could be achieved. By specific techniques, various parameters including stress intensity factor, J -integral, T -stress and S -stress can be further calculated once the cracked body is analyzed using finite element method.

For a plane strain case that semi-infinite crack with a point force f applied at the crack tip in the direction parallel to the crack plane, shown in Figure 2.5, using superscript ' L ' to designate the stress and displacement fields, the analytical solutions are given (Timoshenko and Goodier, 1970) as

$$\sigma_{11}^L = -\frac{f}{\pi r} \cos^3 \theta, \quad \sigma_{22}^L = -\frac{f}{\pi r} \cos \theta \sin^2 \theta, \quad \sigma_{12}^L = -\frac{f}{\pi r} \cos^2 \theta \sin \theta \quad (2.20)$$

and

$$u_1^L = -\frac{1-\nu^2}{E} \frac{f}{\pi} \left\{ \ln\left(\frac{r}{d}\right) + \frac{\sin^2 \theta}{2(1-\nu)} \right\} \quad (2.21a)$$

$$u_2^L = -\frac{1+\nu^2}{2E} \frac{f}{\pi} \{(1-2\nu)\theta + \cos \theta \sin \theta\} \quad (2.21b)$$

where the subscripts 1 and 2 suggest a local Cartesian coordinate system; r and θ are the local polar coordinates; d is a reference distance; E and ν are the Young's modulus and Poisson's ration, respectively. Employing the above solutions, Cardew *et al* (1984) and Kfour (1986) demonstrated that the T -stress for two-dimensional crack problems could be obtained by introducing an interaction J -integral. Nakamura and Parks (1992) extended the method to three-dimensional fracture problems. The interaction integral were defined in the form

$$I(s) = \frac{1}{A_c} \int_{V(s)} \left\{ \left(\sigma_{ij} \frac{\partial u_i^L}{\partial x_k} + \sigma_{ij}^L \frac{\partial u_i}{\partial x_k} \right) \frac{\partial q_k}{\partial x_j} - \sigma_{ij}^L \varepsilon_{ij} \frac{\partial q_k}{\partial x_k} \right\} dV \quad (2.22)$$

where $V(s)$ is a volume which encloses the crack front segment between $s - \varepsilon$ and $s + \varepsilon$ (Figure 2.6); $q_k(s)$ is the spatial weighting function and A_c is the increase in crack area generated by the virtual crack advance; σ_{ij} , ε_{ij} and u_i are the stress, strain and displacement components of the three-dimensional crack problem under consideration.

T -stress at the crack tip then can be calculated by the following equation given by Nakamura and Parks (1992):

$$T(s) = \frac{E}{1-\nu^2} \left\{ \frac{I(s)}{f} + \nu \varepsilon_{33}(s) \right\} \quad (2.23)$$

where $\varepsilon_{33}(s)$ is the extensional strain at point s in the direction tangential to the crack front.

Once T -stress is obtained, S -stress can be calculated by the relationship with T -stress, Nakamura and Parks (1992), expressed as

$$S = E\varepsilon_{33} + \nu T \quad (2.24)$$

2.3 Embedded Elliptical Cracks

An elliptical crack embedded in a solid, illustrated in Figure 2.7, is a common model of three-dimensional analysis. Stress intensity factors had already been well studied and determined analytically.

Green and Sneddon (1950) and Irwin (1962) derived the analytical solutions of stress intensity factor for an elliptical crack embedded in a solid subject to remote tension. For an infinite solid containing a flat elliptical crack with major and minor axes $2a$ and $2b$ respectively, as illustrated in Figure 2.7(b), and subject to uniform tension $\sigma_{zz} = \sigma_0$ at remote ends, the stress intensity factor along the crack front is given by

$$K(\varphi) = \frac{\sigma_0}{E(k)} \left(\frac{b}{a} \right)^{\frac{1}{2}} \left(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi \right)^{\frac{1}{4}} \quad (2.25)$$

where $E(k)$ is the complete elliptic integral of the second kind (Appendix 1).

Shah and Kobayashi (1971) derived the analytical solutions of stress intensity factor for an embedded elliptical crack subject to a polynomial pressure on both crack surfaces using potential function proposed by Segedin (1967). The solution was then used to obtain the stress intensity factor of an elliptical crack in a large beam in pure bending.

However, neither the analytical solutions of T -stress nor the solutions of S -stress have been reported for embedded elliptical cracks. For performing the constraint-based

failure assessment, the solutions of T -stress and S -stress for elliptical cracks embedded in an infinite elastic body subject to uniform tension and bending are to be derived in this thesis.

2.4 Corner Cracks in Finite Thickness Plates

Corner cracks are often found in many engineering structures or components, such as in rivet holes in aeronautics or in nozzle regions of pressure vessels. The residual life of these structures or components are greatly influenced by the fracture and fatigue behavior of the type of cracks, and thus attracted the interest of many researchers. In this thesis, a quarter-elliptical crack in a finite thickness plate, shown in Figure 2.8, is considered.

Using the finite element method, Shiratory and Miyoshi (1992) evaluated the stress intensity factor at two surface points of a quarter-elliptical corner crack in a finite thickness plates, as illustrated in Figure 2.8, with the crack aspect ratio $a/c = 0.2, 0.4, 0.6, 1.0$ and 2.0 , and the relative depth $a/t = 0.1, 0.2, 0.4, 0.6$ to 0.8 . Zheng *et al* (1996) and Kiciak *et al* (1998) evaluated the stress intensity factors at the front point and profile point for corner quarter-elliptical cracks in finite thickness plates by weight function method.

The solutions of T -stress and S -stress for corner cracks are required in order to perform the failure assessment to low constraint geometries. However, the available solutions for corner cracks are very limited. Zhao *et al* (2001) evaluated both the stress

intensity factors K and the T -stress for a special case: a quarter-circular corner crack in a square plate.

In the thesis, T -stress and S -stress for wide range of quarter-elliptical corner cracks in finite thickness plates are evaluated by three-dimensional finite element method. The K values will be also calculated for verification purpose.

Figure 2.1 Three basic modes of fracture

Figure 2.2 Three-dimensional coordinate system for the region at the crack tip

Figure 2.3 The global and local coordinate systems

Figure 2.4 The collapsed 20-node element for modeling the singularity
along the leading edge of the crack

Figure 2.5 A point force f applied at the crack tip, the auxiliary field
for the interaction integral

Figure 2.6 A schematic of element used in the definition of the interaction integral

Figure 2.7 An embedded elliptical crack in a solid under uniaxial loading

(a)

(b)

Figure 2.8 A quarter-elliptical crack in a plate under uniaxial loading

Chapter 3

***T*-stress and *S*-stress for Embedded Elliptical Cracks under Remote Loads**

Based on the potential function generated by Segedin (1967), exact solutions of *T*-stress and *S*-stress are derived for an elliptical crack embedded in an infinite elastic solid which is subject remote loads. These results are verified by three-dimensional finite element analysis. It is noted that in the chapter, Cartesian system (x, y, z) is established so that its origin is at the center of elliptic crack and x -axis is always along the major axis of the ellipse, as shown in Figure 3.1(a).

3.1 Background

In this section, the background information needed in order to solve the embedded elliptical crack problems are summarized, including the ellipsoidal coordinates, harmonic potential function, boundary conditions and the rotating angle in coordinate system transformation.

3.1.1 Ellipsoidal Coordinates

Following Kassir and Sih (1975), the ellipsoidal coordinate system is summarized here. The ellipsoidal coordinates (ξ, η, ζ) are the roots of the equation in s .

$$\frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{s} - 1 = 0 \quad (3.1.1)$$

The ellipsoidal coordinates (ξ, η, ζ) are related to the Cartesian coordinates (x, y, z) by the relations represented in eq. (3.1.2):

$$a^2(a^2 - b^2)x^2 = (a^2 + \xi)(a^2 + \eta)(a^2 + \zeta) \quad (3.1.2a)$$

$$b^2(b^2 - a^2)y^2 = (b^2 + \xi)(b^2 + \eta)(b^2 + \zeta) \quad (3.1.2b)$$

$$a^2b^2z^2 = \xi\eta\zeta \quad (3.1.2c)$$

where

$$-a^2 \leq \zeta \leq -b^2 \leq \eta \leq 0 \leq \xi < \infty \quad (3.1.3)$$

In the plane $z = 0$, $\xi = 0$ corresponds to the points (x, y) inside the ellipse and $\eta = 0$ corresponds to the point (x, y) outside the ellipse. Those points on the periphery of ellipse are identified by both $\xi = 0$ and $\eta = 0$.

Refer to a triply orthogonal systems (n, t, z) whose origin locates at an arbitrary point $P(a\cos\varphi, b\sin\varphi, 0)$ of the periphery of the ellipse as shown in Figure 3.2, where n, t, z are always directed along the principal normal, tangential and binormal of the periphery of the ellipse, the spherical coordinates of any point P_1 are designated by $(r_1, \theta_1, \omega_1)$. The Cartesian coordinates of the point P_1 in Cartesian system (x, y, z) are given by the relations

$$x_1 = a \cos \varphi + r_1 \cos \theta_1 (b \cos \varphi \cos \omega_1 - a \sin \varphi \sin \omega_1) (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.1.4a)$$

$$y_1 = b \sin \varphi + r_1 \cos \theta_1 (a \sin \varphi \cos \omega_1 + b \cos \varphi \sin \omega_1) (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.1.4b)$$

$$z_1 = r_1 \sin \theta_1 \quad (3.1.4c)$$

The ellipsoidal coordinates can be related to the spherical coordinates $(r_1, \theta_1, \omega_1)$ as

$$\xi = abr_1 \cos \theta_1 (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \lambda (1 + \kappa) \quad (3.1.5a)$$

$$\eta = abr_1 \cos \theta_1 (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \lambda (1 - \kappa) \quad (3.1.5b)$$

$$\zeta = -(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi) \quad (3.1.5c)$$

where the parameters λ and κ are defined to be

$$\lambda = \cos \omega_1 + \frac{b^2 - a^2}{ab} \sin \omega_1 \sin \varphi \cos \varphi \quad \text{for } \lambda > 0 \quad (3.1.6a)$$

$$\kappa = (1 + \lambda^{-2} \tan^2 \theta_1)^{\frac{1}{2}} \quad (3.1.6b)$$

If P_1 is in plane $z = 0$, where $\theta_1 = 0$ and $\omega_1 = 0$, eq. (3.1.4a) and (3.1.4b) become

$$x_1 = a \cos \varphi + br_1 \cos \varphi (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.1.7a)$$

$$y_1 = b \sin \varphi + ar_1 \sin \varphi (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.1.7b)$$

and eq. (3.1.5a) to eq. (3.1.5c) become

$$\xi = 2abr_1 (b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.1.8a)$$

$$\eta = 0 \quad (3.1.8b)$$

$$\zeta = -(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi) \quad (3.1.8c)$$

3.1.2 Harmonic Function Φ and Boundary Conditions

Consider a flat elliptical crack embedded in an elastic infinite body as show in Figure 3.1(a). The boundary of the elliptical crack is described by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } z = 0 \quad (3.1.9)$$

Consider the remote loading $\sigma_{zz} = p(x, y)$, where $p(x, y)$ are for the three cases:

$$p(x, y) = p_0 \quad \text{tension (Figure 3.1(b), } \lambda = 0 \text{)}$$

$$p(x, y) = -p_0 \left(\frac{x}{a} \right) \quad \text{bending about } y\text{-axis, (Figure 3.1(c), } \lambda = 0 \text{)}$$

$$p(x, y) = p_0 \left(\frac{y}{b} \right) \quad \text{bending about } x\text{-axis, (Figure 3.1(d), } \lambda = 0 \text{)}$$

From the point of view of elasticity, the cracked body under load $\sigma_{zz} = p(x, y)$ at remote ends can be represented by superimposing a cracked body under the pressure $\sigma_{zz} = -p(x, y)$ acting on the crack surfaces symmetrically to the upper and lower surfaces (Type I) and a non-cracked body under load $\sigma_{zz} = p(x, y)$ at far ends (Type II) (See Figure 3.3). For the simplicity of analysis, stresses of Type I are calculated first, then the stress for the original problem are obtained by superposition with the stresses of Type II.

The boundary conditions for Type I are

$$\sigma_{zz} = -p(x, y) \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, z = 0 \right) \quad (3.1.10a)$$

$$w = 0 \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, z = 0 \right) \quad (3.1.10b)$$

$$\tau_{xz} = \tau_{yz} = 0 \quad (z = 0) \quad (3.1.10c)$$

The displacement and stress components can be expressed by harmonic potential function Φ as discussed in Chapter 2. From eq. (2.8e) and (2.8f), boundary condition (3.1.10c) are automatically satisfied on $z = 0$ plane.

Let

$$p(x, y) = \sum_{i=0}^p \sum_{j=0}^q A_{ij} x^i y^j \quad (3.1.11)$$

and substitute eq. (2.7c) and (2.8c) to eq. (3.1.10a) and (3.1.10b), the boundary conditions (3.1.10a) and (3.1.10b) become

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\sum_{i=0}^p \sum_{j=0}^q A_{ij} x^i y^j}{2G} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, z = 0 \right) \quad (3.1.12a)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, z = 0 \right) \quad (3.1.12b)$$

According to Segedin (1967), the general form of potential function Φ can be written as

$$\Phi = \sum_{i=0}^p \sum_{j=0}^q c_{ij} \frac{\partial^{i+j} V^{(i+j+1)}}{\partial x^i \partial y^j} \quad (3.1.13)$$

where

$$V^{(n)} = \int_z^\infty \frac{\omega^n(s) ds}{\sqrt{Q(s)}} \quad (3.1.14a)$$

is the potential function $V^{(n)}$, and n is a positive integer. $\omega(s)$ and $Q(s)$ are defined as :

$$\omega(s) = 1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{s} \quad (3.1.14b)$$

$$Q(s) = s(a^2 + s)(b^2 + s) \quad (3.1.14c)$$

Thus,

$$\frac{\partial \Phi}{\partial z} = -2z \sum_{i=0}^p \sum_{j=0}^q \left[(i+j+1) c_{ij} \int_z^\infty \frac{1}{s\sqrt{Q(s)}} \frac{\partial^{i+j} (\omega^{i+j}(s))}{\partial x^i \partial y^j} ds \right] \quad (3.1.15)$$

It can be found from eq. (3.1.15) that $\frac{\partial \Phi}{\partial z} = 0$ on the plane $z = 0$ and outside the elliptical crack ($\eta = 0, \xi \neq 0$), as shown by Shah and Kobayashi (1971). Therefore,

boundary condition (3.1.12b) is satisfied. And the last boundary condition that the harmonic function Φ should satisfy is the eq. (3.1.12a), written as

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial z^2} \Big|_{\xi=0} &= - \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) \Big|_{\xi=0} \\ &= - \sum_{i=0}^p \sum_{j=0}^q c_{ij} \left(\frac{\partial^{i+j+2} V^{(i+j+1)}}{\partial x^i \partial y^j \partial x^2} + \frac{\partial^{i+j+2} V^{(i+j+1)}}{\partial x^i \partial y^j \partial y^2} \right) \Big|_{\xi=0} = \frac{1}{2G} \sum_{i=0}^p \sum_{j=0}^q A_{ij} x^i y^j \end{aligned} \quad (3.1.16)$$

Shah and Kobayashi (1971) considered the cases in eq. (3.1.16) of $i + j \leq 3$. Solving the system of equations (3.1.16), one can obtain c_{ij} . In the present work, three combination cases of i and j are considered, namely, $i = 0, j = 0$ (tension); $i = 1, j = 0$ (bending about y -axis); $i = 0, j = 1$ (bending about x -axis). Once Φ is obtained, stresses in the Cartesian system (x, y, z) can be calculated from eq. (2.8). Denote $\bar{n}$ and $\bar{t}$ are unit normal and tangential vectors at point $P(x, y, 0)$ on the leading edge of the crack. A new Cartesian system (x_1, x_2, x_3) is established such that the origin is at point $P(x, y, 0)$ or $P(a \cos \varphi, b \sin \varphi, 0)$ in terms of parametric equations of the ellipse; x_1 and x_2 are along $\bar{n}$ and $\bar{t}$, respectively, shown in Figure 3.4. The stress components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{13}$ and τ_{23} in Cartesian system (x_1, x_2, x_3) , i.e., $\sigma_{nn}, \sigma_{zz}, \sigma_{nt}, \tau_{nz}, \tau_{nt}$ and τ_{zt} in Cartesian system (n, t, z) , can be calculated through the system transformation by rotating angle α from x axis.

3.1.3 The Rotating Angle α

If the Cartesian system (n, t, z) has the origin at point $P(a \cos \varphi, b \sin \varphi, 0)$ on the periphery of an ellipse, and n and t are normal (x_1) and tangential (x_2) direction at point P , shown in Figure 3.4, then the angle α can be correlated to the parametric angle φ and the

complementary modulus k' . From the geometry, $\tan \alpha$ can be expressed by the parametric angle φ as

$$\tan \alpha = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{\left(\frac{dx}{d\varphi}\right)}{\left(\frac{dy}{d\varphi}\right)} = \frac{a \sin \varphi}{b \cos \varphi} \quad (3.1.17)$$

From the property of trigonometry,

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} \quad (3.1.18)$$

$$\sin^2 \alpha = 1 - \cos^2 \alpha = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} \quad (3.1.19)$$

$$\sin \alpha \cos \alpha = \tan \alpha \cos^2 \alpha = \tan \alpha \frac{1}{1 + \tan^2 \alpha} \quad (3.1.20)$$

Substituting equation (3.1.17) into equation (3.1.18), (3.1.19) and (3.1.20),

$$\cos^2 \alpha = \frac{k'^2 \cos^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.1.21)$$

$$\sin^2 \alpha = \frac{\sin^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.1.22)$$

$$\sin \alpha \cos \alpha = \frac{k' \sin \varphi \cos \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.1.23)$$

where $k'^2 = 1 - k^2$. k is the modulus of the ellipse and $k^2 = 1 - \left(\frac{b}{a}\right)^2$. k' is called the complementary modulus of the ellipse.

In the next following sections, the T -stress and S -stress of elliptical cracks embedded in elastic solid under tension and bending conditions are derived.

3.2 T -stress and S -stress for Embedded Elliptical Cracks under Uniform Tension

3.2.1 Harmonic Function Φ and Boundary Conditions

Assume that the solid is under the tension corresponding to the nominal stress field $p(x,y) = p_0$, as shown in Figure 3.1(a), then $p = q = 0$ in eq. (3.1.11). From eq. (3.1.13), the harmonic function Φ is expressed by

$$\Phi = \Phi_{00} = c_{00} V^{(1)} = c_{00} \int_{\xi}^{\infty} \frac{\omega(s) ds}{\sqrt{Q(s)}} \quad (3.2.1)$$

From eq. (3.1.16) the boundary condition that the function Φ has to satisfy is

$$\left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{z=0} = \frac{p_0}{2G} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, z = 0 \right) \quad (3.2.2)$$

From eq. (3.2.2), c_{00} in eq. (3.2.1) has been determined by Shah and Kobayashi (1971),

$$c_{00} = \frac{p_0 a b^2}{8GE(k)} \quad (3.2.3)$$

where $E(k)$ is the complete elliptic integral of the second kind (Appendix 1).

3.2.2 Stress Components

As discussed in Section 2.2.1, the determination of T -stress and S -stress requires the solution of stress components σ_{nn} , σ_{ll} and σ_{zz} . These stress components are determined in this section.

Substituting the expressions of σ_{xx} , τ_{xy} and σ_{yy} in terms of harmonic function Φ , eq. (2.8) into eq. (2.15) and (2.17), and employing eq. (3.1.21) to eq. (3.1.23), the expression for σ_{nn} and σ_{tt} at $z = 0$ plane can be written as:

$$\sigma_{nn} = A_1 \frac{\partial^2 \Phi}{\partial x^2} + A_2 \frac{\partial^2 \Phi}{\partial y^2} + A_3 \frac{\partial^2 \Phi}{\partial x \partial y} \quad (3.2.4)$$

$$\sigma_{tt} = A_2 \frac{\partial^2 \Phi}{\partial x^2} + A_1 \frac{\partial^2 \Phi}{\partial y^2} - A_3 \frac{\partial^2 \Phi}{\partial x \partial y} \quad (3.2.5)$$

where

$$A_1 = 2G \frac{(k'^2 \cos^2 \varphi + 2\nu \sin^2 \varphi)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.2.6a)$$

$$A_2 = 2G \frac{(\sin^2 \varphi + 2\nu k'^2 \cos^2 \varphi)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.2.6b)$$

$$A_3 = 4G \frac{(1 - 2\nu) k' \sin \varphi \cos \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.2.6c)$$

Note that at the $z = 0$ plane, following eq. (2.8c), the σ_{zz} is:

$$\sigma_{zz} = -2G \frac{\partial^2 \Phi}{\partial z^2} \quad (3.2.7)$$

It can be seen from eq. (3.2.4), eq. (3.2.5) and eq. (3.2.7) that stress components can be determined once the harmonic function Φ and its derivatives are obtained for specific loads. For the uniform tension case, the potential function Φ is given in eq. (3.2.1) and eq. (3.2.3), and the derivatives of Φ are then found as follows, Kassir and Sih (1975):

$$\frac{\partial^2 \Phi}{\partial x^2} = c_{00} \frac{\partial^2 V^{(1)}}{\partial x^2} = c_{00} \left(-2 \int_{\xi}^{\infty} \frac{ds}{(a^2 + s) \sqrt{Q(s)}} + \frac{4x^2 \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right) \quad (3.2.8)$$

$$\frac{\partial^2 \Phi}{\partial y^2} = c_{00} \frac{\partial^2 V^{(1)}}{\partial y^2} = c_{00} \left(-2 \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} + \frac{4y^2 \sqrt{Q(\xi)}}{(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right) \quad (3.2.9)$$

$$\frac{\partial^2 \Phi}{\partial x \partial y} = c_{00} \frac{\partial^2 V^{(1)}}{\partial x \partial y} = c_{00} \left(\frac{4xy \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)(\xi - \eta)(\xi - \zeta)} \right) \quad (3.2.10)$$

and

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial z^2} = c_{00} \frac{\partial^2 V^{(1)}}{\partial z^2} = c_{00} & \left(2 \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)\sqrt{Q(s)}} + 2 \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} \right. \\ & \left. - \frac{4x^2 \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} - \frac{4y^2 \sqrt{Q(\xi)}}{(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right) \end{aligned} \quad (3.2.11)$$

The integrals in equations (3.2.8), (3.2.9) and (3.2.11) can be further evaluated with the help of Byrd *et al* (1971), by taking $a^* = 0$, $b^* = -b^2$ and $c^* = -a^2$ in eq. (A.37) (see Appendix 1) and eq. (A.38), and employing eq. (A.41) and eq. (A.43). This leads to

$$\int_{\xi}^{\infty} \frac{ds}{(a^2 + s)\sqrt{Q(s)}} = \frac{2}{a^3 k^2} [u_1 - E(u_1)] \quad (3.2.12)$$

$$\int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} = \frac{2}{a^3 k^2 k'^2} [E(u_1) - k'^2 u_1 - k^2 snu_1 cdu_1] \quad (3.2.13)$$

where u_1 , $E(u_1)$, snu_1 , cdu_1 are Jacobian elliptical functions, outlined in Appendix 1. Now, substituting the above eq. (3.2.8) to eq. (3.2.11) into eq. (3.2.4), eq. (3.2.5) and eq. (3.2.7), the stress components at $\theta = 0$, $z = 0$, can be simplified into:

$$\begin{aligned} \sigma_{nn} = & \frac{p_0 ab^2}{E(k)(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left\{ \frac{b^2 \cos^2 \varphi + 2\nu a^2 \sin^2 \varphi}{a^3 k^2} [E(u_1) - u_1] \right\} \\ & - \frac{p_0 ab^2}{E(k)(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left\{ \frac{a^2 \sin^2 \varphi + 2\nu b^2 \cos^2 \varphi}{a^3 k^2 k'^2} [E(u_1) - k'^2 u_1 - k^2 snu_1 cdu_1] \right\} \\ & + \frac{p_0 ab^2}{E(k)\sqrt{Q(\xi)}} \end{aligned} \quad (3.2.14)$$

$$\begin{aligned}\sigma_{\eta} = & \frac{p_0 ab^2}{E(k)(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left\{ \frac{a^2 \sin^2 \varphi + 2vb^2 \cos^2 \varphi}{a^3 k^2} [E(u_1) - u_1] \right\} \\ & - \frac{p_0 ab^2}{E(k)(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left\{ \frac{b^2 \cos^2 \varphi + 2va^2 \sin^2 \varphi}{a^3 k^2 k'^2} [E(u_1) - k'^2 u_1 - k^2 snu_1 cdu_1] \right\} \\ & + \frac{2vp_0 ab^2}{E(k)\sqrt{Q(\xi)}}\end{aligned}\quad (3.2.15)$$

and

$$\sigma_{\xi} = \frac{p_0}{E(k)} [-E(u_1) + snu_1 cdu_1] + \frac{p_0 ab^2}{E(k)\sqrt{Q(\xi)}} \quad (3.2.16)$$

Note that these solutions are for Type I discussed in Section 3.1.2, after superimposing the stress solutions for Type II under tension, we have the stress solutions for remote tension:

$$\sigma_z = \frac{p_0}{E(k)} [-E(u_1) + snu_1 cdu_1] + \frac{p_0 ab^2}{E(k)\sqrt{Q(\xi)}} + p_0 \quad (3.2.17)$$

while σ_{nn} and σ_{η} will remain as eq. (3.2.14) and eq. (3.2.15), respectively.

3.2.3 Determination of T -stress

The T -stress can be obtained from eq. (2.18). Considering the eq. (3.1.5a), with θ and ω being zero, the relationship between r and ξ is expressed by

$$\xi = 2abr(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)^{-\frac{1}{2}} \quad (3.2.18)$$

Therefore, eq. (2.18) becomes

$$T = \lim_{\xi \rightarrow 0} (\sigma_{nn}|_{z=0} - \sigma_{\xi}|_{z=0}) \quad (3.2.19)$$

Substituting the stress components of eq. (3.2.14) and (3.2.17) into eq. (3.2.19), and take the limit of $\xi \rightarrow 0$, the exact T -stress solution under tension is obtained. Note that if

the limit $\xi \rightarrow 0$, u_1 becomes the complete elliptic integral of first kind $K(k)$, $E(u_1)$ becomes the complete elliptic integral of second kind $E(k)$, and $snu_1cd u_1$ vanishes. The T -stress solution is expressed by

$$T = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} \{f_1 - f_2\} \quad (3.2.20)$$

or

$$t = \frac{T}{p_0} = \frac{k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} \{f_1 - f_2\} \quad (3.2.21)$$

where t is the normalized T -stress and p_0 is nominal stress, and

$$f_1 = \frac{k'^2 \cos^2 \varphi + 2\nu \sin^2 \varphi}{k^2} [E(k) - K(k)] \quad (3.2.22a)$$

$$f_2 = \frac{\sin^2 \varphi + 2\nu k'^2 \cos^2 \varphi}{k^2 k'^2} [E(k) - k'^2 K(k)] \quad (3.2.22c)$$

3.2.4 Determination of S -stress

S -stress in eq. (2.19) can be re-written into the following eq. (3.2.23) when the relationship between r and ξ presented by eq. (3.2.18) is employed:

$$S = \lim_{\xi \rightarrow 0} (\sigma_{rr}|_{z=0} - 2\nu \sigma_{zz}|_{z=0}) \quad (3.2.23)$$

Substituting eq. (3.2.15) and eq. (3.2.17) into eq. (3.2.23), and take the limit of $\xi \rightarrow 0$, the expression of S -stress under tension loads is obtained:

$$S = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} (h_1 - h_2) \quad (3.2.24)$$

or

$$s = \frac{S}{p_0} = \frac{k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} (h_1 - h_2) \quad (3.2.25)$$

where s is the normalized S -stress, and

$$h_1 = \frac{\sin^2 \varphi + 2\nu k'^2 \cos^2 \varphi}{k^2} [E(k) - K(k)] \quad (3.2.26a)$$

$$h_2 = \frac{2\nu \sin^2 \varphi + k'^2 \cos^2 \varphi}{k^2 k'^2} [E(k) - k'^2 K(k)] \quad (3.2.26b)$$

From eq. (3.2.20) and (3.2.24), it can be found that for three-dimensional problems, both T -stress and S -stress depend on the material property parameter ν . The distributions of normalized T -stress (eq. (3.2.21)) and normalized S -stress (eq. (3.2.25)) along the crack front for $\nu = 0.3$ are shown in Figure 3.5 and Figure 3.6, for different aspect ratios of b/a .

It is observed that T -stress and S -stress have the different trends. T -stress is always negative and reaches its minimum value at $\varphi = 90^\circ$. S -stress is always negative too, but it reaches its maximum value at $\varphi = 90^\circ$. For a two-dimensional crack problem, it is recognized that the sign of the T -stress determines the constraint levels near the crack tip, i.e., positive T -stress means high constraint and negative value means low constraint. However, there are two constraint parameters for three-dimensional crack cases: T -stress and S -stress. The effect of T -stress on constraint levels is widely accepted while the effect of S -stress is still under investigation. Therefore, only the sign of T -stress is checked for constraint levels for elliptical cracks. From this point, it can be found that the constraint level is low for elliptical cracks under tension.

It is also noted that for penny shaped crack ($b/a = 1$) the modulus of ellipse, k , becomes zero and the complete elliptic integral of the second kind, $E(k)$, has to be zero, too. Therefore, equations derived here for T -stress and S -stress cannot be used directly for evaluating the T -stress and S -stress of penny-shaped crack. However, the normalized T -stress calculated by eq. (3.2.21) using $b/a = 0.9999$ converges to the result of Wang (2004) for penny-shaped cracks.

3.3 T -stress and S -stress Subject to Bending Loads $p(x, y) = -p_0 \left(\frac{x}{a} \right)$ at Far Ends

In this section, the same procedure as Section 3.2 is applied to calculate the T -stress and S -stress for the remote bending about y -axis, shown in Figure 3.1(c) (biaxial ratio $\lambda = 0$).

3.3.1 Harmonic Function Φ and Boundary Conditions

For this case, as the geometry and loadings illustrated in Figure 3.1(c) (biaxial ratio $\lambda = 0$), $p = 1$, $q = 0$ and the harmonic function Φ is given by eq. (3.1.13), expressed as

$$\Phi = \Phi_{10} = c_{10} \frac{\partial V^{(2)}}{\partial x} = -4c_{10}x \int_{\xi}^{\infty} \frac{\omega(s)ds}{(a^2 + s)\sqrt{Q(s)}} \quad (3.3.1)$$

The boundary condition (3.1.16) becomes

$$\frac{\partial^2 \Phi}{\partial z^2} \Big|_{\xi=0} = -\frac{p_0}{2G} \left(\frac{x}{a} \right) \quad (3.3.2)$$

where c_{10} was calculated by Shah and Kobayashi (1971),

$$c_{10} = -\frac{p_0 a^2 b^2 k^2}{32G[(1-2k^2)E(k) - k'^2 K(k)]} \quad (3.3.3)$$

3.3.2 Stress Components

The stress components σ_{nn} , σ_{tt} and σ_{zz} at $\theta = 0$, $z = 0$, will be calculated in the next. For the potential function Φ is given in eq. (3.3.1) and eq. (3.3.3), the derivatives of Φ are then found as follows, Kassir and Sih (1975):

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= c_{10} \frac{\partial^2}{\partial x^2} \left(\frac{\partial V^{(2)}}{\partial x} \right) \\ &= c_{10} \left[24x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)^2 \sqrt{Q(s)}} - \frac{16x^3 \sqrt{Q(\xi)}}{(a^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} \right] \end{aligned} \quad (3.3.4)$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial y^2} &= c_{10} \frac{\partial^2}{\partial y^2} \left(\frac{\partial V^{(2)}}{\partial x} \right) \\ &= c_{10} \left[8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right] \end{aligned} \quad (3.3.5)$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x \partial y} &= c_{10} \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial V^{(2)}}{\partial x} \right) \\ &= c_{10} \left[8y \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16x^2 y \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (b^2 + \xi)(\xi - \eta)(\xi - \zeta)} \right] \end{aligned} \quad (3.3.6)$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial z^2} &= -c_{10} \left[24x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)^2 \sqrt{Q(s)}} + 8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} \right] \\ &\quad + c_{10} \left[\frac{16x^3 \sqrt{Q(\xi)}}{(a^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} + \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right] \end{aligned} \quad (3.3.7)$$

The integrals in above equations were further evaluated with the help of Byrd *et al* (1975), by substituting $a^* = 0$, $b^* = -b^2$ and $c^* = -a^2$ into eq. (A.37) and eq. (A.40) and employing eq. (A.42) and eq. (A.45).

$$\int_{\xi}^{\infty} \frac{ds}{(a^2 + s)^2 \sqrt{Q(s)}} = \frac{2}{3a^5 k^4} \left[(2 + k^2) u_1 - 2(1 + k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \quad (3.3.8)$$

$$\int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s) \sqrt{Q(s)}} = \frac{2}{a^5 k'^2 k^4} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.9)$$

Substituting eq. (3.3.4) to eq. (3.3.7) into eq. (3.2.4), the stress at $\theta = 0, z = 0$ and in normal direction can be simplified in the form of

$$\sigma_{nn} = 32Gc_{10}(B_1 + B_2 + B_3 + B_4 + B_5 + D_1) \quad (3.3.10)$$

where

$$B_1 = \frac{ab^2 \cos^3 \varphi}{a^5 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(2 + k^2) u_1 - 2(1 + k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \quad (3.3.11a)$$

$$B_2 = \frac{2\nu ab^2 \cos^3 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.11b)$$

$$B_3 = \frac{2(1 - 2\nu)ab^2 \sin^2 \varphi \cos \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.11c)$$

$$B_4 = \frac{a^3 \sin^2 \varphi \cos \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.11d)$$

$$B_5 = \frac{2\nu a^3 \sin^2 \varphi \cos \varphi}{a^5 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(2 + k^2) \mu_1 - 2(1 + k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \quad (3.3.11e)$$

$$D_1 = - \frac{\sqrt{Q(\xi)}}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)(\xi - \eta)(\xi - \zeta)} \times \left(\frac{b^2 \cos^5 \varphi}{a^3} + \frac{2 \cos^3 \varphi \sin^2 \varphi}{a} + \frac{a \cos \varphi \sin^4 \varphi}{b^2} \right) \quad (3.3.11f)$$

Applying eq. (3.2.5), the stress at $\theta = 0$, $z = 0$ and in tangential direction can be represented in the form of

$$\sigma_{\theta} = 32 G c_{10} (B_1' + B_2' + B_3' + B_4' + B_5' + D_1') \quad (3.3.12)$$

where

$$B_1' = \frac{a^3 \sin^2 \varphi \cos \varphi}{a^5 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(2 + k^2) \mu_1 - 2(1 + k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \quad (3.3.13a)$$

$$B_2' = \frac{2\nu a^3 \sin^2 \varphi \cos \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.13b)$$

$$B_3' = - \frac{2(1 - 2\nu) a b^2 \sin^2 \varphi \cos \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.13c)$$

$$B_4' = \frac{a b^2 \cos^3 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.13d)$$

$$B_5 = \frac{2\nu ab^2 \cos^3 \varphi}{a^5 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(2+k^2) \mu_1 - 2(1+k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \quad (3.3.13e)$$

$$D_1 = -\frac{2\nu \sqrt{Q(\xi)} \cos \varphi}{a^3 b^2 (\xi - \eta)(\xi - \zeta)} (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi) \quad (3.3.13f)$$

and from eq. (3.2.7), the stress in z direction for Type I only is

$$\sigma_{zz} = 32Gc_{10} (B_6 + D_2) \quad (3.3.14a)$$

where

$$B_6 = \frac{\cos \varphi}{a^4 k^4} \left[(2+k^2) \mu_1 - 2(1+k^2) E(u_1) + k^2 s n u_1 c n u_1 d n u_1 \right] \\ + \frac{\cos \varphi}{a^4 k'^2 k^4} \left[(1+k'^2) E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.3.14b)$$

$$D_2 = -\frac{\sqrt{Q(\xi)}}{(\xi - \eta)(\xi - \zeta)} \left(\frac{\cos^3 \varphi}{a^3} + \frac{\cos \varphi \sin^2 \varphi}{ab^2} \right) \quad (3.3.14c)$$

After superimposing the stress components of Type I and Type II sub-problems, the stress components σ_{nn} and σ_{tt} do not change but the stress in z direction becomes

$$\sigma_{zz} = 32Gc_{10} (B_6 + D_2) - p_0 \cos \varphi \quad (3.3.15)$$

3.3.3 Determination of T -stress

Substituting (3.3.10) and (3.3.15) into eq. (3.2.19), the exact T -stress solution for embedded elliptical crack in an infinite body subject to bending about y -axis at far ends is given by

$$T = \lim_{\xi \rightarrow 0} (\sigma_{mm}|_{z=0} - \sigma_{zz}|_{z=0}) = \frac{32Gc_{10}(1-2\nu)a \cos \varphi}{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} (F_1 + F_2) + p_0 \cos \varphi \quad (3.3.16)$$

where c_{10} is represented by eq. (3.3.3), and

$$F_1 = \frac{\sin^2 \varphi}{a^3 k^4} [2(1+k^2)E(k) - (2+k^2)K(k)] \quad (3.3.17a)$$

$$F_2 = \frac{b^2(2\sin^2 \varphi - \cos^2 \varphi)}{a^5 k'^2 k^4} [(1+k'^2)E(k) - 2k'^2 K(k)] \quad (3.3.17b)$$

or

$$t = \frac{T}{p_0} = (1-2\nu)C_0[F_3E(k) - F_4K(k)] + \cos \varphi \quad (3.3.18)$$

where

$$C_0 = -\left(\frac{k'}{k}\right)^2 \frac{\cos \varphi}{(\sin^2 \varphi + k'^2 \cos^2 \varphi)[(1-2k^2)E(k) - k'^2 K(k)]} \quad (3.3.19a)$$

$$F_3 = 2(1+k^2)\sin^2 \varphi + (1+k'^2)(2-3\cos^2 \varphi) \quad (3.3.19b)$$

$$F_4 = (2+k^2)\sin^2 \varphi + 2k'^2(2-3\cos^2 \varphi) \quad (3.3.19c)$$

3.3.4 Determination of S -stress

Substituting eq. (3.3.12) and eq. (3.3.15) into eq. (3.2.23), the expression of S -stress under bending loads is obtained:

$$S = \lim_{\xi \rightarrow 0} (\sigma_{xx}|_{z=0} - 2\nu\sigma_{zz}|_{z=0}) = \frac{p_0 C_1 (h_3 - h_4)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + 2\nu p_0 \cos \varphi \quad (3.3.20)$$

or

$$s = \frac{S}{p_0} = \frac{C_1 (h_3 - h_4)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + 2\nu \cos \varphi \quad (3.3.21)$$

where

$$C_1 = -\frac{(1-2\nu)\cos \varphi}{(1-2k^2)E(k) - k'^2 K(k)} \left(\frac{k'}{k}\right)^2 \quad (3.3.22a)$$

$$h_3 = [(2 + k^2)K(k) - 2(1 + k^2)E(k)]\sin^2 \varphi \quad (3.3.22b)$$

$$h_4 = (2 - 3\cos^2 \varphi)[(1 + k'^2)E(k) - 2k'^2 K(k)] \quad (3.3.22c)$$

The results of eq. (3.3.18) and eq. (3.3.21) are plotted in Figure 3.7 and Figure 3.8 for $\nu = 0.3$ and various b/a ratios. It is observed that generally T -stress and S -stress drop from positive values at $\varphi = 0^\circ$ to negative values at $\varphi = 180^\circ$. Both T -stress and S -stress reach zero at $\varphi = 90^\circ$. This trend of T -stress shows that the constraint level is high from $\varphi = 0^\circ$ to $\varphi = 90^\circ$ and low from $\varphi = 90^\circ$ to $\varphi = 180^\circ$. The normalized T -stress calculated by eq. (3.3.18) when $b/a = 0.9999$ converges to the result of Wang (2004) for penny-shaped cracks.

3.4 T -stress and S -stress under Bending Loads $p(x, y) = p_0 \left(\frac{y}{b} \right)$ at Far Ends

Similarly, T -stress and S -stress of embedded elliptical cracks in infinite solids under bending about x -axis are derived in the section, shown in Figure 3.1(d) (biaxial ratio $\lambda = 0$).

3.4.1 Harmonic Function Φ and Boundary conditions

For the geometry and bending about x -axis, as illustrated in Figure 3.1(d) (biaxial ratio $\lambda = 0$), the function Φ is presented in the form of

$$\Phi = \Phi_{01} = c_{01} \frac{\partial V^{(2)}}{\partial y} = -4c_{01}y \int_{\epsilon}^{\infty} \frac{\omega(s)ds}{(b^2 + s)\sqrt{Q(s)}} \quad (3.4.1)$$

and the boundary condition (3.1.16) becomes

$$\left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{\xi=0} = \frac{p_0}{2G} \left(\frac{y}{b} \right) \quad (3.4.2)$$

$\left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{\xi=0}$ was represented by Shah and Kobayashi (1971) as

$$\left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{\xi=0} = -\frac{16c_{01}y}{ab^4k^2} \left[(1+k^2)E(k) - k'^2 K(k) \right] \quad (3.4.3)$$

Substituting (3.4.3) into (3.4.2), c_{10} can be represented by

$$c_{01} = -\frac{ab^3k^2p_0}{32G[(1+k^2)E(k) - k'^2 K(k)]} \quad (3.4.4)$$

3.4.2 Stress Components

The stress components σ_{nn} , $\sigma_{\eta\eta}$ and $\sigma_{\xi\xi}$ at $\theta = 0$, $z = 0$, will be derived in the next. The potential function Φ is given in eq. (3.4.1) and eq. (3.4.4), and the derivatives of Φ are then found as follows, from Kassir and Sih (1975):

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= c_{01} \frac{\partial^2}{\partial x^2} \left(\frac{\partial V^{(2)}}{\partial y} \right) \\ &= c_{01} \left[8y \int_{\xi}^{\infty} \frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} - \frac{16x^2y\sqrt{Q(\xi)}}{(a^2+\xi)^2(b^2+\xi)(\xi-\eta)(\xi-\zeta)} \right] \end{aligned} \quad (3.4.5)$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial y^2} &= c_{01} \frac{\partial^2}{\partial y^2} \left(\frac{\partial V^{(2)}}{\partial y} \right) = c_{01} \left[24y \int_{\xi}^{\infty} \frac{ds}{(b^2+s)^2\sqrt{Q(s)}} - \frac{16y^3\sqrt{Q(\xi)}}{(b^2+\xi)^3(\xi-\eta)(\xi-\zeta)} \right] \end{aligned} \quad (3.4.6)$$

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial x \partial y} &= c_{01} \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial V^{(2)}}{\partial y} \right) \\
&= c_{01} \left[8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \right]
\end{aligned} \tag{3.4.7}$$

and

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial z^2} &= -c_{01} \left[8y \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} + 24y \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)^2 \sqrt{Q(s)}} \right] \\
&\quad + c_{01} \left[\frac{16x^2 y \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (b^2 + \xi)(\xi - \eta)(\xi - \zeta)} + \frac{16y^3 \sqrt{Q(\xi)}}{(b^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} \right]
\end{aligned} \tag{3.4.8}$$

One of the integrals in above equations has already been evaluated by eq. (3.3.9).

Another integral is further evaluated with the help of Byrd *et al* (1975), by substituting

$a^* = 0$, $b^* = -b^2$ and $c^* = -a^2$ into eq. (A.38) and employing eq. (A.44).

$$\begin{aligned}
\int_{\xi}^{\infty} \frac{ds}{(b^2 + s)^2 \sqrt{Q(s)}} &= \frac{2}{3a^5 k^4 k'^4} \left[2(2k^2 - 1)E(u_1) + k'^2 (2 - 3k^2)u_1 \right] \\
&\quad - k^2 snu_1 cdu_1 \frac{2}{3a^5 k^4 k'^4} (k'^2 nd^2 u_1 + 4k^2 - 2)
\end{aligned} \tag{3.4.9}$$

Substituting eq. (3.4.5) - eq. (3.4.8) into eq. (3.2.4), the stress at $\theta = 0$, $z = 0$ and in normal direction can be simplified in the form of

$$\sigma_{nn} = 32Gc_{01}(L_1 + L_2 + L_3 + L_4 + L_5 + M_1) \tag{3.4.10}$$

where

$$L_1 = \frac{b^3 \sin \varphi \cos^2 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2)E(u_1) - 2k'^2 u_1 - k^2 snu_1 cdu_1 \right] \tag{3.4.11a}$$

$$L_2 = \frac{2\nu b^3 \sin \varphi \cos^2 \varphi}{a^5 k^4 k'^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[2(2k^2 - 1)E(u_1) + k'^2 (2 - 3k^2)u_1 \right] \\ - k^2 s n u_1 c d u_1 \frac{2\nu b^3 \sin \varphi \cos^2 \varphi}{a^5 k^4 k'^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} (k'^2 n d^2 u_1 + 4k^2 - 2) \quad (3.4.11b)$$

$$L_3 = \frac{2(1 - 2\nu) a^2 b \sin \varphi \cos^2 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2)E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.4.11c)$$

$$L_4 = \frac{a^2 b \sin^3 \varphi}{a^5 k^4 k'^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[2(2k^2 - 1)E(u_1) + k'^2 (2 - 3k^2)u_1 \right] \\ - k^2 s n u_1 c d u_1 \frac{a^2 b \sin^3 \varphi}{a^5 k^4 k'^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} (k'^2 n d^2 u_1 + 4k^2 - 2) \quad (3.4.11d)$$

$$L_5 = \frac{2\nu a^2 b \sin^3 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2)E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.4.11e)$$

$$M_1 = - \frac{\sqrt{Q(\xi)}}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)(\xi - \eta)(\xi - \zeta)} \\ \times \left(\frac{b \sin \varphi \cos^4 \varphi}{a^2} + \frac{2 \sin^3 \varphi \cos \varphi}{b} + \frac{a^2 \sin^5 \varphi}{b^3} \right) \quad (3.4.11f)$$

Using eq. (3.2.5), the stress at $\theta = 0$, $z = 0$ and in tangential direction can be simplified in the form of

$$\sigma_{\theta} = 32 G c_{01} (\dot{L}_1 + \dot{L}_2 + \dot{L}_3 + \dot{L}_4 + \dot{L}_5 + \dot{M}_1) \quad (3.4.12)$$

where

$$\dot{L}_1 = \frac{a^2 b \sin^3 \varphi}{a^5 k'^2 k^4 (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)} \left[(1 + k'^2)E(u_1) - 2k'^2 u_1 - k^2 s n u_1 c d u_1 \right] \quad (3.4.13a)$$

$$L_2 = \frac{2va^2b\sin^3\varphi}{a^5k^4k'^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}[2(2k^2-1)E(u_1)+k'^2(2-3k^2)u_1] \\ - k^2snu_1cdu_1 \frac{2va^2b\sin^3\varphi}{a^5k^4k'^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}(k'^2nd^2u_1+4k^2-2) \quad (3.4.13b)$$

$$L_3 = -\frac{2(1-2v)a^2b\sin\varphi\cos^2\varphi}{a^5k'^2k^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}[(1+k'^2)E(u_1)-2k'^2u_1-k^2snu_1cdu_1] \quad (3.4.13c)$$

$$L_4 = \frac{b^3\sin\varphi\cos^2\varphi}{a^5k^4k'^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}[2(2k^2-1)E(u_1)+k'^2(2-3k^2)u_1] \\ - k^2snu_1cdu_1 \frac{b^3\sin\varphi\cos^2\varphi}{a^5k^4k'^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}(k'^2nd^2u_1+4k^2-2) \quad (3.4.13d)$$

$$L_5 = \frac{2vb^3\sin\varphi\cos^2\varphi}{a^5k'^2k^4(a^2\sin^2\varphi+b^2\cos^2\varphi)}[(1+k'^2)E(u_1)-2k'^2u_1-k^2snu_1cdu_1] \quad (3.4.13e)$$

$$M_1 = -\frac{2v\sqrt{Q(\xi)}(a^2\sin^2\varphi+b^2\cos^2\varphi)\sin\varphi}{a^2b^3(\xi-\eta)(\xi-\zeta)} \quad (3.4.13f)$$

and the stress in z direction obtained from eq. (3.2.7) for Type I is:

$$\sigma_z = 32Gc_{01}(L_6 + M_2) \quad (3.4.14a)$$

where

$$L_6 = 32Gc_{01} \frac{b\sin\varphi}{a^5k'^2k^4} [(1+k'^2)E(u_1)-2k'^2u_1-k^2snu_1cdu_1] \\ + 32Gc_{01} \frac{b\sin\varphi}{a^5k'^4k^4} [2(2k^2-1)E(u_1)+k'^2(2-3k^2)u_1-k^2snu_1cdu_1(k'^2nd^2u_1+4k^2-2)] \quad (3.4.14b)$$

$$M_2 = -\frac{32Gc_{01}\sqrt{Q(\xi)}}{(\xi-\eta)(\xi-\zeta)} \left[\frac{\cos^2 \varphi \sin \varphi}{a^2 b} + \frac{\sin^3 \varphi}{b^3} \right] \quad (3.4.14c)$$

After superimposing the stress components of Type I and Type II sub-problems, the stress components σ_{nn} and σ_{tt} do not change but the stress in z direction becomes

$$\sigma_z = 32Gc_{01}(L_6 + M_2) + p_0 \sin \varphi \quad (3.4.15)$$

3.4.3 Determination of T -stress

Substituting (3.4.10) and (3.4.15) into eq. (3.2.19), the exact T -stress solution for embedded elliptical crack in an infinite body subject to bending about x -axis at far ends is given by

$$\begin{aligned} T &= \lim_{\xi \rightarrow 0} (\sigma_{nn}|_{z=0} - \sigma_{zz}|_{z=0}) \\ &= -\left(\frac{k'}{k}\right)^2 \frac{p_0(1-2\nu)\sin \varphi}{(\sin^2 \varphi + k'^2 \cos^2 \varphi)[(1+k^2)E(k) - k'^2 K(k)]} (g_1 - g_2) - p_0 \sin \varphi \end{aligned} \quad (3.4.16)$$

or

$$t = \frac{T}{p_0} = -\left(\frac{k'}{k}\right)^2 \frac{(1-2\nu)\sin \varphi}{(\sin^2 \varphi + k'^2 \cos^2 \varphi)[(1+k^2)E(k) - k'^2 K(k)]} (g_1 - g_2) - \sin \varphi \quad (3.4.17)$$

where

$$g_1 = \{(1+k'^2)(2-3\sin^2 \varphi) - 2(2k^2-1)\cos^2 \varphi\}E(k) \quad (3.4.18a)$$

$$g_2 = \{2k'^2(2-3\sin^2 \varphi) + k'^2(2-3k^2)\cos^2 \varphi\}K(k) \quad (3.4.18b)$$

3.4.4 Determination of S -stress

Substituting eq. (3.4.12) and eq. (3.4.15) into eq. (3.2.23), the expression of S -stress under bending loads is obtained:

$$S = \lim_{\xi \rightarrow 0} (\sigma_{xx}|_{z=0} - 2\nu\sigma_{zz}|_{z=0}) = \frac{C_2(h_5 + h_6)p_0}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu p_0 \sin \varphi \quad (3.4.19)$$

or

$$s = \frac{S}{p_0} = \frac{C_2(h_5 + h_6)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \sin \varphi \quad (3.4.20)$$

where

$$C_2 = -\frac{(1-2\nu)\sin \varphi}{(1+k^2)E(k) - k'^2 K(k)} \left(\frac{k'}{k}\right)^2 \quad (3.4.21a)$$

$$h_5 = (\sin^2 \varphi - 2\cos^2 \varphi) [(1+k'^2)E(k) - 2k'^2 K(k)] \quad (3.4.21b)$$

$$h_6 = [2(2k^2 - 1)E(k) + k'^2 (2 - 3k^2)K(k)] \cos^2 \varphi \quad (3.4.21c)$$

The results from eq. (3.4.17) and eq. (3.4.20) are plotted in Figure 3.9 and Figure 3.10 for $\nu = 0.3$ and different b/a ratios. It is observed that both T -stress and S -stress are negative from $\varphi = 0^\circ$ to $\varphi = 180^\circ$ and reaches their minimum values at $\varphi = 90^\circ$ corresponding to the compression stress. Due to the behavior of the bending, it can be concluded that from $\varphi = 0^\circ$ to $\varphi = -180^\circ$ both T -stress and S -stress are positive and reach their maximum values at $\varphi = -90^\circ$. From the trend of T -stress, the constraint level is low from $\varphi = 0^\circ$ to $\varphi = 180^\circ$ and is high from $\varphi = 0^\circ$ to $\varphi = -180^\circ$.

3.5 T -stress and S -stress for Embedded Elliptical Crack under Biaxial Tension and Bending

The T -stress and S -stress solutions for an elliptical crack embedded in an infinite body subject to remote uniaxial tension and bending loads are derived in Section 3.2, Section 3.3 and Section 3.4. But many engineering structures are under biaxial loads. It is necessary to determine the T -stress and S -stress solutions for an elliptical crack embedded in an infinite body subject to remote biaxial loads. Two kinds of loads, remote biaxial tension and remote bending about two orthogonal axes (bi-moments), shown in Figure 3.1(b), (c) and (d), are considered in the following.

3.5.1 T -stress and S -stress Calculations for Remote Biaxial Loads

From the view of elasticity, the T -stress solutions for a cracked body under biaxial loads can be obtained by superposition method as demonstrated below. The T -stress for this problem (denoted by T_{total}) (Figure 3.11(a)) can be divided into two parts: the T -stress for the same crack under the pressure acting on upper and lower crack surfaces (denoted by $T_{crack_pressure}$) (Figure 3.11(b)), and the T -stress of an un-cracked body subject to the same biaxial loadings as the original problem (denoted by $T_{uncracked}$) (Figure 3.11(c)). The relationship at a general point $P (a\cos\varphi, b\sin\varphi)$ can be expressed by

$$T_{total}(P) = T_{crack_pressure}(P) + T_{uncracked}(P) \quad (3.5.1)$$

Similarly, if S_{total} , $S_{crack_pressure}$ and $S_{uncracked}$ are the S -stresses for Figure 3.11(a), Figure 3.11(b) and Figure 3.11(c), these S -stresses at the point P have the relationship of

$$S_{total}(P) = S_{crack_pressure}(P) + S_{uncracked}(P) \quad (3.5.2)$$

It is noted that the $T_{crack_pressure}(P)$ in eq. (3.5.1) and $S_{crack_pressure}(P)$ in eq. (3.5.2) are the T -stress and S -stress for Type I sub-problem discussed earlier, respectively. These two terms can be obtained from the stress components obtained in Section 3.2, Section 3.3 and Section 3.4 for remote tension and bending conditions. For the un-cracked sub-problem, the stress intensity factor is zero because there is no stress singularity along the crack front. However, it cannot be said for T -stress and S -stress. Applying eq. (3.2.19) and eq. (3.2.23), the T -stress and S -stress for un-cracked sub-problem are represented by

$$T_{uncracked}(P) = (\sigma_{nn} - \sigma_{zz}) \Big|_P \quad (3.5.3)$$

$$S_{uncracked}(P) = (\sigma_{tt} - 2\nu\sigma_{zz}) \Big|_P \quad (3.5.4)$$

Substituting eq. (3.5.3) into eq. (3.5.1) and substituting eq. (3.5.4) into eq. (3.5.2), the T -stress and S -stress for the biaxial problem (Figure 3.11(a)) are then

$$T_{total}(P) = T_{crack_pressure}(P) + (\sigma_{nn} - \sigma_{zz}) \Big|_P \quad (3.5.5)$$

$$S_{total}(P) = S_{crack_pressure}(P) + (\sigma_{tt} - 2\nu\sigma_{zz}) \Big|_P \quad (3.5.6)$$

From eq. (3.5.5) and eq. (3.5.6), it can be seen that T -stress and S -stress can be determined once the T -stress and S -stress at point P under crack pressure are obtained together with the stress field at the corresponding point for un-cracked body. The procedure of determining these T -stress and S -stress components is demonstrated in the next sections for biaxial tension and bending conditions.

3.5.2 T -stress and S -stress for Embedded Elliptical Crack Subject to Biaxial Remote

Tension ($\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$)

The crack geometry and loading conditions are shown in Figure 3.1(b) for this case. Substituting eq. (3.2.14) and eq. (3.2.16) into eq. (3.2.19), T -stress for Type I sub-problem under uniform tension, shown in Figure 3.11(b), is expressed by

$$T_{crack_pressure}(\varphi) = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} \{f_1 - f_2\} + p_0 \quad (3.5.7)$$

where

$$f_1 = \frac{k'^2 \cos^2 \varphi + 2\nu \sin^2 \varphi}{k^2} [E(k) - K(k)] \quad (3.5.8a)$$

$$f_2 = \frac{\sin^2 \varphi + 2\nu k'^2 \cos^2 \varphi}{k^2 k'^2} [E(k) - k'^2 K(k)] \quad (3.5.8b)$$

For the un-cracked body subject to biaxial tension corresponding to the nominal stress components $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, shown in Figure 3.11(c), $T_{uncracked}$ can be obtained from eq. (3.5.3) by employing eq. (2.15), eq. (2.16) and eq. (3.1.21).

$$(\sigma_{nn} - \sigma_{zz})_P = p_0 \left(\frac{\lambda k'^2 \cos^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 1 \right) \quad (3.5.9)$$

Substituting eq. (3.5.7) and eq. (3.5.9) into eq. (3.5.5), T -stress for biaxial loading conditions with $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$ can be expressed by

$$T_{total}(\varphi) = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} \{f_1 - f_2\} + \frac{\lambda p_0 k'^2 \cos^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.10a)$$

or

$$t(\varphi) = \frac{T_{total}(\varphi)}{p_0} = \frac{k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} \{f_1 - f_2\} + \frac{\lambda k'^2 \cos^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.10b)$$

where $t(\varphi)$ is the normalized T -stress. f_1 and f_2 are represented by eq. (3.5.8a) and eq. (3.5.8b).

Similarly, substituting eq. (3.2.15) and eq. (3.2.16) into eq. (3.2.23), S -stress for Type I sub-problems under uniform tension, shown in Figure 3.11(b), are represented by

$$S_{crack_pressure}(\varphi) = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} (h_1 - h_2) + 2\nu p_0 \quad (3.5.11)$$

where

$$h_1 = \frac{\sin^2 \varphi + 2\nu k'^2 \cos^2 \varphi}{k^2} [E(k) - K(k)] \quad (3.5.12a)$$

$$h_2 = \frac{2\nu \sin^2 \varphi + k'^2 \cos^2 \varphi}{k^2 k'^2} [E(k) - k'^2 K(k)] \quad (3.5.12b)$$

For the un-cracked body subject to biaxial tension corresponding to the nominal stress $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, shown in Figure 3.11(c), it can be obtained from eq. (3.5.4) by employing eq. (2.16), eq. (2.17) and eq. (3.1.22),

$$(\sigma_{xx} - 2\nu \sigma_{zz})_P = p_0 \left(\frac{\lambda \sin^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \right) \quad (3.5.13)$$

Substituting eq. (3.5.11) and eq. (3.5.13) into eq. (3.5.6), S -stress for biaxial loading conditions with $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$ can be expressed by

$$S_{total}(\varphi) = \frac{p_0 k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} (h_1 - h_2) + \frac{\lambda p_0 \sin^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.14a)$$

or

$$s(\varphi) = \frac{S_{total}(\varphi)}{p_0} = \frac{k'^2}{E(k)(\sin^2 \varphi + k'^2 \cos^2 \varphi)} (h_1 - h_2) + \frac{\lambda \sin^2 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.14b)$$

where $s(\varphi)$ is the normalized S -stress. h_1 and h_2 are represented by eq. (3.5.12a) and eq. (3.5.1b).

The normalized T -stress is plotted in Figure 3.12, Figure 3.13 and Figure 3.14 and the normalized S -stress is plotted in Figure 3.15, Figure 3.16 and Figure 3.17 for $\nu = 0.3$, $\lambda = 0.3, 0.5$ and 1.0 and aspect ratio $b/a = 0.2, 0.4, 0.6$ and 0.9999 . It is observed that with the biaxial ratio, λ , changes from $0.3, 0.5$ to 1.0 , T -stress does not change at $\varphi = 90^\circ$ while increase from negative to positive at $\varphi = 0^\circ$ and $\varphi = 180^\circ$. This trend shows that the constraint level at $\varphi = 0^\circ$ and $\varphi = 180^\circ$ increased from low to high when the second loading becomes larger. However, S -stress does not change at $\varphi = 0^\circ$ and $\varphi = 180^\circ$ but increased at $\varphi = 90^\circ$ from negative to positive values with the greater λ . These plots also reflect that T -stress and S -stress under biaxial tension have the different trend.

3.5.3 T -stress and S -stress for Embedded Elliptical Crack Subject to Bi-Moments

with Remote Bending Loads $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$

The crack geometry and loading conditions are shown in Figure 3.1(c) for this case. Substituting eq. (3.3.10) and eq. (3.3.14a) into eq. (3.2.19), the T -stress for Type I sub-problem (crack pressure) under bending corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$, shown in Figure 3.11(b), can be represented by

$$T_{crack_pressure}(\varphi) = \frac{32Gc_{10}(1-2\nu)a \cos \varphi}{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} (F_1 + F_2) \quad (3.5.15)$$

where

$$c_{10} = -\frac{p_0 a^2 b^2 k^2}{32G[(1-2k^2)E(k) - k'^2 K(k)]} \quad (3.5.16a)$$

$$F_1 = \frac{\sin^2 \varphi}{a^3 k^4} [2(1+k^2)E(k) - (2+k^2)K(k)] \quad (3.5.16b)$$

$$F_2 = \frac{b^2(2\sin^2 \varphi - \cos^2 \varphi)}{a^5 k'^2 k^4} [(1+k'^2)E(k) - 2k'^2 K(k)] \quad (3.5.16c)$$

For the un-cracked body subject to bi-moments corresponding to the nominal stress

$\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, shown in Figure 3.11(c), the corresponding T -stress

can be obtained from eq. (3.5.3) by employing eq. (2.15), eq. (2.16) and eq. (3.1.22)

$$(\sigma_{nn} - \sigma_{xx})_P = -p_0 \left(\frac{\lambda \sin^2 \varphi \cos \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - \cos \varphi \right) \quad (3.5.17)$$

T -stress for bi-moments corresponding to nominal stress $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and

$\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, shown in Figure 3.11(a), is expressed by eq. (3.5.18) after eq. (3.5.15)

and eq. (3.5.17) are substituted into eq. (3.5.5).

$$T_{total}(\varphi) = \frac{32Gc_{10}(1-2\nu)a \cos \varphi}{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} (F_1 + F_2) + p_0 \cos \varphi - \frac{\lambda p_0 \sin^2 \varphi \cos \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.18a)$$

where c_{10} , F_1 and F_2 are represented by eq. (3.5.16a), eq. (3.5.16b) and eq. (3.5.16c). Or expressed by the normalized T -stress,

$$t(\varphi) = \frac{T_{total}(\varphi)}{p_0} = (1-2\nu)C_0[F_3E(k) - F_4K(k)] + \cos\varphi - \frac{\lambda \sin^2\varphi \cos\varphi}{\sin^2\varphi + k'^2 \cos^2\varphi} \quad (3.5.18b)$$

where

$$C_0 = -\left(\frac{k'}{k}\right)^2 \frac{\cos\varphi}{(\sin^2\varphi + k'^2 \cos^2\varphi)[(1-2k^2)E(k) - k'^2 K(k)]} \quad (3.5.18c)$$

$$F_3 = 2(1+k^2)\sin^2\varphi + (1+k'^2)(2-3\cos^2\varphi) \quad (3.5.18d)$$

$$F_4 = (2+k^2)\sin^2\varphi + 2k'^2(2-3\cos^2\varphi) \quad (3.5.18e)$$

Similarly, substituting eq. (3.3.12) and eq. (3.3.14a) into eq. (3.2.23), S -stress for Type I sub-problems under bending stress $\sigma_{xx} = -p_0\left(\frac{x}{a}\right)$, shown in Figure 3.11(b), are represented by

$$S_{crack_pressure}(\varphi) = \frac{p_0 C_1 (h_3 - h_4)}{\sin^2\varphi + k'^2 \cos^2\varphi} \quad (3.5.19)$$

where

$$C_1 = -\frac{(1-2\nu)\cos\varphi}{(1-2k^2)E(k) - k'^2 K(k)} \left(\frac{k'}{k}\right)^2 \quad (3.5.20a)$$

$$h_3 = [(2+k^2)K(k) - 2(1+k^2)E(k)]\sin^2\varphi \quad (3.5.20b)$$

$$h_4 = (2-3\cos^2\varphi)[(1+k'^2)E(k) - 2k'^2 K(k)] \quad (3.5.20c)$$

For the un-cracked body subject to bi-moments corresponding to the nominal stress $\sigma_{xx} = -p_0\left(\frac{x}{a}\right)$ and $\sigma_{yy} = -\lambda p_0\left(\frac{x}{a}\right)$, shown in Figure 3.11(c), it can be obtained from eq. (3.5.4) by employing eq. (2.16), eq. (2.17) and eq. (3.1.21),

$$(\sigma_{xx} - 2\nu\sigma_{zz})|_P = -p_0 \left(\frac{\lambda k'^2 \cos^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \cos \varphi \right) \quad (3.5.21)$$

Substituting eq. (3.5.19) and eq. (3.5.21) into eq. (3.5.6), S -stress for bi-moments, corresponding to nominal stress $\sigma_{zz} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, shown in Figure 3.11(a), can be expressed by

$$S_{total}(\varphi) = \frac{p_0 C_1 (h_3 - h_4)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + 2\nu p_0 \cos \varphi - \frac{\lambda p_0 k'^2 \cos^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.22a)$$

or

$$s(\varphi) = \frac{S_{total}(\varphi)}{p_0} = \frac{C_1 (h_3 - h_4)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + 2\nu \cos \varphi - \frac{\lambda k'^2 \cos^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} \quad (3.5.22b)$$

where C_1 , h_3 and h_4 are represented by eq. (3.5.20a), eq. (3.5.20b) and eq. (3.5.20c).

The normalized T -stress is plotted in Figure 3.18, Figure 3.19 and Figure 3.20 and the normalized S -stress is plotted in Figure 3.21, Figure 3.22 and Figure 3.23 for $\nu = 0.3$, $\lambda = 0.3, 0.5$ and 1.0 and aspect ratio $b/a = 0.2, 0.4, 0.6$ and 0.9999 . It is observed that T -stress does not change at $\varphi = 0^\circ, 90^\circ, 180^\circ$ with different biaxial ratio λ . It is found that T -stress values are decreasing when $0^\circ \leq \varphi < 90^\circ$ and increasing when $90^\circ < \varphi \leq 180^\circ$ with λ going larger, indicating that the constraint level becomes lower in the range of $0^\circ \leq \varphi < 90^\circ$ and higher in the range of $90^\circ < \varphi \leq 180^\circ$. On the other hand, S -stress does not change at $\varphi = 90^\circ$. It is found that S -stress values are decreasing in the range of $0^\circ < \varphi < 90^\circ$ and increasing in the range of $90^\circ < \varphi < 180^\circ$ when λ becomes larger. It is also observed that for this case, both the maximum and minimum S -stress values are at some points offset from major axial and minor axial points.

3.5.4 T -stress and S -stress for Embedded Elliptical Crack Subject to Bi-Moments

with Remote Bending Loads $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$

The crack geometry and loading conditions are shown in Figure 3.1(d) for this case. Substituting eq. (3.4.10) and eq. (3.4.14a) into eq. (3.2.19), the T -stress for Type I sub-problem (crack pressure) under bending corresponding to $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$, shown in Figure 3.11(b), can be represented by

$$T_{crack_pressure}(\varphi) = - \left(\frac{k'}{k} \right)^2 \frac{p_0(1-2\nu)\sin\varphi}{(\sin^2\varphi + k'^2 \cos^2\varphi)[(1+k^2)E(k) - k'^2 K(k)]} (g_1 - g_2) \quad (3.5.23)$$

where

$$g_1 = \{(1+k'^2)(2-3\sin^2\varphi) - 2(2k^2-1)\cos^2\varphi\}E(k) \quad (3.5.24a)$$

$$g_2 = \{2k'^2(2-3\sin^2\varphi) + k'^2(2-3k^2)\cos^2\varphi\}K(k) \quad (3.5.25b)$$

The un-cracked body subject to bi-moments corresponding to the nominal stress $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, shown in Figure 3.11(c), the un-cracked T -stress can be obtained from eq. (3.5.3) by employing eq. (2.15), eq. (2.16) and eq. (3.1.21),

$$(\sigma_{mm} - \sigma_{zz})_p = p_0 \left(\frac{\lambda k'^2 \cos^2\varphi}{\sin^2\varphi + k'^2 \cos^2\varphi} - 1 \right) \sin\varphi \quad (3.5.26)$$

Substituting eq. (3.5.23) and eq. (3.5.26) into eq. (3.5.5), T -stress for bi-moments corresponding to the nominal stress $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, shown in Figure 3.11(a), is represented by

$$T_{total}(\varphi) = -\left(\frac{k'}{k}\right)^2 \frac{p_0(1-2\nu)(g_1 - g_2)\sin\varphi}{(\sin^2\varphi + k'^2 \cos^2\varphi)[(1+k^2)E(k) - k'^2 K(k)]} + p_0 \sin\varphi \left(\frac{\lambda k'^2 \cos^2\varphi}{\sin^2\varphi + k'^2 \cos^2\varphi} - 1 \right) \quad (3.5.27a)$$

or

$$t(\varphi) = \frac{T_{total}(\varphi)}{p_0} = -\left(\frac{k'}{k}\right)^2 \frac{(1-2\nu)(g_1 - g_2)\sin\varphi}{(\sin^2\varphi + k'^2 \cos^2\varphi)[(1+k^2)E(k) - k'^2 K(k)]} + \sin\varphi \left(\frac{\lambda k'^2 \cos^2\varphi}{\sin^2\varphi + k'^2 \cos^2\varphi} - 1 \right) \quad (3.5.27b)$$

where g_1 and g_2 are expressed in eq. (3.5.24a) and eq. (3.5.24b).

Similarly, substituting eq. (3.4.12) and eq. (3.4.14a) into eq. (3.2.23), S -stress for Type I sub-problems under bending stress $\sigma_{\infty} = p_0 \left(\frac{y}{b} \right)$, shown in Figure 3.11(b), are represented by

$$S_{crack_pressure}(\varphi) = \frac{C_2(h_5 + h_6)p_0}{\sin^2\varphi + k'^2 \cos^2\varphi} \quad (3.5.27)$$

where

$$C_2 = -\frac{(1-2\nu)\sin\varphi}{(1+k^2)E(k) - k'^2 K(k)} \left(\frac{k'}{k} \right)^2 \quad (3.5.28a)$$

$$h_5 = (\sin^2\varphi - 2\cos^2\varphi)[(1+k'^2)E(k) - 2k'^2 K(k)] \quad (3.5.28b)$$

$$h_6 = [2(2k^2 - 1)E(k) + k'^2(2 - 3k^2)K(k)]\cos^2\varphi \quad (3.5.28c)$$

For the un-cracked body subject to bi-moments corresponding to the nominal stress

$\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, shown in Figure 3.11(c), it can be obtained from eq.

(3.5.4) by employing eq. (2.16), eq. (2.17) and eq. (3.1.22),

$$(\sigma_{xx} - 2\nu\sigma_{zz})_P = p_0 \left(\frac{\lambda \sin^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \sin \varphi \right) \quad (3.5.29)$$

Substituting eq. (3.5.27) and eq. (3.5.29) into eq. (3.5.6), S -stress for bi-moments,

corresponding to nominal stress $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, shown in Figure

3.11(a), can be expressed by

$$S_{total}(\varphi) = \frac{C_2(h_5 + h_6)p_0}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + p_0 \left(\frac{\lambda \sin^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \sin \varphi \right) \quad (3.5.30a)$$

or

$$s(\varphi) = \frac{S_{total}(\varphi)}{p_0} = \frac{C_2(h_5 + h_6)}{\sin^2 \varphi + k'^2 \cos^2 \varphi} + \frac{\lambda \sin^3 \varphi}{\sin^2 \varphi + k'^2 \cos^2 \varphi} - 2\nu \sin \varphi \quad (3.5.30b)$$

where C_2 , h_5 and h_6 are represented in eq. (3.5.28a), eq. (3.5.28b) and eq. (3.5.28c).

The T -stress for $\nu = 0.3$, $\lambda = 0.3, 0.5$ and 1.0 and aspect ratio $b/a = 0.2, 0.4, 0.6$ and 0.9999 is plotted in Figure 3.24, Figure 3.25 and Figure 3.26, while the S -stress is plotted in Figure 3.27, Figure 3.28 and Figure 3.29. It is observed that T -stress does not change at $\varphi = 0^\circ, 90^\circ$ and 180° with the increasing of λ . For elliptical cracks, T -stress increases quickly from zero to its peak point when parametric angle φ offsets from $\varphi = 0^\circ$, meaning the constraint level becomes higher. Then T -stress drops to the minimum value at $\varphi = 90^\circ$, indicating low constraint. When the parametric angle φ passes 90° , T -stress increases again to another peak point near $\varphi = 180^\circ$. S -stress does not change at $\varphi = 0^\circ$

and $\varphi = 180^\circ$ with the increasing of λ . But at the point $\varphi = 90^\circ$, S -stress increases substantially from negative values to positive values with λ changing from 0.3, 0.5 to 1.0.

3.6 Finite Element Verifications

To verify the T -stress and S -stress solutions derived, three-dimensional finite element analyses are conducted to calculate the T -stress and S -stress for an elliptical crack embedded in an infinite body, using ABAQUS V6.2 (2001), with 20-node isoparametric three-dimensional solid elements. For modeling the crack tip singularity of square root, three-dimensional prism elements with four mid-side nodes at the quarter points, as illustrated in Figure 2.4, are employed and the separate crack-tip nodes are constrained to have the same displacements, Barsoum (1977).

It has been demonstrated that the T -stress and S -stress for a cracked body under biaxial loadings can be obtained by superimposing a cracked body under uniaxial loadings and an un-cracked body under biaxial loadings, shown in Figure 3.11. Because the non-singular stress distribution for the case shown in Figure 3.11(c) has already been determined, therefore, it is only necessary to verify the analytical solutions derived in Section 3.2, Section 3.3 and Section 3.4 for cracked body under uniaxial remote loading conditions, as shown in Figure 3.11(b), by finite element analysis in the thesis.

Two types of loads at far-ends are considered, i.e. uniform tension and bending about major axis $2a$. For modeling remote loads and an elliptical crack in an infinite body, the b/w , w/h and b/t ratios are set to 1/16, 1 and less than 1/6, respectively, shown in Figure 3.1. An elliptical transformation is used to form the crack tip mesh in order to

ensure the intersection lines of elements around the crack tip are orthogonal Banks-Sills (1991). Considering the symmetry in geometry, only 1/8 of the body is modeled for three-dimensional finite element analysis. Total of 18 elements were used along the crack front. A typical finite element mesh ($b/a = 0.6$) is plotted in Figure 3.30.

T -stress is extracted in current calculations using interaction integral introduced by Nakamura and Parks (1992), as summarized in Section 2.2.2. Once T -stress is obtained, S -stress can be calculated by the relationship with T -stress shown in eq. (2.24), which is $S = E\varepsilon_{33} + \nu T$.

The comparison of T -stresses and S -stresses between the analytical ones obtained in this chapter and the finite element results are conducted. Embedded elliptical cracks with the aspect ratio b/a of 0.4, 0.6 and 1.0 under remote tension and bending conditions are considered. The comparisons are plotted from Figure 3.31 to Figure 3.36 for T -stress and from Figure 3.37 to Figure 3.42 for S -stress. The detailed numerical results of T -stress and S -stress are tabulated in Table 3.1 to Table 3.6.

Under tension, the maximum difference of the T -stress between the finite element analysis and the analytical solution is 2.9% and the maximum difference of the S -stress is 4.2%. For the bending case, because the absolute values of T -stress and S -stress are very small near the neutral axis, the relative difference is larger than that of in the tension case for the points close to neutral axes.

It should be noted that the analytical T -stress solutions and S -stress solutions derived in the chapter cannot be used directly for penny-shaped crack which has a modulus $k = 0$. An aspect ratio of $b/a = 0.9999$ is taken in order to compare with the FEM result for penny-shaped crack ($b/a = 1$) in the thesis.

Table 3.1 T -stress and S -stress for $b/a = 0.4$ under uniform tension

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-9.3081E-01	-9.2000E-01	1.18	-6.6437E-01	-6.8000E-01	2.30
1.25	-9.0400E-01	-9.1930E-01	1.66	-6.6607E-01	-6.8070E-01	2.15
2.50	-9.2784E-01	-9.1710E-01	1.17	-6.6793E-01	-6.8290E-01	2.19
3.75	-8.9928E-01	-9.1370E-01	1.58	-6.7352E-01	-6.8630E-01	1.86
5.00	-9.1949E-01	-9.0900E-01	1.15	-6.7875E-01	-6.9100E-01	1.77
6.25	-8.9039E-01	-9.0320E-01	1.42	-6.8726E-01	-6.9680E-01	1.37
7.50	-9.0682E-01	-8.9650E-01	1.15	-6.9508E-01	-7.0350E-01	1.20
8.75	-8.7814E-01	-8.8900E-01	1.22	-7.0538E-01	-7.1100E-01	0.79
10.00	-8.9093E-01	-8.8090E-01	1.14	-7.1482E-01	-7.1910E-01	0.59
11.25	-8.6370E-01	-8.7240E-01	1.00	-7.2583E-01	-7.2760E-01	0.24
12.50	-8.7308E-01	-8.6360E-01	1.10	-7.3588E-01	-7.3640E-01	0.07
13.75	-8.4814E-01	-8.5460E-01	0.76	-7.4685E-01	-7.4540E-01	0.19
15.00	-8.5470E-01	-8.4570E-01	1.06	-7.5667E-01	-7.5430E-01	0.31
16.25	-8.3212E-01	-8.3680E-01	0.56	-7.6712E-01	-7.6320E-01	0.51
17.50	-8.3628E-01	-8.2800E-01	1.00	-7.7647E-01	-7.7200E-01	0.58
18.75	-8.1634E-01	-8.1950E-01	0.39	-7.8582E-01	-7.8050E-01	0.68
20.00	-8.1961E-01	-8.1130E-01	1.02	-7.9429E-01	-7.8870E-01	0.71
21.25	-8.0182E-01	-8.0340E-01	0.20	-8.0304E-01	-7.9660E-01	0.81
22.50	-8.2061E-01	-7.9580E-01	3.12	-8.1646E-01	-8.0420E-01	1.52
26.25	-7.5806E-01	-7.7520E-01	2.21	-8.3220E-01	-8.2480E-01	0.90
30.00	-7.9023E-01	-7.5780E-01	4.28	-8.4781E-01	-8.4220E-01	0.67
33.75	-7.3430E-01	-7.4330E-01	1.21	-8.6327E-01	-8.5670E-01	0.77
37.50	-7.5187E-01	-7.3130E-01	2.81	-8.7392E-01	-8.6870E-01	0.60
41.25	-7.1682E-01	-7.2130E-01	0.62	-8.8463E-01	-8.7870E-01	0.67
45.00	-7.2750E-01	-7.1310E-01	2.02	-8.9194E-01	-8.8690E-01	0.57
48.75	-7.0435E-01	-7.0630E-01	0.28	-8.9948E-01	-8.9370E-01	0.65
52.50	-7.1185E-01	-7.0070E-01	1.59	-9.0458E-01	-8.9930E-01	0.59
56.25	-6.9578E-01	-6.9600E-01	0.03	-9.0993E-01	-9.0400E-01	0.66
60.00	-7.0164E-01	-6.9220E-01	1.36	-9.1344E-01	-9.0780E-01	0.62
63.75	-6.8978E-01	-6.8900E-01	0.11	-9.1721E-01	-9.1100E-01	0.68
67.50	-6.9489E-01	-6.8640E-01	1.24	-9.1954E-01	-9.1360E-01	0.65
71.25	-6.8574E-01	-6.8440E-01	0.20	-9.2212E-01	-9.1560E-01	0.71
75.00	-6.9064E-01	-6.8280E-01	1.15	-9.2353E-01	-9.1720E-01	0.69
78.75	-6.8323E-01	-6.8150E-01	0.25	-9.2518E-01	-9.1850E-01	0.73
82.50	-6.8826E-01	-6.8070E-01	1.11	-9.2579E-01	-9.1930E-01	0.71
86.25	-6.8207E-01	-6.8020E-01	0.28	-9.2665E-01	-9.1980E-01	0.74
90.00	-6.8753E-01	-6.8000E-01	1.11	-9.2660E-01	-9.2000E-01	0.72

Table 3.2 T -stress and S -stress for $b/a = 0.4$ under bending

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-5.5247E-03	0.0000E+00	Infinity	9.7402E-03	0.0000E+00	Infinity
1.25	-1.4433E-02	-1.4300E-02	0.93	-9.6459E-03	-2.0600E-02	53.18
2.50	-2.2782E-02	-2.8700E-02	20.62	-3.2024E-02	-4.1100E-02	22.08
3.75	-3.9310E-02	-4.2900E-02	8.37	-5.3679E-02	-6.1700E-02	13.00
5.00	-5.3917E-02	-5.7200E-02	5.74	-7.5396E-02	-8.2300E-02	8.39
6.25	-6.7851E-02	-7.1300E-02	4.84	-9.6745E-02	-1.0290E-01	5.98
7.50	-8.2927E-02	-8.5400E-02	2.90	-1.1800E-01	-1.2350E-01	4.45
8.75	-9.5746E-02	-9.9400E-02	3.68	-1.3961E-01	-1.4400E-01	3.05
10.00	-1.1115E-01	-1.1320E-01	1.81	-1.6058E-01	-1.6460E-01	2.44
11.25	-1.2333E-01	-1.2700E-01	2.89	-1.8218E-01	-1.8520E-01	1.63
12.50	-1.3874E-01	-1.4060E-01	1.33	-2.0292E-01	-2.0570E-01	1.35
13.75	-1.5042E-01	-1.5410E-01	2.39	-2.2441E-01	-2.2610E-01	0.75
15.00	-1.6566E-01	-1.6750E-01	1.10	-2.4486E-01	-2.4660E-01	0.70
16.25	-1.7699E-01	-1.8080E-01	2.11	-2.6614E-01	-2.6690E-01	0.28
17.50	-1.9200E-01	-1.9400E-01	1.03	-2.8631E-01	-2.8720E-01	0.31
18.75	-2.0303E-01	-2.0700E-01	1.92	-3.0732E-01	-3.0730E-01	0.01
20.00	-2.1772E-01	-2.1980E-01	0.95	-3.2702E-01	-3.2740E-01	0.12
21.25	-2.2865E-01	-2.3260E-01	1.70	-3.4765E-01	-3.4730E-01	0.10
22.50	-2.5078E-01	-2.4520E-01	2.27	-3.7798E-01	-3.6710E-01	2.96
26.25	-2.7355E-01	-2.8220E-01	3.07	-4.2730E-01	-4.2550E-01	0.42
30.00	-3.2096E-01	-3.1790E-01	0.96	-4.8027E-01	-4.8210E-01	0.38
33.75	-3.4340E-01	-3.5210E-01	2.47	-5.3822E-01	-5.3680E-01	0.27
37.50	-3.8520E-01	-3.8490E-01	0.08	-5.8771E-01	-5.8920E-01	0.25
41.25	-4.0703E-01	-4.1600E-01	2.16	-6.3978E-01	-6.3900E-01	0.12
45.00	-4.4337E-01	-4.4530E-01	0.43	-6.8504E-01	-6.8600E-01	0.14
48.75	-4.6340E-01	-4.7280E-01	1.99	-7.3044E-01	-7.3010E-01	0.05
52.50	-4.9469E-01	-4.9830E-01	0.72	-7.7045E-01	-7.7100E-01	0.07
56.25	-5.1186E-01	-5.2180E-01	1.91	-8.0854E-01	-8.0860E-01	0.01
60.00	-5.3817E-01	-5.4300E-01	0.89	-8.4259E-01	-8.4260E-01	0.00
63.75	-5.5165E-01	-5.6200E-01	1.84	-8.7284E-01	-8.7300E-01	0.02
67.50	-5.7295E-01	-5.7860E-01	0.98	-9.0003E-01	-8.9960E-01	0.05
71.25	-5.8211E-01	-5.9270E-01	1.79	-9.2209E-01	-9.2230E-01	0.02
75.00	-5.9827E-01	-6.0440E-01	1.01	-9.4184E-01	-9.4110E-01	0.08
78.75	-6.0270E-01	-6.1360E-01	1.78	-9.5545E-01	-9.5570E-01	0.03
82.50	-6.1365E-01	-6.2010E-01	1.04	-9.6724E-01	-9.6620E-01	0.11
86.25	-6.1310E-01	-6.2410E-01	1.76	-9.7228E-01	-9.7250E-01	0.02
90.00	-6.1891E-01	-6.2540E-01	1.04	-9.7604E-01	-9.7460E-01	0.15

Table 3.3 T -stress and S -stress for $b/a = 0.6$ under uniform tension

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-8.8198E-01	-8.7330E-01	0.99	-7.3513E-01	-7.2670E-01	1.16
1.25	-8.7137E-01	-8.7310E-01	0.20	-7.3546E-01	-7.2690E-01	1.18
2.50	-8.8108E-01	-8.7250E-01	0.98	-7.3559E-01	-7.2750E-01	1.11
3.75	-8.6988E-01	-8.7150E-01	0.19	-7.3697E-01	-7.2850E-01	1.16
5.00	-8.7867E-01	-8.7020E-01	0.97	-7.3785E-01	-7.2980E-01	1.10
6.25	-8.6699E-01	-8.6850E-01	0.17	-7.3990E-01	-7.3150E-01	1.15
7.50	-8.7481E-01	-8.6650E-01	0.96	-7.4148E-01	-7.3350E-01	1.09
8.75	-8.6281E-01	-8.6420E-01	0.16	-7.4414E-01	-7.3580E-01	1.13
10.00	-8.6966E-01	-8.6160E-01	0.93	-7.4633E-01	-7.3840E-01	1.07
11.25	-8.5752E-01	-8.5880E-01	0.15	-7.4944E-01	-7.4120E-01	1.11
12.50	-8.6338E-01	-8.5570E-01	0.90	-7.5214E-01	-7.4430E-01	1.05
13.75	-8.5130E-01	-8.5240E-01	0.13	-7.5562E-01	-7.4760E-01	1.07
15.00	-8.5630E-01	-8.4890E-01	0.87	-7.5866E-01	-7.5110E-01	1.01
16.25	-8.4442E-01	-8.4530E-01	0.10	-7.6248E-01	-7.5470E-01	1.03
17.50	-8.4845E-01	-8.4160E-01	0.81	-7.6584E-01	-7.5840E-01	0.98
18.75	-8.3712E-01	-8.3770E-01	0.07	-7.6971E-01	-7.6230E-01	0.97
20.00	-8.4077E-01	-8.3380E-01	0.84	-7.7299E-01	-7.6620E-01	0.89
21.25	-8.2951E-01	-8.2990E-01	0.05	-7.7723E-01	-7.7010E-01	0.93
22.50	-8.4758E-01	-8.2600E-01	2.61	-7.8366E-01	-7.7400E-01	1.25
26.25	-7.9906E-01	-8.1420E-01	1.86	-7.9300E-01	-7.8580E-01	0.92
30.00	-8.3435E-01	-8.0280E-01	3.93	-8.0207E-01	-7.9720E-01	0.61
33.75	-7.8096E-01	-7.9210E-01	1.41	-8.1396E-01	-8.0790E-01	0.75
37.50	-8.0645E-01	-7.8230E-01	3.09	-8.2206E-01	-8.1770E-01	0.53
41.25	-7.6554E-01	-7.7350E-01	1.03	-8.3193E-01	-8.2650E-01	0.66
45.00	-7.8443E-01	-7.6550E-01	2.47	-8.3853E-01	-8.3450E-01	0.48
48.75	-7.5281E-01	-7.5850E-01	0.75	-8.4644E-01	-8.4150E-01	0.59
52.50	-7.6774E-01	-7.5240E-01	2.04	-8.5156E-01	-8.4760E-01	0.47
56.25	-7.4287E-01	-7.4700E-01	0.55	-8.5771E-01	-8.5300E-01	0.55
60.00	-7.5545E-01	-7.4240E-01	1.76	-8.6148E-01	-8.5760E-01	0.45
63.75	-7.3553E-01	-7.3850E-01	0.40	-8.6612E-01	-8.6150E-01	0.54
67.50	-7.4681E-01	-7.3530E-01	1.57	-8.6871E-01	-8.6470E-01	0.46
71.25	-7.3029E-01	-7.3260E-01	0.32	-8.7208E-01	-8.6740E-01	0.54
75.00	-7.4103E-01	-7.3040E-01	1.46	-8.7362E-01	-8.6960E-01	0.46
78.75	-7.2695E-01	-7.2880E-01	0.25	-8.7587E-01	-8.7120E-01	0.54
82.50	-7.3776E-01	-7.2760E-01	1.40	-8.7645E-01	-8.7240E-01	0.46
86.25	-7.2535E-01	-7.2700E-01	0.23	-8.7773E-01	-8.7300E-01	0.54
90.00	-7.3673E-01	-7.2670E-01	1.38	-8.7758E-01	-8.7330E-01	0.49

Table 3.4 *T*-stress and *S*-stress for $b/a = 0.6$ under bending

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-1.3188E-02	0.0000E+00	Infinity	-6.5917E-03	0.0000E+00	Infinity
1.25	-1.5211E-02	-1.4800E-02	2.77	-3.8940E-03	-2.0100E-02	80.63
2.50	-2.3332E-02	-2.9600E-02	21.18	-2.6775E-02	-4.0200E-02	33.40
3.75	-4.0397E-02	-4.4400E-02	9.01	-4.8202E-02	-6.0300E-02	20.06
5.00	-5.5059E-02	-5.9100E-02	6.84	-7.0012E-02	-8.0300E-02	12.81
6.25	-7.0250E-02	-7.3800E-02	4.81	-9.0919E-02	-1.0040E-01	9.44
7.50	-8.5355E-02	-8.8400E-02	3.44	-1.1212E-01	-1.2040E-01	6.88
8.75	-9.9608E-02	-1.0300E-01	3.29	-1.3311E-01	-1.4040E-01	5.19
10.00	-1.1493E-01	-1.1750E-01	2.18	-1.5394E-01	-1.6030E-01	3.97
11.25	-1.2857E-01	-1.3190E-01	2.53	-1.7469E-01	-1.8030E-01	3.11
12.50	-1.4391E-01	-1.4620E-01	1.57	-1.9522E-01	-2.0010E-01	2.44
13.75	-1.5704E-01	-1.6040E-01	2.09	-2.1578E-01	-2.1990E-01	1.87
15.00	-1.7228E-01	-1.7440E-01	1.21	-2.3600E-01	-2.3970E-01	1.54
16.25	-1.8495E-01	-1.8840E-01	1.83	-2.5631E-01	-2.5930E-01	1.15
17.50	-2.0009E-01	-2.0220E-01	1.04	-2.7627E-01	-2.7890E-01	0.94
18.75	-2.1235E-01	-2.1590E-01	1.64	-2.9634E-01	-2.9840E-01	0.69
20.00	-2.2712E-01	-2.2950E-01	1.03	-3.1572E-01	-3.1780E-01	0.65
21.25	-2.3901E-01	-2.4290E-01	1.60	-3.3548E-01	-3.3700E-01	0.45
22.50	-2.6113E-01	-2.5610E-01	1.96	-3.6397E-01	-3.5620E-01	2.18
26.25	-2.8659E-01	-2.9500E-01	2.85	-4.1264E-01	-4.1270E-01	0.02
30.00	-3.3676E-01	-3.3230E-01	1.34	-4.6638E-01	-4.6770E-01	0.28
33.75	-3.5883E-01	-3.6800E-01	2.49	-5.2128E-01	-5.2090E-01	0.07
37.50	-4.0526E-01	-4.0200E-01	0.81	-5.7110E-01	-5.7200E-01	0.16
41.25	-4.2452E-01	-4.3420E-01	2.23	-6.2095E-01	-6.2070E-01	0.04
45.00	-4.6621E-01	-4.6450E-01	0.37	-6.6617E-01	-6.6690E-01	0.11
48.75	-4.8269E-01	-4.9280E-01	2.05	-7.0993E-01	-7.1010E-01	0.02
52.50	-5.1931E-01	-5.1900E-01	0.06	-7.4963E-01	-7.5030E-01	0.09
56.25	-5.3258E-01	-5.4310E-01	1.94	-7.8660E-01	-7.8730E-01	0.09
60.00	-5.6393E-01	-5.6480E-01	0.15	-8.2018E-01	-8.2080E-01	0.08
63.75	-5.7343E-01	-5.8420E-01	1.84	-8.4969E-01	-8.5080E-01	0.13
67.50	-5.9944E-01	-6.0120E-01	0.29	-8.7638E-01	-8.7700E-01	0.07
71.25	-6.0467E-01	-6.1570E-01	1.79	-8.9799E-01	-8.9940E-01	0.16
75.00	-6.2526E-01	-6.2760E-01	0.37	-9.1726E-01	-9.1790E-01	0.07
78.75	-6.2578E-01	-6.3690E-01	1.75	-9.3067E-01	-9.3230E-01	0.17
82.50	-6.4091E-01	-6.4360E-01	0.42	-9.4209E-01	-9.4270E-01	0.06
86.25	-6.3640E-01	-6.4760E-01	1.73	-9.4716E-01	-9.4900E-01	0.19
90.00	-6.4627E-01	-6.4900E-01	0.42	-9.5078E-01	-9.5100E-01	0.02

Table 3.5 T -stress and S -stress for $b/a = 1.0$ under uniform tension

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-7.9992E-01	-8.0000E-01	0.01	-8.0111E-01	-8.0000E-01	0.14
2.50	-8.0010E-01	-8.0000E-01	0.01	-8.0152E-01	-8.0000E-01	0.19
5.00	-7.9987E-01	-8.0000E-01	0.02	-8.0086E-01	-8.0000E-01	0.11
7.50	-8.0010E-01	-8.0000E-01	0.01	-8.0151E-01	-8.0000E-01	0.19
10.00	-7.9987E-01	-8.0000E-01	0.02	-8.0086E-01	-8.0000E-01	0.11
12.50	-8.0010E-01	-8.0000E-01	0.01	-8.0151E-01	-8.0000E-01	0.19
15.00	-7.9986E-01	-8.0000E-01	0.02	-8.0085E-01	-8.0000E-01	0.11
17.50	-8.0012E-01	-8.0000E-01	0.02	-8.0151E-01	-8.0000E-01	0.19
20.00	-7.9991E-01	-8.0000E-01	0.01	-8.0083E-01	-8.0000E-01	0.10
22.50	-8.0007E-01	-8.0000E-01	0.01	-8.0151E-01	-8.0000E-01	0.19
25.00	-7.9995E-01	-8.0000E-01	0.01	-8.0083E-01	-8.0000E-01	0.10
27.50	-8.0014E-01	-8.0000E-01	0.02	-8.0151E-01	-8.0000E-01	0.19
30.00	-7.9985E-01	-8.0000E-01	0.02	-8.0083E-01	-8.0000E-01	0.10
32.50	-8.0022E-01	-8.0000E-01	0.03	-8.0148E-01	-8.0000E-01	0.19
35.00	-7.9986E-01	-8.0000E-01	0.02	-8.0083E-01	-8.0000E-01	0.10
37.50	-8.0010E-01	-8.0000E-01	0.01	-8.0149E-01	-8.0000E-01	0.19
40.00	-7.9989E-01	-8.0000E-01	0.01	-8.0081E-01	-8.0000E-01	0.10
42.50	-8.0014E-01	-8.0000E-01	0.02	-8.0149E-01	-8.0000E-01	0.19
45.00	-7.9996E-01	-8.0000E-01	0.01	-8.0081E-01	-8.0000E-01	0.10
47.50	-8.0015E-01	-8.0000E-01	0.02	-8.0149E-01	-8.0000E-01	0.19
50.00	-7.9991E-01	-8.0000E-01	0.01	-8.0080E-01	-8.0000E-01	0.10
52.50	-8.0013E-01	-8.0000E-01	0.02	-8.0148E-01	-8.0000E-01	0.18
55.00	-8.0003E-01	-8.0000E-01	0.00	-8.0080E-01	-8.0000E-01	0.10
57.50	-8.0024E-01	-8.0000E-01	0.03	-8.0147E-01	-8.0000E-01	0.18
60.00	-7.9995E-01	-8.0000E-01	0.01	-8.0079E-01	-8.0000E-01	0.10
62.50	-8.0027E-01	-8.0000E-01	0.03	-8.0147E-01	-8.0000E-01	0.18
65.00	-7.9998E-01	-8.0000E-01	0.00	-8.0081E-01	-8.0000E-01	0.10
67.50	-8.0021E-01	-8.0000E-01	0.03	-8.0147E-01	-8.0000E-01	0.18
70.00	-7.9996E-01	-8.0000E-01	0.00	-8.0080E-01	-8.0000E-01	0.10
72.50	-8.0024E-01	-8.0000E-01	0.03	-8.0145E-01	-8.0000E-01	0.18
75.00	-7.9999E-01	-8.0000E-01	0.00	-8.0080E-01	-8.0000E-01	0.10
77.50	-8.0027E-01	-8.0000E-01	0.03	-8.0146E-01	-8.0000E-01	0.18
80.00	-8.0004E-01	-8.0000E-01	0.00	-8.0080E-01	-8.0000E-01	0.10
82.50	-8.0024E-01	-8.0000E-01	0.03	-8.0146E-01	-8.0000E-01	0.18
85.00	-8.0000E-01	-8.0000E-01	0.00	-8.0080E-01	-8.0000E-01	0.10
87.50	-8.0026E-01	-8.0000E-01	0.03	-8.0146E-01	-8.0000E-01	0.18
90.00	-8.0008E-01	-8.0000E-01	0.01	-8.0104E-01	-8.0000E-01	0.13

Table 3.6 T -stress and S -stress for $b/a = 1.0$ under bending

φ (Deg.)	S/p_0 (FEM)	S/p_0 (Analytical)	Relative Difference (%)	T/p_0 (FEM)	T/p_0 (Analytical)	Relative Difference (%)
0.00	-8.1113E-03	0.0000E+00	Infinity	-5.4364E-03	0.0000E+00	Infinity
2.50	-2.9254E-02	-3.0500E-02	4.09	-2.4352E-02	-3.9300E-02	38.04
5.00	-5.3950E-02	-6.1000E-02	11.56	-6.5480E-02	-7.8400E-02	16.48
7.50	-8.9561E-02	-9.1400E-02	2.01	-1.0726E-01	-1.1750E-01	8.71
10.00	-1.2022E-01	-1.2160E-01	1.13	-1.4808E-01	-1.5630E-01	5.26
12.50	-1.5095E-01	-1.5150E-01	0.36	-1.8697E-01	-1.9480E-01	4.02
15.00	-1.8131E-01	-1.8120E-01	0.06	-2.2640E-01	-2.3290E-01	2.79
17.50	-2.1074E-01	-2.1050E-01	0.11	-2.6468E-01	-2.7060E-01	2.19
20.00	-2.4029E-01	-2.3940E-01	0.37	-3.0265E-01	-3.0780E-01	1.67
22.50	-2.6845E-01	-2.6790E-01	0.20	-3.3952E-01	-3.4440E-01	1.42
25.00	-2.9698E-01	-2.9580E-01	0.40	-3.7604E-01	-3.8040E-01	1.15
27.50	-3.2379E-01	-3.2320E-01	0.18	-4.1137E-01	-4.1560E-01	1.02
30.00	-3.5118E-01	-3.5000E-01	0.34	-4.4622E-01	-4.5000E-01	0.84
32.50	-3.7654E-01	-3.7610E-01	0.12	-4.7976E-01	-4.8360E-01	0.79
35.00	-4.0257E-01	-4.0150E-01	0.27	-5.1276E-01	-5.1620E-01	0.67
37.50	-4.2630E-01	-4.2610E-01	0.05	-5.4430E-01	-5.4790E-01	0.66
40.00	-4.5076E-01	-4.4990E-01	0.19	-5.7522E-01	-5.7850E-01	0.57
42.50	-4.7286E-01	-4.7290E-01	0.01	-6.0455E-01	-6.0800E-01	0.57
45.00	-4.9556E-01	-4.9500E-01	0.11	-6.3317E-01	-6.3640E-01	0.51
47.50	-5.1567E-01	-5.1610E-01	0.08	-6.6009E-01	-6.6360E-01	0.53
50.00	-5.3667E-01	-5.3620E-01	0.09	-6.8620E-01	-6.8940E-01	0.46
52.50	-5.5464E-01	-5.5530E-01	0.12	-7.1050E-01	-7.1400E-01	0.49
55.00	-5.7375E-01	-5.7340E-01	0.06	-7.3396E-01	-7.3720E-01	0.44
57.50	-5.8949E-01	-5.9040E-01	0.15	-7.5547E-01	-7.5910E-01	0.48
60.00	-6.0635E-01	-6.0620E-01	0.03	-7.7607E-01	-7.7940E-01	0.43
62.50	-6.1985E-01	-6.2090E-01	0.17	-7.9464E-01	-7.9830E-01	0.46
65.00	-6.3439E-01	-6.3440E-01	0.00	-8.1227E-01	-8.1570E-01	0.42
67.50	-6.4549E-01	-6.4670E-01	0.19	-8.2775E-01	-8.3150E-01	0.45
70.00	-6.5768E-01	-6.5780E-01	0.02	-8.4224E-01	-8.4570E-01	0.41
72.50	-6.6624E-01	-6.6760E-01	0.20	-8.5452E-01	-8.5840E-01	0.45
75.00	-6.7591E-01	-6.7610E-01	0.03	-8.6579E-01	-8.6930E-01	0.40
77.50	-6.8191E-01	-6.8340E-01	0.22	-8.7479E-01	-8.7870E-01	0.45
80.00	-6.8910E-01	-6.8940E-01	0.04	-8.8274E-01	-8.8630E-01	0.40
82.50	-6.9247E-01	-6.9400E-01	0.22	-8.8837E-01	-8.9230E-01	0.44
85.00	-6.9704E-01	-6.9730E-01	0.04	-8.9296E-01	-8.9660E-01	0.41
87.50	-6.9776E-01	-6.9930E-01	0.22	-8.9520E-01	-8.9920E-01	0.45
90.00	-6.9978E-01	-7.0000E-01	0.03	-8.9663E-01	-9.0000E-01	0.37

Figure 3.1(a) An elliptical crack in a solid

($\lambda = \frac{\sigma_{xx}}{\sigma_{zz}}$ is the biaxial ratio. The loading condition is uniaxial when $\lambda = 0$.)

Figure 3.1(b) An elliptical crack in a solid (biaxial tension)

($\lambda = \frac{\sigma_{yy}}{\sigma_{zz}}$ is the biaxial ratio. The loading condition is uniaxial when $\lambda = 0$.)

Figure 3.1(c) An elliptical crack in a solid
(biaxial bending about y-axis and z-axis)

($\lambda = \frac{\sigma_{xx}}{\sigma_{zz}}$ is the biaxial ratio. The loading condition is uniaxial when $\lambda = 0$.)

Figure 3.1(d) An elliptical crack in a solid
(biaxial bending about x -axis and z -axis)

Figure 3.2 Local spherical coordinates on the crack border

Figure 3.3 Principle of superposition

Figure 3.4 Parameters of an ellipse

Figure 3.5 Normalized T -stress along the crack front under uniform tension, $\nu = 0.3$

Figure 3.6 Normalized S -stress along the crack front under uniform tension, $\nu = 0.3$

Figure 3.7 Normalized T -stress along the crack front under bending.

$$p(x, y) = -p_0 \left(\frac{x}{a} \right), \nu = 0.3$$

Figure 3.8 Normalized S -stress along the crack front under bending.

$$p(x, y) = -p_0 \left(\frac{x}{a} \right), \quad \nu = 0.3$$

Figure 3.9 Normalized T -stress along the crack front under bending,

$$p(x, y) = p_0 \left(\frac{y}{b} \right), \nu = 0.3$$

Figure 3.10 Normalized S -stress along the crack front under bending,

$$p(x, y) = p_0 \left(\frac{y}{b} \right), \nu = 0.3$$

Figure 3.11 An example of a crack under biaxial tension conditions

Figure 3.12 Normalized T -stress for embedded elliptical cracks subject to
biaxial tension, $\sigma_{xx} = p_0$ and $\sigma_{yy} = \lambda p_0$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.13 Normalized T -stress for embedded elliptical cracks subject to biaxial tension, $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.14 Normalized T -stress for embedded elliptical cracks subject to biaxial tension, $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.15 Normalized S -stress for embedded elliptical cracks subject to
biaxial tension, $\sigma_{xx} = p_0$ and $\sigma_{yy} = \lambda p_0$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.16 Normalized S -stress for embedded elliptical cracks subject to
biaxial tension, $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.17 Normalized S -stress for embedded elliptical cracks subject to

biaxial tension, $\sigma_{zz} = p_0$ and $\sigma_{xx} = \lambda p_0$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.18 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.19 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.20 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.21 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.22 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{zz} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.23 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = -p_0 \left(\frac{x}{a} \right)$ and $\sigma_{yy} = -\lambda p_0 \left(\frac{x}{a} \right)$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.24 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.25 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.26 Normalized T -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xy} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.27 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{zz} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xx} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 0.3$, $\nu = 0.3$

Figure 3.28 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{yy} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 0.5$, $\nu = 0.3$

Figure 3.29 Normalized S -stress for embedded elliptical cracks subject to bi-moments

corresponding to $\sigma_{xx} = p_0 \left(\frac{y}{b} \right)$ and $\sigma_{xy} = \lambda p_0 \left(\frac{y}{b} \right)$, $\lambda = 1.0$, $\nu = 0.3$

Figure 3.30 Typical finite element mesh ($b/a = 0.6$)

Figure 3.31 Comparison of T -stress for embedded elliptical cracks under tension,

$$b/a = 0.4, \nu = 0.3$$

Figure 3.32 Comparison of T -stress for embedded elliptical cracks under bending,

$$b/a = 0.4, \nu = 0.3$$

Figure 3.33 Comparison of T -stress for embedded elliptical cracks under tension,

$$b/a = 0.6, \nu = 0.3$$

Figure 3.34 Comparison of T -stress for embedded elliptical cracks under bending.

$$b/a = 0.6, \nu = 0.3$$

Figure 3.35 Comparison of T -stress for embedded elliptical cracks under tension,

$$b/a = 1.0, \nu = 0.3$$

Figure 3.36 Comparison of T -stress for embedded elliptical cracks under bending.

$$b/a = 1.0, \nu = 0.3$$

Figure 3.37 Comparison of S-stress for embedded elliptical cracks under tension,

$$b/a = 0.4, \nu = 0.3$$

Figure 3.38 Comparison of S -stress for embedded elliptical cracks under bending.

$$b/a = 0.4, \nu = 0.3$$

Figure 3.39 Comparison of S -stress for embedded elliptical cracks under tension,

$$b/a = 0.6, \nu = 0.3$$

Figure 3.40 Comparison of S -stress for embedded elliptical cracks under bending,

$$b/a = 0.6, \nu = 0.3$$

Figure 3.41 Comparison of S -stress for embedded elliptical cracks under tension,

$$b/a = 1.0, \nu = 0.3$$

Figure 3.42 Comparison of S -stress for embedded elliptical cracks under bending,

$$b/a = 1.0, \nu = 0.3$$

Chapter 4

***T*-stress and *S*-stress for a Quarter-Elliptical Corner Crack Subject to Remote Tension and Bending**

Corner cracks are often detected in engineering structures. The stress intensity factor for a quarter-elliptical crack embedded in a solid under remote tension and bending conditions has been presented by Shiratori and Miyoshi (1992), Zheng *et al* (1996) and Kiciak *et al* (1998). Although the work of Zhao *et al* (2001) provided the numerical solutions of both SIF and *T*-stress for a quarter-circular corner crack in a square plate, the parametric study on the *T*-stress of a quarter-elliptical corner crack in a finite plate is needed in order to model the more general shape of a corner crack. In this chapter, both *T*-stress and *S*-stress for a quarter-elliptical crack in a finite thickness plate are evaluated through three-dimensional finite element method.

4.1 Finite Element Analysis for Corner Cracks

4.1.1 The Model of Three-Dimensional FEM Analysis

Considering the symmetry in geometry, only one-half of the plate was modeled for finite element analysis using ABAQUS, version 6.2. The geometry and coordinate system are shown in Figure 4.1 and Figure 4.2. The 20-node, three-dimensional and isoparametric prism elements were used. For the elements along the crack front, the four mid-side nodes were moved to one-quarter position in order to model the square root

stress singularity of $r^{-1/2}$. The mesh (as illustrated in Figure 2.4) was extensively refined along the crack front. The interaction integral approach is used to calculate T -stress, as summarized in Section 2.2.2. Once the T -stress is obtained, the S -stress can be calculated by eq. (2.24). A typical finite element mesh ($a/c = 0.4$, $a/t = 0.2$) is shown in Figure 4.3.

The three-dimensional finite element procedure has been verified extensively for T -stress calculations of surface cracks in the work of Wang (2003). In this thesis, similar mesh design is used. But the boundary conditions are modified corresponding to quarter elliptical cracks.

Uniform tension and bending, corresponding to the nominal stress expressed in eq. (4.1) and eq. (4.2), were applied at remote ends for each crack geometry of a/t and a/c (see Figure 4.1).

$$\sigma(z) = \sigma_0 \quad (4.1)$$

$$\sigma(z) = \sigma_0 \left(1 + \frac{2z}{t} \right) \quad (4.2)$$

where $\sigma_0 = p$ is the maximum value of the nominal stress.

The distributions of stress intensity factor K were calculated along the crack front for validation purpose. The boundary factor F at two points $\varphi = 0^\circ$ and $\varphi = 90^\circ$ (Figure 4.2), where the crack front intersects with the two free surfaces, was calculated by finite element analysis. The values then were compared with the results of Shiratori and Miyoshi (1992), shown in Table 4.1 and Table 4.2. The maximum relative difference from Shiratori's results is less than 8.2% for all the crack geometries (a/c , a/t) under tension and bending loads, except four geometries (a/c , a/t) of (0.4, 0.8), (0.6, 0.8), (1.0,

0.6) and (1.0, 0.8) under bending loads. This is due to the fact that for these four cases, the stress intensity factors at the point $\varphi = 90^\circ$ are very small.

4.1.2 The Results of *T*-stress and *S*-stress

The distributions of both *T*-stress and *S*-stress along the crack front for quarter-elliptical cracks with aspect ratios, a/c , of 0.2, 0.4, 0.6 and 1.0 in a finite thickness plate with a/t ratios of 0.2, 0.4, 0.6 and 0.8 have been determined for remote uniform tension and bending conditions. The Poisson's ratio $\nu = 0.3$ was used in current analysis.

The distributions of the *T*-stress and *S*-stress for quarter-elliptical corner cracks are tabulated in Table 4.3 to Table 4.10.

Note that the values of the *T*-stress at or within the vicinity of the free surfaces were unreliable (Nakamura and Parks, 1992), the *T*-stress and *S*-stress for $\varphi \geq 5^\circ$ and $\varphi \leq 86.25^\circ$ are taken into consideration in the chapter. From now on, the points *A* and *B* in Figure 4.2 are referring to the points at $\varphi = 5^\circ$ and $\varphi = 86.25^\circ$, respectively. The *T*-stress and *S*-stress in this range are plotted in Figure 4.4 to Figure 4.19 for $a/c = 0.2, 0.4, 0.6$ and 1.0 under tension and bending, respectively.

It is observed that under remote tension conditions both the *T*-stress (Figure 4.4 to 4.7) and *S*-stress (Figure 4.12 to 4.15) are negative along the crack front for all the crack geometry except one case. The exception is that the *T*-stress for the aspect ratio $a/c = 0.2$ and the relative depth $a/t = 0.8$ becomes positive around the middle portion of the parametric angle φ . The variation of *T*-stress is larger for deep cracks ($a/t = 0.8$) than that for shallow and mid-deep cracks ($a/t = 0.2$ to 0.6), shown in Figure 4.4 to 4.7. Under remote bending conditions, the *T*-stress at point *B* is negative for shallow cracks ($a/t =$

0.2) and it is positive for $a/t = 0.4, 0.6$ and 0.8 . T -stress at point A is always negative for current crack geometries (Figure 4.8 to 4.11). The S -stress at point A is negative for all the geometries. At point B , S -stress is positive for deep cracks ($a/t = 0.8$) and negative for relative depth a/t of $0.2, 0.4$ and 0.6 (Figure 4.16 to 4.19).

It is worth noting that under bending loads, for a few geometries, for example, ($a/c = 0.2$ and $a/t = 0.2$; $a/c = 0.4$ and $a/t = 0.6$), T -stress and S -stress distributions along the crack front have a small but noticeable fluctuation around $\varphi = 22.5^\circ$ (See Figures 4.8, 4.9). Several techniques including untie the crack tip nodes, and the application of hybrid elements are used to investigate this effect. And it is concluded that this is a negligible effect and is attributed to the fact that $\varphi = 22.5^\circ$ is the transition point from small size elements to larger size elements (see Figure 4.3).

4.2 Empirical Equations of T -stress and S -stress

Numerical solutions of T -stress and S -stress have been presented in Section 4.2 for quarter-elliptical corner cracks with the aspect ratio a/c of $0.2, 0.4, 0.6$ and 1.0 and the relative depth a/t of $0.2, 0.4, 0.6$ and 0.8 in a finite thickness plate under remote uniform tension and bending. For the convenience of engineering applications, and for corner cracks with other aspect ratio or relative depth values, solutions in the form of equations are required. Considering that T -stress and S -stress reach the maximum or minimum values near the surface points most of the time, T -stress and S -stress at two points A ($\varphi = 5^\circ$) and B ($\varphi = 86.25^\circ$) under remote tension and bending are fitted into empirical equations.

4.2.1 T -stress Equations for Tension

The T -stress solutions for uniform tension corresponding to the nominal stress of eq. (4.1) are expressed in the following forms:

$$T = \sigma_0 V_{A0} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } A, \varphi = 5^\circ \quad (4.3)$$

$$T = \sigma_0 V_{B0} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } B, \varphi = 86.25^\circ \quad (4.4)$$

where functions V_{A0} and V_{B0} are the normalized T -stress at the two locations, respectively. At the point A ($\varphi = 5^\circ$),

$$V_{A0} = A_0 + A_1 \left(\frac{a}{t} \right)^2 + A_2 \left(\frac{a}{t} \right)^4 + A_3 \left(\frac{a}{t} \right)^6 \quad (4.5)$$

$$A_0 = 0.068163 - 1.542072 \left(\frac{a}{c} \right) + 1.552113 \left(\frac{a}{c} \right)^2 - 0.509153 \left(\frac{a}{c} \right)^3 \quad (4.6a)$$

$$A_1 = 1.258998 - 8.061265 \left(\frac{a}{c} \right) + 11.768568 \left(\frac{a}{c} \right)^2 - 5.111856 \left(\frac{a}{c} \right)^3 \quad (4.6b)$$

$$A_2 = -2.273624 + 29.897371 \left(\frac{a}{c} \right) - 55.709961 \left(\frac{a}{c} \right)^2 + 27.927065 \left(\frac{a}{c} \right)^3 \quad (4.6c)$$

$$A_3 = -4.495365 - 1.544842 \left(\frac{a}{c} \right) + 19.034831 \left(\frac{a}{c} \right)^2 - 12.427597 \left(\frac{a}{c} \right)^3 \quad (4.6d)$$

At the point B ($\varphi = 86.25^\circ$),

$$V_{B0} = B_0 + B_1 \left(\frac{a}{t} \right)^2 + B_2 \left(\frac{a}{t} \right)^4 + B_3 \left(\frac{a}{t} \right)^6 \quad (4.7)$$

$$B_0 = -0.566261 + 0.3793\left(\frac{a}{c}\right) - 0.522871\left(\frac{a}{c}\right)^2 + 0.241643\left(\frac{a}{c}\right)^3 \quad (4.8a)$$

$$B_1 = 0.721867 - 6.908725\left(\frac{a}{c}\right) + 13.232847\left(\frac{a}{c}\right)^2 - 7.140914\left(\frac{a}{c}\right)^3 \quad (4.8b)$$

$$B_2 = -3.8875 + 24.240958\left(\frac{a}{c}\right) - 46.463976\left(\frac{a}{c}\right)^2 + 25.690466\left(\frac{a}{c}\right)^3 \quad (4.8c)$$

$$B_3 = 4.24628 - 32.667462\left(\frac{a}{c}\right) + 58.982825\left(\frac{a}{c}\right)^2 - 31.357215\left(\frac{a}{c}\right)^3 \quad (4.8d)$$

The normalized T -stress, $\frac{T}{\sigma_0}$, at point A and point B was plotted in Figure 4.20 and

Figure 4.21, respectively. The maximum relative difference from the finite element is less than 0.1%.

4.2.2 T -stress Equations for Bending

The T -stress solutions for remote bending corresponding to the nominal stress of eq. (4.2) are expressed in the following forms:

$$T = \sigma_0 V_{A1} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } A, \varphi = 5^\circ \quad (4.9)$$

$$T = \sigma_0 V_{B1} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } B, \varphi = 86.25^\circ \quad (4.10)$$

where functions V_{A1} and V_{B1} are the normalized T -stress at the two locations, respectively. At the point A ($\varphi = 5^\circ$),

$$V_{A1} = C_0 + C_1 \left(\frac{a}{t} \right)^2 + C_2 \left(\frac{a}{t} \right)^4 + C_3 \left(\frac{a}{t} \right)^6 \quad (4.11)$$

$$C_0 = 0.465426 - 3.589634\left(\frac{a}{c}\right) + 4.716148\left(\frac{a}{c}\right)^2 - 2.029984\left(\frac{a}{c}\right)^3 \quad (4.12a)$$

$$C_1 = -4.209479 + 20.903976\left(\frac{a}{c}\right) - 34.872986\left(\frac{a}{c}\right)^2 + 17.910851\left(\frac{a}{c}\right)^3 \quad (4.12b)$$

$$C_2 = 16.467708 - 80.161068\left(\frac{a}{c}\right) + 133.258681\left(\frac{a}{c}\right)^2 - 69.284939\left(\frac{a}{c}\right)^3 \quad (4.12c)$$

$$C_3 = -19.652158 + 94.611108\left(\frac{a}{c}\right) - 154.37748\left(\frac{a}{c}\right)^2 + 79.415458\left(\frac{a}{c}\right)^3 \quad (4.12d)$$

At the point B ($\varphi = 86.25^\circ$),

$$V_{B1} = D_0 + D_1\left(\frac{a}{t}\right)^2 + D_2\left(\frac{a}{t}\right)^4 + D_3\left(\frac{a}{t}\right)^6 \quad (4.13)$$

$$D_0 = -0.3374 + 0.048275\left(\frac{a}{c}\right) + 0.158231\left(\frac{a}{c}\right)^2 - 0.173157\left(\frac{a}{c}\right)^3 \quad (4.14a)$$

$$D_1 = 4.41853 - 4.604203\left(\frac{a}{c}\right) + 6.191547\left(\frac{a}{c}\right)^2 - 2.720901\left(\frac{a}{c}\right)^3 \quad (4.14b)$$

$$D_2 = -9.261866 + 17.04015\left(\frac{a}{c}\right) - 26.757943\left(\frac{a}{c}\right)^2 + 13.212818\left(\frac{a}{c}\right)^3 \quad (4.14c)$$

$$D_3 = 10.011471 - 26.336625\left(\frac{a}{c}\right) + 40.744048\left(\frac{a}{c}\right)^2 - 20.051954\left(\frac{a}{c}\right)^3 \quad (4.14d)$$

The normalized T -stress, $\frac{T}{\sigma_0}$, at point A and point B was plotted in Figure 4.22 and

Figure 4.23, respectively. The maximum relative difference from the finite element is less than 0.1%.

4.2.3 *S*-stress Equations for Tension

The *S*-stress solutions for uniform tension corresponding to eq. (4.1) are expressed in the following forms:

$$S = \sigma_0 W_{A0} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } A, \varphi = 5^\circ \quad (4.15)$$

$$S = \sigma_0 W_{B0} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } B, \varphi = 86.25^\circ \quad (4.16)$$

where functions W_{A0} and W_{B0} are the normalized *S*-stress at the two locations, respectively. At the point *A* ($\varphi = 5^\circ$),

$$W_{A0} = A_0' + A_1' \left(\frac{a}{t} \right)^2 + A_2' \left(\frac{a}{t} \right)^4 + A_3' \left(\frac{a}{t} \right)^6 \quad (4.17)$$

$$A_0' = -1.1828 + 1.247169 \left(\frac{a}{c} \right) - 3.158718 \left(\frac{a}{c} \right)^2 + 2.137027 \left(\frac{a}{c} \right)^3 \quad (4.18a)$$

$$A_1' = 0.271408 - 16.17629 \left(\frac{a}{c} \right) + 44.898993 \left(\frac{a}{c} \right)^2 - 29.403776 \left(\frac{a}{c} \right)^3 \quad (4.18b)$$

$$A_2' = -3.973051 + 36.901682 \left(\frac{a}{c} \right) - 107.25076 \left(\frac{a}{c} \right)^2 + 74.327528 \left(\frac{a}{c} \right)^3 \quad (4.18c)$$

$$A_3' = -3.630503 + 3.847307 \left(\frac{a}{c} \right) + 37.841409 \left(\frac{a}{c} \right)^2 - 38.375612 \left(\frac{a}{c} \right)^3 \quad (4.18d)$$

At the point *B* ($\varphi = 86.25^\circ$),

$$W_{B0} = B_0' + B_1' \left(\frac{a}{t} \right)^2 + B_2' \left(\frac{a}{t} \right)^4 + B_3' \left(\frac{a}{t} \right)^6 \quad (4.19)$$

$$B_0' = -0.884585 + 1.418977 \left(\frac{a}{c} \right) - 6.248539 \left(\frac{a}{c} \right)^2 + 4.405167 \left(\frac{a}{c} \right)^3 \quad (4.20a)$$

$$B_1' = 2.382208 - 32.485399\left(\frac{a}{c}\right) + 81.742743\left(\frac{a}{c}\right)^2 - 52.098278\left(\frac{a}{c}\right)^3 \quad (4.20b)$$

$$B_2' = -10.304028 + 93.760518\left(\frac{a}{c}\right) - 229.609158\left(\frac{a}{c}\right)^2 + 146.767216\left(\frac{a}{c}\right)^3 \quad (4.20c)$$

$$B_3' = 11.457527 - 87.413298\left(\frac{a}{c}\right) + 202.409629\left(\frac{a}{c}\right)^2 - 127.287688\left(\frac{a}{c}\right)^3 \quad (4.20d)$$

The normalized S -stress, $\frac{S}{\sigma_0}$, at point A and point B was plotted in Figure 4.24 and

Figure 4.25, respectively. The maximum relative difference from the finite element is less than 0.1%.

4.2.4 S -stress Equations for Bending

The S -stress solutions for remote bending corresponding to the nominal stress of eq. (4.2) are expressed in the following forms:

$$S = \sigma_0 W_{A1} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } A, \varphi = 5^\circ \quad (4.21)$$

$$S = \sigma_0 W_{B1} \left(\frac{a}{c}, \frac{a}{t} \right) \text{ at the point } B, \varphi = 86.25^\circ \quad (4.22)$$

where functions W_{A1} and W_{B1} are the normalized S -stress at the two locations, respectively. At the point A ($\varphi = 5^\circ$),

$$W_{A1} = C_0' + C_1' \left(\frac{a}{t} \right)^2 + C_2' \left(\frac{a}{t} \right)^4 + C_3' \left(\frac{a}{t} \right)^6 \quad (4.23)$$

$$C_0' = -1.054856 + 0.72926\left(\frac{a}{c}\right) - 2.23632\left(\frac{a}{c}\right)^2 + 1.640573\left(\frac{a}{c}\right)^3 \quad (4.24a)$$

$$C_1' = -0.561488 - 7.20228\left(\frac{a}{c}\right) + 29.895738\left(\frac{a}{c}\right)^2 - 21.740734\left(\frac{a}{c}\right)^3 \quad (4.24b)$$

$$C_2' = -0.051714 + 19.794607\left(\frac{a}{c}\right) - 81.673286\left(\frac{a}{c}\right)^2 + 61.161296\left(\frac{a}{c}\right)^3 \quad (4.24c)$$

$$C_3' = -3.541744 + 2.433384\left(\frac{a}{c}\right) + 39.72478\left(\frac{a}{c}\right)^2 - 38.241916\left(\frac{a}{c}\right)^3 \quad (4.24d)$$

At the point B ($\varphi = 86.25^\circ$),

$$W_{B1} = D_0' + D_1'\left(\frac{a}{t}\right)^2 + D_2'\left(\frac{a}{t}\right)^4 + D_3'\left(\frac{a}{t}\right)^6 \quad (4.25)$$

$$D_0' = -0.650186 + 0.601671\left(\frac{a}{c}\right) - 3.991957\left(\frac{a}{c}\right)^2 + 2.953318\left(\frac{a}{c}\right)^3 \quad (4.26a)$$

$$D_1' = 3.785539 - 18.1941\left(\frac{a}{c}\right) + 54.081067\left(\frac{a}{c}\right)^2 - 35.496076\left(\frac{a}{c}\right)^3 \quad (4.26b)$$

$$D_2' = -8.556085 + 51.37179\left(\frac{a}{c}\right) - 145.173915\left(\frac{a}{c}\right)^2 + 95.84762\left(\frac{a}{c}\right)^3 \quad (4.26c)$$

$$D_3' = 8.465107 - 45.630195\left(\frac{a}{c}\right) + 121.948397\left(\frac{a}{c}\right)^2 - 79.876935\left(\frac{a}{c}\right)^3 \quad (4.26d)$$

The normalized S -stress, $\frac{S}{\sigma_0}$, at point A and point B was plotted in Figure 4.26 and

Figure 4.27, respectively. The maximum relative difference from the finite element is less than 0.1%.

Table 4.1 Comparison of the stress intensity factor at the point $\varphi = 0^\circ$
under uniform tension and bending conditions

a/c	a/t	Tension			Bending		
		F	F_s^*	Relative Difference (%)	F	F_s^*	Relative Difference (%)
0.2	0.2	0.5802	0.5840	0.7	0.5417	0.5520	1.9
	0.4	0.6830	0.6940	1.6	0.6099	0.6044	0.9
	0.6	0.8825	0.9120	3.2	0.6938	0.7116	2.5
	0.8	1.2586	1.2720	1.1	0.8388	0.8240	1.8
0.4	0.2	0.7750	0.8190	5.4	0.7310	0.7654	4.5
	0.4	0.8801	0.9270	5.1	0.7643	0.7942	3.8
	0.6	1.0749	1.1000	2.3	0.8480	0.8324	1.9
	0.8	1.3175	1.4220	7.4	0.8798	0.9164	4.0
0.6	0.2	0.9113	0.9571	4.8	0.8351	0.8996	7.2
	0.4	0.9992	1.0439	4.3	0.8673	0.8942	3.0
	0.6	1.1736	1.2064	2.7	0.9167	0.9539	3.9
	0.8	1.3695	1.4481	5.4	0.9309	0.9774	4.8
1.0	0.2	1.1094	1.1484	3.4	1.0446	1.0762	2.9
	0.4	1.1694	1.2273	4.7	1.0241	1.0704	4.3
	0.6	1.3031	1.3475	3.3	1.0377	1.0680	2.8
	0.8	1.4542	1.5094	3.7	1.0348	1.0682	3.1

* F_s values are from Shiratori and Miyoshi (1992).

Table 4.2 Comparison of the stress intensity factor at the point $\varphi = 90^\circ$

under uniform tension and bending conditions

a/c	a/t	Tension			Bending		
		F	F_s^*	Relative Difference (%)	F	F_s^*	Relative Difference (%)
0.2	0.2	1.1528	1.1020	4.6	0.8622	0.8444	2.1
	0.4	1.3891	1.3180	5.4	0.7397	0.7300	1.3
	0.6	1.7997	1.7120	5.1	0.6529	0.6548	0.3
	0.8	2.2898	2.1170	8.2	0.4438	0.4594	3.4
0.4	0.2	1.1297	1.1040	2.3	0.8331	0.8312	0.2
	0.4	1.2526	1.2200	2.7	0.6250	0.6328	1.2
	0.6	1.4712	1.4280	3.0	0.4365	0.4620	5.5
	0.8	1.7626	1.7230	2.3	0.1775	0.2414	26.5
0.6	0.2	1.1412	1.1202	1.9	0.8251	0.8302	0.6
	0.4	1.2182	1.1947	2.0	0.5796	0.5878	1.4
	0.6	1.3589	1.3460	1.0	0.3402	0.3716	8.5
	0.8	1.5822	1.5732	0.6	0.0578	0.1118	48.3
1.0	0.2	1.1763	1.1580	1.6	0.8414	0.8278	1.6
	0.4	1.2162	1.2213	0.4	0.5355	0.5568	3.8
	0.6	1.2904	1.3288	2.9	0.2365	0.2734	13.5
	0.8	1.4434	1.4312	0.9	-0.0851	-0.0262	224.9

* F_s values are from Shiratori and Miyoshi (1992).

Table 4.3 *T*-stress and *S*-stress distributions for corner cracks
under remote uniform tension, aspect ratio $a/c = 0.2$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.1767	-1.0993	-0.1434	-1.2813	-0.1275	-1.6623	-0.5224	-2.4923
6.25	-0.2443	-0.9864	-0.2221	-1.1451	-0.2317	-1.4300	-0.7773	-2.0805
7.50	-0.2554	-1.0187	-0.2343	-1.1787	-0.2346	-1.5133	-0.7282	-2.2636
8.75	-0.3024	-0.9053	-0.2889	-1.0420	-0.3245	-1.2971	-0.9362	-1.8679
10.00	-0.3201	-0.9251	-0.3090	-1.0614	-0.3355	-1.3475	-0.8644	-1.9850
11.25	-0.3540	-0.8434	-0.3483	-0.9617	-0.4038	-1.1878	-0.9927	-1.6498
12.50	-0.3677	-0.8499	-0.3637	-0.9658	-0.4053	-1.2124	-0.8934	-1.7121
13.75	-0.3917	-0.7873	-0.3912	-0.8886	-0.4550	-1.0850	-0.9366	-1.4385
15.00	-0.3989	-0.7887	-0.3987	-0.8868	-0.4431	-1.1003	-0.8148	-1.4633
16.25	-0.4181	-0.7378	-0.4206	-0.8236	-0.4838	-0.9897	-0.7948	-1.2484
17.50	-0.4176	-0.7419	-0.4190	-0.8255	-0.4591	-1.0090	-0.6636	-1.2449
18.75	-0.4369	-0.6929	-0.4409	-0.7646	-0.4977	-0.8996	-0.6144	-1.0849
20.00	-0.4264	-0.7123	-0.4273	-0.7850	-0.4611	-0.9339	-0.5131	-1.0454
21.25	-0.4519	-0.6484	-0.4567	-0.7064	-0.5029	-0.8146	-0.4531	-0.9266
22.50	-0.4347	-0.6273	-0.4389	-0.6790	-0.4694	-0.7274	-0.4346	-0.9252
26.25	-0.4683	-0.6067	-0.4718	-0.6494	-0.4825	-0.7132	-0.1703	-0.6299
30.00	-0.4706	-0.6058	-0.4714	-0.6430	-0.4488	-0.6936	0.0293	-0.6083
33.75	-0.4832	-0.5651	-0.4838	-0.5925	-0.4332	-0.6207	0.1187	-0.4461
37.50	-0.4858	-0.5616	-0.4846	-0.5856	-0.4094	-0.6056	0.1933	-0.4428
41.25	-0.4920	-0.5390	-0.4923	-0.5569	-0.4162	-0.5734	0.2104	-0.4091
45.00	-0.4933	-0.5339	-0.4941	-0.5487	-0.4242	-0.5629	0.1736	-0.4535
48.75	-0.4952	-0.5200	-0.5000	-0.5303	-0.4541	-0.5477	0.0147	-0.4375
52.50	-0.4962	-0.5124	-0.5041	-0.5194	-0.4841	-0.5361	-0.1510	-0.4496
56.25	-0.4958	-0.5065	-0.5087	-0.5113	-0.5120	-0.5396	-0.2434	-0.4937
60.00	-0.4980	-0.4963	-0.5158	-0.4983	-0.5440	-0.5304	-0.3317	-0.5291
63.75	-0.4950	-0.5022	-0.5164	-0.5059	-0.5585	-0.5487	-0.4044	-0.6003
67.50	-0.5004	-0.4867	-0.5268	-0.4884	-0.5937	-0.5288	-0.5003	-0.6011
71.25	-0.4941	-0.5163	-0.5203	-0.5275	-0.5962	-0.5712	-0.5844	-0.7009
75.00	-0.5102	-0.4832	-0.5407	-0.4923	-0.6377	-0.5041	-0.6991	-0.5889
78.75	-0.4988	-0.5854	-0.5256	-0.6264	-0.6299	-0.6664	-0.7859	-0.8074
82.50	-0.5591	-0.4585	-0.5953	-0.4847	-0.7007	-0.4389	-0.9384	-0.4387
86.25	-0.5180	-0.8652	-0.5579	-1.0019	-0.6750	-1.1639	-0.9600	-1.1678

Table 4.4 *T*-stress and *S*-stress distributions for corner cracksunder remote uniform tension, aspect ratio $a/c = 0.4$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.3454	-1.0909	-0.3447	-1.2313	-0.2823	-1.5146	-0.2975	-1.8934
6.25	-0.3524	-1.0715	-0.3511	-1.2073	-0.3045	-1.4533	-0.2858	-1.7516
7.50	-0.3371	-1.0484	-0.3322	-1.1791	-0.2634	-1.4234	-0.2169	-1.7756
8.75	-0.3374	-1.0014	-0.3312	-1.1240	-0.2668	-1.3322	-0.2373	-1.6007
10.00	-0.3327	-0.9940	-0.3246	-1.1137	-0.2479	-1.3302	-0.1768	-1.6530
11.25	-0.3357	-0.9539	-0.3269	-1.0666	-0.2528	-1.2608	-0.2066	-1.4950
12.50	-0.3368	-0.9513	-0.3270	-1.0618	-0.2452	-1.2600	-0.1504	-1.5478
13.75	-0.3418	-0.9150	-0.3317	-1.0192	-0.2518	-1.1977	-0.1844	-1.4027
15.00	-0.3450	-0.9130	-0.3345	-1.0150	-0.2494	-1.1967	-0.1293	-1.4548
16.25	-0.3514	-0.8804	-0.3409	-0.9766	-0.2592	-1.1403	-0.1710	-1.3155
17.50	-0.3553	-0.8781	-0.3447	-0.9724	-0.2572	-1.1397	-0.1126	-1.3747
18.75	-0.3626	-0.8482	-0.3522	-0.9370	-0.2721	-1.0845	-0.1687	-1.2236
20.00	-0.3646	-0.8470	-0.3540	-0.9338	-0.2618	-1.0932	-0.1007	-1.3035
21.25	-0.3759	-0.8162	-0.3663	-0.8978	-0.2914	-1.0215	-0.1806	-1.1268
22.50	-0.3788	-0.8321	-0.3709	-0.9141	-0.2964	-1.0425	-0.1080	-1.0412
26.25	-0.3964	-0.7485	-0.3885	-0.8160	-0.3163	-0.9180	-0.1787	-0.9754
30.00	-0.4039	-0.7718	-0.3966	-0.8371	-0.3211	-0.9493	-0.1462	-1.0445
33.75	-0.4204	-0.6948	-0.4160	-0.7476	-0.3525	-0.8246	-0.1815	-0.8533
37.50	-0.4250	-0.7056	-0.4219	-0.7558	-0.3574	-0.8372	-0.1794	-0.8873
41.25	-0.4371	-0.6538	-0.4375	-0.6951	-0.3881	-0.7527	-0.2448	-0.7787
45.00	-0.4390	-0.6611	-0.4414	-0.7000	-0.3975	-0.7578	-0.2891	-0.8012
48.75	-0.4482	-0.6241	-0.4547	-0.6567	-0.4305	-0.6984	-0.3733	-0.7313
52.50	-0.4483	-0.6320	-0.4575	-0.6633	-0.4466	-0.7015	-0.4400	-0.7438
56.25	-0.4561	-0.6052	-0.4695	-0.6322	-0.4807	-0.6604	-0.5254	-0.7117
60.00	-0.4556	-0.6148	-0.4720	-0.6419	-0.5007	-0.6624	-0.5998	-0.7241
63.75	-0.4628	-0.5984	-0.4832	-0.6243	-0.5296	-0.6424	-0.6711	-0.7204
67.50	-0.4645	-0.6078	-0.4880	-0.6355	-0.5535	-0.6376	-0.7524	-0.7188
71.25	-0.4712	-0.6135	-0.4984	-0.6446	-0.5734	-0.6598	-0.8119	-0.7506
75.00	-0.4828	-0.6046	-0.5142	-0.6381	-0.6077	-0.6323	-0.9050	-0.6921
78.75	-0.4797	-0.6908	-0.5150	-0.7395	-0.6075	-0.7834	-0.9604	-0.8527
82.50	-0.5392	-0.5710	-0.5851	-0.6124	-0.6980	-0.6161	-1.0827	-0.5491
86.25	-0.4980	-1.0697	-0.5490	-1.1743	-0.6825	-1.3317	-1.0836	-1.4573

Table 4.5 *T*-stress and *S*-stress distributions for corner cracksunder remote uniform tension, aspect ratio $a/c = 0.6$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.4236	-1.1017	-0.4426	-1.1454	-0.4144	-1.3554	-0.3497	-1.6136
6.25	-0.4074	-1.0870	-0.4236	-1.1335	-0.4147	-1.3320	-0.3251	-1.5272
7.50	-0.3931	-1.0498	-0.4065	-1.0966	-0.3852	-1.2716	-0.2688	-1.4934
8.75	-0.3832	-1.0246	-0.3943	-1.0667	-0.3712	-1.2181	-0.2645	-1.3885
10.00	-0.3742	-0.9966	-0.3832	-1.0421	-0.3528	-1.1941	-0.2202	-1.3923
11.25	-0.3678	-0.9724	-0.3750	-1.0155	-0.3426	-1.1565	-0.2216	-1.3099
12.50	-0.3623	-0.9590	-0.3679	-1.0044	-0.3308	-1.1441	-0.1866	-1.3206
13.75	-0.3587	-0.9384	-0.3630	-0.9810	-0.3237	-1.1112	-0.1905	-1.2496
15.00	-0.3556	-0.9300	-0.3585	-0.9745	-0.3156	-1.1048	-0.1615	-1.2662
16.25	-0.3542	-0.9113	-0.3561	-0.9531	-0.3116	-1.0753	-0.1720	-1.1980
17.50	-0.3527	-0.9058	-0.3536	-0.9491	-0.3060	-1.0719	-0.1424	-1.2263
18.75	-0.3528	-0.8886	-0.3528	-0.9292	-0.3052	-1.0443	-0.1652	-1.1468
20.00	-0.3521	-0.8847	-0.3513	-0.9261	-0.2995	-1.0435	-0.1243	-1.2056
21.25	-0.3542	-0.8686	-0.3530	-0.9076	-0.3044	-1.0131	-0.1729	-1.0853
22.50	-0.3529	-0.8797	-0.3528	-0.9228	-0.3023	-1.0372	-0.1125	-1.0729
26.25	-0.3616	-0.8213	-0.3591	-0.8525	-0.3095	-0.9427	-0.1854	-1.0086
30.00	-0.3651	-0.8431	-0.3619	-0.8808	-0.3100	-0.9841	-0.1843	-1.0986
33.75	-0.3769	-0.7829	-0.3748	-0.8069	-0.3321	-0.8817	-0.2359	-0.9378
37.50	-0.3824	-0.7991	-0.3808	-0.8267	-0.3397	-0.9096	-0.2500	-1.0032
41.25	-0.3942	-0.7514	-0.3948	-0.7681	-0.3665	-0.8262	-0.3126	-0.8795
45.00	-0.3991	-0.7650	-0.4012	-0.7828	-0.3779	-0.8465	-0.3435	-0.9230
48.75	-0.4102	-0.7267	-0.4151	-0.7357	-0.4084	-0.7776	-0.4142	-0.8263
52.50	-0.4138	-0.7408	-0.4208	-0.7498	-0.4233	-0.7955	-0.4608	-0.8562
56.25	-0.4241	-0.7097	-0.4344	-0.7121	-0.4555	-0.7387	-0.5386	-0.7873
60.00	-0.4257	-0.7280	-0.4383	-0.7295	-0.4706	-0.7586	-0.5942	-0.8105
63.75	-0.4356	-0.7015	-0.4516	-0.6990	-0.5001	-0.7174	-0.6688	-0.7690
67.50	-0.4358	-0.7301	-0.4543	-0.7251	-0.5155	-0.7401	-0.7361	-0.7758
71.25	-0.4455	-0.7194	-0.4676	-0.7136	-0.5403	-0.7330	-0.7980	-0.7787
75.00	-0.4484	-0.7459	-0.4736	-0.7335	-0.5626	-0.7414	-0.8784	-0.7526
78.75	-0.4546	-0.7947	-0.4838	-0.7868	-0.5761	-0.8480	-0.9151	-0.8935
82.50	-0.5002	-0.7370	-0.5362	-0.7227	-0.6567	-0.7263	-1.0249	-0.6805
86.25	-0.4837	-1.2963	-0.5240	-1.2709	-0.6566	-1.3919	-1.0515	-1.5507

Table 4.6 *T*-stress and *S*-stress distributions for corner cracksunder remote uniform tension, aspect ratio $a/c = 1.0$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.4370	-0.9737	-0.4560	-1.0240	-0.4775	-1.1189	-0.4406	-1.3005
6.25	-0.4317	-0.9352	-0.4496	-0.9829	-0.4635	-1.1078	-0.4227	-1.2265
7.50	-0.4229	-0.9004	-0.4394	-0.9456	-0.4531	-1.0480	-0.4003	-1.1724
8.75	-0.4158	-0.8853	-0.4311	-0.9292	-0.4396	-1.0150	-0.3912	-1.1039
10.00	-0.4113	-0.8563	-0.4255	-0.8981	-0.4319	-0.9785	-0.3759	-1.0850
11.25	-0.4056	-0.8406	-0.4188	-0.8810	-0.4225	-0.9569	-0.3697	-1.0442
12.50	-0.4027	-0.8255	-0.4151	-0.8647	-0.4165	-0.9384	-0.3590	-1.0316
13.75	-0.3987	-0.8144	-0.4101	-0.8525	-0.4098	-0.9222	-0.3541	-1.0029
15.00	-0.3967	-0.8057	-0.4073	-0.8430	-0.4049	-0.9112	-0.3461	-0.9971
16.25	-0.3938	-0.7969	-0.4036	-0.8332	-0.3999	-0.8985	-0.3431	-0.9747
17.50	-0.3926	-0.7921	-0.4017	-0.8278	-0.3960	-0.8926	-0.3360	-0.9757
18.75	-0.3901	-0.7849	-0.3986	-0.8198	-0.3922	-0.8822	-0.3358	-0.9530
20.00	-0.3885	-0.7825	-0.3963	-0.8169	-0.3887	-0.8791	-0.3265	-0.9673
21.25	-0.3873	-0.7764	-0.3946	-0.8100	-0.3860	-0.8700	-0.3325	-0.9287
22.50	-0.3824	-0.7829	-0.3884	-0.8163	-0.3783	-0.8790	-0.3096	-0.9425
26.25	-0.3843	-0.7601	-0.3897	-0.7917	-0.3788	-0.8459	-0.3308	-0.9043
30.00	-0.3819	-0.7817	-0.3861	-0.8132	-0.3731	-0.8746	-0.3255	-0.9655
33.75	-0.3813	-0.7544	-0.3850	-0.7839	-0.3745	-0.8338	-0.3417	-0.8904
37.50	-0.3798	-0.7796	-0.3832	-0.8089	-0.3728	-0.8660	-0.3426	-0.9560
41.25	-0.3802	-0.7525	-0.3838	-0.7800	-0.3785	-0.8244	-0.3687	-0.8783
45.00	-0.3790	-0.7793	-0.3828	-0.8068	-0.3799	-0.8580	-0.3765	-0.9415
48.75	-0.3801	-0.7521	-0.3849	-0.7777	-0.3902	-0.8148	-0.4157	-0.8602
52.50	-0.3791	-0.7798	-0.3847	-0.8055	-0.3945	-0.8501	-0.4358	-0.9214
56.25	-0.3812	-0.7532	-0.3882	-0.7774	-0.4088	-0.8056	-0.4882	-0.8398
60.00	-0.3799	-0.7830	-0.3883	-0.8073	-0.4137	-0.8459	-0.5149	-0.9014
63.75	-0.3841	-0.7561	-0.3942	-0.7790	-0.4316	-0.8012	-0.5733	-0.8269
67.50	-0.3834	-0.7944	-0.3951	-0.8180	-0.4360	-0.8541	-0.6091	-0.8825
71.25	-0.3908	-0.7798	-0.4046	-0.8030	-0.4593	-0.8224	-0.6677	-0.8411
75.00	-0.3879	-0.8191	-0.4035	-0.8435	-0.4653	-0.8761	-0.7144	-0.8806
78.75	-0.4114	-0.8063	-0.4298	-0.8308	-0.4988	-0.8967	-0.7569	-0.9624
82.50	-0.4269	-0.8885	-0.4478	-0.9162	-0.5484	-0.8907	-0.8491	-0.8515
86.25	-0.4727	-1.3264	-0.4974	-1.3701	-0.5939	-1.4334	-0.9095	-1.5694

Table 4.7 *T*-stress and *S*-stress distributions for corner cracks under bending,

aspect ratio $a/c = 0.2$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>
5.00	-0.1234	-1.0228	-0.1772	-1.1205	-0.1581	-1.2744	-0.4084	-1.6144
6.25	-0.2113	-0.9217	-0.2181	-0.9910	-0.2071	-1.0874	-0.4964	-1.3286
7.50	-0.1788	-0.9363	-0.2269	-1.0144	-0.1973	-1.1355	-0.4906	-1.4379
8.75	-0.2521	-0.8367	-0.2438	-0.8846	-0.2332	-0.9627	-0.5487	-1.1736
10.00	-0.2307	-0.8409	-0.2581	-0.8940	-0.2298	-0.9836	-0.5217	-1.2275
11.25	-0.2892	-0.7708	-0.2622	-0.7974	-0.2484	-0.8558	-0.5353	-1.0107
12.50	-0.2692	-0.7633	-0.2713	-0.7923	-0.2388	-0.8551	-0.4820	-1.0222
13.75	-0.3131	-0.7097	-0.2668	-0.7162	-0.2428	-0.7542	-0.4505	-0.8498
15.00	-0.2920	-0.6985	-0.2686	-0.7047	-0.2250	-0.7429	-0.3740	-0.8345
16.25	-0.3257	-0.6552	-0.2606	-0.6426	-0.2211	-0.6585	-0.3129	-0.6987
17.50	-0.3022	-0.6464	-0.2551	-0.6316	-0.1964	-0.6452	-0.2231	-0.6687
18.75	-0.3305	-0.6051	-0.2474	-0.5753	-0.1893	-0.5676	-0.1535	-0.5630
20.00	-0.3026	-0.6087	-0.2338	-0.5737	-0.1590	-0.5579	-0.0731	-0.5176
21.25	-0.3313	-0.5566	-0.2307	-0.5111	-0.1516	-0.4817	-0.0132	-0.4271
22.50	-0.1535	-0.4883	-0.0304	-0.4311	-0.0856	-0.4194	-0.0406	-0.4166
26.25	-0.3448	-0.5042	-0.1795	-0.4187	-0.0556	-0.3464	0.2549	-0.1851
30.00	-0.2650	-0.4694	-0.1667	-0.3915	0.0147	-0.2945	0.4519	-0.1134
33.75	-0.3110	-0.4367	-0.1154	-0.3206	0.0926	-0.2053	0.5652	0.0176
37.50	-0.2606	-0.4095	-0.0952	-0.2929	0.1556	-0.1579	0.6718	0.0745
41.25	-0.2793	-0.3874	-0.0535	-0.2446	0.2096	-0.1010	0.7432	0.1285
45.00	-0.2439	-0.3648	-0.0312	-0.2173	0.2510	-0.0610	0.7898	0.1506
48.75	-0.2486	-0.3480	0.0026	-0.1810	0.2823	-0.0185	0.7791	0.2023
52.50	-0.2226	-0.3275	0.0231	-0.1543	0.3071	0.0174	0.7696	0.2403
56.25	-0.2206	-0.3169	0.0506	-0.1291	0.3328	0.0447	0.7936	0.2649
60.00	-0.2022	-0.2979	0.0672	-0.1049	0.3525	0.0729	0.8251	0.2829
63.75	-0.1963	-0.2975	0.0908	-0.0944	0.3793	0.0883	0.8600	0.2952
67.50	-0.1857	-0.2781	0.1024	-0.0731	0.3934	0.1134	0.8883	0.3198
71.25	-0.1770	-0.2982	0.1246	-0.0875	0.4186	0.1126	0.9095	0.3197
75.00	-0.1789	-0.2701	0.1273	-0.0647	0.4265	0.1489	0.9287	0.3745
78.75	-0.1684	-0.3517	0.1459	-0.1467	0.4447	0.0782	0.9286	0.3249
82.50	-0.2073	-0.2594	0.1187	-0.0794	0.4405	0.1601	0.9330	0.4435
86.25	-0.1844	-0.5899	0.1239	-0.4121	0.4239	-0.2001	0.8963	0.2080

Table 4.8 *T*-stress and *S*-stress distributions for corner cracks under bending,
aspect ratio $a/c = 0.4$

	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
φ (°)	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>	<i>T/p</i>	<i>S/p</i>
5.00	-0.3550	-1.0200	-0.3636	-1.0515	-0.3463	-1.1384	-0.3709	-1.2242
6.25	-0.3541	-0.9974	-0.3544	-1.0226	-0.3238	-1.0854	-0.3259	-1.1236
7.50	-0.3331	-0.9714	-0.3246	-0.9897	-0.2772	-1.0539	-0.2657	-1.1164
8.75	-0.3254	-0.9229	-0.3083	-0.9340	-0.2551	-0.9719	-0.2380	-0.9962
10.00	-0.3142	-0.9113	-0.2888	-0.9159	-0.2214	-0.9584	-0.1856	-1.0042
11.25	-0.3088	-0.8693	-0.2747	-0.8672	-0.2010	-0.8902	-0.1604	-0.8979
12.50	-0.3029	-0.8621	-0.2607	-0.8535	-0.1761	-0.8773	-0.1139	-0.9017
13.75	-0.2992	-0.8240	-0.2484	-0.8088	-0.1556	-0.8182	-0.0873	-0.8080
15.00	-0.2953	-0.8171	-0.2367	-0.7955	-0.1346	-0.8044	-0.0457	-0.8054
16.25	-0.2928	-0.7825	-0.2257	-0.7545	-0.1161	-0.7481	-0.0192	-0.7220
17.50	-0.2897	-0.7754	-0.2151	-0.7409	-0.0978	-0.7305	0.0190	-0.7141
18.75	-0.2878	-0.7432	-0.2048	-0.7032	-0.0785	-0.6799	0.0431	-0.6362
20.00	-0.2835	-0.7366	-0.1936	-0.6897	-0.0662	-0.6792	0.0775	-0.6237
21.25	-0.2848	-0.7049	-0.1860	-0.6530	-0.0429	-0.6259	0.0983	-0.5495
22.50	-0.2346	-0.6987	-0.1831	-0.6341	0.0988	-0.5762	0.4103	-0.4316
26.25	-0.2733	-0.6243	-0.1445	-0.5492	0.0245	-0.4833	0.2192	-0.3703
30.00	-0.2641	-0.6296	-0.1171	-0.5348	0.0644	-0.4532	0.2839	-0.3524
33.75	-0.2543	-0.5486	-0.0850	-0.4411	0.1219	-0.3362	0.3786	-0.1966
37.50	-0.2430	-0.5436	-0.0571	-0.4178	0.1624	-0.3002	0.4300	-0.1577
41.25	-0.2321	-0.4869	-0.0273	-0.3502	0.2107	-0.2134	0.4952	-0.0561
45.00	-0.2193	-0.4800	-0.0001	-0.3271	0.2469	-0.1768	0.5271	-0.0152
48.75	-0.2088	-0.4385	0.0261	-0.2767	0.2856	-0.1115	0.5721	0.0623
52.50	-0.1958	-0.4338	0.0510	-0.2580	0.3150	-0.0783	0.5972	0.1030
56.25	-0.1871	-0.4034	0.0725	-0.2212	0.3438	-0.0308	0.6320	0.1564
60.00	-0.1754	-0.4023	0.0937	-0.2092	0.3660	-0.0043	0.6532	0.1883
63.75	-0.1695	-0.3834	0.1096	-0.1870	0.3880	0.0248	0.6854	0.2255
67.50	-0.1618	-0.3851	0.1248	-0.1823	0.4028	0.0459	0.7044	0.2499
71.25	-0.1592	-0.3871	0.1338	-0.1830	0.4185	0.0436	0.7292	0.2715
75.00	-0.1621	-0.3796	0.1368	-0.1785	0.4220	0.0567	0.7425	0.2931
78.75	-0.1573	-0.4483	0.1404	-0.2407	0.4288	-0.0080	0.7469	0.2737
82.50	-0.1995	-0.3646	0.1081	-0.1880	0.4039	0.0276	0.7506	0.3116
86.25	-0.1772	-0.7515	0.1078	-0.5087	0.3754	-0.2650	0.6942	0.1096

Table 4.9 *T*-stress and *S*-stress distributions for corner cracks under bending,

aspect ratio $a/c = 0.6$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.4410	-1.0274	-0.4555	-0.9742	-0.4383	-1.0231	-0.4328	-1.0612
6.25	-0.4204	-1.0098	-0.4248	-0.9556	-0.4184	-0.9935	-0.3880	-0.9908
7.50	-0.4012	-0.9711	-0.3958	-0.9154	-0.3720	-0.9338	-0.3242	-0.9500
8.75	-0.3854	-0.9431	-0.3707	-0.8809	-0.3412	-0.8806	-0.2915	-0.8665
10.00	-0.3709	-0.9125	-0.3466	-0.8506	-0.3025	-0.8477	-0.2362	-0.8488
11.25	-0.3579	-0.8854	-0.3249	-0.8190	-0.2750	-0.8067	-0.2059	-0.7812
12.50	-0.3461	-0.8682	-0.3040	-0.7997	-0.2408	-0.7818	-0.1577	-0.7651
13.75	-0.3355	-0.8443	-0.2850	-0.7710	-0.2159	-0.7447	-0.1270	-0.7073
15.00	-0.3255	-0.8315	-0.2663	-0.7553	-0.1842	-0.7236	-0.0851	-0.6910
16.25	-0.3167	-0.8094	-0.2495	-0.7284	-0.1621	-0.6896	-0.0552	-0.6398
17.50	-0.3081	-0.7992	-0.2325	-0.7147	-0.1324	-0.6704	-0.0179	-0.6227
18.75	-0.3006	-0.7785	-0.2172	-0.6892	-0.1125	-0.6386	0.0103	-0.5756
20.00	-0.2926	-0.7698	-0.2013	-0.6766	-0.0841	-0.6206	0.0438	-0.5587
21.25	-0.2870	-0.7504	-0.1880	-0.6526	-0.0673	-0.5893	0.0693	-0.5132
22.50	-0.2781	-0.7540	-0.2274	-0.6683	-0.0083	-0.5815	0.1416	-0.4951
26.25	-0.2636	-0.6882	-0.1359	-0.5720	0.0054	-0.4880	0.1770	-0.3789
30.00	-0.2454	-0.6907	-0.0951	-0.5606	0.0993	-0.4547	0.2525	-0.3571
33.75	-0.2341	-0.6248	-0.0663	-0.4820	0.1143	-0.3662	0.3185	-0.2316
37.50	-0.2188	-0.6229	-0.0312	-0.4653	0.1948	-0.3267	0.3819	-0.1961
41.25	-0.2094	-0.5705	-0.0064	-0.4024	0.2063	-0.2556	0.4358	-0.0993
45.00	-0.1954	-0.5675	0.0242	-0.3846	0.2735	-0.2161	0.4864	-0.0563
48.75	-0.1878	-0.5255	0.0450	-0.3344	0.2821	-0.1592	0.5284	0.0212
52.50	-0.1749	-0.5245	0.0713	-0.3195	0.3350	-0.1242	0.5670	0.0648
56.25	-0.1695	-0.4914	0.0876	-0.2809	0.3404	-0.0801	0.5978	0.1228
60.00	-0.1573	-0.4960	0.1095	-0.2722	0.3810	-0.0541	0.6275	0.1601
63.75	-0.1550	-0.4700	0.1203	-0.2445	0.3837	-0.0251	0.6500	0.2012
67.50	-0.1445	-0.4852	0.1367	-0.2462	0.4118	-0.0108	0.6707	0.2302
71.25	-0.1458	-0.4752	0.1407	-0.2363	0.4097	-0.0044	0.6851	0.2514
75.00	-0.1406	-0.4935	0.1479	-0.2438	0.4222	-0.0020	0.6971	0.2651
78.75	-0.1437	-0.5326	0.1439	-0.2771	0.4115	-0.0458	0.6948	0.2514
82.50	-0.1826	-0.4974	0.1200	-0.2545	0.3926	-0.0296	0.6916	0.2463
86.25	-0.1660	-0.9159	0.1084	-0.5472	0.3564	-0.2502	0.6306	0.1226

Table 4.10 *T*-stress and *S*-stress distributions for corner cracks under bending,aspect ratio $a/c = 1.0$

φ (°)	$a/t=0.2$		$a/t=0.4$		$a/t=0.6$		$a/t=0.8$	
	T/p	S/p	T/p	S/p	T/p	S/p	T/p	S/p
5.00	-0.4483	-0.9069	-0.4737	-0.8769	-0.4982	-0.8627	-0.4953	-0.8878
6.25	-0.4361	-0.8667	-0.4537	-0.8332	-0.4663	-0.8422	-0.4551	-0.8219
7.50	-0.4207	-0.8301	-0.4306	-0.7929	-0.4367	-0.7833	-0.4106	-0.7677
8.75	-0.4068	-0.8115	-0.4091	-0.7699	-0.4050	-0.7446	-0.3759	-0.7035
10.00	-0.3953	-0.7800	-0.3900	-0.7343	-0.3778	-0.7030	-0.3367	-0.6725
11.25	-0.3829	-0.7606	-0.3699	-0.7104	-0.3493	-0.6726	-0.3039	-0.6270
12.50	-0.3730	-0.7419	-0.3524	-0.6870	-0.3233	-0.6443	-0.2681	-0.5990
13.75	-0.3621	-0.7269	-0.3340	-0.6672	-0.2971	-0.6178	-0.2359	-0.5617
15.00	-0.3532	-0.7139	-0.3174	-0.6492	-0.2722	-0.5948	-0.2024	-0.5370
16.25	-0.3434	-0.7009	-0.3002	-0.6314	-0.2473	-0.5709	-0.1711	-0.5047
17.50	-0.3351	-0.6914	-0.2844	-0.6168	-0.2232	-0.5514	-0.1392	-0.4819
18.75	-0.3259	-0.6800	-0.2679	-0.6006	-0.1995	-0.5294	-0.1088	-0.4529
20.00	-0.3175	-0.6728	-0.2523	-0.5880	-0.1762	-0.5120	-0.0788	-0.4309
21.25	-0.3095	-0.6623	-0.2369	-0.5728	-0.1533	-0.4909	-0.0489	-0.4045
22.50	-0.2864	-0.6590	-0.2204	-0.5451	-0.0895	-0.4684	0.0250	-0.3779
26.25	-0.2796	-0.6284	-0.1789	-0.5199	-0.0668	-0.4170	0.0631	-0.3071
30.00	-0.2575	-0.6318	-0.1362	-0.5048	-0.0031	-0.3872	0.1450	-0.2675
33.75	-0.2384	-0.5955	-0.0982	-0.4581	0.0525	-0.3254	0.2170	-0.1870
37.50	-0.2188	-0.6022	-0.0600	-0.4464	0.1088	-0.2983	0.2893	-0.1473
41.25	-0.2024	-0.5682	-0.0271	-0.4042	0.1563	-0.2430	0.3507	-0.0770
45.00	-0.1850	-0.5772	0.0064	-0.3954	0.2046	-0.2184	0.4112	-0.0355
48.75	-0.1716	-0.5456	0.0337	-0.3578	0.2429	-0.1701	0.4598	0.0250
52.50	-0.1566	-0.5564	0.0620	-0.3520	0.2818	-0.1486	0.5058	0.0667
56.25	-0.1466	-0.5279	0.0831	-0.3203	0.3104	-0.1084	0.5413	0.1158
60.00	-0.1342	-0.5420	0.1057	-0.3186	0.3403	-0.0929	0.5756	0.1535
63.75	-0.1288	-0.5163	0.1195	-0.2932	0.3587	-0.0626	0.5995	0.1888
67.50	-0.1199	-0.5387	0.1353	-0.3003	0.3789	-0.0568	0.6211	0.2185
71.25	-0.1204	-0.5254	0.1402	-0.2879	0.3853	-0.0410	0.6323	0.2364
75.00	-0.1136	-0.5517	0.1495	-0.3020	0.3940	-0.0463	0.6422	0.2495
78.75	-0.1303	-0.5441	0.1376	-0.2994	0.3844	-0.0567	0.6345	0.2451
82.50	-0.1412	-0.6057	0.1294	-0.3430	0.3714	-0.0636	0.6300	0.2233
86.25	-0.1816	-0.9302	0.0918	-0.5655	0.3349	-0.1985	0.5810	0.2052

Figure 4.1 Geometry and coordinate system

Figure 4.2 A quarter-elliptical crack in a plate

Figure 4.3 A typical finite element mesh (part) ($a/c = 0.4$, $a/t = 0.2$)

Figure 4.4 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.2$ under remote uniform tension

Figure 4.5 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.4$ under remote uniform tension

Figure 4.6 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.6$ under remote uniform tension

Figure 4.7 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 1.0$ under remote uniform tension

Figure 4.8 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.2$ under remote bending

Figure 4.9 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.4$ under remote bending

Figure 4.10 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.6$ under remote bending

Figure 4.11 Normalized T -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 1.0$ under remote bending

Figure 4.12 Normalized S -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.2$ under remote uniform tension

Figure 4.13 Normalized S -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.4$ under remote uniform tension

Figure 4.14 Normalized S-stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.6$ under remote uniform tension

Figure 4.15 Normalized S-stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 1.0$ under remote uniform tension

Figure 4.16 Normalized S -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.2$ under remote bending

Figure 4.17 Normalized S -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.4$ under remote bending

Figure 4.18 Normalized S -stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 0.6$ under remote bending

Figure 4.19 Normalized S-stress vs. the parametric angle ϕ for quarter-elliptical corner cracks with aspect ratio $a/c = 1.0$ under remote bending

Figure 4.20 Comparison of the T -stress from empirical equations
with the finite element results at point A under tension

Figure 4.21 Comparison of the T -stress from empirical equations
with the finite element results at point B under tension

Figure 4.22 Comparison of the T -stress from empirical equations
with the finite element results at point A under bending

Figure 4.23 Comparison of the T -stress from empirical equations
with the finite element results at point B under bending

Figure 4.24 Comparison of the S -stress from empirical equations
with the finite element results at point A under tension

Figure 4.25 Comparison of the S -stress from empirical equations
with the finite element results at point B under tension

Figure 4.26 Comparison of the S-stress from empirical equations
with the finite element results at point A under bending

Figure 4.27 Comparison of the S -stress from empirical equations
with the finite element results at point B under bending

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

The analytical solutions of T -stress and S -stress were derived for an elliptical crack embedded in an infinite solid subject to remote uniaxial tension and bending loads using the potential method. These solutions were verified by three-dimensional finite element methods for elliptical cracks with aspect ratio b/a of 0.4, 0.6 and 1.0. Excellent accuracy was achieved between the T -stresses and S -stresses obtained by analytical solutions and finite element analysis. T -stress and S -stress for embedded elliptical cracks under biaxial loadings are further evaluated using superposition method.

From the analytical solutions, it has been found that both T -stress and S -stress strongly depend on the parametric angles, crack geometry, loading conditions and Poisson's ratio. An embedded crack under tension represents low constraint because the T -stress is always negative, while an embedded crack under bending conditions the constraint varies because the T -stress changes between negative values and positive values along the crack front.

The three-dimensional finite element analyses for a quarter-elliptical corner cracks in a finite thickness plate under remote uniaxial tension and bending were conducted. The T -stress and S -stress were determined for quarter-elliptical corner cracks with the aspect ratio a/c of 0.2, 0.4, 0.6 and 1.0, and the relative depth a/t of 0.2, 0.4, 0.6

and 0.8. The stress intensity factors for these cases were also calculated. The values of stress intensity factors calculated by the finite element analysis were compared with the results from Shiratori and Miyoshi (1992). Satisfactory consistency was achieved. For the convenience of engineering application, the empirical equations of T -stress and S -stress at two points A ($\varphi = 5^\circ$) and B ($\varphi = 86.25^\circ$) were given for quarter-elliptical corner cracks in finite thickness plates under remote uniaxial tension and bending.

It is observed that under remote tension conditions both the T -stress and S -stress are negative along the crack front for all the crack geometry except the T -stress for the aspect ratio $a/c = 0.2$ and the relative depth $a/t = 0.8$. In this exception, T -stress becomes positive around the middle portion of the parametric angle φ . The variation of T -stress is larger for deep cracks ($a/t = 0.8$) than that for shallow and mid-deep cracks ($a/t = 0.2 \sim 0.6$). Under remote bending conditions, the T -stress at the point B ($\varphi = 86.25^\circ$) is negative for shallow cracks ($a/t = 0.2$) and positive for $a/t = 0.4, 0.6$ and 0.8 . T -stress at the point A ($\varphi = 5^\circ$) is always negative for current crack geometries, meaning low constraint. The S -stress at the point A is negative for all the geometries. At the point B , S -stress is positive for deep cracks ($a/t = 0.8$) and negative for relative depth a/t of $0.2, 0.4$ and 0.6 .

When combined with the corresponding K or J solutions, these T -stress and S -stress solutions developed in this thesis are suitable for the analysis of constraint effects for engineering components with embedded cracks and corner cracks.

5.2 Recommendations

In the thesis, only constant and linear loadings were applied at remote ends (corresponding to tension and bending conditions) to calculate the T -stress and S -stress of an elliptical crack in an infinite solid and a quarter-elliptical corner crack in a finite thickness plate. However, it is possible to encounter other types of stress distribution, such as nonlinear stress distributions arisen from residual stresses. The T -stress and S -stress solutions for these nonlinear loading conditions are required. Weight function method provides a powerful tool in calculating fracture parameter such as stress intensity factor and T -stress. Once the weight function for the crack geometry is obtained and the crack pressure corresponding to the given loading conditions is known, T -stress can be evaluated conveniently by integration. Therefore, it is recommended that further research work should be conducted in order to derive the appropriate T -stress weight functions for the embedded elliptical crack and corner crack geometries.

References

ABAQUS Manual, (2001), Version 6.2, Hibbitt, Karlsson & Sorensen, Inc..

Ainsworth, R.A. and O'Dowd, N.P., (1995), "Constraint in the failure assessment diagram approach for fracture assessment", *ASME Journal of Pressure Vessel Technology*, Vol. 117, pp. 260-267.

Banks-Sills, L., (1991), "Application of the finite element method to linear elastic fracture mechanics", *Applied Mechanics Reviews*, Vol. 44, pp. 447-461.

Barsoum, R.S., (1977), "Triangular quarter-point elements as elastic and perfectly-plastic crack tip elements", *International Journal for Numerical Methods in Engineering*, Vol. 11, pp. 85-98.

Betegon, C. and Hancock, J.W., (1991), "Two-parameter characterization of elastic-plastic crack-tip fields", *ASME Journal of Applied Mechanics*, Vol. 58, pp. 104-110.

Byrd, Paul F. and Friedman, Morris D., (1971), *Handbook of Elliptic Integrals for Engineers and Scientists*, Second Edition, Springer-Verlag New York Heidelberg Berlin.

Cardew, G.E., Goldthorpe, M.R., Howard, I.C. and Kfoury, A.P., (1984), "On the elastic *T*-term", In *Fundamentals of Deformation and Fracture*, edited by Bilby B.A., Miller, K.J. and Willis J.R., Cambridge University Press, pp. 465-476.

Cotterell, B. and Rice, J.R., (1980), "Slightly curved or kinked cracks", *International Journal of Fracture*, Vol. 16, pp. 155-169.

Delorenzi, H.G., (1985), "Energy release rate calculations by the finite element method", *Engineering Fracture Mechanics*, Vol. 21, pp. 129-143.

Du, Z.Z. and Hancock, J.W., (1991), "The effect of non-singular stresses on crack-tip constraint", *Journal of Mechanical Physics and Solids*, Vol. 39, pp. 555-567.

Gdoutos, E.E., (1993), *Fracture Mechanics: an introduction*, Kluwer Academic Publishers, pp. 4.

Green, A.E. and Sneddon, I.N., (1950), "The distribution of stress in the neighborhood of a flat elliptical crack in an elastic solid", *Proceedings of the Cambridge Philosophical Society*, Vol. 46, pp. 159-163.

Griffith, A.A., (1921), "The phenomena of rupture and flow in solids", *Philosophical Transactions of the Royal Society of London A221*, pp. 163-198.

Hall, L.R., (1971), "Fracture toughness testing at cryogenic temperatures", ASTM-STP-496, American Society for Testing and materials, Philadelphia, pp. 20.

Hancock, J.W., Reuter, W.G. and Parks, D.M., (1993), "Constraint and toughness parameterized by T ", In *Constraint Effects in Fracture*, ASTM STP-1171, edited by Hackett E.M., Schwalbe K.H. and Dodds R.H., American Society of Testing and Materials, Philadelphia, pp. 21-40.

Irwin, G.R., (1962), "Crack extension force for a part-through crack in a plate", *Journal of Applied Mechanics*, pp. 651-654.

Kassir, M.K. and Sih, G.C., (1975), *Three-dimensional crack problems*, Noordhoff International Publishing, Leyden, The Netherlands, pp. 74-76.

Kfourri, A.P., (1986), "Some evaluations of the elastic T -term using Eshelby's method", *International Journal of Fracture*, Vol. 30, pp. 301-315.

Kiciak, A., Glinka, G. and Eman, M., (1998), "Weight functions and stress intensity factors for corner quarter-elliptical crack in finite thickness plate subjected to in-plane loading", *Engineering fracture mechanics*, Vol. 60, No. 2, pp. 221-238.

Kobayahi, A.S. and Enetanya, A.N., (1976), “Stress Intensity factor of a corner crack”, *Mechanics of Crack Growth*, ASTM STP 590, American Society for Testing and Materials, pp. 477-495.

Larsson S.G. and Carlsson A.J., (1973), “Influence of non-singular stress terms and specimen geometry on small-scale yielding at crack-tips in elastic-plastic materials”, *Journal of Mechanical Physics and Solids*, Vol. 21, pp. 447-473.

Leevers, P.S. and Radon, J.C., (1982), “Inherent stress biaxiality in various fracture specimen geometries”, *International Journal of Fracture*, Vol., 19, pp. 311-325.

Li, F.Z., Shih, C.F. and Needleman, A., (1985), “A comparison of methods for calculating energy release rates”, *Engineering Fracture Mechanics*, Vol. 21, pp. 405-421.

Nakamura, T. and Parks, D.M., (1992), “Determination of elastic T -stress along three-dimensional crack fronts using an interaction integral”, *International Journal of Solids Structures*, Vol. 29, No. 13, pp. 1597-1611.

Newman, J.C. and Raju, I.S., (1983), “Stress-intensity factors equations for cracks in three-dimensional finite bodies”, In *Fracture Mechanics: Fourteenth Symposium – Volume I: Theory and Analysis*, ASTM STP 791, edited by Lewis, J.C. and Sines, G.. American Society for Testing and Materials, pp. I-238 – I-265.

O'Dowd, N.P. and Shih, C.F., (1991), "Family of crack-tip fields characterized by a triaxiality parameter: part I – structure of fields", *Journal of Mechanical Physics and Solids*, Vol. 39, pp. 989-1015.

Parks, D.M., (1991), *Defect assessment in components – fundamentals and applications*, Mechanical Engineering Publications, London, UK, pp. 205-231.

Raju, I.S. and Newman, Jr., J.C., (1979), "Stress-intensity factors for a wide range of semi-elliptical surface cracks in finite-thickness plates", *Engineering Fracture Mechanics*, Vol. 11, pp. 817-829.

Rice, J.R., (1967), "Mechanics of crack tip deformation and extension by fatigue", *Fatigue crack propagation*, ASTM STP-415, American Society for Testing and Materials, Philadelphia, pp. 247-309.

Rice, J.R., (1974), "Limitations to the small scale yielding approximation for crack tip plasticity", *Journal of Mechanical Physics and Solids*, Vol. 22, pp. 17-26.

Shah, R.C. and Kobayashi, A.S., (1971), "Stress intensity factor for an elliptical crack under arbitrary normal loading", *Engineering Fracture Mechanics*, Vol. 3, pp. 71-96.

Segedin, C.M., (1967), "Some three-dimensional mixed boundary value problems in elasticity", University of Washington, Department of Aeronautics and Astronautics Report No. 67-3.

Sham, T.L., (1991), "The determination of the elastic T -term using higher order weight functions", *International Journal of Fracture*, Vol. 48, pp. 81-102.

Sherry, A.H., France C.C. and Goldthorpe, M.R., (1995), "Compendium of T -stress solutions for two and three dimensional cracked geometries", *Fatigue and Fracture of Engineering Materials and Structures*, Vol. 18, pp. 141-155.

Shih, C.F. and German, M.D., (1981), "Requirements for a one parameter characterization of crack tip fields by the HRR singularity", *International Journal of Fracture*, Vol. 17, pp. 27-43.

Shih, C.F., Moran, B. and Nakamura, T., (1986), "Energy release rate along a three-dimensional crack front in a thermally stressed body", *International Journal of Fracture*, Vol. 30, pp. 79-102.

Shih, C.F., O'Dowd, N.P. and Kirk, M.T., (1993), "A framework for quantifying crack tip constraint", In *Constraint Effects in Fracture*, ASTM STP-1171, edited by Hackett E. M., Schwalbe K.H. and Dodds R.H., American Society of Testing and Materials, Philadelphia, pp. 2-20.

Shiratori, M. and Miyoshi, T., (1992), “Weighting function for a quarter-elliptical crack in a plate under basic mode of stress distribution”, In *Stress intensity factors handbook*, Vol. 3, edited by Murakami *et al*, Pergamon Press, pp. 591-597.

Timoshenko, S.P. and Goodier, J.N., (1970), *Theory of Elasticity*, McGraw-Hill, New York

Tong, J., (2002), “*T*-stress and its implications for crack growth”, *Engineering Fracture Mechanics*, Vol. 69, pp. 1325-1337.

Wang, X., (2003), “Elastic *T*-stress solutions for semi-elliptical surface cracks in finite thickness plates”, *Engineering Fracture Mechanics*, Vol. 70, pp. 731-756.

Wang, X., (2004), “Elastic *T*-stress solutions for penny-shaped cracks under tension and bending”, *Engineering Fracture Mechanics*, Vol.71, pp. 2283-2298.

Wang, X. and Lambert, S.B., (1995), “Local weight functions for semi-elliptical surface cracks in finite thickness plates”, *Theoretical and Applied Fracture Mechanics*, Vol. 23, pp. 199-208.

Williams, M.L., (1957), “On the stress distribution at the base of a stationary crack”, *ASME Journal of Applied Mechanics*, Vol. 24, pp. 111-114.

Zhao, L.G., Tong, J. and Byrne, J., (2001), "Stress intensity factor K and the elastic T -stress for corner cracks", *International Journal of Fracture*, Vol. 109, pp. 209-225.

Zheng, X.J., Glinka, G. and Dubey, R.N., (1996), "Stress intensity factors and weight functions for a corner crack in a finite thickness plate", *Engineering Fracture Mechanics*, Vol. 54, No. 1, pp. 49-61.

Zienkiewicz, O.C. and Taylor, R.L., (1991), *The Finite Element Method*, 4th Edition, McGraw-Hill, London

Appendix

1. Elliptic Integrals and Jacobian Elliptic Functions

The elliptical integrals and Jacobian elliptic functions are summarized here following Byrd (1971).

1.1 Elliptic Integrals

Elliptic integral of the first kind

$$u = F(\varphi, k) = \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \int_0^x \frac{1}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} dv \quad (\text{A.1})$$

Elliptic integral of the second kind

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^x \frac{\sqrt{1 - k^2 v^2}}{\sqrt{1 - v^2}} dv \quad (\text{A.2})$$

where φ is the amplitude, $\varphi = \text{am} u$, and $x = \sin \varphi$; k and k' are called the modulus and the complementary modulus of Jacobian elliptic functions, respectively.

$$k'^2 = \frac{a^2 - b^2}{a^2} \quad k'^2 = 1 - k^2 \quad (a > b > 0) \quad (\text{A.3})$$

For the amplitude $\varphi = \frac{\pi}{2}$, the elliptic integrals are said to be complete.

Complete elliptic integral of the first kind

$$K = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \int_0^1 \frac{1}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} dv \quad (\text{A.4})$$

Complete elliptic integral of the second kind

$$E = E\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^1 \frac{\sqrt{1 - k^2 v^2}}{\sqrt{1 - v^2}} dv \quad (\text{A.5})$$

1.2 Jacobian Elliptic Functions

$$\begin{aligned} x &= \sin(amu) = \operatorname{sn}u \\ \sqrt{1 - x^2} &= \cos(amu) = \operatorname{cn}u \\ \sqrt{1 - k^2 x^2} &= \sqrt{1 - k^2 \operatorname{sn}^2 u} = \operatorname{dn}u \end{aligned} \quad (\text{A.6})$$

$$\operatorname{sc}u = \frac{\operatorname{sn}u}{\operatorname{cn}u} \quad \operatorname{sd}u = \frac{\operatorname{sn}u}{\operatorname{dn}u} \quad \operatorname{cd}u = \frac{\operatorname{cn}u}{\operatorname{dn}u} \quad (\text{A.7})$$

2. Derivatives

2.1 Partial Derivatives of ellipsoidal Coordinates

From the relationship of ellipsoidal coordinates and Cartesian coordinates as summarized in Section 3.1.1, the partial derivatives of ellipsoidal coordinates are derived.

$$\frac{\partial \xi}{\partial x} = \frac{2\xi(b^2 + \xi)x}{(\xi - \eta)(\xi - \zeta)} \quad (\text{A.8})$$

$$\frac{\partial \xi}{\partial y} = \frac{2\xi(a^2 + \xi)y}{(\xi - \eta)(\xi - \zeta)} \quad (\text{A.9})$$

$$\frac{\partial \xi}{\partial z} = \frac{2z(a^2 + \xi)(b^2 + \xi)}{(\xi - \eta)(\xi - \zeta)} \quad (\text{A.10})$$

2.2 Partial Derivatives of Potential Functions

Following Kassir and Sih (1975), the general form of potential function $V^{(n)}$ is given by (A.11).

$$V^{(n)} = \int_{\xi}^{\infty} \frac{\omega^n(s) ds}{\sqrt{Q(s)}} \quad (\text{A.11})$$

where n is a positive integer. $\omega(s)$ and $Q(s)$ are defined as :

$$\omega(s) = 1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{s} \quad (\text{A.12})$$

$$Q(s) = s(a^2 + s)(b^2 + s) \quad (\text{A.13})$$

The partial derivatives of some of these potential functions and ellipsoidal coordinates are list below.

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (\text{A.14})$$

$$V^{(1)} = \int_{\xi}^{\infty} \frac{\omega(s) ds}{\sqrt{Q(s)}} = \int_{\xi}^{\infty} \left[1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{s} \right] \frac{ds}{\sqrt{Q(s)}} \quad (\text{A.15})$$

$$\frac{\partial V^{(1)}}{\partial x} = -2x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)\sqrt{Q(s)}} \quad (\text{A.16})$$

$$\frac{\partial^2 V^{(1)}}{\partial x^2} = -2 \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)\sqrt{Q(s)}} + \frac{4x^2 \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.17})$$

$$\frac{\partial V^{(1)}}{\partial y} = -2y \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} \quad (\text{A.18})$$

$$\frac{\partial^2 V^{(1)}}{\partial y^2} = -2 \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} + \frac{4y^2 \sqrt{Q(\xi)}}{(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.19})$$

$$\frac{\partial^2 V^{(1)}}{\partial x \partial y} = \frac{4xy \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)(\xi - \eta)(\xi - \zeta)} \quad (\text{A.20})$$

$$\begin{aligned} -\frac{\partial^2 V^{(1)}}{\partial z^2} = \nabla_1^2 V^{(1)} = & -2 \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)\sqrt{Q(s)}} - 2 \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)\sqrt{Q(s)}} \\ & + \frac{4x^2 \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} + \frac{4y^2 \sqrt{Q(\xi)}}{(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \end{aligned} \quad (\text{A.21})$$

$$V^{(2)} = \int_{\xi}^{\infty} \frac{\omega(s)^2 ds}{\sqrt{Q(s)}} \quad (\text{A.22})$$

$$\frac{\partial V^{(2)}}{\partial x} = -4x \int_{\xi}^{\infty} \frac{\omega(s) ds}{(a^2 + s)\sqrt{Q(s)}} \quad (\text{A.23})$$

$$\frac{\partial}{\partial x} \left(\frac{\partial V^{(2)}}{\partial x} \right) = -4 \int_{\xi}^{\infty} \left[1 - \frac{3x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{s} \right] \frac{ds}{(a^2 + s)\sqrt{Q(s)}} \quad (\text{A.24})$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial V^{(2)}}{\partial x} \right) = 24x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)^2 \sqrt{Q(s)}} - \frac{16x^3 \sqrt{Q(\xi)}}{(a^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.25})$$

$$\frac{\partial^2}{\partial x \partial y} \left(\frac{\partial V^{(2)}}{\partial x} \right) = 8y \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16x^2 y \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (b^2 + \xi)(\xi - \eta)(\xi - \zeta)} \quad (\text{A.26})$$

$$\frac{\partial}{\partial y} \left(\frac{\partial V^{(2)}}{\partial x} \right) = 8xy \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} \quad (\text{A.27})$$

$$\frac{\partial^2}{\partial y^2} \left(\frac{\partial V^{(2)}}{\partial x} \right) = 8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.28})$$

$$\begin{aligned} -\frac{\partial^2}{\partial z^2} \left(\frac{\partial V^{(2)}}{\partial x} \right) = \nabla_1^2 \left(\frac{\partial V^{(2)}}{\partial x} \right) = & 24x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)^2 \sqrt{Q(s)}} + 8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} \\ & - \frac{16x^3 \sqrt{Q(\xi)}}{(a^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} - \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \end{aligned} \quad (\text{A.29})$$

$$\frac{\partial V^{(2)}}{\partial y} = -4y \int_{\xi}^{\infty} \frac{\omega(s)ds}{(b^2 + s)\sqrt{Q(s)}} \quad (\text{A.30})$$

$$\frac{\partial}{\partial x} \left(\frac{\partial V^{(2)}}{\partial y} \right) = 8xy \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} \quad (\text{A.31})$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial V^{(2)}}{\partial y} \right) = 8y \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16x^2 y \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (b^2 + \xi)(\xi - \eta)(\xi - \zeta)} \quad (\text{A.32})$$

$$\frac{\partial^2}{\partial x \partial y} \left(\frac{\partial V^{(2)}}{\partial y} \right) = 8x \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} - \frac{16xy^2 \sqrt{Q(\xi)}}{(a^2 + \xi)(b^2 + \xi)^2 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.33})$$

$$\frac{\partial}{\partial y} \left(\frac{\partial V^{(2)}}{\partial y} \right) = -4 \int_{\xi}^{\infty} \frac{\omega(s)ds}{(b^2 + s)\sqrt{Q(s)}} + 8y^2 \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)^2 \sqrt{Q(s)}} \quad (\text{A.34})$$

$$\frac{\partial^2}{\partial y^2} \left(\frac{\partial V^{(2)}}{\partial y} \right) = 24y \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)^2 \sqrt{Q(s)}} - \frac{16y^3 \sqrt{Q(\xi)}}{(b^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} \quad (\text{A.35})$$

$$\begin{aligned} -\frac{\partial^2}{\partial z^2} \left(\frac{\partial V^{(2)}}{\partial y} \right) &= \nabla_1^2 \left(\frac{\partial V^{(2)}}{\partial y} \right) = 8y \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}} + 24y \int_{\xi}^{\infty} \frac{ds}{(b^2 + s)^2 \sqrt{Q(s)}} \\ &\quad - \frac{16x^2 y \sqrt{Q(\xi)}}{(a^2 + \xi)^2 (b^2 + \xi)(\xi - \eta)(\xi - \zeta)} - \frac{16y^3 \sqrt{Q(\xi)}}{(b^2 + \xi)^3 (\xi - \eta)(\xi - \zeta)} \end{aligned} \quad (\text{A.36})$$

3. Integrals

In order to calculate the integrals in the formula of the derivatives of potential functions, some elliptic integrals have to be evaluated first. Fortunately, some basic integrals were given by Byrd *et al* (1971) and are shown below.

$$\int_y^{\infty} \frac{dt}{(t - c^*)^m \sqrt{(t - a^*)(t - b^*)(t - c^*)}} = \frac{g^*}{(a^* - c^*)^m} \int_0^{u_1} s n^{2m} u du \quad (\text{A.37})$$

$$\int_y^{\infty} \frac{dt}{(t-b^*)^m \sqrt{(t-a^*)(t-b^*)(t-c^*)}} = \frac{g^*}{(a^*-c^*)^m} \int_0^{u_1} s d^{2m} u du \quad (A.38)$$

$$\int_y^{\infty} \frac{dt}{(t-a^*)^m \sqrt{(t-a^*)(t-b^*)(t-c^*)}} = \frac{g^*}{(a^*-c^*)^m} \int_0^{u_1} t n^{2m} u du, \quad y \neq a \quad (A.39)$$

$$\int_y^{\infty} \frac{dt}{(t-c^*)(t-b^*) \sqrt{(t-a^*)(t-b^*)(t-c^*)}} = \frac{g^*}{(a^*-c^*)^2} \int_0^{u_1} s n^2 u s d^2 u du \quad (A.40)$$

$$\int s n^2 u du = \frac{1}{k^2} [u - E(u)] \quad (A.41)$$

$$\int s n^4 u du = \frac{1}{3k^4} [(2+k^2)u - 2(1+k^2)E(u) + k^2 s n u c n u d n u] \quad (A.42)$$

$$\int s d^2 u du = \frac{1}{k^2 k'^2} [E(u) - k'^2 u - k^2 s n u c d u] \quad (A.43)$$

$$\int s d^4 u du = \frac{1}{3k^4 k'^4} [2(2k^2-1)E(u) + k'^2 (2-3k^2)u - k^2 s n u c d u (k'^2 n d^2 u + 4k^2 - 2)] \quad (A.44)$$

$$\int s n^2 u s d^2 u du = \frac{1}{k'^2 k^4} [(1+k'^2)E(u) - 2k'^2 u - k^2 s n u c d u] \quad (A.45)$$

$$\int t n^2 u du = \frac{1}{k'^2} [d n u t n u - E(u)] \quad (A.46)$$

where

$$s n^2 u = \frac{a^*-c^*}{t-c^*}, \quad k^2 = \frac{b^*-c^*}{a^*-c^*}, \quad g^* = \frac{2}{\sqrt{a^*-c^*}} \quad (A.47a)$$

$$\varphi = a m u_1 = \sin^{-1} \sqrt{\frac{a^*-c^*}{y-c^*}}, \quad s n u_1 = \sin \varphi, \quad \infty > y \geq a^* > b^* > c^* \quad (A.47b)$$