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TILTED ALGEBRAS OF TYPE

$\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{BC}_n$

by

© Oscar E. Roldan, B.Sc., Licenciatura

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfilment
of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

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January 31, 1983

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in partial fulfilment of the degree of
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ABSTRACT

Let A be a finite dimensional algebra over a commutative field K .
 A (finite-dimensional) A -module T_A is said to be a tilting module if:

- (1) $\text{pd } T_A \leq 1$.
- (2) $\text{Ext}_A^1(T, T) = 0$.
- (3) There is an exact sequence $0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$ with T' and T'' direct sums of summands of T_A .

If A is a hereditary algebra, the algebras of the form $\text{End } T_A$, where T_A is a tilting module, are called tilted algebras. Here, we study the special case of A being the tensor algebra $T(M)$ of a K -modulation M , where the valued graph of M is $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$, and $\tilde{BC}_n$. For these cases, we classify the tilted algebras in terms of their bounden K -modulations.

We also show that the simply connected algebras (finite-dimensional, basic, of finite representation type and with a simply connected Auslander-Reiten graph) arising from admissible grading on the graphs $\tilde{B}_n, \tilde{C}_n, \tilde{B}_n, \tilde{C}_n$ and $\tilde{BC}_n$ coincide with the tilted algebras of these types. The cases $\tilde{B}_n$ and $\tilde{C}_n$ were already considered in a joint paper with I. Assem [6].

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$\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$, AND $\tilde{BC}_n$

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INTRODUCTION

Let A be a finite dimensional algebra over a commutative field K and let $\text{mod } A$ denote the category of finite-dimensional right A -modules. A module T in $\text{mod } A$ is called a tilted module [27] if:

- (1) $\text{pd } T_A \leq 1$.
- (2) $\text{Ext}_A^1(T, T) = 0$.
- (3) There is an exact sequence $0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$ with T', T'' being direct sums of summands of T_A .

The subcategory $T(T_A)$ of all modules generated by T_A and the subcategory $F(T_A)$ of all modules M satisfying $\text{Hom}(T_A, M_A) = 0$ give a torsion theory $(T(T_A), F(T_A))$ in $\text{mod } A$, and by considering the subcategories of $\text{mod } B$, defined by,

$$X(T_A) = \{N_B / N_B \otimes T = 0\} \text{ and } V(T_A) = \{N_B / \text{Tor}_1^B(N, T) = 0\}$$

we have the category equivalences $X(T_A) \cong F(T_A)$ and $V(T_A) \cong T(T_A)$.

Moreover ${}_B T$ is a tilting module and there is a canonical isomorphism $A \cong \text{End}_B T_A$. These properties of tilting modules were proved by D.Happel and C.M. Ringel [27]; they were obtained in a more restrictive context, by Brenner and Butler [16].

If A is a hereditary algebra, the algebras of the form $\text{End } T_A$, where T_A is a tilting module, are called tilted algebras. This notion was introduced by D.Happel and C.M. Ringel and seems to be of ever increasing importance. (see [12],[14],[15],[17],[29],[31]).

For this type of algebras the torsion theory $(X(T_A), V(T_A))$ is splitting, that is to say, every indecomposable B -module is either in $X(T_A)$ or in $V(T_A)$ [27]. Using this property, a rather good description of $\text{mod } B$ and the Auslander-Reiten graph of B is possible.

The notion of grading of a tree and some of its properties appears in [14], where Bongartz and Gabriel introduced this notion in order to deal with finite-dimensional, basic, connected algebras of finite representation type with a simply connected Auslander-Reiten graph.

Let (T, d) be a valued tree. A grading of T is a function $g: T \rightarrow \mathbb{N}$ such that:

- i) $g^{-1}(0) \neq \emptyset$
- ii) $g(i) - g(j) \equiv 1 \pmod{2}$, whenever i, j are neighbours in T ,

First, using the methods and results of [3] and [6], we prove that the simply connected algebras arising from admissible grading of the tree $T \in \{B_n, C_n, \tilde{B}_n, \tilde{C}_n, B\tilde{C}_n\}$ are exactly the tilted algebras of type T with a complete slice in the preprojective component. The Dynkin trees B_n and C_n were already considered in [6].

The second result of this thesis is the classification, in terms of their bounden K -modulations, of those tilted algebras of types $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$, and $B\tilde{C}_n$. The methods and results of [3] and [6] are used here again.

The results are collected in the following theorems.

Theorem (1): The following assertions are equivalent:

- (I) The finite-dimensional algebra A arises from a grading of finite representation type of the tree B_n .
- (II) The bounden K -modulation of the algebra A satisfies the property (B) of [3] (see also (II.1.5.1)) and moreover, does not contain

a bounden K-submodulation of one of the forms:

$$a) \quad E \xrightarrow{X} F_1 \rightarrow E \cdots \cdots E \xrightarrow{Y} F_j \rightarrow E$$

where X and Y equal either E^E_E or 0, and $t \geq 4$.

$$b) \quad E \leftarrow E \leftarrow E \cdots \cdots E \xrightarrow{E^E_E} F \rightarrow E \cdots \cdots E \leftarrow E \leftarrow E$$

$$c) \quad F \cdots \cdots E \rightarrow E \cdots \cdots E \leftarrow E \leftarrow E \leftarrow E$$

↓
E

$$d) \quad F \cdots \cdots E \leftarrow E \cdots \cdots E \rightarrow E \rightarrow E \rightarrow E$$

↑
E

where E and F are as in [3], and we assume that there is no other relation besides the relations shown.

Theorem (2): The following assertions are equivalent:

(I) The finite-dimensional algebra A arises from a grading of finite representation type of type C_n .

(II) A is a tilted algebra of type C_n .

(III) The bounden K-modulation of the algebra A satisfies the property (γ) of [3] (see also (II.1.5.3)) and moreover, does not contain any bounden K-submodulation of one of the forms:

$$a) \quad F \xrightarrow{X} F_1 \rightarrow F \cdots \cdots F \xrightarrow{Y} F_j \rightarrow F$$

where X and Y equal either F^F_F or 0.

$$b) \quad F \leftarrow F \leftarrow F \cdots \cdots F \xrightarrow{F^F_F} E \rightarrow F \cdots \cdots F \leftarrow F \leftarrow F$$

$$c) \quad F \cdots \cdots E \rightarrow E \cdots \cdots E \leftarrow E \leftarrow E \leftarrow E$$

↓
E

$$d) \quad F \cdots \cdots E \leftarrow E \cdots \cdots E \rightarrow E \rightarrow E \rightarrow E$$

↑
E

where E and F are as in [3], and we assume that there is no other relation besides the relations shown,

Theorem (3): Let A be a finite-dimensional K -algebra, then the following statements are equivalent:

(I) A is a tilted algebra of type $\tilde{A}_n$ with $n \geq 2$.

(II) The bounden K -modulation $(M, R) = (F_{i, i}, M_j, R)$ of A satisfies the following properties:

($\tilde{\alpha}_1$) G has either one non-oriented cycle C with at least three vertices and one distinguished arrow denoted by $i' \rightarrow i''$ or a proper full subgraph $i' \xrightarrow{(2,2)} i''$ (but not both).

($\tilde{\alpha}_2$) For all i , the F_i are equal: $F_i = E$. For all $i \neq i', j \neq i''$, $i' M_j = E E$ or 0 . If G contains C , $i' M_{i''} = E E \epsilon E$, while if G contains $i' \xrightarrow{(2,2)} i''$, then $i' M_{i''} = E E \oplus E E \epsilon E$, where in both cases the right actions ϵE is given by a field automorphism ϵ of E .

($\tilde{\alpha}_3$) Each vertex of G has at most four neighbours.

($\tilde{\alpha}_4$) If G contains a subgraph $i' \xrightarrow{(2,2)} i''$ and if i' or i'' has two neighbours, then either

is a bounden K -submodulation of (M, R) .

($\tilde{\alpha}_5$) If $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) . Moreover if G contains $i' \xrightarrow{(2,2)} i''$, then $i \neq i'$ and $i \neq i''$.

(α_6) If $i \in G$ has three neighbours, then either

is a bounden K-submodulation of (M, R) . If G contains a subgraph $i' \xrightarrow{(2,2)} i''$, and i and j are the neighbours of i' or i'' then either

is a bounden K-submodulation of (M, R) . *

(α_7) All relations are of length two and all are zero-relations or (ϕ) -relations (see (α_4)). In addition, in C , the number of oriented clockwise relations equals the number of oriented counterclockwise relations.

(α_0) There is no bounden K-modulation of (M, R) of one of forms :

$$a) F_1 \xrightarrow{X} F_2 \xrightarrow{Y} F_3 \cdots F_{t-2} \xrightarrow{Y} F_{t-1} \xrightarrow{X} F_t,$$

where X and Y equal either $E E$ or 0 , and $t \geq 4$.

$$b) F_i \cdots F_{j_1} \xrightarrow{E} F_{j_2} \xrightarrow{E} F_{j_3} \text{ nor}$$

$$F_i \xrightarrow{E} F_{j_1} \xrightarrow{E} F_{j_2} \xrightarrow{E} F_{j_3}.$$

where either i belongs to C or to $i' \xrightarrow{(2,2)} i''$ and j_2 does not belong to either one.

* We always assume that $i' R_{i''} \oplus j' R_{i''} = i' M_{i''}$ or $i' R_i \oplus i'' R_j = i' M_{i''}$.

when there are no relations on C and i_1, i_2, i_3, i_4 belong to C .

In all cases we assume that there is no other relation besides the relation shown.

Theorem (4): Let A be a finite-dimensional K -algebra. Then the following statements are equivalent:

(I) A is a tilted algebra of type $\tilde{B}_n$.

(II) The algebra A or A^{op} arises from an admissible grading of the tree $\tilde{B}_n$.

(III) The bounden K -modulation $(M, R) = (F_i, {}_iM_j, R)$ of A satisfies the following properties:

($\tilde{\beta}_1$) G is a tree with two distinguished vertices i' and i'' . If the non-oriented path C from i' to i'' has length one, G has a unique subgraph $i' \xrightarrow{(2,2)} i''$.

($\tilde{\beta}_2$) For all $i \neq i', i''$, the F_i are equal: $F_i = E$; furthermore, $[E: F_{i'},] = [E: F_{i''], } = 2$. For all i, j , ${}_iM_j = F_j E_{F_j}$ or 0 .

($\tilde{\beta}_3$) Each vertex of G has at most four neighbours.

($\tilde{\beta}_4$) The vertex i' , or i'' has at most two neighbours, in which case either

$E \xrightarrow{E} F_{i'} \xrightarrow{E} E$ or $E \xrightarrow{E} F_{i''} \xrightarrow{E} E$ is a bounden K -submodulation of (M, R) .

($\tilde{\beta}_5$) If a vertex $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) .

($\tilde{\beta}_6$) If $i \in G$ has three neighbours, then either

is a bounden K-submodulation of (M, R) .

($\tilde{\beta}_7$) All relations are of length two and all are zero-relations,

(μ)-relations or ($\tilde{\mu}$)-relations. (see ($\tilde{\beta}_4$)).

($\tilde{\beta}_0$) There is no bounden K-submodulation of (M, R) of one of the forms:

where X and Y equal either $E^E E$, $E^E F_{i_1}$, $F_{i_1} E^E$ or 0, and $t \geq 4$.

where X and Y equal either $E^E E$ or 0.

where i is on the path C and j_2 is not.

where $i = i'$ or i'' .

In all cases we assume that there is no other relation besides the relations shown.

Theorem (5): Let A be a finite-dimensional K -algebra, then the following statements are equivalent:

(I) A is a tilted algebra of type $\tilde{C}_n$.

(II) The algebra A or A^{op} arises from an admissible grading of the tree $\tilde{C}_n$.

(III) The bounden K -modulation $(M, R) = (F_i, {}_iM_j, R)$ of A satisfies the following properties:

($\tilde{\gamma}_1$) G is a tree with two distinguished vertices i' and i'' .

If the non-oriented path C from i' to i'' has length one, G has a unique subgraph $i' \xrightarrow{(2,2)} i''$.

($\tilde{\gamma}_2$) For all $i \neq i', i''$, the F_i are equal: $F_i = F$, furthermore

$[F_{i'}, : F] = [F_{i''}, : F] = 2$. For all $i \neq i', i''$; $j \neq i', i''$, ${}_iM_j = {}_F F_F$

or 0, for all $i \neq i', i''$; $j = i', i''$, $iM_j = F(F_j)F_j$ or 0, for all $j \neq i', i''$; $i = i', i''$, $iM_j = F_1(F'_1)F$ or 0. Finally, if G contains $i' \xrightarrow{(2,2)} i''$, then $iM_{i''} = F_{i'}(F_{i'})F \otimes F(F_{i''})F_{i''}$.

($\tilde{\gamma}_3$) Each vertex of G has at most four neighbours.

($\tilde{\gamma}_4$) The vertex i' or i'' has at most two neighbours, in which case, if $iM_{i''} = 0$ then either

$F \xrightarrow{F} F_{i'} \xrightarrow{F} F$ or $F \xrightarrow{F} F_{i''} \xrightarrow{F} F$ is a bounden K-submodulation of (M, R) .

If $iM_{i''} \neq 0$ then either

$F_{i'} \xrightarrow{F(F_{i'})} F \xrightarrow{F} F_{i''}$ or $F \xrightarrow{F(F_{i''})} F_{i''} \xrightarrow{F} F_{i'}$ is a bounden

K-submodulation of (M, R) .

($\tilde{\gamma}_5$) If a vertex $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) .

($\tilde{\gamma}_6$) If $i \in G$ has three neighbours, then either

is a bounden K-submodulation of (M, R) .

($\tilde{\gamma}_7$) All relations are of length two and all are zero-relations,

(π)-relations or ($\tilde{\pi}$)-relations. (see ($\tilde{\gamma}_4$)).

($\tilde{\gamma}_0$) There is no bounden K-submodulation of (M, R) of one the forms: c) , d) , e) of Theorem (4) ,

$$a) \quad F_{i_1} \xrightarrow{\quad X \quad} F_{i_2} \xrightarrow{\quad} F_{i_3} \xrightarrow{\quad} \dots \xrightarrow{\quad} F_{i_{t-2}} \xrightarrow{\quad Y \quad} F_{i_{t-1}} \xrightarrow{\quad} F_{i_t} \quad , \text{ where}$$

X and Y equal either to $F_{i'}$, $(F_{i'})_F$, $F^{(F_{i''})}_{i''}$, F^F_F or 0, and $t \geq 4$

$$b) \quad F_{i_1} \xrightarrow{\quad X \quad} F_{i_2} \xrightarrow{\quad} F_{i_3} \xrightarrow{\quad} \dots \xrightarrow{\quad} F_{i_{r-1}} \xrightarrow{\quad F^F_F \quad} F_{i_r} \xrightarrow{\quad} F_{i_{r+1}} \xrightarrow{\quad} \dots \xrightarrow{\quad} F_{i_{t-2}} \xrightarrow{\quad Y \quad} F_{i_{t-1}} \xrightarrow{\quad} F_{i_t}$$

where X and Y equal either F^F_F or 0 .

In all cases we assume that there is no other relation besides the relations shown.

Theorem (6): Let A be a finite-dimensional K-algebra, then the following statements are equivalent:

(I) A is a tilted algebra of type $\tilde{BC}_n$.

(II) The algebra A or A^{op} arises from an admissible grading of the tree $\tilde{BC}_n$.

(III) The bounden K-modulation $(M, R) = (F_i, {}_iM_j, R)$ of A satisfies the following properties:

($\tilde{\gamma}_1$) G is a tree with two distinguished vertices i' and i'' .

If the non oriented path C from i' to i'' has length one, G has a unique subgraph $i' \xrightarrow{(1,4)} i''$ or $i' \xleftarrow{(1,4)} i''$.

($\tilde{\gamma}_2$) For all $i \neq i', i''$, the F_i are equal: $F_i = E$, and furthermore $[F_{i''}:E] = [E:F_{i''}] = 2$. For all $i \neq i''$, $j \neq i''$, ${}_iM_j = F_i E_{F_j}$ or 0. For

all i , ${}_iM_{i''} = F_i^{(F_{i''})}_{F_{i'}}$ or 0, ${}_{i''}M_i = F_{i''}^{(F_{i''})}_{F_{i'}}$.

($\tilde{\gamma}_3$) Each vertex of G has at most four neighbours.

($\tilde{\gamma}_4$) The vertex i' or i'' , has at most two neighbours, in

which case, if $_{i_1}M_{i_1}''=0$ and $_{i_1}M_{i_1}'=0$, then either

$F_{i_1} \xrightarrow{E} F_{i_1}' \rightarrow F_{i_j}$ or $F_{i_1} \xrightarrow{E} F_{i_1}'' \rightarrow F_{i_j}$ is a bounden K-submodulation of (M, R) .

If $_{i_1}M_{i_1}'' \neq 0$ or $_{i_1}M_{i_1}' \neq 0$, then either

$F_{i_1} \xrightarrow{(F_{i_1}'')} F_{i_1}' \rightarrow F_{i_j}$ or $F_{i_1} \xrightarrow{(F_{i_1}')} F_{i_1}'' \rightarrow F_{i_j}$ is a bounden K-submodulation of (M, R) .

($\tilde{B}_5$) If a vertex $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) .

($\tilde{B}_6$) If $i \in G$ has three neighbours, then either

$F_{i_1} \xrightarrow{F_{i_3}} F_{i_1}' \rightarrow F_{i_2}$ or $F_{i_1} \xrightarrow{F_{i_3}} F_{i_1}'' \rightarrow F_{i_2}$ is a bounden K-submodulation of (M, R) .

($\tilde{B}_7$) All relations are of length two and all are zero-relations, (π)-relations, (μ)-relations, (ζ)-relations or (ξ)-relations.

($\tilde{B}_8$) There is no bounden K-submodulation of (M, R) of one of the forms: c), d), e) of Theorem (4),

a) $F_{i_1} \xrightarrow{X} F_{i_2} \rightarrow F_{i_3} \dots F_{i_{t-2}} \xrightarrow{Y} F_{i_{t-1}} \rightarrow F_{i_t}$ where X and Y

equal either $E^E E$, $E^{(F_1'')} F_1''$, $E^{(F_1')} F_1'$, $F_1''^{(F_1'')} E$,
 $F_1'^{(F_1')} E$ or 0, and $t \geq 4$.

$$b) \overset{\text{X}}{\text{---}} F_{i_1} \rightarrow F_{i_2} \rightarrow F_{i_3} \text{---} \dots \text{---} F_{i_{r-1}} \leftarrow F_{i_r} \leftarrow F_{i_{r+1}} \text{---} \dots \text{---} \overset{\text{Y}}{\text{---}} F_{i_{t-2}} \rightarrow F_{i_{t-1}} \rightarrow F_{i_t}$$

where X and Y equal $E^E E$ or 0.

This thesis is divided into four chapters. The first is concerned with presenting the notions and results used in the sequel. Gradings on valued trees are studied in the second chapter and the general properties of tilting modules of hereditary algebras of types $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{BC}_n$ are discussed in the third. The main results are given in the fourth chapter.

CHAPTER I

PRELIMINARIES

In this chapter, we shall introduce the notations and collect certain well-known results and definitions that will be used in the sequel.

0. NOTATION.

(0.1) Throughout this work, we fix a commutative base field K ; algebras will be defined over K , and will be finite-dimensional and associative with unity, modules will usually be right modules and module homomorphisms will be written on the opposite side of scalars.

For any module X , we shall denote by $\text{End } X$ its endomorphism ring, by $\text{rad } X$ its radical, by $\text{soc } X$ its socle, by $|X|$ its length, and by $\text{pd } X$ its projective dimension. The class of all modules which are direct sums of direct summands of X will be denoted by $\text{add } X$. If X and Y are two modules, and $f: X \rightarrow Y$ is a module homomorphism, we shall simply call f a map. If X and Y are isomorphic modules, we write $X \simeq Y$. Finally, we shall denote by $[X]$ the isomorphism class of X .

(0.2) Given a K -algebra A , we shall denote the simple A -modules by $S(i)$, where i runs through a fixed index set, which in most of the cases will be $\{1, \dots, n\}$. $P(i)$ will be the projective A -module with $P(i)/\text{rad } P(i) \simeq S(i)$, and $I(i)$ will be the injective A -module with $\text{soc } I(i) \simeq S(i)$. On the occasions when it is desirable to call attention to the particular K -algebra under consideration, we shall write X_A for an A -module X .

(0.3) Given a module X , its dimension vector $\underline{\dim} X$ is defined as follows:

$$(\underline{\dim} X)_i = |\text{Hom}(P(i), X)_{\text{End } P(i)}|$$

We have that $(\dim X)_i$ is the number of composition factors of X isomorphic to $S(i)$ in any fixed composition series of X . Thus, $\dim X$ is an n -tuple of elements of $\mathbb{N}$ and is usually considered as a element of the vector space $\mathbb{Q}^n$ (or $\mathbb{R}^n$). The support $\text{Supp } X$ of X is the set of those indices i such that $(\dim X)_i \neq 0$.

(0.4) Let F and D be skew-fields containing a field K in their centers and finite dimensional over K . Let ${}_D M_F$ be a D - F -bimodule, K acting centrally on it. Let X and Y be finite-dimensional D and F spaces, respectively, and $({}_D M_F)^*$ the dual of ${}_D M_F$ (see (2.1)). To the F -linear map $\psi: X \otimes_D M \rightarrow Y$, we can attach a D -linear map $\bar{\psi}: X \rightarrow Y \otimes_F M^*$, which we call the adjoint of ψ (see [20]) ..

If $F \subseteq E$, $D \subseteq E$ for a skew-field E , ${}_F E_D$ will denote the bimodule with the canonical actions (given by multiplication) and ${}_F E_{\alpha D}$ will denote the bimodule where the right action by D is given by a field automorphism α of D , that is, for $x \in E$ and $d \in D$, $xd = x\alpha(d)$.

1. VALUED GRAPHS

(1.1) A valued graph (G, d) ([18], [20]) is a set of points, or vertices, together with a set $\underline{d} = \{(d_{ij}, d_{ji}) / d_{ij} \in \mathbb{N}, i, j \in G\}$ satisfying:

i) $d_{ii} = 0$ for all $i \in G$

ii) for every $i \in G$ there exists $f_i \in \mathbb{N}$ such that

$$d_{ij} f_j = d_{ji} f_i \text{ for all } i, j \in G.$$

The set $\underline{d}$ is called the valuation of G .

Note that $d_{ij} \neq 0$ if and only if $d_{ji} \neq 0$. In this case we call such a pair (i, j) an edge of value (d_{ij}, d_{ji}) of the graph and the points i, j are called neighbours. We denote an edge of value (d_{ij}, d_{ji}) by $i \xrightarrow{(d_{ij}, d_{ji})} j$ or simply $i \longleftrightarrow j$ if $d_{ij} = 1 = d_{ji}$.

We shall always assume $(G, \underline{d})$ to be connected, in the sense that for any $i, j \in G$, there is a sequence $i = i_0, i_1, \dots, i_k = j$ such that i_t and i_{t+1} are neighbours in G for $0 \leq t \leq k-1$.

A subgraph of $(G, \underline{d})$ is a valued graph $(G', \underline{d}')$ such that $G' \subseteq G$, and $d'_{ij} = d_{ij}$ for all $i, j \in G'$ whenever $d'_{ij} \neq 0$ in $(G', \underline{d}')$.

(1.2) An orientation of the valued graph $(G, \underline{d})$ is given by prescribing for each edge an order, indicated by an arrow: $i \xleftarrow{(d_{ij}, d_{ji})} j$ or $i \xrightarrow{(d_{ij}, d_{ji})} j$. A vertex $k \in G$ is said to be a sink (respectively a source) with respect to an orientation if, for every arrow $i \xrightarrow{(d_{ij}, d_{ji})} j$ we have $i \neq k$ (respectively $j \neq k$). We call an orientation admissible if there are no oriented cycles [20].

2. K - MODULATIONS AND REPRESENTATIONS

(2.1) Let K be a field and $(G, \underline{d})$ an oriented valued graph. A K-modulation M of $(G, \underline{d})$ is a set of skew-fields $F_i, i \in G$, together with an $F_i - F_j$ -bimodule ${}_i M_j$ for every arrow $i \xrightarrow{(d_{ij}, d_{ji})} j$, such that:

- i) K is contained in the center of F_i for each $i \in G$, and F_i is finite dimensional over K .
- ii) K operates centrally on every ${}_i M_j, (i, j \in G)$.
- iii) $\dim ({}_i M_j)_{F_j} = d_{ij}$, and $\dim ({}_i M_j)_{F_i} = d_{ji}$ are finite.

Since K is a central field, we have a bimodule isomorphism [20]

$$\text{Hom}_{F_1}(M_{ij}, F_1) \simeq \text{Hom}_{F_j}(M_{ij}, F_j),$$

and we consider the dual M_{ij}^* of M_{ij} as the F_1 - F_j -bimodule

$$\text{Hom}_{F_1}(M_{ij}, F_1).$$

(2.2) Let $M = (F_i, M_{ij})$ be a K -modulation of the valued graph $(G, \underline{d})$.

A representation (or module) $\underline{X} = (X_i, j \downarrow i)$, $\underline{X} = (X_i, i \downarrow j)_{i,j \in G}$ of M

is a set of finite dimensional right F_i -spaces X_i , $i \in G$, and

F_j -linear mappings: $j \downarrow i: X_i \otimes_{F_i} M_{ij} \rightarrow X_j$ for all arrows $i \xrightarrow{(d_{ij}, d_{ji})} j$.

A morphism $\underline{\alpha} = (\alpha_i)_{i \in G}$ from the representation $\underline{X} = (X_i, i \downarrow j)$ to the representation $\underline{Y} = (Y_i, i \downarrow j)$ is given by a set of F_j -linear mappings $\alpha_j: X_j \rightarrow Y_j$ such that, for every arrow $i \xrightarrow{(d_{ij}, d_{ji})} j$,

the diagram

$$\begin{array}{ccc} X_i \otimes_{F_i} M_{ij} & \xrightarrow{j \downarrow i} & X_j \\ \alpha_i \otimes i \downarrow & & \downarrow \alpha_j \\ Y_i \otimes_{F_i} M_{ij} & \xrightarrow{j \downarrow i} & Y_j \end{array}$$

is commutative.

The category of all finite dimensional representation of M will be denoted by $L(M)$, [18], [20].

3. TENSOR ALGEBRAS

(3.1) Let M be a K -modulation of $(G, \underline{d})$. The tensor algebra $T(M)$ is

constructed as follows [18], [20]: set $\Lambda = \prod_{i \in G} F_i$, $M = \bigoplus_{i,j} M_{ij}$ and

consider M as a Λ - Λ -bimodule via the projections $\Lambda \rightarrow F_i$ ($i \in G$).

Now, let $M^{(0)} = \Lambda$ and inductively $M^{(t+1)} = M^{(t)} \otimes M$, and put $T(M) = \bigoplus_{t \geq 0} M^{(t)}$.

Multiplication in $T(M)$ is defined through the canonical isomorphisms $M^{(i)} \otimes M^{(j)} \longrightarrow M^{(i+j)}$ and extended by distributivity.

(3.2) Let us note some well-known facts about tensor algebras:

$T(M)$ is a hereditary K -algebra, $L(M)$ is equivalent to the category $\text{mod } T(M)$ of finite-dimensional right $T(M)$ -modules. If A is a finite-dimensional K -algebra which is basic in the sense that the semi-simple part $\Lambda = A/\text{rad } A$ is the product of skew-field F_i :

$$\Lambda = F_1 \times F_2 \times \dots \times F_n,$$

then the Λ - Λ bimodule ${}_{\Lambda}\Lambda/\text{rad}^2 A$ can be uniquely decomposed as a direct sum of F_i - F_j -bimodules ${}_{i,j}M_j$ on which K acts centrally, $M = \bigoplus_{i,j} {}_{i,j}M_j$.

Each ${}_{i,j}M_j$ is a finite-dimensional left F_i -space of dimension $\dim_{F_i} ({}_{i,j}M_j) = d_{ij}$, and a finite-dimensional right F_j -space of dimension $\dim ({}_{i,j}M_j)_{F_j} = d_{ji}$.

In this way we have a K -modulation $M_A = (F_i, {}_{i,j}M_j)_{i,j \in G}$, where G is the oriented graph with valued edge $i \xrightarrow{(d_{ij}, d_{ji})} j$. Finally, note that

$$M_{T(M)} = M.$$

(3.3) A finite dimensional K -algebra A is said to be of finite representation type, if its category of finite-dimensional right modules $\text{mod } A$ has only a finite number of non-isomorphic indecomposable objects.

We shall say that A is of tame representation type provided A is not of finite representation type, whereas for any dimension d , there is a finite number of embedding functors $F_i: \text{mod } K[X] \rightarrow \text{mod } A$ such that all but a finite number of indecomposable A -modules of dimension d are of the form $F_i(M)$,

for some i and some indecomposable $K[X]$ -module M .

(3.4) Some valued graphs are of particular importance. These are the so-called Dynkin graphs:

and Euclidean graphs

The subscript in the Dynkin graph always refers to the number of vertices, while in Euclidean graphs it refers to one less than the number of vertices. The following result shows why these graph are significant.

(3.4.1) Theorem [18],[20] Let $(G, \underline{d})$ be a valued graph with a K -modulation M , then:

- a) $L(M)$ is of finite representation type if and only if $(G, \underline{d})$ is a Dynkin graph.
- b) $L(M)$ is of tame representation type if and only if $(G, \underline{d})$ is a Euclidean graph.

4. BOUNDEN K-MODULATION

(4.1) Let $(G, \underline{d})$ be an oriented valued graph, $M = (F_i, {}_iM_j)$ a K -modulation of $(G, \underline{d})$ and $T(M)$ its tensor algebra. An ideal $R \subseteq \text{rad}^2 T(M)$ is called a relation ideal, and the algebra $A = T(M)/R$ is then said to be given by the bounden K -modulation (M, R) of $(G, \underline{d})$. [1]. For each pair (i, j) of elements of G , the F_i - F_j bimodule ${}_iR_j = F_i R F_j$ is a relation on M , and is said to be of length t if ${}_iR_j \subseteq \text{rad}^t T(M)$ but ${}_iR_j \not\subseteq \text{rad}^{t+1} T(M)$. It is minimal if it is of length t where t is the length of the shortest path from i to j .

Let $i, j \in G$ be arbitrary and $P = (i \rightarrow i_1 \rightarrow i_2 \dots i_{t-1} \rightarrow j)$ be a path of G . We shall call bimodule of the path P the F_i - F_j -bimodule

$${}_iM_j^P = {}_iM_{i_1} \otimes {}_{i_1}M_{i_2} \otimes \dots \otimes {}_{i_{t-1}}M_j$$

We shall also agree to put ${}_iM_j^P = F_i$ if P is the trivial path at i (that is a path of length zero).

Let $\mathcal{P}(i, j)$ be the set of all paths from i to j . Then the (composed) bimodule from i to j is the F_i - F_j -bimodule

$${}_iM_j = \bigoplus_{P \in \mathcal{P}(i, j)} {}_iM_j^P$$

For a given pair of points $(i, j) \in G \times G$, the subbimodule ${}_i R_j$ of ${}_i \tilde{M}_j$ determines uniquely a bimodule epimorphism

$${}_i e_j : {}_i \tilde{M}_j \rightarrow {}_i \tilde{M}_j / {}_i R_j$$

which can be used to define the relation ${}_i R_j$: if ${}_i R_j$ is a (minimal) relation on M of length t from i to j , we denote this fact by a dotted line from i to j in (G, d) (with the K -modulation M), attached to which is the subbimodule ${}_i N_j = {}_i \tilde{M}_j / {}_i R_j$. We say that the submodulation of $P(i, j)$ induced by M has a relation of length t . In particular, if $i \rightarrow i_1 \rightarrow i_2 \dots i_{t-1} \rightarrow j$ is a subgraph of G and ${}_i N_j = 0$, we shall write

$$i \overset{0}{\curvearrowright} i_1 \rightarrow \dots \rightarrow i_{t-1} \rightarrow j$$

or just

$$i \overset{\cdot}{\curvearrowright} i_1 \rightarrow \dots \rightarrow i_{t-1} \rightarrow j$$

In the case that ${}_i N_j \neq 0$, we shall write

$$i \overset{{}_i N_j}{\curvearrowright} i_1 \rightarrow \dots \rightarrow i_{t-1} \rightarrow j$$

or more precisely

$$F_i \rightarrow F_{i_1} \rightarrow \dots \overset{{}_i N_j}{\curvearrowright} F_{i_{t-1}} \rightarrow F_j$$

If ${}_i R_j = \bigoplus_{R \in P} {}_i M_j^R$ for a subset P of $P(i, j)$, ${}_i R_j$ is called a zero-relation.

(4.2) Let (M, R) be a bounden K -modulation of the oriented valued graph (G, d) . Let $\underline{X} = (X_i, {}_i\psi_j)_{i,j \in G}$ be a representation of M . Let i and j be two vertices of G and $P = (i \rightarrow i_1 \rightarrow i_2 \dots i_{t-1} \rightarrow j)$ an element of $P(i, j)$. The map ${}_j\psi_i^P : X_i \otimes_{{}_iM_j}^P \rightarrow X_j$ is the composition:

$${}_j\psi_i^P = {}_j\psi_{i_{t-1}} ({}_{i_{t-1}}\psi_{i_{t-2}} \otimes 1) \dots ({}_{i_2}\psi_{i_1} \otimes 1 \otimes \dots \otimes 1) ({}_{i_1}\psi_i \otimes 1 \otimes \dots \otimes 1).$$

We then obtain the summation map ${}_j\tilde{\psi}_i = \sum_{P \in P(i, j)} {}_j\psi_i^P$.

The representation $\underline{X}$ is said to be bound by R , or to be a bound representation, if the associated $T(M)$ -module is annihilated by the ideal

$$R = \bigoplus_{i, j} {}_iR_j, \text{ in other words, if}$$

$${}_j\tilde{\psi}_i (X_i \otimes_{{}_iM_j} R_j) = 0$$

for all $i, j \in G$. We then have an equivalence between $\text{mod}(T(M)/R)$ and the category $L(M, R)$ of all representations of M bound by R .

The indecomposable projective bound representations of (M, R) are given as follows [1] [20]: let $i \in G$, and define $\underline{P}(i)$ by

$$\underline{P}(i)_j = {}_iM_j / {}_iR_j \text{ for } j \in G$$

The map ${}_j\psi_k : \underline{P}(i)_k \otimes_{{}_kM_j} \rightarrow \underline{P}(i)_j$ is the canonical map $({}_iM_k / {}_iR_k) \otimes_{{}_kM_j} \rightarrow {}_iM_j / {}_iR_j$.

The top of $\underline{P}(i)$ is identified with the simple representation $\underline{S}(i)$ defined

by $\underline{S}(i)_i = F_i$, and $\underline{S}(i)_j = 0$ if $j \neq i$. In this way we have a complete

list of the isomorphism classes of indecomposable projective, and simple bound

representations of (M, R) .

The indecomposable injective bound representations $\underline{I}(1)$ of (M, R) are constructed in a similar manner.

5. AUSLANDER-REITEN SEQUENCES

(5.1) Let X and Y be two non-zero A -modules. A map $f: X \rightarrow Y$ is said to be irreducible if [7]:

1) f is neither a section (split monomorphism), nor a retraction (split epimorphism).

ii) For any factorization $X \xrightarrow{g} Z \xrightarrow{h} Y$ of f either h is a retraction or g is a section.

Note that an irreducible map f is always either a monomorphism or an epimorphism, and if f is an irreducible epimorphism, then its kernel $\ker(f)$ is indecomposable. Similarly, if f is an irreducible monomorphism, its cokernel $\operatorname{coker}(f)$ is indecomposable.

(5.2) Thus, irreducible maps are those nonsplittable maps which have no non-trivial factorization. If X and Y are indecomposable A -modules, $\operatorname{rad}(X, Y)$ denotes the set of non-invertible maps from X to Y , and if $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$, with X_i, Y_j indecomposable, then we define $\operatorname{rad}(X, Y) = \bigoplus_{i,j} \operatorname{rad}(X_i, Y_j)$. Let $\operatorname{rad}^2(X, Y)$ be the set of maps of the form fg , where $g \in \operatorname{rad}(X, Z)$, $f \in \operatorname{rad}(Z, Y)$ for some Z .

Let $\operatorname{Irr}(X, Y) = \operatorname{rad}(X, Y) / \operatorname{rad}^2(X, Y)$; then this is an $\operatorname{End} Y$ - $\operatorname{End} X$ -bimodule which is annihilated on the left by $\operatorname{rad}(Y, Y)$, and on the right by $\operatorname{rad}(X, X)$. Thus $\operatorname{Irr}(X, Y)$ is an $F(Y) - F(X)$ -bimodule, where $F(Z) = \operatorname{End} Z / \operatorname{rad}(Z, Z)$ for an indecomposable A -module Z . It is called the bimodule of irreducible maps [35].

(5.3) Let X, Y and Z be A -modules. The short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \quad (1)$$

is called an Auslander-Reiten sequence [8] if and only if both X and Z are indecomposable and it satisfies the following two properties:

i) If $h: X \rightarrow X'$ is a map which is not a section, then there exists $h^1: Y \rightarrow X^1$ with $h = h^1 f$.

ii) If $k: Z^1 \rightarrow Z$ is a map which is not a retraction, then there exists $k^1: Z^1 \rightarrow Y$ with $k = g k^1$.

The existence and uniqueness of these sequences has been shown by Auslander and Reiten: given a nonprojective indecomposable A-module Z (or an indecomposable noninjective A-module X), there exists a unique Auslander-Reiten sequence (1). We shall call the sequence (1) the Auslander-Reiten sequence starting with X (or ending with Z), and the number of summands in a direct sum decomposition of Y into indecomposable summands will be denoted by $\alpha(Z)$.

In the sequence (1) we call X the translate of Z and write $X = \tau Z$. Conversely, Z is called the inverse translate of X and we write $Z = \tau^{-1} X$.

(5.4.) There is an explicit construction for τ , namely; we consider a minimal projective resolution of Z

$$P_1 \xrightarrow{f} P_0 \rightarrow Z \rightarrow 0$$

and apply the functor $* = \text{Hom}_A(-, A)$, then $\text{Tr} Z = \text{Cok } f^*$.

Let now $D = \text{Hom}_K(-, K)$ be the ordinary duality with respect to the base field K, then $\tau = D \text{Tr}$ and $\tau^{-1} = \text{Tr} D$.

We then have the Auslander-Reiten formula

$$D \underline{\text{Hom}}_A(\tau^{-1} X, Z) = \text{Ext}_A^1(Z, X) = D \overline{\text{Hom}}_A(X, \tau Z)$$

where $\underline{\text{Hom}}_A$ and $\overline{\text{Hom}}_A$ are the factor groups of homomorphisms modulo the subgroup of those homomorphisms which factor through a projective or an injective module, respectively.

(5.5.) Let Z be an indecomposable A -module. A map $g: Y \rightarrow Z$ is called a minimal right almost split map if:

- i) g is not a retraction.
- ii) If $h: Y \rightarrow Z$ is not a retraction, then there exists $h': Y' \rightarrow Y$ such that $gh^1 = h$.
- iii) Y is of the smallest possible length such that i) and ii) are satisfied.

For an indecomposable module Z there is a unique minimal right almost split map given as follows:

i) If Z is projective, the inclusion map of $\text{rad } Z$ into Z is minimal right almost split.

ii) If Z is not projective, the map $g: Y \rightarrow Z$ is minimal right almost split if and only if

$$0 \rightarrow \text{Ker } g \rightarrow Y \rightarrow Z \rightarrow 0$$

is an Auslander-Reiten sequence.

Let $g: Y \rightarrow Z$ be a non-zero map, then g is irreducible if and only if there exists $g^1: Y^1 \rightarrow Z$ such that

$$(g, g^1): Y \oplus Y^1 \rightarrow Z$$

is minimal right almost split. That is to say, g is irreducible if and only if g is not zero and there is a map $g^1: Y^1 \rightarrow Z$ such that we have an Auslander-Reiten sequence

$$0 \rightarrow X \rightarrow Y \oplus Y^1 \xrightarrow{(g, g^1)} Z \rightarrow 0$$

Minimal left almost split maps are defined and constructed in a dual manner.

6. AUSLANDER-REITEN GRAPHS

(6.1) Let A be an algebra. The Auslander-Reiten graph Γ_A of A has as a set of vertices Γ_0 , the set of isomorphism classes of indecomposable A -modules, and there is an arrow (oriented edge) $\alpha: [X] \rightarrow [Y]$ whenever there is an irreducible map from X to Y . This arrow is endowed with a valuation (d_α, d'_α) defined as follows: $d_\alpha = \dim_{\text{End } Y} \text{Irr}(X, Y)$ and $d'_\alpha = \dim_{\text{End } X} \text{Irr}(X, Y)$. By considering the Auslander-Reiten translation $\tau = D\text{Tr}$, we have that each arrow $\alpha: [X] \rightarrow [Y]$ with Y non-projective is paired with an arrow $\sigma\alpha: [\tau Y] \rightarrow [X]$. A topology is defined on Γ_A by considering it as a two-dimensional cell complex [32]. The existence of minimal (right and left) almost split maps, and their properties, show that Γ_A is locally finite, and thus its connected components contain at most countable many vertices. If A is connected, and of finite representation type, then Γ_A is connected. If A is connected and not of finite representation type, then no component of its Auslander-Reiten graph is finite.

A component C of Γ_A will be called preprojective provided there is no oriented cycle in C and any module of C is of the form $\tau^{-t}P$, for some $t \in \mathbb{N}$ and some indecomposable projective module P . If Y is a module in a preprojective component, then we have that Y has only finitely many non-isomorphic predecessors, where X is called a predecessor of Y if there exists an oriented path from $[X]$ to $[Y]$ in Γ_A . As a consequence, if X, Y are indecomposable modules, $\text{Hom}(X, Y) \neq 0$, and Y belongs to a preprojective component (or has only finitely many predecessors), then there is a path from $[X]$ to $[Y]$, and X and Y belong to the same component.

A path $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_m$ in Γ_A will be called a sectional path provided $X_{i-1} \neq \tau X_{i+1}$ for all $1 \leq i \leq m-1$.

If the algebra A is (finite dimensional) basic and connected of finite representation type with a simply - connected Auslander-Reiten graph, we shall call A a simply connected algebra.

(6.2) We consider for the graph Γ_A a K -modulation M_A defined as follows: to each vertex $i = [X]$ is associated the skew-field $F_i = \text{End} X / \text{rad} \text{End} X$, and to each arrow $\alpha: i \rightarrow j$ where $i = [X]$ and $j = [Y]$, the bimodule ${}_i M_j = ({}_F \text{Irr}(X, Y) {}_F)_i^*$, see [37].

(6.3) A translation graph (Γ_0, τ) is given by an oriented graph Γ_0 , and an injective function $\tau: \Gamma'_0 \rightarrow \Gamma_0$ defined on a subset Γ'_0 of Γ satisfying $(\tau X)^t = X^-$, where we denote by Z^+ the set of endpoints of arrows with starting point Z , and by Z^- the set of starting points of arrows with endpoint Z . ([14], [32]). The function τ is called the translation of (Γ_0, τ) . The vertices of Γ_0 on which τ is not defined are called projective; those on which τ^{-1} is not defined are called injective. If Y is not projective and there is an arrow $\alpha: X \rightarrow Y$, then there is a unique arrow $\tau X \rightarrow Y$ and this arrow will be denoted by $\sigma \alpha$.

A valued translation graph $(\Gamma_0, \underline{d}, \tau)$ is given by a translation graph (Γ_0, τ) together with a valuation $\underline{d}$ on Γ_0 such that $d_{\tau j i} = d_{j i}$, $d_{i \tau j} = d_{i j}$ whenever j is non-projective.

(6.4) A translation K -modulation $\tilde{\Gamma} = (\Gamma_0, F, M, \tau, D)$ is defined by [37]:

- i) A translation graph (Γ_0, τ) .
- ii) A map F associating to each vertex $i \in \Gamma_0$ a skew-field F_i such

that K is contained in the center of F_1 .

iii) A map M associating to each arrow $\alpha: i \rightarrow j$ an $F_i - F_j$ bimodule ${}_i M_j := M(\alpha)$, finite-dimensional on both sides and such that K acts centrally on each ${}_i M_j$.

iv) A map associating to each non-projective vertex i an isomorphism $F_i \xrightarrow{\sim} F_{\tau i}$.

v) A map associating to each arrow $\alpha: i \rightarrow j$, with j non-projective, an isomorphism of $F_i - F_j$ bimodules:

$${}_i D_j : {}_i M_j \rightarrow \text{Hom}_{F_j}({}_{\tau j} M_i, F_j(F_j)_{F_i}) \xrightarrow{\sim} \text{Hom}_{F_i}({}_{\tau} M_i, F_i(F_i)_{F_i})$$

where ${}_{\tau j} M_i$ is considered as a left F_j -module by means of the isomorphism $F_j \xrightarrow{\sim} F_{\tau j}$.

It is easy to verify that $M(\Gamma) = (F_i, {}_i M_j)_{i,j \in \Gamma_0}$ yields a K -modulation of $(\Gamma_0, \underline{d})$ where $d_{ij} = \dim_{F_i}({}_i M_j)$ and $d_{ji} = \dim({}_i M_j)_{F_j}$. Moreover, we have that ${}_i M_j$ is the dual bimodule of ${}_{\tau j} M_i$ whenever j is non-projective. Therefore for each arrow $\alpha: i \rightarrow j$, which j non-projective, there exists a non-degenerate bilinear form [20]

$$\chi_\alpha: F_j({}_{\tau j} M_i)_{F_i} \otimes_{F_i} F_i({}_i M_j)_{F_j} \rightarrow F_j(F_j)_{F_j}$$

where, again, ${}_{\tau j} M_i$ is considered as a left F_j -module using iv).

Let us note that $(\Gamma_0, \underline{d}, \tau)$ is a valued translation graph and that the bilinear form χ_α determines an element $c_i^j \in M(\sigma\alpha) \otimes M(\alpha)$ as follows: let $x_1, \dots, x_m$ be a basis of ${}_{F_i} M(\alpha)$, and $\xi_1, \xi_2, \dots, \xi_m$ be the dual basis of $M(\sigma\alpha)_{F_i}$ with respect to the bilinear form χ_α , then $c_i^j = \sum_{t=1}^m \xi_t \otimes x_t$ is

called the canonical element [24].

(6.5) Let $i, j \in \Gamma_0$, and $P = (i \xrightarrow{d_1} i_1 \xrightarrow{d_2} i_2 \rightarrow \dots \rightarrow i_m \xrightarrow{d_m} j)$ be a path in Γ_0 , then we have an $F_i - F_j$ -bimodule ${}_i M_j^P$, and hence an $F_i - F_j$ -bimodule ${}_i \tilde{M}_j$, see (4.1). We can thus define the tensor category $\mathcal{Q} \Gamma$ of Γ as the category having Γ_0 as its set of objects, and ${}_i \tilde{M}_j$ as the set of morphisms from i to j . The mesh category $\diamond \Gamma$ of Γ is the factor category of $\mathcal{Q} \Gamma$ modulo the ideal generated by the elements $\sum c_i^j$, the sum being taken over all arrows $i \rightarrow j$ with j non-projective.

Thus, the Auslander-Reiten graph Γ_A of an algebra A yields, in an obvious way, a translation K -modulation called the Auslander-Reiten K -modulation of A [37], and a valued translation graph.

7. TILTING MODULES AND TILTED ALGEBRAS

(7.1) Let A be an algebra. A module T_A is called a tilting module [27] if:

$$i) \text{ pd } T_A \leq 1$$

$$ii) \text{ Ext}_A^1(T, T) = 0$$

$$iii) \text{ There exists a short exact sequence } 0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$$

with T' and T'' direct sums of summands of T_A .

In order to verify in practice that an A -module T is a tilting module, we have the following results:

(7.1.1.) Lemma [13] Let T be an A -module, then $\text{pd } T \leq 1$ if and only if $\text{Hom}_A(I, \tau T) = 0$ for all injective I in $\text{mod } A$.

(7.1.2.) Lemma [27] Let T be an A -module such that $\text{pd } T \leq 1$, then $\text{DExt}_A^1(T, X) \xrightarrow{\sim} \text{Hom}_A(X, \tau T)$.

(7.1.3.) Proposition [13] Let T be an A -module satisfying i) and ii)

above, then T is a tilting module if and only if the number of isomorphism classes of indecomposable summands of T is equal to the number of isomorphism classes of simple A -modules.

(7.1.4.) Proposition [13] Let T be an A -module satisfying i) and ii), then there is some $X \in \text{mod } A$ such that $T \oplus X$ is a tilting module.

(7.1.5.) Lemma [13] Let T_A be a tilting module, then every indecomposable projective-injective module is a direct summand of T_A .

(7.1.6.) Definition: Let X be an indecomposable A -module, then the prefactor set of X is the set:

$P_+(X) = \{Y/Y \text{ is indecomposable, } \text{Hom}_A(Y, X) \neq 0, \text{Hom}_A(Y, \tau X) = 0\}$ and the postfactor set of X is the set:

$$P_-(X) = \{Y/Y \text{ is indecomposable, } \text{Hom}_A(X, Y) \neq 0, \text{Hom}(\tau^{-1}X, Y) = 0\}$$

This definition is motivated by the following observation: suppose that $B = \text{End } T_A$ for a tilting A -module T , and i, j are two vertices on the graph of B , with $T(i), T(j)$ the corresponding indecomposable summands of T_A , then if there is a non-zero path from i to j (that is, a path from i to j with non-zero relations), we obtain that

$$T(j) \in P_+(T(i)) \text{ and } T(i) \in P_-(T(j)).$$

(7.2.) We now consider an A -module T and denote by $\mathcal{T} = \mathcal{T}(T)$ the full subcategory generated by T , and by $\mathcal{F}$ the full subcategory of those A -module Y such that $\text{Hom}_A(X, Y) = 0$ for all A -modules X in $\mathcal{T}$. If T satisfies i) and ii) in (7.1.), then $(\mathcal{T}, \mathcal{F})$ forms a torsion theory for $\text{mod } A$. Recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod } A$ forms a torsion theory

in mod A if:

- i) For $X \in T$ and $Y \in F$, $\text{Hom}_A(X, Y) = 0$
- ii) $\text{Hom}_A(X, Y) = 0$ for all $Y \in F$ implies $X \in T$.
- iii) $\text{Hom}_A(X, Y) = 0$ for all $X \in T$ implies $Y \in F$.

We have that T is closed under extensions and epimorphic images, and F under submodules and extensions. For any A -module X , there is a unique maximal submodule ${}_T X$ of X in T , and $X/{}_T X$ in F . If for any A -module X , ${}_T X$ is a direct summand of X , the torsion theory (T, F) is called a splitting torsion theory.

Let now T_A be a tilting module and $B = \text{End } T_A$. There are two corresponding full subcategories of mod B defined by:

$$\begin{aligned} X(T_A) &= \{X_B / X_B \otimes_B T_A = 0\} \\ Y(T_A) &= \{Y_B / \text{Tor}_1^B(Y, T) = 0\} \end{aligned}$$

We then have the next result, which is due to Brenner and Butler [16].

(7.2.1.) Theorem of Brenner-Butler: Let T_A be a tilting A -module, and $B = \text{End } T_A$. Then ${}_B T$ is also a tilting B -module and $A = \text{End}_B T$, canonically. The subcategories $T(T_A)$ and $Y(T_A)$ are equivalent under the restriction of the functions $F = \text{Hom}_A(T, -)$ and $G = - \otimes_B T$ which are mutually inverse to each other, and similarly, the subcategories $F(T_A)$ and $X(T_A)$ are equivalent under the restriction of the functors $F' = \text{Ext}_A^1(T, -)$ and $G' = \text{Tor}_1^B(-, T)$ which are again mutually inverse to each other.

(7.3.) There is a characterization of Auslander-Reiten sequences in mod B when T_A is a splitting tilting module, that is, when the torsion theory $(X(T_A), Y(T_A))$ splits. Before giving the characterization, let us

recall the following result:

(7.3.1.) Connecting lemma ([13], [27]) Let T_A be a tilting module, with $B = \text{End } T_A$. Let I_A be an indecomposable injective module, and P_A be a projective module with $\text{soc } I_A \cong P_A / \text{rad } P_A$. Then $\tau^{-1} F(I) = F'(P)$.

We then have the mentioned characterization:

(7.3.2.) Theorem ([2], [13]) Let T_A be a splitting tilting module, and $B = \text{End } T_A$. Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be an Auslander-Reiten sequence in $\text{mod } B$. We have three possible cases:

- i) It is an Auslander-Reiten sequence in $\mathcal{V}(T_A)$.
- ii) It is an Auslander-Reiten sequence in $\mathcal{X}(T_A)$.
- iii) It is a connecting sequence, that is, a sequence of the form:

$$0 \rightarrow F(I) \rightarrow F(I/\text{soc } I) \rightarrow F'(P/\text{rad } P) \rightarrow F'(P) \rightarrow 0,$$

where P is a projective module not in $\text{add } T_A$, and I the injective envelope of $P/\text{rad } P$.

(7.4.) An algebra B is called a tilted algebra if there is a tilting module T_A over a hereditary algebra A such that $B = \text{End } T_A$. B is called tilted of type Δ , for a (non-oriented) valued graph Δ , if A is the tensor algebra of an oriented valued graph, with non-oriented underlying graph Δ .

(7.4.1.) Theorem ([27]). Let A be a hereditary algebra, T_A a tilting A -module, and $B = \text{End } T_A$. Then $(\mathcal{X}(T_A), \mathcal{V}(T_A))$ is a splitting torsion theory in $\text{mod } B$.

(7.4.2) Definition [27]. Let B be an algebra, C a component of Γ_B . A complete slice S in C consists of a set of indecomposable B -modules in C with the following properties:

- i) Given any indecomposable module X , S contains precisely one module

from the orbit $\{\tau^m X/m \in \mathbb{Z}\}$

ii) If $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_r$ is a chain of non-zero maps and indecomposable modules, and X_0, X_r belong to S , then all X_i belong to S .

iii) There is no oriented cycle of irreducible maps

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_r \rightarrow X_0 \text{ with all } X_i \in S.$$

(7.4.5.) Lemma. Let A be a finite dimensional tilted K -algebra of finite representation type. If P is indecomposable projective, I indecomposable injective and there exists a chain of irreducible maps from I to P , then such a path is sectional.

Proof:

Since A is a tilted algebra, there exists a complete slice S in Γ_A . Then, there exist m and n such that $[\tau^m I], [\tau^{-n} P]$ belong to S . Using the chain of irreducible maps from I to P , we obtain a chain of irreducible maps from $\tau^m I$ to $\tau^{-n} P$:

$$\tau^m I \rightarrow \dots \rightarrow I \rightarrow \dots \rightarrow P \rightarrow \dots \rightarrow \tau^{-n} P$$

Since S is a complete slice, $n = 0 = m$ and any oriented path from $[I]$ to $[P]$ must be sectional. This finishes the proof.

CHAPTER II

GRADING OF SOME GRAPHS

The notion of grading of a tree was introduced by Bongartz and Gabriel in [14]. They proved that there exists a bijection between the isomorphism classes of graded trees of finite representation type and the isomorphism classes of simply connected algebras (see (1.6.1)). A description of the grading of finite representation type for the Dynkin graph A_n is given in [14]. Here, using the methods and results of [3], we describe these gradings for the Dynkin graph B_n and C_n in terms of their bounden K -modulation. We also prove that the admissible gradings for a Euclidean graph $G \in \{\tilde{B}_n, \tilde{C}_n, \tilde{B}\tilde{C}_n\}$ are tilted algebras of type G .

1. GRADINGS

(1.1) Let A be a simply connected algebra, we associate to Γ_A an oriented valued graph $Gr(\Gamma_A)$ as follows [14]: the vertices of $Gr(\Gamma_A)$ are the τ -orbits of the isomorphism classes of indecomposable A -modules, and the valued edges correspond to the σ -orbits of the arrows of Γ_A : if $\alpha: [X] \rightarrow [Y]$ is an arrow in Γ_A , with Y projective, there exists an arrow $\tau^\mathbb{Z}[X] \rightarrow \tau^\mathbb{Z}[Y]$ in $Gr(\Gamma_A)$ endowed with the same valuation (d_α, d'_α) (cf. 1.6.1).

Clearly the underlying graph of $\text{Gr}(\Gamma_A)$ is homotopically equivalent to Γ_A , and in particular it is a tree.

The K -modulation of Γ_A induces a K -modulation $m(\Gamma_A)$ for $\text{Gr}(\Gamma_A)$, as follows: to each vertex $i = \tau[X]$ of $\text{Gr}(\Gamma_A)$, we associate the skew-field $F_i = \text{End } X / \text{rad } \text{End } X$ and for each arrow $\tau[X] \rightarrow \tau[Y]$ (where we assume that the representations X and Y are chosen such that Y is projective, and there is an arrow $\alpha: [X] \rightarrow [Y]$), we put $M_{ij} = (\text{Irr}(X, Y))^*$.

(1.2) Definition [14]. Let (T, d) be a valued tree. A grading of T is a function $g: T \rightarrow \mathbb{N}$ such that

- i) $g^{-1}(0) \neq \emptyset$
- ii) $g(i) - g(j) \equiv 1 \pmod{2}$, whenever i, j are neighbours in T .

We shall assume, without loss of generality, that whenever (T, d) is an oriented valued tree, g satisfies the additional condition [4]:

- iii) If $i \rightarrow j$ is an arrow in (T, d) then $g(i) < g(j)$.

We shall define a natural grading g_A on the tree $\text{Gr}(\Gamma_A)$ associated to the algebra A : let us first consider the translation graph Γ_0 which is isomorphic to the underlying translation valued graph of Γ_A , but with trivial valuation (that is, $d_{ij} = d_{ji} = 1$ for every arrow $i \rightarrow j$ in Γ_0), then there exists a unique

morphism of translation graph $\text{pr}_A: (\Gamma_0, \tau) \rightarrow \mathbb{Z}A_2$ such that $\text{Min}_{x \in \Gamma_0} \text{pr}_A(x) = 0$ [14]. Let us put $g_A([P(i)]) = \text{pr}_A([P(i)])$, where $P(i)$ is the unique indecomposable projective such that $i = \tau^{-1}[P(i)]$. Clearly, g_A is a grading.

(1.3) To the valued tree (T, d) , with a grading g and a K -modulation $m = (F_i, {}_iM_j)_{i,j \in T}$, we associate a translation K -modulation $\Omega = (\Omega_0, H, N, \tau, D)$ (see (6.3)). Let

$\Omega_0 = \{(n, i) \in \mathbb{N} \times T / \frac{1}{2}(n - g(i)) \in \mathbb{N}\}$, and for each valued arrow $i \xrightarrow{(d_{ij}, d_{ji})} j$, define families of valued arrows $(n, i) \xrightarrow{(d_{ij}, d_{ji})} (n+1, j)$ and $(n+1, j) \xrightarrow{(d_{ji}, d_{ij})} (n+2, i)$ with $n \in \mathbb{N}$, whenever both end-points lie in Ω_0 .

Thus, we obtain a valued graph endowed with a translation:

the projective vertices are the pairs $(g(i), i)$, with $i \in T$, and

the translate of a non-projective vertex is defined by $\tau(n, i) = (n-2, i)$.

The mapping H is defined by $H(n, i) = F_i$, the mapping N by

$N((n, i), (n+1, j)) = {}_iM_j$ and $N((n+1, j), (n+2, i)) = {}_jM_i = ({}_iM_j)^*$,

while the bimodule isomorphism D follows from the existence of the central field K . [20]

(1.3) Let $(n, i) \in \Omega_0$, and define $v(n, i)$ to be the set $\{j \in T / d_{ij} \neq 0 \text{ and } g(j) < n\}$. It is obviously possible to define inductively on Ω a unique mapping $d: \Omega_0 \rightarrow \mathbb{N}^{(\text{Card } T)}$

such that:

a) For every projective vertex $(g(i), i)$ such that all $j \in v(g(i), i)$ satisfy $d(g(i)-1, j) > 0$, we have:

$$d(g(i), i) = \delta_i + \sum_{j \in v(g(i), i)} d_{ij} d(g(i)-1, j)$$

where, δ_i is the Kronecker delta-function.

b) For every non-projective vertex (n, i) such that $d(n-2, i) > 0$, and moreover

$$\sum_{j \in v(n, i)} d_{ij} d(n-1, j) - d(n-2, i) > 0, \text{ we have:}$$

$$d(n, i) = \sum_{j \in v(n, i)} d_{ij} d(n-1, j) - d(n-2, i).$$

c) For every other $(n, i) \in \Omega_0$, $d(n, i) = 0$.

If $(T, m, g) = (Gr(\Gamma_A), m(\Gamma_A), g_A)$, the uniqueness of d implies that $d = \underline{\dim}$, where $\underline{\dim}$ is the mapping associating to each vertex $[X]$ of Γ_A its dimension vector $\underline{\dim} X$. By analogy d is called the dimension map of Ω . Let now $\Gamma = \Gamma_{(T, m, g)}$ be the full subgraph of Ω given by those vertices (n, i) of Ω_0 such that $d(n, i) > 0$.

(1.3.1) Definition: The grading g is admissible if Γ contains all the projective vertices $(g(i), i)$ of Ω .

(1.3.2) Definition: The grading g is of finite representation type if it is admissible and Γ is finite.

For instance, g_A is of finite representation type and $\Gamma = \Gamma(\text{Gr}(\Gamma_A), m(\Gamma_A), g_A)$ can be identified to Γ_A .

(1.4) Let (T, m, g) be a K -modulated tree, that is a tree T with a K -modulation m , endowed with a grading of finite representation type, and consider the algebra:

$$A = A_{(T, m, g)} = \bigoplus_{i, j \in T} \text{Hom}_{\diamond \Gamma}((g(i), i), (g(j), j)).$$

We have the following theorem.

(1.4.1) Theorem (Bongartz-Gabriel). The map $(T, m, g) \rightarrow A_{(T, m, g)}$ induces a bijection between the isomorphism classes of modulated trees equipped with a grading of finite representation type and the isomorphism classes of simply connected algebras.

Indeed, the proof of [14], done under the assumption that K is algebraically closed, carries over to the general case with only the obvious changes.

(1.5) Example: Let F, E be two skew-fields, finite dimensional over the central subfield K , and such that $\dim E_F = 2$.

Let A be the algebra given by the bounden K -modulated graph:

Then A has the following Auslander-Reiten graph Γ_A :

(where modules are represented by their dimension-vectors). Thus

$\text{Gr}(\Gamma_A)$ is here the Euclidean graph $\tilde{\mathcal{C}}_2$: $h \xrightarrow{(2,1)} i \xrightarrow{(1,2)} j$

and the grading g_A is given by $g_A(h) = 0$, $g_A(i) = 1$, $g_A(j) = 4$.

$\Omega = \Omega(\text{Gr}(\Gamma_A), m(\Gamma_A), g_A)$ is then given by:

and $\Gamma = \Gamma_A$. In fact, A is a tilted algebra of type $\tilde{\mathcal{C}}_2$. We

also note that g_A is a grading for the oriented graph

(1.5) In this subsection we shall recall some definitions and results of [3]. Let A be an algebra, and Γ_A its Auslander-Reiten graph. A vertex x of Γ_A is called a border point if:

- i) There exists at most one arrow α of source x .
- ii) There exists at most one arrow β of target x .
- iii) If α and β both exist, then $\beta = \sigma\alpha$.

(1.5.1) Definition: A K -modulation $m = (F_i, {}_iM_j)$ over an oriented valued graph G with a relation ideal R is said to satisfy the property (β) if:

(β_1) G is a tree

(β_2) There is a vertex i_0 such that $F_{i_0} = F$, and for all $i \neq i_0$, $F_i = E$, and furthermore $[E: F] = 2$. Also for all $i, j \in G$ ${}_iM_j = F_i {}^E F_j$ or 0 .

(β_3) i_0 has at most two neighbours, and, if it is so, we have a submodulation with a relation of length two

called a (μ) -relation.

(β_4) All relations are of length two, and the only relations besides (μ) are the zero-relations.

(β_5) Each vertex of G has at most four neighbours.

(β_6) If a vertex k has four neighbours, then we have a submodulation of the form:

(β_7) If a vertex F_k has three neighbours, there is a sub-modulation of one of the forms:

Then we have the next result:

(1.5.2) Theorem [3]. For a finite-dimensional K -algebra A , the following statements are equivalent:

- i) A is iterated tilted of type B_n .
- ii) A is given by a bounden K -modulation satisfying the property (β).
- iii) A is simply-connected and Γ_A satisfies the following properties:

(Γ_1) There are at most two arrows in Γ_A with a prescribed source or target. For any arrow $\alpha: [X] \rightarrow [Y]$ in Γ_A , we have $d_\alpha \leq 2$ and $d'_\alpha \leq 2$. Moreover, there exists a unique τ -orbit β of Γ_A , entirely consisting of border points, such that $d_\alpha \leq 2$ if and only if $[Y] \in \beta$ and $d'_\alpha \leq 2$ if and only if $[X] \in \beta$.

(Γ_2) If P_A is an indecomposable projective module and $[R] \rightarrow [P]$ is an arrow of Γ_A , there exists at most one arrow of target $[R]$. Dually, if I_A is an indecomposable injective module

and $[I] \rightarrow [J]$ is an arrow of Γ_A , then there exists at most one arrow of source $[J]$.

(1.5.3) Definition: A K -modulation $m = (F_i, {}_iM_j)$ over an oriented valued graph G with a relation ideal R is said to satisfy the property (γ) if it satisfies the properties (β_1) , (β_4) , (β_5) , (β_6) and (β_7) (now renamed respectively (γ_1) , (γ_4) , (γ_5) , (γ_6) and (γ_7)) and:

(γ_2) There is a vertex i_0 such that $F_{i_0} = E$, and, for all $i \neq i_0$, $F_i = F$, where $[E:F] = 2$. Also, for all $i \neq i_0$, $j \neq i_0$, ${}_iM_j = {}_F F_F$ or 0 and if $i_0 \in \{i, j\}$, then ${}_iM_j = {}_{F_i} E_{F_j}$.

(γ_3) The vertex i_0 has at most two neighbours, and, if it is so, there is a submodulation with a relation of length two

called a (π) -relation.

Then we have:

(1.5.4) Theorem [3]: For a finite-dimensional K -algebra A , the following statements are equivalent:

- i) A is iterated tilted of type C_n .
- ii) A is given by a bounden K -modulation satisfying the property (γ) .

iii) A is simply-connected and Γ_A satisfies property (Γ_2) of (1.5.1) and the following property:

(Γ'_1) There are at most two arrows in Γ_A with a prescribed source or target. For any arrow $\alpha: [X] \rightarrow [Y]$ in Γ_A , we have $d_\alpha \leq 2$ and $d'_\alpha \leq 2$. Moreover, there exists a unique τ -orbit $\mathcal{B}'$ of Γ_A , entirely consisting of border points and such that $d_\alpha = 2$ if and only if $[X] \in \mathcal{B}'$, and $d'_\alpha = 2$ if and only if $[Y] \in \mathcal{B}'$.

2. ADMISSIBLE GRADING OF B_n

(2.1) Our object is to give a description of algebras of finite representation type of the form $A_{(T, m, g)}$ (see (1.4)) where the underlying graph of T is the tree

(2.1.1) Lemma. Assume $a = b = 1$ and $A = A_{(T, m, g)}$ for an admissible grading g . If Γ_A contains a complete slice S , then for any vertex $x_1 = (t_1, 1)$ in Γ_A , there exists a complete slice containing it.

Proof:

We first define the following sets:

$$p_1(x_1) = \{(t_1 - m - 1, m) \in \Gamma_A / (t_1 - j - 1, j) \in \Gamma_A \text{ for } j \leq m\}$$

$$p_2(x_1) = \{(t_1 + m - 1, m) \in \Gamma_A / (t_1 + j - 1, j) \in \Gamma_A \text{ for } j \leq m\}.$$

Note that if $p_1(x_1) \cup p_2(x_1)$ contains a vertex of the form (t, n) , then one of $p_1(x_1)$ and $p_2(x_1)$ is a complete slice. So we may assume that $(t, n) \notin p_1(x_1) \cup p_2(x_1)$.

Let m_1 and m_2 be maximal such that $(t_1 - m_1 - 1, m_1) \in p_1(x_1)$ and $(t_1 + m_2 - 1, m_2) \in p_2(x_1)$ and let $S = \{(s_i, i) / 1 \leq i \leq n\}$ be a complete slice of Γ_A .

If $(s_j, j) \in S \cap p_1(x_1)$ for some j , the set

$$\{(s_i, i) / j \leq i \leq n\} \cup \{(t_1 - i - 1, i) / 1 \leq i \leq j\}$$

is a complete slice of Γ_A . Similarly, if $(s_j, j) \in S \cap p_2(x_1)$ we then have a complete slice containing x_1 . Also note that if for some i , $t_1 - i - 1 < s_i < t_1 + i + 1$, then $S \cap (p_1(x_1) \cup p_2(x_1)) \neq \emptyset$.

Thus we can assume that for every i , $s_i < t_1 - i - 1$. In particular none of the vertices in $p_1(x_1)$ is projective. Therefore there exists a path λ from one projective $(g(j), j)$ to x_1 , moreover λ must factor over one vertex $(t_1 - m - 1, m)$ with $m > 1$, in $p_1(x_1)$. That is, λ has the form:

$$(g(j), j) \rightarrow \dots \rightarrow (t_1 - m - 2, m - 1) \rightarrow (t_1 - m - 1, m) \rightarrow \dots \rightarrow x_1.$$

In the mesh category $\diamond \Gamma_A$ the morphism induced for λ and the path λ' :

$$(g(j), j) \rightarrow \dots \rightarrow (t_1^{-m-2}, m-1) \rightarrow (t_1^{-m-1}, m-2) \rightarrow \dots \rightarrow (t_1^{-1}, 2) \rightarrow x_1$$

are the same. However, $\lambda' = 0$ in Γ_A . Therefore $\text{Hom}_A(P, x_1) = 0$ for every projective vertex P . We then get a contradiction to the connectness of A . This finishes the proof.

The following corollary will be used later:

(2.1.2) Corollary. If $a = b = c = d = 1$ and Γ_A contains a complete slice S , then, for any pair of vertices $x_1 = (t_1, 1)$ and $x_n = (t_n, n)$ in Γ_A , there exists a complete slice containing them.

Proof: According to the lemma, there exists a complete slice S containing x_n , and so using the proof of the lemma we can construct a complete slice containing x_n and x_1 .

(2.2) We then have the following result:

(2.2.1) Theorem: Let $A = A_{(T, m, g)}$ for an admissible grading on the tree T . Then A is a tilted algebra of type T with a complete slice in its preprojective component.

Proof: We use induction on n . For $n \leq 2$, the result is trivial. Now let $n > 2$, $a = \max \{g(i) / i \in T\}$ and i_0 be such

that $a = g(i_0)$. The graph $T \setminus \{i_0\}$ is the union of at most two graphs T_1 and T_2 .

Let j_1 and j_2 be the neighbouring vertices of i_0 , let $a_i = \min\{g(j) / j \in T_i\}$ (for $i = 1, 2$) and define $g_i = g|_{T_i} - a_i$. Then g_i is a grading on the tree T_i (for $i = 1, 2$), (if $T_i \neq \emptyset$).

Using the K -modulation m_i induced by m on T_i we have algebras $A_i = A_{(t_i, m_i, g_i)}$ $i = 1, 2$; and by (2.1.1) we know that there is a complete slice S_i in the preprojective component of Γ_{A_i} containing the vertex $(g(i_0) - 1 - a_i, j_i)$. Now we construct the preprojective component of A by using the set of all predecessors of S_i (for $i = 1, 2$) and inserting $(g(i_0), i_0)$. Let

$$S = S_1 \cup \{(g(i_0), i_0)\} \cup S_2.$$

Then it is clear that S is a complete slice in the preprojective component of Γ_A , so we conclude that A is a tilted algebra of type T with a complete slice in its preprojective component.

This proves that the algebras arising from admissible gradings of one of the graphs $A_n, B_n, C_n, \tilde{B}_n, \tilde{C}_n$ and BC_n are tilted of that type.

(2.2.2) In particular, when $a = 1$ and $b = 2$, we have that (m, R) satisfies the property (B) of (1.5), and we can

construct $(\mathfrak{M}, R)$ using proposition (4.2) of [3]: Let $\underline{i} = (g(i), i)$ and $\underline{j} = (g(j), j)$ be two projective points, then there is an arrow $\underline{i} \rightarrow \underline{j}$ in G if and only if one of the following two conditions is satisfied:

- (1) $\underline{i} = (g(i), i)$ belongs to the maximal sectional path $\{(m, h) | h > j, m - h = g(j) - j\}$ and, moreover, no other projective $\underline{k} = (g(k), k)$ lies on this path between $\underline{j}$ and $\underline{i}$.
- (2) $\underline{i} = (g(i), i)$ belongs to the composite of the two maximal sectional paths $\{(m, h) | h < j, m + h = g(j) + j\}$ and $\{(m, h) | m - h = g(j) + j - 2\}$, and, moreover, no other projective $\underline{k} = (g(k), k)$ lies on this path between $\underline{j}$ and $\underline{i}$.

In the first case, the arrow $\underline{i} \rightarrow \underline{j}$ is called a $(+)$ -arrow, and in the second, a $(-)$ -arrow.

This allows us to construct G . The K -modulation of A is defined by:

- (M1) $F_1 = F$ and $F_i = E$ for $i \neq 1$.
- (M2) $M_{i,j}$ equals $E E$ if $i, j \neq 1$, $F E$ if $i = 1$ and $E F$ if $j = 1$.

We now describe the relation ideal R by its minimal

generators:

(R1) All relations are of length two.

(R2) If we have $\underline{i} \rightarrow \underline{k} \rightarrow \underline{j}$ ($k \neq 1$) and $\underline{k} \rightarrow \underline{k}$ is a (+)-arrow (respectively, (-)-arrow), while $\underline{k} \rightarrow \underline{j}$ is a (-)-arrow (respectively, (+)-arrow), we have a zero-relation $\underline{i} R_j = \underline{i} M_k \otimes M_j$.

(R3) If we have $\underline{i} \rightarrow \underline{1} \rightarrow \underline{j}$, then necessarily, $\underline{i} \rightarrow \underline{1}$ is a (+)-arrow and $\underline{1} \rightarrow \underline{j}$ is a (-)-arrow. In this case, $\underline{i} R_j$ is the kernel of the epimorphism $\mu: M_{\underline{i}1} \otimes M_{1j} = E_F^E \otimes F_E^E \rightarrow E_E^E$.

Moreover, the properties (β) imply that $\mathcal{Q}$ is a tree such that every vertex is the source (or the target) of at most one (+)-arrow and one (-)-arrow.

(2.2.3) Lemma: Let g be a grading of finite representation type on the modulated tree $(B_n, \mathbb{M})$, then the bounden k -modulation $(\mathbb{M}, R)$ of the associated algebra A contains no full connected submodulation of one of the forms:

$$d) \quad F \leftarrow E \leftarrow E \text{ --- } E \text{ --- } \dots \text{ --- } E \leftarrow E \leftarrow E$$

$$e) \quad E \leftarrow E \leftarrow E \text{ --- } E \text{ --- } \dots \text{ --- } E \leftarrow E \leftarrow E$$

$$f) \quad E \leftarrow E \leftarrow E \text{ --- } E \text{ --- } \dots \text{ --- } \overset{E}{\overset{E}{\overset{E}{\curvearrowright}}} E \rightarrow F \rightarrow E \text{ --- } \dots \text{ --- } E \leftarrow E \leftarrow E$$

$$g) \quad F \text{ --- } E \text{ --- } \dots \text{ --- } E \rightarrow \overset{E}{\uparrow} E \text{ --- } \dots \text{ --- } E \leftarrow E \leftarrow E$$

$$h) \quad F \text{ --- } E \text{ --- } \dots \text{ --- } E \leftarrow \overset{E}{\downarrow} E \text{ --- } \dots \text{ --- } E \rightarrow E \rightarrow E$$

where it is assumed that there is no other relation besides the relations shown.

Proof: Let us first observe that the cases b), d) and h) can be deduced respectively from a), c) and g) by passing to the opposite algebra. We shall thus assume that $(\mathfrak{m}, R)$ contains a full connected submodulation of one of the forms a), c), e), f) or g) and show that in each case, an injective module lies on the left of a maximal sectional path of Γ_A ending at a projective module, from which we shall deduce the contradiction $\text{Gr}(\Gamma_A) \neq B_n$.

a) Suppose that $(\mathfrak{m}, R)$ contains a full connected submodulation of the form:

$$E_{i_1} \leftarrow E_{i_2} \leftarrow E_{i_3} \text{ --- } E_{i_4} \text{ --- } \dots \text{ --- } E_{i_{t-2}} \leftarrow \overset{E}{\overset{E}{\overset{E}{\curvearrowright}}} F_{i_{t-1}} \leftarrow E_{i_t}$$

where $t \geq 4$, $E_{i_j} = E$ for $j \neq t-1$ and $F_{i_{t-1}} = F$. Let us define $P_{i_{t-1}}(i_t)$ to be the unique submodule of $P(i_t)$ with support $\text{Supp } P_{i_{t-1}}(i_t) = \text{Supp } P(i_t) \cap \text{Supp } P(i_{t-1})$. Obviously, $P_{i_{t-1}}(i_t)$ is indecomposable. Dually, we can define $I^{i_2}(i_1)$ by $\text{Supp } I^{i_2}(i_1) = \text{Supp } I(i_1) \cap \text{Supp } I(i_2)$. This is an indecomposable image of $I(i_1)$. Let us now consider the representation M defined by:

$$\text{Supp } M = \{i_2, i_3, \dots, i_{t-1}\} \cup \{\text{Supp } I(i_1) \cap \text{Supp } I(i_2)\} \cup \{\text{Supp } P(i_t) \cap \text{Supp } P(i_{t-1})\},$$

the coordinate vector spaces are $M_{i_{t-1}} = E_F$, $M_{i_j} = E_E \oplus E_E$ if $2 \leq j \leq t-2$ is such that i_j belongs to $\text{Supp } P(i_{t-1})$; and $M_h = E_E$ if $h \in \text{Supp } M \setminus \text{Supp } P(i_{t-1})$, and the maps between the coordinate vector spaces are the obvious ones. It is clear that M is indecomposable except when $\{i_2, i_3, \dots, i_{t-1}\} \subseteq \text{Supp } P(i_{t-1})$ and the only arrow of target i_2 has source i_3 , in which case $M \cong P(i_{t-1}) \oplus P(i_{t-1})$. On the other hand, it is easy to see that $I^{i_2}(i_1)$ is a submodule of M , while $P_{i_{t-1}}(i_t)$ is an image of M . Thus we have an oriented path in Γ_A defined by the sequence of maps $I(i_1) \xrightarrow{I^{i_2}(i_1)} \dots \rightarrow M \rightarrow \dots \rightarrow P_{i_{t-1}}(i_t) \rightarrow P(i_t)$ (or $I(i_1) \xrightarrow{I^{i_2}(i_1)} \dots \rightarrow P(i_{t-1}) \rightarrow \dots \rightarrow P_{i_{t-1}}(i_t) \rightarrow P(i_t)$ if M is decomposable).

Since $\text{Hom}_A(P(i_{t-1}), P(i_t)) \neq 0$, $P(i_{t-1})$ belongs to the set of prefactors of $P(i_t)$. On the other hand, $i_{t-1} = 1$ implies that $[P(i_{t-1})] \in \beta$, hence there exists a sectional path L of source $[P(i_{t-1})]$ and target $[P(i_t)]$. Since $M_{i_{t-1}} \neq 0$, $[M] \in L$, and since $I(i_1)_{i_{t-1}} = 0$, $I(i_1)$ lies on its left.

Let us show that this implies $\text{Gr}(\Gamma_A) \neq B_n$. By hypothesis, $i_{t-1} = 1$. Consequently, L induces a full connected subgraph G_0 of $\text{Gr}(\Gamma_A)$ of the form:

where $h = \tau^{\mathbb{Z}}[P_{i_{t-1}}(i_t)]$. Since the injective module $I(i_1)$ lies on the left of L , no module of the form $\tau^s I(i_1)$ ($s \geq 0$) lies on L , and hence $i_1 \notin G_0$. On the other hand, the oriented path $I(i_1) \rightarrow \dots \rightarrow M \rightarrow \dots \rightarrow P_{i_{t-1}}(i_t)$ (or $I(i_1) \rightarrow \dots \rightarrow P(i_{t-1}) \rightarrow \dots \rightarrow P_{i_{t-1}}(i_t)$ if M is decomposable) induces a path of $\text{Gr}(\Gamma_A)$ of the form:

Clearly, this path cannot contain i_t , and is not a subgraph of G_0 ; hence there exists $1 \leq k \leq h$ such that

is a full connected subgraph of $\text{Gr}(\Gamma_A)$, which gives the wanted contradiction.

c) Let us now assume that $(\mathfrak{M}, R)$ contains a full connected submodulation of the form:

where $F_{i_t} = F$ and $E_{i_j} = E$ for $j \neq t$. Let us define M to be the indecomposable representation of support:

$$\text{Supp } M = \{i_2, \dots, i_{t-1}\} \cup \{\text{Supp } I(i_1) \cap \text{Supp } I(i_2)\} \cup \{\text{Supp } P(i_t) \cap \text{Supp } P(i_{t-1})\}$$

and such that $M_h = E_E$ for all $h \in \text{Supp } M$, the only mappings

between the coordinate vectors spaces being the obvious ones. Then the rest of the proof of the case a) applies here.

e) and f) We now consider the case where $(\mathfrak{M}, R)$ contains the full connected submodulation

(where, if $t \geq 7$, we may have a j such that $4 \leq j \leq t-3$ and $i_j = 1$ and then we are in the case f)). We shall construct the module M as follows:

$$\text{Supp } M = \{i_2, \dots, i_{t-1}\} \cup \{\text{Supp } I(i_1) \cap \text{Supp } I(i_2)\} \cup \{\text{Supp } P(i_t) \cap \text{Supp } P(i_{t-1})\}$$

(i) If $1 \notin \text{Supp } M$, we put $M_h = E_E$ for all $h \in \text{Supp } M$, with the obvious maps.

(ii) If we have a $4 \leq j \leq t-3$ such that $i_j = 1$, we put $M_{i_j} = E_F$, and $M_h = E_E$ if $h \in \text{Supp } M$, $h \neq 1$. The maps between the coordinate vector spaces are the identity, the zero maps, and $\mu: M_1 \otimes_F E_E \rightarrow E_E$.

(iii) If $1 \in \text{Supp } I(i_1) \cap \text{Supp } I(i_2)$, we put $M_1 = E_F$, and $M_h = E_E$ if $h \in \text{Supp } M$, $h \neq 1$, the maps between the coordinate vector spaces being the obvious ones.

In all the previous cases, M is an indecomposable module lying on the prefactor set of $P(i_t)$, while $I(i_1)$ lies on its left. Let us observe that this prefactor set consists in general of three sectional paths. In all cases, however, we obtain a contradiction. If there exists a sectional path joining $[P(i_{t-1})]$ to $[P(i_t)]$, this contradiction is obtained as in a). Otherwise, there is a path from $[P(i_{t-1})]$ to $[P(i_t)]$ consisting of two

sectional paths factoring over $[N] \in \beta$, and the sectional path L from $[N]$ to $[P(i_t)]$ induces a full connected subgraph G_0 of $\text{Gr}(\Gamma_A)$ of the form

where $h = \tau^{\mathbb{Z}} [P_{i_{t-1}}(i_t)]$, the latter submodule being defined as in a). Assume that $i_1 \in G_0$, then there exists an $s \geq 0$ such that $[\tau^s I(i_1)] \in L$. This, however, is impossible, since the existence of a chain of irreducible maps $\tau^s I(i_1) \rightarrow \dots \rightarrow I(i_1) \rightarrow \dots \rightarrow M \rightarrow \dots \rightarrow P(i_t)$ implies that $[I(i_1)] \in L$, in particular $I(i_1)_{i_{t-1}} \neq 0$, a contradiction. Thus $i_1 \notin G_0$ and the proof proceeds as in a).

g) Finally, assume that we have a full connected submodulation of the form:

where $t \geq 4$, $F_{i_1} = F$ and $E_h = E$ for $h \neq i_1$. We define M by

$$\begin{aligned} \text{Supp } M = \{i_1, i_2, \dots, i_{t-1}\} \cup \{ \text{Supp } I(i_s') \cap \text{Supp } I(i_s) \} \cup \\ \{ \text{Supp } P(i_{t-1}) \cap \text{Supp } P(i_t) \} \end{aligned}$$

and put $M_1 = F$, $M_h = E_E$ for $h \in \text{Supp } M$, $h \neq 1$, together with the obvious maps between the coordinate vector spaces. As in a), we have that $I(i'_s)$ lies on the left of a sectional path ending at $P(i'_t)$, hence the contradiction $\text{Gr}(\Gamma_A) \neq B_n$.

(2.3) Theorem: The following conditions are equivalent:

(I) $A = A_{(B_n, m, g)}$ for a grading g of finite representation type on the oriented valued graph B_n with a K -modulation m .

(II) The bounden K -modulation of A satisfies the property (B) of (1.5) and moreover, does not contain any full connected submodulation of one of the forms:

a) $E \leftarrow E \leftarrow E \leftarrow \dots \leftarrow E \leftarrow \overset{E}{\underset{E}{\text{F}}} \leftarrow E$

b) $E \leftarrow \overset{E}{\underset{E}{\text{F}}} \leftarrow E \leftarrow \dots \leftarrow E \leftarrow E \leftarrow E$

c) $E \leftarrow E \leftarrow E \leftarrow \dots \leftarrow E \leftarrow E \leftarrow F$

d) $F \leftarrow E \leftarrow E \leftarrow \dots \leftarrow E \leftarrow E \leftarrow E$

e) $E \leftarrow E \leftarrow E \leftarrow \dots \leftarrow E \leftarrow E \leftarrow E$

f) $E \leftarrow E \leftarrow E \leftarrow \dots \leftarrow E \rightarrow \overset{E}{\underset{E}{\text{F}}} \rightarrow E \leftarrow \dots \leftarrow E \leftarrow E \leftarrow E$

g) $F - E - \dots E \rightarrow E \downarrow \dots - E \leftarrow E \leftarrow E$

h) $F \rightarrow E \rightarrow \dots \rightarrow E \leftarrow E \rightarrow \dots \rightarrow E \rightarrow E \rightarrow E$

where we assume that there is no other relation besides the relations shown.

(III) A is a tilted algebra of type B_n .

Proof: We have already shown that (III) implies (I) and that (I) implies (II). In order to complete the proof of the theorem, it suffices to show that (II) implies (III).

(2.3.1) Let A be an algebra whose bounden K -modulation (M, R) satisfies the conditions of (II). In order to show that A is tilted, we shall construct a complete slice in Γ_A . Let $s_1, \dots, s_m$ be the set of sources of G , and $P(s_1), \dots, P(s_m)$ be the corresponding indecomposable projective A -modules. Define S to be the full connected subgraph of Γ_A consisting of those points $[M]$ such that if there exists an oriented path from $[M]$ to $[P(s_i)]$, for some i , then this path is sectional.

Since Γ_A has no oriented cycles, and no indecomposable projective A -module lies on the right of the subsection $\cdot S$, it suffices to show that no indecomposable injective A -module lies on

the left of $S[1]$. Let thus $I(i)$ and $P(s)$ be respectively an indecomposable injective and an indecomposable projective A -module, with s a source, such that there exists an oriented path in $\Gamma_A: [I(i)] \rightarrow [M_1] \rightarrow [M_2] \rightarrow \dots \rightarrow [M_m] \rightarrow [P(s)]$. Our aim is to show that such a path is sectional (thus, $\{I(i)\} \in S$). We may obviously assume that i is a sink.

(2.3.2) Let w be the (non-oriented) path of G joining i and s , and A_w be the algebra of the submodulation with graph w , together with the inherited relations. We shall also let $I_w(i)$ and $P_w(s)$ be respectively the restrictions of $I(i)$ and $P(s)$ to w , considered as A -modules via the natural embedding [13]. Thus we have homomorphisms of A -modules:

$$I_w(i) \rightarrow I(i) \rightarrow M_1 \rightarrow \dots \rightarrow M_m \rightarrow P(s) \rightarrow P_w(s).$$

We claim that it might be assumed that $\text{Supp } M_j \cap w \neq \emptyset$ for all $1 \leq j \leq m$. Let us denote by G_a the branch of the tree G , attached at the vertex a of w :

If $\text{Supp } M_j \cap w = \emptyset$ for some $1 \leq j \leq m$ then there exist $h_1 < h_2$ such that all M_h ($h_1 < h < h_2$) have their supports not intersecting w while the supports of M_{h_1} and M_{h_2} intersect w . Since the M_h are indecomposable, there exists a vertex $a \in w$ such that $\text{Supp } M_{h_1+1} \subseteq G_a$. For the same reason, $\text{Supp } M_h \subseteq G_a$ for all $h_1 < h < h_2$. However, $\text{Hom}_A(M_{h_1}, M_{h_1+1}) \neq 0$ and $\text{Hom}_A(M_{h_2-1}, M_{h_2}) \neq 0$ imply that a belongs to the supports of M_{h_1} and M_{h_2} , and thus $\text{Hom}_A(P(a), M_{h_1}) \neq 0$, $\text{Hom}_A(P(a), M_{h_2}) \neq 0$. Now, either there exists no indecomposable A -module N such that both $N_1 \neq 0$ and $N_a \neq 0$, and the set of those indecomposable A -modules M with $M_a \neq 0$ is given by a rectangle in Γ_A :

(then, clearly, if M_{h_1} and M_{h_2} lie in this rectangle, so do all the M_h , for $h_1 < h < h_2$, a contradiction) or else there exists such an N , and the set of those indecomposable A -modules M with $M \neq 0$ is given by the region Γ_a of Γ_A shown below:

Then there exist sectional paths σ_1 and σ_2 , lying entirely within Γ_a , of source $[M_{h_1}]$ (respectively, target $[M_{h_2}]$) and target $[L_1]$ on β (respectively, source $[L_2]$ on β). We have two cases:

(i) L_2 lies on the left of L_1 : then necessarily the sectional paths σ_1 and σ_2 intersect at a point L , say, and the oriented path of Γ_A given by $[M_{h_1}] \rightarrow \dots \rightarrow [L] \rightarrow \dots \rightarrow [M_{h_2}]$ lies entirely within Γ_a (and thus the supports of all its modules intersect w).

(ii) L_2 lies on the right of L_1 , say $L_2 = \tau^{-p}L_1$ with $p \geq 0$, then the oriented path of Γ_A given by $[M_{h_1}] \rightarrow \dots \rightarrow [L_1] \rightarrow * \rightarrow [\tau^{-1}L_1] \rightarrow \dots \rightarrow [\tau^{-p}L_1] = [L_2] \rightarrow \dots \rightarrow [M_{h_2}]$ lies entirely within Γ_a .

This completes the proof of our claim.

(2.3.3) We may thus consider the induced path of Γ_{A_w} given by $[I_w(i)] \rightarrow [M'_1] \rightarrow [M'_2] \rightarrow \dots \rightarrow [M'_m] \rightarrow [P_w(s)]$. The path w has necessarily one of the following two forms:

where only the zero relations are represented by dotted lines.

Then, by the conditions (II):

i) There is no relation between i_j and i_{j+1} for all $1 \leq j \leq r-1$.

ii) There is no zero-relation between i_0, i_1 and between i_r, i_{r+1} but there may be the relation (μ) and in the direction opposite to the adjacent zero-relation.

iii) Two consecutive zero-relations are oriented in opposite directions.

iv) Non oriented edges can be oriented arbitrarily.

We may thus write $w = \bigcup_{j=0}^r w_j$ where w_j is the full connected subpath $i_j \cdots i_{j+1}$. Let A_j be the algebra given by the bounden K -modulation of w_j . For $j \neq 0, r$, A_j is hereditary of type A_n and A_0, A_r are either hereditary of type A_n or tilted of type B_n (if and only if 1 belongs to w_0 or w_r , for then, A_0 or A_r has an indecomposable sincere representation [27]). Every indecomposable A_w -module is in fact an indecomposable A_j -module for some $0 \leq j \leq r$, and Γ_{A_w} has the following form:

(a)

(b)

where $\Gamma_{A_j} \cap \Gamma_{A_{j+1}} = \{[s(i_{j+1})]\} [1]$.

The existence of an oriented path of Γ_{A_w} from $I_w(i_0)$ to $P_w(i_{r+1})$ implies that $r \geq 1$, for, otherwise A_w has an indecomposable sincere representation M_w , and hence we have non-zero mappings $P_w(i_{r+1}) \rightarrow M_w$ and $M_w \rightarrow I_w(i_0)$, which gives a contradiction. Also, w cannot be of type (b), or of type (a) with $r \geq 1$. Consequently, w has the following form:

Moreover, the oriented path from $I_w(i_0)$ to $P_w(i_2)$ in Γ_{A_w} factors necessarily through the simple module $S_w(i_1)$, which is injective when considered as an A_0 -module, and projective when considered as an A_1 -module. We claim that no a_t ($1 \leq t \leq m_1$) is a source, and dually, no b_t ($1 \leq t \leq m_2$) is a sink. This implies that all arrows are oriented to the left, and also, that if $1 \in w$, then either $i_0 = 1$ or $i_2 = 1$.

Indeed, assume inductively that no a_s is a source for $s < t$, while a_t is a source:

$$i_0 \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_t \rightarrow a_{t+1} \rightarrow \cdots \rightarrow i_1$$

We then have a sectional path in Γ_{A_w} given by a sequence of irreducible maps

$$I_w(i_0) \rightarrow I_w(a_1) \rightarrow I_w(a_2) \rightarrow \dots \rightarrow I_w(a_t) = S_w(a_t)$$

and thus we have no oriented path from $I_w(i_0)$ to $S_w(i_1)$. The proof of the second assertion is dual.

2.3.4 We have proved that the path w has the form:

In particular, $\text{Hom}_A(I(i), I(j)) \neq 0$ and $\text{Hom}_A(P(j), P(s)) \neq 0$, whence $I(j)$ belongs to the postfactor set of $I(i)$ and $P(j)$ to the prefactor set of $P(s)$.

We claim that either there exists a sectional path from $I(i)$ to $I(j)$ or from $P(j)$ to $P(s)$. In the first case the existence of oriented paths from $I(i)$ to $P(s)$ and from $P(s)$ to $I(j)$ implies that $P(s)$ lies on this path, and hence the path from $I(i)$ to $P(s)$ is sectional. The second case is dual.

A simple combinatorial argument shows that the graph G has the following form:

where $1 \notin G_{a_t}, G_{b_r}$ for $1 \leq t \leq m_1, 1 \leq r \leq m_2$. We have three cases to consider according as 1 belongs to G_1, G_j or G_s . The cases $1 \in G_1$ and $1 \in G_s$ are dual.

Let us thus assume that $1 \in G_1$, and that there is no sectional path from $I(i)$ to $I(j)$ (in particular, $i \neq 1$). Then there exists an indecomposable A -module M_A such that $[M] \in \mathcal{B}$, $\text{Hom}_A(I(i), M) \neq 0$ and $\text{Hom}_A(M, I(j)) \neq 0$. Obviously, $\text{Hom}_A(I(i), M) \neq 0$ implies $\text{Hom}_A(M, I(i)) = 0$. Thus $M_1 = 0$, $M_j \neq 0$ and also $M_1 \neq 0$ (because $[M] \in \mathcal{B}$). But this contradicts the fact that the support of an indecomposable module must be connected. Thus, in this case, there is a sectional path from $I(i)$ to $I(j)$.

Finally, Suppose that $1 \in G_j$ and let w' be the (non-oriented) path of G joining 1 and j . If there is no sectional path from $I(i)$ to $I(j)$, there exists an indecomposable module

M_A such that $[M] \in \theta$, $\text{Hom}_A(I(1), M) \neq 0$ and $\text{Hom}_A(M, I(j)) \neq 0$.

Now it is easy to see that $[M] \in \theta$ implies that $\dim M_a \leq 1$ for

any $a \in G$, and also that $M_1 \neq 0$. Thus $\dim M_1 = 1$, $\dim M_j = 1$,

whence $\dim M_a = 1$ for all $a \in w'$. But this means in particular

that there is no zero-relation on the path w' . However, in this

case, we can construct an indecomposable module N_A such that

$\text{Hom}_A(P(s), N) \neq 0$ and $\text{Hom}_A(N, I(1)) \neq 0$, which will give an

oriented path from $P(s)$ to $I(1)$, a contradiction. Let us

define N as follows:

$$\begin{aligned} N_a &= E_E && \text{if } a \in w, a \neq j \\ N_a &= E_E \oplus E_E && \text{if } a \in w', a \neq 1 \\ N_a &= E_F && \text{if } a = 1 \\ N_a &= 0 && \text{otherwise.} \end{aligned}$$

The maps between the coordinate vector spaces are the canonical

inclusion $\begin{bmatrix} 1 \\ 0 \end{bmatrix}: E_E \rightarrow E_E \oplus E_E$, the projection $\begin{bmatrix} 0 & 1 \end{bmatrix}$:

$$E_E \oplus E_E \rightarrow E_E, \text{ the map } \varpi: E_F \otimes_F E_E \xrightarrow{\sim} E_E \oplus E_E \xrightarrow{\begin{bmatrix} F & 0 \\ 1 & 1 \end{bmatrix}} E_E \oplus E_E,$$

or its adjoint $\bar{\varpi}$ (cf. [18] or [20]). It is easy to see, using

the methods of [18], that N is indecomposable, and it is clear

that $\text{Hom}_A(P(s), N) \neq 0$ and $\text{Hom}_A(N, I(1)) \neq 0$.

We have thus completed the proof of our claim, and hence

of the theorem.

3. ADMISSIBLE GRADING OF $\mathbb{C}_n$.

(3.1) As in (2.3), we can obtain a result for the tree $\mathbb{C}_n$:

Here E and F are two skew-fields, finite-dimensional over K and such that $[E:F] = 2$.

Theorem: The following conditions are equivalent:

(I) $A = A_{(\mathbb{C}_n, \mathfrak{m}, g)}$ for a grading g of finite representation type on the oriented valued graph $\mathbb{C}_n$ with a K -modulation $\mathfrak{m}$.

(II) The bounden K -modulation of A satisfies the property (Y) of (1.5) and moreover, does not contain any full connected submodulation of one of the forms:

$$f) \quad \overset{\curvearrowright}{F \leftarrow F \leftarrow F} \dots \overset{\overset{F}{\curvearrowright}}{F \rightarrow E \rightarrow F} \dots \overset{\curvearrowright}{F \leftarrow F \leftarrow F}$$

$$g) \quad E \leftarrow F \dots \overset{\overset{F}{\curvearrowright}}{F \rightarrow F} \dots \overset{\curvearrowright}{F \leftarrow F \leftarrow F}$$

$$h) \quad E \leftarrow F \dots \overset{\overset{F}{\curvearrowright}}{F \leftarrow F} \dots \overset{\curvearrowright}{F \rightarrow F \rightarrow F}$$

where we assume that there is not another relation besides the relation shown

(III) A is a tilted algebra of type $\mathbb{C}_n$.

CHAPTER III

TILTING MODULES

In chapter I we discussed various properties of tilted algebras in the general case. In this chapter we collect some basic results about tilted algebras of type Δ , where Δ is a Euclidean graph and in particular for $\Delta \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n\}$.

1. GENERAL PROPERTIES:

(1.1) By A we shall always mean a hereditary finite dimensional K -algebra, and by T_A a tilting module with $B = \text{End } T_A$. The algebra B is called a tilted algebra of type Δ , if A is the tensor algebra corresponding to some K -modulation M of an orientation of the valued graph Δ .

Let A be any hereditary algebra, then there are only finitely many tilting modules with basic endomorphism ring of finite representation type. If in addition A is of tame representation type, there are only finitely many basic algebras which are endomorphism rings of tilting modules (c.f. [28]). Clearly, these algebras are of either finite or tame representation type.

Let A be a hereditary tame algebra and T_A a tilting module, then T_A has an indecomposable summand which is preprojective or preinjective; that is, a tilting module cannot be regular ([28]).

Given a tilting module T_A , we have a criterion for deciding B is of finite representation type. If T_A has both non-zero preprojective and non-zero preinjective direct summands, then B is of finite representation type when A is a tame hereditary algebra. Conversely, for any hereditary algebra A , if B is of finite representation type, T_A must

have non-zero preprojective and preinjective direct summands. We have the following proposition, due to D. Happel and R.M. Ringel [28]:

(1.2) Proposition: Let A be a connected tame hereditary algebra.

Let T_A be a tilting module and $B = \text{End } T_A$. Then the following properties are equivalent:

- (I) $T(T_A)$ is infinite
- (II) $T(T_A)$ contains infinitely many indecomposable preprojective modules, and the modules $\text{Hom}_A({}_B T_A, X_A)$ with X_A indecomposable preprojective and in $T(T_A)$ form a component of the Auslander-Reiter graph of B .
- (III) All preinjective modules belong to $T(T_A)$.
- (IV) T_A has no non-zero preinjective direct summands.
- (V) If R is indecomposable and regular, then there exists i with $\tau^i R$ in $T(T_A)$.
- (VI) There exists a simple homogeneous module which belongs to $T(T_A)$.
- (VII) All homogeneous modules belong to $T(T_A)$.

Note that we also have the dual proposition for $F(T_A)$.

(1.3) Example: Let A be the hereditary algebra given by the following R-modulation (where C_2, C_3 and C_4 are copies of C ; and R_1 and R_3 are copies of R).

$$M = (R_1, C_2, C_3, C_4, R_5; R_5 C_4, C_4 C_3, C_3 C_2, C_2 C_{R_1})$$

the graph G will be

$$T = P(1) \oplus P(4) \oplus P(5) \oplus \tau^{-1}S(1) \oplus \tau^{-1}S(2)$$

(where S corresponds to the simple and simple regular representation $S(2)$) is a-tilting module, and the algebra B is given by the next bounded R -modulation with the relations shown.

Then Γ_B is given by

- i) The preprojective component of the algebra $A_0 = A/\langle e_2, e_3 \rangle$, which contains three projectives.
- ii) The preinjective component of the algebra A together with the Auslander-Reiter sequences ending with $I(2)$ and $I(3)$, which are connecting sequences.
- iii) The homogeneous components of Γ_A .
- iv) One component containing the projective modules $P(3)$ and $P(2)$, which looks as follows:

where the dotted lines have to be identified in order to form a cylinder.

Note that Γ_B has a complete slice in the preinjective component.

(1.4) Lemma. Let B be a tilted algebra of type Δ , where Δ is any graph. If B is of finite representation type, then Γ_B has no oriented cycles.

Proof: This follows from the existence of a complete slice and Proposition 8.1 of [27]. That is, there exists a component C of Γ_B with a complete slice, and therefore without oriented cycles; and since B is of finite representation type $C = \Gamma_B$.

Among other results from this lemma, we have that the indecomposable B -modules are uniquely determined by their dimension vectors. ([26], [27]).

(1.5) Lemma. Let B be a tilted algebra of finite representation type, where Δ is a tree. Then Γ_B is simply connected.

Proof: Again, by the existence of a complete slice in Γ_B of type Δ , we have that B arises from a grading of finite representation type of the tree Δ , and therefore, Γ_B is simply connected. (see (II.1.4.1).

2. REGULAR COMPONENTS

(2.1) We now study the indecomposable regular representations of a hereditary algebra. We first collect some results which will be used in the description of some sets of regular representations.

It is known that the set R of regular representations of a hereditary tame algebra A is an abelian exact subcategory of $\text{mod } A$, and the structure of this subcategory has been completely determined ([18], [20], [33]). There is a quadratic form q associated to A , which is, in this case, positive semi-definite. It vanishes precisely on a one-dimensional subspace N of Q^n which is generated by a vector $\underline{h} = (h_i)$ with coefficients $h_i \in N$, such that at least one of the h_i is equal to one. An indecomposable regular representation

X is called homogeneous if the dimension type of each of its simple composition factors (in R) lies in N . The full subcategory H of R of all homogeneous representations is also an abelian exact subcategory of $\text{mod } A$ closed under extensions. Finally, note the following result [20]:

2.2 Theorem: Let A be the hereditary tame algebra given by a Euclidean valued graph (G, d) with a K -modulation M . Then there exists a finite number of simple regular non-homogeneous representations $E_1, \dots, E_r$ such that

$$\{c^s E_t / 0 \leq s < n_t, 1 \leq t \leq r\}$$

(where n_t is the length of the orbit $\dim E_t$ under the Coxeter transformation c) is a complete list of all (mutually non-isomorphic) simple regular non-homogeneous representations $L(M) \pmod{A}$. Moreover

$$R = R(M) = R^{(1)} \times \dots \times R^{(r)} \times H(M)$$

where $R^{(t)}$ ($1 \leq t \leq r$) is uniserial with the simple regular representations given by $O^{(t)} = \{c^s E_t / 0 \leq s < n_t\}$. The indecomposable representations in $R^{(t)}$ are uniserial with the composition factor from $O^{(t)}$ ordered in a descending sequence corresponding to the action of c . Each of them is determined by its (simple regular) socle and its (regular) length, which can both be arbitrary.

If X is in R we denote by $\text{soc}_R(X)$ its socle inside R and by $|X|_R$ its regular length. Given E simple regular, and $n \in \mathbb{N}$, there exists a unique indecomposable regular representation $E^{(n)}$ with $\text{Soc}_R(E^{(n)}) = E$ and regular length n . Note that every regular representation is of the form $E^{(n)}$, for some E and some n . ([18], [20]).

It is important to note that the number of simple (regular) representations in $R^{(t)}$ is finite and that $r \leq 3$ in (2.2).

Moreover, for a simple regular representation $E \in R^{(t)}$, the representations $E^{(m)}$, for $m \geq n_t$, have $\dim E^{(m)} = h + x$, with $x \geq 0$. Therefore for any preprojective representation P and any preinjective representation I , $\text{Hom}(P, E^{(n)}) \neq 0 \neq \text{Hom}(E^{(n)}, I)$.

2.3 Let us now describe the Auslander-Reiter sequences in R [36].

Let X be an indecomposable regular representation, then $X = E^{(n)}$, where $E = \text{Soc}_R(X)$ and $n = \text{length of } X \text{ in } R$. We have a chain

$$X_1 \subset X_2 \subset \dots \subset X_n = X$$

of irreducible monomorphisms [34], where $X_i = E^{(i)}$ for $1 \leq i \leq n$.

This corresponds to a composition series inside the category R . The factor

X_i / X_{i-1} , which is simple regular, is isomorphic to $E^{(i)}$ for

$1 \leq i \leq n$, and for $1 \leq j \leq i \leq n-1$, the sequences:

$$0 \rightarrow E^{(i)}/E^{(j-1)} \rightarrow E^{(i)}/E^{(j)} \rightarrow E^{(i+1)}/E^{(j-1)} \rightarrow E^{(i+1)}/E^{(j)} \rightarrow 0$$

are Auslander-Reiter sequences [34]. Moreover, we have that

$$X = E^{(n)} \rightarrow E^{(n)}/E^{(1)} \rightarrow \dots \rightarrow E^{(n)}/E^{(n-1)}$$

is a chain of irreducible epimorphisms.

The indecomposable modules in any R_t form a component C_t of the Auslander-Reiter graph, and the underlying graph of this component only depends on n_t .

(The dotted lines have to be identified in order to form a cylinder)

2.4 We now study some chain of irreducible maps in each subcategory $R^{(t)}$ ($1 \leq t \leq r$). We fix the simple regular representation E_t (2.2) for $t = 1, \dots, r$, and therefore every regular representation X in $R^{(t)}$ is of the form $\tau^{-1}(E_t^{(m)})$ for some $m \geq 0$ and $i \in \mathbb{Z}$. We shall denote X by $[i, m]$. Of course $[i + n_t, m] = [i, m] = [i - n_t, m]$.

For the regular representation $[i, n]$, we define the following sets of regular representations:

$$S_1(i, m) = \{[i + j, m - j] / 0 \leq j < m\}$$

$$S_2(i, m) = \{[i, j] / m \leq j\}$$

$$S_3(i, m) = \{[i - j, m + j] / 0 \leq j\}$$

$$S_4(i, m) = \{[i, j] / 1 \leq j \leq m\}$$

They induce a subgraph of the component C_t that we illustrate in the next figure.

(2.4.1) Lemma: For $m \leq n_t - 1$ and $i \in \mathbb{Z}$, we have

a) $[j, s] \in S_k(i, m)$ for some $k \in \{1, 2\}$ if and only if

$\text{Hom}([i,m], [j,s]) \neq 0$ and $\text{Hom}([i+1,m], [j,s]) = 0$.

b) $[j,s] \in S_k(1,m)$ for some $k \in \{3,4\}$ if and only if $\text{Hom}([j,s], [i,m]) \neq 0$ and $\text{Hom}([j,s], [i-1,m]) \neq 0$.

Proof: If $[j,s] \in S_k(1,m)$ for $k = 1(2)$, then there exists a chain of irreducible epimorphisms (monomorphisms) from $[i,m]$ to $[j,s]$, and therefore $\text{Hom}([i,m], [j,s]) \neq 0$.

Consider now $f \in \text{Hom}([i+1,m], [j,s])$. If $f \neq 0$, there must exist a chain of irreducible maps from $[i+1,m]$ to $[j,s] = [j+n_t, s]$, and then such a path can be factorized over the representations $[i+m, 1], [i+m, 2], [i+m+1, 1], \dots, [i+n_t, 1], [i+n_t, m]$, and since $\text{Hom}([i+m, 1], [i+m+1, 1]) = 0$ (cf. [20]), we obtain that $f = 0$. Hence $\text{Hom}([i+1,m], [j,s]) = 0$.

Conversely, assume $\text{Hom}([i,m], [j,s]) \neq 0$ and $\text{Hom}([i+1,m], [j,s]) = 0$. Then there exists a sequence σ of irreducible maps from $[i,m]$ to $[j,s]$. Such a sequence factors over a representation $[j', s']$ in $S_k(1,m)$. If $[j', s'] \neq [j,s]$, then using a representation $[j'', s''] \in \{[j', s'+1], [j'+1, s'-1]\}$, we obtain a sequence σ' of irreducible map from $[i+1,m]$ to $[j,s]$. Since σ factors through σ' , the sequence σ' has a non-zero composition, hence $\text{Hom}([i+1,m], [j,s]) \neq 0$. But this contradicts our assumption, and therefore $[j', s'] = [j,s]$ and $[j,s] \in S_k(1,m)$ for some $k \in \{1,2\}$.

A similar argument proves b).

(2.5) Here, we shall concentrate on the maps between preprojective representations and regular representations in $R^{(t)}$, where

$$G \in \{ \bar{A}_n, \bar{B}_n, \bar{C}_n, \bar{BC}_n, \bar{BD}_n, \bar{CD}_n, \bar{D}_n \}$$

Using the tables in [20], we can see that for each $i \in G$ there exists at least one and at most two simple regular non-homogeneous representations E such that $(\dim E)_i \neq 0$, and by applying, if necessary, powers of τ , it is easy to see that for any preprojective representation P , there is at least one (and at most two) simple regular non-homogeneous representation E such that $\text{Hom}(P(i), E) \neq 0$. If there is only one such representation E , we denote E by E_1 and put $\ell_P(E_1) = n_t$. If there are two such representations E , we denote them by E_1 and E_2 . Note that there exist two integers t_1 and t_2 such that $\tau^{t_1} E_1 = E_2$ and $\tau^{t_2} E_2 = E_1$. We assume that $t_1 \geq t_2$ and let $\ell_P(E_i) = t_i$ for the minimal value of $t_i \geq 0$ ($i=1,2$).

Let P be a preprojective representation and put

$$S_1(P) = \{\tau^j E_1^{(j)} / 0 \leq j \leq \ell_P(E_1) - 2\}$$

(2.5.1) Lemma: Let P be a preprojective representation, and $N(P)$ the set of indecomposable regular representation X satisfying the property (N): $X \in R^{(t)}$, $\text{Hom}(P, X) \neq 0$ and $\text{Hom}(P, \tau X) = 0$.

$$\text{Then } N(P) = \bigcup_{1 \leq i \leq 2} S_i(P).$$

Proof: Let $X \in N(P)$ of regular length d with the composition factors $S, \tau S, \dots, \tau^{d-1} S$ (for a simple regular representation S). Since $\text{Hom}(P, X) \neq 0$, there exists

j ($0 \leq j \leq d-1$) such that $\text{Hom}(P, \tau^j S) \neq 0$. Since the composition factors of τX are $\tau^1 S, \tau^2 S, \tau^3 S, \dots, \tau^d S$ and $\text{Hom}(P, \tau X) = 0$ we have that $\text{Hom}(P, \tau^k S) = 0$ for $1 \leq k \leq d$, and hence $j = 0$ and S must be isomorphic to E_i for some i . Thus $X \in \bigcup_{1 \leq i \leq 2} S_i(P)$.

Conversely, let $X \in \bigcup_{1 \leq i \leq 2} S_i(P)$, then the composition factors of X are: $E_1, \tau E_1, \dots, \tau^{d-1} E_1$ where $d \leq \ell_P(E_1) - 2$; and by definition $\text{Hom}(P, \tau^k E_1) = 0$ for $k = 1, \dots, d$. Therefore $X \in N(P)$.

Note that $S_i(P)$ is a presection of Γ_A ending at E_i , and for every j , a non-zero map $f: P \rightarrow \tau^{j_1} E_i^{(j)}$ must factor over $\tau^{j_1} E_i$ where $j_1 = \ell_P(E_i) - 2$. Moreover

$$\dim(\text{Hom}(P, \tau^{j_1} E_i^{(j)})_{\text{End}(P)}) \leq 2.$$

Finally, we mention that for every regular representation $X \in R^{(t)}$, such that $\text{Hom}(P, \tau X) = 0$, its composition factors belong to the sets $\{E_1, \tau E_1, \dots, \tau^{j_1} E_1\}$ for $1 \leq i \leq 2$.

(2.52) Corollary: Let A be a tame hereditary algebra of type $G \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{E}_6, \tilde{F}_4, \tilde{G}_2\}$. Then for each indecomposable preprojective P the set $R(P) = \{M \in R / \text{Hom}(P, M) \neq 0, \text{Hom}(P, \tau M) = 0\}$ is the union of at most three linear subgraphs in regular components of Γ_A . (There are three linear subgraphs only in the case $\tilde{D}_n$).

2.6. Remarks:

- a) Assume we have a chain (non-oriented) of non-zero maps
- $$P \rightarrow R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow \dots \rightarrow R_t \rightarrow I$$
- where P is an indecomposable projective, I is an indecomposable injective and R_i is an indecomposable regular ($i = 1, \dots, t$) such that $\text{Ext}(M, M) = 0$.

where $M = P \otimes \bigotimes_{i=1}^t R_i \otimes I$. Hence if $E, \tau E, \dots, \tau^d E$ are the composition factors of R_1 , the composition factors of R_i ($i = 1, \dots, t$) are $\tau^{r_i} E, \dots, \tau^{s_i} E$, for some $1 \leq r_i \leq s_i \leq d$, because $\text{Ext}(R_i, P) = 0$, $\text{Hom}(P, \tau^i E) = 0$ for $1 \leq i \leq d+1$.

Using the same argument for the preinjective I , we obtain that

$$R_1 = R_t.$$

b) Let P_1 and P_2 be two preprojective representations and R be regular, then if $\text{Ext}(P_1 \otimes P_2 \otimes R, P_1 \otimes P_2 \otimes R) = 0$, $\text{Hom}(P_1, R) = 0$ and $\text{Hom}(P_2, R) \neq 0$, then $R \in (S_1(P_1) \cup S_2(P_2)) \cap (S_1(P_2) \cup S_2(P_2))$, and therefore the simple regular representations E_1 corresponding to P_1 and E_2 corresponding to P_2 (see (2.5)) are isomorphic. However, this is possible only when

$\text{Hom}(P_1, P_2) \neq 0$ or $\text{Hom}(P_2, P_1) \neq 0$. So given a preprojective representation P and a regular representation $R \in S_1(P) \cup S_2(P)$, the set

$$\{P' \text{ preprojective representation} / \text{Hom}(P', R) \neq 0, \text{Hom}(P', \tau R) = 0\}$$

is contained in one and only one sectional path containing P .

c) Let R be an indecomposable regular representation in $R^{(t)}$.

Assume $R = [i, m]$ (see (2.4)). If $M = [j, s] \in S_1(i, m)$ and $N = [j', s'] \in S_2(j, s)$, then $\text{Ext}(N, R) = D \text{Hom}(R, \tau N) \neq 0$. Therefore if $\text{Ext}(X', R) = 0$, then $N' \in S_4(j, s)$ or $N' \in s, (j, s)$. In the first case $\text{Hom}(N, M) \neq 0$ and in the second $\text{Hom}(R, N') \neq 0$. By induction we have that for any sequence of regular indecomposable $R_1, \dots, R_k$ with either $\text{Hom}(R_i, R_{i+1}) \neq 0$ or $\text{Hom}(R_{i+1}, R_i) \neq 0$, and

$$\text{Ext} \left(\bigotimes_{i=1}^k R_i, \bigotimes_{i=1}^k R_i \right) = 0, \text{ a sequence of non-zero maps } R_{i_1} \rightarrow R_{i_2} \rightarrow \dots$$

.. $R_{i_{s-1}} \rightarrow R_{i_s}$, with either $|R_{i_1}| > |R_{i_j}|$ ($2 \leq j \leq s$) or $|R_{i_j}| < |R_{i_s}|$ ($1 \leq j \leq s-1$)

has non-zero composition. (see figure below)

3. THE PREPROJECTIVE COMPONENT

In this section we study the preprojective component of $L(M)$, where $M = (F_1, M_1)$ is a K -modulation for the graph $G \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n\}$.

(3.1) We shall denote by $P(q, r)$ the representation $\tau^{-r} P(q)$. Let us first present the following well-known result:

(3.1.1) Lemma ([24],[34]). Assume that X and Y are indecomposable representations and that there exists an irreducible map $X \rightarrow Y$. If one of the modules X, Y is preprojective, then both are. Furthermore, if $X = P(q, r)$ and $Y = P(q', s)$, then either $s = r$ and $q' \rightarrow q$ or $s = r + 1$ and $q \rightarrow q'$.

(3.1.2.) Lemma Let M be a preprojective representation and

$$0 \rightarrow M \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix}} \bigoplus_{i=1}^t M_i \xrightarrow{\begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}} N \rightarrow 0$$

be the Auslander-Reiter sequence starting with M , with the M_i indecomposables. Then

- i) If $t = 2$ the maps f_1, f_2, g_1 and g_2 are irreducible monomorphisms.
- ii) If $t = 3$ with M not a simple projective, then the maps g_1, g_2 , and g_3 are irreducible monomorphisms, two of the maps f_1, f_2 , and f_3 are irreducible epimorphisms, and the third one is an irreducible monomorphism. In this case $M_1 \xrightarrow{\sim} M_2$ and $M_3 \xrightarrow{\sim} M_1$.

Proof: Let $M = P(q, r)$. We use induction on the position $\text{pos}(M)$ of M , which is defined by $\text{pos}(M) = (n+1)r + q$.

The first case we have to consider is $r = 0$ and $q = 1$. Consider the Auslander-Reiter sequence starting with M ,

$$(1) \quad 0 \rightarrow M \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix}} \bigoplus_{i=1}^t M_i \xrightarrow{\begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}} \tau^{-1} M \rightarrow 0$$

with M_i indecomposable.

Since M is a simple projective, the M_i ($i = 1, \dots, t$) are projectives. If $t \geq 2$ then

$$|\tau^{-1}M| = \sum_{i=1}^t |M_i| - |M| > |M_j| \quad \text{for } 1 \leq j \leq t$$

and the result is true for $M = P(1,0)$. We may assume it holds for

$$M = P(q,0) \text{ for any sink } q.$$

Now, we prove the result for $M = P(q+1, 0)$ for $q > 1$. If M_1 is projective the map f_1 is an irreducible monomorphism. If $M_1 = P(j,1)$ for some j , then by lemma 1 $q+1 > j$ and the result is valid for M_1 .

We have that the Auslander-Reiter sequence starting at $P(j, 0)$ is:

$$0 \rightarrow P(j,0) \xrightarrow{(f'_1)} P(q+1, 0) \xrightarrow{\oplus_{i=1}^{t'-1} (g'_i)} M'_1 \rightarrow P(j, 1).$$

Let us consider the case $t' \geq 2$. The

maps $(f'_1), (g'_1)$ are monomorphisms, in particular $|P(q+1, 0)| < |P(j, 1)|$.

The last inequality implies that the map f_1 is a monomorphism, and then $|\tau^{-1}M| > |M_1|$, from which it follows that all the maps (g_i) are monomorphisms.

Now, consider the case $t' = 1$, hence $|P(q+1, 0)| > |P(j, 1)|$ and the Auslander-Reiter sequence (1) is now:

$$0 \rightarrow P(q+1, 0) \xrightarrow{(f_1)} P(j,1) \oplus P(j,1) \oplus M_3 \xrightarrow{(g_1)} P(q+2, 0) \rightarrow 0,$$

$$\text{and } |P(j, 1)| = |P(q+1, 0)| - |P(j, 0)| = |P(q+1, 0)| - 1$$

(here $P(j, 0)$ is simple projective).

Then the maps f_1 and f_2 are epimorphisms, and f_3 is a monomorphism either by the induction hypothesis over M_3 or because M_3 is projective. Finally, we have that

$$|P(q+2, 0)| = |M_3| + 2|P(j,1)| - |P(q+1, 0)| > 2|P(j,1)|$$

$$\text{and since } |P(q+1, 0)| = |P(j,1)| + 1; |P(q+2,0)| > |M_3|$$

Therefore the maps g_1, g_2, g_3 are monomorphism. This finishes the proof for $M = P(q+1, 0)$ and hence for every projective. Note the inequality $|\tau M| > 2 |M_1|$ when $t = 3$. ($M_1 \cong M_2$ and $M_3 \neq M_1$).

Now assume the result true for $P(q, r)$ with $r = 1, \dots, n+1$. We shall show that it holds for $P(q+1, r)$.

Consider the following Auslander-Reiter sequences:

$$0 \rightarrow P(q, r) \xrightarrow{(f'_1)} \bigoplus_{i=1}^t M_i \xrightarrow{(g'_1)} P(q, r+1) \rightarrow 0$$

and

$$0 \rightarrow P(q, r+1) \xrightarrow{(f_1)} \bigoplus_{i=1}^t \tau^{-1} M_i \xrightarrow{(g_1)} P(q, r+2) \rightarrow 0$$

If $t \geq 2$ the maps (g'_i) are monomorphisms. Hence $|P(q, r+1)| > |M_1|$, and $t = 2$ implies that $\alpha(\tau^{-1} M_1) = 2$ and the result follows, because $|M_1| < |P(q, r+1)| < |\tau^{-1} M_1|$ for $i = 1, 2$.

In the case that $t = 3$, we may assume $M_1 \cong M_2 \neq M_3$ and $\alpha(M_3) = 2$, $\alpha(M_1) = 1$. By the induction hypothesis we have the inequalities:

$$|P(q, r+1)| > |M_1|, |P(q, r+1)| < |\tau^{-1} M_3|$$

and $|P(q, r+1)| > |\tau^{-1} M_1|$. Now

$$2 |\tau^{-1} M_1| + |\tau M_3| - |P(q, r+1)| > 2 |\tau^{-1} M_1| > |\tau^{-1} M_1|$$

$$\begin{aligned} \text{and } |P(q, r+2)| &= 2 |\tau^{-1} M_1| - |P(q, r+1)| + |\tau^{-1} M_3| \\ &= 2 |P(q, r+1)| - 2 |M_1| - |P(q, r+1)| + |\tau^{-1} M_3| \\ &= |P(q, r+1)| - 2 |M_1| + |\tau^{-1} M_3| > |\tau^{-1} M_3|. \end{aligned}$$

This gives our desired result.

(3.2) Now, we apply the above result to obtain one of the main properties about these graphs.

(3.2.1) Lemma Let M, N be two indecomposable preprojective representations. If there exists a chain of irreducible maps from M to N ,

then $\text{Hom}(M, N) \neq 0$.

Proof:

Let γ be the chain of irreducible maps

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \rightarrow M_{t-1} \xrightarrow{f_t} M_t = N.$$

If all the maps $f_1, \dots, f_t$ are monomorphisms then $f_t \dots f_1 \neq 0$ and $\text{Hom}(M, N) \neq 0$.

Let us now assume that there is j such that f_j is an irreducible epimorphism and $f_t \dots f_1 = 0$. Choose j as small as possible. If $j < t$, then we have, by lemma (3.1.2), an Auslander-Reiter sequence

$$0 \rightarrow M_{j-1} \xrightarrow{(f_j, f'_j, f''_j)} M_j \oplus M_j \oplus M''_j \xrightarrow{(f_{j+1}, f'_{j+1}, f''_{j+1})} \tau^{-1} M_j = M_{j+1} \rightarrow 0$$

where the maps f''_j and f''_{j+1} are irreducible monomorphisms. Thus, we have a chain of irreducible maps

$$M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \rightarrow M_{j-1} \xrightarrow{f''_j} M''_j \xrightarrow{f''_{j+1}} M_{j+1} \rightarrow \dots \rightarrow M_{t-1} \rightarrow M_t = N$$

of the same length. Using induction we obtain that $j = t > 1$.

Again the Auslander-Reiter sequence starting with M_{t-1} is:

$$0 \rightarrow M_{t-1} \xrightarrow{(f_t, f'_t, f''_t)} M_t \oplus M_t \oplus M''_t \xrightarrow{(g_1, g_2, g_3)} \tau^{-1} M_{t-1} \rightarrow 0$$

where f''_t and g_3 are monomorphisms. Let $h = (f_{t-1} f_{t-2} \dots f_1)$, then

$$0 = (g_1 f_t + g_2 f'_t + g_3 f''_t) h = g_2 f'_t h + g_3 f''_t h$$

It follows that

$$g_2 f'_t h = -g_3 f''_t h \neq 0$$

because g_3, f''_t and h are monomorphisms, so that $f'_t h \neq 0$

so we conclude that $\text{Hom}(M, N) \neq 0$.

Note that we also have the dual result for (3.1.1), (3.1.2), (3.1.3) and also for the next result.

(3.2.2) Corollary Let M and N be two indecomposable preprojective representations and suppose that $\text{Hom}(M, N) \neq 0$. Then $\text{Ext}(\tau^{-i}N, M) \neq 0$ for $i \geq 1$

Proof:

Since $\text{Ext}(\tau^{-i}N, M) \cong D \text{Hom}(M, \tau^{-i+1}N)$, and we have the following chain of irreducible maps:

$$M \rightarrow \dots \rightarrow N \rightarrow * \rightarrow \tau^{-1}N \rightarrow \dots \rightarrow \tau^{-i+1}N,$$

it follows that $\text{Ext}(\tau^{-i}N, M) \neq 0$.

(3.2.3) Corollary Let i, j be two vertices of G such that there exists an oriented path from i to j . Then:

$$\text{Ext}(\tau^\ell I_j, \tau^{-k}P_i) \neq 0 \text{ for } k \geq 1 \text{ and } \ell \geq 1.$$

(3.2.4) Corollary:

Let M and N be two indecomposable representations.

If M is preprojective, N preinjective and $\text{Ext}(N, M) = 0$, then $\text{Hom}(M, N) = 0$.

Proof: Using shifts by powers of the Auslander-Reiter translation τ , we can assume that $M = P(i)$ for some i .

Now $\text{Hom}(P(i), N) \neq 0$ implies that

$$\text{Hom}(N, I(i)) \neq 0 \text{ and by the dual of (3.2.2), } \text{Hom}(\tau N, I(i)) \neq 0.$$

Therefore $\text{Ext}(N, P(i)) \cong D \text{Hom}(P(i), \tau N) \neq 0$. This contradicts the hypothesis, so we conclude that $\text{Hom}(M, N) = 0$.

(3.3) Let A be a finite dimensional hereditary algebra of type G .

In this section we shall show some partial results about the tilted algebras of type G . Also we shall investigate the preprojective

(preinjective) direct summands of a tilting A -module.

(3.3.1) Proposition: Let A be a finite dimensional hereditary algebra of Type $G \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{E}_n\}$. Then every preprojective (preinjective) tilting module T_A is a slice module and therefore $\text{End } T_A$ is hereditary.

Proof:

Let $T = \bigoplus_{j=1}^n \tau^{-i_j} P(k_j)$. By (3.2.2.) $k_j \neq k_\ell$ if $j \neq \ell$.

Let $k_\ell \rightarrow k_j$ be an arrow of G , then $i_\ell = i_j$ or $i_\ell = i_j + 1$.

Thus it follows that $\{\tau^{-i_j} P(k_j) / j=1, \dots, n\}$ is a complete slice of Γ_A .

(3.3.2) Proposition

Let A be a finite dimensional hereditary algebra of type $G \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{E}_n\}$ and T_A a tilting module without regular direct summands, then T_A is a slice module and $B = \text{End } T_A$ is a hereditary algebra of type G .

Proof: Let $T_A = \bigoplus_{i \in I} X_i \oplus \bigoplus_{j \in J} Y_j$, where X_i is preprojective $i \in I$, and Y_j is preinjective for $j \in J$.

We know that $\text{Hom}(\tau^{-r} P(\ell), \tau^s I(\ell)) \neq 0$, therefore if $X_i = \tau^{-r_i} P(\ell_i)$ and $Y_j = \tau^{r_j} I(\ell_j)$, $\{\ell_i / i \in I \cup J\} = \{1, \dots, n\}$.

In order to prove the result, assume $I \neq \emptyset \neq J$, then there exist $i \in I$ and $j \in J$ such that either $\ell_i \rightarrow \ell_j$ or $\ell_j \rightarrow \ell_i$ is an arrow in G .

Suppose $\ell_i \rightarrow \ell_j$, then $\text{Hom}(P(\ell_j), \tau^{-t} P(\ell_i)) \neq 0$ for $t \geq 0$ and $\text{Hom}(\tau^{-t} P(\ell_i), I(\ell_j)) \neq 0$, so that $\text{Ext}(I(\ell_j), \tau^{-t} P(\ell_i)) \neq 0$ for $t \geq 1$.

It follows that $\text{Ext}(Y_j, X_i) \neq 0$ for some i, j . We obtain a similar result if we suppose $\ell_j \rightarrow \ell_i$.

Therefore either $I = \emptyset$ or $J = \emptyset$. In any case the result follows from (3.3.2).

(3.3.3) Proposition

Let A be an hereditary algebra of type G and P a preprojective (preinjective) representation. Then the set of preprojective (preinjective) representations M such that

$$(*) \quad \text{Hom}(P, M) \neq 0 \quad \text{and} \quad \text{Hom}(\tau^{-1}P, M) = 0 \quad (\text{Hom}(P, \tau M) = 0)$$

is the union of at most two sectional paths starting at P in the preprojective (preinjective) component of Γ_A .

Proof:

Assume M satisfies condition $(*)$, then there exists a chain of irreducible maps γ from P to M :

$$\gamma : P = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t = M.$$

Assume γ is not a sectional path, then let t_0 be as small as possible such that the path

$$P = M_0 \xrightarrow{f_1} \dots \xrightarrow{f_{t_0}} M_{t_0}$$

is not a sectional path. Then $M_{t_0} = \tau^{-1}M_{t_0-2}$ and since we have a path

of irreducible maps from $\tau^{-1}P$ to $\tau^{-1}M_{t_0-2}$, we obtain a path of

irreducible maps from $\tau^{-1}P$ to M , and by (3.2.1), $\text{Hom}(\tau^{-1}P, M) \neq 0$.

However this is a contradiction. Then every path from P to M must be sectional.

Finally, since there are at most two sectional paths starting at P for any preprojective (preinjective), the result follows.

Similarly we have the next result.

(3.3.4) Proposition:

Let A be an hereditary algebra of type G and P a preprojective (preinjective) representation. Then the set of preprojective (preinjective) representations M such that

$$(**) \quad \text{Hom}(M, P) \neq 0 \quad \text{and} \quad \text{Hom}(M, \tau P) = 0 \quad (\text{Hom}(\tau^{-1}M, P) = 0)$$

is the union of at most two sectional paths, ending at P in the preprojective (preinjective) component of Γ_A .

(3.3.5) Definition: Let P be a preprojective representation, then the sectional paths of (3.3.3) will be denoted by $S'_1(P)$ and $S'_2(P)$ (one of them may be empty), and those of the (3.3.4) by $s_1(P)$ and $s_2(P)$.

For a preinjective I we shall denote by $s_1(I)$ and $s_2(I)$ the subgraphs from (3.3.3) and by $S'_1(I)$ and $S'_2(I)$ those from (3.3.4).

(3.4) Here, we study the maps between modules in $S'_1(P) \cup S'_2(P)$ and $S_1(P) \cup S_2(P)$ for a preprojective representation P . Using the notation in [20] and the tables there, we conclude that for $i = 1, n+1$, in each regular component $R^{(t)}$, there is only one simple regular representation E with $\text{Hom}(P(i), E) \neq 0$. If $G \neq \tilde{A}_n$, we have that $R(P(i)) = S_1(P(i))$ for $i = 1, n+1$. Similarly, it is easy to see that $R(P(i)) = S_1(P(i))$ for $i = 2, n$. Finally, for $2 < i < n$, $R(P(i)) = S_1(P(i)) \cup S_2(P(i))$. Of course, the same result can be obtained for preprojective and preinjective representations.

(3.4.1) Lemma:

Let P be a preprojective representation.

(a) If for $N \in S'_1(P)$ $\text{Hom}(N, M) \neq 0$ for some $M \in S_1(P)$, then $\text{Hom}(N, M') = 0$ for any $M' \in S_2(P)$.

(b) If ~~for~~ $N \in s_1(P)$, $\text{Hom}(N, M) \neq 0$ for some $M \in S_1(P)$

then $\text{Hom}(M, M') = 0$ for any $M' \in S_2(P)$.

Proof: By applying power of τ , we may assume that P is projective, and using reflection functors [20], we may in fact assume that P and N are projectives. Moreover if $P = P(i)$ and $N = P(j)$ we may assume that the orientation of Q is such that there is only one source and one sink, that is we have the orientation given in the tables of [20].

It then becomes easy to check the result for each case.

(3.4.2) Definition:

Let M be an indecomposable representation.

Then

$$P_+(M) = \{N / \text{Hom}(N, M) \neq 0, \text{Hom}(N, \tau M) = 0\}$$

$$P_-(M) = \{N / \text{Hom}(M, N) \neq 0, \text{Hom}(\tau^{-1}M, N) = 0\}$$

are called respectively the prefactor set and postfactor sets of M .

From our lemmas above we have the following result:

(3.4.3) Proposition

(a) Let P be a preprojective representation, then:

$$P_+(P) = s_1(P) \cup s_2(P)$$

$$P_-(P) = S_1^1(P) \cup S_2^1(P) \cup S_1(P) \cup S_2(P).$$

(b) Let I be a preinjective representation,

then:

$$P_+(I) = s_1(I) \cup s_2(I) \cup S_1^1(I) \cup S_2^1(I)$$

$$P_-(I) = s_1(I) \cup s_2(I).$$

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$P_*(P)$ is shown in some cases, in the following figures.

CHAPTER IV

TILTED ALGEBRAS OF TYPE

$\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$ AND $\tilde{BC}_n$

In this chapter we shall give a classification of tilted algebras of types $G \in \{\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{BC}_n\}$ in terms of their bounden K-modulations. We shall use the description of their postfactor and prefactor sets given in the previous chapter.

(1.) TILTED ALGEBRAS OF TYPE $\tilde{A}_n$

(1.1) In order to describe the bounden K-modulations appearing in our main result, we need to define a relation on the (oriented, valued) graph:

$$1 \xrightarrow{(2,2)} 2 \longrightarrow 3$$

with the following K-modulation M :

$$E \xrightarrow{E^E_E \oplus E^E_{\epsilon E}} E \xrightarrow{E^E_E} E$$

where E is a skew field, finite dimensional over the central field K and ϵ is an automorphism of E . Let $\phi: (E^E_E \oplus E^E_{\epsilon E}) \otimes E^E_E \longrightarrow E^E_E$ be an epimorphism and put ${}_1R_3 = \text{Ker}\phi$. We call the resulting relation a ϕ -relation.

For example [21] [33] we can consider a proper subbimodule ${}_E N_E$ of

$$(E^E_E \oplus E^E_{\epsilon E}) \otimes E^E_E \xrightarrow{\sim} E^E_E \oplus E^E_{\epsilon E} \text{ and consider the cononical epimorphism:}$$

$$E^E_E \oplus E^E_{\epsilon E} \longrightarrow (E^E_E \oplus E^E_{\epsilon E}) / {}_E N_E$$

As in (I.4.1), we shall denote this relation by

$$1 \xrightarrow{E^E_E} 2 \longrightarrow 3$$

A similar relation can be defined on the K-modulation

$$E \xrightarrow{E^E E} E \xrightarrow{E^E E \oplus E^E \epsilon E} E$$

(1.2) Definition. A K-modulation $M=(F_i, {}_iM_j)$ over an oriented valued graph G with a relation ideal R is said to satisfy the property (α) if:

($\tilde{\alpha}_1$) G has either one non-oriented cycle C with at least three vertices and one distinguished arrow denoted by $i' \longrightarrow i''$, or a proper full subgraph $i' \xrightarrow{(2,2)} i''$ (but not both).

($\tilde{\alpha}_2$) For all i, the F_i are equal $F_i = E$. For all $i \neq i', j \neq i''$,

$${}_iM_j = E^E E \text{ or } 0. \text{ If G contains C, } {}_{i'}M_{i''} = E^E \epsilon E.$$

If G contains $i' \xrightarrow{(2,2)} i''$, ${}_iM_{i''} = E^E E \oplus E^E \epsilon E$, where in both cases the right action ϵE is given by a field automorphism ϵ of E.

($\tilde{\alpha}_3$) Each vertex of G has at most four neighbours.

($\tilde{\alpha}_4$) If G contains a subgraph $i' \xrightarrow{(2,2)} i''$ and if i' or i'' has two neighbours, then

K-submodulation of (M, R) .

($\tilde{\alpha}_5$) If $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) . Moreover if G contains

$i' \xrightarrow{(2,2)} i''$, then $i \nmid i'$ and $i \nmid i''$.

($\tilde{\alpha}_6$) If $i \in G$ has three neighbours; then either

a subgraph $i' \xrightarrow{(2,2)} i''$ with $i \in \{i', i''\}$ then either

is a bounden K-submodulation of (M, R) . *

($\tilde{\alpha}_7$) All relations are of length two and all are zero-relations or ϕ -relations. In addition, in C , the number of oriented counter-clockwise relations equals the number of oriented clockwise relations.

Moreover, the bounden K-modulation (M, R) is said to satisfy property ($\tilde{\alpha}_7$) if, in addition to ($\tilde{\alpha}_1$) - ($\tilde{\alpha}_7$), it satisfies:

($\tilde{\alpha}_0$) There is no bounden K-submodulation of (M, R) of the form:

where i belongs either to C or to $i' \xrightarrow{(2,2)} i''$ and j_2 does not belong to either one.

* We always assume that $i_1^R i_1'' \oplus i_2^R i_1'' = i_1^M i_1''$ or $i_1^R i_1 \oplus i_2^R i_1 = i_1^M i_1''$

when there are no relations in C and i_1, i_2, i_3 , and i_4 belong to C .

In all cases we assume that there is no other relation besides the relation shown.

(1.3) In order to illustrate the property $(\tilde{\alpha})$, let us consider the following four bounden K -modulations. Here we assume that condition $(\tilde{\alpha}_2)$ is satisfied):

M_1 does not satisfy $(\tilde{\alpha})$ because $1''$ has four neighbours, and by $(\tilde{\alpha}_5)$, $1'$ and $1''$ may have at most three neighbours. Since M_2 has only one relation on the submodulation given by the non-oriented cycle, we see that M_2 does not satisfy $(\tilde{\alpha})$. Note that, its Auslander-Reiten graph contains oriented cycles. It is easy to check that M_3 satisfies $(\tilde{\alpha})$, but contains a submodulation of the type described in $(\tilde{\alpha}_0) c)$, and therefore $(\tilde{\alpha}_7)$ fails. The Auslander-Reiten graph of M_3 has only seven projectives (respectively, injectives) in its preprojective (respectively, preinjective) component and any other component has oriented cycles. Thus M_3 is not the bounded K -modulation of a tilted K -algebra. Finally, we have that M_4 satisfies $(\tilde{\alpha}_7)$ and as we shall see, the algebra A given by M_4 and the relations shown is a tilted algebra of type $\bar{A}_8$. Moreover A is of finite representation type; its Auslander-Reiten graph has the following form:

where the dotted lines have to be identified in order to form a cylinder and we put $\underline{1} = [P(1)]$.

(1.4) Theorem: The following conditions are equivalent:

- (I) The bounden K-modulation of B satisfies the property $(\tilde{\alpha}_T)$.
- (II) B is a tilted algebra of type $\bar{A}_n$ with $n \geq 2$.

Proof:

(1.4.1) We first show that (II) implies (I).

Let A be a hereditary algebra of type $\bar{A}$ and T_A be a tilting module such that $B = \text{End } T_A$. We shall denote by $T(i)$ the direct summand of T_A corresponding to the vertex i of the oriented value graph G of the bounden K-modulation (M, R) of B. We shall prove that B satisfies the property $(\tilde{\alpha}_T)$ in the following way: first, we shall prove conditions $(\tilde{\alpha}_2)$, $(\tilde{\alpha}_3)$, $(\tilde{\alpha}_4)$, $(\tilde{\alpha}_5)$, $(\tilde{\alpha}_6)$ and $(\tilde{\alpha}_7)$, and then $(\tilde{\alpha}_0)$ and $(\tilde{\alpha}_1)$.

$(\tilde{\alpha}_2)$ Since $A \xrightarrow{\sim} \text{End}_B T$, the semisimple part of B is isomorphic to the semisimple part of A, and therefore each skew-field F_i in the K-modulation of A must appear in (M, R) . We have:

$${}_i M_j \xrightarrow{\sim} \text{End } P_B(i) \text{Hom}_B(P(j), P(i)) \text{End } P_B(j) \xrightarrow{\sim} \text{End } T_{(i)} \text{Hom}_A(T(j), T(i)) \text{End } T(j)$$

Thus by considering the postfactor and prefactor sets, we conclude that ${}_i M_j$ satisfies the conditions given in $(\tilde{\alpha}_2)$.

$(\tilde{\alpha}_3)$ Let i be a vertex of G and j a neighbour of i . Then $T(j)$ must be one of the sectional paths on $T(i)$, because $\text{Ext}_1(T(i), T(j)) = 0$ and one of the sets $\text{Hom}_A(T(i), T(j))$ and $\text{Hom}(T(j), T(i))$ is non-zero, that is, $T(j)$ belongs to $P_+(T(i)) \cup P_-(T(i))$ (see (3.4.3)). So if k is

another neighbour of i , $T(k)$ and $T(j)$ must either be in different sectional paths, or in the same path but with $T(i)$ between them. So we see that there are only three possibilities for k , and i has at most four neighbours.

($\tilde{\alpha}_4$) If G contains a subgraph of the form $i' \xrightarrow{(2,2)} i''$, we may assume that $T(i'')$ is preprojective, and therefore so is $T(i')$. Using the postfactor set $P_-(T(i'))$ of $T(i')$, it is clear that $T(i')$ and $T(i'')$ are on a complete slice of type $\tilde{A}_n$ with $T(i'')$ as a source and $T(i')$ as a sink, and hence $\text{Hom}_A(T(i''), T(i')) = E_E \oplus E_{\epsilon E}$. If in addition i' has another neighbour j_0 , $T(j_0)$ must be regular. However, we know that if $\text{Hom}_A(X, T(j_0)) \neq 0$ for $X \in S_1(T(i''))$, then $\text{Hom}(Y, T(j_0)) = 0$ for $Y \in S_2(T(i''))$ (see IIL (3.4)), that is

$$\text{End } T(j_0)^{\text{Hom}(T(i''), T(j_0))} \text{End } T(i'') \xrightarrow{\sim} E_E.$$

Thus, there is a relation on the submodulation

$$E \xrightarrow{E_E \oplus E_{\epsilon E}} E \rightarrow E$$

given by an epimorphism $\phi: (E_E \oplus E_{\epsilon E}) \oplus E_E \rightarrow E_E$; that is

$$\begin{array}{c} \text{---} E_E \text{---} \\ \nearrow \quad \searrow \\ j_1 \longrightarrow i' \longrightarrow i'' \end{array}$$

is a bounden K -submodulation of (M, R) .

The case that $T(i'')$ is preinjective is dual. This finishes the proof of ($\tilde{\alpha}_4$) since $T(i'')$ can not be a regular representation (see (III.2.5) and [20])

(α_5) Assume that i has four neighbours, then two of them belong to the postfactor sets of $T(i)$ and the other two to the prefactor sets of $T(i)$. Therefore there must be two zero-relations. Note that if j is a neighbour of i , then $T(j)$ belongs to only one of the sets $s_1(T(i))$, $s_2(T(i))$, $s'_1(T(i))$, and $s'_2(T(i))$ when $T(i)$ is not regular. That is to say, (M, R) can contain the bounden K -submodulation $i' \xrightarrow{(2,2)} i''$ with $i'=i$ or $i''=i$. If $T(i)$ is regular, we conclude from (III. 2.5) that such a graph is not a subgraph of G .

(α_6) Let us suppose that G contains the subgraph:

then $T(i')$, $T(i'')$ are preinjectives, and $T(i_1)$, $T(i_2)$ are regulars. Therefore, $T(i_1) \in s_1(T(i''))$, $T(i_2) \in s_2(T(i''))$ and $T(i'') \in s_1(T(i_1)) \cap s_2(T(i_2))$. Hence

$$\begin{array}{l} \text{Hom}(T(i_1), T(i'')) \quad \text{End } T(i_1) \xrightarrow{\sim} E^E_E, \text{ and} \\ \text{End } T(i') \end{array}$$

$$\begin{array}{l} \text{Hom}(T(i_2), T(i'')) \quad \text{End } T(i_2) \xrightarrow{\sim} E^E_E \\ \text{End } T(i') \end{array}$$

Thus

is a bounden K -submodulation of (M, R) . Using the same argument that we used to prove (α_5) we prove that condition (α_6) is satisfied.

(α_7) Assume we have a bounden K-submodulation in (M, R) of the form:

$$\begin{array}{c} \text{---} X \text{---} \\ j_s \rightarrow \dots \rightarrow j_2 \rightarrow j_1 \end{array} \quad (1)$$

Since there is no relation between preprojective direct summands of T , we must consider two cases. First, if $T(j_1), \dots, T(j_2)$ are preprojectives and $T(j_{r+1})$ is regular (note that $T(j_{r+1})$ cannot be preinjective), then if $T(j_{r+2})$ is preinjective, there is a zero-relation between j_r and j_{r+2} , and therefore $r=1$ and $s=3$. Second, if $T(j_{r+2})$ is regular, we know that (cf. III.2.6) $T(j_k)$ is regular for $r < k \leq s$, and that there is a sequence of epimorphisms:

$$T(j_{r+1}) \longrightarrow T(j_{r+2}) \longrightarrow \dots \longrightarrow T(j_r) \longrightarrow R$$

where R is a simple regular representation with $\text{Hom}(T(j_r), R) \neq 0$.

Moreover, $\text{Hom}(T(j_r), T(j_k)) \xrightarrow{\sim} \text{Hom}(j_r, R)$ for $r < k \leq s$. Thus $r=1, s=3$. In both cases, it follows that

$$\begin{array}{c} \text{Hom}(T(j_1), T(j_3)) \\ \text{End } T(j_3) \end{array} \xrightarrow{\sim} \begin{array}{c} \text{End } T(j_1) \\ E^E E \end{array}$$

Therefore, we obtain that the bounden K-submodulation (i) has a relation of length two and since $X=E$ or 0 such a relation, if non-trivial, is either a zero-relation or a ϕ -relation.

Let us now consider a (non-linear) relation of the form:

$$\begin{array}{ccccc} & j_s & \longrightarrow & \dots & \longrightarrow & j_1 \\ b & \swarrow & & & & \searrow & a \\ & i_r & \longrightarrow & \dots & \longrightarrow & i_1 \end{array} \quad (2)$$

As before, we may assume that $T(a)$, $T(i_1)$, ..., $T(i_q)$, $T(j_1)$, ..., $T(j_p)$ are preprojective, and $T(i_{q+1})$, $T(j_{p+1})$ are regular. If $T(i_{q+2})$ were regular, by (III.2.6), all of them would also be regular. This is a contradiction to our hypothesis. Similarly, $T(j_{p+2})$ is not regular.

Thus $T(i_{q+2})$, $T(j_{p+2})$ are preinjective and there are two zero-relations between i_q and i_{q+2} , and between j_p and j_{p+2} , respectively. (Note that the modules $T(k)$, $k \in \{i_{q+1}, \dots, i_r, j_{p+1}, \dots, j_r, b\}$, are preinjective). We conclude that there are two zero-relations between a and b , one relation on each path from b to a .

If all the modules $T(k)$ ($k=a, b, i_1, \dots, i_r, j_1, \dots, j_r$) are preprojective, they lie on a complete slice and therefore there is no relation.

Since for each zero-relation $i \xrightarrow{\quad} j \xrightarrow{\quad} k$ on C , the modules $T(i)$, $T(j)$ and $T(k)$ belong to different components of Γ_A , we conclude that the number of (zero)-relations on C must be even.

We now prove condition $(\tilde{\alpha}_0)$.

(a) Assume that (M, R) contains a bounden K -submodulation of the form:

$$F_{j_1} \xleftarrow{\quad X \quad} F_{j_2} \xleftarrow{\quad} F_{j_3} \xleftarrow{\quad} \dots \xleftarrow{\quad} F_{j_{t-2}} \xleftarrow{\quad Y \quad} F_{j_{t-1}} \xleftarrow{\quad} F_{j_t}$$

where X and Y equal either E or 0 , $t \geq 4$, and we suppose that there is no other relation besides the relation shown.

We have to consider two cases:

I) $t > 4$

i) If $T(i_1)$ is preprojective and $T(i_2)$ regular, then $T(i_3)$ must be preinjective. Since there is no relation between i_2 and i_{t-1} , all the modules $T(j_i)$ $3 \leq i \leq t-1$ must be preinjective. This implies that $T(j_{t-1})$ and $T(j_t)$ must be preinjectives too, (here we use that $\text{Hom}(T(j_{t-2}), T(j_{t-1})) \neq 0$), a contradiction.

ii) If $T(j_1)$ and $T(j_2)$ are preprojective, then $T(j_3)$ must be regular and $T(j_4)$ preinjective, so $\text{Hom}(T(j_2), T(j_4)) = 0$ and $t=4$.

iii) For $T(j_1)$ preinjective, it follows immediately that such a relation does not exist. (All modules are preinjective).

iv) If $T(j_1)$ is regular, we just use (III(2.6)) and the postfactor set of $T(j_1)$ to obtain a contradiction.

II) $t=4$.

In this case, we shall see that B has global dimension of at least 3. However, this is impossible, because a tilted algebra has at most global dimension 2. [28]

First, we note that B is of finite representation type, because T_4 contains non-zero preprojective and non-zero preinjective direct summands. Therefore G does not contain the subgraph $1 \xrightarrow{(2,2)} 1$ and so it follows from the proof of [1] Proposition (IV.3.3) that B has global dimension at least 3.

b) We now consider the case where (M, R) contains the bounden K -submodulation

$$P_{j_1} \xrightarrow{\quad} P_{j_2} \xrightarrow{\quad} \dots \xrightarrow{\quad} P_{j_{t-2}} \xrightarrow{\quad} P_{j_{t-1}} \xrightarrow{\quad} P_{j_t}$$

(A dashed arc connects $P_{j_{t-2}}$ and P_{j_t} above the main sequence of arrows.)

where j_1 is on the path C joining i' and i'' , and $j_2, \dots, j_t$ are not.

i) If $T(j_1)$ is regular, then $T(j_2), \dots, T(j_t)$ must be regular because for any other neighbour a of j_1 $T(a)$ must be either preinjective or preprojective. Then using (III.2.6) we obtain a contradiction.

ii) If $T(j_1), \dots, T(j_r)$ are preprojective for $1 \leq r$, and $T(j_{r+1})$ regular we apply (III.2.6) and get a contradiction.

iii) Finally, the case when $T(j_1)$ is preinjective is treated in the same way.

c) The proof for this case is similar to the previous one.

Notice however that for any path starting at a vertex lying in a path joining i' to i'' (in this case), the corresponding summands of T_A are either all preprojective, all regular or all preinjective.

($\bar{\alpha}_1$) Assume that G_B contains two non-oriented cycles λ_1 and λ_2 , and let ω be a path joining them. Denote by i and j respectively a vertex in $\omega \cap \lambda_1$, and a vertex in $\omega \cap \lambda_2$

We have shown that (M, R) satisfies condition $(\bar{\alpha}_0)$, and since a and b have three neighbours, each one of them is the middle vertex of a zero-relation so we obtain a contradiction with $(\bar{\alpha}_0)b$. We note that if the zero-relations lie on cycles, the cycles must contain at least two relations each and the contradiction with $(\bar{\alpha}_0)b$ follows.

(1.4.2) Now we want to prove the converse. Assume that (M, R) satisfies $(\bar{\alpha}_f)$. We shall prove that Γ_B contains a complete slice in one of its components (preprojective or preinjective).

(1) We first consider the case when there exist zero-relations on the cycle C . We shall prove that in this case B is of finite representation type.

The graph G of (M, R) has the following form:

where only one relation is shown. In order to see that B is of finite representation type, we show that there exists an algebra B' and a function ϕ with the following properties:

- a) B' is of finite representation type
- b) The function $\phi: \text{mod } B' \rightarrow \text{mod } B$ induces a surjective between $\text{ind } (B')$ and $\text{ind } (B)$.

First, we define G' to be the graph of the form:

where G_1' and G_1'' are isomorphic to G_0 . Let B' be the algebra arising from the induced bounden K -modulation (M', R') of G' , then B' is iterated tilted of type $A_{n+2} [1]$, hence it is of finite representation type. Moreover, for every representation M and every edge $i \rightarrow j$ such that $M_i \neq 0$ and $M_j \neq 0$, the map $M_i \rightarrow M_j$ is an isomorphism.

Now, let M be indecomposable in $\text{mod } B$ and M_0 its restriction to the subgraph:

Then $M_0 = M'_0 \oplus M''_0$ where $(M'_0)_{a_3} \neq 0$ and $(M''_0)_{a_3} = 0$; and since the map $(M_0)_{a_1} \rightarrow (M_0)_{a_3}$ is a monomorphism, we obtain that $\text{Im}(M_{a_2} \rightarrow M_{a_1}) \subseteq (M''_0)_{a_1}$.

Consider the canonical function $\phi: \text{mod } (B') \rightarrow \text{mod } (B)$ which associates to a representation $N = (N_{i,j}, \psi_{i,j})_{i,j \in I'}$ of G' the representation M which $M|_{Q_i} = N|_{Q_i}$, where $Q_i = G \setminus G_1$ and $Q_1' = G' \setminus G_1'$, $M_1 = N_1 \oplus N_1''$ for $1 \in G_1'$, $1' \in G_1''$, and $1 \in G$. For i, j we take the direct sums of the given linear transformation.

If $M \in \text{mod } B$ we construct the representation N of G' as follows:

$N|_{Q_1'} = M|_{Q_1'}$, $N|_{G_1'} = M'_0$, $N|_{G_1''} = M''_0$, together with the maps $(M)_{a_2} \rightarrow (M)_{a_1}$ and

$(M)_{a_1} \longrightarrow (M)_{a_3}$. We then have $\phi(N)=M$. This is the unique representation in the case $(M)_{a_3} \neq 0$ or $(M)_{a_2} \neq 0$. If both $(M)_{a_3} = 0 = (M)_{a_2}$, then $M'_0 = 0$, and there is a second representation N' of G' with $\phi(N')=M$, namely with $N'|_{G'_1} = M_0$. This proves that ϕ has the properties a) and b) and therefore our claim is true.

By the above proof, we have, in particular, that for every indecomposable representation M of (M, R) and any zero-relation ${}_i R_j$ on M either $M_i = 0$ or $M_j = 0$.

(11) Next, we show that Γ_B has non-oriented cycles. Assume Γ_B contains an oriented cycle.

$$(1) \quad M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_t} M_t \xrightarrow{\phi_t} M_0$$

Since we have assumed that there are at least two zero-relations on C , the graph G has the form illustrated in the next figure:

Note that for each a_i ($i=1,2,3,4,5,6$) there exist j_i such that $a_i \in \text{Supp}(M_{j_i})$, otherwise the cycle (1) would correspond to a cycle in the Auslander-Reiten graph of a tilted algebra of type A_m (for some m) which is a contradiction.

Thus we may assume that $\text{Supp}(M_0) \subseteq Q_1 \cup Q_2 \cup Q_3$ and $a_1 \in \text{Supp}(M_1) \subseteq Q_1 \cup Q_4 \cup Q_5$. Since ϕ_i ($i=0, \dots, t$) is either a monomorphism or epimorphism and ϕ_0

is not a epimorphism, $\text{Supp}(M_0) \subseteq Q_1 \cup Q_3$.

Let us assume that $\text{Supp}(M_a) \subseteq Q_1$. From the commutative diagram

$$\begin{array}{ccc} (M_0)_{a_2} & \xrightarrow{\phi_{1a2}} & (M_1)_{a_2} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & (M_1)_{a_1} \end{array} \quad M_1(a_2, a_3)$$

we obtain either $M_2(a_2, a_1) = 0$ or $(M_1)_{a_2} = 0$. But in the first case we have that M_1 is decomposable, and in the second cases $\text{Supp } M_1 \cap \text{Supp } M_0 = \emptyset$. However, both cases give us contradictions. We have thus proved our claim.

(iii) Now we show that Γ_B contains a complete slice of type $\tilde{A}_n$. Let $s_1, \dots, s_t$ be the sources of C such that C contains a full subgraph of the form:

$$0 \leftarrow \cdots \leftarrow 0 \xrightarrow{s_1} 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0$$

for each $i=1, \dots, t$. We consider $P(s_1), \dots, P(s_t)$ the corresponding indecomposable projective B -modules, and define the set S to be the set of those vertices $[M]$ of Γ_B such that if there exists an oriented path from $[M]$ to $[P(s_i)]$ for some i , then this path is sectional. We shall show that S contains one and only one module of the form $\tau^{-n} P_i$ for some $n \in \mathbb{N}$ and some projective module P_i .

(a) For every indecomposable projective B -module P , there exists an oriented path from P to S .

Let P be the indecomposable projective B -module corresponding to the vertex j . If j is not on the cycle C , then by the condition $(\tilde{\alpha}_0)$, there exists a non-oriented path from j to some source s_j of C with one of the following forms:

here we put $s = 1$.

in all cases b_j ($0 \leq s \leq p$) are on the cycle C , b_p is a source of C and there is no other relation besides the relations shown.

Let us now consider the representation M defined by

$$\text{Supp } M = \{a_1, \dots, a_l, b_1, \dots, b_s\} \cup \{\text{Supp } P(b_p) \cap \text{Supp } P(b_s)\}.$$

The coordinate sector spaces are $M_i = E$ for every $i \in \text{Supp } M$, and the maps between them are the obvious ones. It is clear that M is indecomposable,

$\text{Hom}(P(j), M) \neq 0$, and $\text{Hom}(M, P(b_p)) \neq 0$. Thus we have an oriented path in Γ_B defined by the sequence of maps:

$$P(j) \longrightarrow \dots \longrightarrow M \longrightarrow \dots \longrightarrow P(b_p).$$

Similarly, if j is on the cycle C , we have that G contains a full subgraph of one of the following forms:

$$\text{I)} \quad \begin{array}{ccccccc} 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & \dots & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 \\ j=b_1 & & & & & & b_{s-1} & & b_s & & b_{s+1} & & b_{s+2} & & b_{p-1} & & b_p \end{array}$$

$$\text{II)} \quad \begin{array}{ccccccccccc} 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & \dots & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\ j=b_1 & & b_2 & & b_3 & & b_s & & b_{s+1} & & a_1 & & a_{r-1} & & a_r & & a_r \end{array}$$

where b_p and b_{s+1} are sources in each case and there is no other relation besides the relations shown. Note that in both cases, b_p or b_{s+1} is one of the sources used to defined S . We put $p=s+1$ in the case b).

We define the representation M such that

$$\text{Supp } M = \{b_1, \dots, b_s\} \cup (\text{Supp } P(b_p) \cap \text{Supp } P(b_s))$$

with the obvious coordinate spaces and maps between them.

It is easy to see that M is indecomposable and that we obtain an oriented path from $P(j)$ to $P(b_p)$ or $P(b_{s+1})$ induced by the sequence of maps:

$$P(j) \longrightarrow \dots \longrightarrow M \longrightarrow \dots \longrightarrow P(b_p)$$

or

$$P(j) \longrightarrow \dots \longrightarrow M \longrightarrow \dots \longrightarrow P(b_{s+1})$$

In all cases we see that there is a path from $P(j)$ to S . This finishes the prove of our claim.

(b) We show that if there is a path from one indecomposable injective $I(i)$ to S , such a path is sectional.

Let us assume that for the indecomposable injective $I(i)$, there exists an oriented path in Γ_B :

$$[I(i)] = [M_0] \longrightarrow [M_1] \longrightarrow \dots \longrightarrow [M_t] \longrightarrow [P(s_1)]$$

for some source s_1 . Then M_t is a direct summand of the radical of $P(s_1)$. Also $(M_t)_x \neq 0$ for one neighbour x of s_1 and $(M_t)_y = 0$ for any other neighbour $y \neq x$ of s_1 . Since s_1 is on the cycle C , it has at least two neighbours. By considering the edge $s_1 \longrightarrow y$ we define the algebra A' given by the bounden K -modulation of G' , where G' is defined as follows: the set of vertices of G' is the same as that of G and the set of edges is given by the set of edges of G except the edge $s_1 \longrightarrow y$.

Thus, the algebra A' is a tilted algebra of type A_{n+1} . Since Γ_B has no oriented cycles, $\text{Hom}(P(s_1), M_j) = 0$ for $j = 0, \dots, t$, hence M_j ($j = 0, \dots, t$) are A' -modules, and the path

$$[I_{A'}(i)] = [M_0] \longrightarrow [M_1] \longrightarrow \dots \longrightarrow [M_t] \longrightarrow [P_{A'}(s_1)]$$

is, by lemma (I.7.4.5), a sectional path in $\Gamma_{A'}$. Hence, if the path

$$[M_0] \longrightarrow [M_1] \longrightarrow \dots \longrightarrow [M_t] \longrightarrow [P_B(s_1)]$$

in Γ_B is not sectional, we have that $P(s_1) \cong \tau^{-1} M_j$ for some j . However, this is impossible. This finishes the proof.

(c) Finally, we show that B is a tilted algebra of type A_n .

Since there is a complete slice S in Γ_B , we only have to show that the underlying (unoriented) graph of S is $\tilde{A}_n$.

Let $[M_1]$ be a sink of S , then by construction $M_1 = P(s_1)$ for some source of C , and G contains a full subgraph of one of the forms:

$$I) \quad \begin{array}{cccccccccccccccc} s_1 & a_1 & a_2 & a_3 & a_{r-1} & a_r & a_{r+1} & a_{t-2} & a_{t-1} & a_t & s_j \\ 0 & \rightarrow & 0 & \rightarrow & 0 & \dots & 0 & \rightarrow & 0 & \leftarrow & 0 & \dots & 0 & \leftarrow & 0 & \leftarrow & 0 \end{array}$$

[illegible]

where s_j is a source and there is no other relation besides the relations shown.

We now consider the algebra A' defined in (b) by a neighbour $y + a_2$ of s_1 . Note that every B -module $M (\neq P_B(s_1))$ predecessor of S can be considered as an A' -module. Therefore Γ_B is obtained by using the set of all predecessors of $S \setminus \{P_B(s_1)\}$ in Γ_A , and inserting $P_A(s_1)$. In particular we obtain that every predecessor of $S \setminus \{P_B(s_1)\}$ satisfies the following properties

[1]:

(1) If M is not projective, the Auslander-Reiten sequence ending at M has at most two middle terms.

(2) If M is projective, then for every direct summand M_1 of $\text{rad}(M)$ with a minimal right almost split map $g: N \rightarrow M_1$, N is indecomposable.

Note that for every indecomposable projective B -module P , $\text{rad}(P)$ is the direct sum of at most two indecomposable B -modules. This shows that any indecomposable B -module in S has at most two neighbours.

For the sinks $[P(s_1)]$ and $[P(s_j)]$ of S we define a module M_{1j} as follows:

$$\text{I) } \text{Supp } M_{1j} = \{\text{Supp } P(s_1) \cup \text{Supp } P(s_j)\} \cap \text{Supp } P(a_r)$$

$$\text{II) } \text{Supp } M_{1j} = \{a_r, a_{r+1}, \dots, a_{s-1}, a_s\} \cup \{\text{Supp } P(s_1) \cap \text{Supp } P(a_r)\} \cup$$

$$\cup \{\text{Supp } P(s_j) \cap \text{Supp } P(a_s)\}.$$

In both cases the coordinate vector spaces are $(M_{1j})_k = E$ for every $k \in \text{Supp } M_{1j}$, and the maps between them are the obvious ones. We also have that M_{1j} is indecomposable, $\text{Hom}(M_{1j}, P(s_1)) \neq 0$ and $\text{Hom}(M_{1j}, P(s_j)) \neq 0$. By using the algebra A' again, we see that the paths

$$[M_{1j}] \rightarrow \dots \rightarrow [P(s_1)], [M_{1j}] \rightarrow \dots \rightarrow [P(s_j)]$$

in Γ_B are sectional paths.

Now, $[M_2] = [P(s_j)]$ is a sink of S , and, using the same argument as before we can find a sink $[M_3] = [P(s_k)]$ (in general $M_3 \neq M_1$) and a module M_{jk} such that there are sectional paths from $[M_{jk}]$ to $[M_3]$ and to $[M_2]$. By

induction we obtain a sequence $M_1, \dots, M_t$ of sinks of S and a sequence of $M_{12}, M_{23}, \dots, M_{t1}$ of B -modules (sources of S) together with sectional paths

Hence the complete slice contains a non-oriented cycle. Thus, S has as underlying graph $\tilde{A}_n$. This completes the proof in the case where there exist zero-relations on the cycle C .

(iv) Let us now assume that there is no relation on C .

Let a be a source of C . By the condition $(\tilde{\alpha}_0)$ we may assume that every edge with only one vertex in C has its target in C . Thus a is a source of G , and there exist b and c vertices of G such that each of them is connected to a by one edge. Thus the graph G is of the form:

Then we define the graph G' to be the oriented graph of the form:

with the same G_2 . Note that G_2 has an underlying (unoriented) graph which is a tree. For G' we define a bounden K -modulation $M' = (F'_1, M'_j)$ as

follows: $F'_1 = F_1$ if $1 \in G_2$, $F'_a = F_a = F_{a''}$; if $1, j \in G_2$ then ${}_1M'_j = {}_1M_j$; ${}_bM'_a = {}_bM_a$; ${}_cM_{a''} = {}_cM_a$, and ${}_1M_j = 0$ otherwise. The relations on G' are the same relations as on G_2 and there are no relations starting from a' or a'' .

Let A' be the algebra given by the bounden K -modulation of G' . Then A' is a tilted algebra of type A_{n+2} . Every indecomposable A' -module M with $M_a = M_{a''} = 0$ can be considered as an indecomposable B -module.

We shall show that there is a complete slice of S' in $\Gamma_{A'}$, containing $[P_{A'}(a')]$ and $[P_{A'}(a'')]$ and that each one of them has only one neighbour in $\Gamma_{A'}$. Then we construct the preprojective component of B using the predecessors of S' in $\Gamma_{A'}$, and identifying $P_{A'}(a')$ and $P_{A'}(a'')$ with $P_B(a)$. In this way we obtain a complete slice through the preprojective component of B of type A_n .

Since A' is a tilted algebra, it arises from a grading g of finite representation type of the modulated tree (A_{n+2}, M'') for some modulation M'' of A_{n+2} . Hence $\Gamma_{A'}$ is equal to $\Gamma_{(A_{n+2}, M'', g)}$. We want to see that $[P_{A'}(a')] = (g(1), 1)$ or $[P_{A'}(a'')] = (g(n+2), n+2)$.

Let us assume that $[P_{A'}(a')] = (g(1), 1)$ for some $1 \neq 1, n+2$. Note that $\text{rad } P_{A'}(a')$ is indecomposable, and therefore $\text{rad } P_{A'}(a') = (\ell, 1-1)$ for some ℓ . Let $P_{A'}(j)$ be the indecomposable projective such that $(g(i+1), i+1) = [P_{A'}(j)]$, then $g(i+1) > g(1)$, and we have an oriented path from $P_{A'}(a')$ to $P_{A'}(j)$.

Using the fact that G satisfies the conditions $(\tilde{\alpha}_0)$, we deduce that the (non-oriented) path of G' joining a' and j has one of the forms described below. We shall show that in each case we obtain a contradiction.

$$(I) \quad \begin{array}{ccccccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \dots & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ a'_1=a_1 & & a_2 & & a_3 & & a_{r-1} & & a_r & & a_{r+1} & & a_{r+2} & & a_{s-1} & & a_s=j \end{array}$$

Let us consider the representation M defined by

$$\text{Supp } M = \{a_r, \dots, a_s\} \cup \{\text{Supp } P_A(a') \cap \text{Supp } P_A(a_r)\}$$

and put $M_i = E$ for $i \in \text{Supp } M$ together with the obvious maps between the coordinate vector spaces. It is clear that M is indecomposable, $\text{Hom}(M, P_A(a')) \neq 0$, and $\text{Hom}(P_A(j), M) \neq 0$. Thus we have an oriented path in Γ_A , defined by the sequence of maps

$$P_A(j) \longrightarrow \dots \longrightarrow M \longrightarrow \dots \longrightarrow P_A(a').$$

The existence of a path from $P_A(a')$ to $P_A(j)$ implies that Γ_A contains an oriented cycle, which gives the wanted contradiction.

$$(II) \quad \begin{array}{ccccccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \dots & 0 & \longrightarrow & 0 & \longleftarrow & 0 & \longrightarrow & 0 & \dots & 0 & \longrightarrow & 0 \\ a'_1=a_1 & & a_2 & & a_3 & & a_{r-1} & & a_r & & a_{r+1} & & a_{r+2} & & a_{s-2} & & a_{s-1} & & a_s=j \end{array}$$

In this case we define the indecomposable representation M of support

$$\text{Supp } M = \{a_1, \dots, a_s\}$$

and such that $M_i = E$ for all $i \in \text{Supp } M$, the maps between the coordinates vector spaces being the obvious ones. We then have an oriented path

$$[P_A(a')] \longrightarrow \dots \longrightarrow [P_A(j)] \longrightarrow \dots$$

in Γ_A , and since in Γ_A all paths from $[P_A(a')]$ to M induce the same map, and $\text{Hom}(P_A(a'), M) \neq 0$, we conclude that $\text{Hom}(P_A(a'), P_A(j)) \neq 0$. That is to say, there exists an oriented path in G' from j to a' . This contradicts the fact that a' is a source in G' .

$$(III) \quad 0 \rightarrow 0 \dots 0 \rightarrow 0 \leftarrow 0 \leftarrow 0 \quad 0 \xrightarrow{\quad} 0 \xrightarrow{\quad} 0 \xrightarrow{\quad} 0 \dots 0 \xrightarrow{\quad} 0$$

$$a' = a_1 \ a_2 \ \dots \ a_{r-1} \ a_r \ a_{r+1} \ a_{r+2} \ \dots \ a_{s-2} \ a_{s-1} \ a_s \ a_{s+1} \ a_{s+2} \ \dots \ a_{t-1} \ a_t = j$$

We define the indecomposable representations M' and M'' as follows:

$$\text{Supp } M'' = \{a_1, \dots, a_r\}, \text{ Supp } M' = \{a_s, \dots, a_t\}.$$

$M'_1 = E$ for $i \in \text{Supp } M'$, and $M''_1 = E$ for $i \in \text{Supp } M''$. Again the maps between the coordinate vector spaces are in both cases the obvious ones. Using the sequence of maps

$$P_A(a') \rightarrow \dots \rightarrow P_A(j) \rightarrow \dots \rightarrow M' \rightarrow \dots \rightarrow M''$$

and the same argument as in II we obtain a contradiction.

This shows that either $[P_A(a')] = (g(1), 1)$ or $[P_A(a')] = (g(n+2), n+2)$.

The corresponding result for $P_A(a'')$ can be obtained in the same way.

Now, by lemma (II.2.1.1) there is a complete slice S' in Γ_A containing $[P_A(a')]$ and $[P_A(a'')]$. This completes the proof of case (iv).

(v) Finally, assume that G contains a subgraph of the form $1' \xrightarrow{(2,2)} 1''$ (and therefore G is a tree). Again by the condition $(\tilde{\alpha}_0)$ we can assume that G is of the form:

and define the graph G' to be the oriented graph of the form

with the same G_2 . We define a bounden K -modulation M' as we did before.

The relations on G' are those relations on G_2 and at most two new relations:

(I) If i'' has two neighbours i' and c in G , then

is a full subgraph of G' .

(II) If i'' has three neighbours i' , c_1 and c_2 in G , then

is a full subgraph of G'' .

The rest of the proof of the case (iv) applies here.

(2.) TILTED ALGEBRAS OF TYPE $\bar{B}_n$

(2.1) In this section we shall describe the relations that will appear in the bounden K-modulation of tilted algebras of type $\bar{B}_n$.

Let us consider the valued graph $1 \xrightarrow{(2,1)} 2 \xrightarrow{(1,2)} 3$ with the K-modulation M:

$$E \xrightarrow{E_F^E} F \xrightarrow{F_E^E} E$$

where F and E are skew-fields, finite-dimensional over the central field K. Let $\mu: E_F^E \otimes F_E^E \longrightarrow E_E^E$ be an epimorphism and put ${}_1R_3 = \ker \mu$. The resulting relation is called a (μ) -relation [3]. As an example, we have the epimorphism defined by $\mu(x \otimes y) = xy$ where $x \in E_F^E$ and $y \in F_E^E$.

We now consider the valued graph $1 \xrightarrow{(2,2)} 2 \xrightarrow{(1,2)} 3$ with the K-modulation M:

$$F \xrightarrow{F_F^E} F \xrightarrow{F_E^E} E$$

with the same skew-fields as before. If $\bar{\mu}: F_F^E \otimes F_E^E \longrightarrow F_E^E$ is an epimorphism, the relation ${}_1R_3 = \ker \bar{\mu}$ is called a $(\bar{\mu})$ -relation. In order to consider an example, we let $\{1, e\}$ be a (left and right) F-basis of E. Then an epimorphism $\bar{\mu}$ is given by:

$$\bar{\mu}(x_1 + x_2 e) \otimes (y_1 + y_2 e) = x_1 y_1 + x_1 y_2 e$$

(where $x_1, x_2, y_1, y_2 \in F$).

A similar relation can be considered on the K-modulation

$E \xrightarrow{E_F^E} F \xrightarrow{F_F^E} F$ of the valued graph $1 \xrightarrow{(2,1)} 2 \xrightarrow{(2,2)} 3$.

(2.2) Definition. A K-modulation $M=(F_i, {}_iM_j)$ over an oriented valued graph G with relation ideal R is said to satisfy the property

(B) if:

($\tilde{B}_1$) G is a tree with two distinguished vertices i' and i'' . If the non-oriented path C from i' to i'' has length one, G has a unique proper subgraph $i' \xrightarrow{(2,2)} i''$.

($\tilde{B}_2$) For all $i \neq i', i''$, the F_i are equal: $F_i = E$, and furthermore, $[E:F_i] = [E:F_{i''}] = 2$. For all i, j , ${}_iM_j = F_i E_{F_j}$ or 0.

($\tilde{B}_3$) Each vertex of G has at most four neighbours.

($\tilde{B}_4$) The vertex i' or i'' has at most two neighbours, in which case

$i \xrightarrow{E} i' \xrightarrow{E} j$, or $i \xrightarrow{E} i'' \xrightarrow{E} j$ is a bounden K-submodulation

of (M, R) .

($\tilde{B}_5$) If a vertex $i \in G$ has four neighbours, then

is a bounden K-submodulation of (M, R) .

($\tilde{B}_6$) If $i \in G$ has three neighbours, then either

is a bounden K-submodulation of (M, R)

($\tilde{\beta}_7$) All relations are of length two and all are zero-relations, μ -relations or $\bar{\mu}$ -relations.

Moreover, the bounden K-modulation (M, R) is said to satisfy property ($\tilde{\beta}_7$) if, in addition to ($\tilde{\beta}_1$)-($\tilde{\beta}_7$), it satisfies:

($\tilde{\beta}_0$) There is no bounden K-submodulation of (M, R) of one the form:

(a) $i_1 \xrightarrow{\quad} i_2 \xrightarrow{\quad} i_3 \dots i_{t-2} \xrightarrow{\quad} i_{t-1} \xrightarrow{\quad} i_t$ where X and Y equal either to E or 0, and $t \geq 4$.

(b) $i_1 \xrightarrow{\quad} i_2 \xrightarrow{\quad} i_3 \dots i_{r-1} \xleftarrow{\quad} i_r \xleftarrow{\quad} i_{r+1} \dots i_{t-2} \xrightarrow{\quad} i_{t-1} \xrightarrow{\quad} i_t$ where X and Y equal either E or 0.

(c) $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xrightarrow{\quad} j_2 \xrightarrow{\quad} j_3$ nor $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xleftarrow{\quad} j_3$ where i is on the path C and j_2 is not.

(d) $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xrightarrow{\quad} j_2 \xrightarrow{\quad} \dots \xrightarrow{\quad} j_4 \xleftarrow{\quad} j_5 \xleftarrow{\quad} j_6$, or
 $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xleftarrow{\quad} \dots \xleftarrow{\quad} j_4 \xrightarrow{\quad} j_5 \xrightarrow{\quad} j_6$
 where $i = i'$ or i'' .

(e) $i' \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xleftarrow{\quad} \dots \xleftarrow{\quad} j_4 \xrightarrow{\quad} j_5 \xrightarrow{\quad} \dots \xrightarrow{\quad} i''$
 or $i' \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xleftarrow{\quad} \dots \xleftarrow{\quad} j_5 \xleftarrow{\quad} j_6 \xrightarrow{\quad} \dots \xrightarrow{\quad} i''$

In all cases we assume that there is no other relation besides the relations shown.

(2.3) Examples: We want to illustrate the properties $(\tilde{\beta})$ and $(\tilde{\beta}_T)$ given above. Consider the following bounden K-modulations:

It is easy to see that all of them satisfy property $(\tilde{\beta})$. M_1 does not satisfy $(\tilde{\beta}_T)$ since M_1 contains a bounden K-submodulation of the form (c). Similarly, M_2 does not satisfy $(\tilde{\beta}_T)$, as in this case a bounden K-submodulation of the form (c) is contained in M_2 . Finally, we see that M_3 satisfies $(\tilde{\beta}_T)$, moreover, the algebra A given by M_3 has a complete slice in its Auslander-Reiten graph Γ_A which has the following form:

where we have put $\underline{1} = P(1)$.

(2.4) Theorem: The following conditions are equivalent:

- (I) The bounden K-modulation of B satisfies the property $(\tilde{\beta}_7)$.
- (II) B is a tilted algebra of type $\tilde{B}_n$.

Proof:

(2.4.2) The argument is similar to the one in (1.4.1).

Let A be a hereditary algebra of type $\tilde{B}_n$, and T_A a tilting module such that $B = \text{End } T_A$. We shall denote by $T(i)$ the direct summand of T_A corresponding to the vertex i of the oriented valued graph G of the bounden K-modulation (M, R) of B. Here we shall prove that B satisfies the property $(\tilde{\beta}_7)$ by proving conditions $(\tilde{\beta}_4)$, $(\tilde{\beta}_5)$, $(\tilde{\beta}_6)$, $(\tilde{\beta}_7)$ and then $(\tilde{\beta}_0)$ and $(\tilde{\beta}_1)$. For the remaining conditions we refer to the proof of (1.4).

$(\tilde{\beta}_4)$ It is easy to see that i' and i'' have at most two neighbours (cf III 3.4.3). Now, $T(i')$ is not regular because $F_{i'} = E$, so let us assume that $T(i')$ is preprojective. If j and k are the neighbours of i' then one of them is preprojective, say j , and the other, k , regular, and using the postfactor and prefactor sets, we obtain that

$$\text{End } T(k) \xrightarrow{\text{Hom}(T(j), T(k))} \text{End } T(j) \xrightarrow{\sim} F^E_E \quad \text{if } j = i''.$$

and

$$\text{End } T(k) \xrightarrow{\text{Hom}(T(j), T(k))} \text{End } T(j) \xrightarrow{\sim} F^E_F \quad \text{if } j \neq i''.$$

In both cases, we see that the left and the right actions on the images are induced respectively by the ordinary multiplication of

End $T(j_2)$ ($=E$ or F) and F , that is,

is a bounden K -submodulation of (M, R) .

The same argument can be used for $T(i'')$ preinjective.

(β_5) Assume $i \in G$ has four neighbours, then two of them belong to the sectional paths of the postfactor set of $T(i)$, and the other two to the sectional paths of the prefactors set of $T(i)$, therefore there must be a bounden K -submodulation of the form:

(β_6) The proof is similar to the previous one, we just consider three neighbours in the argument.

(β_7) As in the proof of (α_7), if there is a bounden K -submodulation in (M, R) of the form:

with $j_1 R j_s \neq 0$, then $s=3$ and there are three cases to consider:

- (I) $T(j_1)$ is preprojective, $T(j_2)$ is preprojective and $T(j_3)$ is regular.
- (II) $T(j_1)$ is preprojective, $T(j_2)$ is regular and $T(j_3)$ is preinjective.
- (III) $T(j_1)$ is regular, $T(j_2)$ and $T(j_3)$ are preinjective.

In all cases we may assume that $j_2 \neq i'$ or i'' , since they were considered in (β_4) . Therefore $\text{End } T(j_2) \simeq E$.

By (III 3.2.4), in II) we obtain a zero-relation, and in I) we have that

$$\text{End } T(j_3) \text{ Hom } (T(j_1), T(j_3)) \text{ End } T(j_1) \simeq_E^E E \text{ if } j_1 \neq i' \text{ or } i''$$

or

$$\text{End } T(j_3) \text{ Hom } (T(j_1), T(j_3)) \text{ End } T(j_1) \simeq_E^E E \text{ if } j_1 = i' \text{ or } i''$$

or

$$\text{End } T(j_3) \text{ Hom } (T(j_1), T(j_3)) \text{ End } T(j_1) = 0 \text{ otherwise.}$$

We conclude that there is a trivial relation or a zero-relation. Note that μ -relations and $\bar{\mu}$ -relations can appear only in (β_4) . This finishes the proof of (β_7) .

(β_0) To see that there is no bounden K -modulation of the form given in (a) we refer to the proof in the case of $\bar{A}_n$.

(b) Let us now assume that (M, R) contains a bounden K -submodulation of the form (b).

If $T(i_t)$ and $T(i_{t-1})$ are preprojectives, then $T(i_{t-2})$ must be regular, and since there is no relation between i_{t-2} and i_{r+1} , $T(i_{r+1})$ must also be regular. So we have a contradiction (cf(III.2.6)).

Thus, if $T(i_t)$ is preprojective then $T(i_{t-1})$ must be regular and $T(i_j)$ for $r \leq j \leq t-2$ must be preinjective. Since there is a relation between i_{r-1} and i_{r+1} , $T(i_{r-1})$ must be regular, and therefore all the modules

$T(i_j)$ for $i \leq j \leq r-1$ are regular (cf. IV.2.6). Again we have a contradiction.

(c), (e) Here again we refer to the proof in the case of $\bar{A}_n$.

(d) Suppose there is a bounden K -submodulation of the form:

and that $T(i')$ is preprojective. If one of the modules $T(i_k)$ ($1 \leq k \leq r$), $T(j_1)$, $T(j_2)$, $T(j_4)$ is regular we obtain a contradiction (cf. III.2.6). So all of them are preprojectives and since there is a zero-relation between j_1 and j_3 , $T(j_3)$ is not preprojective, however this is a contradiction because $\text{Hom}(T(j_2), P) = 0$ for any preprojective module P .

If $T(i')$ is preinjective, we obtain that all the modules $T(i_k)$ ($1 \leq k \leq r$), $T(j_1)$, $T(j_2)$, $T(j_4)$ are preinjectives, and since there is a relation between j_4 and j_6 , $T(j_6)$ is not preinjective. We obtain a contradiction due to the fact that $\text{Hom}(I, T(j_6)) = 0$ for any preinjective module I .

It is clear that the other case is similar to the previous one. This completes the proof of (d).

We shall now show that the bounden K -modulation of B satisfies (β_1) .

If G is not a tree, it contains a (non-oriented) cycle C . Let j and l be respectively a sink and a source of C and denote by $j_1, \dots, j_2$ and $l_1, \dots, l_s$ the corresponding vertices of the paths between j and l . (See Figure below).

Let us first assume that $T(j_0), \dots, T(j_q), T(l_s), \dots, T(l_p)$ are preprojectives and that $T(j_{t+1})$ and $T(l_{p-1})$ are regulars for some $q \geq 0$ and $p \leq s$. We then have that $T(j_{q+2})$ and $T(j_{p-2})$ are preinjectives, and therefore, there is a relation on one of the (non-oriented) paths from l_1 to j_0 .

Thus the radical of $P(l)$ is decomposable, with at least two indecomposable direct summands, and there are also at least two irreducible maps with domain $P(j)$ and non-isomorphic codomains.

If all the indecomposable projectives are contained in the same connected component C of Γ_B , then we have two non-homotopic paths from $P(j)$ to $P(l)$.

(They factor over one indecomposable summand of $\text{rad } P(l)$). However, this is a contradiction, because in this case B is a tilted algebra of type $\tilde{B}_n$. Then it arises from a graded valued tree and the preprojective component of Γ_B is of the form $N\tilde{B}_n$. Note that we can always assume that such a component exists, otherwise we pass to the opposite algebra.

Thus we have shown that all modules $T(j_i)$ ($0 \leq i \leq t$), $T(l_m)$ ($0 \leq m \leq r$) are preprojectives. Now using their postfactor and prefactor sets we obtain again a contradiction. In the case that $T(j_0)$ is preinjective, we use a similar argument to produce a contradiction. We have thus completed the proof of (β_1) and therefore have shown that (II) implies (I).

(2.4.3) In order to complete the proof of the theorem, we now show that (I) implies (II).

(i) First we assume that there is a zero-relation on the path C joining the vertices i' and i'' . In this case, we shall prove that B is of finite representation type as in the case $\tilde{A}_n$.

The graph G of (M, R) is of the form:

where the underlying graph of G_i ($i=1,2,3$) is a tree, $i' \in G_1$ and $i'' \in G_3$.

Let N be an indecomposable A -module and N_0 its restriction to the subgraph

Then $N_0 = N'_0 \oplus N''_0$, where $(N'_0)_{a_1} = 0$ and $(N''_0)_{a_1} = 0$. If $a_1 \neq i'$ the rest of the proof of the case $\tilde{A}_n$ applies here. If $a_1 = i'$ then for every F_{a_1} -linear map $\delta: N_{a_2} \oplus_{a_2} M_{a_1} \rightarrow N_{a_1}$ we consider the F_{a_2} -linear map

$$\tilde{\delta}: N_{a_2} \rightarrow N_{a_1} \oplus_{a_1} M_{a_2}. \text{ Since the } F_{a_2}\text{-linear map}$$

$$\tilde{\delta}: (N_0)_{a_2} \rightarrow (N_0)_{a_1} \oplus_{a_1} M_{a_2} \text{ is a monomorphism, we obtain that}$$

$$\text{Im}(\delta_{a_2 a_3}) \subseteq (N''_0)_{a_1}. \text{ From here, the proof for } \tilde{A}_n \text{ carries over in this case}$$

with only the obvious changes.

Keeping the same notation for G as above, we shall show that Γ_B has no oriented cycles.

As in the proof of (1.4), if Γ_B has oriented cycles we can find a chain of irreducible maps.

$$N_1 \xrightarrow{\delta_1} N_2 \xrightarrow{\delta_2} \dots \xrightarrow{\delta_t} N_t \xrightarrow{\delta_t} N_1$$

with $\text{Supp } N_1 \subset G_3 \cup G_2$ and $a_1 \in \text{Supp } N_2 \subset G_2 \cup G_1$. Then δ_1 must be a monomorphism, and $\text{Supp } N_1 \subset G_2$. Note that $i' \in G_1$, and therefore for every representation N of (M, R) with support in $G_2 \cup G_1$, the map corresponding to the edge $a_2 \rightarrow a_1$ is a monomorphism of $a_1 + i'$. If $a_1 = i'$ the map $\bar{\delta}: N_{a_2} \rightarrow N_{a_1} \oplus_{a_1} M_{a_2}$ corresponding to the edge $a_2 \rightarrow a_1$ is a monomorphism. From these remarks it follows that the diagram

$$\begin{array}{ccc} (N_1)_{a_2} & \xrightarrow{\quad} & (N_1)_{a_1} \oplus_{a_1} M_{a_2} \\ \downarrow & & \downarrow \\ (N_2)_{a_2} & \xrightarrow{\quad} & (N_2)_{a_1} \oplus_{a_1} M_{a_2} \end{array}$$

commutes if and only if the F_{a_2} -linear map $(N_1)_{a_2} \rightarrow (N_2)_{a_2}$ is zero. Hence, $\text{Hom}(N_1, N_2) = 0$. This contradiction shows that Γ_B has no oriented cycles.

(ii) We shall now construct a complete slice in Γ_B . As in (1.4.3) first consider the sources $s_1, \dots, s_t$ of C such that C contains a full subgraph of the form

for each $i=1, \dots, t$. Here the relation shown can be any of the relations

given in the definition of (β) . We then consider $P(s_1), \dots, P(s_t)$, the corresponding indecomposable projective B -modules, and define S to be the set of those vertexes $[M]$ of Γ_B such that if there exists an oriented path from $[M]$ to $[P(s_1)]$ for some i , then this path is sectional. In order to prove that B is a tilted algebra, it suffices to show that S is a complete slice in Γ_B . By definition S contains at most one module from each τ -orbit of indecomposable modules. Now we show that S contains exactly one module from each τ -orbit.

(a) For each indecomposable projective B -module P there exists an oriented path from P to S .

Let us first consider a vertex j not on the path C , then there exists a path from j to some s_j . This path is of one of the following forms:

where a_s a source

where a_s a source and $a_1=i'$ or i''

where a_s is a source.

$$a_1 = 1' \text{ or } 1''$$

where the a_i ($0 \leq i \leq r$) are on C , while the b_i ($1 \leq i \leq t$) are not. We assume that there is no other relation besides the relations shown, and that a_s is a source of C . We shall construct a module M in each case as follows:

$$\text{I, III and IV: } M_i = E_E \text{ if } i \in \{b_1, \dots, b_t, a_1, \dots, a_r\} \cup \{\text{Supp } P(a_r) \cap \text{Supp } P(a_s)\}$$

$M_i = 0$ otherwise. The maps between the coordinate vector spaces are the obvious ones.

$$\text{II: } M_i = E_E \oplus E_E \text{ if } i \in \{a_2, \dots, a_4\}, M_{a_1} = E_F \text{ and } M_i = 0 \text{ otherwise.}$$

The maps between the coordinates spaces are the canonical inclusion

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}: E_E \longrightarrow E_E \oplus E_E, \text{ the projection } [01]: E_E \oplus E_E \longrightarrow E_E, \text{ the map } \delta: \\ E_F \oplus E_E \longrightarrow E_E \oplus E_E \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} E_E \oplus E_E, \text{ and the identity.}$$

V: $M_1 = E_E$ if $1 \in \{b_1, \dots, b_t\} \cup \{\text{Supp } P(a_r) \cap \text{Supp } P(a_s)\} \setminus \{a_r\}$,
 $M_1 = E_E \oplus E_E$ if $1 \in \{a_1, \dots, a_r\}$, $M_1 = 0$ otherwise. The maps between the
coordinate spaces are given as in II.

VI: $M_1 = E_E$ if $1 \in \{b_1, \dots, b_t\} \cup \{\text{Supp } P(a_r) \cap \text{Supp } P(a_s)\} \setminus \{a_r\}$,
 $M_{a_r} = E_F$ and $M_1 = 0$ otherwise. The maps between the coordinate vector
spaces are the identify, the zero maps and $: M_{a_r} \oplus_{F_{a_r}} E_E \longrightarrow E_E$.

VII: Finally, $M_1 = E_E$ if $1 \in \{b_1, \dots, b_t, a_2, \dots, a_r\} \cup \{\text{Supp } P(a_r) \cap \text{Supp } P(a_s)\}$
 $M_{a_1} = E_F$. The maps are given as in VI.

In all the previous cases, M is indecomposable, $\text{Hom}(P(j), M) = 0$, and
 $\text{Hom}(M, P(a_s)) = 0$. Thus we have an oriented path in Γ_B induced by the
sequence of maps $P(j) \longrightarrow \dots \longrightarrow M \longrightarrow \dots \longrightarrow P(a_2)$, that is, a path from
 $P(j)$ to S .

Let us now consider the case when j is on C . In this situation,
the path joining j with a source s_j has one of the following forms:

where b_p is a source in each case and there is no other relation. In
this situation we define the module M as follows:

$$\text{Supp } M = \{b_1, \dots, b_s\} \cup \{\text{Supp } P(b_p) \cap \text{Supp } P(b_s)\},$$

and $M_i = F_i$ for $i \in \text{Supp } (M)$, the maps between the coordinates spaces being the obvious ones. Then we just use the argument produced in the case of A_n (4.2, (iii)).

(b) Let $I(i_0)$ be the indecomposable injective corresponding to the vertex i_0 . If there is a path from $I(i_0)$ to S , such a path is sectional.

Let $[I(i_0)] = [M_0] \rightarrow [M_1] \rightarrow \dots \rightarrow [M_r] \rightarrow [P(s_{i_0})]$

be an oriented path from $[I(i_0)]$ to $[P(s_{i_0})]$ for some s_{i_0} . Then M_r is a direct summand of the radical of $P(s_{i_0})$, and therefore $(M_r)_x \neq 0$ for one neighbour x of s_{i_0} .

We now define G' to be the full subgraph of G consisting of those vertices y such that the (non-oriented) path joining y and s_{i_0} contains x .

Let $M' = (F'_i, M'_j)_{i,j \in I'}$ be the bounden K -modulation given by the following definition:

$$F'_i = F_i \text{ if } i \in G' \text{ and } i \neq s_{i_0}, \quad M'_j = M_j$$

for i, j in G' and $i \neq s_{i_0} \neq j$, $F_{i_0} = E$ and $s_{i_0} M'_x = E E$. We also define a relation ideal R' of M' as follows: $R'_{ij} = R_{ij}$ for $i \neq s_{i_0} \neq j$ in G' and if there exists a relation starting with s_{i_0} , we put $R'_{i_0 j} = s_{i_0} M_j \oplus j M_{i_0}$, $i \in G'$, for every relation $s_{i_0} R_i$ in (M, R) .

Then the algebra A' given by the bounden K -modulation (M', R') is a tilted algebra of type B_n (cf [6]), for some n . Since Γ_B has no oriented cycles, $\text{Hom}(P(s_{i_0}), M_j) = 0$ for $j = 0, \dots, t_1$. Hence the M_j ($j = 0, \dots, t$) are A' -modules and the path

$$[I_A, (i_0)] = [M_0] \longrightarrow [M_1] \longrightarrow \dots \longrightarrow [M_r] \longrightarrow [P_A, (s_{i_0})]$$

is, by lemma (I.7.4.5), a sectional path in Γ_A . As in the case of $\tilde{A}_n$, we conclude that if there exists an oriented path from $[I(i_0)]$ to S , it must be sectional.

(c) We shall show that S is of type $\tilde{B}_n$.

We have shown that S is a complete slice in Γ_B . Let $[P(s_1)]$ be a sink of S , hence by construction G contains a full subgraph of one of the forms:

$$\text{I) } \circ \longrightarrow \circ \longrightarrow \dots \circ \longrightarrow \circ \longleftarrow \circ \dots \circ \longleftarrow \circ$$

$$s_1 = a_1 \quad a_2 \quad a_{r-1} \quad a_r \quad a_{r+1} \quad a_{s-1} \quad a_s = s_2$$

$$\text{II) } \circ \longrightarrow \circ \longrightarrow \circ \dots \circ \longrightarrow \circ \xrightarrow{\quad} \circ \longrightarrow \circ \xrightarrow{\quad} \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \dots \circ \longleftarrow \circ$$

$$s_1 = a_1 \quad a_2 \quad a_3 \quad a_r \quad a_{r-1} \quad a_r \quad a_{r+1} \quad a_{s-1} \quad a_s \quad a_{s+1} \quad a_{p-1} \quad a_p = s_2$$

where s_2 is a source and there is no other relation besides the zero-relations shown.

Let us first assume that $s_1 = i'$ or i'' . In this case we consider the algebra A'' given by the bounden K -modulation (M'', R'') , where we have that $F_i'' = F_i'$ for each $i \neq s_1$, $F_{s_1}'' = E$, $M_i'' = M_j$ for $i, j \in G$ and $i \neq s_1 \neq j$, and $s_1 M_j = E E$ for a neighbour j of s_1 in G , $i R_j'' = i R_j$ for $i, j \in G$ $i \neq s_1 \neq j$ and $s_1 R_l'' = s_1 M_j \odot_j M_l$ for $l \in G$ and a relation $s_1 R_l$ in (M, R) . Then we have that A'' is a tilted of type B_{n+1} and that Γ_B is obtained by using the set of all predecessors of $S \setminus \{[P_B(s_1)]\}$ in Γ_A .

and inserting $[P_B(s_1)]$.

If $s_1 \neq i', i''$ then we define the algebra A'' using the bounden K -modulation (M'', R'') , where

$$I'' = (I \setminus \{s_1\}) \cup \{s_1', s_1''\}, F''_1 = F_1 \text{ if } i \neq s_1', s_1'', \quad F''_{s_1'} = E, F''_{s_1''} = E,$$

$M''_j = M_j$ if $i, j \in I \setminus \{s_1\}$, $s_1' M_{j_1} = M_{j_1}$, $s_1'' M_{j_2} = M_{j_2}$ where j_1 and j_2 are the neighbours of s_1 in G , $R''_{j_1} = R_{j_1}$ if $i, j \in I \setminus \{s_1\}$, $s_1' R''_{j_1} = R_{j_1}$, $s_1'' R''_{j_2} = R_{j_2}$; otherwise $M_j = 0$ and $R_j = 0$. In this case A'' is a disconnected tilted algebra of type B_m . In fact A'' is the direct product of two tilted algebras of type B_n . Again Γ_B is obtained using the predecessors of $S \setminus \{P_B(s_1)\}$ in $\Gamma_{B''}$ and inserting $[P_B(s_1)]$.

In both cases every predecessor of $S \setminus \{P_B(s_1)\}$ satisfies the following properties [1]:

- 1) If M is not projective, the Auslander-Reiten sequence ending with M has at most two non-isomorphic middle terms.
- 2) If M is projective, for every direct summand M_1 of $\text{rad } M$, with a minimal right almost split map $g: N \rightarrow M_1$, the module N is either indecomposable or the direct sum of two isomorphic modules.

Since $\text{rad } P_B(s_1) = \text{rad } P_{A''}(s_1) + \text{rad } P_{A''}(s_1'')$ as B -modules we have that $P_B(s_1)$ also has the property 2) above.

This result shows that every module in S has at most two neighbours in S and that $S \setminus \{P_B(s_1)\}$ is the union of at most two graphs of type B_m or A_m . If $s_1 \neq i', i''$, $\text{rad } P(s_1)$ is the direct sum of two indecomposable

(non-isomorphic) modules and

$$S = (S \setminus \{P_B(s_1)\}) \cup \{P_{A''}(s'_1), P_{A''}(s''_1)\}$$

If $s_1 = i'$ or i'' , $\text{rad } P(s_1)$ is indecomposable and we have that

$$S = (S \setminus \{P_B(s_1)\}) \cup \{P_{A''}(s'_1)\}.$$

From this we conclude that S has as underlying graph the graph $\tilde{B}_n$.

(iii) Finally we consider the case when there is no relation on the path C joining i' and i'' .

Let s_1 be a source of G on the path C , A'' the algebra defined above, and A_0 the connected algebra given by the bounden K -submodulation of (M'', R'') whose graph G_0 is given by those vertices connected to s'_1 .

We have seen that A_0 is a tilted algebra of type B_m . We shall show that s'_1 is a strong source A_0 .

Let then $P(i)$ and $P(s'_1)$ be indecomposable projective A_0 -modules such that there exists an oriented path in

$\Gamma_{A_0}: [P(s'_1)] \longrightarrow [M_1] \longrightarrow \dots \longrightarrow [M_m] \longrightarrow [P(i)]$. We shall see that this gives us a contradiction.

Using our result from (II.2.3.2) we may assume that if ω is the (non-oriented) path of G_0 joining i and s'_1 , then $\text{Supp } M_j \cap \omega \neq \emptyset$ for all $1 \leq j \leq m$.

Let A_ω be the algebra of the bounden K -submodulation with graph ω together with the inherited relations. We may thus consider the

induced path of Γ_{A_ω} , given by:

$$[P_\omega(s'_1)] \longrightarrow [M'_1] \longrightarrow \dots \longrightarrow [M'_m] \longrightarrow [P_\omega(i)],$$

where $P_\omega(s'_1)$ and $P_\omega(i)$ are respectively restrictions of $P(s'_1)$ and $P(i)$, considered as A_0 -modules via the natural embedding [36]. Note that as A_0 -modules $P_\omega(s'_1) \cong P(s'_1)$, and $\text{Hom}(P(i), P_\omega(i)) \neq 0$.

The path ω has necessarily one of the following forms:

where non-oriented edges can be oriented arbitrarily, and there is no other relation besides the relation shown.

Let ℓ be minimal such that a_ℓ is a source, then we obtain a sectional path in Γ_{A_ω} given by a sequence of irreducible maps:

$$S_\omega(a_\ell) = P_\omega(a_\ell) \longrightarrow \dots \longrightarrow P_\omega(a_1) = P_\omega(1)$$

and thus we have no oriented path from $P_\omega(s_1)$ to $P_\omega(1)$. For otherwise, such a path would factor over the simple projective A_ω -module $S_\omega(a_\ell)$. Thus s'_1 is a strong source in A_0 .

Similarly we can prove that s'' is a strong source in the connected bounden K -submodulation of (M'', R'') in which it is contained. Let us call the algebra given by this bounden K -submodulation A_1 .

Using lemma (II.2.1.1.), there exists a complete slice S_0 in Γ_{A_0} containing $\text{rad } P_{A_0}(s''_1)$ and a complete slice S_1 in Γ_{A_1} containing $\text{rad } P_{A_1}(s''_1)$.

Now by lemma 1.3 in [27], we have that Γ_B has a preprojective component containing all indecomposable projective B -modules, and since

$$\{S_0 \cup S_1 \cup \{[P_B(s_1)]\} \setminus \{[P_{A_0}(s'_1)], [P_{A_1}(s''_1)]\}$$

is a complete slice in that component, we conclude that S has an underlying graph $\tilde{B}_n$.

(3.) FILTED ALGEBRAS OF TYPE $\tilde{c}_n$

(3.1) In this case, the relations that will appear in the bounden K-modulation are as follows.

First consider two skew-fields F and E , finite dimensional over the central field K such that $[E:F] = 2$.

Let M be the K-modulation

$$F \xrightarrow{F^E E} E \xrightarrow{E^E F} F$$

of the valued graph $1 \xrightarrow{(1,2)} 2 \xrightarrow{(2,1)} 3$. Let $\pi: F^E E \otimes E^E F \longrightarrow F^E F$

be an epimorphism and put ${}_1R_3 = \ker \pi$. Then the relation ${}_1R_3$ on M is called a (π) -relation [3].

Now consider the valued graph $1 \xrightarrow{(2,2)} 2 \xrightarrow{(2,1)} 3$ and the K-modulation:

$$E^E F \otimes F^E E \xrightarrow{E^E F} E^E F$$

If $\tilde{\pi}: (E^E F \otimes F^E E) \otimes E^E F \longrightarrow E^E F$ is an epimorphism, we shall call the relation ${}_1R_3$ a $(\tilde{\pi})$ -relation. For instance, if $\{1, e\}$ is a (left and right) F -basis of E , then an epimorphism $\tilde{\pi}$ is given by:

$$\tilde{\pi}((x_1 + y_1 e) \otimes (x_2 + y_2 e) \otimes (x_3 + y_3 e)) = (x_1 x_2 x_3 + x_1 y_2 y_3) + (y_1 x_2 x_3 + y_1 y_2 y_3) e.$$

In a similar manner, we can define a relation on the K-modulation

$$F \xrightarrow{F^E E} E \xrightarrow{E^E F} E \xrightarrow{F^E E} E.$$

(3.2) Definition: A K-modulation $M = (F_i, {}_iM_j)$ over an oriented valued graph G with relation ideal R is said to satisfy the property $(\tilde{Y})$ if:

($\tilde{\gamma}_1$) G is a tree with two distinguished vertices i' and i'' .

If the non-oriented path C from i' to i'' has length one, G has a unique proper subgraph $i' \xrightarrow{(2,2)} i''$.

($\tilde{\gamma}_2$) For all $i \neq i', i''$, the F_i are equal: $F_i = F$, and furthermore $[E_i : F] = [E_{i''} : F] = 2$. For all $i \neq i', i''$, $j \neq i', i''$, $M_{ij} = F F$ or 0, for all $i \neq i', i''$, $j = i', i''$, $M_{ij} = (E_j)_{F_j}$ or 0, for all $j \neq i', i''$, $i = i', i''$, $M_{ij} = (E_i)_{F_i}$ or 0. Finally, if G contains $i' \xrightarrow{(2,2)} i''$, then

$$i' M_{i''} = E_{i'}^{(E_{i'})} F \otimes F^{(E_{i''})} E_{i''}$$

($\tilde{\gamma}_3$) Each vertex of G has at most four neighbours.

($\tilde{\gamma}_4$) The vertices i' and i'' have at most two neighbours, in which

case $i' \xrightarrow{F} i'' \xrightarrow{F} j$, or $i' \xrightarrow{F} i'' \xrightarrow{F} j$ is a bounden K -submodulation of

(M, R) when $i' M_{i''} = 0$. If $i' M_{i''} \neq 0$ $i' \xrightarrow{E} i'' \xrightarrow{E} j$ or $j \xrightarrow{E} i' \xrightarrow{E} i''$ is a bounden K -submodulation of (M, R) .

($\tilde{\gamma}_5$) If a vertex $i \in G$ has four neighbours, then

is a bounden K -submodulation of (M, R) .

($\tilde{\gamma}_6$) If $i \in G$ has three neighbours, then either

is a bounden K-submodulation of (M, R) .

($\tilde{\gamma}_7$) All relations are of length two and all are zero-relations, (π) -relations or $(\tilde{\pi})$ -relations.

Moreover, the bounden K-modulation (M, R) is said to satisfy property ($\tilde{\gamma}_T$), if, in addition to ($\tilde{\gamma}_1$)-($\tilde{\gamma}_7$), it satisfies:

($\tilde{\gamma}_0$) There is no bounden K-submodulation of (M, R) of one of the forms:

where X and Y equal to either E or F or 0, and $t \geq 4$.

where X and Y equal either E or F or 0.

where i is on the path C and j_2 is not.

where $i = i'$ or i'' .

or

or

In all cases we assume that there is no other relation besides the relations shown.

(3.3) Theorem: The following conditions are equivalent:

- (I) The bounded K -modulation of B satisfies the property $(\tilde{\gamma}_T)$.
- (II) B is a tilted algebra of type $\tilde{C}_n$.

The same argument used to prove Theorem (2.2) applies in this case with the obvious changes.

(4.) TILTED ALGEBRAS OF TYPE $\widetilde{B C}_n$

(4.1) The relations in the previous results will appear in this case, however, a different relation has to be described before we state the theorem.

Let M be the K -modulation

$$F \xrightarrow{F^E_E} E \xrightarrow{E^E_G} G$$

of the graph $1 \xrightarrow{(1,4)} 2 \xrightarrow{(2,1)} 3$, where F , G and E are skew-fields, finite dimensional over the central field K , moreover $[G:F]=2=[E:G]$. For an epimorphism

$$\zeta: F^E_E \otimes E^E_F \xrightarrow{\sim} (F^G_G \otimes G^E_E) \otimes E^E_G \xrightarrow{\sim} F^G_G$$

the resulting relation, with ${}_1R_3 = \text{Ker } \zeta$ will be called a (ζ) -relation.

For the K -modulation

$$E \xrightarrow{E^E_F} F \xrightarrow{F^G_G} G$$

of the graph $1 \xrightarrow{(4,1)} 2 \xrightarrow{(1,2)} 3$ we shall call a (ξ) -relation the relation that is obtained by putting ${}_1R_3 = \text{ker } \xi$, where $\xi: E^E_F \otimes F^G_G \xrightarrow{\sim} E^E_G$ is an epimorphism.

Definition: A K -modulation $M=(F_i, {}_iM_j)$ on an oriented valued graph G with relation ideal R is said to satisfy the property (8γ) if:

$(8\gamma_1)$ G is a tree with two distinguished vertices $1'$ and $1''$. If the non-oriented path C from $1'$ to $1''$ has length one, G has a unique proper subgraph $1' \xrightarrow{(1,4)} 1''$.

(BY₂) For all $i \neq i', i''$, the F_i are equal: $F_i = E$, and furthermore
 $[F_{i''}:E] = [E:F_{i'}] = 2$. For all $i \neq i''; j \neq i''; iM_j = F_i E_{F_j}$ or 0, for all i ,
 $iM_{i''} = (F_{i''})_{F_{i''}}$ or 0, $iM_{i'} = (F_{i'})_{F_{i'}}$ or 0. Finally, if G contains
 $i, \underline{(1,4)}, i''$, $iM_{i''} = F_{i'} (F_{i''})_{F_{i''}}$.

(BY₃) Each vertex of G has at most four neighbours.

(BY₄) The vertices i' and i'' have at most two neighbours, in which case ei-

ther $i \xrightarrow{E} i' \xrightarrow{E} j$, or $i \xrightarrow{E} i'' \xrightarrow{E} j$ is a bounden K-submodulation of
 (M, R) when $iM_{i''} = 0$. If $iM_{i''} \neq 0$ either

is a bounden K-submodulation of (M, R) .

(BY₅) If a vertex i of G has four neighbours, then

is a bounden K-submodulation of (M, R) .

(BY₆) If i of G has three neighbours, then either

is a bounden K-submodulation of (M, R)

$(\gamma\beta_7)$ All relations are of length two and all are zero-relations, (π) -relations, (μ) -relations, (ζ) -relations or (ξ) -relations.

Moreover, the bounden K-modulation (M, R) is said to satisfy property $(\beta\gamma_T)$ if, in addition to $(\beta\gamma_1)$ - $(\beta\gamma_7)$, it satisfies:

$(\beta\gamma_0)$ There is no bounden K-submodulation of (M, R) of one of the forms:

(a) $i_1 \xrightarrow{\quad} i_2 \xrightarrow{\quad} i_3 \dots i_{t-2} \xrightarrow{\quad} i_{t-1} \xrightarrow{\quad} i_t$, where X and Y equal

either E or $F_{i'}$, or $F_{i''}$, and $t \geq 4$.

(b) $i_1 \xrightarrow{\quad} i_2 \xrightarrow{\quad} i_3 \dots i_{r-1} \xleftarrow{\quad} i_r \xleftarrow{\quad} i_{r+1} \dots i_{t-2} \xrightarrow{\quad} i_{t-1} \xrightarrow{\quad} i_t$

where X and Y equal either $F_{i'}$, E, or $F_{i''}$ and Z equals either $F_{i'}$, or $F_{i''}$.

(c) $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xrightarrow{\quad} j_2 \xrightarrow{\quad} j_3$ nor $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xleftarrow{\quad} j_3$

where i is on the path C and j_2 is not.

(d) $i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xrightarrow{\quad} j_2 \xrightarrow{\quad} \dots \xrightarrow{\quad} j_4 \xleftarrow{\quad} j_5 \xleftarrow{\quad} j_6$, or

$i \xrightarrow{\quad} \dots \xrightarrow{\quad} j_1 \xleftarrow{\quad} j_2 \xrightarrow{\quad} \dots \xrightarrow{\quad} j_4 \xrightarrow{\quad} j_5 \xrightarrow{\quad} j_6$

where $i=i'$ or i''

where $(j_0, j_7) \in \{(1', 1''), (1'', 1'')\}$ and it is assumed that there is no other relation besides the relations shown.

3. Theorem: The following conditions are equivalent:

- (I) The bounden K-modulation of B satisfies the property $(\tilde{S}\gamma_T)$.
- (II) B is a tilted algebra of type $\tilde{B} \mathbb{C}_n$.

Again, for the proof we just refer to the argument used to prove (2.4).

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