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Integral Tests for Brownian Motion and Some Related Processes

Stanislav Kepřta

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

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Integral Tests for Brownian Motion and Some Related Processes

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in partial fulfillment of the requirements
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Abstract

Let $\{W_r(t); t \geq 0\}$ be an r -dimensional Brownian motion. The problem of interest is its behavior as $t \rightarrow \infty$. The first integral test obtained was the Kolmogorov test which considered the big values of Brownian motion: for a positive, nondecreasing function ψ .

$$\begin{aligned} P\{W_1(t) \geq \sqrt{2} t^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty\} &= 0 \text{ or } 1 \\ &\text{according as } \int_0^\infty \frac{\psi(t)}{t} \exp(-\psi^2(t)) < \infty \text{ or } = \infty. \quad (0.1) \end{aligned}$$

We will prove the Kolmogorov integral test (0.1) assuming only that $t^{1/\alpha} \psi(t)$ or $t^{1/\alpha} / \psi(t)$ is nondecreasing for some $\alpha > 0$. In a similar way we enlarge the classes of weight functions for Hirsch's and Chung's integral tests, which describe how small the big values of Brownian motion are; the integral test of Dvoretzky-Erdős, which describes the rate of escape of Brownian motion in dimension $r \geq 3$; the integral test of Spitzer, which describes the approach to 0 of Brownian motion in dimension two; and the integral test of Khoshnevisan, Lewis, and Shi (1996), which describes how big the small values of Brownian motion are. Chapter 6 gives a brief summary of results for stable processes and subordinators. In Chapter 7 we enlarge the classes of weight functions for integral tests for occupation time and the length of the longest excursion of one-dimensional Brownian motion, and in Chapter 8 we present integral tests for suprema of a Kiefer process. In Chapter 9 we study two exponential functionals of Brownian motion, and in Chapter 10 we give integral tests for processes associated with planar Brownian motion, namely windings, winding clock and Lévy's stochastic area.

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Chapter 1

Introduction

1.1 Definitions and Notations

Let $\{V(t); t \geq 0\}$ be a stochastic process. We define the following classes of functions (cf. Révész (1990), pp. 33–34).

Definition 1.1. The function ψ belongs to the *upper-upper class* of $\{V(t); t \geq 0\}$ ($\psi(t) \in \text{UUC}(V(t))$) if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega) > 0$ such that $V(t) < \psi(t)$ for all $t > t_0$.

Definition 1.2. The function ψ belongs to the *upper-lower class* of $\{V(t); t \geq 0\}$ ($\psi(t) \in \text{ULC}(V(t))$) if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \dots$ with $t_n \rightarrow \infty$ such that $V(t_i) \geq \psi(t_i)$, $i = 1, 2, \dots$.

Definition 1.3. The function ψ belongs to the *lower-upper class* of $\{V(t); t \geq 0\}$ ($\psi(t) \in \text{LUC}(V(t))$) if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \dots$ with $t_n \rightarrow \infty$ such that $V(t_i) \leq \psi(t_i)$, $i = 1, 2, \dots$.

Definition 1.4. The function ψ belongs to the *lower-lower class* of $\{V(t); t \geq 0\}$ ($\psi(t) \in \text{LLC}(V(t))$) if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega) > 0$ such that $V(t) > \psi(t)$ for all $t > t_0$.

We will refer to UUC and ULC as to the *upper classes* and to LLC and LUC as to the *lower classes*.

Assume, for a moment, that a process $\{V(t); t \geq 0\}$ and functions $\psi(t)$ and $\xi(t)$ are positive. If

$$\limsup_{t \rightarrow \infty} \frac{V(t)}{\psi(t)} = 1 \quad \text{a.s.},$$

then for every $\varepsilon > 0$, $(1 + \varepsilon)\psi(t) \in \text{UUC}(V(t))$ and $(1 - \varepsilon)\psi(t) \in \text{ULC}(V(t))$. Similarly, if

$$\liminf_{t \rightarrow \infty} \frac{V(t)}{\xi(t)} = 1 \quad \text{a.s.},$$

then for every $\varepsilon > 0$, $(1 + \varepsilon)\xi(t) \in \text{LUC}(V(t))$ and $(1 - \varepsilon)\xi(t) \in \text{LLC}(V(t))$. But we do not know whether $\psi(t)$ belongs to $\text{UUC}(V(t))$ or to $\text{ULC}(V(t))$ and whether $\xi(t)$ belongs to $\text{LLC}(V(t))$ or to $\text{LUC}(V(t))$. Therefore, the division of functions into UUC and ULC classes gives a stronger result than a $\limsup$ result, and the division of functions into LLC and LUC classes gives a stronger result than a $\liminf$ result.

We will especially study *Brownian motion*.

Definition 1.5. $\{W(t); t \geq 0\}$ is called a (one-dimensional) *Wiener process* or *Brownian motion* on a probability space $(\Omega, \mathcal{F}, P)$ if

- i) $W : [0, \infty) \times \Omega \rightarrow \mathbb{R}$;
- ii) $W(0, \omega) = 0$ for each ω ;

- iii) $W(t, \cdot)$ is $\mathcal{F}$ -measurable for each t ;
- iv) For each $n \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_n$, the increments $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent and normally distributed, with means zero and variances $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$;
- v) $W(\cdot, \omega)$ is continuous for almost all ω .

Let $r \geq 1$ be an integer. Then $\{W_r(t); t \geq 0\}$ is called an *r-dimensional Brownian motion* if $W_r(t) = (W^{(1)}(t), \dots, W^{(r)}(t))$ and $\{W^{(1)}(t); t \geq 0\}, \dots, \{W^{(r)}(t); t \geq 0\}$ are independent one-dimensional Brownian motions.

We will usually write $W(t)$ instead of $W_1(t)$.

We say that a function $\psi(t) : [1, \infty) \rightarrow [0, \infty)$ is positive if

$$\inf_{1 \leq t \leq 1/\delta} \psi(t) > 0 \quad \text{for all } 0 < \delta < 1.$$

Let $\alpha > 0$. Define the following classes of functions.

$\Psi_0 = \{\psi; \psi \text{ is a positive continuous function defined on } [1, \infty) \text{ such that}$

$\psi(t)$ and $t^{1/2}/\psi(t)$ are nondecreasing in a neighborhood of infinity\},

$\Psi_1 = \{\psi; \psi \text{ is a positive function defined on } [1, \infty),$

nondecreasing in a neighborhood of infinity\},

$\Psi_2^\alpha = \{\psi; \psi \text{ is a positive function defined on } [1, \infty) \text{ such that}$

$t^{1/\alpha}/\psi(t)$ is nondecreasing in a neighborhood of infinity\},

$\Psi_3^\alpha = \{\psi; \psi \text{ is a positive function defined on } [1, \infty) \text{ such that}$

$t^{1/\alpha}\psi(t)$ is nondecreasing in a neighborhood of infinity\},

$\Psi_4^\alpha = \Psi_2^\alpha \cap \Psi_3^\alpha$.

Define the following integrals,

$$\mathcal{J}_d(\psi, c) = \int_0^\infty \frac{\max(1, \psi^d(t))}{t} \exp(-c\psi^2(t)) dt, \quad c > 0, \quad d = 0, 1, 2, \dots, \quad (1.1)$$

$$\mathcal{J}_d(\psi) = \int_0^\infty \frac{1}{t\psi^d(t)} dt, \quad d = 1, 2, \dots. \quad (1.2)$$

and

$$\mathcal{J}_0(\psi) = \int_0^\infty \frac{1}{t\psi(t) \log t} dt. \quad (1.3)$$

We will write $\mathcal{J}_d(\psi)$ instead of $\mathcal{J}_d(\psi, 1)$. One of the reasons why we have $\max(1, \psi^d(t))$ instead of simply $\psi^d(t)$ in the definition of $\mathcal{J}_d$ is to eliminate the possibility that $\mathcal{J}_d(\psi) < \infty$ because $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$. Notice that

$$\mathcal{J}_0(\psi) < \infty \iff \mathcal{J}_1(\psi \circ \exp) < \infty \iff \mathcal{J}_1(f) < \infty, \quad (1.4)$$

where $f(t) = \psi(t) \log t$.

Since we will investigate the behavior of stochastic processes as $t \rightarrow \infty$, it is sufficient if ψ is defined in a neighborhood of infinity, not necessarily in $[1, \infty)$.

Define

$$c^*(\psi) = \inf \{c > 0; \mathcal{J}_d(\psi, c) < \infty\}. \quad (1.5)$$

We will show in Lemma 2.3 that, for our classes of functions, $c^*(\psi)$ does not depend on d .

1.2 Summary of Known Results

Let $\{W_r(t); t \geq 0\}$ be a standard r -dimensional Brownian motion. For $t \geq 0$ and $r = 1$ define

$$M^+(t) = \sup_{0 \leq s \leq t} W_1(s),$$

$$M^-(t) = - \inf_{0 \leq s \leq t} W_1(s),$$

$$M_1(t) = \sup_{0 \leq s \leq t} |W_1(s)| = \max(M^+(t), M^-(t)),$$

and for $r \geq 2$ define

$$M_r(t) = \sup_{0 \leq s \leq t} \|W_r(s)\|,$$

where $\|\cdot\|$ denotes the Euclidian norm.

As far as the upper classes of Wiener process are concerned, the following results are known. The first theorem is due to Erdős (1942), Feller (1943), Kolmogorov (1930) and Petrowsky (1935) (originally for $r = 1$ and $\psi \in \Psi_0$), see also Itô and McKean (1965), Sections 1.8 and 4.12, or Knight (1981), Section 5.4.

Theorem 1.A. *Let $\psi \in \Psi_1$ and let $r \geq 1$ be an integer. Then*

$$P \left\{ M_r(t) \geq \sqrt{2} t^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_r(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.6)$$

Remark 1.1. In the case $r = 1$, $M_r(t)$ in the above theorem can be replaced by $W_1(t)$, $-W_1(t)$, $|W_1(t)|$, $M^+(t)$, or $M^-(t)$. In the case $r \geq 2$, $M_r(t)$ can be replaced by $\|W_r(t)\|$.

Example 1.1. Let $\psi^2(t) = \log_2 t + \frac{r+2}{2} \log_3 t + \log_4 t + \cdots + \log_{p-1} t + (1 + \delta) \log_p t$, where $\log_2 t = \log(3 \vee \log t)$, $\log_k t = \log_{k-1}(3 \vee \log t)$, $k = 3, \dots$. Then for $p \geq 4$,

$$\psi(t) \in \begin{cases} \text{UUC}((2t)^{-1/2} M_r(t)) \\ \text{ULC}((2t)^{-1/2} M_r(t)) \end{cases} \quad \text{according as } \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The following theorem is a result of Csörgő, Shao, and Szyszkowicz (1991) (see also Csörgő and Horváth (1993), Theorem 4.1.1, and Csörgő *et al.* (1986)).

Theorem 1.B. *Let $\psi \in \Psi_2^2$. Then*

$$\limsup_{t \rightarrow \infty} \frac{W(t)}{t^{1/2}\psi(t)} = (2c^*(\psi))^{1/2} \quad a.s. \quad (1.7)$$

For the use of results on the asymptotic behavior of weighted Wiener processes in the areas of strong and weak approximations in probability and statistics, we refer, for example, to Csörgő and Horváth (1993), (1997) and Szyszkowicz (1992)–(1997).

Comparing Theorem 1.A and Theorem 1.B we see that the former has a stronger statement whereas the latter seems to have a broader class of weight functions. However, neither the monotonicity assumption of Theorem 1.A implies the monotonicity assumption of Theorem 1.B, nor is the converse true. One of our goals will be to extend the integral test of Theorem 1.A to a class of weight functions similar to that used in Theorem 1.B. To have an integral test, we need some regularity assumptions imposed on the weight function. The question is what the right assumptions are.

In the setting of Theorems 1.A and 1.B, it is more natural to assume that $f(t) = \sqrt{2} t^{1/2} \psi(t)$ is nondecreasing in a neighborhood of infinity than to assume that $\psi(t)$ or $t^{1/2}/\psi(t)$ is nondecreasing. But the condition that $t^{1/2}/\psi(t)$ is nondecreasing was obtained by inverting the result at zero to the result at infinity. By time inversion of Wiener process we have (for more details see Csörgő, Shao, and Szyszkowicz (1991))

$$\begin{aligned} P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} &= P\left\{tW'\left(\frac{1}{t}\right) \geq f(t) \text{ i.o. as } t \rightarrow \infty\right\} \\ &= P\left\{W'(s) \geq sf\left(\frac{1}{s}\right) \text{ i.o. as } s \rightarrow 0+\right\} \\ &= P\{W'(s) \geq q(s) \text{ i.o. as } s \rightarrow 0+\}, \end{aligned}$$

where $s = 1/t$, $q(s) = sf(1/s)$, and

$$W'(s) = \begin{cases} sW\left(\frac{1}{s}\right), & s > 0, \\ 0, & s = 0, \end{cases}$$

is also a standard Wiener process. Here, it is natural to assume that $q(s) = \sqrt{2}s^{1/2}\psi(1/s) = \sqrt{2}t^{-1/2}\psi(t)$ is a nondecreasing function of s in a right neighborhood of zero, ie., a nonincreasing function of t in a neighborhood of infinity.

In Chapter 2, we present a series of lemmas which answer the questions raised about the monotonicity assumptions. First, we will show (Lemma 2.8) that, assuming that $\psi(t)$ is nondecreasing, it is no loss of generality to assume also $t^{1/2}/\psi(t)$ to be nondecreasing, and, therefore, the assumption of Theorem 1.A is more restrictive than the assumption of Theorem 1.B. Second, we will show (Lemmas 2.15 and 2.16) that, assuming that $t^{1/\alpha}/\psi(t)$ or $t^{1/\alpha}\psi(t)$ is nondecreasing for some $\alpha > 0$, it is no loss of generality to assume also $t^{1/2}\psi(t)$ and $t^{1/2}/\psi(t)$ to be nondecreasing. That means that it is sufficient to prove our theorems assuming that $t^{1/2}\psi(t)$ and $t^{1/2}/\psi(t)$ are nondecreasing, and the statements remain true for $\psi(t)$ such that $t^{1/\alpha}\psi(t)$ or $t^{1/\alpha}/\psi(t)$ is nondecreasing for some $\alpha > 0$. We will also show (Lemmas 2.19 and 2.20) that the statements are not changed when $\psi^2(t)$ is replaced by $\psi^2(t) + K$ or $\psi^2(t) - K$, where K is a positive constant, even though the new function might not satisfy some monotonicity assumptions.

The same argument applies to most of the integral tests presented in the thesis.

Now to the lower classes. A classical theorem was proved by Chung (1948) for one-dimensional Brownian motion and $\psi \in \Psi_0$, and Jain and Taylor (1973) (via passage times) and Gruet and Shi (1995b) (directly) proved an analogous theorem for r -dimensional Brownian motion and $\psi \in \Psi_1$.

Theorem 1.C. *Let $\psi \in \Psi_1$ and let $r \geq 1$ be an integer. Then*

$$P \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_2(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.8)$$

where j_ν denotes the smallest positive zero of the Bessel function $J_\nu(x)$ of index $\nu = (r - 2)/2$, eg., for $r = 1$, $\nu = -1/2$ and $j_{-1/2} = \pi/2$.

Example 1.2. Let $\psi^2(t) = \log_2 t + 2 \log_3 t + \log_4 t + \dots + \log_{p-1} t + (1 + \delta) \log_p t$. Then for $p \geq 4$,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t)) \\ \text{LUC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t)) \end{cases} \quad \text{according as } \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The lower classes of $\{M^+(t); t \geq 0\}$ are different from those of $\{M_1(t); t \geq 0\}$. Hirsch (1965) proved the following theorem (see also Csáki (1978)).

Theorem 1.D. *Let $\psi \in \Psi_0$. Then*

$$P \left\{ M^+(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.9)$$

Example 1.3. Let $\psi(t) = \log t \log_2 t \dots \log_{p-1} t \log_p^{1+\delta} t$. Then for $p \geq 1$,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2}M^+(t)) \\ \text{LUC}(t^{-1/2}M^+(t)) \end{cases} \quad \text{according as } \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The main difference between Theorem 1.C and Theorem 1.D is that if $\mathcal{J}_1(\psi) < \infty$ then $\mathcal{J}_1(c\psi) < \infty$ for each $c > 0$. This means that $\liminf_{t \rightarrow \infty} t^{-1/2}M^+(t)\psi(t)$ is

either 0 or ∞ , in contrast to $\liminf_{t \rightarrow \infty} t^{-1/2} M_1(t) \psi(t)$, where we can obtain a positive finite number. A deeper insight into this phenomenon was obtained by Csáki (1978) by studying the joint behavior of $\{M^+(t); t \geq 0\}$ and $\{M^-(t); t \geq 0\}$.

So far we studied the big values of Brownian motion. For the small values we have (see Révész (1990), Chapter 17)

$$P\{W_1(t) = 0 \text{ i.o.}\} = 1, \quad (1.10)$$

$$P\{W_r(t) = 0 \text{ for any } t > 0\} = 0, \quad r \geq 2, \quad (1.11)$$

$$P\{\|W_2(t)\| \leq \varepsilon \text{ i.o.}\} = 1, \quad (1.12)$$

$$P\{\|W_r(t)\| \leq \varepsilon \text{ i.o.}\} = 0, \quad r \geq 3, \quad (1.13)$$

$$\lim_{t \rightarrow \infty} \|W_r(t)\| = \infty \quad \text{a.s.}, \quad r \geq 3. \quad (1.14)$$

That is, Brownian motion is point recurrent in dimension one, neighborhood recurrent but not point recurrent in dimension two, and transient in dimension three and higher. Therefore, we start the Brownian motion $W_r(t)$, $r \geq 3$, at zero, i.e., $W_r(0) = (0, \dots, 0)$, $r \geq 3$, and study how quickly the small values of $\|W_r(t)\|$, $r \geq 3$, approach infinity, and we start $W_2(t)$ at one, i.e., $W_2(0) = (1, 0)$, and study how quickly the small values of $\|W_2(t)\|$ approach zero. To this end we define for $t \geq 0$,

$$m_2(t) = \inf_{0 \leq s \leq t} \|W_2(s)\|$$

and

$$m_r(t) = \inf_{t \leq s} \|W_r(s)\|, \quad r \geq 3.$$

Dvoretzky and Erdős (1951) proved the following theorem.

Theorem 1.E. *Let $\psi \in \Psi_1$ and let $r \geq 3$ be an integer. Then*

$$P \left\{ m_r(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.15)$$

Example 1.4. Let $\psi(t) = (\log t \log_2 t \dots \log_{p-1} t \log_p^{1+\delta} t)^{1/(r-2)}$. Then for $p \geq 1$.

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2} m_r(t)) \\ \text{LUC}(t^{-1/2} m_r(t)) \end{cases} \text{ according as } \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The answer for $r = 2$ was given by Spitzer (1958).

Theorem 1.F. *Let $g \in \Psi_1$. Then*

$$P \left\{ m_2(t) \leq \frac{t^{1/2}}{g(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \int^\infty \frac{1}{t |\log g(t)|} \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.16)$$

Example 1.5. Let $\psi(t) = \log_2 t \dots \log_{p-1} t \log_p^{1+\delta} t$ and $g(t) = t^{\psi(t)}$. Then for $p \geq 2$.

$$g^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2} m_2(t)) \\ \text{LUC}(t^{-1/2} m_2(t)) \end{cases} \text{ according as } \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The upper classes of $\{m_r(t); t \geq 0\}$ were described in Khoshnevisan, Lewis, and Shi (1996) for $r \geq 3$, and in Gruet and Shi (1995b) (see also Hu and Shi (1995)) for $r = 2$.

Theorem 1.G. *Let $\psi \in \Psi_1$ and let $r \geq 3$ be an integer. Then*

$$P \left\{ m_r(t) \geq \sqrt{2} t^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.17)$$

Theorem 1.H. *Let $\psi \in \Psi_1$. Then*

$$P \left\{ m_2(t) \geq \exp \left(-\frac{\log t}{\psi(t)} \right) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.18)$$

Chapter 2

Lemmas about the Integrals

In this chapter we will prove several lemmas, especially concerning our integrals $\mathcal{J}_d$ and $\mathcal{J}_d$, which will be used repeatedly throughout the text.

First some notations. Let $a_n > 0$ and $b_n > 0$ for all n . We will write $a_n \approx b_n$ if there are positive constants c and c' such that

$$ca_n \leq b_n \leq c'a_n \quad \text{for all large enough } n \geq 1.$$

We will also write $\sum_n a_n \approx \sum_n b_n$ if $a_n \approx b_n$.

Lemma 2.1. *Let ψ belongs to Ψ_2^α or Ψ_3^α for some $\alpha > 0$. Let $d \geq 0$ be an integer and let $c > 0$. Then*

$$\mathcal{J}_d(\psi, c) < \infty \implies \psi(t) \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Proof. Assume that $\mathcal{J}_d(\psi, c) < \infty$, and take $T > 1$ large enough and $\lambda > 1$. Then

for $t^{1/\alpha}/\psi(t)$ nondecreasing,

$$\begin{aligned} & \int_T^{\lambda T} \frac{\max(1, \psi^d(t))}{t} \exp(-c\psi^2(t)) dt \\ & \geq \int_T^{\lambda T} \frac{1}{t} \max \left[1, \frac{\psi^d(\lambda T) T^{d/\alpha}}{(\lambda T)^{d/\alpha}} \right] \exp \left(-c \frac{\psi^2(T)}{T^{2/\alpha}} (\lambda T)^{2/\alpha} \right) dt \\ & = \log(\lambda) \max(1, \lambda^{-d/\alpha} \psi^d(\lambda T)) \exp(-c\lambda^{2/\alpha} \psi^2(T)), \end{aligned} \quad (2.1)$$

and for $t^{1/\alpha}\psi(t)$ nondecreasing,

$$\begin{aligned} & \int_T^{\lambda T} \frac{\max(1, \psi^d(t))}{t} \exp(-c\psi^2(t)) dt \\ & \geq \int_T^{\lambda T} \frac{1}{t} \max \left[1, \frac{\psi^d(T) T^{d/\alpha}}{(\lambda T)^{d/\alpha}} \right] \exp \left(-c \frac{\psi^2(\lambda T) (\lambda T)^{2/\alpha}}{T^{2/\alpha}} \right) dt \\ & = \log(\lambda) \max(1, \lambda^{-d/\alpha} \psi^d(T)) \exp(-c\lambda^{2/\alpha} \psi^2(\lambda T)). \end{aligned} \quad (2.2)$$

In both cases (2.1) and (2.2), the left-hand side goes to zero as $T \rightarrow \infty$ and, therefore, $\psi(T) \rightarrow \infty$ as $T \rightarrow \infty$. $\square$

Lemma 2.2. *Let ψ belongs to Ψ_2^α or Ψ_3^α for some $\alpha > 0$. Let $d \geq 1$ be an integer. Then*

$$\mathcal{J}_d(\psi) < \infty \implies \psi(t) \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Proof. Assume that $\mathcal{J}_d(\psi) < \infty$, and take $T > 1$ large enough and $\lambda > 1$. Then for $t^{1/\alpha}/\psi(t)$ nondecreasing,

$$\int_T^{\lambda T} \frac{1}{t\psi^d(t)} dt = \int_T^{\lambda T} \frac{t^{d/\alpha}}{t\psi^d(t)t^{d/\alpha}} dt \geq \frac{\log \lambda}{\psi^d(T)\lambda^{d/\alpha}}, \quad (2.3)$$

and for $t^{1/\alpha}\psi(t)$ nondecreasing,

$$\int_T^{\lambda T} \frac{1}{t\psi^d(t)} dt = \int_T^{\lambda T} \frac{t^{d/\alpha}}{t\psi^d(t)t^{d/\alpha}} dt \geq \frac{\log \lambda}{\psi^d(\lambda T)\lambda^{d/\alpha}}. \quad (2.4)$$

In both cases (2.3) and (2.4), the left-hand side goes to zero as $T \rightarrow \infty$ and, therefore, $\psi(T) \rightarrow \infty$ as $T \rightarrow \infty$. $\square$

Lemma 2.3. *Let ψ belongs to Ψ_2^α or Ψ_3^α for some $\alpha > 0$. The definition of $c^*(\psi)$ in (1.5) does not depend on d .*

Proof. Let k and d be integers, $k > d \geq 0$. Then $J_k(\psi, c) < \infty$ immediately implies $J_d(\psi, c) < \infty$. If $J_d(\psi, c) < \infty$, then $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$ by Lemma 2.1, and hence for $c' > c$ and t sufficiently large, $\psi^{k-d}(t) \exp(-(c' - c)\psi^2(t)) \leq 1$ and, therefore, $J_k(\psi, c') < \infty$ for all $c' > c$. $\square$

We will give a general lemma which allows us, in proving an integral test, to consider only functions that are bounded above and below by two standard functions (cf. Jain and Taylor (1973), Lemma 2.14).

Lemma 2.4. *Let g be an eventually nonincreasing function from $[1, \infty)$ to $[0, \infty)$. Let h from $[1, \infty)$ to $[0, \infty)$ be a measurable function. Let Ψ be a class of positive functions and let $\psi \in \Psi$. Define*

$$J(\psi) = \int_1^\infty g(\psi(t))h(t) dt,$$

which may be finite or infinite. Assume that for each ψ ,

$$\int_1^s g(\psi(t))h(t) dt < \infty \text{ for all } 1 < s < \infty.$$

Also assume that there exist $\psi_1, \psi_2 \in \Psi$, $\psi_1 \leq \psi_2$, such that $J(\psi_1) = \infty$, $J(\psi_2) < \infty$, and that the following implication holds true:

$$\text{if there is a sequence } t_n \nearrow \infty \text{ such that } \psi(t_n) \leq \psi_1(t_n) \text{ then } J(\psi) = \infty. \quad (2.5)$$

Let $\widehat{\psi}$ denote

$$\widehat{\psi} = \min[\max(\psi, \psi_1), \psi_2].$$

Then we have

$$\mathcal{J}(\psi) < \infty \implies \widehat{\psi} \leq \psi \text{ near } \infty \text{ and } \mathcal{J}(\widehat{\psi}) < \infty. \quad (2.6)$$

and

$$\mathcal{J}(\psi) = \infty \implies \mathcal{J}(\widehat{\psi}) = \infty. \quad (2.7)$$

Proof. We may assume without loss of generality that g is nonincreasing over its entire domain. Then $\xi_1 \leq \xi_2$ implies $\mathcal{J}(\xi_1) \geq \mathcal{J}(\xi_2)$. Let $\mathcal{J}(\psi) < \infty$. If there exists a sequence $t_n \nearrow \infty$ such that $\psi(t_n) < \widehat{\psi}(t_n)$, then we have $\psi(t_n) < \psi_1(t_n)$. By assumption (2.5), this implies $\mathcal{J}(\psi) = \infty$, a contradiction. Hence $\widehat{\psi} \leq \psi$ near infinity, which implies that $\psi_1 \leq \psi$ near infinity. Hence for some T , $1 < T < \infty$,

$$\begin{aligned} \int_T^\infty g(\widehat{\psi}(t))h(t) dt &= \int_T^\infty g(\widehat{\psi}(t))h(t)\mathbf{1}\{\psi_1(t) \leq \psi(t) \leq \psi_2(t)\} dt \\ &\quad + \int_T^\infty g(\widehat{\psi}(t))h(t)\mathbf{1}\{\psi(t) > \psi_2(t)\} dt \\ &\leq \mathcal{J}(\psi) + \mathcal{J}(\psi_2) < \infty. \end{aligned}$$

This proves (2.6).

Now let $\mathcal{J}(\psi) = \infty$. If $\widehat{\psi} \leq \psi$ near infinity, then $\mathcal{J}(\widehat{\psi}) = \infty$. On the other hand, if there exists a sequence $t_n \nearrow \infty$ such that $\psi(t_n) < \widehat{\psi}(t_n)$, then $\widehat{\psi}(t_n) = \psi_1(t_n)$. It follows from assumption (2.5) that $\mathcal{J}(\widehat{\psi}) = \infty$. $\square$

Lemma 2.5. *In Lemma 2.4, put $\Psi = \Psi_1$ and assume that*

$$\lim_{s \rightarrow \infty} g(\psi_1(s)) \int_1^s h(t) dt = \infty. \quad (2.8)$$

Then (2.5) holds true.

Proof. Assume $\psi \in \Psi_1$ and that there is a sequence $t_n \nearrow \infty$ such that $\psi(t_n) \leq \psi_1(t_n)$. Then

$$\begin{aligned} \int_1^{t_n} g(\psi(t))h(t) dt &\geq g(\psi(t_n)) \int_1^{t_n} h(t) dt \\ &\geq g(\psi_1(t_n)) \int_1^{t_n} h(t) dt. \end{aligned}$$

By (2.8), $\mathcal{J}(\psi) = \infty$. □

Remark 2.1. For the $\mathcal{J}_d$ integral we have $h(t) = 1/t$, $g(x) = \max(1, x^d) \exp(-x^2)$; hence g is eventually decreasing. For Ψ_1 we can choose $\psi_1(t) = ((1 - \varepsilon) \log \log t)^{1/2}$ and $\psi_2(t) = ((1 + \varepsilon) \log \log t)^{1/2}$, where $0 < \varepsilon < 1$. We easily verify that $\mathcal{J}_d(\psi_1) = \infty$, $\mathcal{J}_d(\psi_2) < \infty$, and

$$\begin{aligned} \lim_{s \rightarrow \infty} \psi_1^d(s) \exp(-\psi_1^2(s)) \int_1^s \frac{1}{t} dt &= \lim_{s \rightarrow \infty} \exp(-\psi_1^2(s) + \log \psi_1^d(s) + \log \log s) \\ &= \lim_{s \rightarrow \infty} \exp\left[\varepsilon \log \log s + \frac{d}{2} \log((1 - \varepsilon) \log \log s)\right] = \infty. \end{aligned}$$

For Ψ_2^α or Ψ_3^α , $\alpha > 0$, we can choose $\psi_1(t) = C$, where C is a positive constant, since, by Lemma 2.1, (2.5) holds true. The upper bound ψ_2 remains as for Ψ_1 .

For the $\mathcal{J}_d$ integral we have $h(t) = 1/t$, $g(x) = 1/x^d$; hence g is decreasing. For Ψ_1 we can choose $\psi_1(t) = (\log t)^{(1-\varepsilon)/d}$ and $\psi_2(t) = (\log t)^{(1+\varepsilon)/d}$, $0 < \varepsilon < 1$. Again, it is easy to verify that $\mathcal{J}_d(\psi_1) = \infty$, $\mathcal{J}_d(\psi_2) < \infty$, and

$$\lim_{s \rightarrow \infty} \frac{1}{\psi_1^d(s)} \int_1^s \frac{1}{t} dt = \lim_{s \rightarrow \infty} \frac{\log s}{(\log s)^{1-\varepsilon}} = \infty.$$

For Ψ_2^α or Ψ_3^α , $\alpha > 0$, we can choose $\psi_1(t) = C$, where C is a positive constant. The upper bound ψ_2 remains the same as for Ψ_1 .

Remark 2.2. Let $\psi \in \Psi_1$, and put $\psi_1(t) = (\frac{1}{2} \log \log t)^{1/2}$, $\psi_2(t) = (2 \log \log t)^{1/2}$ and $\widehat{\psi} = \min[\max(\psi, \psi_1), \psi_2]$. Then, as Jain and Taylor (1973) pointed out in their

Remark 3.3, we have

$$\mathcal{J}_d(\psi) < \infty \iff \mathcal{J}_d(\widehat{\psi}) < \infty \iff \int_1^\infty \frac{(\log \log t)^{d/2}}{t} \exp(-\psi^2(t)) dt < \infty.$$

Lemma 2.6. *Let g be a nonincreasing function from $[1, \infty)$ to $[0, \infty)$. Let $\lambda > 1$. Then*

$$\int_1^\infty \frac{g(t)}{t} dt < \infty \iff \sum_{k=0}^\infty g(\lambda^k) < \infty.$$

Proof. Let

$$\int_1^\infty \frac{g(t)}{t} dt < \infty.$$

Then

$$\int_1^\infty \frac{g(t)}{t} dt = \sum_{k=0}^\infty \int_{\lambda^k}^{\lambda^{k+1}} \frac{g(t)}{t} dt \geq \sum_{k=0}^\infty g(\lambda^{k+1}) \log \lambda,$$

and, therefore, $\sum_{k=0}^\infty g(\lambda^k) < \infty$.

Now let

$$\sum_{k=0}^\infty g(\lambda^k) < \infty.$$

Then

$$\begin{aligned} \sum_{k=0}^\infty g(\lambda^k) &= \frac{1}{\log \lambda} \sum_{k=0}^\infty g(\lambda^k) \log \lambda = \frac{1}{\log \lambda} \sum_{k=0}^\infty g(\lambda^k) \int_{\lambda^k}^{\lambda^{k+1}} \frac{1}{t} dt \\ &\geq \frac{1}{\log \lambda} \sum_{k=0}^\infty \int_{\lambda^k}^{\lambda^{k+1}} \frac{g(t)}{t} dt = \frac{1}{\log \lambda} \int_1^\infty \frac{g(t)}{t} dt, \end{aligned}$$

and, therefore,

$$\int_1^\infty \frac{g(t)}{t} dt < \infty.$$

□

Lemma 2.7. *Let $\infty > a \geq b \geq c > 0$ and $\infty > x \geq y \geq z > 0$. Then*

$$(a - b) \log \frac{x}{y} + (b - c) \log \frac{y}{z} \leq (a - c) \log \frac{x}{z}. \quad (2.9)$$

Proof. Inequality (2.9) is equivalent to

$$\left(\frac{x}{y}\right)^{a-b} \left(\frac{y}{z}\right)^{b-c} \leq \left(\frac{x}{z}\right)^{a-c},$$

which is in turn equivalent to

$$1 \leq \left(\frac{x}{y}\right)^{b-c} \left(\frac{y}{z}\right)^{a-b},$$

which, under our assumptions, holds true. $\square$

In the following lemma we will show that every function $\psi \in \Psi_1$ can be bounded both from above and from below by functions from Ψ_4^α , $\alpha > 0$, without changing the convergence or divergence of the integral. Therefore, any integral test we will prove for the class Ψ_2^α or Ψ_4^α of weight functions will be automatically true for the class Ψ_1 .

Lemma 2.8. *Let g be an eventually decreasing function from $[1, \infty)$ to $[0, \infty)$. Let $\alpha > 0$ and let $\Psi = \Psi_1 \cup \Psi_2^\alpha$, and for $\psi \in \Psi$ define*

$$\mathcal{J}(\psi) = \int_1^\infty \frac{1}{t} g(\psi(t)) dt,$$

which may be finite or infinite. Assume that for each $\psi \in \Psi$,

$$\int_1^s \frac{1}{t} g(\psi(t)) dt < \infty \quad \text{for all } 1 < s < \infty.$$

Also assume that there exist $\psi_1, \psi_2 \in \Psi$, $\psi_1 \leq \psi_2$, such that $\mathcal{J}(\psi_1) = \infty$, $\mathcal{J}(\psi_2) < \infty$, and that $\mathcal{J}(\psi) < \infty$ if and only if $\mathcal{J}(\hat{\psi}) < \infty$, where $\hat{\psi} = \min[\max(\psi, \psi_1), \psi_2]$. Further assume that for $\psi(t) = t^{1/\alpha}$, $\mathcal{J}(\psi)$ is finite, and that $\lim_{t \rightarrow \infty} t^{-1/\alpha} \psi_2(t) = 0$.

Let $\varphi \in \Psi_1$, $\psi_1 \leq \varphi \leq \psi_2$. Then there exist φ_1 and φ_2 such that $\varphi_1, \varphi_2 \in \Psi_4^\alpha$, $\varphi_1 \leq \varphi \leq \varphi_2$, and $\mathcal{J}(\varphi_1)$, $\mathcal{J}(\varphi)$ and $\mathcal{J}(\varphi_2)$ are either all finite or all infinite.

Proof. Assume without loss of generality that $\varphi \notin \Psi_4^\alpha$. The idea of the proof is to replace the parts of φ where $t^{-1/\alpha}\varphi(t)$ is not nonincreasing, ie., the parts where φ jumps up or is increasing too quickly, by a constant multiple of $t^{1/\alpha}$.

Define

$$\underline{\varphi}(t) = \varphi(e^k), \quad e^k \leq t < e^{k+1},$$

and

$$\overline{\varphi}(t) = \varphi(e^{k+1}), \quad e^k \leq t < e^{k+1}.$$

Obviously $\underline{\varphi}(t) \leq \varphi(t) \leq \overline{\varphi}(t)$, and since

$$J(\varphi) < \infty \iff \sum_k g(\varphi(e^k)) < \infty,$$

$J(\underline{\varphi})$, $J(\varphi)$, and $J(\overline{\varphi})$ are all finite or all infinite.

Define $\xi_T(t)$, $T \geq 1$, $t \geq T$, by $\xi_T(t) = K_T t^{1/\alpha}$, where K_T is such that $\xi_T(T) = \underline{\varphi}(T-)$, ie., $K_T = T^{-1/\alpha} \underline{\varphi}(T-)$ and $\xi_T(t) = T^{-1/\alpha} \underline{\varphi}(T-) t^{1/\alpha}$, where $\underline{\varphi}(T-)$ denotes the left-hand side limit of $\underline{\varphi}$ at T . Put $t_1 = s_1 = 1$, and let k_n be such that $\exp(k_n) \leq s_n < \exp(k_n + 1)$. Then put $t_{n+1} = \exp(k_n + 1)$ and

$$s_{n+1} = \inf\{t > t_{n+1}; \xi_{t_{n+1}}(t) = \underline{\varphi}(t)\}.$$

Our assumptions imply that $s_{n+1} < \infty$ for each n . In this way we obtain a sequence $t_1 = s_1 \leq \dots \leq t_n < s_n \leq t_{n+1} < s_{n+1} \leq \dots$. For $t_n < t < s_n$ we have $\xi_{t_n}(t) < \underline{\varphi}(t)$. Now define

$$\varphi_1(t) = \begin{cases} \underline{\varphi}(t), & s_{n-1} \leq t < t_n, \\ \xi_{t_n}(t), & t_n \leq t < s_n. \end{cases}$$

Obviously $\varphi_1 \leq \varphi$ and notice that $s_n/t_n = \underline{\varphi}^\alpha(s_n)/\underline{\varphi}^\alpha(t_n-)$.

Now we have

$$\begin{aligned}
0 \leq \mathcal{I}(\varphi_1) - \mathcal{I}(\underline{\varphi}) &\leq \sum_n \int_{t_n}^{s_n} \frac{1}{t} g(\varphi_1(t)) dt \\
&= \sum_n \int_{t_n}^{s_n} \frac{1}{t} g(\xi_{t_n}(t)) dt \\
&= \sum_n \int_{t_n}^{s_n} \frac{1}{t} g(t_n^{-1/\alpha} \underline{\varphi}(t_n -) t^{1/\alpha}) dt \\
&= \sum_n \int_{\underline{\varphi}^\alpha(t_n -)}^{\underline{\varphi}^\alpha(s_n)} \frac{1}{t} g(t^{1/\alpha}) dt \\
&\leq \int_1^\infty \frac{1}{t} g(t^{1/\alpha}) dt < \infty.
\end{aligned}$$

This means that $\mathcal{I}(\underline{\varphi})$ and $\mathcal{I}(\varphi_1)$ are both finite or both infinite.

In a similar way we construct φ_2 . Define $\zeta_T(t)$, $T \geq 1$, $t \geq T$, by $\zeta_T(t) = C_T t^{1/\alpha}$, where C_T is such that $\zeta_T(T) = \overline{\varphi}(T)$, ie., $C_T = T^{-1/\alpha} \overline{\varphi}(T)$ and $\zeta_T(t) = T^{-1/\alpha} \overline{\varphi}(T) t^{1/\alpha}$. Define, inductively, $r_1 = v_1 = 1$,

$$r_{n+1} = \inf\{T > v_n; \exists t > T \text{ such that } \zeta_T(t) \leq \overline{\varphi}(t)\},$$

$$v_{n+1} = \sup\{t > r_{n+1}; \zeta_{r_{n+1}}(t) = \overline{\varphi}(t)\}.$$

Our assumptions imply that $v_{n+1} < \infty$. In this way we obtain a sequence $r_1 = v_1 \leq \dots \leq r_n < v_n \leq r_{n+1} < v_{n+1} \leq \dots$. For $r_n < t < v_n$ we have $\overline{\varphi}(t) < \zeta_{r_n}(t)$. Now define

$$\varphi_2(t) = \begin{cases} \overline{\varphi}(t), & v_{n-1} \leq t \leq r_n, \\ \zeta_{r_n}(t), & r_n < t < v_n. \end{cases}$$

Obviously $\varphi \leq \varphi_2$ and observe that $v_n/r_n = \overline{\varphi}^\alpha(v_n)/\overline{\varphi}^\alpha(r_n)$.

Now we have

$$\begin{aligned}
 0 \leq \mathcal{J}(\bar{\varphi}) - \mathcal{J}(\varphi_2) &\leq \sum_n \left[\int_{r_n}^{v_n} \frac{1}{t} g(\bar{\varphi}(r_n)) dt - \int_{r_n}^{v_n} \frac{1}{t} g(\bar{\varphi}(v_n)) dt \right] \\
 &\leq \sum_n \left(g(\bar{\varphi}(r_n)) - g(\bar{\varphi}(v_n)) \right) \log \frac{\bar{\varphi}^\alpha(v_n)}{\bar{\varphi}^\alpha(r_n)} \\
 &\leq \sum_n \left(g(\bar{\varphi}(r_n)) - g(\bar{\varphi}(r_{n+1})) \right) \log \frac{\bar{\varphi}^\alpha(r_{n+1})}{\bar{\varphi}^\alpha(r_n)} \\
 &\leq \sum_n \left(g(a_n) - g(a_{n+1}) \right) \log \frac{a_{n+1}^\alpha}{a_n^\alpha},
 \end{aligned}$$

where the last inequality holds true by Lemma 2.7 for any sequence a_n , $0 < a_n \nearrow \infty$.

Take $\lambda > 1$ and put $a_n = \lambda^n$. Since $\mathcal{J}(t^{1/\alpha}) < \infty$,

$$\mathcal{J}(\bar{\varphi}) - \mathcal{J}(\varphi_2) \leq \log(\lambda^\alpha) \cdot \sum_n g(\lambda^n) < \infty$$

by Lemma 2.6. Therefore, $\mathcal{J}(\bar{\varphi})$ and $\mathcal{J}(\varphi_2)$ are both finite or infinite. $\square$

Lemma 2.9. Let $\psi \in \Psi_2^\alpha$ for some $\alpha > 0$. Let $d \geq 1$ be an integer and let $\lambda > 1$.

Then

$$\mathcal{J}_d(\psi) < \infty \iff \sum_k \frac{1}{\psi^d(\lambda^k)} < \infty.$$

Proof. First, let $\mathcal{J}_d(\psi) < \infty$. Then

$$\begin{aligned}
 \int_1^\infty \frac{1}{t\psi^d(t)} dt &= \sum_{k=0}^\infty \int_{\lambda^k}^{\lambda^{k+1}} \frac{1}{t\psi^d(t)} dt \\
 &= \sum_{k=0}^\infty \int_{\lambda^k}^{\lambda^{k+1}} \frac{t^{d/\alpha}}{t\psi(t)t^{d/\alpha}} dt.
 \end{aligned}$$

Take $K \in \mathbb{N}$ sufficiently large such that $t^{1/\alpha}/\psi(t)$ is nondecreasing on $[\lambda^K, \infty)$. Now

$$\begin{aligned}
 \sum_{k \geq K} \int_{\lambda^k}^{\lambda^{k+1}} \frac{t^{d/\alpha}}{t\psi^d(t)t^{d/\alpha}} dt &\geq \sum_{k \geq K} \int_{\lambda^k}^{\lambda^{k+1}} \frac{1}{t} \cdot \frac{\lambda^{kd/\alpha}}{\psi^d(\lambda^k)} \cdot \frac{1}{\lambda^{(k+1)d/\alpha}} dt \\
 &= \lambda^{-d/\alpha} \log(\lambda) \sum_{k \geq K} \frac{1}{\psi^d(\lambda^k)},
 \end{aligned}$$

and, therefore, $\sum_k 1/\psi^d(\lambda^k) < \infty$.

On the other hand, let $\sum_k 1/\psi^d(\lambda^k) < \infty$. Then, taking K as above, we have

$$\begin{aligned}
 \sum_{k \geq K+1} \frac{1}{\psi^d(\lambda^k)} &= \frac{1}{\log \lambda} \sum_{k \geq K+1} \log(\lambda) \cdot \frac{1}{\psi^d(\lambda^k)} \\
 &= \frac{1}{\log \lambda} \sum_{k \geq K+1} \int_{\lambda^{k-1}}^{\lambda^k} \frac{1}{t} \cdot \frac{1}{\psi^d(\lambda^k)} dt \\
 &= \frac{1}{\lambda^{d/\alpha} \log \lambda} \sum_{k \geq K+1} \int_{\lambda^{k-1}}^{\lambda^k} \frac{1}{t} \cdot \frac{\lambda^{kd/\alpha}}{\psi^d(\lambda^k)} \cdot \frac{1}{\lambda^{(k-1)d/\alpha}} dt \\
 &\geq \frac{1}{\lambda^{d/\alpha} \log \lambda} \sum_{k \geq K+1} \int_{\lambda^{k-1}}^{\lambda^k} \frac{1}{t} \cdot \frac{t^{d/\alpha}}{\psi^d(t)} \cdot \frac{1}{t^{d/\alpha}} dt \\
 &= \frac{1}{\lambda^{d/\alpha} \log \lambda} \int_{\lambda^K}^{\infty} \frac{1}{t \psi^d(t)} dt,
 \end{aligned}$$

and, therefore, $\mathcal{J}_d(\psi) < \infty$. □

Lemma 2.10. *Let $\psi \in \Psi_4^\alpha$, $\alpha > 0$, and assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Take t_0 sufficiently large, such that $t^{1/\alpha}\psi(t)$ and $t^{1/\alpha}/\psi(t)$ are nondecreasing for $t \geq t_0$, and $\psi(t) \geq 2$ for $t \geq t_0$. We will write ψ_k instead of $\psi(t_k)$. Define*

$$t_{k+1} = t_k \frac{\psi_k^2}{\psi_k^2 - 1} = t_k \left(1 + \frac{1}{\psi_k^2 - 1} \right). \quad (2.10)$$

Let $d \geq 0$ be an integer. Then

$$\mathcal{J}_d(\psi) < \infty \iff \sum_k \frac{t_{k+1} - t_k}{t_k} \psi_k^d \exp(-\psi_k^2) < \infty. \quad (2.11)$$

Proof. First observe that $t_{k+1}/t_k \rightarrow 1$ as $k \rightarrow \infty$. Since $t_k^{1/\alpha}\psi_k \leq t_{k+1}^{1/\alpha}\psi_{k+1}$ and $t_k^{1/\alpha}/\psi_k \leq t_{k+1}^{1/\alpha}/\psi_{k+1}$, we have

$$\frac{t_k^{1/\alpha}}{t_{k+1}^{1/\alpha}} \leq \frac{\psi_k}{\psi_{k+1}} \leq \frac{t_{k+1}^{1/\alpha}}{t_k^{1/\alpha}},$$

ie.,

$$\psi_k \left(\frac{\psi_k^2 - 1}{\psi_k^2} \right)^{1/\alpha} \leq \psi_{k+1} \leq \psi_k \left(\frac{\psi_k^2}{\psi_k^2 - 1} \right)^{1/\alpha}. \quad (2.12)$$

Since

$$t_{k+1} - t_k = \frac{t_k}{\psi_k^2 - 1}.$$

we have $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Using our monotonicity assumptions we have

$$\int_{t_k}^{t_{k+1}} \frac{1}{t} \psi^d(t) \exp(-\psi^2(t)) dt \geq \frac{t_{k+1} - t_k}{t_{k+1}} t_k^{d/\alpha} \frac{\psi_{k+1}^d}{t_{k+1}^{d/\alpha}} \exp\left(-\frac{\psi_k^2 t_k^{2/\alpha}}{t_{k+1}^{2/\alpha}}\right)$$

and

$$\int_{t_k}^{t_{k+1}} \frac{1}{t} \psi^d(t) \exp(-\psi^2(t)) dt \leq \frac{t_{k+1} - t_k}{t_k} t_{k+1}^{d/\alpha} \frac{\psi_k^d}{t_k^{d/\alpha}} \exp\left(-\frac{\psi_k^2 t_k^{2/\alpha}}{t_{k+1}^{2/\alpha}}\right).$$

This implies

$$\begin{aligned} \frac{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \exp(-\psi_k^2)}{\int_{t_k}^{t_{k+1}} \frac{1}{t} \psi^d(t) \exp(-\psi^2(t)) dt} &\leq \frac{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \exp(-\psi_k^2)}{\frac{t_{k+1} - t_k}{t_{k+1}} t_k^{d/\alpha} \frac{\psi_{k+1}^d}{t_{k+1}^{d/\alpha}} \exp\left(-\frac{\psi_k^2 t_k^{2/\alpha}}{t_{k+1}^{2/\alpha}}\right)} \\ &= \left(\frac{t_{k+1}}{t_k}\right)^{1+d/\alpha} \frac{\psi_k^d}{\psi_{k+1}^d} \exp\left[-\psi_k^2 \left(1 - \frac{\psi_k^{4/\alpha}}{(\psi_k^2 - 1)^{2/\alpha}}\right)\right] \rightarrow e^{2/\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \exp(-\psi_k^2)}{\int_{t_k}^{t_{k+1}} \frac{1}{t} \psi^d(t) \exp(-\psi^2(t)) dt} &\geq \frac{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \exp(-\psi_k^2)}{\frac{t_{k+1} - t_k}{t_k} t_{k+1}^{d/\alpha} \frac{\psi_k^d}{t_k^{d/\alpha}} \exp\left(-\frac{\psi_k^2 t_k^{2/\alpha}}{t_{k+1}^{2/\alpha}}\right)} \\ &= \frac{t_k^{d/\alpha}}{t_{k+1}^{d/\alpha}} \exp\left[-\psi_k^2 \left(1 - \frac{(\psi_k^2 - 1)^{2/\alpha}}{\psi_k^{4/\alpha}}\right)\right] \rightarrow e^{-2/\alpha}. \end{aligned}$$

Thus the lemma is proved. $\square$

Corollary 2.1. *It follows from the proof of Lemma 2.10 that*

$$\int_{t_k}^{t_{k+1}} \frac{\psi^d(t)}{t} \exp(-\psi^2(t)) dt \approx \psi_k^{d-2} \exp(-\psi_k^2) \approx \widehat{\psi}_k^{d-2} \exp(-\widehat{\psi}_k^2), \quad (2.13)$$

where $\widehat{\psi}_k = \psi(t)$ for some $t \in [t_k, t_{k+1}]$.

We will prove in the following lemma that, if $J_d(\psi) = \infty$, then the integral is also equal to infinity when we integrate only over the intervals where ψ does not have a ‘big increase’.

Lemma 2.11. *Under the assumptions and notations of Lemma 2.10, put*

$$\mathcal{K} = \left\{ k: \psi_k^2 > \psi_{k-1}^2 + \frac{1}{\alpha} \right\} \quad (2.14)$$

and $\mathcal{K}^c = \mathbb{N} - \mathcal{K}$. Then

$$J_d(\psi) = \infty \implies \sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty.$$

Proof. By Corollary 2.1, $J_d(\psi) = \infty$ implies that $\sum_{k=1}^{\infty} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$. If $\sum_{k \in \mathcal{K}} \psi_k^{d-2} \exp(-\psi_k^2) < \infty$ then obviously $\sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$. Hence we only need to show that $\sum_{k \in \mathcal{K}} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$ implies $\sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$.

Define

$$\mathcal{T} = \bigcup_{k \in \mathcal{K}} [t_{k-1}, t_k), \quad (2.15)$$

$$\mathcal{K}_1 = \{k; k \in \mathcal{K}, \psi_{k-1}^2 \geq \psi_l^2 \text{ for all } l < k, l \in \mathcal{K}\}, \quad (2.16)$$

$$\mathcal{T}_1 = \bigcup_{k \in \mathcal{K}_1} [t_{k-1}, t_k), \quad (2.17)$$

and

$$\mathcal{K}_2 = \mathcal{K} - \mathcal{K}_1. \quad (2.18)$$

We will repeatedly use Corollary 2.1. First, we will show that

$$\sum_{k \in \mathcal{K}_1} \psi_k^{d-2} \exp(-\psi_k^2) < \infty.$$

Put $\tilde{\psi}(t) = \psi_{k-1} t^{1/\alpha} / t_{k-1}^{1/\alpha}$ for $t \in [t_{k-1}, t_k]$. Then

$$\begin{aligned} \sum_{k \in \mathcal{K}_1} \psi_k^{d-2} \exp(-\psi_k^2) &\approx \sum_{k \in \mathcal{K}_1} \int_{t_{k-1}}^{t_k} \frac{\tilde{\psi}^d(t)}{t} \exp(-\tilde{\psi}^2(t)) dt \\ &= \sum_{k \in \mathcal{K}_1} \int_{\psi_{k-1}^2}^{\psi_{k-1}^2 [\psi_{k-1}^2 / (\psi_{k-1}^2 - 1)]^{2/\alpha}} \frac{\alpha}{2} z^{d/2-1} e^{-z} dz \\ &\leq 3 \int_1^\infty \frac{\alpha}{2} z^{d/2-1} e^{-z} dz < \infty. \end{aligned}$$

since

$$\psi_{k-1}^2 \left(\frac{\psi_{k-1}^2}{\psi_{k-1}^2 - 1} \right)^{2/\alpha} \leq \psi_{k-1}^2 + \frac{3}{\alpha}$$

and, therefore, no interval of values of z is included in the integration domain more than three times. Thus, $\sum_{k \in \mathcal{K}} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$ implies $\sum_{k \in \mathcal{K}_2} \psi_k^{d-2} \exp(-\psi_k^2) = \infty$. For each $k \in \mathcal{K}_2$, at least some of the values which ψ attains in the interval $[t_{k-1}, t_k]$ were attained before, outside $\mathcal{T}$, i.e., for each $k \in \mathcal{K}_2$ there exist $l_k \notin \mathcal{K}$, $l_k < k$, and $s_{1,k}$ and $s_{2,k}$ such that $t_{l_k-1} \leq s_{1,k} \leq t_{l_k}$, $t_{k-1} \leq s_{2,k} \leq t_k$, and $\psi(s_{1,k}) = \psi(s_{2,k})$. Since $\psi^2(t)$ increases on $[t_{k-1}, t_k]$ at least by $1/\alpha$ and decreases on $[t_{l_k-1}, t_{l_k}]$ at most by $2/\alpha$, it is possible to construct a sequence $s_{1,k}$, $k \in \mathcal{K}_2$, in such a way that $s_{1,k}$ is nondecreasing and there are at most two elements in every interval $[t_{l_k-1}, t_{l_k}]$. Therefore,

$$\begin{aligned} \infty &= \sum_{k \in \mathcal{K}_2} \psi_k^{d-2} \exp(-\psi_k^2) \approx \sum_{k \in \mathcal{K}_2} \psi^{d-2}(s_{2,k}) \exp(-\psi^2(s_{2,k})) \\ &= \sum_{k \in \mathcal{K}_2} \psi^{d-2}(s_{1,k}) \exp(-\psi^2(s_{1,k})) \approx \sum_{k \in \mathcal{K}_2} \psi_{l_k}^{d-2} \exp(-\psi_{l_k}^2) \\ &\leq \sum_{l \in \mathcal{K}^c} \psi_l^{d-2} \exp(-\psi_l^2). \end{aligned}$$

□

Similarly, we will prove in the following lemma that, if $\mathcal{J}_d(\psi) = \infty$, the integral

is also equal to infinity when we integrate only over the intervals where ψ does not have a ‘big decrease’.

Lemma 2.12. *Under the assumptions and notations of Lemma 2.10, put*

$$\mathcal{K} = \left\{ k; \psi_k^2 - \frac{1}{\alpha} > \psi_{k+1}^2 \right\} \quad (2.19)$$

and $\mathcal{K}^c = \mathbb{N} - \mathcal{K}$. Then

$$\mathcal{J}_d(\psi) = \infty \implies \sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty.$$

Proof. Assume that $\mathcal{J}_d(\psi) = \infty$. Denote

$$\mathcal{T} = \bigcup_{k \in \mathcal{K}} [t_k, t_{k+1}].$$

For each $k \in \mathcal{K}$, at least some of the values which ψ attains in the interval $[t_k, t_{k+1}]$ will be attained again, outside $\mathcal{T}$, ie., for each $k \in \mathcal{K}$ there exist $l_k \notin \mathcal{K}$, $l_k > k$, and $s_{1,k}$ and $s_{2,k}$ such that $t_k \leq s_{1,k} \leq t_{k+1}$, $t_{l_k} \leq s_{2,k} \leq t_{l_k+1}$ and $\psi(s_{1,k}) = \psi(s_{2,k})$. Since $\psi^2(t)$ decreases on $[t_k, t_{k+1}]$ at least by $1/\alpha$ and increases on $[t_{l_k}, t_{l_k+1}]$ at most by

$$\psi_k^2 \left(\frac{\psi_k^2}{\psi_k^2 - 1} \right)^{2/\alpha} - \psi_k^2 \leq \frac{3}{\alpha},$$

it is possible to construct a sequence $s_{2,k}$, $k \in \mathcal{K}$, in such a way that $s_{2,k}$ is nondecreasing and there are at most three elements in every interval $[t_{l_k}, t_{l_k+1}]$. Therefore,

$$\begin{aligned} \sum_{k \in \mathcal{K}} \psi_k^{d-2} \exp(-\psi_k^2) &\approx \sum_{k \in \mathcal{K}} \psi^{d-2}(s_{1,k}) \exp(-\psi^2(s_{1,k})) \\ &= \sum_{k \in \mathcal{K}} \psi^{d-2}(s_{2,k}) \exp(-\psi^2(s_{2,k})) \\ &\approx \sum_{k \in \mathcal{K}} \psi_{l_k}^{d-2} \exp(-\psi_{l_k}^2) \\ &\leq \sum_{l \in \mathcal{K}^c} \psi_l^{d-2} \exp(-\psi_l^2). \end{aligned}$$

Therefore, $\mathcal{J}_d(\psi) = \infty$ implies that

$$\sum_{l \in \mathcal{K}^c} \psi_l^{d-2} \exp(-\psi_l^2) = \infty,$$

as required. $\square$

Lemma 2.13. *Let ψ be a positive function, $d \geq 0$ be an integer, and $K > 0$. Then for $\tilde{\psi}(t) = \sqrt{\psi^2(t) + K}$,*

$$\mathcal{J}_d(\tilde{\psi}) < \infty \iff \mathcal{J}_d(\psi) < \infty. \quad (2.20)$$

Proof. Since $\psi(t) \leq \tilde{\psi}(t)$, $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\tilde{\psi}) < \infty$. If $\mathcal{J}_d(\psi) = \infty$, then

$$\begin{aligned} \mathcal{J}_d(\tilde{\psi}) &= \int_0^\infty \frac{\max(1, \tilde{\psi}^d(t))}{t} \exp(-\tilde{\psi}^2(t)) dt \\ &= e^{-K} \int_0^\infty \left(\frac{\max(1, \psi^2(t) + K)}{\max(1, \psi^2(t))} \right)^{d/2} \frac{\max(1, \psi^d(t))}{t} \exp(-\psi^2(t)) dt \\ &= \infty. \end{aligned}$$

$\square$

Lemma 2.14. *Let ψ be a positive function, $d \geq 0$ be an integer, $K > 0$, and let $\psi^2(t) > K$ for all t . Then for $\tilde{\psi}(t) = \sqrt{\psi^2(t) - K}$,*

$$\mathcal{J}_d(\tilde{\psi}) < \infty \iff \mathcal{J}_d(\psi) < \infty. \quad (2.21)$$

Proof. Since $\psi(t) \geq \tilde{\psi}(t)$, $\mathcal{J}_d(\psi) = \infty$ implies $\mathcal{J}_d(\tilde{\psi}) = \infty$. If $\mathcal{J}_d(\psi) < \infty$, then

$$\begin{aligned} \mathcal{J}_d(\tilde{\psi}) &= \int_0^\infty \frac{\max(1, \tilde{\psi}^d(t))}{t} \exp(-\tilde{\psi}^2(t)) dt \\ &= e^K \int_0^\infty \left(\frac{\max(1, \psi^2(t) - K)}{\max(1, \psi^2(t))} \right)^{d/2} \frac{\max(1, \psi^d(t))}{t} \exp(-\psi^2(t)) dt \\ &< \infty. \end{aligned}$$

$\square$

We will show in the two following lemmas, that the functions from Ψ_2^α (from Ψ_3^α), $\alpha > 0$, can be bounded from above and from below by functions from $\Psi_3^\beta \cap \Psi_2^\alpha$ (from $\Psi_2^\beta \cap \Psi_3^\alpha$, respectively), $\beta > 0$, preserving the convergence or divergence of $\mathcal{I}_d$ integral. Therefore, an integral test proved for Ψ_4^α , $\alpha > 0$, will remain true for both Ψ_2^β and Ψ_3^β , $\beta > 0$.

Lemma 2.15. *Let $\psi \in \Psi_2^\alpha$ for some $\alpha > 0$. Let $d \geq 0$ be an integer, and let $\beta > 0$. Then there exist functions $\psi_1, \psi_2 \in \Psi_3^\beta \cap \Psi_2^\alpha$ such that $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$ and $\mathcal{I}_d(\psi_1)$, $\mathcal{I}_d(\psi)$, and $\mathcal{I}_d(\psi_2)$ are all finite or infinite.*

Proof. Let t_0 be such that $t^{1/\alpha}/\psi(t)$ is nondecreasing for $t \geq t_0$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq 2 \log \log t$ for all $t \geq t_0$. Put for $t \geq t_0$,

$$\begin{aligned}\psi_1(t) &= \frac{1}{t^{1/\beta}} \inf_{s \geq t} \psi(s) s^{1/\beta}, \\ \psi_2(t) &= \frac{1}{t^{1/\beta}} \sup_{t_0 \leq s \leq t} \psi(s) s^{1/\beta}.\end{aligned}$$

Obviously $\psi_1, \psi_2 \in \Psi_3^\beta \cap \Psi_2^\alpha$, $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$. Hence $\mathcal{I}_d(\psi) < \infty$ implies $\mathcal{I}_d(\psi_2) < \infty$ and $\mathcal{I}_d(\psi) = \infty$ implies $\mathcal{I}_d(\psi_1) = \infty$. We only need to prove that $\mathcal{I}_d(\psi) < \infty$ implies $\mathcal{I}_d(\psi_1) < \infty$ and $\mathcal{I}_d(\psi) = \infty$ implies $\mathcal{I}_d(\psi_2) = \infty$.

First assume that $\mathcal{I}_d(\psi_1) = \infty$ and $\mathcal{I}_d(\psi) < \infty$. Put

$$\begin{aligned}\mathcal{T} &= \{t; t \geq t_0, \psi_1(t) = \psi(t)\}, \\ \tau_-(t) &= \sup\{s; s \in \mathcal{T}, s \leq t\}, \\ \tau_+(t) &= \inf\{s; s \in \mathcal{T}, s \geq t\}.\end{aligned}$$

Let $K > 1$ be a constant. Define

$$\mathcal{T}_1 = \{t \notin \mathcal{T}; \sup_{\tau_-(t) \leq s \leq \tau_+(t)} \psi^2(s) - \psi_1^2(s) \leq K\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt \leq \mathcal{J}_d(\psi) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt \leq \mathcal{J}_d(\sqrt{\psi^2(t) - K}) < \infty,$$

by Lemma 2.14. Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt = \infty.$$

All intervals contained in $\mathcal{T}_2$ are such that $\psi_1(t)$ is decreasing there as a multiple of $t^{-1/\beta}$. However, a similar argument to that in Lemma 2.12 shows that the integral of ψ_1 must be infinity also if we integrate only over the intervals where ψ_1 does not have a ‘big decrease’, i.e., outside of $\mathcal{T}_2$, a contradiction. Therefore, $\mathcal{J}_d(\psi) = \infty$.

Now assume $\mathcal{J}_d(\psi) = \infty$ and $\mathcal{J}_d(\psi_2) < \infty$. Now put

$$\mathcal{T} = \{t; t \geq t_0, \psi_2(t) = \psi(t)\},$$

$$\tau_-(t) = \sup\{s; s \in \mathcal{T}, s \leq t\},$$

$$\tau_+(t) = \inf\{s; s \in \mathcal{T}, s \geq t\}.$$

Let $K > 1$ be a constant. Define

$$\mathcal{T}_1 = \left\{ t \notin \mathcal{T}; \sup_{\tau_-(t) \leq s \leq \tau_+(t)} (\psi_2^2(s) - \psi^2(s)) \leq K \right\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \leq \mathcal{J}_d(\psi_2) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \leq \mathcal{J}_d(\sqrt{\psi_2^2(t) - K}) < \infty,$$

by Lemma 2.14. Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt = \infty.$$

Let $\mathcal{T}_2 = \bigcup_k [a_k, b_k]$, where $a_1 < b_1 \leq a_2 < b_2 \leq \dots$. Then

$$\begin{aligned} \infty &= \int_{\mathcal{T}_2} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \\ &\leq \sum_k \int_{a_k}^{b_k} \frac{1}{t} \max \left[1, \left(\frac{\psi(b_k) t^{1/\alpha}}{b_k^{1/\alpha}} \right)^d \right] \exp \left(-\frac{\psi^2(b_k) t^{2/\alpha}}{b_k^{2/\alpha}} \right) dt \\ &\leq c \int_{\mathcal{T}_2} \frac{\max[1, \psi_2^d(t)]}{t} \exp(-\psi_2^2(t)) dt, \end{aligned}$$

for some constant $c > 0$ dependent on α and β only, a contradiction. Therefore $\mathcal{J}_d(\psi_2) = \infty$ as desired. $\square$

Lemma 2.16. *Let $\psi \in \Psi_3^\alpha$ for some $\alpha > 0$. Let $d \geq 0$ be an integer, and let $\beta > 0$. Then there exist functions $\psi_1, \psi_2 \in \Psi_2^\beta \cap \Psi_3^\alpha$ such that $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$ and $\mathcal{J}_d(\psi_1)$, $\mathcal{J}_d(\psi)$, and $\mathcal{J}_d(\psi_2)$ are all finite or infinite.*

Proof. Let t_0 be such that $t^{1/\alpha}\psi(t)$ is nondecreasing for $t \geq t_0$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq 2 \log \log t$ for all $t \geq t_0$. Put for $t \geq t_0$,

$$\begin{aligned} \psi_1(t) &= t^{1/\beta} \inf_{t_0 \leq s \leq t} \frac{\psi(s)}{s^{1/\beta}}, \\ \psi_2(t) &= t^{1/\beta} \sup_{s \geq t} \frac{\psi(s)}{s^{1/\beta}}. \end{aligned}$$

Obviously $\psi_1, \psi_2 \in \Psi_2^\beta \cap \Psi_3^\alpha$, $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$. Hence $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\psi_2) < \infty$ and $\mathcal{J}_d(\psi) = \infty$ implies $\mathcal{J}_d(\psi_1) = \infty$. We only need to prove that $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\psi_1) < \infty$ and $\mathcal{J}_d(\psi) = \infty$ implies $\mathcal{J}_d(\psi_2) = \infty$.

First assume that $J_d(\psi_1) = \infty$ and $J_d(\psi) < \infty$. Put

$$\begin{aligned}\mathcal{T} &= \{t; t \geq t_0, \psi_1(t) = \psi(t)\}, \\ \tau_-(t) &= \sup\{s; s \in \mathcal{T}, s \leq t\}, \\ \tau_+(t) &= \inf\{s; s \in \mathcal{T}, s \geq t\}.\end{aligned}$$

Let $K > 1$ be a constant. Define

$$\begin{aligned}\mathcal{T}_1 &= \left\{t \notin \mathcal{T}; \sup_{\tau_-(t) \leq s \leq \tau_+(t)} (\psi^2(s) - \psi_1^2(s)) \leq K\right\}, \\ \mathcal{T}_2 &= \mathcal{T}^c - \mathcal{T}_1.\end{aligned}$$

Now

$$\int_{\mathcal{T}} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt \leq J_d(\psi) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt \leq J_d(\sqrt{\psi^2(t) - K}) < \infty,$$

by Lemma 2.14. Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{\max[1, \psi_1^d(t)]}{t} \exp(-\psi_1^2(t)) dt = \infty.$$

All intervals contained in $\mathcal{T}_2$ are such that $\psi_1(t)$ is increasing there as a multiple of $t^{1/\beta}$ there. However, a similar argument to that in Lemma 2.11 shows that the integral of ψ_1 must be infinity also if we integrate only over the intervals where ψ_1 does not have a ‘big increase’, ie., outside of $\mathcal{T}_2$, a contradiction. Therefore, $J_d(\psi) = \infty$.

Now assume $J_d(\psi) = \infty$ and $J_d(\psi_2) < \infty$. Now put

$$\mathcal{T} = \{t; t \geq t_0, \psi_2(t) = \psi(t)\},$$

$$\tau_-(t) = \sup\{s; s \in \mathcal{T}, s \leq t\}.$$

$$\tau_+(t) = \inf\{s; s \in \mathcal{T}, s \geq t\}$$

Let $K > 1$ be a constant. Define

$$\mathcal{T}_1 = \left\{ t \notin \mathcal{T}; \sup_{\tau_-(t) \leq s \leq \tau_+(t)} (\psi_2^2(s) - \psi^2(s)) \leq K \right\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \leq \mathcal{J}_d(\psi_2) < \infty.$$

and

$$\int_{\mathcal{T}_1} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \leq \mathcal{J}_d(\sqrt{\psi_2^2(t) - K}) < \infty,$$

by Lemma 2.14. Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt = \infty.$$

Let $\mathcal{T}_2 = \bigcup_k [a_k, b_k]$, where $a_1 < b_1 \leq a_2 < b_2 \leq \dots$. Then

$$\begin{aligned} \infty &= \int_{\mathcal{T}_2} \frac{\max[1, \psi^d(t)]}{t} \exp(-\psi^2(t)) dt \\ &\leq \sum_k \int_{a_k}^{b_k} \frac{1}{t} \max \left[1, \left(\frac{\psi(a_k) a_k^{1/\alpha}}{t^{1/\alpha}} \right)^d \right] \exp \left(-\frac{\psi^2(a_k) a_k^{2/\alpha}}{t^{2/\alpha}} \right) dt \\ &\leq c \int_{\mathcal{T}_2} \frac{\max[1, \psi_2^d(t)]}{t} \exp(-\psi_2^2(t)) dt, \end{aligned}$$

for some $c > 0$, a contradiction. Therefore $\mathcal{J}_d(\psi_2) = \infty$ as desired. $\square$

Similar arguments work for the $\mathcal{J}_d$ integral.

Lemma 2.17. *Let $\psi \in \Psi_2^\alpha$ for some $\alpha > 0$. Let $d \geq 1$ be an integer, and let $\beta > 0$. Then there exist functions $\psi_1, \psi_2 \in \Psi_3^\beta \cap \Psi_2^\alpha$ such that $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$ and $\mathcal{J}_d(\psi_1)$, $\mathcal{J}_d(\psi)$, and $\mathcal{J}_d(\psi_2)$ are all finite or infinite.*

Proof. Let t_0 be such that $t^{1/\alpha}/\psi(t)$ is nondecreasing for $t \geq t_0$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq \log^{2/d} t$ for all $t \geq t_0$. Put for $t \geq t_0$,

$$\begin{aligned}\psi_1(t) &= \frac{1}{t^{1/\beta}} \inf_{s \geq t} \psi(s) s^{1/\beta}, \\ \psi_2(t) &= \frac{1}{t^{1/\beta}} \sup_{t_0 \leq s \leq t} \psi(s) s^{1/\beta}.\end{aligned}$$

Obviously $\psi_1, \psi_2 \in \Psi_3^\beta \cap \Psi_2^\alpha$, $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$. Hence $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\psi_2) < \infty$ and $\mathcal{J}_d(\psi) = \infty$ implies $\mathcal{J}_d(\psi_1) = \infty$.

For $t_0 < s_1 < s_2$ and $0 < f_1 < f_2$ put $\tilde{\psi}(t) = f_1 s_1^{-1/\alpha} t^{1/\alpha}$, $s_1 \leq t \leq s_2$, and assume that $f_2 = \tilde{\psi}(s_2)$, and put $\hat{\psi}(t) = f_1 s_2^{1/\alpha} t^{-1/\alpha}$, $s_1 \leq t \leq s_2$, and assume that $\hat{\psi}(s_1) = f_2$. Then

$$\int_{s_1}^{s_2} \frac{1}{t \tilde{\psi}^d(t)} dt = \int_{s_1}^{s_2} \frac{1}{t \hat{\psi}^d(t)} dt = \frac{d}{\alpha} \left(\frac{1}{f_1^d} - \frac{1}{f_2^d} \right).$$

Therefore, if $\psi(t)$ increases as a multiple of a positive power from value f_1 to value f_2 , or decreases as a multiple of a negative power from value f_2 to value f_1 , the integral is the same, and depends on the power only via a multiplicative constant.

First assume that $\mathcal{J}_d(\psi_1) = \infty$ and $\mathcal{J}_d(\psi) < \infty$. Take a constant $K > 1$. Put

$$\begin{aligned}\mathcal{T} &= \{t; t \geq t_0, \psi_1(t) = \psi(t)\}, \\ \tau_-(t) &= \sup\{s; s \in \mathcal{T}, s \leq t\}, \\ \tau_+(t) &= \inf\{s; s \in \mathcal{T}, s \geq t\}, \\ \mathcal{T}_1 &= \{t \notin \mathcal{T}; \psi_1(s) > \psi(s)/K \text{ for all } s \in [\tau_-(t), \tau_+(t)]\}, \\ \mathcal{T}_2 &= \mathcal{T}^c - \mathcal{T}_1.\end{aligned}$$

Now

$$\int_{\mathcal{T}} \frac{1}{t\psi_1^d(t)} dt \leq \mathcal{J}_d(\psi) < \infty.$$

and

$$\int_{\mathcal{T}_1} \frac{1}{t\psi_1^d(t)} dt \leq \mathcal{J}_d(\psi/K) < \infty.$$

Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{1}{t\psi_1^d(t)} dt = \infty.$$

All intervals contained in $\mathcal{T}_2$ are such that $\psi_1(t)$ is decreasing there as a multiple of $t^{-1/\beta}$. However, $\psi_1(t)$ has to go through the same values outside of $\mathcal{T}_2$ and, since it cannot grow faster than a multiple of $t^{1/\alpha}$, we have that

$$\int_{\mathcal{T}_2^c} \frac{1}{t\psi_1^d(t)} dt = \infty.$$

a contradiction. Therefore, $\mathcal{J}_d(\psi) = \infty$.

Now assume $\mathcal{J}_d(\psi) = \infty$ and $\mathcal{J}_d(\psi_2) < \infty$. Take a constant $K > 1$. Put

$$\mathcal{T} = \{t; t \geq t_0, \psi_2(t) = \psi(t)\},$$

$$\tau_-(t) = \sup\{s; s \in \mathcal{T}, s \leq t\},$$

$$\tau_+(t) = \inf\{s; s \in \mathcal{T}, s \geq t\},$$

$$\mathcal{T}_1 = \{t \notin \mathcal{T}; \psi_2(s) < K\psi(s) \text{ for all } s \in [\tau_-(t), \tau_+(t)]\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{1}{t\psi^d(t)} dt \leq \mathcal{J}_d(\psi_2) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{1}{t\psi^d(t)} dt \leq \mathcal{J}_d(\psi_2/K) < \infty.$$

Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{1}{t\psi^d(t)} dt = \infty.$$

Let $\mathcal{T}_2 = \bigcup_k [a_k, b_k]$, where $a_1 < b_1 \leq a_2 < b_2 \leq \dots$. Then

$$\begin{aligned} \infty &= \int_{\mathcal{T}_2} \frac{1}{t\psi^d(t)} dt \\ &\leq \sum_k \int_{a_k}^{b_k} \frac{b_k^{1/\alpha}}{t\psi(b_k)t^{1/\alpha}} dt \\ &\leq c \int_{\mathcal{T}_2} \frac{1}{t\psi_2^d(t)} dt, \end{aligned}$$

for some constant $c > 0$ dependent on α and β only, a contradiction. Therefore $\mathcal{J}_d(\psi_2) = \infty$ as desired. $\square$

Lemma 2.18. *Let $\psi \in \Psi_3^\alpha$ for some $\alpha > 0$. Let $d \geq 1$ be an integer, and let $\beta > 0$. Then there exist functions $\psi_1, \psi_2 \in \Psi_2^\beta \cap \Psi_3^\alpha$ such that $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$ and $\mathcal{J}_d(\psi_1)$, $\mathcal{J}_d(\psi)$, and $\mathcal{J}_d(\psi_2)$ are all finite or infinite.*

Proof. Let t_0 be such that $t^{1/\alpha}\psi(t)$ is nondecreasing for $t \geq t_0$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq \log^{2/d} t$ for all $t \geq t_0$. Put for $t \geq t_0$,

$$\begin{aligned} \psi_1(t) &= t^{1/\beta} \inf_{t_0 \leq s \leq t} \frac{\psi(s)}{s^{1/\beta}}, \\ \psi_2(t) &= t^{1/\beta} \sup_{s \geq t} \frac{\psi(s)}{s^{1/\beta}}. \end{aligned}$$

Obviously $\psi_1, \psi_2 \in \Psi_2^\beta \cap \Psi_3^\alpha$, $\psi_1(t) \leq \psi(t) \leq \psi_2(t)$. Hence $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\psi_2) < \infty$ and $\mathcal{J}_d(\psi) = \infty$ implies $\mathcal{J}_d(\psi_1) = \infty$.

First assume that $\mathcal{J}_d(\psi_1) = \infty$ and $\mathcal{J}_d(\psi) < \infty$. Take a constant $K > 1$. Put

$$\mathcal{T} = \{t: t \geq t_0, \psi_1(t) = \psi(t)\},$$

$$\tau_-(t) = \sup\{s: s \in \mathcal{T}, s \leq t\},$$

$$\tau_+(t) = \inf\{s: s \in \mathcal{T}, s \geq t\},$$

$$\mathcal{T}_1 = \{t \notin \mathcal{T}; \psi_1(s) > \psi(s)/K \text{ for all } s \in [\tau_-(t), \tau_+(t)]\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{1}{t\psi_1^d(t)} dt \leq \mathcal{J}_d(\psi) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{1}{t\psi_1^d(t)} dt \leq \mathcal{J}_d(\psi/K) < \infty.$$

Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{1}{t\psi_1^d(t)} dt = \infty.$$

All intervals contained in $\mathcal{T}_2$ are such that $\psi_1(t)$ is increasing there as a multiple of $t^{1/\beta}$. However, that means that

$$\int_{\mathcal{T}_2} \frac{1}{t\psi_1^d(t)} dt = \infty,$$

a contradiction. Therefore, $\mathcal{J}_d(\psi) = \infty$.

Now assume $\mathcal{J}_d(\psi) = \infty$ and $\mathcal{J}_d(\psi_2) < \infty$. Take a constant $K > 1$. Put

$$\mathcal{T} = \{t; t \geq t_0, \psi_2(t) = \psi(t)\},$$

$$\tau_-(t) = \sup\{s: s \in \mathcal{T}, s \leq t\},$$

$$\tau_+(t) = \inf\{s: s \in \mathcal{T}, s \geq t\},$$

$$\mathcal{T}_1 = \{t \notin \mathcal{T}; \psi_2(s) < K\psi(s) \text{ for all } s \in [\tau_-(t), \tau_+(t)]\},$$

$$\mathcal{T}_2 = \mathcal{T}^c - \mathcal{T}_1.$$

Now

$$\int_{\mathcal{T}} \frac{1}{t\psi^d(t)} dt \leq \mathcal{J}_d(\psi_2) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{1}{t\psi^d(t)} dt \leq \mathcal{J}_d(\psi_2/K) < \infty.$$

Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{1}{t\psi^d(t)} dt = \infty.$$

Let $\mathcal{T}_2 = \bigcup_k [a_k, b_k]$, where $a_1 < b_1 \leq a_2 < b_2 \leq \dots$. Then

$$\begin{aligned} \infty &= \int_{\mathcal{T}_2} \frac{1}{t\psi^d(t)} dt \\ &\leq \sum_k \int_{a_k}^{b_k} \frac{t^{1/\alpha}}{t\psi(a_k)a_k^{1/\alpha}} dt \\ &\leq c \int_{\mathcal{T}_2} \frac{1}{t\psi_2^d(t)} dt, \end{aligned}$$

for some constant $c > 0$ dependent on α and β only, a contradiction. Therefore $\mathcal{J}_d(\psi_2) = \infty$ as desired. $\square$

Lemma 2.19. *Let $d \geq 0$ be an integer, and let $K > 0$. If $\psi \in \Psi_4^\alpha$ for some $\alpha > 0$, then $\tilde{\psi}(t) = \sqrt{\psi^2(t) + K} \in \Psi_4^\alpha$ and $\mathcal{J}_d(\tilde{\psi}) < \infty$ if and only if $\mathcal{J}_d(\psi) < \infty$.*

Proof. First, we verify that $\tilde{\psi}$ satisfies the monotonicity conditions. Since $\psi \in \Psi_4^\alpha$,

$$t^{2/\alpha}\tilde{\psi}^2(t) = t^{2/\alpha}\psi^2(t) + Kt^{2/\alpha}$$

is strictly increasing, and

$$\frac{\tilde{\psi}^2(t)}{t^{2/\alpha}} = \frac{\psi^2(t)}{t^{2/\alpha}} + \frac{K}{t^{2/\alpha}}$$

is strictly decreasing. Therefore $\tilde{\psi} \in \Psi_4^\alpha$. The rest follows from Lemma 2.13. $\square$

Lemma 2.20. *Let $d \geq 0$ be an integer, $K > 0$, and let $\psi \in \Psi_4^\alpha$ for some $\alpha > 0$ be such that $\psi^2(t) > K$ for all t . Then there exists a function $\hat{\psi}(t) \in \Psi_4^\alpha$ such that $\hat{\psi}^2(t) \leq \psi^2(t) - K$ and $\mathcal{J}_d(\hat{\psi}) < \infty$ if and only if $\mathcal{J}_d(\psi) < \infty$.*

Proof. By Lemma 2.16, it is sufficient to find $\hat{\psi} \in \Psi_3^\alpha$. Denote $\tilde{\psi} = \sqrt{\psi^2(t) - K}$. By Lemma 2.14, $\mathcal{J}_d(\psi) < \infty$ if and only if $\mathcal{J}_d(\tilde{\psi}) < \infty$.

Let t_0 be such that $t^{1/\alpha}/\psi(t)$ and $t^{1/\alpha}\psi(t)$ are nondecreasing for $t \geq t_0$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq 2 \log \log t$ for all $t \geq t_0$. Put for $t \geq t_0$,

$$\hat{\psi}(t) = \frac{1}{t^{1/\alpha}} \inf_{s \geq t} \tilde{\psi}(s) s^{1/\alpha}.$$

Obviously, $\hat{\psi}(t) \leq \tilde{\psi}(t)$, and we only need to show that $\mathcal{J}_d(\psi) < \infty$ implies $\mathcal{J}_d(\hat{\psi}) < \infty$.

Assume that $\mathcal{J}_d(\hat{\psi}) = \infty$ and $\mathcal{J}_d(\tilde{\psi}) < \infty$. Put

$$\begin{aligned} \mathcal{T} &= \{t; t \geq t_0, \hat{\psi}(t) = \tilde{\psi}(t)\}, \\ \tau_-(t) &= \sup\{s; s \in \mathcal{T}, s \leq t\}, \\ \tau_+(t) &= \inf\{s; s \in \mathcal{T}, s \geq t\}. \end{aligned}$$

Let $C > 1$ be a constant. Define

$$\begin{aligned} \mathcal{T}_1 &= \left\{ t \notin \mathcal{T}; \sup_{\tau_-(t) \leq s \leq \tau_+(t)} (\tilde{\psi}^2(s) - \hat{\psi}^2(s)) \leq C \right\}, \\ \mathcal{T}_2 &= \mathcal{T}^c - \mathcal{T}_1. \end{aligned}$$

Now

$$\int_{\mathcal{T}} \frac{\max[1, \widehat{\psi}^d(t)]}{t} \exp(-\widehat{\psi}^2(t)) dt \leq \mathcal{J}_d(\widetilde{\psi}) < \infty,$$

and

$$\int_{\mathcal{T}_1} \frac{\max[1, \widehat{\psi}^d(t)]}{t} \exp(-\widehat{\psi}^2(t)) dt \leq \mathcal{J}_d(\sqrt{\widetilde{\psi}^2(t) - C}) < \infty,$$

by Lemma 2.14. Therefore, we must have

$$\int_{\mathcal{T}_2} \frac{\max[1, \widehat{\psi}^d(t)]}{t} \exp(-\widehat{\psi}^2(t)) dt = \infty.$$

That implies that

$$\int_{\mathcal{T}_2^c} \frac{\max[1, \widetilde{\psi}^d(t)]}{t} \exp(-\widetilde{\psi}^2(t)) dt = \infty,$$

a contradiction. Therefore, $\mathcal{J}_d(\psi) = \infty$. $\square$

Lemma 2.21. *Let $\psi \in \Psi_2^\alpha$ (Ψ_3^α , respectively) for some $\alpha > 0$, let $K > 0$, and $d \geq 1$ be an integer. Assume that $\psi(t) > 2K$ for all t . Then there exist functions $\psi_1, \psi_2 \in \Psi_2^\alpha$ (Ψ_3^α , respectively) such that $\psi_1(t) \leq \psi(t) - K$, $\psi_2(t) \geq \psi(t) + K$, and $\mathcal{J}_d(\psi_1)$, $\mathcal{J}_d(\psi)$, and $\mathcal{J}_d(\psi_2)$ are all finite or all infinite.*

Proof. Simply put $\psi_1(t) = \frac{1}{2}\psi(t)$ and $\psi_2(t) = \psi(t) + K$. The assertions of the lemma are obvious. $\square$

Remark 2.3. From Lemmas 2.19, 2.20, and 2.21, we see that for $\psi \in \Psi_2^\alpha$ or Ψ_3^α and $\mathcal{J}_d$ or $\mathcal{J}_d$ we can make the function ψ a bit bigger or a bit smaller and still preserve the convergence or divergence of the integral and, in addition, to get strict monotonicities in the definitions of Ψ_2^α and Ψ_3^α . Therefore, we will sometimes assume that the monotonicities are strict without explicitly saying so.

The following lemma is in part similar to Lemma B of Khoshnevisan, Lewis, and Li (1994).

Lemma 2.22. *Let $f(t)$ be a nondecreasing function. Let $x(t)$ be a continuous function defined on $[0, \infty)$ such that $x(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then*

$$x(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty \iff \sup_{0 \leq s \leq t} x(s) \geq f(t) \text{ i.o. as } t \rightarrow \infty, \quad (2.22)$$

$$x(t) \leq f(t) \text{ i.o. as } t \rightarrow \infty \iff \inf_{s \geq t} x(s) \leq f(t) \text{ i.o. as } t \rightarrow \infty. \quad (2.23)$$

Proof. Since

$$\inf_{s \geq t} x(s) \leq x(t) \leq \sup_{0 \leq s \leq t} x(s),$$

the implications ' $\Rightarrow$ ' are obvious.

Let r_t , $0 \leq r_t \leq t$, be a location of supremum, i.e., $x(r_t) = \sup_{0 \leq s \leq t} x(s)$. If there is a sequence $t_n \nearrow \infty$ such that $\sup_{0 \leq s \leq t_n} x(s) \geq f(t_n)$, then also $x(r_{t_n}) \geq f(t_n) \geq f(r_{t_n})$, since f is nondecreasing. Since $x(t) \rightarrow \infty$ as $t \rightarrow \infty$, $r_{t_n} \nearrow \infty$.

Similarly, if now r_t , $r_t \geq t$, denotes a location of future infimum, i.e., $x(r_t) = \inf_{s \geq t} x(s)$, and $t_n \nearrow \infty$ is a sequence such that $\inf_{s \geq t_n} x(s) \leq f(t_n)$, we have $x(r_{t_n}) \leq f(t_n) \leq f(r_{t_n})$. $\square$

We will often use the following generalization of the Borel-Cantelli lemma.

Lemma 2.23. *Let B_k , $k = 1, \dots$ be arbitrary events. If $\sum_k P(B_k) < \infty$, then $P(B_k \text{ i.o.}) = 0$. If $\sum_k P(B_k) = \infty$ and*

$$\liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{l=1}^n P(B_k \cap B_l)}{\left(\sum_{k=1}^n P(B_k) \right)^2} \leq C \quad (C \geq 1),$$

then $P(B_k \text{ i.o.}) \geq 1/C$.

Chapter 3

Big Values of Brownian Motion

3.1 How Big Are the Big Values?

As promised in Section 1.2, we will prove the integral test of Theorem 1.A under weaker monotonicity assumptions.

Theorem 3.1. *Let $\psi \in \Psi_3^2$ and let $r \geq 1$ be an integer. Then*

$$P \left\{ M_r(t) \geq \sqrt{2} t^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_r(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (3.1)$$

or, equivalently,

$$\psi(t) \in \begin{cases} \text{UUC}((2t)^{-1/2} M_r(t)) \\ \text{ULC}((2t)^{-1/2} M_r(t)) \end{cases} \quad \text{according as } \mathcal{J}_r(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 3.1.

$$\limsup_{t \rightarrow \infty} \frac{M_r(t)}{t^{1/2} \psi(t)} = (2c^*(\psi))^{1/2} \quad a.s.$$

By Lemmas 2.15 and 2.16 we immediately have

Corollary 3.2. *The integral test (3.1) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 3.1. $M_1(t)$ in the above theorem and corollaries can be replaced by $W_1(t)$, $-W_1(t)$, $|W_1(t)|$, $M^-(t)$, or $M^+(t)$ (see Lemma 3.3); $M_r(t)$, $r \geq 2$, can be replaced by $\|W_r(t)\|$ (Lemma 2.22). By Lemma 2.3, it is sufficient to use $J_0(\psi)$ to determine the limit supremum.

Theorem 3.1 obviously implies Theorem 1.A and, via Lemma 2.15, Theorem 1.B.

Example 3.1. This example demonstrates that our Theorem 3.1 is indeed more general than Theorem 1.A. We will construct a function ψ that cannot be put between $((1 - \varepsilon) \log \log t)^{1/2}$ and $((1 + \varepsilon) \log \log t)^{1/2}$.

Let f be a positive, nondecreasing function from $[1, \infty)$ to $(0, \infty)$, such that $f(t) \rightarrow \infty$ as $t \rightarrow \infty$, $t^{1/2}/f(t)$ is nondecreasing, and $f(t) < (2 \log \log t)^{1/2}$ for all t . Let $S_1 < S'_1 < T_1 < T'_1 < S_2 < S'_2 < T_2 < T'_2 < \dots$ and ψ be defined in the following way,

$$\psi(t) = \begin{cases} (2 \log \log t)^{1/2}, & S_n < t \leq S'_n, \\ \psi(S'_n) S_n'^{1/2} t^{-1/2}, & S'_n < t \leq T_n, \\ f(t), & T_n < t \leq T'_n, \\ \psi(T'_n) T_n'^{-1/2} t^{1/2}, & T'_n < t \leq S_{n+1}, \end{cases}$$

where

$$\begin{aligned} T_n &= \inf\{t > S'_n; f(t) = \psi(S'_n) S_n'^{1/2} t^{-1/2}\}, \\ S_{n+1} &= \inf\{t > T'_n; \psi(T'_n) T_n'^{-1/2} t^{1/2} = (2 \log \log t)^{1/2}\}, \\ \psi(S_n) &\leq \psi(T_n). \end{aligned}$$

Then $J_r(\psi) < \infty$ if and only if

$$\sum_n \int_{T_n}^{T'_n} \frac{\psi^r(t)}{t} \exp(-\psi^2(t)) = \sum_n \int_{T_n}^{T'_n} \frac{f^r(t)}{t} \exp(-f^2(t)) < \infty.$$

Hence, f can go to infinity arbitrarily slowly, and convergence or divergence of the integral is decided on the intervals where ψ is equal to f .

We will prove several lemmas needed in the proof of Theorems 3.1. However, instead of Theorem 3.1, we will prove the equivalent Theorem 4.1 in Chapter 4.

The following lemma is similar to Lemma 3.1 of Jain and Taylor (1973).

Lemma 3.1. *To prove Theorem 3.1, it is sufficient to consider ψ satisfying*

$$2 \leq \psi(t) \leq (2 \log \log t)^{1/2}.$$

Proof. Put $J(\psi) = J_r(\psi)$, $\psi_1(t) = 2$, $\psi_2(t) = (2 \log \log t)^{1/2}$, and

$$\widehat{\psi} = \min[\max(\psi, \psi_1), \psi_2].$$

We will show that, if (3.1) is true for $\widehat{\psi}$, then it is also true for ψ . If $J(\psi) < \infty$, then by Lemma 2.4 we have $J(\widehat{\psi}) < \infty$ and $\widehat{\psi} \leq \psi$ near infinity. Since $\widehat{\psi} \in \text{UUC}((2t)^{-1/2}M_r(t))$, we also have $\psi \in \text{UUC}((2t)^{-1/2}M_r(t))$. If $J(\psi) = \infty$, then by Lemma 2.4 we have also $J(\widehat{\psi}) = \infty$, and, therefore, $\widehat{\psi} \in \text{ULC}((2t)^{-1/2}M_r(t))$. This means that for almost all ω there exists a sequence $t_n \nearrow \infty$ such that $M_r(t_n) \geq (2t_n)^{1/2}\widehat{\psi}(t_n)$. Since $J(\psi_2) < \infty$, we have $\psi_2 \in \text{UUC}((2t)^{-1/2}M_r(t))$, and, therefore, we must have $\widehat{\psi}(t_n) < \psi_2(t_n)$ for all large enough n . This implies that $\psi(t_n) \leq \widehat{\psi}(t_n)$. Hence $\psi \in \text{ULC}((2t)^{-1/2}M_r(t))$. $\square$

Lemma 3.2. *Let f be a positive, nondecreasing function. Then*

$$P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = 0 \text{ or } 1.$$

Proof. Denote

$$\mathcal{A}_t = \sigma\{W(s); s \geq t\} \quad \text{and} \quad \mathcal{A} = \bigcap_{t \geq 0} \mathcal{A}_t. \quad (3.2)$$

By Blumenthal's zero-one law (see also Revuz and Yor (1994), Exercise (2.28) on p. 94, and Exercise (2.32) on p. 96), $P(A) = 0$ or 1 for all $A \in \mathcal{A}$. Therefore for

$$\begin{aligned} A &= \{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} \\ &= \lim_{T \rightarrow \infty} \{W(t) \geq f(t) \text{ for some } t \geq T\} \\ &= \bigcap_{T \geq 0} \bigcup_{t \geq T} \{W(t) \geq f(t)\} \in \mathcal{A}, \end{aligned}$$

we have $P(A) = 0$ or 1 . □

Lemma 2.22 tells us that we may replace $M_r(t)$ by $\|W_r(t)\|$ in Theorem 3.1. In the following lemma we will prove that for $r = 1$ the upper classes are the same for $W(t)$, $-W(t)$, $|W(t)|$, $M^+(t)$, $M^-(t)$, and $M_1(t)$.

Lemma 3.3. *Let f be a nondecreasing positive function. Then*

$$\begin{aligned} P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} &= P\{-W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} \\ &= P\{|W(t)| \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = P\{M^+(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} \quad (3.3) \\ &= P\{M^-(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = P\{M_1(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\}. \end{aligned}$$

Proof. First notice that

$$W(t) \leq |W(t)| \leq M_1(t) \quad (3.4)$$

and

$$W(t) \leq M^+(t) \leq M_1(t), \quad (3.5)$$

and that

$$\{-W(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{W(t); t \geq 0\}$$

and

$$\{M^+(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{M^-(t); t \geq 0\}.$$

By Lemma 3.2, either

$$P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = 0,$$

or

$$P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = 1.$$

Let

$$P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = 0.$$

This means that $W(t) < f(t)$ a.s. and $-W(t) < f(t)$ a.s. for $t \geq t_0 = t_0(\omega)$. Therefore $|W(t)| = \max(W(t), -W(t)) < f(t)$ a.s. for $t \geq t_0$. Since both $M^+(t)$ and $f(t)$ are nondecreasing, also $M^+(t) = \sup_{0 \leq s \leq t} W(s) < f(t)$ a.s. for $t \geq t_0$, which again implies $M^-(t) < f(t)$ a.s. for $t \geq t_0$. Also $M_1(t) = \sup_{0 \leq s \leq t} |W(s)| = \max(M^+(t), M^-(t)) < f(t)$ a.s. for $t \geq t_0$. Hence, in this case, all six probabilities in (3.3) are equal to 0.

Now let

$$P\{W(t) \geq f(t) \text{ i.o. as } t \rightarrow \infty\} = 1.$$

This means that there is a sequence $t_n \nearrow \infty$, $t_n = t_n(\omega)$, such that $f(t_n) \leq W(t_n)$. By (3.4) and (3.5) we immediately get $f(t_n) \leq |W(t_n)| \leq M_1(t_n)$ and $f(t_n) \leq M^+(t_n)$. Therefore, in this case, all six probabilities in (3.3) are equal to 1. $\square$

3.2 How Small Are the Big Values?

The following theorem characterizes the lower classes of $\{M_r(t); t \geq 0\}$. The lower classes of $\{M^+(t); t \geq 0\}$ are described in the next section. We enlarge the classes of weight functions (cf. Theorems 1.C and 1.D).

Theorem 3.2. *Let $\psi \in \Psi_2^2$ and let $r \geq 1$ be an integer. Then*

$$\mathbb{P} \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_2(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (3.6)$$

or, equivalently,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t)) \\ \text{LUC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t)) \end{cases} \quad \text{according as } \mathcal{J}_2(\psi) \begin{cases} < \infty \\ = \infty, \end{cases}$$

where j_ν denotes the smallest positive zero of the Bessel function $J_\nu(x)$ of index $\nu = (r-2)/2$, eg., for $r = 1$, $\nu = -1/2$ and $j_{-1/2} = \pi/2$.

Corollary 3.3.

$$\liminf_{t \rightarrow \infty} \frac{M_r(t)}{t^{1/2}} \psi(t) = \left(\frac{j_\nu^2}{2c^*(\psi)} \right)^{1/2} \quad a.s.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 3.4. *The integral test (3.6) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Example 3.2. Let ψ be constructed as in Example 3.1. Then $\mathcal{J}_2(\psi) < \infty$ if and only if

$$\sum_n \int_{T_n}^{T'_n} \frac{\psi^2(t)}{t} \exp(-\psi^2(t)) = \sum_n \int_{T_n}^{T'_n} \frac{f^2(t)}{t} \exp(-f^2(t)) < \infty.$$

Lemma 3.4. *To prove Theorem 3.2, it is sufficient to consider ψ satisfying*

$$2 \leq \psi(t) \leq (2 \log \log t)^{1/2}.$$

Proof. Put $J(\psi) = J_2(\psi)$, $\psi_1(t) = 2$, $\psi_2(t) = (2 \log \log t)^{1/2}$, and

$$\widehat{\psi} = \min[\max(\psi, \psi_1), \psi_2].$$

We will show that, if (3.6) holds true for $\widehat{\psi}$, then it is also true for ψ . If $J(\psi) < \infty$, then by Lemma 2.4 we have $J(\widehat{\psi}) < \infty$ and $\widehat{\psi} \leq \psi$ near infinity. Since $\widehat{\psi}^{-1} \in \text{LLC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t))$, we also have $\psi^{-1} \in \text{LLC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t))$. If $J(\psi) = \infty$, then by Lemma 2.4 we have also $J(\widehat{\psi}) = \infty$, and, therefore, $\widehat{\psi}^{-1} \in \text{LUC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t))$. This means that for almost all ω there exists a sequence $t_n \nearrow \infty$ such that $M_r(t_n) \leq (j_\nu/\sqrt{2})t_n^{1/2}/\widehat{\psi}(t_n)$. Since $J(\psi_2) < \infty$, we have $\psi_2^{-1} \in \text{LLC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t))$, and, therefore, we must have $\widehat{\psi}(t_n) < \psi_2(t_n)$ for all large enough n . This implies that $\psi(t_n) \leq \widehat{\psi}(t_n)$. Hence $\psi^{-1} \in \text{LUC}((\sqrt{2}/j_\nu)t^{-1/2}M_r(t))$. $\square$

Lemma 3.5. *Let f be a positive, nondecreasing function, such that $f(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then*

$$P\{M_r(t) \leq f(t) \text{ i.o. as } t \rightarrow \infty\} = 0 \text{ or } 1.$$

Proof. Put

$$\mathcal{A}_t = \sigma\{W_r(s); s \geq t\},$$

$$\mathcal{A} = \bigcap_{t \geq 0} \mathcal{A}_t.$$

Then

$$\begin{aligned} & \{M_r(t) \leq f(t) \text{ for an unbounded set of } t \geq T\} \\ &= \{M_r(t) - M_r(T) \leq f(t) - M_r(T) \text{ for an unbounded set of } t \geq T\} \in \mathcal{A}_T. \end{aligned}$$

Therefore

$$A = \{M_r(t) \leq f(t) \text{ i.o. as } t \rightarrow \infty\} \in \mathcal{A}$$

and $P(A) = 0$ or 1 . □

Proof of Theorem 3.2. By Lemmas 2.15 and 2.16, it is sufficient to consider $\psi \in \Psi_4^2$.

Let $0 < j_{\nu,1} < j_{\nu,2} < \dots$ be the positive zeros of the Bessel function J_ν , where $\nu = (r-2)/2$, and we will write j_ν instead of $j_{\nu,1}$. The distribution function G_r of $M_r(1)$ is given by (see Gruet and Shi (1995b) and Ciesielski and Taylor (1962))

$$G_r(x) = P\left\{\frac{M_r(t)}{t^{1/2}} < x\right\} = \frac{2^{1-\nu}}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{1-\nu} J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2x^2}\right). \quad (3.7)$$

The positive zeros of $J_\nu(x)$ are interlaced with those of $J_{\nu+1}(x)$ (see Watson (1966), p. 479), ie., $0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \dots$. Since the first term in the sum in (3.7) is dominant, we have

$$\alpha_\nu \exp\left(-\frac{j_\nu^2}{2} \cdot \frac{1}{x^2}\right) \leq G_r(x) \leq \alpha'_\nu \exp\left(-\frac{j_\nu^2}{2} \cdot \frac{1}{x^2}\right) \quad \text{for all } x > 0, \quad (3.8)$$

where

$$\alpha_\nu = \frac{2^{1-\nu}}{\Gamma(\nu+1)} \left(\frac{1}{j_{\nu,1}^{1-\nu} J_{\nu+1}(j_{\nu,1})} + \frac{1}{j_{\nu,2}^{1-\nu} J_{\nu+1}(j_{\nu,2})} \right),$$

and

$$\alpha'_\nu = \frac{2^{1-\nu}}{\Gamma(\nu+1) j_{\nu,1}^{1-\nu} J_{\nu+1}(j_{\nu,1})}.$$

For $\nu = -1/2$ we have $\alpha_{-1/2} = 8/(3\pi)$ and $\alpha'_{-1/2} = 4/\pi$.

Assume that $J_2(\psi) < \infty$. By Lemma 2.1, $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10. Denote

$$A_k = \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Since both $M_r(t)$ and $t^{1/2}/\psi(t)$ are nondecreasing, we obtain

$$\begin{aligned} P(A_k) &\leq P \left\{ M_r(t_k) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t_k^{1/2}}{\psi_{k+1}} \right\} \\ &= P \left\{ \frac{M_r(t_k)}{t_k^{1/2}} \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{1}{\psi_{k+1}} \cdot \left(\frac{t_{k+1}}{t_k} \right)^{1/2} \right\} \\ &\approx \exp \left(-\psi_{k+1}^2 \frac{t_k}{t_{k+1}} \right) \\ &= \exp \left(-\frac{\psi_{k+1}^2 (\psi_k^2 - 1)}{\psi_k^2} \right) \\ &\leq \exp \left(-\frac{(\psi_k^2 - 1)^2}{\psi_k^2} \right) \\ &\approx \exp(-\psi_k^2), \end{aligned} \tag{3.9}$$

which sums by Lemma 2.10. Therefore, by the Borel–Cantelli lemma, $P(A_k \text{ i.o.}) = 0$, which proves the convergent part of the theorem.

Now assume that $J_2(\psi) = \infty$. Since

$$\begin{aligned} P \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ for some } t \geq T \right\} &\geq P \left\{ M_r(T) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{T^{1/2}}{\psi(T)} \right\} \\ &= G_r \left(\frac{j_\nu}{\sqrt{2}} \cdot \frac{1}{\psi(T)} \right) \\ &\geq \alpha_\nu \exp(-\psi^2(T)), \end{aligned}$$

we have that $\psi(T) \rightarrow \infty$ as $T \rightarrow \infty$ is necessary for

$$P \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = 0.$$

Therefore, if $\psi(T) \nrightarrow \infty$ as $T \rightarrow \infty$, we have

$$P \left\{ M_r(t) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = 1.$$

In what follows, assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as above. By Lemma 3.4, we may assume, without loss of generality, that $2 \leq \psi(t) \leq (2 \log \log t)^{1/2}$. Define $\mathcal{K} = \{k; \psi_k^2 > \psi_{k-1}^2 + \frac{1}{2}\}$ and its complement $\mathcal{K}^c = \mathbb{N} - \mathcal{K}$. Now denote

$$A_k = \left\{ M_r(t_k) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t_k^{1/2}}{\psi_k} \right\},$$

and notice that

$$\mathbb{P}(A_k) \approx \exp(-\psi_k^2). \quad (3.10)$$

We will use the Borel–Cantelli lemma to show that A_k , $k \in \mathcal{K}^c$, occur infinitely often with probability one. According to Lemma 2.10, $J_2(\psi) = \infty$ implies that $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$. Moreover, by Lemma 2.11, $\sum_{k \in \mathcal{K}^c} \mathbb{P}(A_k) = \infty$.

Let $k < l$. By Gruet and Shi (1995b), p. 244,

$$\mathbb{P}(A_k \cap A_l) \leq \mathbb{P}(A_k) \mathbb{P} \left\{ M_r(t_l - t_k) \leq \frac{j_\nu}{\sqrt{2}} \cdot \frac{t_l^{1/2}}{\psi_l} \right\},$$

and now, by (3.8), we have

$$\mathbb{P}(A_k \cap A_l) \leq \alpha_\nu'^2 \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right). \quad (3.11)$$

For a fixed k ,

$$\begin{aligned} \frac{\mathbb{P}(A_k \cap A_l)}{\mathbb{P}(A_k) \mathbb{P}(A_l)} &\leq \frac{\alpha_\nu'^2 \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right)}{\alpha_\nu^2 \exp(-\psi_k^2) \exp(-\psi_l^2)} \\ &= \frac{\alpha_\nu'^2}{\alpha_\nu^2} \exp \left(\psi_l^2 \frac{t_k}{t_l} \right) \\ &\searrow \frac{\alpha_\nu'^2}{\alpha_\nu^2}, \end{aligned}$$

as $l \rightarrow \infty$, since $\psi_l^2/t_l \searrow 0$ as $l \rightarrow \infty$.

Define $l_0 = l_0(k)$ by

$$l_0 = \sup \left\{ l; l > k, \psi_l^2 \frac{t_k}{t_l} > 1 \right\}.$$

Then for $l > l_0$,

$$\frac{P(A_k \cap A_l)}{P(A_k) P(A_l)} \leq \frac{\alpha_\nu'^2}{\alpha_\nu^2} e.$$

We want to show that there is a constant C , independent of k , such that

$$\sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \frac{P(A_k \cap A_l)}{P(A_k)} \leq C. \quad (3.12)$$

To this end, we will show that for $l > k$, $l \in \mathcal{K}^c$,

$$\begin{aligned} \frac{P(A_k \cap A_l)}{P(A_k)} &\leq \frac{\alpha_\nu'^2}{\alpha_\nu^2} \exp \left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right) \\ &\leq K \int_{z_l}^{z_{l-1}} \frac{1}{z} e^z \xi^2(z) \exp(-\xi^2(z)) dz, \end{aligned} \quad (3.13)$$

where K is a universal constant and $z = t_k \psi^2(t)/t$, $z_l = t_k \psi_l^2/t_l$, and $\xi(z) = \psi(t)$.

Notice that $t \psi^2(t) = t_k \xi^4(z)/z$ is nondecreasing as t increases. ie., as z decreases.

To prove (3.13), we write

$$\begin{aligned}
& \int_{z_l}^{z_{l-1}} \frac{1}{z} e^z \xi^2(z) \exp(-\xi^2(z)) dz \\
& \geq (z_{l-1} - z_l) \frac{\exp(z_l)}{z_{l-1}^{1/2}} \cdot \frac{\xi^2(z_{l-1})}{z_{l-1}^{1/2}} \exp\left(-\frac{\xi^2(z_l)}{z_{l-1}^{1/2}} z_{l-1}\right) \\
& = \left(\frac{\psi_{l-1}^2 t_k}{t_{l-1}} - \frac{\psi_l^2 t_k}{t_l}\right) \exp\left(\psi_l^2 \frac{t_k}{t_l}\right) \psi_{l-1}^2 \frac{t_{l-1}}{\psi_{l-1}^2 t_k} \exp\left(-\frac{\psi_l^2 t_l^{1/2}}{\psi_l t_k^{1/2}} \cdot \frac{\psi_{l-1} t_k^{1/2}}{t_{l-1}^{1/2}}\right) \\
& = \exp\left(\psi_l^2 \frac{t_k}{t_l}\right) \left(\psi_{l-1}^2 - \frac{\psi_l^2(\psi_{l-1}^2 - 1)}{\psi_{l-1}^2}\right) \exp\left(-\frac{\psi_l \psi_{l-1}^2}{\sqrt{\psi_{l-1}^2 - 1}}\right) \\
& = \exp\left(\psi_l^2 \frac{t_k}{t_l}\right) \exp(-\psi_l^2) \frac{\psi_{l-1}^4 - \psi_l^2(\psi_{l-1}^2 - 1)}{\psi_{l-1}^2} \exp\left(-\frac{\psi_l \psi_{l-1}^2}{\sqrt{\psi_{l-1}^2 - 1}} + \psi_l^2\right) \\
& \geq \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l}\right) \frac{\psi_{l-1}^4 - (\psi_{l-1}^2 + \frac{1}{2})(\psi_{l-1}^2 - 1)}{\psi_{l-1}^2} \\
& \quad \times \exp\left(-\frac{\sqrt{\psi_{l-1}^2 + \frac{1}{2}}}{\sqrt{\psi_{l-1}^2 - 1}} \psi_{l-1}^2 + \psi_{l-1}^2 - 1\right) \\
& = \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l}\right) \left(\frac{1}{2} + \frac{1}{2\psi_{l-1}^2}\right) \exp\left(-\sqrt{\frac{\psi_{l-1}^2 + \frac{1}{2}}{\psi_{l-1}^2 - 1}} \psi_{l-1}^2 + \psi_{l-1}^2 - 1\right) \\
& \geq c \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l}\right)
\end{aligned}$$

for some $0 < c < 1/(2e^2)$. Setting $K = \alpha_\nu^2/(c\alpha_\nu)$ proves (3.13).

Observe that $\xi^2(z_k) = \psi_k^2$ and $z_{l_0} > 1$. From (3.13) we have

$$\begin{aligned} \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \frac{P(A_k \cap A_l)}{P(A_k)} &\leq K \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \int_{z_l}^{z_{l-1}} \frac{1}{z} e^z \xi^2(z) \exp(-\xi^2(z)) dz \\ &\leq K \int_1^{\psi_k^2} \frac{1}{z} e^z \xi^2(z) \exp(-\xi^2(z)) dz \\ &\leq K \int_1^{\psi_k^2} \frac{1}{z} e^z \psi_k \sqrt{z} \exp(-\psi_k \sqrt{z}) dz \\ &\leq 3K = C, \end{aligned}$$

where we replaced $\xi^2(z)$ by $\psi_k \sqrt{z}$, the smallest possible function satisfying $\xi^2(\psi_k^2) = \psi_k^2$ and $\xi^2(z)/\sqrt{z}$ is nonincreasing, and, since xe^{-x} is decreasing for $x > 1$, made the integral as big as possible.

Now we have

$$\begin{aligned} \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=1 \\ l \in \mathcal{K}^c}}^N P(A_k \cap A_l) &= \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N P(A_k) + 2 \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^N P(A_k \cap A_l) \\ &= \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N P(A_k) + 2 \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} P(A_k \cap A_l) + 2 \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=l_0+1 \\ l \in \mathcal{K}^c}}^N P(A_k \cap A_l) \\ &\leq \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N P(A_k) + 2C \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N P(A_k) + 2e \frac{\alpha_\nu'^2}{\alpha_\nu^2} \sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=l_0+1 \\ l \in \mathcal{K}^c}}^N P(A_k) P(A_l), \end{aligned}$$

which implies

$$\liminf_{N \rightarrow \infty} \frac{\sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N \sum_{\substack{l=1 \\ l \in \mathcal{K}^c}}^N P(A_k \cap A_l)}{\left[\sum_{\substack{k=1 \\ k \in \mathcal{K}^c}}^N P(A_k) \right]^2} \leq \frac{\alpha_\nu'^2}{\alpha_\nu^2} e,$$

which, by Lemma 2.23, implies

$$P(A_k \text{ i.o.}) \geq \frac{\alpha_\nu^2}{e \alpha_\nu'^2} > 0.$$

By Lemma 3.5, $P(A_k \text{ i.o.}) = 1$ and the divergent part of the theorem is proved. $\square$

3.3 Lower Classes of $M^+(t)$

Theorem 3.3. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ M^+(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (3.14)$$

or, equivalently,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2}M^+(t)) \\ \text{LUC}(t^{-1/2}M^+(t)) \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 3.5.

$$\liminf_{t \rightarrow \infty} \frac{M^+(t)}{t^{1/2}} \psi(t) = \begin{cases} \infty \\ 0 \end{cases} \text{ a.s. according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 3.6. *The integral test (3.14) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Example 3.3. Let f be a positive, nondecreasing function from $[1, \infty)$ to $(0, \infty)$, such that $f(t) \rightarrow \infty$ as $t \rightarrow \infty$, $t^{1/2}/f(t)$ is nondecreasing, and $f(t) < \log^2 t$ for all t . Let $S_1 < S'_1 < T_1 < T'_1 < S_2 < S'_2 < T_2 < T'_2 < \dots$ and ψ be defined in the

following way,

$$\psi(t) = \begin{cases} \log^2 t, & S_n < t \leq S'_n, \\ \psi(S'_n)S_n'^{1/2}t^{-1/2}, & S'_n < t \leq T_n, \\ f(t), & T_n < t \leq T'_n, \\ \psi(T'_n)T_n'^{-1/2}t^{1/2}, & T'_n < t \leq S_{n+1}, \end{cases}$$

where

$$\begin{aligned} T_n &= \inf\{t > S'_n; f(t) = \psi(S'_n)S_n'^{1/2}t^{-1/2}\}, \\ S_{n+1} &= \inf\{t > T'_n; \psi(T'_n)T_n'^{-1/2}t^{1/2} = \log^2 t\} \\ \psi(S_n) &\leq \psi(T_n). \end{aligned}$$

Then $\mathcal{J}_1(\psi) < \infty$ if and only if

$$\sum_n \int_{T_n}^{T'_n} \frac{1}{t\psi(t)} = \sum_n \int_{T_n}^{T'_n} \frac{1}{tf(t)} < \infty.$$

To prove Theorem 3.3, we follow Csáki (1978). However, a much easier proof will be presented in Chapter 4, where we prove the equivalent Theorem 4.3.

Lemma 3.6. *To prove Theorem 3.3, it is sufficient to consider ψ satisfying*

$$2 \leq \psi(t) \leq (\log t)^2.$$

Proof. Put $\mathcal{J}(\psi) = \mathcal{J}_1(\psi)$, $\psi_1(t) = 2$, $\psi_2(t) = (\log t)^2$, and $\widehat{\psi} = \min[\max(\psi, \psi_1), \psi_2]$. We will show that, if (3.14) is true for $\widehat{\psi}$, then it is also true for ψ . If $\mathcal{J}(\psi) < \infty$, then by Lemma 2.4 we have $\mathcal{J}(\widehat{\psi}) < \infty$ and $\widehat{\psi} \leq \psi$ near infinity. Since $\widehat{\psi}^{-1} \in \text{LLC}(t^{-1/2}M^+(t))$, we also have $\psi^{-1} \in \text{LLC}(t^{-1/2}M^+(t))$. If $\mathcal{J}(\psi) = \infty$, then by Lemma 2.4 we have also $\mathcal{J}(\widehat{\psi}) = \infty$, and, therefore, $\widehat{\psi}^{-1} \in \text{LUC}(t^{-1/2}M^+(t))$. This means that for almost all ω there exists a sequence $t_n \nearrow \infty$ such that $M^+(t_n) \leq$

$t_n^{1/2}/\widehat{\psi}(t_n)$. Since $\mathcal{J}(\psi_2) < \infty$, we have $\psi_2^{-1} \in \text{LLC}(t^{-1/2}M^+(t))$, and, therefore, we must have $\widehat{\psi}(t_n) < \psi_2(t_n)$ for all large enough n . This implies that $\psi(t_n) \leq \widehat{\psi}(t_n)$. Hence $\psi^{-1} \in \text{LUC}(t^{-1/2}M^+(t))$. $\square$

As you can see, Lemmas 3.1, 3.4, and 3.6 are very similar. Therefore, we will not write down similar lemmas for other processes for which we prove integral tests.

For the proof of the divergent part of Theorem 3.3 we need the joint distribution of $(M^+(t_1), M^+(t_2))$ (see Lemma 2.2 of Csáki (1978)).

Lemma 3.7. *Let $t_1 < t_2$ and $0 < a_1 t_1^{1/2} < a_2 t_2^{1/2}$. Then*

$$\begin{aligned} & \mathbb{P} \left\{ M^+(t_1) < a_1 t_1^{1/2}, M^+(t_2) < a_2 t_2^{1/2} \right\} \\ &= \iint_T \frac{1}{(t_1(t_2 - t_1))^{1/2}} \varphi\left(\frac{x}{t_1^{1/2}}\right) \varphi\left(\frac{y}{(t_2 - t_1)^{1/2}}\right) dx dy, \end{aligned} \quad (3.15)$$

where $\varphi(x)$ denotes the standard normal density, $T = T_1 \cup T_2 \cup T_3$ and the domains T_i are given by

$$\begin{aligned} T_1 &= \{-a_1 t_1^{1/2} < x < a_1 t_1^{1/2}, x - a_2 t_2^{1/2} < y < a_2 t_2^{1/2} - x\}, \\ T_2 &= \{x < -a_1 t_1^{1/2}, a_2 t_2^{1/2} - 2a_1 t_1^{1/2} < y + x < a_2 t_2^{1/2}\}, \\ T_3 &= \{x < -a_1 t_1^{1/2}, -a_2 t_2^{1/2} < y - x < -a_2 t_2^{1/2} + 2a_1 t_1^{1/2}\}. \end{aligned} \quad (3.16)$$

Proof of Theorem 3.3. First we prove the convergent part of (3.14). Put $t_k = 2^k$. Since both $M^+(t)$ and $t^{1/2}/\psi(t)$ are nondecreasing, we have for $t \in [t_k, t_{k+1})$,

$$M^+(t) \geq M^+(t_k) \quad \text{and} \quad t^{1/2}/\psi(t_{k+1}) \geq t^{1/2}/\psi(t).$$

Therefore, it is sufficient to show that

$$\mathbb{P} \left\{ M^+(t_k) \leq \frac{t_{k+1}^{1/2}}{\psi(t_{k+1})} \text{ i.o.} \right\} = 0. \quad (3.17)$$

The distribution of $M^+(t)$ is given by

$$\mathbb{P} \left\{ \frac{M^+(t)}{t^{1/2}} \leq x \right\} = 2\Phi(x) - 1, \quad x \geq 0, \quad t > 0,$$

where Φ is the standard normal distribution function. Using the inequality

$$2\Phi(x) - 1 \leq \sqrt{\frac{2}{\pi}}x, \quad x \geq 0,$$

we get

$$\begin{aligned} \mathbb{P} \left\{ M^+(t_k) \leq \frac{t_{k+1}^{1/2}}{\psi(t_{k+1})} \right\} &= \mathbb{P} \left\{ \frac{M^+(t_k)}{t_k^{1/2}} \leq \frac{\sqrt{2}}{\psi(t_{k+1})} \right\} \\ &= 2\Phi \left(\frac{\sqrt{2}}{\psi(t_{k+1})} \right) - 1 \\ &\leq \frac{2}{\sqrt{\pi}\psi(t_{k+1})}. \end{aligned}$$

According to Lemma 2.9, $\mathcal{J}_1(\psi) < \infty$ implies $\sum_k 1/\psi(t_{k+1}) < \infty$, and thus by the Borel–Cantelli lemma we obtain (3.17).

Now assume that $\mathcal{J}_1(\psi) = \infty$. Since

$$\begin{aligned} \mathbb{P} \left\{ M^+(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ for some } t \geq T \right\} &\geq \mathbb{P} \left\{ M^+(T) \leq \frac{T^{1/2}}{\psi(T)} \right\} \\ &= 2\Phi \left(\frac{1}{\psi(T)} \right) - 1, \end{aligned}$$

we have that $\psi(T) \rightarrow \infty$ as $T \rightarrow \infty$ is necessary for

$$\mathbb{P} \left\{ M^+(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = 0.$$

Therefore, if $\psi(T) \nrightarrow \infty$ as $T \rightarrow \infty$, we have

$$\mathbb{P} \left\{ M^+(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = 1.$$

In what follows, assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Put again $t_k = 2^k$ and

$$B_k = \left\{ M^+(t_k) \leq \frac{t_k^{1/2}}{\psi(t_k)} \right\}.$$

Take $0 < C < \sqrt{2/\pi}$. Then for sufficiently large k , say $k \geq k_0$,

$$C \frac{1}{\psi(t_k)} \leq P(B_k) \leq \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\psi(t_k)}.$$

By Lemma 2.9, $\sum_k P(B_k) = \infty$.

To use the Borel–Cantelli lemma, we have to estimate $P(B_k \cap B_l)$. Let $s_1 < s_2$ and $0 < a_1 s_1^{1/2} < a_2 s_2^{1/2}$ and put

$$A_1 = \left\{ M^+(s_1) < a_1 s_1^{1/2} \right\}.$$

$$A_2 = \left\{ M^+(s_2) < a_2 s_2^{1/2} \right\}.$$

Using the notations from Lemma 3.7 we have

$$\begin{aligned} & \iint_{T_1} \frac{1}{(s_1(s_2 - s_1))^{1/2}} \varphi\left(\frac{x}{s_1^{1/2}}\right) \varphi\left(\frac{y}{(s_2 - s_1)^{1/2}}\right) dx dy - P(A_1)P(A_2) \\ &= \int_{-a_1}^{a_1} \int_{T'_1(x)} \varphi(x) \varphi(y) dx dy - \int_{-a_1}^{a_1} \varphi(x) dx \int_{-a_2}^{a_2} \varphi(y) dy \\ &= \int_{-a_1}^{a_1} \varphi(x) \left(\int_{T'_1(x)} \varphi(y) dy - \int_{-a_2}^{a_2} \varphi(y) dy \right) dx \\ &\leq 2 \left[\int_{-a_1}^{x_1} \varphi(x) \varphi(a_2) \left(a_2 \frac{s_2^{1/2} - (s_2 - s_1)^{1/2}}{(s_2 - s_1)^{1/2}} - x \frac{s_1^{1/2}}{(s_2 - s_1)^{1/2}} \right) dx \right. \\ &\quad \left. - \int_{x_1}^{a_1} \varphi(x) \varphi(a_2) \left(-a_2 \frac{s_2^{1/2} - (s_2 - s_1)^{1/2}}{(s_2 - s_1)^{1/2}} + x \frac{s_1^{1/2}}{(s_2 - s_1)^{1/2}} \right) dx \right] \quad (3.18) \\ &= 2 \int_{-a_1}^{a_1} \varphi(x) \varphi(a_2) \left(a_2 \frac{s_2^{1/2} - (s_2 - s_1)^{1/2}}{(s_2 - s_1)^{1/2}} - x \frac{s_1^{1/2}}{(s_2 - s_1)^{1/2}} \right) dx \\ &\leq 4a_1 a_2 \varphi^2(0) \frac{s_2^{1/2} - (s_2 - s_1)^{1/2}}{(s_2 - s_1)^{1/2}} \\ &= \frac{2}{\pi} a_1 a_2 \frac{s_2^{1/2} - (s_2 - s_1)^{1/2}}{(s_2 - s_1)^{1/2}}, \end{aligned}$$

where

$$x_1 = a_2 \frac{s_2^{1/2}}{(s_2 - s_1)^{1/2}}$$

is the point of intersection of the lines $y = a_2$ and $y = a_2(s_2/(s_2 - s_1))^{1/2} - x(s_1/(s_2 - s_1))^{1/2}$, and

$$T'_1(x) = \left\{ y; x \frac{s_1^{1/2}}{(s_2 - s_1)^{1/2}} - a_2 \frac{s_2^{1/2}}{(s_2 - s_1)^{1/2}} < y < a_2 \frac{s_2^{1/2}}{(s_2 - s_1)^{1/2}} - x \frac{s_1^{1/2}}{(s_2 - s_1)^{1/2}} \right\}.$$

The double integral over T_2 is equal to $P\{U < -a_1 s_1^{1/2}, a_2 s_2^{1/2} - 2a_1 s_1^{1/2} < U + V < a_2 s_2^{1/2}\}$, where U and V are independent normal random variables with mean zero and variance s_1 and $s_2 - s_1$, respectively. Hence $U + V$ is an $N(0, s_2)$ random variable and the above probability can be estimated by

$$\begin{aligned} P\{a_2 s_2^{1/2} - 2a_1 s_1^{1/2} < U + V < a_2 s_2^{1/2}\} \\ = \Phi(a_2) - \Phi\left(a_2 - 2a_1 \frac{s_1^{1/2}}{s_2^{1/2}}\right) \leq \frac{2a_1}{\sqrt{2\pi}} \cdot \frac{s_1^{1/2}}{s_2^{1/2}}. \end{aligned} \quad (3.19)$$

The same holds for the integral over T_3 .

From Lemma 3.7 and the estimates (3.18) and (3.19), we have with $a_1 = 1/\psi(t_k)$, $a_2 = 1/\psi(t_l)$, $s_1 = t_k$, and $s_2 = t_l$ for $k < l$,

$$P(B_k \cap B_l) \leq P(B_k) P(B_l) + K_1 P(B_k) \frac{1}{\psi(2^l)} 2^{-(l-k)} + K_2 P(B_k) 2^{-(l-k)/2},$$

where K_1 and K_2 are positive constants, and, therefore,

$$\begin{aligned} \sum_{k \geq k_0} \sum_{l \geq k_0} P(B_k \cap B_l) &= \sum_{k \geq k_0} P(B_k) + 2 \sum_{l > k \geq k_0} P(B_k \cap B_l) \\ &\leq \sum_{k \geq k_0} P(B_k) + 2 \sum_{l > k \geq k_0} P(B_k) P(B_l) \\ &\quad + K'_1 \sum_{k \geq k_0} P(B_k) + K'_2 \sum_{k \geq k_0} P(B_k), \end{aligned} \quad (3.20)$$

where K'_1 and K'_2 are positive constants. By the Borel–Cantelli lemma, the divergent part is proved. $\square$

Chapter 4

Integral Tests for Passage Times

4.1 Introduction

Define the first passage time processes

$$S^+(u, \omega) = \inf\{t \geq 0; W(t, \omega) = u\} = \inf\{t \geq 0; M^+(t, \omega) = u\},$$

and

$$S_r(u, \omega) = \inf\{t \geq 0; \|W_r(t, \omega)\| = u\} = \inf\{t \geq 0; M_r(t, \omega) = u\}.$$

We mention some basic properties of these processes, mainly using Chapter 6 of Karatzas and Shreve (1988).

Let $\{B(t); t \geq 0\}$ be a standard one-dimensional Wiener process starting at 0. Then

$$W(t) = - \int_0^t \operatorname{sgn}(B(s)) \, dB(s), \quad t \geq 0,$$

is itself a standard one-dimensional Wiener process. The corresponding process $\{M^+(t); t \geq 0\}$ is continuous, nondecreasing, and has time derivative almost everywhere equal to zero; it is very much like the Cantor function. It is constant during

excursions of the reflected Wiener process $\{|B(s)|; t \geq 0\}$ away from the origin. These excursions correspond to the jumps of the process $\{S^+(u); u \geq 0\}$. The process $\{S^+(u); u \geq 0\}$ is the left-continuous inverse of the process $\{M^+(t); t \geq 0\}$ and is therefore strictly increasing. Moreover, it increases by jumps only, i.e.,

$$S^+(u) = \sum_{x < u} (S^+(x+) - S^+(x)), \quad u \geq 0.$$

It has stationary and independent increments, with probability density function of $S^+(v) - S^+(u)$ being

$$\frac{v-u}{\sqrt{2\pi t^3}} \exp\left(-\frac{(v-u)^2}{2t}\right), \quad 0 < u < v, \quad t > 0. \quad (4.1)$$

Define also the local time process $\{L(t); t \geq 0\}$ of $\{B(t); t \geq 0\}$ as

$$L(t) = \lim_{\varepsilon \rightarrow 0+} \frac{1}{2\varepsilon} \text{meas}\{0 \leq s \leq t; |B(s)| \leq \varepsilon\},$$

where meas is a Lebesgue measure. It is well known that

$$P\{|B(t)| = M^+(t) - W(t), \quad L(t) = M^+(t) \text{ for all } t \geq 0\} = 1.$$

For each $t > 0$, the time $s \in [0, t]$, for which $W(s) = M^+(t)$ is almost surely unique. Hence, for almost all ω we can define

$$\gamma(t, \omega) = \sup\{s \in [0, t]; B(s, \omega) = 0\} = \sup\{s \in [0, t]; W(s, \omega) = M^+(t, \omega)\},$$

$$\beta(t, \omega) = \inf\{s \in [t, \infty); B(s, \omega) = 0\} = \inf\{s \in [t, \infty); W(s, \omega) = M^+(t, \omega)\}.$$

For t such that $B(t, \omega) \neq 0$, the length of the excursion interval $(\gamma(t, \omega), \beta(t, \omega))$ is the length of the flat interval of $M^+(t, \omega)$, the size of the jump $S^+(M^+(t, \omega)+) - S^+(M^+(t, \omega))$, and the length of the zero free interval of $B(s, \omega)$ straddling t . Therefore, the results (namely Theorems 3.1 and 3.3) obtained for $\{M^+(t); t \geq 0\}$ remain valid when $\{M^+(t); t \geq 0\}$ is replaced by $\{L(t); t \geq 0\}$, and the results for the

length of the longest excursion obtained in Chapter 7 remain valid for the length of the longest flat interval of $\{M^+(t); t \geq 0\}$ or $\{L(t); t \geq 0\}$ (see also (5.4)).

$\{S_r(u); u \geq 0\}$ is a nondecreasing process with independent, nonstationary increments. The distribution of $S_r(u)$ can be obtained from (3.7), since

$$P\{S_r(u) > t\} = P\{M_r(t) < u\}.$$

and the Laplace transform of increments $S_r(b) - S_r(a)$, $0 < a < b < \infty$, is (see Kent (1978))

$$\left(\frac{b}{a}\right)^\nu \frac{I_\nu(a(2s)^{1/2})}{I_\nu(b(2s)^{1/2})}, \quad s > 0,$$

where $\nu = (r - 2)/2$, and I_ν is the Bessel function.

4.2 Theorems

Since the big values of $\{S^+(u); u \geq 0\}$ and $\{S_r(u); u \geq 0\}$ correspond to the small values of $\{M^+(t); t \geq 0\}$ and $\{M_r(t); t \geq 0\}$, respectively, and the small values of $\{S^+(u); u \geq 0\}$ and $\{S_r(u); u \geq 0\}$ correspond to the big values $\{M^+(t); t \geq 0\}$ and $\{M_r(t); t \geq 0\}$, respectively, we will obtain the lower and upper classes of $\{S^+(u); u \geq 0\}$ and $\{S_r(u); u \geq 0\}$ from Theorems 3.1, 3.2, and 3.3.

Define the following classes of functions,

$$\Xi_1 = \{\xi; \xi \text{ is a positive function defined on } [1, \infty),$$

$$\text{nondecreasing in a neighborhood of infinity}\} = \Psi_1,$$

$$\Xi_2 = \{\xi; \xi \text{ is a positive function defined on } [1, \infty) \text{ such that}$$

$$u\xi(u) \text{ is nondecreasing in a neighborhood of infinity}\} = \Psi_3^1,$$

$$\Xi_3 = \{\xi; \xi \text{ is a positive function defined on } [1, \infty) \text{ such that}$$

$$u/\xi(u) \text{ is nondecreasing in a neighborhood of infinity}\} = \Psi_2^1,$$

$$\Xi_4 = \{\xi; \xi \text{ is a positive function defined on } [1, \infty) \text{ such that}$$

$$u/\xi^2(u) \text{ is nondecreasing in a neighborhood of infinity}\} = \Psi_2^2$$

$$\Xi_4^* = \{\xi; \xi \text{ is a positive function defined on } [1, \infty) \text{ such that}$$

$$u\xi^2(u) \text{ is nondecreasing in a neighborhood of infinity}\} = \Psi_3^2.$$

Theorem 4.1. *Let $\xi \in \Xi_3$ and let $r \geq 1$ be an integer. Then*

$$P \left\{ S_r(u) \leq \frac{1}{2} \cdot \frac{u^2}{\xi^2(u)} \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_r(\xi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (4.2)$$

or, equivalently,

$$\xi^{-2}(u) \in \begin{cases} \text{LLC}(2u^{-2}S_r(u)) \\ \text{LUC}(2u^{-2}S_r(u)) \end{cases} \quad \text{according as } J_r(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 4.1.

$$\liminf_{u \rightarrow \infty} \frac{S_r(u)}{u^2} \xi^2(u) = \frac{1}{2c^*(\xi)} \quad a.s.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 4.2. *The integral test (4.2) remains true for $\xi \in \Psi_2^\alpha$ and $\xi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 4.1. $S_1(u)$ can be replaced by $S^+(u)$ in the above theorem and corollaries.

Theorem 4.2. *Let $\xi \in \Xi_2$ and let $r \geq 1$ be an integer. Then*

$$P \left\{ S_r(u) \geq \frac{2}{j_\nu^2} u^2 \xi^2(u) \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_2(\xi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (4.3)$$

or, equivalently,

$$\xi^2(u) \in \begin{cases} \text{UUC}((j_\nu^2/2)u^{-2}S_r(u)) \\ \text{ULC}((j_\nu^2/2)u^{-2}S_r(u)) \end{cases} \quad \text{according as } \mathcal{I}_2(\xi) \begin{cases} < \infty \\ = \infty, \end{cases}$$

where j_ν denotes the smallest positive zero of the Bessel function J_ν of index $\nu = (r-2)/2$.

Corollary 4.3.

$$\limsup_{u \rightarrow \infty} \frac{S_r(u)}{u^2 \xi^2(u)} = \frac{2}{j_\nu^2} c^*(\xi) \quad \text{a.s.}$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 4.4. *The integral test (4.3) remains true for $\xi \in \Psi_2^\alpha$ and $\xi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Theorem 4.3. *Let $\xi \in \Xi_2$. Then*

$$P \left\{ S^+(u) \geq u^2 \xi^2(u) \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (4.4)$$

or, equivalently,

$$\xi^2(u) \in \begin{cases} \text{UUC}(u^{-2}S^+(u)) \\ \text{ULC}(u^{-2}S^+(u)) \end{cases} \quad \text{according as } \mathcal{I}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 4.5.

$$\limsup_{u \rightarrow \infty} \frac{S^+(u)}{u^2 \xi^2(u)} = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s. according as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 4.6. *The integral test (4.4) remains true for $\xi \in \Psi_2^\alpha$ and $\xi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

To prove equivalence of Theorems 4.1, 4.2, and 4.3 to Theorems 3.1, 3.2, and 3.3, we only need to show that the integrals of the ξ 's are finite if and only if the integrals of the corresponding ψ 's are finite. This will be done in Lemmas 4.1, 4.2, and 4.3. (For a similar treatment of $\psi \in \Psi_1$ see Lemma 2.13 of Jain and Taylor (1973).) As explained in Remark 2.3, we will assume, without loss of generality, that the monotonicities are strict.

Lemma 4.1. *Let $\psi \in \Psi_4^2$ and let $d \geq 0$ be an integer. Let a function ξ be defined by*

$$\xi(u) = \psi(t),$$

where

$$t = t(u) = \frac{u^2}{\lambda^2 \xi^2(u)}, \quad \text{and} \quad u = u(t) = \lambda t^{1/2} \psi(t),$$

and λ is a positive constant. Then $\xi \in \Xi_4$ and

$$\mathcal{J}_d(\psi) < \infty \iff \mathcal{J}_d(\xi) < \infty.$$

Proof. Since $\psi \in \Psi_4^2$,

$$\frac{t^{1/2}}{\psi(t)} = \frac{u}{\lambda \xi^2(u)}$$

is a nondecreasing function, and since $u(t)$ is nondecreasing, its inverse $t(u)$ is also nondecreasing. Therefore $\xi \in \Xi_4$.

First we prove that $J_d(\psi) < \infty$ implies that $J_d(\xi) < \infty$. Finiteness of $J_d(\psi)$ gives, by Lemma 2.1, $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10, and put $u_k = u(t_k)$ and $\xi_k = \xi(u_k)$. Observe that

$$\frac{u_k}{u_{k+1}} = \frac{\lambda t_k^{1/2} \psi_k}{\lambda t_{k+1}^{1/2} \psi_{k+1}} = \frac{\sqrt{\psi_k^2 - 1}}{\psi_{k+1}}, \quad (4.5)$$

and hence

$$1 \geq \frac{u_k}{u_{k+1}} \geq \frac{\psi_k^2 - 1}{\psi_k^2} \rightarrow 1 \quad \text{as } k \rightarrow \infty. \quad (4.6)$$

Moreover,

$$\frac{\frac{u_{k+1} - u_k}{\frac{t_{k+1} - t_k}{t_k}}}{\frac{u_k}{t_k}} = (\psi_{k+1} - \sqrt{\psi_k^2 - 1}) \sqrt{\psi_k^2 - 1} \leq 1. \quad (4.7)$$

We have

$$\begin{aligned} \frac{\int_{u_k}^{u_{k+1}} \frac{\xi^d(u)}{u} \exp(-\xi^2(u)) du}{\int_{t_k}^{t_{k+1}} \frac{\psi^d(t)}{t} \exp(-\psi^2(t)) dt} &\leq \frac{\frac{u_{k+1} - u_k}{u_k} \xi_k^d \frac{u_{k+1}^d}{u_k^d} \exp\left(-\frac{\xi_{k+1}^2 u_k^2}{u_{k+1}^2}\right)}{\frac{t_{k+1} - t_k}{t_{k+1}} \psi_{k+1}^d \frac{t_k^{d/2}}{t_{k+1}^{d/2}} \exp\left(-\frac{\psi_k^2 t_{k+1}}{t_k}\right)} \\ &= \frac{\frac{u_{k+1} - u_k}{u_k} \psi_k^d \frac{u_{k+1}^d}{u_k^d}}{\frac{t_{k+1} - t_k}{t_k} \psi_{k+1}^d \frac{t_k^{1+d/2}}{t_{k+1}^{1+d/2}}} \exp\left(-(\psi_k^2 - 1) + \frac{\psi_k^4}{\psi_k^2 - 1}\right), \end{aligned}$$

which, by (4.6) and (4.7), is bounded from above by an expression which goes to e^2 as $k \rightarrow \infty$. Therefore, $J_d(\xi) < \infty$.

Now assume that $J_d(\psi) = \infty$. If $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$ then also $\xi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and, by Lemma 2.1, $J_d(\xi) = \infty$. Therefore, we may assume that $\psi(t) \rightarrow \infty$

as $t \rightarrow \infty$. Define t_k , u_k , ψ_k , and ξ_k as above. Then by Lemma 2.12,

$$\sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty,$$

where

$$\mathcal{K} = \left\{ k; \psi_k^2 - \frac{1}{2} > \psi_{k+1}^2 \right\}.$$

For $k \in \mathcal{K}^c$ we have

$$\frac{\frac{u_{k+1} - u_k}{t_{k+1} - t_k}}{\frac{u_k}{t_k}} \geq (\sqrt{\psi_k^2 - \frac{1}{2}} - \sqrt{\psi_k^2 - 1}) \sqrt{\psi_k^2 - 1} \rightarrow \frac{1}{4} \quad \text{as } k \rightarrow \infty. \quad (4.8)$$

Now we have

$$\begin{aligned} \frac{\int_{u_k}^{u_{k+1}} \frac{\xi^d(u)}{u} \exp(-\xi^2(u)) du}{\int_{t_k}^{t_{k+1}} \frac{\psi^d(t)}{t} \exp(-\psi^2(t)) dt} &\geq \frac{\frac{u_{k+1} - u_k}{u_{k+1}} \xi_{k+1}^d \frac{u_k^d}{u_{k+1}^d} \exp\left(-\frac{\xi_k^2 u_{k+1}^2}{u_k^2}\right)}{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \frac{t_{k+1}^{d/2}}{t_k^{d/2}} \exp\left(-\frac{\psi_k^2 t_k}{t_{k+1}}\right)} \\ &= \frac{\frac{u_{k+1} - u_k}{u_k} \psi_{k+1}^d \frac{u_k^{d+1}}{u_{k+1}^{d+1}}}{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \frac{t_{k+1}^{d/2}}{t_k^{d/2}}} \exp\left(-\frac{\psi_k^2 \psi_{k+1}^2}{\psi_k^2 - 1} + \psi_k^2 - 1\right), \end{aligned}$$

which, by (4.6) and (4.8), is bounded from below by an expression which goes to $\frac{1}{4}e^{-3}$ as $k \rightarrow \infty$. Therefore, $\mathcal{J}_d(\xi) = \infty$. $\square$

Lemma 4.2. Let $\psi \in \Psi_4^2$ and let $d \geq 0$ be an integer. Let a function ξ be defined by

$$\xi(u) = \psi(t),$$

where

$$u = u(t) = \lambda \frac{t^{1/2}}{\psi(t)} \quad \text{and} \quad t = t(u) = \frac{1}{\lambda^2} u^2 \xi^2(u),$$

and λ is a positive constant. Then $\xi \in \Xi_4^*$ and

$$\mathcal{J}_d(\psi) < \infty \iff \mathcal{J}_d(\xi) < \infty.$$

Proof. The proof goes along the lines of the proof of Lemma 4.1. Since $\psi \in \Psi_4^2$,

$$t^{1/2}\psi(t) = \frac{1}{\lambda}u\xi^2(u)$$

is a nondecreasing function, and since $u(t)$ is nondecreasing, its inverse $t(u)$ is also nondecreasing. Therefore $\xi \in \Xi_4^*$.

Assume that $\mathcal{J}_d(\psi) < \infty$. By Lemma 2.1, $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10, and put $u_k = u(t_k)$ and $\xi_k = \xi(u_k)$. Observe that

$$\frac{u_k}{u_{k+1}} = \frac{\frac{\lambda t_k^{1/2}}{\psi_k}}{\frac{\lambda t_{k+1}^{1/2}}{\psi_{k+1}}} = \frac{\psi_{k+1} \sqrt{\psi_k^2 - 1}}{\psi_k^2}, \quad (4.9)$$

and hence

$$1 \geq \frac{u_k}{u_{k+1}} \geq \frac{\psi_k^2 - 1}{\psi_k^2} \rightarrow 1 \quad \text{as } k \rightarrow \infty. \quad (4.10)$$

Moreover,

$$\frac{\frac{u_{k+1} - u_k}{u_k}}{\frac{t_{k+1} - t_k}{t_k}} = \frac{\psi_k^2 \sqrt{\psi_k^2 - 1}}{\psi_{k+1}} - (\psi_k^2 - 1) \leq 1. \quad (4.11)$$

We have

$$\begin{aligned} & \frac{\int_{u_k}^{u_{k+1}} \frac{\xi^d(u)}{u} \exp(-\xi^2(u)) du}{\int_{t_k}^{t_{k+1}} \frac{\psi^d(t)}{t} \exp(-\psi^2(t)) dt} \leq \frac{\frac{u_{k+1} - u_k}{u_k} \xi_{k+1}^d \frac{u_{k+1}^d}{u_k^d} \exp\left(-\frac{\xi_k^2 u_k^2}{u_{k+1}^2}\right)}{\frac{t_{k+1} - t_k}{t_{k+1}} \psi_{k+1}^d \frac{t_k^{d/2}}{t_{k+1}^{d/2}} \exp\left(-\frac{\psi_k^2 t_{k+1}}{t_k}\right)} \\ &= \frac{\frac{u_{k+1} - u_k}{u_k} \cdot \frac{u_{k+1}^d}{u_k^d}}{\frac{t_{k+1} - t_k}{t_k} \cdot \frac{t_k^{1+d/2}}{t_{k+1}^{1+d/2}}} \exp\left(-\frac{\psi_{k+1}^2(\psi_k^2 - 1)}{\psi_k^2} + \frac{\psi_k^4}{\psi_k^2 - 1}\right), \end{aligned}$$

which, by (4.10) and (4.11), is bounded from above by an expression which goes to e^3 as $k \rightarrow \infty$. Therefore, $J_d(\xi) < \infty$.

Now assume that $J_d(\psi) = \infty$. The case $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$ is obvious, hence assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$, and define t_k , u_k , ψ_k , and ξ_k as above. Then by Lemma 2.11,

$$\sum_{k \in \mathcal{K}^c} \psi_k^{d-2} \exp(-\psi_k^2) = \infty.$$

where

$$\mathcal{K} = \left\{ k: \psi_k^2 > \psi_{k-1}^2 + \frac{1}{2} \right\}.$$

For $k \in \mathcal{K}^c$ we have

$$\frac{\frac{u_{k+1} - u_k}{t_{k+1} - t_k}}{\frac{u_k}{t_k}} \geq \frac{\psi_k^2 \sqrt{\psi_k^2 - 1}}{\sqrt{\psi_k^2 + \frac{1}{2}}} - (\psi_k^2 - 1) \rightarrow \frac{1}{4} \quad \text{as } k \rightarrow \infty. \quad (4.12)$$

Now we have

$$\begin{aligned} \frac{\int_{u_k}^{u_{k+1}} \frac{\xi^d(u)}{u} \exp(-\xi^2(u)) du}{\int_{t_k}^{t_{k+1}} \frac{\psi^d(t)}{t} \exp(-\psi^2(t)) dt} &\geq \frac{\frac{u_{k+1} - u_k}{u_{k+1}} \xi_k^d \frac{u_k^d}{u_{k+1}^d} \exp\left(-\frac{\xi_{k+1}^2 u_{k+1}^2}{u_k^2}\right)}{\frac{t_{k+1} - t_k}{t_k} \psi_k^d \frac{t_{k+1}^{d/2}}{t_k^{d/2}} \exp\left(-\frac{\psi_k^2 t_k}{t_{k+1}}\right)} \\ &= \frac{\frac{u_{k+1} - u_k}{t_{k+1} - t_k} \cdot \frac{u_k^{d+1}}{u_{k+1}^{d+1}}}{\frac{t_{k+1} - t_k}{t_k} \cdot \frac{t_{k+1}^{d/2}}{t_k^{d/2}}} \exp\left(-\frac{\psi_k^4}{\psi_k^2 - 1} + \psi_k^2 - 1\right), \end{aligned}$$

which, by (4.10) and (4.12), is bounded from below by an expression which goes to $\frac{1}{4}e^{-2}$ as $k \rightarrow \infty$. Therefore, $J_d(\xi) = \infty$. $\square$

Lemma 4.3. *Let $\psi \in \Psi_2^\alpha$ for some $\alpha > 0$, and let ξ be defined by*

$$\xi(u) = \psi(t),$$

where

$$u = u(t) = \frac{t^{1/\alpha}}{\psi(t)} \quad \text{and} \quad t = t(u) = u^\alpha \xi^\alpha(u).$$

Then $\xi \in \Xi_2$ and

$$\mathcal{J}_1(\psi) < \infty \iff \mathcal{J}_1(\xi) < \infty.$$

Proof. Obviously, $\xi \in \Xi_2$. Take $\lambda > 1$ and put $u_k = u(\lambda^k) = \lambda^{k/\alpha}/\psi(\lambda^k)$. Take $K \in \mathbb{N}$ sufficiently large such that $t^{1/\alpha}/\psi(t)$ is nondecreasing on $[\lambda^K, \infty)$.

Assume $\mathcal{J}_1(\psi) < \infty$. By Lemma 2.9, $\sum_{k \geq K} 1/\psi(\lambda^k) < \infty$. Now

$$\begin{aligned} \int_{u_K}^{\infty} \frac{1}{u\xi(u)} du &= \sum_{k \geq K} \int_{u_k}^{u_{k+1}} \frac{1}{u\xi(u)} du \\ &\leq \sum_{k \geq K} \frac{u_{k+1} - u_k}{u_k \xi(u_k)} \\ &= \sum_{k \geq K} \left(\frac{\lambda^{1/\alpha}}{\psi(\lambda^{k+1})} - \frac{1}{\psi(\lambda^k)} \right) \\ &\leq (\lambda^{1/\alpha} - 1) \sum_{k \geq K} \frac{1}{\psi(\lambda^k)} < \infty. \end{aligned}$$

Now assume that $\mathcal{J}_1(\psi) = \infty$, ie., $\sum_{k \geq K} 1/\psi(\lambda^k) = \infty$. Then

$$\begin{aligned} \int_{u_K}^{\infty} \frac{1}{u\xi(u)} du &= \sum_{k \geq K} \int_{u_k}^{u_{k+1}} \frac{1}{u\xi(u)} du \\ &\geq \sum_{k \geq K} \frac{u_{k+1} - u_k}{u_{k+1} \xi(u_{k+1})} \\ &= \sum_{k \geq K} \left(\frac{1}{\psi(\lambda^{k+1})} - \frac{1}{\lambda^{1/\alpha} \psi(\lambda^k)} \right) \\ &= \left(1 - \frac{1}{\lambda^{1/\alpha}} \right) \left(\sum_{k \geq K+1} \frac{1}{\psi(\lambda^k)} \right) - \frac{1}{\lambda^{1/\alpha} \psi(\lambda^K)} = \infty. \end{aligned}$$

□

4.3 Proof of Theorem 4.1

We have not proved Theorem 3.1 yet. Actually, we will prove now the equivalent Theorem 4.1 (see also Theorem 6.C).

We need to estimate the distribution of increments of $\{S_r(u); u \geq 0\}$. We closely follow Lemmas 3.1–3.3 of Khoshnevisan, Lewis, and Shi (1996) given, in our notation, for $\{s_r(u); u \geq 0\}$ defined in (5.2). Let $G_r(x)$ be the distribution function of $S_r(1)$. By Lemmas 2.7 and 2.8 in Jain and Taylor (1973), for all $t > 0$ and all $x > 0$,

$$\mathbb{P} \left\{ \frac{M_r(t)}{t^{1/2}} \geq x \right\} \approx (\max(1, x))^{r-2} \exp \left(-\frac{1}{2}x^2 \right). \quad (4.13)$$

(Notice the similarity with the integrand in (1.1).) From (4.13) we have

$$\begin{aligned} G_r(x) &= \mathbb{P}\{S_r(1) \leq x\} = \mathbb{P}\{M_r(x) \geq 1\} \\ &= \mathbb{P} \left\{ M_r(1) \geq \frac{1}{x^{1/2}} \right\} \approx (\min(1, x))^{1-r/2} \exp \left(-\frac{1}{2x} \right). \end{aligned} \quad (4.14)$$

Lemma 4.4. *Let $x, y > 0$ and $0 < u < 1$. Then*

$$\mathbb{P}\{S_r(1) - S_r(u) \leq x\} \leq \frac{G_r(x+y)}{G_r(yu^{-2})}. \quad (4.15)$$

Proof. Inequality (4.15) follows from

$$\{\omega; S_r(1) - S_r(u) \leq x\} \cap \{\omega; S_r(u) \leq y\} \subset \{\omega; S_r(1) \leq x+y\},$$

the scaling property of $\{S_r(u); u \geq 0\}$, ie., $S_r(u) \stackrel{D}{=} u^{-2}S_r(1)$, and the independence of the increments. $\square$

The next lemma provides an estimate for the distribution of $S_r(1) - S_r(u)$ when u is small.

Lemma 4.5. *Take $\delta > 0$. Then there exists a positive constant $c = c(\delta, r)$, such that for all $0 \leq x \leq 1/\delta$ and $0 \leq u \leq \delta x$,*

$$P\{S_r(1) - S_r(u) \leq x\} \leq cG_r(x). \quad (4.16)$$

Proof. Notice that $P\{S_r(1) - S_r(u) \leq x\}$ is a nondecreasing function of u and, therefore, it is sufficient to proof (4.16) for $u = \delta x$. Let μ be the median of $S_r(1)$, i.e., $G_r(\mu) = 1/2$. Applying (4.15) with $y = \mu\delta^2x^2$ we obtain

$$\begin{aligned} P\{S_r(1) - S_r(\delta x) \leq x\} &\leq \frac{G_r(x + \mu\delta^2x^2)}{G_r(\mu)} \\ &= 2G_r(x + \mu\delta^2x^2) \\ &\approx (x + \mu\delta^2x^2)^{1-r/2} \exp\left(-\frac{1}{2(x + \mu\delta^2x^2)}\right), \end{aligned}$$

since $x + \mu\delta^2x^2 \leq 1/\delta + \mu$. Observe that

$$(x + \mu\delta^2x^2)^{1-r/2} \leq \begin{cases} x^{1-r/2}, & r \geq 2, \\ x^{1-r/2}(1 + \mu\delta)^{1-r/2}, & r < 2, \end{cases}$$

and

$$\exp\left(-\frac{1}{2(x + \mu\delta^2x^2)}\right) \leq \exp\left(-\frac{1}{2x}\right) \exp\left(\frac{\mu\delta^2}{2}\right),$$

and, therefore,

$$G_r(x + \mu\delta^2x^2) \leq cG_r(x)$$

for some $c > 0$. □

The following lemma gives estimates for the distribution of the increment $S_r(1) - S_r(u)$ when u is not necessarily small. Put $x^* = 1$ for $r \leq 2$ and

$$x^* = \sup\{x: 0 < x \leq e^{-1}, 4(r-2)x \log(x^{-1}) \leq 1/2\}$$

for $r > 2$.

Lemma 4.6. *Let $\varepsilon > 0$. Then there exists a positive constant $c = c(\varepsilon, r)$, such that for all $x \leq x^* \wedge 1/(2\varepsilon)$ and $0 \leq u \leq 1 - x\varepsilon$,*

$$P\{S_r(1) - S_r(u) \leq x\} \leq c \exp\left(-\frac{(1-u)^2}{4x}\right). \quad (4.17)$$

and, if, in addition, $x\varepsilon \leq u$, then

$$P\{S_r(1) - S_r(u) \leq x\} \leq cu^{1-r/2} \exp\left(-\frac{(1-u)^2}{2x}\right). \quad (4.18)$$

Proof. First, we will consider $x\varepsilon \leq u \leq 1 - x\varepsilon$. We want to apply Lemma 4.4 with $y = xu/(1-u)$. Observe that

$$\begin{aligned} x + y &= \frac{x}{1-u} \leq \frac{1}{\varepsilon}, \\ yu^{-2} &= \frac{x}{u} + \frac{x}{1-u} \leq \frac{2}{\varepsilon}, \\ -\frac{1}{2(x+y)} + \frac{1}{2yu^{-2}} &= -\frac{(1-u)^2}{2x}, \\ \frac{x+y}{yu^{-2}} &= u. \end{aligned}$$

By (4.14) and (4.15) we obtain

$$\begin{aligned} P\{S_r(1) - S_r(u) \leq x\} &\leq \frac{G_r(x+y)}{G_r(yu^{-2})} \\ &\approx \frac{(x+y)^{1-r/2} \exp\left(-\frac{1}{2(x+y)}\right)}{(yu^{-2})^{1-r/2} \exp\left(-\frac{1}{2yu^{-2}}\right)} \\ &= u^{1-r/2} \exp\left(-\frac{(1-u)^2}{2x}\right). \end{aligned}$$

Therefore,

$$P\{S_r(1) - S_r(u) \leq x\} \leq c_1 u^{1-r/2} \exp\left(-\frac{(1-u)^2}{2x}\right)$$

for some $c_1 > 0$. This proves (4.18).

In the second case, $0 < u \leq x\varepsilon$, we apply (4.18) with $u = x\varepsilon$,

$$\begin{aligned} P\{S_r(1) - S_r(u) \leq x\} &\leq P\{S_r(1) - S_r(x\varepsilon) \leq x\} \\ &\leq c_1(x\varepsilon)^{1-r/2} \exp\left(-\frac{(1-x\varepsilon)^2}{2x}\right). \end{aligned}$$

Since $0 < u \leq x\varepsilon$,

$$\exp\left(-\frac{(1-x\varepsilon)^2}{2x}\right) \leq e^\varepsilon \exp\left(-\frac{(1-u)^2}{2x}\right).$$

and we obtain

$$P\{S_r(1) - S_r(u) \leq x\} \leq c_2 x^{1-r/2} \exp\left(-\frac{(1-u)^2}{2x}\right), \quad (4.19)$$

where $c_2 = c_1 \varepsilon^{1-r/2} e^\varepsilon$.

If $r \leq 2$, then we are done, since u and x are bounded by 1. If $r > 2$, then first consider $1/2 \leq u \leq 1 - x\varepsilon$ and bound $u^{1-r/2}$ by $2^{-1+r/2}$ in (4.18) to obtain the desired result. Second, if $0 < u \leq 1/2$, then, by (4.19),

$$\begin{aligned} P\{S_r(1) - S_r(u) \leq x\} &\leq c_2 \exp\left(-\frac{(1-u)^2}{2x} + \frac{r-2}{2} \log \frac{1}{x}\right) \\ &= c_2 \exp\left[-\frac{(1-u)^2}{2x} \left(1 - \frac{(r-2)x}{(1-u)^2} \log \frac{1}{x}\right)\right]. \end{aligned}$$

However, since $x \leq x^*$ and $0 < u \leq 1/2$, it follows that

$$1 - \frac{(r-2)x}{(1-u)^2} \log \frac{1}{x} \geq \frac{1}{2},$$

as required. This proves (4.17). □

Proof of Theorem 4.1. By Lemmas 2.15 and 2.16, we may assume that $\xi \in \Psi_4^2$. Let $\mathcal{I}_r(\xi) < \infty$. Then, by Lemma 2.1, $\xi(u) \rightarrow \infty$ as $u \rightarrow \infty$. Define u_k and ξ_k as in Lemma 2.10, ie., $u_{k+1} = u_k \xi_k^2 / (\xi_k^2 - 1)$ and $\xi_k = \xi(u_k)$. Put

$$A_k = \left\{ S_r(u) \leq \frac{u^2}{2\xi^2(u)} \text{ for some } u \in [u_k, u_{k+1}) \right\}.$$

Now

$$\begin{aligned}
P(A_k) &\leq P \left\{ S_r(u_k) \leq \frac{u_{k+1}^2}{2\xi_{k+1}^2} \right\} \\
&= P \left\{ S_r(1) \leq \frac{u_{k+1}^2}{2\xi_{k+1}^2 u_k^2} \right\} \\
&= P \left\{ S_r(1) \leq \frac{\xi_k^4}{2\xi_{k+1}^2 (\xi_k^2 - 1)^2} \right\} \\
&\leq P \left\{ S_r(1) \leq \frac{\xi_k^4}{2(\xi_k^2 - 1)^3} \right\} \\
&\approx P \left\{ S_r(1) \leq \frac{1}{2\xi_k^2} \right\} \\
&\approx \xi_k^{r-2} \exp(-\xi_k^2),
\end{aligned}$$

which sums by Lemma 2.10, and the Borel–Cantelli lemma finishes the proof of the convergent part of the theorem.

Now assume that $J_r(\xi) = \infty$. The case $\xi(u) \nrightarrow \infty$ as $u \rightarrow \infty$ is obvious, hence we may assume that $\xi(u) \rightarrow \infty$ as $u \rightarrow \infty$ and, by an argument similar to that in Lemma 3.1, that $2 \leq \xi^2(u) \leq 2 \log \log u$. Define u_k and ξ_k as above and now put

$$A_k = \left\{ S_r(u_k) \leq \frac{u_k^2}{2\xi_k^2} \right\}.$$

Since $P(A_k) \approx \xi_k^{r-2} \exp(-\xi_k^2)$, $\sum_k P(A_k) = \infty$ by Lemma 2.10.

Recollect that $\{S_r(u); u \geq 0\}$ is nondecreasing and has independent increments. Thus, we have for all $0 \leq u < v$ and $0 < x, y$,

$$P\{S_r(u) \leq x, S_r(v) \leq y\} \leq P\{S_r(u) \leq x\} P\{S_r(v) - S_r(u) \leq y\}.$$

Consider now $P(A_k \cap A_l)$ for $l > k$,

$$\begin{aligned}
P(A_k \cap A_l) &\leq P(A_k) P \left\{ S_r(u_l) - S_r(u_k) \leq \frac{u_l^2}{2\xi_l^2} \right\} \\
&= P(A_k) P \left\{ S_r(1) - S_r \left(\frac{u_k}{u_l} \right) \leq \frac{1}{2\xi_l^2} \right\}.
\end{aligned}$$

Denote $p_k = P(A_k)$ and $p_{kl} = P(A_k \cap A_l)$. We will prove that for $k < l$,

$$p_{kl} \leq C_1 a_{l-k}(p_k + p_l) + C_2 p_k p_l, \quad (4.20)$$

where C_1 and C_2 are positive constants and $a_j, j = 1, 2, \dots$, is a summable sequence.

It is then easily verified that

$$\liminf_{N \rightarrow \infty} \frac{\sum_{k=1}^N \sum_{l=1}^N p_{kl}}{\left(\sum_{k=1}^N p_k \right)^2} \leq C_2,$$

and the Borel–Cantelli lemma finishes the proof.

Since $\xi_l^2 u_k / u_l \searrow 0$ as $l \rightarrow \infty$, we can define

$$l_0 = l_0(k) = \sup \left\{ l > k: \xi_l^2 \frac{u_k}{u_l} \geq 1 \right\}.$$

For $l \geq l_0$, by Lemma 4.5,

$$P \left\{ S_r(1) - S_r \left(\frac{u_k}{u_l} \right) \leq \frac{1}{2\xi_l^2} \right\} \leq c G_r \left(\frac{1}{2\xi_l^2} \right),$$

for some $c > 0$, and, therefore,

$$p_{kl} \leq C p_k p_l$$

for some $C > 0$, and (4.20) is satisfied in this case.

Assume now that $l < l_0$. We will follow closely the proof of Theorem 1 in Pruitt (1990). We will consider three cases. First, assume that

$$|\xi_l^2 - \xi_k^2| \geq 2 \log(l - k). \quad (4.21)$$

If $\xi_k^2 \geq \xi_l^2 + 2\log(l-k)$, then

$$\begin{aligned}
 p_{kl} &\leq p_k \approx \xi_k^{r-2} \exp(-\xi_k^2) \\
 &\leq (\xi_l^2 + 2\log(l-k))^{(r-2)/2} \exp(-\xi_l^2 - 2\log(l-k)) \\
 &= \left(\frac{\xi_l^2 + 2\log(l-k)}{\xi_l^2} \right)^{(r-2)/2} \xi_l^{r-2} \exp(-\xi_l^2 - 2\log(l-k)) \\
 &\leq c p_l \frac{(\log(l-k))^{(r-2)/2}}{(l-k)^2}
 \end{aligned}$$

for some $c > 0$. If $\xi_k^2 \leq \xi_l^2 - 2\log(l-k)$, then

$$\begin{aligned}
 p_{kl} &\leq p_l \approx \xi_l^{r-2} \exp(-\xi_l^2) \\
 &\leq (\xi_k^2 + 2\log(l-k))^{(r-2)/2} \exp(-\xi_k^2 - 2\log(l-k)) \\
 &= \left(\frac{\xi_k^2 + 2\log(l-k)}{\xi_k^2} \right)^{(r-2)/2} \xi_k^{r-2} \exp(-\xi_k^2 - 2\log(l-k)) \\
 &\leq c p_k \frac{(\log(l-k))^{(r-2)/2}}{(l-k)^2}
 \end{aligned}$$

for some $c > 0$. In both cases, (4.20) is satisfied.

Second, assume that

$$\begin{aligned}
 |\xi_l^2 - \xi_k^2| &< 2\log(l-k), \\
 \xi_l^2 \frac{u_l - u_k}{u_l} &\geq C_3 \log(l-k).
 \end{aligned} \tag{4.22}$$

We put $\varrho = \frac{3}{4}$ and $C_3 = 32\varrho/(1-\varrho) = 96$. Also denote

$$\begin{aligned}
 \beta_l &= \frac{u_l^2}{2\xi_l^2}, \\
 \sqrt{\lambda_l} &= \frac{\sqrt{2}\xi_l^2}{u_l} = \frac{u_l}{\sqrt{2}\beta_l}, \\
 \alpha &= \frac{u_k^2}{2(\xi_k^2 + 2\log(l-k))}, \\
 \sqrt{\mu} &= \frac{u_k}{\sqrt{2}\alpha} = \frac{\sqrt{2}}{u_k} (\xi_k^2 + 2\log(l-k)).
 \end{aligned}$$

Obviously $\alpha < \beta_k \leq \beta_l$. Now

$$\begin{aligned} p_{kl} &\leq P\{S_r(u_k) \leq \alpha\} + P\{\alpha < S_r(u_k) \leq \beta_k, S_r(u_l) \leq \beta_l\} \\ &\leq P\{S_r(u_k) \leq \alpha\} + p_k P\{S_r(u_l) - S_r(u_k) \leq \beta_l - \alpha\}. \end{aligned} \quad (4.23)$$

Since

$$\begin{aligned} P\{S_r(u_k) \leq \alpha\} &= P\left\{S_r(1) \leq \frac{1}{2(\xi_k^2 + 2\log(l-k))}\right\} \\ &\approx (\xi_k^2 + 2\log(l-k))^{(r-2)/2} \exp(-\xi_k^2 - 2\log(l-k)) \\ &= \left(\frac{\xi_k^2 + 2\log(l-k)}{\xi_k^2}\right)^{(r-2)/2} \xi_k^{r-2} \exp(-\xi_k^2)(l-k)^{-2} \\ &\leq C p_k \frac{(\log(l-k))^{(r-2)/2}}{(l-k)^2} \end{aligned}$$

for some $C > 0$, the first term in (4.23) satisfies (4.20).

For the second term, we will use (4.17) to obtain

$$\begin{aligned} P\{S_r(u_l) - S_r(u_k) \leq \beta_l - \alpha\} &= P\left\{S_r(1) - S_r\left(\frac{u_k}{u_l}\right) \leq \frac{\beta_l - \alpha}{u_l^2}\right\} \\ &\leq C \exp\left(-\frac{(u_l - u_k)^2}{4(\beta_l - \alpha)}\right) \\ &= C \exp\left(-\frac{\sqrt{2}}{4}(u_l - u_k)\sqrt{\nu}\right) \end{aligned}$$

for some $C > 0$, where

$$\sqrt{\nu} = \frac{u_l - u_k}{\sqrt{2}(\beta_l - \alpha)}.$$

It is now sufficient to show that

$$\frac{\sqrt{2}}{4}(u_l - u_k)\sqrt{\nu} \geq 2\log(l-k).$$

Put

$$\delta = \frac{1 - \varrho}{4\varrho} = \frac{1}{12}.$$

If

$$\sqrt{\nu} \geq \delta \sqrt{\lambda_l},$$

then

$$\frac{\sqrt{2}}{4}(u_l - u_k)\sqrt{\nu} \geq \delta(u_l - u_k)\sqrt{\lambda_l} \geq \frac{C_3\delta}{2} \log(l - k) = 4 \log(l - k).$$

Now assume that

$$\sqrt{\nu} < \delta \sqrt{\lambda_l} < \sqrt{\varrho \lambda_l}.$$

Define η by

$$\frac{1}{\sqrt{\eta}} = \frac{u_l - u_k}{u_k} \frac{1}{\sqrt{\lambda_l}} \left(\frac{1}{\sqrt{\varrho}} - 1 \right) + \frac{1}{\sqrt{\mu}}.$$

Notice that

$$(u_l - u_k) \left(\frac{1}{\sqrt{\nu}} - \frac{1}{\sqrt{\varrho \lambda_l}} \right) = u_k \left(\frac{1}{\sqrt{\lambda_l}} - \frac{1}{\sqrt{\eta}} \right).$$

Obviously,

$$\nu < \varrho \lambda_l < \lambda_l < \eta < \mu.$$

Now

$$(u_l - u_k)(\sqrt{\varrho \lambda_l} - \sqrt{\nu}) = \sqrt{\frac{\nu \varrho}{\eta}} u_k (\sqrt{\eta} - \sqrt{\lambda_l}) \leq \varrho u_k (\sqrt{\eta} - \sqrt{\lambda_l}).$$

and

$$(u_l - u_k)(\sqrt{\lambda_l} - \sqrt{\varrho \lambda_l}) = \sqrt{\frac{\varrho}{\eta \mu}} \lambda_l u_k (\sqrt{\mu} - \sqrt{\eta}) \leq u_k (\sqrt{\mu} - \sqrt{\eta}).$$

This gives

$$\begin{aligned}
(1 - \delta)(u_l - u_k)\sqrt{\lambda_l} &\leq (u_l - u_k)(\sqrt{\lambda_l} - \sqrt{\nu}) \\
&\leq u_k[\sqrt{\mu} - \sqrt{\eta} + \varrho(\sqrt{\eta} - \sqrt{\lambda_l})] \\
&= u_k[\sqrt{\mu} - \sqrt{\lambda_l} - (1 - \varrho)(\sqrt{\eta} - \sqrt{\lambda_l})] \\
&= u_k[\sqrt{\lambda_k} - \sqrt{\lambda_l} - (1 - \varrho)(\sqrt{\eta} - \sqrt{\lambda_l})] + 2\sqrt{2}\log(l - k) \\
&\leq (u_l - u_k)\sqrt{\lambda_l} - (1 - \varrho)u_k(\sqrt{\eta} - \sqrt{\lambda_l}) + 4\sqrt{2}\log(l - k).
\end{aligned}$$

Therefore,

$$(1 - \varrho)u_k(\sqrt{\eta} - \sqrt{\lambda_l}) \leq \delta(u_l - u_k)\sqrt{\lambda_l} + 4\sqrt{2}\log(l - k).$$

We also have

$$\begin{aligned}
(u_l - u_k)(\sqrt{\varrho\lambda_l} - \sqrt{\nu}) &\leq \varrho u_k(\sqrt{\eta} - \sqrt{\lambda_l}) \\
&\leq \frac{\varrho}{1 - \varrho}[\delta(u_l - u_k)\sqrt{\lambda_l} + 4\sqrt{2}\log(l - k)].
\end{aligned}$$

Finally, this gives, by the definitions of δ and C_3 ,

$$\begin{aligned}
\frac{\sqrt{2}}{4}(u_l - u_k)\sqrt{\nu} &\geq \frac{\sqrt{2}}{4}(u_l - u_k)\left[\sqrt{\varrho\lambda_l} - \delta\frac{\varrho}{1 - \varrho}\sqrt{\lambda_l}\right] - 2\frac{\varrho}{1 - \varrho}\log(l - k) \\
&\geq \frac{\sqrt{2}}{4}(u_l - u_k)\sqrt{\lambda_l}\left[\frac{1}{2} - \delta\frac{\varrho}{1 - \varrho}\right] - 2\frac{\varrho}{1 - \varrho}\log(l - k) \\
&\geq \log(l - k)\left[\frac{1}{8}C_3 - 2\frac{\varrho}{1 - \varrho}\right] \\
&= 6\log(l - k).
\end{aligned}$$

Hence, we proved (4.20) in case (4.22).

Third, consider

$$\begin{aligned}
|\xi_l^2 - \xi_k^2| &< 2\log(l - k), \\
\xi_l^2 \frac{u_l - u_k}{u_l} &< C_3 \log(l - k).
\end{aligned} \tag{4.24}$$

Since

$$\xi_k^2 \frac{u_{l+1} - u_k}{u_k} - \xi_k^2 \frac{u_l - u_k}{u_k}$$

is close to one, we have

$$\begin{aligned} \frac{l-k}{2} &\leq \xi_k^2 \frac{u_l - u_k}{u_k} \\ &= (u_l - u_k) \left(\frac{\xi_l^2}{u_l} - \frac{\xi_k^2}{u_k} \right) + (u_l - u_k) \frac{\xi_l^2}{u_l} \\ &\leq (u_l - u_k) \left(\frac{\xi_l^2 (u_l - u_k)}{u_l u_k} + 2 \frac{\log(l-k)}{u_k} \right) + C_3 \log(l-k) \\ &\leq \frac{C_3(C_3 + 2) \log^2(l-k)}{u_k \frac{\xi_l^2}{u_l}} + C_3 \log(l-k). \end{aligned}$$

Thus we have

$$u_k \frac{\xi_l^2}{u_l} = O \left(\frac{1}{l-k} \log^2(l-k) \right). \quad (4.25)$$

Define α , β_l , λ_l , μ , and ν as above. Put $\varrho = \log(l-k)$ (assuming $l-k \geq 3$) and define ζ by

$$\frac{1}{\sqrt{\zeta}} = \frac{1}{\sqrt{\nu}} - \frac{u_k}{u_l - u_k} \left(\frac{1}{\sqrt{\lambda_l}} - \frac{1}{\sqrt{\varrho \lambda_l}} \right) = \frac{1}{\sqrt{\lambda_l}} + \frac{u_k}{u_l - u_k} \left(\frac{1}{\sqrt{\varrho \lambda_l}} - \frac{1}{\sqrt{\mu}} \right).$$

Using (4.25) we have

$$\begin{aligned} (u_l - u_k)(\sqrt{\zeta} - \sqrt{\nu}) &= \sqrt{\frac{\nu \zeta}{\varrho \lambda_l^2}} u_k \sqrt{\lambda_l} (\sqrt{\varrho} - 1) \\ &\leq u_k \varrho^2 \sqrt{\lambda_l} \\ &= O \left(\frac{\log^4(l-k)}{l-k} \right) = O(1), \end{aligned}$$

and (recollect that $\varrho = \log(l - k)$)

$$\begin{aligned}
 (u_l - u_k)(\sqrt{\lambda_l} - \sqrt{\zeta}) &= \sqrt{\frac{\zeta}{\varrho\mu}} u_k (\sqrt{\mu} - \sqrt{\varrho\lambda_l}) \\
 &\leq \frac{1}{\varrho} u_k (\sqrt{\mu} - \sqrt{\lambda_l}) \\
 &= \frac{1}{\varrho} (u_k \sqrt{\lambda_k} + 2\sqrt{2} \log(l - k) - u_k \sqrt{\lambda_l}) \\
 &= \frac{1}{\varrho} ((u_l - u_k) \sqrt{\lambda_l} + 4\sqrt{2} \log(l - k)) \\
 &\leq \frac{1}{\varrho} (C_3 \sqrt{2} + 4\sqrt{2}) \log(l - k) = (C_3 + 4) \sqrt{2}.
 \end{aligned}$$

Combining these last two facts shows that $(u_l - u_k)(\sqrt{\lambda_l} - \sqrt{\nu})$ is bounded. Thus

$$(u_l - u_k) \sqrt{\nu/2} = \frac{(u_l - u_k)^2}{2(\beta_l - \alpha)} = \xi_l^2 + O(1). \quad (4.26)$$

By (4.18),

$$p = P\{S_r(u_l) - S_r(u_k) \leq \beta_l - \alpha\} \leq C \left(\frac{u_k}{u_l}\right)^{1-r/2} \exp\left(-\frac{(u_l - u_k)^2}{2(\beta_l - \alpha)}\right),$$

for some $C > 0$. Since $p_l \approx \xi_l^{r-2} \exp(-\xi_l^2)$, we have for $r \geq 2$, using (4.26),

$$\frac{p}{p_l} \leq C'$$

for some $C' > 0$, since, for $l < l_0$ also

$$\left(\frac{u_k}{u_l}\right)^{1-r/2} \frac{1}{\xi_l^{r-2}} = \left(\frac{u_l}{\xi_l^2 u_k}\right)^{(r-2)/2} \leq 1.$$

This finishes the proof of the theorem for $r \geq 2$.

For $r = 1$, we will, at this moment, rather work with $\{S^+(u); u \geq 0\}$,

$$\begin{aligned}
 P\{S^+(u_l) - S^+(u_k) \leq \beta_l - \alpha\} &= P\left\{S^+(1) \leq \frac{\beta_l - \alpha}{(u_l - u_k)^2}\right\} \\
 &\approx \left(\frac{\beta_l - \alpha}{(u_l - u_k)^2}\right)^{1/2} \exp\left(-\frac{(u_l - u_k)^2}{2(\beta_l - \alpha)}\right) \\
 &\approx \xi_l^{-1/2} \exp(-\xi_l^2) \approx p_l.
 \end{aligned}$$

This finishes the proof of the theorem for $\{S^+(u); u \geq 0\}$ and, therefore, also for $\{S_1(u); u \geq 0\}$. $\square$

4.4 Proof of Theorem 4.3

As promised in Section 3.3, we now present a direct proof of Theorem 4.3, which, via Lemma 4.3, implies Theorem 3.3. See also (5.3) and Theorem 6.B. First, we need a lemma.

Lemma 4.7. *Let $\xi \in \Xi_2$ and let $\lambda > 1$. Then*

$$\mathcal{J}_d(\xi) < \infty \iff \sum_k \frac{1}{\xi^d(\lambda^k)} < \infty.$$

Proof. We have for large enough K ,

$$\begin{aligned} \int_{\lambda^K}^{\infty} \frac{1}{u \xi^d(u)} du &= \sum_{k=K}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \frac{1}{u \xi^d(u)} du \\ &= \sum_{k=K}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \frac{u^d}{u \xi^d(u) u^d} du \\ &\geq \sum_{k \geq K} \frac{(\lambda^{k+1} - \lambda^k) \lambda^{kd}}{\lambda^{k+1} \xi^d(\lambda^{k+1}) \lambda^{(k+1)d}} \\ &= \frac{\lambda - 1}{\lambda^{d+1} \xi^d(\lambda^{k+1})}, \end{aligned}$$

and, therefore, $\mathcal{J}_d(\xi) < \infty$ implies $\sum_k 1/\xi^d(\lambda^k) < \infty$.

On the other hand,

$$\begin{aligned} \sum_{k=K}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \frac{u^d}{u \xi^d(u) u^d} du &\leq \sum_{k=K}^{\infty} \frac{(\lambda^{k+1} - \lambda^k) \lambda^{(k+1)d}}{\lambda^k \xi^d(\lambda^k) \lambda^{kd}} \\ &= \frac{(\lambda - 1) \lambda^d}{\xi^d(\lambda^k)}, \end{aligned}$$

and, therefore, $\mathcal{J}_d(\xi) = \infty$ implies $\sum_k 1/\xi^d(\lambda^k) = \infty$. $\square$

Proof of Theorem 4.3. Assume that $\mathcal{J}_1(\xi) < \infty$. By Lemma 2.2, $\xi(u) \rightarrow \infty$ as $u \rightarrow \infty$. Denote

$$A_k = \{S^+(u) \geq u^2 \xi^2(u) \text{ for some } u \in [2^k, 2^{k+1})\}.$$

Since both $S^+(u)$ and $u^2 \xi^2(u)$ are nondecreasing, we have

$$\begin{aligned} P(A_k) &\leq P\{S^+(2^{k+1}) \geq 2^{2k} \xi^2(2^k)\} \\ &= P\{S^+(1) \geq \tfrac{1}{4} \xi^2(2^k)\} \\ &\approx \frac{1}{\xi(2^k)}, \end{aligned}$$

which sums by Lemma 4.7, and the Borel–Cantelli lemma finishes the proof of the convergent part.

Now assume that $\mathcal{J}_1(\xi) = \infty$. The case $\xi(u) \not\rightarrow \infty$ as $u \rightarrow \infty$ is obvious, hence assume $\xi(u) \rightarrow \infty$ as $u \rightarrow \infty$. Define

$$A_k = \{S^+(2^k) \geq 2^{2k} \xi^2(2^k)\}$$

and

$$B_k = \{S^+(2^k) - S^+(2^{k-1}) \geq 2^{2k} \xi^2(2^k)\}.$$

Obviously $B_k \subset A_k$. Since B_k , $k = 1, 2, \dots$, are independent, it is sufficient to prove $\sum_k P(B_k) = \infty$ and then the Borel–Cantelli lemma gives $P(B_k \text{ i.o.}) = 1$. Since $S^+(2^k) - S^+(2^{k-1}) \stackrel{\mathcal{D}}{=} S^+(2^{k-1})$, we have

$$\begin{aligned} P(B_k) &= P\{S^+(2^{k-1}) \geq 2^{2k} \xi^2(2^k)\} \\ &= P\{S^+(1) \geq 4 \xi^2(2^k)\} \\ &\approx \frac{1}{\xi(2^k)}, \end{aligned}$$

the sum of which diverges by Lemma 4.7. This finishes the proof of the theorem. $\square$

Chapter 5

Small Values of Brownian Motion

5.1 How Small Are the Small Values?

Let $r \geq 3$. Then with probability one, $\|W_r(t)\|$ tends to infinity as $t \rightarrow \infty$. The rate of escape to infinity is described in the following theorem, which generalizes Theorem 1.E (see also Theorem 6.1).

Theorem 5.1. *Let $\psi \in \Psi_2^2$ and let $r \geq 3$ be an integer. Then*

$$P \left\{ m_r(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \quad (5.1)$$

or, equivalently,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2}m_r(t)) \\ \text{LUC}(t^{-1/2}m_r(t)) \end{cases} \quad \text{according as } \mathcal{J}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 5.1.

$$\liminf_{t \rightarrow \infty} \frac{m_r(t)}{t^{1/2}} \psi(t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. according as } \mathcal{J}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 5.2. *The integral test (5.1) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

We define an inverse of $\mathbf{m}_r(t)$. Put

$$\mathbf{s}_r(u) = \sup\{t > 0; \|W_r(t)\| \leq u\}. \quad (5.2)$$

By Gettoor (1979), the process $\{\mathbf{s}_r(u); u \geq 0\}$ is nondecreasing, continuous in probability, and has independent increments. The density $f(x; u)$ of $\mathbf{s}_r(u)$ is given by

$$f(x; u) = \frac{u^{r-2}}{2^{(r-2)/2} \Gamma((r-2)/2)} \cdot \frac{1}{x^{r/2}} \exp\left(-\frac{u^2}{2x}\right), \quad x > 0, u > 0.$$

The increments are stationary if and only if $r = 3$. The Laplace transform of the increments $\mathbf{s}_r(b) - \mathbf{s}_r(a)$, $0 < a < b < \infty$, is

$$\left(\frac{b}{a}\right)^\nu \frac{K_\nu(b(2s)^{1/2})}{K_\nu(a(2s)^{1/2})}, \quad s > 0,$$

where $\nu = (r-2)/2$, and K_ν is the Bessel function.

Notice that for $r = 3$, Theorem 5.1 exactly corresponds to Theorem 3.3. This is not surprising, since it is clear from (4.1) and the properties of $\{S^+(u); u \geq 0\}$ given in the beginning of Chapter 4 that

$$\{\mathbf{s}_3(u); u \geq 0\} \stackrel{\mathcal{D}}{=} \{S^+(u); u \geq 0\}, \quad (5.3)$$

and, therefore, also

$$\{\mathbf{m}_3(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{M^+(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{L(t); t \geq 0\}. \quad (5.4)$$

Thus, Theorem 3.3 is a special case of Theorem 5.1.

For a generalization of Theorem 1.E to Gaussian fields see Theorem 5.1 and Proposition 5.2 in Khoshnevisan and Shi (1996a), which could be also stated with $\psi \in \Psi_2^2$ instead of only with $\psi \in \Psi_1$.

Proof of Theorem 5.1. The density of $m_r(1)$ is given by (see Gettoor (1979) or Kent (1978))

$$P(m_r(1) \in dx) = \frac{2}{\Gamma((r-2)/2)2^{(r-2)/2}} x^{r-3} \exp\left(-\frac{x^2}{2}\right) dx, \quad x > 0. \quad (5.5)$$

Assume that $\psi \in \Psi_2^2$ and $\mathcal{J}_{r-2}(\psi) < \infty$. Then $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Take $\lambda > 1$ and denote

$$A_k = \left\{ m_r(t) \leq \frac{t^{1/2}}{\psi(t)} \text{ for some } t \in [\lambda^k, \lambda^{k+1}) \right\}.$$

Then

$$\begin{aligned} P(A_k) &\leq P\left\{ m_r(\lambda^k) \leq \frac{\lambda^{(k+1)/2}}{\psi(\lambda^{k+1})} \right\} \\ &= P\left\{ m_r(1) \leq \frac{\lambda^{1/2}}{\psi(\lambda^{k+1})} \right\} \\ &= \int_0^{\lambda^{1/2}/\psi(\lambda^{k+1})} \frac{2}{\Gamma((r-2)/2)2^{(r-2)/2}} x^{r-3} \exp\left(-\frac{x^2}{2}\right) dx \\ &\approx \frac{1}{\psi^{r-2}(\lambda^{k+1})}, \end{aligned}$$

which sums by Lemma 2.9. By the Borel–Cantelli lemma, $P(A_k \text{ i.o.}) = 0$.

Now assume that $\mathcal{J}_{r-2}(\psi) = \infty$. The case $\psi(t) \nrightarrow \infty$ as $t \rightarrow \infty$ is obvious, so assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Take $\lambda > 1$ and now put

$$A_k = \left\{ \|W_r(t)\| \leq \frac{\lambda^{k/2}}{\psi(\lambda^k)} \text{ for some } t \in [\lambda^k, \lambda^{k+1}) \right\}.$$

Clearly, the event A_k implies the event

$$\left\{ \|W_r(t)\| \leq \frac{t^{1/2}}{\psi(t)} \text{ for some } t \in [\lambda^k, \lambda^{k+1}) \right\}.$$

By Proposition 5.2 of Khoshnevisan and Shi (1996a),

$$P(A_k) \geq C_1 \frac{1}{\psi^{r-2}(\lambda^k)}$$

for some constant $C_1 > 0$. By Lemma 2.9, $\sum_k P(A_k) = \infty$.

Consider now $l > k + 1$. Then

$$\begin{aligned} P(A_k \cap A_l) &\leq P \left\{ A_k, \|W_r(t) - W_r(\lambda^{k+1}) + W_r(\lambda^{k+1})\| \leq \frac{\lambda^{l/2}}{\psi(\lambda^l)} \right. \\ &\quad \left. \text{for some } t \in [\lambda^l, \lambda^{l+1}) \right\} \\ &\leq P(A_k) \sup_{a \in \mathbb{R}^r} P \left\{ \|W_r(t) - W_r(\lambda^{k+1}) - a\| \leq \frac{\lambda^{l/2}}{\psi(\lambda^l)} \text{ for some } t \in [\lambda^l, \lambda^{l+1}) \right\} \\ &\leq C_2 P(A_k) P(A_l), \end{aligned}$$

for some $C_2 > 0$, where the last inequality follows from Proposition 5.2 of Khoshnevisan and Shi (1996a). It is now straightforward to verify that

$$\liminf_{N \rightarrow \infty} \frac{\sum_{k=1}^N \sum_{l=1}^N P(A_k \cap A_l)}{\left(\sum_{k=1}^N P(A_k) \right)^2} \leq C_2,$$

and, therefore, by the Borel–Cantelli lemma, $P(A_k \text{ i.o.}) > 0$, and, by the zero-one law, $P(A_k \text{ i.o.}) = 1$. $\square$

For $r = 2$ we will generalize Theorem 1.F. We could also remove the unnecessary monotonicity condition from an analogous Theorem 5.2 for Gaussian fields in Khoshnevisan and Shi (1996b). We will write the integral test in a slightly different form (cf. (1.4)) than in Theorem 1.F.

Theorem 5.2. *Let $\psi \circ \exp \in \Psi_3^1$. Then*

$$P \{ m_2(t) \leq \exp(-\psi(t) \log t) \text{ i.o. as } t \rightarrow \infty \} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (5.6)$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 5.3. *The integral test (5.6) remains true for $\psi \circ \exp \in \Psi_2^\alpha$ and $\psi \circ \exp \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 5.1. The above theorem implies Theorem 1.F by putting $g(t) = t^{1/2+\psi(t)}$.

Proof. Define for $0 < s_1 < s_2$ and $x > 0$,

$$G(s_1, s_2; x) = P \left\{ \inf_{s_1 \leq t \leq s_2} \|W_2(t)\| < x \mid \|W_2(0)\| = 0 \right\}.$$

According to Spitzer (1958),

$$G(s_1, s_2; x) \approx \frac{\log s_2 - \log s_1}{\log x^{-2}}$$

for small enough x , and by Brownian scaling, $G(s_1, s_2; x) = G(c^2 s_1, c^2 s_2; cx)$, $c > 0$.

Assume that $\mathcal{J}_0(\psi) < \infty$. Then $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Take $\lambda > 1$ and denote $t_k = \exp(\lambda^k)$, $\psi_k = \psi(t_k)$, and $f_k = \exp(-\psi_k \log t_k)$. Define

$$A_k = \{ \|W_2(t)\| \leq \exp(-\psi(t) \log t) \text{ for some } t \in [t_k, t_{k+1}) \}.$$

Then

$$\begin{aligned} P(A_k) &\leq P \{ \|W_2(t)\| \leq f_k \text{ for some } t \in [t_k, t_{k+1}) \} \\ &= G(t_k, t_{k+1}; f_k) = G \left(1, \frac{t_{k+1}}{t_k}; \frac{f_k}{t_k^{1/2}} \right) \\ &\approx \frac{\log t_{k+1} - \log t_k}{\log t_k^{1+2\psi_k}} = \frac{\lambda^{k+1} - \lambda^k}{\lambda^k(1 + 2\psi_k)} \\ &= \frac{\lambda - 1}{1 + 2\psi_k} \approx \frac{1}{\psi_k}, \end{aligned}$$

which sums by (1.4) and Lemma 2.9, and the Borel–Cantelli lemma gives the convergent part of the theorem.

Now assume that $\mathcal{J}_0(\psi) = \infty$. The case $\psi(t) \nrightarrow$ as $t \rightarrow \infty$ is obvious and, therefore, assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Take λ , t_k , ψ_k , and f_k as above and now put

$$A_k = \{\|W_2(t)\| \leq f_{k+1} \text{ for some } t \in [t_k, t_{k+1})\}.$$

Clearly, the event A_k implies the event

$$\{\|W_2(t)\| \leq \exp(-\psi(t) \log t) \text{ for some } t \in [t_k, t_{k+1})\}.$$

Now

$$\begin{aligned} P(A_k) &= G(t_k, t_{k+1}; f_{k+1}) = G\left(1, \frac{t_{k+1}}{t_k}; \frac{f_{k+1}}{t_k^{1/2}}\right) \\ &\approx \frac{\log t_{k+1} - \log t_k}{\log(t_k t_{k+1}^{2\psi_{k+1}})} = \frac{\lambda^{k+1} - \lambda^k}{\lambda^k + 2\lambda^{k+1}\psi_{k+1}} \\ &= \frac{\lambda - 1}{1 + 2\lambda\psi_{k+1}} \approx \frac{1}{\psi_{k+1}}. \end{aligned}$$

By (1.4) and Lemma 2.9, $\sum_k P(A_k) = \infty$.

Consider now $l > k + 1$. Then

$$\begin{aligned} P(A_k \cap A_l) &\leq P\{A_k, \|W_2(t) - W_2(t_{k+1}) + W_2(t_{k+1})\| \leq f_{l+1} \text{ for some } t \in [t_l, t_{l+1})\} \\ &\leq P(A_k) \sup_{a \in \mathbb{Z}^2} P\{\|W_2(t) - W_2(t_{k+1}) - a\| \leq f_{l+1} \text{ for some } t \in [t_l, t_{l+1})\} \\ &= P(A_k) P\{\|W_2(t) - W_2(t_{k+1})\| \leq f_{l+1} \text{ for some } t \in [t_l, t_{l+1})\} \\ &= P(A_k) P\{\|W_2(t)\| \leq f_{l+1} \text{ for some } t \in [t_l - t_{k+1}, t_{l+1} - t_{k+1})\} \\ &\leq C P(A_k) P(A_l), \end{aligned}$$

where C is a positive constant. The last inequality follows from

$$\begin{aligned}
 & P\{\|W_2(t)\| \leq f_{l+1} \text{ for some } t \in [t_l - t_{k+1}, t_{l+1} - t_{k+1}]\} \\
 &= G(t_l - t_{k+1}, t_{l+1} - t_{k+1}; f_{l+1}) \\
 &\approx \frac{\log(t_{l+1} - t_{k+1}) - \log(t_l - t_{k+1})}{\log(t_l - t_{k+1}) + 2\psi_{l+1} \log t_{l+1}} \\
 &= \frac{\lambda^{l+1} - \lambda^l + \log\left(1 - \frac{t_{k+1}}{t_{l+1}}\right) - \log\left(1 - \frac{t_{k+1}}{t_l}\right)}{\lambda^l + \log\left(1 - \frac{t_{k+1}}{t_l}\right) + 2\lambda^{l+1}\psi_{l+1}} \\
 &\leq \frac{\lambda^{l+1} - \lambda^l + 0 - \log\left(1 - \frac{t_{k+1}}{t_{k+2}}\right)}{\lambda^l + \log\left(1 - \frac{t_{k+1}}{t_{k+2}}\right) + 2\lambda^{l+1}\psi_{l+1}} \\
 &\approx \frac{1}{\psi_{l+1}} \approx P(A_l).
 \end{aligned}$$

It is now easy to check that

$$\liminf_{N \rightarrow \infty} \frac{\sum_{k=1}^N \sum_{l=1}^N P(A_k \cap A_l)}{\left(\sum_{k=1}^N P(A_k)\right)^2} \leq C,$$

and, therefore, by the Borel–Cantelli lemma, $P(A_k \text{ i.o.}) > 0$, and, by the zero-one law, $P(A_k \text{ i.o.}) = 1$. □

5.2 How Big Are the Small Values?

In the preceding section we described the lower classes of $\{m_r(t); t \geq 0\}$ for $r = 2$ and $r \geq 3$. As far as the upper classes given in Theorem 1.G are concerned, again, we were able to enlarge the class of weight functions.

Theorem 5.3. *Let $\psi \in \Psi_3^2$ and $r \geq 3$ be an integer. Then*

$$P \left\{ m_r(t) \geq \sqrt{2} t^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_{r-2}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (5.7)$$

By Lemmas 2.15 and 2.16 we immediately have

Corollary 5.4. *The integral test (5.7) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Notice the correspondence of Theorem 3.1 for $\{M_{r-2}(t); t \geq 0\}$ and Theorem 5.3 for $\{m_r(t); t \geq 0\}$, $r \geq 3$. This is, of course, obvious for $r = 3$ by (5.4).

To prove Theorem 5.3, we will use the same approach as for Theorem 3.1. By Lemma 4.1, Theorem 5.3 is equivalent to

Theorem 5.4. *Let $\xi \in \Xi_3$ and $r \geq 3$ be an integer. Then*

$$P \left\{ s_r(u) \leq \frac{1}{2} \cdot \frac{u^2}{\xi^2(u)} \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_{r-2}(\xi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (5.8)$$

Comparing processes $\{S_{r-2}(u); u \geq 0\}$ and $\{s_r(u); u \geq 0\}$, we see that both have independent increments, the same scaling property, ie., $u^{-2}S_r(u) \stackrel{\mathcal{D}}{=} S_r(1)$ and $u^{-2}s_r(u) \stackrel{\mathcal{D}}{=} s_r(1)$, and the same asymptotic distribution, ie.,

$$P\{s_r(1) \leq x\} \approx (\min(1, x))^{2-r/2} \exp\left(-\frac{1}{2x}\right) \approx P\{S_{r-2}(1) \leq x\}.$$

No other properties of $\{S_r(u); u \geq 0\}$ were used in proving Lemmas 4.4–4.6 and Theorem 4.1 (actually, Lemmas 4.4–4.6 were adapted from Khoshnevisan, Lewis, and Shi (1996), were they were proved for $\{s_r(u); u \geq 0\}$). Therefore, the proof of Theorem 5.4 is identical to the proof of Theorem 4.1.

Chapter 6

Stable Processes and Subordinators

Most of the results presented for Brownian motion were attempted for stable processes. We will give very brief summary and point out some interesting connections. For more results and proofs see expository works of Taylor (1973), Fristedt (1974), and Mijneer (1975), and papers of Breiman (1968), Pruitt and Taylor (1983), and Pruitt (1989), (1990) and (1991).

Let U be a random variable with a strictly stable distribution of index α , $0 < \alpha \leq 2$, and $|\beta| \leq 1$, ie., its characteristic function $f(t)$ is given by

$$\log f(t) = \begin{cases} -|t|^\alpha [1 + i\beta \operatorname{sgn}(t) \tan(\pi\alpha/2)], & \alpha \neq 1, \\ -|t| [1 - i\beta \operatorname{sgn}(t)(2/\pi) \log|t|], & \alpha = 1. \end{cases}$$

The case $\beta = 0$ is called *symmetric*, the case $|\beta| = 1$ is called *completely asymmetric*. Hence, the characteristic function of a strictly stable symmetric distribution is simply $f(t) = \exp(-|t|^\alpha)$.

Definition 6.1. $\{X(t); t \geq 0\}$ is called a (one-dimensional) *strictly stable process* on a probability space $(\Omega, \mathcal{F}, P)$ if

- i) $X : [0, \infty) \times \Omega \rightarrow \mathbb{R}$;
- ii) $X(0, \omega) = 0$ for each ω ;
- iii) $X(t, \cdot)$ is $\mathcal{F}$ -measurable for each t ;
- iv) the increments are stationary, independent, with distribution $X(t) \stackrel{\mathcal{D}}{=} t^{1/\alpha}U$, $\alpha \neq 1$, or $X(t) \stackrel{\mathcal{D}}{=} tU + (2/\pi)\beta \log t$, $\alpha = 1$.
- v) $X(\cdot, \omega)$ is right-continuous and has finite left-hand side limits for almost all ω .

$\{X_r(t); t \geq 0\}$ is a standard r -dimensional symmetric strictly stable process in $\mathbb{R}^r$ of index α if its components are independent one-dimensional symmetric strictly stable processes of index α .

Standard examples of symmetric strictly stable processes are (after some standardization)

- Brownian motion $\{W_r(t); t \geq 0\}$ ($\alpha = 2, r \geq 1$);
- First passage time process $\{S^+(u); u \geq 0\}$ ($\alpha = 1/2, r = 1$);
- Cauchy process $\{C(t); t \geq 0\}$ ($\alpha = 1, r = 1$).

The distribution of the Cauchy process is given by

$$P\{C(1) \leq x\} = \frac{1}{2} + \frac{1}{\pi} \arctan x.$$

The Cauchy process can be represented by two independent Brownian motions. Take $\{W_2(t) = (W^{(1)}(t), W^{(2)}(t)); t \geq 0\}$ and let $\{S^+(u); u \geq 0\}$ be the first

passage time process corresponding to $W^{(1)}(t)$. Then

$$C(t) = W^{(2)}(S^+(t)), \quad t \geq 0, \quad (6.1)$$

is a standard Cauchy process.

Put $R_r(t) = \|X_r(t)\|$, $t \geq 0$, where $\{X_r(t); t \geq 0\}$ is an r -dimensional symmetric strictly stable process of index α . Then $\{R_r(t); t \geq 0\}$ is (cf. (1.10)–(1.14))

- point recurrent if $\alpha > r$;
- neighborhood recurrent but not point recurrent if $\alpha = r$;
- transient if $\alpha < r$.

The first passage time process $S^+(u)$ is, in addition to being a strictly stable process of index $1/2$, a *subordinator* (see Chapter 6 of Karatzas and Shreve (1988)).

Definition 6.2. A real valued process $\{N(t); t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a *subordinator* if it has stationary, independent increments, and if almost every path of N is nondecreasing, is right-continuous, and satisfies $N(0) = 0$.

The moment generating function of a subordinator is given by

$$E \exp(-\lambda N(t)) = \exp \left\{ -t \left[b\lambda + \int_0^\infty (1 - \exp(-\lambda x)) \mu(dx) \right] \right\}, \quad t \geq 0, \lambda \geq 0, \quad (6.2)$$

where $b \geq 0$ is a constant and μ is a σ -finite measure on $(0, \infty)$. For $S^+(t)$, we have

$$\mu(dx) = \frac{dx}{\sqrt{2\pi x^3}}, \quad x > 0,$$

$$E \exp(-\lambda S^+(t)) = \exp(-t\sqrt{2\lambda}), \quad 0 \leq \lambda, t < \infty.$$

Now we quote a few theorems from Fristedt (1974).

Theorem 6.A. *Let $\{X(t); t \geq 0\}$ be a strictly stable subordinator with*

$$\mathbb{E} \exp(-\lambda X(t)) = \exp(-tb\lambda^\alpha), \quad \alpha < 1.$$

Let $\xi \in \Xi_1$. Then

$$\mathbb{P} \left\{ X(t) \leq \alpha(1-\alpha)^{(1-\alpha)/\alpha} b^{1/\alpha} \frac{t^{1/\alpha}}{\xi^{2(1-\alpha)/\alpha}(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases}$$

$$\text{according as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

The above theorem (Theorem 6.20 in Fristedt (1974)) applied to $S^+(t)$, i.e., with $\alpha = 1/2$, $b = \sqrt{2}$, gives our Theorem 4.1 with $\xi \in \Xi_1$.

Theorem 11.2 of Fristedt (1974) quoted below says that for strictly stable processes having positive jumps the limit supremum is either 0 or ∞ .

Theorem 6.B. *Let $\{X(t); t \geq 0\}$ be a strictly stable process in $\mathbb{R}$ of index α with Lévy measure $\mu(0, \infty) > 0$. Let $\xi \in \Xi_2$. Then*

$$\limsup_{t \rightarrow \infty} \frac{X(t)}{t^{1/\alpha} \xi^{1/\alpha}(t)} = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s. according as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

The above theorem applied to $\{S^+(t); t \geq 0\}$ gives exactly our Theorem 4.3. The proof is not much more difficult than the proof of Theorem 4.3.

The Cauchy process also satisfies the assumptions of Theorem 6.B, hence we have

$$\limsup_{t \rightarrow \infty} \frac{C(t)}{t\xi(t)} = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s. according as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Theorem 11.5 of Fristedt (1974) contains as special cases the results of Dvoretzky and Erdős (1951) (our Theorem 1.E), Spitzer (1958) (our Theorem 1.F), and an analogous result for the Cauchy process. We will remove the unnecessary monotonicity assumption $\psi \in \Psi_1$ from this theorem.

Theorem 6.1. *Let $\{X_r(t); t \geq 0\}$ be a strictly stable process in $\mathbb{R}^r$ of index $\alpha \leq r$ having no projection which is a subordinator. First, let $\alpha < r$ and $\psi \in \Psi_2^\alpha$. Then*

$$\liminf_{t \rightarrow \infty} \frac{\|X_r(t)\|}{t^{1/\alpha}} \psi(t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. according as } \mathcal{J}_{r-\alpha}(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (6.3)$$

Second, let $\alpha = r$ and let $\psi \circ \exp \in \Psi_3^1$. Then

$$\liminf_{t \rightarrow \infty} \|X_r(t)\| \exp(\psi(t) \log t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (6.4)$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 6.1. *The integral test (6.3) remains true for $\psi \in \Psi_2^\beta$ and $\psi \in \Psi_3^\beta$ for every $\beta > 0$. The integral test (6.4) remains true for $\psi \circ \exp \in \Psi_2^\beta$ and $\psi \circ \exp \in \Psi_3^\beta$ for every $\beta > 0$.*

Theorem 6.1 contains as special cases Theorem 5.1, Theorem 5.2, and for Cauchy process it gives

$$\liminf_{t \rightarrow \infty} |C(t)| \exp(\psi(t) \log t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Define

$$C^+(t) = \sup_{0 \leq s \leq t} C(s),$$

$$C^*(t) = \sup_{0 \leq s \leq t} |C(s)|.$$

It would be also interesting to study the lower classes of $\{C^+(t); t \geq 0\}$ and of $\{C^*(t); t \geq 0\}$ as we studied the lower classes of $\{M^+(t); t \geq 0\}$ and $\{M_1(t); t \geq 0\}$. As far as we are aware, the answers to these problems are not known, except the following law of the iterated logarithm (see Fristedt (1974), Theorem 11.6),

$$\liminf_{t \rightarrow \infty} \frac{C^*(t)}{t} \log \log t = c \quad \text{a.s.},$$

where the value of c is not known explicitly.

Proof of Theorem 6.1. The proof is analogous to the proofs of Theorems 5.1 and 5.2. so we only give a short sketch. Take $\lambda > 1$ and put

$$A_k = \{\|X_r(t)\| \leq h(t) \text{ for some } t \in [\lambda^k, \lambda^{k+1})\},$$

where $h(t) = t^{1/\alpha}/\psi(t)$, $\alpha < r$, or $h(t) = \exp(-\psi(t) \log t)$, $\alpha = r$. Then $\sum_k P(A_k)$ converges or diverges according as the appropriate integral converges or diverges. For $l > k + 1$ it is possible to show that

$$P(A_k \cap A_l) \leq C P(A_k) P(A_l),$$

for some positive constant C . An application of the Borel–Cantelli lemma finishes the proof. $\square$

A very general result for subordinators was obtained by Pruitt (1991), namely,

Theorem 6.C. *Assume that $\{X(t); t \geq 0\}$ is a nongenerate subordinator which satisfies*

$$\liminf_{x \rightarrow \infty} \frac{\int_{y \leq x} y^2 \mu(dy)}{x^2 \mu(x, \infty)} > 0. \quad (6.5)$$

Let $h(t)$ be a function defined for small t such that $h(t)/t$ is nondecreasing. Then

$$P\{X(t) \leq h(t) \text{ i.o. as } t \rightarrow 0+\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \int_0^1 \frac{1}{t} \log \left(\frac{3}{p_t} \right) p_t dt \begin{cases} < \infty \\ = \infty. \end{cases}$$

where $p_t = P\{X(t) \leq h(t)\}$. The convergent part is valid without the assumption (6.5).

For stable subordinators of index α , the ratio (6.5) is equal to $\alpha/(2-\alpha)$ for every x . We apply Theorem 6.C to $\{S^+(u); u \geq 0\}$ and $h(u) = u^2/(2\xi^2(u))$, $\xi \in \Xi_4$. Since $\xi \in \Xi_4$,

$$\frac{h(u)}{u} = \frac{u}{2\xi^2(u)}$$

is nondecreasing, and notice that

$$\begin{aligned} p_u &= P \left\{ S^+(u) \leq \frac{u^2}{2\xi^2(u)} \right\} \\ &= 2[1 - \Phi(\sqrt{2}\xi(u))] \\ &\approx \frac{1}{\max(1, \xi(u))} \exp(-\xi^2(u)). \end{aligned}$$

Moreover,

$$\log \frac{1}{p_u} \sim \xi^2(u).$$

Theorem 6.C gives

$$P \left\{ S^+(u) \leq \frac{u^2}{2\xi^2(u)} \text{ i.o. as } u \rightarrow 0+ \right\} = \begin{cases} 0 \\ 1 \end{cases}$$

$$\text{according as } \int_{0+} \frac{\max(1, \xi(u))}{u} \exp(-\xi^2(u)) \, du \begin{cases} < \infty \\ = \infty. \end{cases}$$

Transforming the statement from a neighborhood of zero to a neighborhood of infinity, we obtain exactly the statement of Theorem 4.1 with $\xi \in \Xi_4$.

Chapter 7

Occupation Time and Excursions

In this chapter we will study positive and negative excursions of Brownian motion. Let $\{W(t); t \geq 0\}$ be a one-dimensional Brownian motion. Define the total time $\Gamma_+(t)$ of positive excursions up to time t ,

$$\Gamma_+(t) = \int_0^t \mathbf{1}\{W(s) \geq 0\} ds, \quad t \geq 0,$$

and the total time $\Gamma_-(t)$ of negative excursions up to time t ,

$$\Gamma_-(t) = \int_0^t \mathbf{1}\{W(s) < 0\} ds = t - \Gamma_+(t), \quad t \geq 0.$$

The distribution of $\Gamma_+(t)$ is well-known to be

$$\mathbf{P}\{\Gamma_+(t) \leq x\} = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}, \quad 0 \leq x \leq t.$$

For more details about the properties of the processes $\{\Gamma_+(t); t \geq 0\}$ and $\{\Gamma_-(t); t \geq 0\}$ and their relationship see Chapter 6 of Karatzas and Shreve (1988). We will study the upper and lower classes of $\{\Gamma_+(t); t \geq 0\}$. We have

Theorem 7.1. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ \Gamma_+(t) \leq \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (7.1)$$

For $\psi \in \Psi_1$, the above theorem was proved by Chung and Erdős (1952) and Hobson (1994). For theorems about occupation time of more general sets and in more than one dimension see Bertoin (1995) and Meyre and Werner (1995).

Since

$$\{\Gamma_+(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{\Gamma_-(t); t \geq 0\} = \{t - \Gamma_+(t); t \geq 0\},$$

we can replace $\Gamma_+(t)$ in Theorem 7.1 by $\Gamma_-(t)$ and we immediately obtain the following characterization of the upper classes of $\{\Gamma_+(t); t \geq 0\}$.

Theorem 7.2. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ \Gamma_+(t) \geq t - \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (7.2)$$

We also want to describe the upper and lower classes of the longest excursion up to time t (and, therefore, also of the longest flat interval of M^+ up to time t). Define for all $t \geq 0$,

$$z_-(t) = \sup\{u; 0 \leq u < t, W(u) = 0\},$$

$$z_+(t) = \inf\{u; u > t, W(u) = 0\},$$

$$z(t) = t - z_-(t),$$

$$z^*(t) = \sup\{z(u); 0 \leq u < t\}.$$

Thus $z_-(t)$ is the last zero of W before t , $z_+(t)$ is the first zero of W after t , $z(t)$ is the length of the excursion starting at $z_-(t)$ up to time t , and $z^*(t)$ is the length of the longest excursion up to time t .

Theorem 7.3. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ z(t) \geq t - \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (7.3)$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 7.1. *The integral tests (7.1), (7.2), and (7.3) remain true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Theorem 7.4. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ z^*(t) \leq \beta \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_2(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \quad (7.4)$$

where $\beta \doteq 0.85403$ is the root of the equation

$$\sum_{k=1}^{\infty} \frac{\beta^k}{k!(2k-1)} = 1.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 7.2. *The integral test (7.4) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

For $\psi \in \Psi_1$, Theorem 7.3 was proved in Chung and Erdős (1952), and Theorem 7.4 in Csáki, Erdős, and Révész (1985).

Proof of Theorems 7.1, 7.2, and 7.3. We have already explained how Theorem 7.2 can be deduced from Theorem 7.1 and vice versa. Since $z(t) \leq \max[\Gamma_+(t), \Gamma_-(t)]$, the convergent part of Theorem 7.2 implies the convergent part of Theorem 7.3 and the divergent part of Theorem 7.3 implies the divergent part of Theorem 7.2.

First we prove the convergent part of Theorem 7.2. Assume that $\mathcal{J}_1(\psi) < \infty$. Take $\lambda > 1$. Put

$$A_k = \left\{ \Gamma_+(t) \geq t - \frac{t}{\psi^2(t)} \text{ for some } t \in [\lambda^k, \lambda^{k+1}) \right\}.$$

Now

$$\begin{aligned} P(A_k) &= P \left\{ \Gamma_+(t) \leq \frac{t}{\psi^2(t)} \text{ for some } t \in [\lambda^k, \lambda^{k+1}) \right\} \\ &\leq P \left\{ \Gamma_+(\lambda^k) \leq \frac{\lambda^{k+1}}{\psi^2(\lambda^{k+1})} \right\} \\ &= \frac{2}{\pi} \arcsin \sqrt{\frac{\lambda}{\psi^2(\lambda^{k+1})}} \\ &\approx \frac{1}{\psi(\lambda^{k+1})}, \end{aligned}$$

which sums by Lemma 2.9, and an application of the Borel–Cantelli Lemma finishes the proof of the convergent part.

Now we prove the divergent part of Theorem 7.3. By Problem 3, p. 28, and Problem 1, p. 78, of Itô and McKean (1965), we have

$$P\{z_-(t) \leq x\} = \frac{1}{\pi} \int_0^{x/t} \frac{1}{\sqrt{z}\sqrt{1-z}} dz = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}, \quad 0 \leq x \leq t, \quad (7.5)$$

$$P\{z_+(t) \in db \mid z_-(t) = a\} = \frac{1}{2} \sqrt{\frac{t-a}{(b-a)^3}} db, \quad b \geq t, \quad (7.6)$$

$$P\{z(t) > b\} = P\{z_-(t) < t-b\} = \frac{2}{\pi} \arccos \sqrt{\frac{b}{t}}, \quad 0 \leq b \leq t, \quad (7.7)$$

$$P\{z(t) > b \mid z(s) = a\} = \int_0^{(t-s-b) \vee 0} \frac{\sqrt{a}}{2(u+a)^{3/2}} \frac{2}{\pi} \arccos \sqrt{\frac{b}{t-s-u}} du$$

$$+ \mathbf{1}\{t - s > b - a\} \sqrt{\frac{a}{t - s + a}}, \quad a \leq s \leq t, \quad b \leq t. \quad (7.8)$$

The last probability can be derived by conditioning on $z_+(s)$. If $z_+(s) \geq t$, then $z(t) = t - s + z(s)$, and if $z_+(s) < t$, then $z(t) \stackrel{\mathcal{D}}{=} z(t - z_+(s))$.

Assume that $\mathcal{J}_1(\psi) = \infty$. The case $\psi(t) \not\rightarrow \infty$ as $t \rightarrow \infty$ is obvious. Therefore assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and we may assume without loss of generality that $2 \leq \psi(t) \leq \log^2 t$. Take $\lambda > 1$ and write ψ_k for $\psi(\lambda^k)$. Denote

$$A_k = \left\{ z(\lambda^k) \geq \lambda^k \left(1 - \frac{1}{\psi_k^2} \right) \right\}.$$

Take $\varepsilon > 0$ and k large enough. Then

$$\begin{aligned} P(A_k) &= \frac{2}{\pi} \arccos \sqrt{\frac{\lambda^k(1 - \psi_k^{-2})}{\lambda^k}} \\ &= \frac{2}{\pi} \arcsin \sqrt{\frac{1}{\psi_k^2}} \geq \frac{2}{\pi} \cdot \frac{1}{\psi_k}. \end{aligned}$$

By Lemma 2.9, $\sum_k P(A_k) = \infty$.

Consider $l > k$. Then

$$\begin{aligned} P(A_l \cap A_k) &= \int_{\lambda^k(1 - \psi_k^{-2})}^{\lambda^k} P \left\{ z(\lambda^l) \geq \lambda^l \left(1 - \frac{1}{\psi_l^2} \right) \mid z(\lambda^k) = z \right\} P\{z(\lambda^k) \in dz\} \\ &= \int_{z=\lambda^k(1 - \psi_k^{-2})}^{\lambda^k} \left[\int_{u=0}^{(\lambda^l \psi_l^{-2} - \lambda^k) \vee 0} \frac{1}{2} \frac{\sqrt{z}}{(u + z)^{3/2}} \frac{2}{\pi} \arccos \sqrt{\frac{\lambda^l(1 - \psi_l^{-2})}{\lambda^l - \lambda^k - u}} du \right. \\ &\quad \left. + \mathbf{1}\{\lambda^l - \lambda^k > \lambda^l(1 - \psi_l^{-2}) - z\} \frac{\sqrt{z}}{\sqrt{\lambda^l - \lambda^k + z}} \right] \frac{1}{\pi \sqrt{z} \sqrt{\lambda^k - z}} dz. \end{aligned}$$

Now we split the expression into two parts. The second part is

$$\begin{aligned} &\int_{z=\lambda^k(1 - \psi_k^{-2})}^{\lambda^k} \mathbf{1}\{\lambda^l - \lambda^k > \lambda^l(1 - \psi_l^{-2}) - z\} \frac{\sqrt{z}}{\sqrt{\lambda^l - \lambda^k + z}} \frac{1}{\pi \sqrt{z} \sqrt{\lambda^k - z}} dz \\ &= \frac{1}{\pi} \int_{w=0}^{\lambda^k \psi_k^{-2}} \frac{1}{\sqrt{w} \sqrt{\lambda^l - w}} dw = \frac{2}{\pi} \arcsin \sqrt{\frac{\lambda^k}{\lambda^l \psi_k^2}} \\ &\leq \frac{2}{\pi} (1 + \varepsilon) \frac{1}{\psi_k \lambda^{(l-k)/2}}. \end{aligned} \quad (7.9)$$

For the first part we will use integration by parts and omit the second term.

$$\begin{aligned}
 & \int_{w=0}^{\lambda^k \psi_k^{-2}} \left[\int_{u=0}^{(\lambda^l \psi_l^{-2} - \lambda^k) \vee 0} \frac{1}{\pi^2} \arccos \left(\frac{\lambda^l (1 - \psi_l^{-2})}{\lambda^l - \lambda^k - u} \right)^{1/2} \frac{1}{(u + \lambda^k - w)^{3/2}} du \right] \frac{1}{\sqrt{w}} dw \\
 &= \int_{w=0}^{\lambda^k \psi_k^{-2}} \frac{1}{\pi^2 \sqrt{w}} \left\{ \left[-\frac{2}{\sqrt{u + \lambda^k - w}} \arccos \sqrt{\frac{\lambda^l (1 - \psi_l^{-2})}{\lambda^l - \lambda^k - u}} \right]_{u=0}^{(\lambda^l \psi_l^{-2} - \lambda^k) \vee 0} \right. \\
 &\quad \left. - \int_{u=0}^{(\lambda^l \psi_l^{-2} - \lambda^k) \vee 0} \sqrt{\frac{\lambda^l (1 - \psi_l^{-2})}{\lambda^l \psi_l^{-2} - \lambda^k - u}} \frac{1}{(\lambda^l - \lambda^k - u)} \frac{1}{\sqrt{u + \lambda^k - w}} du \right\} dw \\
 &\leq \int_{w=0}^{\lambda^k \psi_k^{-2}} \frac{2}{\pi^2} \frac{1}{\sqrt{w} \sqrt{\lambda^k - w}} \arccos \sqrt{\frac{\lambda^l (1 - \psi_l^{-2})}{\lambda^l - \lambda^k}} dw \\
 &= \frac{4}{\pi^2} \arcsin \frac{1}{\psi_k} \arcsin \frac{1}{\psi_l} \sqrt{\frac{\lambda^l - \psi_l^2 \lambda^k}{\lambda^l - \lambda^k}} \\
 &\leq \frac{4}{\pi^2} (1 + \varepsilon)^2 \frac{1}{\psi_k \psi_l}
 \end{aligned} \tag{7.10}$$

in the case $\lambda^l \psi_l^{-2} - \lambda^k > 0$, zero otherwise.

Hence, for large enough k and all $l > k$ we have

$$P(A_k \cap A_l) \leq (1 + \varepsilon) P(A_k) \lambda^{-(l-k)/2} + (1 + \varepsilon)^2 P(A_k) P(A_l),$$

which easily implies

$$\liminf_{N \rightarrow \infty} \frac{\sum_{k=1}^N \sum_{l=1}^N P(A_k \cap A_l)}{\left[\sum_{k=1}^N P(A_k) \right]^2} = 1.$$

An application of the Borel–Cantelli lemma finishes the proof. $\square$

Now we will prove Theorem 7.4. The problem is a bit more difficult than the other theorems in this chapter but most of the work has been already done by Csáki, Erdős, and Révész (1985) and in our Theorem 3.2.

Lemma 7.1.

$$P\{z^*(t) \leq tx\} = P\{z^*(1) \leq x\} \approx \exp\left(-\frac{\beta}{x}\right),$$

$0 < x < \infty$, where β is given in Theorem 7.4.

Proof. Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables with

$$P\{X_1 = 1\} = P\{X_1 = -1\} = \frac{1}{2}$$

and let $S_0 = 0$, $S_n = X_1 + \dots + X_n$, $n = 1, 2, \dots$, be a corresponding random walk.

Let ϱ_i , $i = 0, 1, \dots$, be zeros of $\{S_n; n = 0, 1, \dots\}$, ie., $\varrho_0 = 0$ and

$$\varrho_i = \inf\{k: k > \varrho_{i-1}, S_k = 0\}, \quad i = 1, 2, \dots$$

Let $z_{rw}^*(n)$ denote the length of the longest excursion of the random walk $\{S_k; k = 0, \dots, n\}$, ie.,

$$z_{rw}^*(n) = \max\{\varrho_1, \varrho_2 - \varrho_1, \dots, n - \varrho_{k^*}\},$$

where ϱ_{k^*} is the last zero smaller than n . By Lemma 3 of Csáki, Erdős, and Révész (1985),

$$P\{z_{rw}^*(2n) \leq 2a\} \approx \exp\left(-\beta \frac{n}{a}\right)$$

provided that $n \leq a^{3/2}$.

Take $0 < x < \infty$. Then $n \leq (nx)^{3/2}$ for large enough n . Now

$$\begin{aligned} P\{z^*(1) \leq x\} &= \lim_{n \rightarrow \infty} P\left\{\frac{z_{rw}^*(2n)}{2n} \leq x\right\} \\ &= \lim_{n \rightarrow \infty} P\{z_{rw}^*(2n) \leq 2nx\} \\ &\approx \exp\left(-\frac{\beta}{x}\right). \end{aligned}$$

The rest of the lemma follows by Brownian scaling. □

Proof of Theorem 7.4. By Lemmas 2.15 and 2.16, it is sufficient to prove the theorem for $\psi \in \Psi_4^2$.

Assume that $J_2(\psi) < \infty$. Then $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10. Put

$$A_k = \left\{ z^*(t) \leq \beta \frac{t}{\psi^2(t)} \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Then, since both $z^*(t)$ and $t/\psi^2(t)$ are nondecreasing, we have, by Lemma 7.1,

$$\begin{aligned} P(A_k) &\leq P \left\{ z^*(t_k) \leq \beta \frac{t_{k+1}}{\psi_{k+1}^2} \right\} \\ &= P \left\{ \frac{z^*(t_k)}{t_k} \leq \beta \frac{t_{k+1}}{t_k \psi_{k+1}^2} \right\} \\ &= P \left\{ z^*(1) \leq \beta \frac{\psi_k^2}{(\psi_k^2 - 1) \psi_{k+1}^2} \right\} \\ &\approx \exp \left(-\frac{(\psi_k^2 - 1) \psi_{k+1}^2}{\psi_k^2} \right) \approx \exp(-\psi_k^2), \end{aligned}$$

which sums by Lemma 2.10, and the Borel–Cantelli lemma gives the convergent part of the theorem.

Now assume $J_2(\psi) = \infty$. If $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$, then obviously

$$P \left\{ z^*(t) \leq \beta \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = 1.$$

Therefore assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. We may assume without loss of generality that $2 \leq \psi^2(t) \leq 2 \log \log t$. Define t_k and ψ_k as above and now put

$$A_k = \left\{ z^*(t_k) \leq \beta \frac{t_k}{\psi_k^2} \right\}.$$

Since

$$P(A_k) \approx \exp(-\psi_k^2), \tag{7.11}$$

we have, by Lemma 2.10, $\sum_k P(A_k) = \infty$.

We need to estimate $P(A_k \cap A_l)$ for $k < l$. Let $z^*(t_k, t_l)$ be the length of the longest excursion of $\{W(t); t_k \leq t \leq t_l\}$, ie.,

$$z^*(t_k, t_l) = \sup_{t_k \leq t \leq t_l} [t - \max(t_k, z_-(t))].$$

Now (cf. Lemma 7 of Csáki, Erdős, and Révész (1985))

$$\begin{aligned} P(A_k \cap A_l) &\leq P \left\{ z^*(t_k) \leq \beta \frac{t_k}{\psi_k^2}, z^*(t_k, t_l) \leq \beta \frac{t_l}{\psi_l^2} \right\} \\ &= \int_{-\infty}^{\infty} P \left\{ z^*(t_k) \leq \beta \frac{t_k}{\psi_k^2}, z^*(t_k, t_l) \leq \beta \frac{t_l}{\psi_l^2} \mid W(t_k) = a \right\} P\{W(t_k) \in da\} \\ &= \int_{-\infty}^{\infty} P \left\{ z^*(t_k) \leq \beta \frac{t_k}{\psi_k^2} \mid W(t_k) = a \right\} P \left\{ z^*(t_k, t_l) \leq \beta \frac{t_l}{\psi_l^2} \mid W(t_k) = a \right\} \\ &\quad \times P\{W(t_k) \in da\} \\ &\leq \int_{-\infty}^{\infty} P \left\{ z^*(t_k) \leq \beta \frac{t_k}{\psi_k^2} \mid W(t_k) = a \right\} P \left\{ z^*(t_l - t_k) \leq \beta \frac{t_l}{\psi_l^2} \right\} P\{W(t_k) \in da\} \\ &= P(A_k) P \left\{ z^*(1) \leq \beta \frac{t_l}{(t_l - t_k)\psi_l^2} \right\} \\ &\leq C \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right), \end{aligned} \tag{7.12}$$

where C is a positive constant. The first inequality in (7.12) follows from the fact that $z^*(t_k, t_l) \leq z^*(t_l)$; the second one from the fact that $z^*(t_k, t_l)$ is smallest when $W(t_k) = 0$, and when $W(t_k) = 0$, $z^*(t_k, t_l) \stackrel{D}{=} z^*(t_l - t_k)$; and the third inequality follows from Lemma 7.1. We also used conditional independence of $z^*(t_k)$ and $z^*(t_k, t_l)$ given $W(t_k)$.

Notice that the estimates (7.11) and (7.12) of $P(A_k)$ and $P(A_k \cap A_l)$ correspond exactly to those in (3.10) and (3.11) in Theorem 3.2. Therefore, the rest of the proof follows exactly the lines of the proof of Theorem 3.2 and is omitted. $\square$

Chapter 8

Integral Tests for Suprema of a Kiefer Process

8.1 Introduction

Let F be a nondegenerate distribution function on $\mathbb{R}^d$, $d \geq 1$. A Kiefer process $K_F(\mathbf{x}, t)$ on $\mathbb{R}^d \times [0, +\infty)$ associated with the distribution function F is a separable, zero-mean, Gaussian process with $K_F(\mathbf{x}, 0) = 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and

$$\mathbb{E} K_F(\mathbf{x}, t) K_F(\mathbf{y}, s) = (F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x})F(\mathbf{y}))(s \wedge t),$$

where $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \dots, \min(x_d, y_d))$, $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$.

We will study the upper classes of $\Delta(t) = \sup_{\mathbf{x} \in \mathbb{R}^d} |K_F(\mathbf{x}, t)|$. The following results are known. Adler and Brown (1986) proved (cf. their Theorem 5.2)

Theorem 8.A. *Let F be a continuous distribution function defined on $\mathbb{I}^d = [0, 1]^d$ with uniform marginals. Let ψ be a positive, nondecreasing, continuous function. If*

$\mathcal{I}_{2d}(\psi) < \infty$, then

$$\mathbb{P} \left\{ \sup_{\mathbf{x} \in \mathbb{I}^d} |K_F(\mathbf{x}, t)| \geq \left(\frac{1}{2} t \right)^{1/2} \psi(t) \text{ for only a bounded set of } t \right\} = 1. \quad (8.1)$$

Suppose, in addition, that there exist an $\mathbf{x}^* \in \mathbb{I}^d$, a neighborhood $\mathcal{N}$ of $\mathbf{x}^*$ and a constant $\gamma > 0$ satisfying $F(\mathbf{x}^*) = \frac{1}{2}$, and throughout $\mathcal{N}$, F possesses continuous first-order partial derivatives $\gamma_i = \partial F / \partial x_i$ satisfying

$$\inf_i \inf_{\mathcal{N}} \gamma_i(\mathbf{x}) = \gamma > 0.$$

Then $\mathcal{I}_{2d}(\psi) = \infty$ implies

$$\mathbb{P} \left\{ \sup_{\mathbf{x} \in \mathbb{I}^d} |K_F(\mathbf{x}, n)| \geq \left(\frac{1}{2} n \right)^{1/2} \psi(n) \text{ for an unbounded set of } n \in \mathbb{N} \right\} = 1. \quad (8.2)$$

Example 8.1. Let $\psi^2(t) = \log_2 t + (d+1) \log_3 t + \log_4 t + \cdots + \log_{p-1} t + (1+\delta) \log_p t$, $p \geq 4$. If $\delta > 0$, then (8.1) holds true, if $\delta \leq 0$, then (8.2) holds true.

Keprta (1997) improved a result of Csörgő, Horváth, and Szyszkowicz (1994) and proved

Theorem 8.1. Let $\psi \in \Psi_2^2$. Denote

$$\tau = \sup_{\mathbf{x} \in \mathbb{R}^d} F(\mathbf{x})(1 - F(\mathbf{x})).$$

Then

$$\limsup_{t \rightarrow \infty} \frac{\Delta(t)}{t^{1/2} \psi(t)} = (2\tau c^*(\psi))^{1/2} \quad a.s. \quad (8.3)$$

Similarly to the case of Wiener process, Theorem 8.A has a stronger statement whereas Theorem 8.1 has a broader class of weight functions.

8.2 Upper Classes of Suprema of a Kiefer Process

Our goal will be to prove

Theorem 8.2. *Let $\psi \in \Psi_3^2$. Denote*

$$\tau = \sup_{\mathbf{x} \in \mathbb{R}^d} F(\mathbf{x})(1 - F(\mathbf{x})).$$

Then

$$\mathcal{J}_{2d}(\psi) < \infty \implies \mathbb{P} \{ \Delta(t) \geq (2\tau t)^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \} = 0, \quad (8.4)$$

and

$$\mathcal{J}_1(\psi) = \infty \implies \mathbb{P} \{ \Delta(t) \geq (2\tau t)^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \} = 1, \quad (8.5)$$

or, equivalently,

$$\mathcal{J}_{2d}(\psi) < \infty \implies \psi(t) \in \text{UUC}((2\tau t)^{-1/2} \Delta(t)),$$

and

$$\mathcal{J}_1(\psi) = \infty \implies \psi(t) \in \text{ULC}((2\tau t)^{-1/2} \Delta(t)).$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 8.1. *The integral tests (8.4) and (8.5) remain true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Theorem 8.2 implies (8.1) and, via Lemma 2.15, Theorem 8.1.

Example 8.2. Let ψ be constructed as in Example 3.1. Then $\mathcal{J}_{2d}(\psi) < \infty$ if and only if

$$\sum_n \int_{T_n}^{T'_n} \frac{\psi^{2d}(t)}{t} \exp(-\psi^2(t)) = \sum_n \int_{T_n}^{T'_n} \frac{f^{2d}(t)}{t} \exp(-f^2(t)) < \infty.$$

We need an estimate of tail probability for Kiefer process. The following generalization of Theorem 4.1 of Adler and Brown (1986) (cf. Lemma 2.2 in Csörgő, Horváth, and Szyszkowicz (1994)) is due to Kepřta (1997) (see his Theorem 2).

Theorem 8.3. *Let F be a continuous distribution function on $\mathbb{I}^d = [0, 1]^d$ with uniform one-dimensional marginals. Let $\{W_F(\mathbf{x}); \mathbf{x} \in \mathbb{I}^d\}$ be a separable Gaussian process with zero mean and $E W_F(\mathbf{x}) W_F(\mathbf{y}) = F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x}) F(\mathbf{y})$. Denote $\sigma^2(\mathbf{x}) = \text{var } W_F(\mathbf{x}) = F(\mathbf{x})(1 - F(\mathbf{x}))$. Let $\eta \in (0, 1/4]$. Then there exist constants c_d , $d \geq 1$, independent of F , such that*

$$P\left\{\sup[W_F(\mathbf{x}); \sigma^2(\mathbf{x}) \leq \eta] > u\right\} \leq c_d u^{2(d-1)} \exp\left(-\frac{u^2}{2\eta}\right)$$

for large enough u .

Corollary 8.2. *Let $\{K_F(\mathbf{x}, t); \mathbf{x} \in \mathbb{R}^d, t \geq 0\}$ be a Kiefer process as defined above and*

$$\tau = \sup_{\mathbf{x} \in \mathbb{R}^d} F(\mathbf{x})(1 - F(\mathbf{x})).$$

Then there exist constants c_d , $d \geq 1$, independent of F , such that for all t

$$P\{\Delta(t)/t^{1/2} > u\} \leq c_d u^{2(d-1)} \exp\left(-\frac{u^2}{2\tau}\right)$$

for large enough u .

Remark 8.1. Talagrand (1994) (see also Adler (1990)) has similar inequalities for tails of suprema of Gaussian processes. An attempt to adapt these to the problem at hand would require a similar analysis of their proofs to that we carry out here for Theorem 4.1 of Adler and Brown (1986).

The additional assumptions on F to obtain (8.2) are needed to guarantee that the tail probability is $O(u^{2(d-1)} \exp(-2u^2))$ (since F is continuous in Theorem 8.A,

$\tau = 1/4$). An example is given on p. 13 in Adler and Brown (1986) where F is defined on $\mathbb{I}^2$ but the tail probability is $O(u \exp(-2u^2))$. Other powers of u are probably possible as well. Therefore, without additional assumptions on F , we cannot do better than in Theorem 8.2.

To prove their Theorem 4.1, Adler and Brown (1986) used several auxiliary lemmas (Lemmas 4.1 – 4.3 in Adler and Brown (1986)), which we state now here.

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{I}^d$, $\mathbf{x}_1 \leq \mathbf{x}_2$, $u > 0$. The inequality $\mathbf{x}_1 \leq \mathbf{x}_2$ is meant component-wise. Denote $A(u) = A(\mathbf{x}_1, \mathbf{x}_2, u) = \{\sup[W_F(\mathbf{x}); \mathbf{x}_1 \leq \mathbf{x} \leq \mathbf{x}_2] > u\}$.

Lemma 8.1. *Take $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{I}^d$, $\mathbf{x}_1 \leq \mathbf{x}_2$, $1/4 < \alpha < 1/2$ and $u > 1$. If*

$$\alpha \leq F(\mathbf{x}_1) \leq F(\mathbf{x}_2) \leq 1 - \alpha$$

and

$$F(\mathbf{x}_2) - F(\mathbf{x}_1) \leq \frac{\alpha^2}{2u^2},$$

then

$$P(A(u)) \leq O\left(\frac{1}{u} \exp\left(-\frac{u^2}{2\hat{\sigma}^2}\right)\right) \quad \text{as } u \rightarrow \infty.$$

where $\hat{\sigma}^2 = \inf\{\sigma^2(\mathbf{x}); \mathbf{x}_1 \leq \mathbf{x} \leq \mathbf{x}_2\}$.

Remark 8.2. $1/4$ of $1/4 < \alpha < 1/2$ can be replaced by any δ , $0 < \delta < 1/2$.

Lemma 8.2. *If $\alpha \in (0, 1/2)$, $\beta \in (1, (4\alpha(1 - \alpha))^{-1})$ and*

$$B(u) = \left\{ \sup[W_F(\mathbf{x}); F(\mathbf{x}) < \alpha \text{ or } F(\mathbf{x}) > 1 - \alpha] > u \right\},$$

then

$$P(B(u)) \leq O(\exp(-2\beta u^2)) \quad \text{as } u \rightarrow \infty.$$

Proof of Theorem 8.3. We divide $\mathbb{I}^d$ into a large number of small cubes and separate these cubes into two groups. We use Lemma 8.2 for the cubes over which W_F has small variance, and Lemma 8.1 for the cubes over which W_F has large variance. Write $\eta = a(1 - a)$, $0 < a \leq 1/2$. Choose $\alpha \in (0, a)$, $u > 1$, put $\gamma = \alpha^2/(2du^2)$ and denote $\mathbf{1} = (1, \dots, 1)'$. Let $\mathbf{x} \in \mathbb{I}^d$ be such that also $\mathbf{x} + \gamma\mathbf{1} \in \mathbb{I}^d$. Then the uniformity of marginals of F implies

$$F(\mathbf{x} + \gamma\mathbf{1}) \leq F(\mathbf{x}) + \gamma d = F(\mathbf{x}) + \frac{\alpha^2}{2u^2}. \quad (8.6)$$

Consider the lattice of points of the form $\gamma(n_1, \dots, n_d)$ where $n_i = 0, 1, \dots, [\gamma^{-1}]$. Each of these points has a unique expression as $\mathbf{p} + j\gamma\mathbf{1}$, where $\mathbf{p} \in \pi$ and π is the set of $\gamma(n_1, \dots, n_d)$ with $\min\{n_i; i = 1, \dots, d\} = 0$. For each $\mathbf{p} \in \pi$ define, inductively,

$$\begin{aligned} j_1 &= j_1(\mathbf{p}) = \max\{j; F(\mathbf{p} + j\gamma\mathbf{1}) < \alpha\}, \\ j_i &= j_i(\mathbf{p}) = \max\left\{j; F(\mathbf{p} + j\gamma\mathbf{1}) - F(\mathbf{p} + j_{i-1}\gamma\mathbf{1}) \leq \frac{\alpha^2}{2u^2}\right\}, \end{aligned}$$

and define

$$\begin{aligned} J_1 &= J_1(\mathbf{p}) = \min\{i; F(\mathbf{p} + j_i\gamma\mathbf{1}) > \alpha\}, \\ J_2 &= J_2(\mathbf{p}) = \max\{i; F(\mathbf{p} + j_i\gamma\mathbf{1}) < 1 - \alpha\}, \\ J &= J(\mathbf{p}) = \min\{i; F(\mathbf{p} + j_i\gamma\mathbf{1}) > 1 - \alpha\}. \end{aligned}$$

Now (8.6) implies that $j_i - j_{i-1} \geq 1$ for all i and $\mathbf{p}$. Furthermore,

$$\begin{aligned} 0 < \alpha - \frac{\alpha^2}{2u^2} &\leq F(\mathbf{p} + j_i\gamma\mathbf{1}) \leq \alpha + \frac{\alpha^2}{2u^2}, & \text{for } i = 1, \dots, J_1, \\ 1 - \alpha - \frac{\alpha^2}{2u^2} &\leq F(\mathbf{p} + j_i\gamma\mathbf{1}) \leq 1 - \alpha + \frac{\alpha^2}{2u^2} < 1, & \text{for } i = J_2, \dots, J. \end{aligned}$$

Denote

$$I(\mathbf{p}) = \{i; 1 \leq i \leq J_1(\mathbf{p}) - 1\} \cup \{i; J_2(\mathbf{p}) \leq i \leq J(\mathbf{p}) - 1\}$$

and

$$S^*(\mathbf{p}) = \bigcup_{i \in I(\mathbf{p})} \{\mathbf{x}; \mathbf{p} + j_i \gamma \mathbf{1} \leq \mathbf{x} \leq \mathbf{p} + j_{i+1} \gamma \mathbf{1}\}.$$

Since

$$\{\mathbf{x}; \alpha(1 - \alpha) \leq \sigma^2(\mathbf{x}) \leq \eta\} \subset \bigcup_{\mathbf{p} \in \pi} S^*(\mathbf{p}),$$

we have

$$\begin{aligned} & P\left\{\sup[W_F(\mathbf{x}); \alpha(1 - \alpha) \leq \sigma^2(\mathbf{x}) \leq \eta] > u\right\} \\ & \leq P\left\{\sup\left[W_F(\mathbf{x}); \mathbf{x} \in \bigcup_{\mathbf{p} \in \pi} S^*(\mathbf{p})\right] > u\right\} \\ & \leq \sum_{\mathbf{p} \in \pi} \sum_{i \in I(\mathbf{p})} P(A(\mathbf{p} + j_i \gamma \mathbf{1}, \mathbf{p} + j_{i+1} \gamma \mathbf{1}, u)) \\ & \leq \sum_{\mathbf{p} \in \pi} O\left(\sum_{i \in I(\mathbf{p})} \frac{1}{u} \exp\left(-\frac{u^2}{2\hat{\sigma}^2(\mathbf{p} + j_i \gamma \mathbf{1})}\right)\right), \end{aligned}$$

where $\hat{\sigma}^2(\mathbf{p} + j_i \gamma \mathbf{1}) = \inf\{\sigma^2(\mathbf{x}); \mathbf{p} + j_i \gamma \mathbf{1} \leq \mathbf{x} \leq \mathbf{p} + j_{i+1} \gamma \mathbf{1}\}$.

Note that for all i and $\mathbf{p}$

$$F(\mathbf{p} + j_{i+2} \gamma \mathbf{1}) - F(\mathbf{p} + j_i \gamma \mathbf{1}) > \frac{\alpha^2}{2u^2}.$$

Define

$$I^* = \min\left\{i; a - i \frac{\alpha^2}{2u^2} < \alpha\right\} = \min\left\{i; 1 - a + i \frac{\alpha^2}{2u^2} > 1 - \alpha\right\}.$$

Then the sequence $\{\hat{\sigma}^2(\mathbf{p} + j_i \gamma \mathbf{1}); i \in I(\mathbf{p})\}$ is dominated by the sequence

$$\{a_0, a_0, a_0, a_0, a_1, a_1, a_1, a_1, \dots, a_{I^*}, a_{I^*}, a_{I^*}, a_{I^*}\},$$

where

$$a_i = \eta - \left(\frac{i\alpha^2}{2u^2}\right)^2.$$

The new sequence might have more elements and, after rearranging, dominates the former sequence termwise.

Now

$$\begin{aligned}
 \sum_{i \in I(\mathbf{p})} \frac{1}{u} \exp \left(-\frac{u^2}{2\hat{\sigma}^2(\mathbf{p} + j_i \gamma \mathbf{1})} \right) &\leq 4 \sum_{i=0}^{I^*} \frac{1}{u} \exp \left[-u^2 \left[2 \left(\eta - \frac{1}{4} \left(\frac{i\alpha^2}{u^2} \right)^2 \right) \right]^{-1} \right] \\
 &\leq 4 \sum_{i=0}^{I^*} \frac{1}{u} \exp \left[-\frac{u^2}{2\eta} \left(1 + \frac{1}{4\eta} \left(\frac{i\alpha^2}{u^2} \right)^2 \right) \right] \\
 &= 4 \exp \left(-\frac{u^2}{2\eta} \right) \sum_{i=0}^{I^*} \frac{1}{u} \exp \left(-\frac{i^2 \alpha^4}{8\eta^2 u^2} \right) \\
 &= \exp \left(-\frac{u^2}{2\eta} \right) O \left(\int_0^\infty \exp \left(-\frac{\alpha^4}{8\eta^2} y \right) dy \right) \\
 &= O \left(\exp \left(-\frac{u^2}{2\eta} \right) \right).
 \end{aligned}$$

Note that $\boldsymbol{\pi}$ has at most $d(2 + 2du^2/\alpha^2)^{d-1} = O(u^{2(d-1)})$ points. Therefore

$$\begin{aligned}
 &\mathbb{P} \left\{ \sup [W_F(\mathbf{x}); \sigma^2(\mathbf{x}) \leq \eta] > u \right\} \\
 &\leq \mathbb{P} \left\{ \sup [W_F(\mathbf{x}); \alpha(1 - \alpha) \leq \sigma^2(\mathbf{x}) \leq \eta] > u \right\} \\
 &\quad + \mathbb{P} \left\{ \sup [W_F(\mathbf{x}); F(\mathbf{x}) < \alpha \text{ or } F(\mathbf{x}) > 1 - \alpha] > u \right\} \\
 &\leq O \left(u^{2(d-1)} \exp \left(-\frac{u^2}{2\eta} \right) \right) + O(\exp(-2\beta u^2)) \\
 &= O \left(u^{2(d-1)} \exp \left(-\frac{u^2}{2\eta} \right) \right).
 \end{aligned}$$

□

Proof of Corollary 8.2. We will use the same argument as Csörgö, Horváth, and Szyszkowicz (1994) in their Lemma 2.2. First notice that for every t , $\Delta(t)/t^{1/2} \stackrel{\mathcal{D}}{=} \Delta(1)$. Let $F_1, \dots, F_d$ denote the marginal distribution functions of F . For $\mathbf{x} =$

$(x_1, \dots, x_d) \in \mathbb{R}^d$ define $L(\mathbf{x}) = (F_1(x_1), \dots, F_d(x_d))$. There is a distribution function $G(\mathbf{u})$, $\mathbf{u} \in \mathbb{I}^d = [0, 1]^d$, such that its marginals are uniform on $[0, 1]$ and $G(L(\mathbf{x})) = F(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$. Therefore

$$\{\Delta(t); 0 \leq t < \infty\} \stackrel{\mathcal{D}}{=} \left\{ \sup_{\mathbf{x} \in \mathbb{R}^d} |K_{G \circ L}(\mathbf{x}, t)|; 0 \leq t < \infty \right\},$$

and

$$\mathbb{P} \left\{ \sup_{\mathbf{x} \in \mathbb{R}^d} |K_{G \circ L}(\mathbf{x}, t)| \geq u \right\} \leq \mathbb{P} \left\{ \sup_{\mathbf{u} \in \mathbb{I}^d} |K_G(\mathbf{u}, t)| \geq u \right\}$$

for all $u \geq 0$. Hence the corollary immediately follows from Theorem 8.3. $\square$

Proof of Theorem 8.2. Assume $\mathcal{I}_{2d}(\psi) < \infty$. By Lemma 2.1, $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10. The process $\{K_F(\mathbf{x}, t); \mathbf{x} \in \mathbb{R}^d, t \geq 0\}$ has independent increments in t . This means that $\{K_F(\mathbf{x}, t_n) - K_F(\mathbf{x}, t_{n-1}); \mathbf{x} \in \mathbb{R}^d\}, \dots, \{K_F(\mathbf{x}, t_2) - K_F(\mathbf{x}, t_1); \mathbf{x} \in \mathbb{R}^d\}$ are independent processes for all $t_n > t_{n-1} > \dots > t_1$, $n = 2, \dots$. Hence (see Csörgő, Horváth, and Szyszkowicz (1994), p. 7, or Adler and Brown (1986), p. 26),

$$\mathbb{P} \left\{ \sup_{t_k \leq t \leq t_{k+1}} \Delta(t) > u \right\} \leq 2 \mathbb{P} \{ \Delta(t_{k+1}) > u \}$$

for all $u > 0$.

Using Corollary 8.2 we have

$$\begin{aligned}
 & \mathbb{P} \left\{ \sup_{t_k \leq t \leq t_{k+1}} \frac{\Delta(t)}{(2\tau t)^{1/2} \psi(t)} > 1 \right\} \leq \mathbb{P} \left\{ \sup_{t_k \leq t \leq t_{k+1}} \frac{\Delta(t)}{(2\tau t_k)^{1/2} \psi_k} > 1 \right\} \\
 & = \mathbb{P} \left\{ \sup_{t_k \leq t \leq t_{k+1}} \Delta(t) > (2\tau t_k)^{1/2} \psi_k \right\} \\
 & \leq 2 \mathbb{P} \left\{ \Delta(t_{k+1}) > (2\tau t_k)^{1/2} \psi_k \right\} \\
 & = 2 \mathbb{P} \left\{ \frac{\Delta(t_{k+1})}{t_{k+1}^{1/2}} > \frac{(2\tau t_k)^{1/2}}{t_{k+1}^{1/2}} \psi_k \right\} \\
 & \leq 2c_d \left[\frac{(2\tau t_k)^{1/2}}{t_{k+1}^{1/2}} \psi_k \right]^{2(d-1)} \exp \left(-\frac{1}{2\tau} \cdot \frac{2\tau t_k}{t_{k+1}} \psi_k^2 \right) \\
 & = 2c_d (2\tau)^{d-1} \left(\frac{t_k}{t_{k+1}} \right)^{d-1} \psi_k^{2d-2} \exp \left(-\frac{t_k}{t_{k+1}} \psi_k^2 \right) \\
 & \approx \psi_k^{2d-2} \exp(-\psi_k^2),
 \end{aligned}$$

which, by Lemma 2.10, sums. Therefore, by the Borel–Cantelli Lemma,

$$\mathbb{P} \left\{ \Delta(t) \geq (2\tau t)^{1/2} \psi(t) \text{ i.o. as } t \rightarrow \infty \right\} = 0.$$

Now assume that $J_1(\psi) = \infty$. Since for $\mathbf{x}_0$ such that $0 < F(\mathbf{x}_0) < 1$, $\{ [F(\mathbf{x}_0)(1 - F(\mathbf{x}_0))]^{-1/2} K_F(\mathbf{x}_0, t); t \geq 0 \}$ is a standard Wiener process, we get from Theorem 3.1 that there is a sequence $t_n \nearrow \infty$ such that

$$[F(\mathbf{x}_0)(1 - F(\mathbf{x}_0))]^{-1/2} |K_F(\mathbf{x}_0, t_n)| \geq \sqrt{2} t_n^{1/2} \psi(t_n). \quad (8.7)$$

Taking the supremum over $\mathbf{x}_0$ we get

$$\Delta(t_n) \geq (2\tau t_n)^{1/2} \psi(t_n).$$

□

Chapter 9

Integral Tests for Two Exponential Functionals of Brownian Motion

9.1 Introduction

Define

$$Y(t) = \int_0^\infty \exp\left(W(u) - \frac{u}{t}\right) du, \quad t > 0, \quad (9.1)$$

and

$$Z(t) = \int_0^t \exp(W(u)) du, \quad t > 0. \quad (9.2)$$

The study of these processes was motivated by several problems in financial mathematics (see Gruet and Shi (1995a) and Dufresne (1990)). The upper functions of these two processes were studied in Gruet and Shi (1995a). They proved the following theorems.

Theorem 9.A. *Let $\psi \in \Psi_1$. Then*

$$P \left\{ Y(t) \geq \exp \left(\frac{1}{2} t \psi^2(t) \right) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Theorem 9.B. *Let $\psi \in \Psi_1$. Then*

$$P \left\{ Z(t) \geq \exp \left((2t)^{1/2} \psi(t) \right) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Again, we will try to enlarge the classes of weight functions.

9.2 The $Z(t)$ process

Theorem 9.1. *Let $\psi \in \Psi_3^2$. Then*

$$P \left\{ Z(t) \geq \exp \left((2t)^{1/2} \psi(t) \right) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (9.3)$$

or, equivalently,

$$\psi(t) \in \begin{cases} \text{UUC}((2t)^{-1} \log Z(t)) \\ \text{ULC}((2t)^{-1} \log Z(t)) \end{cases} \quad \text{according as } J_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 9.1.

$$\limsup_{t \rightarrow \infty} \frac{\log Z(t)}{t^{1/2} \psi(t)} = (2c^*(\psi))^{1/2} \quad a.s.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 9.2. *The integral test (9.3) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Theorem 9.2. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ Z(t) \leq \exp \left(\frac{t^{1/2}}{\psi(t)} \right) \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \quad (9.4)$$

or, equivalently,

$$\psi^{-1}(t) \in \begin{cases} \text{LLC}(t^{-1/2} \log Z(t)) \\ \text{LUC}(t^{-1/2} \log Z(t)) \end{cases} \text{ according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 9.3.

$$\liminf_{t \rightarrow \infty} \frac{\log Z(t)}{t^{1/2}} \psi(t) = \begin{cases} \infty \\ 0 \end{cases} \text{ a.s. according as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 9.4. *The integral test (9.4) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

We will need a theorem on the large increments of Wiener process (cf. Theorem 1.2.1 in Csörgő and Révész (1981)).

Theorem 9.C. *Let $\{W(t); t \geq 0\}$ be a Wiener process and $0 < a_T \leq T$ be a nondecreasing function of T . Assume that $T^{-1}a_T$ is nonincreasing. Then*

$$\begin{aligned} \limsup_{T \rightarrow \infty} \beta_T a_T^{-1/2} \sup_{0 \leq t \leq T-a_T} (W(t+a_T) - W(t)) &= 1 \quad a.s., \\ \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq a_T} \beta_T (W(T+t) - W(T)) &= 1 \quad a.s., \\ \limsup_{T \rightarrow \infty} \beta_T (W(T+a_T) - W(T)) &= 1 \quad a.s., \end{aligned}$$

where

$$\beta_T = (2(\log T a_T^{-1} + \log \log T))^{-1/2}.$$

If we also assume that

$$\lim_{T \rightarrow \infty} \frac{\log T a_T^{-1}}{\log \log T} = \infty,$$

then

$$\lim_{T \rightarrow \infty} \beta_T a_T^{-1/2} \sup_{0 \leq t \leq T-a_T} (W(t+a_T) - W(t)) = 1 \quad a.s.$$

Theorem 9.1 and Theorem 9.2 immediately follow from Theorem 3.1 and Theorem 3.3 via the following invariance proved by Csáki *et al.* (1996).

Theorem 9.D.

$$|\log Z(t) - M^+(t)| = O(\log t) \quad a.s. \text{ as } t \rightarrow \infty.$$

Proof. Let u_0 be a place of maxima, ie., $W(u_0) = M^+(t)$. By Theorem 9.C we have

$$\sup_{0 \leq u \leq 1} \sup_{0 \leq s \leq t} |W(s+u) - W(s)| = O(\sqrt{\log t}) \quad a.s.$$

Hence,

$$W(u) \geq W(u_0) - O(\sqrt{\log t}), \quad u_0 - 1 \leq u \leq u_0.$$

Therefore,

$$Z(t) \geq \int_{u_0-1}^{u_0} \exp(W(u)) \, du \geq \exp(W(u_0) - O(\sqrt{\log t})),$$

and clearly

$$Z(t) \leq t \exp(M^+(t)),$$

giving

$$M(t) - O(\sqrt{\log t}) \leq \log Z(t) \leq M^+(t) + \log t.$$

□

9.3 The $Y(t)$ process

For $\psi \in \Psi_3^2$, we were able to prove a bit weaker result than Theorem 9.A.

Theorem 9.3. *Let $\psi \in \Psi_3^2$. Then*

$$J_2(\psi) < \infty \implies P \{ Y(t) \geq \exp(\tfrac{1}{2}t\psi^2(t)) \text{ i.o. as } t \rightarrow \infty \} = 0, \quad (9.5)$$

and

$$J_1(\psi) = \infty \implies P \{ Y(t) \geq \exp(\tfrac{1}{2}t\psi^2(t)) \text{ i.o. as } t \rightarrow \infty \} = 1. \quad (9.6)$$

Corollary 9.5.

$$\limsup_{t \rightarrow \infty} \frac{\log Y(t)}{t\psi^2(t)} = \frac{1}{2}c^*(\psi) \quad \text{a.s.}$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 9.6. *The integral tests (9.5) and (9.6) remain true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Csáki *et al.* (1996) also proved an invariance theorem for $\{Y(t); t \geq 0\}$.

Theorem 9.E. *Denote*

$$\{Q(t); t > 0\} = \left\{ \sup_{0 \leq u < \infty} \left(W(u) - \frac{u}{t} \right); t > 0 \right\}.$$

Then

$$|\log Y(t) - Q(t)| = O(\log t) \quad \text{a.s. as } t \rightarrow \infty.$$

Proof. Let u_0 be a place of maxima, ie.,

$$W(u_0) - \frac{u_0}{t} = Q(t).$$

Further let u_1 be such that

$$W(u) < \frac{u^{2/3}}{2}, \quad u > u_1.$$

Let $t > u_1$ and $u > t^3$. Then

$$W(u) < \frac{u}{2t}$$

and $u_0 < t^3$. Now

$$\begin{aligned} Y(t) &= \int_0^{t^3} \exp \left(W(u) - \frac{u}{t} \right) du + \int_{t^3}^{\infty} \exp \left(W(u) - \frac{u}{t} \right) du \\ &\leq t^3 \exp(Q(t)) + \int_0^{\infty} \exp \left(-\frac{u}{2t} \right) du \\ &= t^3 \exp(Q(t)) + 2t \\ &= \exp(Q(t) + O(\log t)). \end{aligned}$$

Conversely, via Theorem 9.C,

$$\begin{aligned} Y(t) &\geq \int_{u_0-1}^{u_0} \exp \left(W(u) - \frac{u}{t} \right) du \\ &\geq \exp \left(W(u_0) - \frac{u_0}{t} - O(\sqrt{\log t}) \right) \\ &= \exp(Q(t) + O(\log t)). \end{aligned}$$

□

Theorem 9.E implies that Theorem 9.3 (and also Theorem 9.5 given below) remains true if we replace $\log Y(t)$ by $Q(t)$, ie.,

Theorem 9.4. *Let $\psi \in \Psi_3^2$. If $J_2(\psi) < \infty$ then for all large enough t ,*

$$W(u) < \frac{u}{t} + \frac{1}{2}t\psi^2(t) \quad \text{for all } u > 0. \quad (9.7)$$

and, if $J_1(\psi) = \infty$ then there is a sequence $t_n \nearrow \infty$ and a sequence u_n such that

$$W(u_n) \geq \frac{u_n}{t_n} + \frac{1}{2}t_n\psi^2(t_n). \quad (9.8)$$

In Theorem 9.4, the upper envelop for $\{W(u); u \geq 0\}$ is constructed as a collection of straight lines (indexed by t) with slope $1/t$ and intercept $\frac{1}{2}t\psi^2(t)$. The integral test determines whether $\{W(u); u \geq 0\}$ is below the envelop or not. Quite interestingly, the integral test is the same (at least for $\psi \in \Psi_1$ and very likely also for $\psi \in \Psi_3^2$) as in Theorem 3.1, where the envelop is constructed as $(2t)^{1/2}\psi(t)$.

To prove Theorem 9.3, we need an auxiliary lemma.

Lemma 9.1. *Let $\psi \in \Psi_4^2$ and $\lambda > 0$. Then there is a unique function ξ satisfying*

$$\xi(s) = \psi\left(\frac{\lambda s^{1/2}}{\xi(s)}\right). \quad (9.9)$$

Moreover, $\xi \in \Psi_4^2$ and

$$J_1(\psi) < \infty \iff J_1(\xi) < \infty.$$

Proof. Put

$$s(t) = \frac{1}{\lambda^2}t^2\psi^2(t).$$

Since $t\psi^2(t)$ is nondecreasing, $s(t)$ is strictly increasing and we can find its strictly increasing inverse $t(s)$ and then define

$$\xi(s) = \psi(t(s)).$$

Thus,

$$t(s) = \frac{\lambda s^{1/2}}{\psi(t)} = \frac{\lambda s^{1/2}}{\xi(s)},$$

and therefore (9.9) is satisfied.

Observe that for $s_1 \leq s_2$,

$$\xi(s_2) = \psi \left(\frac{\lambda s_2^{1/2}}{\xi(s_2)} \right) \leq \xi(s_1) \left[\frac{\lambda s_2^{1/2}}{\xi(s_2)} \cdot \frac{\xi(s_1)}{\lambda s_1^{1/2}} \right]^{1/2} = \xi(s_1)^{3/2} \frac{1}{\xi(s_2)^{1/2}} \left(\frac{s_2}{s_1} \right)^{1/4},$$

ie.,

$$\xi(s_2) \leq \xi(s_1) \left(\frac{s_2}{s_1} \right)^{1/6},$$

and

$$\xi(s_2) = \psi \left(\frac{\lambda s_2^{1/2}}{\xi(s_2)} \right) \geq \xi(s_1) \left[\frac{\lambda s_1^{1/2}}{\xi(s_1)} \cdot \frac{\xi(s_2)}{\lambda s_2^{1/2}} \right]^{1/2} = \xi(s_1)^{1/2} \xi(s_2)^{1/2} \left(\frac{s_1}{s_2} \right)^{1/4},$$

ie.,

$$\xi(s_2) \geq \xi(s_1) \left(\frac{s_1}{s_2} \right)^{1/2}.$$

Therefore $\xi \in \Psi_4^2$.

If $\psi(t) \nrightarrow \infty$ as $t \rightarrow \infty$, then $\xi(t) \nrightarrow \infty$ as $t \rightarrow \infty$, and $J_1(\psi) = \infty$ and $J_1(\xi) = \infty$. Assume that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10 and put

$$s_k = \frac{1}{\lambda^2} t_k^2 \psi_k^2.$$

By Lemma 2.10,

$$J_1(\psi) < \infty \iff \sum_k \frac{\psi_k}{\psi_k^2 - 1} \exp(-\psi_k^2) = \sum_k \frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k)) < \infty.$$

Hence, we only need to show that

$$J_1(\xi) < \infty \iff \sum_k \frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k)) < \infty.$$

which will be implied by

$$\frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k)) \approx \int_{s_k}^{s_{k+1}} \frac{\xi(s)}{s} \exp(-\xi^2(s)) \, ds. \quad (9.10)$$

Observe that

$$\begin{aligned} s_{k+1} - s_k &= \frac{1}{\lambda^2} t_{k+1}^2 \psi_{k+1}^2 - \frac{1}{\lambda^2} t_k^2 \psi_k^2 \\ &= \frac{1}{\lambda^2} t_k^2 \psi_k^2 \left(\frac{\psi_k^2 \psi_{k+1}^2}{(\psi_k^2 - 1)^2} - 1 \right) \\ &= s_k \left(\frac{\psi_k^2 \psi_{k+1}^2}{(\psi_k^2 - 1)^2} - 1 \right), \end{aligned}$$

and

$$\frac{s_k}{s_{k+1}} = \frac{\frac{1}{\lambda^2} t_k^2 \psi_k^2}{\frac{1}{\lambda^2} t_k^2 \frac{\psi_k^4}{(\psi_k^2 - 1)^2} \psi_{k+1}^2} = \frac{(\psi_k^2 - 1)^2}{\psi_k^2 \psi_{k+1}^2}.$$

Recollect that

$$\psi_k^2 - 1 \leq \psi_{k+1}^2 \leq \frac{\psi_k^4}{\psi_k^2 - 1}.$$

First we estimate the integral in (9.10) from above.

$$\begin{aligned}
 \frac{\int_{s_k}^{s_{k+1}} \frac{\xi(s)}{s} \exp(-\xi^2(s)) ds}{\frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k))} &\leq \frac{\frac{s_{k+1} - s_k}{s_k} \xi(s_k) \exp\left(-\frac{\xi^2(s_k)s_k}{s_{k+1}}\right)}{\frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k))} \\
 &= \frac{\psi_k^2 \psi_{k+1}^2 - (\psi_k^2 - 1)^2}{\psi_k^2 - 1} \exp\left[\psi_k^2 \left(1 - \frac{s_k}{s_{k+1}}\right)\right] \\
 &= \frac{\psi_k^2 \psi_{k+1}^2 - (\psi_k^2 - 1)^2}{\psi_k^2 - 1} \exp\left[\psi_k^2 \left(1 - \frac{(\psi_k^2 - 1)^2}{\psi_k^2 \psi_{k+1}^2}\right)\right] \\
 &\leq \frac{\psi_k^2 \frac{\psi_k^4}{\psi_k^2 - 1} - (\psi_k^2 - 1)^2}{\psi_k^2 - 1} \exp\left[\psi_k^2 - \frac{(\psi_k^2 - 1)^3}{\psi_k^4}\right] \\
 &= \frac{3\psi_k^4 - 3\psi_k^2 + 1}{(\psi_k^2 - 1)^2} \exp\left[\frac{3\psi_k^4 - 3\psi_k^2 + 1}{\psi_k^4}\right] \\
 &\rightarrow 3e^3.
 \end{aligned}$$

(9.11)

Now we estimate the integral in (9.10) from below.

$$\begin{aligned}
 \frac{\int_{s_k}^{s_{k+1}} \frac{\xi(s)}{s} \exp(-\xi^2(s)) ds}{\frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k))} &\geq \frac{\frac{s_{k+1} - s_k}{s_{k+1}} \xi(s_{k+1}) \exp\left(-\frac{\xi^2(s_k)s_{k+1}}{s_k}\right)}{\frac{\xi(s_k)}{\xi^2(s_k) - 1} \exp(-\xi^2(s_k))} \\
 &= \left(1 - \frac{(\psi_k^2 - 1)^2}{\psi_k^2 \psi_{k+1}^2}\right) \frac{(\psi_k^2 - 1)\psi_{k+1}}{\psi_k} \exp\left[-\psi_k^2 \frac{s_{k+1} - s_k}{s_k}\right] \\
 &= \left(\frac{(\psi_k^2 - 1)\psi_{k+1}}{\psi_k} - \frac{(\psi_k^2 - 1)^3}{\psi_k^3 \psi_{k+1}}\right) \exp\left[-\psi_k^2 \frac{\psi_k^2 \psi_{k+1}^2 - (\psi_k^2 - 1)^2}{(\psi_k^2 - 1)^2}\right] \\
 &\geq \left(\frac{(\psi_k^2 - 1)\sqrt{\psi_k^2 - 1}}{\psi_k} - \frac{(\psi_k^2 - 1)^3}{\psi_k^3 \sqrt{\psi_k^2 - 1}}\right) \exp\left[-\psi_k^2 \frac{\psi_k^2 \frac{\psi_k^4}{\psi_k^2 - 1} - (\psi_k^2 - 1)^2}{(\psi_k^2 - 1)^2}\right] \\
 &= \frac{(\psi_k^2 - 1)^2}{\psi_k^3 \sqrt{\psi_k^2 - 1}} \exp\left[-\psi_k^2 \frac{3\psi_k^4 - 3\psi_k^2 + 1}{(\psi_k^2 - 1)^3}\right] \\
 &\rightarrow e^{-3}.
 \end{aligned}$$

(9.12)

Now (9.11) and (9.12) imply (9.10). $\square$

Proof of Theorem 9.3. By Lemmas 2.15 and 2.16, it is sufficient to prove the theorem for $\psi \in \Psi_4^2$.

Let $J_2(\psi) < \infty$. By Lemma 2.1, $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10. Put

$$A_k = \left\{ Y(t) \geq \exp\left(\frac{1}{2}t\psi^2(t)\right) \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Since $2/Y(t)$ has a Gamma distribution with parameter $2/t$ (see Dufresne (1990) or Yor (1992)), we have

$$\begin{aligned} P(A_k) &\leq P\left\{ Y(t_{k+1}) \geq \exp\left(\frac{1}{2}t_k\psi_k^2\right) \right\} \\ &= P\left\{ \frac{2}{Y(t_{k+1})} \leq 2\exp\left(-\frac{1}{2}t_k\psi_k^2\right) \right\} \\ &= \frac{1}{\Gamma(2/t_{k+1})} \int_0^{2\exp[-\frac{1}{2}t_k\psi_k^2]} x^{-1+2/t_{k+1}} e^{-x} dx \\ &\leq \frac{1}{\Gamma(2/t_{k+1})} \left[\frac{t_{k+1}}{2} x^{2/t_{k+1}} \right]_{x=0}^{2\exp[-\frac{1}{2}t_k\psi_k^2]} \\ &= \frac{2^{2/t_{k+1}}}{\Gamma(1+2/t_{k+1})} \exp\left(-\frac{t_k}{t_{k+1}}\psi_k^2\right) \\ &\approx \exp(-\psi_k^2), \end{aligned}$$

which sums by Lemma 2.10, and the Borel–Cantelli lemma finishes the proof of the convergent part.

Now we will prove the divergent part of the theorem. Let $J_1(\psi) = \infty$. Define ξ by

$$\xi(s) = \psi(t),$$

where $t = (2s)^{1/2}/\xi(s)$. By Lemma 9.1, $\xi \in \Psi_4^2$ and $J_1(\xi) = \infty$. Therefore, by Theorem 3.1, for almost all ω there is a sequence $s_k = s_k(\omega)$ such that $s_k \rightarrow \infty$ as

$k \rightarrow \infty$ and

$$W(s_k) \geq (2s_k)^{1/2} \xi(s_k). \quad (9.13)$$

Put

$$t_k = \frac{(2s_k)^{1/2}}{\xi(s_k)}.$$

We want to show that

$$Y(t_k) \geq \exp \left(\frac{1}{2} t_k \psi^2(t_k) \right).$$

We could follow the proof of the divergent part in Gruet and Shi (1995a), but it is shorter to prove the divergent part of Theorem 9.4 instead. Notice that

$$\frac{s_k}{t_k} + \frac{1}{2} t_k \psi^2(t_k) = \sqrt{2} s_k^{1/2} \xi(s_k) \quad (9.14)$$

and, therefore from (9.13) and (9.14),

$$W(s_k) \geq \frac{s_k}{t_k} + \frac{1}{2} t_k \psi^2(t_k).$$

as desired. $\square$

For the lower classes of $\{Y(t); t \geq 0\}$ we were able to obtain only a convergent part.

Theorem 9.5. *Let $\psi \in \Psi_2^2$. If $\mathcal{J}_2(\psi) < \infty$, then*

$$\mathbb{P} \left\{ Y(t) \leq \exp \left(\frac{t}{\psi^2(t)} \right) \text{ i.o. as } t \rightarrow \infty \right\} = 0. \quad (9.15)$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 9.7. *The integral test (9.15) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Proof. Assume that $\mathcal{J}_2(\psi) < \infty$. Then $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and we may assume, without loss of generality, that $\psi(t) \leq \log t$.

Take $\lambda > 1$. Using the fact that $2/Y(t)$ has a Gamma distribution with parameter $2/t$ (see Dufresne (1990) or Yor (1992)), the inequality $1 - x \leq e^{-x} \leq 1/x$, $x > 0$, and the monotonicity of $Y(t)$ and $t/\psi^2(t)$, we have, for large enough k ,

$$\begin{aligned}
 & \mathbb{P}\{Y(t) \leq \exp(t\psi^{-2}(t)) \text{ for some } t \in [\lambda^k, \lambda^{k+1})\} \\
 & \leq \mathbb{P}\{Y(\lambda^k) \leq \exp(\lambda^{k+1}\psi^{-2}(\lambda^{k+1}))\} \\
 & = \mathbb{P}\left\{\frac{2}{Y(\lambda^k)} \geq 2\exp(-\lambda^{k+1}\psi^{-2}(\lambda^{k+1}))\right\} \\
 & = 1 - \frac{1}{\Gamma(2\lambda^{-k})} \int_0^{2\exp[-\lambda^{k+1}\psi^{-2}(\lambda^{k+1})]} x^{2\lambda^{-k}-1} e^{-x} dx \\
 & \leq 1 - \frac{1}{\Gamma(2\lambda^{-k})} \exp[-2\exp(-\lambda^{k+1}\psi^{-2}(\lambda^{k+1}))] \int_0^{2\exp[-\lambda^{k+1}\psi^{-2}(\lambda^{k+1})]} x^{2\lambda^{-k}-1} dx \\
 & = 1 - \frac{1}{\Gamma(2\lambda^{-k})2\lambda^{-k}} 2^{2\lambda^{-k}} \exp[-2\exp(-\lambda^{k+1}\psi^{-2}(\lambda^{k+1})) - 2\lambda\psi^{-2}(\lambda^{k+1})] \\
 & \leq 1 - [1 - 2\exp(-\lambda^{k+1}\psi^{-2}(\lambda^{k+1})) - 2\lambda\psi^{-2}(\lambda^{k+1})] \\
 & \leq 2\frac{\psi^2(\lambda^{k+1})}{\lambda^{k+1}} + 2\frac{\lambda}{\psi^2(\lambda^{k+1})} \\
 & \leq 3\frac{\lambda}{\psi^2(\lambda^{k+1})},
 \end{aligned} \tag{9.16}$$

which sums by Lemma 2.9 and, therefore, by the Borel–Cantelli lemma,

$$\mathbb{P}\{Y(t) \leq \exp(t\psi^{-2}(t)) \text{ i.o. as } t \rightarrow \infty\} = 0.$$

□

Chapter 10

Planar Brownian Motion

10.1 The Skew Product

In this chapter we will study processes related to planar Brownian motion. First we introduce the so-called *skew product*. We use especially Pitman and Yor (1986) and Section 5.5 of Durrett (1984) in our presentation.

Let $\{U(t); t \geq 0\}$ be a complex Brownian motion starting at zero, ie.,

$$U(t) = W^{(1)}(t) + iW^{(2)}(t), \quad U(0) = 0, \quad (10.1)$$

where $\{W^{(1)}(t); t \geq 0\}$ and $\{W^{(2)}(t); t \geq 0\}$ are independent one-dimensional Brownian motions. Put

$$V(t) = \exp(U(t)), \quad t \geq 0, \quad (10.2)$$

$$\beta(t) = \int_0^t |V(s)|^2 ds, \quad t \geq 0, \quad (10.3)$$

and define the inverse $H(u)$ of $\beta(t)$ by

$$H(u) = \inf\{t > 0; \beta(t) \geq u\}. \quad (10.4)$$

Then $\{B(t); t \geq 0\}$, where $B(t) = V(H(t))$, is a complex Brownian motion starting at 1. It can be written as

$$B(t) = B^{(1)}(t) + iB^{(2)}(t) = \varrho(t) \exp(i\theta(t)), \quad (10.5)$$

where $\{B^{(1)}(t); t \geq 0\}$ and $\{B^{(2)}(t); t \geq 0\}$ are two independent one-dimensional Brownian motions, $B^{(1)}(0) = 1$, $B^{(2)}(0) = 0$; $\varrho(t) = |B(t)|$ is the radial part and $\theta(t)$ records the total winding of B around 0 up to time t . Since $\{B(t); t \geq 0\}$ almost surely never hits 0, $\theta(t)$ is correctly defined, has continuous paths, $\theta(0) = 0$, and $\sin \theta(t) = B^{(2)}(t)/\varrho(t)$.

Since we have constructed $B(t)$ as $V(H(t))$, we easily obtain the so-called skew product representation,

$$\varrho(t) = \exp[W^{(1)}(H(t))], \quad t \geq 0, \quad (10.6)$$

$$\theta(t) = W^{(2)}(H(t)), \quad t \geq 0. \quad (10.7)$$

Moreover,

$$\beta(t) = \int_0^t |V(s)|^2 ds = \int_0^t \exp(2W^{(1)}(s)) ds, \quad (10.8)$$

and, therefore, $\{H(t); t \geq 0\}$ and $\{W^{(2)}(t); t \geq 0\}$ are independent. The process $\{H(t); t \geq 0\}$ is called the winding clock process.

Notice that, if we define the inverse $H_Z(u)$ of the process $Z(t)$ defined in (9.2) by

$$H_Z(u) = \inf\{t > 0; Z(t) \geq u\} = \inf\left\{t > 0; \int_0^t \exp(W(s)) ds \geq u\right\}, \quad (10.9)$$

then, by the scaling property of Brownian motion,

$$\{H(u); u \geq 0\} \stackrel{\mathcal{D}}{=} \left\{\frac{1}{4}H_Z(4u); u \geq 0\right\}. \quad (10.10)$$

Let $\{S^+(u); u \geq 0\}$ be the first passage time process of $\{W^{(1)}(t); t \geq 0\}$, ie..

$$S^+(u) = \inf\{t \geq 0; W^{(1)}(t) \geq u\}, \quad u \geq 0.$$

By (6.1), $\{C(t); t \geq 0\} = \{W^{(2)}(S^+(t)); t \geq 0\}$ is a Cauchy process. Spitzer (1958) showed that

$$\frac{2\theta(t)}{\log t} \xrightarrow{\mathcal{D}} C(1) \quad \text{as } t \rightarrow \infty. \quad (10.11)$$

The idea is to show that $H(\exp(2t))$ is ‘close’ to $S^+(t)$ (cf. Theorem 9.D), and then

$$\frac{\theta(\exp(2t))}{t} = \frac{W^{(2)}[H(\exp(2t))]}{t}$$

is ‘close’ to

$$\frac{W^{(2)}(S^+(t))}{t} \xrightarrow{\mathcal{D}} C(1),$$

see Durrett (1984), pp. 136–138, for more details.

In the following three sections we will study the upper and lower classes of $\{H(t); t \geq 0\}$, $\{\theta(t); t \geq 0\}$, and of Lévy’s stochastic area, respectively.

10.2 Integral Tests for the Winding Clock

In this section we will use the relationship (10.10) also pointed out in Gruet and Shi (1995a) between the process $\{Z(t); t \geq 0\}$ and the Brownian winding clock to investigate the upper and lower classes of the latter. We can get the lower and upper classes of $\{H(u); u \geq 0\}$ from Theorems 9.1 and 9.2 using (10.10) and Lemmas 4.1 and 4.3.

Theorem 10.1. *Let ξ be such that $\xi \circ \exp \in \Xi_3$. Then*

$$P \left\{ H(u) \leq \frac{1}{8} \cdot \frac{\log^2 u}{\xi^2(u)} \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_1(\xi \circ \exp) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (10.12)$$

Corollary 10.1.

$$\liminf_{u \rightarrow \infty} \frac{H(u)}{\log^2 u} \xi^2(u) = \frac{1}{8c^*(\xi \circ \exp)}.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 10.2. *The integral test (10.12) remains true for $\xi \circ \exp \in \Psi_2^\alpha$ and $\xi \circ \exp \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 10.1. Notice that

$$J_1(\xi \circ \exp) < \infty \iff \int_0^\infty \frac{\max(1, \xi(s))}{s \log s} \exp(-\xi^2(s)) ds < \infty.$$

Theorem 10.2. *Let ξ be such that $\xi \circ \exp \in \Xi_2$. Then*

$$P \left\{ H(u) \geq \log^2(u) \xi^2(u) \text{ i.o. as } u \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } J_0(\xi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (10.13)$$

Corollary 10.3.

$$\limsup_{u \rightarrow \infty} \frac{H(u)}{\log^2(u) \xi^2(u)} = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s. according as } J_0(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

By Lemmas 2.17 and 2.18 we immediately obtain

Corollary 10.4. *The integral test (10.13) remains true for $\xi \circ \exp \in \Psi_2^\alpha$ and $\xi \circ \exp \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 10.2. If we denote $f(u) = \log(u)\xi(u)$ in Theorem 10.2, then $P\{H(u) \geq f^2(u) \text{ i.o. as } u \rightarrow \infty\} = 0 \text{ or } 1$ according as $\mathcal{J}_1(f)$ converges or diverges.

Remark 10.3. Theorem 10.1 was proved in Gruet and Shi (1995a), assuming, in addition, that $\xi \circ \exp \in \Xi_1$ (see their Theorem 3). Theorem 10.2 was proved by Bertoin and Werner (1994b) with $\xi \circ \exp \in \Xi_2$ (see their Theorem 2).

10.3 Integral Tests for Windings

Consider the winding process $\{\theta(t); t \geq 0\}$ and define

$$\theta^+(t) = \sup_{0 \leq s \leq t} \theta(s), \quad t \geq 0, \quad (10.14)$$

$$\theta^*(t) = \sup_{0 \leq s \leq t} |\theta(s)|, \quad t \geq 0. \quad (10.15)$$

Bertoin and Werner (1994a,b) proved, using the connection of $\theta(t)$ with Cauchy process, the following theorems.

Theorem 10.A. *Let $\psi \circ \exp \in \Psi_3^1$. Then*

$$\limsup_{t \rightarrow \infty} \frac{\theta(t)}{\psi(t) \log t} = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s. according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Theorem 10.B. *Let $\psi \circ \exp \in \Psi_2^1$. Then*

$$\liminf_{t \rightarrow \infty} \frac{\theta^+(t)}{\log t} \psi(t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. according as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

The lower classes of $\{\theta^*(t); t \geq 0\}$ were studied in Shi (1994), assuming that $\psi \in \Psi_1$. We remove this condition and prove

Theorem 10.3. *Let $\psi \circ \exp \in \Psi_2^2$. Then*

$$P \left\{ \theta^*(t) \leq \frac{\pi}{4} \cdot \frac{\log t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_2(\psi \circ \exp) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (10.16)$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 10.5. *The integral test (10.16) remains true for $\psi \circ \exp \in \Psi_2^\alpha$ and $\psi \circ \exp \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Corollary 10.6.

$$\liminf_{t \rightarrow \infty} \frac{\theta^*(t)}{\log t} \psi^2(t) = \frac{\pi}{4c^*(\psi)} \quad a.s.$$

Remark 10.4.

$$\mathcal{I}_2(\psi \circ \exp) < \infty \iff \int^\infty \frac{\max(1, \psi^2(t))}{t \log t} \exp(-\psi^2(t)) dt < \infty.$$

We will need two lemmas from Shi (1994).

Lemma 10.1. *Let $\{W(t); t \geq 0\}$ be a standard Brownian motion, and $\{D(t); t \geq 0\}$ be a positive nondecreasing continuous process independent of $\{W(t); t \geq 0\}$. Let*

$$W^*(D(t)) = \sup_{0 \leq s \leq t} |W(D(s))|.$$

Then for all $0 \leq s \leq t$ and $0 < x \leq y$,

$$\frac{8}{3\pi} E \exp \left[-\frac{\pi^2}{8x^2} D(t) \right] \leq P \{ W^*(D(t)) < x \} \leq \frac{4}{\pi} E \exp \left[-\frac{\pi^2}{8x^2} D(t) \right], \quad (10.17)$$

$$P\{W^*(D(s)) < x, W^*(D(t)) < y\} \leq \frac{16}{\pi^2} E \exp \left[-\frac{\pi^2}{8x^2} D(s) - \frac{\pi^2}{8y^2} (D(t) - D(s)) \right]. \quad (10.18)$$

The above estimates are obtained by conditioning on $\{D(t); t \geq 0\}$ and using (3.8).

Lemma 10.2. *For all $u \geq v > 0$, $s > 0$, and $t > 0$, we have*

$$E \exp \left[-\frac{u^2}{2} H(t) \right] \leq 2 \exp \left(-\frac{u}{2} \log t \right), \quad (10.19)$$

$$E \exp \left[-\frac{u^2}{2} H(t) - \frac{v^2}{2} (H(t+s) - H(t)) \right] \leq 4 \exp \left(-\frac{u}{2} \log s - \frac{u-v}{2} \log t \right). \quad (10.20)$$

If, moreover, $0 \leq u \leq 1$, then

$$E \exp \left[-\frac{u^2}{2} H(t) \right] \geq \frac{1}{3} \exp \left(-\frac{u}{2} \log t \right) \exp \left(-\frac{1}{2t} \right). \quad (10.21)$$

From these two lemmas we easily get

Lemma 10.3. *For $t > 1$ and $x > 0$,*

$$P \left\{ \theta^*(t) \leq \frac{\pi}{4} \cdot \frac{\log t}{x^2} \right\} \leq \frac{8}{\pi} \exp(-x^2), \quad (10.22)$$

and for $t > 1$ and $x^2 \leq \frac{1}{2} \log t$,

$$P \left\{ \theta^*(t) \leq \frac{\pi}{4} \cdot \frac{\log t}{x^2} \right\} \geq \frac{8}{15\pi} \exp(-x^2). \quad (10.23)$$

Proof. From the skew product representation $\theta(t) = W^{(2)}(H(t))$ and Lemmas 10.1 and 10.2 we have for $y > 0$,

$$\begin{aligned} P \left\{ \sup_{0 \leq s \leq t} |W^{(2)}(H(s))| < y \right\} &\leq \frac{4}{\pi} E \exp \left[-\frac{\pi^2}{8y^2} H(t) \right] \\ &\leq \frac{8}{\pi} \exp \left[-\frac{\pi}{4y} \log t \right]. \end{aligned}$$

Putting $y = (\pi/4) \log t/x^2$ we obtain (10.22).

Using Lemmas 10.1 and 10.2 again, we have for $y \geq \pi/2$.

$$\begin{aligned} P\left\{\sup_{0 \leq s \leq t} |W^{(2)}(H(s))| < y\right\} &\geq \frac{8}{3\pi} E \exp\left[-\frac{\pi^2}{8y^2} H(t)\right] \\ &\geq \frac{8}{15\pi} \exp\left[-\frac{\pi}{4y} \log t\right]. \end{aligned}$$

since $\exp(-1/(2t)) \geq 0.6$ for $t > 1$. Putting $y = (\pi/4) \log t/x^2$ and noticing that $x^2 \leq \frac{1}{2} \log t$ implies $y \geq \pi/2$, we obtain (10.23). $\square$

Proof of Theorem 10.3. By Lemmas 2.15 and 2.16, it is sufficient to consider $\psi \circ \exp \in \Psi_4^2$.

Put $z = \log t$ and $\varphi(z) = \psi(e^z)$. Assume that $J_2(\varphi) < \infty$. By Lemma 2.1, $\varphi(z) \rightarrow \infty$ as $z \rightarrow \infty$. Define z_k and φ_k as in Lemma 2.10, ie.,

$$\begin{aligned} z_{k+1} &= z_k \frac{\varphi_k^2}{\varphi_k^2 - 1}, \quad \varphi_k = \varphi(z_k), \\ \log t_{k+1} &= \log t_k \cdot \frac{\psi_k^2}{\psi_k^2 - 1}, \quad \psi_k = \psi(t_k) = \varphi_k. \end{aligned}$$

Put

$$A_k = \left\{ \theta^*(t) \leq \frac{\pi}{4} \cdot \frac{\log t}{\psi^2(t)} \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Since both $\theta^*(t)$ and $\log t/\psi^2(t) = z/\varphi^2(z)$ are nondecreasing and using (10.22) we have

$$\begin{aligned} P(A_k) &\leq P\left\{\theta^*(t_k) \leq \frac{\pi}{4} \cdot \frac{\log t_{k+1}}{\psi_{k+1}^2}\right\} \\ &\leq \frac{8}{\pi} \exp\left(-\frac{\log t_k}{\log t_{k+1}} \psi_{k+1}^2\right) \approx \exp(-\psi_k^2), \end{aligned}$$

which sums by Lemma 2.10, and the Borel–Cantelli lemma yields the convergent part of the theorem.

Now assume that $J_2(\varphi) = \infty$. The case $\varphi(z) \rightarrow \infty$ as $z \rightarrow \infty$ is obvious, so we only need to treat the case $\varphi(z) \rightarrow \infty$ as $z \rightarrow \infty$. We may assume, without loss of generality, that $2 \leq \varphi^2(z) \leq 2 \log \log z$. Now put

$$A_k = \left\{ \theta^*(t_k) \leq \frac{\pi}{4} \cdot \frac{\log t_k}{\psi_k^2} \right\}.$$

Since $\frac{1}{2} \log t_k \geq \psi_k^2$, we have from (10.23),

$$P(A_k) \geq \frac{8}{15\pi} \exp(-\psi_k^2).$$

Thus

$$P(A_k) \approx \exp(-\psi_k^2), \quad (10.24)$$

and, therefore, $\sum_k P(A_k) = \infty$.

For $0 \leq s \leq t$ and $0 < x \leq y$, Lemmas 10.1 and 10.2 give,

$$\begin{aligned} P \left\{ \sup_{0 \leq u \leq s} |W^{(2)}(H(u))| < x, \sup_{0 \leq u \leq t} |W^{(2)}(H(u))| < y \right\} \\ \leq \frac{16}{\pi^2} E \exp \left[-\frac{\pi^2}{8x^2} H(s) - \frac{\pi^2}{8y^2} (H(t) - H(s)) \right] \\ \leq \frac{64}{\pi^2} \exp \left[-\frac{\pi}{4y} \log(t-s) - \frac{\pi}{4} \left(\frac{1}{x} - \frac{1}{y} \right) \log s \right]. \end{aligned}$$

Hence, for $l > k$,

$$\begin{aligned} P(A_k \cap A_l) &\leq C \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{\log(t_l - t_k) - \log t_k}{\log t_l} \right) \\ &= C \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{\log t_l - \log t_k}{\log t_l} \right) \exp \left(\psi_l^2 \frac{\log t_l - \log(t_l - t_k)}{\log t_l} \right) \end{aligned}$$

for some $C > 0$. Since $\psi^2(t)/\log t$ is nonincreasing and $\log t - \log(t - t_k)$ is nonincreasing for $t > t_k$, we have

$$\begin{aligned} \frac{\psi_l^2}{\log t_l} (\log t_l - \log(t_l - t_k)) &\leq \frac{\psi_{k+1}^2}{\log t_{k+1}} (\log t_{k+1} - \log(t_{k+1} - t_k)) \\ &= \frac{\psi_{k+1}^2}{\log t_{k+1}} \log \left(1 + \frac{1}{t_k^{1/(\psi_k^2 - 1)} - 1} \right). \end{aligned}$$

Since $\psi_k^2 / \log t_k \searrow 0$ as $k \rightarrow \infty$, we have for a positive constant c and large enough k ,

$$\begin{aligned} \frac{\psi_{k+1}^2}{\log t_{k+1}} &< c, \\ \frac{1}{t_k^{1/(\psi_k^2-1)} - 1} &< c, \\ \exp \left[\frac{\psi_{k+1}^2}{\log t_{k+1}} (\log t_{k+1} - \log(t_{k+1} - t_k)) \right] &< \exp(c^2), \end{aligned}$$

and, therefore,

$$P(A_k \cap A_l) \leq C' \exp(-\psi_k^2) \exp \left(-\psi_l^2 \frac{\log t_l - \log t_k}{\log t_l} \right)$$

for some positive constant C' , all sufficiently large k , and $l > k$. This estimate exactly corresponds to (3.11), so the proof can be finished along the lines of Theorem 3.2. $\square$

10.4 Integral Test for Lévy's Stochastic Area

Shi (1994) also studied lower classes of Lévy's stochastic area $\sigma(t)$, where

$$\sigma(t) = \int_0^t B^{(1)}(u) dB^{(2)}(u) - B^{(2)}(u) dB^{(1)}(u), \quad t \geq 0. \quad (10.25)$$

Put

$$\sigma^*(t) = \sup_{0 \leq s \leq t} |\sigma(s)|. \quad (10.26)$$

Theorem 10.4. *Let $\psi \in \Psi_2^2$. Then*

$$P \left\{ \sigma^*(t) \leq \frac{\pi}{2} \cdot \frac{t}{\psi^2(t)} \text{ i.o. as } t \rightarrow \infty \right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as } \mathcal{I}_2(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \quad (10.27)$$

Corollary 10.7.

$$\liminf_{t \rightarrow \infty} \frac{\sigma^*(t)}{t} \psi^2(t) = \frac{\pi}{2c^*(\psi)} \quad a.s.$$

By Lemmas 2.15 and 2.16 we immediately obtain

Corollary 10.8. *The integral test (10.27) remains true for $\psi \in \Psi_2^\alpha$ and $\psi \in \Psi_3^\alpha$ for every $\alpha > 0$.*

Remark 10.5. Shi (1994) proved the above theorem assuming that ψ and $t/\psi^2(t)$ are nondecreasing.

Proof. By Lemmas 2.15 and 2.16, it is sufficient to consider $\psi \in \Psi_4^2$. Using probability estimates in Shi (1994) and our Theorem 3.2, the proof will be quite easy.

Shi (1994) represented $\sigma(t)$ as

$$\sigma(t) = W(D(t)), \quad (10.28)$$

where

$$D(t) = \int_0^t \varrho^2(u) du,$$

and $\{W(t); t \geq 0\}$ is a standard Brownian motion, independent of $\{D(t); t \geq 0\}$, and calculated the estimates (for positive numbers s, t, u , and v)

$$\exp(-ut) \leq E \exp \left[-\frac{u^2}{2} D(t) \right] \leq 2 \exp(-ut), \quad (10.29)$$

$$E \exp \left[-\frac{u^2}{2} D(t) - \frac{v^2}{2} (D(t+s) - D(t)) \right] \leq 4 \exp(-ut - vs). \quad (10.30)$$

Assume that $J_2(\psi) < \infty$. Then we know that $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Define t_k and ψ_k as in Lemma 2.10. Put

$$A_k = \left\{ \sigma^*(t) \leq \frac{\pi}{2} \cdot \frac{t}{\psi^2(t)} \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Then, by monotonicity of $\sigma^*(t)$ and $t/\psi^2(t)$, (10.28), (10.29), and Lemma 10.1, we obtain,

$$\begin{aligned} P(A_k) &\leq P\left\{\sigma^*(t_k) \leq \frac{\pi}{2} \cdot \frac{t_{k+1}}{\psi_{k+1}^2}\right\} \\ &\leq C \exp(-\psi_{k+1}^2), \end{aligned}$$

where C is a positive constant. By Lemma 2.10, $\sum_k P(A_k) < \infty$. The Borel-Cantelli lemma finishes the proof of the convergent part of the theorem.

Now assume that $J_2(\psi) = \infty$. We may consider only the case $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Put

$$A_k = \left\{\sigma^*(t_k) \leq \frac{\pi}{2} \cdot \frac{t_k}{\psi^2(t_k)}\right\}.$$

From (10.28), (10.29), and Lemma 10.1, we have

$$P(A_k) \approx \exp(-\psi_k^2).$$

and using also (10.30) we obtain for $l > k$,

$$P(A_k \cap A_l) \leq C_1 \exp(-\psi_k^2) \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l}\right),$$

where C_1 is a positive constant. The proof can be finished along the lines of the proof of Theorem 3.2. □

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IMAGE EVALUATION TEST TARGET (QA-3)

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