**DETERMINATION OF T -STRESS SOLUTIONS FOR
CRACKED 2D ANISOTROPIC ELASTIC BODIES USING
THE BOUNDARY ELEMENT METHOD**

By

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B.E. (Honours), M.E.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

Ottawa-Carleton Institute for Mechanical and Aerospace Engineering

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Carleton University

Ottawa, Ontario, Canada

June, 2007

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ISBN: 978-0-494-33521-5

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ISBN: 978-0-494-33521-5

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ABSTRACT

The importance of the two-parameter approach in linear elastic fracture mechanics (LEFM) analysis is increasingly being recognized for fracture assessments in engineering applications. The consideration of the second parameter, namely, the elastic T -stress, together with the stress intensity factor represents the near crack-tip field more accurately and provides a better explanation to various fracture behaviour. Although numerical and analytical schemes to obtain stress intensity factors for cracks in anisotropic solids are well established, the same cannot be said for obtaining the T -stress. In the present work, path independent M -integral schemes are developed to evaluate this quantity in two-dimensional anisotropic elasticity. They include the treatment of cracks in homogeneous solids as well as bi-material interface cracks. The Boundary Element Method (BEM) is employed as the numerical tool due to its efficiency for treating problems in LEFM. A self-regularisation technique is implemented into the BEM algorithm to obtain accurate interior point field solutions for the M -integral evaluation. Following the validation of these schemes, T -stress solutions for several problems of engineering significance are obtained. The results show a significant effect of geometry as well as material properties on the T -stress; they are expected to serve as reference solutions for future studies in this area. The weight function method to obtain the T -stress is also developed here; weight-function solutions provide an efficient means to evaluate the T -stress in applications with complex loading conditions. The veracity of this approach is demonstrated with some examples.

*In the loving memory of my grandfather, Bhakta Bahadur Singh Thakuri who showed me
the path to truth, patience, humanity and knowledge.*

ACKNOWLEDGEMENTS

First of all, nothing would be possible without the blessings of the almighty.

I would like to thank my parents, Gen. Shambhu D. Shah and Mrs. Manju Shah, for always reminding me that the knowledge is the greatest asset in this world. Thanks to my father for sparking my interest in science since my childhood and to my mother for being the pillar of strength in every endeavour I embarked on. I am grateful to them for the never ending supply love and support. Warm thanks to my brother Prabhav D. Shah who has always given me unconditional support and has helped me through the toughest decisions. Lots of love to my daughter Pratha who never complained my unavailability during the thesis write-up and always remembers to brighten my day with a sweet smile. I want to thank my wife, Pradha Shah, who has always been my better half and supported me through thick and thin. It was the everlasting sweet memories of my grandparents, aunt Maiyya and Akshat that helped propel me throughout this work.

I owe my sincere gratitude to my supervisors, Dr. Choon-Lai Tan and Dr. Xin Wang, for their continuous support and excellent advice during these four years. I am grateful to them for going that extra mile to help and guide me throughout my PhD work. I want to thank Dr. Song at the University of New South Wales, Australia for verifying some of the numerical results here using an alternative method, namely, the scaled boundary finite-element method. I would also like to thank Dr. Jonathan C. Beddoes for his undivided support. Sincere appreciation goes to Marlene, Nancy and Christie who were always there whenever I needed help with administrative issues in the Department. I would also like to thank my colleagues for making my experience at Carleton University an excellent one.

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NOMENCLATURE

a	crack length
$\mathbf{a}$	reduced elastic compliance matrix
a_{mn}	elastic compliances, $m, n = 1, 2, \dots, 6$
a'_{mn}	transformed elastic compliances in rotated co-ordinates, $m, n = 1, 2, \dots, 6$
B	biaxiality ratio
BEM	boundary element method
BIE	boundary integral equation
$\mathbf{C}$	stiffness matrix
CCP	centre cracked plate
$C^{(m)}$	material specific coefficient for T -stress.
C_{ij}	reduced elastic stiffness coefficients, $i, j = 1, 2, \dots, 6$
C_{ijks}	elastic stiffness coefficients, $i, j, k, s = 1, 2$
$\mathbf{C}^0$	principal submatrix of stiffness matrix
DECP	double edge cracked plate
D_i	weight function parameters for T -stress
D_{kij}	kernel of Somigliana's identity for strain
$\overline{D_{kij}}$	kernel of Somigliana's identity for stress
E	Young's modulus for isotropic solids
E'	Young's modulus for plane stress or plane strain conditions
f	point load
$\mathbf{f}$	force vector

FEM	finite element method
G	energy release rate
G	shear modulus for isotropic elastic solids
G₁₂	shear modulus in x_1 - x_2 plane.
G_C	critical energy release rate
G_I	energy release rates for mode I crack extension
G_{II}	energy release rates for mode II crack extension
H	Barnett-Lothe tensor
H	height of the cracked geometry
Im (.....)	imaginary component of the complex terms in the parenthesis
J	<i>J</i> -integral
J(ξ)	Jacobian of transformation from the coordinate x_i to the intrinsic coordinate ξ
K	stress intensity factor
K	complex stress intensity factor for bimaterial interface cracks
K^c_{mat}	constraint based fracture toughness
K_I	mode I stress intensity factor
K_{IC}	mode I fracture toughness for plane strain conditions
K_{II}	mode II stress intensity factor
K_{IIC}	mode II fracture toughness for plane strain conditions
K_{mat}	material fracture toughness
K₀	stress intensity factor modulus for bimaterial interface cracks
L	characteristic length

L	Barnett-Lothe tensor
LEFM	linear elastic fracture mechanics
m	material designation for bimaterial interface
M	Mutual integral; number of boundary elements
$m(x_1)$	weight function for mode I loading
m_j	direction cosines, $j = 1, 2$
N	fundamental elasticity matrix
N	number of nodes along the boundary
$n(x_1)$	weight function for mode II loading
$N^c(\xi)$	shape function for quadratic isoparametric element
n_j	unit outward normal and tangent for $j = 1, 2$, respectively
P	source point along the boundary in BEM analysis
p	interior source point in BEM analysis
$P^{d(b,c)}$	c th node of the b th element
P^a	a th node of the mesh
Q	field point along the boundary in BEM analysis
Q, R, T	elasticity matrices
r	radius for polar co-ordinates; distance between source point and field point; radius
Re (.....)	real component of the complex terms in the parenthesis
S	boundary of the problem domain
S	Barnett-Lothe tensor
SBFEM	scaled boundary finite-element method

SECP	single edge cracked plate
S_{kij}	kernel of Somigliana's identity for strain
$\overline{S_{kij}}$	kernel of Somigliana's identity for stress
SSY	small-scale yielding
T	T -stress
t_i	traction, $i, j = 1, 2$
$\bar{t}_i$	traction coefficient, $i, j = 1, 2$
T_{jk}	fundamental solution for traction, $j, k = 1, 2$
$\mathbf{u}$	displacement vector
u_i	displacement, $i, j = 1, 2$
$u_{i,j}$	displacement derivative, $i, j = 1, 2$
U_{jk}	fundamental solution for displacement, $j, k = 1, 2$
W	width of the cracked geometry
x_i	local co-ordinate axes
X_i	global co-ordinate axes
X'_i	displaced global co-ordinate axes
Φ	stress function
Φ	stress function vector
Γ	contour
ΔK	stress intensity factor range for fatigue loading
α	real component of the complex parameter; order of polynomial; angle

α_{ij}	coefficients for J -integral and stress intensity factor relation, $i, j = 1, 2$
β	imaginary component of the complex parameter
χ	Airy's stress function
ε	circular contour radius; oscillatory index
ε_{ij}	Strain tensor, $i, j = 1, 2$
ϕ	angle for polar co-ordinates
γ	bimaterial constant
$\eta_{jk,l}$	the coefficient of mutual influence of the first kind, $j, k, l = 1, 2$
$\eta_{l,jk}$	the coefficient of mutual influence of the second kind, $j, k, l = 1, 2$
λ	angle
μ_{ij}	"complex parameter", $i, j = 1, 2$
ν	Poisson's ratio for isotropic solids
ν_{jk}	Poisson's ratio for anisotropic solids: the ratio of compressive strain in x_k direction due to a unit extensional strain in x_j direction
σ_{ij}	Stress tensor, $i, j = 1, 2$
ω	angle of crack inclination with the displaced global co-ordinate axes
ξ	intrinsic co-ordinates
ψ	angle between material principal axes with displaced global Cartesian co-ordinate axes; phase angle for stress intensity factors for bimaterial interface crack

CHAPTER 1

INTRODUCTION

Cracks often form in mechanical and structural components in engineering applications which are subjected to various types of loading and constraints under service conditions. Assessing the structural integrity of these components with cracks is therefore very important. To this end, linear elastic fracture mechanics, abbreviated as LEFM, has been well established as an important analysis tool. Conventional LEFM analysis is primarily focused on the stress intensity factor, K , which characterises the severity of the singular stresses in the vicinity of the crack tip; failure by fracture is predicted to occur when the stress intensity factor reaches a critical value which is assumed to be a material property constant. K is well known to be the leading singular term in Williams' (1957) series expansion for the elastic stress field ahead of a crack tip. Its use and the methods to obtain its numerical values for fracture assessments are also well established in the literature. However, there are still some shortcomings in using this parameter alone in conventional fracture mechanics to accurately predict and explain some key fracture phenomena such as, crack tunnelling, crack path stability and discrepancies in the fracture toughness in various test specimens of the same material. A number of studies over the years by various researchers, e.g., Larsson and Carlsson (1973), Rice (1974),

Cotterell and Rice (1980), have suggested that the additional consideration of the leading non-singular parameter in the Williams' series expansion could offer better explanations and predictions for the above mentioned crack behaviour. This leading non-singular term is referred in the literature as " T -stress"; it provides a measure of the crack-tip stress constraint level. Studies have suggested that the importance of T -stress becomes more relevant for the cases with low crack-tip constraint; its inclusion along with the stress intensity factor can then provide more accurate and realistic assessment of the crack behaviour. In problems with low crack-tip stress constraint, the consideration of only the stress intensity factor to predict the failure load is over-conservative, see, e.g., Larsson and Carlsson (1973).

Over the years, numerous works have been carried out to further establish the significance of the T -stress in LEFM analysis. Several analytical and numerical schemes have also been developed to evaluate T -stress for isotropic cracked solids. However, the corresponding work for anisotropic cracked solids remains very limited, even though some experimental and numerical studies have suggested similar significance of this additional parameter in LEFM analysis. Given the fact that anisotropic materials are increasingly being used in modern engineering applications, e.g., composites, single crystal alloys and bicrystals, the study of T -stress for cracks in such materials is of great importance. This includes cracks in homogeneous anisotropic solids as well as those lying at the interfaces between multi-phase materials. Indeed, to the best of the author's knowledge, no numerical scheme based on the boundary element method for such crack problems in anisotropy have been reported in the open literature until recently, Shah *et al.* (2006a,b) and Shah *et al.* (2006), based on work presented in this thesis.

The boundary element method (BEM), also known as the boundary integral equation (BIE) method, is well established as an efficient numerical tool in elastostatics to analyse problems with rapidly varying stress distributions, e.g., the problems in fracture mechanics. One of the advantages of this method over several other methods, e.g., finite element method, is that only the boundary of the problem domain needs to be modelled. In spite of its established use for fracture mechanics analysis (see, e.g., Cruse, 1988, Aliabadi, 2002, Tan and Gao, 1992b, Denda and Dong, 1999, Denda and Marante, 2004), it is surprising to note that the BEM has not been much employed for T -stress analysis. It has been used fairly recently for isotropy, but remains extremely limited in anisotropy.

Among the various methods available to extract T -stress in two-dimensional isotropy, the method based on the conservation M -integral (see, e.g., Sladek *et al.*, 1997b) in conjunction with the BEM is of relevance to the present work. Unlike other schemes which depend upon the near crack-tip field solutions, this method relies on field solutions at points remote from the crack tip. This reduces numerical errors introduced by the crack-tip stress singularity. An objective of this study is to develop numerical schemes based on the M -integral and BEM to obtain T -stress solutions for crack problems in two-dimensional anisotropy. In addition, the method based on weight functions for isotropy (Wang, 2002a,b) will also be adopted here and extended to anisotropy. The weight function method has evolved as an efficient numerical tool to accurately predict T -stress for cracks under complex stress fields.

This thesis is organised as follows. In the next chapter, the basics of linear elastic fracture mechanics (LEFM) is reviewed with focus on the significance of the T -stress

parameter as well as the analytical and numerical schemes which have been developed to evaluate it. The boundary element method applicable to two-dimensional anisotropy is discussed in Chapter 3. The conventional BEM suffers a drawback in accurately obtaining the field solutions for interior points close to the boundary; this issue has been addressed using a “self-regularising” technique. The procedure of Richardson and Cruse (1999) for isotropic elasticity is adopted and extended to anisotropy with some validation examples. This is followed by the development of the M -integral to evaluate T -stress for cracks in homogeneous anisotropic solids, in Chapter 4. The successful implementation of the algorithm is validated by some examples for which the solutions exist in the literature. Additional results are obtained for a comprehensive study of some standard cracked specimens to understand the effect of various geometric and material parameters on the T -stress. Chapter 5 covers the work on the extraction of T -stress solutions from M -integral for interface cracks between dissimilar anisotropic solids. The veracity of this approach is demonstrated by comparing the solutions for crack problems in isotropy obtained from this method with those available in the literature. This is because no corresponding solutions for anisotropy exist in the literature. Results obtained for some problems of engineering significance are then presented, some of which were later verified by Song (2006) using a different approach based on the scaled boundary finite-element (SBFEM) method. In Chapter 6, the reference weight function method for obtaining T -stress in isotropy is successfully extended to two-dimensional anisotropy. Finally, the work undertaken in this thesis is summarised in Chapter 7.

CHAPTER 2

LINEAR ELASTIC FRACTURE MECHANICS AND THE T -STRESS

2.0 Introduction

The theory of linear elastic fracture mechanics (LEFM) is first reviewed here. Expressions to characterize the near crack-tip fields will be presented for isotropic and anisotropic materials, as well as for cracks at bimaterial interfaces. The significance of the two-parameter LEFM approach, which includes an important fracture mechanics term, namely, the T -stress, will be discussed. The various numerical schemes which are available in the literature to extract the T -stress will then be reviewed.

2.1 Linear Elastic Fracture Mechanics (LEFM)

The failure assessment of engineering structures involves well-established methodologies, one of which is linear elastic fracture mechanics (LEFM). LEFM is based on the assumption that the material under consideration behaves linearly, even in the vicinity of the crack tip. Thus, it is valid only if the inelastic deformation at the crack tip is small in comparison with the crack size, commonly referred to as the small-scale yielding (SSY).

The crack in a continuum can be studied under LEFM using two alternative but equivalent approaches, namely, the energy release rate approach and the stress intensity factor approach. The first approach states that a crack grows when the energy release rate G (after Griffith, 1920) is equal to or greater than the critical energy value G_C , which is the resistance provided by the material. The second approach incorporates a parameter termed the stress intensity factor, K , which measures the severity of the stress field at the crack tip. When this parameter attains a magnitude equal to or greater than the critical stress intensity factor K_C , also termed as the fracture toughness of the material, failure by fracture occurs (see, e.g., Sih and Liebowitz, 1968). In two-dimensional analysis, there are two modes of crack deformation thereby giving the corresponding mode I and mode II stress intensity factors. The mode I stress intensity factor, K_I , corresponds to the crack opening deformation about the plane of the crack as given in figure (2.1). The second mode stress intensity factor, K_{II} , corresponds to the sliding or shear mode where the deformation is anti-symmetric about a plane normal to the crack front and is tangential along the crack plane as in figure (2.2). Corresponding material fracture toughness exists for both modes, K_{IC} for mode I and K_{IIc} for mode II. The equivalence of both approaches has been established to be as follows for plane stress and plane strain conditions under mixed-mode crack deformation,

$$G = G_I + G_{II} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'}, \quad (2.1)$$

$$G_C = \frac{K_{IC}^2}{E'} + \frac{K_{IIc}^2}{E'}. \quad (2.2)$$

where G_I and G_{II} are the energy release rates for mode I and mode II crack extensions, respectively, $E' = E$ for plane stress and $E' = \frac{E}{(1-\nu^2)}$ for plane strain condition, E being the Young's modulus for the isotropic material.

In the presence of a crack in a medium, there is a discontinuity at the crack tip. The expression for the near crack-tip stress field for isotropic materials was first given by Williams (1957), as follows for mixed-mode loading,

$$\sigma_{ij}(r, \phi) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^I(\phi) + \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\phi) + T \delta_{ij} + g_{ij}(\phi) r^{1/2} + \dots \quad (2.3)$$

where r and ϕ are polar co-ordinates with the origin at the crack-tip as shown in figure (2.3); the terms $f_{ij}^I(\phi)$, $f_{ij}^{II}(\phi)$ and $g_{ij}(\phi)$ are the angular functions; and T is the T -stress.

In traditional LEFM analysis only the first two singular terms in equation (2.3) consisting of K_I and K_{II} are taken into consideration to characterize the stress field in the vicinity of the crack tip. This approach proposes that fracture in brittle materials and in elastic-plastic materials, provided the SSY condition is maintained in the latter case, can be predicted with reference to the critical stress intensity factor or fracture toughness, K_{IC} , which is assumed to be an invariant material property. This conventional approach is also termed as the one-parameter approach, the parameter being stress intensity factor K (for brittle material in which SSY can be attained) or J -integral (for brittle as well as ductile material). The J -integral (Rice, 1968) is a path independent line integral around the crack tip and its definition stems from the law of conservation applicable to a deformed elastic medium. Rice (1968) showed that for linear elastic solids, the J -integral represents the

strain energy release rate (strain energy per unit area). For a contour around a crack tip, as shown in figure (2.4), J -integral is expressed as,

$$J = \int_{\Gamma} (W n_1 - t_i u_{i,1}) d\Gamma, \quad (2.4)$$

where

$i, j = 1, 2$, W is the strain energy density given by $W = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}$, t_i is the traction vector,

Γ is an arbitrary contour around the crack tip and n_1 is the x_1 -component of the unit outward vector normal to the contour Γ . Γ is equal to,

$$\Gamma = \Gamma_0 + \Gamma_{c+} + \Gamma_{c-} \quad (2.5)$$

where Γ_0 is the section of the contour excluding the crack faces and Γ_{c+} and Γ_{c-} are the sections along the upper and lower crack faces, respectively. For the cases when the crack-face load is absent, Γ_{c+} and Γ_{c-} vanish. Similarly, for the case of symmetric crack-face load, since t_i and n_1 with respect to the local co-ordinates, x_i , are equal and opposite for the upper and lower crack faces, these components vanish. If a non-symmetric crack load exists, this component is non zero and should be taken into consideration.

For elastic solids under small scale yielding, the J -integral can be expressed in terms of Griffith's energy release rate, G , and thus the stress intensity factor K can be expressed as (Rice, 1968),

$$J = G = \frac{K_I^2}{E'} \text{ for single mode (in this case, mode I) crack extension and} \quad (2.6.a)$$

$$J = G_I + G_{II} = \frac{1}{E'} (K_I^2 + K_{II}^2) \text{ for mixed-mode crack extension.} \quad (2.6.b)$$

In order to characterize the near crack-tip fields, the conventional one-parameter approach ignores the contribution from higher order terms in the Williams' series expansion. The significance of the leading non-singular parameter in the Williams' series expansion, T -stress, in fracture assessment was studied more widely somewhat later. These studies gave rise to a two-parameter approach being established in which the stress intensity factor together with T -stress are employed to characterize the near crack-tip fields. The T -stress is the stress acting parallel to the crack plane. It is the only non-singular term in the Williams' series expansion (equation 2.3) which does not vanish at the crack tip due to its independence on the distance r from the crack tip. As will be further discussed below, the T -stress quantifies the level of stress constraint at the crack tip. A positive T -stress implies a rise in the crack-tip stress triaxiality and elevation in the crack-tip stress constraint level. Conversely, a negative T -stress suggests a drop in the crack-tip triaxiality and thus a decrease in crack-tip stress constraint level. The stress triaxiality at the crack tip can be expressed in terms of the ratio between the hydrostatic or mean stress, $\sigma_m = \frac{1}{3}\sigma_{kk}$, and the von Mises stress, $\bar{\sigma}$, as $\sigma_m/\bar{\sigma}$. The nomenclature for the leading non-singular term, T -stress, in Williams' series expansion can be attributed to Rice (1974). In general, the T -stress is expressed in a dimensionless form normalised by the applied stress, σ_0 , as,

$$\bar{T} = \frac{T}{\sigma_0}. \quad (2.7.a)$$

Alternatively, the T -stress can also be normalised with respect to K_I , thus giving a biaxiality ratio B , introduced by Leever and Radon (1982) as,

$$B = T \frac{\sqrt{\pi a}}{K_I}. \quad (2.7.b)$$

Sih *et al.* (1965) has shown that the stress variation near the crack tip in a homogeneous anisotropic material follows a similar singular trend as in the case of isotropy. The only difference between the Williams' series expansion for isotropy and anisotropy lies in the expressions for angular functions $f_{ij}^I(\phi)$ and $f_{ij}^{II}(\phi)$. Thus the definition of the stress intensity factor remains valid in the case of a homogeneous anisotropic cracked body. The explicit expressions for the stress field along the crack plane near a crack tip in a homogeneous anisotropic medium are given as follows (Sih *et al.*, 1965),

$$\begin{aligned} \sigma_{11}(r, \phi) = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left\{ \frac{\mu_2}{\sqrt{\cos \phi + \mu_2 \sin \phi}} - \frac{\mu_1}{\sqrt{\cos \phi + \mu_1 \sin \phi}} \right\} \right] \\ & + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} \left\{ \frac{(\mu_2)^2}{\sqrt{\cos \phi + \mu_2 \sin \phi}} - \frac{(\mu_1)^2}{\sqrt{\cos \phi + \mu_1 \sin \phi}} \right\} \right] + T, \end{aligned} \quad (2.8.a)$$

$$\begin{aligned} \sigma_{22}(r, \phi) = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} \left\{ \frac{\mu_1}{\sqrt{\cos \phi + \mu_2 \sin \phi}} - \frac{\mu_2}{\sqrt{\cos \phi + \mu_1 \sin \phi}} \right\} \right] \\ & + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} \left\{ \frac{1}{\sqrt{\cos \phi + \mu_2 \sin \phi}} - \frac{1}{\sqrt{\cos \phi + \mu_1 \sin \phi}} \right\} \right], \end{aligned} \quad (2.8.b)$$

$$\begin{aligned} \sigma_{12}(r, \phi) = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left\{ \frac{1}{\sqrt{\cos \phi + \mu_1 \sin \phi}} - \frac{1}{\sqrt{\cos \phi + \mu_2 \sin \phi}} \right\} \right] \\ & + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} \left\{ \frac{\mu_1}{\sqrt{\cos \phi + \mu_1 \sin \phi}} - \frac{\mu_2}{\sqrt{\cos \phi + \mu_2 \sin \phi}} \right\} \right]. \end{aligned} \quad (2.8.c)$$

where Re denotes the real part of the quantity within the parentheses and μ_i are material property specific (complex) variables in the direction parallel to the crack plane.

For a bimaterial interface crack, i.e., a crack lying along the interface between dissimilar materials, however, the stress singularity is of oscillatory nature and has been a matter of extensive discussion and study over the years, see, e.g., Williams (1959), Erdogan (1963), Comninou (1977, 1990), Rice (1988), Ting (1990), Suo (1990). The conventional definition of mode I and mode II stress intensity factors does not hold true here as K_I and K_{II} are always coupled. Because of this coupling, the stress intensity factors are often expressed in a complex form as $\mathbf{K} = K_I + iK_{II}$, where $i = \sqrt{-1}$. The crack closure imparted due to this oscillatory singularity can often be ignored considering the small scale contact zone (Rice, 1988) for many situations. The definition of $\mathbf{K}$ in the interfacial cracks is complete with the consideration of a characteristic length L and phase angle $\phi = \tan^{-1}(K_{II}/K_I)$ as it has been shown to be scale dependent (Rice, 1988). The stress field in the vicinity of a bimaterial interface crack tip can be expressed as follows,

$$\sigma_{ij}^{(m)} = f(\mathbf{K}, r, \gamma, \theta) + C^{(m)} \delta_{ij1} T + O(r^\alpha), \quad (2.9)$$

where γ is a bimaterial constant often known as Dundur's parameter, m denotes the material under consideration between the materials forming the interface, $\alpha > 0$ and $C^{(m)}$ is a material specific coefficient.

2.2 Significance of T -stress

A considerable amount of work has been carried out over the years to understand the significance of T -stress in linear elastic fracture mechanics. In particular, the influence of its magnitude and sign on such fracture assessment factors as the fracture toughness,

crack shape development, crack tip constraint level, crack growth direction and crack growth rate has been studied by various researchers. They are briefly reviewed here.

2.2.1 Effect of the sign of T -stress on crack-tip constraint

The evidence that T -stress has an important effect on the crack-tip constraint was first presented by Larsson and Carlsson (1973) and Rice (1974). They showed the effect of the leading non-singular term on the plastic zone shape and size around a crack tip. Specifically, the work by Larsson and Carlsson (1973) proposed a modified “boundary layer formulation” that incorporates the T -stress in addition to the stress intensity factor to obtain the stress field near the crack tip. It should be further noted that Cotterell and Rice (1980) also established that the T -stress has no influence on the value of J -integral.

Betegon and Hancock (1991) and Betegon *et al.* (1993) observed that the tensile or positive T -stress showed a minimal elevation on the stress triaxiality at the crack tip and J -dominance remained prominent for zero or positive value of this parameter. However, a compressive T -stress was found to significantly reduce the stress triaxiality and the constraint level, thus facilitating the development of plasticity around the crack (figure 2.5). Du and Hancock (1991) have further shown that the decrease in the crack-tip constraint in the plastic zone was attributed to the decrease in the hydrostatic stress by an amount as a function of T -stress. In conformance with the findings of Larsson and Carlsson (1973) and Rice (1974), Shih *et al.* (1993) observed a significant effect of compressive T -stress on crack-tip plastic zone size (figure 2.6) while the effect of positive T -stress on the plastic zone size was less significant (figure 2.7) as the plastic deformation around the crack tip was limited. The correlation between T -stress and the

hydrostatic stress, similar to the one obtained by Du and Hancock (1991) was observed by O'Dowd and Shih (1991). Bilby *et al.* (1986) also showed the reduction in the stress triaxiality at the crack tip by a compressive T -stress. The effect of the negative T -stress is to lower the crack-tip stress constraint level thus lowering the crack tip opening stress produced for the same applied K_I or J . Hence, in order to facilitate cleavage fracture, higher J will be required for a structure with higher compressive T -stress, thus causing a loss in the J -dominance. Due to this loss in the J -dominance, the consideration of two-parameter approach to study the crack-tip stress constraint level is justified, especially in structures with negative T -stress. Thus, while positive T -stress increases the stress triaxiality at the crack tip, thus promoting higher crack-tip stress constraint, the reverse is true for negative T -stress.

2.2.2 Influence of T -stress on fracture toughness

Conventional LEFM is based on the assumption that the fracture toughness is a material specific property only. It should be noted that in order to determine the material fracture toughness, a compact tension specimen or occasionally a three point bending specimen is usually considered in practice and the result obtained is assumed to be valid for other specimens too. However, in the study conducted by Hancock *et al.* (1993), a noticeable difference between the fracture toughness, K_{IC} , was observed between various cracked specimens of the same material; centre cracked plate, single edge cracked plate and three point bending specimen with shallow crack showed higher toughness while three point bending specimen with deep crack and compact tension specimen showed lower toughness levels. This observed difference in fracture toughness can be attributed to the

difference between the geometry based crack-tip stress constraint level in different specimens.

Similarly, Hall (1971), in experimental studies of cracked aluminium and titanium specimens, and Shih and German (1981), in finite element analysis of bending and centre-cracked specimens, observed similar effect of the constraint on the fracture toughness. In the experimental study by Hancock *et al.* (1993) with centre-cracked specimens, an increase in the fracture toughness with negative T -stress was observed. These observations on the effect of the constraint level or T -stress in fracture toughness gave rise to the consideration of the modified constraint based fracture toughness K_{mat}^c . As discussed earlier, the effect of a high constraint level being minimal on the K_I or J dominance, the material fracture toughness K_{mat} or K_{IC} for high constraint geometries, such as compact tension and three point bending specimens, remains valid. However, for specimens with low constraint geometries and thus, compressive T -stress, e.g., centre cracked and double edge cracked plates, the K_{IC} will be lower or more conservative than the more accurate modified K_{mat}^c , as shown in figure (2.8). The effect of the negative T -stress on facilitating plastic deformation at the crack tip increases the toughness of the specimen. It can be inferred from the above discussion that the specimens of the same material demonstrating similar constraint level should have same level of toughness and this was shown by Hancock *et al.* (1993). This dependence of the fracture toughness on the constraint level is the basis for the modified failure assessment diagram proposed by Ainsworth and O'Dowd (1995) which is discussed in the next section.

2.2.3 *T*-stress in failure assessment diagram (FAD)

Since its introduction (Dowling and Townley, 1975) in R6 procedure (Harrison *et al.*, 1976) to predict the integrity of a structure with flaws, the failure assessment diagram (FAD) has been included in standard industrial design codes such as ASME Boiler and Pressure Vessel Code and British Standards Published Document PD6493 (1991). The conventional failure assessment diagram used in R6 procedure (Milne *et al.*, 1986, 1988) is as shown in figure (2.9). It comprises of a curve obtained by plotting K_r as a function of L_r where,

$$K_r = K(P, a) / K_{mat}, \quad (2.10)$$

$$L_r = P / P_L(a, \sigma_Y). \quad (2.11)$$

In the above equations, K is the stress intensity factor as a function of the applied load P and crack size a , K_{mat} is K_{IC} , the fracture resistance or toughness of the material obtained from deep cracked bend specimen and P_L is the plastic collapse load corresponding to the yield stress σ_Y . This two-parameter failure assessment diagram envelops the safe region by a curve, $K_r \leq f(L_r)$, such that, if the (K_r, L_r) point obtained from equations (2.10) and (2.11) lies within it, failure does not occur. The FAD excludes failure by plastic collapse by a cut-off at the right of the curve.

Recognising the significance of the *T*-stress in fracture analysis, Ainsworth and O'Dowd (1995) proposed the framework to include the constraint based parameter in the FAD diagram, resulting in a modified FAD curve. It has also been introduced in SINTAP (Ainsworth *et al.*, 2000) procedure, where K_r expressed as a function of L_r is modified to incorporate the constraint effect as,

$$K_r = f(L_r) \left(\frac{K_{mat}^c}{K_{IC}} \right). \quad (2.12)$$

The constraint based FAD is given in figure (2.10). It is evident from this figure that for lower crack-tip stress constraint level, there is an upward shift in the FAD curve thus making it less conservative for such conditions.

2.2.4 Effect of T -stress on crack shape development (crack tunnelling)

The corner notch (CN) specimen is widely used to study fatigue crack growth in turbine discs in aerospace applications, see e.g., Pickard *et al.* (1983). In such specimens subjected to quasi-static and static loading conditions and high temperatures, Byrne *et al.* (1993, 1997) and Tong *et al.* (1999) have observed higher crack growth rate at the mid plane than the regions near the free surface (figure 2.11). This phenomenon is termed as “crack tunnelling”. The crack-front shape development is influenced by the crack-tip stress constraint level and the load applied. It is established that high crack-tip constraint facilitates the crack growth by limiting the plastic deformation about the crack tip. Inclusion of the T -stress in the analysis has helped to provide a better understanding of the mechanics of the crack growth behaviour in the specimen.

Zhao *et al.* (2001) studied the stress intensity factor K and T -stress values for a three dimensional CN crack front using domain integral technique (Delorenzi, 1985, Li *et al.*, 1985, Shih *et al.*, 1986) and interaction integral technique (Nakamura and Parks, 1992), respectively, in conjunction with the finite element method. In the study, the CN specimens with both quarter circular as well as idealized tunnelled corner cracks were

considered with varying crack sizes. In this discussion, only the results pertaining to the tunnelled corner cracks are presented due to its relevance.

In the case of the tunnelled corner crack models, various geometries were considered as per convention given in figure (2.12) with $a_0/W = 0.2$ and 0.4 and the corresponding η values, which define the degree of tunnelling (Tong *et al.*, 1999), were varied from 0 to 1.4 for $a_0/W = 0.2$ and 0 to 0.98 for $a_0/W = 0.4$, where a_0 is the initial crack length. The term, η , is given in terms of the crack geometry as follows,

$$\eta = \frac{a_{\max} - a_f}{a_f - a_0}, \quad (2.13)$$

where $a_{\max}$ is the maximum length at the tunnelled crack front and a_f is the final crack length at the surface (figure 2.12). Tong *et al.* (1999) have shown experimentally that η varies from 0 to 2. It should be noted that the case $\eta = 0$ (no tunnelling), represents a case of quarter-circular crack of length a_0 , where, $a_{\max} = a_f = a_0$.

In the numerical study, Zhao *et al.* (2001) obtained the normalised values of K and T -stress along the crack front. The results indicate that K is lower at the mid-plane than near the free surfaces. Also, with an increase in the degree of crack tunnelling, η , the difference between K value near the mid plane and near the surfaces becomes more prominent. In contrast to the K behaviour along the crack front, the crack advance observed is higher near the mid-plane zone than near the surface. Thus, K alone is not sufficient to adequately explain the crack tunnelling phenomenon. If one considers the variation of the T -stress, it is observed to increase at the mid-plane region and gradually reduces near the free surfaces. The increase in T -stress near the mid-plane region is intensified with the increasing stress intensity factor K and the degree of tunnelling η . At

the mid-plane of the crack front, the degree of constraint is at its highest. Conversely, near the free surface, where T -stress is comparatively lower, the constraint level at the crack front also is lower. Even though K near the mid-plane region is lower than that near the surface, the higher value of T -stress near the mid-plane facilitates faster crack growth in that region thus giving rise to the crack tunnelling.

2.2.5 Role of T -stress in directional stability

The directional stability of a crack growth is defined in terms of the degree of its excursion from the original or ideal crack path. As shown in figure (2.13), a stable crack growth is directed towards the ideal path, thus making it more predictable. Unstable crack growth, also termed as out-of-plane crack growth, occurs when the crack advances gradually away from the original crack plane (indicated by the ideal path), thus making it less predictable. Cotterell (1966) devised a directional stability argument for crack growth based on the stress distribution (tangential stress σ_θ) near the crack tip, expressed in terms of power series expansion as,

$$\sigma_\theta = a_1 \left(\frac{a}{r} \right)^{1/2} \left(\frac{3}{4} \cos \left(\frac{\theta}{2} \right) + \frac{1}{4} \cos \left(\frac{3\theta}{2} \right) \right) + a_2 \sin^2 \theta + a_3 \left(\frac{r}{a} \right)^{1/2} \left(5 \cos \theta - \cos \left(\frac{5\theta}{2} \right) \right) + O \left(\frac{r}{a} \right). \quad (2.14)$$

With reference to figure (2.13), for small growth S in the direction $d\theta$, the stability of path is dependent upon the relative magnitudes of $d\theta$ and $d\phi$. When $d\theta < d\phi$, corresponding to figure (2.13.a), the path is stable and the crack progresses towards the stable ideal path. Otherwise, when $d\theta > d\phi$ as represented in figure (2.13.b), the path deviates further from the ideal one and thus the crack growth is unstable. The comparative magnitudes of $d\theta$ and $d\phi$ can be determined by considering local mode I and

mode II components at the branch crack tip assuming the direction of crack growth is dictated by the maximum tangential stress (MTS) criterion, (Erdogan and Sih, 1963) as,

$$\frac{d\phi}{d\theta} = 1 - \left(\frac{S}{a}\right)^{1/2} \left[\frac{4}{\pi} \left(\frac{a_2}{a_1}\right) + \left(\frac{a_3}{a_1}\right) \left(\frac{S}{a}\right)^{1/2} \right] \text{ for } S \ll a. \quad (2.15)$$

It is apparent from equation (2.15) that the ratio $d\phi/d\theta$ not only depends upon S , the excursion, but also on the sign of a_2 , which is the T -stress, the leading non singular term in equation (2.14). For positive large values of a_2 , $d\phi/d\theta$ will always be less than unity and thus would influence unstable crack growth. Whereas, a negative a_2 value, will induce a stable crack growth as $d\phi/d\theta$ is more than unity in this case.

Cotterell (1970) also determined the values of the coefficients a_i ($i = 3$) for a compact tension (CT) specimen and plotted $d\phi/d\theta$ ratio against the excursion S for different crack sizes. For such a specimen, he showed that $d\phi < d\theta$ always holds true and thus unstable crack growth is observed. The high positive T -stress values obtained in highly constrained CT specimens further corroborates the above observation. In case of CN specimens, the negative value of T -stress influences stable crack growth and out-of-plane crack growth is not observed. Similar effect of positive T -stress on crack path stability was reported by Cotterell and Rice (1980). However, the ratio $d\phi/d\theta$ is necessary but not the sufficient factor dictating the crack growth stability. The crack growth deviation from the ideal path is also influenced by factors such as material properties, microstructure, crack growth mechanism, etc. For example, Tong (2002) reported that the out-of-plane crack growth was not observed in specimens made up of conventional cast and wrought alloys where the plastic zone size is in the same order as that of the grain size.

Observations similar to those in isotropic solids have also been reported in anisotropic solids. The perturbation analysis by Gao and Chiu (1992) on slightly kinked or curved cracks in anisotropic solids showed that in the case of orthotropic materials, the crack extension is stable for $T < 0$ and unstable for $T > 0$. Considering a crack along one of the material principal axes in an infinite orthotropic solid, they showed that with increasing or decreasing stiffness in the direction parallel or perpendicular, respectively, to the crack, the T -stress becomes more compressive and helps stabilizing the path. This is consistent with the observation that crack tends to grow in direction parallel to fibres in case of unidirectional fibre reinforced composites. Numerical studies performed by Yuan and Yang (2001) on stitched carbon warp-knit fabric composites showed that the large tensile T -stress may inhibit the crack extension in the same plane, thus inducing crack kinking.

Similarly, Lim *et al.* (2001) studied the effect of the biaxial load and the non-singular stress terms on the crack growth direction in anisotropic solids. Their work was based on the normal stress ratio theory (Buczek and Herakovich, 1985) which states that the crack extension takes place in the direction along which the normal stress ratio R_0 is maximum. R_0 for a particular point (r_0, θ) around the crack tip, $0 < r_0 \ll 1$ being a reference distance from the crack tip, is determined from the ratio between the normal stress $\sigma_{\theta\theta}$ acting on the radial plane and tensile strength $T_{\theta\theta}$ in the direction. In the study r_0 was taken to be 0.01 times the crack size. They conducted the study with varying levels of biaxial loading and tensile strength ratio and concluded that the predicted crack growth direction depends upon the in-plane load biaxiality as well as the stress intensity factor.

Thus, the inclusion of singular as well as the non-singular stress parameters to predict the crack path holds equal importance in the case of anisotropic cracked bodies too.

2.2.6 Effect of T -stress on fatigue crack growth (FCG) rate

Besides the crack tunnelling phenomenon in fatigue crack growth discussed earlier, various studies have suggested that the crack-tip stress constraint has also a significant effect on the fatigue crack growth (FCG) rate. A significant difference in the fatigue crack growth rate between two specimens, namely, compact tension, CT, and corner notched, CN, made up of Waspaloy and subjected to temperature of 650°C was observed by Tong *et al.* (1999). A similar behaviour was previously observed in the study by Brown and Hicks (1982) on nickel-based alloys at 200°C. Tong (2002) has also reported that an investigation on PM alloy, U720Li, yielded similar trend at room temperature, and established that the geometry, not the temperature, is the reason for the observed crack growth rate discrepancy. Tong (2002) further reported a considerable variation in FCG rate between CT, single edge tension SENT and centre cracked tension CCT specimens of mild steel with same thickness and initial crack lengths subjected to fatigue loading under the same range of the stress intensity factor, ΔK . Among these specimens, CT showed the highest FCG rate followed by SENT and CCT, although the rates for the latter two were comparable. In order to explain these discrepancies between the crack growth rates, the argument of crack closure has been often brought forward. However, the phenomenon of crack closure is often observed only at the threshold loading region (Elber, 1970) whereas, in the discussed work by Tong *et al.* (1999), the crack growth rate difference is minimal for CT and CN specimen at the threshold and only becomes

significant for higher values of ΔK . This discrepancy can however be explained in terms of the crack-tip stress constraint. Results from the studies by various researchers, as presented in Tong *et al.* (1999), have shown that the T -stress is positive for a CT specimen and negative and comparable for both SENT and CCT specimens, although it is higher for SENT than for CCT. This suggests a correlation between the T -stress and the fatigue crack growth rate as well. For CT specimens with positive T -stress, the FCG rate is high. On the other hand negative T -stress in SENT and CCT specimens corresponds to lower FCG rate. Negative T -stress facilitates the development of plasticity and thus lowers the FCG rate.

2.3 Numerical Evaluation of T -stress

Over the years, there has been a number of studies devoted to the numerical determination of the T -stress in cracked bodies. Most of them deal with two-dimensional and three-dimensional isotropic elastic solids. The corresponding work in anisotropy is very scarce indeed and thus provides the motivation for the present study. In the following discussion, the methodologies available to extract T -stress will be briefly reviewed; only developments relevant to the current work will be discussed in greater detail. The discussion is categorized into the methods available in isotropy, those available in anisotropy and finally, development for cracks between dissimilar elastic media. It will be confined to two-dimensional elastic media under static mechanical loads.

2.3.1 Developments in two-dimensional isotropic cracked solids

In the case of homogeneous isotropic cracked solids, numerous schemes have been developed to extract the T -stress, both numerically and analytically. Larsson and Carlsson (1973) calculated the T -stress for cracked isotropic geometries by subtracting the value of stress σ_{11} acting parallel to the crack plane, obtained by conventional boundary layer solution, from σ_{11} obtained using finite element analysis of the specimen. The difference between these two values of σ_{11} was attributed to the T -stress, as the boundary layer solution ignores the contribution from the leading non-singular component of the Williams' series expansion. Leever and Radon (1982) evaluated the biaxiality parameter using variational principles and a matrix formulation for the coefficients of Williams' series expansion; they were implemented in conjunction with the finite element method. Cardew *et al.* (1984) and later Kfoury (1986) successfully implemented the method of path independent J -integral, based on the work by Eshelby (1975) to evaluate the T -stress solutions using finite element method. Eischen (1987) also developed a method to obtain the non-singular stress component in cracked geometries using J -integral approach in conjunction with finite element method.

The method of higher order weight functions (second order) obtained using work-conjugate integral to calculate T -stress was implemented by Sham (1991) in conjunction with finite element method. Nakamura and Parks (1992) implemented the interaction integral, in conjunction with the finite element method, in order to extract T -stress values in three-dimensional geometries which is also applicable to two-dimensional problems. Preliminary work in conjunction with boundary element method to evaluate T -stress in cracked specimens was first conducted by Olsen (1994) where J -integrals and L -integrals,

stemming from J -integral definition, were used. Chen (1994) adopted the method of superposition for evaluating T -stress in multiple cracks in an infinite plate. Hallback and Jonsson (1996) calculated the T -stress in mixed-mode loading condition using path-independent integrals by Eshelby (1975), in conjunction with finite element method.

Sladek *et al.* (1997a) obtained the contour integral based on Betti's reciprocal work theorem in conjunction with the boundary element method to evaluate the non-singular stress term in cracks under mixed-mode load conditions. Fett (1997, 2002), Fett and Rizzi (2004) and Wang (2002a,b) obtained the T -stress solutions for various cracked geometries using weight functions. Similar work was done by Hooton *et al.* (1998) on weight function to obtain T -stress solutions. Ayatollahi *et al.* (1998) developed crack flank nodal displacement method using finite element method to determine T -stress in mixed-mode loading. Smith *et al.* (2001) have presented a closed form analytical solution for T -stress for inclined cracks in infinite plates under uniaxial and biaxial loads. Yang and Chander (1999) have implemented the stress difference method (SDM) to obtain the T -stress in terms of stress field ahead of the crack tip. Tan and Wang (2003) have also developed a direct method based on the nodal displacements along the crack-tip quarter point elements (crack flank) from a boundary element method analysis.

In addition to the methods mentioned above, the developments in two-dimensional isotropy relevant to the current work will be discussed in greater detail below.

2.3.1.1 Mutual-integral or conservation integral method

It is noted that among the numerical schemes for obtaining the T -stress, except for the ones based on the conservation integral, all rely on the field values near to the crack tip. These schemes are therefore less reliable due to inaccuracies introduced by the singularity at the crack tip, and the inefficiencies in the mesh design to account for the effects of the singularity. The schemes which are based on the conservation integrals, on the other hand, rely on the field solutions at points remote from the crack tip such that the effect of the singular field is minimal. This leads to more accurate results and enables the numerical modelling to be much more relaxed and efficient. Based on the notion of path independent J -integral by Eshelby (1975) and Kfoury (1986), Sladek *et al.* (1997b) and Sladek and Sladek (1997a) have developed the Mutual integral, or M -integral method to determine elastostatics, elastodynamic and thermoelastic T -stresses in two-dimensional isotropic cracked solids in conjunction with boundary element method. This methodology, which will be extended to two-dimensional anisotropy in the current work, is discussed below.

In a cracked body, the path independent J -integral, Rice (1968), along a contour Γ_0 can be expressed as,

$$J = \int_{\Gamma_0} (W n_1 - t_i u_{i,1}) d\Gamma = \int_{\Gamma_0} \left(\frac{1}{2} \sigma_{ij} \varepsilon_{ij} n_1 - \sigma_{ij} n_j u_{i,1} \right) d\Gamma, \quad (2.16)$$

where the terms have their conventional definitions as discussed in equation (2.4).

Consider two independent equilibrium states $(\sigma_{ij}^A, \varepsilon_{ij}^A, u_i^A)$ and $(\sigma_{ij}^B, \varepsilon_{ij}^B, u_i^B)$. The state (A) corresponds to the boundary value problem to be analysed for the unknown T -stress. The second state (B), referred also as the auxiliary field, corresponds to the

solution of a semi-infinite crack loaded by a point force f (per unit thickness) applied at the crack tip in the direction parallel to the crack plane, as shown in figure (2.14). The solutions for the auxiliary field (B) at any point at (r, ϕ) from the local coordinate with origin at the crack tip, as shown in figure (2.14), are given in Timoshenko and Goodier (1970) as follows:

Stresses:

$$\begin{aligned}\sigma_{11}^B(r, \phi) &= -\frac{f}{\pi r} \cos^3 \phi, \\ \sigma_{12}^B(r, \phi) &= -\frac{f}{\pi r} \cos^2 \phi \sin \phi, \\ \sigma_{22}^B(r, \phi) &= -\frac{f}{\pi r} \cos \phi \sin^2 \phi.\end{aligned}\tag{2.17}$$

Displacements:

$$\begin{aligned}u_1^B(r, \phi) &= -\frac{f}{\pi E} (1 - \nu^2) \ln \left(\frac{r}{d} \right) - \frac{f}{2\pi E} (1 + \nu) \sin^2 \phi, \\ u_2^B(r, \phi) &= -\frac{f}{2\pi E} (1 + \nu) (1 - 2\nu) \phi + \frac{f}{2\pi E} (1 + \nu) \sin \phi \cos \phi.\end{aligned}\tag{2.18}$$

The derivatives of the displacements are

$$\begin{aligned}u_{1,1}^B(r, \phi) &= -\frac{f}{\pi E r} (1 - \nu^2) \left(\cos \phi - \frac{1}{1 - \nu} \sin^2 \phi \cos \phi \right), \\ u_{2,1}^B(r, \phi) &= \frac{f}{\pi E r} (1 + \nu) (1 - 2\nu - \cos 2\phi) \sin \phi, \\ u_{1,2}^B(r, \phi) &= -\frac{f}{\pi E r} (1 + \nu) (1 - \nu + \cos^2 \phi) \sin \phi, \\ u_{2,2}^B(r, \phi) &= -\frac{f}{2\pi E r} (1 + \nu) (1 - 2\nu - \cos 2\phi) \cos \phi.\end{aligned}\tag{2.19}$$

The corresponding J -integrals with respect to the local coordinates, x_i , for states (A) and (B) can now be written as,

$$J^{(A)} = \int_{\Gamma_0} \left(\frac{1}{2} \sigma_{ij}^A \varepsilon_{ij}^A n_i - \sigma_{ij}^A n_j u_{i,1}^A \right) d\Gamma, \quad (2.20.a)$$

$$J^{(B)} = \int_{\Gamma_0} \left(\frac{1}{2} \sigma_{ij}^B \varepsilon_{ij}^B n_i - \sigma_{ij}^B n_j u_{i,1}^B \right) d\Gamma. \quad (2.20.b)$$

Next, if J -integrals corresponding to both states are superimposed, the result is,

$$J^{(A+B)} = \int_{\Gamma_0} \left[\frac{1}{2} (\sigma_{ij}^A + \sigma_{ij}^B) (\varepsilon_{ij}^A + \varepsilon_{ij}^B) n_i - (\sigma_{ij}^A + \sigma_{ij}^B) n_j (u_{i,1}^A + u_{i,1}^B) \right] d\Gamma. \quad (2.21)$$

The mutual M -integral in local coordinates, x_i , is given in terms of the J -integrals as (Sladek *et al.*, 1997b and Sladek and Sladek, 1997a),

$$M = J^{(A+B)} - J^{(A)} - J^{(B)} = \int_{\Gamma_0} \left[\frac{1}{2} (\sigma_{ij}^A \varepsilon_{ij}^B + \sigma_{ij}^B \varepsilon_{ij}^A) n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right] d\Gamma. \quad (2.22)$$

Since, in the elastic medium, the stiffness remains the same for either loading states,

$$\sigma_{ij}^A \varepsilon_{ij}^B = \sigma_{ij}^B \varepsilon_{ij}^A. \quad (2.23)$$

Thus, equation (2.22) becomes,

$$M = \int_{\Gamma_0} (\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A) d\Gamma. \quad (2.24)$$

As the M -integral inherits the property of J -integral of path independence, equation (2.24) can be expressed in terms of a circular contour with radius ε around the crack tip such that the radius ε is shrunk to zero, as follows:

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A) d\Gamma \quad (2.25)$$

The asymptotic stresses and displacements corresponding to state (4) can be split into the singular and non-singular components as,

$$\sigma_{ij}^A = \sigma_{ij}^s + \sigma_{ij}^T, \quad (2.26)$$

$$u_i^A = u_i^s + u_i^T. \quad (2.27)$$

In the above equations, the terms with superscript s are those containing the stress intensity factor and are singular in nature, whereas those with superscript T are the leading non-singular components of the asymptotic expansion and are proportional to the T -stress. Due to the boundedness of the J -integral and thus the M -integral, the contribution from the singular terms in equation (2.25) vanishes. Also, if the expression for the M -integral in equation (2.25) in a circular contour with angle $\phi = -\pi$ to $+\pi$ is analysed, the terms with singular terms cancel out (Sladek *et al.*, 1997b) further supporting the boundedness of the M -integral. Thus, it is established that only non-vanishing contribution to the M -integral is due to the terms corresponding to T -stress. The M -integral in equation (2.25) then reduces to,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{1k} - \sigma_{ik}^T u_{i,1}^B - \sigma_{ik}^B u_{i,1}^T \right] n_k d\Gamma, \quad (2.28)$$

where

$$\sigma_{ij}^T = T \delta_{i1} \delta_{j1}. \quad (2.29)$$

Substituting equation (2.29) into equation (2.28),

$$\begin{aligned} M &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (T \delta_{i1} \delta_{j1} \varepsilon_{ij}^B \delta_{1k} - T \delta_{i1} \delta_{k1} u_{i,1}^B - \sigma_{ik}^B u_{i,1}^T) n_k d\Gamma \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (T \varepsilon_{11}^B \delta_{1k} - T \delta_{k1} u_{1,1}^B - \sigma_{ik}^B u_{1,1}^T \delta_{i1}) n_k d\Gamma \end{aligned} \quad (2.30)$$

Since, $u_{1,I} = \varepsilon_{11}$,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (T \varepsilon_{11}^B \delta_{1k} - T \varepsilon_{11}^B \delta_{1k} - \sigma_{ik}^B \varepsilon_{11}^T \delta_{il}) n_k d\Gamma. \quad (2.31)$$

Using the constitutive relation $\varepsilon_{11}^T = \left(\frac{1}{E'} \right) T$ where

$$E' = E \text{ for plane stress and } E' = \frac{E}{(1-\nu^2)} \text{ for plane strain conditions,} \quad (2.32)$$

the M -integral becomes,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} -\sigma_{ik}^B \left(\frac{1}{E'} \right) T \delta_{il} n_k d\Gamma$$

or,

$$\begin{aligned} M &= \left(\frac{1}{E'} \right) T \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} -\sigma_{ik}^B \delta_{il} n_k d\Gamma \\ &= \left(\frac{1}{E'} \right) T \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} -\sigma_{1k}^B n_k d\Gamma \\ &= \left(\frac{1}{E'} \right) T f. \end{aligned} \quad (2.33)$$

Rearranging equation (2.33) gives the T -stress in terms of M -integral as,

$$T = \frac{ME'}{f}. \quad (2.34)$$

In order to evaluate the M -integral used in equation (2.34), equation (2.24) is used for any arbitrary contour. This method of M -integral developed for isotropy will serve as a reference for the corresponding development in two-dimensional anisotropy presented in this thesis.

2.3.1.2 Weight function method

Often in engineering applications, complex stress fields are developed in a solid structure due to the presence of stress concentrations, residual stresses, thermal and complex mechanical loads. The weight function method, initially introduced by Bueckner (1970) for obtaining the stress intensity factor, provides a powerful tool to predict the T -stress values for any arbitrary loading condition once the reference weight function solutions for that particular cracked geometry have been obtained. Based on the principle of superposition, Chen (2000), showed that the T -stress for a cracked body under loading by a remote stress field Q can be expressed as the sum of two T -stress values, as in figures (2.15). The first one is the T -stress for the cracked specimen due to a crack-face pressure, figure (2.15.b); this crack-face pressure corresponds to the stress $\sigma(x_1)$ produced by the field Q in the same specimen in the absence of the crack. The T -stress for this case can be obtained by integrating the product of the stress distribution $\sigma(x_1)$ and the weight function $l(x_1)$ over the crack face. The weight function $l(x_1)$ which must first be determined, is essentially the T -stress solution for the geometry with equal and opposite point loads acting at a distance x_1 from the reference point (figure 2.16). The second component in the superposition is the T -stress in an uncracked geometry under given field Q , figure (2.15.c). Thus,

$$T = T_{crack\ pressure} + T_{uncracked}, \quad (2.35)$$

where

$$T_{crack\ pressure} = \int_0^a \sigma(x_1) l(x_1) dx_1. \quad (2.36)$$

Chen (2000) has shown that in an uncracked geometry, unlike the stress intensity factor, T -stress is non zero and has a finite value given in terms of the stress field as,

$$T_{uncracked} = (\sigma_{11} - \sigma_{22})_{crack\ tip}. \quad (2.37)$$

The following two-term form for the weight function $t(x_1, a)$ was proposed by Wang (2002a), based on the form used by Fett (1997) for the computation of the stress intensity factor,

$$t(x_1, a) = \frac{2}{\pi a} \left[D_1 \left(1 - \frac{x_1}{a} \right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a} \right)^{\frac{3}{2}} \right]. \quad (2.38)$$

It is apparent from equation (2.38) that once the constants D_1 and D_2 for a particular cracked geometry are determined, the weight function can be evaluated. In order to obtain the two unknowns D_1 and D_2 , Wang (2002a,b) proposed an engineering approach of using two reference T -stress solutions from which two sets of simultaneous equations on D_i can be obtained. It involves evaluating the T -stress solutions for the cracked geometry under two loading conditions such that there is no contribution of $T_{uncracked}$ in those cases. This can be achieved when the cracked geometry is subjected to a normal crack-face pressure with the following stress distribution,

$$\sigma_{22}(x_1) = (-\sigma_o) \cdot \left(1 - \frac{x_1}{a} \right)^n, \quad (2.39)$$

where $n = 0$ and 1 corresponds to uniform and linearly varying pressures, respectively. The distance x_1 in equation (2.39) may be defined as shown in figure (2.16), for the single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP). Wang (2002a,b) has used the interaction integral method in conjunction with the finite element method to obtain the reference solutions. Utilizing the same approach, Li *et*

al. (2005) extracted the weight functions for internal edge cracks in thick-walled cylinders using boundary element method while Wang *et al.* (2006) estimated the T -stress for small cracks at notches using finite element method. Although the method of reference T -stress solutions was originally developed for mode I loading, it can be used for mixed-mode loading as well.

It should be noted that although weight function solutions for stress intensity factor in anisotropic cracked bodies exist in the literature, e.g., Deukman (1987), Sham and Zhou (1989), Gao (1992), the corresponding weight function solutions for T -stress are extremely scarce indeed. The weight function approach based on Wang (2002a,b) for isotropy will be extended in this thesis to develop the weight function for anisotropy using the boundary element method.

2.3.2 Developments in two-dimensional anisotropic cracked solids

In comparison with the number of studies related to the evaluation of T -stress in isotropy, those for orthotropy and general anisotropy remain relatively few to date. They include the analytical solution presented by Gao and Chiu (1992) which is applicable to a crack in infinite orthotropic plate, provided one of the orthotropic material principal axes and crack plane are parallel. The T -stress for a crack in such an orthotropic plate subjected to remote stresses, σ_{11} and σ_{22} , can be obtained by,

$$T = \sigma_{11} - \left(\sqrt{\frac{a_{22}}{a_{11}}} \right) \sigma_{22}, \quad (2.40)$$

where a_{22} and a_{11} are the first and second diagonal elements of the compliance matrix for the material.

Using the Stroh formalism in anisotropic elasticity, Yang and Yuan (2000a) developed a path-independent conservation integral and employed Betti's reciprocal theorem to obtain the T -stress for anisotropic solids. They have also applied their technique to kinked crack problems in orthotropy (Yang and Yuan, 2000b) as well as for crack in fabric composites (Yang and Yuan, 2001). The latter work was carried out using the finite element method as the numerical tool.

The non-singular term for multiple cracks in anisotropic solids was investigated by Ma *et al.* (1997). The development was specifically for cracks in an infinite medium and a limited set of result was presented for the orthotropic case. Lim *et al.* (2001) also studied the effect of the non-singular parameter in crack extension by developing an analytical solution to determine the non-singular term for biaxial loading in an infinite plate. The work is more focused in studying the effect of various parameters on the direction of crack extension rather than the determination of the value of T -stress, *per se*. Recently, Kim and Paulino (2004) have obtained the expressions to evaluate the T -stress using the conservation integral (M -integral) applicable to functionally graded and orthotropic materials, based on both Lekhnitskii and Stroh formalisms for anisotropic elasticity. The numerical results were obtained by the finite element method. Song (2004) has also presented a relatively new and novel numerical tool of the scaled boundary finite-element method (Song and Wolf, 2002), also termed as SBFEM, for obtaining the T -stress for anisotropic elasticity. However, no numerical solutions on T -stress for anisotropic specimens were reported in that work. Using another relatively new numerical technique, namely, the fractal finite element method (Panagiotopolous, 1992), Su and Sun (2005) have evaluated T -stress for crack in a finite anisotropic plate as well.

2.3.3 Developments in cracks along bimaterial interface

With the ever-increasing use of modern engineered materials such as composites, bicrystals, thin-film coatings and other multi-phase material systems, the study of cracks along the interface between dissimilar materials has become more important. Although, there has been a significant number of works on the determination of the stress intensity factor for the bimaterial interface crack, those for the T -stress remains very limited.

Among the earliest works on the determination of T -stress for isotropic bimaterial interface crack are the ones by Boem and Earmme (1993) and Cho *et al.* (1994). They implemented the conservation integral to obtain closed-form T -stress solutions for interfacial crack in an infinite medium subjected to a concentrated load or a couple. Sladek and Sladek (1997b) extended their own work involving M -integral in conjunction with the boundary element method for homogeneous isotropy to the bimaterial interface crack. Using a similar analytical approach as Boem and Earmme (1993), Moon and Earmme (1998) evaluated the T -stress for a crack in infinite dissimilar solids under in-plane and anti-plane loading. Recently, Fett and Rizzi (2004) used the weight function approach in conjunction with the finite element method to obtain the T -stress solutions for isotropic bimaterial interface cracks.

Extending the work on isotropy by Moon and Earmme (1998), Kim *et al.* (2001) developed a method based on the M -integral for the extraction of T -stress solution for an interface crack between dissimilar anisotropic bodies of infinite extents. No numerical results were presented in the work, however. Very recently, Song (2004, 2005) utilized the scaled boundary finite-element method to obtain the T -stress values for interface cracks between dissimilar isotropic media. Although the development is capable of

obtaining the T -stress solutions for bimaterial interface cracks between anisotropic solids, no numerical solution was reported in the work. There is indeed paucity in the development of numerical methods to extract the T -stress for a bimaterial interface crack between anisotropic elastic media and in particular, using the boundary element method and conservation integral; hence, the motivation of the current work. In order to provide a background for the development undertaken in the current thesis, the work by Sladek and Sladek (1997b) will be reviewed next.

2.3.3.1 *Mutual-integral method for bimaterial interface crack in two-dimensional isotropy*

As mentioned previously, Sladek and Sladek (1997b) extended their work on homogeneous isotropy (Sladek *et al.*, 1997b and Sladek and Sladek, 1997a) to bimaterial interface crack to evaluate T -stress term by developing the appropriate M -integral for the system. The singular stress field near an interface crack tip is oscillatory in nature, it can be written as follows:

$$\sigma_{kj}^{(m)}(r, \phi) = F_{kj}^{(m)}(\phi) \operatorname{Re} \{ \mathbf{K} r^{-1/2+i\varepsilon} \} + H_{kj}^{(m)}(\phi) \operatorname{Im} \{ \mathbf{K} r^{-1/2+i\varepsilon} \} + C^{(m)} T \delta_{k1} \delta_{j1} + O(r^\alpha), \quad (2.41)$$

where $\alpha > 0$. The superscript $m = 1$ or 2 denotes the material (figure 2.17), with material 1 corresponding to $(0 < \phi < \pi)$ and material 2 corresponding to $(-\pi < \phi < 0)$. Due to the oscillatory stress singularity at the crack tip, the stress intensity factor $\mathbf{K}$ is written in complex form. The explicit expressions for the angular functions $F_{kj}^{(m)}(\phi)$ and $H_{kj}^{(m)}(\phi)$ for isotropy are available in, e.g., Sladek and Sladek (1997b). The parameter ε is a bimaterial constant, also known as the oscillatory index, given as,

$$\varepsilon = \frac{1}{2\pi} \ln \left[\left(\frac{k_1}{G_1} + \frac{1}{G_2} \right) / \left(\frac{k_2}{G_2} + \frac{1}{G_1} \right) \right], \quad (2.42)$$

where $k_m = (3 - 4\nu_m)$ for plane strain and $k_m = (3 - 4\nu_m)/(1 + \nu_m)$ for plane stress conditions, G_m is the shear modulus and ν_m is the Poisson's ratio for material m . The material specific coefficient $C^{(m)}$ in equation (2.41) is given as,

$$C^{(1)} = c_2/c_1 \text{ and } C^{(2)} = 1 \quad (2.43)$$

where

$$c_m = \frac{1 + k_m}{G_m}. \quad (2.44)$$

Similar to the derivation of M -integral expression by Sladek *et al.* (1997b) for homogeneous isotropy, discussed in Section (2.3.1.1), the M -integral for a contour Γ_0 around an interface crack (figure 2.17) can be expressed in terms of the two independent equilibrium states $(\sigma_{ij}^A, \varepsilon_{ij}^A, u_i^A)$ and $(\sigma_{ij}^B, \varepsilon_{ij}^B, u_i^B)$ as,

$$M = \int_{\Gamma_0} (\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A) d\Gamma. \quad (2.45)$$

The state corresponding to (A) is the boundary value problem being analysed for the unknown T -stress and state (B) corresponds to the auxiliary field. Sladek and Sladek (1997b) have derived the expression for state (B) based on the Muskhelishvili complex potentials applied to a semi-infinite crack along the dissimilar materials interface subjected to a point force f at the crack tip along the direction of the crack plane, see figure (2.18). The explicit solutions for the stresses and displacement derivatives required in equation (2.45) are given as follows for a point at (r, ϕ) in material m :

Stresses:

$$\begin{aligned}
 \sigma_{11}^{B(m)}(r, \phi) &= -C^{(m)} \left(\frac{c_1}{c_1 + c_2} \right) \left(\frac{2f}{\pi r} \right) \cos^3 \phi, \\
 \sigma_{12}^{B(m)}(r, \phi) &= -C^{(m)} \left(\frac{c_1}{c_1 + c_2} \right) \left(\frac{2f}{\pi r} \right) \cos^2 \phi \sin \phi, \\
 \sigma_{22}^{B(m)}(r, \phi) &= -C^{(m)} \left(\frac{c_1}{c_1 + c_2} \right) \left(\frac{2f}{\pi r} \right) \cos \phi \sin^2 \phi.
 \end{aligned} \tag{2.46}$$

Displacement derivatives:

$$\begin{aligned}
 u_{1,1}^{B(m)}(r, \phi) &= -C^{(m)} \left(\frac{c_1}{c_1 + c_2} \right) \left(\frac{2f}{\pi r} \right) \left(\frac{1 + \nu_m}{E_m} \right) (1 - \nu_m - \sin^2 \phi) \cos \phi, \\
 u_{2,1}^{B(m)}(r, \phi) &= C^{(m)} \left(\frac{c_1}{c_1 + c_2} \right) \left(\frac{f}{\pi r} \right) \left(\frac{1 + \nu_m}{E_m} \right) (1 - 2\nu_m - \cos 2\phi) \sin \phi.
 \end{aligned} \tag{2.47}$$

The strains for the auxiliary field can then be obtained using the displacement derivative solutions.

As before, the M -integral being path independent, can be evaluated along an arbitrary circular contour Γ_ε with radius ε . The M -integral corresponding to the condition when the radius ε approaches zero becomes,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A) d\Gamma. \tag{2.48}$$

Due to the boundedness of M -integral, the singular terms as well as the vanishing terms in the contour integral do not contribute to it. This leaves the M -integral expression in terms of the function of the non-singular T -stress as,

$$M = \frac{Tf}{E_2}. \tag{2.49}$$

E'_2 corresponds to material 2 and is given by equation (2.32).

Rearranging equation (2.49) gives the expression for evaluating the T -stress using local M -integral with respect to material 2 as follows:

$$T = \frac{ME'_2}{f}. \quad (2.50)$$

Thus using equation (2.50), the T -stress for an interface crack between dissimilar elastic isotropic solids can be obtained. The M -integral required to evaluate the T -stress is computed using equation (2.45).

2.4 Closing Remarks

The definition and the significance of T -stress in fracture mechanics analysis have been discussed. The literature review of the schemes to obtain this quantity has also been given. From this, it is evident that no previous work has been carried out to evaluate the T -stress using the BEM for cracks in anisotropic solids and no T -stress solutions have been reported in the literature for interfacial cracks between dissimilar anisotropic solids. A review of BEM and two-dimensional anisotropic elasticity is presented next, in the following chapter.

CHAPTER 3

BOUNDARY ELEMENT METHOD IN TWO-DIMENSIONAL ANISOTROPIC ELASTICITY

3.0 Introduction

In this chapter, the numerical analysis of an anisotropic elastic body with the boundary element method (BEM) is discussed. The theory of two-dimensional anisotropic elasticity is briefly reviewed first. Then, the BEM applicable to two-dimensional anisotropy is discussed including the treatment of crack problems. Finally, an approach to overcome a difficulty in evaluating certain interior point field solutions using the conventional BEM is discussed; its implementation in two-dimensional anisotropy is presented. In what follows, Einstein's indicial notation will be used; the indices in the two-dimensional Cartesian co-ordinate system will, unless otherwise indicated, range from 1 to 2, in general.

3.1 Two-Dimensional Anisotropic Elasticity

The constitutive equation, i.e., the generalized Hooke's law, for a homogeneous anisotropic material can be expressed for plane stress in two-dimensional co-ordinates, x_i , as follows (see, e.g., Lekhnitskii, 1963):

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \quad (3.1)$$

where σ_{ij} and ε_{ij} are the stress and strain tensors, respectively, a_{mn} are the elastic compliances of the material ($m, n = 1, 2, \dots, 6$) and are elements of the reduced compliance matrix **a**. The elastic compliances can be expressed in terms of the engineering constants as follows (Lekhnitskii, 1963):

$$\begin{aligned} a_{11} &= \frac{1}{E_{11}}, \\ a_{22} &= \frac{1}{E_{22}}, \\ a_{66} &= \frac{1}{G_{12}}, \\ a_{12} &= -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}}, \\ a_{16} &= \frac{\eta_{12,1}}{E_{11}} = \frac{\eta_{1,12}}{G_{12}}, \\ a_{26} &= \frac{\eta_{12,2}}{E_{22}} = \frac{\eta_{2,12}}{G_{12}}. \end{aligned} \quad (3.2)$$

where E_{kk} is the Young's modulus in x_k direction, G_{12} is the shear modulus in x_1 - x_2 plane, ν_{jk} is the Poisson's ratio given by the ratio of compressive strain in x_k direction due to a unit extensional strain in x_j direction, $\eta_{jk,l}$ and $\eta_{l,jk}$ are the coefficients of mutual influence

of the first and the second kind, respectively, and provide the coupling between normal strains and shear stress. In the case of orthotropic materials, the coefficients of mutual influence of the first kind and the second kind are zero.

If plane strain conditions prevail, equation (3.1) remains valid provided that a_{jk} is replaced by the reduced compliance b_{jk} such that,

$$b_{jk} = a_{jk} - \frac{a_{j3}a_{k3}}{a_{33}}, \quad (j,k = 1, 2, 6), \quad (3.3)$$

where

$$\begin{aligned} a_{j3} &= -\frac{\nu_{j3}}{E_{jj}} = -\frac{\nu_{3j}}{E_{33}}, \\ a_{33} &= \frac{1}{E_{33}}, \\ a_{63} &= \frac{\eta_{12,3}}{E_{33}} = \frac{\eta_{3,12}}{G_{12}}. \end{aligned} \quad (3.4)$$

where the index 3 refers to the direction along x_3 co-ordinate axis.

Similarly, when expressed in terms of the principal submatrix $\mathbf{C}^0$ of the stiffness matrix $\mathbf{C}$, equation (3.1) can be expressed as,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = [\mathbf{C}^0] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}. \quad (3.5)$$

The stiffness and the reduced compliance matrices are inverse of each other such that,

$$\mathbf{C}^0 \mathbf{a} = \mathbf{I} = \mathbf{a} \mathbf{C}^0, \quad (3.6)$$

where $\mathbf{I}$ is the identity matrix.

There are two main formalisms which have been proposed to treat two-dimensional deformations in anisotropic materials. One was proposed by Lekhnitskii (1950, 1957) and is widely adopted in the engineering community due to the ease involved with it for practical interpretation. The Lekhnitskii formalism is an extension and generalization of the Muskhelishvili (1953) treatment for the solution of two-dimensional deformations in isotropy. The approach is based on stresses and it assumes that the stresses are dependent upon x_i for a given set of boundary conditions. On the other hand, the Stroh (1958, 1962) formalism is based on displacements and assumes their dependence on x_i . Thus, the treatment of anisotropic solids by Lekhnitskii formalism involves the reduced elastic compliances and that by Stroh formalism involves the elastic stiffnesses. The Stroh formalism is increasingly being appreciated within the engineering community due to its mathematical elegance. In the following sections, only the key developments of both formalisms relevant to the present work will be discussed in more detail. Although independent, both formalisms can be shown to be equivalent to each other (Barnett and Kirchner, 1997).

3.1.1 Lekhnitskii formalism

For two-dimensional stress system, σ_{ij} , the equations of equilibrium for plane problems, in the absence of body forces, can be expressed as,

$$\sigma_{ij,j} = 0. \quad (3.7)$$

Equation (3.7) is satisfied by expressing the stresses in terms of the Airy's stress function, χ , where

$$\sigma_{11} = \frac{\partial^2 \chi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \chi}{\partial x_1^2} \quad \text{and} \quad \sigma_{12} = -\frac{\partial^2 \chi}{\partial x_1 \partial x_2}. \quad (3.8)$$

The condition of compatibility of deformation can be expressed by the compatibility equation in terms of strains ε_{ij} as,

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}. \quad (3.9)$$

Utilizing equations (3.1), (3.8) and (3.9), the governing equation for anisotropic solids in two-dimensional elasticity can be obtained in terms of the stress function as,

$$a_{22} \frac{\partial^4 \chi}{\partial x_1^4} - 2a_{26} \frac{\partial^4 \chi}{\partial x_1^3 \partial x_2} + (2a_{12} + a_{66}) \frac{\partial^4 \chi}{\partial x_1^2 \partial x_2^2} - 2a_{16} \frac{\partial^4 \chi}{\partial x_2^3 \partial x_1} + a_{11} \frac{\partial^4 \chi}{\partial x_2^4} = 0. \quad (3.10)$$

The general solution for equation (3.10) can be expressed in terms of a complex co-ordinate:

$$z = x_1 + \mu x_2, \quad (3.11)$$

where

$$\mu = \alpha + i\beta; \quad i = \sqrt{-1}, \quad (3.12)$$

and α and β depend on the material properties.

Substituting equation (3.11) into equation (3.10), the characteristic equation for two-dimensional anisotropy is obtained as follows:

$$\left\{ a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} \right\} \frac{\partial^4 \chi}{\partial z^4} = 0. \quad (3.13)$$

For a non-trivial solution of equation (3.13), the terms inside the brackets must be identically zero for two-dimensional anisotropic material. Thus the characteristic equation in terms of the reduced elastic compliances and μ is obtained as follows:

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \quad (3.14)$$

Lekhnitskii (1963) has shown that the roots of the above characteristic equation cannot be real and must be either purely imaginary or complex. Thus the roots μ_j can be expressed as two pairs of complex conjugates as,

$$\mu_j = \alpha_j + i\beta_j, \quad (3.15.a)$$

$$\bar{\mu}_j = \alpha_j - i\beta_j, \quad (3.15.b)$$

where $\beta > 0$.

The roots are distinct for an anisotropic material. These roots, also termed as the “complex parameters”, can be considered to quantify the degree of anisotropy for plane problems. For the isotropic case, a_{16} and a_{26} are non existent, $a_{11} = a_{22}$ and a_{66} and a_{12} can be expressed in terms of a_{11} ; equation (3.14) can then be expressed in terms of the engineering constants as,

$$\frac{1}{E}\mu^4 + \left(-\frac{2\nu}{E} + \frac{2(1+\nu)}{E}\right)\mu^2 + \frac{1}{E} = 0. \quad (3.16)$$

Thus for isotropic case,

$$\mu_1 = \mu_2 = i, \quad \bar{\mu}_1 = \bar{\mu}_2 = -i, \quad \alpha = 0 \quad \text{and} \quad \beta = 1. \quad (3.17)$$

For an orthotropic material, a_{16} and a_{26} are equal to zero and thus either,

$$\mu_j = i\beta_j, \quad \mu_1 = \mu_2 = i\beta \quad \text{or,}$$

$$\mu_1 = \alpha + i\beta \quad \text{and} \quad \mu_2 = -\alpha + i\beta. \quad (3.18)$$

Once the complex parameters are obtained, equation (3.13) can be successively integrated and the stress function can be obtained using the boundary conditions. Hence,

the solutions for the stress fields can be obtained from which the displacement fields can be evaluated.

3.1.2 Stroh formalism

Using the strain-displacement relations, the general form of Hooke's law may be written as,

$$\sigma_{ij} = C_{ijks} u_{k,s}, \quad (3.19)$$

where $i, j, k, s = 1, 2$ for two-dimensional system, C_{ijks} are the elastic stiffness which are elements of the stiffness matrix $\mathbf{C}$. $u_{k,s}$ are the displacement derivatives and are related to the strains as,

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (3.20)$$

If there is no body force, the general solution for the displacement vector $\mathbf{u}$ and the stress function vector, Φ , are given by the Stroh formalism as follows, Ting (1996):

$$\mathbf{u} = \text{Re} [\mathbf{A} f(z) \mathbf{d}], \quad (3.21.a)$$

$$\Phi = \text{Re} [\mathbf{B} f(z) \mathbf{d}]. \quad (3.21.b)$$

where

$$\sigma_{i1} = -\Phi_{i,2}, \quad \sigma_{i2} = \Phi_{i,1}, \quad (3.22.a)$$

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2], \quad (3.22.b)$$

$$f(z) = \text{diag} [f(z_1), f(z_2)], \quad (3.22.c)$$

$$z_k = x_1 + \mu_k x_2, \text{ as in equation (3.11), such that } \text{Im}(\mu_k) > 0. \quad (3.22.d)$$

In equation (3.22), $f(z)$ is the diagonal matrix of an arbitrary function; $\mathbf{d}$ is an unknown complex vector to be determined; μ_k , $\mathbf{a}_k$ and $\mathbf{b}_k$ are the Stroh eigenvalues and eigenvectors, respectively, which are functions of the elastic constants only. These can be obtained by solving the eigenvalue problem as follows (Ting, 1996):

$$\mathbf{N}\xi = \mu\xi, \quad (3.23)$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad (3.24.a)$$

$$\xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \quad (3.24.b)$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \quad (3.24.c)$$

$\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ for two-dimensional elasticity are 2×2 matrices, the elements of which are given by,

$$Q_{ij} = C_{i1j1}, \quad R_{ij} = C_{i1j2}, \quad T_{ij} = C_{i2j2}. \quad (3.25)$$

In expanded form, they may be expressed as

$$\mathbf{Q} = \begin{bmatrix} C_{1111} & C_{1121} \\ C_{1121} & C_{2121} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{16} \\ C_{16} & C_{66} \end{bmatrix}, \quad (3.26.a)$$

$$\mathbf{R} = \begin{bmatrix} C_{1112} & C_{1122} \\ C_{2112} & C_{2122} \end{bmatrix} = \begin{bmatrix} C_{16} & C_{12} \\ C_{66} & C_{26} \end{bmatrix}, \quad (3.26.b)$$

$$\mathbf{T} = \begin{bmatrix} C_{1212} & C_{1222} \\ C_{2212} & C_{2222} \end{bmatrix} = \begin{bmatrix} C_{66} & C_{26} \\ C_{26} & C_{22} \end{bmatrix}. \quad (3.26.c)$$

The 4×4 $\mathbf{N}$ matrix for two-dimensional elasticity is the equivalent of 6×6 $\mathbf{N}$ matrix for three-dimensional elasticity which is also known as the fundamental elasticity

matrix, first introduced by Ingebrigtsen and Tønning (1969). In equations (3.24), $\mathbf{N}_1$ is dimensionless, $\mathbf{N}_2$ has the dimension of compliance and $\mathbf{N}_3$ has the dimension of stress or stiffness. The eigenvalue problem in equations (3.24) gives four eigenvalues and four associated eigenvectors ($\mathbf{a}$, $\mathbf{b}$). Since $\mathbf{N}$ is asymmetric and the strain energy is positive definite (Ting, 1996), there are two pairs of complex conjugates for μ as follows:

$$\mu_{\alpha+2} = \bar{\mu}_{\alpha}, \quad \mathbf{a}_{\alpha+2} = \bar{\mathbf{a}}_{\alpha}, \quad \mathbf{b}_{\alpha+2} = \bar{\mathbf{b}}_{\alpha}, \quad \alpha = 1, 2 \quad \text{Im}[\mu_{\alpha}] > 0. \quad (3.27)$$

This provides the background for the Stroh formalism. Additional identities due to Stroh (Ting, 1996) useful in solving problems in anisotropy will be presented in the relevant sections where they are utilized.

3.1.3 Two-dimensional transformation of elastic constants and identities

While dealing with homogeneous anisotropic solids, due to the variation in the elastic properties with orientation, it is important to obtain the elastic properties in a particular direction. In this section, transformation rules applicable to a few elastic constants relevant to the current work are presented below.

With reference to figure (3.1), let the elastic compliance a_{mn} with respect to coordinate axes x_1 and x_2 is now denoted by a'_{mn} with respect to new coordinate axes x'_1 and x'_2 . When the angle of rotation between x'_i and x_i is ϕ , the transformed compliances, a'_{ij} , are given as follows, Lekhnitski (1963, 1968):

$$a'_{11} = a_{11} \cos^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + a_{22} \sin^4 \phi + (a_{16} \cos^2 \phi + a_{26} \sin^2 \phi) \sin 2\phi,$$

$$a'_{12} = a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \phi \cos^2 \phi + \frac{1}{2}(a_{26} - a_{16}) \sin 2\phi \cos 2\phi,$$

$$\begin{aligned}
a'_{16} &= \left[a_{22} \sin^2 \phi - a_{11} \cos^2 \phi + \frac{1}{2} (2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi \\
&\quad + (a_{16} \cos^2 \phi)(\cos^2 \phi - 3 \sin^2 \phi) + (a_{26} \sin^2 \phi)(3 \cos^2 \phi - \sin^2 \phi), \\
a'_{22} &= a_{11} \sin^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + a_{22} \cos^4 \phi - (a_{16} \sin^2 \phi + a_{26} \cos^2 \phi) \sin 2\phi, \\
a'_{26} &= \left[a_{22} \cos^2 \phi - a_{11} \sin^2 \phi - \frac{1}{2} (2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi \\
&\quad + (a_{16} \sin^2 \phi)(3 \cos^2 \phi - \sin^2 \phi) + (a_{26} \cos^2 \phi)(\cos^2 \phi - 3 \sin^2 \phi), \\
a'_{66} &= a_{66} + 4(a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \phi \cos^2 \phi + 2(a_{26} - a_{16}) \sin 2\phi \cos 2\phi. \tag{3.28}
\end{aligned}$$

Another important elastic parameter in two-dimensional anisotropy is the complex parameter μ_k . For the new axes x'_i , the complex parameter μ'_k can be expressed in terms of μ_k in the original axes, x_i , as follows (Voigt, 1928):

$$\mu'_k = \frac{\mu_k \cos \phi - \sin \phi}{\cos \phi + \mu_k \sin \phi}. \tag{3.29}$$

Similarly, the elastic constants **Q**, **R** and **T** obey the following formulae for transformation about an angle ϕ ,

$$\begin{aligned}
\mathbf{Q}(\phi) &= \mathbf{Q} \cos^2 \phi + (\mathbf{R} + \mathbf{R}^T) \sin \phi \cos \phi + \mathbf{T} \sin^2 \phi, \\
\mathbf{R}(\phi) &= \mathbf{R} \cos^2 \phi + (\mathbf{T} - \mathbf{Q}) \sin \phi \cos \phi - \mathbf{R}^T \sin^2 \phi, \\
\mathbf{T}(\phi) &= \mathbf{T} \cos^2 \phi - (\mathbf{R} + \mathbf{R}^T) \sin \phi \cos \phi + \mathbf{Q} \sin^2 \phi. \tag{3.30}
\end{aligned}$$

3.2 BEM in Two-dimensional Anisotropic Elasticity

The BEM development in two-dimensional anisotropy is similar to that for the isotropic case, see, e.g., Cruse and Swedlow (1971), Cruse (1988), Tan and Gao (1992a,b). The integral expressions used in BEM are derived from the treatment of differential equations

for elastostatics using a suitable fundamental solution and a reciprocal relation. In the case of elastostatics, the Navier's equation of equilibrium serves as the differential equation and the reciprocal relation is obtained from the Betti-Rayleigh's reciprocal work theorem. With appropriate mathematical operations and the use of suitable fundamental solutions for anisotropy, the boundary integral equation (BIE) can be represented in terms of displacements u_j and tractions t_j along the modelled boundary S of the homogeneous elastic domain as follows:

$$C_{jk}(P)u_k(P) + \int_S T_{jk}(P,Q)u_k(Q)dS(Q) = \int_S U_{jk}(P,Q)t_k(Q)dS(Q), \quad (3.31)$$

where $C_{jk}(P)$ depends upon the local geometry of the boundary S at point P , $U_{jk}(P,Q)$ and $T_{jk}(P,Q)$ are the fundamental solutions representing displacements and tractions, respectively, in the x_k direction at point Q due to a unit line load acting in x_j direction at point P in a plane homogeneous infinite body (figure 3.2).

The explicit forms of the fundamental solutions for anisotropic elastic domain have been given by Cruse (1988) as follows,

$$U_{jk} = 2 \mathbf{Re} [r_{k1} A_{j1} \ln(z_1) + r_{k2} A_{j2} \ln(z_2)], \quad (3.32)$$

$$T_{j1} = 2n_1 \mathbf{Re} [\mu_1^2 A_{j1}/z_1 + \mu_2^2 A_{j2}/z_2] - 2n_2 \mathbf{Re} [\mu_1 A_{j1}/z_1 + \mu_2 A_{j2}/z_2], \quad (3.33.a)$$

$$T_{j2} = -2n_1 \mathbf{Re} [\mu_1 A_{j1}/z_1 + \mu_2 A_{j2}/z_2] + 2n_2 \mathbf{Re} [A_{j1}/z_1 + A_{j2}/z_2]. \quad (3.33.b)$$

where $\mathbf{Re}$ stands for the real part of the terms within the parenthesis, and

$$r_{1j} = a_{11} \mu_j^2 + a_{12} - a_{16} \mu_j, \quad (3.34.a)$$

$$r_{2j} = a_{12} \mu_j + a_{22}/\mu_j - a_{26}. \quad (3.34.b)$$

In the above equations, n_j are the unit outward normal components at Q with respect to the x_i co-ordinate system and A_{jk} are complex constants that can be obtained from the following equations given in a matrix form as:

$$[\text{Im}\{\mathbf{B}_1\} \text{Re}\{\mathbf{B}_1\} \text{Im}\{\mathbf{B}_2\} \text{Re}\{\mathbf{B}_2\}] \begin{bmatrix} \text{Re}\{A_{j1}\} \\ \text{Im}\{A_{j1}\} \\ \text{Re}\{A_{j2}\} \\ \text{Im}\{A_{j2}\} \end{bmatrix} = \begin{bmatrix} -(1/4\pi) \delta_{j2} \\ (1/4\pi) \delta_{j1} \\ 0 \\ 0 \end{bmatrix}. \quad (3.35)$$

In equation (3.35) δ_{jk} is the Kronecker delta and

$$\{\mathbf{B}_k\} = [i \ \mu_k \ r_{1k} \ r_{2k}]^T \quad k = 1, 2. \quad (3.36)$$

Once the BIE has been solved for the boundary values of the displacements and tractions, the solution for any interior point can be obtained using the Somigliana's identities for displacements $u_i(p)$, displacement derivatives $u_{i,j}(p)$, strains $\varepsilon_{ij}(p)$ and stresses $\sigma_{ij}(p)$. These solutions at p can be evaluated as follows:

$$u_j(p) = \int_S t_i(Q) U_{ji}(p, Q) dS(Q) - \int_S u_i(Q) T_{ji}(p, Q) dS(Q), \quad (3.37)$$

from which the displacement derivative can be evaluated as,

$$\frac{\partial u_i(p)}{\partial x_j} = \int_S u_k(Q) \frac{\partial T_{ik}(p, Q)}{\partial x_j} dS(Q) - \int_S t_k(Q) \frac{\partial U_{ik}(p, Q)}{\partial x_j} dS(Q), \quad (3.38)$$

thus, giving strains as,

$$\begin{aligned} 2\varepsilon_{ij}(p) &= \frac{1}{2} (u_{i,j}(p) + u_{j,i}(p)) \\ &= \int_S u_k(Q) \left[\frac{\partial T_{ik}(p, Q)}{\partial x_j} + \frac{\partial T_{jk}(p, Q)}{\partial x_i} \right] dS(Q) - \int_S t_k(Q) \left[\frac{\partial U_{ik}(p, Q)}{\partial x_j} + \frac{\partial U_{jk}(p, Q)}{\partial x_i} \right] dS(Q) \\ &= \int_S u_k(Q) S_{kij}(p, Q) dS(Q) - \int_S t_k(Q) D_{kij}(p, Q) dS(Q) \end{aligned} \quad (3.39)$$

and stresses as,

$$\sigma_{ij}(p) = - \int_S u_k(Q) \overline{S_{kij}}(p, Q) dS(Q) + \int_S t_k(Q) \overline{D_{kij}}(p, Q) dS(Q). \quad (3.40)$$

The kernels S_{kij} and D_{kij} are given as, see, e.g., Cruse and Swedlow (1971), Tan and Gao (1992):

$$S_{kij} = 2\text{Re} \left\{ R_{i1} Q_{j1} (\mu_1 n_1 - n_2) A_{k1} / (z_1)^2 + R_{i2} Q_{j2} (\mu_2 n_1 - n_2) A_{k2} / (z_2)^2 \right\} \\ + 2\text{Re} \left\{ R_{k1} Q_{j1} (\mu_1 n_1 - n_2) A_{i1} / (z_1)^2 + R_{k2} Q_{j2} (\mu_2 n_1 - n_2) A_{i2} / (z_2)^2 \right\}, \quad (3.41.a)$$

$$D_{kij} = -2\text{Re} \left\{ R_{i1} P_{j1} A_{k1} / z_1 + R_{i2} P_{j2} A_{k2} / z_2 \right\} \\ - 2\text{Re} \left\{ R_{k1} P_{j1} A_{i1} / z_1 + R_{k2} P_{j2} A_{i2} / z_2 \right\}. \quad (3.41.b)$$

where

$$R_{1k} = 1, \quad (3.42.a)$$

$$R_{2k} = \mu_k, \quad (3.42.b)$$

$$P_{1k} = a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k, \quad (3.43.a)$$

$$P_{2k} = a_{12}\mu_k + \frac{a_{22}}{\mu_k} - a_{26} \text{ and} \quad (3.43.b)$$

$$Q_{ik} = \begin{bmatrix} \mu_i & \mu_k \\ -1 & -1 \end{bmatrix}. \quad (3.44)$$

The kernels $\overline{S_{kij}}$ and $\overline{D_{kij}}$ present in the Somigliana's identity for stresses $\sigma_{ij}(p)$ are derived with the application of the constitutive equation for two-dimensional anisotropic elasticity given by equation (3.1).

To solve the boundary integral equation (3.31) numerically for an elastic domain, the boundary of the problem is discretized into M discrete elements with N number of distinct nodes. The quadratic isoparametric element formulation is implemented in the

present study as its efficiency has been well established in BEM analysis. The quadratic isoparametric formulation assumes the geometry and the primary solution variables vary in a quadratic manner along each of the three-noded elements. The distribution of the variables along an element, as given in figure (3.3) can be defined in terms of the shape functions, $N^c(\xi)$, of the intrinsic coordinate ξ and the respective nodal values, as follows:

$$x_j(\xi) = N^c(\xi) x_j^c, \quad (3.45.a)$$

$$u_j(\xi) = N^c(\xi) u_j^c, \quad (3.45.b)$$

$$t_j(\xi) = N^c(\xi) t_j^c; \quad c = 1, 2 \text{ and } 3. \quad (3.45.c)$$

The shape function variables $N^c(\xi)$ are given as (figure, 3.4),

$$N^1(\xi) = -\frac{1}{2}\xi(1-\xi), \quad (3.46.a)$$

$$N^2(\xi) = (1-\xi^2), \quad (3.46.b)$$

$$N^3(\xi) = \frac{1}{2}\xi(1+\xi). \quad (3.46.c)$$

Substituting equations (3.45) into the boundary integral equation (3.31) results in

$$\begin{aligned} C_{jk}(P^a)u_k(P^a) + \sum_{b=1}^M \sum_{c=1}^3 u_k(P^{d(b,c)}) \int_{-1}^{+1} T_{jk}(P^a, Q(\xi)) N^c(\xi) |J(\xi)| d\xi \\ = \sum_{b=1}^M \sum_{c=1}^3 t_k(P^{d(b,c)}) \int_{-1}^{+1} U_{jk}(P^a, Q(\xi)) N^c(\xi) |J(\xi)| d\xi, \end{aligned} \quad (3.47)$$

where P^a is the a th node of the mesh and ranges from 1 to N , $P^{d(b,c)}$ is the c th node of the b th element and $J(\xi)$ is the Jacobian of transformation from the coordinate x_i to the intrinsic coordinate ξ . The Jacobian of transformation is given by

$$|J(\xi)| = \frac{dS}{d\xi} = \sqrt{\left(\frac{dx_1}{d\xi}\right)^2 + \left(\frac{dx_2}{d\xi}\right)^2} = \sqrt{\left(\frac{dN^c(\xi)}{d\xi} x_1^c\right)^2 + \left(\frac{dN^c(\xi)}{d\xi} x_2^c\right)^2}. \quad (3.48)$$

The problem domain after discretisation is shown schematically in figure (3.2). Similar discretisation of the Somigliana's identities leads to solutions for interior points. It should be noted that in the above discussion, the body force and thermal effects are ignored. Their presence in the elastic domain will give rise to additional integral terms in the BIE, see, e.g., Banerjee and Butterfield (1981), Tan (1987). In order to evaluate the integrands in equation (3.47), Gaussian quadrature is used along the elements. The final form of the discretized BIE, equation (3.47), represents a set of $2N$ linear algebraic equations in terms of the unknown displacements u_k and tractions t_k at the boundary nodes of the solution domain. This set of linear equations can be expressed in a matrix form as follows:

$$\mathbf{F} \mathbf{u} = \mathbf{b}, \quad (3.49)$$

where matrix $\mathbf{F}$ represents the coefficient matrix, column vector $\mathbf{u}$ contains the unknown values of the tractions and displacements and vector $\mathbf{b}$ contains the known values of the tractions and displacements. Solving equations (3.49) gives the required traction and displacement solutions along the boundary of the elastic domain from which the required stress and strain values can be obtained.

3.2.1 BEM fracture mechanics analysis

One commonly adopted scheme to model the crack in the problem domain in BEM is the “subregioning technique” (see, e.g., Tan and Gao, 1990). With this scheme, the problem domain is divided into subregions such that the crack plane lies along the common

boundaries of the corresponding subregions. For example, a rectangular plate with a central crack, as shown in figure (3.5.a), is modelled as shown in figure (3.5.b) using this technique.

With reference to the Williams' series expansion, the stress distribution near the crack tip has a singularity of order $(r)^{-1/2}$. The conventional isoparametric element does not account for this singular behavior, thus a relatively large number of elements is needed near the crack tip. Henshell & Shaw (1975) and Barsoum (1976) developed a simple procedure for finite element (FE) analysis to avoid this need. By placing the mid-node of the isoparametric element of length l adjacent to the crack-tip to a position $1/4l$ from the crack tip, as shown in figures (3.6) and (3.7), they showed that the displacement, u_j , will be of the proper form for the near-tip field:

$$u_j = A_j^{(1)} + A_j^{(2)} \sqrt{\frac{r}{l}} + A_j^{(3)} \frac{r}{l}. \quad (3.50)$$

The stresses in displacement-based finite element formulations are also of the proper form for the near-tip field along the quarter-point element as they are secondary solution variables obtained via differentiating equation (3.50) for the strains. In BEM, however, both displacements and tractions are primary solution variables. As such the tractions, t_j , along the quarter-point will vary as

$$t_j = A_j^{(1)} + A_j^{(2)} \sqrt{\frac{r}{l}} + A_j^{(3)} \frac{r}{l}, \quad (3.51)$$

where the superscripts denote the nodes of the element as shown in figure (3.6). This is not the required $(r)^{-1/2}$ singularity. In order to obtain the proper traction variation along this crack-tip element on the crack plane, an established practice in BEM is to multiply

the shape function by non-dimensional parameter $\sqrt{\frac{l}{r}}$ such that (see, e.g., Cruse and Wilson, 1977, Martinex and Dominguez, 1984. Tan and Gao, 1990),

$$t_i = \bar{t}_i^{(1)} \cdot N^{(1)}(\xi) \cdot \left(\sqrt{\frac{l}{r}}\right) + \bar{t}_i^{(2)} \cdot N^{(2)}(\xi) \cdot \left(\sqrt{\frac{l}{r}}\right) + \bar{t}_i^{(3)} \cdot N^{(3)}(\xi) \cdot \left(\sqrt{\frac{l}{r}}\right). \quad (3.52)$$

The computed traction coefficients $\bar{t}_i$ at any node along the singular crack-tip element can then be related to the corresponding physical values, t_i , as,

$$\bar{t}_i^{(3)} = t_i^{(3)} \text{ at node (3) where } r = l, \quad (3.53.a)$$

$$\bar{t}_i^{(2)} = \frac{t_i^{(2)}}{2} \text{ at node (2) where } r = l/4, \quad (3.53.b)$$

$$\bar{t}_i^{(1)} = \lim_{r \rightarrow 0} t_i^{(1)} \sqrt{\frac{r}{l}} \text{ at node (1) where } r = 0. \quad (3.53.c)$$

If the crack lies along the interface between two dissimilar solids, the traction singularity at the interface crack-tip is of oscillatory nature. By introducing a new modified shape function which incorporates this oscillatory nature of the singularity, the variation of the singular traction over the quarter-point crack-tip element can be expressed as follows (Tan *et al.*, 1992):

$$t_i = N^{(1)}(\xi) \cdot \left(\sqrt{\frac{l}{r}}\right) \cdot \mathbf{R}' \left[\left(\frac{r}{l}\right)^{i\gamma} \right] \cdot \bar{t}_i^{(1)} + N^{(2)}(\xi) \cdot \left(\sqrt{\frac{l}{r}}\right) \cdot \mathbf{R}' \left[\left(\frac{r}{l}\right)^{i\gamma} \right] \cdot \bar{t}_i^{(2)} \\ + N^{(3)}(\xi) \cdot \mathbf{R}' \left[\left(\frac{r}{l}\right)^{i\gamma} \right] \cdot \left(\sqrt{\frac{l}{r}}\right) \cdot \bar{t}_i^{(3)}, \quad (3.54)$$

where

$$\mathbf{R}' \left[\left(\frac{r}{l}\right)^{-i\gamma} \right] = \mathbf{Re} \left[\left(\frac{r}{l}\right)^{-i\gamma} \right] \mathbf{I} + \mathbf{Im} \left[\left(\frac{r}{l}\right)^{-i\gamma} \right] \mathbf{P}, \quad (3.55)$$

In equation (3.55), γ is the oscillatory index, $\mathbf{I}$ is the identity matrix and $\mathbf{P}$ is dependant upon the elastic constants of the bimaterial, the explicit equation for these terms is given in, e.g., Wu (1990), Tan *et al.* (1992), Ting (1996).

For a bimaterial interface crack, the physical tractions and the computed tractions can be related as follows:

$$\bar{t}_i^{(3)} = t_i^{(3)}, \quad (3.56.a)$$

$$\bar{t}_i^{(2)} = \frac{1}{2} \mathbf{R}' \left[\left(\frac{1}{2} \right)^{-i\gamma} \right] \cdot t_i^{(2)}, \quad (3.56.b)$$

$$\bar{t}_i^{(1)} = \lim_{r \rightarrow 0} \sqrt{\frac{r}{l}} \cdot \mathbf{R}' \left[\left(\frac{r}{l} \right)^{-i\gamma} \right] \cdot t_i^{(1)}. \quad (3.56.c)$$

By equating the corresponding traction fields ahead of the crack-tip with the assumed variation along the quarter-point element, relatively simple formulae for evaluating the stress intensity factors directly from the computed traction coefficient at the crack-tip node may be obtained. In homogeneous anisotropic bodies, they are (Tan and Gao, 1992a,b):

$$K_I = \bar{t}_2^{(1)} \sqrt{2\pi l}, \quad (3.57.a)$$

$$K_{II} = \bar{t}_1^{(1)} \sqrt{2\pi l}. \quad (3.57.b)$$

For a bimaterial interface crack, they are (Tan *et al.*, 1992):

$$K_I(L) = \sqrt{2\pi l} \cdot \mathbf{R}' \left[\left(\frac{L}{l} \right)^{i\gamma} \right] \cdot \bar{t}_2^{(1)}, \quad (3.58.a)$$

$$K_{II}(L) = \sqrt{2\pi l} \cdot \mathbf{R}' \left[\left(\frac{L}{l} \right)^{i\gamma} \right] \cdot \bar{t}_1^{(1)}. \quad (3.58.b)$$

The stress intensity factors for a bimaterial interface crack are scale-dependent, see, e.g., Rice (1988). Stress intensity factors for two different characteristic dimensions, L_1 and L_2 , can be related as (Wu, 1990) follows:

$$K_i(L_1) = \mathbf{R}' \left[\left(\frac{L_1}{L_2} \right)^{i'} \right] \cdot K_i(L_2). \quad (3.59)$$

The stress intensity factors for such cracks can also be expressed in a non-oscillatory form, termed as the modulus K_0 , as follows (Tan and Gao, 1990):

$$K_0 = \sqrt{K_I^2 + K_{II}^2}, \quad (3.60)$$

the definition of which is complete with the corresponding phase angle, ψ , given as,

$$\psi = \tan^{-1} \left(\frac{K_{II}}{K_I} \right). \quad (3.61)$$

These “traction formulae” for the evaluation of stress intensity factors have been shown to be relatively insensitive to BEM mesh designs and provide a quick means for these fracture parameters to be evaluated. Unfortunately, similar schemes to directly relate the T -stress to the traction coefficients by a formula have not been successful. A formula based on computed crack-flank displacements have been developed for isotropic elasticity by Tan and Wang (2003), however, the accuracy of the solutions obtained can be sensitive to the mesh design employed. The M -contour integral is therefore adopted in this study to this end. It requires, however, field solutions along a contour around the crack-tip within the domain.

3.3 Self-regularisation for Interior Point Solutions

The conventional BEM treatment using Somigliana's identities to obtain field solutions at interior points give accurate results for points sufficiently far away from the boundary. However, as the interior point under consideration approaches the boundary, the field solution for that point becomes increasingly less accurate unless mesh refinement or higher order integration is adopted. This is due to the singular nature of the kernels U_{ji} , T_{ji} , D_{kij} , S_{kij} , $\bar{D}_{kij}$ and $\bar{S}_{kij}$ of order $\ln r$, r^{-1} , r^{-1} , r^{-2} , r^{-1} and r^{-2} , respectively, where r is the distance between the field and load points. It gives rise to near-singular nature of the integrals for a point in the vicinity of the boundary as r approaches zero. Experience with BEM modelling has shown that this effect is more evident when the distance between the source point p and the nearest element becomes less than the size of the element.

In order to minimize this effect at points near the boundary, Li (2003) used an adaptive meshing scheme in which the size of the element closest to the interior point was refined relative to the distance between them. For example, the numerical modelling for a point p near the boundary S of an elastic body (figure 3.8.a) is done such that the elements near p are gradually refined (figure 3.8.b). This practice yields good accuracy of the results but it demands a refined meshing scheme, making the modelling and the solution scheme less efficient.

One relatively simple approach to overcome this near-singularity effect without the need for excessive mesh refinement was proposed by Richardson and Cruse (1999) for isotropy. The approach is termed as "self-regularisation" in which the singular nature of the Somigliana's kernels is either removed or reduced making it weakly singular. In their work,

they proposed a scheme to obtain a continuous form of the Somigliana's displacement identity by regularizing the singular kernels in it as follows,

$$u_j(p) - u_j(P) = \int_S t_i(Q) U_{ji}(p, Q) dS(Q) - \int_S [u_i(Q) - u_i(P)] T_{ji}(p, Q) dS(Q), \quad (3.62)$$

where P corresponds to a point along the boundary S closest to the interior point p . In the above expression, the first integrand is weakly singular at the boundary when p approaches P and the second integrand is continuous.

Similarly, a weakly singular form of the stress identity is obtained by a relatively simple procedure. The method is equivalent to subtracting and adding back a simple solution corresponding to a state of constant stress in the body that is equal to the boundary stress at point P close to the interior point p , provided that the stress at the boundary point P is continuous. The self-regular stress identity can then be expressed as follows:

$$\begin{aligned} \sigma_{ij}(p) = \sigma_{ij}(P) - \int_S [u_k(Q) - u_k^L(P, Q)] \overline{S_{kij}}(p, Q) dS(Q) \\ + \int_S [t_k(Q) - t_k^L(P, Q)] \overline{D_{kij}}(p, Q) dS(Q) \end{aligned} \quad (3.63)$$

In equation (3.63), u_k^L and t_k^L are the linear state displacements and tractions related to a constant stress state in the body; the latter corresponds to the stress at the boundary point P as shown in equations (3.64) below:

$$u_k^L(P, Q) = u_k(P) + u_{k,m}(P) \{x_m(Q) - x_m(P)\}, \quad (3.64.a)$$

$$t_k^L(P, Q) = \sigma_{mk}(P) n_m(Q). \quad (3.64.b)$$

The traction and displacements at any point P on the boundary S can be obtained by interpolating the boundary solutions available from the BEM analysis. In order to

obtain the displacement derivative $u_{k,m}$ at point P , a set of four equations are required, the first two of which are derived from the constitutive equation as follows:

$$t_j = \sigma_{ij} n_i = \left\{ \frac{2G\nu}{1-2\nu} u_{i,i} \delta_{ij} + G(u_{i,j} + u_{j,i}) \right\} n_i. \quad (3.65)$$

The displacement derivatives with respect to the boundary S and the intrinsic coordinate ξ can be expressed as,

$$\frac{\partial u_k}{\partial x_1} \frac{\partial x_1}{\partial S} + \frac{\partial u_k}{\partial x_2} \frac{\partial x_2}{\partial S} = \frac{du_k}{dS} \approx \frac{du_k(\xi)}{d\xi} \frac{d\xi}{dS} = \frac{1}{J(\xi)} \sum_{i=1}^3 N'_i(\xi) u_k^i, \quad (3.66)$$

where N'_i is the derivative of the corresponding shape function and $i, j, k = 1, 2$.

or,

$$\frac{\partial u_k}{\partial x_1} (-n_2) + \frac{\partial u_k}{\partial x_2} n_1 = \frac{1}{J(\xi)} \sum_{i=1}^3 N'_i(\xi) u_k^i, \quad (3.67)$$

where n_1 and n_2 are unit outward normal and tangent to the boundary S , respectively.

Hence,

$$\sum_{i=1}^3 N'_i(\xi) u_k^i = -J(\xi) n_2 u_{k,1} + J(\xi) n_1 u_{k,2}. \quad (3.68)$$

Equations (3.65) and (3.68) provide a set of equations with unknown $u_{k,m}$. This set of equations when inversed using a symbolic processor gives the displacement derivative at the source point P in terms of the nodal tractions and displacements.

3.3.1 Self-regularisation in two-dimensional anisotropy

In the present work, the self-regularisation for two-dimensional isotropy is extended to two-dimensional anisotropy. In the implementation, the regularisation point P is chosen

to be a nodal point closest to the interior point p . In the special case of a cracked geometry when P is the crack-tip node, P is shifted to the corresponding quarter-point node. This will avoid the error introduced by the singular stresses $\sigma_{ij}(P)$ at the crack tip. Similar to the development in isotropy, the displacement derivatives in anisotropy are obtained as discussed next.

Following the constitutive equation for two-dimensional anisotropy, the traction can now be represented in terms of the displacement derivatives as,

$$\begin{aligned} t_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 \\ &= \{C_{11}n_1 + C_{16}n_2\}u_{1,1} + \{C_{12}n_1 + C_{26}n_2\}u_{2,2} + \{C_{16}n_1 + C_{66}n_2\}u_{1,2} + \{C_{16}n_1 + C_{66}n_2\}u_{2,1} \end{aligned} \quad (3.69.a)$$

$$\begin{aligned} t_2 &= \sigma_{12}n_1 + \sigma_{22}n_2 \\ &= \{C_{16}n_1 + C_{12}n_2\}u_{1,1} + \{C_{26}n_1 + C_{22}n_2\}u_{2,2} + \{C_{66}n_1 + C_{26}n_2\}u_{1,2} + \{C_{66}n_1 + C_{26}n_2\}u_{2,1} . \end{aligned} \quad (3.69.b)$$

The set of equations (3.68) and (3.69), when inversed using a symbolic processor, gives expressions for the displacement derivatives at the point P .

3.4 Test Problems and Validations

The self-regularisation technique discussed above was implemented in the BEM formulation in this study. The veracity of this development was tested against the available analytical solutions for two simple cases presented below. Although the first test case is one of isotropic elasticity, it is nevertheless analysed through the BEM algorithm for anisotropy as discussed above.

3.4.1 Bending of an isotropic cantilever beam

The first test problem considered was bending of a cantilever beam. The beam considered was 10 units in length and 1 unit in width with a uniform square cross section. It was fixed at one end and a unit shear stress was applied at the other end as shown in figure (3.9.a). The beam was treated as isotropic with $E = 1000$ units and $\nu = 0$, and plane stress conditions were assumed. This allows for a direct comparison with the analytical solution from simple beam theory.

For the BEM analysis, the problem boundary was discretized into 22 elements as shown in figure (3.9.b). The flexural stresses, σ_{22} , were obtained along the interior points at a distance of 4 units from the fixed end. The interior points under consideration were at a distance of 0.1 to 0.4 units from the neutral axis on the tensile side. BEM analysis was carried out using the self-regularised and the non-regularised, conventional interior point solution scheme with the same mesh.

The results from the BEM analysis and simple flexure formula are compared in table (3.1). The self-regularised solutions are in an excellent agreement with the analytical solutions with a maximum difference of 0.5%. On the other hand, non-regularised solutions show very large deviations for points close to the boundary. It is evident that even for the points very close to the boundary, the self-regularisation technique provides excellent results with a relatively coarse mesh. In order to attain such accuracy using non-regularised schemes, very refined mesh along the boundary is required.

3.4.2 Anisotropic plate under uniform tension

In order to test the veracity of the current development in anisotropy, an orthotropic, square plate with each side 10 units in length was considered. The plate was subjected to a uniform tensile stress, σ_0 , of unit magnitude and plane stress conditions were considered. The material chosen was a graphite/epoxy composite with the following properties (Tan and Gao, 1992b):

$$E_{11} = 144.8 \text{ GPa}; \quad E_{22} = 11.7 \text{ GPa}; \quad G_{12} = 9.66 \text{ GPa}; \quad \nu_{12} = 0.21.$$

The material principal axes were oriented at 30° with respect to the global axes to impart general anisotropy, as shown in figure (3.10.a). A mesh with the lowest number of boundary elements possible, one for each side of the plate, was employed as shown in figure (3.10.b). Each of the 4 elements used was 10 units in size. The interior points were chosen at a distance of 0.001, 0.01, 0.05, 0.1 and 5 units from the segment AB, as shown in figure (3.10.b).

In table (3.2), the values of the stresses σ_{22} at the interior points, obtained from the self-regularised and conventional non-regularised BEM analysis, are compared with the exact solution. The employment of the self-regularisation scheme yielded excellent results with a maximum difference of 0.3% even when the interior point was very close to the boundary, unlike the conventional BEM approach.

3.5 Closing Remarks

Two-dimensional anisotropic elasticity has been briefly reviewed with two different but equivalent formalisms. The BEM treatment of anisotropic elastic domain has also been presented. A simple but novel approach of self-regularisation to accurately obtain the

interior point solutions near the boundary in isotropy has been discussed and this approach is successfully extended to anisotropy. This has been verified with some test examples. With the employment of this scheme to obtain the interior point solutions accurately, the implementation of the M -contour integral to obtain the T -stress for an anisotropic elastic cracked body will be discussed in the next chapter.

CHAPTER 4

EVALUATION OF T -STRESS IN ANISOTROPIC ELASTIC CRACKED BODIES

4.0 Introduction

In this chapter, the development of the M -integral approach with BEM to evaluate T -stress for cracked homogeneous anisotropic elastic bodies is presented. The scheme is validated by comparing the results with those available in the literature, namely, for isotropic and orthotropic specimens, via the algorithm for anisotropy. This is mainly because of the scarcity of the solutions in the literature for anisotropic specimens. Parametric studies on some geometries of practical significance are then performed to investigate the effects of various material and geometric parameters on the T -stress.

4.1 M -integral for Cracks in Two-dimensional Anisotropic Elastic Solids

The M -integral approach to evaluate T -stress solutions in two-dimensional isotropy, as discussed in Section (2.3.1.1), is extended to anisotropy here. It may be recalled from

equation (2.24) that the M -integral based on two independent equilibrium states, $A(\sigma_{ij}^A, \varepsilon_{ij}^A, u_i^A)$ and $B(\sigma_{ij}^B, \varepsilon_{ij}^B, u_i^B)$, is as follows:

$$M = \int_{\Gamma_0} (\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A) d\Gamma, \quad (4.1)$$

where state (A) contains the field values due to the stress intensity factor K and T -stress as given in equation (2.8). The equilibrium state (B), also referred to as the auxiliary field, corresponds to the solution to the problem of an anisotropic solid with a semi-infinite crack subjected to a point load f acting along the crack plane in the anisotropic solid, as shown in figure (4.1). The required auxiliary field solutions for an anisotropic body are obtained using both the Lekhnitskii and Stroh formalisms here and are presented in the following section. Due to the equivalence of the Lekhnitskii and Stroh formalisms (Barnett and Kirchner, 1997), the current development using both approaches will provide a cross check for the results obtained.

4.1.1 Auxiliary field solutions using Lekhnitskii formalism

Lekhnitskii (1963, 1968) has developed a solution for a point load acting at the apex of a wedge shaped beam with unit thickness comprising of a homogeneous anisotropic elastic material. The wedge profile can be extended such that the wedge angle can be increased to the value of 2π making the free surfaces as the faces of a crack in the solid. With reference to figure (4.1), the field solutions at any point $p(r, \phi)$ in terms of the polar coordinates are given as:

$$\sigma_{rr}^B = \frac{C \cos \phi + D \sin \phi}{rL(\phi)}, \quad (4.2.a)$$

$$\sigma_{\phi\phi}^B = \sigma_{r\phi}^B = 0, \quad (4.2.b)$$

where

$$L(\phi) = a_{11} \cos^4 \phi + 2a_{16} \sin \phi \cos^3 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + 2a_{26} \sin^3 \phi \cos \phi + a_{22} \sin^4 \phi, \quad (4.3)$$

and C and D are determined by solving the following equations:

$$C \int_{-\psi_1}^{\psi_2} \frac{\cos^2 \phi}{L(\phi)} d\phi + D \int_{-\psi_1}^{\psi_2} \frac{\sin \phi \cos \phi}{L(\phi)} d\phi = -f \cos \omega, \quad (4.4.a)$$

$$C \int_{-\psi_1}^{\psi_2} \frac{\sin \phi \cos \phi}{L(\phi)} d\phi + D \int_{-\psi_1}^{\psi_2} \frac{\sin^2 \phi}{L(\phi)} d\phi = -f \sin \omega. \quad (4.4.b)$$

The generalized Hooke's law for two-dimensional elasticity in the polar co-ordinate system is given as,

$$\begin{aligned} \varepsilon_{rr}^B &= a'_{11} \sigma_{rr}^B + a'_{12} \sigma_{\phi\phi}^B + a'_{16} \sigma_{r\phi}^B, \\ \varepsilon_{\phi\phi}^B &= a'_{12} \sigma_{rr}^B + a'_{22} \sigma_{\phi\phi}^B + a'_{26} \sigma_{r\phi}^B, \\ \varepsilon_{r\phi}^B &= a'_{16} \sigma_{rr}^B + a'_{26} \sigma_{\phi\phi}^B + a'_{66} \sigma_{r\phi}^B. \end{aligned} \quad (4.5)$$

From equations (4.2) and (4.5),

$$\begin{aligned} \varepsilon_{rr}^B &= a'_{11} \sigma_{rr}^B, \\ \varepsilon_{\phi\phi}^B &= a'_{12} \sigma_{rr}^B, \\ \varepsilon_{r\phi}^B &= a'_{16} \sigma_{rr}^B. \end{aligned} \quad (4.6)$$

where a'_{ij} are transformed values of a_{ij} along direction ϕ as given in equation (3.28); a_{ij} are defined with respect to the displaced global co-ordinate system, X'_i , as shown in figure (4.1). Hence, the auxiliary stress and strain fields in this Cartesian co-ordinate system, X'_i , can be expressed as follows:

Stresses:

$$\begin{aligned}
\sigma_{11}^B &= \sigma_{rr}^B \cos^2 \phi = \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \cos^2 \phi, \\
\sigma_{22}^B &= \sigma_{rr}^B \sin^2 \phi = \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \sin^2 \phi, \\
\sigma_{12}^B &= \sigma_{rr}^B \sin \phi \cos \phi = \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \sin \phi \cos \phi.
\end{aligned} \tag{4.7}$$

Strains:

$$\begin{aligned}
\varepsilon_{11}^B &= a_{11}\sigma_{11}^B + a_{12}\sigma_{22}^B + a_{13}\sigma_{33}^B + a_{14}\sigma_{23}^B + a_{15}\sigma_{13}^B + a_{16}\sigma_{12}^B \\
&= a_{11}\sigma_{11}^B + a_{12}\sigma_{22}^B + a_{16}\sigma_{12}^B, \\
\varepsilon_{22}^B &= a_{12}\sigma_{11}^B + a_{22}\sigma_{22}^B + a_{23}\sigma_{33}^B + a_{24}\sigma_{23}^B + a_{25}\sigma_{13}^B + a_{26}\sigma_{12}^B \\
&= a_{12}\sigma_{11}^B + a_{22}\sigma_{22}^B + a_{26}\sigma_{12}^B, \\
\varepsilon_{12}^B &= a_{16}\sigma_{11}^B + a_{26}\sigma_{22}^B + a_{36}\sigma_{33}^B + a_{46}\sigma_{23}^B + a_{56}\sigma_{13}^B + a_{66}\sigma_{12}^B \\
&= a_{16}\sigma_{11}^B + a_{26}\sigma_{22}^B + a_{66}\sigma_{12}^B.
\end{aligned} \tag{4.8}$$

The displacement derivatives for the auxiliary field (B) at (r, ϕ) can be expressed as follows:

$$\begin{aligned}
u_{1,1}^B &= \varepsilon_{11}^B = a_{11}\sigma_{11}^B + a_{12}\sigma_{22}^B + a_{16}\sigma_{12}^B \\
&= \left(a_{11} \cos^2 \phi + a_{12} \sin^2 \phi + a_{16} \sin \phi \cos \phi \right) \frac{C \cos \phi + D \sin \phi}{rL(\phi)},
\end{aligned} \tag{4.9.a}$$

$$\begin{aligned}
u_{2,2}^B &= \varepsilon_{22}^B = a_{12}\sigma_{11}^B + a_{22}\sigma_{22}^B + a_{26}\sigma_{12}^B \\
&= \left(a_{12} \cos^2 \phi + a_{22} \sin^2 \phi + a_{26} \sin \phi \cos \phi \right) \frac{C \cos \phi + D \sin \phi}{rL(\phi)},
\end{aligned} \tag{4.9.b}$$

$$u_{2,1}^B = 2\varepsilon_{12}^B - u_{1,2}^B. \tag{4.9.c}$$

The derivation of $u_{1,2}^B$ will be discussed next.

The Cartesian displacement along X'_1 in terms of polar co-ordinates can be expressed in a differential form as,

$$\frac{\partial u_1^B}{\partial r} = \frac{\partial u_r^B}{\partial r} \cos \phi - \frac{\partial u_\phi^B}{\partial r} \sin \phi. \tag{4.10}$$

Applying the chain rule of differentiation, $u_{1,1}$, and $u_{1,2}$ are obtained as follows:

$$\begin{aligned} u_{1,1}^B &= \frac{\partial u_1^B}{\partial r} \frac{\partial r}{\partial X'_1} + \frac{\partial u_1^B}{\partial \phi} \frac{\partial \phi}{\partial X'_1} \\ &= \frac{\partial u_1^B}{\partial r} \cos \phi - \frac{\partial u_1^B}{\partial \phi} \frac{\sin \phi}{r}, \end{aligned} \quad (4.11.a)$$

$$\begin{aligned} u_{1,2}^B &= \frac{\partial u_1^B}{\partial r} \frac{\partial r}{\partial X'_2} + \frac{\partial u_1^B}{\partial \phi} \frac{\partial \phi}{\partial X'_2} \\ &= \frac{\partial u_1^B}{\partial r} \sin \phi + \frac{\partial u_1^B}{\partial \phi} \frac{\cos \phi}{r}. \end{aligned} \quad (4.11.b)$$

From equation (4.6),

$$\varepsilon_{rr}^B = \frac{\partial u_r^B}{\partial r} = a'_{11} \frac{C \cos \phi + D \sin \phi}{rL(\phi)}, \quad (4.12.a)$$

$$\varepsilon_{\phi\phi}^B = \frac{u_r^B}{r} + \frac{\partial u_\phi^B}{r \partial \phi} = a'_{12} \sigma_{rr}^B = a'_{12} \frac{C \cos \phi + D \sin \phi}{rL(\phi)}, \quad (4.12.b)$$

$$\varepsilon_{r\phi}^B = \frac{1}{2} \left(\frac{\partial u_r^B}{r \partial \phi} + \frac{\partial u_\phi^B}{\partial r} - \frac{u_\phi^B}{r} \right) = a'_{16} \sigma_{rr}^B = a'_{16} \frac{C \cos \phi + D \sin \phi}{rL(\phi)}. \quad (4.12.c)$$

Integrating equation (4.12.a) with respect to r yields,

$$u_r^B = \left(a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} \right) \ln r + g(\phi), \quad (4.13)$$

where $g(\phi)$ is a function of ϕ .

From equations (4.12.b) and (4.13)

$$u_\phi^B = \int_\phi a'_{12} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi - \ln r \int_\phi a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi - \int_\phi g(\phi) d\phi + h(r), \quad (4.14)$$

where $h(r)$ is a function of r .

Thus,

$$\frac{\partial u_\phi^B}{\partial r} = -\frac{1}{r} \int_\phi a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi + \frac{\partial h(r)}{\partial r}. \quad (4.15)$$

When $r \rightarrow \infty$,

$$\varepsilon_{r\phi}^B = a'_{16} \frac{C \cos \phi + D \sin \phi}{rL(\phi)} = 0. \quad (4.16)$$

Thus as $r \rightarrow \infty$, equation (4.12.c) becomes,

$$\frac{\partial u_\phi^B}{\partial r} = \frac{1}{r} \left(-\frac{\partial u_r^B}{\partial \phi} + u_\phi^B \right) = 0. \quad (4.17)$$

This implies

$$\frac{\partial h(r)}{\partial r} = 0. \quad (4.18)$$

From equations (4.15) and (4.18)

$$\frac{\partial u_\phi^B}{\partial r} = -\frac{1}{r} \int_\phi a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi. \quad (4.19)$$

Substituting equations (4.12.a) and (4.19) into equation (4.10),

$$\frac{\partial u_1^B}{\partial r} = a'_{11} \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \cos \phi + \frac{1}{r} \int_\phi a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi \sin \phi. \quad (4.20)$$

Let

$$I(\phi) = \int_\phi a'_{11} \frac{C \cos \phi + D \sin \phi}{L(\phi)} d\phi. \quad (4.21)$$

Thus, equation (4.20) becomes

$$\frac{\partial u_1^B}{\partial r} = a'_{11} \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \cos \phi + \frac{1}{r} I(\phi) \sin \phi. \quad (4.22)$$

From equation (4.11.a)

$$u_{1,1}^B = \frac{\partial u_1^B}{\partial r} \cos \phi - \frac{\partial u_1^B}{\partial \phi} \frac{\sin \phi}{r},$$

or,

$$\frac{\partial u_1^B}{\partial \phi} = \frac{r}{\sin \phi} \left(\frac{\partial u_1^B}{\partial r} \cos \phi - u_{1,1}^B \right). \quad (4.23)$$

Substituting equation (4.23) into equation (4.11.b) gives,

$$\begin{aligned} u_{1,2}^B &= \frac{\partial u_1^B}{\partial r} \sin \phi + \frac{r}{\sin \phi} \left(\frac{\partial u_1^B}{\partial r} \cos \phi - u_{1,1}^B \right) \frac{\cos \phi}{r} \\ &= \frac{\partial u_1^B}{\partial r} \left(\sin \phi + \frac{\cos^2 \phi}{\sin \phi} \right) - u_{1,1}^B \frac{\cos \phi}{\sin \phi} \\ &= \frac{\partial u_1^B}{\partial r} \frac{1}{\sin \phi} - u_{1,1}^B \frac{\cos \phi}{\sin \phi} \\ &= \frac{1}{\sin \phi} \left(\frac{\partial u_1^B}{\partial r} - u_{1,1}^B \cos \phi \right). \end{aligned} \quad (4.24.a)$$

Using equations (4.24.a), (4.9.a), (4.9.c), (4.8) and (3.28),

$$\begin{aligned} u_{2,1}^B &= 2\varepsilon_{12}^B - \frac{1}{\sin \phi} \left(\frac{\partial u_1^B}{\partial r} - u_{1,1}^B \cos \phi \right) \\ &= 2 \left(a_{16} \cos^2 \phi + a_{26} \sin^2 \phi + a_{66} \sin \phi \cos \phi \right) \left(\frac{C \cos \phi + D \sin \phi}{rL(\phi)} \right) \\ &\quad - \left\{ a_{11} \cos^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + a_{22} \sin^4 \phi \right\} \frac{C \cos \phi + D \sin \phi \cos \phi}{rL(\phi) \sin \phi} \\ &\quad - \frac{1}{r} I(\phi) + \frac{\cos \phi}{\sin \phi} \left\{ (a_{11} \cos^2 \phi + a_{12} \sin^2 \phi + a_{16} \sin \phi \cos \phi) \frac{C \cos \phi + D \sin \phi}{rL(\phi)} \right\}, \end{aligned} \quad (4.24.b)$$

$$u_{1,1}^B = \varepsilon_{11}^B, \quad (4.24.c)$$

$$u_{2,2}^B = \varepsilon_{22}^B. \quad (4.24.d)$$

Expressions (4.7) and (4.24) provide the stresses and displacement derivatives, respectively, for the auxiliary fields.

4.1.2 Auxiliary field solutions using Stroh formalism

The derivation of auxiliary field solutions for cracks in homogeneous anisotropic elastic solid using Stroh formalism, as discussed below, is a specific case of the solutions for bimaterial interface cracks to be discussed in Chapter 5; for a crack in a homogeneous solid, the materials forming the interface are the same. Thus, in the following discussion, only the expressions relevant to the homogeneous solids will be presented.

Utilizing the Stroh formalism for two-dimensional anisotropy, Chung and Ting (1995) presented the expressions for displacements and stress function at a point (r, ϕ) in a wedge comprising of anisotropic composite materials subjected to a concentrated force $\mathbf{f}$ at the wedge apex (figure 4.1). In the case of a homogeneous wedge comprising of only one type of material, the displacements and stress function, respectively, reduce to,

$$\mathbf{u} = -\frac{\ln r}{\pi} \mathbf{h} - \mathbf{S}(\phi) \mathbf{h} + \mathbf{H}(\phi) \mathbf{g} + \mathbf{u}_0, \quad (4.25.a)$$

$$\Phi = \frac{\ln r}{\pi} \mathbf{g} + \mathbf{S}^T(\phi) \mathbf{g} + \mathbf{L}(\phi) \mathbf{h}. \quad (4.25.b)$$

where vector $\mathbf{u}_0$ is a constant and $\mathbf{S}(\phi)$, $\mathbf{H}(\phi)$ and $\mathbf{L}(\phi)$ are the Barnett-Lothe tensors (Ting, 1996) given as,

$$\mathbf{S}(\phi) = \frac{1}{\pi} \int_0^\phi \mathbf{N}_1(\lambda) d\lambda, \quad (4.26.a)$$

$$\mathbf{H}(\phi) = \frac{1}{\pi} \int_0^\phi \mathbf{N}_2(\lambda) d\lambda, \quad (4.26.b)$$

$$\mathbf{L}(\phi) = -\frac{1}{\pi} \int_0^\phi \mathbf{N}_3(\lambda) d\lambda, \quad (4.26.c)$$

where $\mathbf{N}_i$ is the i th element of the fundamental elasticity matrix as given in equations (3.24). In order to obtain these identities for an angle λ , the following relations should be used:

$$\mathbf{N}_1(\lambda) = -\mathbf{T}^{-1}(\lambda) \mathbf{R}^T(\lambda), \quad (4.27.a)$$

$$\mathbf{N}_2(\lambda) = \mathbf{T}^{-1}(\lambda), \quad (4.27.b)$$

$$\mathbf{N}_3(\lambda) = \mathbf{R}(\lambda) \mathbf{T}^{-1}(\lambda) \mathbf{R}^T(\lambda) - \mathbf{Q}(\lambda). \quad (4.27.c)$$

For the wedge, the tractions are null at the free surfaces, thus, in polar co-ordinates,

$$(t_\phi)_j = \frac{\partial \Phi_j}{\partial r} = \frac{1}{\pi r} g_j = 0 \quad (j = 1, 2), \quad (4.28)$$

and thus,

$$\mathbf{g} = 0. \quad (4.29)$$

g_j being the element of vector $\mathbf{g}$.

The vector $\mathbf{h}$ is given as

$$\mathbf{h} = \underline{\mathbf{L}}^{-1} \mathbf{f}, \quad (4.30)$$

where

$$\underline{\mathbf{L}} = \frac{1}{\pi} \int_0^\alpha -\mathbf{N}_3(\lambda) d\lambda, \quad (4.31)$$

α being the angle between the free surfaces of the wedge which is equal to 2π , i.e., $(\psi_1 + \psi_2)$, as in figure (4.1) for a pair of crack surfaces.

In equation (4.30) $\mathbf{f}$ is the point force vector applied at the wedge apex (figure 4.1), that is,

$$\mathbf{f} = \begin{Bmatrix} f \cos \omega \\ f \sin \omega \end{Bmatrix}, \quad (4.32)$$

where ω is the angle between the crack and the global co-ordinate axes, X'_i .

Substituting equation (4.29) into equations (4.25) results in,

$$\mathbf{u} = -\frac{\ln r}{\pi} \mathbf{h} - \mathbf{S}(\phi) \mathbf{h} + \mathbf{u}_0, \quad (4.33.a)$$

$$\Phi = \mathbf{L}(\phi) \mathbf{h}. \quad (4.33.b)$$

The displacement derivatives with respect to X'_i are obtained as follows,

$$\begin{aligned} u_{i,j} &= \frac{\partial u_i}{\partial r} \frac{\partial r}{\partial X'_j} + \frac{\partial u_i}{\partial \phi} \frac{\partial \phi}{\partial X'_j} \\ &= \frac{\partial u_i}{\partial r} n_j(\phi) + \frac{\partial u_i}{\partial \phi} \frac{1}{r} m_j(\phi), \\ &= \frac{1}{\pi r} \left\{ -h_i n_j(\phi) - (\mathbf{N}_1(\phi) \mathbf{h})_i m_j(\phi) \right\}, \end{aligned} \quad (4.34)$$

where

$$n_1(\phi) = \cos(\phi), \quad n_2(\phi) = \sin(\phi), \quad m_1(\phi) = -\sin(\phi), \quad m_2(\phi) = \cos(\phi). \quad (4.35)$$

The stresses are obtained as follows:

$$\begin{aligned} \sigma_{1j} &= -\Phi_{j,2} \\ &= -\frac{\partial \Phi_j}{\partial X'_2} = -\left[\frac{\partial \Phi_j}{\partial r} \frac{\partial r}{\partial X'_2} + \frac{\partial \Phi_j}{\partial \phi} \frac{\partial \phi}{\partial X'_2} \right] \\ &= -\frac{\partial \Phi_j}{\partial \phi} \frac{1}{r} m_2(\phi) = \frac{1}{\pi r} [\mathbf{N}_3(\phi) \mathbf{h}]_j m_2(\phi), \end{aligned} \quad (4.36.a)$$

and

$$\sigma_{2j} = \Phi_{j,1} = -\frac{1}{\pi r} [\mathbf{N}_3(\phi) \mathbf{h}]_j m_1(\phi). \quad (4.36.b)$$

Equations (4.34) and (4.36) provide the required auxiliary field solutions for the evaluation of M -integral in equation (4.1). Alternative but equivalent expressions for these auxiliary field solutions for general anisotropy have also been proposed by Yang and Yuan (2000a) and Kim *et al.* (2001).

4.1.3 Extraction of T -stress from M -integral

As in the case of isotropy, the M -integral in anisotropy inherits the path independent property from J -integral. Thus, the integration contour can be arbitrarily chosen, say, a circle with radius ε , which is then shrunk to zero. The M -integral then reduces to,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right] d\Gamma. \quad (4.37)$$

It can be recalled from equations (2.26) and (2.27) that for equilibrium state (A), the asymptotic displacements and stresses may be separated into singular and non-singular components. The boundedness of the M -integral leaves only the non-vanishing contribution to it from the non-singular terms containing T -stress functions. Thus, equation (4.37) reduces to

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{1k} - \sigma_{ik}^T u_{i,1}^B - \sigma_{ik}^B u_{i,1}^T \right] n_k d\Gamma, \quad (4.38)$$

where,

$$\sigma_{ij}^T = T \delta_{ij} \delta_{j,1}. \quad (4.39)$$

The elastic strain ε_{11}^T , along the local co-ordinates x_i under plane stress conditions is given in terms of the T -stress as:

$$\varepsilon_{11}^T = a'_{11} T. \quad (4.40)$$

Thus,

$$u_{i,1}^T = u_{i,1}^T \delta_{i1} = \varepsilon_{11}^T \delta_{i1} = a'_{11} T \delta_{i1}. \quad (4.41)$$

Thus, the M -integral reduces to

$$\begin{aligned} M &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{1k} - \sigma_{ik}^T \varepsilon_{1,1}^B \delta_{1i} - \sigma_{ik}^B u_{i,1}^T \right] n_k d\Gamma \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[T \delta_{i1} \delta_{j1} \varepsilon_{ij}^B \delta_{1k} n_k - T \delta_{i1} \delta_{k1} \varepsilon_{1,1}^B \delta_{1i} n_k - \sigma_{ik}^B u_{i,1}^T n_k \right] d\Gamma \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[T \varepsilon_{11}^B n_1 - T \varepsilon_{11}^B n_1 - \sigma_{ik}^B u_{i,1}^T n_k \right] d\Gamma \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_o} \left[\sigma_{ik}^B u_{i,1}^T \right] n_k d\Gamma = - \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_o} \left[\sigma_{ij}^B u_{i,1}^T \right] n_j d\Gamma \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^B n_j \varepsilon_{11}^T \delta_{i1} \right] d\Gamma = - \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{1j}^B n_j a'_{11} T \right] d\Gamma \\ &= -a'_{11} T \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{1j}^B n_j \right] d\Gamma \\ &= a'_{11} T f. \end{aligned} \quad (4.42)$$

Equation (4.42) may be re-arranged as,

$$T = M \frac{1}{a'_{11} f}. \quad (4.43)$$

Equation (4.43) provides the means to evaluate T -stress from the M -integral solution in local co-ordinates, x_i , obtained from equation (4.1). In order to evaluate the T -stress for plane strain conditions, a'_{11} in equation (4.43) should be replaced by b'_{11} as in equation (3.3). In the present implementation, in order to obtain the M -integral in local co-ordinates, x_i , it is first obtained in the global co-ordinates, X'_i , and then transformed

into the local co-ordinates. This is done as the auxiliary field solutions are readily available in the global co-ordinates. The co-ordinate transformation of the M -integral follows that of the J -integral (Kishimoto *et al.*, 1980) as follows:

$$M_{(Local)} = M_{1(Global)} \cos \omega + M_{2(Global)} \sin \omega, \quad (4.44)$$

where

$$M_{k(Global)} = \int_{\Gamma_0} \left[(\sigma_{ij}^A \varepsilon_{ij}^B n_k - \sigma_{ij}^A n_j u_{i,k}^B - \sigma_{ij}^B n_j u_{i,k}^A) \right] d\Gamma. \quad (4.45)$$

4.2 Stress intensity factor using M -integral

Although the main focus of the present study is not on the stress intensity factor, it is also evaluated in the present work along with the T -stress solutions for completeness. An approach based on the M -integral as developed by Wang *et al.* (1980) to compute stress intensity factors in two-dimensional homogeneous anisotropy is employed in the present study. The arguments presented earlier on the accuracy of T -stress solutions using contour integral formulation apply to stress intensity factors as well. Using this approach, the stress intensity factors and the corresponding T -stress solutions could be computed from the same contour in one run. The stress intensity factors computed with this approach can be checked with those obtained using BEM traction formula by Tan and Gao (1992a,b) as given in equations (3.57). The development based on M -integral, given in equation (4.1), will be briefly reviewed here.

Based on the energy release rate, the J -integral can be related to the stress intensity factors for a cracked homogeneous anisotropic solid as

$$J = \alpha_{11} K_I^2 + \alpha_{12} K_I K_{II} + \alpha_{22} K_{II}^2, \quad (4.46)$$

where

$$\alpha_{11} = -\frac{a_{22}}{2} \mathbf{Im} \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right), \quad (4.47.a)$$

$$\alpha_{22} = \frac{a_{11}}{2} \mathbf{Im} (\mu_1 + \mu_2) \text{ and} \quad (4.47.b)$$

$$\alpha_{12} = -\frac{a_{22}}{2} \mathbf{Im} \left(\frac{1}{\mu_1 \mu_2} \right) + \frac{a_{11}}{2} \mathbf{Im} (\mu_1 \mu_2). \quad (4.47.c)$$

Considering two independent equilibrium states (*A*) and (*B*), the *M*-integral is obtained in terms of the stress intensity factors as follows:

$$M = J^{(A+B)} - J^{(A)} - J^{(B)} = 2\alpha_{11} K_I^A K_I^B + \alpha_{12} (K_I^A K_{II}^B + K_I^B K_{II}^A) + 2\alpha_{22} K_{II}^A K_{II}^B. \quad (4.48)$$

While the state (*A*) corresponds to the boundary value problem, two sets of auxiliary field (*B*) solutions are required to compute the mixed-mode stress intensity factors from *M*-integral. The first set, denoted by superscript (*B1*), corresponds to the case of mode I deformation only, such that,

$$K_I^{B1} = 1 \text{ and } K_{II}^{B1} = 0, \quad (4.49)$$

and the corresponding *M*-integral in equation (4.48) becomes

$$M^{(1)} = 2\alpha_{11} K_I^A + \alpha_{12} K_{II}^A = \int_{\Gamma_0} (\sigma_{ij}^A \varepsilon_{ij}^{B1} n_i - \sigma_{ij}^A n_j u_{i,1}^{B1} - \sigma_{ij}^{B1} n_j u_{i,1}^A) d\Gamma. \quad (4.50)$$

Similarly the second set of the auxiliary field, denoted by superscript (*B2*), for the case of pure mode II deformation corresponds to,

$$K_I^{B2} = 0 \text{ and } K_{II}^{B2} = 1. \quad (4.51)$$

The corresponding *M*-integral is given as

$$M^{(2)} = \alpha_{12} K_I^A + 2\alpha_{22} K_{II}^A = \int_{\Gamma_0} (\sigma_{ij}^A \varepsilon_{ij}^{B2} n_i - \sigma_{ij}^A n_j u_{i,1}^{B2} - \sigma_{ij}^{B2} n_j u_{i,1}^A) d\Gamma. \quad (4.52)$$

Once the M -integrals in equations (4.50) and (4.52) are computed, the mode I and mode II stress intensity factors can be evaluated by solving these two equations. The explicit solutions for auxiliary stresses, σ_{ij}^{Bk} , and displacements, u_i^{Bk} ($k = 1, 2$), derived from the crack-tip field solutions by Sih *et al.* (1965) are given in Wang and Corten (1980). In order to obtain the required displacement derivatives, $u_{i,j}^{Bk}$, from u_i^{Bk} , the following expression was used in the present work:

$$u_{i,j}^{Bk} = \frac{\partial u_i^{Bk}}{\partial r} \frac{\partial r}{\partial x_j} + \frac{\partial u_i^{Bk}}{\partial \phi} \frac{\partial \phi}{\partial x_j} = \frac{\partial u_i^{Bk}}{\partial r} n_j(\phi) + \frac{\partial u_i^{Bk}}{\partial \phi} m_j(\phi), \quad (4.53)$$

where

$$n_1(\phi) = \cos(\phi), \quad n_2(\phi) = \sin(\phi), \quad m_1(\phi) = -\sin(\phi), \quad m_2(\phi) = \cos(\phi). \quad (4.54)$$

It should be noted that all the terms in the M -integrals associated with stress intensity factors are directly computed in the local co-ordinate system, x_i . This is due to the readily available required auxiliary field expressions in the local co-ordinates.

4.3 Test Problems and Validation

The validation for the implementation of M -integral method in this work to derive stress intensity factors in two-dimensional anisotropy will first be presented. Then the veracity of this development to compute the T -stress using the M -integral approach will be illustrated by a few examples. Due to paucity of T -stress solutions for general anisotropy, the validation was carried out by considering the problems in isotropy and orthotropy for which solutions are available in the literature. The test problem comprised of an inclined centre crack in infinite plate subjected to remote uniform tension, σ_0 . Plane stress

conditions were assumed in both the cases. Although the main focus of the present work is on the T -stress, the corresponding solutions for the stress intensity factors obtained using both the M -integral and traction formula are presented for completeness and verification.

In the course of verifying the implementation of the formulations discussed above with some test problems with known solutions, some modelling guidelines were also established. This concerns the appropriate relative size of the quarter-point crack-tip elements, l/a , and the relative radius of the circular contour, r/a , as defined in figure (4.2), used to evaluate the M -integral for obtaining accurate results of the stress intensity factor and the T -stress. A fairly extensive parametric study was carried out to this end; it was found that the discrepancies of the BEM computed stress intensity factors with established solutions in the literature were generally less than 1 percent when l/a was in the range of $0.05 \leq l/a \leq 0.25$, and similarly for T -stress solutions, when $r/a \geq 0.4$. Throughout this thesis work therefore, the adopted value of l/a was 0.15, unless otherwise stated. Also at least two different contour radii were used for each problem, with $r/a = 0.4$, 0.5 or 0.6, to ensure the consistency of the M -integral values by virtue of the path independence. In addition, both the solutions using the approaches based on the Lekhnitskii and Stroh formalisms were obtained and the discrepancy was generally within 0.2%. In the results presented throughout this thesis, only the T -stress solutions obtained using the M -integral approach based on the Stroh formalism are shown.

4.3.1 Inclined centre crack in an anisotropic plate under uniform tension

In order to test the veracity of the current implementation of *M*-integral approach using BEM to compute stress intensity factors, a glass/epoxy plate under uniform tension σ_0 was first investigated. This is shown in figure (4.3.a); the geometric parameters considered were: $a/W = 0.2$, $H/W = 2$ and the angle of inclination of the crack $\omega = 45^\circ$. Plane stress conditions were assumed. The stress intensity factor solutions for this problem had been obtained by Gandhi (1972) using a modified mapping collocation technique, Tan and Gao (1992a), and Sollero and Aliabadi (1993) using BEM, whose results all showed excellent agreement with each other. The elastic properties along the principal material axes of glass/epoxy were taken to be as follows:

$$E_{11} = 48.26 \text{ GPa}; \quad E_{22} = 17.24 \text{ GPa}; \quad G_{12} = 6.89 \text{ GPa}; \quad \nu_{12} = 0.29.$$

The angle between the material principal axes and the global co-ordinate axes, ψ , was varied from 0° to 180° , with 15° increments. The BEM mesh employed for this study is shown in figure (4.3.b).

The results obtained for normalised stress intensity factors, $\frac{K_I}{\sigma_0 \sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$,

from the current BEM study using *M*-integral and traction formulae are compared with the reference solutions in table (4.1). The results obtained using BEM traction formula exhibit very good agreement with the reference solutions with a difference of less than 1.5%; those using *M*-integral method showed even better agreement with a discrepancy of less than 0.5%. This confirms the accuracy of the present BEM implementation using *M*-integral for stress intensity factors. Also included in table (4.1) are the computed *T*-stress solutions normalised with respect to the applied load, σ_0 . It can be noted that for

this geometry, the magnitudes of the normalised T -stress are relatively small. They are all positive except for the cases of $\psi = 45^\circ$ and 90° where some loss of crack-tip stress constraint occurs.

4.3.2 A centre crack in an infinitely long isotropic plate under remote uniform tension

The case of a centre, inclined crack in an infinitely long isotropic plate under remote tension, σ_0 , was treated next. For the purpose of the BEM analysis, with reference to figure (4.4.a), the geometric parameters considered were: $H/W = 5$ and $a/W = 0.1$ and the angle of inclination of the crack, $\omega = 0^\circ$ to 75° , in increments of 15° . Figure (4.4.c) shows a typical BEM mesh employed. The T -stress solutions were obtained for both crack tips and their difference were generally less than 0.05%. Similarly, the discrepancy in the T -stress solutions for different contour radii was generally within 0.05%. The normalised T -stress solutions, T/σ_0 , along with the stress intensity factors obtained in this study are compared with the closed-form solutions given by Smith *et al.* (2001) in table (4.2). The results exhibit an excellent agreement between the BEM and the reference solutions with discrepancy of generally less than 1%.

4.3.3 A centre crack in an infinite orthotropic plate under remote uniform tension

The problem considered in the previous example was investigated here with orthotropic material properties (figure 4.4.b). The same BEM mesh shown in figure (4.4.c) was employed. The T -stress solutions for an orthotropic centre cracked plate (CCP) with $a/W = 0.1$ under remote uniform tension have been obtained by Kim and Paulino (2004) and

they were used as the reference solutions for comparison. The values for the material properties used in their analysis were as follows:

$$E_{11} = 10,000; \quad E_{22} = 1,000; \quad G_{12} = 1216; \quad \nu_{12} = 0.3.$$

The T -stress solutions for both crack tips were in excellent agreement, with deviations of generally less than 0.05%; discrepancies in the results for different contour radii considered were generally within 0.05% as well. The BEM computed results and the corresponding reference solutions are listed in table (4.3). The solutions are in excellent agreement, with deviations of generally less than 1%.

4.4 Numerical Examples

Two sets of problems in general anisotropy were investigated in this study. The first set comprised of a comprehensive parametric study of the effect of various geometric and material parameters on the T -stress. To this end, standard specimens, namely single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP) were analysed. The second set of problem considered was a crack emanating from a circular hole in a rectangular plate. These sets of analyses are discussed individually in the following sections. In order to facilitate the discussion that follows it is worthwhile to revisit equations (2.8) for the stress field ahead of the crack tip in anisotropy. Along the crack plane, where $\theta = 0^\circ$, the stress field solutions reduce to,

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} (\mu_2 - \mu_1) \right] + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} \{ (\mu_2)^2 - (\mu_1)^2 \} \right] + T, \quad (4.55.a)$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{\mu_1 - \mu_2} (\mu_1 - \mu_2) \right], \quad (4.55.b)$$

$$\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi r}} \mathbf{Re} \left[\frac{1}{\mu_1 - \mu_2} (\mu_1 - \mu_2) \right]. \quad (4.55.c)$$

Eliminating stress intensity factors in equations (4.55), the T -stress can be represented in terms of the stress field ahead of the crack tip as follows,

$$T = \sigma_{11} + \mathbf{Re}(\mu_1 \mu_2) \sigma_{22} + \mathbf{Re}(\mu_1 + \mu_2) \sigma_{12}, \quad (4.56)$$

where the stresses and μ_i in the above equation are in local co-ordinates x_i . In equation (4.56) the terms with complex parameters, μ_i , can be related to the elastic compliances, a'_{mn} , of the material in local co-ordinates as follows (Ting, 1996):

$$\mathbf{Re}(\mu_1 \mu_2 \bar{\mu}_1 \bar{\mu}_2) = \frac{a'_{22}}{a'_{11}}, \quad (4.57.a)$$

$$\mathbf{Re}(\mu_1 + \mu_2) = \frac{a'_{16}}{a'_{11}}. \quad (4.57.b)$$

For the case of orthotropy, when material principal axes coincide with x_i , equations (4.57) reduce to,

$$\mathbf{Re}(\mu_1 \mu_2) = -\sqrt{\frac{a'_{22}}{a'_{11}}}, \quad (4.58.a)$$

$$\mathbf{Re}(\mu_1 + \mu_2) = 0. \quad (4.58.b)$$

From these equations, it is evident that in addition to geometry and loads that affect the stress field ahead of the crack-tip, the crack-tip stress constraint level is also affected by the elastic material properties, unlike in the isotropic case.

4.4.1 Effect of crack geometry and material parameters on T -stress

A parametric study was conducted to investigate the influence of degree of orthotropy, crack size, crack inclination angle and orientation of the material principal axes on the T -stress for three commonly analysed plate specimens. They are single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP) specimens. The plates were subjected to remote uniform tension and plane stress conditions were assumed. The degree of orthotropy of the material properties as introduced by Bowie and Freese (1972) was adopted in which these properties are defined in terms of purely imaginary roots of the characteristic equation denoted by β_1 and β_2 , as follows:

$$\beta_1\beta_2 = \left(\frac{E_{11}}{E_{22}} \right)^{1/2}, \quad (4.59.a)$$

$$\beta_1 + \beta_2 = \sqrt{2 \left[\left(\frac{E_1}{E_2} \right)^{1/2} + \frac{E_1}{2G_{12}} - \nu_{12} \right]}. \quad (4.59.b)$$

Due to a wide range of orthotropic materials available, the definition of such materials in terms of these two parameters provides a relatively convenient and organised representation of the material properties. The values of these parameters analysed are listed in table (4.4), with β_1 ranging from 1.5 to 6 and two values of β_2 , namely 0.5 and 0.75; the different combinations of the values of β_1 and β_2 are denoted by Case (1) to Case (8). These values cover a relatively wide range of commonly used orthotropic materials in engineering structural applications, as shown in table (4.5). In this parametric investigation, different orientations of the material principal axes were considered to impart general anisotropy in the analysis. It should be noted that if an orthotropic body contains a crack which does not lie along a plane of elastic symmetry, the analytical

relations are similar to those for a generally anisotropic body (Sih and Liebowitz, 1968). The angle between the global co-ordinate axes and the principal material axes, ψ , was varied from 0° to 90° . The angle ψ was defined such that it is the orientation of the direction of first material principal axis with respect to the direction of the applied load, as shown in figures (4.5.a), (4.6.a) and (4.7.a) for SECP, CCP and DECP specimens, respectively. For all the cases, the geometric parameters considered were: relative crack size, $a/W = 0.1$ to 0.5 , crack inclination angle, $\omega = 0^\circ$ to 45° and $H/W = 5$. Only this H/W ratio was considered because it was found that beyond H/W of 4, there is no significant change in the T -stress and stress intensity factor results. The results from the study performed to study the effect of H/W on T -stress and stress intensity factors is included in digital medium (Appendix B).

Analysis was carried out for both the crack tips in the case of CCP and DECP specimens and excellent agreement, with deviations of generally within 0.5%, for T -stress and stress intensity factor solutions for both the crack tips were observed. Also, for all the specimens studied, the T -stress values obtained for different contour radii was found to deviate generally by less than 0.5%.

More than three thousand cases were analysed in this study. For completeness, the corresponding stress intensity factor solutions were also obtained using the M -integral. Due to the voluminous amount of these results, only a few representative sets are presented in the following sections. The complete sets of results obtained in this study are included in digital medium (Appendix B). Instructions on accessing the results in Appendix B are given in Appendix A. In the results presented, the T -stress values are normalised with respect to the applied load, σ_0 , as T/σ_0 and the stress intensity factors are

normalised as $\frac{K_I}{\sigma_o \sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_o \sqrt{\pi a}}$, where a is the crack size. The observations and discussions on the effect of various geometric and material parameters on T -stress for the specimens considered as presented below apply directly to the cases shown here. Some of the qualitative features will remain the same while others may change somewhat depending on the geometric and material parameters. What is clearly evident is that material anisotropy and isotropy has a significant effect on the crack-tip stress constraint level manifested by the magnitude and sign of the T -stress.

The effect of the crack size on T -stress for SECP, CCP and DECP is illustrated here for a selected case of $\omega = 45^\circ$ and $\psi = 30^\circ$ in figures (4.8), (4.9) and (4.10), respectively. It can be seen from these figures that for the cases of SECP and DECP, there is an elevation in the crack-tip stress constraint level with an increase in crack size. However, for the case of CCP, the constraint level generally lowers with increasing crack length, except for the cases corresponding to the material Cases (4) and (8). The cases considered generally demonstrate a high crack-tip stress constraint level as the T -stress is tensile for most of the cases, except for a few. It is interesting to note that for all the specimens, the T -stress gradually increases with an increase in the β_1 for a given β_2 .

To demonstrate the effect of material orientation on T -stress, cases corresponding to $\omega = 45^\circ$ and $a/W = 0.1$ are arbitrarily chosen here. They are shown for the SECP, CCP and DECP specimens in figures (4.11), (4.12) and (4.13), respectively; the corresponding results are also presented in tables (4.6), (4.7) and (4.8). It is apparent from these examples that the variation of T -stress with ψ exhibits sinusoidal behaviour which is in agreement with the transformation law for the elastic constants of the material. This is

also consistent with the angular transformation of the complex parameters present in the equation (4.56) given by equation (3.29). Also evident is that for the material Case (5), the effect of ψ on T -stress is minimal. This is due to the fact that for this material case, the difference between the compliances along the orthogonal directions, one of which lies along the crack plane, is the smallest; the effect of ψ on the compliances and thus the compliance ratio in either direction is smaller than for the other cases.

The angle of crack inclination, ω , also exhibited a significant influence on the crack-tip stress constraint level. As seen in figures (4.14), (4.15) and (4.16) for SECP, CCP and DECP specimens with $\psi = 45^\circ$ and $a/W = 0.1$, the effect of ω on T -stress is more or less similar for these three geometries for a given material set. In general, except a few cases, the T -stress gradually increases with crack inclination and attains a tensile value for $\omega = 45^\circ$.

Based on the examples presented above, it is apparent that ω , ψ and crack size indeed have a significant effect on the T -stress; this also holds true for the rest of the results obtained in this parametric study which are included in Appendix B. These results can be interpreted in terms of the geometric and material parameters with the help of equation (4.56).

4.4.2 A crack emanating from a circular hole in a composite plate

A crack emanating from a circular hole in a rectangular graphite/epoxy plate was analysed. This example has important applications in, for example, the aerospace industry. The plate was subjected to remote uniform tension, σ_0 , and plane stress conditions were assumed. The problem analysed is shown in figure (4.17.a) with $H/W = 2$ and $r/W = 0.1$.

The engineering elastic constants for graphite/epoxy composite were chosen to be as follows (Tan and Gao, 1992b):

$$E_{11} = 144.8 \text{ GPa}; \quad E_{22} = 11.7 \text{ GPa}; \quad G_{12} = 9.66 \text{ GPa}; \quad \nu_{12} = 0.21.$$

The relative crack size, a/r , was varied from 0.05 to 1.0; within this range, the effects of the stress concentration on the T -stress would be more pronounced. In order to study the effect of material orientation on T -stress, the angle ψ (see figure 4.17.a) was varied from 0° to 90° , in 15° increments. It can be recalled from Section (4.3) that the relative crack-tip element size, l/a , between 0.05 and 0.25 in the current BEM modelling was found to yield good results for the T -stress and the stress intensity factors. To illustrate that the crack-tip element size can be relaxed without compromising the accuracy, $l/a = 0.25$ was considered here, since comparable solutions for some of the cases studied were later provided by Song (2006) using SBFEM. A typical BEM mesh employed in this study is shown in figure (4.17.b).

The solutions obtained for the normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0 \sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$, are listed in tables (4.9.a), (4.9.b) and (4.9.c). It is evident from these results that the T -stress is compressive for all the cases, indicating low stress constraint conditions at the crack tip. The effect of the relative crack size, a/r , on normalised T -stress is presented in figures (4.18.a) and (4.18.b). It can be seen in figure (4.18.a) that for $\psi < 75^\circ$, the change in the absolute magnitudes of T -stress with a/r is relatively small whereas for $\psi \geq 75^\circ$, this is not the case. The magnified representation of the variations for the cases of $\psi \leq 45^\circ$ is shown in figure (4.18.b), where the changes in the values of T -stress with crack size are elaborated. From these observations, it is

apparent that the variations of T -stress are more rapid when the cracks are shorter than when they grow longer. This can be attributed to the effect of stress concentration which is more pronounced near the edge of the hole. This is also manifested in the stress intensity factor solutions. It can be seen from tables (4.9) that, except for the cases when $\psi = 45^\circ$ and 90° , the mode-mixity exists for shorter cracks near the stress concentration whereas mode II stress intensity factor gradually diminishes for longer cracks.

The corresponding effect of material orientation, ψ , on the T -stress was also found to be significant, as shown in figure (4.19). It can be seen from this figure that for all the crack sizes investigated, changes in the value of the T -stress were relatively small and gradual from $\psi = 0^\circ$ to about $\psi = 45^\circ$. Beyond $\psi = 45^\circ$, this parameter increased in magnitude very rapidly. This is consistent with the fact that for $\psi < 45^\circ$, the compliance ratio, $\frac{a'_{22}}{a'_{11}}$, varies gradually, whereas for $\psi > 45^\circ$, the variation becomes significant.

4.5 Closing Remarks

The M -integral to evaluate T -stress solutions for homogeneous anisotropic elastic cracked solids has been presented in this chapter. The implementation of this method into BEM has been validated by comparing the results obtained in the present work with those available in the literature for a few specific cases of isotropy and orthotropy. In addition, an expression for T -stress in terms of the stress field ahead of a crack tip has been derived for two-dimensional anisotropy; this expression shows the dependency of the T -stress on both geometry and material. Results obtained from a comprehensive parametric study conducted in this work have also been presented and it revealed a noticeable effect of

various geometric and material parameters on the T -stress. Finally a problem involving a crack originating from a stress concentration was investigated, some results for which was later verified by Song (2006) using an independent approach based on SBFEM. This further strengthened the veracity of the current development based on the M -integral and BEM. The results for all the problems analysed clearly showed that besides the geometric parameter, the normalised T -stress are also affected by material properties, in particular the relative orientation of the material principal axes with the crack plane. The development and part of the results discussed here have also been published in Shah *et al.* (2006a) and Shah *et al.* (2006).

CHAPTER 5

M-INTEGRAL FOR INTERFACE CRACKS BETWEEN DISSIMILAR ANISOTROPIC MATERIALS

5.0 Introduction

In this Chapter, the development of the *M*-contour integral for the determination of *T*-stress for a crack lying along the interface between dissimilar anisotropic elastic media using the BEM is presented. Due to the paucity of available solutions for anisotropic elasticity in the literature, an isotropic bimaterial interface crack problem was considered for its validation. The problem was however solved through the algorithm developed for the anisotropic formulation. Here, the validation for this problem is first presented followed by the analysis of a bimaterial orthotropic specimen. Finally, analysis was carried out for anisotropic bimaterial interface cracks in some applications of engineering significance. Collaboration was later sought with Song (2006), who had indicated, in his paper (Song, 2005) that the SBFEM was able to solve the problem for anisotropic bimaterial interface cracks but with no results presented. Through this collaboration, independent and successful verification of the two numerical approaches for these problems was done. The results obtained here have been published recently in Shah *et al.* (2006b) and Shah *et al.* (2006), the latter includes those from the collaboration.

5.1 *M*-integral for Interfacial Crack Between Dissimilar Anisotropic Materials

Following their work for homogeneous isotropic cracked bodies, Sladek and Sladek (1997b) have also presented a scheme for the evaluation of *T*-stress for an interface crack between two isotropic media based on the *M*-integral. As presented in equation (2.45) for an interfacial crack, it can be recalled to be as follows:

$$M = \int_{\Gamma_0} \left(\sigma_{ij}^A \varepsilon_{ij}^B n_i - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right) d\Gamma. \quad (5.1)$$

The development by Sladek and Sladek (1997b) is extended to the corresponding interface crack problem for anisotropic elasticity here. As before, the evaluation of *M*-integral requires two independent equilibrium states, $A(\sigma_{ij}^A, \varepsilon_{ij}^A, u_i^A)$ and $B(\sigma_{ij}^B, \varepsilon_{ij}^B, u_i^B)$. The state (*A*) represents the boundary value problem to be analysed using the BEM. The auxiliary state, (*B*), corresponds to the solution of a semi-infinite interfacial crack loaded by a point force *f* acting along the bimaterial interface at the crack tip. Before discussing the derivation of this auxiliary field, the material specific coefficient $C^{(m)}$ from equation (2.41), for material *m*, will be derived for anisotropy as follows.

The equilibrium state (*A*) can be considered to be a superposition of a singular term denoted by superscript *s* and a non-singular term denoted by superscript *T* as follows:

$$\sigma_{ij}^A = \sigma_{ij}^s + \sigma_{ij}^T, \quad (5.2.a)$$

$$u_i^A = u_i^s + u_i^T. \quad (5.2.b)$$

The displacement, $u_i^{A(m)}$, for material $m = 1, 2$ along the bonded interface is continuous and thus,

$$u_{i,1}^{A(1)} = u_{i,1}^{A(2)}. \quad (5.3)$$

Hence,

$$u_{1,1}^{A(1)} = u_{1,1}^{A(2)}. \quad (5.4)$$

Along the interface, if only non-singular components due to T -stress are considered, then,

$$u_{1,1}^{T(m)} \Rightarrow \varepsilon_{11}^{T(m)} = a_{11}'^{(m)} \sigma_{11}^{T(m)}. \quad (5.5)$$

Using the continuity of strain along the interface,

$$a_{11}'^{(2)} \sigma_{11}^{T(2)} = a_{11}'^{(1)} \sigma_{11}^{T(1)} \quad (5.6)$$

or,

$$\sigma_{11}^{T(1)} = \frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} \sigma_{11}^{T(2)}. \quad (5.7)$$

Hence, the material coefficient $C^{(m)}$ for anisotropy are given as follows:

$$C^{(2)} = 1, \quad (5.8.a)$$

$$C^{(1)} = \frac{a_{11}'^{(2)}}{a_{11}'^{(1)}}. \quad (5.8.b)$$

5.1.1 Auxiliary field solutions for interface cracks

The auxiliary field (B) solution, for a semi-infinite crack along the interface between dissimilar anisotropic materials with a point force $\mathbf{f}$ acting at the crack tip along the direction of the crack plane, can be obtained from the work by Chung and Ting (1995). They have presented the solutions for an anisotropic composite wedge subjected to a concentrated force, $\mathbf{f}$, at the wedge apex. The wedge angle can be extended to represent a semi-infinite crack in the bimaterial interface, as shown in figure (5.1). For the sake of

clarity, the superscript (B) representing the auxiliary field is dropped here. Based on the Stroh formalism, the displacement vector, $\mathbf{u}$, and the stress function vector, Φ , at a point (r, ϕ) in material (m) for such a system are given as,

$$\mathbf{u}^{(m)} = -\frac{\ln r}{\pi} \mathbf{h}^{(m)} - \mathbf{S}^{(m)}(\phi) \mathbf{h}^{(m)} + \mathbf{H}^{(m)}(\phi) \mathbf{g}^{(m)} + \mathbf{u}_0^{(m)}, \quad (5.9.a)$$

$$\Phi^{(m)} = \frac{\ln r}{\pi} \mathbf{g}^{(m)} + \mathbf{S}^{T(m)}(\phi) \mathbf{g}^{(m)} + \mathbf{L}^{(m)}(\phi) \mathbf{h}^{(m)}. \quad (5.9.b)$$

where vector $\mathbf{u}_0^{(m)}$ is a constant and $\mathbf{S}^{(m)}(\phi)$, $\mathbf{H}^{(m)}(\phi)$ and $\mathbf{L}^{(m)}(\phi)$ are the Barnett-Lothe tensors (Ting, 1996), for material m , previously given in equation (4.26). The Barnett-Lothe tensors are evaluated using equations (4.26) and (4.27). While evaluating these tensors for a composite wedge, care should be taken while considering the field values; they should correspond to the material, m , at which they are being evaluated.

It can be recalled from equation (4.29) that based on zero tractions along the free surfaces of the wedge,

$$\mathbf{g}^{(m)} = 0. \quad (5.10)$$

The vector $\mathbf{h}^{(m)}$ is given as,

$$\mathbf{h}^{(m)} = \underline{\mathbf{L}}^{-1} \mathbf{f}, \quad (5.11)$$

where

$$\underline{\mathbf{L}} = \frac{1}{\pi} \int_{\theta_0}^{\theta_n} -\mathbf{N}_3(\lambda) d\lambda. \quad (5.12)$$

The term $\underline{\mathbf{L}}$ is constant for the composite wedge.

The point force vector $\mathbf{f}$ applied at the wedge apex can be expressed as,

$$\mathbf{f} = \begin{Bmatrix} f \cos \omega \\ f \sin \omega \end{Bmatrix}, \quad (5.13)$$

where ω is the angle of crack inclination with the displaced global co-ordinate axes, X'_j .

Because of the continuity of displacement along the bonded interface, vector $\mathbf{h}$ is invariant for all sections in the composite wedge. Thus, in what follows, $\mathbf{h}^{(m)}$ is replaced by invariant $\mathbf{h}$. Hence, substituting the value of $\mathbf{g}^{(m)}$ from equation (5.10), equations (5.9) reduce to,

$$\mathbf{u}^{(m)} = -\frac{\ln r}{\pi} \mathbf{h} - \mathbf{S}^{(m)}(\phi) \mathbf{h} + \mathbf{u}_0^{(m)}, \quad (5.14.a)$$

$$\Phi^{(m)} = \mathbf{L}^{(m)}(\phi) \mathbf{h}. \quad (5.14.b)$$

The displacement derivatives with respect to the axes, X'_j , are then obtained as follows,

$$\begin{aligned} u_{i,j}^{(m)} &= \frac{\partial u_i^{(m)}}{\partial r} \frac{\partial r}{\partial X'_j} + \frac{\partial u_i^{(m)}}{\partial \phi} \frac{\partial \phi}{\partial X'_j} \\ &= \frac{\partial u_i^{(m)}}{\partial r} n_j(\phi) + \frac{\partial u_i^{(m)}}{\partial \phi} \frac{1}{r} m_j(\phi) \\ &= \frac{1}{\pi r} \left\{ -h_i n_j(\phi) - (\mathbf{N}_1(\phi) \mathbf{h})_i m_j(\phi) \right\}. \end{aligned} \quad (5.15)$$

where

$$n_1(\phi) = \cos(\phi), \quad n_2(\phi) = \sin(\phi), \quad m_1(\phi) = -\sin(\phi) \text{ and } m_2(\phi) = \cos(\phi). \quad (5.16)$$

The corresponding stresses are:

$$\begin{aligned} \sigma_{1j}^{(m)} &= -\Phi_{j,2}^{(m)} \\ &= -\frac{\partial \Phi_j^{(m)}}{\partial X'_2} = -\left[\frac{\partial \Phi_j^{(m)}}{\partial r} \frac{\partial r}{\partial X'_2} + \frac{\partial \Phi_j^{(m)}}{\partial \phi} \frac{\partial \phi}{\partial X'_2} \right] \\ &= -\frac{\Phi_j^{(m)}}{\partial \phi} \frac{1}{r} m_2(\phi) = \frac{1}{\pi r} [\mathbf{N}_3(\phi) \mathbf{h}]_j m_2(\phi), \end{aligned} \quad (5.17.a)$$

and, similarly,

$$\sigma_{2j}^{(m)} = \Phi_{j,1}^{(m)} = -\frac{1}{\pi r} [N_3(\phi) \mathbf{h}]_j m_1(\phi). \quad (5.17.b)$$

Equations (5.15) and (5.17) provide the required auxiliary field solutions for the evaluation of the M -integral in equation (5.1). In the present implementation, the integral in equation (5.12) is carried out with limits ϕ_0 to $\phi_1 = \pi + \phi_0$ and ϕ_1 to $\phi_2 = 2\pi + \phi_0$ (figure 5.1). For simplicity, the force $\mathbf{f}$ applied at the crack tip is arbitrarily taken to be of unit magnitude.

5.1.2 Extraction of the T -stress from M -integral

As before, the path independent M -integral can be assumed to have a circular contour of radius ε , which when shrunk to zero reduces to,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^A \varepsilon_{ij}^B n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right] d\Gamma. \quad (5.18)$$

Because the M -integral is bounded, consideration of the non-vanishing contribution from the non-singular terms from equations (5.2) reduces equation (5.18) to the following form as,

$$M = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{1k} - \sigma_{ik}^T u_{i,1}^B - \sigma_{ik}^B u_{i,1}^T \right] n_k d\Gamma, \quad (5.19)$$

where

$$\sigma_{ij}^T = \frac{a_{11}^{(2)}}{a_{11}^{(1)}} T \delta_{i1} \delta_{j1}, \quad \text{for } 0 \leq \theta \leq \pi \quad (\text{material 1}) \quad (5.20.a)$$

$$\sigma_{ij}^T = T \delta_{i1} \delta_{j1}, \quad \text{for } -\pi \leq \theta \leq 0 \quad (\text{material 2}). \quad (5.20.b)$$

The M -integral can further be expressed as a sum of integrals along the contour in material 1 and material 2 as follows:

$$\begin{aligned}
M &= \lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{lk} - \sigma_{ik}^T u_{i,l}^B - \sigma_{ik}^B u_{i,l}^T \right] n_k \varepsilon d\theta + \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[\sigma_{ij}^T \varepsilon_{ij}^B \delta_{lk} - \sigma_{ik}^T u_{i,l}^B - \sigma_{ik}^B u_{i,l}^T \right] n_k \varepsilon d\theta \\
&= \lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} T \delta_{il} \delta_{jl} \varepsilon_{ij}^B \delta_{lk} n_k - \frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} T \delta_{il} \delta_{kl} \varepsilon_{l,i}^B \delta_{li} n_k - \sigma_{ik}^B u_{i,l}^T n_k \right] n_k \varepsilon d\theta \\
&\quad + \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[T \delta_{il} \delta_{jl} \varepsilon_{ij}^B \delta_{lk} n_k - T \delta_{il} \delta_{kl} \varepsilon_{l,i}^B \delta_{li} n_k - \sigma_{ik}^B u_{i,l}^T n_k \right] n_k \varepsilon d\theta \\
&= \lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} T \varepsilon_{11}^B n_1 - \frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} T \varepsilon_{11}^B n_1 - \sigma_{ik}^B u_{i,l}^T n_k \right] d\Gamma \\
&\quad + \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[T \varepsilon_{11}^B n_1 - T \varepsilon_{11}^B n_1 - \sigma_{ik}^B u_{i,l}^T n_k \right] d\Gamma \\
&= -\lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\sigma_{1j}^B n_j a_{11}'^{(1)} \sigma_{11}^T \right] \varepsilon d\theta - \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[\sigma_{1j}^B n_j a_{11}'^{(2)} \sigma_{11}^T \right] \varepsilon d\theta \\
&= -\lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\sigma_{1j}^B n_j a_{11}'^{(1)} \frac{a_{11}'^{(2)}}{a_{11}'^{(1)}} T \right] \varepsilon d\theta - \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[\sigma_{1j}^B n_j a_{11}'^{(2)} T \right] \varepsilon d\theta \\
&= -T a_{11}'^{(2)} \left\{ \lim_{\varepsilon \rightarrow 0} \int_0^\pi \left[\sigma_{1j}^B n_j \right] \varepsilon d\theta + \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^0 \left[\sigma_{1j}^B n_j \right] \varepsilon d\theta \right\} \\
&= -T a_{11}'^{(2)} \left\{ \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[\sigma_{1j}^B n_j \right] d\Gamma \right\} \\
&= -T a_{11}'^{(2)} \{-f\} \\
&= T a_{11}'^{(2)} f.
\end{aligned} \tag{5.21}$$

Thus, for plane stress conditions, the T -stress can be expressed in terms of the M -integral as follows:

$$T = M \frac{1}{a'_{11}{}^{(2)} f}. \quad (5.22)$$

The M -integral in equation (5.22) is evaluated using the contour integral given in equation (5.1) in the local co-ordinates, x_i . For the evaluation of T -stress in plane strain conditions, a'_{11} in equation (5.22) should be replaced by b'_{11} given in equation (3.3). Similar to the case of homogeneous anisotropy, the M -integral is first obtained in the global co-ordinates, X'_j , and later transformed into the local coordinates as follows:

$$M_{(Local)} = M_{1(Global)} \cos \omega + M_{2(Global)} \sin \omega, \quad (5.23)$$

where

$$M_{k(Global)} = \int_{\Gamma_0} \left[(\sigma_{ij}^A \varepsilon_{ij}^B n_k - \sigma_{ij}^A n_j u_{i,k}^B - \sigma_{ij}^B n_j u_{i,k}^A) \right] d\Gamma. \quad (5.24)$$

It should be noted that in the case of a bimaterial interface crack, there are two in-plane T -stresses, depending on which material region it is defined for. The two values are proportional to each other as discussed below. In the preceding development, the T -stress is defined with respect to material 2 (figure 5.1). The T -stress with respect to material 1 can be obtained by setting $C^{(1)}$ to unity. This leads to,

$$C^{(2)} = \frac{a'_{11}{}^{(1)}}{a'_{11}{}^{(2)}}. \quad (5.25)$$

From equation (5.7), it is evident that the T -stress values in material 1 and 2 can be related as,

$$T^{(1)} a'_{11}{}^{(1)} = T^{(2)} a'_{11}{}^{(2)}. \quad (5.26)$$

In the present work, the T -stress is defined along the interface in material 2 and will simply be denoted as T instead of $T^{(2)}$.

5.2 Numerical Examples

In order to test the veracity of the above formulations, a problem of an interfacial crack between dissimilar isotropic media was first considered. In the second example, the study was conducted for dissimilar orthotropic materials with the geometry and load conditions from the first example. This would show the influence of material orthotropy on the T -stress for such cracks. In the third example, the effect of the material orientation and the geometry on the T -stress for a cracked anisotropic silicon disc with dissimilar halves was investigated. The fourth and the last examples present the variation of T -stress with geometry and material orientation for interface cracks between dissimilar anisotropic solids in systems of engineering significance; they are a fault crack between anisotropic reservoir rocks and a crack along bicrystal junction between YBCO crystals, respectively. The presentation of the T -stress values in the following discussions are accompanied with the stress intensity factors, obtained using the BEM traction formula (Tan *et al.*, 1992), for completeness. The characteristic length L for the computation of the stress intensity factors (see, e.g., Tan *et al.*, 1992) was chosen to be $2a$, unless stated otherwise.

5.2.1 Test problem: an isotropic bimaterial centre cracked plate under uniform remote tension (Problem A)

Figure (5.2.a) shows a rectangular plate consisting of two dissimilar isotropic materials (E_1, ν_1) and (E_2, ν_2), respectively, with a centre crack along the material interface. The centre cracked plate (CCP) is subjected to a remote uniform tensile load σ_0 . With reference to figure (5.2.a), the geometric cases considered were $a/W = 0.15$ and 0.5 and $H/W = 2$. The material properties considered were $E_1/E_2 = 1, 2, 5$ and 10 , with $\nu_1 = \nu_2$

$= 0.3$. In all the cases, plane strain conditions were assumed. The solutions for these cases exist in the literature by Sladek and Sladek (1997b) using the M -integral approach with BEM and by Song (2005) using SBFEM. Figure (5.2.b) shows a typical BEM mesh considered in this study. Absolute agreement in the T -stress values for both crack tips was observed for each case. The results obtained for the normalised T -stress, T/σ_0 , are listed in table (5.1), together with the solutions from the above-mentioned authors. It can be seen from the table that the results obtained from the present BEM treatment are in excellent agreement with those obtained by Song (2005), within 0.5% deviation, even if slightly less so with those by Sladek and Sladek (1997b). However, it should be noted that the values attributed to Sladek and Sladek (1997b) reported here are the digitized values obtained by Song (2005) from the graphical results presented by the former. The results presented in table (5.1) indicate a rapid decrease in the negative magnitude of the T -stress with increasing ratio E_1/E_2 , for a given crack size, signifying increasing crack-tip stress constraint.

5.2.2 An orthotropic bimaterial centre cracked plate under remote tension (Problem B)

A CCP specimen with the same geometry, except for the crack size, used in the previous example was analysed with dissimilar orthotropic materials. The relative crack length, a/W , considered was varied from 0.1 to 0.5. The material properties considered are given as follows:

Material 1: $E_{11} = 1000 \text{ GPa}$, $E_{22} = 500 \text{ GPa}$, $G_{12} = 100.1 \text{ GPa}$, $\nu_{12} = 0.3$.

Material 2: $E_{11} = 200 \text{ GPa}$, $E_{22} = 60 \text{ GPa}$, $G_{12} = 15.7 \text{ GPa}$, $\nu_{12} = 0.3$.

The cracked plate under consideration was subjected to a uniform remote tensile load σ_0 (figure 5.2.a) and plane stress conditions were assumed.

The numerical results obtained for the normalised T -stress, T/σ_0 , are presented in table (5.2). The T -stress solutions for both crack tips were considered and they showed excellent agreement. As in the previous example for isotropy, the numerical values for the T -stress are negative and gradually become more negative with the crack size (figure 5.3). This suggests a decrease in the crack-tip stress constraint level with an increase in the crack size. For comparison, the corresponding results obtained for the case when materials 1 and 2 are isotropic with the Young's moduli ratio $E_1/E_2 = 500/60$ and the Poisson's ratios $\nu_1 = \nu_2 = 0.3$ are included in table (5.2). It is apparent from the comparison that the orthotropy has a profound influence on the value of T -stress.

5.2.3 A cracked anisotropic bimaterial disc under uniform radial tension (Problem C)

A centre cracked anisotropic bimaterial disc subjected to uniform radial tension, σ_0 , was investigated (figure 5.4.a). The material considered was single crystal silicon in [110] plane. Single crystal silicon has been widely used in, e.g., electronic devices. The engineering constants for silicon in plane [110], as given below, were calculated from its elastic compliances available in Simmons and Wang (1971):

$$E_{11} = 169.1 \text{ GPa}, \quad E_{22} = 130.1 \text{ GPa}, \quad G_{12} = 79.6 \text{ GPa}, \quad \nu_{12} = 0.362.$$

In the study, the angle of orientation, ψ , of the material principal axes with the displaced global Cartesian axes was varied from 0° to 90° , in 15° increments, for material 1; for material 2 the material principal axes were rotated in the reverse sense, i.e., $-\psi$, in each

case by the same magnitude, simultaneously. The cases when angle $\psi = 0^\circ$ and 90° represent the materials as orthotropic whereas for other intermediate angles, the materials behave as anisotropic. The relative crack lengths analysed were $a/r = 0.1, 0.2, 0.3, 0.4$ and 0.5 and plane stress conditions were assumed. For the purpose of comparison, the analysis was also carried out for a homogeneous isotropic disc with the same geometric parameters and loading conditions. A typical BEM mesh employed for this analysis is given in figure (5.4.b).

The results for the normalised T -stress, T/σ_0 , obtained from this analysis are presented in tables (5.3.a) and (5.3.b) for crack tips A and B, respectively, for varying ψ and a/r . Included in the table also are the corresponding normalised stress intensity factors which were calculated with characteristic length $L = a$. In this study, the imaginary component of the complex stress intensity factor, K_{II} , was observed to be equal to zero for all the cases. This can be attributed to the symmetric loading and material orientation about the interface. The values for the case when $a/r = 0.5$ were latter verified by Song (2006) using SBFEM and are also included in tables (5.3). The effect of ψ on the normalised T -stress values for either crack tips is presented in figures (5.5.a) and (5.5.b) for different values of a/r . It is apparent from these figures that there is a significant effect of the material orientation on the T -stress. With a few exceptions, it is generally observed that for a given crack size, the T -stress increases with increasing ψ . In figures (5.6.a) and (5.6.b), the variation of the normalised T -stress with a/r for different ψ values considered, are presented for both crack tips. Also included in these figures in dotted line are the corresponding results for the isotropic disc analysed. The variation of the normalised T -stress with a/r for both anisotropic and isotropic cases, although similar in

trend, are different in values. This again confirms the significant effect of the degree of anisotropy on T -stress. For all the cases considered, it can be seen that the crack-tip stress constraint level gradually falls with increasing crack size.

5.2.4 A fault crack between anisotropic rocks (Problem D)

Fault cracks existing along the interface between different layers of rock are of great significance in civil and geotechnical engineering studies. For illustration, here, a fault crack along the interface between sandstone reservoir rock (material 1) and dolomite reservoir rock (material 2) in a large domain was investigated. Sandstone and dolomite both exhibit general anisotropic mechanical behaviour. Their elastic properties for these materials along their respective principal axes are listed in table (5.4).

As shown in figure (5.7.a), the numerical solution domain for the analysis was chosen as a rectangular plate with $H/W = 2$, and the relative size of the interface crack $a/W = 0.1$. Plane strain conditions were assumed. The bimaterial plate was subjected to a remote uniform tension, σ_0 . Principal axes for both of the materials were chosen parallel to the global Cartesian axes. Various cases were investigated with the angle of inclination of the interface between the rocks, ω , where the crack lie, varying from 0° to 60° , in 15° increments. A typical BEM mesh employed in this study is given in figure (5.7.b). The analysis was carried out for both crack tips A and B.

The results obtained for the normalised T -stress, T/σ_0 , for varying ω are listed in tables (5.5.a) and (5.5.b) for crack tips A and B, respectively. The results obtained in the present work have been independently verified by Song (2006) using SBFEM and are included in the table for comparison, the difference being, in general, less than one

percent. Also included in the table are the normalised stress intensity factors calculated with the characteristic length, $L = a$. The variation of the normalised T -stress with the angle of inclination, ω , at tips A and B is given in figures (5.8.a) and (5.8.b), respectively. It is evident from the figures that the normalised T -stress becomes less negative with increasing ω and attains tensile value when $\omega \geq 45^\circ$, indicating an increase in the stress constraint level at both of the crack-tips.

5.2.5 A crack along bicrystal interface junction (Problem E)

In electronic applications, bicrystal interface materials are widely used for their excellent superconductive properties; for example, bicrystal junction between YBCO ($\text{YBa}_2\text{Cu}_3\text{O}_7$) crystals is used in Josephson junctions in super-sensitive magnetometers (SQUIDS). In this example, a single edge crack was introduced along the bicrystal interface or junction between the YBCO crystals as shown in figures (5.9.a) and (5.9.b). The geometric parameters analysed were: $H/W = 4$, $a/W = 0.1, 0.2, 0.3, 0.4$ and 0.5 . Plane stress conditions were assumed and the specimen was subjected to pure bending (figure 5.9.a) and uniform tension (figure 5.9.b). The elastic properties available for YBCO crystal in [100] plane at 300K were considered; they are (Ramakrishnan and Krishnamurthy, 1991): $E_{11} = 118.33$ GPa, $E_{22} = 91.99$ GPa, $G_{12} = 42.31$ GPa and $\nu_{12} = 0.268$.

such that the material principal axes are oriented at $+\psi$ and $-\psi$ with respect to the Cartesian axes for material 1 and material 2, respectively, as shown in these figures. The values of ψ investigated were 0° to 90° , in 15° increments. Figure (5.9.c) shows a typical BEM mesh employed for this analysis.

The results obtained for the normalised T -stress, T/σ_0 , and the normalised stress intensity factors are listed in tables (5.6.a) and (5.6.b) for the cases of pure bending and uniform remote tension, respectively. Also included in these tables are the corresponding results for the normalised T -stress obtained by Song (2006) using SBFEM. Excellent agreement between the results from these two independent numerical methods is obtained with a discrepancy of within 0.5%. The variation of T -stress with crack size is given in figures (5.10.a) and (5.10.b) for the cases of pure bending and remote tension, respectively. It can be seen from figure (5.10.a) that for the case of pure bending, the T -stress increases with increasing crack size ultimately attaining a tensile value when $a/W \geq 0.4$. On the other hand for the case of remote tension, as shown in figure (5.10.b), generally, the T -stress gradually decreases with crack size till $a/W = 0.2$ or 0.3 and then rapidly increases as a/W approaches 0.5 ; however, for this case of loading, T -stress is always compressive. Figures (5.11.a) and (5.11.b) for the case of pure bending and remote uniform tension, respectively, show the effect of the orientation of the material principal axes on the T -stress. It is interesting to note that for both the cases of loading, the effect of ψ on T -stress becomes more prominent as the crack size increases. Similar to the observation in Problem (C), the imaginary component, K_{II} , of the stress intensity factor was non-existent for all the cases in current problem. This again can be attributed to the symmetry of loading and material orientation at either halves of the bimaterial continuum.

5.3 Closing Remarks

An approach to obtain the T -stress for an interface crack between dissimilar anisotropic elastic materials using the M -integral in conjunction with BEM has been presented. The veracity of this development has been demonstrated by several examples. The numerical results have also been validated by an independent numerical method namely, SBFEM. The study has established that the geometry, material properties and material orientation in interface cracks between dissimilar anisotropic solids have a significant effect on the T -stress. These problems and solutions could serve as benchmark reference cases for future studies in this area.

CHAPTER 6

WEIGHT FUNCTION METHOD FOR EVALUATING T -STRESS IN TWO-DIMENSIONAL ANISOTROPY

6.0 Introduction

In this chapter, the weight function method to evaluate T -stress in homogeneous cracked solids under mixed-mode loading is presented. The reference weight function approach developed for isotropic solids is extended to anisotropy. The veracity of the approach is established by comparing the results obtained with those obtained directly by the M -integral method discussed in Chapter 4.

6.1 Weight Function Method for Mixed-mode Loading in Anisotropy

The weight function method to determine T -stress in isotropic cracked solids applicable to single mode loading by Wang (2002a,b) has been discussed in Section (2.3.1.3). For mixed-mode loading, the definition of the weight functions remains the same as in the case of single-mode loading, except for the need to define them separately for mode I and mode II. For mixed-mode loading, the integral in equation (2.38) can be further expanded as,

$$T_{crack\ pressure} = \int_0^a \sigma_{22}(x_1) m(x_1) dx_1 + \int_0^a \sigma_{12}(x_1) n(x_1) dx_1, \quad (6.1)$$

where, $\sigma_{22}(x_1)$ and $\sigma_{12}(x_1)$ are the normal pressure and shear stress, respectively, along the prospective crack plane in an uncracked body under stress field Q , as shown in figure (6.1); and $m(x_1)$ and $n(x_1)$ are the weight functions for mode I and mode II loadings, respectively. The form for the weight functions $m(x_1)$ and $n(x_1)$ can be approximated by two-term expressions similar to those in equation (2.38) for single-mode loading as follows:

$$m(x_1) = \frac{2}{\pi a} \left[D_1 \left(1 - \frac{x_1}{a} \right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a} \right)^{\frac{3}{2}} \right], \quad (6.2.a)$$

$$n(x_1) = \frac{2}{\pi a} \left[D_3 \left(1 - \frac{x_1}{a} \right)^{\frac{1}{2}} + D_4 \left(1 - \frac{x_1}{a} \right)^{\frac{3}{2}} \right], \quad (6.2.b)$$

where D_i is determined using four reference T -stress solutions, determination of which will be discussed in the next section.

The second term, $T_{uncracked}$, in equation (2.35) in case of isotropy has been expressed as the difference between the direct stresses at the prospective crack-tip position in the uncracked body as given in equation (2.37). However, this relation is not directly applicable to the case of anisotropic cracked bodies because the T -stress now also depends on the material properties. In order to apply the weight function method for cracks in such materials, the $T_{uncracked}$ is required. With reference to equation (4.56) and figure (6.1), $T_{uncracked}$ can be expressed in terms of the stress field at a point, $x_1 = a$, in the uncracked geometry as follows:

$$T_{uncracked} = \left[\sigma_{11} + \mathbf{Re}(\mu_1 \mu_2) \sigma_{22} + \mathbf{Re}(\mu_1 + \mu_2) \sigma_{12} \right] \Big|_{x_1=a}. \quad (6.3)$$

In equation (6.3), point $x_1 = a$ is the point in the uncracked geometry where the crack tip would lie in the corresponding cracked geometry. Thus, once μ_1 and μ_2 are evaluated in the local coordinates along the crack plane, the $T_{uncracked}$ can be obtained using the stress field solutions.

6.1.1 Reference T -stress solutions

From equations (6.2), it has been noted above that the weight function for a particular crack geometry and material can be derived from a set of four reference T -stress solutions. To this end, as discussed in Section (2.3.1.2), the reference T -stress solutions can be conveniently chosen to be from those load conditions on the cracked geometry for which there are no contributions to $T_{uncracked}$. Thus, two reference solutions can be obtained for the cases when the cracked geometry is subjected to normal crack-face pressures, and two other solutions can be obtained when it is under shear loads on the crack faces; these correspond to the cracked geometry being under mode I and mode II loadings, respectively. In the present work, the following stress distributions are considered for $\sigma_{22}(x_1)$ and $\sigma_{12}(x_1)$ applied along the crack faces:

$$\sigma_{22}(x_1) = (-\sigma_{22}^*) \cdot \left(1 - \frac{x_1}{a}\right)^n, \quad (6.4.a)$$

$$\sigma_{12}(x_1) = \sigma_{12}^* \left(1 - \frac{x_1}{a}\right)^n, \quad (6.4.b)$$

where $n = 0$ and 1 , which corresponds to uniform and linearly varying loads, respectively.

The distance x_1 in equations (6.4) may be defined as shown in figures (2.16), for the

single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP). In equation (6.4.a), the negative value of σ_{22}^* is considered as $\sigma_{22}(x_1)$ is the crack-face pressure. Also, care should be taken while applying the crack-face shear stress $\sigma_{12}(x_1)$ such that it assumes a positive value along both the upper and lower crack faces.

From equations (2.35), (6.1) and (6.4), a set of four simultaneous equations are obtained, as follows,

$$T_1 = \int_0^a \frac{2(-\sigma_{22}^*)}{\pi a} \left[D_1 \left(1 - \frac{x_1}{a}\right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a}\right)^{\frac{3}{2}} \right] dx_1, \quad (6.5.a)$$

$$T_2 = \int_0^a \frac{2}{\pi a} (-\sigma_{22}^*) \left(1 - \frac{x_1}{a}\right) \left[D_1 \left(1 - \frac{x_1}{a}\right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a}\right)^{\frac{3}{2}} \right] dx_1, \quad (6.5.b)$$

$$T_3 = \int_0^a \frac{2\sigma_{12}^*}{\pi a} \left[D_3 \left(1 - \frac{x_1}{a}\right)^{\frac{1}{2}} + D_4 \left(1 - \frac{x_1}{a}\right)^{\frac{3}{2}} \right] dx_1, \quad (6.5.c)$$

$$T_4 = \int_0^a \frac{2}{\pi a} \sigma_{12}^* \left(1 - \frac{x_1}{a}\right) \left[D_3 \left(1 - \frac{x_1}{a}\right)^{\frac{1}{2}} + D_4 \left(1 - \frac{x_1}{a}\right)^{\frac{3}{2}} \right] dx_1. \quad (6.5.d)$$

In equations (6.5), T_1 and T_2 are the reference T -stress solutions for normal crack-face pressure $\sigma_{22}(x_1)$ with $n = 0$ and 1 , respectively; similarly, T_3 and T_4 are the corresponding reference T -stress solutions for shear crack-face loading, $\sigma_{12}(x_1)$. Solving equations (6.5) yields the weight function parameters, D_i , from which the weight functions $m(x_1)$ and $n(x_1)$ can be determined for the crack geometry and material combination using equations (6.2). Carrying out the indicated integrations analytically

and solving equations (6.5), the D_i can be expressed in terms of the reference T -stress solutions as shown below:

$$D_1 = \left(\frac{105}{16}\right)\pi T_2 - \left(\frac{75}{16}\right)\pi T_1, \quad (6.6.a)$$

$$D_2 = \left(\frac{105}{16}\right)\pi T_1 - \left(\frac{175}{16}\right)\pi T_2, \quad (6.6.b)$$

$$D_3 = \left(-\frac{105}{16}\right)\pi T_4 + \left(\frac{75}{16}\right)\pi T_3, \quad (6.6.c)$$

$$D_4 = \left(-\frac{105}{16}\right)\pi T_3 + \left(\frac{175}{16}\right)\pi T_4. \quad (6.6.d)$$

For the determination of the reference T -stress solutions, the conservative M -integral approach using BEM for cracked anisotropic bodies, as presented in Chapter 4, is employed. In the following section, some numerical examples are presented to demonstrate the veracity of this weight function based development for two-dimensional anisotropic cracked bodies.

6.2 Test Problems and Validation

The efficacy of the present development on the weight function method to evaluate the T -stress for cracks in homogeneous anisotropic solids under mixed-mode loading was verified with two sets of problems. The first set of problems investigated consisted of anisotropic SECP, CCP and DECP plates made of graphite/epoxy with the material constants the same as those considered for the problem in Section (4.4.2). For the second set of problem, the one from Section (4.4.2), which has a crack emanating from a stress concentration, is re-visited using weight function approach. The solutions obtained using

the weight function approach for these examples were compared with those from the direct M -integral approach. Plane stress conditions were assumed for all the cases. The integration required, as in equation (6.1), was carried out by Gaussian quadrature. While carrying out the integration along the crack length, a , proper signs for the stresses, $\sigma_{22}(x_1)$ and $\sigma_{12}(x_1)$, were adopted with reference to the upper crack face.

6.2.1 SECP, CCP and DECP under uniform remote tension

Here, rectangular graphite/epoxy plates with standard cracked geometries, namely SECP, CCP and DECP, subjected to remote uniform tension, σ_0 , were investigated. The geometries analysed were similar to those considered in Section (4.4.1) but with $H/W = 4$; see figures (4.5.a), (4.6.a) and (4.7.b) for SECP, CCP and DECP specimens, respectively. The BEM mesh employed were similar to those in figures (4.5.b), (4.6.b) and (4.7.b) for these three specimens. The relative crack size, a/W , was varied from 0.1 to 0.5. The angle of inclination of the crack, ω , was varied as 0° , 22.5° and 45° while the angle of orientation of the material principal axes, ψ , with respect to the global Cartesian axes was varied as 0° , 30° and 60° .

The computed values of the reference T -stresses, T_1 , T_2 , T_3 and T_4 are listed in tables (6.1), (6.2) and (6.3), respectively, for SECP, CCP and DECP specimens; from these values, the weight function parameter, D_i , for the crack problems were obtained and are also included in these tables. In table (6.4), the values for $\text{Re}(\mu_1\mu_2)$ and $\text{Re}(\mu_1+\mu_2)$ in local coordinates, x_i , are presented for different values of ψ and ω considered in this study. Normalised T -stress results, T/σ_0 , obtained using the weight function approach are listed with those obtained from the direct method in tables (6.5),

(6.6) and (6.7). As can be seen from the results, the agreement between the corresponding two sets of solutions is excellent with deviations generally less than 1.5%, 2% and 1% for SECP, CCP and DECP specimens, respectively; exceptions being the cases where the solutions are small in magnitude. These results for the cases when $\omega = 0^\circ$, 22.5° and 45° are also shown in figures (6.2), (6.3) and (6.4) for these specimens.

6.2.2 Crack emanating from a circular hole in an infinite graphite/epoxy plate under remote tension

The problem of a crack emanating from a circular hole in an infinite graphite/epoxy plate under remote uniform tension from Section (4.4.2) was re-analysed here using the weight function method. As before, the geometric parameters considered were, $r = 0.1W$, $a/r = 0.1$ to 1.0 with an increment of 0.1 and $H/W = 2$ (figure 4.17.a). The orientation of the material principal axes, ψ , was arbitrarily chosen as 30° . For this material orientation and crack geometry, $\text{Re}(\mu_1\mu_2) = -0.3835$ and $\text{Re}(\mu_1+\mu_2) = 0.4784$.

As discussed previously in Section (4.4.2), mode-mixity exists for this problem, as is evident from the stress intensity factor solutions presented in table (4.9). Thus, while evaluating the T -stress solutions using the weight functions, this fact was also taken into consideration. In table (6.8) the reference T -stress solutions and the corresponding weight function parameters, D_{Is} , are tabulated for the different values of a/r considered.

With reference to equation (6.1), in order to evaluate $T_{uncracked}$, the stresses, $\sigma_{22}(x_1)$ and $\sigma_{12}(x_1)$, at the prospective crack planes of the uncracked specimens under the same loads are required. These values were interpolated from the relation obtained by curve fitting the corresponding results at a set of arbitrary collinear points from a separate BEM

analysis of the uncracked specimen. A sixth order polynomial was used for the curve fit.

The approximated polynomials used for the stresses are as follows:

$$\begin{aligned}\sigma_{11}(x_1) = & 11.621 (x_1)^6 - 28.2 (x_1)^5 + 16.67 (x_1)^4 + 8.6457 (x_1)^3 \\ & - 12.396 (x_1)^2 + 3.7603 (x_1) + 0.0199,\end{aligned}\tag{6.7.a}$$

$$\begin{aligned}\sigma_{22}(x_1) = & -70.652 (x_1)^6 + 246.77 (x_1)^5 - 335.24 (x_1)^4 + 218.9 (x_1)^3 \\ & - 65.323 (x_1)^2 + 3.9149 (x_1) + 2.8239,\end{aligned}\tag{6.7.b}$$

$$\begin{aligned}\sigma_{12}(x_1) = & 13.843 (x_1)^6 - 61.112 (x_1)^5 + 105.31 (x_1)^4 - 89.095 (x_1)^3 \\ & + 37.334 (x_1)^2 - 6.2703 (x_1) - 0.0234.\end{aligned}\tag{6.7.c}$$

While implementing the above expressions to obtain the integrals in equation (6.1), as discussed earlier, negative values of the stresses computed are used whereas, to evaluate the $T_{uncracked}$, the signs are kept unchanged. The weight function solutions for normalised T -stress, T/σ_0 , are shown along with the solutions obtained directly in Section (4.4.2) in table (6.9). These results shown are also plotted in figure (6.5) where it can be seen that there is a good agreement between the two sets of solutions, the maximum discrepancy being approximately 4%.

6.3 Closing Remarks

A weight function method based on simple reference solutions for evaluating T -stress in plane isotropic cracked elastic bodies has been successfully extended to the general anisotropic case. In this development, a simple expression to obtain the contribution to the T -stress for the cracked body from the corresponding stress field in the uncracked anisotropic body has been presented. The veracity of the development has been validated

for the cases of an inclined crack in rectangular plates under remote tension as well as a crack occurring at a stress concentration. Comparison of T -stress results obtained using the weight function approach with those directly computed using the M -integral approach in Chapter 4 showed a good agreement indeed, establishing the efficacy of this method.

The significance of two-parameter LEFM approach to study crack behaviours in elastic solids is well-established in the literature. These parameters are stress intensity factor and T -stress; they correspond to the leading singular and non-singular terms, respectively, in the Williams' series expansion for the elastic stress field in the vicinity of a crack tip. There has been an extensive amount of development on the analytical and numerical schemes to compute these parameters for crack problems in isotropy. In case of anisotropy, although the schemes to evaluate the stress intensity factor are readily available, the corresponding work for T -stress remains scarce indeed, even more so when in conjunction with the BEM. Further, there was no solution available in the literature for this parameter for bimaterial interface cracks in anisotropy, except the ones published recently in Shah *et al.* (2006) and Shah *et al.* (2006b) based on the work presented in this thesis. The objective of this study, thus, was to develop schemes based on BEM to evaluate T -stresses for cracks in anisotropic solids, including those along the interfaces between dissimilar media.

To achieve this objective, the M -integral approach to evaluate T -stress for isotropic cracked solids was successfully extended to two-dimensional anisotropy using

BEM. The M -integral was also formulated for cracks along the interface between dissimilar anisotropic solids. Using these schemes, the T -stress was obtained for several crack problems and the effect of various geometric and material parameters on it was investigated. In addition, the successful implementation of the weight function method for cracked problems in homogeneous anisotropy was also achieved.

The concepts of linear elastic fracture mechanics were first reviewed, in Chapter 2, and the notion of the two-parameter approach in fracture assessment has been discussed. This involves the use of the stress intensity factor and T -stress. The significance of T -stress in fracture mechanics has been presented followed by the methodologies available to compute it.

In Chapter 3, the boundary element method treatment for two-dimensional anisotropic cracked bodies was presented with a brief review on anisotropic elasticity and the BEM. The implementation of self-regularisation technique into BEM was discussed for two-dimensional anisotropy, supported by a few validation problems. This allowed the accurate evaluation of the field solutions for interior points close to the boundary, even with coarser mesh.

The derivation of M -integral for extracting T -stress solutions in cracked homogeneous anisotropic elastic solids was presented in Chapter 4. The veracity of this work was given by comparing the results for some problems with those available in the literature. This scheme was utilized to obtain the T -stress solutions for an extensive set of standard cracked geometries with commonly used material properties and a problem of engineering significance. The study revealed a significant effect of geometry and material on the T -stress. A relation between T -stress and the stress field ahead of a crack tip was

also derived for the case of homogeneous anisotropy; this allows a better insight into the effect of the geometric and material parameters on the T -stress. Also, the M -integral treatment to obtain the stress intensity factor (Wang *et al.*, 1980) in such materials was briefly reviewed; this approach has been implemented into the BEM code and a few test problems were considered to validate this.

The derivation of the M -integral to obtain T -stress for an interface crack between dissimilar anisotropic solids was presented in Chapter 5. This approach was verified with a problem in isotropy due to paucity of such solutions for anisotropy. T -stress solutions were obtained for orthotropic bimaterial interface cracks and anisotropic bimaterial interface cracks in some applications of engineering significance. This work led to collaboration with Song (2006), through which the current solutions were verified by a different numerical scheme based on SBFEM.

Finally, in Chapter 6, the reference weight function method to compute T -stress in homogeneous anisotropic cracked solids under mixed-mode loading was presented. The development was verified by comparing the solutions for some crack problems obtained by this method with those from the direct M -integral approach, as discussed in Chapter 4. The comparison showed a very good agreement further establishing the efficiency of this method to accurately predict T -stress in two-dimensional anisotropy.

The current work on the numerical schemes to extract T -stress in two-dimensional anisotropy is expected to help relieve the existing scarcity for such methods in the literature; the results obtained are expected to act as benchmark reference for future study in this area. These methodologies based on the M -contour integral have been demonstrated to produce very accurate results. The developments and solutions from this

thesis work have also been published in Shah *et al.* (2006) and Shah *et al.* (2006a), (2006b) and (2007). These publications, in “pdf” file format, can be accessed in Appendix C in the digital medium included at the end of this thesis.

Based on these methodologies presented to extract T -stress solutions for cracks in two-dimensional anisotropic solids under static mechanical loading, future work can be directed towards the treatment of such problems with thermal loading, body forces and fatigue. Also, this work can be extended to three-dimensional domains which could help in the study of many crack behaviour, which cannot be adequately modelled in two dimensions.

APPENDICES

APPENDIX A

INSTRUCTIONS TO ACCESS DIGITAL MEDIUM (CD)

A digital medium (CD) is included at the end of this thesis. The CD contains the same instructions as detailed below to access the files in Appendices B and C; Appendix B provides the complete set of the numerical results obtained for the cases analysed in Section (4.4.1) in Chapter 4, while in Appendix C all the publications arisen out of the work in this thesis are given in pdf format. The instructions to open the CD and access the files in it are as follows:

Opening Appendices B and C:

1. Insert the CD into a CD-ROM drive.
2. The main page ("Appendices.html") will automatically open.

If the page does not load automatically, it can be accessed at the root folder of the CD.

3. Once in the main page, click on the corresponding link to open Appendix B or C.

Appendix B:

- (a) The main page for Appendix B ("AppendixB.html") opens in a new window. The results for normalised T -stress and stress intensity factors for the problems investigated in Section (4.4.1) are organised under three separate headings for the

three specimens, namely, single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP).

- (b) The legend for a particular specimen can be viewed by clicking on the image on the right side of the main window; a new window with zoomed view of the geometry opens.
- (c) For each specimen, the results for the normalised T -stress and stress intensity factors are organised and presented with respect to the aspect ratio (H/W), the relative crack size (a/W), the angle of crack inclination (ω) and the angle between the global co-ordinates axes and material principal axes (ψ).
 - (i) Two types of files are presented for each set:
 - the Excel file (.xls) contains the results in a formatted tabular form and graphs (for selected cases) and can be opened using Ms Excel.
 - the text file (.txt) contains the results in an unformatted form (comma delimited) and can be opened using the notepad, wordpad, Internet Explorer or Ms Excel.
- (d) In order to open a file, click on the link. In order to save the file to the local disc, right click on the link and choose "save target as" and specify the location.
- (e) Accessing data in the Excel file:

The main page in the Excel file contains a list of the links to more specific data sets as follows:

 - (i) clicking on the first link opens the Excel tab containing a complete result-set for that particular specimen and parameter combination.

- (ii) clicking on the links presented in tabular form, opens the Excel tabs containing graphs for the variation of the normalised T -stress with the chosen parameter for some representative cases.
- (f) These data files can also be directly accessed at "CD root folder:\ APPENDIXB\ Data\".

Appendix C:

- (a) The pdf files are organised in two categories:
 - publication in refereed journals
 - conference proceedings
- (b) To open a particular "pdf" file click on the corresponding link.
- (c) These data files can also be directly accessed at "CD root folder:\ APPENDIXC\ Files\".

Troubleshooting:

1. The main page does not open automatically when the CD is loaded into the drive:
 - a. Check that compatible system and browser are available (see minimum requirements).
 - b. To open the main html file manually, go into the root folder of the CD drive and double- click on "Appendices.html".
2. Contents and links are not displayed properly:
 - a. Check if the browser used is compatible (see minimum requirements).
 - b. Allow JavaScript in the browser.

3. The Excel, text or pdf files do not open:
 - a. Check if the correct version of Ms Excel is available (see minimum requirements).
 - b. Allow the pop up window (unblock the setting) in the browser.

Minimum requirements:

1. A CD-ROM drive capable of reading CD-R media.
2. Browsers in the order of preference :
 - a. Internet Explorer (version 5 or newer)
 - b. Mozilla Firefox (version 2 or newer)
 - c. Netscape (version 7 or newer)
3. Microsoft Office Excel 2000
4. Windows 2000, Windows XP (SP1, SP2 or newer versions)
5. Adobe pdf reader (version 6 or newer)

APPENDIX B

COMPLETE SET OF RESULTS FOR THE CASES ANALYSED IN SECTION (4.4.1)

This appendix is in the digital format in the CD included with this thesis. It contains all the numerical results for the cases (more than three thousand) analysed in Section (4.4.1) for, namely, single edge cracked plate (SECP), centre cracked plate (CCP) and double edge cracked plate (DECP). The results are presented for normalised T -stress and stress intensity factors.

APPENDIX C

PUBLICATIONS STEMMING FROM THE WORK OF THIS THESIS

The following publications based on the work in this thesis are available in the digital format (pdf) in the CD included with this thesis.

- Shah, P.D.; Song, C.; Tan, C.L.; Wang, X. (2006). Numerical evaluation of T -stress solutions for cracks in plane anisotropic bodies. *Structural Durability & Health Monitoring*, 2(4), 225-237.
- Shah, P.D.; Tan, C.L.; Wang, X. (2006a). T -stress solutions for two-dimensional crack problems in anisotropic elasticity using the Boundary Element Method. *Fatigue and Fracture of Engineering Materials and Structures*, 29, 343-356.
- Shah, P.D.; Tan, C.L.; Wang, X. (2006b). Evaluation of T -stress for an interface crack between dissimilar anisotropic materials using the Boundary Element Method. *Computer Modeling in Engineering & Sciences*, 13(3), 185-197.
- Shah, P.D.; Tan, C.L.; Wang, X. (2007). Evaluation of T -stress for an interface crack lying between dissimilar anisotropic solids using BEM. *Advances in Computational & Experimental Engineering and Sciences, Proceedings of ICCES'07, Miami*, (editors: Han, Z.; Lee, S.W.; Nakagaki, M.), 675-680.
- Shah, P.D.; Tan, C.L.; Wang, X. (to appear). Determination of weight functions for the elastic T -stress in two-dimensional anisotropy using BEM. *Advances in Boundary Element Techniques, Proceedings of Boundary Element Techniques VIII, 24-26th July, 2007, Naples, Italy*, (editors: Selvadurai, A.P.S.; Tan, C.L.; Aliabadi, M. H.).

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TABLES

Table 3.1 Interior point stress solutions in a cantilever beam using self-regularised and non-regularised BEM.

Points	Distance from the Neutral Axis x_1	σ_{22} (Regularised)	σ_{22} (Non-regularised)	σ_{22} (Analytical)
1	0.49	35.1000	-9169.9000	35.2800
2	0.48	34.4600	-2950.2000	34.5600
3	0.47	33.7310	143.5700	33.8400
4	0.46	32.9890	518.2600	33.1200
5	0.45	32.2610	356.3200	32.4000
6	0.4	28.6880	16.2280	28.8000
7	0.3	21.5150	21.7890	21.6000
8	0.2	14.3430	14.3520	14.4000
9	0.1	7.1718	7.1692	7.2000

Table 3.2 Stresses, σ_{22} , at interior points in a graphite/epoxy plate under uniform tension using self-regularised and non-regularised BEM.

Points	Distance from AB	σ_{22} (Regularised)	σ_{22} (Non-regularised)	σ_{22} (Exact)
1	0.001	1.0003	-3.2644	1.00
2	0.01	1.0003	-3.2605	1.00
3	0.05	1.0003	-3.2443	1.00
4	0.1	1.0003	-3.2284	1.00
5	5.0	1.0002	1.0064	1.00

Table 4.1 Normalized stress intensity factors, $\frac{K_I}{\sigma_0 \sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$, and T -stress, T/σ_0 , for a CCP glass/epoxy plate under unit tension; $\sigma_0 = 1$, $a/W = 0.2$ and $\omega = 45^\circ$.

ψ°	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$					$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$					T/σ_0 (BEM)
	Gandhi (1972)	BEM (Traction formula)	% Diff	BEM (M - integral)	% Diff	Gandhi (1972)	BEM (Traction formula)	% Diff	BEM (M - integral)	% Diff	
0	0.522	0.529	-1.34	0.522	0	0.507	0.514	-1.38	0.507	0	0.1892
45	0.515	0.522	-1.36	0.515	0	0.505	0.511	-1.19	0.505	0	-0.3705
90	0.513	0.519	-1.17	0.512	0.19	0.509	0.514	-0.98	0.507	0.39	-0.1479
105	0.517	0.523	-1.16	0.516	0.19	0.510	0.517	-1.37	0.510	0	0.0658
120	0.524	0.530	-1.15	0.524	0	0.512	0.519	-1.37	0.511	0.20	0.1788
135	0.532	0.538	-1.13	0.531	0.19	0.511	0.518	-1.37	0.511	0	0.1771
180	0.522	0.529	-1.34	0.522	0	0.507	0.514	-1.38	0.507	0	0.1892

Table 4.2 Normalized T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0 \sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$ for an isotropic CCP under remote uniform tension, σ_0 , with an inclined crack;
Ref.: Smith *et al.* (2001).

ω°	T/σ_0			$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$			$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$		
	(BEM)	(Ref.)	% Diff	(BEM)	(Ref.)	% Diff	(BEM)	(Ref.)	% Diff
0	-1.0062	-1.0000	-0.62	1.0058	1.0000	-0.58	0	0	0
15	-0.8729	-0.8660	-0.80	0.9387	0.9330	-0.61	0.2502	0.2500	-0.08
30	-0.5071	-0.5000	-1.42	0.7554	0.7500	-0.72	0.4338	0.4330	-0.18
45	-0.0067	0	-----	0.5043	0.5000	-0.86	0.5016	0.5000	-0.33
60	0.4959	0.5000	0.83	0.2525	0.2500	-0.99	0.4350	0.4330	-0.45
75	0.8646	0.8660	0.16	0.0678	0.0670	-1.16	0.2508	0.2500	-0.34

Table 4.3 Normalized T -stresses, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for an orthotropic CCP under remote uniform tension, σ_0 , with an inclined crack.

ω°	T/σ_0			$\frac{K_I}{\sigma_0\sqrt{\pi a}}$ (BEM)	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$ (BEM)
	(BEM)	Kim and Paulino (2004)	% Diff		
0	-3.185	-3.164	-0.66	1.0059	0
15	-1.661	-1.643	-1.10	0.9403	0.2480
30	0.031	0.031	0	0.7598	0.4317
45	0.723	0.718	-0.70	0.5103	0.5020
60	0.938	0.935	-0.32	0.2565	0.4366
75	0.992	0.988	-0.40	0.0697	0.2506

Table 4.4 Different cases of materials considered for the orthotropic cases analysed in the parametric study; $\nu_{12} = 0.3$.

Materials	β_1	β_2
Set (1)		
Case 1	1.5	0.5
Case 2	3.0	0.5
Case 3	4.5	0.5
Case 4	6.0	0.5
Set (2)		
Case 5	1.5	0.75
Case 6	3.0	0.75
Case 7	4.5	0.75
Case 8	6.0	0.75

Table 4.5 Typical material properties for some common orthotropic materials, and the corresponding values of β_1 and β_2 (Cassell, 2002).

Material	E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	ν_{12}	β_1	β_2
E-glass/ 470-36 E-glass/ Vinylester	24.40	6.87	2.89	0.32	2.71	0.70
High Modulus Graphite/ Epoxy	221.00	6.90	4.80	0.30	6.69	0.85
Kevlar 49/ Epoxy	86.00	5.50	2.10	0.34	6.32	0.63
Glass/ Epoxy	38.60	8.27	4.14	0.26	2.96	0.73
Boron/ Epoxy	204.00	18.50	5.59	0.23	5.98	0.55
γ -Alumina/ Epoxy	137.00	18.00	7.25	0.30	4.23	0.65
S-glass/ Epoxy	55.00	16.00	7.60	0.28	2.47	0.75

Table 4.6.a Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a SECP specimen under remote uniform tension, σ_0 for material Set (1); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
1	0	0.4139	0.7495	0.3372
	15	0.4847	0.7336	0.3396
	30	0.4946	0.7237	0.3467
	45	0.3082	0.7108	0.3660
	60	0.0895	0.6931	0.3934
	75	0.0598	0.6768	0.4135
	90	0.1561	0.6785	0.4217
2	0	0.1013	0.6492	0.4396
	15	0.3480	0.7122	0.4386
	30	0.5665	0.8039	0.4086
	45	0.5660	0.8522	0.3642
	60	0.3733	0.8386	0.3330
	75	0.2820	0.7876	0.3259
	90	0.3557	0.7403	0.3344
3	0	0.0312	0.6127	0.4743
	15	0.3389	0.7322	0.4838
	30	0.6202	0.8937	0.4420
	45	0.6743	0.9782	0.3626
	60	0.4917	0.9589	0.2915
	75	0.3681	0.8768	0.2649
	90	0.4319	0.7897	0.2753
4	0	0.0071	0.5949	0.4923
	15	0.3498	0.7627	0.5167
	30	0.6594	0.9770	0.4670
	45	0.7333	1.0914	0.3616
	60	0.5668	1.0637	0.2565
	75	0.4157	0.9539	0.2154
	90	0.4667	0.8344	0.2267

Table 4.6.b Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a SECP specimen under remote uniform tension, σ_0 for material Set (2); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
5	0	0.2432	0.7056	0.3967
	15	0.3086	0.7194	0.3947
	30	0.3626	0.7386	0.3864
	45	0.3765	0.7519	0.3753
	60	0.3600	0.7533	0.3661
	75	0.3498	0.7454	0.3614
	90	0.3656	0.7358	0.3604
6	0	0.0194	0.6312	0.4736
	15	0.2365	0.7140	0.4720
	30	0.4219	0.8325	0.4418
	45	0.5357	0.9148	0.3791
	60	0.5818	0.9331	0.3077
	75	0.5966	0.8984	0.2573
	90	0.6442	0.8339	0.2397
7	0	-0.0195	0.6025	0.4986
	15	0.2426	0.7400	0.5116
	30	0.4645	0.9317	0.4757
	45	0.6021	1.0570	0.3807
	60	0.6703	1.0782	0.2641
	75	0.6821	1.0166	0.1836
	90	0.7405	0.9086	0.1573
8	0	-0.0297	0.5883	0.5114
	15	0.2586	0.7742	0.5428
	30	0.4967	1.0219	0.5021
	45	0.6375	1.1838	0.3816
	60	0.7261	1.2050	0.2254
	75	0.7283	1.1158	0.1236
	90	0.7798	0.9745	0.0894

Table 4.7.a Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a CCP specimen under remote uniform tension, σ_0 for material Set (1); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
1	0	0.1156	0.5049	0.5017
	15	0.1780	0.5044	0.5017
	30	0.1109	0.5042	0.5016
	45	-0.1745	0.5041	0.5013
	60	-0.3686	0.5039	0.5012
	75	-0.2831	0.5035	0.5012
	90	-0.0971	0.5033	0.5015
2	0	-0.0837	0.5026	0.5014
	15	0.1914	0.5037	0.5024
	30	0.3197	0.5065	0.5029
	45	0.1592	0.5088	0.5025
	60	-0.0722	0.5091	0.5014
	75	-0.0609	0.5070	0.5011
	90	0.1125	0.5046	0.5018
3	0	-0.0964	0.5019	0.5011
	15	0.2331	0.5042	0.5033
	30	0.4031	0.5099	0.5041
	45	0.2699	0.5151	0.5033
	60	0.0265	0.5157	0.5012
	75	0.0212	0.5110	0.5005
	90	0.2026	0.5054	0.5022
4	0	-0.0889	0.5026	0.5022
	15	0.2651	0.5038	0.5040
	30	0.4471	0.5127	0.5048
	45	0.3231	0.5236	0.5040
	60	0.0762	0.5230	0.5009
	75	0.0630	0.5141	0.5002
	90	0.2500	0.5058	0.5026

Table 4.7.b Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a CCP specimen under remote uniform tension, σ_0 for material Set (2); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
5	0	-0.0550	0.5039	0.5016
	15	0.0240	0.5042	0.5018
	30	0.0621	0.5047	0.5019
	45	0.0489	0.5051	0.5019
	60	0.0206	0.5052	0.5017
	75	0.0202	0.5050	0.5017
	90	0.0485	0.5046	0.5017
6	0	-0.1627	0.5023	0.5013
	15	0.0815	0.5039	0.5024
	30	0.2232	0.5073	0.5034
	45	0.2715	0.5106	0.5035
	60	0.2707	0.5116	0.5026
	75	0.2876	0.5099	0.5020
	90	0.3498	0.5068	0.5021
7	0	-0.1446	0.5017	0.5011
	15	0.1309	0.5045	0.5034
	30	0.2873	0.5109	0.5048
	45	0.3454	0.5175	0.5048
	60	0.3503	0.5194	0.5029
	75	0.3787	0.5148	0.5018
	90	0.4708	0.5081	0.5027
8	0	-0.1234	0.5026	0.5023
	15	0.1645	0.5043	0.5042
	30	0.3222	0.5135	0.5055
	45	0.3808	0.5265	0.5058
	60	0.3889	0.5284	0.5029
	75	0.4242	0.5181	0.5022
	90	0.5329	0.5087	0.5032

Table 4.8.a Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a DECP plate under remote uniform tension, σ_0 for material Set (1); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
1	0	0.4038	0.7313	0.3291
	15	0.4731	0.7166	0.3318
	30	0.4832	0.7080	0.3393
	45	0.3010	0.6955	0.3583
	60	0.0868	0.6770	0.3845
	75	0.0579	0.6600	0.4033
	90	0.1517	0.6616	0.4113
2	0	0.0979	0.6312	0.4276
	15	0.3397	0.6960	0.4288
	30	0.5552	0.7876	0.4004
	45	0.5545	0.8355	0.3568
	60	0.3631	0.8194	0.3258
	75	0.2757	0.7670	0.3174
	90	0.3464	0.7205	0.3252
3	0	0.0295	0.5939	0.4601
	15	0.3313	0.7165	0.4735
	30	0.6078	0.8759	0.4334
	45	0.6618	0.9595	0.3547
	60	0.4739	0.9354	0.2864
	75	0.3607	0.8525	0.2580
	90	0.4206	0.7675	0.2666
4	0	0.0061	0.5754	0.4765
	15	0.3422	0.7471	0.5062
	30	0.6450	0.9579	0.4581
	45	0.7219	1.0713	0.3530
	60	0.5374	1.0330	0.2558
	75	0.4052	0.9212	0.2138
	90	0.4562	0.8077	0.2205

Table 4.8.b Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a DECP plate under remote uniform tension, σ_0 for material Set (2); $\omega = 45^\circ$, $a/W = 0.1$.

Case	ψ°	T/σ_0	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
5	0	0.2372	0.6890	0.3874
	15	0.3010	0.7029	0.3859
	30	0.3541	0.7222	0.3779
	45	0.3677	0.7354	0.3670
	60	0.3514	0.7363	0.3579
	75	0.3413	0.7281	0.3531
	90	0.3566	0.7186	0.3521
6	0	0.0183	0.6142	0.4611
	15	0.2310	0.6982	0.4616
	30	0.4133	0.8149	0.4325
	45	0.5243	0.8955	0.3707
	60	0.5665	0.9111	0.3012
	75	0.5844	0.8763	0.2506
	90	0.6291	0.8134	0.2329
7	0	-0.0195	0.5845	0.4840
	15	0.2374	0.7247	0.5011
	30	0.4548	0.9120	0.4659
	45	0.5905	1.0347	0.3715
	60	0.6501	1.0500	0.2600
	75	0.6692	0.9902	0.1786
	90	0.7232	0.8851	0.1514
8	0	-0.0292	0.5695	0.4955
	15	0.2531	0.7593	0.5323
	30	0.4851	1.0007	0.4919
	45	0.6263	1.1598	0.3716
	60	0.6946	1.1671	0.2271
	75	0.7141	1.0786	0.1253
	90	0.7649	0.9446	0.0873

Table 4.9.a Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a crack emanating from a circular hole in a rectangular graphite/epoxy plate under remote tension, σ_0 , for $\psi = 0^\circ, 15^\circ$ and 30° .

ψ°	a/r	T/σ_0 (BEM)	T/σ_0 (SBFEM)	% Diff	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
0	0.05	-0.3612	-0.3664	1.42	4.0697	0
	0.1	-0.2343	-0.2383	1.68	3.2265	0
	0.2	-0.1649	-0.1669	1.20	2.4405	0
	0.3	-0.1552	-0.1561	0.58	2.0465	0
	0.4	-0.1607	-0.1607	0	1.8089	0
	0.5	-0.1706	-0.1699	-0.41	1.6446	0
	0.6	-0.1801	-0.1802	0.06	1.5413	0
	0.7	-0.1904	-0.1903	-0.05	1.4507	0
	0.8	-0.2001	-0.1997	-0.20	1.3778	0
	0.9	-0.2087	-0.2082	-0.24	1.3197	0
	1	-0.2166	-0.2160	-0.28	1.2718	0
15	0.05	-0.3824	N.A.	-----	3.9945	-0.4322
	0.1	-0.2409	N.A.	-----	3.3095	-0.3584
	0.2	-0.1562	N.A.	-----	2.5100	-0.2165
	0.3	-0.1504	N.A.	-----	2.0910	-0.1333
	0.4	-0.1615	N.A.	-----	1.8403	-0.0851
	0.5	-0.1762	N.A.	-----	1.6682	-0.0542
	0.6	-0.1888	N.A.	-----	1.5605	-0.0376
	0.7	-0.2019	N.A.	-----	1.4672	-0.0238
	0.8	-0.2142	N.A.	-----	1.3927	-0.0137
	0.9	-0.2247	N.A.	-----	1.3338	-0.0071
	1	-0.2343	N.A.	-----	1.2853	-0.0021
30	0.05	-0.3054	-0.3067	0.42	3.5303	-0.6872
	0.1	-0.2528	-0.2563	1.37	3.2865	-0.6876
	0.2	-0.1207	-0.1244	2.97	2.6961	-0.5382
	0.3	-0.0977	-0.0977	0.00	2.2339	-0.3623
	0.4	-0.1247	-0.1222	-2.05	1.9368	-0.2396
	0.5	-0.1599	-0.1558	-2.63	1.7351	-0.1581
	0.6	-0.1868	-0.1870	0.11	1.6091	-0.1118
	0.7	-0.2139	-0.2139	0.00	1.5044	-0.0750
	0.8	-0.2379	-0.2367	-0.51	1.4216	-0.0474
	0.9	-0.2568	-0.2560	-0.31	1.3589	-0.0302
	1	-0.2736	-0.2725	-0.40	1.3067	-0.0167

Table 4.9.b Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a crack emanating from a circular hole in a rectangular graphite/epoxy plate under remote tension, σ_0 , for $\psi = 45^\circ$ and 60° .

ψ°	a/r	T/σ_0 (BEM)	T/σ_0 (SBFEM)	% Diff	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
45	0.05	-0.2755	N.A.	-----	3.1204	-0.8204
	0.1	-0.2505	N.A.	-----	2.9802	-0.8209
	0.2	-0.1706	N.A.	-----	2.6893	-0.7758
	0.3	-0.0866	N.A.	-----	2.3878	-0.6700
	0.4	-0.0508	N.A.	-----	2.1113	-0.5328
	0.5	-0.0720	N.A.	-----	1.8812	-0.3988
	0.6	-0.1153	N.A.	-----	1.7218	-0.3016
	0.7	-0.1719	N.A.	-----	1.5900	-0.2196
	0.8	-0.2258	N.A.	-----	1.4861	-0.1558
	0.9	-0.2727	N.A.	-----	1.4089	-0.1121
	1	-0.3136	N.A.	-----	1.3472	-0.0790
60	0.05	-0.5623	-0.5585	-0.68	2.8048	-0.8025
	0.1	-0.4947	-0.4893	-1.10	2.6776	-0.7895
	0.2	-0.3816	-0.3782	-0.90	2.4535	-0.7548
	0.3	-0.2913	-0.2904	-0.31	2.2571	-0.7064
	0.4	-0.2229	-0.2229	0.00	2.0856	-0.6475
	0.5	-0.1771	-0.1770	-0.06	1.9313	-0.5793
	0.6	-0.1573	-0.1552	-1.35	1.8080	-0.5158
	0.7	-0.1614	-0.1582	-2.02	1.6903	-0.4440
	0.8	-0.1902	-0.1836	-3.59	1.5867	-0.3734
	0.9	-0.2335	-0.2262	-3.23	1.4995	-0.3106
	1	-0.2894	-0.2797	-3.47	1.4248	-0.2532

Table 4.9.c Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a crack emanating from a circular hole in a rectangular graphite/epoxy plate under remote tension, σ_0 , for $\psi = 75^\circ$ and 90° .

ψ°	a/r	T/σ_0 (BEM)	T/σ_0 (SBFEM)	% Diff	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$
75	0.05	-1.8273	N.A.	-----	2.5517	-0.5341
	0.1	-1.6304	N.A.	-----	2.4318	-0.5216
	0.2	-1.3325	N.A.	-----	2.2289	-0.4960
	0.3	-1.1249	N.A.	-----	2.0615	-0.4687
	0.4	-0.9824	N.A.	-----	1.9238	-0.4410
	0.5	-0.8848	N.A.	-----	1.8055	-0.4122
	0.6	-0.8228	N.A.	-----	1.7134	-0.3868
	0.7	-0.7863	N.A.	-----	1.6280	-0.3592
	0.8	-0.7690	N.A.	-----	1.5528	-0.3315
	0.9	-0.7706	N.A.	-----	1.4875	-0.3050
	1	-0.7846	N.A.	-----	1.4300	-0.2791
90	0.05	-4.0512	-4.0187	-0.81	2.4455	0
	0.1	-3.6562	-3.6282	-0.77	2.3287	0
	0.2	-3.0616	-3.0416	-0.66	2.1331	0
	0.3	-2.6502	-2.6375	-0.48	1.9740	0
	0.4	-2.3684	-2.3580	-0.44	1.8446	0
	0.5	-2.1744	-2.1660	-0.39	1.7352	0
	0.6	-2.0458	-2.0368	-0.44	1.6511	0
	0.7	-1.9633	-1.9536	-0.50	1.5734	0
	0.8	-1.9154	-1.9044	-0.58	1.5058	0
	0.9	-1.8923	-1.8807	-0.62	1.4473	0
	1	-1.8898	-1.8761	-0.73	1.3962	0

Table 5.1 Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for a cracked isotropic bi-material CCP specimen in Problem (A).

a/W	$\frac{E_1}{E_2}$	T/σ_0					$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$	$\tan^{-1}\left(\frac{K_{II}}{K_I}\right)$
		Present	Song (2005)	% Diff.	Sladek & Sladek (1997b)	% Diff.			
0.15	1	-1.016	-1.015	-0.1	-1.043	2.6	1.026	0.000	0.00
	2	-0.681	-0.681	0.0	-0.706	3.5	1.023	0.051	2.84
	5	-0.348	-0.347	-0.3	-0.366	4.9	1.011	0.104	5.87
	10	-0.193	-0.192	-0.5	-0.204	5.4	0.999	0.129	7.37
0.5	1	-1.257	-1.260	0.2	-1.272	1.2	1.202	0.000	0.00
	2	-0.846	-0.847	0.1	-0.861	1.7	1.197	0.046	2.22
	5	-0.436	-0.437	0.2	-0.450	3.1	1.180	0.093	4.52
	10	-0.244	-0.244	0.0	-0.260	6.2	1.163	0.114	5.62

Table 5.2 Normalised T -stress, T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for different relative crack lengths, a/W , for Problem (B); corresponding results for an isotropic bi-material specimen is given.

a/W	T/σ_0		$\frac{K_I}{\sigma_0\sqrt{\pi a}}$		$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$		$\tan^{-1}\left(\frac{K_{II}}{K_I}\right)$	
	Orthotropic	Isotropic	Ortho.	Iso.	Ortho.	Iso.	Ortho.	Iso.
0.1	-0.5225	-0.2217	1.0285	1.0189	0.1745	0.1672	9.63	9.32
0.2	-0.5343	-0.2272	1.0436	1.0388	0.1725	0.1619	9.39	8.86
0.3	-0.5561	-0.2375	1.0697	1.0713	0.1732	0.1598	9.20	8.49
0.4	-0.5912	-0.2546	1.1134	1.1222	0.1775	0.1616	9.06	8.19
0.5	-0.6507	-0.2829	1.1758	1.1957	0.1855	0.1682	8.96	8.01

Table 5.3.a Normalised T -stress, T/σ_0 , and stress intensity factor, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$, for varying material orientation, ψ , and relative crack size, a/r , in a centre cracked bimaterial silicon disc in Problem (C) for crack tip A.

a/r	ψ°	T/σ_0	T/σ_0 Song (2006)	% Diff.	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$
0.1	0	-0.1601	N.A.	----	1.0286
	15	-0.1443	N.A.	----	1.0290
	30	-0.0922	N.A.	----	1.0300
	45	-0.0103	N.A.	----	1.0311
	60	0.0599	N.A.	----	1.0318
	75	0.0949	N.A.	----	1.0318
	90	0.1043	N.A.	----	1.0317
0.2	0	-0.2195	N.A.	----	1.0700
	15	-0.1998	N.A.	----	1.0712
	30	-0.1475	N.A.	----	1.0749
	45	-0.0676	N.A.	----	1.0791
	60	0.0019	N.A.	----	1.0818
	75	0.0386	N.A.	----	1.0825
	90	0.0502	N.A.	----	1.0824
0.3	0	-0.3187	N.A.	----	1.1405
	15	-0.2879	N.A.	----	1.1420
	30	-0.2302	N.A.	----	1.1490
	45	-0.1530	N.A.	----	1.1577
	60	-0.0879	N.A.	----	1.1642
	75	-0.0516	N.A.	----	1.1673
	90	-0.0375	N.A.	----	1.1682
0.4	0	-0.4565	N.A.	----	1.2404
	15	-0.4060	N.A.	----	1.2413
	30	-0.3343	N.A.	----	1.2507
	45	-0.2570	N.A.	----	1.2645
	60	-0.1998	N.A.	----	1.2770
	75	-0.1681	N.A.	----	1.2851
	90	-0.1538	N.A.	----	1.2884
0.5	0	-0.6464	-0.6426	-0.59	1.3763
	15	-0.5665	-0.5635	-0.53	1.3751
	30	-0.4687	-0.4666	-0.45	1.3855
	45	-0.3841	-0.3820	-0.54	1.4046
	60	-0.3364	-0.3338	-0.79	1.4253
	75	-0.3149	-0.3119	-0.95	1.4416
	90	-0.3061	-0.3030	-1.02	1.4495

Table 5.3.b Normalised T -stress, T/σ_0 , and stress intensity factor, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$, for varying material orientation, ψ , and relative crack size, a/r , in a centre cracked bimaterial silicon disc in Problem (C) for crack tip B.

a/r	ψ°	T/σ_0	T/σ_0 Song (2006)	% Diff.	$\frac{K_I}{\sigma_0\sqrt{\pi a}}$
0.1	0	-0.1601	N.A.	----	1.0288
	15	-0.1457	N.A.	----	1.0293
	30	-0.0941	N.A.	----	1.0303
	45	-0.0113	N.A.	----	1.0317
	60	0.0601	N.A.	----	1.0321
	75	0.0954	N.A.	----	1.0318
	90	0.1043	N.A.	----	1.0318
0.2	0	-0.2195	N.A.	----	1.0703
	15	-0.2103	N.A.	----	1.0722
	30	-0.1618	N.A.	----	1.0764
	45	-0.0757	N.A.	----	1.0803
	60	0.0025	N.A.	----	1.0824
	75	0.0415	N.A.	----	1.0825
	90	0.0502	N.A.	----	1.0826
0.3	0	-0.3187	N.A.	----	1.1410
	15	-0.3204	N.A.	----	1.1456
	30	-0.2753	N.A.	----	1.1540
	45	-0.1803	N.A.	----	1.1621
	60	-0.0894	N.A.	----	1.1668
	75	-0.0449	N.A.	----	1.1684
	90	-0.0375	N.A.	----	1.1686
0.4	0	-0.4565	N.A.	----	1.2405
	15	-0.4738	N.A.	----	1.2483
	30	-0.4307	N.A.	----	1.2612
	45	-0.3206	N.A.	----	1.2738
	60	-0.2123	N.A.	----	1.2830
	75	-0.1609	N.A.	----	1.2878
	90	-0.1538	N.A.	----	1.2885
0.5	0	-0.6464	-0.6426	-0.59	1.3766
	15	-0.6840	-0.6796	-0.64	1.3881
	30	-0.6413	-0.6369	-0.69	1.4051
	45	-0.5104	-0.5064	-0.79	1.4232
	60	-0.3796	-0.3761	-0.93	1.4385
	75	-0.3175	-0.3144	-1.00	1.4482
	90	-0.3061	-0.3030	-1.02	1.4499

Table 5.4 Elastic mechanical properties of sandstone and dolomite reservoir rock, Rasolofosaon and Zinszner (2002).

Elastic Constants	Sandstone (Material 1)	Dolomite (Material 2)
E_{11} (GPa)	16.59	61.46
E_{22} (GPa)	17.95	47.61
E_{33} (GPa)	20.39	55.6
G_{12} (GPa)	7.67	20
G_{23} (GPa)	8.23	23.3
G_{31} (GPa)	7.53	24.38
ν_{12}	0.167	0.153
ν_{13}	0.142	0.173
ν_{23}	0.156	0.171
$\eta_{12,1}$	0.025	- 0.142
$\eta_{12,2}$	-0.137	0.033
$\eta_{12,3}$	-0.022	-0.015
$\mu_{31,23}$	-0.087	-0.064

Table 5.5.a Normalised T -stress T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for crack tip A of an interface crack in a large rectangular domain comprising sandstone and dolomite, for Problem (D).

ω°	T/σ_0			$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$	$\tan^{-1}\left(\frac{K_{II}}{K_I}\right)$
	BEM	SBFEM	% Diff			
0	-1.7318	-1.7317	-0.01	1.0325	-0.0702	-3.89
15	-1.5450	-1.5450	0.00	0.9676	0.3097	17.74
30	-0.8272	-0.8273	0.01	0.8181	0.4077	26.49
45	0.0180	0.0175	-3.03	0.5743	0.4828	40.06
60	0.7912	0.7879	-0.42	0.2662	0.4728	60.61

Table 5.5.b Normalised T -stress T/σ_0 , and stress intensity factors, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$ and $\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$, for crack tip B of an interface crack in a large rectangular domain comprising sandstone and dolomite, for Problem (D).

ω°	T/σ_0			$\frac{K_I}{\sigma_0\sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0\sqrt{\pi a}}$	$\tan^{-1}\left(\frac{K_{II}}{K_I}\right)$
	BEM	SBFEM	% Diff			
0	-1.7309	-1.7308	0.00	1.0370	0.0684	3.77
15	-1.5538	-1.5534	-0.03	0.9744	0.2346	13.54
30	-0.8303	-0.8303	0.00	0.7064	0.5811	39.43
45	0.0333	0.0326	-2.09	0.4413	0.6144	54.33
60	0.8128	0.8097	-0.38	0.2877	0.4600	57.98

Table 5.6.a Normalised T -stress, T/σ_0 , and stress intensity factor, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$, for an interface crack along YCBO bicrystal junction in Problem (E) for pure bending.

ω°	a/W	T/σ_0			$\frac{K_I}{\sigma_0\sqrt{\pi a}}$
		BEM	SBFEM	%Diff	
0	0.1	-0.3327	-0.3331	-0.12	1.0562
	0.2	-0.2099	-0.2100	-0.05	1.0615
	0.3	-0.0698	-0.0698	0.00	1.1265
	0.4	0.1062	0.1065	-0.28	1.2576
	0.5	0.3494	0.3504	-0.29	1.4838
15	0.1	-0.3012	-0.3015	-0.10	1.0380
	0.2	-0.1750	-0.1751	-0.06	1.0483
	0.3	-0.0243	-0.0243	0.00	1.1154
	0.4	0.1716	0.1720	-0.23	1.2470
	0.5	0.4502	0.4513	-0.24	1.4726
30	0.1	-0.2859	-0.2862	-0.10	1.0243
	0.2	-0.1533	-0.1534	-0.07	1.0385
	0.3	0.0103	0.0103	0.00	1.1072
	0.4	0.2277	0.2282	-0.22	1.2393
	0.5	0.5426	0.5439	-0.24	1.4643
45	0.1	-0.2927	-0.2930	-0.10	1.0189
	0.2	-0.1515	-0.1516	-0.07	1.0345
	0.3	0.0248	0.0249	-0.4	1.1036
	0.4	0.2610	0.2616	-0.23	1.2361
	0.5	0.6052	0.6067	-0.25	1.4611
60	0.1	-0.3241	-0.3244	-0.09	1.0232
	0.2	-0.1739	-0.1740	-0.06	1.0377
	0.3	0.0116	0.0117	-0.69	1.1063
	0.4	0.2582	0.2588	-0.23	1.2386
	0.5	0.6154	0.6169	-0.24	1.4637
75	0.1	-0.3745	-0.3748	-0.08	1.0367
	0.2	-0.2177	-0.2178	-0.05	1.0474
	0.3	-0.0303	-0.0302	0.33	1.1146
	0.4	0.2134	0.2139	-0.23	1.2463
	0.5	0.5599	0.5613	-0.25	1.4717
90	0.1	-0.4281	-0.4285	-0.09	1.0562
	0.2	-0.2700	-0.2701	-0.04	1.0614
	0.3	-0.0898	-0.0898	0.00	1.1263
	0.4	0.1367	0.1371	-0.26	1.2572
	0.5	0.4496	0.4508	-0.27	1.4833

Table 5.6.b Normalised T -stress, T/σ_0 , and stress intensity factor, $\frac{K_I}{\sigma_0\sqrt{\pi a}}$, for an interface crack along YCBO bicrystal junction in Problem (E) for uniform remote tension.

ω°	a/W	T/σ_0			$\frac{K_I}{\sigma_0\sqrt{\pi a}}$
		BEM	SBFEM	% Diff	
0	0.1	-0.4852	-0.4851	0.02	1.2040
	0.2	-0.5194	-0.5193	0.02	1.3830
	0.3	-0.5382	-0.5381	0.02	1.6754
	0.4	-0.5100	-0.5098	0.04	2.1245
	0.5	-0.3725	-0.3719	0.16	2.8274
15	0.1	-0.4569	-0.4567	0.04	1.2096
	0.2	-0.4840	-0.4838	0.04	1.3853
	0.3	-0.4872	-0.4871	0.02	1.6741
	0.4	-0.4277	-0.4275	0.05	2.1193
	0.5	-0.2271	-0.2263	0.35	2.8172
30	0.1	-0.4473	-0.4470	0.07	1.2136
	0.2	-0.4697	-0.4694	0.06	1.3869
	0.3	-0.4611	-0.4609	0.04	1.6732
	0.4	-0.3757	-0.3753	0.11	2.1156
	0.5	-0.1198	-0.1188	0.83	2.8097
45	0.1	-0.4634	-0.4632	0.04	1.2153
	0.2	-0.4851	-0.4847	0.08	1.3876
	0.3	-0.4713	-0.4711	0.04	1.6728
	0.4	-0.3714	-0.3709	0.13	2.1141
	0.5	-0.0813	-0.0801	1.48	2.8067
60	0.1	-0.5073	-0.5070	0.06	1.2136
	0.2	-0.5328	-0.5324	0.08	1.3869
	0.3	-0.5230	-0.5228	0.04	1.6729
	0.4	-0.4261	-0.4256	0.12	2.1152
	0.5	-0.1358	-0.1347	0.81	2.8090
75	0.1	-0.5684	-0.5680	0.07	1.2094
	0.2	-0.6020	-0.6016	0.07	1.3852
	0.3	-0.6060	-0.6057	0.05	1.6738
	0.4	-0.5320	-0.5316	0.08	2.1187
	0.5	-0.2823	-0.2814	0.32	2.8163
90	0.1	-0.6243	-0.6239	0.06	1.2043
	0.2	-0.6683	-0.6680	0.04	1.3830
	0.3	-0.6923	-0.6922	0.01	1.6751
	0.4	-0.6561	-0.6558	0.05	2.1240
	0.5	-0.4790	-0.4783	0.15	2.8267

Table 6.1.a Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy SECP specimen for $\omega = 0^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.1235	0.0868	0	0.0001	-0.0290	-0.4365	-0.0004	0.0009
	0.2	0.1130	0.0804	0.0001	0.0001	-0.0054	-0.4346	-0.0006	0.0014
	0.3	0.1085	0.0777	0.0002	0.0002	0.0031	-0.4314	-0.0009	0.0025
	0.4	0.1205	0.0852	0.0006	0.0005	-0.0185	-0.4425	-0.0026	0.0066
	0.5	0.1700	0.1169	0.0013	0.0012	-0.0923	-0.5136	-0.0055	0.0143
30	0.1	0.2818	0.2005	0.2570	0.1591	-0.0155	-1.0807	0.5052	0.1674
	0.2	0.2846	0.2024	0.2510	0.1557	-0.0180	-1.0877	0.4854	0.1765
	0.3	0.3096	0.2186	0.2262	0.1417	-0.0528	-1.1279	0.4101	0.2049
	0.4	0.3832	0.2665	0.1690	0.1090	-0.1473	-1.2591	0.2427	0.2592
	0.5	0.5629	0.3847	0.0623	0.0476	-0.3579	-1.6139	-0.0638	0.3510
60	0.1	0.6447	0.4617	0.5114	0.3156	0.0256	-2.5743	1.0249	0.3000
	0.2	0.6541	0.4680	0.4967	0.3073	0.0152	-2.5941	0.9789	0.3190
	0.3	0.7214	0.5114	0.4432	0.2768	-0.0807	-2.6986	0.8210	0.3721
	0.4	0.9113	0.6347	0.3290	0.2110	-0.3345	-3.0212	0.4952	0.4668
	0.5	1.3696	0.9354	0.1188	0.0897	-0.8845	-3.9043	-0.1002	0.6335

Table 6.1.b Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy SECP specimen for $\omega = 22.5^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.2117	0.1456	0.1489	0.0945	-0.1143	-0.6406	0.2453	0.1760
	0.2	0.2027	0.1404	0.1470	0.0936	-0.0912	-0.6441	0.2353	0.1853
	0.3	0.1970	0.1373	0.1375	0.0887	-0.0702	-0.6568	0.1967	0.2123
	0.4	0.2019	0.1413	0.1150	0.0765	-0.0601	-0.6929	0.1156	0.2588
	0.5	0.2320	0.1623	0.0743	0.0542	-0.0715	-0.7920	-0.0244	0.3323
30	0.1	0.4287	0.3031	0.4306	0.2656	-0.0644	-1.5764	0.8661	0.2475
	0.2	0.4522	0.3180	0.4145	0.2563	-0.1021	-1.6055	0.8190	0.2627
	0.3	0.4979	0.3480	0.3761	0.2341	-0.1584	-1.6913	0.7122	0.2901
	0.4	0.5819	0.4040	0.3025	0.1912	-0.2391	-1.8864	0.5135	0.3320
	0.5	0.7457	0.5145	0.1746	0.1168	-0.3739	-2.3054	0.1634	0.4132
60	0.1	0.7218	0.5177	0.6789	0.4135	0.0443	-2.9084	1.4738	0.2097
	0.2	0.7809	0.5549	0.6499	0.3967	-0.0587	-2.9687	1.3921	0.2319
	0.3	0.9082	0.6369	0.5847	0.3589	-0.2440	-3.1599	1.2099	0.2795
	0.4	1.1103	0.7698	0.4685	0.2906	-0.4796	-3.5608	0.9090	0.3249
	0.5	1.5283	1.0461	0.2685	0.1751	-0.9387	-4.4373	0.3444	0.4806

Table 6.1.c Normalised values of reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy SECP specimen for $\omega = 45^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.6984	0.4688	0.4131	0.2580	-0.6197	-1.7096	0.7638	0.3491
	0.2	0.6978	0.4689	0.4092	0.2560	-0.6087	-1.7257	0.7478	0.3607
	0.3	0.6980	0.4703	0.3958	0.2488	-0.5836	-1.7685	0.7000	0.3876
	0.4	0.6995	0.4735	0.3670	0.2327	-0.5393	-1.8480	0.6082	0.4276
	0.5	0.7077	0.4825	0.3170	0.2043	-0.4745	-1.9883	0.4569	0.4833
30	0.1	1.1045	0.7651	0.8023	0.4870	-0.4906	-3.5198	1.7740	0.1939
	0.2	1.1389	0.7874	0.7757	0.4714	-0.5385	-3.5749	1.7046	0.2053
	0.3	1.1958	0.8250	0.7297	0.4442	-0.6016	-3.6933	1.5882	0.2183
	0.4	1.2789	0.8808	0.6600	0.4026	-0.6738	-3.8990	1.4193	0.2261
	0.5	1.4060	0.9675	0.5552	0.3403	-0.7585	-4.2572	1.1610	0.2452
60	0.1	1.5641	1.0902	1.0843	0.6425	-0.5558	-5.2157	2.7216	-0.2778
	0.2	1.6250	1.1298	1.0371	0.6139	-0.6369	-5.3198	2.6156	-0.2866
	0.3	1.7458	1.2090	0.9646	0.5708	-0.7842	-5.5487	2.4362	-0.2723
	0.4	1.8934	1.3063	0.8593	0.5051	-0.9516	-5.8495	2.2419	-0.3620
	0.5	2.1689	1.4884	0.7076	0.4137	-1.2533	-6.4283	1.8915	-0.3737

Table 6.2.a Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy CCP specimen for $\omega = 0^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	-0.0018	-0.0011	0	0	0.0038	0.0007	0	0
	0.2	-0.0082	-0.0048	0.0001	0.0001	0.0218	-0.0041	-0.0006	0.0014
	0.3	-0.0200	-0.0116	0.0003	0.0002	0.0554	-0.0137	0.0003	0.0007
	0.4	-0.0392	-0.0227	0.0009	0.0005	0.1093	-0.0282	0.0029	-0.0014
	0.5	-0.0701	-0.0403	0.0020	0.0011	0.2015	-0.0605	0.0068	-0.0034
30	0.1	-0.0056	-0.0034	-0.0013	-0.0008	0.0124	0.0014	-0.0027	-0.0007
	0.2	-0.0238	-0.0140	-0.0051	-0.0030	0.0619	-0.0096	-0.0133	0.0021
	0.3	-0.0531	-0.0309	-0.0151	-0.0090	0.1449	-0.0330	-0.0368	0.0021
	0.4	-0.0948	-0.0547	-0.0374	-0.0223	0.2683	-0.0749	-0.0910	0.0048
	0.5	-0.1491	-0.0850	-0.0799	-0.0475	0.4433	-0.1533	-0.1973	0.0151
60	0.1	-0.0166	-0.0098	-0.0032	-0.0019	0.0424	-0.0055	-0.0080	0.0007
	0.2	-0.0625	-0.0363	-0.0120	-0.0071	0.1720	-0.0412	-0.0303	0.0034
	0.3	-0.1369	-0.0793	-0.0374	-0.0221	0.3811	-0.0976	-0.0951	0.0117
	0.4	-0.2438	-0.1407	-0.0901	-0.0535	0.6895	-0.1917	-0.2238	0.0192
	0.5	-0.3847	-0.2196	-0.1906	-0.1127	1.1377	-0.3855	-0.4833	0.0570

Table 6.2.b Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy CCP specimen for $\omega = 22.5^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	-0.0029	-0.0017	-0.0015	-0.0007	0.0077	-0.0014	-0.0077	0.0069
	0.2	-0.0096	-0.0055	-0.0063	-0.0023	0.0280	-0.0089	-0.0454	0.0509
	0.3	-0.0283	-0.0164	-0.0134	-0.0076	0.0786	-0.0199	-0.0406	0.0151
	0.4	-0.0545	-0.0315	-0.0276	-0.0159	0.1532	-0.0412	-0.0786	0.0227
	0.5	-0.0953	-0.0551	-0.0525	-0.0306	0.2674	-0.0715	-0.1423	0.0309
30	0.1	-0.0071	-0.0041	-0.0008	-0.0005	0.0200	-0.0055	-0.0015	-0.0007
	0.2	-0.0262	-0.0151	-0.0033	-0.0021	0.0745	-0.0213	-0.0053	-0.0041
	0.3	-0.0584	-0.0339	-0.0142	-0.0086	0.1611	-0.0392	-0.0318	-0.0027
	0.4	-0.0977	-0.0564	-0.0401	-0.0240	0.2760	-0.0763	-0.0957	0.0021
	0.5	-0.1444	-0.0824	-0.0886	-0.0526	0.4276	-0.1457	-0.2203	0.0192
60	0.1	-0.0168	-0.0098	0.0015	0.0008	0.0454	-0.0096	0.0056	-0.0034
	0.2	-0.0534	-0.0329	-0.0029	-0.0019	0.1081	0.0296	-0.0035	-0.0055
	0.3	-0.1207	-0.0700	-0.0153	-0.0095	0.3343	-0.0832	-0.0295	-0.0110
	0.4	-0.1836	-0.1055	-0.0545	-0.0330	0.5287	-0.1601	-0.1222	-0.0103
	0.5	-0.2344	-0.1321	-0.1234	-0.0729	0.7284	-0.2934	-0.3143	0.0392

Table 6.2.c Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy CCP specimen for $\omega = 45^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	-0.0052	-0.0032	-0.0016	-0.0010	0.0106	0.0027	-0.0029	-0.0014
	0.2	-0.0211	-0.0124	-0.0057	-0.0022	0.0551	-0.0089	-0.0386	0.0419
	0.3	-0.0487	-0.0284	-0.0138	-0.0061	0.1317	-0.0282	-0.0775	0.0749
	0.4	-0.0909	-0.0529	-0.0288	-0.0138	0.2480	-0.0564	-0.1396	0.1196
	0.5	-0.1510	-0.0880	-0.0536	-0.0255	0.4094	-0.0893	-0.2636	0.2288
30	0.1	-0.0065	-0.0038	-0.0007	-0.0003	0.0174	-0.0034	-0.0041	0.0041
	0.2	-0.0272	-0.0157	-0.0016	-0.0005	0.0769	-0.0213	-0.0133	0.0158
	0.3	-0.0583	-0.0338	-0.0046	-0.0020	0.1617	-0.0405	-0.0265	0.0261
	0.4	-0.0958	-0.0556	-0.0205	-0.0119	0.2645	-0.0646	-0.0565	0.0137
	0.5	-0.1354	-0.0781	-0.0511	-0.0249	0.3838	-0.1079	-0.2392	0.1979
60	0.1	-0.0140	-0.0082	0.0024	0.0013	0.0371	-0.0069	0.0085	-0.0048
	0.2	-0.0444	-0.0257	0.0091	0.0040	0.1240	-0.0323	0.0515	-0.0502
	0.3	-0.0869	-0.0506	0.0046	0.0016	0.2365	-0.0529	0.0348	-0.0399
	0.4	-0.1174	-0.0675	-0.0104	-0.0061	0.3372	-0.1010	-0.0274	0.0048
	0.5	-0.1254	-0.0697	-0.0432	-0.0210	0.4097	-0.1904	-0.2032	0.1691

Table 6.3.a Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy DECP specimen for $\omega = 0^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.1311	0.0913	0	0	-0.0472	-0.4360	0	0
	0.2	0.1301	0.0908	0	0	-0.0440	-0.4376	0	0
	0.3	0.1282	0.0897	0	0	-0.0378	-0.4405	0	0
	0.4	0.1261	0.0888	0	0	-0.0266	-0.4510	0	0
	0.5	0.1245	0.0884	0	0	-0.0124	-0.4684	0	0
30	0.1	0.2844	0.2022	0.2566	0.1589	-0.0202	-1.0833	0.5016	0.1715
	0.2	0.2893	0.2052	0.2552	0.1581	-0.0307	-1.0849	0.4975	0.1729
	0.3	0.3013	0.2125	0.2511	0.1558	-0.0556	-1.0904	0.4843	0.1787
	0.4	0.3207	0.2249	0.2426	0.1511	-0.0870	-1.1145	0.4573	0.1904
	0.5	0.3443	0.2403	0.2296	0.1438	-0.1161	-1.1586	0.4157	0.2087
60	0.1	0.6529	0.4669	0.5113	0.3156	0.0108	-2.5820	1.0222	0.3042
	0.2	0.6677	0.4759	0.5087	0.3142	-0.0218	-2.5856	1.0140	0.3075
	0.3	0.7034	0.4977	0.5009	0.3098	-0.0975	-2.5998	0.9894	0.3180
	0.4	0.7557	0.5310	0.4852	0.3010	-0.1811	-2.6659	0.9404	0.3380
	0.5	0.8178	0.5717	0.4584	0.2858	-0.2567	-2.7838	0.8580	0.3702

Table 6.3.b Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy DECP specimen for $\omega = 22.5^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.2181	0.1495	0.1482	0.0940	-0.1301	-0.6397	0.2434	0.1762
	0.2	0.2197	0.1505	0.1473	0.0935	-0.1321	-0.6425	0.2405	0.1775
	0.3	0.2208	0.1514	0.1445	0.0920	-0.1296	-0.6509	0.2310	0.1827
	0.4	0.2209	0.1519	0.1397	0.0894	-0.1202	-0.6670	0.2136	0.1927
	0.5	0.2202	0.1523	0.1333	0.0860	-0.1026	-0.6937	0.1896	0.2075
30	0.1	0.4235	0.2999	0.4335	0.2672	-0.0545	-1.5722	0.8752	0.2438
	0.2	0.4318	0.3050	0.4295	0.2649	-0.0709	-1.5775	0.8632	0.2477
	0.3	0.4494	0.3160	0.4214	0.2603	-0.1035	-1.5925	0.8391	0.2565
	0.4	0.4788	0.3346	0.4080	0.2527	-0.1525	-1.6262	0.7997	0.2696
	0.5	0.5191	0.3606	0.3879	0.2411	-0.2095	-1.6893	0.7416	0.2872
60	0.1	0.7082	0.5093	0.6845	0.4166	0.0716	-2.9004	1.4915	0.2023
	0.2	0.7250	0.5197	0.6770	0.4123	0.0375	-2.9096	1.4696	0.2093
	0.3	0.7579	0.5401	0.6625	0.4039	-0.0248	-2.9348	1.4291	0.2200
	0.4	0.8126	0.5746	0.6395	0.3906	-0.1207	-2.9900	1.3648	0.2366
	0.5	0.8967	0.6284	0.6040	0.3697	-0.2511	-3.1030	1.2719	0.2521

Table 6.3.c Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for graphite/epoxy DECP specimen for $\omega = 45^\circ$.

ψ°	a/W	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0	0.1	0.6992	0.4694	0.4127	0.2579	-0.6186	-1.7149	0.7592	0.3552
	0.2	0.7038	0.4722	0.4114	0.2572	-0.6286	-1.7161	0.7557	0.3561
	0.3	0.7144	0.4788	0.4083	0.2555	-0.6490	-1.7240	0.7460	0.3602
	0.4	0.7295	0.4885	0.4026	0.2523	-0.6717	-1.7452	0.7287	0.3667
	0.5	0.7458	0.4994	0.3930	0.2468	-0.6860	-1.7852	0.6994	0.3776
30	0.1	1.0907	0.7581	0.8102	0.4930	-0.4320	-3.5633	1.7672	0.2365
	0.2	1.1004	0.7642	0.8030	0.4888	-0.4481	-3.5742	1.7474	0.2409
	0.3	1.1163	0.7744	0.7915	0.4821	-0.4722	-3.5965	1.7166	0.2472
	0.4	1.1401	0.7898	0.7759	0.4730	-0.5068	-3.6325	1.6751	0.2551
	0.5	1.1747	0.8121	0.7554	0.4609	-0.5546	-3.6885	1.6221	0.2628
60	0.1	1.4542	1.0215	1.0467	0.6279	-0.3535	-5.1213	2.4676	-0.0023
	0.2	1.4718	1.0326	1.0356	0.6215	-0.3859	-5.1366	2.4370	0.0052
	0.3	1.5053	1.0540	1.0146	0.6093	-0.4390	-5.1798	2.3803	0.0172
	0.4	1.5417	1.0770	0.9977	0.5998	-0.5008	-5.2197	2.3265	0.0405
	0.5	1.5983	1.1150	0.9482	0.5694	-0.5499	-5.3601	2.2244	0.0164

Table 6.4 Values for the real components of $(\mu_1\mu_2)$ and $(\mu_1+\mu_2)$ along the local co-ordinates, x_i , in a graphite/epoxy composite.

ω°	ψ°	Re $(\mu_1\mu_2)$	Re $(\mu_1+\mu_2)$
0	0	-0.2843	0
	30	-0.3835	0.4784
	60	-0.9198	1.2745
22.5	0	-0.3377	-0.3429
	30	-0.2899	0.1084
	60	-0.4499	0.6334
45	0	-0.5463	-0.8142
	30	-0.3074	-0.2212
	60	-0.3074	0.2212

Table 6.5.a Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct (M -integral) methods for SECP under remote tension for $\omega = 0^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.1606	-0.1608	-0.12	1.1648	0
	0.2	-0.1711	-0.1713	-0.12	1.3454	0
	0.3	-0.1755	-0.1757	-0.11	1.6360	0
	0.4	-0.1636	-0.1637	-0.06	2.0805	0
	0.5	-0.1141	-0.1143	-0.18	2.7790	0
30	0.1	-0.1016	-0.1018	-0.20	1.3418	-0.2285
	0.2	-0.0986	-0.0989	-0.30	1.5365	-0.2652
	0.3	-0.0734	-0.0739	-0.68	1.8644	-0.3310
	0.4	0.0006	-0.0004	-----	2.3796	-0.4402
	0.5	0.1816	0.1794	1.21	3.2043	-0.6229
60	0.1	-0.2752	-0.2752	0.00	1.2940	-0.3580
	0.2	-0.2659	-0.2657	0.08	1.4917	-0.4199
	0.3	-0.1982	-0.1984	-0.10	1.8137	-0.5277
	0.4	-0.0085	-0.0085	0.00	2.3165	-0.7060
	0.5	0.4501	0.4498	0.07	3.1117	-1.0016

Table 6.5.b Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct (M -integral) methods for SECP under remote tension for $\omega = 22.5^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.1348	-0.1350	-0.15	1.0010	0.3651
	0.2	-0.1417	-0.1420	-0.20	1.1368	0.4135
	0.3	-0.1428	-0.1435	-0.48	1.3518	0.4893
	0.4	-0.1301	-0.1313	-0.91	1.6705	0.6000
	0.5	-0.0887	-0.0912	-2.88	2.1490	0.7645
30	0.1	0.1514	0.1510	0.23	1.2890	0.2005
	0.2	0.1771	0.1767	0.23	1.4527	0.2221
	0.3	0.2297	0.2293	0.16	1.7251	0.2545
	0.4	0.3275	0.3270	0.16	2.1460	0.2990
	0.5	0.5124	0.5121	0.06	2.7903	0.3602
60	0.1	0.3647	0.3625	0.62	1.3666	-0.0531
	0.2	0.4247	0.4231	0.37	1.5515	-0.0686
	0.3	0.5492	0.5549	-1.04	1.8583	-0.0997
	0.4	0.7636	0.7684	-0.63	2.3210	-0.1585
	0.5	1.1826	1.1960	-1.13	3.0296	-0.2544

Table 6.5.c Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct methods (M -integral) for SECP under remote tension for $\omega = 45^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.0374	-0.0375	-0.48	0.6092	0.5027
	0.2	-0.0357	-0.0359	-0.68	0.6614	0.5447
	0.3	-0.0288	-0.0291	-1.20	0.7414	0.6082
	0.4	-0.0131	-0.0140	-6.55	0.8530	0.6953
	0.5	0.0167	0.0152	8.97	1.0040	0.8119
30	0.1	0.3873	0.3868	0.13	0.9066	0.4789
	0.2	0.4178	0.4173	0.12	0.9790	0.5152
	0.3	0.4694	0.4688	0.12	1.0981	0.5730
	0.4	0.5458	0.5452	0.12	1.2670	0.6525
	0.5	0.6617	0.6611	0.09	1.5035	0.7613
60	0.1	0.6960	0.6968	-0.11	1.0927	0.2704
	0.2	0.7526	0.7509	0.23	1.1832	0.2892
	0.3	0.8497	0.8475	0.26	1.3402	0.3178
	0.4	0.9815	0.9740	0.77	1.5444	0.3511
	0.5	1.1897	1.1876	0.18	1.8400	0.3961

Table 6.6.a Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct (M -integral) methods for CCP under uniform remote tension for $\omega = 0^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.2858	-0.2861	-0.11	1.0046	0
	0.2	-0.2920	-0.2925	-0.19	1.0211	0
	0.3	-0.3034	-0.3042	-0.27	1.0506	0
	0.4	-0.3225	-0.3235	-0.31	1.0967	0
	0.5	-0.3534	-0.3544	-0.26	1.1657	0
30	0.1	-0.3892	-0.3891	0.03	1.0116	-0.0038
	0.2	-0.4084	-0.4074	0.24	1.0482	-0.0150
	0.3	-0.4396	-0.4367	0.66	1.1097	-0.0329
	0.4	-0.4836	-0.4783	1.08	1.1991	-0.0574
	0.5	-0.5426	-0.5327	1.83	1.3218	-0.0878
60	0.1	-0.9373	-0.9364	0.10	1.0102	-0.0059
	0.2	-0.9852	-0.9823	0.29	1.0401	-0.0216
	0.3	-1.0633	-1.0567	0.62	1.0913	-0.0480
	0.4	-1.1755	-1.1637	1.00	1.1679	-0.0831
	0.5	-1.3255	-1.3045	1.58	1.2747	-0.1282

Table 6.6.b Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct (M -integral) methods for CCP under uniform remote tension for $\omega = 22.5^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.2647	-0.2649	-0.10	0.8572	0.3547
	0.2	-0.2709	-0.2713	-0.14	0.8710	0.3601
	0.3	-0.2820	-0.2824	-0.15	0.8935	0.3659
	0.4	-0.2994	-0.2998	-0.11	0.9288	0.3770
	0.5	-0.3257	-0.3258	-0.04	0.9800	0.3935
30	0.1	-0.0684	-0.0685	-0.11	0.8677	0.3542
	0.2	-0.0847	-0.0848	-0.10	0.9108	0.3573
	0.3	-0.1075	-0.1075	-0.05	0.9818	0.3617
	0.4	-0.1321	-0.1319	0.12	1.0753	0.3698
	0.5	-0.1548	-0.1546	0.13	1.1997	0.3819
60	0.1	-0.0287	-0.0285	0.65	0.8717	0.3500
	0.2	-0.0640	-0.0639	0.23	0.9281	0.3348
	0.3	-0.1113	-0.1112	0.09	1.0088	0.3237
	0.4	-0.1512	-0.1511	0.04	1.1154	0.3072
	0.5	-0.1703	-0.1700	0.15	1.2459	0.2940

Table 6.6.c Normalised T -stress solutions, T/σ_0 , obtained from the weight function and the direct (M -integral) methods for CCP under uniform remote tension for $\omega = 45^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.1819	-0.1820	-0.10	0.5015	0.5009
	0.2	-0.1877	-0.1879	-0.09	0.5080	0.5055
	0.3	-0.1975	-0.1976	-0.07	0.5188	0.5130
	0.4	-0.2113	-0.2113	0.01	0.5344	0.5240
	0.5	-0.2291	-0.2289	0.10	0.5555	0.5391
30	0.1	0.2326	0.2328	-0.07	0.5070	0.5037
	0.2	0.2228	0.2229	-0.07	0.5340	0.5173
	0.3	0.2087	0.2089	-0.06	0.5820	0.5410
	0.4	0.1979	0.1980	-0.05	0.6389	0.5700
	0.5	0.1935	0.1936	-0.04	0.7080	0.6063
60	0.1	0.4486	0.4487	-0.04	0.5175	0.5019
	0.2	0.4301	0.4302	-0.02	0.5594	0.5161
	0.3	0.4108	0.4111	-0.08	0.6278	0.5232
	0.4	0.4032	0.4034	-0.05	0.6996	0.5443
	0.5	0.4155	0.4158	-0.09	0.8020	0.5649

Table 6.7.a Normalised T -stress solutions, T/σ_0 , obtained from weight function and the direct (M -integral) methods for DECP under uniform remote tension for $\omega = 0^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.1529	-0.1532	-0.21	1.0877	0
	0.2	-0.1538	-0.1542	-0.20	1.0833	0
	0.3	-0.1559	-0.1561	-0.10	1.0903	0
	0.4	-0.1582	-0.1581	0.07	1.1127	0
	0.5	-0.1605	-0.1597	0.45	1.1553	0
30	0.1	-0.0987	-0.0991	-0.36	1.2795	-0.2165
	0.2	-0.0940	-0.0942	-0.27	1.2966	-0.2140
	0.3	-0.0821	-0.0823	-0.28	1.3134	-0.2036
	0.4	-0.0626	-0.0628	-0.40	1.3259	-0.1832
	0.5	-0.0389	-0.0392	-0.93	1.3422	-0.1536
60	0.1	-0.2668	-0.2669	-0.06	1.2273	-0.3369
	0.2	-0.2521	-0.2522	-0.02	1.2435	-0.3336
	0.3	-0.2168	-0.2164	0.18	1.2591	-0.3176
	0.4	-0.1642	-0.1641	0.04	1.2743	-0.2853
	0.5	-0.1021	-0.1020	0.13	1.2941	-0.2376

Table 6.7.b Normalised T -stress solutions, T/σ_0 , obtained from weight function and the direct (M -integral) methods for DECP under uniform remote tension for $\omega = 22.5^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.1289	-0.1292	-0.24	0.9469	0.3462
	0.2	-0.1274	-0.1276	-0.17	0.9457	0.3489
	0.3	-0.1258	-0.1257	0.05	0.9490	0.3555
	0.4	-0.1246	-0.1239	0.52	0.9609	0.3667
	0.5	-0.1238	-0.1222	1.24	0.9858	0.3832
30	0.1	0.1461	0.1455	0.39	1.2455	0.1942
	0.2	0.1545	0.1540	0.34	1.2815	0.2015
	0.3	0.1724	0.1719	0.25	1.3232	0.2145
	0.4	0.2023	0.2018	0.27	1.3567	0.2354
	0.5	0.2444	0.2432	0.49	1.3731	0.2670
60	0.1	0.3492	0.3488	0.12	1.3114	-0.0490
	0.2	0.3662	0.3658	0.11	1.3530	-0.0495
	0.3	0.3994	0.3990	0.10	1.4069	-0.0466
	0.4	0.4545	0.4539	0.14	1.4602	-0.0343
	0.5	0.5382	0.5383	-0.01	1.4966	-0.0028

Table 6.7.c Normalised T -stress solutions, T/σ_0 , obtained from weight function and the direct (M -integral) method for DECP under uniform remote tension for $\omega = 45^\circ$.

ψ°	a/W	T/σ_0 (Direct)	T/σ_0 (Weight function)	%Diff	$\frac{K_I}{\sigma_0 \sqrt{\pi a}}$	$\frac{K_{II}}{\sigma_0 \sqrt{\pi a}}$
0	0.1	-0.0366	-0.0369	-0.90	0.5925	0.4893
	0.2	-0.0340	-0.0340	-0.21	0.5985	0.4954
	0.3	-0.0277	-0.0272	1.81	0.6044	0.5034
	0.4	-0.0181	-0.0168	7.19	0.6091	0.5132
	0.5	-0.0063	-0.0038	39.43	0.6131	0.5252
30	0.1	0.3769	0.3760	0.24	0.8882	0.4701
	0.2	0.3848	0.3844	0.11	0.9080	0.4804
	0.3	0.3986	0.3981	0.13	0.9411	0.4970
	0.4	0.4179	0.4178	0.03	0.9832	0.5184
	0.5	0.4457	0.4454	0.08	1.0315	0.5450
60	0.1	0.6617	0.6607	0.15	1.0679	0.2674
	0.2	0.6766	0.6750	0.23	1.0932	0.2738
	0.3	0.7028	0.7023	0.08	1.1361	0.2824
	0.4	0.7327	0.7289	0.52	1.1872	0.2953
	0.5	0.7827	0.7820	0.10	1.2543	0.3084

Table 6.8 Normalised reference T -stress solutions, T_1/σ_{22}^* , T_2/σ_{22}^* , T_3/σ_{12}^* and T_4/σ_{12}^* , and weight function parameters, D_i , for different sizes of cracks emanating from a circular hole in graphite/epoxy plate when $\psi = 30^\circ$.

a/r	$\frac{T_1}{\sigma_{22}^*}$	$\frac{T_2}{\sigma_{22}^*}$	$\frac{T_3}{\sigma_{12}^*}$	$\frac{T_4}{\sigma_{12}^*}$	D_1	D_2	D_3	D_4
0.1	0.2336	0.1665	0.2210	0.1382	-0.0081	-0.9040	0.4061	0.1909
0.2	0.1773	0.1278	0.1813	0.1150	0.0240	-0.7362	0.3001	0.2118
0.3	0.1277	0.0937	0.1394	0.0898	0.0506	-0.5859	0.2011	0.2121
0.4	0.0939	0.0696	0.1114	0.0729	0.0524	-0.4562	0.1379	0.2076
0.5	0.0725	0.0545	0.0913	0.0622	0.0559	-0.3780	0.0612	0.2565
0.6	0.0590	0.0448	0.0779	0.0539	0.0547	-0.3227	0.0358	0.2463
0.7	0.0477	0.0368	0.0684	0.0477	0.0557	-0.2802	0.0229	0.2304
0.8	0.0404	0.0313	0.0604	0.0426	0.0497	-0.2414	0.0112	0.2183
0.9	0.0333	0.0262	0.0552	0.0390	0.0506	-0.2149	0.0094	0.2012
1.0	0.0286	0.0225	0.0491	0.0350	0.0443	-0.1859	0.0012	0.1909

Table 6.9 Normalised T -stress, T/σ_0 , for a crack emanating from a circular hole in a graphite/epoxy plate subjected to uniform remote tension for $\psi = 30^\circ$; comparison of the results obtained from the weight function method and the direct M -integral method.

a/r	T/σ_0 (Direct approach)	T/σ_0 (Weight function)	%Diff
0.1	-0.2528	-0.2438	3.55
0.2	-0.1207	-0.1221	-1.15
0.3	-0.0977	-0.1005	-2.90
0.4	-0.1247	-0.1281	-2.66
0.5	-0.1599	-0.1601	-0.12
0.6	-0.1868	-0.1871	-0.14
0.7	-0.2139	-0.2140	-0.04
0.8	-0.2379	-0.2348	1.31
0.9	-0.2568	-0.2549	0.75
1	-0.2736	-0.2709	0.99

FIGURES

Figure 2.1 Mode I (Opening mode).

Figure 2.2 Mode II (Sliding mode).

Figure 2.3 Distribution of σ_{22} along $\phi = 0^\circ$.

Figure 2.4 J -integral contour around a crack tip.

Figure 2.5 The variation of the peak triaxiality in front of a plane strain blunting crack with respect to the normalised T -stress, T/σ_0 , σ_0 being the yield stress. (Hancock *et al.*, 1993).

Figure 2.6 Plastic zone around the crack tip obtained from modified boundary analysis for normalised negative T -stresses, T/σ_0 , σ_0 being the yield stress. (Shih *et al.*, 1993).

Figure 2.7 Plastic zone around the crack tip obtained from modified boundary analysis for normalised positive T -stresses, T/σ_0 , σ_0 being the yield stress. (Shih *et al.*, 1993).

Figure 2.8 Effect of T -stress on the fracture toughness K_{mat}^c (Ainsworth *et al.*, 2000).

Figure 2.9 The R6 failure assessment diagram (FAD) without the consideration of the constraint (Ainsworth and O'Dowd, 1995).

Figure 2.10. Constraint based FAD (Ainsworth *et al.*, 2000).

Figure 2.11 Crack tunnelling phenomenon in a CN specimen under fatigue loading (Waspaloy, 650°C); Byrne *et al.* (1993).

Figure 2.12 Tunnelled corner crack in CN specimen.

Figure 2.13 Directional stability in crack growth.

Figure 2.14 Point load f applied at the crack tip parallel to the crack plane.

Figure 2.15 The superposition principle for T -stress evaluation.

Figure 2.16 Weight function $t(x_1')$ at $x_1 = a$ due to point forces $P = 1$ for (a) a single edge cracked plate (SECP); (b) centre cracked plate (CCP); and (c) double edge cracked plate (DECP).

Figure 2.17 Contour Γ_0 around the crack tip and material designation for the bimaterial interface crack.

Figure 2.18 Point load f applied at the crack tip parallel to the crack plane between dissimilar materials.

Figure 3.1 Transformation of the co-ordinate system.

Figure 3.2 A typical BEM mesh along the boundary S ; contribution of point Q_m on element m on node P_n .

Figure 3.3 Intrinsic coordinates for a quadratic isoparametric element.

Figure 3.4 Shape functions for a quadratic isoparametric element.

Figure 3.5 (a) A centre cracked plate and (b) the subregioning technique applied for its BEM model.

Figure 3.6 Quarter-point crack-tip element, crack tip node being node 1.

Figure 3.7 Crack-tip elements along the crack flank and crack plane.

Figure 3.8 (a) An interior point p near the boundary S and (b) the conventional refined BEM meshing scheme along the boundary.

Figure 3.9 (a) A slender cantilever beam with a unit shear stress, σ_{12} , at the free end, (b) BEM mesh employed and (c) enlarged view of the interior points close to the boundary.

Figure 3.10 (a) An anisotropic plate with tensile loading, σ_0 , of unit magnitude. (b) BEM mesh employed; $\psi = 30^\circ$.

Figure 4.1 Auxiliary field (B) problem: a point force f applied at the crack tip of a homogeneous anisotropic body along the crack plane.

Figure 4.2 A crack model with the definition of crack length, a , quarter-point crack-tip element length, l , and contour radius, r .

Figure 4.3 (a) An inclined centre crack in a glass/epoxy plate under uniform unit tension, (b) BEM mesh employed; $\sigma_0 = 1$, $a/W = 0.2$, $H/W = 2$.

Figure 4.4 A CCP with an inclined crack under remote uniform tension, σ_0 , with (a) isotropic and (b) orthotropic materials. (c) A typical BEM mesh employed.

Figure 4.5 A single edge cracked plate (SECP) under remote load σ_0 , (b) BEM mesh implemented.

Figure 4.6 A centre cracked plate (CCP) under remote load σ_0 , (b) BEM mesh implemented.

Figure 4.7 A double edge cracked plate (DECP) under remote load σ_0 , (b) BEM mesh implemented.

(a)

(b)

Figure 4.8 Variation of normalised T -stress, T/σ_0 , with the relative crack size, a/W , for a SECP under remote tension, σ_0 ; $\omega = 45^\circ$ and $\psi = 30^\circ$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.9 Variation of normalised T -stress, T/σ_0 , with the relative crack size, a/W , for a CCP under remote tension, σ_0 ; $\omega = 45^\circ$ and $\psi = 30^\circ$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.10 Variation of normalised T -stress, T/σ_0 , with the relative crack size, a/W , for a DECP under remote tension, σ_0 ; $\omega = 45^\circ$ and $\psi = 30^\circ$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.11 Variation of normalised T -stress, T/σ_0 , with material orientation, ψ , for a SECP under remote tension, σ_0 ; $\omega = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.12 Variation of normalised T -stress, T/σ_0 , with material orientation, ψ , for a CCP under remote tension, σ_0 ; $\omega = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.13 Variation of normalised T -stress, T/σ_0 , with material orientation, ψ , for a DECP under remote tension, σ_0 ; $\omega = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.14 Variation of normalised T -stress, T/σ_0 , with the crack inclination, ω , for a SECP under remote tension, σ_0 ; $\psi = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.15 Variation of normalised T -stress, T/σ_0 , with the crack inclination, ω , for a CCP under remote tension, σ_0 ; $\psi = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

(a)

(b)

Figure 4.16 Variation of normalised T -stress, T/σ_0 , with the crack inclination, ω , for a DECP under remote tension, σ_0 ; $\psi = 45^\circ$ and $a/W = 0.1$ (see table 4.4 for material case numbers).

Figure 4.17 (a) Crack emanating from a circular hole in a rectangular graphite/epoxy plate; $H/W = 2$, $r/W = 1$. (b) A typical BEM mesh employed.

(a)

(b)

Figure 4.18 (a) Variation of normalised T -stress, T/σ_0 , with relative crack size, a/r , for a crack emanating from a circular hole in a graphite/epoxy plate under remote tension, σ_0 , for different material orientations, ψ , (b) zoomed view of the cases when $\psi \leq 45^\circ$.

Figure 4.19 Variation of normalised T -stress, T/σ_0 , with material orientation, ψ , in a crack emanating from a circular hole in a graphite/epoxy plate under remote tension, σ_0 , for different relative crack sizes, a/r .

Figure 5.1 A composite wedge comprising of two anisotropic materials and point force f acting on the apex; auxiliary field problem.

Figure 5.2 Problem (A): (a) a centre cracked plate (CCP) under remote load σ_0 (b) BEM mesh employed for $a/W = 0.5$.

Figure 5.3 Variation of the normalised T -stress, T/σ_0 , with the relative crack length, a/W , for Problem (B), for the cases when both of the dissimilar materials are orthotropic and isotropic.

Figure 5.4 Problem (C): (a) Silicon [110] disc with dissimilar halves containing a central crack along the interface, under uniform radial tension, σ_0 (b) BEM mesh and contour integral paths for $a/r = 0.5$.

(a)

(b)

Figure 5.5 Variation of the normalised T -stress, T/σ_0 , with angle, ψ , in a centrally cracked bimaterial silicon [110] disc, for Problem (C): (a) at crack tip A (b) at crack tip B.

Figure 5.6 Variation of the normalised T -stress, T/σ_0 , with relative crack size, a/r , in centrally cracked bimaterial silicon [110] disc, for Problem (C): (a) at crack tip A and (b) at crack tip B.

Figure 5.7 Problem (D): (a) a large plate with a central crack located along the interface between dissimilar anisotropic rocks (b) typical BEM mesh employed.

(a)

(b)

Figure 5.8 Variation of the normalised T -stress, T/σ_0 , with the angle of the interface crack inclination, ω , for a large rectangular domain comprising sandstone and dolomite, for Problem (D), (a) at crack tip A and (b) at crack tip B.

Figure 5.9 Problem (E): a bicrystal with an edge crack under (a) pure bending and (b) uniform tension. (c) A typical BEM mesh employed; the left half is considered as material 2 for T -stress calculation.

(a)

(b)

Figure 5.10 Variation of the normalised T -stress, T/σ_0 , with relative crack size, a/W , for different material orientations, ψ , of the interface crack along a YCBO bicrystal junction for the cases of (a) pure bending (b) uniform remote tension; Problem (E).

(a)

(b)

Figure 5.11 Variation of the normalised T -stress, T/σ_0 , with angle, ψ , for different relative crack sizes, a/W , of the interface crack along a YCBO bicrystal junction for the cases of (a) pure bending (b) uniform remote tension; Problem (E).

Figure 6.1 The superposition principle for T -stress evaluation for mixed-mode crack opening.

Figure 6.2.a Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for SECP under uniform remote tension for different crack sizes when $\omega = 0^\circ$.

Figure 6.2.b Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for SECP under uniform remote tension for different crack sizes when $\omega = 22.5^\circ$.

Figure 6.2.c Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for SECP under uniform remote tension for different crack sizes when $\omega = 45^\circ$.

Figure 6.3.a Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for CCP under uniform remote tension for different crack sizes when $\omega = 0^\circ$.

Figure 6.3.b Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for CCP under uniform remote tension for different crack sizes when $\omega = 22.5^\circ$.

Figure 6.3.c Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for CCP under uniform remote tension for different crack sizes when $\omega = 45^\circ$.

Figure 6.4.a Comparison of the normalised T -stress solutions, T/σ_o , obtained from the weight function method and the direct method (M -integral) for DECP under uniform remote tension for different crack sizes when $\omega = 0^\circ$.

Figure 6.4.b Comparison of the normalised T -stress solutions, T/σ_o , obtained from the weight function method and the direct method (M -integral) for DECP under uniform remote tension for different crack sizes when $\omega = 22.5^\circ$.

Figure 6.4.c Comparison of the normalised T -stress solutions, T/σ_0 , obtained from the weight function method and the direct method (M -integral) for DECP under uniform remote tension for different crack sizes when $\omega = 45^\circ$.

Figure 6.5 Variation of the normalised T -stress, T/σ_0 , with a/r for $\psi = 30^\circ$ for crack emanating from a circular hole in graphite/epoxy plate under remote tension, σ_0 ; comparison of the results from weight function method and the direct M -integral approach.