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**Light front formulation of QED - application to
certain tree level processes.**

**Light front formulation of the electro-weak
theory**

by

Marcelo J. Valdés

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Science

Ottawa-Carleton Institute for Physics
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Ottawa, Ontario, Canada

July, 2003

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Thesis Supervisor

Chair, Department of Physics

Carleton University

Date: Sept 16, 2003

Abstract

In the light front formulation of Quantum Field Theory there are present linear infrared singularities which must be addressed. In works by Zhang and Harindranath a prescription is given for the treatment of these singularities. They show that the application of this prescription leads to the cancelation of the linear infrared singularities in loop diagrams. In this work we do a second order perturbative calculation and calculate the cross sections for the following Quantum Electrodynamics processes: $e\gamma \rightarrow e\gamma$, $e^+e^- \rightarrow \mu^+\mu^-$, $e^+e^- \rightarrow e^+e^-$, $e\mu \rightarrow e\mu$ and apply the prescription to cancel the linear infrared singularity, whenever present. We show that the well known expressions of the different cross sections for these processes are exactly reproduced. We also develop the light front formulation of electro-weak unification using the Higgs mechanism and Goldstone bosons for spontaneous breaking of gauge symmetry, in the form of a generalized Hamiltonian system with constraints. We quantize this theory using the one-form method of Faddeev and Jackiw.

Acknowledgments

I wish to thank my thesis supervisor Dr. M. K. Sundaresan for his patience, support and guidance before and during the development of this thesis. My understanding, of modern physics in general and Quantum Field Theory in particular, has evolved and deepened during our very many discussions. He has shown great tolerance when guiding me out of my many misunderstandings of the subject of this thesis.

I wish to thank my wife for her support and encouragement. For lending her attentive ear when I voiced the difficulties I have encountered, during this work, and my bewilderment with the things I have managed to learn. I dedicate this work to her.

I wish to thank my friend Dr. A. Durakovic for his support and encouragement when I decided to return to school, after so many years in the working force, and undertake these studies after my retirement. I dedicate this work to him.

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Chapter 1

Introduction

1.1 Motivation

Theoretical studies of elementary particle processes at high energies are carried out using relativistic Quantum Field Theory (abbreviated as QFT hereafter). As is well known, Feynman, Schwinger, Tomonaga, Dyson and others developed Quantum Electrodynamics (abbreviated as QED hereafter), which has been successfully applied to a number of physical processes and the results are in extremely good agreement with experiments. As the electron-photon coupling is weak, characterized by the fine structure constant $\alpha = e^2/4\pi \approx 1/137$, a fully covariant perturbation theory has been developed, which allows one to calculate processes to as high an order as calculational resources allow. Notwithstanding its many successes, there is an area where not much progress has been made. This concerns bound states of elementary particles. Such

problems are beyond the reach of perturbation theory and non-perturbative treatments have to be developed. One of the attempts in this direction has been the development of methods using light front quantization of gauge field theories, notably by S. Brodsky (SLAC) and coworkers [15], and by K. Wilson (Ohio State University) and coworkers [18]. Although a beginning has been made, there are many problems which have to be solved before one can treat bound states successfully. In this thesis we will be concerned with the treatment of certain singularities associated with momentum zero sector, which arise in working with light front formulation of gauge field theories. We explain below how these singularities arise and describe a prescription that has been proposed to handle them. We check how well the prescription works by using it in the calculation of cross sections for a number of specific physical processes for which the results are well known by using the usual Feynman diagrammatic techniques. In a search of the published literature we have not found any detailed calculations of cross sections for tree level processes using the LF formulation. We fill this gap in this thesis. The fact that we reproduce well known results by light front methods confirms the validity of the proposed prescription.

After presenting our notations and setting the groundwork in chapter 1, we show in chapter 2 details of the light front formulation of QED. Cross sections for Compton scattering (Klein-Nishina formula), Bhabha scattering (scattering of electrons on positrons), $e^+e^- \rightarrow \mu^+\mu^-$, and for $e\mu$ scattering are all derived in chapter 3 and shown to agree

with well known results. In chapter 4 of the thesis, we develop the light front formulation of electro-weak unification using the Higgs mechanism and Goldstone bosons for spontaneous breaking of gauge symmetry.

1.1.1 Historical Overview and Introduction to Light Front Coordinates

In the usual formulation of field theories, fields are functions of the coordinates x^μ ($x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$), where c is the velocity of light (which adopting the natural system of units, $\hbar = c = 1$, we will take to be unity) and t the time coordinate, x , y , z are the usual three dimensional coordinates. The procedure to quantize the classical field theory follows the procedure in particle mechanics. We briefly review what we do there.

In particle mechanics, one starts with a Lagrangian L (taken to be the difference of T , the kinetic energy of the particles, and V , the potential energy of the interaction of the particles), which is a function of the generalized coordinates q_i and the generalized velocities $\dot{q}_i$, and the time t . The time integral of L is the *action* having the dimension of $\hbar$ (which will be dimensionless in the natural system of units we adopt). Extremum of the action gives the Euler-Lagrange equations, which are the dynamical equations of motion for the system. These are second order differential equations in time whose solutions give the generalized coordinates for the dynamical system. Canonical momenta conjugate to the generalized coordinates are defined by taking derivatives of

the Lagrangian with respect to the generalized velocities. A Hamiltonian H is defined through a Legendre transformation of the Lagrangian using the canonical momenta. H the Hamiltonian turns out to be equal to the sum of T and V , in other words, the total energy of the system. An equivalent formulation of the dynamical equations are obtained through the Hamiltonian. Instead of one second order differential equation for the generalized coordinates with respect to time, one gets two first order differential equations, one for the time derivative of the generalized coordinate and another for the time derivative of the conjugate momentum. These are referred to as Hamilton's equations. The quantization of the theory is effected by treating the coordinates and conjugate momenta as operators on which commutation relations are imposed. The quantum equations of motion for the coordinates and the momenta, that is, the time derivatives of the coordinates and momenta are found from the commutators of the Hamiltonian with the coordinates and the momenta respectively.

Field theory represents a generalization of discrete particle mechanics to a system with continuous degrees of freedom. The Lagrangian is found from the volume integral of a Lagrangian density, $\mathcal{L}$. The Lagrangian density is a function of the field function and its space and time derivatives. The *action* in this case will involve the integral over the volume as well as over time of the Lagrangian density and hence can be relativistically invariant if the Lagrangian density is chosen to be relativistically invariant. Extremum of the action gives the Euler-Lagrange equations for the field. A suitable choice of

Lagrangian density has to be made so as to get the desired field equation. Thus, for example, in the case of a scalar field ϕ of mass m , we would choose an expression for the Lagrangian density such that the resulting Euler-Lagrange equation will be the Klein-Gordon equation for ϕ . Such a Lagrangian density is

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2), \quad (1.1)$$

and the resulting Klein-Gordon equation is $\partial^\mu \partial_\mu \phi + m^2 \phi = 0$. Similarly, for the electromagnetic field, described by the four-vector potentials $A^\mu(x)$, choosing the Lagrangian density as,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (1.2)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, will give Maxwell's equations as the Euler-Lagrange equations. Canonical momenta conjugate to the field functions are defined by taking the derivatives of the Lagrangian density with respect to the time derivative of the fields. For the scalar field ϕ , for example, the conjugate momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad (1.3)$$

A Legendre transformation of the Lagrangian density is carried out using the conjugate momenta, and a Hamiltonian density $\mathcal{H}$ is derived. The quantization of the theory is achieved by considering the field functions and canonical momenta as operators, and at a particular time t , commutation relations are set up between them. Since the quantization condition involves a commutation relation between the field and its

canonical conjugate at the same time t , this quantization is referred to as “equal time quantization” or the “instant form”.

As in many things, we owe to Dirac [5] the idea of using an alternate method, namely, light front (abbreviated as LF hereafter) coordinates for quantizing field theories, which he called the “front form”, and which is known also as “light-front quantization”. To explain the features of this method, we now introduce LF coordinates and the notations we will be using in this work. We remind the reader that we work in natural units, that is, $c = \hbar = 1$.

The LF coordinate system is defined as:

$$x^\mu = C^\mu_\nu \hat{x}^\nu, \quad (1.4)$$

where x^μ are the LF coordinates (x^+, x^1, x^2, x^-) and $\hat{x}^\nu$ (with carets) are the usual coordinates $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$. The non-zero entries of the C and C^{-1} matrices are:

$$C_0^+ = C_3^+ = C_0^- = -C_3^- = \frac{1}{\sqrt{2}}, \quad C_1^1 = C_2^2 = 1, \quad (1.5)$$

$$(C^{-1})_+^0 = (C^{-1})_-^0 = (C^{-1})_+^3 = -(C^{-1})_-^3 = \frac{1}{\sqrt{2}}, \quad (C^{-1})_1^1 = (C^{-1})_2^2 = 1. \quad (1.6)$$

The non-zero entries for the metric tensor of the LF coordinates system are:

$$g^{+-} = g^{-+} = -g^{11} = -g^{22} = 1. \quad (1.7)$$

On the other hand Minkowski's metric tensor has the following non-zero entries:

$$\hat{g}^{00} = -\hat{g}^{11} = -\hat{g}^{22} = -\hat{g}^{33} = 1. \quad (1.8)$$

According to the above definition:

$$x^+ = C_\nu^+ \hat{x}^\nu = \frac{1}{\sqrt{2}}(\hat{x}^0 + \hat{x}^3), \quad (1.9)$$

$$x^- = C_\nu^- \hat{x}^\nu = \frac{1}{\sqrt{2}}(\hat{x}^0 - \hat{x}^3), \quad (1.10)$$

$$x^i = \hat{x}^i, \quad i = 1, 2, \quad (1.11)$$

$$x_+ = x^-, \quad x_- = x^+, \quad (1.12)$$

$$x_i = -x^i, \quad i = 1, 2. \quad (1.13)$$

In LF coordinates the product of two 4-vectors becomes:

$$x.y = x_\mu y^\mu = g_{\mu\nu} x^\nu y^\mu = x^- y^+ + x^+ y^- - x_\perp y_\perp = x^- y^+ + x^+ y^- - x^i y^i, \quad (1.14)$$

where repeated indices are to be summed over here and in further work below.

The differential of the volume becomes:

$$d^4x = d\hat{x}^0 d\hat{x}^1 d\hat{x}^2 d\hat{x}^3 = dx^+ d^2x_\perp dx^-. \quad (1.15)$$

Other results of interest to us are:

$$\begin{aligned} \frac{\partial}{\partial x_+} &= \partial^+ = \frac{\partial}{\partial x^-} = \partial_-, \quad \frac{\partial}{\partial x_-} = \partial^- = \frac{\partial}{\partial x^+} = \partial_+, \quad \partial^i = \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i}, \\ \partial_\mu \partial^\mu &= 2 \partial^+ \partial^- - \partial_\perp^2. \end{aligned} \quad (1.16)$$

The relationship between the partial derivatives in the usual and the LF coordinates is given by,

$$\hat{x}^0 = \frac{1}{\sqrt{2}}(x^+ + x^-),$$

$$\begin{aligned}
\hat{x}^3 &= \frac{1}{\sqrt{2}}(x^+ - x^-), \\
\frac{\partial}{\partial x^+} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \hat{x}^0} + \frac{\partial}{\partial \hat{x}^3}\right) = \frac{1}{\sqrt{2}}(\partial^0 - \partial^3), \\
\frac{\partial}{\partial x^-} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \hat{x}^0} - \frac{\partial}{\partial \hat{x}^3}\right) = \frac{1}{\sqrt{2}}(\partial^0 + \partial^3).
\end{aligned}$$

Thus, in the LF coordinates, we have that:

$$\partial^\mu = \left(\frac{\partial}{\partial x^+}, \partial^1, \partial^2, \frac{\partial}{\partial x^-}\right) = \left(\frac{\partial^0 - \partial^3}{\sqrt{2}}, \partial^1, \partial^2, \frac{\partial^0 + \partial^3}{\sqrt{2}}\right) = (\partial^-, \partial^1, \partial^2, \partial^+). \quad (1.17)$$

We will choose the following definition of the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^{j=1,2} = \begin{pmatrix} -i\sigma^j & 0 \\ 0 & i\sigma^j \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}. \quad (1.18)$$

Let us define

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3) = \begin{pmatrix} 0 & 0 \\ \sqrt{2}i & 0 \end{pmatrix}, \quad \gamma^- = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3) = \begin{pmatrix} 0 & -\sqrt{2}i \\ 0 & 0 \end{pmatrix}. \quad (1.19)$$

We now define the LF *projection* operators:

$$\Lambda^+ = \frac{1}{\sqrt{2}}\gamma^0\gamma^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda^- = \frac{1}{\sqrt{2}}\gamma^0\gamma^- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad (1.20)$$

where I is the 2 by 2 unit matrix. It is simply to verify that:

$$(\Lambda^\pm)^\dagger = \Lambda^\pm, \quad (\Lambda^\pm)^2 = \Lambda^\pm, \quad \Lambda^+ + \Lambda^- = I. \quad (1.21)$$

We will also define the *chiral* operators which will filter out the right-handed and left-handed components, respectively, of a four-component spinor:

$$L = \frac{1 - \gamma^5}{2} = \frac{1}{2} \begin{pmatrix} 1 - \sigma^3 & 0 \\ 0 & 1 + \sigma^3 \end{pmatrix}, \quad R = \frac{1 + \gamma^5}{2} = \frac{1}{2} \begin{pmatrix} 1 + \sigma^3 & 0 \\ 0 & 1 - \sigma^3 \end{pmatrix}. \quad (1.22)$$

Let

$$\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (1.23)$$

then

$$\psi_+ = \Lambda^+ \psi = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \psi_- = \Lambda^- \psi = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \xi^- \end{pmatrix}, \quad (1.24)$$

$$\psi_L = L\psi = \begin{pmatrix} 0 \\ b \\ c \\ 0 \end{pmatrix}, \quad \psi_R = R\psi = \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix}, \quad (1.25)$$

of course ψ_L and ψ_R can also be decomposed into plus and minus components.

The LF formulation of quantum field theories (referred to as LFQFT hereafter) in the LF coordinates is historically related to the use of the infinite-momentum reference frame. Fubini [6] first used the concept of infinite-momentum frame, which is not a Lorentz frame, but can be viewed as the limit of a reference frame Lorentz transformed to almost the speed of light. (Such Lorentz transformations between uniformly moving

reference frames are referred to as “boosts”). By convention the boost takes place along the longitudinal coordinate. Weinberg [7] showed that the use of such a frame will lead, in the limit approaching c , to a simplification of the vacuum structure of the scalar field, in a theory called the ϕ^3 model, involving the interaction of three scalar fields. He noticed that a large number of possible Feynman diagrams are suppressed, in particular, those diagrams leading to a “dressing” of the vacuum involving vacuum to vacuum transitions. Susskind [8, 9], Bardakci and Halpern [10] viewed the infinite-momentum frame as essentially a change of variables, from the usual variables $(x^0, \vec{x})$ to the light-front variables $(x^+, x_\perp, x^-)$. It should be stressed that although the motivation for the LF coordinate system came initially from considerations of infinite momentum frame, it is not necessary to restrict oneself to this particular frame. Viewed as essentially a change of variables one has a clearly fully relativistic field theory with the added benefit of a simplified vacuum structure.

Work by Chang and Ma [1], Kogut and Soper [2], Bjorken, Kogut and Soper [3], Neville and Rohrlich [4] have shown that QED can be successfully formulated using the light-front coordinates. Kogut and Soper [2] postulated the commutation relations, which Chang, Root and Yan [11] later derived them by using Schwinger’s action principle. Bjorken, Kogut and Soper [3] applied the LFQFT formulation to various processes of scattering in an external field. Thus, the instant form and the light-front formulations of QED have been shown to be fully equivalent, with the infinite momentum

frame being considered as the bridge connecting the two. Sawicki [12] has shown that the transformation from equal-time perturbation theory to the light-front perturbation theory can be achieved without any reference to an infinite momentum frame, as a simple transformation of coordinates.

Starting from the work on QED by Kogut and Soper [2] and the quantization of massless Yang-Mills fields by Tomboulis [14], Zhang and Harindranath [16, 17] and Wilson, Harindranath, et al [18] have developed a formalism for an x^+ perturbation theory of Quantum Chromodynamics (QCD) in the light-front gauge.

1.1.2 Benefits of LF Coordinates - Poincaré Generators

Dirac [5] pointed out a very important benefit of using LF coordinates. Consider inhomogeneous Lorentz transformations of the form, $\hat{x}_\mu = \Lambda_\mu^\nu \hat{x}_\nu + \hat{a}_\mu$. These generate, in addition to translations, rotations and boosts. They are members of the Poincaré group of transformations. There are 10 Poincaré generators, which are defined by: $\hat{J}_i = \epsilon_{ijk} \hat{L}_{jk}$ (Rotations), $\hat{K}_i = \hat{L}_{0i}$ (Boosts) and $\hat{P}^\mu = (i\partial^0, \vec{\hat{P}})$ for $i, j, k = 1, 2, 3$. Where $\vec{\hat{P}} = (\hat{P}^1, \hat{P}^2, \hat{P}^3)$ is a 3-vector. In natural units, P 's have dimensions of 1/length, while the J 's are dimensionless.

In the usual coordinates, $x^\mu = (x^0, \vec{x})$, six of these generators, $\hat{P}^1, \hat{P}^2, \hat{P}^3, \hat{J}_1, \hat{J}_2$ and $\hat{J}_3$ acting on a scalar field are kinematic in the sense that the transformations these generate do not leave the $x^0 = 0$ surface. The remaining four generators $\hat{P}_0, \hat{K}_i$ ($i = 1, 2, 3$) acting on a scalar field generate transformations which leave the

$x^0 = 0$ surface, that is, they involve time evolution, and hence can be characterized as dynamic. Considering a field with some intrinsic spin (a multi-component function) will only mean inclusion of terms in the generators which act on the components and not on the space-time variables of the function, and so does not change the division into kinematic and dynamic generators.

In LF coordinates $x^\mu = (x^+, \vec{x}_\perp, x^-)$, we can see immediately, P_i , ($i = 1, 2$) and $P^+ = i\partial/\partial x^-$ are kinematic, since they generate displacements only within the $x^+ = 0$ surface. It is also seen that $L_{ij} = i(x_i P_j - x_j P_i)$ ($i, j = 1, 2$) and L_{-i} ($i = 1, 2$) are kinematic. The remaining momentum component $P^- = i\partial/\partial x^+$, generates displacement in LF time x^+ , is hence dynamic. The remaining generators that could be dynamic are L_{+i} and L_{+-} , since these are the only generators which contain ∂_+ . However, $L_{+-} = i(x_+ P_- - x_- P_+)$ is not dynamic, even though it contains $P_+ = P^-$; this is because $x_- = x^+$ and $x^+ = 0$ on the surface. Hence in LF coordinates, seven generators, P_i , P^+ , L_{-i} , L_{ij} , L_{+-} , are kinematic and the three generators P^- , L_{+i} are dynamic - one more kinematic generator than in the instant form coordinates. Written in terms of the angular momentum and boosts, six of the seven kinematic generators are, P_i , P^+ , $J_3 = L_{12}$, $K^1 + J^2$, $K^2 - J^1$. Unlike in the instant form where all three components of angular momentum are kinematic, in the LF formulation only one of them, J_3 , is kinematic. In LF formulation, angular momentum cannot be separated from dynamics. P^- serves to generate displacements in LF time x^+ and

hence can be identified with the Hamiltonian for the system.

We further note that in LF, $p^\mu x_\mu = x^- p^+ + x^+ p^- - p_\perp x_\perp$. Further since x^+ is adopted as the time coordinate in LF, we will adopt p^- as the energy in LF, and p^+ as the longitudinal component of the momentum. The mass shell relation $p^\mu p_\mu = m^2$ translates in LF as the relation

$$p^- = \frac{p_\perp^2 + m^2}{2p^+}. \quad (1.26)$$

For positive p^+ , the sign of p^- is positive only, so there is no ambiguity in the sign of the energy in LF formulation. However, for a particle of mass zero and $p_\perp = 0$, the $p^+ = 0$ sector leads to the problem of indeterminate p^- . Even for massive particles, one has to give a prescription to get around the singularity at $p^+ = 0$. This is addressed further below.

The Lagrangian density for a Dirac field is given by

$$\mathcal{L}_F = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi. \quad (1.27)$$

The equation of motion is then $(i\gamma_\mu \partial^\mu - m)\psi = 0$. Kogut et al [2] have shown that the spinors satisfying Dirac's equation are the helicity eigenspinors measured in a frame boosted to nearly the speed of light along the z-axis. That is to say: For

$$\hat{h}_0 = \frac{1}{2} \frac{\vec{p} \cdot \vec{\sigma}}{|\vec{p}|}, \quad (1.28)$$

the helicity operator in the laboratory, boost the operator along the z-axis and determine the eigenspinors. Therefore, when $\hat{h}_0 \rightarrow \hat{h}_{\omega \rightarrow \infty}$, the eigenspinors of $\hat{h}_{\omega \rightarrow \infty}$ are solutions

to the Dirac equation, where the frame of reference is moving in the z direction with a velocity equal to $-\tanh\omega$.

1.1.3 The singularity at $p^+ = 0$ in LF formulation

In this subsection, we consider the simple example of a self coupled scalar field Lagrangian, to illustrate the general procedure of constructing a Hamiltonian from it, and to bring to the fore, the problem with the $p^+ = 0$ sector. Let us start with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2) + \lambda\phi^4. \quad (1.29)$$

Our object is to cast this theory as a generalized Hamiltonian system possessing constraints following a procedure similar to that of Zhang and Harindranath [16, 17]. We rewrite the Lagrangian density in LF variables as

$$\mathcal{L} = \frac{1}{2}(\partial^+\phi\partial^-\phi + \partial^-\phi\partial^+\phi) - \frac{1}{2}\vec{\partial}_\perp\phi\cdot\vec{\partial}_\perp\phi - \frac{1}{2}m^2\phi^2 + \lambda\phi^4. \quad (1.30)$$

We define the conjugate momentum to the field variable ϕ ,

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial^-\phi)} = \partial^+\phi. \quad (1.31)$$

This shows that the conjugate momentum $\Pi = \partial^+\phi$ is a dependent variable. In the expression given above for the Lagrangian density, we write $\partial^-\phi\partial^+\phi = \partial^-(\phi\partial^+\phi) - \phi(\partial^-\partial^+\phi)$ and $\vec{\partial}_\perp\phi\cdot\vec{\partial}_\perp\phi = \vec{\partial}_\perp\cdot(\phi\vec{\partial}_\perp\phi) - \phi(\vec{\partial}_\perp^2\phi)$, and introducing the Hamiltonian density $\mathcal{H} = \vec{\partial}_\perp\phi\cdot\vec{\partial}_\perp\phi + m^2\phi^2$, we get

$$\mathcal{L} = \frac{1}{2}\partial^+\phi\partial^-\phi - \mathcal{H} + \frac{1}{2}\partial^+(\phi\partial^-\phi) + \frac{1}{2}\vec{\partial}^2(\phi\vec{\partial}^2\phi) - \frac{1}{2}\phi[\partial^-\partial^+\phi + \vec{\partial}^2\phi + m^2\phi - 2\lambda\phi^3]. \quad (1.32)$$

We have re-written the Lagrangian density in this form to exhibit explicitly the Hamiltonian density and also the dynamical variables and constraints. The first term contains the LF time derivative. The Hamiltonian density is the next term, a function of the dynamical variable ϕ (and of $\partial^i \phi (i = 1, 2)$). The third and the fourth terms are total derivatives (which will become surface terms) and hence ignorable. The last term is a constraint term, where ϕ occurs as a Lagrange multiplier. The square bracketed term in the product when set to zero gives the constraint equation which $\partial^- \Pi$ satisfies. This is typical of the procedure we will follow in dealing with QED and electro-weak unification in the later chapters.

The constraint equation following from the above is

$$\partial^+(\partial^-\phi) + \partial_\perp^2 \phi + m^2 \phi - 2\lambda \phi^3 = 0. \quad (1.33)$$

To solve this equation for $\partial^-\phi$, we introduce the operator $\frac{1}{\partial^+}$,

$$\frac{1}{\partial^+} f(x^-, x^+, x_\perp) = \frac{1}{2} \int_{-\infty}^{+\infty} dx_1^- \epsilon(x^- - x_1^-) f(x_1^-, x^+, x_\perp) + C(x^+, x_\perp), \quad (1.34)$$

where x_1^- is a dummy variable, $\epsilon(a) = 1, 0, -1$ for $a > 0, = 0, < 0$ respectively, and $C(x^+, x_\perp)$ is an integration constant (independent of x^-). We thus write formally,

$$\partial^-\phi = \frac{1}{\partial^+} [-m^2 \phi - \partial_\perp^2 \phi + 2\lambda \phi^3]. \quad (1.35)$$

Now to quantize this scalar field theory, we have to derive the commutation relations for the canonical variables ϕ and Π , where Π is a dependent field variable obtained from a solution of the constraint equation. It requires that we impose *a priori*

boundary conditions on the physical variables. A boundary condition suitable for this purpose follows from the requirement of uniqueness of the solution of the initial value problem at $x^+ = 0$ [22],[23]. This means that we choose the arbitrary integration constant $C(x^+, x_\perp) = 0$. This choice is equivalent to imposing antisymmetric boundary conditions for the physical variables at $\pm$ infinity in the longitudinal direction.

Using the constraint equation, and the definition of $\frac{1}{\partial^+}$ given above, we can determine $\partial^- \phi$ in terms of the field ϕ and its transverse derivatives. The unique determination of the initial values of ϕ and $\partial_\perp \phi$ necessitates that they satisfy antisymmetric boundary conditions at infinite longitudinal boundary, that is, $C(x^+, x_\perp) = 0$ in the definition of $\frac{1}{\partial^+}$.

A further fact can be noted from the definition of the operation $\frac{1}{\partial^+}$ given above. Its Fourier transform has the form

$$\mathcal{F}\left[\frac{1}{\partial^+}f(x^-)\right] = \frac{1}{2}\left(\frac{1}{p^+ + i\epsilon} + \frac{1}{p^+ - i\epsilon}\right)f(p^+) \equiv \frac{1}{[p^+]}f(p^+). \quad (1.36)$$

This amounts to a prescription of how to treat the singularity at $p^+ = 0$. We shall use the notation $\frac{1}{[p^+]}$ to signify the prescription contained in the middle expression of (1.36). If the field is massless, we have pointed out before that the energy p^- becomes indeterminate when p^+ goes to zero for $p_\perp = 0$. Thus the prescription deals with the infra red problem associated with $p^+ = 0$, $p_\perp = 0$ in the massless case. This will be of importance in gauge field theories where the quanta have zero mass.

Chapter 2

Light front formulation of QED.

2.1 Abelian gauge fields.

In this section we develop the LF formulation of QED, an Abelian gauge theory. We will carry out the work in such a way that crucial points are highlighted, and we can use the same approach when we later on develop the LF formulation of the electro-weak unification.

In QED the coupling constant g is equal to the electric charge of the fermion. In order to show the formal parallelism between the formulation of QED and that of the electro-weak unification we will retain g as the representation of the electromagnetic coupling constant. $\mathcal{L}_{QED}$, the Lagrangian density function for QED is a scalar function. From the equal-time formulation of QFT, we know that $\mathcal{L}_{QED}$ is composed of a gauge field component (GB) plus a fermionic component (F). We start with $\mathcal{L}_{GB}$, the

Lagrangian density of a massless real gauge field. First, we define

$$F^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu, \quad (2.1)$$

where C^μ are the gauge potentials. Then, the Lagrangian density for the gauge fields $\mathcal{L}_{GB}$ is given by

$$\mathcal{L}_{GB} = -\frac{1}{4}(F^{\mu\nu}F_{\mu\nu}), \quad (2.2)$$

where again repeated indices indicate a summation over all their values. In order to illustrate the methodology we will calculate in detail the expression for $\mathcal{L}_{QED}$. Thus,

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{+\nu}F_{\nu}^- + F^{-\nu}F_{\nu}^+ - F^{i\nu}F_{\nu}^i, \\ &= -2F^{+-}F^{+-} - 4F^{i+}F^{i-} + F^{ij}F^{ij}. \end{aligned} \quad (2.3)$$

When expressed in the LF coordinates $\mathcal{L}_{GB}$ becomes:

$$\mathcal{L}_{GB} = \frac{1}{2}(F^{+-})^2 + F^{+i}F^{-i} - \frac{1}{4}(F^{ij}F^{ij}). \quad (2.4)$$

In order to be able to draw parallels between the LF and the equal time formulations, we introduce the concepts of electric (E^μ) and magnetic (B^μ) fields. Thus, let

$$F^{+i} = -E^i, \quad F^{+-} = -E^-, \quad B^- = F^{ij}, \quad i \neq j. \quad (2.5)$$

We also define the gauge condition for the LF formulation of QED to be:

$$C^+ = 0. \quad (2.6)$$

From the definition of canonical conjugate we have that:

$$\frac{\partial \mathcal{L}_{GB}}{\partial(\partial^- C_\mu)} = -F^{+\mu}(x) = E^\mu(x). \quad (2.7)$$

Thus, we can easily derive the expressions:

$$\mathcal{L}_{GB} = \frac{1}{2}(E^-)^2 - \frac{1}{2}(B^-)^2 - E^i(\partial^- C^i - \partial^i C^-), \quad (2.8)$$

$$\begin{aligned} &= -E^i \partial^- C^i - \frac{1}{2}(E^-)^2 - \frac{1}{2}(B^-)^2 - [\partial^+(E^- C^-) - \partial^i(E^i C^-)] \\ &\quad + C^-[\partial^+ E^- - \partial^i E^i], \end{aligned} \quad (2.9)$$

$$= F^{+i} \partial^- C^i - \mathcal{H}_{GB} + C^-[\partial^+ E^- - \partial^i E^i], \quad (2.10)$$

where the Hamiltonian density of the gauge field is given by the expression:

$$\mathcal{H}_{GB} = \frac{1}{2}(E^-)^2 + \frac{1}{2}(B^-)^2 + [\partial^+(E^- C^-) - \partial^i(E^i C^-)]. \quad (2.11)$$

At this point it is worthwhile making the following observation. While making the calculations that lead to expression (2.10), no assumption was made regarding the nature (dependent or independent field variable) of the components of the gauge field. $\mathcal{L}_{GB}$ has been cast in such a way that the first term involves the “time” derivative of the transverse component of the gauge potential, C^i , and its canonical conjugate $E^i = -F^{+i}$, the second term is the generalized Hamiltonian density, and the last term involving C^- , the longitudinal component of the gauge potential playing the role of a Lagrange multiplier, represents the constraint term. Setting the coefficient of C^- to zero, shows that the longitudinal component of the gauge field E^- is a dependent variable determined by E^i , the transverse components of the gauge field. This feature of the constraint term determining which field variable are independent and which are dependent will be a recurring theme in the further developments below.

In the Introduction we introduced the concepts and the methodology of the generalized Hamiltonian formulation. In (1.32) we took the coefficient of ϕ , the field variable, and made it zero in order to obtain the equation satisfied by $\partial^-\Pi$. Since ∂^-C^- is not present in the Lagrangian density of QED, the Euler-Lagrange equation for C^- does not yield an equation of motion (there is no time evolution). Instead we have a constraint equation.

2.2 Fermionic fields.

In the previous section we presented the gauge fields. Here we present the Dirac fields needed in QED. Again the approach we will use while doing the development is intended to lay down the ground work for when we develop the electro-weak unification in LF.

First, we introduce the massless Lagrangian density for the fermion,

$$\mathcal{L}_F = \bar{\psi} i \gamma^\mu \partial_\mu \psi. \quad (2.12)$$

We know from the equal time formulation of QFT, that in order to have gauge invariance we must replace the partial derivatives by covariant derivatives, with their associated gauge fields. This covariant derivative is given by the expression:

$$D_\mu = \frac{1}{2} \overleftrightarrow{\partial}_\mu - i g C_\mu, \quad (2.13)$$

where the $\overleftrightarrow{\partial}_\mu$ operator is defined as follows:

$$A \overleftrightarrow{\partial}_\mu B = A (\partial_\mu B) - (\partial_\mu A) B. \quad (2.14)$$

And as we already know, the coupling constant is denoted by g , and it represents the bare electric charge ($-e$). Thus, $\mathcal{L}_F$ becomes:

$$\mathcal{L}_F = \bar{\psi} i \gamma^\mu D_\mu \psi. \quad (2.15)$$

The inclusion of a mass term $m \bar{\psi} \psi$ does not affect gauge invariance as can readily be shown. The reason for not including such a term from the outset is to illustrate the procedure that will be followed when developing the LF formulation of the electro-weak unification theory. Using the γ and Λ matrices, as defined in (1.18) to (1.20), we can easily derive the expression for $\mathcal{L}_F$ in the LF coordinates. Let

$$\alpha^i = \gamma^0 \gamma^i, \quad \psi_+ = \Lambda^+ \psi, \quad \psi_- = \Lambda^- \psi, \quad \psi = \psi_+ + \psi_-. \quad (2.16)$$

Then the expression for the Lagrangian density becomes

$$\begin{aligned} \mathcal{L}_F &= i \psi^\dagger \left\{ \gamma^0 \gamma^+ D^- + \gamma^0 \gamma^- D^+ - \gamma^0 \gamma^i D^i \right\} \psi, \\ &= \frac{i}{\sqrt{2}} \psi_+^\dagger \partial^- \psi_+ - \frac{i}{\sqrt{2}} (\partial^- \psi_+^\dagger) \psi_+ - \mathcal{H}_F + \\ &\quad + C^- (g \sqrt{2} \psi_+^\dagger \psi_+) + \frac{1}{2} \left\{ \psi_-^\dagger \mathcal{A}_F + \mathcal{A}_F^\dagger \psi_- \right\}, \end{aligned} \quad (2.17)$$

where $\mathcal{A}_F$ and $\mathcal{H}_F$ are as follows:

$$\mathcal{A}_F = i \sqrt{2} \partial^+ \psi_- - (i \alpha^i \partial^i + g \alpha^i C^i) \psi_+, \quad (2.18)$$

$$\mathcal{H}_F = \frac{1}{2} \left\{ \psi_+^\dagger (i \alpha^i \partial^i + g \alpha^i C^i) \psi_- + [(i \alpha^i \partial^i + g \alpha^i C^i) \psi_-]^\dagger \psi_+ \right\}. \quad (2.19)$$

The last two terms within (2.17) represent the constraint terms. From (2.17), (2.18) and (2.19) we can see that $\mathcal{L}_F$ is now expressed in term of:

$$\psi_+, \psi_-, \psi_+^\dagger, \psi_-^\dagger, C^-, C^i. \quad (2.20)$$

Furthermore, only ψ_+ and $\psi_+^\dagger$ have a time derivative. From the definition of canonical conjugate we have that:

$$\Pi_{\psi_+} = \frac{\partial \mathcal{L}_F}{\partial(\partial^- \psi_+)} = i \frac{1}{\sqrt{2}} \psi_+^\dagger, \quad (2.21)$$

$$\Pi_{\psi_+^\dagger} = \frac{\partial \mathcal{L}_F}{\partial(\partial^- \psi_+^\dagger)} = -i \frac{1}{\sqrt{2}} \psi_+. \quad (2.22)$$

Thus, we can express $\mathcal{L}_F$ in terms of the dynamical variables and their canonical conjugates,

$$\mathcal{L}_F = \Pi_{\psi_+} \partial^- \psi_+ + (\partial^- \psi_+)^{\dagger} \Pi_{\psi_+^\dagger} + C^- (g \sqrt{2} \psi_+^\dagger \psi_+) - \mathcal{H}_F + \frac{1}{2} \{ \psi_-^\dagger \mathcal{A}_F + \mathcal{A}_F^\dagger \psi_- \}. \quad (2.23)$$

In QED there is no need to consider left-handed and right-handed fermions separately. Nevertheless, we will introduce the L and R operators to explore the structure $\mathcal{L}_F$ would assume if we were to consider left-handed and right-handed fermions. Let

$$\psi_L = L\psi = \frac{1 - \gamma^5}{2} \psi, \quad \psi_R = R\psi = \frac{1 + \gamma^5}{2} \psi. \quad (2.24)$$

We can now express $\mathcal{L}_F$ in terms of these components,

$$\mathcal{L}_F = \bar{\psi} i \gamma^\mu D_\mu \psi = \bar{\psi}_L i \gamma^\mu D_\mu \psi_L + \bar{\psi}_R i \gamma^\mu D_\mu \psi_R. \quad (2.25)$$

It is simple to verify that the structure of the Lagrangian density has not changed except that now we have two similar structures one for the left-handed component and one for

the right-handed one.

We define $\mathcal{L}_{QED}$, the Lagrangian density of the system as $\mathcal{L}_{QED} = \mathcal{L}_{GB} + \mathcal{L}_F$. Thus,

$$\begin{aligned} \mathcal{L}_{QED} = & F^{+i}\partial^-C^i + \frac{i}{\sqrt{2}}\psi_+^\dagger\partial^-\psi_+ - \frac{i}{\sqrt{2}}(\partial^-\psi_+)^\dagger\psi_+ - \mathcal{H}_{QED} + \\ & + C^-\left\{\partial^+E^- - \partial^iE^i + g\sqrt{2}\psi_+^\dagger\psi_+\right\} + \frac{1}{2}\left\{\psi_-^\dagger\mathcal{A}_F + \mathcal{A}_F^\dagger\psi_-\right\}, \end{aligned} \quad (2.26)$$

where the Hamiltonian density $\mathcal{H}_{QED}$ is given by,

$$\begin{aligned} \mathcal{H}_{QED} = & \frac{1}{2}(E^-)^2 + \frac{1}{2}(B^-)^2 + [\partial^+(E^-C^-) - \partial^i(E^iC^-)] + \\ & + \frac{1}{2}\left\{\psi_+^\dagger(i\alpha^i\partial^i + g\alpha^iC^i)\psi_- + [(i\alpha^i\partial^i + g\alpha^iC^i)\psi_-]^\dagger\psi_+\right\}. \end{aligned} \quad (2.27)$$

We now introduce a mass term. The existence of this term precludes chiral symmetry, but does not affect gauge invariance. By applying the projection operators we obtain,

$$m\bar{\psi}\psi = m\psi_+^\dagger\gamma^0\psi_- + m\psi_-^\dagger\gamma^0\psi_+. \quad (2.28)$$

In accordance with (1.27), the mass terms (2.28) are to be subtracted from $\mathcal{L}_{QED}$ and thus, $\mathcal{A}_F$, $\mathcal{A}_F^\dagger$ and $\mathcal{H}_{QED}$ will have to be redefined and include mass terms. They become:

$$\mathcal{A}_F = i\sqrt{2}\partial^+\psi_- - (i\alpha^i\partial^i + g\alpha^iC^i + m\gamma^0)\psi_+, \quad (2.29)$$

$$\begin{aligned} \mathcal{H}_{QED} = & \frac{1}{2}\left\{(E^-)^2 + (B^-)^2 + [\partial^+(E^-C^-) - \partial^i(E^iC^-)] + \right. \\ & + \left[\psi_+^\dagger(i\alpha^i\partial^i + g\alpha^iC^i + m\gamma^0)\psi_- + \right. \\ & \left. \left. + [(i\alpha^i\partial^i + g\alpha^iC^i + m\gamma^0)\psi_-]^\dagger\psi_+\right]\right\}. \end{aligned} \quad (2.30)$$

Equation (2.26) shows again that C^i , ψ_+ , are independent degrees of freedom, and C^- , ψ_- represent Lagrange multipliers, and setting their coefficients to zero determines them in terms of the independent degrees C^i , ψ_+ . The constraint equations are:

$$\partial^+ E^- - \partial^i E^i + g \sqrt{2} \psi_+^\dagger \psi_+ = 0, \quad (2.31)$$

$$\mathcal{A}_F = i \sqrt{2} \partial^+ \psi_- - (i \alpha^i \partial^i + g \alpha^i C^i + m \gamma^0) \psi_+ = 0, \quad (2.32)$$

$$\mathcal{A}_F^\dagger = 0. \quad (2.33)$$

By solving these constraint equations, we find the Lagrange multipliers, C^- , ψ_- . We get,

$$\psi_- = \frac{1}{i \sqrt{2} \partial^+} [(i \alpha^i \partial^i + g \alpha^i C^i + m \gamma^0) \psi_+], \quad (2.34)$$

$$E^- = \frac{1}{\partial^+} (\partial^i E^i - g \sqrt{2} \psi_+^\dagger \psi_+) = -\partial^i C^i - g \sqrt{2} \frac{1}{\partial^+} (\psi_+^\dagger \psi_+), \quad (2.35)$$

$$C^- = -\frac{1}{\partial^+} E^- = \frac{1}{\partial^+} [\partial^i C^i + g \sqrt{2} \frac{1}{\partial^+} (\psi_+^\dagger \psi_+)]. \quad (2.36)$$

In (1.34) we defined the integral operator $\frac{1}{\partial^+}$. Using that definition E^- will be antisymmetric at the longitudinal boundaries. We will have,

$$E^-(x) |_{x^-=\pm\infty} = -\partial^i C^i |_{x^-=\pm\infty} \mp g \sqrt{2} \int_{-\infty}^{+\infty} dx' (\psi_+^\dagger \psi_+). \quad (2.37)$$

This means that C^i , C^- , E^- , and E^i must all be antisymmetric at the longitudinal boundaries. Physical states must have a vanishing energy density at the boundaries.

This means that $E^-(x) |_{x^-=\pm\infty} = 0$, which in turn means that

$$\partial^i C^i |_{x^-=\pm\infty} = \mp g \sqrt{2} \int_{-\infty}^{+\infty} dx' (\psi_+^\dagger \psi_+). \quad (2.38)$$

Thus the transverse gauge fields at the longitudinal boundaries are generated by the fermionic fields, and are non-vanishing. Since x^- is absent from the integral, the interaction is non-local. These integrals that determine the value of the field variables at the longitudinal boundaries are called boundary integrals.

We will now show that the term $\partial^+(E^-C^-) - \partial^i(E^iC^-)$ of $\mathcal{H}_{QED}$ is a surface term and can thus be ignored. Two Hamiltonian densities that lead to the same Hamiltonian are considered to be equivalent, and we can drop one in favor of the other. We have just shown that C^- and E^- are antisymmetric, therefore when we integrate the first term in dx^- the result is

$$(E^-C^-) |_{x^- \rightarrow +\infty} - (E^-C^-) |_{x^- \rightarrow -\infty} = 0. \quad (2.39)$$

We can further assume that C^i vanishes as $O(\frac{1}{r})$ while C^- and E^i vanish as $O(\frac{1}{r^2})$ when $r = |x_\perp| \rightarrow \infty$. Therefore E^iC^- will vanish as $O(\frac{1}{r^3})$. On integrating $d^2x_\perp$ we will obtain:

$$\int dx^2 \int_{-\infty}^{+\infty} dx^1 [\partial^1(E^1C^-)] + \int dx^1 \int_{-\infty}^{+\infty} dx^2 [\partial^2(E^2C^-)] \rightarrow 0. \quad (2.40)$$

Thus, the Hamiltonian density of QED becomes after ignoring the surface term,

$$\begin{aligned} \mathcal{H}_{QED} = \frac{1}{2} \Big\{ (E^-)^2 + (B^-)^2 + \Big[\psi_+^\dagger (i\alpha^i \partial^i + g\alpha^i C^i + m\gamma^0) \psi_- + \\ + [(i\alpha^i \partial^i + g\alpha^i C^i + m\gamma^0) \psi_-]^\dagger \psi_+ \Big] \Big\}. \end{aligned} \quad (2.41)$$

In (1.24) we introduced the 2-component spinors ξ and ξ^- .

$$\psi_+ = \frac{1}{\sqrt[4]{2}} \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \xi^- \end{pmatrix}. \quad (2.42)$$

from these definitions it follows that

$$\alpha^i \psi_+ = \frac{1}{\sqrt[4]{2}} \begin{pmatrix} 0 \\ \sigma^i \xi \end{pmatrix}, \quad m \gamma^0 \psi_+ = \frac{1}{\sqrt[4]{2}} \begin{pmatrix} 0 \\ i m \xi \end{pmatrix}, \quad (2.43)$$

$$\alpha^i \psi_- = \begin{pmatrix} \sigma^i \xi^- \\ 0 \end{pmatrix}, \quad m \gamma^0 \psi_- = \begin{pmatrix} -i m \xi^- \\ 0 \end{pmatrix}. \quad (2.44)$$

Thus the relationship between ξ and ξ^- is given by

$$\psi_- = \begin{pmatrix} 0 \\ \xi^- \end{pmatrix} = \frac{1}{\sqrt[4]{8}} \begin{pmatrix} 0 \\ \frac{1}{i\partial^+} (i\sigma^i \partial^i + g\sigma^i C^i + i m) \xi \end{pmatrix}. \quad (2.45)$$

The factor $\sqrt[4]{2}$ is supplied in order to obtain consistent expressions for the free Hamiltonian density. We must now consider the representation of anti-fermions. The charge conjugation matrix C satisfies the following relationship [19], which is independent of the representation of γ^μ :

$$C \gamma^{\mu T} C^{-1} = -\gamma^\mu. \quad (2.46)$$

It is simple to show that

$$C = \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} \quad (2.47)$$

is a particular representation satisfying (2.46). So, for the anti-fermion we have that

$$\psi_c = C \bar{\psi}^T = C (\gamma^0)^T \psi^*, \quad (2.48)$$

with a phase factor of 1 [19]. Thus

$$\psi_c = \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt[4]{2}} \xi^* \\ \xi^{-*} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt[4]{2}} \sigma^1 \xi^* \\ \sigma^1 \xi^{-*} \end{pmatrix} = (\psi_c)_+ + (\psi_c)_-, \quad (2.49)$$

from which we can see that there is a helicity flip in the 2-component spinor. That is to say, $\xi_c = \sigma^1 \xi^*$. This shows that a 2-component spinor can also be used when representing an anti-particle. The fact that both the particle and its anti-particle can be represented by a 2-component spinor will become relevant later.

We now expand (2.41), using (2.34), (2.35), (2.36), (2.42) and (2.45). After some straightforward algebra, we obtain:

$$\begin{aligned} \mathcal{H}_{QED} = & \frac{1}{2} \left[\partial^i C^i + g \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) \right] \left[\partial^j C^j + g \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) \right] + \frac{1}{2} (\partial^i C^j - \partial^j C^i)^2 + \\ & + \frac{1}{2} \left\{ \xi^\dagger \left[i\sigma^i \partial^i + g\sigma^i C^i - i m \right] \left[\frac{1}{i 2 \partial^+} (i\sigma^j \partial^j + g\sigma^j C^j + i m) \xi \right] + \right. \\ & \left. + H.C. \right\}. \end{aligned} \quad (2.50)$$

The Hamiltonian of the system is

$$H_{QED} = \int dx^- d^2 x_\perp \mathcal{H}_{QED}. \quad (2.51)$$

We will now divide the Hamiltonian density of QED into a free Hamiltonian density and an interaction Hamiltonian density. To obtain the free part, we set the coupling constant to zero. Consider the following integrals:

$$\begin{aligned} \int dx^1 dx^2 \partial^1 C^1 \partial^2 C^2 &= \int dx^1 \partial^1 C^1 (C^2|_{x^2=\pm\infty}) - \int dx^1 dx^2 C^2 \partial^2 (\partial^1 C^1), \\ \int dx^1 dx^2 \partial^1 C^2 \partial^2 C^1 &= \int dx^2 \partial^2 C^1 (C^2|_{x^1=\pm\infty}) - \int dx^1 dx^2 C^2 \partial^1 (\partial^2 C^1). \end{aligned} \quad (2.52)$$

Therefore, since $C^i|_{x^1=\pm\infty} = C^i|_{x^2=\pm\infty} = 0$ then

$$\partial^i C^i \partial^j C^j + (\partial^i C^j - \partial^j C^i)^2 = (\partial^i C^j)^2. \quad (2.53)$$

Summed independently over i and j ($i, j = 1, 2$). Thus, under volume integration and based on the chosen boundary conditions the free Hamiltonian density becomes:

$$\mathcal{H}_{QED}^0 = \frac{1}{2} (\partial^i C^j)^2 + \frac{1}{2} \left\{ \xi^\dagger \left\{ \frac{1}{i 2 \partial^+} (-\partial_\perp^2 + m^2) \xi \right\} + H.C. \right\}. \quad (2.54)$$

The free fields C^i and ξ satisfy the free field equations and the solutions are plane waves $\sim e^{iq \cdot x}$. In order to simplify the expression for $\mathcal{H}_{QED}^0$ we look into the values of $\frac{1}{i\partial^+}\psi$ and $(\frac{1}{i\partial^+}\psi)^\dagger$. We apply the definition of the integral operator given in (1.34) to $e^{iq \cdot x}$. For this purpose we use the integral representation of the step function θ ,

$$\theta(x^- - x'^-) = -\frac{1}{2\pi i} \int dk^+ \frac{1}{k^+ + i\varepsilon} e^{-ik^+(x^- - x'^-)}, \quad (2.55)$$

$$\theta(x'^- - x^-) = -\frac{1}{2\pi i} \int dk^+ \frac{1}{k^+ + i\varepsilon} e^{ik^+(x^- - x'^-)}. \quad (2.56)$$

Then the anti-symmetric step function is given by the expression:

$$\epsilon(x^- - x'^-) = \theta(x^- - x'^-) - \theta(x'^- - x^-), \quad (2.57)$$

$$= -\frac{1}{2\pi i} \int dk^+ \left[\frac{e^{-ik^+(x^- - x'^-)}}{k^+ + i\varepsilon} - \frac{e^{ik^+(x^- - x'^-)}}{k^+ + i\varepsilon} \right]. \quad (2.58)$$

Next, we calculate the expression

$$\begin{aligned} \left(\frac{1}{\partial^+} \right) e^{iq \cdot x} &= -\frac{1}{2\pi i} \frac{1}{2} \int dx'^- dk^+ \left[\frac{e^{-ik^+(x^- - x'^-)}}{k^+ + i\varepsilon} - \frac{e^{ik^+(x^- - x'^-)}}{k^+ + i\varepsilon} \right] \\ &\quad \cdot e^{i(q^+ x'^- + q^- x^+ - q_\perp \cdot x_\perp)}, \\ &= \frac{e^{i(q^- x^+ - q_\perp \cdot x_\perp)}}{2i} \int dk^+ \left[\frac{1}{k^+ - i\varepsilon} + \frac{1}{k^+ + i\varepsilon} \right] e^{ik^+ x^-} \delta(k^+ - q^+), \\ &= \frac{e^{i(q^+ x^- + q^- x^+ - q_\perp \cdot x_\perp)}}{2i} \left[\frac{1}{q^+ - i\varepsilon} + \frac{1}{q^+ + i\varepsilon} \right] = \frac{e^{iq \cdot x}}{i[q^+]}. \end{aligned} \quad (2.59)$$

Applying (2.59) we find that

$$\begin{aligned} \left(\frac{1}{i\partial^+} \right) e^{-iq \cdot x} &= \frac{e^{-iq \cdot x}}{[q^+]}, \\ \left[\left(\frac{1}{i\partial^+} \right) e^{-iq \cdot x} \right]^\dagger &= \frac{e^{iq \cdot x}}{[q^+]^\dagger} = \frac{e^{iq \cdot x}}{[q^+]}. \end{aligned} \quad (2.60)$$

Taking into account the hermitian conjugate term in (2.54), it can be written as:

$$\mathcal{H}_{QED}^0 = \frac{1}{2} (\partial^i C^j)^2 + \xi^\dagger \left\{ \frac{1}{i2\partial^+} (-\partial_\perp^2 + m^2) \xi \right\}. \quad (2.61)$$

We now define $\mathcal{H}_{QED}^{int} = \mathcal{H}_{QED} - \mathcal{H}_{QED}^0 = \mathcal{H}_{QED}^{FPF} + \mathcal{H}_{QED}^{FFFF} + \mathcal{H}_{QED}^{FPPF}$, where

$$\begin{aligned} \mathcal{H}_{QED}^{FPF} &= g (\partial^i C^i) \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) + \frac{1}{2} g \left\{ \xi^\dagger (i\sigma^i \partial^i - im) \left[\left(\frac{1}{i2\partial^+} \right) (\sigma^j C^j) \xi \right] + \right. \\ &\quad \left. + \xi^\dagger \sigma^i C^i \left[\left(\frac{1}{i2\partial^+} \right) (i\sigma^j \partial^j + im) \xi \right] + H.C. \right\}, \end{aligned} \quad (2.62)$$

$$\mathcal{H}_{QED}^{FFFF} = \frac{1}{2} g^2 \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi), \quad (2.63)$$

$$\mathcal{H}_{QED}^{FPPF} = g^2 \xi^\dagger \left\{ \sigma^i C^i \left(\frac{1}{i2\partial^+} \right) (\sigma^j C^j) \xi \right\}, \quad (2.64)$$

where F stands for fermions and P stands for photons. $\mathcal{H}_{QED}^{FPF}$ is of first order in g , while $\mathcal{H}_{QED}^{FFFF}$ and $\mathcal{H}_{QED}^{FPPF}$ are of second order in g . These involve operators $\frac{1}{\partial^+}$ and $(\frac{1}{\partial^+})^2$.

We see that under dx^- integration the following equality holds:

$$g (\partial^i C^i) \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) = -g \xi^\dagger \xi \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\}. \quad (2.65)$$

Also, as x^- and x^i are independent variables, it follows that,

$$\xi^\dagger (\sigma^i \partial^i - m) \left\{ \left(\frac{1}{2\partial^+} \right) (g \sigma^j C^j) \xi \right\} = \xi^\dagger \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (g \sigma^j C^j) \xi \right\}. \quad (2.66)$$

Thus, we have the following expression for $\mathcal{H}_{QED}^{FPF}$:

$$\begin{aligned} \mathcal{H}_{QED}^{FPF} = & -g \xi^\dagger \xi \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} + g \xi^\dagger \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (\sigma^j C^j) \xi \right\} + \\ & + g \xi^\dagger \sigma^i C^i \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^j \partial^j + m) \xi \right\}. \end{aligned} \quad (2.67)$$

The Hamiltonian densities for QED given by the expressions (2.62) to (2.64) are equivalent to the expressions (3.4) to (3.7) of Zhang and Harindranath [17], for QCD.

2.3 Boundary Integrals and their role in the cancellation of infrared singularities.

Here we take a look at some of the improper integrals that appear in the LF formulation of QED.

Let ρ be a charge density,

$$\rho(x^+, x_\perp, x'^-) = \xi^\dagger(x^+, x_\perp, x'^-) \xi(x^+, x_\perp, x'^-). \quad (2.68)$$

We can express $\mathcal{H}_{QED}^{FFFF}$ in terms of the charge density,

$$\mathcal{H}_{QED}^{FFFF} = \frac{1}{8} g^2 \int_{-\infty}^{+\infty} dx'^- dx''- \epsilon(x^- - x'^-) \epsilon(x^- - x''-) \rho(x^+, x_\perp, x'^-) \rho(x^+, x_\perp, x''-). \quad (2.69)$$

By using the identity [16], [20],

$$\frac{1}{2} \int_{-\lambda/2}^{\lambda/2} dx^- \epsilon(x^- - x'^-) \epsilon(x^- - x''-) = -|x'^- - x''-| + \frac{1}{2}\lambda, \quad (2.70)$$

where λ is a parameter that denotes the length of the interval of integration, it can be

shown that H_{QED}^{FFFF} (involving an integral over dx^- of the density) can be expressed as:

$$\begin{aligned} H_{QED}^{FFFF} = & -\frac{1}{4}g^2 \int_{-\infty}^{+\infty} d^2\vec{x}_\perp dx'^- dx''^- \rho(x^+, x_\perp, x'^-) |x'^- - x''^-| \\ & \cdot \rho(x^+, \vec{x}_\perp, x''^-) + \\ & + \lim_{\lambda \rightarrow \infty} \lambda \frac{1}{8} g^2 \int_{-\infty}^{+\infty} d^2\vec{x}_\perp \left\{ \int_{-\infty}^{+\infty} dx'^- \rho(x^+, \vec{x}_\perp, x'^-) \right\}^2. \end{aligned} \quad (2.71)$$

This last term obviously represents a singularity. The term contains the square of the boundary integral given by (2.38). Next, let us consider the expression:

$$\left(\frac{1}{\partial^+}\right)^2 f(x^-) = \frac{1}{4} \int_{-\infty}^{+\infty} dx'^- dx''^- \epsilon(x^- - x'^-) \epsilon(x'^- - x''^-) f(x^+, x_\perp, x''^-). \quad (2.72)$$

Integrating (2.72) over $dx^- d^2\vec{x}_\perp$, we obtain:

$$\begin{aligned} \int_{-\infty}^{+\infty} dx^- d^2\vec{x}_\perp \left(\frac{1}{\partial^+}\right)^2 f(x^-) = & \int_{-\infty}^{+\infty} dx^- dx''^- d^2\vec{x}_\perp f(x^+, \vec{x}_\perp, x''^-) \\ & \cdot \int_{-\infty}^{+\infty} dx'^- \epsilon(x^- - x'^-) \epsilon(x'^- - x''^-). \end{aligned} \quad (2.73)$$

This shows that $\left(\frac{1}{\partial^+}\right)^2 f(x^-)$ leads to a boundary integral and vice-versa. There are two ways of defining the Fourier transform of $\left(\frac{1}{\partial^+}\right)^2 f(x^-)$. In one case we exclude the λ term in (2.71), in the other we include this term. The Fourier transforms of these are as follows. If the λ term in (2.71) is excluded, we have,

$$\mathcal{F} \left[\left(\frac{1}{\partial^+}\right)^2 f(x^-) \right] = \frac{1}{2} \left(\frac{1}{(p^+ + i\epsilon)^2} + \frac{1}{(p^+ - i\epsilon)^2} \right) f(p^+) \equiv \frac{1}{[p^+]^2} f(p^+) \quad (2.74)$$

while, if the λ term in (2.71) is included, we have,

$$\mathcal{F} \left[\left(\frac{1}{\partial^+}\right)^2 f(x^-) \right] = \left[\frac{1}{2} \left(\frac{1}{p^+ + i\epsilon} + \frac{1}{p^+ - i\epsilon} \right) \right]^2 f(p^+) \equiv \frac{1}{[p^+]^2} f(p^+). \quad (2.75)$$

These represent different ways of treating the operator $(\frac{1}{\partial^+})^2$.

Zhang and Harindranath [16], have shown that the inclusion of the term in λ will lead to a principal value prescription in momentum space that is necessary to eliminate the linear infrared singularities. Exclusion of the term in λ will lead to uncanceled linear infrared divergencies. In Chapter 3 we will use this prescription to cancel linear infrared singularities. In Appendix A we exhibit the Fourier expansion of the terms leading to these infrared singularities.

2.4 Quantization of QED.

The next step is the creation of the commutation relations; that is to say we quantize, we leap from classical fields to quantum fields.

From (2.26) and following the procedure presented by Faddeev and Jackiw [24] applicable to a generalized Hamiltonian system with constraints, we calculate the commutation relations for QED. Following that reference, the Lagrangian one-form $\mathcal{L}dx^+$ is given by:

$$\mathcal{L}dx^+ = \mathcal{E}^i dC^i + \Pi_{\psi_+(x)} d\psi_+ + d\psi_+^\dagger \Pi_{\psi_+^\dagger(x)} - \mathcal{H}dx^+, \quad (2.76)$$

where $\mathcal{E}^i = \partial^+ C^i$, $\Pi_{\psi_+} = i \frac{1}{\sqrt{2}} \psi_+^\dagger$, and $\Pi_{\psi_+^\dagger} = -i \frac{1}{\sqrt{2}} \psi_+$. The constraints have been used in the determination of the dependent variables, and have therefore been eliminated. Finally we obtain, after a total x^+ derivative has been dropped,

$$\begin{aligned} \mathcal{L}dx^+ &= \frac{1}{2} \left[\mathcal{E}^i dC^i + \Pi_{\psi_+(x)} d\psi_+ + d\psi_+^\dagger \Pi_{\psi_+^\dagger(x)} - C^i d\mathcal{E}^i - d\Pi_{\psi_+(x)} \psi_+ - \psi_+^\dagger d\Pi_{\psi_+^\dagger(x)} \right] - \\ &\quad - \mathcal{H}dx^+. \end{aligned} \quad (2.77)$$

Next, we recast the one-form, so that terms corresponding to the products of spinors are reordered and neighboring terms have the same structure. In particular, we are referring to the third and sixth terms in (2.77). This is required to be able to create anti-commutation relations and their hermitian conjugates in a consistent manner. Thus, the one-form becomes

$$\mathcal{L}dx^+ = \frac{1}{2} \left[\mathcal{E}^i dC^i + \Pi_{\psi_+(x)} d\psi_+ - \psi_+^\dagger d\Pi_{\psi_+^\dagger(x)} - C^i d\mathcal{E}^i - d\Pi_{\psi_+(x)} \psi_+ + d\psi_+^\dagger \Pi_{\psi_+^\dagger(x)} \right] - \mathcal{H}dx^+. \quad (2.78)$$

The one-form has the general expression:

$$\mathcal{L}dx^+ = \frac{1}{2} f^\alpha \Gamma_{\alpha\beta} df^\beta - \mathcal{H}dx^+, \quad (2.79)$$

where the f 's satisfy the commutation relation:

$$[f^\beta, f^\alpha] = i [\Gamma_{\alpha\beta}]^{-1} = i [\Gamma^{-1}]_{\beta\alpha}. \quad (2.80)$$

From (2.78) we can extract the entries of the $\Gamma_{\alpha\beta}$ matrix.

$$\Gamma_{\alpha\beta} = \begin{matrix} & \mathcal{E}^i & \Pi_{\psi_+} & \Pi_{\psi_+^\dagger} & C^i & \psi_+ & \psi_+^\dagger \\ \begin{matrix} \mathcal{E}^i \\ \Pi_{\psi_+} \\ \Pi_{\psi_+^\dagger} \\ C^i \\ \psi_+ \\ \psi_+^\dagger \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \quad (2.81)$$

where we define $f^\alpha = (\mathcal{E}^i, \Pi_{\psi_+}, \Pi_{\psi_+^\dagger}, C^i, \psi_+, \psi_+^\dagger)$. For equation (2.80) we need

$[\Gamma^{-1}]_{\beta\alpha}$, which is the inverse of $\Gamma_{\alpha\beta}$,

$$[\Gamma^{-1}]_{\beta\alpha} = \begin{matrix} & \mathcal{E}^i & \Pi_{\psi_+} & \Pi_{\psi_+^\dagger} & C^i & \psi_+ & \psi_+^\dagger \\ \begin{matrix} \mathcal{E}^i \\ \Pi_{\psi_+} \\ \Pi_{\psi_+^\dagger} \\ C^i \\ \psi_+ \\ \psi_+^\dagger \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}. \quad (2.82)$$

We include a $\delta^{ij} \delta(x^- - y^-) \delta^2(x_\perp - y_\perp)$ for all relations, and for the anti-commutation relations we include a Λ^+ to filter the 2-spinor contributions in the anti-commutation

relations. The results are as follows:

$$[C^i, \mathcal{E}^j]_{x^+=y^+} = i \delta^{ij} \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.83)$$

$$\{\psi_+, \Pi_{\psi_+}\}_{x^+=y^+} = i \Lambda^+ \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.84)$$

$$\{\psi_+^\dagger, \Pi_{\psi_+^\dagger}\}_{x^+=y^+} = -i \Lambda^+ \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.85)$$

$$[\psi, C^i]_{x^+=y^+} = [\psi^\dagger, C^i]_{x^+=y^+} = 0, \quad (2.86)$$

$$[\psi(x), \psi(y)]_{x^+=y^+} = 0. \quad (2.87)$$

In the instant form formulation of QFT, the anti-commutation relation of two ψ spinors involves the presence of a unit matrix, which in LF is replaced by a Λ^+ matrix as here we are dealing with ψ_+ instead of full ψ 's. The commutation relations can also be expressed as:

$$[C^i(x), \partial^+ C^j(y)]_{x^+=y^+} = i \delta^{ij} \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.88)$$

$$[C^i(x), C^j(y)]_{x^+=y^+} = -i \frac{\delta^{ij}}{2} \epsilon(x^- - y^-) \delta^2(x_\perp - y_\perp). \quad (2.89)$$

This commutator is not zero because $\mathcal{E}^i$ is a function of C^i . If the constant of integration in the integral operator ($\frac{1}{\partial^+}$) would not have been chosen 0, then the commutator would have been undetermined. From equation (2.84) and (2.87), we have

$$\{\psi_+(x), \psi_+^\dagger(y)\}_{x^+=y^+} = \Lambda^+ \sqrt{2} \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.90)$$

$$\{\xi(x), \xi^\dagger(y)\}_{x^+=y^+} = 2I \delta(x^- - y^-) \delta^2(x_\perp - y_\perp), \quad (2.91)$$

$$\{\xi(x), \xi(y)\}_{x^+=y^+} = 0. \quad (2.92)$$

We have now quantized the fields, and from now on we are dealing with field operators.

Here the interval for which $x^+ = y^+$ is either $-(x_\perp - y_\perp)^2$ and therefore space-like or light-like if $x_\perp = y_\perp$, never time-like, so causality is maintained here also.

2.5 Equations of Motion.

We now verify the consistency of the quantization conditions and the equations of motion for the field operators. Operators $C^i(x)$ and $\psi(x)$ satisfy the quantum equations of motion:

$$i \partial^- C^i(x) = [C^i(x), H_0], \quad (2.93)$$

$$i \partial^- \psi(x) = [\psi(x), H_0], \quad (2.94)$$

where H_0 , the free Hamiltonian, is given by

$$H_0 = \int_{-\infty}^{+\infty} dx^- d^2 \vec{x}_\perp \left[\frac{1}{2} \partial^i C^j \partial^i C^j - \xi^\dagger \left(\frac{1}{i 2 \partial^+} \right) (\partial_\perp^2 - m^2) \xi \right]. \quad (2.95)$$

These equations are relatively easy to solve. Thus, from (2.93) and (2.95), and taking into account commutation relations (2.86) and (2.89), and the fact that $x'^+ = x^+$ then

$$\begin{aligned} \partial^- C^k(x') &= \frac{1}{2i} \int dx^- d^2 \vec{x}_\perp \left[C^k(x') \partial^i C^j(x) \partial^i C^j(x) - \partial^i C^j(x) \partial^i C^j(x) C^k(x') \right], \\ &= -\frac{1}{2} \delta^{kj} \int dx^- d^2 \vec{x}_\perp \epsilon(x'^- - x^-) \left[\partial^i \left(\delta^2(\vec{x}'_\perp - \vec{x}_\perp) \right) \right] \partial^i C^j(x). \end{aligned} \quad (2.96)$$

Based on the assumption made in (2.40), that C^i vanishes with order $O(\frac{1}{r})$ when $|r_\perp| \rightarrow 0$, then $\left[\partial^1 C^k(x) \right]_{x_1=\pm\infty} = \left[\partial^2 C^k(x) \right]_{x_2=\pm\infty} = 0$. Thus, the equation of

motion (2.93) becomes

$$\partial^- C^k(x') = \frac{1}{2} \int dx^- \epsilon(x'^- - x^-) \partial_\perp^2 C^k(x). \quad (2.97)$$

Similarly, from (2.94) and (2.95) we have that,

$$i \partial^- \xi(x') = \int dx^- d^2 \vec{x}_\perp \left\{ \xi(x') \xi^\dagger(x) + \xi^\dagger(x) \xi(x') \right\} \left(\frac{i}{2 \partial^+} \right) (\partial_\perp^2 - m^2) \xi(x). \quad (2.98)$$

Again, taking into account commutation relations (2.86) and (2.89), and (2.91) and the fact that $x'^+ = x^+$ then

$$\begin{aligned} \partial^- \xi(x) &= \int dx'^- d^2 \vec{x}'_\perp \delta(x'^- - x^-) \delta^2(\vec{x}'_\perp - \vec{x}_\perp) \\ &\quad \cdot \frac{1}{2} \int dx''^- \epsilon(x'^- - x''^-) (\partial_\perp^2 - m^2) \xi(x'', x'^-, \vec{x}'_\perp), \\ &= \frac{1}{2} \int dx''^- \epsilon(x^- - x''^-) (\partial_\perp^2 - m^2) \xi(x'', x^+, x_\perp). \end{aligned} \quad (2.99)$$

The solutions for these equations are:

$$C^i(x) = \sum_\lambda \int \frac{1}{(2\pi)^3} d^2 \vec{q}_\perp \frac{dq^+}{2[q^+]} \left[\epsilon_\lambda^i c(q, \lambda) e^{-iqx} + \epsilon_\lambda^{*i} c^\dagger(q, \lambda) e^{iqx} \right], \quad (2.100)$$

$$\begin{aligned} \xi(x) &= \sum_\lambda \chi_\lambda \int \frac{1}{(2\pi)^3} d^2 \vec{p}_\perp \frac{dp^+}{2[p^+]} \left[\sqrt{2p^+} b(p, \lambda) e^{-ipx} + \right. \\ &\quad \left. + \sqrt{2p^+} d^\dagger(p, -\lambda) e^{ipx} \right], \end{aligned} \quad (2.101)$$

where $c(q, \lambda)$, $c^\dagger(q, \lambda)$ are annihilation and creation operators respectively for photons while $b(p, \lambda)$, $d^\dagger(p, -\lambda)$ are annihilation and creation operators for the fermions. The left-hand-side (abbreviated as l.h.s. hereafter) of (2.97) is

$$\sum_\lambda \int \frac{1}{(2\pi)^3} d^2 \vec{q}_\perp \frac{dq^+}{2[q^+]} \left[-i q^- \epsilon_\lambda^i c(q, \lambda) e^{-iqx} + i q^- \epsilon_\lambda^{*i} c^\dagger(q, \lambda) e^{iqx} \right]. \quad (2.102)$$

While the right-hand-side (abbreviated as r.h.s. hereafter) is

$$\sum_{\lambda} \int \frac{1}{(2\pi)^3} d^2 \vec{q}_{\perp} \frac{dq^+}{2[q^+]} \frac{(\vec{q}_{\perp})^2}{2} \left[-i \epsilon_{\lambda}^i c(q, \lambda) \frac{e^{-iqx}}{[q^+]} + i \epsilon_{\lambda}^{*i} c^{\dagger}(q, \lambda) \frac{e^{iqx}}{[q^+]} \right]. \quad (2.103)$$

Therefore (2.100) satisfies (2.97) iff:

$$q^- = \frac{(\vec{q}_{\perp})^2}{2q^+}. \quad (2.104)$$

Similarly, for the fermions, the l.h.s. of (2.99) is equal to

$$\sum_{\lambda} \chi_{\lambda} \int \frac{1}{(2\pi)^3} d^2 \vec{p}_{\perp} \frac{dp^+}{2[p^+]} \left[-i p^- \sqrt{2p^+} b(p, \lambda) e^{-ipx} + i p^- \sqrt{2p^+} d^{\dagger}(p, -\lambda) e^{ipx} \right]. \quad (2.105)$$

While the r.h.s. of (2.99) is equal to

$$\begin{aligned} \sum_{\lambda} \chi_{\lambda} \int \frac{1}{(2\pi)^3} d^2 \vec{p}_{\perp} \frac{dp^+}{2[p^+]} \left[-\frac{(\vec{p}_{\perp})^2 + m^2}{2[p^+]} \sqrt{2p^+} b(p, \lambda) e^{-ipx} + \right. \\ \left. + \frac{(\vec{p}_{\perp})^2 + m^2}{2[p^+]} \sqrt{2p^+} d^{\dagger}(p, -\lambda) e^{ipx} \right]. \end{aligned} \quad (2.106)$$

Therefore (2.101) satisfies (2.99) iff:

$$p^- = \frac{(\vec{p}_{\perp})^2 + m^2}{2p^+}. \quad (2.107)$$

This means that the particles created/annihilated are on their mass-shell. In the above expressions λ is defined as the helicity of the particle. For photons it can take the values ± 1 and for fermions $\pm \frac{1}{2}$. The photon polarization vectors are: $\epsilon_1^i = \frac{1}{\sqrt{2}}(1, i)$ and $\epsilon_{-1}^i = \frac{1}{\sqrt{2}}(1, -i)$, while the fermion spinors are just the eigenspinors of a non-relativistic particle of spin- $\frac{1}{2}$. Thus $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that even

though non-relativistic spinors arise, the theory is fully relativistic. We have obtained the commutation relations for the field operators, we shall now find their expression for the c , $c^\dagger$, b , $b^\dagger$, d and $d^\dagger$ operators. We start with the anti-commutation relations for the fermionic fields. The canonical anti-commutation relations for the Dirac spinors are given at the level of the their components [29], that is to say:

$$\begin{aligned} \{ \xi_\alpha, \xi_\beta^\dagger \} &= \delta_{\alpha\beta} \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2 \vec{q}'_\perp \frac{1}{(2\pi)^3} \frac{dq^+}{2[q^+]} d^2 \vec{q}_\perp \sqrt{2q^+ 2q'^+} \\ &\quad \left[\{ b^\dagger, b \} e^{-i(q'x - qy)} + \{ b^\dagger, d^\dagger \} e^{i(q'x + qy)} + \right. \\ &\quad \left. + \{ d^\dagger, d \} e^{i(q'x - qy)} + \{ b, d \} e^{-i(q'x + qy)} \right], \end{aligned} \quad (2.108)$$

where $\delta_{\alpha\beta} = \chi_\alpha \chi_\beta^\dagger$. Usually the sub-indices are suppressed. Therefore, for $x^+ = y^+$ we have that:

$$\begin{aligned} \{ \xi, \xi^\dagger \} &= I \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2 \vec{q}'_\perp \frac{1}{(2\pi)^3} \frac{dq^+}{2[q^+]} d^2 \vec{q}_\perp \sqrt{2q^+ 2q'^+} \\ &\quad \left[\{ b^\dagger, b \} e^{-i(q'x - qy)} + \{ b^\dagger, d^\dagger \} e^{i(q'x + qy)} + \right. \\ &\quad \left. + \{ d^\dagger, d \} e^{i(q'x - qy)} + \{ b, d \} e^{-i(q'x + qy)} \right]. \end{aligned} \quad (2.109)$$

$\delta(x^- - y^-) \delta^2(\vec{x}_\perp - \vec{y}_\perp)$ can only be produced by $e^{\pm i(q'x - qy)}$ therefore the coefficients of $e^{\pm i(q'x + qy)}$ have to be zero. Therefore, $\{ b, d \} = \{ b^\dagger, d^\dagger \} = 0$.

$$\begin{aligned} \{ \xi, \xi^\dagger \} &= I \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2 \vec{q}'_\perp \frac{1}{(2\pi)^3} \frac{dq^+}{2[q^+]} d^2 \vec{q}_\perp \sqrt{2q^+ 2q'^+} \\ &\quad \left[\{ b^\dagger, b \} e^{-i(q'x - qy)} + \{ d^\dagger, d \} e^{i(q'x - qy)} \right]. \end{aligned} \quad (2.110)$$

We now postulate the anti-commutation relations for the fermionic creation and annihilation operators. Let

$$\begin{aligned} \{b^\dagger(q, \lambda), b(q', \lambda')\} &= \{d^\dagger(q', -\lambda'), d(q, -\lambda)\}, \\ &= (2\pi)^3 2q^+ \delta_{\lambda, \lambda'} \delta(q'^+ - q^+) \delta^2(\vec{q}'_\perp - \vec{q}_\perp). \end{aligned} \quad (2.111)$$

Then since $x^+ = y^+$, (2.109) becomes:

$$\begin{aligned} \{\xi, \xi^\dagger\} &= I \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2\vec{q}'_\perp \frac{1}{(2\pi)^3} \frac{dq^+}{2[q^+]} d^2\vec{q}_\perp \sqrt{2q^+ 2q'^+} \\ &\quad \left[\{b^\dagger, b\} e^{-i(q'x - qy)} + \{d^\dagger, d\} e^{i(q'x - qy)} \right], \\ &= I \int \frac{1}{(2\pi)^3} dq^+ d^2\vec{q}_\perp \left[e^{-iq(x-y)} + e^{iq(x-y)} \right], \\ &= 2I \delta(x^- - y^-) \delta^2(\vec{x}_\perp - \vec{y}_\perp). \end{aligned} \quad (2.112)$$

Once it is proven (via the δ functions) that $q'^+ = q^+$ and $\vec{q}'_\perp = \vec{q}_\perp$ it follows that $q'^- = q^-$ because we are dealing with identical fermions. Therefore, $q' = q$. In a similar manner we can calculate the commutation relations for the photon operators $c, c^\dagger$. For this we will use the commutations relations for the photon field. For $x^+ = y^+$ we have that:

$$\begin{aligned} [C^i(x), C^j(y)] &= \delta^{ij} \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2\vec{q}'_\perp \frac{1}{(2\pi)^3} \frac{dq^+}{2[q^+]} d^2\vec{q}_\perp \\ &\quad \cdot \left\{ [c^\dagger, c] e^{-i(q'y - qx)} + [c^\dagger, c^\dagger] e^{i(q'y + qx)} \right. \\ &\quad \left. + [c, c^\dagger] e^{i(q'y - qx)} + [c, c] e^{-i(q'y + qx)} \right\}, \end{aligned} \quad (2.113)$$

where $\delta^{ij} = \epsilon_\lambda^i (\epsilon_{\lambda'}^j)^*$. For reasons similar to the ones used for $\{\xi, \xi^\dagger\}$ we have that $[c, c] = [c^\dagger, c^\dagger] = 0$. We will now postulate the commutation relations for the

creation and annihilation operators of the photon fields. Let

$$[c(q, \lambda), c^\dagger(q', \lambda')] = (2\pi)^3 2q^+ \delta_{\lambda, \lambda'} \delta(q'^+ - q^+) \delta^2(\vec{q}'_\perp - \vec{q}_\perp). \quad (2.114)$$

Then since $x^+ = y^+$, (2.113) becomes:

$$\begin{aligned} [C^i(x), C^j(y)] &= -\delta^{ij} \int \frac{1}{(2\pi)^3} \frac{dq'^+}{2[q'^+]} d^2\vec{q}'_\perp dq^+ d^2\vec{q}_\perp \left[e^{-i(q'y - qx)} - e^{i(q'y - qx)} \right] \\ &\quad \cdot \delta(q'^+ - q^+) \delta^2(\vec{q}'_\perp - \vec{q}_\perp). \end{aligned} \quad (2.115)$$

After integration using the $\delta(q'^+ - q^+)$ and $\delta^2(\vec{q}'_\perp - \vec{q}_\perp)$ functions the exponents of the exponentials are given by: $\mp \left[(y^- - x^-)q^+ - (\vec{y}_\perp - \vec{x}_\perp) \cdot \vec{q}_\perp \right]$ and

$$\begin{aligned} [C^i(x), C^j(y)] &= -i\delta^{ij} \frac{1}{4} \delta^2(\vec{x}_\perp - \vec{y}_\perp) \int \frac{1}{2\pi i} dq^+ \left[\frac{1}{q^+ + i\epsilon} + \frac{1}{q^+ - i\epsilon} \right] \\ &\quad \cdot \left[e^{i(x^- - y^-)q^+} - e^{-i(x^- - y^-)q^+} \right]. \end{aligned} \quad (2.116)$$

Taking into consideration the definitions (2.55), (2.56) and (2.57) we have that (2.116)

becomes:

$$[C^i(x), C^j(y)] = -i\delta^{ij} \frac{1}{2} \delta^2(\vec{x}_\perp - \vec{y}_\perp) \epsilon(x^- - y^-). \quad (2.117)$$

This completes the verification of the consistency of the commutation relations and the quantum equations of motion.

2.6 QED Hamiltonian Densities.

Here we regroup the components of the interaction Hamiltonian density into terms representing different types of interactions.

From (2.67), (2.63) and (2.64) we have the following expressions for QED's Hamiltonian densities:

$$\begin{aligned} \mathcal{H}_{QED}^{FPF} = & -g \xi^\dagger \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} \xi + g \xi^\dagger \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (\sigma^j C^j) \xi \right\} + \\ & + g \xi^\dagger \sigma^i C^i \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^j \partial^j + m) \xi \right\}, \end{aligned} \quad (2.118)$$

$$\mathcal{H}_{QED}^{FFFF} = \frac{1}{2} g^2 \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi), \quad (2.119)$$

$$\mathcal{H}_{QED}^{FPPF} = g^2 \xi^\dagger \left\{ \sigma^i C^i \left(\frac{1}{i 2\partial^+} \right) (\sigma^j C^j) \xi \right\}. \quad (2.120)$$

It can easily be seen that in $\mathcal{H}_{QED}^{FPF}$, the fermion field appears twice and the photon field once, that means that it represents a family of vertices with three lines. On the other hand, $\mathcal{H}_{QED}^{FFFF}$ represents a family of vertices with four lines, all of them corresponding to fermions. Finally, $\mathcal{H}_{QED}^{FPPF}$ also represents vertices with four lines, two fermions and two photons. In each one of the expressions we substitute the Fourier expansions (2.100) and (2.101) of the field operators. In the following we list the generic vertices which will be needed for the tree level processes we consider in Chapter 3 (see Appendix A for the detailed calculations). As in the equal time formulation of QED, an anti-particle with $p^\mu = (p^-, p_\perp, p^+)$ is represented as $-p^\mu$. As shown in Appendix A, the matrix element for a vertex includes a string of creation and annihilation operators, which when used in conjunction with the commutation relations and the initial state will yield the sign associated with the vertex. There is also an exponential which when integrated over the whole space will yield the δ functions present in expressions (2.122) to (2.127). We integrate the Hamiltonian density over the whole space in order to obtain

the Hamiltonian.

All vertices composed of three lines can be generically represented by the expression:

$$\Gamma^i(p_1 = p_{input}, p_2 = p_{output}; q_{photon}) = -\frac{q^i}{q^+} + \frac{\sigma \cdot p_{2\perp} - i m}{2 p_2^+} \sigma^i + \sigma^i \frac{\sigma \cdot p_{1\perp} + i m}{2 p_1^+}. \quad (2.121)$$

For an anti-particle we will substitute p^μ by $-p^\mu$ in expression (2.121). Appendix A shows that this substitution applies to the Γ functions, but it does not apply to the associated δ functions.

There are four basic three-line vertices that will be used in Chapter 3.

(1) Emission of a photon by a fermion, represented by the diagram:

Diagram 1.

To this diagram corresponds the rule:

$$g(2\pi)^3 \delta(p_2^+ + q^+ - p_1^+) \delta^2(\vec{p}_{2\perp} + \vec{q}_\perp - \vec{p}_{1\perp}) \sqrt{2 p_1^+} \sqrt{2 p_2^+} \chi^\dagger(\lambda_2) \Gamma^i(p_1, p_2; q) \epsilon^{*i}(\lambda) \chi(\lambda_1). \quad (2.122)$$

(2) Absorption of a photon by a fermion, represented by the diagram:

Diagram 2.

To this diagram corresponds the rule:

$$g(2\pi)^3 \delta(p_2^+ - q^+ - p_1^+) \delta^2(\vec{p}_{2\perp} - \vec{q}_\perp - \vec{p}_{1\perp}) \sqrt{2 p_1^+} \sqrt{2 p_2^+} \chi^\dagger(\lambda_2) \Gamma^i(p_1, p_2; q) \epsilon^i(\lambda) \chi(\lambda_1). \quad (2.123)$$

(3) Pair production, represented by the diagram:

To this vertex corresponds the expression:

$$g (2\pi)^3 \delta(q^+ - p_1^+ - p_2^+) \delta^2(\vec{q}_\perp - \vec{p}_{1\perp} - \vec{p}_{2\perp}) \sqrt{2p_1^+} \sqrt{2p_2^+} \chi^\dagger(\lambda_1) \Gamma^i(-p_2, p_1; q) \epsilon^i(\lambda) \chi(-\lambda_2). \quad (2.124)$$

(4) Pair annihilation into a photon, represented by the diagram:

To this diagram corresponds the rule:

$$g (2\pi)^3 \delta(p_1^+ + p_2^+ - q^+) \delta^2(\vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{q}_\perp) \sqrt{2p_1^+} \sqrt{2p_2^+} \chi^\dagger(-\lambda_2) \Gamma^i(p_1, -p_2; q) \epsilon^{*i}(\lambda) \chi(\lambda_1). \quad (2.125)$$

Next we consider the four-fermion line vertex, which is represented by the diagram:

To this diagram corresponds the rule:

$$g^2 (2\pi)^3 \delta(p_1^+ + p_2^+ - p_3^+ - p_4^+) \delta^2(\vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{p}_{3\perp} - \vec{p}_{4\perp}) \sqrt{2p_1^+} \sqrt{2p_2^+} \cdot \chi^\dagger(\lambda_3) \chi(\lambda_1) \chi^\dagger(\lambda_4) \chi(\lambda_2) \frac{1}{[p_1^+ - p_3^+]^2} \sqrt{2p_3^+} \sqrt{2p_4^+}. \quad (2.126)$$

The comment previously made about anti-particles is also applicable to this diagram. Finally we have the vertex represented by:

Diagram 6.

To this diagram corresponds the rule:

$$g^2 (2\pi)^3 \delta(p_2^+ + k_2^+ - p_1^+ - k_1^+) \delta^2(\vec{p}_{2\perp} + \vec{k}_{2\perp} - \vec{p}_{1\perp} - \vec{k}_{1\perp}) \sqrt{2p_1^+} \sqrt{2p_2^+} \cdot \chi^\dagger(s_2) \sigma \cdot \epsilon(\lambda_1) \sigma \cdot \epsilon^*(\lambda_2) \chi(s_1) \frac{1}{[p_1^+ - k_2^+]}. \quad (2.127)$$

Here, too the comment previously made about anti-particles is also applicable to this diagram.

These are the basic vertices that are produced from the interaction Hamiltonian, of which diagrams 1, 2, 3 and 4 are of first order in the coupling constant g , whereas diagrams 5 and 6 correspond to second order in g . Diagrams 5 and 6 are referred to as instantaneous diagrams because the intermediate photon or fermion line is represented by a vertical line, at a single value of x^+ , and hence represented with a stroke through it.

2.7 Normalization.

In this section we want to determine the normalization convention for the gauge field operator based on the Fourier expansion (2.100) and commutation relation (2.89). That is to say we want to determine the vacuum expectation value

$$\langle 0 | \int d^3x (C^{i\dagger} C^i) | 0 \rangle. \quad (2.128)$$

Since C^i ($i = 1, 2$) are the only physical degrees of freedom in this formulation, we normalize only them. We have

$$\begin{aligned} C^{i\dagger} C^i &= \sum_{\lambda, \lambda'} \int \frac{1}{(2\pi)^3} d^2 \vec{q}_\perp \frac{dq^+}{2[q^+]} \frac{1}{(2\pi)^3} d^2 \vec{q}'_\perp \frac{dq'^+}{2[q'^+]} \left[\epsilon_\lambda^{*i} \epsilon_{\lambda'}^i c^\dagger(q, \lambda) c(q', \lambda') e^{ix(q-q')} + \right. \\ &\quad + \epsilon_\lambda^{*i} \epsilon_{\lambda'}^{*i} c^\dagger(q, \lambda) c^\dagger(q', \lambda') e^{ix(q+q')} + \epsilon_\lambda^i \epsilon_{\lambda'}^i c(q, \lambda) c(q', \lambda') e^{-ix(q+q')} + \\ &\quad \left. + \epsilon_\lambda^i \epsilon_{\lambda'}^{*i} c(q, \lambda) c^\dagger(q', \lambda') e^{-ix(q-q')} \right]. \end{aligned} \quad (2.129)$$

Next, we calculate the vacuum expectation value for C^i ,

$$\begin{aligned} \langle 0 | \int d^3 x (C^{i\dagger} C^i) | 0 \rangle &= \sum_{\lambda, \lambda'} \epsilon_\lambda^i \epsilon_{\lambda'}^{*i} \int d^3 x \frac{1}{(2\pi)^3} d^2 \vec{q}_\perp \frac{dq^+}{2[q^+]} \frac{1}{(2\pi)^3} d^2 \vec{q}'_\perp \frac{dq'^+}{2[q'^+]} \\ &\quad \cdot \int d^3 x \langle 0 | c(q, \lambda) c^\dagger(q', \lambda') | 0 \rangle e^{-ix(q-q')} \\ &= \frac{1}{2[q^+]}. \end{aligned} \quad (2.130)$$

Next, we determine the normalization convention for the fermion field operator based on the Fourier expansion (2.101) and commutation relation (2.91). That is to say we want to determine the vacuum expectation value

$$\langle 0 | \int d^3 x (\xi^\dagger \xi) | 0 \rangle. \quad (2.131)$$

We calculate the vacuum expectation value of ξ and not that of ψ because the LF Lagrangian and Hamiltonian densities are given in terms of ξ .

$$\begin{aligned} \xi^\dagger \xi &= \sum_{\lambda', \lambda} \chi_{\lambda'}^\dagger \chi_\lambda \int \frac{1}{(2\pi)^3} d^2 \vec{p}'_\perp \frac{dp'^+}{2[p'^+]} \frac{1}{(2\pi)^3} d^2 \vec{p}_\perp \frac{dp^+}{2[p^+]} \sqrt{2p'^+} \sqrt{2p^+} \\ &\quad \cdot \left[b^\dagger(p', \lambda') b(p, \lambda) e^{i(p'-p) \cdot x} + d(p, -\lambda') b(p, \lambda) e^{-i(p'+p) \cdot x} + \right. \\ &\quad \left. + b^\dagger(p', \lambda') d^\dagger(p, -\lambda) e^{i(p'+p) \cdot x} + d(p, -\lambda') d^\dagger(p, -\lambda) e^{-i(p'-p) \cdot x} \right]. \end{aligned} \quad (2.132)$$

and again we find the vacuum expectation value, this time for ξ ,

$$\begin{aligned} \langle 0 | \int d^3 x \xi^\dagger \xi | 0 \rangle &= \sum_{\lambda, \lambda'} \chi_\lambda^\dagger \chi_{\lambda'} \int \frac{1}{(2\pi)^6} \frac{dp'^+ d^2 \vec{p}'_\perp}{2[p'^+]} \frac{dp^+ d^2 \vec{p}_\perp}{2[p^+]} \sqrt{2p^+ 2p'^+} \\ &\quad \cdot \int d^3 x \langle 0 | d(p', \lambda') d^\dagger(p, -\lambda) | 0 \rangle e^{ix(p-p')}, \\ &= 1. \end{aligned} \quad (2.133)$$

Thus, the gauge bosons are normalized to $\frac{1}{2[q^+]}$, while the fermions are normalized to 1.

2.8 Calculation of flux factor and $d\sigma$ in the LF formulation.

Part of the work of this thesis is the calculation of cross sections. Thus, we must know how to represent the flux factor in LF as well as the expression of a cross section in LF.

Let us consider the two particle to two particle process $p_A p_B \rightarrow p_1 p_2$. Let k_A and k_B be the momentum of the wave packets with $\phi(k_A)$ and $\phi(k_B)$ distributions peaked about p_A and p_B respectively. We define the initial state to be:

$$|\phi(k_A)\phi(k_B)\rangle_{in} = \int \frac{dk_A^+}{2(2\pi)^3 k_A^+} \frac{dk_B^+}{2(2\pi)^3 k_B^+} d^2\vec{k}_{A\perp} d^2\vec{k}_{B\perp} \phi(k_A)\phi(k_B) \cdot e^{-i\vec{b}\cdot\vec{k}_{B\perp}} |k_A k_B\rangle_{in}. \quad (2.134)$$

To derive the flux factor, we adopt the method used in reference [25]. According to this $d\sigma$ is given by

$$d\sigma = \prod_{i=1,2} \frac{1}{(2\pi)^3} dp_i \int d^2b \frac{dk_A}{(2\pi)^3} \frac{dk_B}{(2\pi)^3} \frac{d\vec{k}_A}{(2\pi)^3} \frac{d\vec{k}_B}{(2\pi)^3} \mathcal{M}(k_A, k_B \rightarrow p_1, p_2) \cdot \mathcal{M}^\dagger(\vec{k}_A, \vec{k}_B \rightarrow p_1, p_2) e^{i\vec{b}\cdot(\vec{k}_{B\perp} - \vec{k}_{A\perp})} (2\pi)^4 \delta^4(k_A + k_B - p_1 - p_2) \cdot (2\pi)^4 \delta^4(\vec{k}_A + \vec{k}_B - p_1 - p_2) \phi(k_A)\phi(k_B)\phi^*(\vec{k}_A)\phi^*(\vec{k}_B), \quad (2.135)$$

where $\mathcal{M}$ is the matrix element for the two particle to two particle process, $\mathcal{M}^\dagger$ is its Hermitian adjoint, dk stands for $\frac{1}{2k^+} dk^+ d^2\vec{k}_\perp$ and

$$\delta^4(k_A + k_B - p_1 - p_2) = \delta(k_A^+ + k_B^+ - p_1^+ - p_2^+) \delta(k_A^- + k_B^- - p_1^- - p_2^-) \cdot \delta^2(\vec{k}_{A\perp} + \vec{k}_{B\perp} - \vec{p}_{1\perp} - \vec{p}_{2\perp}). \quad (2.136)$$

Because of cylindrical symmetry $\vec{b}$ has no longitudinal component. By performing the d^2b integration we obtain the factor $(2\pi)^2 \delta^2(\vec{k}_{B\perp} - \vec{k}_{A\perp})$. Next, we integrate over $d\vec{k}_A$ and $d\vec{k}_B$. This gives us

$$\int \frac{d\vec{k}_A}{(2\pi)^3} \frac{d\vec{k}_B}{(2\pi)^3} (2\pi)^2 \delta^2(\vec{k}_{B\perp} - \vec{k}_{A\perp}) \mathcal{M}^\dagger(\vec{k}_A, \vec{k}_B \rightarrow p_1, p_2) (2\pi)^4 \cdot \delta^4(\vec{k}_A + \vec{k}_B - p_1 - p_2) \phi^*(\vec{k}_A)\phi^*(\vec{k}_B). \quad (2.137)$$

Integrating over $d\vec{k}_B$ we obtain:

$$\begin{aligned} & \int d\vec{k}_A \frac{1}{2(q_1^+ - \vec{k}_A^+)} \delta^2(\vec{k}_{A\perp} + \vec{k}_{B\perp} - \vec{q}_{1\perp}) \mathcal{M}^\dagger(\vec{k}_A, q_1^+ - \vec{k}_A^+, k_{B\perp} \rightarrow p_1, p_2) \\ & \cdot \phi^*(\vec{k}_A) \phi^*(\vec{k}_B) \\ & \cdot \delta\left(\frac{(\vec{k}_{A\perp})^2 + m_A^2}{2\vec{k}_A^+} + \frac{(\vec{k}_{B\perp})^2 + m_B^2}{2(\vec{q}_1^+ - \vec{k}_A^+)} - \frac{(\vec{p}_{1\perp})^2 + m_1^2}{2p_1^+} - \frac{(\vec{p}_{2\perp})^2 + m_2^2}{2p_2^+}\right), \end{aligned} \quad (2.138)$$

where $q_1^+ = p_1^+ + p_2^+$ and $\vec{q}_{1\perp} = \vec{p}_{1\perp} + \vec{p}_{2\perp}$. Integrate $d^2\vec{k}_{A\perp}$ using $\delta^2(\vec{k}_{A\perp} + \vec{k}_{B\perp} - \vec{q}_{1\perp})$. This will give $\vec{k}_{A\perp} = \vec{q}_{1\perp} - \vec{k}_{B\perp}$, a similar calculation for $\mathcal{M}$ will give that $\vec{k}_{A\perp} = \vec{k}_{A\perp}$. The $d\vec{k}_A^+$ integration follows,

$$\begin{aligned} & \int \frac{d\vec{k}_A^+}{2\vec{k}_A^+} \frac{1}{2(q_1^+ - \vec{k}_A^+)} \mathcal{M}^\dagger(\vec{k}_A^+, k_{A\perp}, q_1^+ - \vec{k}_A^+, k_{B\perp} \rightarrow p_1, p_2) \phi^*(\vec{k}_A) \phi^*(\vec{k}_B) \\ & \cdot \delta\left(\frac{(\vec{k}_{A\perp})^2 + m_A^2}{2\vec{k}_A^+} + \frac{(\vec{k}_{B\perp})^2 + m_B^2}{2(\vec{q}_1^+ - \vec{k}_A^+)} - \frac{(\vec{p}_{1\perp})^2 + m_1^2}{2p_1^+} - \frac{(\vec{p}_{2\perp})^2 + m_2^2}{2p_2^+}\right). \end{aligned} \quad (2.139)$$

Taking into consideration the properties of the δ function, the integral becomes:

$$\int d\vec{k}_A^+ \mathcal{M}^\dagger(\vec{k}_A^+, k_{A\perp}, q_1^+ - \vec{k}_A^+, k_{B\perp} \rightarrow p_1, p_2) \phi^*(\vec{k}_A) \phi^*(\vec{k}_B) \delta(f(\vec{k}_A^+)), \quad (2.140)$$

where $f(\vec{k}_A^+)$ is given by

$$\begin{aligned} f(\vec{k}_A^+) &= 2(\vec{q}_1^+ - \vec{k}_A^+)((\vec{k}_{A\perp})^2 + m_A^2) + 2\vec{k}_A^+((\vec{k}_{B\perp})^2 + m_B^2) - \\ & - 2(\vec{q}_1^+ - \vec{k}_A^+)2(\vec{k}_A^+)\left(\frac{(\vec{p}_{1\perp})^2 + m_1^2}{2p_1^+} - \frac{(\vec{p}_{2\perp})^2 + m_2^2}{2p_2^+}\right) \end{aligned} \quad (2.141)$$

If we let $f(\vec{k}_{A_0}^+) = 0$ then the value of the integral is:

$$\left[\frac{df(\vec{k}_A^+)}{d\vec{k}_A^+}\right]_{\vec{k}_A^+ = \vec{k}_{A_0}^+}^{-1} \mathcal{M}^\dagger(\vec{k}_{A_0}^+, k_{A\perp}, q_1^+ - \vec{k}_{A_0}^+, k_{B\perp} \rightarrow p_1, p_2) \phi^*(\vec{k}_A) \phi^*(\vec{k}_B) \quad (2.142)$$

In order to find the value $\vec{k}_{A_0}^+$, we must solve the quadratic equation:

$$4(\vec{k}_A^+)^2 Q^- + (\vec{k}_A^+)(2(\vec{k}_{B\perp})^2 + 2m_B^2 - 4q_1^+ Q^- - 2(\vec{k}_{A\perp})^2 - 2m_A^2) + 2q_1^+((\vec{k}_{A\perp})^2 + m_A^2) = 0, \quad (2.143)$$

where $Q^- = \frac{(\vec{p}_1)^2 + m_1^2}{2p_1^+} + \frac{(\vec{p}_2)^2 + m_2^2}{2p_2^+}$. The solution of the equation (2.143) is given by:

$$\begin{aligned} \bar{k}_{A_0}^+ &= \frac{1}{8Q^-} \left\{ - \left[2(\vec{k}_{B\perp})^2 + 2m_B^2 - 4q_1^+ Q^- - 2(\vec{k}_{A\perp})^2 - 2m_A^2 \right] \right. \\ &\quad \pm \left[\left\{ 2(\vec{k}_{B\perp})^2 + 2m_B^2 - 4q_1^+ Q^- - 2(\vec{k}_{A\perp})^2 - 2m_A^2 \right\}^2 - \right. \\ &\quad \left. \left. - 32q_1^+ Q^- \{ (\vec{k}_{A\perp})^2 + m_A^2 \} \right]^{\frac{1}{2}} \right\} \end{aligned} \quad (2.144)$$

Substituting these values into $\frac{df(\bar{k}_A^+)}{d\bar{k}_A^+}$ we obtain,

$$\begin{aligned} \left[\frac{df(\bar{k}_A^+)}{d\bar{k}_A^+} \right] &= \pm \left[\left\{ 2(\vec{k}_{B\perp})^2 + 2m_B^2 - 4q_1^+ Q^- - 2(\vec{k}_{A\perp})^2 - 2m_A^2 \right\}^2 - \right. \\ &\quad \left. - 32q_1^+ Q^- \{ (\vec{k}_{A\perp})^2 + m_A^2 \} \right]^{\frac{1}{2}}, \end{aligned} \quad (2.145)$$

evaluated at $\bar{k}_A^+ = \bar{k}_{A_0}^+$.

Let $\Lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ then

$$\left[\frac{df(\bar{k}_A^+)}{d\bar{k}_A^+} \right]_{\bar{k}_A^+ = \bar{k}_{A_0}^+} = \Lambda^{\frac{1}{2}}(4q_1^+ Q^-, 2((\vec{k}_{A\perp})^2 + m_A^2), 2((\vec{k}_{B\perp})^2 + m_B^2)) \quad (2.146)$$

Thus, the flux factor is given by

$$\Lambda^{-\frac{1}{2}}(4q_1^+ Q^-, 2((\vec{k}_{A\perp})^2 + m_A^2), 2((\vec{k}_{B\perp})^2 + m_B^2)). \quad (2.147)$$

We have assumed that the wave packets are peaked about p_A and p_B . We evaluate the smooth functions of k_A and k_B at p_A and p_B , and take them outside the sign of the integral. The argument of the δ function also includes k_A and k_B . Here too we replace them by p_A and p_B . So the integral becomes:

$$\begin{aligned} d\sigma &= \prod_{i=1,2} \frac{1}{(2\pi)^3} dp_i \Lambda^{-\frac{1}{2}}(4q_1^+ Q^-, 2((\vec{k}_{A\perp})^2 + m_A^2), 2((\vec{k}_{B\perp})^2 + m_B^2)) \\ &\quad | \mathcal{M}(p_A, p_B \rightarrow p_1, p_2) |^2 (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2), \end{aligned} \quad (2.148)$$

where the δ^4 function is given by (2.136). This is the general expression of the differential cross section for a two particle to two particle process, in the LF coordinates.

2.9 x^+ perturbation theory.

If the free field state is represented by $|\Phi\rangle$, then to second order perturbation $|\Psi\rangle$ is given by:

$$|\Psi\rangle = |\Phi\rangle + \sum_{n_1} \frac{|n_1\rangle \langle n_1| H_{int}(0) |\Phi\rangle}{p^- - p_{n_1}^- + i\epsilon}, \quad (2.149)$$

where n_1 represent intermediate states and $H_{int}(0)$ is the interaction Hamiltonian. The matrix elements to second order in g , are then given by the expression,

$$\begin{aligned} \langle \Psi_f | H_{int}(0) | \Psi_i \rangle = & \langle \Phi_f | H_{int}(0) | \Phi_i \rangle + \\ & + \sum_{n_1} \frac{\langle \Phi_f | H_{int}(0) | n_1 \rangle \langle n_1 | H_{int}(0) | \Phi_i \rangle}{p_f^- - p_{n_1}^- + i\epsilon} \end{aligned} \quad (2.150)$$

For physical processes for which the matrix element is computed to second order in g , the two terms in (2.150) have to be computed to the same order in g . This means, for example, that the term (2.118) of the interaction Hamiltonian density $\mathcal{H}_{QED}^{int}$, will contribute to the second term (2.150) with a sum over various intermediate states n_1 , while (2.119) and (2.120) will contribute to the first term of (2.150). In (2.150) p_f^- is the total final (initial) energy while $p_{n_1}^-$ is the total energy of the intermediate state n_1 . The general rules for the creation of the matrix elements of an x^+ perturbative expansion directly from the diagrams in momentum space are:

1. Draw all topologically distinct x^+ - ordered diagrams,
2. For each intermediate state include a factor

$$\frac{1}{p_f^- - p_{n_1}^- + i\epsilon}, \quad (2.151)$$

3. Include an over-all factor of

$$-2\pi i \delta(p_f^- - p_i^-), \quad (2.152)$$

4. For each internal line sum over spins and integrate with

$$\frac{1}{(2\pi)^3} \int d^2 p_\perp \int \frac{dp^+}{2[p^+]} \theta(p^+), \quad (2.153)$$

5. For each vertex,

- (a) include a factor

$$(2\pi)^3 \delta(p_{out}^+ - p_{in}^+) \delta^2(\vec{p}_{out\perp} - \vec{p}_{in\perp}), \quad (2.154)$$

- (b) for each fermion, whether entering or leaving the vertex, include a factor

$$\sqrt{2p^+}, \quad (2.155)$$

- (c) include the matrix element corresponding to the vertex.

We will use these rules in the following chapter to generate matrix elements for certain tree level physical processes and calculate their cross sections. We will need to be careful about the treatment of $p^+ = 0$ singularities mentioned earlier.

Chapter 3

Tree level processes.

In this chapter we present four tree level processes to illustrate the procedure to calculate differential cross sections and total cross sections. We show how the application of the prescription given in expression (1.36) results in the cancelation of the linear infrared singularities in tree level processes. We also verify that the LF expressions for the cross sections are equal to the familiar expressions obtained using the equal time formulation of QFT. As we saw in chapter 1, an energy-momentum four vector which in the equal time form is represented by:

$$p^\mu = (p^0, \ p^1, \ p^2, \ p^3). \quad (3.1)$$

in the LF formulation is represented by:

$$p^\mu = \left(\frac{p^0 - p^3}{\sqrt{2}}, \ p^1, \ p^2, \ \frac{p^0 + p^3}{\sqrt{2}} \right) = (p^-, \ p^1, \ p^2, \ p^+), \quad (3.2)$$

where all particles satisfying the equations of motion, in the LF formulation, are on their mass-shell, thus

$$p^- = \frac{\vec{p}_\perp^2 + m^2}{2p^+}. \quad (3.3)$$

In the LF formulation we use x^+ -oriented diagrams. The “time” variable x^+ , in this, case runs from left to right.

The examples included in this chapter are $e\gamma$ scattering, where no linear infrared singularity can occur, $e^+e^- \rightarrow \mu^+\mu^-$ production where the singularity occurs in an unphysical sector and the $e^+e^- \rightarrow e^+e^-$ and $e\mu \rightarrow e\mu$ processes where cancellations takes place. A linear infrared singularity can occur only in cases where there is an exchange of a virtual photon. In this case, diagram 5 participates in the process and it is its presence that ensures the cancelation of the linear infrared singularity.

3.1 $e\gamma \rightarrow e\gamma$ (Compton scattering).

In this section we derive the expression for the $e\gamma$ scattering cross section using the LF variables. In Appendix A we calculate the components of the Hamiltonian density in terms of the creation and annihilation operators. The normal procedure is to select an initial and a final state and with the help of the commutation relations, select the elements of the Hamiltonian relevant to the problem. That is to say those that allow an overlap between initial and final states. The diagrams relevant for the $e\gamma$ scattering in second order perturbation are

Diagram 7.

Diagram 8.

Diagram 9.

Diagram 10.

Diagrams 7, 8 and 9 include a vertical line slicing through, and highlighting, the particles present in the intermediate state. In accordance with (2.150), we have to calculate the energy denominator given by (2.151). In diagram 7 the total initial (final) energy is given by

$$p_i^- = p_f^- = p_1^- + k_1^- = p_2^- + k_2^-, \quad (3.4)$$

and the total energy of the intermediate state is q_1^- . Thus the energy denominator d_1 is given by

$$d_1 = p_1^- + k_1^- - q_1^- = p_2^- + k_2^- - q_1^-. \quad (3.5)$$

In a similar manner we determine from diagram 8, that the total energy for the intermediate state is

$$k_1^- + q_2^- + k_2^-. \quad (3.6)$$

Thus, the energy denominator d_2 is given by

$$d_2 = p_1^- + k_1^- - q_2^- - k_2^- - k_1^- = p_2^- + k_2^- - q_2^- - k_2^- - k_1^- = p_2^- - q_2^- - k_1^-. \quad (3.7)$$

It is easy to see that in diagrams 8 and 9, $q_2^- = -q_3^-$. From diagram 9, the total energy of the intermediate state is given by

$$p_2^- + q_3^- + p_1^-. \quad (3.8)$$

Then, the energy denominator d_3 is given by

$$d_3 = p_1^- + k_1^- - q_3^- - p_2^- - p_1^- = p_2^- + k_2^- - q_3^- - p_2^- - p_1^- = k_2^- + q_2^- - p_1^-. \quad (3.9)$$

This is typical of the method we will use to calculate the energy denominators for all diagrams which are not a contact or instantaneous diagram. In any frame of reference,

$$q_1^+ = p_1^+ + k_1^+ = p_2^+ + k_2^+, \quad q_2^+ = p_1^+ - k_2^+, \quad q_3^+ = k_2^+ - p_1^+, \quad m_1 = m_2 = m. \quad (3.10)$$

Let us choose to work in a frame where $p_{1\perp} = k_{1\perp} = 0$. For each diagram relevant to the $e\gamma$ scattering process there is an associated matrix element. Matrix element M_1 corresponds to diagram 7, M_2 corresponds to diagram 8, M_3 corresponds to diagram

9 and M_4 corresponds to diagram 10. These matrix elements are as follows:

$$\begin{aligned}
M_1 = & e^2 \sum_{s'} \int \frac{dq_1^+ d^2 \vec{q}_{1\perp}}{(2\pi)^3 2q_1^+} \theta(q_1^+) \chi_{s_2}^\dagger \Gamma^j(q_1, p_2; k_2) \epsilon^{j*}(\lambda_2) \chi_{s'} \sqrt{2p_2^+ 2q_1^+} \\
& \cdot \frac{1}{d_1} \sqrt{2q_1^+ 2p_1^+} \chi_{s'}^\dagger \Gamma^i(p_1, q_1; k_1) \epsilon^i(\lambda_1) \chi_{s_1} (2\pi)^3 \delta(p_1^+ + k_1^+ - q_1^+) \\
& \cdot \delta^2(\vec{p}_{1\perp} + \vec{k}_{1\perp} - \vec{q}_{1\perp}) (2\pi)^3 \delta(p_2^+ + k_2^+ - q_1^+) \\
& \cdot \delta^2(\vec{p}_{2\perp} + \vec{k}_{2\perp} - \vec{q}_{1\perp}) (-2\pi i) \delta(p_1^- + k_1^- - p_2^- - k_2^-), \tag{3.11}
\end{aligned}$$

$$\begin{aligned}
M_2 = & e^2 \sum_{s'} \int \frac{dq_2^+ d^2 \vec{q}_{2\perp}}{(2\pi)^3 2q_2^+} \theta(q_2^+) \chi_{s_2}^\dagger \Gamma^i(q_2, p_2; k_1) \epsilon^i(\lambda_1) \chi_{s'} \sqrt{2p_2^+ 2q_2^+} \\
& \cdot \frac{1}{d_2} \sqrt{2q_2^+ 2p_1^+} \chi_{s'}^\dagger \Gamma^j(p_1, q_2; k_2) \epsilon^{j*}(\lambda_2) \chi_{s_1} (2\pi)^3 \delta(p_2^+ - q_2^+ - k_1^+) \\
& \cdot \delta^2(\vec{p}_{2\perp} - \vec{q}_{2\perp} - \vec{k}_{1\perp}) (2\pi)^3 \delta(k_2^+ + q_2^+ - p_1^+) \\
& \cdot \delta^2(\vec{k}_{2\perp} + \vec{q}_{2\perp} - \vec{p}_{1\perp}) (-2\pi i) \delta(p_1^- + k_1^- - p_2^- - k_2^-), \tag{3.12}
\end{aligned}$$

$$\begin{aligned}
M_3 = & e^2 \sum_{s'} \int \frac{dq_3^+ d^2 \vec{q}_{3\perp}}{(2\pi)^3 2q_3^+} \theta(q_3^+) \chi_{s_2}^\dagger \Gamma^i(q_3, p_2; k_1) \epsilon^i(\lambda_1) \chi_{-s'} \sqrt{2p_2^+ 2q_3^+} \\
& \cdot \frac{1}{d_3} \sqrt{2q_3^+ 2p_1^+} \chi_{-s'}^\dagger \Gamma^j(p_1, q_3; k_2) \epsilon^{j*}(\lambda_2) \chi_{s_1} (2\pi)^3 \delta(p_2^+ + q_3^+ - k_1^+) \\
& \cdot \delta^2(\vec{p}_{2\perp} + \vec{q}_{3\perp} - \vec{k}_{1\perp}) (2\pi)^3 \delta(k_2^+ - p_1^+ - q_3^+) \\
& \cdot \delta^2(\vec{k}_{2\perp} - \vec{p}_{1\perp} - \vec{q}_{3\perp}) (-2\pi i) \delta(p_1^- + k_1^- - p_2^- - k_2^-), \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
M_4 = & e^2 \chi_{s_2}^\dagger \sigma \cdot \epsilon(\lambda_1) \frac{1}{2[p_1^+ - k_2^+]} \sigma \cdot \epsilon^*(\lambda_2) \chi_{s_1} \sqrt{2p_1^+ 2p_2^+} \\
& \cdot (2\pi)^3 (-2\pi i) \delta^4(p_2 + k_2 - p_1 - k_1), \tag{3.14}
\end{aligned}$$

where in accordance with the rules for the fermion-fermion-photon vertex described by (2.121), the Γ terms are given by the expressions,

$$\begin{aligned}
\Gamma^j(q_1, p_2; k_2) &= \frac{k_2^j}{k_2^+} - \frac{\sigma \cdot p_2 - im}{2p_2^+} \sigma^j - \sigma^j \frac{\sigma \cdot q_1 + im}{2q_1^+}, \\
\Gamma^i(p_1, q_1; k_1) &= \frac{k_1^i}{k_1^+} - \frac{\sigma \cdot q_1 - im}{2q_1^+} \sigma^i - \sigma^i \frac{\sigma \cdot p_1 + im}{2p_1^+}, \\
\Gamma^i(q_2, p_2; k_1) &= \frac{k_1^i}{k_1^+} - \frac{\sigma \cdot p_2 - im}{2p_2^+} \sigma^i - \sigma^i \frac{\sigma \cdot q_2 + im}{2q_2^+}, \\
\Gamma^j(p_1, q_2; k_2) &= \frac{k_2^j}{k_2^+} - \frac{\sigma \cdot q_2 - im}{2q_2^+} \sigma^j - \sigma^j \frac{\sigma \cdot p_1 + im}{2p_1^+}, \\
\Gamma^j(p_1, q_3; k_2) &= \Gamma^j(p_1, -q_2; k_2) = \frac{k_2^j}{k_2^+} - \frac{\sigma \cdot q_3 - im}{2q_3^+} \sigma^j - \sigma^j \frac{\sigma \cdot p_1 + im}{2p_1^+},
\end{aligned}$$

$$\Gamma^i(q_3, p_2; k_1) = \Gamma^i(-q_2, p_2; k_1) = \frac{k_1^i}{k_1^+} - \frac{\sigma \cdot p_2 - im}{2p_2^+} \sigma^i - \sigma^i \frac{\sigma \cdot q_3 + im}{2q_3^+}. \quad (3.15)$$

In the chosen frame of reference we have that

$$\begin{aligned} \frac{1}{d_1} &= \frac{1}{p_1^- + k_1^- - q_1^-} = \frac{p_1^+ q_1^+}{2m^2 k_1^+}, \\ \frac{1}{d_2} &= \frac{1}{p_2^- - k_1^- - q_2^-} = \frac{-2p_2^+ q_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)}, \\ \frac{1}{d_3} &= \frac{1}{k_2^- - p_1^- + q_2^-} = \frac{2p_2^+ q_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)} = -\frac{1}{d_2}, \\ \frac{1}{2q_2^+} \frac{1}{d_2} &= \frac{-p_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)} = \frac{1}{2q_3^+} \frac{1}{d_3}. \end{aligned} \quad (3.16)$$

We note that $d_2 = -d_3$, this together with the minus sign for M_3 from the commutation relations makes the signs of M_2 and M_3 equal. Since $\theta(p_1^+ - p_2^+) + \theta(p_2^+ - p_1^+) = 1$, then $M_2 + M_3$ is equal to M_2 without the θ function. Since $p^+ \geq 0$, a $\theta(p^+)$ must be present for each particle, as indicated in (2.153). From equation (2.148), the differential cross section is given by the expression:

$$d\sigma = \frac{dp_2^+}{(2\pi)^3 2p_2^+} d^2\vec{p}_{2\perp} \frac{dk_2^+}{(2\pi)^3 2k_2^+} d^2\vec{k}_{2\perp} \frac{|M|^2 (2\pi)^4 \delta^4(p_1 + k_1 - p_2 - k_1)}{\Lambda^{\frac{1}{2}}(4q_1^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m^2], 2(\vec{k}_{1\perp})^2)}, \quad (3.17)$$

where the Λ function and Q^- are given in this case by the expressions:

$$Q^- = p_1^- + k_1^- = p_2^- + k_2^- = \frac{m^2}{2p_1^+}, \quad (3.18)$$

$$\Lambda\left(4q_1^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m^2], 2(\vec{k}_{1\perp})^2\right) = 16(q_1^+)^2 \left(Q^- - \frac{m^2}{2q_1^+}\right)^2 = 4m^4 \left(\frac{k_1^+}{p_1^+}\right)^2, \quad (3.19)$$

where the δ^4 notation from (2.136) is used and $|M|^2$ is the square of the modulus of the amplitude of the process summed over final states and averaged over the initial states,

$$|M|^2 = \frac{1}{4} \sum_{s_2, \lambda_2, s_1, \lambda_1} |M_1 + M_2 + M_3 + M_4|^2.$$

For the phase-space integrals, we use the $\delta(p_1^+ + k_1^+ - p_2^+ - k_2^+)$ and $\delta^2(\vec{p}_{1\perp} + \vec{k}_{1\perp} - \vec{p}_{2\perp} - \vec{k}_{2\perp})$ functions when performing the integration for one of the final particles, say p_2 .

Because of the δ functions, the integration on dp_2^+ and $d^2\vec{p}_{2\perp}$ makes $p_2^+ = q_1^+ - k_2^+$

and $\vec{p}_{1\perp} + \vec{k}_{1\perp} = \vec{p}_{2\perp} + \vec{k}_{2\perp} = 0$, hence $\vec{p}_{2\perp} = -\vec{k}_{2\perp}$. We write $d^2\vec{k}_{2\perp} = k_{2\perp} dk_{2\perp} d\varphi = \frac{1}{2}d(k_{2\perp}^2)d\varphi = \frac{1}{2}d(\vec{k}_{2\perp}^2)d\varphi$. From the definition of p^- we have that

$$\delta(p_1^- + k_1^- - p_2^- - k_2^-) = \left[2 \frac{k_2^+(q_1^+ - k_2^+)}{q_1^+} \right] \delta \left(m^2 \left[\frac{k_2^+(k_1^+ - k_2^+)}{p_1^+ q_1^+} \right] - \vec{k}_{2\perp}^2 \right). \quad (3.20)$$

This means that the value of $(\vec{k}_{2\perp})^2$, is fixed by this δ function. Introducing (3.19) in (3.17), we get

$$d\sigma = \frac{dk_2^+}{2k_2^+} \frac{1}{2} d(k_{2\perp}^2) d\varphi \frac{dp_2^+}{(2\pi)^2 2p_2^+} d^2\vec{p}_{2\perp} \frac{p_1^+}{2m^2 k_1^+} |M|^2 \delta^4(p_1 + k_1 - p_2 - k_1). \quad (3.21)$$

We integrate the differential cross section on dp_2^+ and $d^2\vec{p}_{2\perp}$, taking into account (3.20), and the result is

$$d\sigma = \int dk_2^+ d(k_{2\perp}^2) d\varphi \frac{A^2}{8(2\pi)^2} \frac{p_1^+}{m^2 k_1^+} \frac{1}{q_1^+} \delta \left(m^2 \left[\frac{k_2^+(k_1^+ - k_2^+)}{p_1^+ q_1^+} \right] - \vec{k}_{2\perp}^2 \right), \quad (3.22)$$

where $A^2 = |M|^2_{p_2^+ = p_1^+ + k_1^+ - k_2^+, \vec{p}_{2\perp} = \vec{p}_{1\perp} + \vec{k}_{2\perp} - \vec{k}_{1\perp}}$. The argument of the δ function constrains k_1^+ to be $\geq k_2^+$, because $\vec{k}_{2\perp}^2 \geq 0$. We now introduce the dimensionless variables x and y . Let

$$x = \frac{k_1^+}{q_1^+}, \quad y = \frac{k_2^+}{q_1^+} \quad (3.23)$$

then,

$$p_1^+ = (1-x)q_1^+, \quad p_2^+ = (1-y)q_1^+, \quad k_{2\perp}^2 = \frac{m^2 y(x-y)}{(1-x)} \quad (3.24)$$

which demands that $x > y$, because $\vec{k}_{2\perp}^2$ is manifestly positive or zero. Then there is the $\theta(q_2^+) = \theta(p_1^+ - k_2^+)$ function, which demands that $(1-x) \geq y$. On the other hand there is the $\theta(-q_2^+) = \theta(-p_1^+ + k_2^+)$ function, which demands that $(1-x) \leq y$.

These conditions complement each other. For $x = \frac{1}{2}$, we have the boundary between the two θ functions. Therefore for $x < \frac{1}{2}$, we obtain $0 \leq y \leq (1-x)$ while for $x > \frac{1}{2}$, we obtain $(1-x) \leq y \leq x$. Combining both domains we obtain that $0 \leq y \leq x$. From Appendix B, (B.10) we see that:

$$A^2 = \frac{e^4}{x^2} \left\{ (8-8x+2x^2-2x^3) + 2yx^2 + \frac{(8-16x+8x^2)}{(1-x+y)^2} + \frac{(-16+24x-6x^2)}{(1-x+y)} \right\}. \quad (3.25)$$

The differential cross section is then given by the expression:

$$\frac{d^2\sigma}{dyd\varphi} = \frac{\alpha^2}{2m^2} \left[-2 \frac{(-4 + 4x - x^2 + x^3)}{x^2} + 2y + 8 \frac{(1 - 2x + x^2)}{x^2(1 - x + y)^2} - 2 \frac{(8 - 12x + 3x^2)}{x^2(1 - x + y)} \right] \cdot \left[\frac{1 - x}{x} \right]. \quad (3.26)$$

We can now calculate the expression for the angular dependence of the differential cross section. This requires that we know the functional dependence of y in $\cos \theta$

$$d\Omega = d(\cos \theta) d\varphi, \quad (3.27)$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{d^2\sigma}{dyd\varphi} \frac{dy}{d(\cos \theta)}, \\ &= \frac{\alpha^2}{2m^2} \left[-2 \frac{(-4 + 4x - x^2 + x^3)}{x^2} + 2y + 8 \frac{(1 - 2x + x^2)}{x^2(1 - x + y)^2} - 2 \frac{(8 - 12x + 3x^2)}{x^2(1 - x + y)} \right] \cdot \left[\frac{1 - x}{x} \right] \frac{dy}{d(\cos \theta)}. \end{aligned} \quad (3.28)$$

We will consider two frames of reference, the laboratory and the CMS (Center of Momentum System).

Laboratory reference frame:

We will assume that the electron is at rest. Under this assumption the energy-momentum four vectors of the initial particles, in the equal time formulation, are

$$p_1^\mu = (m, 0, 0, 0), \quad k_1^\mu = (\omega, 0, 0, \omega) \quad (3.29)$$

while for the final photon we have: $k_2^\mu = (\omega', k_2^1, k_2^2, \omega' \cos \theta)$. From these we obtain

$$p_1^+ = \frac{1}{\sqrt{2}}m, \quad k_1^+ = \sqrt{2}\omega, \quad k_2^+ = \frac{1}{\sqrt{2}}\omega'(1 + \cos \theta). \quad (3.30)$$

Letting $\gamma = \omega/m$, we have

$$\begin{aligned} x &= \frac{2\gamma}{2\gamma + 1}, \quad \gamma = \frac{x}{2(1 - x)}, \quad \frac{\omega'}{\omega} = \frac{1}{1 + \gamma(1 - \cos \theta)}, \\ y &= \frac{\gamma(1 + \cos \theta)}{(2\gamma + 1)(1 + \gamma(1 - \cos \theta))}, \end{aligned}$$

where m is the mass of the electron, ω is the energy of the incident photon and ω' is the energy of the scattered photon. To calculate angular dependence we consider

$$y = \frac{x(1 + \cos \theta)}{2(1 + \gamma(1 - \cos \theta))}, \quad \frac{dy}{d(\cos \theta)} = \frac{\gamma}{(1 + \gamma(1 - \cos \theta))^2} = \left(\frac{\omega'}{\omega}\right)^2 \frac{\omega}{m}$$

The angular dependence of the differential cross section is

$$\frac{d\sigma}{d(\cos \theta)} = \pi \frac{\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2 \theta \right]. \quad (3.31)$$

This expression for the angular dependence of the differential cross section coincides with expression (5.91) of Peskin and Schroeder [25].

To calculate the total cross section we integrate either the expression given in (3.26), for $0 \leq \varphi \leq 2\pi$ and $0 \leq y \leq x$ or we can integrate the expression given in (3.31) for $0 \leq \theta \leq \pi$. We choose to integrate (3.26), because the results obtained this way are easier to compare with the expression of the cross section given in Heitler [26]. The Thomson cross section is given by:

$$\sigma_0 = \frac{8\pi}{3} \frac{\alpha^2}{m^2} \quad (3.32)$$

then we have:

$$\begin{aligned} \sigma &= \int_0^x \pi \frac{\alpha^2}{m^2} \left[2 \frac{(4 - 4x + x^2 - x^3)}{x^2} + 2y + 8 \frac{(1 - 2x + x^2)}{x^2(1 - x + y)^2} - 2 \frac{(8 - 12x + 3x^2)}{x^2(1 - x + y)} \right] \\ &\quad \cdot \left[\frac{1 - x}{x} \right] dy, \\ &= \pi \frac{\alpha^2}{m^2} \frac{1}{x^3} [16x - 32x^2 + 18x^3 - 3x^4 + x^5 + (16 - 40x + 30x^2 - 6x^3) \ln(1 - x)], \\ &= \frac{3}{4} \sigma_0 \left\{ \frac{1 + \gamma}{\gamma^3} \left[\frac{2\gamma(1 + \gamma)}{1 + 2\gamma} - \ln(1 + 2\gamma) \right] + \frac{1}{2\gamma} \ln(1 + 2\gamma) - \frac{1 + 3\gamma}{(1 + 2\gamma)^2} \right\}. \end{aligned} \quad (3.33)$$

This expression for the total cross section agrees with the expression given in Heitler [26].

In the CMS reference frame:

The energy-momentum four vectors for the initial particles, in the equal time formulation, are

$$p_1^\mu = (E_e, 0, 0, -\omega), \quad k_1^\mu = (\omega, 0, 0, \omega) \quad (3.34)$$

while for the final photon we have: $k_2^\mu = (\omega, k_2^1, k_2^2, \omega \cos \theta)$, from which we obtain

$$p_1^+ = \frac{1}{\sqrt{2}}(E_e - \omega), \quad k_1^+ = \sqrt{2}\omega, \quad k_2^+ = \frac{1}{\sqrt{2}}\omega(1 + \cos \theta). \quad (3.35)$$

We calculate the dimensionless variables x and y .

$$x = \frac{2\omega}{E_e + \omega}, \quad y = \frac{\omega(1 + \cos \theta)}{E_e + \omega}, \quad \frac{dy}{d(\cos \theta)} = \frac{\omega}{E_e + \omega}. \quad (3.36)$$

The differential cross section is given by the expression:

$$\begin{aligned} \frac{d\sigma}{d\Omega} = & \frac{\omega\alpha^2}{E_e + \omega} \left[\frac{1 + \cos \theta}{2(E_e + \omega)^3} + \frac{(E_e - \omega)^2(E_e + \omega)}{2\omega^3(E_e + \omega \cos \theta)^2} + \frac{(E_e^3 + E_e^2\omega - 2\omega^3)}{2\omega^3(E_e + \omega)^2} \right. \\ & \left. - \frac{(E_e + \omega)^2(8 + \frac{12\omega^2}{(E_e + \omega)^2} - \frac{24\omega}{E_e + \omega})}{8\omega^3(E_e + \omega \cos \theta)} \right]. \end{aligned} \quad (3.37)$$

At high energy and when $\theta \approx \pi$ then the following approximation applies.

$$\frac{d\sigma}{d(\cos \theta)} = \frac{\pi\alpha^2}{2\omega^2(1 + \cos \theta)}. \quad (3.38)$$

This expression is equal to the one given on page 164 of Peskin and Schroeder [25].

3.2 $e^+ e^- \rightarrow \mu^+ \mu^-$ production.

This process involves the annihilation of a particle with its anti-particle. The anti-particle with energy-momentum four vector p^μ is represented by a particle of energy-momentum four vector $-p^\mu$, as indicated in section 2.6. The diagrams relevant for the $e^+ e^- \rightarrow \mu^+ \mu^-$ production process in second order perturbation are

There are two diagrams involved. A third diagram that would seem applicable is forbidden because involves three particles being created from the vacuum. The diagram we are referring to is the following:

Diagram 13.

In diagram 13 $\delta(k_1^+ + p_3^+ + p_4^+)$ will always be zero.

In diagram 11 the total initial (final) energy is given by

$$p_i^- = p_f^- = p_1^- + p_2^- = p_3^- + p_4^-, \quad (3.39)$$

and the total energy of the intermediate state is k^- . Thus, the energy denominator d_1 is given by

$$d_1 = p_1^- + p_2^- - k^- = p_3^- + p_4^- - k^-. \quad (3.40)$$

In any frame of reference $k^+ = p_1^+ + p_2^+ = p_3^+ + p_4^+$. Let us choose to work in a frame where $\vec{p}_{1\perp} = \vec{p}_{2\perp} = 0$. Here matrix element M_1 corresponds to diagram 11 and matrix element M_2 corresponds to diagram 12. These matrix elements are as follows:

$$\begin{aligned} M_1 = & e^2 \int \frac{dk^+ d^2 \vec{k}_\perp}{(2\pi)^3 2k^+} \theta(k^+) \sum_\lambda \chi_{s_3}^\dagger \Gamma^i(-p_4, p_3; k) \chi_{-s_4} \epsilon^i(\lambda) \frac{1}{d_1} \chi_{-s_2}^\dagger \\ & \cdot \Gamma^j(p_1, -p_2; k) \chi_{s_1} \epsilon^{j*}(\lambda) (2\pi)^3 \delta(p_3^+ + p_4^+ - k^+) \delta^2(\vec{p}_{3\perp} + \vec{p}_{4\perp} - \vec{k}_\perp) \\ & \cdot (2\pi)^3 \delta(p_2^+ + p_1^+ - k^+) \delta^2(\vec{p}_{2\perp} + \vec{p}_{1\perp} - \vec{k}_\perp) \sqrt{2p_3^+ 2p_4^+} \\ & \cdot \sqrt{2p_1^+ 2p_2^+} (-2i\pi) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \end{aligned} \quad (3.41)$$

$$M_2 = e^2 \chi_{-s_2}^\dagger \chi_{s_1} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ + p_2^+]^2} \chi_{s_3}^\dagger \chi_{-s_4} \sqrt{2p_3^+ 2p_4^+}. \quad (3.42)$$

The Γ terms are given by the expressions,

$$\Gamma^i(-p_4, p_3; k) = \frac{k^i}{k^+} - \frac{\sigma \cdot p_3 - im_\mu}{2p_3^+} \sigma^i - \sigma^i \frac{\sigma \cdot (-p_4) + im_\mu}{-2p_4^+},$$

$$\Gamma^j(p_1, -p_2; k) = \frac{k^j}{k^+} - \frac{\sigma(-p_2) - im_e}{-2p_2^+} \sigma^j - \sigma^j \frac{\sigma p_1 + im_e}{2p_1^+}. \quad (3.43)$$

We remind ourselves that, as indicated in section 2.6, we use $-p^\mu$ for the Γ of an anti-particle, but we always use p^μ for the associated δ functions.

In the chosen frame of reference

$$\frac{1}{d_1} = \frac{1}{p_1^- + p_2^- - k^-} = \frac{2}{m_e^2(\frac{1}{p_1^+} + \frac{1}{p_2^+})}. \quad (3.44)$$

We will make the approximation $m_\mu \gg m_e$, or $m_e = 0$ (whenever possible). In the LF formulation of QED, the possibility of an infrared singularity arises when a photon is exchanged between two vertices. In our case there is a virtual photon being exchanged, but there is no infrared singularity. We can see that there is a term $\frac{k^2}{[k^+]^2}$ from M_1 and a term $\frac{4}{[p_1^+ + p_2^+]}$ from the instantaneous term, with $k^+ = p_1^+ + p_2^+$. An infrared singularity could happen when $p_1^+ + p_2^+ = 0$, but that implies $p_1^+ = p_2^+ = 0$ as in the LF formulation $p^+ \geq 0$. We have assumed that $m_e = 0$ which means that $E_e = p_e = 0$, so the muons cannot be created. This means that the sector $p_1^+ + p_2^+ = 0$, is unphysical. However because of crossing-symmetries, an unphysical sector in one process becomes physical in another. In Appendix C we show that a cancelation of these $\frac{1}{[k^+]^2}$ terms takes place, and that the amplitude is well behaved in the $k^+ = 0$ sector. Also, the calculations in Appendix C show that for the conditions $\vec{p}_{1\perp} = \vec{p}_{2\perp} = 0$, there is no interference between diagrams 11 and 12. And the fact that we consider $m_e = 0$ eliminates the contact term altogether. As in the previous example, we use the $\delta(p_1^- + p_2^- - p_3^- - p_4^-)$ to determine the boundaries of integration. From equation (2.148), the differential cross section is given by the expression:

$$d\sigma = \frac{dp_3^+ d^2\vec{p}_{3\perp}}{(2\pi)^3 2 p_3^+} \frac{dp_4^+ d^2\vec{p}_{4\perp}}{(2\pi)^3 2 p_4^+} \frac{|M|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{\Lambda^{\frac{1}{2}}(4k^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m_1^2], 2[(\vec{p}_{2\perp})^2 + m_2^2])}, \quad (3.45)$$

where the Λ function and Q^- are given in this case by the expressions:

$$Q^- = p_3^- + p_4^- = p_1^- + p_2^-,$$

$$\Lambda\left(4k^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m_1^2], 2[(\vec{p}_{2\perp})^2 + m_2^2]\right) = 4m_e^4 \left(\frac{(p_1^+)^2 - (p_2^+)^2}{p_1^+ p_2^+}\right)^2, \quad (3.46)$$

where the δ^4 notation from (2.136) is used and $|M|^2$ is the square of the modulus of the amplitude of the process summed over final states and averaged over the initial states,

$$|M|^2 = \frac{1}{4} \sum_{s_4, s_3, s_2, s_1} |M_1 + M_2|^2.$$

For the phase-space integrals, we use the $\delta(p_1^+ + p_2^+ - p_3^+ - p_4^+)$ and $\delta^2(\vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{p}_{3\perp} - \vec{p}_{4\perp})$ functions when performing the integration for one of the final particles, say p_4 . Because of the δ functions, the integration on dp_4^+ and $d^2\vec{p}_{4\perp}$ makes $p_3^+ = k^+ - p_4^+$ and $\vec{p}_{1\perp} + \vec{p}_{2\perp} = \vec{p}_{3\perp} + \vec{p}_{4\perp} = 0$. Hence $\vec{p}_{3\perp} = -\vec{p}_{4\perp}$. We write $d^2\vec{p}_{3\perp} = p_{3\perp} dp_{3\perp} d\varphi = \frac{1}{2}d(p_{3\perp}^2)d\varphi = \frac{1}{2}d(\vec{p}_{3\perp})^2 d\varphi$. From the definition of p^- we have:

$$\delta(p_1^- + p_2^- - p_3^- - p_4^-) = 2 \frac{p_3^+ p_4^+}{p_1^+ + p_2^+} \delta\left((\vec{p}_{3\perp})^2 + m_\mu^2 - m_e^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right). \quad (3.47)$$

This δ function constrains the value of $(\vec{p}_{3\perp})^2$. Then the differential cross section is given by the expression:

$$d\sigma = \frac{dp_3^+}{2p_3^+} \frac{1}{2} d(p_{3\perp}^2) d\varphi \frac{dp_4^+ d^2\vec{p}_{4\perp}}{(2\pi)^2 2p_4^+} \frac{p_1^+ p_2^+}{2m_e^2[(p_1^+)^2 - (p_2^+)^2]} |M|^2 \delta^4(p_1 + p_2 - p_3 - p_4). \quad (3.48)$$

We integrate the differential cross section on dp_4^+ and $d^2\vec{p}_{4\perp}$, taking into account (3.47), and the result is

$$d\sigma = dp_3^+ d(p_{3\perp}^2) d\varphi \frac{A^2 p_1^+ p_2^+}{8(2\pi)^2 m_e^2 (p_1^+ - p_2^+)(p_1^+ + p_2^+)^2} \delta\left((\vec{p}_{3\perp})^2 + m_\mu^2 - m_e^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right), \quad (3.49)$$

where $A^2 = |M|^2_{p_4^+ = p_1^+ + p_2^+ - p_3^+, \vec{p}_{4\perp} = \vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{p}_{3\perp}}$, (see Appendix C for a detailed calculation of A^2). From Appendix C, expression (C.22) we see that:

$$A^2 = \frac{2e^4}{m_e^2 (p_1^+ + p_2^+)^2} \left\{ m_\mu^2 \left(\frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right)^2 + \frac{(\vec{p}_{3\perp})^2}{[p_3^+]^2} + \frac{(\vec{p}_{4\perp})^2}{[p_4^+]^2} \right\} p_1^+ p_2^+ p_3^+ p_4^+ \quad (3.50)$$

Thus, the differential cross section becomes

$$\begin{aligned} d\sigma = & dp_3^+ d(p_{3\perp}^2) d\varphi \frac{e^4}{(4\pi)^2} \frac{p_1^+ p_2^+ (p_1^+ p_2^+ p_3^+ p_4^+)}{m_e^4 (p_1^+ - p_2^+)(p_1^+ + p_2^+)^4} \\ & \cdot \left\{ m_\mu^2 \left(\frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right)^2 + \frac{(\vec{p}_{3\perp})^2}{[p_3^+]^2} + \frac{(\vec{p}_{4\perp})^2}{[p_4^+]^2} \right\} \\ & \cdot \delta\left((\vec{p}_{3\perp})^2 + m_\mu^2 - m_e^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right). \end{aligned} \quad (3.51)$$

As pointed out before, since $(\vec{p}_{3\perp})^2 = (\vec{p}_{4\perp})^2 \geq 0$, then the argument of the δ function in $(\vec{p}_{3\perp})^2$,

$$\delta\left((\vec{p}_{3\perp})^2 + m_\mu^2 - m_e^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right) = \delta\left((\vec{p}_{3\perp})^2 + m_\mu^2 - m_e^2 \frac{p_3^+(p_1^+ + p_2^+ - p_3^+)}{p_1^+ p_2^+}\right), \quad (3.52)$$

shows that the boundaries of integration over p_3^+ are determined by the equation

$$(p_3^+)^2 - (p_1^+ + p_2^+) p_3^+ + \frac{m_\mu^2}{m_e^2} p_1^+ p_2^+ = 0. \quad (3.53)$$

From this we extract the boundaries of integration for dp_3^+ to be

$$p_{3min}^+ = \frac{1}{2} \left\{ p_1^+ + p_2^+ - \sqrt{(p_1^+ + p_2^+)^2 - 4 \frac{m_\mu^2}{m_e^2} p_1^+ p_2^+} \right\}, \quad (3.54)$$

$$p_{3max}^+ = \frac{1}{2} \left\{ p_1^+ + p_2^+ + \sqrt{(p_1^+ + p_2^+)^2 - 4 \frac{m_\mu^2}{m_e^2} p_1^+ p_2^+} \right\}. \quad (3.55)$$

We now integrate on $dp_{3\perp}^2$ to obtain the differential cross section.

$$\begin{aligned} \frac{d^2\sigma}{dp_3^+ d\varphi} &= \frac{e^4}{(4\pi)^2} \frac{p_1^+ p_2^+ (p_1^+ p_2^+ p_3^+ p_4^+)}{m_e^4 (p_1^+ - p_2^+) (p_1^+ + p_2^+)^4} \\ &\quad \cdot \left\{ m_\mu^2 \left(\frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right)^2 + \frac{\vec{p}_{3\perp}^2}{[p_3^+]^2} + \frac{\vec{p}_{4\perp}^2}{[p_4^+]^2} \right\}, \\ &= \alpha^2 \frac{(p_1^+ p_2^+)^2}{m_e^4 (p_1^+ - p_2^+) (p_1^+ + p_2^+)^4} \\ &\quad \cdot \left\{ 2 m_\mu^2 + \frac{m_e^2}{p_1^+ p_2^+} ([p_3^+]^2 + [p_4^+]^2) \right\}. \end{aligned} \quad (3.56)$$

We can express the differential cross section as a function of the dimensionless variables x and y , where these variables are defined as follows:

$$x = \frac{p_1^+}{k^+}, \quad y = \frac{p_3^+}{k^+}. \quad (3.57)$$

The differential cross section when expressed in terms of these variables becomes:

$$\frac{d^2\sigma}{dy d\varphi} = \alpha^2 \frac{x^2 (1-x)^2}{m_e^4 (2x-1)} \left\{ 2 m_\mu^2 + \frac{m_e^2}{x(1-x)} (y^2 + (1-y)^2) \right\}. \quad (3.58)$$

Under this representation the boundaries of integration become:

$$y_{min} = \frac{1}{2} \left\{ 1 - \sqrt{1 - 4 \frac{m_\mu^2}{m_e^2} x (1-x)} \right\}, \quad (3.59)$$

$$y_{max} = \frac{1}{2} \left\{ 1 + \sqrt{1 - 4 \frac{m_\mu^2}{m_e^2} x (1-x)} \right\}. \quad (3.60)$$

This expression is totally independent of any frame of reference. However, to show the angular dependence of the differential cross section we must choose a frame of reference. As in the previous example we can choose to work in either laboratory frame of reference or the CMS frame of reference. Let us choose the CMS reference frame. In CMS $E_e = E_\mu = E$. Whenever possible we will also make the approximation $p_e \approx E$ or $m_e \approx 0$.

Let

$$\begin{aligned} p_1^+ &= \frac{1}{\sqrt{2}}(E + p_e), \quad p_2^+ = \frac{1}{\sqrt{2}}(E - p_e), \quad p_3^+ = \frac{1}{\sqrt{2}}(E + p_\mu \cos \theta), \quad p_1^+ p_2^+ = \frac{1}{2} m_e^2, \\ p_4^+ &= \frac{1}{\sqrt{2}}(E - p_\mu \cos \theta), \quad \beta_e = \frac{p_e}{E} = \sqrt{1 - \frac{m_e^2}{E^2}}, \quad \beta_\mu = \frac{p_\mu}{E} = \sqrt{1 - \frac{m_\mu^2}{E^2}}, \\ p_{3min}^+ &= \sqrt{2}E \{ 1 - \beta_\mu \}, \quad p_{3max}^+ = \sqrt{2}E \{ 1 + \beta_\mu \}, \quad \frac{dp_3^+}{d(\cos \theta)} = \frac{p_\mu}{\sqrt{2}}. \end{aligned} \quad (3.61)$$

We now calculate the angular dependence of the differential cross section, first in terms of the dimensionless variables,

$$x = \frac{1}{2}(1 + \beta_e), \quad y = \frac{1}{2}(1 + \beta_\mu \cos \theta), \quad \frac{dy}{d(\cos \theta)} = \frac{\beta_\mu}{2}, \quad (3.62)$$

$$\frac{d\sigma}{d\Omega} = \alpha^2 \frac{x^2 (1-x)^2}{m_e^4 (2x-1)} \frac{\beta_\mu}{2} \left\{ 2m_\mu^2 + \frac{m_e^2}{x(1-x)} (y^2 + (1-y)^2) \right\}. \quad (3.63)$$

The expression does not explicitly show the angular dependence as it is hidden inside the y 's. We must, therefore, replace the dimensionless variables by their values given above. Instead, we choose to start from (3.56). Thus,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2 (p_1^+ p_2^+)^2}{m_e^4 (p_1^+ - p_2^+) (p_1^+ + p_2^+)^4} \left\{ 2m_\mu^2 + \frac{m_e^2}{p_1^+ p_2^+} ([p_3^+]^2 + [p_4^+]^2) \right\} \frac{1}{\sqrt{2}} p_\mu, \\ &= \frac{\alpha^2}{4 E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right\}, \end{aligned} \quad (3.64)$$

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{2E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right\}. \quad (3.65)$$

This expression for the angular dependence of the differential cross section agrees with formula (5.12) of Peskin and Schroeder [25]. As in the previous example we can either integrate (3.56) for the boundaries $0 \leq \varphi \leq 2\pi$ and $p_{3min}^+ \leq p_3^+ \leq p_{3max}^+$ or we can integrate the expression given in (3.65) for $0 \leq \theta \leq \pi$. Both methods must and do give the same result. In this case we choose to integrate (3.65).

$$\begin{aligned} \sigma &= \int_0^\pi \frac{\pi\alpha^2}{2E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right\} d(\cos\theta), \\ &= \frac{4\pi\alpha^2}{3E_{cm}^2} \left\{ 1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right\} \sqrt{1 - \frac{m_\mu^2}{E^2}} \end{aligned} \quad (3.66)$$

This expression of the total cross section agrees with formula (5.13) of Peskin and Schroeder [25]. For a high energy process $E \gg m_\mu^2$ and $E = p_e = p_\mu$.

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{2E_{cm}^2} \{1 + \cos^2\theta\}. \quad (3.67)$$

This expression is identical with formula (5.14) of Peskin and Schroeder [25]. For the non-relativistic limit $E \approx m_\mu$, then:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}}. \quad (3.68)$$

Again this expression agrees with formula (5.33) of Peskin and Schroeder [25].

3.3 $e^+ e^- \rightarrow e^+ e^-$ (Bhabha scattering).

This example clearly shows the contribution of the boundary integrals to the cancellation of the infrared singularities, which has to be explicitly counter-balanced and canceled by the instantaneous term. There are five contributing diagrams:

Diagram 14.

Diagram 15.

Diagram 16.

Diagram 17.

Diagram 18.

In diagram 14 the total initial (final) energy is given by

$$p_i^- = p_f^- = p_1^- + p_2^- = p_3^- + p_4^-, \quad (3.69)$$

and the total energy of the intermediate state is $p_2^- + q_1^- + p_3^-$. Thus, the energy denominator d_1 is given by

$$d_1 = p_1^- - p_3^- - q_1^- = p_4^- - p_2^- - q_1^-. \quad (3.70)$$

In a similar manner we determine from diagram 15, that the total energy for the intermediate state is

$$p_1^- + q_2^- + p_4^-, \quad (3.71)$$

and then the energy denominator d_2 is given by

$$d_2 = p_2^- - q_2^- - p_4^-. \quad (3.72)$$

It is easy to see that in diagrams 14 and 15, $q_1^- = -q_2^-$. For diagram 17, the total energy of the intermediate state is given by q_3^- , and the energy denominator d_3 is given by

$$d_3 = p_1^- + p_2^- - q_3^-. \quad (3.73)$$

Let us choose to work in a frame of reference where $\vec{p}_{1\perp} = \vec{p}_{2\perp} = 0$. In any frame of reference $q_1^+ = p_1^+ - p_3^+$, $q_2^+ = p_3^+ - p_1^+ = -q_1^+$, $q_3^+ = p_1^+ + p_2^+$. In this example matrix elements M_1 , M_2 and M_3 correspond to diagrams 14, 15 and 17 respectively. Matrix elements M_4 and M_5 correspond to diagram 16 and 18 respectively. These matrix elements are as follows:

$$\begin{aligned}
M_1 = & e^2 \int \frac{dq_1^+ d^2 \vec{q}_{1\perp}}{(2\pi)^3 2 q_1^+} \theta(q_1^+) \sum_{\lambda} \chi_{-s_2}^\dagger \Gamma^j(-p_4, -p_2; q_1) \epsilon^j(\lambda) \chi_{-s_4} \chi_{s_3}^\dagger \\
& \cdot \Gamma^i(p_1, p_3; q_1) \epsilon^{i*}(\lambda) \chi_{s_1} \frac{1}{d_1} \sqrt{2 p_2^+ 2 p_4^+} \sqrt{2 p_1^+ 2 p_3^+} (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \\
& \cdot \delta^2(\vec{q}_{1\perp} - \vec{p}_{4\perp} + \vec{p}_{2\perp}) (2\pi)^3 \delta(p_1^+ - q_1^+ - p_3^+) \\
& \cdot \delta^2(\vec{p}_{1\perp} - \vec{q}_{1\perp} - \vec{p}_{3\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{3.74}
\end{aligned}$$

$$\begin{aligned}
M_2 = & e^2 \int \frac{dq_1^+ d^2 q_{1\perp}}{(2\pi)^3 2 q_2^+} \theta(q_2^+) \sum_{\lambda'} \chi_{-s_2}^\dagger \Gamma^j(-p_4, -p_2; q_2) \epsilon^{j*}(\lambda') \chi_{-s_4} \chi_{s_3}^\dagger \\
& \cdot \Gamma^i(p_1, p_3; q_2) \epsilon^i(\lambda) \chi_{s_1} \frac{1}{d_2} \sqrt{2 p_2^+ 2 p_4^+} \sqrt{2 p_1^+ 2 p_3^+} (2\pi)^3 \delta(p_2^+ - q_1^+ - p_4^+) \\
& \cdot \delta^2(\vec{p}_{2\perp} - \vec{q}_{1\perp} - \vec{p}_{4\perp}) (2\pi)^3 \delta(p_3^+ - q_1^+ - p_1^+) \\
& \cdot \delta^2(\vec{p}_{3\perp} - \vec{q}_{1\perp} - \vec{p}_{1\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{3.75}
\end{aligned}$$

$$\begin{aligned}
M_3 = & e^2 \int \frac{dq_3^+ d^2 q_{3\perp}}{(2\pi)^3 2 q_3^+} \theta(q_3^+) \sum_{\lambda''} \chi_{s_3}^\dagger \Gamma^i(-p_4, p_3; q_3) \epsilon^i(\lambda'') \chi_{-s_4} \chi_{s_2}^\dagger \\
& \cdot \Gamma^j(p_1, -p_2; q_3) \epsilon^{j*}(\lambda'') \chi_{s_1} \frac{1}{d_3} \sqrt{2 p_3^+ 2 p_4^+} \sqrt{2 p_1^+ 2 p_2^+} (2\pi)^3 \\
& \cdot \delta(p_3^+ + p_4^+ - q_3^+) \delta^2(\vec{p}_{3\perp} + \vec{p}_{4\perp} - \vec{q}_{3\perp}) (2\pi)^3 \delta(q_3^+ - p_1^+ - p_2^+) \\
& \cdot \delta^2(\vec{q}_{3\perp} - \vec{p}_{1\perp} - \vec{p}_{2\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{3.76}
\end{aligned}$$

$$\begin{aligned}
M_4 = & e^2 \chi_{s_3}^\dagger \chi_{s_1} \sqrt{2 p_1^+ 2 p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{-s_2}^\dagger \chi_{-s_4} \sqrt{2 p_3^+ 2 p_4^+} \\
& \cdot (2\pi)^3 (-2\pi i) \delta^4(p_3 + p_4 - p_1 - p_2), \tag{3.77}
\end{aligned}$$

$$\begin{aligned}
M_5 = & e^2 \chi_{-s_2}^\dagger \chi_{s_1} \sqrt{2 p_1^+ 2 p_2^+} \frac{1}{[p_1^+ + p_2^+]^2} \chi_{s_3}^\dagger \chi_{-s_4} \sqrt{2 p_3^+ 2 p_4^+} \\
& \cdot (2\pi)^3 (-2\pi i) \delta^4(p_3 + p_4 - p_1 - p_2). \tag{3.78}
\end{aligned}$$

The Γ terms are given by the expressions,

$$\begin{aligned}
\Gamma^i(-p_4, -p_2; q_1) &= \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot p_4 - im}{[p_4^+]} + \frac{\sigma \cdot p_2 + im}{[p_2^+]} \sigma^i - \frac{2q_1^i}{[q_1^+]} \right\}, \\
\Gamma^j(p_1, p_3; q_1) &= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im}{[p_1^+]} + \frac{\sigma \cdot p_3 - im}{[p_3^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\},
\end{aligned}$$

$$\begin{aligned}
\Gamma^i(-p_4, p_3; q_3) &= \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot (-p_4) + im}{[-p_4^+]} + \frac{\sigma \cdot p_3 - im}{[p_3^+]} \sigma^i - \frac{2q_3^i}{[q_3^+]} \right\}, \\
\Gamma^j(p_1, -p_2; q_3) &= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im}{[p_1^+]} + \frac{\sigma \cdot (-p_2) - im}{[-p_2^+]} \sigma^j - \frac{2q_3^j}{[q_3^+]} \right\}. \quad (3.79)
\end{aligned}$$

In the chosen frame of reference we have that

$$\begin{aligned}
\frac{1}{d_1} &= \frac{1}{p_1^- - p_3^- - q_1^-} = \frac{-2p_1^+ p_3^+ (p_1^+ - p_3^+)}{m^2(p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2}, \\
\frac{1}{d_2} &= \frac{1}{p_2^- - p_4^- - q_2^-} = -\frac{1}{d_1}, \\
\frac{1}{d_3} &= \frac{2}{m^2 \left(\frac{1}{p_1^+} + \frac{1}{p_2^+} \right) - \frac{q_3^2}{q_3^+}}, \quad (3.80)
\end{aligned}$$

$$\frac{1}{2q_1^+} \frac{1}{d_1} = \frac{-p_1^+ p_3^+}{m^2(p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2} = \frac{1}{2q_2^+} \frac{1}{d_2}, \quad (3.81)$$

$$\frac{1}{2q_3^+} \frac{1}{d_3} = \frac{p_1^+ p_2^+}{m^2(p_1^+ + p_2^+)^2}. \quad (3.82)$$

Since in 14 and 15 we have $q_1 = -q_2$, we will write $-q_1$ instead of q_2 . Here too the two diagrams complement each other due to the presence of the θ functions. So instead of calculating the amplitudes for both M_1 and M_2 , we do the calculations for only one of the two, valid for all q_1^+ , as we did in the case of the $e\gamma$ scattering. M_1 , M_2 and M_4 have a minus sign from the anti commutation relations, while M_3 and M_5 have a plus sign. There are 2 instantaneous terms, M_4 and M_5 .

Term M_4 has an infrared singularity when $p_1^+ = p_3^+$. First, we will verify that the linear infrared singularity is canceled (see Appendix D). We will ignore a common factor $\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}$. For $[s_1, s_2, s_3, s_4] = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ or $[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$. Consider the amplitude:

$$M_1 + M_4 = \frac{2}{2q_1^+ d_1} \left\{ 2 \frac{(\vec{q}_{1\perp})^2}{[q_1^+]^2} - \frac{q_{1+}}{[q_1^+]} \left[\frac{p_{3-}}{[p_3^+]} + \frac{p_{4-}}{[p_4^+]} \right] \right\} (-1) + \frac{4}{[q_1^+]^2} (-1), \quad (3.83)$$

where $q_{1+} = q_1^1 + iq_1^2$. A term in $\frac{1}{[q_1^+]^2}$ will lead to a linear infrared singularity if it cannot be compensated. In our case, the two terms involved are such that the singularity is canceled.

$$\frac{-4}{2q_1^+ d_1} \frac{(\vec{q}_{1\perp})^2}{[q_1^+]^2} - \frac{4}{[q_1^+]^2} = -\frac{4}{[q_1^+]^2} \left\{ \frac{m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 p_1^+ q_1^+}{m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 (p_1^+)^2} \right\},$$

$$= -\frac{4}{[q_1^+]} \left\{ \frac{m^2(q_1^+) + (\vec{q}_{1\perp})^2 p_1^+}{m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 (p_1^+)^2} \right\}. \quad (3.84)$$

As we note, there is no linear infrared singularity left! We now combine this result with the rest of the initial expression, to obtain $M_1 + M_4$. The resulting amplitude is given by:

$$\begin{aligned} M_1 + M_4 &= -\frac{2}{[q_1^+]} \left\{ \frac{2p_4^+ \{ m^2(q_1^+) + (\vec{q}_{1\perp})^2 p_1^+ \} + p_1^+ (\vec{q}_{1\perp})^2 (p_3^+ - p_4^+)}{p_4^+ \{ m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 (p_1^+)^2 \}} \right\}, \\ &= -\frac{2}{[q_1^+]} \left\{ \frac{2m^2(q_1^+)(p_4^+) + (\vec{q}_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{p_4^+ \{ m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 (p_1^+)^2 \}} \right\}. \end{aligned} \quad (3.85)$$

Similarly, for $[s_1, s_2, s_3, s_4] = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ or $[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$. The resulting amplitude is given by:

$$M_1 + M_4 = -\frac{2}{[q_1^+]} \left\{ \frac{2m^2(q_1^+)(p_4^+) + (\vec{q}_{1\perp})^2 p_1^+ (p_1^+ + p_4^+)}{p_4^+ \{ m^2(q_1^+)^2 + (\vec{q}_{1\perp})^2 (p_1^+)^2 \}} \right\}, \quad (3.86)$$

again, there is no linear singularity left!

From the properties of the δ function and the definition of p^- we have that

$$\delta(p_1^- + p_2^- - p_3^- - p_4^-) = \frac{2p_3^+ p_4^+}{p_3^+ + p_4^+} \delta\left((\vec{p}_{3\perp})^2 + m^2 - m^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right). \quad (3.87)$$

Since $(\vec{p}_{3\perp})^2$ is manifestly positive or zero, this implies that $p_3^+ p_4^+ \geq p_1^+ p_2^+$. We now introduce the dimensionless variables:

$$x = \frac{p_1^+}{p_1^+ + p_2^+}, \quad y = \frac{p_3^+}{p_1^+ + p_2^+}. \quad (3.88)$$

From equation (2.148), the differential cross section is given by the expression:

$$d\sigma = \frac{dp_3^+}{(2\pi)^3 2p_3^+} d^2\vec{p}_{3\perp} \frac{dp_4^+}{(2\pi)^3 2p_4^+} d^2\vec{p}_{4\perp} \frac{|M|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{\Lambda^{\frac{1}{2}}(4q^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m^2], 2[(\vec{p}_{2\perp})^2 + m^2])}, \quad (3.89)$$

where the Λ function, q^+ and Q^- are given in this case by the expressions:

$$\begin{aligned} q^+ &= p_1^+ + p_2^+ = p_3^+ + p_4^+, \quad Q^- = p_3^- + p_4^- = p_1^- + p_2^-, \\ \Lambda\left(4q^+ Q^-, 2[(\vec{p}_{1\perp})^2 + m^2], 2[(\vec{p}_{2\perp})^2 + m^2]\right) &= 4m^4 \left(\frac{(p_1^+)^2 - (p_2^+)^2}{p_1^+ p_2^+} \right)^2, \end{aligned} \quad (3.90)$$

where the δ^4 notation from (2.136) is used and $|M|^2$ is the square of the modulus of the amplitude of the process summed over final states and averaged over the initial states,

$$|M|^2 = \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} |M_1 + M_2 + M_3 + M_4 + M_5|^2.$$

For the phase-space integrals, we use the $\delta(p_1^+ + p_2^+ - p_3^+ - p_4^+)$ and $\delta^2(\vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{p}_{3\perp} - \vec{p}_{4\perp})$ functions when performing the integration for one of the final particles, say p_4 . Because of the δ functions, the integration on dp_4^+ and $d^2\vec{p}_{4\perp}$ makes $p_3^+ = q_1^+ - p_4^+$ and $\vec{p}_{1\perp} + \vec{p}_{2\perp} = \vec{p}_{3\perp} + \vec{p}_{4\perp} = 0$. Hence, $\vec{p}_{3\perp} = -\vec{p}_{4\perp}$. We write $d^2p_{3\perp} = p_{3\perp} dp_{3\perp} d\varphi = \frac{1}{2}d(p_{3\perp}^2)d\varphi = \frac{1}{2}d(\vec{p}_{3\perp})^2d\varphi$. Then the differential cross section is given by the expression:

$$d\sigma = \frac{dp_3^+}{2p_3^+} \frac{1}{2} d(p_{3\perp}^2) d\varphi \frac{dp_4^+ d^2\vec{p}_{4\perp}}{(2\pi)^2 2p_4^+} \frac{p_1^+ p_2^+}{2m^2[(p_1^+)^2 - (p_2^+)^2]} |M|^2 \delta^4(p_1 + p_2 - p_3 - p_4). \quad (3.91)$$

We integrate the differential cross section on dp_4^+ and $d^2\vec{p}_{4\perp}$, taking into account (3.87) and the result is

$$d\sigma = dp_3^+ d(p_{3\perp}^2) d\varphi \frac{A^2 p_1^+ p_2^+}{8(2\pi)^2 m^2 (p_1^+ - p_2^+)(p_1^+ + p_2^+)^2} \delta\left((\vec{p}_{3\perp})^2 + m^2 - m^2 \frac{p_3^+ p_4^+}{p_1^+ p_2^+}\right), \quad (3.92)$$

where $A^2 = |M|^2_{p_4^+ = p_1^+ + p_2^+ - p_3^+, \vec{p}_{4\perp} = \vec{p}_{1\perp} + \vec{p}_{2\perp} - \vec{p}_{3\perp}}$.

Let us choose to work in the CMS reference frame. In this case,

$$\begin{aligned} p_1^\mu &= (E, 0, 0, p), & p_2^\mu &= (E, 0, 0, -p), & p_3^\mu &= (E, \dots, \dots, p \cos \theta), \\ p_4^\mu &= (E, \dots, \dots, -p \cos \theta), & p_1^+ &= \frac{1}{\sqrt{2}}(E + p), & p_2^+ &= \frac{1}{\sqrt{2}}(E - p) \\ p_3^+ &= \frac{1}{\sqrt{2}}(E - p \cos \theta), & p_4^+ &= \frac{1}{\sqrt{2}}(E + p \cos \theta), & q^+ &= \sqrt{2}E, \\ \beta &= \frac{\sqrt{E^2 - m^2}}{E} = \frac{p}{E}. \end{aligned} \quad (3.93)$$

As we already saw, there is the constraint $p_3^+ p_4^+ \geq p_1^+ p_2^+$. In terms of the dimensionless variables x and y given by expression (3.88), we have

$$\begin{aligned} x &\geq \frac{1}{2}, & y(1-y) &\geq x(1-x), & p_1^+ p_2^+ &= (q^+)^2 x(1-x) = \frac{m^2}{2}, \\ q^+ &= \frac{m}{\sqrt{2x(1-x)}}, \\ p_{3\perp}^2 &= (\vec{p}_{3\perp})^2 = 2(q^+)^2 \{y(1-y) - x(1-x)\} \geq 0. \end{aligned} \quad (3.94)$$

The equation

$$y(1-y) - x(1-x) = 0 \quad (3.95)$$

has two solutions, and they determine the boundaries of integration for y ,

$$y = x, \quad y = 1 - x. \quad (3.96)$$

Since $x \geq \frac{1}{2}$, we have $(1-x) \leq y \leq x$. The flux factor is given by the expression:

$$\frac{p_1^+ p_2^+}{2m^2 [(p_1^+)^2 - (p_2^+)^2]} = \frac{x(1-x)}{2m^2(2x-1)}. \quad (3.97)$$

In order to calculate the expression for the angular dependence of the differential cross section, let us consider the following alternative expressions for the dimensionless variables:

$$x = \frac{1}{2}(1+\beta), \quad y = \frac{1}{2}(1+\beta(1-\zeta)), \quad \zeta = 1 - \cos \theta, \quad \frac{dy}{d(\cos \theta)} = \frac{1}{2}\beta. \quad (3.98)$$

We now integrate expression (3.92) in $p_{3\perp}^2$ and convert the expression to the dimensionless variables:

$$d\sigma = dp_3^+ d\varphi \frac{A^2}{8(2\pi)^2} \frac{p_1^+ p_2^+}{m^2(p_1^+ - p_2^+)(p_1^+ + p_2^+)^2} = dy d\varphi \frac{A^2}{2(4\pi)^2} \frac{x(1-x)}{m^2(2x-1)}. \quad (3.99)$$

Appendix D contains the detailed calculations of A^2 . From (D.18) of Appendix D we see that:

$$\begin{aligned} A^2 = & 4(4\pi)^2 \alpha^2 x(1-x) \left\{ \frac{-3-2x+3x^2}{2(x-1)x} + \frac{(-1+2x)y}{x} - \frac{(-1+2x)^2 y^2}{2(x-1)x} + \right. \\ & \left. + \frac{-1+4x-8x^2+8x^3-4x^4}{2(x-1)x(2x-1)^2(y-x)^2} + \frac{-1+2x^2-4x^3+2x^4}{(x-1)x(2x-1)(y-x)} \right\}. \end{aligned} \quad (3.100)$$

Substituting (3.100) in (3.99) we obtain

$$\begin{aligned} d\sigma = & dy d\varphi \frac{4x^2(1-x)^2}{m^2(2x-1)} \alpha^2 \left\{ \frac{-3-2x+3x^2}{2(x-1)x} + \frac{(-1+2x)y}{x} - \frac{(-1+2x)^2 y^2}{2(x-1)x} + \right. \\ & \left. + \frac{-1+4x-8x^2+8x^3-4x^4}{2(x-1)x(2x-1)^2(y-x)^2} + \frac{-1+2x^2-4x^3+2x^4}{(x-1)x(2x-1)(y-x)} \right\}. \end{aligned} \quad (3.101)$$

The angular dependence of the cross section is given by:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma}{dy} \frac{dy}{d(\cos \theta)}, \\
&= \frac{\alpha^2}{2E^2} \frac{m^2}{E^2} \frac{1}{4} \left\{ \frac{-3 - 2x + 3x^2}{2(x-1)x} + \frac{(-1+2x)y}{x} - \frac{(-1+2x)^2 y^2}{2(x-1)x} \right. \\
&\quad \left. + \frac{-1 + 4x - 8x^2 + 8x^3 - 4x^4}{2(x-1)x(2x-1)^2(y-x)^2} + \frac{-1 + 2x^2 - 4x^3 + 2x^4}{(x-1)x(2x-1)(y-x)} \right\}, \\
&= \frac{\alpha^2}{2E^2} \left\{ \frac{5}{4} - \frac{8E^4 - m^4}{4E^2(E^2 - m^2)(1 - \cos \theta)} + \frac{(2E^2 - m^2)^2}{2(E^2 - m^2)^2(1 - \cos \theta)^2} + \right. \\
&\quad \left. + \frac{2E^4(-1 + 2\cos \theta + \cos^2 \theta)}{16E^4} + \right. \\
&\quad \left. + \frac{E^2 m^2(1 - \cos \theta)(2 + \cos \theta) + 2m^4 \cos^2 \theta}{16E^4} \right\}. \tag{3.102}
\end{aligned}$$

The expression is identical with the one given in Itzykson and Zuber [27], page 281, except for the absence of the 4 in the denominator of the term with the factor $8E^4 - m^4$.

We have verified this to be a misprint in the book [27].

3.4 $e\mu \rightarrow e\mu$ scattering.

The previous example includes diagrams in which the initial particles are annihilated and similar particles are created. It also includes diagrams in which one particle emits a photon that is then absorbed by the other particle. Here since the initial particles are not a particle-antiparticle pair, no annihilation can take place. Thus, in this example we will retain those diagrams in which there is the emission-absorption of a photon. We will use some of the results of the calculations from the Bhabha scattering keeping in mind that the initial particles have different masses. There are three relevant diagrams, and they are the same ones that participated in the cancelation of the linear infrared singularity. We again verify that the cancelation takes place, even though we have already verified the cancelation in the previous example. These diagrams are

Diagram 19.

Diagram 20.

Diagram 21.

In diagram 19 the total initial (final) energy is given by

$$p_i^- = p_f^- = p_1^- + p_2^- = p_3^- + p_4^-, \quad (3.103)$$

and the total energy of the intermediate state is $p_2^- + q_1^- + p_3^-$. Thus, the energy denominator d_1 is given by

$$d_1 = p_1^- - p_3^- - q_1^- = p_4^- - p_2^- - q_1^-. \quad (3.104)$$

In a similar manner we determine from diagram 20, that the total energy for the intermediate state is

$$p_1^- + q_2^- + p_4^-, \quad (3.105)$$

and then the energy denominator d_2 is given by

$$d_2 = p_2^- - q_2^- - p_4^-. \quad (3.106)$$

It is easy to see that in diagrams 19 and 20, $q_1^- = -q_2^-$. Again we choose to work in a frame of reference where $\vec{p}_{1\perp} = \vec{p}_{2\perp} = 0$. In any frame of reference $q_1^+ = p_1^+ - p_3^+$, $q_2^+ = p_3^+ - p_1^+ = -q_1^+$, $q^+ = p_1^+ + p_2^+$. As before, for each diagram relevant to the $e\mu$ scattering process, there is an associated matrix element. In this case matrix element M_1 , M_2 and M_3 correspond to diagrams 19, 20 and 21, respectively. These matrix elements are as follows:

$$\begin{aligned} M_1 = & e^2 \int \frac{dq_1^+ d^2 \vec{q}_{1\perp}}{(2\pi)^3 2q_1^+} \theta(q_1^+) \sum_{\lambda} \chi_{s_4}^\dagger \Gamma^j(p_2, p_4; q_1) \epsilon^j(\lambda) \chi_{s_2} \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_1) \\ & \cdot \epsilon^{i*}(\lambda) \chi_{s_1} \frac{1}{d_1} \sqrt{2p_1^+ 2p_3^+} \sqrt{2p_2^+ 2p_4^+} (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \\ & \cdot \delta^2(\vec{q}_{1\perp} - \vec{p}_{4\perp} + \vec{p}_{2\perp}) (2\pi)^3 \delta(p_1^+ - q_1^+ - p_3^+) \\ & \cdot \delta^2(\vec{p}_{1\perp} - \vec{q}_{1\perp} - \vec{p}_{3\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \end{aligned} \quad (3.107)$$

$$\begin{aligned}
M_2 = & e^2 \int \frac{dq_1^+ d^2 \vec{q}_{1\perp}}{(2\pi)^3 2q_2^+} \theta(q_2^+) \sum_{\lambda'} \chi_{s_4}^\dagger \Gamma^j(p_2, p_4; q_2) \epsilon^{j*}(\lambda') \chi_{s_2} \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_2) \\
& \cdot \epsilon^i(\lambda) \chi_{s_1} \frac{1}{d_2} \sqrt{2p_2^+ 2p_4^+} \sqrt{2p_1^+ 2p_3^+} (2\pi)^6 \delta(p_2^+ - q_1^+ - p_4^+) \\
& \cdot \delta^2(\vec{p}_{2\perp} - \vec{q}_{1\perp} - \vec{p}_{4\perp}) \delta(p_3^+ - q_1^+ - p_1^+) \\
& \cdot \delta^2(\vec{p}_{3\perp} - \vec{q}_{1\perp} - \vec{p}_{1\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{3.108}
\end{aligned}$$

$$\begin{aligned}
M_3 = & e^2 \chi_{s_3}^\dagger \chi_{s_1} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{s_4}^\dagger \chi_{s_2} \sqrt{2p_3^+ 2p_4^+} \\
& \cdot (2\pi)^3 (-2\pi i) \delta^4(p_3 + p_4 - p_1 - p_2). \tag{3.109}
\end{aligned}$$

The Γ terms are given by the expressions,

$$\Gamma^i(p_2, p_4; q_1) = \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot p_2 + im_\mu}{[p_2^+]} + \frac{\sigma \cdot p_4 - im_\mu}{[p_4^+]} \sigma^i - \frac{2q_1^i}{[q_1^+]} \right\}, \tag{3.110}$$

$$\Gamma^j(p_1, p_3; q_1) = \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im_e}{[p_1^+]} + \frac{\sigma \cdot p_3 - im_e}{[p_3^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\}. \tag{3.111}$$

In the chosen frame of reference we have that

$$\frac{1}{d_1} = \frac{1}{p_1^- - p_3^- - q_1^-} = \frac{-2p_1^+ p_3^+ (p_1^+ - p_3^+)}{m_e^2 (p_1^+ - p_3^+)^2 + (\vec{p}_{3\perp})^2 (p_1^+)^2} = -\frac{1}{d_2}, \tag{3.112}$$

$$\frac{1}{2q_1^+} \frac{1}{d_1} = \frac{-p_1^+ p_3^+}{m_e^2 (p_1^+ - p_3^+)^2 + (\vec{p}_{3\perp})^2 (p_1^+)^2} = \frac{1}{2q_2^+} \frac{1}{d_2}. \tag{3.113}$$

Since in diagrams 19 and 20 we have $q_1 = -q_2$, we will write $-q_1$ instead of q_2 . As in the previous examples diagrams M_1 and M_2 complement each other. Thus we will retain only M_1 with a $\theta(q_1^+)$ equal to 1 for $p_1^+ \geq p_3^+$ and $p_1^+ \leq p_3^+$. Since Γ^j in this example is the same one as in the Bhabha scattering, we can construct the amplitudes directly from the previous example (see Appendix E for detailed calculations). Here too we have to verify that the infrared singularity is canceled by the interference between diagrams M_1 and M_3 . In connection with the energies δ function we have:

$$\begin{aligned}
p_1^- + p_2^- - p_3^- - p_4^- &= \frac{m_e^2}{2p_1^+} + \frac{m_\mu^2}{2p_2^+} - \frac{(\vec{p}_{3\perp})^2 + m_e^2}{2p_3^+} - \frac{(\vec{p}_{3\perp})^2 + m_\mu^2}{2p_4^+}, \\
\delta(p_1^- + p_2^- - p_3^- - p_4^-) &= \frac{2p_3^+ p_4^+}{p_3^+ + p_4^+} \\
&\cdot \delta \left[(\vec{p}_{3\perp})^2 - m_e^2 \frac{p_4^+ (p_3^+ - p_1^+)}{p_1^+ (p_3^+ + p_4^+)} - m_\mu^2 \frac{p_3^+ (p_4^+ - p_2^+)}{p_2^+ (p_3^+ + p_4^+)} \right], \\
(\vec{p}_{3\perp})^2 &= m_e^2 \frac{p_4^+ (p_3^+ - p_1^+)}{p_1^+ (p_3^+ + p_4^+)} + m_\mu^2 \frac{p_3^+ (p_4^+ - p_2^+)}{p_2^+ (p_3^+ + p_4^+)} \geq 0. \tag{3.114}
\end{aligned}$$

Expression (3.114) is a constraint equation on the values of p_3^+ and p_4^+ , which will determine the limits of integration. Since $p_4^+ = p_1^+ + p_2^+ - p_3^+$, then the fact that $(\vec{p})_{3\perp}^2 \geq 0$ leads to

$$m_e^2 [p_2^+ p_4^+ (p_3^+ - p_1^+)] + m_\mu^2 [p_1^+ p_3^+ (p_4^+ - p_2^+)] \geq 0, \quad (3.115)$$

$$(p_1^+ - p_3^+) \left\{ -m_e^2 p_2^+ (p_1^+ + p_2^+ - p_3^+) + m_\mu^2 p_1^+ p_3^+ \right\} \geq 0. \quad (3.116)$$

This shows that the limits of integration are

$$p_3^+ = p_1^+, \quad (3.117)$$

$$p_3^+ = \frac{m_e^2 p_2^+ (p_1^+ + p_2^+)}{m_\mu^2 p_1^+ + m_e^2 p_2^+}. \quad (3.118)$$

It is easier to determine the relative values of these limits if we analyze them using the dimensionless variables. Again,

$$x = \frac{p_1^+}{p_1^+ + p_2^+} = \frac{p_1^+}{q^+}, \quad y = \frac{p_3^+}{p_1^+ + p_2^+} = \frac{p_3^+}{q^+}. \quad (3.119)$$

From the previous example we know that $x \geq \frac{1}{2}$, and since $m_\mu \gg m_e$, then the limits of integration are

$$y_{max} = x, \quad (3.120)$$

$$y_{min} = \frac{m_e^2 (1 - x)}{m_\mu^2 x + m_e^2 (1 - x)}. \quad (3.121)$$

Let

$$Q^- = p_1^- + p_2^- = p_3^- + p_4^- = \frac{1}{2} \left\{ \frac{m_e^2 p_2^+ + m_\mu^2 p_1^+}{p_1^+ p_2^+} \right\} = \frac{1}{2} \left\{ \frac{m_e^2 (1 - x) + m_\mu^2 x}{x(1 - x)q^+} \right\}, \quad (3.122)$$

$$q^+ Q^- = \frac{m_e^2 (1 - x) + m_\mu^2 x}{2x(1 - x)}. \quad (3.123)$$

As before the flux factor is given by the expression:

$$\Lambda^{-\frac{1}{2}}(4q^+ Q^-, 2m_e^2, 2m_\mu^2) = \frac{x(1 - x)}{2 \{ m_e^2 (1 - x)^2 - m_\mu^2 x^2 \}}. \quad (3.124)$$

As in the previous examples we calculate $|M|^2$, the square of the modulus of the amplitude of the process summed over final states and averaged over the initial states, which in our case is

$$|M|^2 = \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} |M_1 + M_2 + M_3|^2. \quad (3.125)$$

If in expression (3.99) of the Bhabha scattering we replace (3.97) by (3.124) and (3.100) by (3.125), we obtain the expression for the differential cross section of the $e\mu$ scattering. Thus,

$$d\sigma = dy d\varphi \frac{A^2}{(4\pi)^2} \frac{x(1-x)}{2\{m_e^2(1-x)^2 - m_\mu^2 x^2\}}, \quad (3.126)$$

where $A^2 = |M|^2$ with the conditions

$$\begin{aligned} p_4^+ &= p_1^+ + p_2^+ - p_3^+, \\ \vec{p}_{4\perp} &= -\vec{p}_{3\perp}, \\ (\vec{p}_{3\perp})^2 &= m_e^2 \frac{p_4^+(p_3^+ - p_1^+)}{p_1^+(p_3^+ + p_4^+)} + m_\mu^2 \frac{p_3^+(p_4^+ - p_2^+)}{p_2^+(p_3^+ + p_4^+)}. \end{aligned} \quad (3.127)$$

From Appendix E, expression (E.13) we can see that:

$$\begin{aligned} A^2 = e^4 & \frac{2x(-1+y)}{(m_e^2(-1+x)^2 - m_\mu^2 x^2)^2 (x-y)^2 y} \\ & \left\{ \frac{m_e^2(-1+x)^2(-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x} + \right. \\ & + \frac{(-1+x)(m_\mu^2 xy + m_e^2(-1+x)(-1+y))^2}{(-1+y)^2} + \\ & + \frac{m_e^2(-1+x)^4(-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x(-1+y)^2} + \\ & + \frac{m_e^2 m_\mu^2(-1+x)(x-y)^4}{(-1+y)^2} + \\ & + \frac{(-1+x)(m_e^2(-1+x)(-1+x-y)(-1+y) + m_\mu^2 xy(-1-x+y))^2}{(-1+y)^2} + \\ & - \frac{m_\mu^2 x^3(x-y)(m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} + \\ & \left. - \frac{m_\mu^2 x(x-y)y^2(m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} \right\} \end{aligned} \quad (3.128)$$

and the exact differential cross section is given by the expression:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} = & \alpha^2 \frac{(-1+x)x^2(1-y)}{(m_e^2(-1+x)^2 - m_\mu^2 x^2)^3 (x-y)^2 y} \\
& \left\{ \frac{m_e^2(-1+x)^2(-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x} + \right. \\
& + \frac{(-1+x)(m_\mu^2 xy + m_e^2(-1+x)(-1+y))^2}{(-1+y)^2} + \\
& + \frac{m_e^2(-1+x)^4(-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x(-1+y)^2} + \\
& + \frac{m_e^2 m_\mu^2(-1+x)(x-y)^4}{(-1+y)^2} + \\
& + \frac{(-1+x)(m_e^2(-1+x)(-1+x-y)(-1+y) + m_\mu^2 xy(-1-x+y))^2}{(-1+y)^2} + \\
& - \frac{m_\mu^2 x^3(x-y)(m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} + \\
& \left. - \frac{m_\mu^2 x(x-y)y^2(m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} \right\}. \tag{3.129}
\end{aligned}$$

In order to compare with some known results, we make the approximation $m_e \approx 0$.

The value of A^2 in this approximation is then given by the expression:

$$\begin{aligned}
A^2 = & e^4 \frac{2x}{x^3(x-y)^2(-1+y)} \left\{ -x^4 + x^3(xy+y) + xy^2(xy+y) + \right. \\
& \left. + x^2(xy - 2xy^2 - y^2) - xy(2 - 2y + y^2) \right\} \tag{3.130}
\end{aligned}$$

and the differential cross section is then given by the expression:

$$\begin{aligned}
d\sigma = & dy d\varphi \frac{\alpha^2(-1+x)x}{m_\mu^2 x^4(x-y)^2(-1+y)} \left\{ -x^4 + x^3(xy+y) + xy^2(xy+y) + \right. \\
& \left. + x^2(xy - 2xy^2 - y^2) - xy(2 - 2y + y^2) \right\}. \tag{3.131}
\end{aligned}$$

We will now work in the CMS reference frame. In this reference frame we have:

$$p_1^+ = \frac{E_e + p}{\sqrt{2}}, \quad p_2^+ = \frac{E_\mu - p}{\sqrt{2}}, \quad p_3^+ = \frac{E_e + p \cos \theta}{\sqrt{2}}, \tag{3.132}$$

$$x = p_4^+ = \frac{E_\mu - p \cos \theta}{\sqrt{2}}, \quad \frac{E_e + p}{E_e + E_\mu}, \quad y = \frac{E_e + p \cos \theta}{E_e + E_\mu}. \tag{3.133}$$

Since we are making the approximation $m_e \approx 0$, then

$$x = \frac{2p}{p + E_\mu}, \quad y = \frac{p(1 + \cos \theta)}{p + E_\mu}. \tag{3.134}$$

The differential cross section is then given by the expression:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 (E_\mu + p)^2 + (E_\mu + p \cos \theta)^2 - m_\mu^2 (1 - \cos \theta)}{2 p^2 (E_\mu + p)^2 (1 - \cos \theta)^2}. \quad (3.135)$$

The differential cross section for this process given in equation (5.64) of the reference [25], contains an obvious error as can be easily seen from their earlier equations (5.61) and (5.62). Once that error is corrected, the expression we have derived matches the expression in the above mentioned textbook.

This concludes our examination of a number of tree level processes from the point of view of cancelation of linear infrared divergencies using the suggested prescription for dealing with this type of singularities.

All the initial evaluations of the amplitudes, involving helicity eigenstates and Pauli matrices, were manually performed. They were then fed to tailor-made Mathematica programs where symbolic manipulations were carried out to obtain the final results. In cases where linear infrared singularities were present, their cancelations were manually performed and Mathematica programs were used for further work.

Chapter 4

LF form of the electro-weak unification.

4.1 Yang-Mills gauge fields.

We would like to point out that a different procedure has been followed in establishing the LF formulation of the standard model by Srivastava and Brodsky [30]. Here we develop a generalized Hamiltonian approach with constraints for electro-weak unification parallel to the one we have used for developing the LF formulation of QED.

We start with the development of the massless Lagrangian density of the Yang-Mills gauge fields. Let

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon^{abc}A_b^\mu A_c^\nu, \quad (4.1)$$

$$F^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu, \quad (4.2)$$

where A_a^μ and C^μ are real fields for $(a, b, c = 1, 2, 3)$ and $(\mu, \nu = 0, 1, 2, 3)$. The former are $SU(2)$ gauge boson fields and the latter are $U(1)$ gauge boson fields. Here we choose to work in the LF gauge: $A_a^+ = C^+ = 0$, $(a = 1, 2, 3)$. The gauge fields Lagrangian density now contains two components, one belonging to $SU(2)$ and the other to $U(1)$

$$\mathcal{L}_{GB} = -\frac{1}{4} G_{\mu\nu a} G_a^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (4.3)$$

The presence of the coupling constant g indicates that the Lagrangian density contains the interaction of the Yang-Mills gauge fields among themselves. This interaction points to the existence of an $SU(2)$ charge for these fields. We also know that the C^μ gauge fields do not interact with each other. Photons have no $U(1)$ charge. When we express (4.1) and (4.2) in terms of the LF coordinates we find that

$$F_{\mu\nu}F^{\mu\nu} = 2F^{+-}F^{-+} - 4F^{i+}F^{i-} + F^{ij}F^{ij}, \quad (4.4)$$

$$G^{\mu\nu a}G_{\mu\nu a} = 2G^{+-a}G^{-+a} - 4G^{i+a}G^{-ia} + G^{ija}G^{ija}. \quad (4.5)$$

Thus, the Lagrangian density in the LF formulation is given by the expression:

$$\mathcal{L}_{GB} = \frac{1}{2}G_a^{+-}G_a^{-+} + G_a^{i+}G_a^{i-} - \frac{1}{4}G_a^{ij}G_a^{ij} - \frac{1}{2}F^{+-}F^{-+} + F^{i+}F^{i-} - \frac{1}{4}F^{ij}F^{ij}. \quad (4.6)$$

Here too, as in QED, we introduce the electric (E^μ) and magnetic (B^μ) fields. Let

$$G_a^{+i} = -E_a^i, \quad G_a^{+-} = -E_a^-, \quad G_a^{ij} = -B_a^-, \quad (4.7)$$

$$F^{+i} = -E^i, \quad F^{+-} = -E^-, \quad F^{ij} = -B^-. \quad (4.8)$$

Then the expression of the Lagrangian density becomes

$$\mathcal{L}_{GB} = \frac{1}{2}(E_a^-)^2 - \frac{1}{2}(B_a^-)^2 - E_a^i G_a^{-i} + \frac{1}{2}(E^-)^2 - \frac{1}{2}(B^-)^2 - E^i F^{-i}. \quad (4.9)$$

We rewrite this expression, following the same procedure outlined previously for QED, exhibiting the time derivatives, Hamiltonian density and constraint terms

$$\begin{aligned} \mathcal{L}_{GB} = & G_a^{+i}\partial^- A_a^i + F^{+i}\partial^- C^i - \mathcal{H}_{GB} + \\ & + A_a^- \{ \partial^+ E_a^- - \partial^i E_a^i - g\epsilon^{abc} A_b^i E_c^i \} + C^- \{ \partial^+ E^- - \partial^i E^i \}, \end{aligned} \quad (4.10)$$

where the Hamiltonian density $\mathcal{H}_{GB}$ is given by the expression

$$\begin{aligned} \mathcal{H}_{GB} = & \frac{1}{2}(E_a^-)^2 + \frac{1}{2}(B_a^-)^2 + \{ \partial^+(E_a^- A_a^-) - \partial^i(E_a^i A_a^-) \} + \\ & + \frac{1}{2}(E^-)^2 + \frac{1}{2}(B^-)^2 + \{ \partial^+(E^- C^-) - \partial^i(E^i C^-) \}. \end{aligned} \quad (4.11)$$

From the definition of canonical conjugate to a field, we find that the canonical conjugates to fields $A_{a\mu}$ and C_μ are

$$E_a^\mu(x) = \frac{\partial \mathcal{L}_{GB}}{\partial(\partial^- A_{a\mu})} = -G_a^{+\mu}(x), \quad E^\mu(x) = \frac{\partial \mathcal{L}_{GB}}{\partial(\partial^- C_\mu)} = -F^{+\mu}(x). \quad (4.12)$$

We note that the field variables A_a^- and C^- are shown explicitly in equation (4.10). The terms $\partial^+(E_a^- A_a^-) - \partial^i(E_a^i A_a^-)$ and $\partial^+(E^- C^-) - \partial^i(E^i C^-)$ are surface terms, and for reasons similar to the ones in QED they can be discarded. Here too $\partial^- A_a^-$ and $\partial^- C^-$ do not appear in $\mathcal{L}_{GB}$ but $\partial^- A_a^i$ and $\partial^- C^i$ do. This shows that A_a^i and C^i are the only dynamical variables. For reasons similar to the one in QED, A_a^- and C^- are Lagrange multipliers. As in QED, $\partial^+ E_a^- - \partial^i E_a^i - g\epsilon^{abc} A_b^i E_c^i = 0$ and $\partial^+ E^- - \partial^i E^i = 0$ are constraint equations, which are consistent with the equations of motion. By solving them we find A_a^- and C^- as in QED.

We already know from the equal time formulation of QFT that when we break gauge symmetry in order to give mass to the gauge fields, a new set of gauge fields, denoted $W^\pm$, Z^0 and A , arises. We will convert the expression (4.10) of the gauge boson Lagrangian density from the present gauge fields to these new gauge fields. From here on we will refer to the Z^0 fields as simply Z fields. We will use:

$$A_1^i = \frac{1}{\sqrt{2}}(W^{i(+)} + W^{i(-)}), \quad A_2^i = \frac{i}{\sqrt{2}}(W^{i(+)} - W^{i(-)}), \quad (4.13)$$

$$A_1^- = \frac{1}{\sqrt{2}}(W^{-(+)} + W^{-(-)}), \quad A_2^- = \frac{i}{\sqrt{2}}(W^{-(+)} - W^{-(-)}), \quad (4.14)$$

$$A_3^i = Z^i \cos \theta_w + A^i \sin \theta_w, \quad A_3^- = Z^- \cos \theta_w + A^- \sin \theta_w, \quad (4.15)$$

$$C^i = A^i \cos \theta_w - Z^i \sin \theta_w, \quad C^- = A^- \cos \theta_w - Z^- \sin \theta_w. \quad (4.16)$$

The conversion of $\mathcal{L}_{GB}$ (4.10), after dropping the surface terms, to the new gauge fields $W^{i(\pm)}$, Z^i and A^i , requires some algebraic manipulations, at the end of which we obtain,

$$\begin{aligned}
\mathcal{L}_{GB} = & \partial^+ W^{i(+)} \partial^- W^{i(-)} + \partial^+ W^{i(-)} \partial^- W^{i(+)} + \partial^+ A^i \partial^- A^i + \partial^+ Z^i \partial^- Z^i + \\
& + W^{-(+)} \left[-\partial^+ \partial^+ W^{-(+)} + \partial^i \partial^+ W^{i(-)} - i g \left\{ W^{i(-)} \partial^+ (A^i \sin \theta_w + \right. \right. \\
& \quad \left. \left. + Z^i \cos \theta_w) - (A^i \sin \theta_w + Z^i \cos \theta_w) \partial^+ W^{i(-)} \right\} \right] + \\
& + W^{(-)} \left[-\partial^+ \partial^+ W^{-(+)} + \partial^i \partial^+ W^{i(+)} + i g \left\{ W^{i(+)} \partial^+ (A^i \sin \theta_w + \right. \right. \\
& \quad \left. \left. + Z^i \cos \theta_w) - (A^i \sin \theta_w + Z^i \cos \theta_w) \partial^+ W^{i(+)} \right\} \right] + \\
& + A^- \left[-\partial^+ \partial^+ A^- + \partial^i \partial^+ A^i - i g \sin \theta_w \left\{ W^{i(+)} \partial^+ W^{i(-)} - \right. \right. \\
& \quad \left. \left. - W^{i(-)} \partial^+ W^{i(+)} \right\} \right] + \\
& + Z^- \left[-\partial^+ \partial^+ Z^- + \partial^i \partial^+ Z^i - i g \cos \theta_w \left\{ W^{i(+)} \partial^+ W^{i(-)} - \right. \right. \\
& \quad \left. \left. + W^{i(-)} \partial^+ W^{i(+)} \right\} \right] - \mathcal{H}_{GB}, \tag{4.17}
\end{aligned}$$

where $\mathcal{H}_{GB}$ is given by the expression

$$\begin{aligned}
\mathcal{H}_{GB} = & \partial^+ W^{-(+)} \partial^+ W^{(-)} + \partial^i W^{j(+)} \partial^i W^{j(-)} + \\
& + \frac{1}{2} (\partial^+ A^-)^2 + \frac{1}{2} (\partial^i A^i)^2 + \frac{1}{2} (\partial^+ Z^-)^2 + \frac{1}{2} (\partial^i Z^i)^2 \tag{4.18}
\end{aligned}$$

for $i, j = 1, 2$ respectively, but $i \neq j$.

4.2 Fermionic fields.

We have twelve elementary fermions: the leptons ($\nu_e, e, \nu_\mu, \mu, \nu_\tau, \tau$) and the quarks (u, d, c, s, b, t). These fermions are organized into three generations of leptons and three generations of quarks. Each generation is represented as a pair, that is to say, they are represented as: (ν_e, e), (ν_μ, μ), (ν_τ, τ), (u, d), (c, s), (t, b). A crucial property of the weak-interactions is the fact that the W gauge bosons couple only to left-handed helicity states of relativistic leptons and quarks.

Let us consider the Lagrangian density for a massless fermion field: $\mathcal{L}_F = \bar{\psi} (i\gamma_\mu \partial^\mu) \psi$. Let us choose the same representations of the γ matrices (1.18) and (1.19), projection

operators (1.20), and of the chiral operators R and L (1.22) we used in QED. As we already saw in (2.25),

$$\mathcal{L}_F = \bar{\psi}_L (i\gamma_\mu \partial^\mu) \psi_L + \bar{\psi}_R (i\gamma_\mu \partial^\mu) \psi_R. \quad (4.19)$$

If we want to represent two or more non-interacting fermion fields, we just need to add expressions like (4.19) together, one per fermion field. We note that (4.19) represents fermions that possess both a left and a right chiral component. Let us assume, for example, that we want to represent just two fermion fields, field ψ_1 without the right chiral component while field ψ_2 has both left and right chiral components. For this example (4.19) would become:

$$\mathcal{L}_F = \bar{\psi}_{L_1} (i\gamma_\mu \partial^\mu) \psi_{L_1} + \bar{\psi}_{L_2} (i\gamma_\mu \partial^\mu) \psi_{L_2} + \bar{\psi}_{R_2} (i\gamma_\mu \partial^\mu) \psi_{R_2}. \quad (4.20)$$

We can regroup the terms of (4.20) in a more compact form. Let

$$\psi_L = \begin{pmatrix} \psi_{L_1} \\ \psi_{L_2} \end{pmatrix} \quad (4.21)$$

be an $SU(2)$ doublet, of left-handed fermion fields and ψ_{R_2} a singlet under $SU(2)$, of right-handed fermion fields (for now we will denote R_2 by just R). These are two different representations of $SU(2)$. In this case (4.20) can be expressed as:

$$\mathcal{L}_F = \bar{\psi}_L (i\gamma_\mu \partial^\mu) \psi_L + \bar{\psi}_R (i\gamma_\mu \partial^\mu) \psi_R. \quad (4.22)$$

We will consider here only one generation of leptons, the electron and electron-neutrino (ν_e, e) and one generation of quarks, the u -quark and d -quark (u, d). That is to say we are considering four fermion fields which, initially, are non-interacting. Other generations of fermions can be added, by simply adding terms similar to the ones we have already shown. What we say for the u -quark and d -quark is applicable to the u -type quarks and d -type quarks.

By demanding gauge invariance of the Lagrangian density we are forced to couple the fermion fields to gauge fields and to introduce covariant derivatives, one for ψ_L and one for ψ_R . Since ψ_L and ψ_R belong to different representations of $SU(2)$ their

covariant derivatives will be different; this implies different sets of couplings. If the covariant derivative for the right-handed fermion does not include any reference to the W gauge boson, this will ensure that the right-handed fermion does not couple to the W boson. The covariant derivative for an $SU(2) \otimes U(1)$ symmetry group, is defined as:

$$D_\mu = \frac{1}{2} \overleftrightarrow{\partial}_\mu - i g T_a (A_a)_\mu - i g' T_Y C_\mu, \quad (4.23)$$

where T_a ($a = 1, 2, 3$) is the isospin generator of $SU(2)$, while $T_Y = Y/2$, where Y is the hypercharge, is the $U(1)$ generator. Next, we define the electric charge operator Q as

$$Q = T_3 + T_Y. \quad (4.24)$$

When applied to the doublets, the operator Q is a two by two matrix, thus the T_Y operator multiplies the unit matrix. On the other hand, when applied to the singlets ($T_3 = 0$), the Q operator is number or one by one matrix. If the electric charge of a particle is known, then once we specify the value of T_3 the corresponding value of T_Y is automatically determined. And, therefore, so is the value of Y . This means that particles for which we specify the same value of T_3 , and which have different electric charges will have different values for their hypercharge. For our purpose we have two possible values for the $SU(2)$ generator T^a : $T^a = (\tau^a)/2$, for the $SU(2)$ doublets, and $T^a = 0$, for the $SU(2)$ singlets. When we express the covariant derivative (4.23) in terms of the gauge bosons $W^{(\pm)}$, Z and A it becomes:

$$D_\mu = \frac{1}{2} \overleftrightarrow{\partial}_\mu - i \frac{g}{\sqrt{2}} \left\{ T^+ W_\mu^{(+)} + T^- W_\mu^{(-)} \right\} - i A_\mu \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y \right\} - i Z_\mu \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\}, \quad (4.25)$$

where the $SU(2)$ generators are combined into

$$T^\pm = T^1 \pm i T^2 = \frac{\tau^1 \pm i \tau^2}{2} = \tau^\pm. \quad (4.26)$$

τ^i is the i -th Pauli matrix. Of course, when T^a is zero, the above τ 's are also zero. The requirement that the W bosons do not couple to right-handed fermions results in different assignments of T^a for left-handed and right-handed fermions. For the right-handed covariant derivative if we make $T^a = 0$, we ensure no coupling between the

right-handed fermions and the W gauge boson. Thus, for the right-handed fermion $Q = T_Y$. This means that for a right-handed electron (hereby denoted as e_R) $Y = -2$, whereby $Y = +4/3$ for a right-handed u -quark field and $Y = -2/3$ for a right-handed d -quark field. The left-handed components being organized into $SU(2)$ doublets and, thus, having $T^a = (\tau^a)/2$, must have a value for Y such that the electric charges of the two particles are reproduced. That is to say, since $T_3 = \pm 1/2$ then the Y for the doublet is different from that for the singlets. For the doublet of left-handed leptons hereby denoted E_L and the doublet of left-handed quarks hereby denoted Q_L we make the following assignments of Y

$$E_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \quad Y = -1, \quad Q_L = \begin{pmatrix} u \\ d \end{pmatrix}, \quad Y = \frac{1}{3}. \quad (4.27)$$

The upper member of either doublet has $T_3 = +1/2$, while the lower member has $T_3 = -1/2$. When $Y = -1$ then charges are 0 for ν_e and -1 for e_L . On the other hand, when $Y = 1/3$, then the charges are $2/3$ for the u -quark and $-1/3$ for the d -quark. Since it is known that $g \sin \theta_w = g' \cos \theta_w = e$, we have $g \sin \theta_w T_3 + g' \cos \theta_w T_Y = eQ$.

In expression (4.21) we denoted a doublet by the symbol ψ_L . In a change of notation, let $E_L = \psi_L^{(1)}$ and $Q_L = \psi_L^{(2)}$. Thus, superscripts 1 and 2 will refer to the leptons and quarks, respectively. Hence, $\psi_L^{(f)}$ stands for a doublet of left-handed fermions. Further, let $\psi_{L_1}^{(f)}$ represent the top member of the doublet with suffix 1 and $\psi_{L_2}^{(f)}$ represent the bottom member of a doublet with suffix 2. For instance, $\psi_{L_1}^{(1)} = \nu_e$ and $\psi_{L_2}^{(1)} = e_L$. We can easily extend this convention to include the right-handed singlets. Again, superscript 1 and 2 will refer to the leptons and quarks, respectively. So $\psi_{R_1}^{(1)}$ will represent a right-handed ν_e , which does not exist in the standard model, while $\psi_{R_2}^{(1)}$ will represent a right-handed electron. Similarly, $\psi_{R_1}^{(2)}$ will represent a u_R and $\psi_{R_2}^{(2)}$ will represent a d_R . If we set $\psi_{R_1}^{(1)}$ to be identically zero, we can use a very compact naming convention. Because of the different values for T_Y , we must introduce an addition to the naming convention. We will use $T_Y^{(f)}$ for the doublets, and $T_{Y(1)}^{(f)}$ and $T_{Y(2)}^{(f)}$ for the singlets, where the subindex identifies whether is the first or the second member of the pair. So, after we introduce the naming convention (4.25) becomes

$$D_\mu = \frac{1}{2} \overleftrightarrow{\partial}_\mu - i \frac{g}{\sqrt{2}} \left\{ T^+ W_\mu^{(+)} + T^- W_\mu^{(-)} \right\} - i A_\mu \left\{ g \sin \theta_w T_3 + \right. \\ \left. + g' \cos \theta_w T_Y^{(f)} \right\} - i Z_\mu \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\}, \quad (4.28)$$

$$D_\mu = \frac{1}{2} \overleftrightarrow{\partial}_\mu - i g' T_{Y(1,2)}^{(f)} \left\{ A_\mu \cos \theta_w - Z_\mu \sin \theta_w \right\}, \quad (4.29)$$

(4.28) for the doublets and (4.29) for the singlets.

We have so far considered massless fermions. One cannot introduce a mass term in the fermion Lagrangian density $\mathcal{L}_F$ without violating the $SU(2)$ symmetry. Masses for fermions are generated by introducing Yukawa couplings respecting $SU(2)$ symmetry as shown in section 4.4.

We will now follow a parallel approach to QED's, taking into consideration the fact that we are working with two generations of fermions. When including one generation of leptons and one generation of quarks, the Lagrangian density (4.22) becomes,

$$\mathcal{L}_F = \overline{\psi_L^{(f)}} (i \not{D}) \psi_L^{(f)} + \overline{\psi_{R_1}^{(f)}} (i \not{D}) \psi_{R_1}^{(f)} + \overline{\psi_{R_2}^{(f)}} (i \not{D}) \psi_{R_2}^{(f)}, \quad (4.30)$$

where $\not{D} = \gamma_\mu D^\mu$, the values of T^a and $T_Y^{(f)}$ are determined by the $SU(2)$ representation of the fermion fields to which the $\not{D}$ operator applies. As usual $\overline{\psi_L^{(f)}} = (\psi_L^{(f)})^\dagger \gamma^0$. Let us first consider the first term of (4.30) belonging to the $SU(2)$ doublet. For a doublet, the project operators $\Lambda^\pm$ will act on the individual component fields of the doublet. In the case of the singlet there should be no confusion as to their effect. Therefore, let

$$\Lambda^+ \psi_L^{(f)} = \begin{pmatrix} \Lambda^+ \psi_{L_1}^{(f)} \\ \Lambda^+ \psi_{L_2}^{(f)} \end{pmatrix} = \psi_{L_+}^{(f)}, \quad \Lambda^- \psi_L^{(f)} = \psi_{L_-}^{(f)}. \quad (4.31)$$

Thus, the first term of (4.30) becomes

$$\begin{aligned} \overline{\psi_L^{(f)}} (i \not{D}) \psi_L^{(f)} &= i (\psi_L^{(f)})^\dagger \left\{ \gamma^0 \gamma^+ D^- + \gamma^0 \gamma^- D^+ - \gamma^0 \gamma^i D^i \right\} \psi_L^{(f)}, \\ &= i \sqrt{2} (\psi_{L_+}^{(f)})^\dagger D^- \psi_{L_+}^{(f)} + i \sqrt{2} (\psi_{L_-}^{(f)})^\dagger D^+ \psi_{L_-}^{(f)} - \\ &\quad - (\psi_{L_+}^{(f)})^\dagger i \alpha^i D^i \psi_{L_-}^{(f)} + (\psi_{L_-}^{(f)})^\dagger i \alpha^i D^i \psi_{L_+}^{(f)}. \end{aligned} \quad (4.32)$$

Adding also the terms for the singlets, (4.30) finally becomes

$$\begin{aligned}
\mathcal{L}_F = & (\psi_{L+}^{(f)})^\dagger i\sqrt{2}D^- \psi_{L+}^{(f)} + (\psi_{L-}^{(f)})^\dagger i\sqrt{2}D^+ \psi_{L-}^{(f)} - (\psi_{L+}^{(f)})^\dagger i\alpha^i D^i \psi_{L-}^{(f)} + \\
& + (\psi_{L-}^{(f)})^\dagger i\alpha^i D^i \psi_{L+}^{(f)} + (\psi_{R1+}^{(f)})^\dagger i\sqrt{2}D^- \psi_{R1+}^{(f)} + (\psi_{R1-}^{(f)})^\dagger i\sqrt{2}D^+ \psi_{R1-}^{(f)} - \\
& - (\psi_{R1+}^{(f)})^\dagger i\alpha^i D^i \psi_{R1-}^{(f)} + (\psi_{R1-}^{(f)})^\dagger i\alpha^i D^i \psi_{R1+}^{(f)} + (\psi_{R2+}^{(f)})^\dagger i\sqrt{2}D^- \psi_{R2+}^{(f)} + \\
& + (\psi_{R2-}^{(f)})^\dagger i\sqrt{2}D^+ \psi_{R2-}^{(f)} - (\psi_{R2+}^{(f)})^\dagger i\alpha^i D^i \psi_{R2-}^{(f)} + \\
& + (\psi_{R2-}^{(f)})^\dagger i\alpha^i D^i \psi_{R2+}^{(f)}, \tag{4.33}
\end{aligned}$$

where α^i acts on the individual ψ 's representing the fermion fields. We can now expand (4.33) which, after considering the gauge condition $A^+ = C^+ = 0$ becomes,

$$\begin{aligned}
\mathcal{L}_F = & \frac{i}{\sqrt{2}} (\psi_{L+}^{(f)})^\dagger (\partial^- \psi_{L+}^{(f)}) - \frac{i}{\sqrt{2}} (\partial^- \psi_{L+}^{(f)})^\dagger \psi_{L+}^{(f)} + \frac{i}{\sqrt{2}} (\psi_{R1+}^{(f)})^\dagger (\partial^- \psi_{R1+}^{(f)}) - \\
& - \frac{i}{\sqrt{2}} (\partial^- \psi_{R1+}^{(f)})^\dagger \psi_{R1+}^{(f)} + \frac{i}{\sqrt{2}} (\psi_{R2+}^{(f)})^\dagger (\partial^- \psi_{R2+}^{(f)}) - \frac{i}{\sqrt{2}} (\partial^- \psi_{R2+}^{(f)})^\dagger \psi_{R2+}^{(f)} + \\
& + W^{-(+)} [g (\psi_{L+}^{(f)})^\dagger T_+ \psi_{L+}^{(f)}] + W^{(-)} [g (\psi_{L+}^{(f)})^\dagger T_- \psi_{L+}^{(f)}] + \\
& + A^- \sqrt{2} [(\psi_{L+}^{(f)})^\dagger (g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)}) \psi_{L+}^{(f)} + \\
& + g' \cos \theta_w (\psi_{R1+}^{(f)})^\dagger T_{Y(1)}^{(f)} \psi_{R1+}^{(f)} + g' \cos \theta_w (\psi_{R2+}^{(f)})^\dagger T_{Y(2)}^{(f)} \psi_{R2+}^{(f)}] + \\
& + Z^- \sqrt{2} [(\psi_{L+}^{(f)})^\dagger (g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)}) \psi_{L+}^{(f)} - \\
& - g' \sin \theta_w (\psi_{R1+}^{(f)})^\dagger T_{Y(1)}^{(f)} \psi_{R1+}^{(f)} - g' \sin \theta_w (\psi_{R2+}^{(f)})^\dagger T_{Y(2)}^{(f)} \psi_{R2+}^{(f)}] + \\
& + \frac{1}{2} \{ (\psi_{L-}^{(f)})^\dagger P^{(f)} + (P^{(f)})^\dagger \psi_{L-}^{(f)} \} + \frac{1}{2} \{ (\psi_{R1-}^{(f)})^\dagger S_1^{(f)} + (S_1^{(f)})^\dagger \psi_{R1-}^{(f)} \} + \\
& + \frac{1}{2} \{ (\psi_{R2-}^{(f)})^\dagger S_2^{(f)} + (S_2^{(f)})^\dagger \psi_{R2-}^{(f)} \} - \mathcal{H}_F. \tag{4.34}
\end{aligned}$$

$P^{(f)}$, $S_1^{(f)}$ and $S_2^{(f)}$ are equivalent to the $\mathcal{A}_F$ of QED and are given by the following expressions:

$$\begin{aligned}
P^{(f)} = & i\sqrt{2}\partial^+ \psi_{L-}^{(f)} - \alpha^i \left[i\partial^i + \frac{g}{\sqrt{2}} \{ \tau^+ W^{i(+)} + \tau^- W^{i(-)} \} + \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \} + \\
& \left. + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \} \right] \psi_{L+}^{(f)}, \tag{4.35}
\end{aligned}$$

$$S_1^{(f)} = i\sqrt{2}\partial^+ \psi_{R1-}^{(f)} - \alpha^i \left[i\partial^i + g' T_{Y(1)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R1+}^{(f)}, \tag{4.36}$$

$$S_2^{(f)} = i\sqrt{2}\partial^+ \psi_{R2-}^{(f)} - \alpha^i \left[i\partial^i + g' T_{Y(2)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R2+}^{(f)}. \tag{4.37}$$

The Hamiltonian density $\mathcal{H}_F$ is given by the expression

$$\begin{aligned}
\mathcal{H}_F = & \frac{1}{2} (\psi_{L+}^{(f)})^\dagger \alpha^i \left[i \partial^i + \frac{g}{\sqrt{2}} \{ \tau^+ W^{i(+)} + \tau^- W^{i(-)} \} + \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \} + \\
& + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \} \left. \right] \psi_{L-}^{(f)} + \\
& + \frac{1}{2} (\psi_{R1+}^{(f)})^\dagger \alpha^i \left[i \partial^i + g' T_{Y(1)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R1-}^{(f)} + \\
& + \frac{1}{2} (\psi_{R2+}^{(f)})^\dagger \alpha^i \left[i \partial^i + g' T_{Y(2)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R2-}^{(f)} + \\
& + H.C.
\end{aligned} \tag{4.38}$$

As in QED, the presence of the longitudinal components of the gauge bosons is an indication that this Lagrangian density is not complete and it must be complemented with the gauge bosons Lagrangian density $\mathcal{L}_{GB}$. As in QED, here too the minus components of the fermion fields $\psi_{L-}^{(f)}$, $(\psi_{L-}^{(f)})^\dagger$, $\psi_{R-}^{(f)}$ and $(\psi_{R-}^{(f)})^\dagger$ are dependent variables. And when the constraint equations are solved, the Lagrange multipliers are determined in terms of the independent variables $\psi_{L+}^{(f)}$, $\psi_{R1+}^{(f)}$ and $\psi_{R2+}^{(f)}$. As in QED we use the integral operator $(\frac{1}{\partial^+})$, defined in (1.34), to solve the constraint equations and obtain the solutions for the dependent variables or Lagrange multipliers. Since P , S_1 and S_2 represent constraints then, as in QED, by solving the equations $P = 0$, $S_1 = 0$ and $S_2 = 0$ we will find the solutions for the dependent variables: $\psi_{L-}^{(f)}$, $\psi_{R1-}^{(f)}$ and $\psi_{R2-}^{(f)}$. These are

$$\begin{aligned}
\psi_{L-}^{(f)} = & \frac{1}{i \sqrt{2} \partial^+} \alpha^i \left[i \partial^i + \frac{g}{\sqrt{2}} \{ \tau^+ W^{i(+)} + \tau^- W^{i(-)} \} + \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \} + \\
& + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \} \left. \right] \psi_{L+}^{(f)},
\end{aligned} \tag{4.39}$$

$$\psi_{R1-}^{(f)} = \frac{1}{i \sqrt{2} \partial^+} \alpha^i \left[i \partial^i + g' T_{Y(1)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R1+}^{(f)}, \tag{4.40}$$

$$\psi_{R2-}^{(f)} = \frac{1}{i \sqrt{2} \partial^+} \alpha^i \left[i \partial^i + g' T_{Y(2)}^{(f)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \psi_{R2+}^{(f)}. \tag{4.41}$$

When we substitute the dependent field variables in (4.38) by their solutions (4.39) to (4.41), we obtain the Hamiltonian density $\mathcal{H}_F$ in terms of the independent field variables. Thus,

$$\begin{aligned}
\mathcal{H}_F = & \frac{1}{2} (\psi_{L+}^{(f)})^\dagger \alpha^i \left[i \partial^i + \frac{g}{\sqrt{2}} \left\{ \tau^+ W^{i(+)} + \tau^- W^{i(-)} \right\} + \right. \\
& + A^i \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \right\} + \\
& \left. + Z^i \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\} \right] \\
& \cdot \frac{1}{i \sqrt{2} \partial^+} \alpha^j \left[i \partial^j + \frac{g}{\sqrt{2}} \left\{ \tau^+ W^{j(+)} + \tau^- W^{j(-)} \right\} + \right. \\
& + A^j \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \right\} + \\
& \left. + Z^j \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\} \right] \psi_{L+}^{(f)} + \\
& + \frac{1}{2} (\psi_{R1+}^{(f)})^\dagger \alpha^i \left[i \partial^i + g' T_{Y(1)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \\
& \cdot \frac{1}{i \sqrt{2} \partial^+} \alpha^j \left[i \partial^j + g' T_{Y(1)}^{(f)} \left\{ A^j \cos \theta_w - Z^j \sin \theta_w \right\} \right] \psi_{R1+}^{(f)} + \\
& + \frac{1}{2} (\psi_{R2+}^{(f)})^\dagger \alpha^i \left[i \partial^i + g' T_{Y(2)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \\
& \cdot \frac{1}{i \sqrt{2} \partial^+} \alpha^j \left[i \partial^j + g' T_{Y(2)}^{(f)} \left\{ A^j \cos \theta_w - Z^j \sin \theta_w \right\} \right] \psi_{R2+}^{(f)} \\
& + H.C.
\end{aligned} \tag{4.42}$$

We note that these are very compact expressions where, when $f = 1$, we are dealing with leptons, with the understanding that the field for a right-handed neutrino is identically zero everywhere, and when $f = 2$, we are dealing with quarks. If we sum over $f = 1, 2$ we are dealing with both leptons and quarks and the convention on summation applies, and again, with the understanding that the field for a right-handed neutrino is identically zero everywhere.

As in QED we now move from a four-component spinor to a two-component spinor representation. Here too we must introduce the factor of $\sqrt{2}$ as we did in QED. In (1.24) and (1.25) we defined the relationship between the two-component left-handed spinors and the four-component left-handed spinors, here we just extend that definition to cover the doublets of left-handed spinors. As in QED the two-component spinors are similar to the non-relativistic spinors. For the left-handed fermions we define the

two-component spinors ξ of helicity -1 .

$$\xi_{1,2}^{(f)} = \begin{pmatrix} 0 \\ \sqrt[4]{2} b_{1,2}^{(f)} \end{pmatrix}, \quad \psi_{L+}^{(f)} = \begin{pmatrix} \frac{\xi_1^{(f)}}{\sqrt[4]{2}} \\ 0 \\ \frac{\xi_2^{(f)}}{\sqrt[4]{2}} \\ 0 \end{pmatrix}, \quad \xi^{(f)} = \begin{pmatrix} \xi_1^{(f)} \\ \xi_2^{(f)} \end{pmatrix}. \quad (4.43)$$

Similarly, for the right-handed singlets we define the two-component spinors κ of helicity $+1$

$$\kappa_{1,2}^{(f)} = \begin{pmatrix} \sqrt[4]{2} a_{1,2}^{(f)} \\ 0 \end{pmatrix}, \quad \psi_{R1+,2+}^{(f)} = \begin{pmatrix} \frac{\kappa_{1,2}^{(f)}}{\sqrt[4]{2}} \\ 0 \end{pmatrix}, \quad (4.44)$$

The ξ 's will always stand for a left-handed object, whether a doublet or elements of a doublet, while κ 's will always stand for right-handed singlets. It is easy to see that if $\psi_{R1+}^{(1)} \equiv 0$ everywhere then the requirement that $\psi_{R1}^{(1)} \equiv 0$ everywhere will be met. The presence of the α^i matrices in (4.39) to (4.41) ensures a flip of the helicity of the ξ spinors (via a multiplication by the σ^i matrices). After some algebra it can be shown that the structure of $\psi_{L-}^{(f)}$ and $\psi_{R-}^{(f)}$ obtained from (4.39) to (4.41), does satisfy the structure of a minus component. It is easy to verify that when expressed in terms of two-component spinors, the plus and minus components of ψ have opposite helicities. The application of the charge conjugation operator shows, as in QED, that the anti-fermions can also be represented by two-component spinors. We extend our naming convention to include the dependent variables. Let

$$\psi_{L-}^{(f)} = \begin{pmatrix} 0 \\ \xi_1^{(f)-} \\ 0 \\ \xi_2^{(f)-} \end{pmatrix}, \quad \xi^{(f)-} = \begin{pmatrix} \xi_1^{(f)-} \\ \xi_2^{(f)-} \end{pmatrix}, \quad \psi_{R1-}^{(f)} = \begin{pmatrix} 0 \\ \kappa_1^{(f)-} \end{pmatrix}, \quad \psi_{R2-}^{(f)} = \begin{pmatrix} 0 \\ \kappa_2^{(f)-} \end{pmatrix} \quad (4.45)$$

The ξ 's and κ 's without the minus index represent the independent field variables, while the ξ^- 's and κ^- 's with the minus index represent the dependent field variables.

But before we can present the relationship between the dependent and independent field variables, we must convert (4.34) to its two-component spinors representation. Then, the Lagrangian density, $\mathcal{L}_F$ becomes:

$$\begin{aligned}
\mathcal{L}_F = & \frac{i}{2} (\xi^{(f)})^\dagger \partial^- \xi^{(f)} - \frac{i}{2} (\partial^- \xi^{(f)})^\dagger \xi^{(f)} + \frac{i}{2} (\kappa_1^{(f)})^\dagger \partial^- \kappa_1^{(f)} - \\
& - \frac{i}{2} (\partial^- \kappa_1^{(f)})^\dagger \kappa_1^{(f)} + \frac{i}{2} (\kappa_2^{(f)})^\dagger \partial^- \kappa_2^{(f)} - \frac{i}{2} (\partial^- \kappa_2^{(f)})^\dagger \kappa_2^{(f)} + \\
& + W^{-(+)} \left[\frac{g}{\sqrt{2}} (\xi^{(f)})^\dagger T_+ \xi^{(f)} \right] + W^{(-)} \left[\frac{g}{\sqrt{2}} (\xi^{(f)})^\dagger T_- \xi^{(f)} \right] + \\
& + A^- \left[(\xi^{(f)})^\dagger (g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)}) \xi^{(f)} + \right. \\
& \quad \left. + g' \cos \theta_w (\kappa_1^{(f)})^\dagger T_{Y(1)}^{(f)} \kappa_1^{(f)} + g' \cos \theta_w (\kappa_2^{(f)})^\dagger T_{Y(2)}^{(f)} \kappa_2^{(f)} \right] + \\
& + Z^- \left[(\xi^{(f)})^\dagger (g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)}) \xi^{(f)} - \right. \\
& \quad \left. - g' \sin \theta_w (\kappa_1^{(f)})^\dagger T_{Y(1)}^{(f)} \kappa_1^{(f)} - g' \sin \theta_w (\kappa_2^{(f)})^\dagger T_{Y(2)}^{(f)} \kappa_2^{(f)} \right] + \\
& + \frac{1}{2} \left\{ (\xi^{(f)-})^\dagger \mathcal{P}^{(f)} + (\mathcal{P}^{(f)})^\dagger \xi^{(f)-} \right\} + \frac{1}{2} \left\{ (\kappa_1^{(f)-})^\dagger \mathcal{R}_1^{(f)} + (\mathcal{R}_1^{(f)})^\dagger \kappa_1^{(f)-} \right\} + \\
& + \frac{1}{2} \left\{ (\kappa_2^{(f)-})^\dagger \mathcal{R}_2^{(f)} + (\mathcal{R}_2^{(f)})^\dagger \kappa_2^{(f)-} \right\} - \mathcal{H}_F
\end{aligned} \tag{4.46}$$

In (4.46) the constraint terms are now given by the expressions:

$$\begin{aligned}
\mathcal{P}^{(f)} = & i \sqrt{2} \partial^+ \xi^{(f)-} - \sigma^i \left[i \partial^i + \frac{g}{\sqrt{2}} \left\{ T^+ W^{i(+)} + T^- W^{i(-)} \right\} + \right. \\
& + A^i \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y \right\} + \\
& \left. + Z^i \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y \right\} \right] \xi^{(f)} \frac{1}{\sqrt{2}},
\end{aligned} \tag{4.47}$$

$$\mathcal{R}_1^{(f)} = i \sqrt{2} \partial^+ \kappa_1^{(f)-} - \sigma^i \left[i \partial^i + g' T_Y^1 \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \kappa_1^{(f)} \frac{1}{\sqrt{2}}, \tag{4.48}$$

$$\mathcal{R}_2^{(f)} = i \sqrt{2} \partial^+ \kappa_2^{(f)-} - \sigma^i \left[i \partial^i + g' T_Y^2 \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \kappa_2^{(f)} \frac{1}{\sqrt{2}}. \tag{4.49}$$

We remind the reader that we use the Pauli matrix notation $T^\pm = \tau^\pm$ for the weak-isospin and σ^i for the two-component spinors. By solving the constraint equations we again obtain the solutions for the dependent field variables, this time in terms of two-component spinors.

$$\begin{aligned}\xi^{(f)-} &= \frac{1}{i\sqrt[4]{8}\partial^+} \sigma^i \left[i\partial^i + \frac{g}{\sqrt{2}} \left\{ \tau^+ W^{i(+)} + \tau^- W^{i(-)} \right\} + \right. \\ &\quad + A^i \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \right\} + \\ &\quad \left. + Z^i \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\} \right] \xi^{(f)},\end{aligned}\quad (4.50)$$

$$\kappa_1^{(f)-} = \frac{1}{i\sqrt[4]{8}\partial^+} \sigma^i \left[i\partial^i + g' T_{Y(1)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \kappa_1^{(f)}, \quad (4.51)$$

$$\kappa_2^{(f)-} = \frac{1}{i\sqrt[4]{8}\partial^+} \sigma^i \left[i\partial^i + g' T_{Y(2)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \kappa_2^{(f)}. \quad (4.52)$$

In the two component form these solutions relate the dependent field variables in terms of the independent field variables.

A straightforward calculation shows that we can obtain an expression for $\mathcal{H}_F$ in terms of the doublet $\xi^{(f)}$ and the singlets $\kappa_1^{(f)}$ and $\kappa_2^{(f)}$ defined in (4.43) to (4.45). We simply replace α^i by σ^i , $\psi_L^{(f)}$ by $\xi^{(f)}$, $\psi_{L_{j+}}^{(f)}$ by $\xi_j^{(f)}$, $\psi_{R_{j+}}^{(f)}$ by $\kappa_j^{(f)}$ ($j = 1, 2$) and the dependent variables ξ_j^- and κ_j^- ($j = 1, 2$), by their corresponding solutions. Thus,

$$\begin{aligned}\mathcal{H}_F &= \frac{1}{2} (\xi^{(f)})^\dagger \sigma^i \left[i\partial^i + \frac{g}{\sqrt{2}} \left\{ T^+ W^{i(+)} + T^- W^{i(-)} \right\} + \right. \\ &\quad + A^i \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \right\} + \\ &\quad \left. + Z^i \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\} \right] \\ &\quad \cdot \frac{1}{2i\partial^+} \sigma^j \left[i\partial^j + \frac{g}{\sqrt{2}} \left\{ \tau^+ W^{j(+)} + \tau^- W^{j(-)} \right\} + \right. \\ &\quad + A^j \left\{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)} \right\} + \\ &\quad \left. + Z^j \left\{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)} \right\} \right] \xi^{(f)} + \\ &\quad + \frac{1}{2} (\kappa_1^{(f)})^\dagger \sigma^i \left[i\partial^i + g' T_{Y(1)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \\ &\quad \cdot \frac{1}{2i\partial^+} \sigma^j \left[i\partial^j + g' T_{Y(1)}^{(f)} \left\{ A^j \cos \theta_w - Z^j \sin \theta_w \right\} \right] \kappa_1^{(f)} + \\ &\quad + \frac{1}{2} (\kappa_2^{(f)})^\dagger \sigma^i \left[i\partial^i + g' T_{Y(2)}^{(f)} \left\{ A^i \cos \theta_w - Z^i \sin \theta_w \right\} \right] \\ &\quad \cdot \frac{1}{2i\partial^+} \sigma^j \left[i\partial^j + g' T_{Y(2)}^{(f)} \left\{ A^j \cos \theta_w - Z^j \sin \theta_w \right\} \right] \kappa_2^{(f)} + H.C. \quad (4.53)\end{aligned}$$

By taking this Hamiltonian density and dropping all terms in g and g' , we obtain the

free Hamiltonian density. We denote it by $\mathcal{H}_F^0$. That is to say,

$$\begin{aligned} \mathcal{H}_F^0 = & \frac{1}{2} \begin{pmatrix} \xi_1^{(f)\dagger} & \xi_2^{(f)\dagger} \end{pmatrix} \frac{1}{2i\partial^+} [-\partial_\perp^2] \begin{pmatrix} \xi_1^{(f)} \\ \xi_2^{(f)} \end{pmatrix} + \frac{1}{2} (\kappa_1^{(f)})^\dagger \frac{1}{2i\partial^+} [-\partial_\perp^2] \kappa_1^{(f)} \\ & + \frac{1}{2} (\kappa_2^{(f)})^\dagger \frac{1}{2i\partial^+} \sigma^j [-\partial_\perp^2] \kappa_2^{(f)} + H.C. \end{aligned} \quad (4.54)$$

In the next section we discuss the Higgs-Goldstone mechanism in the LF formulation. The breaking of symmetry results in three bosons acquiring mass and the fourth gauge boson, corresponding to the photon, is left massless.

4.3 Higgs-Goldstone mechanism for the breaking of symmetry spontaneously.

We use the Higgs-Goldstone mechanism to spontaneously break the $SU(2) \otimes U(1)$ symmetry down to $U(1)_{em}$, in order to give mass to the gauge bosons. However, a new subtlety is introduced when we apply the procedure in the LF formulation. This is due to the fact that only the transverse components of the gauge fields are independent variables, and are, therefore, the only ones present in $\mathcal{L}_{GB}$. If we apply the Higgs-Goldstone mechanism with only the physical neutral Higgs $h(x)$ introduced, it will give mass only to the $W^{i(\pm)}$ and Z^i components (called the transverse components in LF), for $i = 1, 2$. To have the $W^{-(\pm)}$ and Z^- components (called the longitudinal components in LF) also acquire mass it is necessary to introduce charged $G^\pm$ and neutral G^0 , Goldstone bosons.

We start with the Lagrangian density for a new scalar field. Consider

$$\mathcal{L}_H = (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi), \quad (4.55)$$

where $\Phi = \begin{pmatrix} G^+ \\ \chi^0 \end{pmatrix}$ is an $SU(2)$ doublet of scalars, $V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$ and D^μ given below in (4.56). We will use Φ when breaking the symmetry in order, as we already have said, to give mass to the gauge fields. The Lagrangian density (4.55) uses

the following definition of the covariant derivative:

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} \{ W_\mu^{(+)} T^+ + W_\mu^{(-)} T^- \} - \frac{igZ_\mu}{\cos \theta_w} \{ T^3 - Q \sin^2 \theta_w \} - ieQA_\mu, \quad (4.56)$$

which follows from (4.25) when expressed in terms of the operator Q .

In the Higgs-Goldstone mechanism the charge of the G^+ is $+1$ while the charge of χ^0 is zero. Since $Q = T_3 + T_Y$ and $T_Y = Y/2$, then for a hypercharge $Y(\Phi) = 1$, we get correctly the charges $+1$ for G^+ and zero for χ^0 . We choose G^+ and χ^0 to be complex scalar fields, with

$$G^+ = -\frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \quad (4.57)$$

$$\chi^0 = \frac{v + h(x) + iG^0}{\sqrt{2}}, \quad (4.58)$$

where ϕ_1 , ϕ_2 , $h(x)$ and G^0 are real scalar fields, and v is a real constant. Introducing

$$v = \sqrt{\frac{\mu^2}{\lambda}}, \quad M_w = \frac{gv}{2}, \quad M_z = \frac{v\sqrt{g^2 + g'^2}}{2} = \frac{gv}{2 \cos \theta_w} \quad (4.59)$$

then $D_\mu \Phi$ in (4.55) becomes

$$D_\mu \Phi = \begin{pmatrix} \partial_\mu G^+ - iM_w W_\mu^{(+)} - \frac{ig}{2}(h(x) + iG^0)W_\mu^{(+)} - \frac{ig \cos 2\theta_w}{2 \cos \theta_w} Z_\mu G^+ - ieA_\mu G^+ \\ \partial_\mu \left(\frac{h(x) + iG^0}{\sqrt{2}} \right) + \frac{i}{\sqrt{2}} M_z Z_\mu - \frac{ig}{\sqrt{2}} W_\mu^{(-)} G^+ + \frac{ig}{2 \cos \theta_w} \left(\frac{h(x) + iG^0}{\sqrt{2}} \right) Z_\mu \end{pmatrix}. \quad (4.60)$$

Defining

$$g(\theta_w) = \frac{g \cos 2\theta_w}{2 \cos \theta_w}, \quad g'(\theta_w) = \frac{g}{2 \cos \theta_w}. \quad (4.61)$$

Writing $(D^\mu\Phi)^\dagger(D_\mu\Phi)$ in the LF formulation, we obtain

$$\begin{aligned}
(D^\mu\Phi)^\dagger D_\mu\Phi = & (\partial^+G^-) \left\{ \partial^-G^+ - iM_w W^{(+)-} - \frac{ig}{2}(h(x) + iG^0)W^{(+)-} - \right. \\
& \left. - ig(\theta_w)Z^-G^+ - ieA^-G^+ \right\} + \left\{ \partial^-G^- + iM_w W^{(-)-} \right. \\
& \left. + \frac{ig}{2}(h(x) - iG^0)W^{(-)-} + ig(\theta_w)Z^-G^- + ieA^-G^- \right\} (\partial^+G^+) \\
& + \frac{1}{2} \left\{ \partial^+(h - iG^0) \right\} \left\{ \partial^-(h + iG^0) + iM_z Z^- - igW^{(-)-}G^+ + \right. \\
& \left. + ig'(\theta_w)(h + iG^0)Z^- \right\} + \frac{1}{2} \left\{ \partial^-(h - iG^0) - iM_z Z^- + \right. \\
& \left. + igW^{(+)-}G^- - ig'(\theta_w)(h - iG^0)Z^- \right\} \left\{ \partial^+(h + iG^0) \right\} - \\
& - (D^i\Phi)^\dagger(D^i\Phi). \tag{4.62}
\end{aligned}$$

The potential term V becomes

$$\begin{aligned}
V(\Phi) = & \lambda \left[G^-G^+ + vh(x) + \frac{h^2(x) + (G^0)^2}{2} \right]^2 - \lambda \frac{v^4}{4} \\
\simeq & \lambda \left[G^-G^+ + vh(x) + \frac{h^2(x) + (G^0)^2}{2} \right]^2, \tag{4.63}
\end{aligned}$$

where the term $\lambda v^4/4$ being a constant does not have any effect and can be ignored.

On combining (4.62) and (4.63), we obtain the following LF form expression for $\mathcal{L}_H$:

$$\begin{aligned}
\mathcal{L}_H = & (\partial^+G^-)(\partial^-G^+) + (\partial^-G^-)(\partial^+G^+) + (\partial^+h)(\partial^-h) + (\partial^+G^0)(\partial^-G^0) + \\
& + W^{(-)-} \left\{ i(\partial^+G^+) \left[M_w + \frac{g}{2}(h - iG^0) \right] - \frac{ig}{2}G^+\partial^+(h - iG^0) \right\} + \\
& + W^{(+)-} \left\{ -i(\partial^+G^-) \left[M_w + \frac{g}{2}(h + iG^0) \right] + \frac{ig}{2}G^-\partial^+(h + iG^0) \right\} + \\
& + Z^- \left\{ ig(\theta_w)G^-(\partial^+G^+) - ig(\theta_w)G^+(\partial^+G^-) + \right. \\
& \quad + \frac{i}{2} \left[M_z + g'(\theta_w)(h + iG^0) \right] \left[\partial^+(h - iG^0) \right] - \\
& \quad - \frac{i}{2} \left[M_z + g'(\theta_w)(h - iG^0) \right] \left[\partial^+(h + iG^0) \right] \left. \right\} + \\
& + A^- \left\{ ieG^-(\partial^+G^+) - ieG^+(\partial^+G^-) \right\} - \mathcal{H}_H, \tag{4.64}
\end{aligned}$$

where $\mathcal{H}_H = (D^i\Phi)^\dagger(D^i\Phi) + V(\Phi)$. After some simplifications we find that the Hamiltonian density is given by the expression,

$$\begin{aligned}
\mathcal{H}_H = & (\partial^i G^-)(\partial^i G^+) + i M_w \left(W^{i(-)} \partial^i G^+ - W^{i(+)} \partial^i G^- \right) + M_w^2 W^{i(-)} W^{i(+)} + \\
& - i \left[\partial^i G^- + i M_w W^{i(-)} \right] \left[\frac{g}{2} (h + i G^0) W^{i(+)} + \{ g(\theta_w) Z^i + e A^i \} G^+ \right] + \\
& + i \left[\partial^i G^+ - i M_w W^{i(+)} \right] \left[\frac{g}{2} (h - i G^0) W^{i(-)} + \{ g(\theta_w) Z^i + e A^i \} G^- \right] + \\
& + \left[\frac{g}{2} (h - i G^0) W^{i(-)} + \{ g(\theta_w) Z^i + e A^i \} G^- \right] \\
& \quad \cdot \left[\frac{g}{2} (h + i G^0) W^{i(+)} + \{ g(\theta_w) Z^i + e A^i \} G^+ \right] + \\
& + \frac{1}{2} \left\{ \partial^i (h - i G^0) \partial^i (h + i G^0) + 2 M_z Z^i \partial^i G^0 + M_z^2 Z^i Z^i + \right. \\
& \quad - i (\partial^i (h - i G^0) - i M_z Z^i) (g G^+ W^{i(-)} - g'(\theta_w) (h + i G^0) Z^i) + \\
& \quad + i (\partial^i (h + i G^0) + i M_z Z^i) (g G^- W^{i(+)} - g'(\theta_w) (h - i G^0) Z^i) + \\
& \quad + \left[g G^- W^{i(+)} - g'(\theta_w) (h - i G^0) Z^i \right] \\
& \quad \cdot \left[g G^+ W^{i(-)} - g'(\theta_w) (h + i G^0) Z^i \right] \left. \right\} + \\
& + \lambda \left(G^- G^+ + v h(x) + \frac{h^2(x) + (G^0)^2}{2} \right)^2. \tag{4.65}
\end{aligned}$$

We now direct our attention towards the term $\lambda v^2 h^2(x)$. For future reference we may mention that it is possible to express λ in terms of g , m_h and M_w as

$$\lambda v^2 = \frac{1}{2} m_h^2, \quad \lambda v = \frac{g m_h^2}{4 M_w}, \quad \lambda = \frac{g^2}{8} \left(\frac{m_h}{M_w} \right)^2. \tag{4.66}$$

Then, we notice in (4.65) the presence of mass terms for the gauge fields as well as the scalar Higgs field. Here too we define the free Hamiltonian density by making the coupling constants and λ zero. We get for $\mathcal{H}_H^0$,

$$\begin{aligned}
\mathcal{H}_H^0 = & M_w^2 W^{i(-)} W^{i(+)} + \frac{1}{2} M_z^2 Z^i Z^i + \\
& + (\partial^i G^-)(\partial^i G^+) + \frac{1}{2} (\partial^i h(x))^2 + \frac{1}{2} (\partial^i G^0)^2 + \frac{1}{2} m_h^2 h(x)^2. \tag{4.67}
\end{aligned}$$

At this point we have three Lagrangian densities $\mathcal{L}_{GB}$, $\mathcal{L}_F$ and $\mathcal{L}_H$. The Lagrangian density of the now massive gauge bosons fields is given by $\mathcal{L} = \mathcal{L}_{GB} + \mathcal{L}_F + \mathcal{L}_H$. To see this we add $\mathcal{H}_{GB}$ from (4.18) to $\mathcal{H}_H^0$ in (4.67), and we see that the transverse fields W^i and Z^i acquire masses M_w and M_z respectively. The constraint terms, which now include contributions from all three Lagrangian densities, are found to be:

$$\begin{aligned}
& W^{-(+)} \left[\partial^+ (-\partial^+ W^{(-)} + \partial^i W^{i(-)} - i M_w G^-) + J_1 \right] + \\
& W^{-(-)} \left[\partial^+ (-\partial^+ W^{(+)} + \partial^i W^{i(+)} + i M_w G^+) + J_1^* \right] + \\
& Z^- \left[\partial^+ (-\partial^+ Z^- + \partial^i Z^i + M_z G^0) + J_2 \right] + \\
& A^- \left[\partial^+ (-\partial^+ A^- + \partial^i A^i) + J_3 \right].
\end{aligned} \tag{4.68}$$

where the quantities J_1 , J_2 and J_3 are given by the expressions,

$$\begin{aligned}
J_1 = & -i g \left\{ W^{i(-)} \partial^+ (A^i \sin \theta_w + Z^i \cos \theta_w) - \right. \\
& \left. - (A^i \sin \theta_w + Z^i \cos \theta_w) \partial^+ W^{i(-)} \right\} + \frac{g}{\sqrt{2}} (\xi^{(f)})^\dagger (T_+) \xi^{(f)} - \\
& - \frac{i g}{2} \left[(\partial^+ G^-) [(h(x) + i G^0)] - G^- \partial^+ [h(x) + i G^0] \right],
\end{aligned} \tag{4.69}$$

$$\begin{aligned}
J_2 = & -i g \cos \theta_w \left\{ W^{i(+)} \partial^+ W^{i(-)} - W^{i(-)} \partial^+ W^{i(+)} \right\} + \\
& + (\xi^{(f)})^\dagger (g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(f)}) \xi^{(f)} - \\
& - g' \sin \theta_w [(\kappa_1^{(f)})^\dagger T_{Y(1)}^{(f)} \kappa_1^{(f)} + (\kappa_2^{(f)})^\dagger T_{Y(2)}^{(f)} \kappa_2^{(f)}] + \\
& + i g (\theta_w) [G^- (\partial^+ G^+) - G^+ (\partial^+ G^-)] + \\
& + g' (\theta_w) [h(x) \partial^+ G^0 - G^0 \partial^+ h(x)],
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
J_3 = & -i g \sin \theta_w \left\{ W^{i(+)} \partial^+ W^{i(-)} - W^{i(-)} \partial^+ W^{i(+)} \right\} + \\
& + (\xi^{(f)})^\dagger (g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(f)}) \xi^{(f)} + \\
& + i e [G^- (\partial^+ G^+) - G^+ (\partial^+ G^-)] + \\
& + g' \cos \theta_w [(\kappa_1^{(f)})^\dagger T_{Y(1)}^{(f)} \kappa_1^{(f)} + (\kappa_2^{(f)})^\dagger T_{Y(2)}^{(f)} \kappa_2^{(f)}].
\end{aligned} \tag{4.71}$$

J_1^* is the hermitian adjoint of J_1 .

In the absence of interactions, that is to say when the coupling constants g and g' are zero, the constraint equations become,

$$\partial^+ (-\partial^+ W^{(-)} + \partial^i W^{i(-)} - i M_w G^-) = 0, \tag{4.72}$$

$$\partial^+ (-\partial^+ W^{(+)} + \partial^i W^{i(+)} + i M_w G^+) = 0, \tag{4.73}$$

$$\partial^+ (-\partial^+ Z^- + \partial^i Z^i + M_z G^0) = 0, \tag{4.74}$$

$$\partial^+ (-\partial^+ A^- + \partial^i A^i) = 0. \tag{4.75}$$

Equations (4.72), (4.73), (4.74) and (4.72) are satisfied if,

$$\partial^+ W^{(-)} - \partial^i W^{i(-)} = \partial^\mu W_\mu^{(-)} = -i M_w G^-, \quad (4.76)$$

$$\partial^+ W^{(+)} - \partial^i W^{i(+)} = \partial^\mu W_\mu^{(+)} = i M_w G^+, \quad (4.77)$$

$$\partial^+ Z^- - \partial^i Z^i = \partial^\mu Z_\mu = M_z G^0, \quad (4.78)$$

$$\partial^+ A^- - \partial^i A^i = \partial^\mu A_\mu = 0. \quad (4.79)$$

These are 't Hooft's conditions for the gauge fields. Typically they relate the minus component of the gauge field, say $W^{(-)}$ to a combination of the physical fields $W^{i(+)}$ and G^+ , similarly for the other fields. Note that the electromagnetic field A^μ still satisfies the massless Lorentz condition. Now note that if M_w and M_z were zero, the minus components are entirely dependent on the two physical components labeled by $i = 1, 2$. But now with M_w and M_z not zero the minus component of a field becomes an independent new field. Thus, the massive gauge fields acquire three components of polarization instead of two and the massless electromagnetic field still retains two polarization states.

There are two alternative ways of handling these three polarization states. The first, is to rely on the physical components $i = 1, 2$ and introduce explicitly the scalars $G^\pm$ and G^0 to give the extra degrees of freedom. In this approach the physical fields are those which manifest themselves through the time derivative terms in the Lagrangian density. This is seen in (4.64) where the LF time derivatives $\partial^- G^\pm$ and $\partial^- G^0$ appear in the Lagrangian density explicitly. Such a method corresponds, in equal time formulation, to working in the Feynman-'t Hooft's gauge, where the gauge propagator is simple but one has to introduce additional diagrams involving $G^\pm$ and G^0 explicitly. The second, is to eliminate $G^\pm$ and G^0 in favor of the minus components through equations (4.76) to (4.78) and re-express the Lagrangian density in terms of the minus field components and $i = 1, 2$ field components only.

Thus, one has two possibilities: either treat $W^{i(\pm)}$, $W^{-(\pm)}$ and Z^i for $i = 1, 2$ as physical, in which case the $G^\pm$ and G^0 will be eliminated as unphysical, or, $W^{i(\pm)}$, $G^\pm$, Z^i , G^0 for $i = 1, 2$ as physical, in which case W^- and Z^- will not appear explicitly.

These correspond, in the equal time formulation, to working with the unitary gauge and renormalization gauge, respectively.

The constraint equations have as their solutions,

$$\partial^+ W^{-(+)} = \partial^i W^{i(+)} + i M_w G^+ + \frac{1}{\partial^+} J_1^*, \quad (4.80)$$

$$\partial^+ W^{(-)} = \partial^i W^{i(-)} - i M_w G^- + \frac{1}{\partial^+} J_1, \quad (4.81)$$

$$\partial^+ Z^- = \partial^i Z^i + M_z G^0 + \frac{1}{\partial^+} J_2, \quad (4.82)$$

$$\partial^+ A^- = \partial^i A^i + \frac{1}{\partial^+} J_3. \quad (4.83)$$

These are the values of the longitudinal derivatives of the longitudinal components of the gauge bosons fields that will be used in conjunction with (4.18) to calculate $\mathcal{H}_{GB}$ in terms of the independent field variables. Thus, after inserting the expressions from (4.80) to (4.83) into (4.18) we obtain

$$\begin{aligned} \mathcal{H}_{GB} = & \left[\partial^i W^{i(-)} - i M_w G^- + \frac{1}{\partial^+} J_1 \right] \left[\partial^j W^{j(+)} + i M_w G^+ + \frac{1}{\partial^+} J_1^* \right] + \\ & + \frac{1}{2} \left[\partial^i A^i + \frac{1}{\partial^+} J_3 \right] \left[\partial^j A^j + \frac{1}{\partial^+} J_3 \right] + \\ & + \frac{1}{2} \left[\partial^i Z^i + M_z G^0 + \frac{1}{\partial^+} J_2 \right] \left[\partial^j Z^j + M_z G^0 + \frac{1}{\partial^+} J_2 \right] + \\ & + \partial^i W^{j(+)} \partial^i W^{j(-)} + \frac{1}{2} (\partial^i A^j)^2 + \frac{1}{2} (\partial^i Z^j)^2. \end{aligned} \quad (4.84)$$

Summed independently over i and j ($i, j = 1, 2$). Here we do as we did in (2.53). This Hamiltonian density (4.84) is to be complemented with the Higgs Hamiltonian density (4.65) and together they will form the Hamiltonian density for the massive gauge bosons fields plus the scalar fields. We define the free Hamiltonian density $\mathcal{H}_{GB}^0$ using the same approach as before. Thus,

$$\mathcal{H}_{GB}^0 = \partial^i W^{j(-)} \partial^i W^{j(+)} + M_w^2 G^+ G^- + \frac{1}{2} [\partial^i A^j]^2 + \frac{1}{2} [\partial^i Z^j]^2 + \frac{1}{2} M_z^2 [G^0]^2. \quad (4.85)$$

The free Hamiltonian density of the massive gauge and scalar fields is given by:

$$\begin{aligned} \mathcal{H}_{GB}^0 + \mathcal{H}_H^0 = & \partial^i W^{j(-)} \partial^i W^{j(+)} + M_w^2 W^{i(-)} W^{i(+)} + \frac{1}{2} [\partial^i Z^j]^2 + \frac{1}{2} M_z^2 Z^i Z^i + \\ & + \frac{1}{2} [\partial^i h(x)]^2 + \frac{1}{2} m_h^2 h^2(x) + (\partial^i G^-)(\partial^i G^+) + M_w^2 G^+ G^- + \\ & + \frac{1}{2} [\partial^i G^0]^2 + \frac{1}{2} M_z^2 [G^0]^2 + \frac{1}{2} [\partial^i A^j]^2. \end{aligned} \quad (4.86)$$

As we can see, only the photon field has remained massless.

4.4 Yukawa Lagrangian; generation of mass for the fermions.

So far we have used the Higgs-Goldstone system to generate mass for the gauge bosons breaking $SU(2) \otimes U(1)$ symmetry. We generate mass for the fermions through the Yukawa coupling of the Higgs-Goldstone bosons to the fermions. The details of this are presented in this section.

We start with the doublet Φ introduced earlier with $Y(\Phi) = 1$, and introduce $\tilde{\Phi} = i\sigma_2 \Phi^*$ with hypercharge $Y(\tilde{\Phi}) = -1$, which will be necessary to give mass to the u -quark. The $SU(2)$ doublet of scalars Φ will be used to give mass to the electron and the d -quark. As required, the electron-neutrino will remain massless. Consider the Lagrangian density $\mathcal{L}_Y$ [28] in which the doublets of scalar fields Φ and $\tilde{\Phi}$ interact with the fermions and which is invariant under $SU(2) \otimes U(1)$:

$$\mathcal{L}_Y = -\lambda_e \bar{\psi}_L^{(1)} \Phi \psi_{R_2}^{(1)} - \lambda_u \bar{\psi}_L^{(2)} \tilde{\Phi} \psi_{R_1}^{(2)} - \lambda_d \bar{\psi}_L^{(2)} \Phi \psi_{R_2}^{(2)} + H.C. \quad (4.87)$$

From (1.18), (1.21), (1.22) and (4.31) we have that in general:

$$\bar{\psi}_{L+} \psi_{R+} = \bar{\psi}_{L-} \psi_{R-} = \bar{\psi}_{R+} \psi_{L+} = \bar{\psi}_{R-} \psi_{L-} = 0. \quad (4.88)$$

This means that (4.87), when expanded, is reduced to:

$$\begin{aligned} \mathcal{L}_Y = & -\lambda_e G^+ \bar{\psi}_{L1-}^{(1)} \psi_{R2+}^{(1)} - \lambda_e G^+ \bar{\psi}_{L1+}^{(1)} \psi_{R2-}^{(1)} - \\ & -\lambda_e \chi^0 \bar{\psi}_{L2-}^{(1)} \psi_{R2+}^{(1)} - \lambda_e \chi^0 \bar{\psi}_{L2+}^{(1)} \psi_{R2-}^{(1)} - \\ & -\lambda_d G^+ \bar{\psi}_{L1-}^{(2)} \psi_{R2+}^{(2)} - \lambda_d G^+ \bar{\psi}_{L1+}^{(2)} \psi_{R2-}^{(2)} - \\ & -\lambda_d \chi^0 \bar{\psi}_{L2-}^{(2)} \psi_{R2+}^{(2)} - \lambda_d \chi^0 \bar{\psi}_{L2+}^{(2)} \psi_{R2-}^{(2)} - \\ & -\lambda_u (\chi^0)^* \bar{\psi}_{L1-}^{(2)} \psi_{R1+}^{(2)} - \lambda_u (\chi^0)^* \bar{\psi}_{L1+}^{(2)} \psi_{R1-}^{(2)} + \\ & + \lambda_u G^- \bar{\psi}_{L2-}^{(2)} \psi_{R1+}^{(2)} + \lambda_u G^- \bar{\psi}_{L2+}^{(2)} \psi_{R1-}^{(2)} + H.C. \end{aligned} \quad (4.89)$$

Since $\mathcal{L}_F$ (4.46) is given in terms of ξ 's, it will be convenient to convert $\mathcal{L}_Y$ also to terms of ξ 's. It is easy to verify that for the relationships given in (4.43) we have:

$$\overline{\psi^{(f)}}_{L_{j+}} = \frac{1}{\sqrt[4]{2}} (0 \quad i \{\xi_j^{(f)}\}^\dagger), \quad \overline{\psi^{(f)}}_{L_{j-}} = (i \{\xi_j^{(f)-}\}^\dagger \quad 0). \quad (4.90)$$

and that this result is applicable to both elements of the doublets of spinors. By using (4.90) we can further simplify (4.89) with the result,

$$\begin{aligned} \mathcal{L}_Y = & -\lambda_e G^+ \left\{ i(\xi_1^{(1)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_2^{(1)}) - \lambda_e G^+ \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_1^{(1)})^\dagger \right\} (\kappa_2^{(1)-}) - \\ & -\lambda_e \chi^0 \left\{ i(\xi_2^{(1)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_2^{(1)}) - \lambda_e \chi^0 \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_2^{(1)})^\dagger \right\} (\kappa_2^{(1)-}) - \\ & -\lambda_d G^+ \left\{ i(\xi_1^{(2)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_2^{(2)}) - \lambda_d G^+ \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_1^{(2)})^\dagger \right\} (\kappa_2^{(2)-}) - \\ & -\lambda_d \chi^0 \left\{ i(\xi_2^{(2)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_2^{(2)}) - \lambda_d \chi^0 \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_2^{(2)})^\dagger \right\} (\kappa_2^{(2)-}) - \\ & -\lambda_u (\chi^0)^* \left\{ i(\xi_1^{(2)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_1^{(2)}) - \lambda_u (\chi^0)^* \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_1^{(2)})^\dagger \right\} (\kappa_1^{(2)-}) + \\ & + \lambda_u G^- \left\{ i(\xi_2^{(2)-})^\dagger \right\} \frac{1}{\sqrt[4]{2}} (\kappa_1^{(2)}) + \lambda_u G^- \left\{ -i \frac{1}{\sqrt[4]{2}} (\xi_2^{(2)})^\dagger \right\} (\kappa_1^{(2)-}) + \\ & + H.C. \end{aligned} \quad (4.91)$$

where (4.45) allows us to calculate ξ^- and κ^- . Using the v term from (4.58), for χ^0 and $(\chi^0)^*$ in (4.91), terms that have the structure of the mass term of a Dirac Lagrangian are produced. They are

$$\begin{aligned} & -\lambda_e \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_2^{(1)-})^\dagger \right\} (\kappa_2^{(1)}) + \lambda_e \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_2^{(1)})^\dagger \right\} (\kappa_2^{(1)-}) - \\ & -\lambda_d \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_2^{(2)-})^\dagger \right\} (\kappa_2^{(2)}) + \lambda_d \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_2^{(2)})^\dagger \right\} (\kappa_2^{(2)-}) - \\ & -\lambda_u \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_1^{(2)-})^\dagger \right\} (\kappa_1^{(2)}) + \lambda_u \frac{v}{\sqrt[4]{8}} \left\{ i(\xi_1^{(2)})^\dagger \right\} (\kappa_1^{(2)-}) + \\ & + H.C. \end{aligned} \quad (4.92)$$

Introducing

$$m_e = \lambda_e \frac{v}{\sqrt{2}}, \quad m_u = \lambda_u \frac{v}{\sqrt{2}}, \quad m_d = \lambda_d \frac{v}{\sqrt{2}}, \quad (4.93)$$

we see from (4.92) that the electron, u -quark and d -quark all acquired masses m_e , m_u and m_d respectively. As required there is no mass term for the electron-neutrino. Here we are considering only the first generation of leptons and the first generation of

quarks. We can also include additional terms for the other two known generations of leptons, a muon-neutrino and muon pair or a tau-neutrino and tau pair, and create the corresponding mass terms by giving different values to the coupling constant, similarly for quarks. Then, (4.91) can be written as

$$\begin{aligned}
\mathcal{L}_Y = & -\lambda_e G^+ i(\xi_1^{(1)-})^\dagger \frac{1}{\sqrt{2}}(\kappa_2^{(1)}) + \lambda_e G^+ i\frac{1}{\sqrt{2}}(\xi_1^{(1)})^\dagger (\kappa_2^{(1)-}) - \\
& -\lambda_e \frac{h(x) + iG^0}{\sqrt{8}} i(\xi_2^{(1)-})^\dagger (\kappa_2^{(1)}) + \lambda_e \frac{h(x) + iG^0}{\sqrt{8}} i(\xi_2^{(1)})^\dagger (\kappa_2^{(1)-}) - \\
& -\lambda_d G^+ i(\xi_1^{(2)-})^\dagger \frac{1}{\sqrt{2}}(\kappa_2^{(2)}) + \lambda_d G^+ i\frac{1}{\sqrt{2}}(\xi_1^{(2)})^\dagger (\kappa_2^{(2)-}) - \\
& -\lambda_d \frac{h(x) + iG^0}{\sqrt{8}} i(\xi_2^{(2)-})^\dagger (\kappa_2^{(2)}) + \lambda_d \frac{h(x) + iG^0}{\sqrt{8}} i(\xi_2^{(2)})^\dagger (\kappa_2^{(2)-}) - \\
& -\lambda_u \frac{h(x) - iG^0}{\sqrt{8}} i(\xi_1^{(2)-})^\dagger (\kappa_1^{(2)}) + \lambda_u \frac{h(x) - iG^0}{\sqrt{8}} i(\xi_1^{(2)})^\dagger (\kappa_1^{(2)-}) + \\
& +\lambda_u G^- i(\xi_2^{(2)-})^\dagger \frac{1}{\sqrt{2}}(\kappa_1^{(2)}) - \lambda_u G^- i\frac{1}{\sqrt{2}}(\xi_2^{(2)})^\dagger (\kappa_1^{(2)-}) - \\
& -m_e \frac{i}{\sqrt{2}} (\xi_2^{(1)-})^\dagger (\kappa_2^{(1)}) + m_e \frac{i}{\sqrt{2}} (\xi_2^{(1)})^\dagger (\kappa_2^{(1)-}) - \\
& -m_d \frac{i}{\sqrt{2}} (\xi_2^{(2)-})^\dagger (\kappa_2^{(2)}) + m_d \frac{i}{\sqrt{2}} (\xi_2^{(2)})^\dagger (\kappa_2^{(2)-}) - \\
& -m_u \frac{i}{\sqrt{2}} (\xi_1^{(2)-})^\dagger (\kappa_1^{(2)}) + m_u \frac{i}{\sqrt{2}} (\xi_1^{(2)})^\dagger (\kappa_1^{(2)-}) + H.C. \tag{4.94}
\end{aligned}$$

We note that (4.93) allows us to define the Yukawa coupling constants in terms of experimentally measured quantities. That is to say,

$$\lambda_e = \frac{g}{\sqrt{2}} \frac{m_e}{M_W}, \quad \lambda_u = \frac{g}{\sqrt{2}} \frac{m_u}{M_W}, \quad \lambda_d = \frac{g}{\sqrt{2}} \frac{m_d}{M_W}. \tag{4.95}$$

We also note that the Lagrangian density (4.94) contains no time derivatives. It is simple to see that when we combine this Lagrangian density with that for the fermions, that the presence of the mass terms leads to the fermion acquiring masses. Adding $\mathcal{L}_Y$ to the previous Lagrangian density (4.46) will affect the constraint terms and the Hamiltonian density. To exhibit the terms associated with the constraints, we expand the expressions and add the mass terms from $\mathcal{L}_Y$ that relate to the constraints, and, finally, obtain the expression:

$$\begin{aligned}
& + \frac{1}{2} (\xi^{(1)-})^\dagger \mathcal{P}^{(1)} + \frac{1}{2} (\xi^{(2)-})^\dagger \mathcal{P}^{(2)} + \\
& + \frac{1}{2} (\kappa_1^{(2)-})^\dagger \mathcal{R}_1^{(2)} + \frac{1}{2} (\kappa_2^{(1)-})^\dagger \mathcal{R}_2^{(1)} + \frac{1}{2} (\kappa_2^{(2)-})^\dagger \mathcal{R}_2^{(2)} - \\
& - \frac{1}{2} m_e \frac{i}{\sqrt[4]{2}} (\xi_2^{(1)-})^\dagger \kappa_2^{(1)} - \frac{1}{2} m_e \frac{i}{\sqrt[4]{2}} (\kappa_2^{(1)-})^\dagger \xi_2^{(1)} - \\
& - \frac{1}{2} m_d \frac{i}{\sqrt[4]{2}} (\xi_2^{(2)-})^\dagger \kappa_2^{(2)} - \frac{1}{2} m_d \frac{i}{\sqrt[4]{2}} (\kappa_2^{(2)-})^\dagger \xi_2^{(2)} - \\
& - \frac{1}{2} m_u \frac{i}{\sqrt[4]{2}} (\xi_1^{(2)-})^\dagger \kappa_1^{(2)} - \frac{1}{2} m_u \frac{i}{\sqrt[4]{2}} (\kappa_1^{(2)-})^\dagger \xi_1^{(2)} + H.C. \quad (4.96)
\end{aligned}$$

where $\mathcal{P}^{(f)}$ and $\mathcal{R}^{(f)}$ are defined in (4.47) to (4.49). We see that the mass terms modify the constraint terms and, therefore, will modify the solutions to the dependent variables. The simplest way to make this fact evident is to construct two column vectors, and use for convenience and compactness, the notation,

$$\mathcal{V} = \frac{i}{\sqrt[4]{2}} \begin{pmatrix} 0 \\ m_e \kappa_2^{(1)} \end{pmatrix}, \quad \mathcal{G} = \frac{i}{\sqrt[4]{2}} \begin{pmatrix} m_u \kappa_1^{(2)} \\ m_d \kappa_2^{(2)} \end{pmatrix}. \quad (4.97)$$

Thus, (4.96) can be written as,

$$\begin{aligned}
& + \frac{1}{2} (\xi^{(1)-})^\dagger [\mathcal{P}^{(1)} - \mathcal{V}] + \frac{1}{2} (\xi^{(2)-})^\dagger [\mathcal{P}^{(2)} - \mathcal{G}] + \\
& + \frac{1}{2} (\kappa_1^{(2)-})^\dagger [\mathcal{R}_1^{(2)} - m_u \frac{i}{\sqrt[4]{2}} \xi_1^{(2)}] + \frac{1}{2} (\kappa_2^{(1)-})^\dagger [\mathcal{R}_2^{(1)} - m_e \frac{i}{\sqrt[4]{2}} \xi_2^{(1)}] + \\
& + \frac{1}{2} (\kappa_2^{(2)-})^\dagger [\mathcal{R}_2^{(2)} - m_d \frac{i}{\sqrt[4]{2}} \xi_2^{(2)}] + H.C. + \quad (4.98)
\end{aligned}$$

This means that solutions (4.50) to (4.52) are modified accordingly. Thus, they become:

$$\begin{aligned}
\kappa_2^{(1)-} &= \frac{1}{i \sqrt[4]{8} \partial^+} \left\{ \sigma^i \left[i \partial^i + g' T_{Y(2)}^{(1)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_2^{(1)} + m_e i \xi_2^{(1)} \right\}, \\
\kappa_1^{(2)-} &= \frac{1}{i \sqrt[4]{8} \partial^+} \left\{ \sigma^i \left[i \partial^i + g' T_{Y(1)}^{(2)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_1^{(2)} + m_u i \xi_1^{(2)} \right\}, \\
\kappa_2^{(2)-} &= \frac{1}{i \sqrt[4]{8} \partial^+} \left\{ \sigma^i \left[i \partial^i + g' T_{Y(2)}^{(2)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_2^{(2)} + m_d i \xi_2^{(2)} \right\}, \\
\xi^{(1)-} &= \frac{1}{i \sqrt[4]{8} \partial^+} \left\{ \sqrt[4]{8} \mathcal{V} + \sigma^i \left[i \partial^i + \frac{g}{\sqrt{2}} \{ T^+ W^{i(+)} + T^- W^{i(-)} \} + \right. \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(1)} \} + \\
& \left. \left. + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(1)} \} \right] \xi^{(1)} \right\},
\end{aligned}$$

$$\begin{aligned}
\xi^{(2)-} = & \frac{1}{i\sqrt[4]{8}\partial^+} \left\{ \sqrt[4]{8}\mathcal{G} + \sigma^i \left[i\partial^i + \frac{g}{\sqrt{2}} \{ T^+ W^{i(+)} + T^- W^{i(-)} \} + \right. \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(2)} \} + \\
& \left. \left. + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(2)} \} \right] \xi^{(2)} \right\}. \quad (4.99)
\end{aligned}$$

Next, we modify expression (4.53), the Hamiltonian density. We must add the relevant mass terms, and we must use the solutions of the dependent field variables given by (4.99). We use the matrix notation,

$$\mathcal{D} = \frac{i}{\sqrt[4]{2}} \begin{pmatrix} 0 \\ i m_e \kappa_2^{(1)-} \end{pmatrix}, \quad \mathcal{K} = \frac{i}{\sqrt[4]{2}} \begin{pmatrix} i m_u \kappa_1^{(2)-} \\ i m_d \kappa_2^{(2)-} \end{pmatrix}, \quad (4.100)$$

and write,

$$\begin{aligned}
\mathcal{H}_F = & \frac{1}{2} \frac{1}{\sqrt[4]{2}} \xi^{(1)\dagger} \left\{ -\sqrt[4]{2}\mathcal{D} + \sigma^i \left[i\partial^i + \frac{g}{\sqrt{2}} \{ T^+ W^{i(+)} + T^- W^{i(-)} \} + \right. \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(1)} \} + \\
& \left. \left. + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(1)} \} \right] \xi^{(1)-} \right\} + \\
& + \frac{1}{2} \frac{1}{\sqrt[4]{2}} \xi^{(2)\dagger} \left\{ -\sqrt[4]{2}\mathcal{K} + \sigma^i \left[i\partial^i + \frac{g}{\sqrt{2}} \{ T^+ W^{i(+)} + T^- W^{i(-)} \} + \right. \right. \\
& + A^i \{ g \sin \theta_w T_3 + g' \cos \theta_w T_Y^{(2)} \} + \\
& \left. \left. + Z^i \{ g \cos \theta_w T_3 - g' \sin \theta_w T_Y^{(2)} \} \right] \xi^{(2)-} \right\} + \\
& + \frac{1}{2} \frac{1}{\sqrt[4]{2}} \kappa_2^{(1)\dagger} \left\{ \sigma^i \left[i\partial^i + g' T_{Y(2)}^{(1)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_2^{(1)-} - \right. \\
& \quad \left. - m_e i \xi_2^{(1)-} \right\} + \\
& + \frac{1}{2} \frac{1}{\sqrt[4]{2}} \kappa_1^{(2)\dagger} \left\{ \sigma^i \left[i\partial^i + g' T_{Y(1)}^{(2)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_1^{(2)-} - \right. \\
& \quad \left. - m_u i \xi_1^{(2)-} \right\} + \\
& + \frac{1}{2} \frac{1}{\sqrt[4]{2}} \kappa_2^{(2)\dagger} \left\{ \sigma^i \left[i\partial^i + g' T_{Y(2)}^{(2)} \{ A^i \cos \theta_w - Z^i \sin \theta_w \} \right] \kappa_2^{(2)-} - \right. \\
& \quad \left. - m_d i \xi_2^{(2)-} \right\} + H.C. \quad (4.101)
\end{aligned}$$

where the appropriate T_3 , $T^\pm$ and T_Y and the constraint equations (4.99) for the dependent fields have to be used. We note that, as expected, the mass terms are

associated with flips on the helicity of the particles. We also note that mass square terms appear due to the Yukawa coupling when the dependent fields are eliminated, terms which we must isolate and add to the free Hamiltonian density of the fermions.

4.5 Hamiltonian densities in the electro-weak unification.

We have constructed a massless gauge boson Lagrangian density $\mathcal{H}_{GB}$ and a massless Dirac Lagrangian density $\mathcal{H}_F$, for the first generation of both leptons and quarks. We then used the Higgs-Goldstone mechanism to break the $SU(2) \otimes U(1)$ symmetry down to $U(1)_{em}$ in the process creating mass terms for the gauge bosons and constructing the longitudinal components of the bosons acquiring a mass term. We have used a Yukawa coupling to give mass to the fermions. We constructed free Hamiltonian densities from these Lagrangians, but we held back adding the Yukawa mass terms because they do not lend themselves to a compact notation. The Hamiltonian density for the electro-weak theory is the sum

$$\mathcal{H}_{ew} = \mathcal{H}_{GB} + \mathcal{H}_F + \mathcal{H}_H + \mathcal{H}_Y, \quad (4.102)$$

out of which the free Hamiltonian density is given by:

$$\begin{aligned} \mathcal{H}_{ew}^0 = & \partial^i W^{j(+)} \partial^i W^{j(-)} + M_W^2 W^{i(-)} W^{i(+)} + \partial^i G^- \partial^i G^+ + M_W^2 G^- G^+ + \\ & + \frac{1}{2} [\partial^i Z^j]^2 + \frac{1}{2} M_Z^2 Z^i Z^i + \frac{1}{2} [\partial^i G^0]^2 + \frac{1}{2} M_Z^2 (G^0)^2 + \\ & + \frac{1}{2} [\partial^i A^j]^2 + \frac{1}{2} [\partial^i h(x)]^2 + \frac{1}{2} m_h h^2(x) + \\ & + \left\{ \frac{1}{2} \xi_e^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_e^2] \xi_e + \frac{1}{2} \kappa_e^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_e^2] \kappa_e + \right. \\ & + \frac{1}{2} \xi_u^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_u^2] \xi_u + \frac{1}{2} \kappa_u^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_u^2] \kappa_u + \\ & + \frac{1}{2} \xi_d^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_d^2] \xi_d + \frac{1}{2} \kappa_d^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2 + m_d^2] \kappa_d + \\ & \left. + \frac{1}{2} \xi_{\nu_e}^\dagger \left(\frac{1}{2i\partial^+} \right) [-\partial_\perp^2] \xi_{\nu_e} + H.C. \right\}. \end{aligned} \quad (4.103)$$

If we compare the structures of the expressions for the different field variables with those present in the QED free Hamiltonian density we will notice the similarity in

the structures. This justifies the Fourier expansions for the free field variables that we present in the next section. We can now partition the Hamiltonian density into the free Hamiltonian density and the interaction Hamiltonian density, that is to say: $\mathcal{H}_{ew} = \mathcal{H}_{ew}^0 + \mathcal{H}_{ew}^{int}$. From (4.103) we can see that the structure of the free Hamiltonian matches the structure of the free Hamiltonian for QED, as far as the fermions are concerned, and that for the gauge bosons the differences that arise are due to the mass terms, the longitudinal components being generated by the scalar fields G .

Rather than creating a massive expression for $\mathcal{H}_{ew}$, we will refer the reader to the expressions of the different Hamiltonian densities. Thus, (4.84) corresponds to $\mathcal{H}_{GB}$, (4.101) corresponds to $\mathcal{H}_F$, (4.65) corresponds to $\mathcal{H}_H$ and (4.94) corresponds to $\mathcal{H}_Y$.

4.6 Expansion of the field variables in terms of creation and annihilation operators.

We know from 't Hooft's conditions that $G^\pm$ and G^0 are intimately connected with the longitudinal components of $W^{-(\pm)}$ and Z^- . Because of the parallelism we have seen to exist between the formulation of QED and that of the electro-weak unification, it is reasonable to expect that the Fourier expansion of the fields will also show a similar parallelism once we take into consideration the fact that three of the gauge fields are now massive. That is to say, they will have three polarizations states when massive and two otherwise. We start with the Fourier expansion of the initial Yang-Mills gauge fields A_a^μ and C^μ .

$$A_a^\mu(x) = \sum_{(\alpha)} \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[\epsilon_{(\alpha)}^\mu a_a(q, \alpha) e^{-iqx} + \epsilon_{(\alpha)}^{*\mu} a_a^\dagger(q, \alpha) e^{iqx} \right], \quad (4.104)$$

$$C^\mu(x) = \sum_{(\alpha)} \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[\epsilon_{(\alpha)}^\mu c(q, \alpha) e^{-iqx} + \epsilon_{(\alpha)}^{*\mu} c^\dagger(q, \alpha) e^{iqx} \right], \quad (4.105)$$

where $\mu = 1, 2, -$ and $\alpha = 1, 2, 3$. The four-vectors $\epsilon_{(\alpha)}^\mu$ are the polarization vectors, while $a_a(q, \alpha)$ and $c(q, \alpha)$ are the annihilation operators for the A_a^μ and C^μ respectively. Because of the gauge condition, $\epsilon_{(\alpha)}^+ = 0$. To determine the polarization vectors we proceed as follows.

We remind ourselves that $W^{\mu(\pm)} = (A_1^\mu \mp iA_2^\mu)/\sqrt{2}$, and that $G^\pm$ is a complex scalar, $G^\pm = \mp i(\phi_1 \mp i\phi_2)/\sqrt{2}$, and $G^0 = \phi_3$. We have derived 't Hooft's conditions equations (4.76) to (4.79) for the $W^{(\pm)}$ fields. When expressed in terms of A_α^μ and C^μ fields, the conditions become:

$$\partial^+ A_1^- - \partial^i A_1^i = -M_W \phi_1, \quad \partial^+ A_2^- - \partial^i A_2^i = -M_W \phi_2, \quad (4.106)$$

$$\partial^+ A_3^- - \partial^i A_3^i = M_W \cos \theta_W \phi_3, \quad \partial^+ C^- - \partial^i C^i = -M_W \sin \theta_W \phi_3. \quad (4.107)$$

As we can see 't Hooft's conditions are of exactly the same form as those derived for the massive gauge fields. It is obvious that the polarizations for the W fields will be linear combinations of those for A_1^μ and A_2^μ such that they will be circularly polarized if the $A_{1,2}^\mu$ fields are linearly polarized, and similarly for the other fields. Let us consider 't Hooft's condition for field A_1^μ . The Fourier expansion of field A_1^μ is given by (4.104) and the one for the scalar field ϕ_1 follows:

$$\phi_1(x) = \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[b_1(q) e^{-iqx} + b_1^\dagger(q) e^{iqx} \right], \quad (4.108)$$

We now apply the relevant 't Hooft's condition to these fields, and we get the following equation for the operators.

$$\sum_{\alpha=1,2,3} (q^+ \epsilon_{(\alpha)}^- - q^i \epsilon_{(\alpha)}^i) a_1(q, \alpha) + i M_W b_1(q) = 0. \quad (4.109)$$

First, we postulate that $a_1(q, 3) = i b_1(q)$. Since the operators are independent from each other, we obtain the following system of linear equations:

$$\begin{aligned} q^+ \epsilon_{(\perp)}^- - q^i \epsilon_{(\perp)}^i &= 0, \\ q^+ \epsilon_{(3)}^- - q^i \epsilon_{(3)}^i + M_W &= 0, \end{aligned} \quad (4.110)$$

where $\perp = 1, 2$. A solution for this system of equations is given by:

$$\epsilon_{(1)}^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{q^1}{q^+} \end{pmatrix}, \quad \epsilon_{(2)}^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{q^2}{q^+} \end{pmatrix}, \quad \epsilon_{(3)}^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{M_W}{q^+} \end{pmatrix}. \quad (4.111)$$

The first two vectors have norm -1 , and clearly $q^\mu \cdot \epsilon_{(1)}^\mu = q^\mu \cdot \epsilon_{(2)}^\mu = 0$, while the third one is a null vector and $q^\mu \cdot \epsilon_{(3)}^\mu \neq 0$. Thus, the first two are transverse, while the third one is not. Let us separate $\epsilon_{(3)}^\mu$ into a component transverse to q^μ and another component parallel to q^μ . Since we know the longitudinal polarization to be: q^μ / M_W we write $\epsilon_{(3)}^{\mu(L)} = -q^\mu / M_W$ and $\epsilon_{(3)}^{\mu(T)} = \epsilon_{(3)}^\mu - \epsilon_{(3)}^{\mu(L)}$. It is easy to verify that:

$$\epsilon_{(3)}^{\mu(T)} = \begin{pmatrix} \frac{q^+}{M_W} \\ \frac{q^1}{M_W} \\ \frac{q^2}{M_W} \\ -\frac{M_W}{q^+} + \frac{q^-}{M_W} \end{pmatrix}, \quad \epsilon_{(3)}^{\mu(L)} = \begin{pmatrix} -\frac{q^+}{M_W} \\ -\frac{q^1}{M_W} \\ -\frac{q^2}{M_W} \\ -\frac{q^-}{M_W} \end{pmatrix}, \quad \epsilon_{(3)}^{\mu(T)} + \epsilon_{(3)}^{\mu(L)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{M_W}{q^+} \end{pmatrix} = \epsilon_{(3)}^\mu. \quad (4.112)$$

It is also easy to verify that $q_\mu \cdot \epsilon_{(3)}^{\mu(T)} = 0$. Thus, the third polarization is neither transverse to nor longitudinal with respect to q^μ .

Next we deal with the fermion fields. Here we have only one polarization for the left-handed fermions and one polarization for the right-handed fermions. That being the case, let

$$\xi_k^{(f)}(x) = \chi_L \int \frac{1}{(2\pi)^3} d^2 p_\perp \frac{dp^+}{2[p^+]} \left[\sqrt{2p^+} s_k^{(f)}(p, \lambda_L) e^{-ipx} + \sqrt{2p^+} (u_k^{(f)})^\dagger(p, -\lambda_L) e^{ipx} \right], \quad (4.113)$$

$$\kappa_k^{(f)}(x) = \chi_R \int \frac{1}{(2\pi)^3} d^2 p_\perp \frac{dp^+}{2[p^+]} \left[\sqrt{2p^+} v_k^{(f)}(p, \lambda_R) e^{-ipx} + \sqrt{2p^+} (w_k^{(f)})^\dagger(p, -\lambda_R) e^{ipx} \right], \quad (4.114)$$

where $s_k^{(f)}(p, \lambda_L)$ and $(u_k^{(f)})^\dagger(p, -\lambda_L)$ are annihilation and creation operators of the left-handed fermion and anti-fermion fields, and $v_k^{(f)}(p, \lambda_R)$ and $(w_k^{(f)})^\dagger(p, -\lambda_R)$ are annihilation and creation operators of the right-handed fermion and anti-fermion fields.

The expressions for the non-relativistic spinors λ_L , and λ_R are

$$\lambda_L = -1, \quad \lambda_R = +1, \quad \chi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4.115)$$

For $f = 1$ and $k = 1$, (4.113) gives the expression corresponding to the left-handed electron-neutrino, while for $f = 1$ and $k = 2$, (4.113) plus (4.114) give the two-components expression corresponding to the full electron field. Given the fact that QED's Hamiltonian density is embedded into the electro-weak Hamiltonian density, these expressions will allow the exact reproduction of QED vertices within the electro-weak unification. If we wish to construct the four component expression we first obtain $\psi_+^{(f)}$ from $\xi_k^{(f)} + \kappa_k^{(f)}$ for the corresponding value of k and then obtain $\psi_-^{(f)}$ from $\psi_+^{(f)}$. Identical fermion fields are to be defined for the quarks. In the case of the quarks both the u and d quarks have left-handed as well as right-handed components. Finally, there are the real scalar fields $h(x)$, G^0 and the complex scalar field G^+ . Let

$$h(x) = \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[h_1(q) e^{-iqx} + h_1^\dagger(q) e^{iqx} \right], \quad (4.116)$$

$$G^0(x) = \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[g_{(0)} e^{-iqx} + g_{(0)}^\dagger e^{iqx} \right], \quad (4.117)$$

$$G^+(x) = \int \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \left[g_{(+)} e^{-iqx} + g_{(-)}^\dagger e^{iqx} \right], \quad (4.118)$$

These operators are defined keeping in mind that $G^+ = -i(\phi_1 - i\phi_2)/\sqrt{2}$, and $G^0 = \phi_3$. And that there is a relationship between them and the operators associated with third polarization vector of the gauge fields. By this we mean a relationship of the type $a_1(q, 3) = i b_1(q)$. Taking into consideration (4.107), we should include the $\cos \theta_w$ and $\sin \theta_w$ in the definition of the operators and not in the definition of the third polarization vector.

4.7 Quantization of the electro-weak unification.

We quantized QED using the procedure presented by Faddeev and Jackiw. We started by creating a Lagrangian one-form and following the procedure, we defined the commutation relations (2.90) between the dynamical variable ψ_+ and its canonical conjugate, we then defined (2.91) for the ξ variable. In the electro-weak formulation we will exploit the formal parallelism between QED and the electro-weak unification and

follow, exactly, the same procedure. We start by adding (4.17), (4.34), (4.64) and (4.91) to obtain $\mathcal{L}$, the Lagrangian density for the electro-weak theory for the massive gauge fields. We have dropped all the constraints which have been used when determining the solutions of the dependent variables. Thus,

$$\begin{aligned}
\mathcal{L} = & \partial^+ W^{i(+)} \partial^- W^{i(-)} + \partial^+ W^{i(-)} \partial^- W^{i(+)} + \partial^+ A^i \partial^- A^i + \partial^+ Z^i \partial^- Z^i + \\
& + (\partial^+ G^-)(\partial^- G^+) + (\partial^- G^-)(\partial^+ G^+) + (\partial^+ h)(\partial^- h) + (\partial^+ G^0)(\partial^- G^0) + \\
& + \frac{i}{\sqrt{2}} (\psi_{L+}^{(f)})^\dagger (\partial^- \psi_{L+}^{(f)}) - \frac{i}{\sqrt{2}} (\partial^- \psi_{L+}^{(f)})^\dagger \psi_{L+}^{(f)} + \frac{i}{\sqrt{2}} (\psi_{R1+}^{(f)})^\dagger (\partial^- \psi_{R1+}^{(f)}) - \\
& - \frac{i}{\sqrt{2}} (\partial^- \psi_{R1+}^{(f)})^\dagger \psi_{R1+}^{(f)} + \frac{i}{\sqrt{2}} (\psi_{R2+}^{(f)})^\dagger (\partial^- \psi_{R2+}^{(f)}) - \\
& - \frac{i}{\sqrt{2}} (\partial^- \psi_{R2+}^{(f)})^\dagger \psi_{R2+}^{(f)} - \mathcal{H}
\end{aligned} \tag{4.119}$$

We note that we have expressed the Lagrangian density in term of the four component spinors. We want the commutation relations for these variables, as we did in QED. The commutation relations for the two component spinors can easily be derived from the commutation relations for the four components as we saw in QED. Next, we express the Lagrangian density in terms of field variables and their canonical conjugates. We can infer from previous examples that each term multiplying a ∂^- is the canonical conjugate of the variable being derived. Thus, from the definition of canonical conjugates we have:

$$\Pi_{W^{i(-)}} = \partial^+ W^{i(+)}, \quad \Pi_{W^{i(+)}} = \partial^+ W^{i(-)}, \tag{4.120}$$

$$\Pi_{Z^i} = \partial^+ Z^i, \quad \Pi_{A^i} = \partial^+ A^i, \tag{4.121}$$

$$\Pi_{G^+} = \partial^+ G^-, \quad \Pi_{G^-} = \partial^+ G^+, \tag{4.122}$$

$$\Pi_{G^0} = \partial^+ G^0, \quad \Pi_h = \partial^+ h, \tag{4.123}$$

$$\Pi_{\psi_{L+}^{(f)}} = \frac{i}{\sqrt{2}} (\psi_{L+}^{(f)})^\dagger, \quad \Pi_{(\psi_{L+}^{(f)})^\dagger} = -\frac{i}{\sqrt{2}} \psi_{L+}^{(f)}, \tag{4.124}$$

$$\Pi_{\psi_{R1+}^{(f)}} = \frac{i}{\sqrt{2}} (\psi_{R1+}^{(f)})^\dagger, \quad \Pi_{(\psi_{R1+}^{(f)})^\dagger} = -\frac{i}{\sqrt{2}} \psi_{R1+}^{(f)}, \tag{4.125}$$

$$\Pi_{\psi_{R2+}^{(f)}} = \frac{i}{\sqrt{2}} (\psi_{R2+}^{(f)})^\dagger, \quad \Pi_{(\psi_{R2+}^{(f)})^\dagger} = -\frac{i}{\sqrt{2}} \psi_{R2+}^{(f)}. \tag{4.126}$$

Next, we create the Lagrangian one-form

$$\begin{aligned}
\mathcal{L}dx^+ = & \frac{1}{2} \left[\Pi_{W^{i(-)}} dW^{i(-)} + \Pi_{W^{i(+)}} dW^{i(+)} + \Pi_{Z^i} dZ^i + \Pi_{A^i} dA^i + \Pi_{G^+} dG^+ + \right. \\
& + \Pi_{G^-} dG^- + \Pi_{G^0} dG^0 + \Pi_h dh + \Pi_{\psi_{L+}^{(f)}} d\psi_{L+}^{(f)} + d(\psi_{L+}^{(f)})^\dagger \Pi_{(\psi_{L+}^{(f)})^\dagger} + \\
& + \Pi_{\psi_{R1+}^{(f)}} d\psi_{R1+}^{(f)} + d(\psi_{R1+}^{(f)})^\dagger \Pi_{(\psi_{R1+}^{(f)})^\dagger} + \Pi_{\psi_{R2+}^{(f)}} d\psi_{R2+}^{(f)} + d(\psi_{R2+}^{(f)})^\dagger \Pi_{(\psi_{R2+}^{(f)})^\dagger} - \\
& - W^{i(-)} d\Pi_{W^{i(-)}} - W^{i(+)} d\Pi_{W^{i(+)}} - Z^i d\Pi_{Z^i} - A^i d\Pi_{A^i} - G^+ d\Pi_{G^+} - \\
& - G^- d\Pi_{G^-} - G^0 d\Pi_{G^0} - h d\Pi_h - d\Pi_{\psi_{L+}^{(f)}} \psi_{L+}^{(f)} - (\psi_{L+}^{(f)})^\dagger d\Pi_{(\psi_{L+}^{(f)})^\dagger} - \\
& - d\Pi_{\psi_{R1+}^{(f)}} \psi_{R1+}^{(f)} - (\psi_{R1+}^{(f)})^\dagger d\Pi_{(\psi_{R1+}^{(f)})^\dagger} - d\Pi_{\psi_{R2+}^{(f)}} \psi_{R2+}^{(f)} - \\
& \left. - (\psi_{R2+}^{(f)})^\dagger d\Pi_{(\psi_{R2+}^{(f)})^\dagger} \right] - \mathcal{H}dx^+. \tag{4.127}
\end{aligned}$$

Next, we repeat what was done in (2.77) and (2.78), we recast the one-form so all terms in the first half have the same structure and those in the second half have the same structure. We are referring in particular to the 10th, 12th, 14th, 23rd, 25th and 27th terms. We also expand the doublet of left-handed components for obvious reasons. The one-form becomes

$$\begin{aligned}
\mathcal{L}dx^+ = & \frac{1}{2} \left[\Pi_{W^{i(-)}} dW^{i(-)} + \Pi_{W^{i(+)}} dW^{i(+)} + \Pi_{Z^i} dZ^i + \Pi_{A^i} dA^i + \Pi_{G^+} dG^+ + \right. \\
& + \Pi_{G^-} dG^- + \Pi_{G^0} dG^0 + \Pi_h dh + \Pi_{\psi_{L+}^{(f)}} d\psi_{L+}^{(f)} - (\psi_{L+}^{(f)})^\dagger d\Pi_{(\psi_{L+}^{(f)})^\dagger} + \\
& + \Pi_{\psi_{R1+}^{(f)}} d\psi_{R1+}^{(f)} - (\psi_{R1+}^{(f)})^\dagger d\Pi_{(\psi_{R1+}^{(f)})^\dagger} + \Pi_{\psi_{R2+}^{(f)}} d\psi_{R2+}^{(f)} - (\psi_{R2+}^{(f)})^\dagger d\Pi_{(\psi_{R2+}^{(f)})^\dagger} - \\
& - W^{i(-)} d\Pi_{W^{i(-)}} - W^{i(+)} d\Pi_{W^{i(+)}} - Z^i d\Pi_{Z^i} - A^i d\Pi_{A^i} - G^+ d\Pi_{G^+} - \\
& - G^- d\Pi_{G^-} - G^0 d\Pi_{G^0} - h d\Pi_h - d\Pi_{\psi_{L+}^{(f)}} \psi_{L+}^{(f)} + d(\psi_{L+}^{(f)})^\dagger \Pi_{(\psi_{L+}^{(f)})^\dagger} - \\
& - d\Pi_{\psi_{R1+}^{(f)}} \psi_{R1+}^{(f)} + d(\psi_{R1+}^{(f)})^\dagger \Pi_{(\psi_{R1+}^{(f)})^\dagger} - d\Pi_{\psi_{R2+}^{(f)}} \psi_{R2+}^{(f)} - \\
& \left. + d(\psi_{R2+}^{(f)})^\dagger \Pi_{(\psi_{R2+}^{(f)})^\dagger} \right] - \mathcal{H}dx^+, \tag{4.128}
\end{aligned}$$

$$= \frac{1}{2} f^\alpha \Gamma_{\alpha\beta} df^\beta - \mathcal{H}dx^+. \tag{4.129}$$

From (4.128) we can directly create the $\Gamma_{\alpha\beta}$ matrix as we did in QED. From the $[\Gamma^{-1}]_{\alpha\beta}$ we extract, directly, the value of the commutators as we did in QED. Except, that in QED we deal with a six by six matrix, while here we deal with a thirty two

(once we expand the doublets) by thirty two matrix. If we compare (4.129) with (2.78) we can easily see that the structures of the matrices are identical. That is to say,

$$\Gamma_{\alpha\beta} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \quad (4.130)$$

where A stands for a diagonal sixteen by sixteen matrix of ± 1 's. As before it is easy to verify that

$$[\Gamma^{-1}]_{\beta\alpha} = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}, \quad (4.131)$$

where the f 's in (4.129) satisfy the commutation relation:

$$[f^\beta, f^\alpha] = i [\Gamma_{\alpha\beta}]^{-1} = i [\Gamma^{-1}]_{\beta\alpha}, \quad (4.132)$$

as in (2.80), where f^α in this case is defined as:

$$f^\alpha = \left(\Pi_{W^{i(-)}}, \Pi_{W^{i(+)}} , \Pi_{Z^i}, \Pi_{A^i}, \Pi_{G^+}, \Pi_{G^-}, \Pi_{G^0}, \Pi_h, \Pi_{\psi_{L+}^{(f)}}, (\psi_{L+}^{(f)})^\dagger, \right. \\ \left. \Pi_{\psi_{R1+}^{(f)}}, (\psi_{R1+}^{(f)})^\dagger, \Pi_{\psi_{R2+}^{(f)}}, (\psi_{R2+}^{(f)})^\dagger, W^{i(-)}, W^{i(+)}, Z^i, A^i, G^+, \right. \\ \left. G^-, G^0, h, \psi_{L+}^{(f)}, \Pi_{(\psi_{L+}^{(f)})^\dagger}, \psi_{R1+}^{(f)}, \Pi_{(\psi_{R1+}^{(f)})^\dagger}, \psi_{R2+}^{(f)}, \Pi_{(\psi_{R2+}^{(f)})^\dagger} \right)$$

We must remember to expand the doublets (which means thirty two members as opposed to the twenty eight listed). In QED, we have one vector boson and two spinors, here we have four vector bosons, four scalar bosons and four spinors. The structure of the canonical conjugates to the field variables is the same as in QED. Since the structure and composition of both matrices is identical we can simply predict similar, if not identical, commutation relations. However, because of 't Hooft's conditions the transverse fields and the G 's will not commute. We do not go into details on that subject here. As in QED one could use the plane wave expansion for all the fields in terms of their corresponding creation and annihilation operators and derive commutation relations for the creation and annihilation operators of the relevant fields from (4.132). We do not go into details on that subject here.

Chapter 5

Conclusion.

Kogut and Soper [2] refer to the isomorphism that exists between a subgroup of the Poincaré group and the Galilean symmetry group of non-relativistic quantum mechanics (NRQM) in two dimensions. As a result of this isomorphism the structure of quantum mechanics in the LF formulation has a non-relativistic structure although the theory is fully relativistic. We see this reflected in the structure of the LF formulations of QED and electro-weak unification. By working in the LF formulation we lose explicit covariance but we gain the simplicity of a two dimensional NRQM. The Hamiltonian density resulting from such formulation includes only the transverse components of the gauge fields and the ψ_+ component of the fermion fields. For massive gauge fields in the gauge in which the plus components are zero, the contributions of the minus components is expressed in terms of the transverse fields and of a scalar field.

The generalized Hamiltonian approach that Wilson and his co-workers have developed has been applied here to verify that the linear infrared singularities do not present a problem provided their rule for the treatment of the zero momentum singularities is followed. This effectively makes the vacuum trivial even for mass zero gauge theories like QED and QCD. Since the electro-weak formulation closely parallels that of QED,

the same prescriptions for handling the singularities should work, although we have not explicitly verified this to be the case.

Chapter 6

Appendices

Appendix A

Expansion of Hamiltonian densities.

In this appendix we want to obtain the expressions of the Hamiltonian density of QED in momentum space. Expressions (2.100) and (2.101) give us the Fourier expansions of the field operators:

$$C^i(x) = \sum_{\lambda} \int \frac{1}{(2\pi)^3} d^2 q_{\perp} \frac{dq^+}{2[q^+]} \left[\epsilon_{\lambda}^i c(q, \lambda) e^{-iqx} + \epsilon_{\lambda}^{*i} c^{\dagger}(q, \lambda) e^{iqx} \right], \quad (\text{A.1})$$

$$\begin{aligned} \xi(x) = \sum_{\lambda} \chi_{\lambda} \int \frac{1}{(2\pi)^3} d^2 p_{\perp} \frac{dp^+}{2[p^+]} \left[\sqrt{2p^+} b(p, \lambda) e^{-ipx} + \right. \\ \left. + \sqrt{2p^+} d^{\dagger}(p, -\lambda) e^{ipx} \right]. \end{aligned} \quad (\text{A.2})$$

We want to obtain the Fourier expansion of:

$$\begin{aligned} \mathcal{H}_{QED}^{FPF} = -g \xi^{\dagger} \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} \xi + g \xi^{\dagger} \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (\sigma^j C^j) \xi \right\} + \\ + g \xi^{\dagger} \sigma^i C^i \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^j \partial^j + m) \xi \right\}. \end{aligned} \quad (\text{A.3})$$

We start with the term:

$$g \xi^{\dagger} \sigma^i C^i \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^j \partial^j + m) \xi \right\}. \quad (\text{A.4})$$

We will partition the expression in order to avoid repetition of calculations. Thus, we first expand

$$\begin{aligned} \left(\frac{1}{\partial^+}\right)(\sigma^j \partial^j + m)\xi &= \sum_{\lambda_1} \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \sqrt{2p_1^+} \left[\frac{(-i\sigma \cdot p_1 + m)}{-i[p_1^+]} \right. \\ &\quad \left. \cdot b(p_1, \lambda_1) e^{-ip_1 x} + \frac{(i\sigma \cdot p_1 + m)}{i[p_1^+]} d^\dagger(p_1, -\lambda_1) e^{ip_1 x} \right]. \end{aligned} \quad (\text{A.5})$$

And then we expand

$$\begin{aligned} \xi^\dagger \sigma^i C^i &= \sum_{\lambda, \lambda_2} \chi_{\lambda_2}^\dagger \int \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_2^+} \left[b^\dagger(p_2, \lambda_2) \right. \\ &\quad \cdot \sigma \cdot \epsilon_\lambda c(q, \lambda) e^{(-iqx + ip_2 x)} + b^\dagger(p_2, \lambda_2) \sigma \cdot \epsilon_\lambda^* c^\dagger(q, \lambda) e^{(iqx + ip_2 x)} + \\ &\quad \left. + d(p_2, -\lambda_2) \sigma \cdot \epsilon_\lambda c(q, \lambda) e^{(-iqx - ip_2 x)} + d(p_2, -\lambda_2) \sigma \cdot \epsilon_\lambda^* c^\dagger(q, \lambda) e^{(iqx - ip_2 x)} \right]. \end{aligned} \quad (\text{A.6})$$

Which, when recombined together give the result,

$$\begin{aligned} g \xi^\dagger \sigma^i C^i \left(\frac{1}{2\partial^+}\right)(\sigma^j \partial^j + m)\xi &= \\ g \sum_{\lambda, \lambda_1, \lambda_2} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} &\int \frac{1}{(2\pi)^9} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\ &\left[b^\dagger(p_2, \lambda_2) c(q, \lambda) b(p_1, \lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 - m)}{i2[p_1^+]} e^{ix(-p_1 - q + p_2)} + \right. \\ &+ b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 - m)}{i2[p_1^+]} e^{ix(-p_1 + q + p_2)} + \\ &+ d(p_2, -\lambda_2) c(q, \lambda) b(p_1, \lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 - m)}{i2[p_1^+]} e^{ix(-p_1 - q - p_2)} + \\ &+ d(p_2, -\lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 - m)}{i2[p_1^+]} e^{ix(-p_1 + q - p_2)} + \\ &+ b^\dagger(p_2, \lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 - q + p_2)} + \\ &+ b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 + q + p_2)} + \\ &+ d(p_2, -\lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 - q - p_2)} + \\ &\left. + d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 + q - p_2)} \right], \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} &+ b^\dagger(p_2, \lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 - q + p_2)} + \\ &+ b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 + q + p_2)} + \\ &+ d(p_2, -\lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 - q - p_2)} + \\ &+ d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 + q - p_2)} \Big], \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} &+ d(p_2, -\lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 - q - p_2)} + \\ &+ d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* \frac{(i\sigma \cdot p_1 + m)}{i2[p_1^+]} e^{ix(p_1 + q - p_2)} \Big], \end{aligned} \quad (\text{A.9})$$

where terms (A.7) and (A.8) can be eliminated because we cannot create or annihilate three particles at the same vertex. Next, we consider the term $g \xi^\dagger \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (\sigma^j C^j) \xi \right\}$, we now reuse some of the previous calculations. First we calculate

$$\begin{aligned} \sigma^i C^i \xi = & \sum_{\lambda, \lambda_1} \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \left[c(q, \lambda) b(p_1, \lambda_1) \right. \\ & \cdot \sigma \cdot \epsilon_\lambda e^{(-iqx - ip_1 x)} + c^\dagger(q, \lambda) b(p_1, \lambda_1) \sigma \cdot \epsilon_\lambda^* e^{(iqx - ip_1 x)} + \\ & + c(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda e^{(-iqx + ip_1 x)} + \\ & \left. + c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_\lambda^* e^{(iqx + ip_1 x)} \right] \end{aligned} \quad (\text{A.10})$$

And then we reuse the results of (A.5) after changing the sign of the mass term

$$\begin{aligned} \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) \sigma^i C^i \xi = & \sum_{\lambda, \lambda_1} \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \\ & \sqrt{2p_1^+} \left[c(q, \lambda) b(p_1, \lambda_1) \frac{-i\sigma \cdot (q + p_1) - m}{-i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda e^{(-iqx - ip_1 x)} + \right. \\ & + c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{i\sigma \cdot (q - p_1) - m}{i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{(iqx - ip_1 x)} + \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} & + c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{-i\sigma \cdot (q - p_1) - m}{-i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda e^{(-iqx + ip_1 x)} + \\ & \left. + c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{i\sigma \cdot (q + p_1) - m}{i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{(iqx + ip_1 x)} \right]. \end{aligned} \quad (\text{A.12})$$

And finally after recombining the terms we obtain

$$\begin{aligned}
& g\xi^\dagger \left\{ \left(\frac{1}{2\partial^+} \right) (\sigma^i \partial^i - m) (\sigma^j C^j) \xi \right\} = \\
& g \sum_{\lambda, \lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^9} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\
& \left[b^\dagger(p_2, \lambda_2) c(q, \lambda) b(p_1, \lambda_1) \frac{-i\sigma \cdot (q + p_1) - m}{-i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda e^{ix(-q-p_1+p_2)} + \right. \\
& + b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{i\sigma \cdot (q - p_1) - m}{i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{ix(q-p_1+p_2)} +; \\
& + b^\dagger(p_2, \lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{-i\sigma \cdot (q - p_1) - m}{-i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda e^{ix(-q+p_1+p_2)} + \\
& \left. + b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{i\sigma \cdot (q + p_1) - m}{i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{ix(q+p_1+p_2)} + \right] \quad (A.13)
\end{aligned}$$

$$+ d(p_2, -\lambda_2) c(q, \lambda) b(p_1, \lambda_1) \frac{-i\sigma \cdot (q + p_1) - m}{-i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda e^{ix(-q-p_1-p_2)} + \quad (A.14)$$

$$\begin{aligned}
& + d(p_2, -\lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{i\sigma \cdot (q - p_1) - m}{i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{ix(q-p_1-p_2)} + \\
& + d(p_2, -\lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{-i\sigma \cdot (q - p_1) - m}{-i2[q^+ - p_1^+]} \sigma \cdot \epsilon_\lambda e^{ix(-q+p_1-p_2)} + \\
& + d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{i\sigma \cdot (q + p_1) - m}{i2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda^* e^{ix(q+p_1-p_2)} \Big]. \quad (A.15)
\end{aligned}$$

Terms (A.13) and (A.14) can be eliminated because we cannot create or annihilate three particles at the same vertex. Finally, we consider the term $-g\xi^\dagger \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} \xi$,

First, we deal with

$$\begin{aligned}
\left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} \xi &= \sum_{\lambda, \lambda_1} \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \\
& \left[c(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{(-iqx - ip_1 x)} + \right. \\
& + c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{(iqx - ip_1 x)} + \\
& + c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{(-iqx + ip_1 x)} + \\
& \left. + c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{(iqx + ip_1 x)} \right]. \quad (A.16)
\end{aligned}$$

and then finally, we deal with the whole term

$$\begin{aligned}
-g \xi^\dagger \left\{ \left(\frac{1}{\partial^+} \right) \partial^i C^i \right\} \xi &= -g \sqrt{2} \xi^\dagger \sum_{\lambda, \lambda_2} \chi_{\lambda_2} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \\
&\quad \sqrt{2p_1^+} \left[c(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{(-iqx - ip_1 x)} + \right. \\
&\quad \left. + c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{(iqx - ip_1 x)} + \right. \\
&\quad \left. + c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{(-iqx + ip_1 x)} + \right. \\
&\quad \left. + c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{(iqx + ip_1 x)} \right] = \\
-g \sum_{\lambda, \lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^9} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\
&\quad \left[b^\dagger(p_2, \lambda_2) c(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{ix(-q - p_1 + p_2)} + \right. \\
&\quad \left. + b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{ix(q - p_1 + p_2)} + \right. \\
&\quad \left. + b^\dagger(p_2, \lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{ix(-q + p_1 + p_2)} + \right. \\
&\quad \left. + b^\dagger(p_2, \lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{ix(q + p_1 + p_2)} + \right. \tag{A.17} \\
&\quad \left. + d(p_2, -\lambda_2) c(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{ix(-q - p_1 - p_2)} + \right. \tag{A.18} \\
&\quad \left. + d(p_2, -\lambda_2) c^\dagger(q, \lambda) b(p_1, \lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{ix(q - p_1 - p_2)} + \right. \\
&\quad \left. + d(p_2, -\lambda_2) c(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda}{[q^+]} e^{ix(-q + p_1 - p_2)} + \right. \\
&\quad \left. + d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1) \frac{q \cdot \epsilon_\lambda^*}{[q^+]} e^{ix(q + p_1 - p_2)} \right]. \tag{A.19}
\end{aligned}$$

Terms (A.17) and (A.18) can be eliminated because we cannot create or annihilate three particles at the same vertex. Next we group (A.9), (A.15) and (A.19) according to the sequence of operators or the exponentials, respecting the order shown. For instance, let us group together the terms corresponding to:

$d(p_2, -\lambda_2) c^\dagger(q, \lambda) d^\dagger(p_1, -\lambda_1)$, we will obtain the expression:

$$\sum_{\lambda, \lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^9} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\ g \left\{ \frac{-q^j}{[q^+]} + \frac{\sigma \cdot (q^+ + p_1^+) + i m}{2[q^+ + p_1^+]} \sigma^j + \sigma^j \frac{(\sigma \cdot p_1 - i m)}{2[p_1^+]} \right\} \epsilon_\lambda^{j*} e^{ix(q+p_1-p_2)}. \quad (\text{A.20})$$

This vertex corresponds to the emission of a photon by a positron. The initial and final kets are:

$$d^\dagger(p_{\text{initial}}) | 0 > \quad c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) | 0 >. \quad (\text{A.21})$$

The amplitude we seek is related to the sequence of operators given by:

$$d(p_1) c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) d^\dagger(p_{\text{initial}}) | 0 >. \quad (\text{A.22})$$

Obviously $p_{\text{final}} \neq p_1$, because a photon is being emitted as well, then taking into consideration (2.111) and (2.114), we have that:

$$d(p_1) c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) d^\dagger(p_{\text{initial}}) | 0 > = c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) d(p_1) d^\dagger(p_{\text{initial}}) | 0 > = \\ c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) \left[(2\pi)^3 2 p_1^+ \delta_{\lambda_1, \lambda_{\text{initial}}} \delta(p_1^+ - p_{\text{initial}}^+) \delta^2(p_{1\perp} - p_{\text{initial}\perp}) - \right. \\ \left. d^\dagger(p_{\text{initial}}) d(p_1) \right] | 0 > = \\ c^\dagger(q_{\text{final}}) d^\dagger(p_{\text{final}}) \left[(2\pi)^3 2 p_1^+ \delta_{\lambda_1, \lambda_{\text{initial}}} \delta(p_1^+ - p_{\text{initial}}^+) \delta^2(p_{1\perp} - p_{\text{initial}\perp}) \right] | 0 >.$$

Thus, (A.20) becomes:

$$\sum_{\lambda, \lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^6} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\ g \left\{ \frac{-q \cdot \epsilon_\lambda^*}{[q^+]} + \frac{\sigma \cdot (q^+ + p_1^+) + i m}{2[q^+ + p_1^+]} \sigma \cdot \epsilon_\lambda^* + \sigma \cdot \epsilon_\lambda^* \frac{(\sigma \cdot p_1 - i m)}{2[p_1^+]} \right\} e^{ix(q+p_1-p_2)}. \quad (\text{A.23})$$

Next, we find the Fourier expansion for:

$$\mathcal{H}_{QED}^{FFFF} = g^2 \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi) \left(\frac{1}{\partial^+} \right) (\xi^\dagger \xi). \quad (\text{A.24})$$

Again we segment the expression to minimize the repetition of calculations. First, we expand the expression

$$\begin{aligned}
\xi^\dagger \xi = & \sum_{\lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\
& \cdot \left[b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1) e^{ip_2 x - ip_1 x} + b^\dagger(p_2, \lambda_2) d^\dagger(p_1, -\lambda_1) e^{ip_2 x + ip_1 x} + \right. \\
& + d(p_2, -\lambda_2) b(p_1, \lambda_1) e^{-ip_2 x - ip_1 x} + \\
& \left. + d(p_2, -\lambda_2) d^\dagger(p_1, -\lambda_1) e^{-ip_2 x + ip_1 x} \right]. \tag{A.25}
\end{aligned}$$

Next, using the result from (A.25) we calculate

$$\begin{aligned}
\left(\frac{1}{\partial^+}\right)(\xi^\dagger \xi) = & \sum_{\lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \sqrt{2p_1^+} \sqrt{2p_2^+} \\
& \left[b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1) \frac{e^{ip_2 x - ip_1 x}}{i[p_2^+ - p_1^+]} + \right. \\
& + b^\dagger(p_2, \lambda_2) d^\dagger(p_1, -\lambda_1) \frac{e^{ip_2 x + ip_1 x}}{i[p_2^+ + p_1^+]} + \\
& + d(p_2, -\lambda_2) b(p_1, \lambda_1) \frac{e^{-ip_2 x - ip_1 x}}{-i[p_2^+ + p_1^+]} + \\
& \left. + d(p_2, -\lambda_2) d^\dagger(p_1, -\lambda_1) \frac{e^{-ip_2 x + ip_1 x}}{-i[p_2^+ - p_1^+]} \right]. \tag{A.26}
\end{aligned}$$

Using the result from (A.26) we finally obtain

$$\begin{aligned}
\left(\frac{1}{\partial^+}\right)(\xi^\dagger \xi) \left(\frac{1}{\partial^+}\right)(\xi^\dagger \xi) = & \sum_{\lambda_4, \lambda_3, \lambda_2, \lambda_1} \chi_{\lambda_4}^\dagger \chi_{\lambda_3} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{3\perp} \frac{dp_3^+}{2[p_3^+]} \\
& \cdot \frac{1}{(2\pi)^3} d^2 p_{4\perp} \frac{dp_4^+}{2[p_4^+]} \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \\
& \cdot \left[b^\dagger(p_4, \lambda_4) b(p_3, \lambda_3) \frac{e^{ip_4 x - ip_3 x}}{i[p_4^+ - p_3^+]} + b^\dagger(p_4, \lambda_4) d^\dagger(p_3, -\lambda_3) \frac{e^{ip_4 x + ip_3 x}}{i[p_4^+ + p_3^+]} + \right. \\
& + d(p_4, -\lambda_4) b(p_3, \lambda_3) \frac{e^{-ip_4 x - ip_3 x}}{-i[p_4^+ + p_3^+]} + d(p_4, -\lambda_4) d^\dagger(p_3, -\lambda_3) \frac{e^{-ip_4 x + ip_3 x}}{-i[p_4^+ - p_3^+]} \left. \right] \\
& \cdot \left[b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1) \frac{e^{ip_2 x - ip_1 x}}{i[p_2^+ - p_1^+]} + b^\dagger(p_2, \lambda_2) d^\dagger(p_1, -\lambda_1) \frac{e^{ip_2 x + ip_1 x}}{i[p_2^+ + p_1^+]} + \right. \\
& + d(p_2, -\lambda_2) b(p_1, \lambda_1) \frac{e^{-ip_2 x - ip_1 x}}{-i[p_2^+ + p_1^+]} + d(p_2, -\lambda_2) d^\dagger(p_1, -\lambda_1) \frac{e^{-ip_2 x + ip_1 x}}{-i[p_2^+ - p_1^+]} \left. \right] \\
& \cdot \sqrt{2p_1^+} \sqrt{2p_2^+} \sqrt{2p_3^+} \sqrt{2p_4^+}. \tag{A.27}
\end{aligned}$$

(A.27) includes sixteen terms similar in nature. We will consider one of them, for

instance the term:

$$\begin{aligned}
& - \int \frac{1}{(2\pi)^3} d^2 p_{3\perp} \frac{dp_3^+}{2[p_3^+]} \frac{1}{(2\pi)^3} d^2 p_{4\perp} \frac{dp_4^+}{2[p_4^+]} \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \\
& \quad \sqrt{2p_1^+} \sqrt{2p_2^+} \sqrt{2p_3^+} \sqrt{2p_4^+} \chi_{\lambda_4}^\dagger \chi_{\lambda_3} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \\
& \quad b^\dagger(p_4, \lambda_4) b(p_3, \lambda_3) b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1) \frac{e^{ix(p_2 - p_1 + p_4 - p_3)}}{[p_2^+ - p_1^+][p_4^+ - p_3^+]} \quad (A.28)
\end{aligned}$$

This expression corresponds to a four fermion lines vertex and we consider it a full term, an instantaneous or contact term, represented by diagram 5. Since here we are dealing with a contact or instantaneous term, we have to consider both an initial and a final state. As in the case of (A.20), we have to consider the effect of the commutation relation (2.111), also we have that there are two streams to consider. Streams $b^\dagger(p_4, \lambda_4) b(p_3, \lambda_3) b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1)$ and $b^\dagger(p_2, \lambda_2) b(p_1, \lambda_1) b^\dagger(p_4, \lambda_4) b(p_3, \lambda_3)$ give the same results, this gives a factor of 2 which cancels the $1/2$ in the Hamiltonian density. Therefore, (A.28) becomes:

$$\begin{aligned}
& - \chi_{\lambda_4}^\dagger \chi_{\lambda_3} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{3\perp} \frac{dp_3^+}{2[p_3^+]} \frac{1}{(2\pi)^3} d^2 p_{4\perp} \frac{dp_4^+}{2[p_4^+]} \frac{e^{ix(p_2 - p_1 + p_4 - p_3)}}{[p_2^+ - p_1^+][p_4^+ - p_3^+]} \\
& \quad \sqrt{2p_1^+} \sqrt{2p_2^+} \sqrt{2p_3^+} \sqrt{2p_4^+}. \quad (A.29)
\end{aligned}$$

Finally, we will Fourier expand the Hamiltonian density:

$$\mathcal{H}_{QED}^{FPPF} = g^2 \xi^\dagger \left\{ \sigma^i C^i \left(\frac{1}{i2\partial^+} \right) (\sigma^j C^j) \xi \right\}. \quad (A.30)$$

As before we will segment the expression. From (A.6) we obtain the expression for $\xi^\dagger \sigma^i C^i$. Next we calculate

$$\begin{aligned}
\left(\frac{1}{i2\partial^+} \right) \sigma^i C^i \xi &= \sum_{\lambda, \lambda_1} \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q'_\perp \frac{dq'^+}{2[q'^+]} \sqrt{2p_1^+}; \\
& \cdot \left[c(q', \lambda') b(p_1, \lambda_1) \sigma_{\epsilon\lambda'} \frac{e^{(-iq'x - ip_1x)}}{2[q'^+ + p_1^+]} + \right. \\
& \quad + c^\dagger(q', \lambda') b(p_1, \lambda_1) \sigma_{\epsilon\lambda'}^* \frac{e^{(iq'x - ip_1x)}}{-2[q'^+ - p_1^+]} + \\
& \quad + c(q', \lambda') d^\dagger(p_1, -\lambda_1) \sigma_{\epsilon\lambda'} \frac{e^{(-iq'x + ip_1x)}}{2[q'^+ - p_1^+]} + \\
& \quad \left. + c^\dagger(q', \lambda') d^\dagger(p_1, -\lambda_1) \sigma_{\epsilon\lambda'}^* \frac{e^{(iq'x + ip_1x)}}{-2[q'^+ + p_1^+]} \right]. \quad (A.31)
\end{aligned}$$

Both expressions when combined in the proper order give,

$$\begin{aligned}
& \sum_{\lambda, \lambda', \lambda_2, \lambda_1} \chi_{\lambda_2}^\dagger \chi_{\lambda_1} \int \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \frac{1}{(2\pi)^3} d^2 q_\perp \frac{dq^+}{2[q^+]} \sqrt{2p_2^+} \sqrt{2p_1^+} \\
& \cdot \frac{1}{(2\pi)^3} d^2 p_{1\perp} \frac{dp_1^+}{2[p_1^+]} \frac{1}{(2\pi)^3} d^2 q'_\perp \frac{dq'^+}{2[q'^+]} \\
& \cdot \left[b^\dagger(p_2, \lambda_2) \sigma \cdot \epsilon_\lambda c(q, \lambda) e^{(-iqx + ip_2 x)} + b^\dagger(p_2, \lambda_2) \sigma \cdot \epsilon_\lambda^* c^\dagger(q, \lambda) e^{(iqx + ip_2 x)} + \right. \\
& \quad \left. + d(p_2, -\lambda_2) \sigma \cdot \epsilon_\lambda c(q, \lambda) e^{(-iqx - ip_2 x)} + d(p_2, -\lambda_2) \sigma \cdot \epsilon_\lambda^* c^\dagger(q, \lambda) e^{(iqx - ip_2 x)} \right] \\
& \cdot \left[c(q', \lambda') b(p_1, \lambda_1) \sigma \cdot \epsilon_{\lambda'} \frac{e^{(-iq'x - ip_1 x)}}{2[q'^+ + p_1^+]} + \right. \\
& \quad + c^\dagger(q', \lambda') b(p_1, \lambda_1) \sigma \cdot \epsilon_{\lambda'}^* \frac{e^{(iq'x - ip_1 x)}}{-2[q'^+ - p_1^+]} + \\
& \quad + c(q', \lambda') d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_{\lambda'} \frac{e^{(-iq'x + ip_1 x)}}{2[q'^+ - p_1^+]} + \\
& \quad \left. + c^\dagger(q', \lambda') d^\dagger(p_1, -\lambda_1) \sigma \cdot \epsilon_{\lambda'}^* \frac{e^{(iq'x + ip_1 x)}}{-2[q'^+ + p_1^+]} \right] \tag{A.32}
\end{aligned}$$

This expression is composed of sixteen terms of similar structure. We will consider one of them, for instance

$$-\chi_{\lambda_2}^\dagger \chi_{\lambda_1} b^\dagger(p_2, \lambda_2) \sigma \cdot \epsilon_\lambda c(q, \lambda) c^\dagger(q', \lambda') b(p_1, \lambda_1) \sigma \cdot \epsilon_{\lambda'}^* \frac{e^{ix(q' - p_1 - q + p_2)}}{2[q'^+ - p_1^+]}. \tag{A.33}$$

This expression corresponds to a two fermion lines, two photon lines vertex and we consider it a full term, an instantaneous or contact term, represented by diagram 6. We must now choose one initial state and one final state and determine the coefficients of these vertices. We then intercalate the operator between the final bra and initial ket. Again we take into account (2.111) and (2.114), and also the fact that here too we have two contributors $b^\dagger(p_2, \lambda_2) c(q, \lambda) c^\dagger(q', \lambda') b(p_1, \lambda_1)$ and $c^\dagger(q', \lambda') b(p_1, \lambda_1) b^\dagger(p_2, \lambda_2) c(q, \lambda)$ this gives a factor of 2 which cancels the 2 in the denominator of (A.33). So the final result will be of the form:

$$-\chi_{\lambda_2}^\dagger \chi_{\lambda_1} \sigma \cdot \epsilon_\lambda \sigma \cdot \epsilon_{\lambda'}^* \int \frac{1}{(2\pi)^3} d^2 p_{2\perp} \frac{dp_2^+}{2[p_2^+]} \sqrt{2p_2^+} \sqrt{2p_1^+} \frac{1}{(2\pi)^3} d^2 q'_\perp \frac{dq'^+}{2[q'^+]} \frac{e^{ix(q' - p_1 - q + p_2)}}{[q'^+ - p_1^+]} \tag{A.34}$$

Appendix B

$|M|^2$ for $e\gamma \rightarrow e\gamma$.

The structure of the matrix is as follows: $[s_2, \lambda_2, s_1, \lambda_1]$. We will first give the matrix of amplitudes in the field variables followed by the square of the modulus of the same entry in the dimensionless variables.

$$M[s_2, \lambda_2, s_1, \lambda_1] = M_1[s_2, \lambda_2, s_1, \lambda_1] + M_2[s_2, \lambda_2, s_1, \lambda_1] + M_3[s_2, \lambda_2, s_1, \lambda_1] + M_4[s_2, \lambda_2, s_1, \lambda_1], \quad (\text{B.1})$$

where M_i for $i = 1, 2, 3, 4$ is the amplitude of the corresponding diagram. We will calculate in detail the amplitude for $[s_2, \lambda_2, s_1, \lambda_1] = [\frac{1}{2}, 1, \frac{1}{2}, 1]$, to show the techniques used in calculating the amplitudes. All other amplitudes are calculated in a similar way. Let us choose a frame of reference in which $p_{1\perp} = p_{2\perp} = 0$. In any frame of reference

$$q_1^+ = p_1^+ + k_1^+ = p_2^+ + k_2^+, \quad q_2^+ = p_1^+ - k_2^+ = p_2^+ - k_1^+. \quad (\text{B.2})$$

$$\begin{aligned} A_{s_2 s'}(\lambda_2) &= \chi_{s_2}^\dagger \Gamma^j(q_1, p_2, k_2) \epsilon^{j*}(\lambda_2) \chi_{s'} \\ &= \chi_{s_2}^\dagger \left(\frac{k_2^j}{k_2^+} - \frac{\sigma \cdot p_2 - im}{2p_2^+} \sigma^j - \sigma^j \frac{\sigma \cdot q_1 + im}{2q_1^+} \right) \epsilon^{j*}(\lambda_2) \chi_{s'}, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned}
B_{s's_1}(\lambda_1) &= \chi_{s'}^\dagger \Gamma^i(p_1, q_1, k_1) \epsilon^i(\lambda_1) \chi_{s_1} \\
&= \chi_{s'}^\dagger \left(\frac{k_1^i}{k_1^+} - \frac{\sigma \cdot q_1 - im}{2q_1^+} \sigma^i - \sigma^i \frac{\sigma \cdot p_1 + im}{2p_1^+} \right) \epsilon^i(\lambda_1) \chi_{s_1}, \\
C_{s_2 s''}(\lambda_1) &= \chi_{s_2}^\dagger \Gamma^i(q_2, p_2, k_1) \epsilon^i(\lambda_1) \chi_{s''} \\
&= \chi_{s_2}^\dagger \left(\frac{k_1^i}{k_1^+} - \frac{\sigma \cdot p_2 - im}{2p_2^+} \sigma^i - \sigma^i \frac{\sigma \cdot q_2 + im}{2q_2^+} \right) \epsilon^i(\lambda_1) \chi_{s''}, \quad (\text{B.4}) \\
D_{s'' s_1}(\lambda_2) &= \chi_{s''}^\dagger \Gamma^i(p_1, q_2, k_2) \epsilon^{i*}(\lambda_2) \chi_{s_1} \\
&= \chi_{s''}^\dagger \left(\frac{k_2^j}{k_2^+} - \frac{\sigma \cdot q_2 - im}{2q_2^+} \sigma^j - \sigma^j \frac{\sigma \cdot p_1 + im}{2p_1^+} \right) \epsilon^{i*}(\lambda_2) \chi_{s_1}, \\
E_{s_2 s_1}(\lambda_1, \lambda_2) &= \chi_{s_2}^\dagger \sigma \cdot \epsilon(\lambda_1) \frac{1}{2[p_1^+ - k_2^+]} \sigma \cdot \epsilon^*(\lambda_2) \chi_{s_1}, \quad (\text{B.5})
\end{aligned}$$

$$\begin{aligned}
\epsilon(1) &= \frac{1}{\sqrt{2}} (1 \ i), \quad \epsilon(\lambda) = \epsilon^*(-\lambda), \\
k^i \cdot \epsilon^i(1) &= \frac{1}{\sqrt{2}} (k^1 + ik^2) = \frac{1}{\sqrt{2}} k_+, \quad \sigma \cdot \epsilon(1) = \sqrt{2} \sigma^+, \\
k^j \cdot \epsilon^j(-1) &= \frac{1}{\sqrt{2}} (k^1 - ik^2) = \frac{1}{\sqrt{2}} k_-, \quad \sigma \cdot \epsilon(-1) = \sqrt{2} \sigma^-, \\
p_\pm &= p^1 \pm ip^2 = p_\perp e^{\pm i\varphi}, \quad \sigma \cdot p = \sigma^+ p_- + \sigma^- p_+, \\
\chi_{\frac{1}{2}}^\dagger &= (1 \ 0), \quad \chi_{-\frac{1}{2}}^\dagger = (0 \ 1).
\end{aligned}$$

We now calculate the sum over polarizations.

$$\begin{aligned}
A_{\frac{1}{2}\frac{1}{2}}(1) &= \frac{1}{\sqrt{2}} e^{-i\varphi} \frac{k_{2\perp} q_1^+}{k_2^+ p_2^+}, \quad B_{\frac{1}{2}\frac{1}{2}}(1) = 0, \quad C_{\frac{1}{2}\frac{1}{2}}(1) = \frac{1}{\sqrt{2}} \frac{k_{2\perp}}{q_2^+} e^{i\varphi}, \\
D_{\frac{1}{2}\frac{1}{2}}(1) &= \frac{1}{\sqrt{2}} e^{-i\varphi} \frac{k_{2\perp} p_1^+}{k_2^+ q_2^+}, \\
A_{\frac{1}{2}-\frac{1}{2}}(1) &= 0, \quad B_{\frac{1}{2}-\frac{1}{2}}(1) = 0, \quad C_{\frac{1}{2}-\frac{1}{2}}(1) = -\frac{1}{\sqrt{2}} im \frac{q_2^+ - p_2^+}{p_2^+ q_2^+}, \\
D_{\frac{1}{2}-\frac{1}{2}}(1) &= \frac{1}{\sqrt{2}} im \frac{p_1^+ - q_2^+}{p_1^+ q_2^+},
\end{aligned}$$

Next we calculate the products

$$\begin{aligned}
A_{\frac{1}{2}\frac{1}{2}}(1) * B_{\frac{1}{2}\frac{1}{2}}(1) + A_{\frac{1}{2}-\frac{1}{2}}(1) * B_{\frac{1}{2}-\frac{1}{2}}(1) &= 0, \\
C_{\frac{1}{2}\frac{1}{2}}(1) * D_{\frac{1}{2}\frac{1}{2}}(1) + C_{\frac{1}{2}-\frac{1}{2}}(1) * D_{\frac{1}{2}-\frac{1}{2}}(1) &= \frac{k_{2\perp}^2 p_1^+}{2 k_2^+ (q_2^+)^2} - \\
&\quad - \frac{m^2 (p_1^+ - q_2^+) (q_2^+ - p_2^+)}{2 p_1^+ p_2^+ (q_2^+)^2}. \quad (\text{B.6})
\end{aligned}$$

Next, we multiply (B.6) by the energy denominator factor.

$$\frac{1}{d_2} = \frac{1}{p_2^- - k_1^- - q_2^-} = \frac{-2p_2^+ q_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)}, \quad (\text{B.7})$$

$$\left(\frac{k_{2\perp}^2 p_1^+}{2k_2^+(q_2^+)^2} - \frac{m^2(p_1^+ - q_2^+)(q_2^+ - p_2^+)}{2p_1^+ p_2^+(q_2^+)^2} \right) * \frac{-2p_2^+ q_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)}. \quad (\text{B.8})$$

We now add the contribution from the instantaneous term for $E_{\frac{1}{2}\frac{1}{2}}(1, 1) = \frac{1}{q_2^+}$.

$$\frac{1}{q_2^+} - \left(\frac{k_{2\perp}^2 p_1^+}{k_2^+(q_2^+)^2} - \frac{m^2(p_1^+ - q_2^+)(q_2^+ - p_2^+)}{p_1^+ p_2^+(q_2^+)^2} \right) * \frac{p_2^+ q_2^+}{(k_1^+)(k_{2\perp}^2 + m^2)}. \quad (\text{B.9})$$

And finally we multiply by the common factor $e^2 \sqrt{2p_1^+ 2p_2^+}$ and we obtain the entry A_1 below, where e stands for the charge of the fermion. To avoid any confusion we will omit this factor from here on and re-introduce it at the end of this appendix. As indicated before each and every amplitude is calculated in the same way, for different polarization and/or helicities.

$$\begin{aligned} A_1 &= M\left[\frac{1}{2}, 1, \frac{1}{2}, 1\right] = \sqrt{p_1^+ p_2^+} \left\{ \frac{2}{q_2^+} - \frac{p_2^+ q_2^+ \left\{ \frac{2k_{2\perp}^2 p_1^+}{k_2^+(q_2^+)^2} - \frac{2m^2(p_1^+ - q_2^+)(q_2^+ - p_2^+)}{p_1^+ p_2^+(q_2^+)^2} \right\}}{k_1^+(m^2 + k_{2\perp}^2)} \right\}, \\ A_2 &= M\left[\frac{1}{2}, 1, -\frac{1}{2}, 1\right] = -\frac{2ie^{i\varphi} k_{2\perp} m \sqrt{p_1^+ p_2^+} (p_1^+ - q_2^+)}{k_1^+(m^2 + k_{2\perp}^2) p_1^+}, \\ A_3 &= M\left[-\frac{1}{2}, 1, \frac{1}{2}, 1\right] = -2ie^{-i\varphi} \sqrt{p_1^+ p_2^+} \left\{ \frac{k_{2\perp} q_1^+}{m k_2^+ p_2^+} + \frac{k_{2\perp} m (q_2^+ - p_2^+)}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)} \right\}, \\ A_4 &= M\left[-\frac{1}{2}, 1, -\frac{1}{2}, 1\right] = \sqrt{p_1^+ p_2^+} \left\{ \frac{2}{q_1^+} + \frac{2k_2^+}{p_2^+ q_1^+} - \frac{2k_{2\perp}^2 q_2^+}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)} \right\}, \\ A_5 &= M\left[\frac{1}{2}, 1, \frac{1}{2}, -1\right] = -\frac{2e^{2i\varphi} k_{2\perp}^2 p_2^+ \sqrt{p_1^+ p_2^+}}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)}, \\ A_6 &= M\left[\frac{1}{2}, 1, -\frac{1}{2}, -1\right] = 0, \\ A_7 &= M\left[-\frac{1}{2}, 1, \frac{1}{2}, -1\right] = -2ie^{i\varphi} \sqrt{p_1^+ p_2^+} \left\{ \frac{k_{2\perp}}{m k_2^+} + \frac{m p_2^+}{k_1^+ (m^2 + k_{2\perp}^2)} \right. \\ &\quad \left. \cdot \left(\frac{k_{2\perp} (p_1^+ - q_2^+)}{p_1^+ q_2^+} + \frac{k_{2\perp} p_1^+ (q_2^+ - p_2^+)}{k_2^+ p_2^+ q_2^+} \right) \right\}, \\ A_8 &= M\left[-\frac{1}{2}, 1, -\frac{1}{2}, -1\right] = -2e^{2i\varphi} k_{2\perp}^2 p_1^+ \sqrt{p_1^+ p_2^+} \frac{1}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)}, \end{aligned}$$

$$\begin{aligned}
A_9 &= M\left[\frac{1}{2}, -1, \frac{1}{2}, 1\right] = -2e^{-2i\varphi} k_{2\perp}^2 p_1^+ \sqrt{p_1^+ p_2^+} \frac{1}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)}, \\
A_{10} &= M\left[-\frac{1}{2}, 1, \frac{1}{2}, -1\right] = -2ie^{-i\varphi} \sqrt{p_1^+ p_2^+} \left\{ \frac{k_{2\perp}}{m k_2^+} + \frac{m p_2^+}{k_1^+ (m^2 + k_{2\perp}^2)} \right. \\
&\quad \left. \cdot \left(\frac{k_{2\perp} (p_1^+ - q_2^+)}{p_1^+ q_2^+} + \frac{k_{2\perp} p_1^+ (q_2^+ - p_2^+)}{k_2^+ p_2^+ q_2^+} \right) \right\}, \\
A_{11} &= M\left[-\frac{1}{2}, -1, \frac{1}{2}, 1\right] = 0, \\
A_{12} &= M\left[-\frac{1}{2}, -1, -\frac{1}{2}, 1\right] = -\frac{2e^{-2i\varphi} k_{2\perp}^2 p_2^+ \sqrt{p_1^+ p_2^+}}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)}, \\
A_{13} &= M\left[-\frac{1}{2}, 1, -\frac{1}{2}, 1\right] = \sqrt{p_1^+ p_2^+} \left\{ \frac{2}{q_1^+} + \frac{2k_2^+}{p_2^+ q_1^+} - \frac{2k_{2\perp}^2 q_2^+}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)} \right\}, \\
A_{14} &= M\left[\frac{1}{2}, -1, -\frac{1}{2}, -1\right] = -2ie^{i\varphi} \sqrt{p_1^+ p_2^+} \left\{ \frac{k_{2\perp} q_1^+}{m k_2^+ p_2^+} + \frac{k_{2\perp} m (q_2^+ - p_2^+)}{k_1^+ k_2^+ (m^2 + k_{2\perp}^2)} \right\}, \\
A_{15} &= M\left[-\frac{1}{2}, -1, \frac{1}{2}, -1\right] = -\frac{2ie^{-i\varphi} k_{2\perp} m \sqrt{p_1^+ p_2^+} (p_1^+ - q_2^+)}{k_1^+ (m^2 + k_{2\perp}^2) p_1^+}, \\
A_{16} &= M\left[-\frac{1}{2}, -1, -\frac{1}{2}, -1\right] = \sqrt{p_1^+ p_2^+} \left\{ \frac{2}{q_2^+} - \right. \\
&\quad \left. - \frac{p_2^+ q_2^+ \left\{ \frac{2k_{2\perp}^2 p_1^+}{k_2^+ (q_2^+)^2} - \frac{2m^2 (p_1^+ - q_2^+) (q_2^+ - p_2^+)}{p_1^+ p_2^+ (q_2^+)^2} \right\}}{k_1^+ (m^2 + k_{2\perp}^2)} \right\}.
\end{aligned}$$

In view that the following symmetries are present, we will list only one member of each pair:

$$\begin{aligned}
|A_1|^2 &= |A_{16}|^2, \quad |A_2|^2 = |A_{15}|^2, \quad |A_3|^2 = |A_{14}|^2, \quad |A_4|^2 = |A_{13}|^2, \\
|A_5|^2 &= |A_{12}|^2, \quad |A_6|^2 = |A_{11}|^2, \quad |A_7|^2 = |A_{10}|^2, \quad |A_8|^2 = |A_9|^2
\end{aligned}$$

The square of the modulus of the above amplitudes are:

$$\begin{aligned}
|A_1|^2 &= \frac{4(x-1)y^2}{x^2(x-y-1)^2(y-1)}, \\
|A_3|^2 &= \frac{4(x-1)(x-y)y}{(1-x)(x-y-1)^2(y-1)}, \\
|A_5|^2 &= \frac{4(x-1)(x-y)^2(y-1)}{x^2(x-y-1)^2}, \\
|A_7|^2 &= \frac{4(x-1)^3(x-y)^3y}{(1-x)x^2(x-y-1)^2(y-1)},
\end{aligned}$$

$$\begin{aligned}
|A_9|^2 &= \frac{4(x-1)^3(x-y)^2}{x^2(x-y-1)^2(y-1)}, \\
|A_{11}|^2 &= 0, \\
|A_{13}|^2 &= \frac{4(x-1)(x-y+1)^2y^2}{x^2(x-y-1)^2(y-1)}, \\
|A_{15}|^2 &= \frac{4(x-1)(x-y)y^3}{(1-x)x^2(x-y-1)^2(y-1)}.
\end{aligned}$$

So when we will add the square of the modulus of the amplitudes we will add from 1 to 8 and divide by 2 instead of 4. There is a factor of e^4 which we have so far omitted and which now we include. Thus

$$\begin{aligned}
A^2 &= \frac{e^4}{2} \left\{ |A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 + |A_5|^2 + |A_6|^2 + \right. \\
&\quad \left. + |A_7|^2 + |A_8|^2 \right\}, \\
&= e^4 \left[-\frac{2(-4+4x-x^2+x^3)}{x^2} + 2y + \frac{8(1-2x+x^2)}{x^2(1-x+y)^2} - \right. \\
&\quad \left. - \frac{2(8-12x+3x^2)}{x^2(1-x+y)} \right]. \tag{B.10}
\end{aligned}$$

Appendix C

$$| M |^2 \text{ for } e^+ e^- \rightarrow \mu^+ \mu^-.$$

The structure of the matrix is as follows: $[s_4, s_3, s_2, s_1]$. We will first give the matrix of amplitudes in the field variables, and evaluate the result for the selected values of the transverse components. The table shows that there is no interference between diagrams 11 and 12. Of the sixteen combinations of helicities for diagram 11, only 6 are not zero. In the case of diagram 12 only 4 are not zero.

$$M[s_4, s_3, s_2, s_1] = M_1[s_4, s_3, s_2, s_1] + M_2[s_4, s_3, s_2, s_1]. \quad (\text{C.1})$$

Here again we show the calculation of the amplitude for one combination of helicities, the rest are calculated in a similar way. We first calculate the expression for the energy denominator:

$$\frac{1}{d_1} = \frac{1}{p_1^- + p_2^- - k^-} = \frac{1}{\frac{p_{1\perp}^2 + m^2}{2p_1^+} + \frac{p_{2\perp}^2 + m^2}{2p_2^+} - \frac{k_\perp^2}{2k^+}} = \frac{2}{\frac{p_{1\perp}^2 + m^2}{p_1^+} + \frac{p_{2\perp}^2 + m^2}{p_2^+} - \frac{k_\perp^2}{k^+}}. \quad (\text{C.2})$$

The Γ 's for the vertices are given by the expressions:

$$\begin{aligned} \Gamma^i(-p_4, p_3; k) &= \frac{k^i}{k^+} - \frac{\sigma \cdot p_3 - im}{2p_3^+} \sigma^i - \sigma^i \frac{\sigma \cdot (-p_4) + im}{-2p_4^+}, \\ \Gamma^j(p_1, -p_2; k) &= \frac{k^j}{k^+} - \frac{\sigma \cdot (-p_2) - im}{-2p_2^+} \sigma^j - \sigma^j \frac{\sigma \cdot p_1 + im}{2p_1^+}. \end{aligned} \quad (\text{C.3})$$

The amplitudes for the related processes are given by the expressions

$$\begin{aligned}
 M_1 = & e^2 \int \frac{dk^+ d^2 k_\perp}{(2\pi)^3 2k^+} \theta(k^+) \sum_\lambda \chi_{s_3}^\dagger \Gamma^i(-p_4, p_3; k) \chi_{-s_4} \epsilon^i(\lambda) \frac{1}{d_1} \chi_{-s_2}^\dagger \\
 & \cdot \Gamma^j(p_1, -p_2; k) \chi_{s_1} \epsilon^{j*}(\lambda) (2\pi)^3 \delta(p_3^+ + p_4^+ - k^+) \delta^2(p_{3\perp} + p_{4\perp} - k_\perp) \\
 & \cdot (2\pi)^3 \delta(p_2^+ + p_1^+ - k^+) \delta^2(p_{2\perp} + p_{1\perp} - k_\perp) \sqrt{2p_3^+ 2p_4^+} \sqrt{2p_1^+ 2p_2^+} \\
 & \cdot (-2i\pi) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{C.4}
 \end{aligned}$$

$$M_2 = e^2 \chi_{-s_2}^\dagger \chi_{s_1} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ + p_2^+]^2} \chi_{s_3}^\dagger \chi_{-s_4} \sqrt{2p_3^+ 2p_4^+}. \tag{C.5}$$

When considering the anti-commutation relations we will note that the diagram 11 has an overall positive sign and diagram 12 has an overall positive sign too.

$$\begin{aligned}
 \epsilon(1) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix}, \quad \epsilon(\lambda) = \epsilon^*(-\lambda), \\
 k^i \cdot \epsilon^i(1) &= \frac{1}{\sqrt{2}} (k^1 + ik^2) = \frac{1}{\sqrt{2}} k_+, \quad \sigma \cdot \epsilon(1) = \sqrt{2} \sigma^+, \\
 k^j \cdot \epsilon^j(-1) &= \frac{1}{\sqrt{2}} (k^1 - ik^2) = \frac{1}{\sqrt{2}} k_-, \quad \sigma \cdot \epsilon(-1) = \sqrt{2} \sigma^-, \\
 p_\pm &= p^1 \pm ip^2 = p_\perp e^{\pm i\varphi}, \quad \sigma \cdot p = \sigma^+ p_- + \sigma^- p_+, \\
 \chi_{\frac{1}{2}}^\dagger &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}}^\dagger = \begin{pmatrix} 0 & 1 \end{pmatrix}.
 \end{aligned}$$

$$\begin{aligned}
 \Gamma^i(-p_4, p_3; k) \epsilon^i(1) &= \frac{1}{\sqrt{2}} \frac{k_+}{[k^+]} - \frac{\sigma^+ p_{3-} + \sigma^- p_{3+} - im_\mu}{[p_3^+]} \frac{1}{\sqrt{2}} \sigma^+ \\
 &\quad - \frac{1}{\sqrt{2}} \sigma^+ \frac{\sigma^+ p_{4-} + \sigma^- p_{4+} - im_\mu}{[p_4^+]}, \\
 &= \frac{1}{\sqrt{2}} \left\{ \frac{k_+}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \sigma^- \sigma^+ - \frac{p_{4+}}{[p_4^+]} \sigma^+ \sigma^- \right. \\
 &\quad \left. + im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \sigma^+ \right\}, \tag{C.6}
 \end{aligned}$$

$$\begin{aligned}
 \Gamma^i(-p_4, p_3; k) \epsilon^i(-1) &= \frac{1}{\sqrt{2}} \left\{ \frac{k_-}{[k^+]} - \frac{p_{3-}}{[p_3^+]} \sigma^+ \sigma^- - \frac{p_{4-}}{[p_4^+]} \sigma^- \sigma^+ \right. \\
 &\quad \left. + im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \sigma^- \right\}, \tag{C.7}
 \end{aligned}$$

$$\begin{aligned}
 \Gamma^j(p_1, -p_2; k) \epsilon^{j*}(1) &= \frac{1}{\sqrt{2}} \frac{k_-}{[k^+]} - \frac{\sigma^+ p_{2-} + \sigma^- p_{2+} + im_e}{[p_2^+]} \frac{1}{\sqrt{2}} \sigma^- \\
 &\quad - \frac{1}{\sqrt{2}} \sigma^- \frac{\sigma^+ p_{1-} + \sigma^- p_{1+} + im_e}{[p_1^+]},
 \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{k_-}{[k^+]} - \frac{p_{2-}}{[p_2^+]} \sigma^+ \sigma^- - \frac{p_{1-}}{[p_1^+]} \sigma^- \sigma^+ \right. \\ \left. - im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \sigma^- \right\}, \quad (C.8)$$

$$\Gamma^j(p_1, -p_2; k) \epsilon^{j*}(-1) = \frac{1}{\sqrt{2}} \left\{ \frac{k_+}{[k^+]} - \frac{p_{2+}}{[p_2^+]} \sigma^- \sigma^+ - \frac{p_{1+}}{[p_1^+]} \sigma^+ \sigma^- \right. \\ \left. - im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \sigma^+ \right\}. \quad (C.9)$$

Next, we calculate the amplitude $M_1[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$.

$$M_1[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = \left\{ \chi_{\frac{1}{2}}^\dagger \Gamma^i(-p_4, p_3; k) \chi_{-\frac{1}{2}} \epsilon^i(1) \chi_{-\frac{1}{2}}^\dagger \Gamma^j(p_1, -p_2; k) \chi_{\frac{1}{2}} \epsilon^{j*}(1) + \right. \\ \left. + \chi_{\frac{1}{2}}^\dagger \Gamma^i(-p_4, p_3; k) \chi_{-\frac{1}{2}} \epsilon^i(-1) \chi_{-\frac{1}{2}}^\dagger \Gamma^j(p_1, -p_2; k) \chi_{\frac{1}{2}} \epsilon^{j*}(-1) \right\} \\ \cdot \frac{1}{p_1^- + p_2^- - k^-} \sqrt{2p_1^+ 2p_2^+ 2p_3^+ 2p_4^+}, \quad (C.10)$$

$$= \sqrt{2}(im_\mu) \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \sqrt{2}(-im_e) \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \\ \cdot \frac{1}{p_1^- + p_2^- - k^-} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\ = 4m_e^2 m_\mu^2 \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{p_1^- + p_2^- - k^-}. \quad (C.11)$$

This result is to be multiplied by the factor

$$\frac{1}{2[k^+]} (2\pi)^3 (-2\pi i) \delta^4(p_1 + p_2 - p_3 - p_4). \quad (C.12)$$

In the tables that follow there are contributions by the instantaneous or contact term, which are of the form $\frac{1}{[k^+]^2}$. They represent possible LF infrared singularities. For this process the sector $k^+ = 0$ is unphysical, no $\mu^+\mu^-$ pair can be produced. Nevertheless, we must show that an analytical continuation of the amplitude function will be well behaved. This is a requirement that has its origin in the crossing symmetry, an unphysical sector for one process is a physical sector for another. There are four combinations of helicities for which the amplitudes contain $\frac{1}{[k^+]^2}$ terms and they are:

$$M[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}], \quad M[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}], \quad M[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}], \\ M[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]. \quad (C.13)$$

The amplitude for any of these four combinations contains two terms in $\frac{1}{[k^+]^2}$. These terms are:

$$\left\{ \frac{k_\perp^2}{[k^+]^2} \frac{1}{2[k^+](p_1^- + p_2^- - k^-)} + \frac{1}{[p_1^+ + p_2^+]^2} \right\} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \quad (\text{C.14})$$

We first consider the first of the two terms,

$$\frac{k_\perp^2}{[k^+]^2} \frac{1}{2[k^+](p_1^- + p_2^- - k^-)} = \frac{k_\perp^2}{[k^+]^2} \frac{1}{p_{1\perp}^2 \left(\frac{k^+}{p_1^+}\right) + p_{2\perp}^2 \left(\frac{k^+}{p_2^+}\right) + m_e^2 \left(\frac{(k^+)^2}{p_1^+ p_2^+}\right) - (p_{1\perp}^2 + p_{2\perp}^2)^2} \quad (\text{C.15})$$

If $k_\perp^2$ is finite, then as $k^+ \rightarrow 0$ this term cancels the second term in (C.14). If k^+ is finite then as $k_\perp^2 \rightarrow 0$ this term is zero and only the instantaneous term contributes. So the amplitude is non-singular in the $k^+ = 0$ sector as required. Choosing a frame of reference such that:

$$p_{1\perp} = p_{2\perp} = k_\perp = 0. \quad (\text{C.16})$$

We calculate the amplitudes, which are given by the expressions,

$$\begin{aligned} A_1 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = 4m_e m_\mu \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \\ &\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\ A_2 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = 4im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{1-}}{[p_1^+]} \right\} \\ &\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\ A_3 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = 4im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{2-}}{[p_2^+]} \right\} \\ &\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\ A_4 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = 0, \\ A_5 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \\ &\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \end{aligned}$$

$$\begin{aligned}
A_6 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{1-}}{[p_1^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]} + \frac{4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{[p_1^+ + p_2^+]^2} + \\
&\quad + 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{2+}}{[p_2^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_7 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{2-}}{[p_2^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]} + \frac{4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{[p_1^+ + p_2^+]^2} + \\
&\quad + 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{1+}}{[p_1^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_8 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_9 &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{4+}}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_{10} &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{4+}}{[p_4^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{1-}}{[p_1^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]} + \frac{4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{[p_1^+ + p_2^+]^2} + \\
&\quad + 4 \left\{ \frac{k_-}{[k^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{2+}}{[p_2^+]} \right\} \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]},
\end{aligned}$$

$$\begin{aligned}
A_{11} &= M[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}] = 4 \left\{ \frac{k_+}{[k^+]} - \frac{p_{4+}}{[p_4^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{2-}}{[p_2^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]} + \frac{4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{[p_1^+ + p_2^+]^2} + \\
&\quad + 4 \left\{ \frac{k_-}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{1+}}{[p_1^+]} \right\} \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}, \\
A_{12} &= M[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}, \\
A_{13} &= M[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = 0, \\
A_{14} &= M[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = 4im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{2+}}{[p_2^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}, \\
A_{15} &= M[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}] = 4im_\mu \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{1+}}{[p_1^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}, \\
A_{16} &= M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = 4m_e m_\mu \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}.
\end{aligned}$$

We note the symmetries between 1 and 16, 2 and 15, ..., 8 and 9. Therefore we will square the modulus of the amplitudes and add 1 to 8 and divide by 2 instead of 4. Given the fact that we have chosen $p_{1\perp} = p_{2\perp} = k_\perp = 0$, then

$$p_{1+} = p_{1-} = p_{2+} = p_{2-} = p_{3+} = p_{3-} = p_{4+} = p_{4-} = k_+ = k_- = 0. \quad (\text{C.17})$$

and the amplitudes become

$$\begin{aligned}
A_1 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = 4m_em_\mu \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_2 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = 0, \\
A_3 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = 0, \\
A_4 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = 0, \\
A_5 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_6 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = \frac{4}{[p_1^+ + p_2^+]^2} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\
A_7 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = \frac{4}{[p_1^+ + p_2^+]^2} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\
A_8 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_9 &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_+}{[k^+]} - \frac{p_{4+}}{[p_4^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_{10} &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = \frac{4}{[p_1^+ + p_2^+]^2} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\
A_{11} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = \frac{4}{[p_1^+ + p_2^+]^2} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\
A_{12} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = -4im_e \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{k_-}{[k^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \\
&\quad \cdot \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 \left[\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right]}, \\
A_{13} &= M\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = 0,
\end{aligned}$$

$$\begin{aligned}
A_{14} &= M[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = 0, \\
A_{15} &= M[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}] = 0, \\
A_{16} &= M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = 4m_e m_\mu \left\{ \frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right\} \left\{ \frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right\} \\
&\quad \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{2[k^+] m_e^2 [\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]}]}. \tag{C.18}
\end{aligned}$$

The next step is to calculate M^2 , the sum of the square of the modulus of the amplitudes of the process. First we consider the contributions to the amplitude when $i = 6, 7, 10, 11$. These terms represent the contribution of the instantaneous diagrams, and thus they do not include the energy denominator as a factor. Thus,

$$\begin{aligned}
n_{i=6,7,10,11} &= e^2 \frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{[p_1^+ + p_2^+]^2} (2\pi)^3 \delta(p_3^+ + p_4^+ - p_1^+ - p_2^+) \\
&\quad \cdot \delta^2(p_{3\perp} + p_{4\perp} - p_{1\perp} - p_{2\perp}) (-2\pi i) \delta(p_3^- + p_4^- - p_1^- - p_2^-). \tag{C.19}
\end{aligned}$$

In the CMS frame of reference $p_1^+ p_2^+ \propto m_e^2 \approx 0$ means that the term $\sqrt{p_1^+ p_2^+}$ will give zero (unless neutralized by a $\frac{1}{m_e}$ term). In the case of the instantaneous terms there is no such cancelation, and the term is zero, so the instantaneous term does not contribute to the above summation. Next we find the sum of the square of the modulus of A_i .

$$\begin{aligned}
N^2 &= \sum_{i=1, \dots, 16} |A_i|^2, \\
&= \frac{1}{(p_1^+ + p_2^+)^2} \frac{e^4}{m_e^4 (\frac{1}{p_1^+} + \frac{1}{p_2^+})^2} \left\{ 8 m_e^2 m_\mu^2 \left(\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right)^2 \left(\frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right)^2 + \right. \\
&\quad \left. + 8 m_e^2 \left(\frac{1}{[p_1^+]} + \frac{1}{[p_2^+]} \right)^2 \left(\frac{(p_{3\perp})^2}{[p_3^+]^2} + \frac{(p_{4\perp})^2}{[p_4^+]^2} \right) \right\} \\
&\quad \cdot (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) (2\pi)^4 \delta^4(0). \tag{C.20}
\end{aligned}$$

We want to calculate the volume, time and helicity average of N^2 . Let $VT = (2\pi)^4 \delta^4(0)$ with

$$K^2 = \frac{1}{4VT} N^2, \tag{C.21}$$

$$\begin{aligned}
 &= \frac{2e^4}{m_e^2(p_1^+ + p_2^+)^2} \left\{ m_\mu^2 \left(\frac{1}{[p_3^+]} + \frac{1}{[p_4^+]} \right)^2 + \frac{(p_{3\perp})^2}{[p_3^+]^2} + \frac{(p_{4\perp})^2}{[p_4^+]^2} \right\} p_1^+ p_2^+ p_3^+ p_4^+ \\
 &\quad \cdot (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2), \tag{C.22}
 \end{aligned}$$

$$= |M|^2 (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2). \tag{C.23}$$

Appendix D

$|M|^2$ for Bhabha scattering.

This example is similar in many ways to the $e^+ e^- \rightarrow \mu^+ \mu^-$ example. The flux factor, for instance, is the same. There are, however, additional diagrams that have to be taken into consideration. Here too we will show the detailed calculations for one combination of helicities. The rest are all calculated the same way. Again, we will give the final results as a function of the dimensionless variables, x and y . Let $p_{1\perp} = p_{2\perp} = 0$ then $(q_{1\perp})^2 = (p_{3\perp})^2$.

$$\begin{aligned}
\Gamma^i(-p_4, -p_2; q_1) &= \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot p_4 - im}{[p_4^+]} + \frac{\sigma \cdot p_2 + im}{[p_2^+]} \sigma^i - \frac{2q_1^i}{[q_1^+]} \right\}, \\
\Gamma^j(p_1, p_3; q_1) &= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im}{[p_1^+]} + \frac{\sigma \cdot p_3 - im}{[p_3^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\}, \\
\Gamma^i(-p_4, p_3; q_3) &= \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot (-p_4) + im}{[-p_4^+]} + \frac{\sigma \cdot p_3 - im}{[p_3^+]} \sigma^i - \frac{2q_3^i}{[q_3^+]} \right\}, \\
\Gamma^j(p_1, -p_2; q_3) &= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im}{[p_1^+]} + \frac{\sigma \cdot (-p_2) - im}{[-p_2^+]} \sigma^j - \frac{2q_3^j}{[q_3^+]} \right\}, \quad (D.1) \\
\frac{1}{d_1} &= \frac{1}{p_1^- + p_2^- - (p_3^- + p_2^- + q_1^-)}, \\
\frac{1}{d_3} &= \frac{1}{p_1^- + p_2^- - q_3^-},
\end{aligned}$$

$$\frac{1}{2q_1^+} \frac{1}{d_1} = \frac{-p_1^+ p_3^+}{m^2(p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2} = \frac{1}{2q_2^+} \frac{1}{d_2}, \quad (\text{D.2})$$

$$\frac{1}{2q_3^+} \frac{1}{d_3} = \frac{p_1^+ p_2^+}{m^2(p_1^+ + p_2^+)^2}. \quad (\text{D.3})$$

Let

$$\begin{aligned} F_1(\lambda) &= \Gamma^i(-p_4, -p_2, q_1) \cdot \epsilon^i(\lambda), \\ &= \frac{1}{2} \left\{ \sigma^i \frac{\sigma \cdot p_4 - im}{[p_4^+]} + \frac{\sigma \cdot p_2 + im}{[p_2^+]} \sigma^i - \frac{2q_1^i}{[q_1^+]} \right\} \cdot \epsilon^i(\lambda), \\ F_2(\lambda) &= \Gamma^j(p_1, p_3, q_1) \cdot \epsilon^{*j}(\lambda), \\ &= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im}{[p_1^+]} + \frac{\sigma \cdot p_3 - im}{[p_3^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\} \cdot \epsilon^{*j}(\lambda), \\ F_1(1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^+ - \frac{p_{4+}}{[p_4^+]} \sigma^+ \sigma^- \right\}, \\ F_1(-1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^- - \frac{p_{4-}}{[p_4^+]} \sigma^- \sigma^+ \right\}, \\ F_2(1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^- - \frac{p_{3-}}{[p_3^+]} \sigma^+ \sigma^- \right\}, \\ F_2(-1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^+ - \frac{p_{3+}}{[p_3^+]} \sigma^- \sigma^+ \right\}. \end{aligned} \quad (\text{D.4})$$

As indicated in section 3.3, we combine amplitudes $M_1 + M_2$ since they complement each other.

$$\begin{aligned} M_1 &= e^2 \int \frac{dq_1^+ d^2 q_{1\perp}}{(2\pi)^3 2q_1^+} \theta(q_1^+) \sum_{\lambda} \chi_{-s_2}^\dagger \Gamma^j(-p_4, -p_2; q_1) \epsilon^j(\lambda) \chi_{-s_4} \left(\frac{1}{d_1} \right) \\ &\quad \cdot \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_1) \epsilon^{i*}(\lambda) \chi_{s_1} \sqrt{2p_2^+ 2p_4^+} (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \\ &\quad \cdot \delta^2(q_{1\perp} - p_{4\perp} + p_{2\perp}) \sqrt{2p_1^+ 2p_3^+} (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \\ &\quad \cdot \delta^2(q_{1\perp} - p_{4\perp} + p_{2\perp}) (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \\ M_4 &= e^2 \chi_{s_2}^\dagger \chi_{s_1} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{-s_2}^\dagger \chi_{-s_4} \sqrt{2p_3^+ 2p_4^+} \\ &\quad \cdot (2\pi)^3 (-2\pi i) \delta^4(p_3 + p_4 - p_1 - p_2). \end{aligned} \quad (\text{D.5})$$

Let

$$\sum_{\lambda} \chi_{-s_2}^\dagger \Gamma^j(-p_4, -p_2; q_1) \epsilon^j(\lambda) \chi_{-s_4} \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_1) \epsilon^{i*}(\lambda) \chi_{s_1}$$

$$\begin{aligned}
&= \chi_{-s_2}^\dagger F_1(1) \chi_{-s_4} \chi_{s_3}^\dagger F_2(1) \chi_{s_1} + \chi_{-s_2}^\dagger F_1(-1) \chi_{-s_4} \chi_{s_3}^\dagger F_2(-1) \chi_{s_1}, \\
&= A_1(s_1, s_3) B_1(s_2, s_4) + A_{-1}(s_1, s_3) B_{-1}(s_2, s_4).
\end{aligned} \tag{D.6}$$

Again we will calculate the amplitude for the helicities $[s_1, s_2, s_3, s_4] = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned}
A_1\left(\frac{1}{2}, \frac{1}{2}\right) &= -(1 \ 0) \frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^- - \frac{p_{3-}}{[p_3^+]} \sigma^+ \sigma^- \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} \right\}, \\
A_{-1}\left(\frac{1}{2}, \frac{1}{2}\right) &= -(1 \ 0) \frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^+ - \frac{p_{3+}}{[p_3^+]} \sigma^- \sigma^+ \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} \right\}, \\
B_1\left(\frac{1}{2}, \frac{1}{2}\right) &= -(1 \ 0) \frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^+ - \frac{p_{4+}}{[p_4^+]} \sigma^+ \sigma^- \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} \right\}, \\
B_{-1}\left(\frac{1}{2}, \frac{1}{2}\right) &= -(1 \ 0) \frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^- - \frac{p_{4-}}{[p_4^+]} \sigma^- \sigma^+ \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{4-}}{[p_4^+]} \right\}.
\end{aligned} \tag{D.8}$$

We combine the above results as indicated in (D.6) and multiply it by $\frac{1}{2q_1^+ d_1}$, leaving out the δ functions and the π 's.

$$\begin{aligned}
M_1 &= \left\{ A_1 B_1 + A_{-1} B_{-1} \right\} \frac{1}{2q_1^+ d_1} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\
&= 2e^2 \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ 2 \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \frac{-p_1^+ p_3^+ \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2}.
\end{aligned} \tag{D.9}$$

We now consider the instantaneous term associated with M_1 . For our chosen helicities M_4 contributes.

$$\begin{aligned}
M_4 &= e^2 \chi_{\frac{1}{2}}^\dagger \chi_{\frac{1}{2}} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{-\frac{1}{2}}^\dagger \chi_{-\frac{1}{2}} \sqrt{2p_3^+ 2p_4^+}, \\
&= 4e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \frac{1}{[q_1^+]^2}.
\end{aligned} \tag{D.10}$$

$M_1 + M_4$ has an infrared singularity that is canceled by the contribution of the instantaneous term, but the cancelation has to be manually done. Other than that the procedure we have followed is general and will be followed for the other combinations of helicities. Both M_1 and M_4 have a minus sign originating in the anti-commutation relations.

$$M_1 + M_4 = -2e^2 \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ 2 \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \frac{-p_1^+ p_3^+ \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2} -$$

$$- 4e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \frac{1}{[q_1^+]^2}, \quad (D.11)$$

$$= e^2 \left\{ -\frac{4}{[q_1^+]^2} \left\{ \frac{-(q_{1\perp})^2 p_1^+ p_3^+}{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2} + 1 \right\} + \right.$$

$$+ \frac{-p_1^+ p_3^+}{m^2 (p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2} \frac{2q_{1+}}{[q_1^+]} \left\{ \frac{p_{3-}}{[p_3^+]} + \frac{p_{4-}}{[p_4^+]} \right\} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+},$$

$$= -2e^2 \frac{2m^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{q_1^+ p_4^+ \{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}. \quad (D.12)$$

As we can see the linear infrared singularity represented by the term in $\frac{1}{[q_1^+]^2}$ has been eliminated. There are 3 other combinations of helicities for which a singularity is possible, but in all cases it is canceled. The combination $[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$ is treated as above. A new case is given by $[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ and $[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$. In these case a similar treatment of the amplitude yields the result:

$$M_1 + M_4 = -2e^2 \frac{2m^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{q_1^+ p_4^+ \{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}. \quad (D.13)$$

We can now proceed to calculate the amplitude for M_3 .

$$M_3 = e^2 \int \frac{dq_3^+ d^2 q_{3\perp}}{(2\pi)^3 2q_3^+} \theta(q_3^+) \sum_{\lambda''} \chi_{s_3}^\dagger \Gamma^i(-p_4, p_3; q_3) \epsilon^i(\lambda'') \chi_{-s_4}$$

$$\cdot \frac{1}{d_3} \chi_{s_2}^\dagger \Gamma^j(p_1, -p_2; q_3) \epsilon^{j*}(\lambda'') \chi_{s_1} \sqrt{2p_3^+ 2p_4^+} \sqrt{2p_1^+ 2p_2^+}$$

$$\cdot (2\pi)^3 \delta(p_3^+ + p_4^+ - q_3^+) \delta^2(p_{3\perp} + p_{4\perp} - q_{3\perp})$$

$$\cdot (2\pi)^3 \delta(q_3^+ - p_1^+ - p_2^+) \delta^2(q_{3\perp} - p_{1\perp} - p_{2\perp})$$

$$\cdot (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \quad (D.14)$$

Let,

$$q = p_1^+ + p_2^+, \quad q^2 = \frac{m^2}{2x(1-x)}, \quad x = \frac{p_1^+}{q}, \quad y = \frac{p_3^+}{q}, \quad q_1^+ = q(x-y),$$

$$q_{1\perp} = m \sqrt{\frac{y(1-y)}{x(1-x)}} - 1, \quad \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} = q^2 \sqrt{x(1-x)y(1-y)}. \quad (\text{D.15})$$

The diagrams M_3 and M_5 are the same diagrams participating in the $e^+ e^- \rightarrow \mu^+ \mu^-$ scattering, so we can use the results from that example replacing m_μ by m , but we must remember that the approximation $m_\mu \gg m_e$ allowed us to drop the contribution of the instantaneous term, which we cannot do here. M_5 does not contain an infrared singularity. Next we add all the contributions for the helicities $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and convert the expression to the dimensionless variables.

$$\sum_{i=1,2,3,4,5} M_i[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = -2 \frac{e^2}{p_4^+} \left\{ \frac{2m^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{q_1^+ \{m^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} - \frac{1}{p_3^+} \right\} \cdot \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \quad (\text{D.16})$$

$$= \frac{-2\sqrt{x(1-x)y(1-y)}}{y(1-y)} \left\{ \frac{1-x-y+2xy}{(2x-1)(x-y)} - 1 \right\}. \quad (\text{D.17})$$

Each one of the entries below is to be multiplied by $4\sqrt{x(1-x)y(1-y)}$.

$$A_1 = M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = \frac{-1}{2y(1-y)} \left\{ \frac{1-x-y+2xy}{(2x-1)(x-y)} - 1 \right\},$$

$$A_2 = M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = \frac{-i}{2(1-y)} \left\{ \frac{x}{(x-y)(2x-1)} - 1 \right\}$$

$$\cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{i\varphi},$$

$$A_3 = M[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}] = \frac{i}{2y} \left\{ \frac{1-x}{(x-y)(2x-1)} - 1 \right\}$$

$$\cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{i\varphi},$$

$$A_4 = M[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}] = 0,$$

$$A_5 = M[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = \frac{-i}{2} \left\{ \frac{x^2}{y(1-y)(x-y)(2x-1)} \right\}$$

$$\cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{-i\varphi},$$

$$A_6 = M[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = -1 + \frac{1 + \frac{(1-x-y)^2}{2y(1-y)}}{(x-y)(2x-1)},$$

$$\begin{aligned}
A_7 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -1 - \frac{(x-y)}{2y(1-y)(2x-1)}, \\
A_8 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = \frac{-i}{2} \frac{(1-x)^2}{y(1-y)(x-y)(2x-1)} \\
&\quad \cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{i\varphi}, \\
A_9 &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = \frac{-i}{2} \frac{(1-x)^2}{y(1-y)(x-y)(2x-1)} \\
&\quad \cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{-i\varphi}, \\
A_{10} &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = -1 - \frac{(x-y)}{2y(1-y)(2x-1)}, \\
A_{11} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -1 + \frac{1 + \frac{(1-x-y)^2}{2y(1-y)}}{(x-y)(2x-1)}, \\
A_{12} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = \frac{-i}{2} \left\{ \frac{x^2}{y(1-y)(x-y)(2x-1)} \right\} \\
&\quad \cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{i\varphi}, \\
A_{13} &= M\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = 0, \\
A_{14} &= M\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = \frac{i}{2y} \left\{ \frac{1-x}{(x-y)(2x-1)} - 1 \right\} \\
&\quad \cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{-i\varphi}, \\
A_{15} &= M\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = \frac{-i}{2(1-y)} \left\{ \frac{x}{(x-y)(2x-1)} - 1 \right\} \\
&\quad \cdot \sqrt{\frac{(x-y)(x+y-1)}{x(1-x)}} e^{-i\varphi}, \\
A_{16} &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = \frac{-1}{2y(1-y)} \left\{ \frac{1-x-y+2xy}{(2x-1)(x-y)} - 1 \right\}.
\end{aligned}$$

Next we calculate the sum of the square of the modulus of the above amplitudes.

$$N^2 = \sum_{i=1, \dots, 16} |A_i|^2,$$

$$\begin{aligned}
= & 16 e^4 \left\{ \frac{(2x-1)^2}{2x(x-1)} + \frac{1-6x+7x^2+18x^3-39x^4+24x^5-4x^6}{2(x-1)^3x(2x-1)^2(y-1)} + \right. \\
& + \frac{-1+2x-7x^2+22x^3-21x^4+4x^6}{2(x-1)x^3(2x-1)^2y} + \frac{1-4x+8x^2-8x^3+4x^4}{2(x-1)^2x^2(2x-1)^2(y-x)^2} + \\
& \left. + \frac{-1+2x-6x^2+12x^3-16x^4+12x^5-4x^6}{2(x-1)^3x^3(2x-1)(y-x)} \right\} x(1-x)y(1-y). \quad (D.18)
\end{aligned}$$

As in the previous examples this expression is to be multiplied by

$$(2\pi)^4 \delta^4(p_1 + p_2 + P_3 + p_4) (2\pi)^4 \delta^4(0). \quad (D.19)$$

Again we want the volume, time and helicity average of N^2 . Let $VT = (2\pi)^4 \delta^4(0)$ with

$$\frac{1}{4} \frac{N^2}{VT} = |M|^2 (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2). \quad (D.20)$$

Appendix E

$|M|^2$ for $e\mu \rightarrow e\mu$.

As in the previous example, let

$$\begin{aligned}
F_1(\lambda) &= \Gamma^i(p_2, p_4; q_1) \cdot \epsilon^i(\lambda), \\
&= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_2 + im_\mu}{[p_2^+]} + \frac{\sigma \cdot p_4 - im_\mu}{[p_4^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\} \cdot \epsilon^i(\lambda), \\
F_2(\lambda) &= \Gamma^j(p_1, p_3; q_1) \cdot \epsilon^{*j}(\lambda), \\
&= \frac{1}{2} \left\{ \sigma^j \frac{\sigma \cdot p_1 + im_e}{[p_1^+]} + \frac{\sigma \cdot p_3 - im_e}{[p_3^+]} \sigma^j - \frac{2q_1^j}{[q_1^+]} \right\} \cdot \epsilon^{*j}(\lambda), \\
F_1(1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im_\mu \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^- - \frac{p_{4-}}{[p_4^+]} \sigma^+ \sigma^- \right\}, \\
F_1(-1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im_\mu \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^+ - \frac{p_{4+}}{[p_4^+]} \sigma^- \sigma^+ \right\}, \\
F_2(1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im_e \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^- - \frac{p_{3-}}{[p_3^+]} \sigma^+ \sigma^- \right\}, \\
F_2(-1) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im_e \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^+ - \frac{p_{3+}}{[p_3^+]} \sigma^- \sigma^+ \right\}. \tag{E.1}
\end{aligned}$$

Let $p_{1\perp} = p_{2\perp} = 0$. Then

$$q_1^+ = p_1^+ - p_3^+, \quad q_2^+ = p_3^+ - p_1^+ = -q_1^+. \tag{E.2}$$

$$\begin{aligned}
M_1 = & e^2 \int \frac{dq_1^+ d^2 q_{1\perp}}{(2\pi)^3 2 q_1^+} \theta(q_1^+) \sum_{\lambda} \chi_{s_4}^\dagger \Gamma^j(p_2, p_4; q_1) \epsilon^j(\lambda) \chi_{s_2} \\
& \cdot \frac{1}{d_1} \sqrt{2 p_1^+ 2 p_3^+} \sqrt{2 p_2^+ 2 p_4^+} \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_1) \epsilon^{i*}(\lambda) \chi_{s_1} \\
& \cdot (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \delta^2(q_{1\perp} - p_{4\perp} + p_{2\perp}) \\
& \cdot (2\pi)^3 \delta(q_1^+ - p_4^+ + p_2^+) \delta^2(q_{1\perp} - p_{4\perp} + p_{2\perp}) \\
& \cdot (-2\pi i) \delta(p_1^- + p_2^- - p_3^- - p_4^-), \tag{E.3}
\end{aligned}$$

$$\begin{aligned}
M_3 = & e^2 \chi_{s_3}^\dagger \chi_{s_1} \sqrt{2 p_1^+ 2 p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{s_4}^\dagger \chi_{s_2} \sqrt{2 p_3^+ 2 p_4^+} \\
& \cdot (2\pi)^3 (-2\pi i) \delta^4(p_3 + p_4 - p_1 - p_2), \tag{E.4}
\end{aligned}$$

$$d_1 = \frac{1}{p_1^- + p_2^- - (p_3^- + p_2^- + q_1^-)} \tag{E.5}$$

$$\frac{1}{2q_1^+} \frac{1}{d_1} = \frac{-p_1^+ p_3^+}{m_e^2 (p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2}. \tag{E.6}$$

Let

$$\begin{aligned}
& \sum_{\lambda} \chi_{s_4}^\dagger \Gamma^j(p_2, p_4; q_1) \epsilon^j(\lambda) \chi_{s_2} \chi_{s_3}^\dagger \Gamma^i(p_1, p_3; q_1) \epsilon^{i*}(\lambda) \chi_{s_1} \\
& = \chi_{s_4}^\dagger F_1(1) \chi_{s_2} \chi_{s_3}^\dagger F_2(1) \chi_{s_1} + \chi_{s_4}^\dagger F_1(-1) \chi_{s_2} \chi_{s_3}^\dagger F_2(-1) \chi_{s_1}, \\
& = A_1(s_1, s_3) B_1(s_2, s_4) + A_{-1}(s_1, s_3) B_{-1}(s_2, s_4), \tag{E.7}
\end{aligned}$$

where we define the A 's and B 's as follows:

$$\begin{aligned}
A_1(s_1, s_3) &= -\chi_{s_3}^\dagger \frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im_e \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^- - \frac{p_{3-}}{[p_3^+]} \sigma^+ \sigma^- \right\} \chi_{s_1}, \\
A_{-1}(s_1, s_3) &= -\chi_{s_3}^\dagger \frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im_e \left(\frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right) \sigma^+ - \frac{p_{3+}}{[p_3^+]} \sigma^- \sigma^+ \right\} \chi_{s_1}, \\
B_1(s_2, s_4) &= -\chi_{s_4}^\dagger \frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - im_\mu \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^+ - \frac{p_{4+}}{[p_4^+]} \sigma^+ \sigma^- \right\} \chi_{s_2}, \\
B_{-1}(s_2, s_4) &= -\chi_{s_4}^\dagger \frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - im_\mu \left(\frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right) \sigma^- - \frac{p_{4-}}{[p_4^+]} \sigma^- \sigma^+ \right\} \chi_{s_2}.
\end{aligned}$$

We combine the above results as indicated in (E.7) and multiply it by $\frac{\sqrt{2 p_1^+ 2 p_2^+ 2 p_3^+ 2 p_4^+}}{2 q_1^+ d_1}$, leaving out the δ functions and the π 's.

$$A_1\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} \right\},$$

$$\begin{aligned}
A_1\left(\frac{1}{2}, -\frac{1}{2}\right) &= \frac{1}{\sqrt{2}}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\}, \\
A_1\left(-\frac{1}{2}, \frac{1}{2}\right) &= 0, \\
A_1\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} \right\}, \\
A_{-1}\left(\frac{1}{2}, \frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} \right\}, \\
A_{-1}\left(\frac{1}{2}, -\frac{1}{2}\right) &= 0, \\
A_{-1}\left(-\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{\sqrt{2}}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\}, \\
A_{-1}\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{3+}}{[p_3^+]} \right\}, \\
B_1\left(\frac{1}{2}, \frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} \right\}, \\
B_1\left(\frac{1}{2}, -\frac{1}{2}\right) &= 0, \\
B_1\left(-\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{\sqrt{2}}im_\mu \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
B_1\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{4+}}{[p_4^+]} \right\}, \\
B_{-1}\left(\frac{1}{2}, \frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{4-}}{[p_4^+]} \right\}, \\
B_{-1}\left(\frac{1}{2}, -\frac{1}{2}\right) &= \frac{1}{\sqrt{2}}im_\mu \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
B_{-1}\left(-\frac{1}{2}, \frac{1}{2}\right) &= 0, \\
B_{-1}\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -\frac{1}{\sqrt{2}} \left\{ \frac{q_{1-}}{[q_1^+]} \right\}.
\end{aligned}$$

By combining these expressions in a suitable manner we obtain the amplitudes for the different helicity eigenstates,

$$\begin{aligned}
A_i &= M[s_1, s_3, s_2, s_4], \\
A_1 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} \right\} + \frac{1}{2} \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{4-}}{[p_4^+]} \right\}, \\
&= \frac{1}{2} \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ 2 \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} - \frac{p_{4-}}{[p_4^+]} \right\},
\end{aligned}$$

$$\begin{aligned}
A_2 &= M\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = -\frac{1}{2}im_\mu \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_3 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{2}im_\mu \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_4 &= M\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = \frac{1}{2} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{4+}}{[p_4^+]} \right\} + \\
&\quad + \frac{1}{2} \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} \right\}, \\
A_5 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{2}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} \right\}, \\
A_6 &= M\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = 0, \\
A_7 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{2}m_e m_\mu \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_8 &= M\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = -\frac{1}{2}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{4+}}{[p_4^+]} \right\}, \\
A_9 &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{2}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{4-}}{[p_4^+]} \right\}, \\
A_{10} &= M\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = -\frac{1}{2}m_e m_\mu \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_{11} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = 0, \\
A_{12} &= M\left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = -\frac{1}{2}im_e \left\{ \frac{1}{[p_1^+]} - \frac{1}{[p_3^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} \right\}, \\
A_{13} &= M\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2} \left\{ \frac{q_{1-}}{[q_1^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} \right\} + \\
&\quad + \frac{1}{2} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} - \frac{p_{4-}}{[p_4^+]} \right\}, \\
A_{14} &= M\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] = -\frac{1}{2}im_\mu \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_{15} &= M\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{2}im_\mu \left\{ \frac{q_{1-}}{[q_1^+]} \right\} \left\{ \frac{1}{[p_2^+]} - \frac{1}{[p_4^+]} \right\}, \\
A_{16} &= M\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] = \frac{1}{2} \left\{ \frac{q_{1-}}{[q_1^+]} \right\} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{4+}}{[p_4^+]} \right\} + \\
&\quad + \frac{1}{2} \left\{ \frac{q_{1+}}{[q_1^+]} - \frac{p_{3+}}{[p_3^+]} \right\} \left\{ \frac{q_{1-}}{[q_1^+]} \right\}, \\
&= \frac{1}{2} \left\{ \frac{q_{1-}}{[q_1^+]} \right\} \left\{ 2 \frac{q_{1+}}{[q_1^+]} - \frac{p_{3+}}{[p_3^+]} - \frac{p_{4+}}{[p_4^+]} \right\}.
\end{aligned}$$

When we square the modulus of the amplitudes, we will take advantage of the symmetries present. We now multiply the above expressions (so they become amplitudes) by

$$\frac{e^2}{2q_1^+} \frac{\sqrt{2p_1^+ 2p_2^+ 2p_3^+ 2p_4^+}}{d_1} = \frac{-e^2 p_1^+ p_3^+ \sqrt{2p_1^+ 2p_2^+ 2p_3^+ 2p_4^+}}{m_e^2 (p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2} = \frac{-e^2 p_1^+ p_3^+ \sqrt{2p_1^+ 2p_2^+ 2p_3^+ 2p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2}. \quad (\text{E.8})$$

We then must consider the four cases for which there is the possibility of a linear infrared singularity. These cases are:

$$M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], \quad M[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}], \quad M[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}], \quad M[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}].$$

The instantaneous term M_3 contributes only for these four combinations of helicities, for the rest its contribution is zero. First we consider $M[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned} M_3 &= e^2 \chi_{\frac{1}{2}}^\dagger \chi_{\frac{1}{2}} \sqrt{2p_1^+ 2p_2^+} \frac{1}{[p_1^+ - p_3^+]^2} \chi_{-\frac{1}{2}}^\dagger \chi_{-\frac{1}{2}} \sqrt{2p_3^+ 2p_4^+}, \\ &= 4e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \frac{1}{[q_1^+]^2}. \end{aligned} \quad (\text{E.9})$$

$M_1 + M_3$ has an infrared singularity that is canceled by the contribution of the instantaneous term, but the cancelation has to be manually done. Other than that the procedure we have followed is general and will be followed for the other combinations of helicities. Both M_1 and M_3 have a minus sign originating in the anti-commutation relations.

$$\begin{aligned} M_1 + M_3 &= -2e^2 \left\{ \frac{q_{1+}}{[q_1^+]} \right\} \left\{ 2 \frac{q_{1-}}{[q_1^+]} - \frac{p_{3-}}{[p_3^+]} - \frac{p_{4-}}{[p_4^+]} \right\} \frac{-p_1^+ p_3^+ \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2} - \\ &\quad - 4e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \frac{1}{[q_1^+]^2}, \end{aligned} \quad (\text{E.10})$$

$$\begin{aligned} &= e^2 \left\{ -\frac{4}{[q_1^+]^2} \left\{ \frac{-(q_{1\perp})^2 p_1^+ p_3^+}{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2} + 1 \right\} + \right. \\ &\quad \left. + \frac{-p_1^+ p_3^+}{m_e^2 (p_1^+ - p_3^+)^2 + (p_{3\perp})^2 (p_1^+)^2} \frac{2q_{1+}}{[q_1^+]} \left\{ \frac{p_{3-}}{[p_3^+]} + \frac{p_{4-}}{[p_4^+]} \right\} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \right\} \\ &= -2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}. \end{aligned} \quad (\text{E.11})$$

As we can see the linear infrared singularity represented by the term in $\frac{1}{[q_1^+]^2}$ has been eliminated. There are 3 other combinations of helicities for which a singularity is

possible, but in all cases it is canceled. The combination $[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$ is treated as above. A new case is given by $[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ and $[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$. In these case a similar treatment of the amplitude yields the result:

$$M_1 + M_3 = -2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_1^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}. \quad (\text{E.12})$$

Thus the amplitudes are:

$$\begin{aligned} A_1 &= 2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_1^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\ A_2 &= -2im_\mu \frac{p_1^+ p_3^+ q_{1\perp}}{p_2^+ p_4^+} \frac{e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2} e^{i\phi}, \\ A_3 &= 2im_\mu (-q_{1\perp}) \frac{(p_1^+)^2 e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{p_2^+ p_4^+ \{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}} e^{-i\phi}, \\ A_4 &= -2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_1^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\ A_5 &= 2im_e \frac{q_{1\perp} e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2} e^{i\phi}, \\ A_6 &= 0, \\ A_7 &= 2m_e m_\mu \frac{q_1^+ e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{p_2^+ p_4^+ \{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}}, \\ A_8 &= -2im_e \frac{q_{1\perp}}{p_4^+} \frac{p_2^+ e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{\{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}} e^{i\phi}, \\ A_9 &= -2im_e \frac{q_{1\perp}}{p_4^+} \frac{p_2^+ e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{\{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}} e^{-i\phi}, \\ A_{10} &= 2m_e m_\mu \frac{q_1^+ e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{p_2^+ p_4^+ \{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}}, \\ A_{11} &= 0, \\ A_{12} &= 2im_e \frac{q_{1\perp} e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2} e^{-i\phi}, \\ A_{13} &= -2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_1^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}, \\ A_{14} &= 2im_\mu (-q_{1\perp}) \frac{(p_1^+)^2 e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{p_2^+ p_4^+ \{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2\}} e^{i\phi}, \end{aligned}$$

$$\begin{aligned}
A_{15} &= -2im_\mu \frac{p_1^+ p_3^+ q_{1\perp}}{p_2^+ p_4^+} \frac{e^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2} e^{-i\phi}, \\
A_{16} &= 2e^2 \frac{2m_e^2 q_1^+ p_4^+ + (q_{1\perp})^2 p_1^+ (p_3^+ + p_4^+)}{q_1^+ p_4^+ \{m_e^2 (q_1^+)^2 + (p_{3\perp})^2 (p_1^+)^2\}} \sqrt{p_1^+ p_2^+ p_3^+ p_4^+}.
\end{aligned}$$

We now convert these expressions to the dimensionless variables, square the modulus of each amplitude and sum them. Since there is an obvious symmetry we convert only expressions A_1 to A_8 .

$$\begin{aligned}
|M|^2 &= \frac{1}{4} \sum_i |A_i|^2, \\
&= e^4 \frac{2x(-1+y)}{(m_e^2(-1+x)^2 - m_\mu^2 x^2)^2 (x-y)^2 y} \\
&\quad \cdot \left\{ \frac{m_e^2(-1+x)^2 (-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x} + \right. \\
&\quad + \frac{(-1+x)(m_\mu^2 xy + m_e^2(-1+x)(-1+y))^2}{(-1+y)^2} + \\
&\quad + \frac{m_e^2(-1+x)^4 (-m_\mu^2 xy + m_e^2(-1+x)(-1+y))(x-y)}{x(-1+y)^2} + \\
&\quad + \frac{m_e^2 m_\mu^2 (-1+x)(x-y)^4}{(-1+y)^2} + \\
&\quad + \frac{(-1+x)(m_e^2(-1+x)(-1+x-y)(-1+y) + m_\mu^2 xy(-1-x+y))^2}{(-1+y)^2} + \\
&\quad - \frac{m_\mu^2 x^3 (x-y)(m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} + \\
&\quad \left. - \frac{m_\mu^2 x(x-y)y^2 (m_\mu^2 xy + m_e^2(-1+x+y-xy))}{(-1+y)^2} \right\}. \tag{E.13}
\end{aligned}$$

We also convert the following expressions to the dimensionless variables x and y for their use in further calculations:

$$\begin{aligned}
p_{3\perp}^2 &= m_e^2 \frac{p_4^+ (p_3^+ - p_1^+)}{p_1^+ (p_3^+ + p_4^+)} + m_\mu^2 \frac{p_3^+ (p_4^+ - p_2^+)}{p_2^+ (p_3^+ + p_4^+)}, \\
&= \frac{(x-y)}{x(1-x)} \{ m_\mu^2 xy - m_e^2 (1-x)(1-y) \}, \tag{E.14}
\end{aligned}$$

$$m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2 = \frac{y(x-y)}{1-x} Q^2 \{ m_\mu^2 x^2 - m_e^2 (1-x)^2 \}, \tag{E.15}$$

$$\frac{\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}}{m_e^2 (q_1^+)^2 + (q_{1\perp})^2 (p_1^+)^2} = \frac{(1-x) \sqrt{x(1-x)y(1-y)}}{y(x-y) \{ m_\mu^2 x^2 - m_e^2 (1-x)^2 \}}. \tag{E.16}$$

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