SAPIENZA
UNIVERSITÀ DI ROMA

Carleton
UNIVERSITY

Department of Mechanical and Aerospace Engineering

Cotutelle Doctoral Program in
Tecnologia Aeronautica e Spaziale, XXVIII Ciclo
and
Aerospace Engineering

On the use of Frequency Response Functions in the finite element model updating

Candidate:
Melissa Arras

Advisors:
Prof. G. Coppotelli
Prof. F. Nitzsche

May 2016

©2016 - *Melissa Arras*

All rights reserved.

Abstract

Finite element model updating methodologies allow the identification of structures by improving the accuracy of an initial finite element model considering data acquired from a vibration test. Usually, numerical predictions and experimental results disagree thus requiring an updated model that best reproduces the dynamic behavior of the actual structure. Several updating methodologies have been developed in the last years such as the *modal-based* and *response-based* ones, which minimize the difference between the measured and predicted modal parameters and Frequency Response Functions, respectively. This thesis investigates a response-based method which iteratively minimizes a residual vector defined on the correlation functions between the Frequency Response Functions, which are directly available from experimental tests, to identify the mismodeled regions of a numerical model and improve the correlation with its experimental counterpart. The minimization problem is generally solved by means of the weighted least-squares approach and starting from a reference formulation available in literature, some enhancements are proposed, validated numerically, and extended to real experimental data using simple structures such as beams and plates. Finally, the proposed

methodology is applied to identify the physical properties of a reduced-scale helicopter blade model that incorporates a semi-active device for vibration control, named Smart Spring. As a further study, the effectiveness of this device to reduce vibration is analytically verified through the development of its mathematical model in the frequency domain that makes use of an open-loop control law. Moreover, the capability of the Smart Spring to modulate the structural properties of a system and alter its dynamic behavior, thus acting as a semi-active device, is experimentally assessed through vibration tests performed at different operative working conditions of the device.

Acknowledgments

I wish to express my deep gratitude to my research supervisors, Prof. Giuliano Coppotelli and Prof. Fred Nietzsche, for their encouragement, guidance, valuable support, and useful recommendations on my work. I would also like to thank Prof. Pier Marzocca who gave me the opportunity to work in his laboratory and Prof. Daniel Feszty for his willingness to welcome me to the Rotorcraft Research Laboratory at Carleton University. I also thank the institutions that supported the research, specifically the University of Rome La Sapienza, through the funds for young researchers, Carleton University, through a Research Assistantship, and Zonta International, through the Amelia Earhart Fellowship program.

Special thanks are due to all my friends and colleagues with whom I had the opportunity to work during these years. I am grateful to my family for the sustained support and presence, despite the distance. Finally, from the bottom of my heart I thank Mirco who always encourages me to never give up.

Contents

List of Abbreviations	V
List of Figures	XIV
List of Tables	XVIII
Introduction	1
I Theoretical Background	6
1 Background on structural dynamics	7
1.1 FE numerical modelling	7
1.1.1 Spatial model	9
1.1.2 Modal model	10
1.1.3 Response model	13
1.1.4 Routes to vibration analysis	15
1.2 Experimental modal analysis	17
1.2.1 Description of the test setup	18
1.2.2 Digital signal processing	20

1.2.3	Derivation of Frequency Response Functions	22
1.2.4	Estimate of the modal model	24
1.3	Numerical-experimental correlation	27
2	State-of-the-art in model updating	31
2.1	Direct and iterative model updating methods	31
2.1.1	Direct methods	32
2.1.2	Iterative methods	34
2.2	Modal-based methods	36
2.3	Response-based methods	37
2.4	The Predictor-Corrector method	40
2.4.1	Choice of weighting matrices	44
II	Theoretical Developments, Numerical and Experimental Validation	46
3	The enhanced Predictor-Corrector model updating technique	47
3.1	Weighting matrices	47
3.1.1	Weighting matrix for frequency points, $[W_f]$	48
3.1.2	Weighting matrix for parameters, $[W_p]$	50
3.2	Choice of regularization parameter	52
3.2.1	Generalized Singular Value Decomposition	54
3.2.2	L-curve criterion	56
3.3	Robustness of the algorithm in case of noisy data	58
3.4	Summary of the enhanced methodology	63

3.5	Implementation of the enhanced methodology	65
4	Numerical test cases	67
4.1	The cantilever beam	68
4.1.1	Case 1: -35% stiffness of first element	71
4.1.2	Case 2: $+25\%$ mass of sixth element	77
4.1.3	Case 3: variation of all stiffness parameters	84
4.2	Discussion of results	88
5	Experimental validation on simple structures	91
5.1	Updating of a cantilever beam FE model	92
5.1.1	Experimental setup	92
5.1.2	Numerical model	94
5.1.3	Numerical-experimental correlation	95
5.1.4	Identification of the stiffness distribution of the beam	97
5.2	Updating of a cantilever plate FE model	106
5.2.1	Experimental setup	106
5.2.2	Numerical model	107
5.2.3	Numerical-experimental correlation	109
5.2.4	Identification of the stiffness distribution of the plate	110
5.3	Discussion of results	115
III	Application to Rotorcraft Technology	117
6	Introduction to vibration control	118

7 Analytical validation of the Smart Spring	122
7.1 The analytical model of the Smart Spring	124
7.1.1 Parametric study of the system response	132
7.2 The base-excited concept of the Smart Spring	141
7.2.1 The analytical model of the base-excited Smart Spring	142
7.2.2 Parametric study of the system response	147
8 Experimental investigation of the Active Pitch Link	151
8.1 Experimental setup	152
8.2 Estimate of the modal parameters	155
8.3 Effect of the Active Pitch Link	159
9 Identification of the blade-APL system	163
9.1 Numerical model	164
9.2 Numerical-experimental correlation	165
9.3 Identification of the system	167
9.4 Discussion of results	174
Conclusions and Suggestions for further work	176
Bibliography	180

List of Abbreviations

APL	Active Pitch Link
ARMA	Auto-Regressive Moving-Average
COMAC	Coordinate Modal Assurance Criterion
DFT	Discrete Fourier Transform
DOF	Degree of Freedom
EMA	Experimental Modal Analysis
ERA	Eigensystem Realization Algorithm
FE	Finite Element
FEA	Finite Element Analysis
FRAC	Frequency Response Assurance Criterion
FRF	Frequency Response Function
GSVD	Generalised Singular Value Decomposition
HHC	Higher Harmonic Control
IBC	Individual Blade COntrol
LSCE	Least Squares Complex Exponential
LSFD	Least Squares Frequency Domain
MAC	Modal Assurance Criterion
MDOF	Multi Degree of Freedom
MIMO	Multiple Input Multiple Output
MLE	Maximum Likelihood Estimator
OMA	Operational Modal Analysis
P-C	Predictor-Corrector
PSD	Power Spectral Density
SDOF	Single Degree of Freedom
SHARCS	Smart Hybrid Active Rotor Control System
SIMO	Single Input Multiple Output
SVD	Singular Value Decomposition

List of Figures

1.1	Block diagram of the Frequency Response Function.	14
1.2	Comparison between theoretical and experimental approaches in vibration analysis, Ref.[4].	17
1.3	Example of leakage, Ref. [4].	21
1.4	Example of window functions, Ref. [4].	22
3.1	Typical L-curve with emphasis on the point of minimum distance from the origin.	58
3.2	Typical curve for the selection of η^{opt}	63
3.3	Flowchart of the enhanced methodology.	64
4.1	Nodes numbering of the numerical cantilever beam.	68

4.2	Numerical cantilever beam case 1: comparison between the measured and predicted natural frequencies.	73
4.3	Numerical cantilever beam case 1: initial correlation between the experimental and numerical mode sets.	74
4.4	Numerical cantilever beam case 1: comparison between the shape correlation functions χ_s	75
4.5	Numerical cantilever beam case 1: comparison between the amplitude correlation functions χ_a	75
4.6	Numerical cantilever beam case 1: comparison between the $H_{2,4}$ FRF. . .	76
4.7	Numerical cantilever beam case 1: comparison between the driving point FRF.	76
4.8	Numerical cantilever beam case 1: comparison between the $H_{6,4}$ FRF. . .	77
4.9	Numerical cantilever beam case 1: comparison between the $H_{8,4}$ FRF. . .	77
4.10	Numerical cantilever beam case 1: comparison between the natural frequency shifts.	78
4.11	Numerical cantilever beam case 1: convergence history of the updating parameters.	78

4.12 Numerical cantilever beam case 2: comparison between the measured and predicted natural frequencies.	80
4.13 Numerical cantilever beam case 2: comparison between the shape corre- lation functions χ_s	81
4.14 Numerical cantilever beam case 2: comparison between the amplitude correlation functions χ_a	81
4.15 Numerical cantilever beam case 2: comparison between the driving point FRF.	82
4.16 Numerical cantilever beam case 2: comparison between the natural fre- quency shifts.	82
4.17 Numerical cantilever beam case 2: convergence history of the updating parameters.	83
4.18 Numerical cantilever beam case 3: comparison between the measured and predicted natural frequencies.	85
4.19 Numerical cantilever beam case 3: comparison between the shape corre- lation functions χ_s	85
4.20 Numerical cantilever beam case 3: comparison between the amplitude correlation functions χ_a	86

4.21 Numerical cantilever beam case 3: comparison between the driving point FRF.	86
4.22 Numerical cantilever beam case 3: comparison between the natural frequency shifts.	87
4.23 Numerical cantilever beam case 3: convergence history of the updating parameters.	87
5.1 Experimental setup for the cantilever beam.	93
5.2 Nodes numbering of the experimental cantilever beam.	94
5.3 Experimental cantilever beam: comparison between the measured and predicted natural frequencies.	96
5.4 Experimental cantilever beam: initial correlation between the experimental and numerical mode sets.	96
5.5 Experimental cantilever beam: comparison between the shape correlation functions χ_s	98
5.6 Experimental cantilever beam: comparison between the amplitude correlation functions χ_a	98
5.7 Experimental cantilever beam: comparison between the driving point FRF.	99

5.8	Experimental cantilever beam: comparison between the $H_{5,3}$ FRF.	100
5.9	Experimental cantilever beam: comparison between the $H_{7,3}$ FRF.	101
5.10	Experimental cantilever beam: comparison between the $H_{9,3}$ FRF.	102
5.11	Experimental cantilever beam: comparison between the $H_{11,3}$ FRF.	103
5.12	Experimental cantilever beam: comparison between the natural frequency shifts.	104
5.13	Experimental cantilever beam: comparison between the final MAC.	104
5.14	Experimental cantilever beam: convergence history of the updating pa- rameters.	105
5.15	Experimental setup for the cantilever plate.	107
5.16	Nodes numbering of the experimental cantilever plate.	108
5.17	Experimental cantilever plate: comparison between the measured and pre- dicted natural frequencies.	109
5.18	Experimental cantilever plate: initial correlation between the experimen- tal and numerical mode sets.	110
5.19	Experimental cantilever plate: comparison between the shape correlation functions χ_s	111

5.20	Experimental cantilever plate: comparison between the amplitude correlation functions χ_a .	111
5.21	Experimental cantilever plate: comparison between the driving point FRF.	112
5.22	Experimental cantilever plate: comparison between the natural frequency shifts.	112
5.23	Experimental cantilever plate: comparison between the final MAC.	113
5.24	Experimental cantilever plate: convergence history of the updating parameters.	113
7.1	Conceptual drawing of the Smart Spring, Ref. [86].	123
7.2	Smart Spring control law.	126
7.3	Parametric study for the 1/rev harmonic component.	133
7.4	Parametric study for the 2/rev harmonic component.	134
7.5	Parametric study for the 3/rev harmonic component.	135
7.6	Parametric study for the 4/rev harmonic component.	136
7.7	Parametric study for the 5/rev harmonic component.	137
7.8	Parametric study for the 6/rev harmonic component.	138

7.9	Parametric study for the 7/rev harmonic component.	139
7.10	Parametric study for the 8/rev harmonic component.	140
7.11	New conceptual drawing of the Smart Spring.	142
7.12	Parametric study for the 1/rev harmonic component, base-excited system.	147
7.13	Parametric study for the 2/rev harmonic component, base-excited system.	148
7.14	Parametric study for the 3/rev harmonic component, base-excited system.	148
7.15	Parametric study for the 4/rev harmonic component, base-excited system.	149
8.1	The Active Pitch Link (APL).	152
8.2	Experimental setup of the non-rotating tests.	153
8.3	Ballast system to keep the blade in the horizontal position.	154
8.4	Particular of sensors positioning.	154
8.5	Mode shapes of the blade-APL system in its reference configuration. . . .	158
8.6	MAC between the mode shapes estimated for the different configurations of the APL.	160
8.7	Effect of the APL stiffness on the torsional mode shape.	161
8.8	Out-of-phase internal mode of the APL.	162

9.1	Finite element model of the blade-APL system.	165
9.2	Blade-APL system: comparison between the measured and predicted natural frequencies.	166
9.3	Blade-APL system: initial correlation between the experimental and numerical mode sets.	166
9.4	Blade-APL system: comparison between the shape correlation functions χ_s	167
9.5	Blade-APL system: comparison between the amplitude correlation functions χ_a	168
9.6	Blade-APL system: comparison between the driving point FRF.	168
9.7	Blade-APL system: comparison between the natural frequency shifts.	169
9.8	Blade-APL system: comparison between the final MAC.	169
9.9	Blade-APL system: convergence history of the updating parameter corresponding to density of the blade.	170
9.10	Blade-APL system: convergence history of the updating parameter corresponding to thickness of the blade-hub connection.	170
9.11	Blade-APL system: convergence history of the updating parameter corresponding to Young's modulus of the blade-hub connection.	171

9.12 Blade-APL system: convergence history of the updating parameter corresponding to mass of the accelerometers. 172

9.13 Blade-APL system: convergence history of the updating parameter corresponding to stiffness of the APL. 173

List of Tables

4.1	Physical properties of the numerical cantilever beam.	68
4.2	Settings of the algorithm used for the numerical cantilever beam.	69
4.3	Numerical cantilever beam case 1: target values of the updating parameters.	71
4.4	Numerical cantilever beam case 1: initial natural frequency shifts.	72
4.5	Numerical cantilever beam case 1: final natural frequency shifts.	73
4.6	Numerical cantilever beam case 1: comparison between the final updating parameters.	79
4.7	Numerical cantilever beam case 2: target values of the updating parameters.	80
4.8	Numerical cantilever beam case 2: initial natural frequency shifts.	80
4.9	Numerical cantilever beam case 2: final natural frequency shifts.	83

4.10 Numerical cantilever beam case 2: comparison between the final updating parameters.	83
4.11 Numerical cantilever beam case 3: target values of the updating parameters.	84
4.12 Numerical cantilever beam case 3: initial natural frequency shifts.	84
4.13 Numerical cantilever beam case 3: final natural frequency shifts.	88
4.14 Numerical cantilever beam case 3: comparison between the final updating parameters.	88
5.1 Physical properties of the experimental cantilever beam.	93
5.2 Experimental cantilever beam: settings of the acquisition parameters. . .	93
5.3 Modal parameters of the experimental cantilever beam.	93
5.4 Numerical modal parameters of the experimental cantilever beam.	95
5.5 Experimental cantilever beam: initial natural frequency shifts.	95
5.6 Experimental cantilever beam: final natural frequency shifts.	103
5.7 Experimental cantilever beam: comparison between the final updating parameters.	105
5.8 Physical properties of the experimental cantilever plate.	107

5.9	Experimental cantilever plate: settings of the acquisition parameters. . .	107
5.10	Modal parameters of the experimental cantilever plate.	107
5.11	Numerical modal parameters of the experimental cantilever plate. . . .	108
5.12	Experimental cantilever plate: initial natural frequency shifts.	109
5.13	Experimental cantilever plate: final natural frequency shifts.	115
5.14	Experimental cantilever plate: comparison between the final updating parameters.	115
8.1	Settings of the acquisition parameters for the blade-APL system.	153
8.2	Modal parameters of the system with the APL in “hard” configuration. .	156
8.3	Modal parameters of the system with the APL in “intermediate” configu- ration.	156
8.4	Modal parameters of the system with the APL in “soft” configuration. . .	157
9.1	Physical properties of the blade.	164
9.2	Physical properties of the blade-hub connection.	165
9.3	Numerical modal parameters of the blade-APL system.	165
9.4	Blade-APL system: initial natural frequency shifts.	166

9.5 Blade-APL system: final natural frequency shifts. 174

9.6 Blade-APL system: comparison between the initial and final identified
system properties. 174

Introduction

Nowadays, having reliable dynamic mathematical models is essential for the design and analysis of complex structures whose operation, integrity, safety, and control depend on their dynamic characteristics. For this purpose, there is a real interest to refine the predicted responses in light of the ones measured on the actual structure. Such activity goes by the name of *model updating*, which is considered one of the most demanding application in the field of structural dynamics.

The dynamic behavior of a structure can be studied through two distinct activities, namely numerical modelling and vibration tests. The finite element (FE) method is a numerical analysis technique for obtaining approximate solutions to a wide variety of engineering problems that are either difficult or impossible to solve by theoretical methods. Refs. [1, 2, 3] provide a comprehensive introduction to the subject. The numerical model deriving from FE modelling consists of $N \times N$ mass and stiffness matrices, being N the number of degrees of freedom (DOFs), from which the predicted modal parameters, i.e. natural frequencies and mode shapes, and Frequency Response Functions (FRFs) of a structure can be estimated. Since the numerical modelling tries to reproduce the

actual dynamic behavior of structures, modal testing has become the classical procedure to validate an existing FE model. According to Ref. [4], the most popular application of modal testing is to provide a direct comparison between the predictions of specific dynamic properties and the corresponding ones actually observed, thus allowing the validation of the numerical model. Specifically, such a comparison can be made if the corresponding data sets are in the same format, that is in terms of modal parameters or FRFs. The degree of similarity between the two models is quantified through the introduction of specific correlation coefficients. Examples are the Modal Assurance Criterion (MAC), Ref. [5], the Coordinate Modal Assurance Criterion (COMAC), Ref. [6], and the Frequency Response Assurance Criterion (FRAC), Ref. [7]. If the correlation is not satisfactory, that is considerable discrepancies between the two dynamic models exist, the numerical model needs adjustments or modifications by means of the model updating methodologies. After this process, the updated model describes the dynamic properties of the actual structure more accurately and it can be used for further analyses, such as to assess structural modifications, optimize the production processes, and predict alternative loading arrangements. Several updating methodologies, based on either modal parameters or FRFs, have been introduced over the past years, Ref. [8]. Research in this field is now focusing on the improvement of algorithms that use FRFs because they are directly available from modal tests, whereas the experimental modal parameters have to be estimated, thus giving the possibility to introduce some errors. Currently, the most used methodology in FRFs-based category is the one introduced by Henning Grafe in

1998, Ref. [9], and it makes use of some correlation functions between the FRFs allowing to have a global correlation in the frequency range of the analysis. Such a correlation is improved by minimizing a non-linear penalty function, and the closed-form solution is obtained through the weighted least-square approach.

Considering the Grafe method as reference, the aim of this thesis is to develop a methodology for the identification of structures that includes the uncertainties affecting experimental data, in order to provide numerical models that are more representative of the actual dynamic behavior. Indeed, it was observed that sometimes the reference method provides updated FE models that are scarcely representative of the physical properties of the analyzed structure, although the errors between the numerical and experimental dynamic parameters are greatly reduced. This fact highlights the feature of the updating methodologies to produce non-unique results, that is different combinations of the mass and stiffness parameters could identify an equivalent dynamic FE model representative of the actual dynamic behavior. The main issue is to understand if the updated FE model is physically meaningful, based on the evidence of the structure under investigation. The inclusion of the uncertainties affecting the experimental data in the updating methodology could be useful to avoid numerical instabilities that may result in a low capability of the procedure to update correctly the numerical model, thus providing acceptable results when using real-world applications and allowing the use of the updated FE models for additional structural analyses.

Outline of the thesis

The thesis is divided into three parts:

1. **Theoretical Background.** An overview of structural dynamics is given in Chapter 1, in which two distinct activities that allow the study of the dynamic behavior of a structure, namely numerical modelling and vibration tests, are described. The state-of-the-art in structural updating is introduced in Chapter 2, where the methodology considered as reference in this thesis is described in detail.
2. **Theoretical Developments, Validation and Application to simple structures.** The methodological developments are discussed in detail in Chapter 3, and the developed methodology is validated through numerical test cases by comparing the results with the ones coming from the reference method, as reported in Chapter 4. Real experimental data are introduced in Chapter 5, where the accuracy of the methodology is assessed using simple structures, such as beams and plates.
3. **Application to Rotorcraft Technology.** In this part, the developed methodology is applied to a reduced-scale helicopter blade model that incorporates a semi-active device, known as Smart Spring, for vibration control. Following the introduction to vibration control methodologies reported in Chapter 6, the effectiveness of the Smart Spring in reducing vibration is analytically verified through the development of its mathematical model in the frequency domain, presented in Chapter

7. In addition, the capability of the semi-active device to alter the dynamics of a system is experimentally assessed through tests performed at different operative working conditions of the Smart Spring, as described in Chapter 8. Finally, the identification of the reduced-scale helicopter blade system that incorporates the Smart Spring device is addressed in Chapter 9.

Part I

Theoretical Background

Chapter 1

Background on structural dynamics

The subject of dynamics is of paramount importance in engineering given the constant challenge of reducing the levels of vibration and noise, increasing the fatigue life of machines, improving the precision in measuring instruments. The dynamic behavior of a structure can be studied through two distinct activities, namely the numerical modelling and vibration tests, whose theoretical bases are illustrated in this chapter.

1.1 FE numerical modelling

The finite element (FE) method is the most widely used numerical analysis technique for obtaining approximate solutions to a variety of engineering problems. It originated in the field of structural analysis and the well known texts by Bathe [2], Zienkiewics and Taylor [10], and many other books available in literature, e.g. Refs. [1, 3, 11] provide a comprehensive introduction to the subject. The method involves discretizing

the physical domain, i.e. the geometry of the actual structure, into small *elements*, each one forming a simple unit whose structural behavior can be readily analyzed, and interconnecting these elements each other by individual points known as *nodes*. Once the elements have been chosen, they have to be assembled into one system representation given by mass and stiffness matrices of dimension N , N being the number of degrees of freedom in the model, and their individual physical properties must be determined in order to perform the dynamic analysis of the structure. The level of discretization, the choice of elements, and the solution algorithms used, influence the result significantly. Inaccuracies or uncertainties may originate from a FE representation such as:

- errors introduced by employing numerical techniques, e.g. the discretization error due to approximating a continuous structure by a finite number of individual elements, truncation and round-off errors, etc.;
- modelling errors due to the assumptions made by the analyst to characterize the mechanical behavior of the physical structure, e.g. simplifications of the structure, choice of elements and their properties, erroneous representation of boundary conditions, erroneous modelling of joints, etc.

1.1.1 Spatial model

The FE representation of a continuous structure leads to a system of N linear second-order differential equations that can be written in matrix form as:

$$[M] \{\ddot{x}(t)\} + [C] \{\dot{x}(t)\} + [K] \{x(t)\} = \{f(t)\} \quad (1.1)$$

where $[M]$, $[C]$, $[K]$ are $N \times N$ mass, damping, and stiffness matrices, respectively representing the system matrices and $\{x(t)\}$, $\{f(t)\}$ are $N \times 1$ vectors of time-varying unknown displacements and forces, respectively. According to Ewins, Ref. [4], the system matrices are referred to as the *Spatial Model*. These matrices derive from the definition of some energetic quantities such as the kinetic energy T , the Rayleigh's dissipation function D , and the potential energy V , Ref. [1]:

$$T = \frac{1}{2} \{\dot{x}\}^T [M] \{\dot{x}\} \quad (1.2)$$

$$D = -\{\dot{x}\}^T [C] \{\dot{x}\} \quad (1.3)$$

$$V = \frac{1}{2} \{x\}^T [K] \{x\} \quad (1.4)$$

where T and D are also generally known as quadratic forms of the velocity, and V as quadratic form of the displacement. Since the kinetic energy is a positive definite quadratic form, $[M]$ is positive definite as well. In contrast, the potential energy is positive semi-definite, so that $[K]$ is positive semi-definite allowing the rigid body motion of the system when it is unrestrained. In order to solve Eq. (1.1), in which the time is the independent variable, two initial conditions have to be imposed to solve it with a

closed form:

$$\{x(t=0)\} = \{x_0\} \quad (1.5)$$

$$\{\dot{x}(t=0)\} = \{\dot{x}_0\}$$

1.1.2 Modal model

Considering the free vibration solution, i.e. $\{f(t)\} = \{0\}$, assuming that a solution exists of the form $\{x(t)\} = \{x\} e^{j\omega t}$, and substituting it into Eq. (1.1) (with no damping), it is possible to introduce the eigenvalue problem associated to the system matrices $[K]$ and $[M]$ as:

$$([K] - \omega^2 [M]) \{x\} e^{j\omega t} = \{0\} \quad (1.6)$$

for which the non-trivial solution could be obtained by finding the roots of the discriminant equation:

$$\det([K] - \omega^2 [M]) = 0 \quad (1.7)$$

that is by evaluating N distinct roots, known as characteristic values, or eigenvalues. The square roots of these quantities are the system natural frequencies ω_n ($n = 1, 2, \dots, N$), and substituting any one of these back into Eq. (1.6) yields a corresponding set of relative values for $\{x\}$, i.e. $\{\psi\}^{(n)}$, the so-called mode shape corresponding to the n -th natural frequency. The complete solution can be then expressed using two $N \times N$ matrices,

known as eigen-matrices, by assembling the eigenvalues and eigenvectors as follows:

$$[\Omega^2] = \begin{bmatrix} \ddots & & \\ & \omega_n^2 & \\ & & \ddots \end{bmatrix} \quad (1.8)$$

$$[\Psi] = \left[\{\psi\}^{(1)} \mid \{\psi\}^{(2)} \mid \dots \mid \{\psi\}^{(N)} \right] \quad (1.9)$$

which constitute the *Modal Model*, Ref. [4]. Such a model possesses the orthogonality properties that allow one to simplify the analysis of the system. The orthogonality of the mode shapes with respect to the mass and stiffness matrices can be stated as:

$$[\Psi]^T [M] [\Psi] = \begin{bmatrix} \ddots & & \\ & m_n & \\ & & \ddots \end{bmatrix} \quad [\Psi]^T [K] [\Psi] = \begin{bmatrix} \ddots & & \\ & k_n & \\ & & \ddots \end{bmatrix} \quad (1.10)$$

from which $\omega_n^2 = k_n/m_n$, where m_n and k_n are referred to as the modal or generalised mass and stiffness of mode n , respectively. It is worthwhile remarking that the eigenvalue matrix, Eq. (1.8), is unique, whereas the eigenvector matrix $[\Psi]$, Eq. (1.9) is not because the mode shapes are subject to unknown scaling factors that only affect their amplitude, while keeping their shape unique. The process of scaling the natural modes to make also their amplitude unique is called normalization, and the mass-normalization is an example. Specifically, the mass-normalized vectors are written as $[\Phi]$ and have the particular property that:

$$[\Phi]^T [M] [\Phi] = [I] \quad (1.11)$$

and thus

$$[\Phi]^T [K] [\Phi] = [\Omega^2] \quad (1.12)$$

The relationship between the matrix $[\Phi]$ containing the mass-normalised mode shapes and its more general form $[\Psi]$ is, simply:

$$[\Phi] = [\Psi] \begin{bmatrix} \ddots & & \\ & \frac{1}{\sqrt{m_n}} & \\ & & \ddots \end{bmatrix} \quad (1.13)$$

From the orthogonality properties of the modal model, Eq. (1.10), it is readily seen that the eigenvectors effectively decouple the equation of motion and that the transformation into the modal domain leads to a new set of coordinates which are independent of each other. Considering the following linear transformation:

$$\{x(t)\} = [\Phi] \{q(t)\} \quad (1.14)$$

Eq. (1.1) can be written as:

$$\{\ddot{q}(t)\} + [\Omega^2] \{q(t)\} = \{g(t)\} \quad (1.15)$$

where $\{q(t)\}$ is the vector of the modal coordinates, and $\{g(t)\} = [\Phi]^T \{f(t)\}$ is the vector of the generalized forces, coming from the projection of the force vector $\{f(t)\}$ on the modal base.

The eigenvalue problem associated to $[K]$ and $[M]$, Eq. (1.6), can be extended to the more general case of damped systems according to:

$$(\lambda_n^2 [M] + \lambda_n [C] + [K]) \{p\}^n = \{0\} \quad (1.16)$$

where λ_n and $\{p\}^n$ are the n-th complex eigenvalue and eigenvector, respectively. The mathematical formulation of the modal model for damped systems is not derived in detail

in this thesis for the sake of brevity. Further details can be found in many books such as Refs. [12, 13, 14].

1.1.3 Response model

The response model of a system is derived taking the Fourier transform of Eq. (1.1) thus writing the governing equation of the system in the frequency domain as:

$$(-\omega^2 [M] + j\omega [C] + [K]) \{x(\omega)\} = \{f(\omega)\} \quad (1.17)$$

which can be rearranged to solve for the unknown vector:

$$\{x(\omega)\} = [H(\omega)] \{f(\omega)\} \quad (1.18)$$

where:

$$[H(\omega)] = (-\omega^2 [M] + j\omega [C] + [K])^{-1} \quad (1.19)$$

is the $N \times N$ Frequency Response Function (FRF) matrix of the system and constitutes its *Response Model*. Precisely, Eq. (1.17) identifies the receptance matrix $[\alpha(\omega)] = [H(\omega)]$, as the ratio of displacement to force. Other forms of FRF matrices are the mobility matrix $[Y(\omega)] = j\omega [H(\omega)]$ and the accelerance matrix $[A(\omega)] = -\omega^2 [H(\omega)]$, that relate the velocity and the acceleration to the input force, respectively. The general element in the receptance matrix $H_{ij}(\omega)$ relates the response of the system at the i -th degree of freedom, x_i , to the excitation at the j -th DOF, as shown in Fig. 1.1. The values of the elements of $[H(\omega)]$ can be clearly determined at any frequency of interest by considering

Figure 1.1: Block diagram of the Frequency Response Function.

the appropriate values into Eq. (1.19). However, this involves the inversion of a system matrix at each frequency and, according to Ref. [4], this has several disadvantages. In particular, it becomes costly for large-order systems and it is inefficient if only a few of the individual FRF expressions are required. To overcome these difficulties, the response model is derived by using the modal properties of the system. Specifically, the linear transformation introduced by Eq. (1.14) is written in the frequency domain and then applied to Eq. (1.17) in order to have, after some manipulations whose detailed description can be found in Ref. [4], the following set of uncoupled equations for the general case of damped systems:

$$\begin{bmatrix} \ddots & & \\ & -\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2 & \\ & & \ddots \end{bmatrix} \{q(\omega)\} = \{g(\omega)\} \quad (1.20)$$

where Eq. (1.11) and Eq. (1.12) have been taken into account, and ζ_n is the modal damping ratio. Solving Eq. (1.20) for the vector of the modal coordinates:

$$\{q(\omega)\} = \begin{bmatrix} \ddots & & \\ & \frac{1}{-\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2} & \\ & & \ddots \end{bmatrix} \{g(\omega)\} \quad (1.21)$$

and coming back to the physical displacement and force vectors through Eq. (1.14):

$$\{x(\omega)\} = [\Phi] \begin{bmatrix} \ddots & & \\ & \frac{1}{-\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2} & \\ & & \ddots \end{bmatrix} [\Phi]^T \{f(\omega)\} \quad (1.22)$$

it is possible to write the analytical expression of the system FRF from its definition given by Eq. (1.18) as:

$$[H(\omega)] = [\Phi] \begin{bmatrix} \ddots & & \\ & \frac{1}{-\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2} & \\ & & \ddots \end{bmatrix} [\Phi]^T \quad (1.23)$$

whose generic element is a complex function of the frequency variable:

$$H_{ij}(\omega) = \sum_{n=1}^N \frac{\phi_i^{(n)} \phi_j^{(n)T}}{(-\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2)} \quad (1.24)$$

It is worthwhile remarking that Eq. (1.24) refers to the case of unitary modal masses.

Recalling Eq. (1.13), the general expression of $H_{ij}(\omega)$ is:

$$H_{ij}(\omega) = \sum_{n=1}^N \frac{\psi_i^{(n)} \psi_j^{(n)T}}{m_n (-\omega^2 + 2j\zeta_n\omega_n\omega + \omega_n^2)} \quad (1.25)$$

It is important to note that the Response model is strictly related to the Spatial and Modal models by Eq. (1.19) and Eq. (1.25), respectively.

1.1.4 Routes to vibration analysis

The mathematical models introduced in the previous sections can be summarized in the theoretical route to vibration analysis introduced by Ewins in Ref. [4]. Fig. 1.2(a) illustrates the three phases through which a typical vibration analysis progresses. Specifically, a first description of the physical characteristics of the structure is given in terms of its mass, stiffness, and damping properties, and this is referred to as the Spatial Model, as described in Section 1.1.1. Then the analytical modal analysis of the spatial model leads

to a description of the dynamic behavior of the structure as a set of vibration modes, known as the Modal Model and introduced in Section 1.1.2. Such a model is defined as a set of natural frequencies with the corresponding vibration mode shapes and modal damping factors. This solution always describes the various ways in which the structure vibrates naturally, i.e. without any external forcing or excitation, and so these are called the natural modes of the structure. The analysis of exactly how the structure will vibrate under given excitation conditions and with what amplitudes is performed in the last phase. Since innumerable solutions that depend on the inherent properties of the structure and on the imposed excitation could be evaluated, it is convenient to present an analysis of the response of the structure to a standard excitation and describe this as the Response Model, Section 1.1.3. The standard excitation is that of a unit-amplitude sinusoidal force applied to each point on the structure individually, and at every frequency within a specified range. Thus, the response model consists of a set of FRFs which are usually considered as the most representative of the system dynamics. As indicated in Fig. 1.2(a), the numerical modelling follows the path from the spatial to the response model. However, the vibration analysis can be undertaken also in the reverse direction, giving the so-called experimental route to vibration analysis shown in Fig. 1.2(b). In this case, the route is from the measurement of the response properties of the system (e.g. the measured FRFs) to the identification of the vibration modes and the synthesis of a representative structural model.

(a) Theoretical Route to Vibration Analysis

(b) Experimental Route to Vibration Analysis

Figure 1.2: Comparison between theoretical and experimental approaches in vibration analysis, Ref.[4].

1.2 Experimental modal analysis

The experimental study of the dynamic properties of a structure follows the reverse path with respect to the numerical modelling, and it basically consists of two steps, that is data acquisition and data analysis for the estimate of modal parameters such as natural frequencies, mode shapes, and damping ratios. According to Ref. [15], the structure is a *black box* that needs to be deciphered. The traditional approach is to perform the structural identification by exciting the structure with a known force input, measuring the response so that the FRFs can be directly derived from the force and response information, then extracting the modal parameters through specific procedures that will be introduced later on. It is worth remarking that the assumption of linear and time invariant system is essential to attain accurate FRF measurements. Differently from the traditional approach, it is possible to measure the responses only and identify the

system by means of Operational Modal Analysis (OMA) techniques, whose importance is continuously increasing, Refs. [16, 17, 18].

1.2.1 Description of the test setup

Paying specific attention to the traditional experimental modal analysis (EMA) where both excitation and response are measured simultaneously, the main items in the measurement chain are:

- **Test structure.** It is the system under investigation and its preparation is the important preliminary to the vibration test. Specifically, the first decision which has to be taken is about the boundary conditions to be considered, that is whether the structure is to be tested in a free or grounded condition.
- **Excitation mechanism.** Although there are different types of excitation equipment, the most common ones are the shaker and the impact hammer. The electromagnetic shaker allows to control the excitation of the structure by means of random, sinusoidal, periodic signals, whereas the impact hammer produces an impulsive force. The shaker belongs to the contacting types of exciters, i.e. it remains attached to the structure throughout the test introducing insertion errors that could be not negligible. In contrast, the hammer is in contact with the structure for a short period, while the excitation is being applied. The choice of the exciter often depends on the equipment available, the type of structure to analyze, and the

amount of energy to impart to the structure in order to produce a satisfactory response. Complementary to the shaker exciters are the signal generator which is the source for the excitation signal, such as sinusoidal, transient, periodic, random, and the power amplifier that allows the structure to be excited with a suitable frequency content.

- **Transducers.** They allow to measure both the excitation force and the output responses. Transducers based on the properties of piezoelectric materials are the most popular, and the basic principle of operation makes use of the fact that a strain in these piezoelectric materials generates an electrical charge which may be converted into a voltage in a suitable range. Piezoelectric load cells and accelerometers are generally used to measure the excitation force and the output responses in terms of accelerations, respectively. However, strain gauges, optical, capacitive or inductive transducers may also be used.
- **Data acquisition system.** The analysis of vibration data is usually performed digitally, requiring the analogue signals from the transducers to be converted to digital signals via A/D (analogue to digital) converters. In particular, the spectrum analyzer records the continuous (analogue) signal as a series of N discretized values. Therefore, issues related to signal processing as the sampling rate, aliasing, leakage, and windowing have to be taken into account (see Section 1.2.2 for further details).
- **Computer unit.** It allows the analyst to drive the test, collect the measured

data, and post-process them in order to perform the modal identification, that is to estimate the modal parameters of the test structure, through specific software. All the acquisition parameters, such as the frequency range of analysis, the transducers sensitivities, the sampling frequency, etc, have to be set by the analyst through this unit.

For the sake of completeness, a brief description of the problems related to the use of digital signals is introduced in the next section.

1.2.2 Digital signal processing

As mentioned before, the spectrum analyzer allows to convert and record analogue signals as a set of N discrete values spaced in the period T , during which the measurement is made, by means of a sampling time T_s introducing some problems to deal with. Assuming that the sample in time T is periodic, a Discrete Fourier Transform (DFT) can be computed to describe the signal by its constituent frequencies and work in the frequency domain. The highest frequency which can be included in the spectrum is given by the Shannon's theorem, that relates the Nyquist frequency, i.e. the maximum frequency f_{max} , with the sampling frequency f_s according to:

$$f_{max} \leq \frac{f_s}{2} \quad (1.26)$$

If Eq. (1.26) is not satisfied, high frequencies in the original signal will appear as low frequencies introducing the problem known as *aliasing*. This causes the distortion of the

spectrum measured via the DFT because the frequency components of the signal above $f_s/2$ will appear reflected or aliased in the range $[0, f_s/2]$. A solution to this problem is the use of an anti-aliasing filter with a sharp cut-off that submits a modified time history to the analyzer.

The DFT assumes that the signal is periodic over the sampling interval chosen, but in general it is not true leading to a problem known as *leakage*, best illustrated in Fig. 1.3. Specifically, when the signal is perfectly periodic in the time window T , the resulting spectrum is a single line at the frequency of the sine wave, as shown in Fig. 1.3(a). However, when the periodicity assumption is not strictly valid and there is a discontinuity at each end of the sample as illustrated in Fig. 1.3(b), the spectrum does not indicate the single frequency but it is spread over several lines close to the true frequency.

One practical solution to the leakage problem involves the use of *windowing* that im-

Figure 1.3: Example of leakage, Ref. [4].

poses a prescribed profile on the time signal prior to performing the Fourier transform and the windows are generally depicted as time function $w(t)$. In Fig. 1.4 some representative windows generally used are depicted, Ref. [4]. In particular, the rectangular

window (a) corresponds to no window, the Hanning (b) or Cosine Taper (c) windows are typically used for random or steady periodic signals, whereas the exponential window (d) is used for transient signals, such as an impulse, where the most important information is concentrated in the initial part of the time record.

Figure 1.4: Example of window functions, Ref. [4].

1.2.3 Derivation of Frequency Response Functions

The first phase of the experimental route to vibration analysis is the estimate of the response model, that is the Frequency Response Functions of the structure as depicted in Fig. 1.2(b). Once the problems related to digital signals have been addressed, the theoretical basis of how the FRFs are estimated by the analyzer needs to be introduced. From the measurement of two output responses in the time domain $x_i(t)$ and $x_k(t)$, $i, k = 1, 2, \dots, N_o$ being N_o the number of outputs it is possible to evaluate the output correlation function $R_{x_i x_k}(\tau)$, as a function of the time lag τ :

$$R_{x_i x_k}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x_i(t) x_k(t + \tau) dt \quad (1.27)$$

Under the hypothesis of Gaussian stationary, ergodic with zero mean value signals, the output Power Spectral Density (PSD) functions are evaluated through the Wiener-Khintchine relations, Ref. [19]:

$$S_{x_i x_k}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{x_i x_k}(\tau) e^{-i\omega\tau} d\tau \quad (1.28)$$

It is usually referred to auto and cross-correlation PSD functions if $i = k$, or $i \neq k$ respectively. Identical definitions are used for two input signals $f_j(t)$ and $f_k(t)$, $j, k = 1, 2, \dots, N_i$ being N_i the number of inputs, giving the input correlation function $R_{f_j f_k}(\tau)$ and the input PSD $S_{f_j f_k}(\omega)$. By combining the output and input PSD functions two different estimates of the FRF, commonly recognized as H_1 and H_2 , can be derived, Ref. [20]:

$$H_1(\omega) = \frac{S_{fx}(\omega)}{S_{ff}(\omega)} \quad (1.29)$$

$$H_2(\omega) = \frac{S_{xx}(\omega)}{S_{xf}(\omega)} \quad (1.30)$$

where the subscripts referring to the input and output points have been omitted for the sake of clarity, and $S_{fx}(\omega)$ is the cross-spectrum of excitation and response. The H_1 estimator minimises the error in the FRF due to noise on the output signal and its use is preferred to estimate the peaks of anti-resonance, whereas the H_2 estimator minimises the error in the FRF due to noise on the input signal preferring its use for the estimate of the peaks of resonance. Another FRF estimator that is unbiased with respect to both input and output noise can be used:

$$H_v(\omega) = \frac{S_{xx} - \kappa S_{ff} + \sqrt{[S_{ff}\kappa - S_{xx}]^2 + 4|S_{fx}|^2\kappa}}{2S_{xf}} \quad (1.31)$$

in which $\kappa = S_{n_x n_x} / S_{n_f n_f}$ is the ratio between the auto-PSD of the noise acting on the output and the auto-PSD of the noise acting on the input. The H_1 and H_2 estimators are related by the coherence function $\gamma^2(\omega)$, defined as follows:

$$\gamma^2(\omega) = \frac{H_1(\omega)}{H_2(\omega)} = \frac{|S_{fx}(\omega)|^2}{S_{ff}(\omega)S_{xx}(\omega)} \quad (1.32)$$

This function assumes values between 0 and 1 and it gives a measure of the quality of the measured data as a function of frequency. High values of coherence imply accurate data.

1.2.4 Estimate of the modal model

Once the FRFs have been measured, the data have to be processed to obtain the modal model, that is natural frequencies, damping ratios, and mode shapes of the structure. Since a detailed description of the methods that allow the modal identification from FRF data is beyond the aim of this thesis, only an overview of them is given in this section. The simplest method is the *peak-picking* one that focuses on the FRF near each resonance and identifies the modal properties of the corresponding single degree of freedom (SDOF) system. This method works adequately for structures whose FRFs exhibit well-separated modes which are not so lightly-damped, thus not allowing accurate measurements at resonance, but which are not so heavily damped that the response at resonance is not dominated by just one mode. It simply consists of picking the peaks of resonance from the modulus plot of the FRF, estimating the damping factors by the half-power bandwidth

method and evaluating the components of the mode shapes from the imaginary parts of the FRF plot, Ref. [4]. When the SDOF assumption cannot be satisfied, there exist several alternative methods which are generally classified as multiple degree of freedom (MDOF) approaches. Among them, time-domain and frequency-domain methods are available.

Time-domain methods transform the FRF data back into the time domain in the form of impulse response functions. One of the first methods in this class was the Ibrahim method that identifies the modal parameters by means of an eigenvalue problem, Ref. [21]. Other methods are the Eigensystem Realisation Algorithms (ERA) Ref. [22] and the auto-regressive moving average (ARMA) models Ref. [23] both using concepts developed in control engineering. Much faster methods are the Least Squares Complex Exponential (LSCE) Ref. [24] and the polyreference-LSCE (p-LSCE) Ref. [25] that fit a weighted sum of damped complex exponentials to the impulse response functions, and the natural frequencies and damping ratios are calculated by solving an eigenproblem. The unknown in MDOF approaches for modal identification is the number of modes in the frequency range of analysis, that is the model order, which is calculated through an iterative procedure introducing the so-called stabilization diagram. It is built by comparing the modal parameters calculated from models of consecutive order and by defining them as stable if the relative error is less than an arbitrary small fixed threshold. The stable poles are then chosen as representative of the modal parameters.

Frequency domain methods generally fit the FRF data to a rational polynomial in the

frequency variable. A classical method belonging to this class is the Least Squares Frequency-Domain (LSFD) algorithm, based on the expression of the response model in terms of the modal parameters as in Eq. (1.25). A further evolution of the LSFD method is the polyreference version of it, the so-called PolyMAX algorithm developed by LMS and implemented in the LMS Test.Lab[®] software as described in Ref. [26]. These methods directly use the measured FRF data as primary data, in contrast to the time-domain methods that require the impulse response functions.

All the above introduced methods belong to the class of deterministic algorithms and it should be noted that stochastic approaches lead to a more realistic description of real-life experiments. Examples in this category are the frequency domain Maximum Likelihood Estimator (MLE) Ref. [27] and the recent PolyMAX plus Ref. [28] that combines the MLE and PolyMAX algorithms.

To close the overview on the EMA, it is worth remarking that as for the numerical modelling, modal testing has also a number of problems that an analyst has to deal with. Specifically, they are related to the limited number of measurement locations, to the impossibility to measure some degrees of freedom, such as rotational and internal ones, to the number of identified modes that is limited by the frequency range of analysis. In addition, the modal analysis usually yields complex mode shapes whereas the numerical analysis is usually based on the normal mode theory. Finally, the test structure may exhibit significant nonlinearities whereas the numerical model assumes linearity.

1.3 Numerical-experimental correlation

As stated by Ewins in Ref. [4], the most popular application of modal testing is to provide a direct comparison between predictions for the dynamic behavior of a structure and those actually observed in practice. Indeed, test data are considered to be more accurate and thus used as reference to assess the quality of the available FE model, and validate it. In the comparison between numerical and experimental models, major problems arise because of the discrepancy between the large number of DOFs in the numerical model and the limited number of measurement points, and modelling inaccuracies, e.g. the omission of damping in the numerical model. If the comparison between the two set of data is satisfactory, according to some correlation coefficients that are introduced later on in this section, the FE model is representative of the actual structure, otherwise some adjustments to the numerical model have to be made in order to bring it closer to the experimental model. This process, known as *model updating*, is of paramount importance in structural dynamics and it is the core of the thesis, so it will be addressed in the next chapter.

Modal-based correlation techniques are the most developed and commonly used for the numerical-experimental correlation. The simplest approach to correlate the two models is the direct comparison between the natural frequencies by plotting the measured values against the predicted ones. If the points lie on or close to a straight line of slope 1, the data are perfectly correlated. A percentage difference between the two sets of data can

be used as a quantitative coefficient and defined as:

$$|\Delta f_i| = \frac{|f_i^X - f_i^A|}{f_i^X} \times 100 \quad (1.33)$$

where f_i^X and f_i^A are the i -th experimental and analytical natural frequencies, respectively. A mean value can be evaluated by:

$$|\overline{\Delta f}| = \frac{\sum_{i=1}^{N_m} |\Delta f_i|}{N_m} \quad (1.34)$$

where N_m is the number of natural frequencies considered in the frequency range of interest.

The correlation between the mode shapes can be performed by plotting each element of the experimental mode shapes vector versus its predicted counterpart, and the individual points on this plot should lie close to a straight line passing through the origin if the two mode sets are well correlated. A more strict coefficient called Modal Assurance Criterion (MAC) is introduced in Ref. [5] to correlate the experimental and numerical mode shapes. It uniquely defines a real scalar between zero and unity and it is defined between mode $\{\phi_i\}$ and mode $\{\phi_j\}$ as:

$$MAC_{ij} = \frac{\left| \{\phi_i^X\}^H \{\phi_j^A\} \right|^2}{\left(\{\phi_i^X\}^H \{\phi_i^X\} \right) \left(\{\phi_j^A\}^H \{\phi_j^A\} \right)} \quad (1.35)$$

where superscripts X and A indicate experimental and analytical mode shapes, respectively, and H is the Hermitian operator. Values close to 1 (0) indicate good (poor) correlation between the experimental and analytical mode shapes. This correlation coefficient is not capable of distinguishing between systematic errors and local discrepancies,

thus it is usually used in conjunction with the Coordinate Modal Assurance Criterion (COMAC) that determines the correlation between two measurement locations for all the paired mode shapes as introduced in Ref. [6]. Specifically, having first evaluated the set of L mode pairs through the MAC, the COMAC calculates the amount of correlation at each coordinate over all correlated mode pairs as:

$$COMAC_i = \frac{\sum_{r=1}^L \left| \{\phi_{ir}^X\}^H \{\phi_{ir}^A\} \right|^2}{\sum_{r=1}^L \{\phi_{ir}^X\}^2 \sum_{r=1}^L \{\phi_{ir}^A\}^2} \quad (1.36)$$

In terms of the response model, the direct correlation between measured and predicted FRFs is less developed because as for many years the correlation between the two models was confined to the modal domain. Nevertheless, more recently attention has been paid to response-based correlation coefficient, such as the Frequency Response Assurance Criterion (FRAC) introduced in Ref. [7]:

$$FRAC(\omega_k) = \frac{\left| \{H_{ij}^X(\omega_k)\}^H \{H_{ij}^A(\omega_k)\} \right|^2}{\left(\{H_{ij}^X(\omega_k)\}^H \{H_{ij}^X(\omega_k)\} \right) \left(\{H_{ij}^A(\omega_k)\}^H \{H_{ij}^A(\omega_k)\} \right)} \quad (1.37)$$

where $\{H_{ij}(\omega_k)\}$ is the FRF vector corresponding to the i -th response and j -th excitation DOFs at each k -th frequency point. It assumes unitary values for identical FRFs. Different response-based coefficients based on two correlation functions between FRFs were introduced by Grafe in Ref. [9]. These correlation functions examine the level of correlation at each k -th frequency point and they are defined as:

$$\chi_s(\omega_k) \equiv \frac{\left| \{H^X(\omega_k)\}^H \{H^A(\omega_k)\} \right|^2}{\left(\{H^X(\omega_k)\}^H \{H^X(\omega_k)\} \right) \left(\{H^A(\omega_k)\}^H \{H^A(\omega_k)\} \right)} \quad (1.38)$$

and

$$\chi_a(\omega_k) \equiv \frac{2 \left| \{H^X(\omega_k)\}^H \{H^A(\omega_k)\} \right|}{\{H^X(\omega_k)\}^H \{H^X(\omega_k)\} + \{H^A(\omega_k)\}^H \{H^A(\omega_k)\}} \quad (1.39)$$

where $\{H^X(\omega_k)\} \in \mathbb{C}^{N_o \times 1}$ and $\{H^A(\omega_k)\} \in \mathbb{C}^{N_o \times 1}$ are the vectors of the measured and predicted frequency response functions at each k -th frequency point, and N_o is the number of the measured output responses. The function defined by Eq. (1.38), named *shape correlation function*, assumes a value between zero and unity indicating a perfect correlation with the unitary value, whereas no correlation exists when $\chi_s(\omega_k) = 0$. Such a function is unable to detect scaling errors and it takes into account only the discrepancies in the overall shape of the frequency response functions, as its name suggests. The lack of sensitivity to scaling of the shape correlation coefficient does not allow the identification of identical FRFs. If only one measured and predicted FRF are correlated, the column vectors reduce to scalars and $H^A(\omega_k) = kH^X(\omega_k)$ is always satisfied (the constant k may be either real or complex), leading to an unitary value of χ_s across the full frequency spectrum also for uncorrelated FRFs. In order to take into account the discrepancies in the amplitude of the FRFs, the function defined by Eq. (1.39), named *amplitude correlation function*, is introduced. It assumes, as $\chi_s(\omega_k)$, a value between zero and unity but this time the correlation measure is more stringent and becomes unity only if $\{H^A(\omega_k)\} = \{H^X(\omega_k)\}$. In other words, all elements of the response vectors must be identical in both phase and amplitude even if only one FRF is considered.

The above introduced correlation functions are the starting point of the updating methodologies considered in this thesis as it will be shown in Section 2.4.

Chapter 2

State-of-the-art in model updating

The aim of this chapter is to give an overview of the model updating methodologies available in literature. More emphasis will be placed on the technique which the method chosen as reference in this thesis belongs to, in order to give a good theoretical basis for the development and the explanation of the proposed updating methodology.

2.1 Direct and iterative model updating methods

Model updating methods are usually divided into two categories, namely direct and iterative methods, whose advantages and disadvantages are described in the following sections.

2.1.1 Direct methods

The direct methods are the earliest generation of FE model updating algorithms and they directly solve for the updated global mass and stiffness matrices. The equations defining the differences between the predicted dynamic characteristics and the experimental ones usually make use of the eigenvalues and eigenvectors, that is natural frequencies and mode shapes. According to Ref. [29], the main advantages of direct methods are:

- assured convergence, since the method does not need any iteration
- the methods try to produce the reference data set exactly
- the CPU time is usually less than that required by iterative methods

However, they have some disadvantages:

- high quality measurements and accurate modal data analysis are needed
- FE model and experimental model need the same size, so reduction or expansion techniques must be introduced
- there is no guarantee for the positiveness of the updated mass and stiffness matrices.

Among the direct methods, those using Lagrange multipliers are the most popular. They try to minimize a penalty function subject to some constraints on the independent variables, such as the system symmetry and the orthogonality conditions. These methods

have been used to correct either the stiffness matrix, Ref. [30], or the mass matrix, Ref. [31], or both stiffness and mass matrices sequentially, Refs. [32, 33]. Other examples are the matrix mixing methods, introduced in Refs. [34, 35] and expanded in Refs. [36, 37]. These methods consider that the mass and stiffness matrices can be constructed directly if one can assume that all vibration modes are measured at all degrees of freedom. This implies two main difficulties. First, the number of numerical degrees of freedom is usually higher than the number of measured modes. Second, the response of the structure can only be measured at a limited number of coordinates. However, this may be overcome by expanding the experimental mode shapes to full size by using mass and stiffness matrices from the finite element model. Different strategies have been proposed to deal with the above introduced difficulties, such as adding some linearly independent vectors to the modal matrix to make it square and invertible, Ref. [34], limiting the number of degrees of freedom to be equal to the number of measured modes, Ref. [35], or applying a pseudo-inverse to incomplete modal matrix, Ref. [38]. Error matrix methods are other examples of direct methods. They use the concept of error matrices, defined as the difference between the analytical and experimental system matrices. These matrices are determined by using an approach based on the contribution of the known modes to the system flexibility, Refs. [4, 39]. Due to the requirements of high quality experimental data, none of the direct methods gives satisfactory results.

2.1.2 Iterative methods

The iterative FE model updating methods seek to improve the correlation between the analytical and experimental models via a penalty function. Since the penalty function is usually non-linear, the problem has to be linearized and optimized iteratively. Unlike direct methods, which focus on the global system matrices, iterative procedures take into account the discretized nature of the FE model, working on an elemental basis. Specifically, FE models are an assembly of individual finite elements and each of them is defined by its design parameters, such as its geometry or structural properties. According to the *p-value* formulation, Refs. [8, 9], the elemental mass and stiffness matrices are parameterized by the design parameters p^m and p^k , respectively that have to be calculated by the iterative procedures:

$$[M(\{p\})] = \sum_{i=1}^{N_m} p_i^m [M_i^e] \quad (2.1)$$

$$[K(\{p\})] = \sum_{i=1}^{N_k} p_i^k [K_i^e] \quad (2.2)$$

where N_m and N_k denote the number of p-values acting on the mass and stiffness matrices, respectively. Iterative methods work together with the parameterized FE model and introduce changes to the pre-defined design parameters that act on the elemental mass and stiffness matrices. Due to the non-linear nature of the penalty function, iterative methods may not converge. However, they have two main advantages that make them the most favourite procedures for structural updating. First, they allow to update simultaneously a wide range of parameters and second, both measured and analytical data

can be weighted according to engineering intuition, offering an updating which is physically more meaningful than that offered by direct methods. Typically, these methods are based on the use of a Taylor series of the dynamic data expanded as a function of the unknown updating parameters. This series is truncated to produce a linear approximation of the form:

$$\{\varepsilon\} = [S] \{\Delta p\} \quad (2.3)$$

where $\{\varepsilon\}$ is the residual vector, representing the difference between the measured and predicted dynamic parameters, $[S]$ is the sensitivity matrix, which contains the first derivatives of the dynamic parameters with respect to the design parameters, and $\{\Delta p\}$ is the unknown vector representing the changes in the updating parameters. Eq. (2.3) is the standard form of the iterative sensitivity-based model updating formulation, Ref. [40], and it is solved for $\{\Delta p\}$ which is then used to obtain the updated mass and stiffness matrices, according to Eq. (2.1) and Eq. (2.2). The process is repeated until convergence is achieved within a specified tolerance. Regarding the residual vector $\{\varepsilon\}$, it can be defined with respect to the modal parameters, that is natural frequencies and mode shapes, generating the so-called *modal-based updating methods*, or with respect to the Frequency Response Functions, which are used in the so-called *response-based updating methods*. A general description of these methods is given in the next sections.

2.2 Modal-based methods

In the modal-based methods, the difference between the eigenvalues and eigenvectors, i.e. natural frequencies and mode shapes, assumes the role of residual vector $\{\varepsilon\}$, and the sensitivity matrix $[S]$ includes the first-order derivatives of these dynamic quantities with respect to the selected updating parameters. The preliminary step of the modal-based updating methods is the identification of pairing modes between the analytical and experimental model. It can be performed by visual means but it is usually supported by quantitative coefficients, e.g. the Modal Assurance Criterion defined by Eq. (1.35). After having identified pairs of correlated modes, Eq. (2.3) is solved for the design parameters changes, $\{\Delta p\}$, and the FE model is updated. Following the eigensolution of the updated FE model, a new mode pairing is performed and the process goes ahead until convergence is achieved. Unlike the eigenvalue sensitivities, which may be determined analytically as suggested in Ref. [41], eigenvectors sensitivities cannot be solved in closed form and must therefore be approximated. Several methods for calculating approximated mode shapes derivatives can be found in literature, Refs. [41, 42, 43].

Usually, the number of measurement points is not equal to that of updating parameters, then the sensitivity matrix $[S]$ in Eq. (2.3) is not square, requiring the introduction of techniques to overcome this difficulty. The case in which there are more design parameters than measurements is addressed in Ref. [44] where extra constraints to turn the parameter estimation problem into an over-determined set of equations are introduced.

On the contrary, the Singular Value Decomposition (SVD) is used in Refs. [45, 46] for a case with more measurements than design parameters.

Modal-based updating methods have been discussed extensively by many authors. An eigensensitivity approach using weighted least-squares solutions where the weights can accommodate engineering intuition, is discussed in Ref. [47] whereas the application to practical examples of the updating formulation can be found in Refs. [48, 49]. Furthermore, neural networks have been applied for modal-based updating purposes, Refs. [50, 51], highlighting that their application is limited to small systems due to the excessive computational effort.

2.3 Response-based methods

Modal-based updating methods need to identify the modal parameters from the measurements and perform the pairing with the analytical ones. Response-based updating methods allow to avoid errors due to modal parameters extraction by using the measured Frequency Response Functions directly. Furthermore, the identification problem can easily be turned into an over-determined set of equations because FRFs can be evaluated at a large number of frequency points. Referring to Eq. (2.3), the residual vector $\{\varepsilon\}$ is defined through the difference between the predicted and measured FRFs and the sensitivity matrix $[S]$ includes the first-order derivatives of the FRFs with respect to the selected updating parameters.

Ref. [52] is considered the first paper in which the idea of using measured data directly is introduced, whereas detailed discussions on the developments are presented in Refs. [53, 54]. The benefits that response-based methods have over modal-based algorithms are stressed in Ref. [55] in which it is stated that experimental modal analysis can introduce errors exceeding the level of accuracy required to update FE models, especially when the tested structure exhibits close modes or regions of high modal density. The potential advantages of this model updating procedure aroused increasing interest of many authors to develop and improve these methods. Refs. [56, 57, 58] provide a good introduction to the subject of response-based updating procedures.

One disadvantage of the FRF-based updating techniques is the requirement of a one-to-one correspondence between the number of degrees of freedom in the FE model and the number of measurement points. From the practical point of view, it is difficult to satisfy as not all analytical DOFs can be measured, then numerical expansion or condensation schemes, e.g. the Guyan reduction Ref. [59], must be introduced. Experience has shown that condensation tools are generally more successful than expansion schemes used for measured data, Refs. [60, 61].

The incompleteness ratio defined by:

$$i_r = \frac{n}{N} \quad (2.4)$$

where n is the number of measurement points and N the number of DOFs in the FE model, is introduced in Ref. [9] to define the success rate of the updating techniques

in presence of incomplete sets of measurements. When dealing with model updating problems of industrial size, it is usual to have very small values of the incompleteness ratio and this is the real challenge of updating large FE models. Other difficulties in FRF-based model updating methods include the ill-conditioning of sensitivity matrix and the selection of frequency points. The treatment of ill-conditioned, noisy systems of equations is a problem central to finite element model updating, then regularization techniques are generally introduced to avoid numerical instabilities, Refs. [62, 63, 64]. Regarding the selection of frequency points, those points that are close to the predicted natural frequencies have to be excluded to prevent numerical instabilities if an undamped FE model is considered, Ref. [9]. Indeed, frequency response functions tend to infinite in correspondence of natural frequencies if damping is not introduced in the analytical model.

In the category of response-based methods, the Predictor-Corrector (P-C) algorithm that uses the correlation functions between FRFs defined by Eqs. (1.38) and (1.39), was introduced in 1998 by Henning Grafe, Ref. [9]. This method effectively removes the built-in requirement of a one-to-one correspondence between the number of measurements and the number of DOFs in the FE model, since the considered correlation functions provide a global picture of correlation. Indeed, they project the measured and predicted responses (usually complex) to a real correlation coefficient between zero and unity at each frequency point. Using all measured frequency points, the resulting two correlation curves uniquely combine the information provided by any number of measurements and

their corresponding predictions. Therefore, the objective of the P-C method is seeking a unit correlation, avoiding the need to update the predicted responses directly. Given the promising features of using the new correlation coefficients, the method introduced in Ref. [9] is considered the starting point of the proposed research, and it is described in detail in the next section.

2.4 The Predictor-Corrector method

According to Ref. [9], the residual vector $\{\varepsilon\}$ in Eq. (2.3) is derived from a first-order Taylor series expansion of the correlation functions $\chi_s(\omega_k)$ and $\chi_a(\omega_k)$ defined by Eq. (1.38) and Eq. (1.39), respectively. Specifically, for a given frequency point ω_k , and evaluating the actual correlation functions (subscript *new*) from the knowledge of the required change of the updating parameters, $\{\Delta p\}$, and from the current correlation function (subscript *old*), it is possible to write for both χ_s and χ_a :

$$\chi(\omega_k)|_{new} = \chi(\omega_k)|_{old} + \left\{ \frac{\partial \chi(\omega_k)}{\partial \{p\}} \right\}^T \{\Delta p\} \quad (2.5)$$

In the structural updating scheme, $\chi(\omega_k)|_{new}$ is constraint to be unitary, that is a perfect correlation is sought, so the residual vector $\{\varepsilon\} \in \mathbb{R}^{2N_f \times 1}$, being N_f the number of

frequency points, is written as:

$$\{\varepsilon\} = \begin{Bmatrix} 1 - \chi_s(\omega_1) \\ \vdots \\ 1 - \chi_s(\omega_{N_f}) \\ 1 - \chi_a(\omega_1) \\ \vdots \\ 1 - \chi_a(\omega_{N_f}) \end{Bmatrix} \quad (2.6)$$

$\{\Delta p\} \in \mathbb{R}^{N_p \times 1}$ is the unknown vector of changes of the N_p design parameters to be obtained iteratively, whereas $\left\{ \frac{\partial \chi(\omega_k)}{\partial \{p\}} \right\}$ in Eq. (2.5) is the k-th row of the sensitivity matrix $[S] \in \mathbb{R}^{2N_f \times N_p}$ defined as:

$$[S] = \begin{bmatrix} \frac{\partial \chi_s(\omega_1)}{\partial p_1} & \cdots & \frac{\partial \chi_s(\omega_1)}{\partial p_{N_p}} \\ \vdots & \cdots & \vdots \\ \frac{\partial \chi_s(\omega_{N_f})}{\partial p_1} & \cdots & \frac{\partial \chi_s(\omega_{N_f})}{\partial p_{N_p}} \\ \frac{\partial \chi_a(\omega_1)}{\partial p_1} & \cdots & \frac{\partial \chi_a(\omega_1)}{\partial p_{N_p}} \\ \vdots & \cdots & \vdots \\ \frac{\partial \chi_a(\omega_{N_f})}{\partial p_1} & \cdots & \frac{\partial \chi_a(\omega_{N_f})}{\partial p_{N_p}} \end{bmatrix} \quad (2.7)$$

which takes into account the sensitivity on the dynamic behaviour of the structure induced by a unit variation of the i-th updating parameter p_i . The sensitivity matrix could be obtained either analytically, from the derivatives of the dynamic stiffness with respect to the i-th updating parameter p_i , if the p-value formulation is adopted, Eq. (2.1) and Eq. (2.2), or numerically, from a finite difference numerical scheme. In this thesis the analytical formulation is considered by making use of the derivatives reported in Ref. [9].

As it can be noted from Eqs. (2.6) and (2.7), the approach uses two equations (one using $\chi_s(\omega_k)$, the other $\chi_a(\omega_k)$) for each of the frequency points representing a prediction and

a correction estimate, respectively. As both equations interact with each other, the formulation is referred to as the Predictor-Corrector method, Ref. [9]. Eq. (2.3), together with the expressions introduced in Eq. (2.6) and Eq. (2.7), can be readily turned into an over-determined system of equations by considering a large number of frequency points. Since generally $2N_f > N_p$, a weighted least-squares approach is proposed to minimize:

$$F_w := \{\varepsilon\}^T [W_f] \{\varepsilon\} \quad (2.8)$$

where $[W_f] \in \mathbb{R}^{2N_f \times 2N_f}$ is a diagonal weighting matrix whose elements act on the frequency points. Eq. (2.8) can be approximated as, Ref. [9]:

$$J(\{\Delta p\}) = J_1(\{\Delta p\}) + k_p J_2(\{\Delta p\}) \quad (2.9)$$

where $J_1(\{\Delta p\})$ is the *Quadratic Approximation* of Eq. (2.8), $J_2(\{\Delta p\})$, named *limiting function*, is a function that limits the parameters in order to satisfy the first-order approximation hypothesis of the correlation functions, Eq. (2.5), and k_p is a regularization parameter that balances the two terms. The quadratic approximation $J_1(\{\Delta p\})$ can be evaluated by considering the second-order Taylor expansion of Eq. (2.8) for values of $\{p\}$ near to $\{p_0\}$:

$$J_1(\{\Delta p\}) = F_{w_0} + \{\Delta p\}^T \{g_0\} + \frac{1}{2} \{\Delta p\}^T [Hess_0] \{\Delta p\} \quad (2.10)$$

in which the gradient vector $\{g_0\}$ and the Hessian matrix $[Hess_0]$ are, respectively:

$$\{g_0\} = \{\nabla F_w|_0\} = -2 [S_0]^T [W_f] \{\varepsilon_0\} \quad (2.11)$$

$$[Hess_0] = [\nabla^2 F_w|_0] \simeq 2 [S_0]^T [W_f] [S_0] = [Hess_0^{app}] \quad (2.12)$$

In the previous equation, the derivatives of the sensitivity matrix with respect to the parameters have been neglected leading to an approximation of the Hessian matrix. The final expression of Eq. (2.10) is:

$$J_1(\{\Delta p\}) = \{\varepsilon_0\}^T [W_f] \{\varepsilon_0\} - 2 \{\Delta p\}^T [S_0]^T [W_f] \{\varepsilon_0\} + \{\Delta p\}^T [S_0]^T [W_f] [S_0] \{\Delta p\} \quad (2.13)$$

Regarding the limiting function $J_2(\{\Delta p\})$, it has to guarantee that the components of the unknown vector are small enough to keep Eq. (2.5) valid, and it can be formulated as, Ref. [9]:

$$J_2(\{\Delta p\}) = \{\Delta p\}^T [W_p] \{\Delta p\} \quad (2.14)$$

where $[W_p] \in \mathbb{R}^{N_p \times N_p}$ is a diagonal weighting matrix whose elements act on the updating parameters. Introducing Eqs. (2.13) and (2.14) into Eq. (2.9):

$$\begin{aligned} J(\{\Delta p\}) &= \{\varepsilon_0\}^T [W_f] \{\varepsilon_0\} - 2 \{\Delta p\}^T [S_0]^T [W_f] \{\varepsilon_0\} + \{\Delta p\}^T [S_0]^T [W_f] [S_0] \{\Delta p\} \\ &\quad + k_p \{\Delta p\}^T [W_p] \{\Delta p\} \end{aligned} \quad (2.15)$$

and imposing its stationarity with respect to the components of $\{\Delta p\}$, that is:

$$\left\{ \frac{\partial J(\{\Delta p\})}{\partial \{\Delta p\}} \right\} = -2 [S_0]^T [W_f] \{\varepsilon_0\} + 2 [S_0]^T [W_f] [S_0] \{\Delta p\} + 2k_p [W_p] \{\Delta p\} = \{0\} \quad (2.16)$$

Eq. (2.16) can be solved for the updating parameter changes, Ref [65]:

$$\{\Delta p\} = \left[[S_0]^T [W_f] [S_0] + k_p [W_p] \right]^{-1} [S_0]^T [W_f] \{\varepsilon_0\} \quad (2.17)$$

2.4.1 Choice of weighting matrices

The weighting matrices in Eq. (2.17) need to be determined. Each diagonal element of the weighting matrices $[W_f]$ and $[W_p]$ represents the level of confidence one has in the frequency points and the design parameter changes, respectively. Therefore, more leverage is placed on those equations where the weighting coefficients are higher.

The identification of weighting matrices is a difficult subject and often statistical properties are employed, Ref. [8]. In the solution procedure suggested in Ref. [9], the correlation functions are used directly:

$$[W_f^{Grafe}]_{2N_f \times 2N_f} \equiv \begin{bmatrix} \begin{bmatrix} \ddots & & \\ & \chi_s(\omega_k) & \\ & & \ddots \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \ddots & & \\ & \chi_a(\omega_k) & \\ & & \ddots \end{bmatrix} \end{bmatrix} \quad (2.18)$$

in which the weighting factors vary between zero and unity, placing more confidence on those equations whose corresponding frequency points are better correlated. The parameter weighting matrix $[W_p]$ should be chosen to reflect the uncertainty in the initial parameter estimates, Refs. [66, 67]. This may be formally related to Bayesian methods where the weighting matrices are the inverse of the variances, Ref. [68]. However, it is rarely possible to know the variance of the initial parameter estimates and the alternative

given in Ref. [65] relates the choice of $[W_p]$ to the inverse of the squared sensitivity matrix:

$$[W_p^{Grafe}]_{N_p \times N_p} \equiv \begin{bmatrix} \ddots & & \\ & \frac{diag([w_p])}{\max(diag([w_p]))} & \\ & & \ddots \end{bmatrix} \quad (2.19)$$

where

$$[w_p]_{N_p \times N_p} = [S_0]^T [W_f^{Grafe}] [S_0]^{-1} \quad (2.20)$$

This definition allows the parameters to be constrained according to their sensitivity, meaning that a parameter remains unchanged if its sensitivity approaches zero. Several criteria have been proposed to choose the regularisation parameters k_p . In Ref. [65] it is suggested that k_p lies in the range between 0 (no regularisation) and 0.3. High values of the parameter are used if there are many insensitive parameters, and the higher the value of k_p the higher is the necessary number of iteration steps to achieve convergence. The following expression was considered in Ref. [9]:

$$k_p^{Grafe} \equiv \|[w_p]\|_2 \quad (2.21)$$

which, referring to Eq. (2.20), shows the dependence of the regularization parameter on the sensitivity matrix. Specifically, it increases when the components of the sensitivity matrix decrease as it happens when the correlation corresponding to a given configuration of updating parameters reaches its maximum value. However, the regularization parameter could tend to infinite even if only one column of the sensitivity matrix tends to zero causing the algorithm to stop, although the exact solution is not reached.

Part II

Theoretical Developments, Numerical and Experimental Validation

Chapter 3

The enhanced Predictor-Corrector model updating technique

Considering the updating method introduced in Section 2.4 as reference, the aim of this chapter is to introduce and describe in detail the developed methodology. Specifically, a novel formulation of the weighting matrices and the regularization parameter, with respect to the reference one given by Eq. (2.18), Eq. (2.19), and Eq. (2.21), respectively, is introduced. Furthermore, a procedure for the numerical relaxation of the solution when dealing with noisy data is proposed.

3.1 Weighting matrices

In the weighted least-squares approach, Ref. [69], weighting matrices act on both equations and unknowns, that is on the components of the residual vector and updating parameters, respectively, when dealing with model updating methodologies as introduced

in Section 2.4.

3.1.1 Weighting matrix for frequency points, $[W_f]$

The weighting matrix for frequency points acts on the components of the residual vector and it should give more weight to those components that show a high correlation, or in other words a low residual. As an intuitive consideration, greater weight can be given to those components of the residual vector that are lower and less uncertain. The variance of a quantity measures how it is far from its expected value, so it can be considered as a measure of its uncertainty. In the developed methodology the following weighting matrix is formulated:

$$[W_f^{var}] = \begin{bmatrix} \ddots & & & \\ & \frac{1}{\varepsilon^2(\omega_k) + \sigma_\varepsilon^2(\omega_k)} & & \\ & & \ddots & \end{bmatrix} \quad (3.1)$$

where $\sigma_\varepsilon^2(\omega_k)$ is the k-th component of the variance vector associated to the residual vector $\{\varepsilon\}$. This component is not a direct measure, but it depends on the experimental FRFs and on the updating parameters, therefore the variance vector $\{\sigma_\varepsilon^2\}$ could be evaluated at each iteration step using the propagation law¹ of the variance over the

¹Let $R = f(x_1, x_2, \dots, x_n)$ be a derived function of $x_i, i = 1, 2, \dots, n$ direct measures, then its variance σ_R^2 can be evaluated through the propagation law according to:

$$\sigma_R^2 = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \sigma_{x_i} \right)^2$$

measured FRFs:

$$\sigma_{\varepsilon_s}^2(\omega_k) = \sum_{l=1}^{N_{FRF}} \left(\frac{\partial \varepsilon_s(\omega_k)}{\partial H_{lk}^X} \right)^2 \sigma_{H_{lk}^X}^2 \quad (3.2)$$

$$\sigma_{\varepsilon_a}^2(\omega_k) = \sum_{l=1}^{N_{FRF}} \left(\frac{\partial \varepsilon_a(\omega_k)}{\partial H_{lk}^X} \right)^2 \sigma_{H_{lk}^X}^2 \quad (3.3)$$

in which H_{lk}^X is the l -th experimental FRF at k -th frequency point, N_{FRF} is the total number of considered FRFs in the updating process, and the subscripts s and a refer to the shape and amplitude correlation function respectively. The partial derivatives of the residual vector, with respect to the experimental data, are evaluated at each iteration step as:

$$\frac{\partial \varepsilon_s(\omega_k)}{\partial H_{jk}^X} = -2 \frac{H_{jk}^A \left(\sum_{l=1}^{N_{FRF}} H_{lk}^A H_{lk}^X \right) \sum_{l=1}^{N_{FRF}} H_{lk}^{A^2} \sum_{l=1}^{N_{FRF}} H_{lk}^{X^2} - H_{jk}^X \left(\sum_{l=1}^{N_{FRF}} H_{lk}^A H_{lk}^X \right)^2 \sum_l H_{lk}^{X^2}}{\left(\sum_{l=1}^{N_{FRF}} H_{lk}^{A^2} \sum_{l=1}^{N_{FRF}} H_{lk}^{X^2} \right)^2} \quad (3.4)$$

$$\frac{\partial \varepsilon_a(\omega_k)}{\partial H_{jk}^X} = -2 \frac{\text{sign} \left(\sum_{l=1}^{N_{FRF}} H_{lk}^X H_{lk}^A \right) H_{jk}^A \left(\sum_{l=1}^{N_{FRF}} H_{lk}^{A^2} + \sum_{l=1}^{N_{FRF}} H_{lk}^{X^2} \right) - 2 \left| \sum_{l=1}^{N_{FRF}} H_{lk}^X H_{lk}^A \right| H_{jk}^X}{\left(\sum_{l=1}^{N_{FRF}} H_{lk}^{A^2} + \sum_{l=1}^{N_{FRF}} H_{lk}^{X^2} \right)^2} \quad (3.5)$$

in which H_{lk}^A is the analytical counterpart of the experimental FRF H_{lk}^X previously introduced. When the numerical analysis is considered, the data variance can be analytically computed, whereas in case of real experimental data the variance matrix can be evaluated at each frequency point from Ref. [19] as:

$$\sigma_{H_{lk}^X}^2(\omega_k) = \frac{1}{2N_a} \frac{1 - \gamma^2(\omega_k)}{\gamma^2(\omega_k)} |H_{lk}^X(\omega_k)|^2 \quad (3.6)$$

where N_a is the number of averages set in the acquisition system and $\gamma^2(\omega_k)$ is the coherence function, Eq. (1.32), evaluated at the k -th frequency point and directly available from tests. The final expression of the residual vector variance is given, at each frequency point, by:

$$\{\sigma_\varepsilon^2\} = \left\{ \begin{array}{c} \sigma_{\varepsilon_s}^2(\omega_1) \\ \vdots \\ \sigma_{\varepsilon_s}^2(\omega_{N_f}) \\ \sigma_{\varepsilon_a}^2(\omega_1) \\ \vdots \\ \sigma_{\varepsilon_a}^2(\omega_{N_f}) \end{array} \right\} \quad (3.7)$$

The weighting matrix $[W_f^{var}]$ could be then normalized as:

$$[W_f] = \frac{[W_f^{var}]}{Tr([W_f^{var}])} \quad (3.8)$$

where $Tr(\cdot)$ denotes the trace operator.

3.1.2 Weighting matrix for parameters, $[W_p]$

The weighting matrix for parameters constrains the rate of change in the design parameters avoiding excessive parameter variations. The formulation proposed in this thesis starts from the second-order Taylor expansion of the quadratic approximation of the penalty function, Eq. (2.10), considering the derivatives of the sensitivity matrix with respect to the updating parameters in the Hessian matrix. These derivatives were neglected in the reference method, Eq. (2.12), and their inclusion could be useful for taking into account significant non-linearities that affect the correlation functions, i.e. the residual vector, with respect to the updating parameters. Referring to Eq. (2.12), the Hessian

matrix is now evaluated as follows:

$$\begin{aligned} [Hess_0] &= 2 [S_0]^T [W_f] [S_0] - 2 \left[\frac{\partial [S_0]^T}{\partial p_1} [W_f] \{\varepsilon_0\}, \dots, \frac{\partial [S_0]^T}{\partial p_{N_p}} [W_f] \{\varepsilon_0\} \right] \\ &= [Hess_{0,1}] + [Hess_{0,2}] \end{aligned} \quad (3.9)$$

where $[Hess_{0,1}]$ refers to the terms not containing derivatives, whereas $[Hess_{0,2}]$ groups the terms containing derivatives all together. In order to reduce the computational time for the calculation of derivatives, it is proposed to introduce the diagonal matrix $[B]$, that approximates $[Hess_{0,2}]$, as:

$$[Hess_0^{app}] = [Hess_{0,1}] + 2 [B] \quad (3.10)$$

Considering a generic variation $\{\Delta p\}$ and the limit $\{\Delta p\} \rightarrow \{0\}$, the Hessian matrix can be written as:

$$[Hess_0] \{\Delta p\} = \{\Delta g_0\} \quad (3.11)$$

which allows to evaluate the diagonal matrix $2 [B]$ by using the backward finite difference (superscript b) at the n -th iteration step, that is:

$$([Hess_{0,1}] + 2 [B]) \{\Delta^b p\} = \{\Delta^b g_0\} \quad (3.12)$$

where $\{\Delta^b g_0\} = \{g_0\}^n - \{g_0\}^{n-1}$ and $\{\Delta^b p\} = \{p\}^n - \{p\}^{n-1}$, and finally Eq. (3.12) is solved for the diagonal matrix $2 [B]$:

$$2 [B] = \text{diag} \left[[P] \left(\{\Delta^b g_0\} - [Hess_{0,1}] \{\Delta^b p\} \right) \right] \quad (3.13)$$

where the absolute value has been introduced to keep the Hessian matrix positive semi-definite, and $[P]$ is evaluated as:

$$[P] \equiv \begin{bmatrix} \ddots & & & \\ & \frac{1}{|\Delta^b p_i|} & & \\ & & \ddots & \end{bmatrix} \quad (3.14)$$

Eq. (3.13) gives an estimate of $[Hess_{0,2}]$, constrained to be diagonal and positive semi-definite, and it is used to define the normalized weighting matrix $[W_p]$:

$$[W_p] = \frac{[B]}{Tr([B])} \quad (3.15)$$

It is worth remarking that $\{\Delta^b p\}$ cannot be defined at the first iteration step, so Eq. (3.15) is introduced starting from the second iteration step, whereas Eq. (2.19) is used in the first one.

The proposed weighting matrix for parameters constrains those parameters for which the quadratic approximation containing only $[Hess_{0,1}]$, i.e. Eq. (2.13), is not adequate to approximate the penalty function F_w introduced in Eq. (2.8). The relative weight between the quadratic approximation and the limiting function, in which $[W_p]$ acts, is quantified by the regularization parameter k_p whose selection criterion is introduced in the next section.

3.2 Choice of regularization parameter

Referring to the minimization problem stated in Eq. (2.9), the contribution of the limiting function $J_2(\{\Delta p\})$ to the overall penalty function $J(\{\Delta p\})$ is measured by the

regularization parameter $k_p > 0$. Among the several criteria available in literature for selecting the optimal value of regularization parameter, Refs. [62, 63, 64], the L-curve criterion is the most promising. It has a wide range of applicability and it is also used in multi-objective optimization. In order to explain how the L-curve criterion works, the following functions called *residue* and *constraint*, respectively, are introduced:

$$R(\{\Delta p\}) \equiv \left\| [W_f]^{1/2} (\{\varepsilon_0\} - [S_0] \{\Delta p\}) \right\|_2 \quad (3.16)$$

$$C(\{\Delta p\}) \equiv \left\| [W_p]^{1/2} \{\Delta p\} \right\|_2 \quad (3.17)$$

Using the above-introduced quantities, the penalty function Eq. (2.9) is rewritten in an equivalent form:

$$J(\{\Delta p\}) = R^2(\{\Delta p\}) + k_p C^2(\{\Delta p\}) \quad (3.18)$$

The solution vector can be easily obtained by imposing the stationarity of the penalty function with respect to $\{\Delta p\}$ for a fixed value of k_p , leading to a parameterized solution of Eq. (2.17). In this way, each value of k_p gives a different solution vector $\{\Delta p\}$, and then of the residue and constraint functions, Eqs. (3.16) and (3.17) respectively. As it can be noted from Eq. (2.17), the solution vector is obtained through an inverse matrix and the computational time required to iteratively solve the minimization problem could be greatly reduced by using the Generalized Singular Value Decomposition (GSVD), Ref. [70], which is introduced in the next section.

3.2.1 Generalized Singular Value Decomposition

The GSVD, Ref. [70], allows to write the solution vector $\{\Delta p\}$ as a sum of rational functions in k_p thus reducing the computational time with respect to the solution given by Eq. (2.17). The GSVD works by introducing the simultaneous diagonalization of two rectangular matrices with an equal number of columns and an arbitrary number of rows. Referring to Eqs. (3.16) and (3.17), the GSVD of the matrix pair $[W_f]^{1/2} [S_0] \in \mathbb{R}^{2N_f \times N_p}$ and $[W_p]^{1/2} \in \mathbb{R}^{N_p \times N_p}$ is:

$$[W_f]^{1/2} [S_0] = [U] [\Sigma] [X]^T \quad (3.19)$$

$$[W_p]^{1/2} = [V] [\Pi] [X]^T \quad (3.20)$$

where

$$[U] \in \mathbb{R}^{2N_f \times N_p}, \quad [\Sigma] \in \mathbb{R}^{N_p \times N_p}, \quad [X] \in \mathbb{R}^{N_p \times N_p}$$

$$[V] \in \mathbb{R}^{N_p \times N_p}, \quad [\Pi] \in \mathbb{R}^{N_p \times N_p}$$

The matrices $[U]$ and $[V]$ are orthogonal, so that

$$[U]^T [U] = [I], \quad [V]^T [V] = [I] \quad (3.21)$$

and $[\Sigma]$ and $[\Pi]$ are diagonal matrices that satisfy

$$[\Sigma]^T [\Sigma] + [\Pi]^T [\Pi] = [I]$$

In order to introduce the GSVD into the solution vector, Eq. (2.17) is written as follows:

$$\{\Delta p(k_p)\} = \left[\left([W_f]^{1/2} [S_0] \right)^T \left([W_f]^{1/2} [S_0] \right) + k_p \left([W_p]^{1/2} \right)^T \left([W_p]^{1/2} \right) \right]^{-1} \cdot \left([W_f]^{1/2} [S_0] \right)^T [W_f]^{1/2} \{\varepsilon_0\}$$

and, introducing Eq. (3.19) and Eq. (3.20) into the previous equation and defining:

$$[X]^{-T} \equiv ([X]^{-1})^T \quad (3.22)$$

$$\{z\} \equiv [U]^T [W_f]^{1/2} \{\varepsilon_0\} \quad (3.23)$$

$$\sigma_r \equiv \Sigma_{rr} \quad (3.24)$$

$$\mu_r \equiv \Pi_{rr} \quad (3.25)$$

it is possible to write

$$\{\Delta p(k_p)\} = \sum_{r=1}^{N_p} \frac{1}{1 + k_p \frac{\mu_r^2}{\sigma_r^2}} \left(\frac{z_r}{\sigma_r} \right) [X]_r^{-T} \quad (3.26)$$

where $[X]_r^{-T}$ is the r -th column of the matrix $[X]^{-T}$. Eq. (3.26) shows that the solution vector can be computed as a sum of rational functions in k_p through the inversion of $[X]^T$, at each iteration. This procedure reduces the computational time with respect to the inversion that appears in Eq. (2.17). Indeed, Eq. (2.17) requires a matrix inversion for each value of k_p at each iteration, whereas Eq. (3.26) requires only one matrix inversion for all values of k_p at each iteration.

3.2.2 L-curve criterion

As mentioned before, the L-curve criterion seems to be the most promising for selecting the optimal value of the regularization parameter k_p . Each value of k_p gives a different solution vector $\{\Delta p\}$, and then different values of the residue and constraint functions, Eqs. (3.16) and (3.17), respectively. One way of obtaining the regularization parameter is to plot the values of the constraint function against the residue ones, obtained for different values of k_p . In Ref. [71] it is showed that the constraint is a monotonically decreasing function of the residue. In particular, it is pointed out that for a reasonable signal-to-noise ratio and satisfaction of the Picard condition, Ref. [72], in the constraint versus residue plot the curve is approximately vertical for $k_p < k_p^{opt}$, and becomes a horizontal line when $k_p > k_p^{opt}$, with a corner near the optimal regularization parameter k_p^{opt} . The curve is called L-curve because of this behavior. It is shown in Ref. [73] that the characteristic L-shaped corner is most pronounced in a log-log plot, then k_p^{opt} corresponds to the point with maximum curvature at the corner of the log-log plot of the L-curve.

In order to apply the L-curve criterion to the minimization problem described by Eq. (3.18), given a range of variability and a multiplicative step of k_p , i.e. $k_p \in [k_p^{low}, k_p^{up}]$, the logarithmic values of the residue and constraint are evaluated:

$$LR(k_p) \equiv \log_{10}(R(k_p)) \quad (3.27)$$

$$LC(k_p) \equiv \log_{10}(C(k_p)) \quad (3.28)$$

and normalized with respect to the difference between their maximum and minimum values within the variability range:

$$LRn(k_p) = \frac{LR(k_p)}{LR(k_p)_{max} - LR(k_p)_{min}} \quad (3.29)$$

$$LCn(k_p) = \frac{LC(k_p)}{LC(k_p)_{max} - LC(k_p)_{min}} \quad (3.30)$$

The L-curve is defined as the set of points $Pt(k_p)$ whose coordinates are:

$$Pt(k_p) \equiv (LRn(k_p), LCn(k_p)) \quad (3.31)$$

After choosing the point of minimum values as:

$$O \equiv \left(\min_{k_p \in [k_p^{low}, k_p^{up}]} (LRn), \min_{k_p \in [k_p^{low}, k_p^{up}]} (LCn) \right)$$

the distance function between the generic point $Pt(k_p)$ and the origin O is defined:

$$d(k_p) \equiv \|Pt(k_p) - O\|_2 \quad (3.32)$$

and the optimal value k_p^{opt} is chosen so that the distance function assumes the minimum value. In Fig. 3.1 a typical L-curve, evaluated for different values of the parameter k_p (red circles) is shown. In the same figure, the minimum distance function is highlighted with the red straight line together with the corresponding value of the optimal value k_p^{opt} .

Figure 3.1: Typical L-curve with emphasis on the point of minimum distance from the origin.

3.3 Robustness of the algorithm in case of noisy data

Experimental data are generally noisy and it could cause the divergence of the updating algorithm. To prevent this behavior, a relaxation of the solution that takes into account the uncertainty associated to each component of the updating parameter vector is introduced. The solution vector is parameterized through a diagonal damping matrix $[\Lambda(\eta)]$ depending on an unknown parameter, denoted as η , that will be evaluated at each iteration step following the L-curve based procedure previously described to estimate k_p^{opt} .

The following expression of the damped solution vector is introduced:

$$\{\Delta p^d(\eta)\} = [\Lambda(\eta)] \{\Delta p(k_p^{opt})\} \quad (3.33)$$

where

$$[\Lambda(\eta)] \equiv \begin{bmatrix} \ddots & & & \\ & \frac{1}{1+\eta \frac{|\delta \Delta p_i(k_p^{opt})|}{|\Delta p_i(k_p^{opt})|}} & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad (3.34)$$

As it can be noted from Eq. (3.33), the damped solution vector is evaluated after $\{\Delta p(k_p^{opt})\}$ and if there are no uncertainties the two solutions coincide, otherwise the unknown damping parameter η reduces the variations of those parameters affected by higher uncertainties, avoiding the divergence of the algorithm. In Eq. (3.34), $\delta \Delta p_i(k_p^{opt})$ represents the uncertainty on the i -th component of the solution vector and it can be evaluated by considering the propagation law. In particular, starting from the uncertainty on the FRFs, which are directly available from the experimental tests, it is possible to evaluate the uncertainty on the residual vector, on the components of $\{z\}$, Eq. (3.23), and finally on the components of the solution vector given by Eq. (3.26). The derivation of all the expressions is now described in detail. The uncertainty on the experimental FRF is evaluated considering its variance, Eq. (3.6), that is:

$$\delta H_{lk}^X = \frac{\sigma_{H_{lk}^X}}{\sqrt{N_a}} \quad (3.35)$$

By considering Eqs. (3.4) and (3.5), and the propagation law, the uncertainties on the mean values of the components of the residual vector $\{\varepsilon\}$, i.e., those calculated in the mean value of the data, can be estimated at each k -th frequency point as:

$$\delta \varepsilon_s(\omega_k) = \sqrt{\sum_{l=1}^{N_{FRF}} \left(\frac{\partial \varepsilon_s(\omega_k)}{\partial H_{lk}^X} \right)^2 \delta H_{lk}^{X^2}} \quad (3.36)$$

$$\delta\varepsilon_a(\omega_k) = \sqrt{\sum_{l=1}^{N_{FRF}} \left(\frac{\partial\varepsilon_a(\omega_k)}{\partial H_{lk}^X} \right)^2 \delta H_{lk}^{X^2}} \quad (3.37)$$

The uncertainty on the components of $\{z\}$ is evaluated as:

$$\delta z_r = \sqrt{\sum_{q=1}^{2N_f} \left(\frac{\partial z_r}{\partial \varepsilon_q} \right)^2 \delta \varepsilon_q^2} \quad (3.38)$$

where the partial derivative is derived analytically considering Eq. (3.23) and the dependence of $[W_f]$ on the residual vector, Eq. (3.1), that is:

$$\frac{\partial z_r}{\partial \varepsilon_q} = U_{qr} W_{fq}^{1/2} - U_{qr} W_{fq}^{3/2} \varepsilon_{0q}^2 \quad (3.39)$$

Finally, the uncertainty on the solution vector can be evaluated by applying the propagation law to the uncertainty on $\{z\}$:

$$\delta \Delta p_i(k_p^{opt}) = \sqrt{\sum_{r=1}^{N_p} \left(\frac{\partial \Delta p_i}{\partial z_r}(k_p^{opt}) \right)^2 \delta z_r^2} \quad (3.40)$$

where, considering Eq. (3.26)

$$\frac{\partial \Delta p_i(k_p^{opt})}{\partial z_r} = \frac{1}{1 + k_p^{opt} \frac{\mu_r^2}{\sigma_r^2}} \frac{1}{\sigma_r} X_{ir}^{-T} \quad (3.41)$$

Once the uncertainty on the solution vector has been estimated, a selection criterion for the damped parameter η has to be introduced in order to calculate the damping matrix, Eq. (3.34) and then the damped solution vector Eq. (3.33).

Following a procedure similar to the one introduced for the selection of k_p^{opt} , the damped parameter is obtained by plotting the values of the damped penalty function against its uncertainty with respect to the residual vector for different values of η in a given

variability range. Considering Eq. (2.9), the damped penalty function is written as:

$$J^d(\{\Delta p^d(\eta)\}) = J_1(\{\Delta p^d(\eta)\}) + k_p^{opt} J_2(\{\Delta p(\eta)^d\}) \quad (3.42)$$

that is, Eq. (2.15) is evaluated considering the damped solution vector Eq. (3.33) and the optimal value of k_p . The uncertainty on the damped penalty function with respect to the residual vector is evaluated by applying the propagation law:

$$\delta J^d(\{\Delta p^d(\eta)\}) = \sqrt{\sum_{q=1}^{2N_f} \left(\frac{\partial J^d(\{\Delta p^d(\eta)\})}{\partial \varepsilon_{0,q}} \right)^2} \delta \varepsilon_{0,q}^2 \quad (3.43)$$

where the partial derivative can be derived analytically considering the equations introduced previously. In order to select the optimal value of η , a range of variability and a multiplicative step, i.e. $\eta \in [\eta^{low}, \eta^{up}]$, are chosen and the following logarithmic values are introduced:

$$LJ^d(\eta) \equiv \log_{10}(J^d(\eta)) \quad (3.44)$$

$$L\delta J^d(\eta) \equiv \log_{10}\left(\frac{\delta J^d}{J^d}(\eta)\right) \quad (3.45)$$

where, for the sake of brevity, the dependence of the functions on $\{\Delta p^d(\eta)\}$ has been replaced with η . Similarly to Eqs. (3.29) and (3.30), the above-introduced functions are normalized with respect to the difference between their maximum and minimum values within the variability range:

$$LJ^d n(\eta) = \frac{LJ^d(\eta)}{LJ^d(\eta)_{max} - LJ^d(\eta)_{min}} \quad (3.46)$$

$$L\delta J^d n(\eta) = \frac{L\delta J^d(\eta)}{L\delta J^d(\eta)_{max} - L\delta J^d(\eta)_{min}} \quad (3.47)$$

The new curve is defined as the set of points $Pt(\eta)$ whose coordinates are:

$$Pt(\eta) \equiv (LJ^d n(\eta), L\delta J^d n(\eta)) \quad (3.48)$$

Following the procedure of the previous section, the point of minimum values is chosen as the origin O :

$$O \equiv \left(\min_{\eta} (LJ^d n), \min_{\eta} (L\delta J^d n) \right) \quad (3.49)$$

and the distance function is defined as the distance of a generic point of the curve $Pt(\eta)$ from the origin O :

$$d(\eta) \equiv \|Pt(\eta) - O\|_2 \quad (3.50)$$

The curve described in Eq. (3.48) is computed for different values of η and its optimal value η^{opt} is chosen so that the distance function assumes its minimum value. In Fig. 3.2 a typical curve parameterized with respect to η (red circles) is shown. The minimum distance function is reported with the straight red dashed line together with the optimal value of the parameter, η^{opt} . Once this parameter is estimated, $\{\Delta p^d\}$ can be calculated at each iteration step from Eq. (3.33) using $\eta = \eta^{opt}$.

Figure 3.2: Typical curve for the selection of η^{opt} .

3.4 Summary of the enhanced methodology

The enhanced updating methodology could be summarized in the following steps that have to be performed iteratively until the convergence of the solution vector $\{\Delta p\}$ is reached:

1. comparison between the numerical and the experimental model through the evaluation of the correlation functions $\chi_s(\omega)$, Eq. (1.38), and $\chi_a(\omega)$, Eq. (1.39);
2. evaluation of the weighting matrix $[W_f]$ for frequency points that gives more weight to the components of the residual vector that are lower and less uncertain, Section 3.1.1;
3. evaluation of the weighting matrix $[W_p]$ for parameters that takes into account the derivatives of the sensitivity matrix with respect to the updating parameters,

Section 3.1.2;

4. evaluation of the optimal value of the regularization parameter k_p^{opt} from the L-curve criterion, through the minimization of a distance function, Section 3.2;
5. evaluation of a damping parameter η^{opt} for the robustness of the algorithm in case of noisy data, Section 3.3;
6. solution vector from Eq. (2.17), considering the damped solution given by Eq. (3.33).

These steps are shown in Fig. 3.3 by means of a flowchart.

Figure 3.3: Flowchart of the enhanced methodology.

3.5 Implementation of the enhanced methodology

Both reference and enhanced identification procedures have been implemented into MatLab® for their validation through simple FE models as those corresponding to beams and plates. In order to apply the developed methodology to more complex structures, it would be helpful to take advantage of a commercial software to perform the dynamic analysis of large FE models, correlate them with their experimental counterparts, and update the numerical models. The commercial software FEMtools® allows to do this. It is a multi-functional, cross-platform and solver-independent software suite providing advanced analyses and scripting solutions for several kinds of applications, Ref. [74]. The software is provided with tools for dynamic analysis, correlation analysis, sensitivity analysis, FE model updating, and optimization. In addition, it can be complemented with add-on products such as NASTRAN Interface for the finite element analysis and UNIVERSAL FILE Interface to import and export experimental models. The functionality of FEMtools® can be customized and extended through FEMtools Script, Ref. [75], which is an integrated programming language that makes use of a library of functions called FEMtools API to communicate with the FEMtools database and perform user interface customization. This feature allowed the creation of a numerical platform for the enhanced identification methodology by interfacing FEMtools® with the MatLab® code in order to apply the method to any kind of structure. Specifically, the interface works in FEMtools® by calling the external MatLab® code as follows:

1. Import of the experimental model in terms of its geometry, modal parameters, and FRFs;
2. Import of the FE model created with NASTRAN, evaluation of its modal parameters and FRFs (in correspondence of measurement points);
3. Evaluation of shape and amplitude correlation functions that define the residual vector, definition of parameters to update, computation of sensitivity matrix, saving these quantities in order to introduce them into the MatLab code;
4. Run the MatLab external code in which the core block of Fig. 3.3 is implemented to provide the solution vector $\{\Delta p\}$;
5. Import of the solution vector into the FEMtools[®] database, evaluation of the updated modal parameters, FRFs, correlation functions, and repetition of the above steps until convergence is reached, through specific criteria introduced by the user.

It is important to note that the FEMtools[®] Model Updating tool in its standard configuration uses an iterative sensitivity-based method whose residual vector can be defined in terms of modal parameters, FRFs, or shape and amplitude correlation functions between FRFs, Ref. [76]. In this last case, the model updating is performed through the Grafe method introduced in Chapter 2, thus giving the opportunity to compare the results from the reference and enhanced updating methodologies.

Chapter 4

Numerical test cases

The enhanced identification methodology introduced in Chapter 3 is now validated by analyzing several numerical test cases in order to verify its capability to identify dynamic models. The numerical validation of a methodology is of paramount importance to assess its accuracy. In general, two nominal FE models are separately considered to represent the *simulated* experimental model and the initial numerical model to be updated. Precisely, the mass and/or stiffness of some elements of the numerical model are modified with respect to the experimental one, considered as reference, according to known changes that have to be identified by the updating methodologies in order to maximize the correlation between the two models. In this way, the solution which the updated numerical model has to tend to is known, thus giving the opportunity to critically analyze the results. The case studies described in this chapter refer to a cantilever beam, a structure that can be easily modeled, reducing computational efforts, and for which there exists a closed-form solution, thus allowing the assessment of the methodol-

ogy. In addition, the same structure will be used in the next chapter for the experimental validation of the methodology, taking advantage of its simplicity when introducing real data for which the solution is not known. The proposed methodology is validated by comparing the results given by the enhanced approach with the ones coming from the reference method introduced in Section 2.4, to show if and how it is effectively improved.

4.1 The cantilever beam

The numerical validation of the enhanced methodology has been performed considering an aluminum cantilever beam, whose physical properties are reported in Table 4.1.

Table 4.1: Physical properties of the numerical cantilever beam.

$L(m)$	$b(m)$	$t(m)$	$E(GPa)$	$\rho(kg/m^3)$
0.8	0.02	0.005	70	2700

Figure 4.1: Nodes numbering of the numerical cantilever beam.

The beam has been modeled as an Euler-Bernoulli beam, i.e., shear and axial strains have been neglected, and discretized with 8 finite elements, Fig. 4.1. Displacements in

the transverse z direction, and the bending rotation around the y -axis, have been taken as the two degrees of freedom available at each node, so the resulting FE model has 16 DOFs in total, considering that the first node is fixed. This FE model represents the reference structure, i.e. the simulated experimental model to be identified, and it assures the numerical convergence of the modal model, that is the number of elements used to discretize the beam are sufficient to well identify all the modal parameters in the frequency range of analysis. The algorithm settings used in this case study are

Table 4.2: Settings of the algorithm used for the numerical cantilever beam.

Range (Hz)	N_f	N_a	N_{FRF}	k_ω
0-256	1024	20	4	0.07

reported in Table 4.2. Specifically, the following parameters have to be set: the frequency range, the number of frequency points N_f , the number of averages N_a that simulate the experimental FRFs acquisition, the number of FRFs N_{FRF} , and the parameter $k_\omega \geq 0$ that allows the exclusion of those frequency points ω_k around the natural frequencies ω_n that could cause numerical instabilities if damping is not taken into account in the numerical model. Such parameter satisfies:

$$\omega_n (1 - k_\omega) < \omega_k < \omega_n (1 + k_\omega) \quad (4.1)$$

The presence of noise has been considered for the experimental model by adding 10% multiplicative and uniformly distributed random noise to each FRF according to:

$$H_{ij}^X(\omega) \rightarrow H_{ij}^X(\omega) (1 + 0.10\epsilon) \quad (4.2)$$

where ϵ is a random number between -1 and 1. Each experimental FRF is obtained by averaging N_a single measures trying to simulate the experimental FRF acquisition. Referring to Fig. 4.1, the frequency response functions $H_{2,4}$, $H_{4,4}$, $H_{6,4}$, $H_{8,4}$ defined on the vertical displacement DOF have been used in the analysis, thus simulating a Single Input Multiple Output (SIMO) test. It is worth remarking that in general, Multiple Input Multiple Output (MIMO) systems can be considered in the analysis. The numerical model to be updated is obtained modifying the stiffness and/or the mass of some elements of the above introduced FE model. Referring to the well-known expressions of the elemental mass and stiffness matrices of a beam, Ref. [77]:

$$[M^e] = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -13L^2 & -22L & 4L^2 \end{bmatrix} \quad (4.3)$$

$$[K^e] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (4.4)$$

a change in the mass could be seen as a change in the density or area of the elemental beam, whereas a change in the stiffness could suggest changes in the Young's modulus or moment of inertia of the elemental beam, given a fixed length of the structure. Several test cases have been considered for the validation of the proposed approach by introducing stiffness and/or mass changes in a reference numerical model, and the most significative ones will be described in detail in the following sections. Specifically, considering the reference stiffness and mass elemental matrices given by Eqs. (4.3) and (4.4), a single change of stiffness and mass, and multiple changes of stiffness have been imposed to

assess the methodology. Further test cases can be found in Refs. [78, 79].

4.1.1 Case 1: -35% stiffness of first element

In this case study, the numerical model to be updated is obtained by reducing the stiffness of the first element by an amount of 35% , a variation that allows to clearly show the discrepancies between the (simulated) experimental model and the initial numerical model. Considering the p-value formulation, Eq. (2.2), the parameter concerning the first elemental stiffness matrix is equal to 0.65 , and the vector of the updating parameter $\{p\}$ is represented by 8 parameters, initially unitary, associated to each elemental stiffness matrix. The aim of the identification methodologies is to find the value of the first parameter equal to $\frac{1}{1-0.35} = 1.54$ keeping unchanged the other parameters, that is with an unitary value. Table 4.3 summarizes the imposed stiffness variation and the values of the parameters that should be identified by the updating methodologies. In the

Table 4.3: Numerical cantilever beam case 1: target values of the updating parameters.

Element #	$\Delta K (\%)$	Target p^k
1	-35	1.54
2	0	1
3	0	1
4	0	1
5	0	1
6	0	1
7	0	1
8	0	1

considered frequency range of analysis, four natural frequencies are present and Table

4.4 shows that the imposed change of stiffness of the first element causes a shift of the numerical natural frequencies f_n^A with respect to the experimental ones f_n^X , considered as the reference values. In addition, Fig. 4.2 shows that the points pairing the measured and predicted natural frequencies do not perfectly lie on the line of slope 1 but they slightly deviate confirming the presence of some differences between the structural properties of the (simulated) experimental model and the numerical one. The correlation between the measured and predicted mode shapes is reported in Fig. 4.3(a) in terms of the Modal Assurance Criterion, whereas Fig. 4.3(b) shows how the individual points, pairing the components of each experimental mode shape against their numerical counterparts, lie on a straight line passing through the origin, thus highlighting the excellent correlation between the mode sets. Both reference and enhanced updating methodologies are carried out to identify the values of the stiffness updating parameters capable to improve the numerical-experimental correlation, thus reducing those natural frequency shifts.

Table 4.4: Numerical cantilever beam case 1: initial natural frequency shifts.

Mode #	f_n^A (Hz)	f_n^X (Hz)	$ \Delta f $ (%)
1	5.80	6.43	9.70
2	38.13	40.27	5.35
3	109.02	112.83	3.38
4	215.63	221.46	2.63

As reported in Fig. 4.4 and Fig. 4.5, the final correlation functions, $\chi_s(\omega)$ and $\chi_a(\omega)$, are almost unitary in all the frequency range of analysis, i.e. both reference and enhanced methods have maximized the correlation between the FE model and the simulated experimental model. Such an improvement of the correlation functions allows

Figure 4.2: Numerical cantilever beam case 1: comparison between the measured and predicted natural frequencies.

Table 4.5: Numerical cantilever beam case 1: final natural frequency shifts.

Mode #	$ \Delta f_n ^i$ (%) Initial	$ \Delta f_n ^u$ (%) Reference method	$ \Delta f_n ^u$ (%) Enhanced method
1	9.70	0.94	1.47 E-4
2	5.35	0.69	3.91 E-2
3	3.38	0.02	0.23 E-2
4	2.63	0.51	0.42 E-2

the identification of the assumed experimental natural frequencies, as shown in Figs. 4.6-4.9 where the initial analytical FRFs, denoted as H^a , move to the updated FRFs, H^u , matching the measured ones, H^x , thus reducing the natural frequency shifts, as it can be observed from Fig. 4.10 and Table 4.5. In this Table, $|\Delta f_n|^i$ and $|\Delta f_n|^u$ indicate the initial and the updated natural frequency shifts, respectively. Although both methodologies improve the numerical-experimental correlation, in terms of FRFs correlation functions, and reduce the natural frequency shifts, they behave differently when considering the updating parameters. Referring to the target values reported in

(a) MAC

(b) Modal Coordinates

Figure 4.3: Numerical cantilever beam case 1: initial correlation between the experimental and numerical mode sets.

Table 4.3, Fig. 4.11(a) shows that the reference method does not identify the exact value of the first parameter, suggesting also not unitary values for the other parameters, that are supposed to be unchanged by the updating methodology, thus providing an equivalent dynamic model that matches the experimental one. On the contrary, the target values of parameters are perfectly reached by the enhanced method as illustrated in Fig. 4.11(b), highlighting a real improvement with respect to the reference method. It should be

Figure 4.4: Numerical cantilever beam case 1: comparison between the shape correlation functions χ_s .

Figure 4.5: Numerical cantilever beam case 1: comparison between the amplitude correlation functions χ_a .

stated that the convergence history of the updating parameters, as the one reported in Fig. 4.11, is a valuable tool to assess the numerical procedure. Indeed, it allows to either verify that convergence is smoothly reached or highlight a fluctuating behavior of the parameters that could cause the wrong identification of the actual dynamic properties.

Figure 4.6: Numerical cantilever beam case 1: comparison between the $H_{2,4}$ FRF.

Figure 4.7: Numerical cantilever beam case 1: comparison between the driving point FRF.

For the sake of completeness, Table 4.6 lists the values of parameters obtained with both methods.

Figure 4.8: Numerical cantilever beam case 1: comparison between the $H_{6,4}$ FRF. Figure 4.9: Numerical cantilever beam case 1: comparison between the $H_{8,4}$ FRF.

4.1.2 Case 2: +25% mass of sixth element

In this case study, the modified structure is obtained by increasing the mass of the sixth element by 25% in order to verify how the proposed methodology behaves when small changes of mass are imposed. Referring to Eq. (2.1), this change acts on the sixth

Figure 4.10: Numerical cantilever beam case 1: comparison between the natural frequency shifts.

Figure 4.11: Numerical cantilever beam case 1: convergence history of the updating parameters.

elemental mass matrix with the corresponding parameter equal to 1.25. In this case, the vector of the updating parameters $\{p\}$ is given by the 8 parameters associated to each elemental mass matrix, and the updating methodologies have to identify the sixth parameter equal to $\frac{1}{1+0.25} = 0.8$ keeping unchanged the others. Table 4.7 summarizes

Table 4.6: Numerical cantilever beam case 1: comparison between the final updating parameters.

Parameter #	Initial value	Target value	Updated Reference method	Updated Enhanced method
1	1	1.54	1.37	1.53
2	1	1	1.10	1
3	1	1	1.02	1
4	1	1	1.03	1
5	1	1	1.01	0.99
6	1	1	1.03	1
7	1	1	0.99	1
8	1	1	1	1

the imposed mass variation and the values of parameters that should be identified by the updating methodologies. The effect of increasing the mass of an element is a slight variation of the numerical natural frequencies with respect to the reference ones, as shown in Table 4.8 and Fig. 4.12. The excellent correlation between the mode shapes in terms of the MAC matrix is not reported for the sake of brevity, being it very similar to the one introduced in the previous section, that is Fig. 4.3(a). The aim of the reference and enhanced updating methodologies is to identify the target values of the mass updating parameters in order to maximize the numerical-experimental correlation.

Because of the low imposed mass variation, the initial correlation between the simulated experimental model and the modified structure, in terms of $\chi_s(\omega)$ and $\chi_a(\omega)$, is unitary in all the frequency range of analysis except for some frequency points in which it is improved by both methods as it can be noted from Figs. 4.13 and 4.14. Such an improvement of the correlation functions allows the updated driving point FRF H^u to overlap the experimental one H^x , as shown in Fig. 4.15. It is worth remarking that all

Table 4.7: Numerical cantilever beam case 2: target values of the updating parameters.

Element #	$\Delta M (\%)$	Target p^m
1	0	1
2	0	1
3	0	1
4	0	1
5	0	1
6	+25	0.8
7	0	1
8	0	1

Table 4.8: Numerical cantilever beam case 2: initial natural frequency shifts.

Mode #	$f_n^A (Hz)$	$f_n^X (Hz)$	$ \Delta f (\%)$
1	6.30	6.43	2.02
2	39.94	40.27	0.82
3	110.27	112.83	2.27
4	219.69	221.46	0.80

Figure 4.12: Numerical cantilever beam case 2: comparison between the measured and predicted natural frequencies.

the FRFs considered in the analysis are well correlated, but only the driving point one is reported to not weigh the writing down. Regarding the final shifts between the natural

Figure 4.13: Numerical cantilever beam case 2: comparison between the shape correlation functions χ_s .

Figure 4.14: Numerical cantilever beam case 2: comparison between the amplitude correlation functions χ_a .

frequencies, Fig. 4.16 and Table 4.9 show that they are greatly reduced. Similarly to the previous case study, Fig. 4.17 and Table 4.10 highlight two different equivalent dynamic models that match the experimental one. Regarding the reference method, Fig. 4.17(a), the error element, that is the sixth parameter, does not assume the expected value of

Figure 4.15: Numerical cantilever beam case 2: comparison between the driving point FRF.

Figure 4.16: Numerical cantilever beam case 2: comparison between the natural frequency shifts.

0.8, whereas it is well identified by the enhanced method, Fig. 4.17(b). Also this case study allows a satisfactory validation of the proposed identification methodology.

Figure 4.17: Numerical cantilever beam case 2: convergence history of the updating parameters.

Table 4.9: Numerical cantilever beam case 2: final natural frequency shifts.

Mode #	$ \Delta f_n ^i$ (%)	$ \Delta f_n ^u$ (%)	$ \Delta f_n ^u$ (%)
	Initial	Reference method	Enhanced method
1	2.02	0.13	0.45 E-2
2	0.82	0.25	1.90 E-2
3	2.27	0.39	1.11 E-2
4	0.80	0.26	1.08 E-2

Table 4.10: Numerical cantilever beam case 2: comparison between the final updating parameters.

Parameter #	Initial value	Target value	Updated Reference method	Updated Enhanced method
1	1	1	1	1
2	1	1	0.99	1
3	1	1	0.98	0.99
4	1	1	0.99	1
5	1	1	0.98	0.99
6	1	0.8	0.88	0.8
7	1	1	0.96	0.99
8	1	1	0.98	0.99

4.1.3 Case 3: variation of all stiffness parameters

The aim of this case study is to validate the proposed methodology by modifying all the stiffness parameters to simulate the very general scenario that could occur when dealing with real-world applications, in which all the elements could be source of mismodeling. Table 4.11 lists the changes of stiffness imposed to each elemental stiffness matrix to create the modified numerical model. The corresponding changes in the natural frequencies are remarkable with respect to the reference values as reported in Table 4.12 and Fig. 4.18, whereas the excellent correlation between the mode shapes is confirmed also for this case study. Figs. 4.19 and 4.20 reveal that both methods maximize the poor initial

Table 4.11: Numerical cantilever beam case 3: target values of the updating parameters.

Element #	ΔK (%)	Target p^k
1	-50	2
2	-20	1.25
3	+10	0.91
4	-80	5
5	+25	0.8
6	-10	1.1
7	+35	0.74
8	+50	0.67

Table 4.12: Numerical cantilever beam case 3: initial natural frequency shifts.

Mode #	f_n^A (Hz)	f_n^X (Hz)	$ \Delta f $ (%)
1	4.77	6.43	25.78
2	28.03	40.27	30.41
3	103.06	112.83	8.66
4	183.66	221.46	17.07

correlation in terms of $\chi_s(\omega)$ and $\chi_a(\omega)$. However, it can be noted from Figs. 4.19(b)

Figure 4.18: Numerical cantilever beam case 3: comparison between the measured and predicted natural frequencies.

Figure 4.19: Numerical cantilever beam case 3: comparison between the shape correlation functions χ_s .

and 4.20(b) that the enhanced method allows the correlation functions to reach the unitary value in all the frequency range of analysis, whereas it does not happen with the reference method, especially at high frequencies, Figs. 4.19(a) and 4.20(a). Regarding the updated FRFs, only the driving point one is reported for the sake of brevity in Fig.

Figure 4.20: Numerical cantilever beam case 3: comparison between the amplitude correlation functions χ_a .

Figure 4.21: Numerical cantilever beam case 3: comparison between the driving point FRF.

4.21, highlighting the capability of the proposed methodology to match the experimental model, as it can be observed from Fig. 4.21(b), and correctly identify the actual natural frequencies, as shown in Fig. 4.22(b) and Table 4.13. Although the natural frequency shifts are greatly reduced also by the reference method, Fig. 4.22(a), it is not capable

Figure 4.22: Numerical cantilever beam case 3: comparison between the natural frequency shifts.

Figure 4.23: Numerical cantilever beam case 3: convergence history of the updating parameters.

to identify the target values of the stiffness parameters as reported in Fig. 4.23(a) and Table 4.14, where it can be noted that only two parameters, specifically the third and sixth ones, reach a value very close to the target one, giving an equivalent dynamic model that does not adequately represent the actual structure. On the contrary, Fig. 4.23(b)

and Table 4.14 display that all the updating parameters reach the expected values with the enhanced method, giving an additional evidence of its capability.

Table 4.13: Numerical cantilever beam case 3: final natural frequency shifts.

Mode #	$ \Delta f_n ^i$ (%) Initial	$ \Delta f_n ^u$ (%) Reference method	$ \Delta f_n ^u$ (%) Enhanced method
1	25.78	2.16	0.25 E-2
2	30.41	0.85	0.54 E-2
3	8.66	1.76	0.15 E-2
4	17.07	2.01	0.55 E-2

Table 4.14: Numerical cantilever beam case 3: comparison between the final updating parameters.

Parameter #	Initial value	Target value	Updated Reference method	Updated Enhanced method
1	1	2	1.44	2
2	1	1.25	2.12	1.25
3	1	0.91	0.95	0.9
4	1	5	3.74	5
5	1	0.8	2.67	0.79
6	1	1.1	1.22	1.11
7	1	0.74	0.48	0.74
8	1	0.67	1	0.66

4.2 Discussion of results

The case studies presented in this chapter have allowed the numerical validation of the methodology developed in Chapter 3. Different scenarios have been introduced to validate the enhanced methodology and compare its findings with the ones given by the reference method, introduced in Section 2.4. Specifically, a simple cantilever beam has been considered in order to critically analyze the results and the numerical model to

be updated has been obtained by imposing a single elemental variation of stiffness and mass, and a multiple elemental variation of stiffness.

In the case of a single variation of the stiffness, Section 4.1.1, or the mass, Section 4.1.2 both methodologies, i.e. the reference and enhanced ones, have identified equivalent dynamic numerical models that match the simulated experimental one in terms of natural frequencies and FRFs. Nevertheless, the values of the updating parameters reached by the two methods have highlighted the main strength of the developed methodology. Indeed, while the final values of the stiffness and mass parameters given by the reference method have not reached the expected values, the enhanced method has identified them perfectly, showing the effectiveness of the proposed theoretical developments. These analyses confirm the peculiarity of the updating methodologies to produce non-unique results: the experimental-numerical correlation can be improved through different combinations of the physical properties of a structure, represented by the updating parameters that act on the mass and stiffness matrices. The simulated cases are useful to determine which methodology gets closer to the known solution, and the analyses presented in the previous sections have shown that the enhanced methodology effectively improves the reference one.

The multiple variation of the stiffness, Section 4.1.3, has confirmed that the reference method allows the improvement of the correlation between the numerical and experimental models by giving an equivalent dynamic numerical model that is not fully representative of the actual structure. On the contrary, the enhanced methodology has

identified the actual values of the modified stiffness thus giving an updated FE model that perfectly matches the experimental one.

The promising results shown in this chapter prepare the ground for the experimental validation of the enhanced updating methodology, that will be the content of next chapter.

Chapter 5

Experimental validation on simple structures

The aim of this chapter is to apply the enhanced methodology introduced in Chapter 3 to real experimental data and validate it against the reference method, Section 2.4. This is carried out by performing modal testings on simple structures, such beams and plates. In this type of analyses, there is not an a priori knowledge of the solution to which the identification procedures should lead, but there is just the evidence of the test structure. Then, the achieved results have to be critically analyzed by the analyst according to his/her engineering intuition. Indeed, updating methodologies are required to provide FE models that well represent the dynamic behavior of the corresponding actual structures, and at the same time values of FE parameters that are physically meaningful.

5.1 Updating of a cantilever beam FE model

The application to a cantilever beam is introduced in this section. Following the description of the experimental setup and the FE representation, the initial correlation between the two models is evaluated and the structural updating performed using both methodologies, that is the Grafe and the enhanced one.

5.1.1 Experimental setup

The structure under investigation is an aluminum cantilever beam, whose physical properties are reported in Table 5.1, tested at the Structural Dynamics Laboratory of the Department of Mechanical and Aerospace Engineering of University of Rome “La Sapienza”. The structure was excited in the frequency range 0 – 256 Hz by an impact hammer and the transverse (z-direction) responses were recorded using 5 uniformly distributed ICP accelerometers, Fig. 5.1. The point close to the clamped end was considered as the excitation point in order to maximize the energy given to the structure while keeping the amplitude of vibrations relatively small. The LMS Scadas *III* hardware and the LMS Test.Lab[®] software were used to carry out the SIMO impact test. The settings of the experimental test are reported in Table 5.2.

Using the LMS PolyMax method, Ref. [26], 3 mode shapes were estimated in the considered frequency range corresponding to the first three out-of-plane bending modes. Table 5.3 lists the estimated experimental modal parameters, that is the natural frequen-

Table 5.1: Physical properties of the experimental cantilever beam.

$L(m)$	$b(m)$	$t(m)$	$E(GPa)$	$\rho(kg/m^3)$
0.41	0.015	0.003	70	2700

Figure 5.1: Experimental setup for the cantilever beam.

Table 5.2: Experimental cantilever beam: settings of the acquisition parameters.

Range (Hz)	N_f	N_a
0-256	512	30

cies f_n , the damping ratios ζ_n , and the description of the mode shapes, where B means *bending*.

Table 5.3: Modal parameters of the experimental cantilever beam.

Mode #	$f_n^X (Hz)$	$\zeta_n (\%)$	Description
1	12.82	0.22	I B
2	79.84	0.31	II B
3	220.48	0.32	III B

5.1.2 Numerical model

The experimental cantilever beam was modeled as an Euler-Bernoulli beam using 10 finite elements for a total of 20 DOFs, Fig. 5.2. Such a representation guarantees the numerical convergence up to the third modal order. The initial numerical model also includes five lumped masses, added to the numerical nodes corresponding to the experimental ones, to take into account the weight of the accelerometers. A value of 10 grams was considered for each lumped mass, as representative of the sensor itself and the connecting cable. Referring to Fig. 5.2, the frequency response functions $H_{3,3}$, $H_{5,3}$, $H_{7,3}$, $H_{9,3}$, $H_{11,3}$ defined on the vertical displacement DOF were used in the identification procedure. In addition, a constant modal damping was included in the FE model. The modal parameters corresponding to the initial numerical model are listed in Table 5.4.

Figure 5.2: Nodes numbering of the experimental cantilever beam.

Table 5.4: Numerical modal parameters of the experimental cantilever beam.

Mode #	$f_n^A (Hz)$	$\zeta_n (\%)$	Description
1	13.72	0.2	I B
2	85.70	0.2	II B
3	239.15	0.2	III B

5.1.3 Numerical-experimental correlation

Once the two models and their dynamic behavior are available, they have to be compared using the classical indicators, i.e. the natural frequency shifts and the Modal Assurance Criterion. As it can be noted from Tables 5.3 and 5.4, the estimated experimental and analytical natural frequencies are different, and the shift between them is reported in Table 5.5. The discrepancies between the predicted and experimental natural frequencies is shown also in Fig. 5.3, where it can be seen that the pairing points do not lie on the line of slope 1. Regarding the mode shapes, Fig. 5.4 shows an excellent correlation between them. Indeed, the MAC matrix is almost diagonal suggesting a very clear pairing between the measured and predicted mode shapes.

The initial shift between the natural frequencies highlights the presence of some

Table 5.5: Experimental cantilever beam: initial natural frequency shifts.

Mode #	$f_n^A (Hz)$	$f_n^X (Hz)$	$ \Delta f (\%)$
1	13.72	12.82	7.05
2	85.70	79.24	8.15
3	239.15	220.48	8.47

mismodeled areas in the numerical model, that have to be identified by the updating methodologies in order to get the numerical modal parameters closer to the experimental

Figure 5.3: Experimental cantilever beam: comparison between the measured and predicted natural frequencies.

Figure 5.4: Experimental cantilever beam: initial correlation between the experimental and numerical mode sets.

ones. The identification procedure was performed by activating the stiffness parameters and results are shown in the next section.

5.1.4 Identification of the stiffness distribution of the beam

The experimental identification of the dynamic properties of the beam was performed by considering a number of stiffness updating parameters equal to the number of elements of the FE model, that is 10, in order to visualize the areas in the model that need changes. Recalling Eq. (4.1), some frequency points in the neighbourhood of the natural frequencies were excluded by considering $k_\omega = 0.05$ to avoid numerical instabilities. In addition, as it is commonly done in structural updating an upper and lower limit to the parameters variation was imposed to obtain a solution more representative of the reality and avoid divergence. Such a procedure means that if a parameter reaches the imposed minimum/maximum value, it assumes that value until the convergence of the solution is reached. Specifically, a limit of $\pm 50\%$ with respect to the reference stiffness value, i.e. Young's modulus or bending moment of inertia, was considered with the aim to obtain a physically meaningful updated FE model. As reported in Fig. 5.5(a), the reference method makes the final shape correlation function worst in all the frequency range of analysis, while improving enough the amplitude correlation function, Fig. 5.6(a). However, such a correlation function remains almost unchanged in the frequency range from about 75 Hz to 100 Hz, that is in the neighbourhood of the second natural frequency, resulting in its wrong identification. Indeed, the updated FRFs, H^u , depicted in Figs. 5.7(a)-5.11(a) in terms of their amplitude and in Figs. 5.7(c)-5.11(c) in terms of their phase, do not match the experimental ones, H^x , in correspondence of the second natural

Figure 5.5: Experimental cantilever beam: comparison between the shape correlation functions χ_s . Figure 5.6: Experimental cantilever beam: comparison between the amplitude correlation functions χ_a .

frequency, where the natural frequency shift is not reduced with respect to the reference experimental value. It is worth remarking that the curves representing the phases of the updated FRFs could not follow exactly the behavior of the experimental curves in all the frequency range of analysis. However, they should overlap in correspondence of the

Figure 5.7: Experimental cantilever beam: comparison between the driving point FRF.

change of phase, which suggests the presence of a natural frequency. Fig. 5.12(a) and Table 5.6 show the final natural frequency shifts, and it can be verified that the reference method reduced the average error from 7.89% to 2.53%. The excellent correlation between the mode shapes is maintained as suggested from the MAC matrix reported in Fig. 5.13(a). Regarding the convergence history of the updating parameters, Fig. 5.14(a) and Table 5.7 show that almost all elements could be considered as a source of

Figure 5.8: Experimental cantilever beam: comparison between the $H_{5,3}$ FRF.

mismodeling regions in the FE model. Moreover, the stiffness variation proposed by the reference method has not given an equivalent dynamic FE model of the actual structure, pointing out the importance of having an alternative formulation for the identification of structures using experimental data.

Figs. 5.5(b) and 5.6(b) show instead that the enhanced methodology improves both the correlation functions in the whole frequency range, allowing the perfect identification

Figure 5.9: Experimental cantilever beam: comparison between the $H_{7,3}$ FRF.

of all the natural frequencies considered in the analysis. As it can be noted from Figs. 5.7(b)-5.11(b), in terms of amplitude, and from Figs. 5.7(d)-5.11(d), in terms of phase, the updated FRFs well match the experimental ones. In addition, the great reduction of the natural frequency shifts is highlighted by the bar plot of Fig. 5.12(b) and Table 5.6, from which it can be verified that the mean error was reduced to 0.69%. Furthermore, Fig. 5.13(b) shows that the updated mode shapes are well correlated with the exper-

Figure 5.10: Experimental cantilever beam: comparison between the $H_{9,3}$ FRF.

imental ones. This good identification of the experimental modal model was achieved by the stiffness parameters as depicted in Fig. 5.14(b), and listed in Table 5.7 where it can be noted that the maximum imposed variation is obtained by the second and tenth element while keeping the other values restrained enough. It should be noted from Fig. 5.14 that the enhanced method required more iteration steps than the reference one to reach convergence, but they behaved similarly in terms of CPU time.

Figure 5.11: Experimental cantilever beam: comparison between the $H_{11,3}$ FRF.

Table 5.6: Experimental cantilever beam: final natural frequency shifts.

Mode	$ \Delta f_n ^i$ (%)	$ \Delta f_n ^u$ (%)	$ \Delta f_n ^u$ (%)
#	Initial	Reference method	Enhanced method
1	7.05	0.41	0.48
2	8.15	6.98	0.90
3	8.47	0.19	0.70

(a) Reference method

(b) Enhanced method

Figure 5.12: Experimental cantilever beam: comparison between the natural frequency shifts.

(a) Reference method

(b) Enhanced method

Figure 5.13: Experimental cantilever beam: comparison between the final MAC.

Figure 5.14: Experimental cantilever beam: convergence history of the updating parameters.

Table 5.7: Experimental cantilever beam: comparison between the final updating parameters.

Parameter #	Initial value	Updated Reference method	Updated Enhanced method
1	1	0.5	0.62
2	1	1.5	1.5
3	1	1.5	0.96
4	1	1.22	0.85
5	1	1.39	1
6	1	1.5	1
7	1	1.5	0.99
8	1	0.55	0.97
9	1	0.59	0.70
10	1	0.96	0.5

5.2 Updating of a cantilever plate FE model

The aim of this section is to extend the validation of the developed methodology to a two-dimensional structure, such as a cantilever plate. Again, the description of the experimental setup and the FE model are firstly introduced, then the correlation between the two models is evaluated and improved by performing the structural updating using the reference and the enhanced identification methodologies.

5.2.1 Experimental setup

The structure under investigation is an aluminum plate, whose physical properties are reported in Table 5.8, tested at the Structural Dynamics Laboratory of the Department of Mechanical and Aerospace Engineering of University of Rome “La Sapienza”. The structure was fixed along one side and excited in the frequency range 0 – 512 Hz by an impact hammer, and the transverse (z-direction) responses were recorded using 4 ICP accelerometers distributed as shown in Fig. 5.15. As in the previous experimental test case, the LMS Scadas *III* hardware and the LMS Test.Lab[®] software were used to carry out the SIMO impact test. The acquisition settings of the experimental test are reported in Table 5.9.

Using the LMS PolyMax method, 4 mode shapes were estimated in the considered frequency range corresponding to the first two bending and torsional modes, as reported in Table 5.10, where B and T mean *bending* and *torsional*, respectively.

Table 5.8: Physical properties of the experimental cantilever plate.

$L(m)$	$b(m)$	$t(m)$	$E(GPa)$	$\rho(kg/m^3)$	ν
0.25	0.20	0.0032	70	2700	0.33

Figure 5.15: Experimental setup for the cantilever plate.

Table 5.9: Experimental cantilever plate: settings of the acquisition parameters.

Range (Hz)	N_f	N_a
0-512	2048	30

Table 5.10: Modal parameters of the experimental cantilever plate.

Mode #	$f_n^X (Hz)$	$\zeta_n (\%)$	Description
1	27.07	0.72	I B
2	100.73	0.14	I T
3	177.38	0.53	II B
4	339.35	0.22	II T

5.2.2 Numerical model

The Kirchhoff plate model was used for the FE model of the structure considering 3 DOFs per node, that is the vertical displacement (z direction) and the two bending

rotations around the x and y-axes. The plate was discretized with 20 elements (10 and 2 along its length and width, respectively) for a total of 90 DOFs, as shown in Fig. 5.16 guaranteeing to represent the numerical modal model accurately in the considered range of analysis. Referring to Fig. 5.16, the frequency response functions $H_{16,16}$, $H_{31,16}$, $H_{33,16}$, $H_{18,16}$ defined on the vertical displacement DOF were used in the identification procedure. In addition, a constant modal damping was included in the FE model. The modal parameters corresponding to the initial numerical model are listed in Table 5.11.

Figure 5.16: Nodes numbering of the experimental cantilever plate.

Table 5.11: Numerical modal parameters of the experimental cantilever plate.

Mode #	$f_n^A (Hz)$	$\zeta_n (\%)$	Description
1	39.72	0.2	I B
2	102.91	0.2	I T
3	245.06	0.2	II B
4	333.58	0.2	II T

5.2.3 Numerical-experimental correlation

The experimental and numerical dynamic models are compared in terms of the shift between the natural frequencies, reported in Table 5.12 and Fig. 5.17, and the MAC index for the mode shapes, shown in Fig. 5.18.

Table 5.12 shows a remarkable initial shift for the first and third natural frequency,

Table 5.12: Experimental cantilever plate: initial natural frequency shifts.

Mode #	f_n^A (Hz)	f_n^X (Hz)	$ \Delta f $ (%)
1	39.72	27.07	46.74
2	102.91	100.73	2.17
3	245.06	177.38	38.15
4	333.58	339.35	1.70

Figure 5.17: Experimental cantilever plate: comparison between the measured and predicted natural frequencies.

corresponding to the two bending modes (see Tables 5.10 and 5.11), whereas the torsional natural frequencies are well identified by the initial FE model. It can be seen also from

Figure 5.18: Experimental cantilever plate: initial correlation between the experimental and numerical mode sets.

Fig. 5.17, where the pairing points corresponding to the torsional natural frequencies perfectly lie on the line of slope 1. Regarding the mode shapes, an excellent initial correlation for all of them can be noted in Fig. 5.18. In order to reduce the gap between the numerical and experimental natural frequencies, thus improving the correlation between the two models, the structural updating was performed using the reference and the enhanced methodology to identify the actual stiffness properties of the plate and validate the developed method when having two-dimensional structures.

5.2.4 Identification of the stiffness distribution of the plate

In this test case, the solution vector $\{p\}$ is represented by 20 parameters, associated to each elemental stiffness matrix. Again, some frequency points were excluded by considering $k_\omega = 0.05$, and an upper and lower limit of $\pm 80\%$ with respect to the reference stiffness value was imposed in order to reduce the very high shift between the bending

natural frequencies, while trying to identify physically meaningful structural properties.

Figs. 5.19 and 5.20 show that the initial correlation functions have a sudden drop

Figure 5.19: Experimental cantilever plate: comparison between the shape correlation functions χ_s .

Figure 5.20: Experimental cantilever plate: comparison between the amplitude correlation functions χ_a .

in the frequency range from about 180 to 270 Hz, and that the reference method is not capable of improving it enough as it can be noted from part (a) of the same figures. It

Figure 5.21: Experimental cantilever plate: comparison between the driving point FRF.

Figure 5.22: Experimental cantilever plate: comparison between the natural frequency shifts.

results in a poor identification of the experimental natural frequencies as it can be noted from the comparison of the updated and experimental amplitude of the driving point FRF (the only one reported for the sake of brevity), Fig. 5.21(a), and the bar plot of Fig. 5.22(a). The numerical values of those shifts are listed in Table 5.13, suggesting

Figure 5.23: Experimental cantilever plate: comparison between the final MAC.

Figure 5.24: Experimental cantilever plate: convergence history of the updating parameters.

that their mean value is reduced from 22.19% to 9.27% by the reference method. In addition, the final correlation between the mode shapes improved slightly, with respect to the initial values, as shown in Fig. 5.23(a). The convergence history of the stiffness parameters is depicted in Fig. 5.24(a) and their behavior could justify the not excellent

improvement of the numerical-experimental correlation. Indeed, most of the parameters remained unchanged at the end of the updating process, see Table 5.14, not representing opportunely the actual stiffness of the plate.

On the other hand, the enhanced methodology permits to globally improve the correlation functions reaching the desired unitary value in almost all the frequency range of analysis, as shown in Figs. 5.19(b) and 5.20(b). High values of $\chi_s(\omega)$ and $\chi_a(\omega)$ mean a high correlation between the FRFs, as suggested by Fig. 5.21(b) where the updated driving point FRF matches its experimental counterpart. In terms of the natural frequency shifts, Fig. 5.22(b) and Table 5.13 show their excellent final reduction, whose mean value is reduced from 22.19% to 2.86%, whereas the final correlation between the corresponding mode shapes is depicted in Fig. 5.23(b). The structural properties of the plate, as identified by the enhanced method, are shown in Fig. 5.24(b) in terms of the updating parameters. They suggest several areas, see Table 5.14 for their numerical values, that were mismodeled in the initial FE model giving the reason of the high initial natural frequency shifts, especially for the bending mode shapes, whose updated values well represent the actual structure. The two-dimensional test case presented in this section further validates the developed methodology as an alternative to the reference method.

Table 5.13: Experimental cantilever plate: final natural frequency shifts.

Mode #	$ \Delta f_n ^i$ (%) Initial	$ \Delta f_n ^u$ (%) Reference method	$ \Delta f_n ^u$ (%) Enhanced method
1	46.74	2.52	4.19
2	2.17	9.09	3.88
3	38.15	16.41	1.67
4	1.70	9.06	1.71

Table 5.14: Experimental cantilever plate: comparison between the final updating parameters.

Parameter #	Initial value	Updated Reference method	Updated Enhanced method
1	1	0.2	0.34
2	1	0.22	0.2
3	1	1	1.8
4	1	1	1.8
5	1	1	0.68
6	1	0.99	1.34
7	1	0.99	0.79
8	1	0.99	1
9	1	0.99	0.2
10	1	0.99	0.38
11	1	0.99	0.52
12	1	0.99	1.8
13	1	1	1.8
14	1	1	1.8
15	1	1	1
16	1	1	1.8
17	1	1	1.49
18	1	1	1.8
19	1	1.1	1.28
20	1	1.12	1.8

5.3 Discussion of results

Two experimental test cases have been presented in this chapter to validate the developed methodology with respect to the reference one by Henning Grafe. The experimental val-

validation of the methodology has been performed on simple structures before going ahead with the application to more complex systems. Specifically, a one-dimensional structure such as a beam has been considered firstly, Section 5.1, then moving on a two-dimensional structure given by a plate, Section 5.2.

Although the solution of the simulated cases presented in Chapter 4 was known, in the case of real experimental data nothing can be said about the values that the identification procedure should suggest to improve the numerical-experimental correlation. Some reasonings could be done based on engineering considerations, however any combination of the updating parameters may provide a FE model representative of the actual structure, in terms of its dynamic characteristics. In the specific cases presented in the previous sections, the identification procedure has taken only the stiffness properties into account because the weight and center of gravity of the FE models were in good agreement with the actual properties of the structures, thus not requiring the identification of the mass properties and restricting the set of possible solutions. In general, disagreements in those quantities could suggest the inclusion of both mass and stiffness properties in the identification procedure. The application of the methodology to real experimental data has shown interesting results and highlighted the power of the enhanced methodology with respect to the reference one. Indeed, the proposed identification methodology has been capable to improve the correlation between the predicted and measured dynamic properties of the structures under investigation, by identifying parameters that could effectively represent the actual structural properties.

Part III

Application to Rotorcraft Technology

Chapter 6

Introduction to vibration control

Suppression of undesired vibrations generated by mechanical systems has been a challenging problem in many engineering applications. Among them, the helicopter machine is one of the most complex system for vibration control. Helicopter rotors operate in a highly complex unsteady aerodynamic environment due to the cyclic variation of the aerodynamic loads on the blade, Ref. [80]. In steady flight, these loads cause a periodic motion of the blade, thus generating periodic forces and moments which are transmitted to the rotor hub, whose reactions contribute mostly to the vibration in a helicopter airframe. Indeed, due to the coupling between the rotor system and the airframe, the vibratory hub loads are transferred throughout the structure. The aerodynamic interaction between the main rotor wake and the airframe surfaces is an additional source of vibration. Specifically, the vortex shedding by the main rotor interacts with the horizontal and vertical tail surfaces exciting the tail of the helicopter. Other sources of vibration come from the tail rotor force fluctuations, the engine, and the transmission.

Understandably, helicopter vibration is a serious concern and the development of techniques for vibration control is one of the most important field of research in helicopter industry. According to Ref. [81], the control of vibration is important for four main reasons:

- to improve crew efficiency, and hence safety of operation;
- to improve the comfort of passengers;
- to improve the reliability of avionic and mechanical equipment;
- to improve the fatigue lives of airframe structural components.

Hence it is very important to control vibration throughout the design, development, and in-service stages of a helicopter project.

Numerous control approaches have been proposed to help alleviate vibration, either *passive* or *active*. As defined in Ref. [82], a passive control system consists of an embedded device that modifies the stiffness or the damping of the structure in an appropriate manner that reduces its vibration response, without requiring an external power source to operate. Examples of passive systems are vibration absorbers, dampers, isolation devices. Although they are relatively inexpensive and do not require energy supply, they are usually effective in suppressing vibration only at a specific frequency. On the contrary, active control systems suppress vibration adaptively by using one or more control actuators, powered by an external source, that apply forces to the structure based on the

response of the system measured by physical sensors. Active systems require external power to perform the control action, and this makes these systems vulnerable to power failure. However, their controllable nature allows vibration reduction over a broadband of frequencies, thus preferring them with respect to the passive systems. Furthermore, the advent of technologies based on smart structures using active materials, specifically piezoelectric materials, allowed a wide extent of the active control techniques as these materials directly convert electrical energy into a mechanical deformation that can be used for structural actuation, Ref. [83]. Active schemes can be classified into two categories, namely direct-active and indirect-active. In direct-active concepts, the vibration reduction is obtained by directly acting against the excitation force. Examples in this category are the Higher Harmonic Control (HHC) and the Individual Blade Control (IBC) schemes. The first acts by generating higher harmonics aerodynamic loads that counteract the original loads, and applying them to all the blade simultaneously through the helicopter swashplate. The second approach uses actuators to control each blade independently and simultaneously in the rotating frame, leading to a more complex system. The main issue in direct-active approaches is the high actuation forces as well as the displacements required to directly counteract the vibratory forces, thus requiring displacement amplification devices or application of very high voltages to achieve the required suppression of vibration. On the other hand, the indirect-active schemes, also known as semi-active, indirectly reduce vibrations by modulating the structural properties such as mass, stiffness, or damping. Because these systems do not directly counteract

the excitation forces, the required actuation forces and displacements are lower than those required by the direct-active approaches. The Smart Spring device belongs to the semi-active schemes for vibration control, and it allows to change the structural impedance of a system adaptively by activating a piezoceramic actuator, Ref. [84]. Its primary advantage is that it requires the actuator to produce micro displacements to generate relatively high actuation forces. Such device has been developed within the Smart Hybrid Active Rotor Control System (SHARCS) international project lead by the Carleton University Rotorcraft Research Group (Ottawa, Canada) Ref. [85], in which the Smart Spring has been inserted between the blade horn and the rotor swashplate, i.e. at the root of the blade, hence replacing the conventional pitch link with the Active Pitch Link (APL). The APL allows to control the vibrations transmitted into the airframe by actively changing the dynamic stiffness of the pitch link. Specifically, it modifies the torsional characteristics of the blade to reduce vibrations. The Smart Spring concept is addressed in this thesis from the analytical, numerical, and experimental point of view with the aim of the identification of a reduced-scale helicopter blade model that incorporates the APL, through the enhanced methodology developed in Chapter 3.

Chapter 7

Analytical validation of the Smart Spring

In this chapter, the capability of the Smart Spring device to reduce vibration by acting on the structural properties of a dynamic system is verified by introducing an open-loop control law for the modulation of the stiffness of the system, and developing its analytical model in the frequency domain. It allows the study of the response of the system in terms of its harmonic components, being it extremely useful when dealing with rotorcraft applications. The Smart Spring device has been extensively studied in the past years, Refs. [86, 87]. However, its innovative concept has required further analytical developments in order to clearly understand the behavior of the system, thus motivating the work that will be presented in the next sections.

The conceptual drawing of the device is shown in Fig. 7.1, where it can be noted that it consists of two mass-spring systems denoted with subscripts 1 and 2. Specifically, k_1 and m_1 represent the main system stiffness and mass, respectively, whereas the secondary

system is characterized by k_2 and m_2 . Activating the piezoelectric actuator, the resulting

Figure 7.1: Conceptual drawing of the Smart Spring, Ref. [86].

stiffness of the system varies in the range $[k_1, k_1 + k_2]$. Furthermore, the main spring is attached to a moving structure that has a built-in sleeve, where the external vibratory force $F(t)$ is considered applied to. The piezoelectric actuator is attached to the secondary system which is inserted into the sleeve. Depending on the actuator configuration, the system behaves in a different way. When the actuator is off, the sleeve can move freely and only the main system determines the dynamic response of the system. Instead, when the actuator is powered on, a normal force $N(t)$, that engages the two systems, is generated. If the resultant friction force $\mu(t)N(t)$, being $\mu(t)$ the friction coefficient that is, in turn, time dependent, is sufficiently large, the two systems are fully coupled and the system is characterized by an equivalent stiffness, k , given by the “parallel” of the stiffness of the two systems, that is $k = k_1 + k_2$, and the mass of the system increases from m_1 to $m = m_1 + m_2$. The friction force applied to the sleeve can be controlled by an external voltage stimulus to the actuator. Therefore the voltage applied to the piezoelectric actuator could continuously provide the control of the resulting stiffness,

damping, and mass of the Smart Spring system.

7.1 The analytical model of the Smart Spring

Referring to Fig. 7.1, the equations of motion of the Smart Spring system are given by:

$$\begin{cases} m_1 \ddot{x}_1(t) + k_1 x_1(t) = F(t) \mp \mu(t)N(t) \\ m_2 \ddot{x}_2(t) + k_2 x_2(t) = \pm \mu(t)N(t) \end{cases} \quad (7.1)$$

where $F(t)$ is the external force applied to the structure and $\mu(t)N(t)$ is the frictional force due to the action of the piezoceramic actuator. When the system is unlocked, that is there is not the frictional force and the two mass-spring systems are uncoupled, the previous equations of motion become:

$$\begin{cases} m_1 \ddot{x}_1(t) + k_1 x_1(t) = F(t) \\ m_2 \ddot{x}_2(t) + k_2 x_2(t) = 0 \end{cases} \quad (7.2)$$

When the system is locked, that is the frictional force allows the coupling of the two systems, $x_1(t) = x_2(t)$ and $\ddot{x}_1(t) = \ddot{x}_2(t)$. Hence it is possible to sum Eqs. (7.1) to have:

$$(m_1 + m_2) \ddot{x}_1(t) + (k_1 + k_2) x_1(t) = F(t) \quad (7.3)$$

Summing Eqs. (7.1) in any case allows the equations of motion to be written as follows:

$$m_1 \ddot{x}_1(t) \left(1 + \frac{m_2 \ddot{x}_2(t)}{m_1 \ddot{x}_1(t)} \right) + k_1 x_1(t) \left(1 + \frac{k_2 x_2(t)}{k_1 x_1(t)} \right) = F(t) \quad (7.4)$$

which can be rearranged as:

$$m(t) \ddot{x}_1(t) + k(t) x_1(t) = F(t) \quad (7.5)$$

where

$$m(t) = m_1 \left(1 + \frac{m_2 \ddot{x}_2(t)}{m_1 \ddot{x}_1(t)} \right) \quad (7.6)$$

$$k(t) = k_1 \left(1 + \frac{k_2 x_2(t)}{k_1 x_1(t)} \right) \quad (7.7)$$

In the assumption to control only the stiffness of the system, i.e. $|m_2 \ddot{x}_2(t)| \ll |m_1 \ddot{x}_1(t)|$, Eq. (7.5) becomes:

$$m_1 \ddot{x}_1(t) + k(t) x_1(t) = F(t) \quad (7.8)$$

that represents the formulation from which the analytical model of the Smart Spring in the frequency-domain is derived. As suggested from Eq. (7.7), the stiffness of the system is a function of time (through the ratio between the displacement of the secondary and main system), and the control law that modulates it from the value of the main spring k_1 to the value of the coupled system, that is $k = k_1 + k_2$, is now introduced. Specifically, the square-wave control law shown in Fig. 7.2 is considered to perform such a modulation of the stiffness. When the control function is equal to 0 (1), the stiffness of the system assumes the value of k_1 ($k = k_1 + k_2$), as it can be verified from Eq. (7.7). In the same figure, the dependence on time of the ratio $\frac{x_2(t)}{x_1(t)}$ is indicated through the new variable α , that is:

$$\alpha = r\psi + \phi \quad (7.9)$$

where r and ϕ are the control harmonic and control phase of the Smart Spring, respectively, whereas $\psi = \Omega t$ being Ω the rotor fundamental frequency. The stiffness of the

Figure 7.2: Smart Spring control law.

system is then modulated according to:

$$k(t) \rightarrow k(\alpha) = k_1 + k_2 f(\alpha) \quad (7.10)$$

where $f(\alpha) = \frac{x_2}{x_1}(\alpha)$. The harmonic balance method provides the mathematical model of the Smart Spring in the frequency domain, thus allowing to study the response of the system in terms of its harmonic components, by assuming:

$$F(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\psi} \quad (7.11)$$

$$x_1(t) = \sum_{n=-\infty}^{+\infty} x_{1n} e^{in\psi} \quad (7.12)$$

where the excitation force and the displacement solution are considered as a combination of harmonics of the multiples of the rotor frequency. Recalling that the square-wave control law can be represented as a Fourier series, Ref. [20]:

$$f(\alpha) = a_0 + \sum_{p=1}^{+\infty} (a_p \cos(p\alpha) + b_p \sin(p\alpha)) \quad (7.13)$$

for which it is well known that $a_0 = 1/2$, $a_p = 0$, and:

$$b_p = \begin{cases} 0 & \text{if } p \text{ even} \\ \frac{2}{p\pi} & \text{if } p \text{ odd} \end{cases} \quad (7.14)$$

Introducing the complex form of Eq. 7.13, the b_p coefficients assume the following equivalent expression:

$$b_p = \frac{1}{i\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{ip\alpha} \quad (7.15)$$

Hence,

$$f(\alpha) = \frac{1}{2} + \frac{1}{i\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{ip\alpha} \quad (7.16)$$

Substituting the above equation into Eq. (7.10), the instantaneous stiffness of the system is modulated as:

$$k(\alpha) = k_0 + \frac{k_2}{i\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{ip\alpha} \quad (7.17)$$

where

$$k_0 = \frac{2k_1 + k_2}{2} \quad (7.18)$$

Using Eqs. (7.11), (7.12), and (7.17) the equation of motion, Eq. (7.8), can be written in the frequency domain:

$$\begin{aligned} -m_1 \Omega^2 \sum_{n=\pm 1, \pm 2, \dots}^{+\infty} n^2 x_{1n} e^{in\psi} + \left[k_0 + \frac{k_2}{i\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{ip\alpha} \right] \\ \cdot \left[x_{10} + \sum_{n=\pm 1, \pm 2, \dots}^{+\infty} x_{1n} e^{in\psi} \right] \\ = F_0 + \sum_{n=\pm 1, \pm 2, \dots}^{+\infty} F_n e^{in\psi} \end{aligned} \quad (7.19)$$

Manipulating the previous equation, it is possible to write¹:

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \left[(-m_1 \Omega^2 n^2 + k_0) x_{1_n} e^{in\psi} + \right. \\ \left. + \frac{k_2}{\pi} \frac{1}{p} x_{1_n} e^{i(pr+n)} e^{i(p\phi - \frac{\pi}{2})} \right] = \\ = \sum_{n=-\infty}^{+\infty} F_n e^{in\psi} \end{aligned} \quad (7.20)$$

Noting that $(pr + n) \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$ and $p \rightarrow \pm\infty$, and that both p and n are dummy indices, it results in:

$$\sum_{n=-\infty}^{+\infty} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} x_{1_n} e^{i(pr+n)\psi} = \sum_{n=-\infty}^{+\infty} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} x_{1_{(pr+n)}} e^{in\psi} \quad (7.21)$$

and Eq. (7.20) is rearranged as follows:

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \left[(-m_1 \Omega^2 n^2 + k_0) x_{1_n} + \right. \\ \left. + \frac{k_2}{p\pi} x_{1_{(pr+n)}} e^{i(p\phi - \frac{\pi}{2})} \right] e^{in\psi} = \\ = \sum_{n=-\infty}^{+\infty} F_n e^{in\psi} \end{aligned} \quad (7.22)$$

or

$$\begin{aligned} (-m_1 \Omega^2 n^2 + k_0) x_{1_n} + \frac{k_2}{\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} x_{1_{(pr+n)}} = F_n \\ n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (7.23)$$

¹Note that the following equivalences have been taken into account:

$$\begin{aligned} \alpha &= r\psi + \phi \\ \frac{1}{i} &= e^{-i\frac{\pi}{2}} \end{aligned}$$

Dividing Eq. (7.23) by k_1 , that is the uncoupled main spring constant:

$$\left[- \left(\frac{\Omega}{\omega_1} \right)^2 n^2 + \frac{k_0}{k_1} \right] x_{1n} + \frac{k_2}{k_1} \frac{1}{\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} x_{1(pr+n)} = \frac{F_n}{k_1} \quad (7.24)$$

$$n = 0, \pm 1, \pm 2, \dots$$

where $\omega_1^2 = \frac{k_1}{m_1}$ is the natural frequency of the main system. Recalling Eq. (7.18) and introducing the non-dimensional parameter $\kappa = \frac{k_2}{k_1}$, the final formulation of the equation of motion in the frequency domain is derived:

$$\left[- \left(\frac{\Omega}{\omega_1} \right)^2 n^2 + 1 + \frac{\kappa}{2} \right] x_{1n} + \frac{\kappa}{\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} x_{1(pr+n)} = \frac{F_n}{k_1} \quad (7.25)$$

$$n = 0, \pm 1, \pm 2, \dots$$

In order to write the system of algebraic equations to solve for the displacement solution expressed in terms of the harmonic components x_{1n} , some manipulations have to be performed on the summation term in Eq. (7.25). Specifically:

$$\begin{aligned} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} &= \dots - \frac{1}{5} e^{i(-5\phi - \frac{\pi}{2})} x_{1-5r+n} + \\ &- \frac{1}{3} e^{i(-3\phi - \frac{\pi}{2})} x_{1-3r+n} + \\ &- e^{i(-\phi - \frac{\pi}{2})} x_{1-r+n} + \\ &+ e^{i(\phi - \frac{\pi}{2})} x_{1r+n} + \\ &+ \frac{1}{3} e^{i(3\phi - \frac{\pi}{2})} x_{13r+n} + \\ &+ \frac{1}{5} e^{i(5\phi - \frac{\pi}{2})} x_{15r+n} + \dots \end{aligned} \quad (7.26)$$

where only the first few terms have been expressed for the sake of brevity. Introducing the Euler's formula in the previous equation, and considering the real and imaginary

parts for each harmonic, the n-th general term of Eq. (7.25) can be written as:

$$\begin{aligned}
 & \left[- \left(\frac{\Omega}{\omega_1} \right)^2 n^2 + 1 + \frac{\kappa}{2} \right] [\text{Re}(x_{1_n}) + i \text{Im}(x_{1_n})] + \frac{2\kappa}{\pi} \left[\dots + \right. \\
 & + \frac{1}{5} \cos(5\phi) \text{Im}(x_{1_{5r+n}}) + \frac{1}{5} \sin(5\phi) \text{Re}(x_{1_{5r+n}}) + \\
 & + \frac{1}{3} \cos(3\phi) \text{Im}(x_{1_{3r+n}}) + \frac{1}{3} \sin(3\phi) \text{Re}(x_{1_{3r+n}}) + \\
 & \left. + \cos(\phi) \text{Im}(x_{1_{r+n}}) + \sin(\phi) \text{Re}(x_{1_{r+n}}) \right] = \\
 & = \frac{1}{k_1} [\text{Re}(F_n) + i \text{Im}(F_n)]
 \end{aligned} \tag{7.27}$$

Defining the vectors:

$$\{z\} = \left\{ \begin{array}{c} \vdots \\ \text{Re}(x_{1_{5r+n}}) \\ \vdots \\ \text{Re}(x_{1_{3r+n}}) \\ \vdots \\ \text{Re}(x_{1_{r+n}}) \\ \vdots \\ \text{Re}(x_{1_n}) \\ \dots\dots\dots \\ \text{Im}(x_{1_n}) \\ \vdots \\ \text{Im}(x_{1_{r+n}}) \\ \vdots \\ \text{Im}(x_{1_{3r+n}}) \\ \vdots \\ \text{Im}(x_{1_{5r+n}}) \\ \vdots \end{array} \right\}_{2N \times 1} \tag{7.28}$$

$$\{b\} = \frac{1}{k_1} \left\{ \begin{array}{c} \vdots \\ \text{Re}(F_{n+1}) \\ \text{Re}(F_n) \\ \dots\dots\dots \\ \text{Im}(F_n) \\ \text{Im}(F_{n+1}) \\ \vdots \end{array} \right\}_{2N \times 1} \quad (7.29)$$

where $N > n + rp_{max}$, Eq. (7.27) can be written as an algebraic system:

$$[A] \{z\} = \{b\} \quad (7.30)$$

and solved for $\{z\}$, where $[A]$ is the $2N \times 2N$ matrix:

$$[A] = [\text{diag} \{\Gamma(n)\}] + \begin{bmatrix} [A_{11}] & [A_{12}] \\ [0] & [0] \end{bmatrix} \quad (7.31)$$

having introduced:

$$\{\Gamma(n)\} = \left\{ \begin{array}{c} \vdots \\ \gamma(n+1) \\ \gamma(n) \\ \dots\dots\dots \\ \gamma(n) \\ \gamma(n+1) \\ \vdots \end{array} \right\}_{2N \times 1} \quad (7.32)$$

where from Eq. (7.27):

$$\gamma(n) = \left[- \left(\frac{\Omega}{\omega_1} \right)^2 n^2 + 1 + \frac{\kappa}{2} \right] \quad (7.33)$$

and

$$[A_{11}]_{N \times N} = \begin{bmatrix} \dots & \ddots & \dots & \ddots & \dots & \ddots & 0 \\ \frac{2\kappa}{5\pi} \sin(5\phi) & \dots & \frac{2\kappa}{3\pi} \sin(3\phi) & \dots & \frac{2\kappa}{\pi} \sin(\phi) & \dots & \dots \\ \dots & \frac{2\kappa}{5\pi} \sin(5\phi) & \dots & \frac{2\kappa}{3\pi} \sin(3\phi) & \dots & \frac{2\kappa}{\pi} \sin(\phi) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (7.34)$$

$$[A_{12}]_{N \times N} = \begin{bmatrix} & & \ddots & & & & & & \\ & 0 & \dots & \frac{2\kappa}{\pi} \cos(\phi) & \dots & \frac{2\kappa}{3\pi} \cos(3\phi) & \dots & \frac{2\kappa}{5\pi} \cos(5\phi) & \\ 0 & \dots & \frac{2\kappa}{\pi} \cos(\phi) & \dots & \frac{2\kappa}{3\pi} \cos(3\phi) & \dots & \frac{2\kappa}{5\pi} \cos(5\phi) & \dots & \end{bmatrix} \quad (7.35)$$

The above algebraic system shows the coupling between the uncontrolled, i.e. $\frac{F_n}{k_1}$, and controlled harmonics, and that the solution depends on three parameters, namely κ , $\frac{\Omega}{\omega_1}$, and the control phase of the Smart Spring ϕ . In a simple open-loop control approach, the control objective is to adjust those parameters in order to reduce the response amplitude, so the vibration of the system. In order to analytically verify the capability of the Smart Spring to reduce vibration, a parametric study is performed by varying the above introduced non-dimensional parameters.

7.1.1 Parametric study of the system response

The mathematical model derived in the previous section has been implemented into MatLab[®] in order to analyze the response of the system, in terms of its harmonics, when κ , $\frac{\Omega}{\omega_1}$, and ϕ vary within a given range. A constant excitation force has been considered and the control harmonic r , see Eq. (7.9), has been set equal to 4 simulating to control the critical harmonic of a four blade rotor. In addition, $p_{max} = 5$ thus allowing the parametric study up to the 22-nd harmonic component of the displacement of the main system. However, only the most significative results up to the 8-th harmonic of the rotor fundamental frequency are shown in Figs. 7.3-7.10. Given an operative working condition, that is a value of $\frac{\Omega}{\omega_1}$, the aim of these numerical analyses is to show how the

amplitude of the response can be properly reduced by modulating the stiffness of the system. In the following analyses, the amplitude of the harmonics is normalized with respect to the maximum value obtained at each simulation.

It is worth remarking that each harmonic depends also on the controlled harmonics,

Figure 7.3: Parametric study for the 1/rev harmonic component.

so referring to Eq. (7.27) it may be observed that with the assumptions made for the simulation, that is $r = 4$ and $p_{max} = 5$, considering the 1/rev harmonic component as

Figure 7.4: Parametric study for the 2/rev harmonic component.

example the corresponding solution is composed also by the controlled harmonics x_{15} , x_{113} , and x_{121} .

Fig. 7.3 shows the results for the amplitude of the 1/rev harmonic component. It can be observed from Fig. 7.3(a) that there is a strong dependence on ϕ for small values of $\frac{\Omega}{\omega_1}$ that is, given a value of κ it is possible to choose the optimal value of the control phase that minimizes the response amplitude by acting on the control law

Figure 7.5: Parametric study for the 3/rev harmonic component.

defined by Eq. (7.17), through α . Such dependence gradually vanishes as $\frac{\Omega}{\omega_1}$ increases and the amplitude of the displacement is almost constant with ϕ . It means that given an operative working condition and identified a system, in terms of $\frac{\Omega}{\omega_1}$ and κ , the amplitude of the response cannot be reduced by modulating the control phase ϕ . However, the simulations suggest to modify the system according to the most suitable value of κ that minimizes the response. It could be observed from Fig. 7.3(b) that initially the higher

Figure 7.6: Parametric study for the 4/rev harmonic component.

the value of κ is considered, the lower is the response amplitude but the area of maximum amplitude tends to higher values of κ as soon as $\frac{\Omega}{\omega_1}$ increases, as shown in Figs. 7.3(c) and 7.3(d). A similar behavior is obtained for the amplitudes of the 2/rev and 3/rev harmonic components as suggested from Figs. 7.4 and 7.5, respectively. However, they start to increase with κ for lower values of $\frac{\Omega}{\omega_1}$.

As regards the 4-th, Fig. 7.6, and 5-th, Fig. 7.7, harmonic components, they show an

Figure 7.7: Parametric study for the 5/rev harmonic component.

almost constant amplitude for a wide range of κ for small values of $\frac{\Omega}{\omega_1}$, then assuming the same behavior observed for the lower components. The amplitudes of the higher harmonics, from the 6-th to the 8-th, Figs. 7.8-7.10, tend to increase with κ also for small values of $\frac{\Omega}{\omega_1}$. It is important to note that for a specific harmonic component, the contour plots follow the same behavior but the reduction of the amplitude becomes lower as soon as the operative working condition, that is $\frac{\Omega}{\omega_1}$, increases. Considering the 7/rev

Figure 7.8: Parametric study for the 6/rev harmonic component.

component depicted in Fig. 7.9 as an example, its amplitude can be reduced by 50% when $\frac{\Omega}{\omega_1} = 0.3$ but the percentage decreases to about 2.2% when $\frac{\Omega}{\omega_1} = 1.4$.

According to Eq. (7.30), the solution vector $\{z\}$ is obtained through the inverse of $[A]$, whose diagonal elements, defined by Eq. (7.33), have to be properly chosen to avoid singularities. In addition, the mathematical formulation allows to understand the behavior assumed by the higher harmonics when $\frac{\Omega}{\omega_1}$ increases. Indeed, as n increases,

Figure 7.9: Parametric study for the 7/rev harmonic component.

that is a higher harmonic is taken into account, the amplitude response is less affected by κ when $\frac{\Omega}{\omega_1} > 1$. As an example, the amplitude of the 5/rev harmonic component reported in Fig. 7.7(c) shows a percentage decrease of 3.5% (referring to the maximum and minimum values assumed by the harmonic component) when κ decreases, whereas a lower harmonic, e.g. the 2/rev harmonic component reported in Fig. 7.4(c), shows a reduction of the amplitude response of about 20% with low values of κ . Such a behavior

Figure 7.10: Parametric study for the 8/rev harmonic component.

can be justified by Eq. (7.33), which suggests that as soon as the first term $-\left(\frac{\Omega}{\omega_1}\right)^2 n^2$ becomes much higher than the second one $\left(1 + \frac{\kappa}{2}\right)$ as n increases, a variation of κ does not cause a significant variation of the amplitude response.

The analysis performed by varying the parameters on which the response of the system depends, allows the optimization of the Smart Spring properties, in terms of its stiffness and mass, and the control input α , defined by Eq. (7.9), which together with Eq. (7.17)

describes the stiffness modulation to reduce the target amplitude response.

7.2 The base-excited concept of the Smart Spring

In this section a base-excited concept of the Smart Spring system is introduced. Such a representation of the device is innovative with respect to the one generally assumed in literature, as in Refs. [86, 87], and analyzed in Section 7.1. It is important to highlight that the Smart Spring device could be realized by combining mass-spring systems into several different ways for different applications. The base-excited model is developed in this thesis with the aim of giving a more realistic representation of the device represented by Fig. 7.1, to analytically demonstrate its capability to reduce the vibration of helicopter blades.

The conceptual drawing of the base-excited model is shown in Fig. 7.11, where the new spring labeled k_3 represents the rotor blade stiffness and $F(t)$ represents the vibration due to aerodynamic loading of the blade during forward flight. The goal of the system is to reduce the vibration transmitted from mass m_b to mass m_1 , considering that the vibration experienced by mass m_1 represents the vibration transmitted to the swashplate on the helicopter. As done in the previous section, the analytical validation of the system is performed by introducing the open-loop control law for the modulation of stiffness and developing the mathematical model in the frequency domain in order to have a harmonic representation of the response of the system.

Figure 7.11: New conceptual drawing of the Smart Spring.

7.2.1 The analytical model of the base-excited Smart Spring

The equations of motion of the system can be derived by writing the kinetic and potential energies of the system. To do it, the generalized coordinates, i.e. the degrees of freedom of the system, have to be introduced. Referring to Fig. 7.11, they correspond to the displacements x_1 , x_2 , and x_b , and the kinetic and potential energies can be written as:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_b\dot{x}_b^2 \quad (7.36)$$

$$U = \frac{1}{2}k_1(x_1 - x_b)^2 + \frac{1}{2}k_2(x_2 - x_b)^2 + \frac{1}{2}k_3x_b^2 \quad (7.37)$$

and the non-conservative forces associated to each degree of freedom are:

$$Q_{x_1} = \mp \mu(t)N(t) \quad (7.38)$$

$$Q_{x_2} = \pm \mu(t)N(t) \quad (7.39)$$

$$Q_{x_b} = F(t) \quad (7.40)$$

By applying the Lagrange's equation², it is possible to write the equations governing the motion of the system:

$$m_1 \ddot{x}_1(t) + k_1 x_1(t) - k_1 x_b(t) = \mp \mu(t) N(t) \quad (7.41)$$

$$m_2 \ddot{x}_2(t) + k_2 x_2(t) - k_2 x_b(t) = \pm \mu(t) N(t) \quad (7.42)$$

$$m_b \ddot{x}_b(t) - k_1 x_1(t) - k_2 x_2(t) + (k_1 + k_2 + k_3) x_b(t) = F(t) \quad (7.43)$$

which can be rearranged in the standard matrix form:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_b \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_b \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_b \end{Bmatrix} = \begin{Bmatrix} \mp \mu(t) N(t) \\ \pm \mu(t) N(t) \\ F(t) \end{Bmatrix} \quad (7.44)$$

Summing Eq. (7.41) and Eq. (7.42), and rearranging Eq. (7.43) it is possible to write the equations of motion as follows:

$$\begin{cases} m_1 \ddot{x}_1(t) \left(1 + \frac{m_2 \ddot{x}_2(t)}{m_1 \ddot{x}_1(t)} \right) + k_1 x_1(t) \left(1 + \frac{k_2 x_2(t)}{k_1 x_1(t)} \right) - (k_1 + k_2) x_b(t) = 0 \\ m_b \ddot{x}_b(t) - k_1 x_1(t) \left(1 + \frac{k_2 x_2(t)}{k_1 x_1(t)} \right) + (k_1 + k_2 + k_3) x_b(t) = F(t) \end{cases} \quad (7.45)$$

Assuming again to control only the stiffness of the system, i.e. $|m_2 \ddot{x}_2(t)| \ll |m_1 \ddot{x}_1(t)|$,

and considering Eq. (7.7), Eq. (7.45) results in:

$$\begin{cases} m_1 \ddot{x}_1(t) + k(t) x_1(t) - (k_1 + k_2) x_b(t) = 0 \\ m_b \ddot{x}_b(t) - k(t) x_1(t) + (k_1 + k_2 + k_3) x_b(t) = F(t) \end{cases} \quad (7.46)$$

which has to be solved for both $x_1(t)$ and $x_b(t)$. Eq. (7.46) can be written in the frequency domain by considering the harmonic formulation for the excitation force and

²The general form of Lagrange's equation for the generalized coordinate q_i is:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_{q_i}$$

the displacement of the main system, Eq. (7.11) and Eq. (7.12), respectively, and the displacement of the base according to:

$$x_b(t) = \sum_{n=-\infty}^{+\infty} x_{b_n} e^{in\psi} \quad (7.47)$$

where $\psi = \Omega t$, being Ω the rotor fundamental frequency. The above equation assumes that the base solution is harmonic on the rotor fundamental frequency. Assuming the square wave control law introduced by Eq. (7.17) for the modulation of stiffness, and introducing the harmonic formulation into Eq. (7.46), the following equations are obtained:

$$\begin{cases} \left[-\left(\frac{\Omega}{\omega_1}\right)^2 n^2 + 1 + \frac{\kappa}{2} \right] x_{1_n} + \frac{\kappa}{\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} x_{1_{(pr+n)}} - (1 + \kappa) x_{b_n} = 0 \\ \left[-\kappa_b \left(\frac{\Omega}{\omega_b}\right)^2 n^2 + 1 + \kappa + \kappa_b \right] x_{b_n} - \left(1 + \frac{\kappa}{2}\right) x_{1_n} + \frac{\kappa}{\pi} \sum_{p=\pm 1, \pm 3, \dots}^{+\infty} \frac{1}{p} e^{i(p\phi - \frac{\pi}{2})} x_{1_{(pr+n)}} = \frac{F_n}{k_1} \\ n = 0, \pm 1, \pm 2, \dots \end{cases} \quad (7.48)$$

where $\omega_b = \sqrt{k_3/m_b}$ is the natural frequency of the base, and the new non-dimensional parameter $\kappa_b = \frac{k_3}{k_1}$ has been introduced. Solving the first equation of Eq. (7.48) for x_{b_n} , substituting it into the second one, and manipulating it is possible to write the n-th

general term of the displacement solution x_{1_n} as:

$$\begin{aligned}
& \left\{ n^4 \left(\frac{\Omega}{\omega_b} \right)^2 \left(\frac{\Omega}{\omega_1} \right)^2 - n^2 \left[\left(\frac{\Omega}{\omega_b} \right)^2 \left(1 + \frac{\kappa}{2} \right) + \left(\frac{\Omega}{\omega_1} \right)^2 \left(1 + \frac{\kappa}{\kappa_b} + \frac{1}{\kappa_b} \right) \right] + \left(1 + \frac{\kappa}{2} \right) \right\} \\
& [\text{Re}(x_{1_n}) + i \text{Im}(x_{1_n})] + \left[- \left(\frac{\Omega}{\omega_b} \right)^2 n^2 + 1 \right] \frac{2\kappa}{\pi} \left[\dots + \right. \\
& + \frac{1}{5} \cos(5\phi) \text{Im}(x_{1_{5r+n}}) + \frac{1}{5} \sin(5\phi) \text{Re}(x_{1_{5r+n}}) + \\
& + \frac{1}{3} \cos(3\phi) \text{Im}(x_{1_{3r+n}}) + \frac{1}{3} \sin(3\phi) \text{Re}(x_{1_{3r+n}}) + \\
& \left. + \cos(\phi) \text{Im}(x_{1_{r+n}}) + \sin(\phi) \text{Re}(x_{1_{r+n}}) \right] = \\
& = \frac{1}{k_1} [\text{Re}(F_n) + i \text{Im}(F_n)] \left(\frac{1 + \kappa}{\kappa_b} \right)
\end{aligned} \tag{7.49}$$

which is different from Eq. (7.27) due to the presence of the terms related to the base response. The above equation can be written in the matrix form given by Eq. (7.30), that is $[A] \{z\} = \{b\}$, by introducing the new forcing vector:

$$\{b\} = \frac{1}{k_1} \left(\frac{1 + \kappa}{\kappa_b} \right) \left\{ \begin{array}{c} \vdots \\ \text{Re}(F_{n+1}) \\ \text{Re}(F_n) \\ \dots\dots\dots \\ \text{Im}(F_n) \\ \text{Im}(F_{n+1}) \\ \vdots \end{array} \right\}_{2N \times 1} \tag{7.50}$$

and the matrix $[A]$ as:

$$[A] = \left[\text{diag} \left\{ \tilde{\Gamma}(n) \right\} \right] + \left[\text{diag} \left\{ \tilde{\beta}(n) \right\} \right] \begin{bmatrix} [A_{11}] & [A_{12}] \\ [0] & [0] \end{bmatrix} \tag{7.51}$$

where

$$\left\{ \tilde{\Gamma}(n) \right\} = \left\{ \begin{array}{c} \vdots \\ \tilde{\gamma}(n+1) \\ \tilde{\gamma}(n) \\ \dots\dots\dots \\ \tilde{\gamma}(n) \\ \tilde{\gamma}(n+1) \\ \vdots \end{array} \right\}_{2N \times 1} \quad (7.52)$$

and from Eq. (7.49):

$$\tilde{\gamma}(n) = n^4 \left(\frac{\Omega}{\omega_b} \right)^2 \left(\frac{\Omega}{\omega_1} \right)^2 - n^2 \left[\left(\frac{\Omega}{\omega_b} \right)^2 \left(1 + \frac{\kappa}{2} \right) + \left(\frac{\Omega}{\omega_1} \right)^2 \left(1 + \frac{\kappa}{\kappa_b} + \frac{1}{\kappa_b} \right) \right] + \left(1 + \frac{\kappa}{2} \right) \quad (7.53)$$

whereas:

$$\left\{ \tilde{\beta}(n) \right\} = \left\{ \begin{array}{c} \vdots \\ \beta(n+1) \\ \beta(n) \\ \dots\dots\dots \\ \beta(n) \\ \beta(n+1) \\ \vdots \end{array} \right\}_{2N \times 1} \quad (7.54)$$

in which, from Eq. (7.49):

$$\beta(n) = \left[- \left(\frac{\Omega}{\omega_b} \right)^2 n^2 + 1 \right] \quad (7.55)$$

The matrices $[A_{11}]$ and $[A_{12}]$ assume the expression given by Eqs. (7.34) and (7.35).

Solving the algebraic system $[A] \{z\} = \{b\}$ allows the harmonic components of the solution vector for the displacement of the main system to be obtained together with the harmonic components of the base displacement, from the first equation of Eq. (7.48).

Differently from the analytical model developed in Section 7.1, the solution now depends on five dimensionless parameters, namely κ , κ_b , $\frac{\Omega}{\omega_1}$, $\frac{\Omega}{\omega_b}$, and the control phase of

the Smart Spring ϕ . In a simple open-loop control approach, the control objective is to adjust these parameters in order to reduce the response amplitude, in terms of the harmonic components of the displacement of the main system and the base.

7.2.2 Parametric study of the system response

The analytical model derived in the previous section highlights the dependence of the solution on five parameters, which are varied in order to evaluate their effect on the harmonic components of the main system and base displacements. Considering the same settings of Section 7.1.1, that is $r = 4$, $p_{max} = 5$, and a constant excitation force, Figs. 7.12-7.15 show how the amplitudes of the responses change with the variation of parameters. In particular, the analyses have been performed by varying three parameters while keeping the other two fixed. To be consistent with the representation introduced

Figure 7.12: Parametric study for the 1/rev harmonic component, base-excited system.

Figure 7.13: Parametric study for the 2/rev harmonic component, base-excited system.

Figure 7.14: Parametric study for the 3/rev harmonic component, base-excited system.

in Section 7.1.1, it has been chosen to vary ϕ and κ_b in each simulation. The results shown in this section refer to a specific operative condition, $\frac{\Omega}{\omega_1}$ and $\frac{\Omega}{\omega_b}$, and a given ratio between the stiffness of the secondary and main spring, κ . Although the parametric study has been performed for all harmonic components up to 22-nd order, only the first,

Figure 7.15: Parametric study for the 4/rev harmonic component, base-excited system.

second, third, and fourth, are reported here for the sake of brevity.

It can be noted from Fig. 7.12 and Fig. 7.13 that the 1/rev and 2/rev harmonic components of both displacement solutions, x_1 and x_b , show a very similar behavior when κ_b and ϕ vary. Specifically, once the system has been identified through a value of κ_b there is lesser dependence on the control phase of the Smart Spring, for higher values of κ_b . The simulations suggest that the responses amplitudes can be reduced by changing the system, through a new value of κ_b . Regarding the 3/rev harmonic component, the behavior of the main spring reported in Fig. 7.14(a) and that of the base, Fig. 7.14(b), are slightly different. While the displacement of the base is mostly constant for all values of κ_b , the displacement of the main spring exhibits variation of the amplitude with κ_b . A very interesting result has been obtained for the 4/rev component as shown in Fig. 7.15. Indeed, the main spring and the base follow the same behavior when the parameters vary.

Actually, the absolute values of the amplitudes are slightly different. On a quality level, the parametric study reported in this section proves that the optimal set of parameters would allow to reduce the responses amplitudes of both the main spring and the base by 50%.

Chapter 8

Experimental investigation of the Active Pitch Link

In this chapter, the activities carried out to experimentally assess the effectiveness of the Active Pitch Link (APL) are presented. As briefly introduced in Chapter 6, the Smart Spring device was developed by the Carleton University Rotorcraft Research Group (Ottawa, Canada) in order to replace the conventional pitch link with an active, or adaptive, pitch link through the insertion of the Smart Spring between the blade horn and the rotor swashplate, Fig. 8.1. The purpose of this device is to reduce the vibration transmitted to the non-rotating frames by actively changing the impedance of the pitch link to vary the torsional characteristic of the blade, which the APL is attached to, throughout its boundary conditions. Results from a non-rotating extensive test campaign performed at Carleton University are investigated in the next sections to show the capability of the active pitch link to modify the eigenproperties of the blade-APL system. Specifically, the solid pitch link is substituted by an elastic pitch link, i.e. the Smart Spring whose

stiffness is modulated by an input voltage to the piezoceramic actuator.

Figure 8.1: The Active Pitch Link (APL).

8.1 Experimental setup

The effects of modulating the stiffness of the Smart Spring, then of the APL, were investigated by performing a traditional input-output modal analysis on a reduced-scale helicopter model that incorporates the active pitch link for vibration control in order to identify the modal parameters of the system and evaluate how the different APL configurations affect the dynamics of the blade. The non-rotating experimental tests were carried out at the Rotorcraft Laboratory of Carleton University, and the solid pitch link was substituted by an elastic pitch link with $k_{APL} = 256 \text{ kN/m}$. Such a stiffness was modulated by electrically controlling the piezoceramic device inside the pitch link through the slip rings. Three input voltages corresponding to 0V, 90V, and 150V were

considered leading to the “hard” (uncoupled systems), “intermediate”, and “soft” (fully coupled systems) configurations of the APL, respectively. Referring to the mathematical model described in Chapter 7, the increase of voltage causes the gradual coupling of the system allowing to evaluate the effects of softening the APL stiffness on the dynamics of the system.

The experimental setup considered for the input-output modal analysis is shown in Fig. 8.2 where the hub, used to host the measurement system, the APL, and the excitation point are highlighted. During the tests, the blade was pulled by means of a ballast system, as shown in Fig. 8.3, in order to keep it in horizontal position and simulate the centrifugal force present in rotating conditions. The system was excited by an

Figure 8.2: Experimental setup of the non-rotating tests.

Table 8.1: Settings of the acquisition parameters for the blade-APL system.

Range (Hz)	N_f	N_a
0-1024	4096	20

Figure 8.3: Ballast system to keep the blade in the horizontal position.

Figure 8.4: Particular of sensors positioning.

impact hammer in the frequency range 0 – 1024 Hz, see Table 8.1, and the responses were recorded by 22 accelerometers through 3 measurement runs (roving procedure). The sensors were placed along the leading and trailing edges of the blade, at the top and bottom of the APL, Fig. 8.4, and on the main shaft all in the vertical direction. The LMS Scadas *III* hardware and the LMS Test.Lab[®] software were used to carry out the impact test. The modal parameters estimated for each configuration of the APL, that is with different input voltages, are reported in the next section in order to verify the capability of the APL to modify the structural properties of the system.

8.2 Estimate of the modal parameters

The modal parameters of the system were estimated using the LMS PolyMax software and the “hard” configuration of the APL, that is the one corresponding to an input voltage of 0V, is considered as reference. Indeed, in this case the APL is not actuated thus representing the solid pitch link. Table 8.2 lists the natural frequencies, the damping ratios, and the type of mode shape, where B and T refer to the out-of-plane bending and torsional modes respectively, estimated for the reference configuration of the system. In the selected frequency range, 9 mode shapes were estimated and three of them, that is the 5-th, 7-th, and 8-th show a bending-torsional coupling. The most significative mode shapes are depicted in Fig. 8.5. The modal parameters estimated with the APL in “intermediate”, i.e. 90V, and “soft”, i.e. 150V, configurations are reported in Table 8.3

Table 8.2: Modal parameters of the system with the APL in “hard” configuration.

Mode #	$f_n (Hz)$	$\zeta_n (\%)$	Description
1	14.73	4.88	I B
2	63.10	3.66	II B
3	165.64	3.74	III B
4	213.04	3.78	I T
5	253.03	0.24	III B + I T
6	293.52	2.31	IV B
7	469.48	1.37	II T + IV B
8	500.59	0.55	IV B + II T
9	555.98	1.82	II T

and Table 8.4 respectively, where it can be noted that the modulation of the stiffness of the APL effectively changes the eigenproperties of the system.

Table 8.3: Modal parameters of the system with the APL in “intermediate” configuration.

Mode #	$f_n (Hz)$	$\zeta_n (\%)$	Description
1	15.68	4.61	I B
2	64.79	2.29	II B
3	167.65	1.30	III B
4	212.80	2.21	II T
5	254.89	0.42	IV B + I T
6	295.18	1.86	IV B
7	470.91	1.04	II T + IV B
8	504.88	0.76	IV B + II T
9	561.48	1.75	II T

Table 8.4: Modal parameters of the system with the APL in “soft” configuration.

Mode #	f_n (Hz)	ζ_n (%)	Description
1	15.55	3.38	I B
2	63.65	2.93	II B
3	167.23	1.95	III B
4	212.94	1.82	II T
5	253.20	0.46	IV B + I T
6	295.98	1.52	IV B
7	470.24	0.85	II T + IV B
8	503.04	0.50	IV B + II T
9	563.22	1.22	II T

Figure 8.5: Mode shapes of the blade-APL system in its reference configuration.

8.3 Effect of the Active Pitch Link

The effect of the APL on the mode shapes of the blade-APL system was monitored by means of the Modal Assurance Criterion between the different configurations. As it can be observed from Fig. 8.6, the mode shapes show a good correlation, although the MAC matrices are not fully diagonal. The main effect of the stiffness modulation from the “hard” to the “soft” configuration of the APL, shown in Fig. 8.6(c), regards the 4-th mode shape, corresponding to the first torsional mode, whose correlation is equal to about 86%. Such a change of the torsional mode shape can be observed in Fig. 8.7. It is important to note that the natural frequency corresponding to the torsional mode shape does not change when actuating the APL (its variation is of about -0.05%), whereas the damping ratio is 52% lower than the reference value suggesting the presence of an energy transfer. See the values reported in Table 8.2 and Table 8.4 for clarity.

On the other hand, the same mode does not show significative changes when the APL is in its “intermediate” configuration, either with respect to the reference configuration Fig. 8.6(a), or with respect to the “soft” configuration Fig. 8.6(b).

In order to better understand the effect of modulating the APL stiffness, the identification of the system was performed by considering only the FRFs corresponding to the measurement points on the APL, shown in Fig. 8.4. This detailed analysis highlighted the presence of an out-of-phase internal mode of the APL that could influence the shape of the first torsional mode of the blade-APL system when the APL is in its

Figure 8.6: MAC between the mode shapes estimated for the different configurations of the APL.

“soft” configuration. The internal mode shape of the APL is depicted in Fig. 8.8 where it can be noted that the APL ends follow the same path, that is they move in-phase, in the reference configuration, see Fig. 8.8(a). In contrast, the two ends start to follow an inverse path when the APL is actuated, Fig. 8.8(b). This effect is emphasized in the “soft” configuration, Fig. 8.8(c), in which the out-of-phase internal mode can be clearly observed.

Figure 8.7: Effect of the APL stiffness on the torsional mode shape.

The test campaign carried out on the blade-APL system has shown the capability of the active pitch link to modulate the structural properties of the system and alter its dynamic behavior, thus experimentally assessing its effectiveness as a semi-active device.

Figure 8.8: Out-of-phase internal mode of the APL.

Chapter 9

Identification of the blade-APL system

In this chapter the reference and the enhanced identification methodologies are applied to the system introduced in Chapter 8, that is the reduced-scale helicopter blade model that incorporates the Active Pitch Link for vibration control. The aim is to identify the structural properties of the system in order to adjust the numerical model in light of the responses measured on the structure. It is worth remarking the intrinsic non-linear dynamic behavior of the APL, requiring the system to be linearized around a prescribed operative working condition in order to apply the identification methodologies. Specifically, the configuration of the APL corresponding to the maximum voltage applied to the actuator, in which the system is completely coupled, is considered. In this way, the identification procedure is performed by taking into account parameters corresponding to the structural properties of the blade and the APL.

9.1 Numerical model

The reduced-scale helicopter blade model was modeled with 269 plate elements and a lumped mass and stiffness were used to model the Active Pitch Link. As shown in Fig. 9.1, different properties were used for the blade (green area) and the connection between the hub and the blade (blue area). The corresponding properties are listed in Table 9.1 and Table 9.2, respectively. It should be noted that the two values of the width of the blade reported in Table 9.1 correspond to the values at its root and tip. In addition, all the accelerometers on the blade, that is 17, were included in the FE model as lumped masses to take into account their weight and inertia. In the frequency range considered for the identification procedure, that is 40-250 Hz, three modal parameters were estimated and they are listed in Table 9.3, where it can be noted that they correspond to the second and third out-of-plane bending modes, and first torsional mode. The choice of the frequency range in which to perform the identification of the system was dictated by the very poor correlation between the numerical and experimental models at low frequencies, probably due to the difficulty to represent numerically the actual boundary condition.

Table 9.1: Physical properties of the blade.

$L(m)$	$b(m)$	$t(m)$	$E(GPa)$	$\rho(kg/m^3)$	ν
0.81	0.04 – 0.08	0.0065	90	1050	0.33

Figure 9.1: Finite element model of the blade-APL system.

Table 9.2: Physical properties of the blade-hub connection.

$L(m)$	$b(m)$	$t(m)$	$E(GPa)$	$\rho(kg/m^3)$	ν
0.12	0.04	0.008	70	2700	0.33

Table 9.3: Numerical modal parameters of the blade-APL system.

Mode #	$f_n^A(Hz)$	$\zeta_n(\%)$	Description
1	57.71	0.2	II B
2	165.29	0.2	III B
3	198.95	0.2	I T

9.2 Numerical-experimental correlation

Referring to Table 8.4, the experimental natural frequencies of the system in the frequency range of interest are different from the analytical ones and their comparison is reported in Table 9.4 and Fig. 9.2. The correlation between the mode shapes was evaluated through the Modal Assurance Criterion, whose excellent values are shown in Fig. 9.3. Table 9.4 suggests that the initial FE model is not representative of the actual dynamic behavior especially for the first and third natural frequency, thus requiring to perform an identification procedure to improve the correlation between the two models.

Table 9.4: Blade-APL system: initial natural frequency shifts.

Mode #	f_n^A (Hz)	f_n^X (Hz)	$ \Delta f $ (%)
1	57.71	63.65	9.33
2	165.29	167.23	1.16
3	198.95	212.94	6.57

Figure 9.2: Blade-APL system: comparison between the measured and predicted natural frequencies.

Figure 9.3: Blade-APL system: initial correlation between the experimental and numerical mode sets.

9.3 Identification of the system

Given the complexity of the system, the FE model updating was performed by activating only the most sensitive parameters to reduce the computational cost. Specifically, the parameters corresponding to the density of the blade, the thickness and the Young's modulus of the blade-hub connection, the mass of some accelerometers, and the lumped stiffness of the APL were separately included into the identification procedure, thus giving some partially updated FE models that led to the final updated FE model representative of the actual dynamic behavior of the blade-APL system. For the sake of brevity, only the results pertaining to the final updated FE model are described in detail. Nevertheless, the convergence history of all the parameters included in the updating procedure is shown, in order to describe how the physical properties of the initial FE have to be changed to have a good representation of the actual system. As

Figure 9.4: Blade-APL system: comparison between the shape correlation functions χ_s .

Figure 9.5: Blade-APL system: comparison between the amplitude correlation functions χ_a .

Figure 9.6: Blade-APL system: comparison between the driving point FRF.

often recalled throughout the thesis, the updating methodologies have to maximize the correlation functions between the FRFs. As it can be noted from Fig. 9.4(a) and Fig. 9.5(a), the reference method improves the shape and amplitude correlation functions in a limited frequency range from about 180 to 220 Hz, that is in the neighbourhood of the

Figure 9.7: Blade-APL system: comparison between the natural frequency shifts.

Figure 9.8: Blade-APL system: comparison between the final MAC.

third natural frequency, while remaining equal (or getting worse) to the initial values in correspondence of the other frequency points. On the contrary, the enhanced methodology allows the improvement of the correlation functions in almost all the frequency range of analysis as depicted in Fig. 9.4(b) and Fig. 9.5(b), suggesting a better identification

Figure 9.9: Blade-APL system: convergence history of the updating parameter corresponding to density of the blade.

Figure 9.10: Blade-APL system: convergence history of the updating parameter corresponding to thickness of the blade-hub connection.

of the experimental dynamic behavior with respect to the one given by the reference method. Indeed, Fig. 9.6(a) shows that the updated amplitude of the driving point FRF (the only one reported for the sake of brevity), red line, does not match perfectly

Figure 9.11: Blade-APL system: convergence history of the updating parameter corresponding to Young's modulus of the blade-hub connection.

the experimental one, green line, when using the reference method whereas it does when using the enhanced method, Fig. 9.6(b) except for the final frequency points in which the third natural frequency is well identified but the FRFs are not overlapping. It could be due to the values reached by the correlation functions in that frequency range, which are not as good as those corresponding to the other frequency points. Fig. 9.7 and Table 9.5 show that both methods reduce the natural frequency shifts. Specifically, the enhanced method behaves better than the reference one in correspondence of the first and third natural frequencies, whereas the reference method allows an excellent identification of the second natural frequency. It is important to note that the enhanced methodology perfectly identifies the torsional mode, that is the one which the APL acts on. In average, the natural frequency shifts are reduced from 5.69% to 2.58% and 1.21% by the reference and enhanced methodology, respectively. Regarding the updated mode shapes,

(a) Reference method

(b) Enhanced method

(c) Reference method

(d) Enhanced method

Figure 9.12: Blade-APL system: convergence history of the updating parameter corresponding to mass of the accelerometers.

the MAC matrix reported in Fig. 9.8 shows that both methods allow them to be well correlated with the corresponding experimental quantities.

As stated at the beginning of this section, the identification of the system was performed through several steps by activating the most sensitive parameters, whose convergence histories are reported from Fig. 9.9 to Fig. 9.13. The reference method maintains the values

Figure 9.13: Blade-APL system: convergence history of the updating parameter corresponding to stiffness of the APL.

of the density of the blade, Fig. 9.9(a) and the thickness of the blade-hub connection, Fig. 9.10(a), almost constant to the initial values whereas the enhanced method, parts (b) of the same figures, suggests a reduction of the density of the blade of about 16% and the thickness of the blade-hub connection of about 4%. As regards the Young's modulus of the connection between the blade and the hub, Fig. 9.11, an increase of about 10% is reached by both methodologies. Recalling that the accelerometers were numerically modeled as lumped masses, it was possible to include the parameters associated to their mass in the identification procedure. Specifically, the most sensitive resulted as those in correspondence of three sensors located at the root of the blade and one at the tip, which are depicted in Fig. 9.12. The most interesting result regards the identification of the stiffness of the APL, whose convergence history is shown in Fig. 9.13. As it can be noted from Fig. 9.13(a), the reference method does not allow a significant identification of that

parameter because the algorithm diverges whereas the enhanced method, Fig. 9.13(b), suggests a physically meaningful value of the parameter associated to the stiffness of the APL. Table 9.6 summarizes the initial and final values of the physical properties included in the identification procedure.

Table 9.5: Blade-APL system: final natural frequency shifts.

Mode #	$ \Delta f_n ^i$ (%) Initial	$ \Delta f_n ^u$ (%) Reference method	$ \Delta f_n ^u$ (%) Enhanced method
1	9.33	4.73	1.78
2	1.16	0.28	1.24
3	6.57	2.72	0.60

Table 9.6: Blade-APL system: comparison between the initial and final identified system properties.

Parameter type	Initial value	Updated Reference method	Updated Enhanced method
ρ_{blade}	1050 kg/m^3	1050 kg/m^3	913.5 kg/m^3
t_{conn}	$8 \cdot 10^{-3} m$	$8 \cdot 10^{-3} m$	$7.76 \cdot 10^{-3} m$
m_1	$2 \cdot 10^{-3} kg$	$0.1 \cdot 10^{-3} kg$	$4.84 \cdot 10^{-2} kg$
m_2	$2 \cdot 10^{-3} kg$	$1.25 \cdot 10^{-2} kg$	$2.8 \cdot 10^{-3} kg$
m_3	$2 \cdot 10^{-3} kg$	$0.1 \cdot 10^{-3} kg$	$0.1 \cdot 10^{-3} kg$
m_{16}	$2 \cdot 10^{-3} kg$	$2.3 \cdot 10^{-3} kg$	$2 \cdot 10^{-3} kg$
E_{conn}	$70 \cdot 10^9 Pa$	$76.3 \cdot 10^9 Pa$	$77 \cdot 10^9 Pa$
k_{APL}	$1 \cdot 10^4 N/m$	$4.43 \cdot 10^9 N/m$	$6.9 \cdot 10^5 N/m$

9.4 Discussion of results

In this chapter, the reference and the enhanced methodologies have been applied to identify the structural properties of a reduced-scale helicopter blade model that incorporates the APL for vibration control. In order to speed up the identification procedure, a se-

lection of the most sensitive parameters among the ones corresponding to the physical properties of the blade, the connection between the blade and the hub, and the lumped parameters associated to the APL, has been performed. In addition, they have been separately included into the updating procedure, thus allowing to identify the FE model representative of the actual system. As illustrated in Section 9.3, the reference method is not capable of maximizing the correlation functions between the FRFs and it results in a poor identification of the experimental model representing the actual structure. On the contrary, the enhanced method has given very good results in terms of reduction of the natural frequency shifts and identification of physically meaningful parameters of the system, thus confirming its capability to estimate an accurate FE model whose dynamic properties are well correlated with the target ones. The main goal of applying the reference and developed methodologies to this complex system was the identification of the stiffness properties of the APL, whose nominal value is $k_{APL} = 256 \text{ kN/m}$ as specified in Chapter 8. The value identified by the enhanced methodology is 170 % higher than the nominal one, and it could suggest the presence of some mechanisms in the APL that have not been accurately modeled. In addition, the nominal stiffness of the spring installed in the APL is not directly representative of the stiffness of the APL during an operative working condition. The results shown in this chapter highlight that the enhanced method can be successfully applied also to complex systems, being an efficient alternative to the Grafe method.

Conclusions and Suggestions for further work

In structural dynamics, model updating can be defined as the process of identifying certain parameters of FE models by using vibration measurements, thus allowing the identification of a numerical model representative of the actual structure. Modal-based updating methodologies, i.e. based on the use of modal parameters such as natural frequencies and mode shapes, have long been used as they are, generally, more suitable to handle large FE models. Over the past years, research in this field has focused on response-based methodologies which directly use the measured responses in terms of FRFs, not necessarily requiring an experimental modal analysis. The interest in investigating FRFs-based methodologies when using real-world applications was the motivation of this research.

Following the description of the state-of-the-art in model updating, more attention has been paid to an iterative sensitivity-based methodology that minimizes a residual vector defined on two correlation functions between FRFs, through a weighted least-squares approach. Such a method has been considered as reference and an enhanced methodol-

ogy has been developed for the purpose of identifying parameters of FE models that are more representative of the physical properties of the structure under investigation.

Specifically, uncertainties affecting experimental data have been taken into account in the enhanced methodology by introducing a new formulation of the weighting matrices acting on the solution, and proposing a numerical relaxation scheme that makes the algorithm more robust with noisy data, as those generally available from vibration tests.

The developed methodology has been implemented and validated through numerical test cases, in which the experimental data are simulated and the solution is known, showing its excellent capability to identify the imposed variation of stiffness and/or mass.

Furthermore, the methodology has been experimentally validated by using simple structures, such as beams and plates, before considering complex systems. When using real experimental data, there is not a priori knowledge of the solution to be found by the identification procedures thus requiring an accurate analysis of the results, that is the updated FE model has to reproduce the actual dynamic behavior together with physically meaningful parameters. The experimental validation of the developed methodology has confirmed its real enhancement with respect to the reference one, which resulted in a poor identification of the actual structures.

Finally, a complex system composed by a reduced-scale helicopter blade model that incorporates a semi-active device for vibration control, named Smart Spring, has been deeply investigated in order to identify its physical properties. Specifically, the effectiveness of the Smart Spring to reduce vibration has been numerically validated, through the

introduction of an open-loop control law and the development of its analytical model. Moreover, the capability of the semi-active device to modulate the structural properties of a system and modify its dynamic behavior has been experimentally assessed. It should be stressed that, unlike the reference method, the enhanced methodology has allowed the identification of a FE model that well represents the actual system, with respect to stiffness and mass distribution.

It can be concluded that the methodology developed in this thesis gives an efficient contribution to the state-of-the-art in model updating, and highlights the importance of including the uncertainties affecting experimental data to improve the correlation between a FE model and its experimental counterpart by identifying physical properties effectively representative of the actual structure.

Suggestions for further work

Although the developed methodology can be considered an important step towards the identification of structures using experimental data, it obviously requires some improvements and modifications. First, since the choice of parameters is a crucial step in model updating, it would be advantageous to develop a procedure for selecting those updating parameters that will effectively produce an improvement in the modelling of the test structure. Currently, the choice is performed on those parameters to which data are sensitive but this is not a sufficient reason that allows the selection of the most suit-

able set of parameters. It could happen that responses are not sensitive to a parameter that could be instead the source of discrepancy between the FE model and the actual structure. Second, the methodology developed in this thesis should be adapted to the use of operational data, e.g. those coming from flight tests, thus giving the opportunity to identify structures under actual working conditions. Referring to the blade system considered in this thesis, the inclusion of data recorded during whirl-tower tests into the developed updating methodology would allow to identify the system in its operative conditions, and develop a closed-loop control law for vibration reduction. Finally, the developed methodology is suited for being parallelised and it could speed up the solution, making possible the identification of FE models of industrial size.

Bibliography

- [1] Zienkiewicz, O. C., *The Finite Element Method in Engineering Science*, McGraw Hill, 1971.
- [2] Bathe, K. J., *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, 1982.
- [3] Hughes, T. J. R., *The Finite Element Method*, Prentice-Hall, 1987.
- [4] Ewins, D. J., *Modal Testing: Theory and Practice*, Research Studies Press, 1984.
- [5] Allemang, R. and Brown, D., “A correlation coefficient for modal vector analysis,” *Proceedings of the 1st IMAC*, 1982.
- [6] Lieven, N. A. and Ewins, D. J., “Spatial correlation of mode shapes, the coordinate modal assurance criterion (COMAC),” *Proceedings of the 6th IMAC*, 1988.
- [7] Heylen, W. and Lammens, S., “FRAC: a consistent way of comparing frequency response functions,” *Proceedings of the Identification in Engineering Systems Conference*, 1996.
- [8] Friswell, M. I. and Mottershead, J. E., *Finite Element Model updating in structural dynamics*, Kluwer Academic Publishers, 1995.
- [9] Grafe, H., *Model updating of large structural dynamics models using measured response functions*, Ph.D. thesis, Imperial College of Science, Technology, and Medicine, London, UK, 1998.
- [10] Zienkiewicz, O. C. and Taylor, R. L., *The Finite Element Method*, McGraw Hill, 1988.
- [11] Nath, B., *Fundamental of Finite Elements for Engineers*, Athlone Press, 1974.
- [12] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, 1967.
- [13] Morino, L. and Mastroddi, F., *Course Notes in Structural Dynamics*, 2012.
- [14] Geradin, M. and Rixen, D., *Mechanical vibrations: theory and application to structural dynamics*, Wiley, 1997.

-
- [15] He, J. and Fu, Z. F., *Modal Analysis*, Butterworth-Heinemann, 2001.
- [16] Brincker, R., Zhang, L., and Andersen, P., “Modal Identification from Ambient Responses Using Frequency Domain Decomposition,” *Proceedings of the 18th IMAC*, 2000.
- [17] Agneni, A., Crema, L. B., and Coppotelli, G., “Time and frequency domain modal parameter estimation by output-only functions,” *Proceedings of the International Forum on Aeroelasticity and Structural Dynamics*, 2003.
- [18] Coppotelli, G., “On the estimate of the FRFs from operational data,” *Mechanical Systems and Signal Processing*, Vol. 23, 2009, pp. 288–299.
- [19] Bendat, J. S. and Piersol, A. G., *Random Data*, John Wiley & Sons, 1971.
- [20] Shin, K. and Hammond, P., *Fundamentals of Signal Processing for Sound and Vibration Engineers*, John Wiley & Sons, 2008.
- [21] Ibrahim, S. and Mikulcik, E., “A method from the direct identification of vibration parameters from the free response,” *The Shock and Vibration Bulletin*, Vol. 47, No. 4, 1977, pp. 183–198.
- [22] Juang, J. and Pappa, R., “An eigensystem realization algorithm for modal parameter identification and model reduction,” *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 5, 1985, pp. 620–627.
- [23] Whittle, P., *Hypothesis testing in time series analysis*, 1951.
- [24] Brown, D., Allemang, R., Zimmerman, R., and Mergeay, M., “Parameter estimation techniques for modal analysis,” *SAE paper*, Vol. 790221, 1979.
- [25] Vold, H., Kundrat, J., Rocklin, G., and Russel, R., “A multi-input modal estimation algorithm for mini-computers,” *SAE paper*, Vol. 820194, 1982.
- [26] LMS, *The LMS Test.Lab Modal Analysis manual*, 2010.
- [27] Basseville, M., Benveniste, A., Goursat, M., Hermans, L., Mevel, L., and der Auwer-aer, H. V., “Identification of linear systems. A practical guide to accurate modeling,” *Transactions of the ASME: Journal of Dynamic Systems, Measurement, and Control*, 2001.
- [28] El-Kafafy, M., Guillaume, P., Peeters, B., Marra, F., and Coppotelli, G., “Advanced Frequency-Domain Modal Analysis for Dealing with Measurement Noise and Parameter Uncertainty,” *Topics in Modal Analysis*, Vol. 5, 2012, pp. 179–199.
- [29] Rad, S. Z., *Methods for updating numerical models in structural dynamics*, Ph.D. thesis, Imperial College of Science, Technology, and Medicine, London, UK, 1997.
- [30] Baruch, M., “Optimization procedure to correct stiffness and flexibility matrices using vibration tests,” *AIAA Journal*, Vol. 16, No. 11, 1978, pp. 1208–1210.

-
- [31] Berman, A., "Mass matrix correction using an incomplete set of measured modes," *AIAA Journal*, Vol. 17, No. 1, 1979, pp. 1147–1148.
- [32] Berman, A. and Nagy, E. J., "Improvement of a large analytical model using test data," *AIAA Journal*, Vol. 21, No. 8, 1983, pp. 1168–1173.
- [33] Baruch, M., "Optimal correction of mass and stiffness matrices using measured modes," *AIAA Journal*, Vol. 20, No. 11, 1982, pp. 1623–1626.
- [34] Ross, R. G., "Synthesis of stiffness and mass matrices from experimental vibration modes," *National Aeronautics and Space Engineering and Manufacturing Meeting*, 1971, pp. 2627–2635.
- [35] Thoren, A. R., "Derivation of mass and stiffness matrices from dynamic test data," *AIAA Conference Paper*, Vol. 1, No. 72, 1972, pp. 346.
- [36] Caesar, B., "Updating system matrices using modal test data," *Proceedings of the 5th IMAC*, 1987, pp. 453–459.
- [37] Link, M., Weiland, M., and Bassagan, J. M., "Direct physical matrix identification as compared to phase resonance testing: An assessment based on practical application," *Proceedings of the 5th IMAC*, 1987, pp. 804–811.
- [38] Luk, Y. W., "Identification of physical mass, stiffness and damping matrices," *Proceedings of the 5th IMAC*, 1987, pp. 679–685.
- [39] Sidhu, J. and Ewins, D. J., "Correlation of finite element and modal test studies of a practical structure," *Proceedings of the 2nd IMAC*, 1984, pp. 756–762.
- [40] Mottershead, J. E., Link, M., and Friswell, M. I., "The sensitivity method in finite element model updating: A tutorial," *Mechanical Systems and Signal Processing*, Vol. 25, 2011, pp. 2275–2296.
- [41] Fox, R. L. and Kapoor, M. P., "Rate of change of eigenvalues and eigenvectors," *AIAA Journal*, Vol. 12, No. 6, 1968, pp. 2426–2429.
- [42] Lim, K. B., "Re-examination of eigenvectors derivatives," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 10, No. 6, 1987, pp. 581–587.
- [43] Chu, Y. and Rudisill, C. S., "Numerical methods for evaluating the derivatives of eigenvalues and eigenvectors," *AIAA Journal*, 1975, pp. 834–837.
- [44] Chen, J. C. and Garba, J. A., "Analytical model improvement using modal test results," *AIAA Journal*, Vol. 18, No. 6, 1980, pp. 684–690.
- [45] Ojalvo, I. U., Ting, T., Pilon, D., and Twomey, W., "Practical suggestions for modifying math models to correlate with actual modal test results," *Proceedings of the 7th IMAC*, 1989, pp. 347–354.

-
- [46] Hart, G. C. and Yao, J. P., "System identification in structural dynamics," *American Society of Civil Engineers, Journal of Engineering Mechanics Division*, 1977, pp. 1089–1104.
- [47] Dascotte, E. and Vanhonacker, P., "Development of an automatic updating program," *Proceedings of the 7th IMAC*, 1989.
- [48] Link, M., "Identification and correction of errors in analytical models using test data - theoretical and practical bounds," *Proceedings of the 8th IMAC*, 1990.
- [49] Dascotte, E., "Practical applications of finite element tuning using experimental modal data," *Proceedings of the 8th IMAC*, 1990.
- [50] Atalla, M. J. and Inman, D. J., "Model updating using neural networks," *Second International Conference: Structural Dynamic Modelling: Test, Analysis and Correlation*, 1996.
- [51] Santamaria, F., Arras, M., and Coppotelli, G., "Using neural networks for F.E. Model Updating of structures in operational conditions," *Proceedings of the 6th International Operational Modal Analysis Conference*, 2015.
- [52] Natke, H. G., "Die Korrektur des Rechenmodells eines elastomechanischen Systems mittels gemessener erzwungener Schwingungen," *Ingenieure-Archiv*, Vol. 46, 1977, pp. 169–184.
- [53] Natke, H. G., *Einfuehrung in Theorie und Praxis der Zeitreihen- und Modal-analyse*, Vieweg Verlag, Wiesbaden, 1983.
- [54] Cottin, N., Felgenhauer, H. P., and Natke, H. G., "On the parameter identification of elastomechanical systems using input and output residuals," *Ingenieure-Archiv*, Vol. 54, No. 5, 1984, pp. 378–387.
- [55] Sestieri, A. and D'ambrogio, W., "Why be modal: how to avoid the use of modes in the modification of vibrating systems," *The International Journal of Analytical and Experimental Modal Analysis*, Vol. 4, No. 1, 1989, pp. 25–30.
- [56] Natke, H. G., "Updating computational models in the frequency domain based on measured data: a survey," *Probabilistic Engineering Mechanics*, Vol. 3, No. 1, 1988, pp. 28–35.
- [57] Imregun, M. and Visser, W. J., "A review of model updating techniques," *The Shock and Vibration Digest*, Vol. 23, No. 1, 1991, pp. 9–20.
- [58] Mottershead, J. E. and Friswell, M. I., "Model updating in structural dynamics: a survey," *Journal of Sound and Vibration*, Vol. 167, 1993, pp. 347–375.
- [59] Guyan, R. J., "Reduction of stiffness and mass matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, pp. 380.
- [60] Gysin, H., "Expansion: The Achilles heel of FE model updating," *15th International Seminar on Modal Analysis*, 1990.

-
- [61] Callahan, J. O., Avitabile, P., and Riemer, R., "System equivalent reduction expansion process," *Proceedings of the 7th IMAC*, 1989.
- [62] Friswell, M. I., Mottershead, J., and Ahmadian, H., "Finite-element model updating using experimental data: parametrization and regularization," *Philosophical Transactions of The Royal Society London*, Vol. 359, 2001, pp. 169–186.
- [63] Ahmadian, H., Friswell, M. I., and Mottershead, J., "Regularization methods for finite-element model updating," *Mechanical Systems and Signal Processing*, Vol. 12, No. 1, 1998, pp. 47–64.
- [64] Titurus, B. and Friswell, M. I., "Regularization in model updating," *International Journal for Numerical Methods in Engineering*, Vol. 75, No. 4, 2008, pp. 440–478.
- [65] Link, M., "Updating analytical models by using local and global parameters and relaxed optimisation requirements," *Mechanical Systems and Signal Processing*, Vol. 12, No. 1, 1998, pp. 7–22.
- [66] Mottershead, J. and Foster, C. D., "On the treatment of ill-conditioning in spatial parameter estimation from measured vibration data," *Mechanical Systems and Signal Processing*, Vol. 5, No. 2, 1991, pp. 139–154.
- [67] Prells, U., "A regularisation method for the linear error localisation of models of elastomechanical systems," *Inverse Problems in Engineering*, 1996, pp. 197–217.
- [68] Friswell, M. I., "The adjustment of structural parameters using a minimum variance estimator," *Mechanical Systems and Signal Processing*, Vol. 3, No. 2, 1989, pp. 143–155.
- [69] Pintelon, R. and Schoukens, J., *System Identification: a frequency domain approach*, IEEE Press, 2001.
- [70] Paige, C. C. and Saunders, M., "Towards a generalized singular value decomposition," *SIAM Journal on Numerical Analysis*, Vol. 8, No. 3, 1981, pp. 398–405.
- [71] Hansen, P. C., "Analysis of discrete ill-posed problems by means of the L-Curve," *SIAM Review*, Vol. 34, No. 4, 1992, pp. 561–580.
- [72] Hansen, P. C., "The discrete Picard condition for discrete ill-posed problems," *BIT*, Vol. 30, 1990, pp. 658–672.
- [73] Hansen, P. C. and O’Leary, D. P., "The use of the L-curve in the regularization of discrete ill-posed problems," *SIAM Journal on Scientific Computing*, Vol. 14, No. 6, 1993, pp. 1487–1503.
- [74] *FEMtools[®] Framework User’s Guide, Version 3.5.2*, Dynamic Design Solutions NV, 2011.
- [75] *FEMtools[®] Programmer’s Guide, Version 3.5.2*, Dynamic Design Solutions NV, 2011.

-
- [76] *FEMtools® Model Updating Theoretical Manual, Version 3.5.2*, Dynamic Design Solutions NV, 2011.
- [77] Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, Dover Publications, 1968.
- [78] Arras, M. and Coppotelli, G., “Finite-Element Structural Updating using Frequency Response Functions,” *Journal of Aircraft*, Vol. 52, No. 5, 2015, pp. 1454–1468.
- [79] Arras, M., Coppotelli, G., Nitzsche, F., and Feszty, D., “Identification of the Smart Spring properties from FRFs measurements,” *Proceedings of the 56th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, 2015, pp. DOI: <http://arc.aiaa.org/doi/abs/10.2514/6.2015-2061>.
- [80] Johnson, W., *Helicopter theory*, Dover Publications, 1994.
- [81] Bramwell, A. R. S., Done, G., and Balmford, D., *Bramwell’s Helicopter Dynamics*, Butterworth-Heinemann, 2001.
- [82] Casciati, F., Magonette, G., and Marazzi, F., *Technology of Semiactive Devices and Applications in Vibration Mitigation*, John Wiley & Sons, 2006.
- [83] Jaffe, B., Cook, W. R., and Jaffe, H. L., *Piezoelectric ceramics*, Academic Press, 1971.
- [84] Nitzsche, F., “Aeroelastic Analysis of a Rotor Blade with Active Impedance Control at the Root,” *Canadian Aeronautics and Space Journal*, Vol. 47, No. 1, 2001.
- [85] Nitzsche, F., Feszty, D., Khomutov, K., Lynch, B., Mander, A., and Ulker, F., “Design and instrumentation of the SHARCS scaled rotor with three independent control systems,” *Proceedings of the 64th Annual Forum of the American Helicopter Society*, 2008.
- [86] Wickramasinghe, V., Yong, C., Zimcik, D., Harold, T., and Nitzsche, F., “Smart Spring: A Novel Adaptive Impedance Control Approach for Active Vibration Suppression Applications,” *Proceedings of SPIE*, Vol. 5390, 2004.
- [87] Young, C., Zimcik, D. G., Wickramasinghe, V. K., and Nitzsche, F., “Development of the Smart Spring for Active Vibration Control of Helicopter Blades,” *Journal of Intelligent Material Systems and Structures*, Vol. 15, 2004, pp. 37–47.