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Evaluation of Stress Intensity Factors
in Multi-Cracked Thick-Walled Cylinders
using Finite Element Methods

by

Kenneth J. Kirkhope

A thesis submitted to the Faculty of
Graduate Studies and Research
in partial fulfillment of the
requirements for the degree of
Master of Engineering

Department of Mechanical and Aeronautical Engineering
Carleton University
Ottawa, Ontario
Canada
April 1988

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ABSTRACT

This thesis reports the results of numerical studies of mode I stress intensity factors for single and multiple radial surface cracks located at the bore of internally pressurized thick-walled cylinders.

The general purpose finite element program ASAS was used for the analyses, which considered both two and three dimensional crack geometries. The two dimensional analyses were used to quantify the cylinder size effect by considering cylinders with radius ratios, $W = R_2/R_1$, ranging from 1.5 to 2.25 and containing up to 40 equispaced cracks. The three dimensional analyses were used to quantify the crack shape effect for semi-elliptical flaws with major to minor crack length ratios, b/a , ranging from 1.0 to 4.0, and results are presented for the first time for a cylinder with $W = 2.0$ containing 1, 2, 4, 8, 16 and 32 cracks. Stress intensity factors are presented in condensed non-dimensionalized form and are compared with published results, where these exist.

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Nomenclature

- a - Crack length (semi-minor axis of semi-elliptical crack)
- a_c - Critical crack length
- a_e - William's effective crack radius
- b - Semi-major axis of semi-elliptical crack
- C - Material constant for Paris law
- $\frac{da}{dN}$ - Crack growth rate
- E - Young's modulus of elasticity
- E_2 - Elliptical integral of the 2nd kind
- EPFM - Elastic-Plastic Fracture Mechanics
- FEM - Finite Element Method
- h_j - Stress intensity factor polynomial influence function
- K - Stress intensity factor
- K_I - Mode I stress intensity factor
- K_{II} - Mode II stress intensity factor
- K_{III} - Mode III stress intensity factor
- K_{IC} - Plane strain fracture toughness
- K_{Ie} - Stress intensity factor for semi-elliptical crack
- K_{Is} - Stress intensity factor for straight-fronted crack
- L - Length of crack-tip element normal to crack front
- LEFM - Linear-Elastic Fracture Mechanics

Nomenclature

- M - Length of crack-tip element parallel to crack front
- M_C - Kobayashi's curvature correction factor
- M_p - Underwood's stress intensity magnification factor
- n - Material power constant for Paris law
- N - Number of cracks
- NDT - Non-destructive testing
- P - Internal pressure load
- r - Coordinate in radial direction
- R_N - Multiple crack load relief factor
- R_1 - Bore radius
- R_2 - Outer wall radius
- t - Cylinder wall thickness
- v - Crack opening displacement
- W - Cylinder radius ratio
- x - Coordinate in the X direction
- y - Coordinate in the Y direction
- Y - Linear elastic shape factor
- z - Coordinate in the axial direction
- 2D - Two-dimensional
- 3D - Three-dimensional
- α - Correlation parameter
- β - Angle along the crack front
- ϕ - Elliptic angle

Nomenclature

- θ - Coordinate in the circumferential (hoop) direction
- σ - Stress (nominal)
- σ_r - Radial stress
- σ_θ - Hoop stress
- σ_z - Axial stress
- $\tau_{r\theta}$ - Shear stress
- ν - Poisson's ratio

CHAPTER 1

INTRODUCTION

1.1 Purpose

The mechanics of brittle fracture and fatigue of thick-walled cylinders subjected to internal pressure-loading is of considerable practical importance as this type of geometry and loading characterizes a wide range of mechanical components. Examples include items such as chemical reaction vessels, high pressure steam and hydraulic piping, isostatic compaction chambers, and gun barrels. In many of the above examples, a thorough understanding of the fatigue and fracture behaviour is vital because the occurrence of failure can be catastrophic and potentially hazardous to personnel working nearby. In this respect, linear elastic fracture mechanics (LEFM) has an important role to play.

LEFM can be applied to thick-walled cylindrical pressure vessels to ensure that an economic and fail-safe design is arrived at by providing a means for predicting fatigue lives and failure modes within certain limitations. However, the practical implementation of LEFM analysis to the design of thick-walled pressure vessels requires knowledge of, among other things, appropriate stress intensity factors. It is with this aspect of LEFM that the work reported on here is concerned. In particular, this thesis deals with the evaluation of the mode I stress intensity factors for internally pressurized, thick-walled cylinders containing radial cracks along the internal bore.

Existing stress intensity factor solutions for pressurized thick cylinders are restricted to a limited range of cylinder and crack geometries. Solutions for the problem of a cylinder containing multiple radial cracks at the bore surface are limited primarily to two dimensional analyses where the crack is assumed to be straight fronted. Stress intensity factor solutions for the more realistic three dimensional semi-elliptical crack problem are largely devoted to single crack and thin-walled vessel geometries. These are not readily applicable to thick-walled vessels which contain appreciable stress gradients.

Since exact solutions for these problems are generally difficult to obtain analytically, various numerical or experimental techniques are employed to obtain estimates of the stress intensity factors. Of these, the finite element method (FEM) has proven to be a most popular and powerful means of analyzing crack problems, and is used in the studies described in this thesis.

1.2 Scope

This thesis describes work which has been carried out using the finite element method to determine stress intensity factors for internally pressurized thick-walled cylinders with radial straight-fronted and semi-elliptical surface cracks at the inner bore, as illustrated in Figure 1.1.

The studies considered a wide range of crack geometries, and crack configurations. These are summarized below:

a) Straight-Fronted (2D) Crack Studies

$W = 1.50, 1.75, 2.00, 2.25$

$a/t = 0.05, 0.10, 0.20, 0.25, 0.30, 0.35, 0.40, 0.50$

$N = 1, 2, 4, 20, 40$ equispaced cracks.

b) Semi-Elliptical (3D) Crack Studies

$W = 2.00$

$a/t = 0.05, 0.10, 0.20, 0.35, 0.50$

$b/a = 1.0, 2.0, 3.0, 4.0$

$N = 1, 2, 4, 8, 16, 32$ equispaced cracks.

Chapter 2 reviews the background and theory of LEFM applied to fracture and fatigue of pressurized cylinders. The need for stress intensity factor (SIF) data in order to use LEFM is described, and previous research on determination of SIF for two and three dimensional crack geometries is discussed in detail.

Chapter 3 briefly describes the finite element program — (ASAS-H) used to conduct the studies.

Chapters 4 and 5 present the results of studies of straight-fronted (2D) and semi-elliptical (3D) crack geometries respectively. Single and multiple identical cracks are considered. In each chapter the requirements and considerations involved in formulating FEM meshes which ensure adequate accuracy while being computationally efficient are discussed in detail. The results of the numerical studies are then presented and compared with those of other researchers where possible. Considerable effort has been made to condense the data into compact non-dimensional form.

Finally, Chapter 6 concludes the thesis with a summary of the important new results obtained.

Figure 1.1 Internally Pressurized Thick-Walled Cylinder
Containing Radial Surface Cracks at the Bore

CHAPTER 2

FRACTURE AND FATIGUE ANALYSIS OF THICK-WALLED CYLINDERS

2.1 Background

The behaviour of thick-walled cylinders subjected to internal pressure loading has warranted a great deal of attention since Lamé and Clapeyron first published their classical analysis of the problem in 1833. Knowledge in this field has acquired considerable practical importance owing to the extensive range of mechanical components characterized by this geometry and loading. Common examples include such items as chemical reaction vessels, high-pressure steam and hydraulic piping, isostatic compaction chambers, and large calibre gun barrels.

In many instances, these components are employed where it is necessary to contain fluids or gases under extremely high pressures. Thick-walled cylindrical vessels used in the manufacture of polyethylene, for instance, are routinely subjected to pressures on the order of 70 MPa to 400 MPa. Similar vessels used in laboratory applications have been designed to contain pressures as high as 800 MPa.

In addition to the high pressure loads commonly encountered, these components are often situated in critical sections of plant processes or machinery systems. In such circumstances, failure may have serious safety and economic implications. Thus a thorough understanding of the fatigue and fracture behaviour of internally pressurized thick-walled cylinders is vital if the functional reliability of these

components is to be maintained throughout the intended service life.

Fatigue in thick-walled pressure vessels may or may not be a problem, depending on the application. For instance, stirred autoclaves for use in the continuous polyethylene process are subjected at most to two or three thousand cycles of pressure in a lifetime. On the other hand, isostatic compaction chambers may be required to operate in service for more than 100,000 cycles.

Gun barrels are a special case. A series of internal grooves, termed rifling, are situated along the bore surface to place a spin on the projectile and aid in the stability of its flight. These grooves, which may number upwards to 40, are prime initiation sites for fatigue crack growth. In fact, it is usually assumed that several cracks are present within the first few firing rounds due to the heat and pressure loads experienced¹. Eventually one or more cracks will break through the cylinder wall, or may reach critical dimensions resulting in brittle fracture of the gun barrel.

An example of the catastrophic and extremely hazardous nature of one such failure was reported by Davidson et al [1]. The vessel in this instance was in fact a 175 mm gun barrel. Failure occurred when a small crack situated at the bore reached critical dimensions under normal service (i.e. firing) conditions. The failure resulting in the scattering of a large number of gun fragments over distances up to 1.25 km from the firing site. Similar, though usually less alarming incidents of brittle fracture have occurred in the commercial and laboratory fields [2, 3, 4, 5].

¹This process is sometimes referred to as "heat checking".

High pressure vessels are normally manufactured from high-strength low alloy steels. For some applications vessels of a multi-layer or composite construction can be used. However, this type is generally not acceptable when cross bores are required. Such applications generally use monobloc forged vessels where, in attempting to ensure uniform and predictable properties, there is a strong incentive to keep the section thickness to a minimum. However the problem arises in the requirement for a material which exhibits both a high tensile strength as well as a strong resistance to fracture. High strength and fracture resistance are material properties which tend to compete against each other. Fast fracture resulting from either manufacturing flaws, or fatigue cracks which reach a critical unstable size, must be avoided at all costs. Criteria for design of such pressure vessels is not based on ductile yield but on fracture or fatigue considerations based on the use of LEFM and which require knowledge of the SIF present at the flaw or tip of the crack.

Sections of the ASME Boiler and Pressure Vessel Code [6] provide guidelines for design against fracture in pressure vessels of various classes. However, there remains considerable debate about some of the underlying assumptions, and of the applicability of these guidelines to relatively thick-walled components [7, 8, 9]. Indeed, none of the National codes of practice appear to cover satisfactorily the design of thick-walled forged vessels although this position may be changing [7]. The essential problem is that the application of rules appropriate for thin-walled vessels become increas-

ingly in error as the radius ratio $W = R_2/R_1$, of the vessel increases, so that further study of the conditions present in thick wall vessels remains desirable.

2.2 Stress Conditions in Pressurized Thick-Walled Cylinders

The equations for the radial equilibrium of a linear, homogeneous and isotropic thick-walled cylinder subjected to internal pressure were first derived by Lamé and Clapeyron [10]. For a simple cylinder of radius ratio, W , and subjected to an internal pressure load, P , these equations reduce to:

$$\sigma_r = P \cdot \left[\frac{1 - (R_2/r)^2}{W^2 - 1} \right] \quad - 2.1$$

$$\sigma_\theta = P \cdot \left[\frac{1 + (R_2/r)^2}{W^2 - 1} \right] \quad - 2.2$$

$$\tau_{r\theta} = 0 \quad - 2.3$$

where σ_r is the radial component of stress, σ_θ is the circumferential or hoop stress and $\tau_{r\theta}$ is the shear stress in this reference coordinate system. The variation of these components of stress are shown in Figure 2.1 for a cylinder of radius ratio $W = 2$. Note that σ_r and σ_θ each are maximum at the cylinder bore ($r = R_1$), and that these maximum values are not dependent on the absolute cylinder dimensions but only on

the radius ratio W . It is also evident that a significant variation in the stress levels occurs across the cylinder wall. This is in contrast with the familiar thin-walled or thin-shelled approximation:

$$\sigma_{\theta} = \frac{P \cdot R}{t} \quad - 2.4$$

$$\sigma_r = 0 \quad - 2.5$$

where t is the vessel wall thickness. The thin-walled expressions become increasingly in error as the radius ratio of the cylinder increases. Even with a radius ratio as small as 1.2, the error in the maximum hoop stress is nearly 10%. The ASME Boiler and Pressure Vessel Code [6] recommends the use of equations 2.1 and 2.2 when the cylinder radius ratio exceeds a value of 1.1.

A third principal stress, σ_z , in the axial direction occurs as a result of the end conditions imposed on the cylinder. The three end conditions most commonly encountered are:

1) Open-Ended $\sigma_z = 0$ - 2.6

2) Closed-Ended $\sigma_z = \frac{P}{W^2 - 1}$ - 2.7

3) Zero Axial Strain $\sigma_z = \frac{-2\nu P}{W^2 - 1}$ - 2.8

as shown in Figure 2.2. In each case the axial stress is considered constant across the wall of the cylinder, and in

the particular case of the closed-ended cylinder is equal to the mean of the hoop and radial stresses at any radius. The zero or plane strain end condition is often assumed for relatively long and opened-ended cylinders, such as gun barrels, where strain in the axial direction is inhibited by the material bulk.

It should be noted that the tensile stresses at the bore of an internally pressurized cylinder can be reduced considerably by employing a technique known as autofrettage. Residual compressive stresses at the cylinder bore are set up by initially subjecting the vessel to a pressure sufficient to cause yielding and subsequent plastic deformation to some extent through the wall thickness. The residual stresses, which are present when the cylinder is under no external loads, effectively counteract the stresses incurred due to the pressurization of the cylinder, resulting in higher allowable service pressures and enhanced fatigue limits. The effect of autofrettage is not considered in the cracked cylinder problems discussed in this thesis.

2.3 Applicable Concepts of LEFM

The technology of fracture mechanics is presently based on two approaches that are somewhat complementary and give similar results. The transition-temperature approach grew from the recognition that materials lose plasticity, hence toughness at low temperatures. This approach is performed primarily by experimental 'energy absorption' testing techniques. The other, more analytic approach, was derived from considerations of the stress field and the plastic zone at the tips of cracks. This approach includes the techniques of

linear-elastic and elastic-plastic fracture mechanics (LEFM and EPFM respectively) and is the approach used to conduct the work in this thesis.

The stress field in an elastic body containing a very sharp crack was solved by Irwin [11] by an Airy stress function approach based on a method described by Westergaard [12]. The stress distributions were solved for three basic 'modes' of crack surface displacement shown in Figure 2.3 and described as follows:

Mode I: Opening or tensile mode, where the crack surfaces move directly apart.

Mode II: Sliding or in-plane shear mode, where the crack surfaces slide over one another in a direction perpendicular to the leading edge of the crack.

Mode III: Tearing or antiplane shear mode, where the crack surfaces move relative to one another and parallel to the leading edge of the crack.

The superposition of these three modes is sufficient to describe the most general cases of crack surface displacement. However, since there is always a tendency for a brittle fracture to propagate in the direction which minimizes the shear loading, the first mode is generally regarded as the most important. It is conventional to add the Roman numerals I, II and III as subscripts to various symbols to indicate the crack displacement mode.

For the notation shown in Figure 2.4, the crack tip stresses for mode I loading are:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) & + \dots \\ \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) & + \dots \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} & + \dots \end{bmatrix} \quad - 2.9$$

$\sigma_z = \nu(\sigma_x + \sigma_y)$ plane strain

$\sigma_z = 0$ plane stress

The parameter, K_I , is the mode I stress intensity factor which characterizes the magnitude of the $1/\sqrt{r}$ stress singularity at the crack tip, and includes the influence of loading stress, crack length and structural geometry. Similar expressions, involving K_{II} and K_{III} are obtained from mode II and mode III crack tip displacements. The rigorous derivation of the stress intensity factor, K , can be found in original publications [14] and in most advanced texts [15 - 18].

Expressions for K are usually given in the general form:

$$K = Y \cdot \sigma \cdot \sqrt{\pi a} \quad -2.10$$

where Y = linear elastic shape factor

σ = nominal stress

a = characteristic crack length

For all geometries, Y is a function of crack size and shape, body size and shape, as well as mode of loading.

The use of a single parameter; K , to define the stress field and consequent onset of unstable fracture in cracked bodies is the basis of LEFM, and can be justified for brittle materials from thermodynamic arguments. Extension of these arguments to more ductile materials requires simplifying assumptions.

It was early realized that materials do not exhibit elastic behaviour and that non linear plastic deformation occurs, especially at the crucial crack tip region. If this zone of plastic deformation is 'small' and well contained within an 'elastic matrix', then LEFM should still suffice to describe the fracture behaviour of the cracked body. Increasingly, however, attention has focussed on the limitations of using LEFM to characterize the onset of crack growth and stable tearing that precedes fracture of ductile materials [19, 20]. The resulting approach, generally referred to as elastic-plastic fracture mechanics (EPFM), has also attempted to describe the behaviour of the cracked body in terms of a single parameter which accounts for the non linear plastic deformations occurring at the crack tip. Several parameters have been proposed; J-integral, crack tip opening displacement, energy releases, and continue to be the subject of intense investigations and correlation.

In general, the designer of a high pressure component is constrained by the initial selection of a material with a certain minimum yield strength. Materials used in high pressure applications are therefore usually of a relatively high tensile strength and can be classified as "quasi-brittle". In such circumstances, LEFM analysis provides a reasonably valid approximation of the fracture behaviour of such components. Unstable fracture is then assumed to occur when the stress intensity reaches some critical value,

denoted K_{IC} , often referred to as the plane strain fracture toughness of the material. A minimum fracture toughness of 120 MPa $\sqrt{m}$ is typical for many high pressure applications. The 'critical' crack length, a_c , can then be defined from equation 2.10 by:

$$a_c = \left(\frac{K_{IC}}{\bar{Y} \cdot \sigma \cdot \sqrt{\pi}} \right)^2 \quad - 2.11$$

LEFM analysis can also be utilized to describe sub-critical crack growth by fatigue loading through the empirical relation proposed by Paris [24]:

$$\frac{da}{dN} = C \cdot \Delta K^n \quad - 2.12$$

where $\frac{da}{dN}$ = increment of crack growth per load cycle or 'crack growth rate'

ΔK = range of stress intensity experienced in a load cycle

C, n = material constants

It should be noted that C and n in turn have been shown to be dependant on environment, frequency of loading, and stress ratio and in general must be determined experimentally for the precise application. The importance of this relationship is that if the material constants and a K calibration are known or assumed, equation 2.12 (or similar expressions [22]) can be integrated to give an estimate of the fatigue life of the component before the onset of brittle fracture (i.e. when $a = a_c$).

A common industry practice is to design the pressure vessel component such that unstable fracture cannot occur. That is, the critical crack length exceeds the vessel wall thickness so that a 'leak before break' situation occurs, reducing the risk of an explosive sudden fracture. However, in some applications, such as in the nuclear and chemical industries, a leak before break criterion may not be acceptable. In such circumstances, inspections of the pressure vessel components must be maintained at regular intervals to insure continuous safe operation. These intervals can be defined through use of the fatigue crack growth law by determining the number of cycles it would take for the maximum undetected flaw to grow to critical proportions. The maximum undetectable, or minimum detectable flaw size depends on the non-destructive testing (NDT) technique used in the inspection. X-ray and ultrasonic testing techniques are among the most widely used methods of flaw detection in high pressure applications.

After these inspections have been performed, special guidelines such as Section XI of the ASME Boiler and Pressure Vessel Code [6] may be used where there is an indication of crack presence. The main body of Section XI gives conservative guidelines for a) determining the size and orientation of the flaw from the NDT examination data and b) deciding whether the flaw should be repaired. An important aspect of this evaluation procedure is the ability to accurately predict stress intensity factors for the range of flaw shapes and sizes commonly encountered in these applications.

2.4 Previous Research on Cracked Cylinder Problems

A considerable volume of literature is devoted to the problem of internally pressurized cylinders containing cracks owing to the practical importance of this problem. The vast majority of these report on straight-fronted or two-dimensional (2D) crack geometries, however, recently much emphasis has been placed on the more realistic (and difficult) problem of semi-elliptical (3D) surface flaws. The following sections review the important findings of previous researchers on the subject of stress intensity factor solutions for cracked cylinders.

2.4.1 Straight-Fronted (2D) Crack Geometries

The earliest stress intensity factor solution to have application to internally pressurized thick-walled cylinders was provided by Bowie in his classical analysis of a circular hole in an infinite plate containing one and two diametrically opposed cracks [23]. Although these solutions correspond to axial cracks of uniform depth in a hollow cylinder of very large diameter ratio, they are of limited practical use for the range of vessel geometries normally used.

Bowie and Freese [24] applied a modified-mapping collocation technique to obtain stress intensity factors for a single radial crack at the bore surface of a ring. Results are presented for a wide range of cylinder radius and crack depth to wall thickness ratios. Shannon [25] also presented similar results for diameter ratios, W , between 1.6 and 3.0 and crack depth to wall thickness ratios, a/t , between 0.1 and 0.9. In both cases the results obtained agree well, within 4 to 8% with experimental results obtained by Shannon

and Austin [26] and by Underwood et al. [27].

Various authors have investigated the problem of multiple cracking in cylinders and rings under the influence of internal pressure. Tracy [28] investigated the problem of radial cracks of equal length emanating from the outer and inner surfaces of a ring using a combination of complex variable theory and mapping collocation method. Results are presented for one to four cracks in circular rings with diameter ratios of 2.0 and 3.0. This work is compared with results obtained by Bowie and Freese [24] for one and two cracks.

Shannon [25] also investigated the problem for two diametrically opposed cracks for a wide range of cylinder radius ratios and crack depths. Results were obtained by the finite element method using large numbers of constant strain elements. Among the conclusions reached was the assertion that two diametrically opposed cracks represents the worst case of multiple cracking in thick-walled cylinders.

Barratta [29] describes an approximate method of obtaining stress intensity factors for multiple cracks by using the concept of a load relief factor R_N originally suggested by Neuber [30]. The load-relief factor can be defined as the stress intensity factor of a multiple cracked body divided by the value for a single cracked body of the same geometry and loading conditions. Symbolically, R_N is defined in terms of Mode I stress intensity factors as:

$$R_N = \frac{(K_I)_{N,\infty}}{(K_I)_{1,\infty}}$$

where $(K_I)_{N,\infty}$ is the Mode I stress intensity factor for the multiple cracked body of infinite geometry and $(K_I)_{1,\infty}$ is the Mode I stress intensity factor for the same body and loading but having a single crack. This approach is used to obtain the stress intensity factor for a finite body with multiple cracks using;

$$(K_I)_{N,f} = R_N \cdot (K_I)_{1,f} \quad \text{---2.14}$$

where the subscript f denotes a finite body. In applying this approach to multiply cracked cylinders, Barratta obtained the required load relief factor from the work of Rooke and Tweed [31] who solved the problem of multiple cracks around a pressurized circular hole in an infinite plate. The finite geometry solutions were then obtained by using the results of Bowie and Freese in the manner explained above. Results are presented for cylinders having 36 and 48 cracks with various diameter ratios from 1.5 to 3.0 and crack depths a/t from 0.0 to 0.5.

Clarke and Morton [32] used an approximate approach similar to Baratta based on the assumption that the load relief factor for an internally pressurized thick-walled cylinder is the same as that for an edge cracked plate with similar crack penetration depths and crack tip separation distances. Again by using the results of Bowie and Freese, solutions were presented for 40 cracks in a cylinder of diameter ratio 2.0. These were compared with solutions presented by Goldthorpe [33] who also used the concept of load relief factor very much similar to that proposed by Baratta. Good agreement was obtained despite the mathematically unjustified assumptions which were made by Clarke and Morton.

Pu and Hussain [34] used the finite element method to compute solutions for multiple cracked circular rings. The models used consisted entirely of twelve noded cubic isoparametry elements, and special collapsed node elements to represent the stress singularity at the crack tip. This crack element is formed by collapsing the quadrilateral element into a triangle and placing the mid-side nodes at the 1/9 and 4/9 positions of the element sides (similar to the quadratic element described in Chapter 3). Stress intensity factors were obtained for cylinders containing 1 to 40 radial cracks, a/t between 0.1 and 0.6 and W between 1.5 and 2.5. The results for one and two radial cracks agreed to within 2% with those of Bowie and Freese. The authors also compared their results with those obtained by the approximate methods of Baratta and Goldthorpe, and determined that Goldthorpe's results were too low and Baratta's results too high.

Hoff [35] used quadratic isoparametric elements (similar to the ones used by this author) to obtain stress intensity factors for a similar range of problems as those presented by Pu and Hussain. Good agreement with the results published by Tracy, and by Pu and Hussain were obtained for smaller numbers of cracks, but discrepancies of up to 14% were obtained for the 40 crack cases. These discrepancies were thought to be due to the method of determining the stress intensity factors from the finite element results. Pu and Hussain substituted the global displacements at the crack tip into the Westergaard equations (see Chapter 3) whereas Hoff used the stiffness derivative method, the latter being generally considered to be more accurate. It was also pointed out that poor mesh design on the part of Pu and Hussain could be an additional factor in these discrepancies, and seems more likely to the author.

Pick et. al [36] used photoelastic techniques to obtain stress intensity factors for single and multiple cracked cylinders. Despite inherent difficulties in producing the models, and the large scatter associated with the measurement techniques, good agreement ($\pm 10\%$) was obtained for the single crack cases compared to the finite element results of Hoff. Poorer agreement was obtained for the two and four crack cases with the photoelastic results being 10 to 25% lower. Although similar trends between the finite element and photoelastic results were noted, it was concluded that the large scatter obtained for the experimental results made it difficult to support or disprove the 'two-crack' effect claimed by the numerical solutions.

2.4.2 Semi-Elliptical (3D) Crack Geometries

A number of authors have studied the problem of semi-elliptical cracks at the bore of internally pressurized cylinders. Much of the work was conducted during the late 1970's, a time at which it was just becoming feasible to employ numerical techniques to solve complex three dimensional crack problems. It is worth noting that most of the papers deal with relatively thin-walled cylinders, (i.e. $W \leq 1.1$), although some authors have attempted extrapolating thin-walled results to estimate thicker-walled vessel geometries.

The earliest important attempt at estimating stress intensity factors for semi-elliptical cracks in a cylinder subjected to internal pressure loading was by Underwood [37]. He proposed a simple combination of pressurized cylinders analysis with solutions for semi-elliptical edge cracks in a

finite thickness plate under uniform tensile loading by the equation:

$$K_{Ie} = M_p \cdot K_{Is} \quad - 2.15$$

where K_{Ie} = SIF for semi-elliptical crack in a pressurized cylinder

K_{Is} = SIF for straight-fronted crack in a pressurized cylinder

$$M_p = \frac{\text{SIF for plate with semi-elliptical crack}}{\text{SIF for plate with straight-fronted crack}}$$

The above equation was assumed to apply for similar crack depth ratios, a/t , and crack shapes, b/a , within the cylinder and plate geometries. Stress intensity factors at the deepest point of crack penetration were presented for $W = 2.0$, $a/t = 0$ to 0.7 , and $b/a = 1.0$ to $50.$, and were claimed to be accurate to within 10% for a/t less than 0.4 . He noted, however, the basic assumption that crack shape effects in a cylinder are similar to the effects in a plate becomes less certain for extreme values of b/a . He also surmised that the constraint of finite crack length, lacking in the straight-fronted crack analysis, could be significant even for large values of b/a .

Blackburn and Hellen [38], using the finite element method, modelled one quarter of the cylinder with approximately 80 twenty-node quadratic brick elements together with fifteen-node special singularity elements at the crack tip. The stress intensity factors were evaluated using the virtual crack extension method of Hellen [39] for a cylinder with

$W = 1.44$, $a/t = 0.4$ and $b/a = 3.33$. They claimed agreement within 1% of known solutions.

Kobayashi [40] estimated stress intensity factors for semi-elliptical cracks in tubes subjected to internal pressure loading via a superposition method. The analogy assumed was that the cracked cylinder geometry could be approximated by a finite thickness plate under similar loading conditions. The latter problem could be further approximated by superimposing uniform pressure and uniform bending moment loads on a semi-elliptical surface crack in an infinite solid and applying appropriate correction factors. Some uncertainty in the modelling of the correction factors led to later revisions in the approach [41, 42, 43].

In the latter paper, Kobayashi et al. used an iterative procedure based on the alternating technique to derive stress intensity factors for semi-elliptical cracks in flat plates. These results were converted to cracked cylinder geometries using a single curvature correction factor, $M_C(\beta)$:

$$\begin{array}{lcl} K_I(\beta) & = & M_C(\beta) \cdot K_I(\beta) \\ \text{cylinder} & & \text{flat plate} \end{array} \quad - 2.16$$

The hypothesis used is that the ratio between the values of stress intensity factor in cylinders and plates (ie. M_C) are equal in two or three dimensions. Results were presented for $W = 1.11$ to 1.5 , $b/a = 1.02$ and 5.0 , $a/t = 0.4$ to 0.8 and $\beta = 0$ to 90° along the crack front. It was suggested that linear interpolation be used for flaw shapes between 1.02 and 5.0 . A comparison with the finite element results of Blackburn et al. [38] indicated reasonable agreement although slightly different geometries were used in the comparison.

Poor agreement with the results of Underwood [37], except for very shallow cracks, was presumed to substantiate the importance of the back surface curvature correction.

Schmitt and Keim [44] studied the two crack problem in a hollow cylinder subjected to temperature and pressure loading using the finite element method. They used 20 node isoparametric elements with 1/4 point singularities and the MARC program to derive stress intensity factors for $W = 1.1$, $a/t = 0.25$, 0.375 , and $b/a = 3.0$. Solutions obtained from crack element displacements were noted to be somewhat (about 7.5%) higher than similar solutions obtained using the J integral approach. However, the latter approach exhibited instabilities at the mid point nodes along the crack tip.

Helliot et al. [45] used the boundary integral equations method to derive stress intensity factor solutions for $W = 1.1$, $b/a = 3$, and $a/t = 0.25$, 0.5 and 0.8 . They developed "polynomial influence functions" by applying superposition and assuming that the stress applied on the crack may be approximated by a polynomial of the form:

$$\sigma(x/t) = \sigma_0 + \sigma_1 (x/t) + \sigma_2 (x/t)^2 + \dots \quad - 2.17$$

The stress intensity factors are evaluated for separate load cases corresponding to each of the stress terms in the polynomial approximation and are presented in the form of polynomial influence functions, h_j , which are functions of geometry only. The stress intensity factor for any arbitrary load case approximated by a polynomial can then be derived by superposition:

$$K_I(\varnothing) = \frac{\sqrt{\pi a}}{E_2} [\cos^2 \varnothing + (a/b)^2 \sin^2 \varnothing]^{\frac{1}{4}} \sum_{j=0}^{\infty} h_j(\varnothing) \cdot \sigma_j - 2.19$$

where E_2 is the elliptic integral
 $\varnothing$ is the elliptic angle

They assessed the errors obtained from the boundary integral equations solutions and concluded that the stress intensity factors were accurate to within 5% for the cracked cylinder geometries considered.

McGowan and Raymond [46] used the finite element method to study the same geometries as Helliot et al and used a similar method of presenting the results. The most encouraging aspect of this study was the good agreement obtained between the boundary integral equations and finite element method results. The results were also compared to the approximate results of Kobayashi et al.. Good agreement was obtained for $a/t = 0.25$, however, for deeper cracks the approximate method greatly underestimated the stress intensity factor, especially towards the free surface.

Atluri and Katherison [47] used a three-dimensional hybrid displacement finite element formulation. Solutions were obtained for external and internal pressurized and unpressurized cracks in cylinders with $W = 2.0$, $a/t = 0.5$ and 0.8 , and $b/a = 1.0$ and 5.0 . The results are compared with the conventional finite element solutions of Blackburn and Hellen and the approximate solutions of Kobayashi et al.. Good agreement was obtained when comparing to the conventional finite element solutions and, in the case of semi-circular

flaws, the approximate solutions of Kobayashi. For deeper semi-elliptical flaws, the method of Kobayashi failed to yield comparable results. The author concluded that a two-dimensional analog employing various curvature correction factors may not be adequate when modelling a complex three-dimensional problem. This conclusion is supported by the findings of other authors, notably Helliot et al. and McGowan and Raymond, and seems, to this author, to raise serious doubts about the validity of the method of Kobayashi.

Tan and Fenner [48] used the boundary integral equations method to obtain stress intensity factors for cylinders containing one crack with $W = 2.0$ and 3.0 , $a/t = 0.2$ to 0.8 and $b/a = 1.25$. The choice of this semi-elliptical flaw shape was based on the results of experiments that showed the flaw shape once established, stayed very close to $b/a = 1.25$ for cylinders with radius ratios between 2 and 3. The solutions obtained were claimed to be accurate to within 5%. They also presented polar plots of the distorted outer surface of the cylinder for various crack depths and concluded that only two planes of symmetry may be assumed for cracks penetrating deeper than 30% of the cylinder wall. They also demonstrated the "excessive" overestimates of K_I obtained from two-dimensional analyses.

Williams [49] presented an approximate method for the determination of stress intensity factors for semi-elliptical flaws in thick-walled cylinders. This method is based on the general analytic solution for a penny shaped crack in an infinite body subjected to an arbitrary axisymmetric load distribution. The approximation suggested for the unsymmetric elliptical flaw is to treat the ellipse as a circle of the same area with an equivalent radius, $a_e = \sqrt{b \cdot a}$, and determin-

ing an equivalent pressure distribution by analogy with the general load distribution. His extension of these simplifying assumptions to the completely unsymmetrical semi-elliptical surface flaw in a thick-walled cylinder is not clearly understood by this author and seems to be totally unjustifiable. However, results obtained are compared to those given by Tan and Fenner [48] and show surprisingly good agreement except for deep cracks, ie. $a/t > 0.6$.

Raju and Newman [50] used three-dimensional finite element analysis to derive stress intensity factors for cylinders containing two cracks with $W = 1.1$ and 1.25 , $a/t = 0.2, 0.5$ and 0.8 , and $b/a = 1.0, 2.5$ and 5.0 . Both inner and outer flaws were studied using a polynomial influence coefficient method, similar to Helliot et al. and McGowan and Raymond. Each model contained approximately 6500 degrees of freedom. The nodal force method was used to determine the stress intensity factors from the finite element results and good agreement was obtained when comparing to the latter authors.

Pick and Burns [36], in an extension of their two-dimensional photoelastic stress freezing experiments, determined stress intensity factors for cylinders containing 2 cracks with $W = 2.0$, $a/t = 0.08, 0.1$ and 0.15 , and $b/a = 2.0, 3.0$ and 4.0 . They compared their experimental results to those of Helliot et al. and McGowan and Raymond, and found the numerical solutions to be low by approximately 21%. This author notes, however, that the cylinder geometries studied by Helliot et al. are primarily for thin-walled cylinders (ie. $W = 1.1$) and that extrapolation of these results to $W = 2.0$ is not warranted. Thus, a conclusion suggesting that the analytic or numerical solutions are non-conservative must

be drawn with caution due to the somewhat limited experimental evidence and the basis by which the comparisons were made.

2.5 Summary

The available literature on stress intensity factor solutions for internally pressurized cylinders suggests that straight-fronted (2D) crack geometries have received considerable study and are well understood. The solutions of Bowie and Freese [24] for one crack provides a benchmark against which further studies of single and multiple crack geometries can be compared. The FEM studies of Pu and Hussain [34] and of Hoff [35] provide the most comprehensive results for multiple crack geometries, however discrepancies of up to 14% were obtained for large arrays of multiple cracks.

The bulk of the work carried out on semi-elliptical (3D) crack geometries has been devoted to relatively thin-walled cylinders. The reliable work on thick-walled cylinders appears to reduce to that of Atluri et al. [47] and Tan et al. [48]. These works considered a fairly limited range of cylinder and crack geometries and only a single crack. The only multiple crack work reported is by Raju et al. [50] for two cracks in relatively thin-walled cylinders, $W = 1.1$ and 1.25 .

A need for further comprehensive studies of 2D multiple, and 3D single and multiple crack geometries is evident.

Figure 2.1 Stress Conditions In An Internally Pressurized Thick-Walled Cylinder ($W = 2$)

a) Open-Ended $\sigma_z = 0$

b) Closed-Ended $\sigma_z = \frac{P}{w^2 - 1}$

c) Zero Axial Strain $\sigma_z = \frac{-2\nu P}{w^2 - 1}$

Figure 2.2 Commonly Encountered End Conditions

a) Mode I

b) Mode II

c) Mode III

Figure 2.3 Basic Modes of Crack Surface Displacement

Figure 2.4 Elastic-Stress-Field Distribution Ahead of Crack

CHAPTER 3

THE FINITE ELEMENT METHOD IN LEFM

3.1 Introduction

The analysis of fracture mechanics problems has been, arguably, one of the most active branches of applied mechanics in the past two decades. The necessity to analyze increasingly more complex and/or practical crack problems, where simple idealizations of geometry, material behaviour, and loading are inapplicable, has focussed much attention on numerical solution techniques. These include the finite difference, boundary integral equations, and finite element methods. The finite element method, in particular, has gained wide acceptance in the engineering community because of its demonstrated versatility and accuracy in the analysis of complex structures, loads and boundary conditions.

A rigorous review of the theory behind the finite element method is inappropriate for the purposes of this thesis. It is noteworthy that the developments in the finite element method have been very extensive indeed, and the available publications number in the thousands. The basic finite element theory is well documented (see for example References 52 and 53). In addition several excellent reviews of FEM techniques in fracture mechanics have been published. The reviews by Chan et al [54], and Gallagher [55], are particularly notable. Benzley and Parks [56] discuss available general purpose FEM packages for fracture analysis and this extremely useful survey has since been updated by Mackerle and Fredrikson [57].

3.2 Crack-Tip Elements

The density of a finite element mesh is primarily dependant on the anticipated stress gradients in the structure. Around crack tips, this is particularly crucial and much work in the application of FEM to LEFM (or fracture problems in general) has focussed on modelling the stress singularity that occurs.

The first attempts to model cracks were simply the use of very large numbers of conventional elements, i.e. no attempt was made to take into account the stress singularity in the element formulation. It was clearly demonstrated [54] that many hundreds of elements are required to achieve perhaps 5% accuracy. This approach was eventually abandoned in favour of elements which take explicit account of the singularities. A review of several of the most important of these formulations, including classical solution based singularity elements, polynomial singularity function elements, and modified isoparametric elements is given in [58]. Only the latter element type will be discussed here as they have been used in the work conducted in this thesis.

3.2.1 Isoparametric Crack-Tip Elements

Isoparametric elements are widely available in commercial finite element programs. Their application to fracture mechanics problems is based upon the ability to represent the $1/\sqrt{r}$ stress singularity occurring in elastic crack problems by a very simple modification to the geometry of the standard isoparametric element. Henshell and Shaw [59] were the first to recognize that by shifting the "mid-side" nodes to the quarter point in a quadratic isoparametric triangular or

quadrilateral element, the required singularity resulted at the nearest adjacent node.

Barsoum [60], in a most important paper, investigated two and three-dimensional quadratic isoparametric elements. He introduced the idea of "collapsing" nodes along one edge of the element, and placing the adjacent nodes at the quarter-point. These collapsed or degenerate elements were later shown [61] to contain the required $1/\sqrt{r}$ stress singularity along any ray from the crack-tip, whereas the simple modified elements exhibit this singularity only along the boundaries of the element. The demonstrated accuracy of the collapsed form of isoparametric elements, together with their wide availability and ease of application, makes them the preferred choice for elastic crack analysis.

3.3 Determination of Stress Intensity Factors

A number of methods are available by which stress intensity factors can be extracted from the results of a finite element analysis of a crack problem. These may be broadly classified as displacement or stress solution methods and energy methods. Only the crack-tip displacement method shall be discussed here.

In this approach the displacements obtained in the FEM are used in conjunction with the classical near-field solutions to extract the stress intensity factor. Tracey [62] made an important contribution by deriving closed form expressions for K_I and K_{II} in terms of the solution displacements of the collapsed node isoparametric elements when applied to 2D problems.

Later, Ingraftea et al [63] applied this approach to the collapsed isoparametric wedge used in 3D problems, and derived closed form expressions for K_I , K_{II} , and K_{III} . These expressions are shown in Figure 3.1 and can be simplified for symmetric cases where only one half of the crack need be analyzed. For Mode I crack opening the simplified expressions are:

- at element end-faces (and for 2-D elements)

$$K_I = \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2L}} (4v_B - v_C) \quad - 3.1$$

$$\text{and } K_I = \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2L}} (4v_E - v_F) \quad - 3.2$$

- at mid-side nodes

$$K_I = \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2L}} (2(v_B + v_E) + v_D - (v_C + v_F)) \quad - 3.3$$

Although the above expressions provide a convenient means for obtaining K_I from the results of a finite element analysis, the accuracy of the solution is very much dependent on the size of the crack-tip element in relation to the length of the crack. This is discussed in more detail in Chapters 4 and 5 when considering the design of 2D and 3D meshes.

3.4 The ASAS Finite Element Program

ASAS is a comprehensive general purpose finite element system designed and distributed by the Atkins Research and Development Group. The system includes many pre-processors

for data and mesh formulation, post-processors for manipulating results obtained from an ASAS analysis, as well as a graphics display program used to display the original structure and, after analysis, to plot the deformed shape of the model [64]. The main program incorporates facilities for linear stress analysis, natural frequency analysis and heat transfer analysis. The system has an extensive library of element types and a wide range of loading and boundary conditions.

ASAS Version H, which was used for this work, was particularly effective since the element library included 2D and 3D collapsed isoparametric crack elements, and there are facilities for sub structure analysis as discussed in the following sections.

3.4.1 LEFM Capabilities of ASAS-H:

The ASAS-H program contains a considerable variety of finite elements which are capable of modelling two- and three-dimensional structures. There is no restriction on the number of element types in an analysis, but few problems require more than three or four different types. The ASAS library includes elements based on displacement assumptions and on stress assumptions. For stress analysis problems, standard quadratic isoparametric membrane elements such as the TRM6 and QUM8 elements, and corresponding 3D brick elements such as the BR15 and BR20 elements are the most applicable and generally provide the most accurate results.

The ASAS-H element library also contains a number of special elements for modelling cracked structures. These

elements are used around the tip of the crack and are formulated to represent the stress singularity at that point. Two of the elements are based on the degenerate isoparametric elements described by Barsoum [61] and reviewed in Section 3.2.1 of this thesis. These are compatible with the standard isoparametric elements and thus they are used in this work.

CTM6: a triangular membrane element with six nodes for two-dimensional analysis is shown in Figure 3.2;

CB15: a wedge shaped brick element with 15 nodes for three-dimensional analysis shown in Figure 3.3.

There is no stress output for these elements because of the singularity at the crack-tip. Stress intensity factors are obtained from the finite element analysis by substituting the crack-tip element displacements into the expressions derived by Ingraffea (see Section 3.3).

3.4.2 Multi-Level Substructuring Capabilities of ASAS-H:

The multi-level substructuring analysis capabilities of ASAS-H greatly extends the flexibility and power of the main program. This facility is described in detail in [64] but a short description is included here for convenience.

The main feature of a substructured analysis is that the structure is subdivided into several parts, each of which may be partially analyzed separately. These components, as they are called, are later assembled together to form either higher level, more complex components or the global struc-

ture. Subsequent processing after the global structure has been formed enables displacements, reactions and stresses to be recovered for each separate component as and when required.

The components are formed as conventional finite element models except that boundaries to be attached to other components are defined as "link" freedoms. The component, which may contain many elements, is run through the ASAS-H program to form a single macro- or super-element with freedoms described only at the link node boundaries. The model of the structure is then formed from the component parts and solved for displacements at the nodes in a similar way to the assembly and solution of a single level finite element model.

In the creation of a structural model the ASAS system provides for rotation, translation and mirroring of components so that the symmetry of the structure may be utilized to the maximum extent. An example of the procedure taken from [64] is shown diagrammatically in Figure 3.4.

The advantages of sub structure analysis are two-fold for the pressurized cylinder problem.

- only that part of the global model which changes need be re-formulated as a substructure; for example the crack mesh model as geometry changes;
- by changing suppressions in the substructure process, global models corresponding to 1, 2, 4, 8, etc. cracks are readily formed.

Details of these processes will be elaborated in Chapters 4 and 5, and are particularly advantageous when considering the complex, 3-dimensional, and thus expensive crack problems.

$$K_I = \frac{E}{4(1-\nu^2)} \sqrt{\left(\frac{\pi}{2L_1}\right)} [2v_B - v_C + 2v_E - v_F + v_D - 2v_{B'} + v_{C'} - 2v_{E'} + v_{F'} - v_{D'} \\ + \frac{1}{2}\eta(-4v_B + v_C + 4v_E - v_F + 4v_{B'} - v_{C'} - 4v_{E'} + v_{F'}) \\ + \frac{1}{2}\eta^2(v_F + v_{C'} - 2v_D - v_{F'} - v_{C'} + 2v_{D'})]$$

$$K_{II} = \frac{E}{4(1-\nu^2)} \sqrt{\left(\frac{\pi}{2L_1}\right)} [2u_B - u_C + 2u_E - u_F + u_D - 2u_{B'} + u_{C'} - 2u_{E'} + u_{F'} - u_{D'} \\ + \frac{1}{2}\eta(-4u_B + u_C + 4u_E - u_F + 4u_{B'} - u_{C'} - 4u_{E'} + u_{F'}) \\ + \frac{1}{2}\eta^2(u_F + u_{C'} - 2u_D - u_{F'} - u_{C'} + 2u_{D'})]$$

and

$$K_{III} = \frac{E}{4(1+\nu)} \sqrt{\left(\frac{\pi}{2L_1}\right)} [2w_B - w_C + 2w_E - w_F + w_D - 2w_{B'} + w_{C'} - 2w_{E'} + w_{F'} - w_{D'} \\ + \frac{1}{2}\eta(-4w_B + w_C + 4w_E - w_F + 4w_{B'} - w_{C'} - 4w_{E'} + w_{F'}) \\ + \frac{1}{2}\eta^2(w_F + w_{C'} - 2w_D - w_{F'} - w_{C'} + 2w_{D'})]$$

Figure 3.1 Equations For SIF From Nodal Displacements [63]

Figure 3.2 ASAS-H CTM6 Element

Figure 3.3 ASAS-H CB15 Element

Figure 3.4 Multi-level Substructuring of Complex Structures [64]

CHAPTER 4

ANALYSIS OF STRAIGHT FRONTED (2D) CRACKS IN A PRESSURIZED CYLINDER

4.1 Introduction

This chapter describes analysis of stress intensity factors for single and multiple straight-fronted cracks occurring at the bore of internally pressurized thick-walled cylinders. For the straight-fronted crack idealization, the analysis reduces to that of a plane (i.e. two-dimensional) problem in which the normalized stress intensity factor, $K_I/P\sqrt{\pi a}$, is dependent on the number of cracks, N , the crack depth to wall thickness ratio, a/t , and the cylinder radius ratio, W , as depicted in Figure 4.1. The stress intensity factors derived from the two dimensional analysis approximate conditions where the crack length along the cylinder bore is long relative to its depth. This is typically the case for cracks originating at tool markings, or along the rifling grooves of a gun barrel. As the cracks penetrate deeper into the cylinder wall and become more irregular in profile, the two-dimensional solutions become less accurate and a more rigorous "three-dimensional" treatment of the problem is required, as undertaken in Chapter 5.

From an analytical viewpoint, the stress intensity factor solutions obtained for straight-fronted cracks are of interest because they represent a physically limiting case (i.e. $b/a \rightarrow \infty$). It can be shown that for single cracks, the straight-fronted crack provides an upperbound solution and as such provides conservative estimates of the stress intensity factor.

From a practical viewpoint, plane analyses of fracture mechanics problems are inherently less complex and expensive to perform than full three-dimensional analyses. This is particularly true if the finite element method is employed as the means of analyzing the problem. Three dimensional finite element meshes are extremely difficult to prepare without the aid of computerized mesh generators, and obtaining the results places large demands on computer core memory and processing time. On the other hand, a two-dimensional finite element mesh is relatively simple to prepare and at least an order of magnitude less costly to process. Thus the two-dimensional analysis becomes an attractive means of investigating effects which are not expected to be strongly dependent on the crack shape. For instance, the relative effect of cylinder radius ratio on the stress intensity factors can be investigated by analyzing a wide range of cylinder geometries and crack configurations using two-dimensional finite element models. Such an approach would be prohibitively complex and expensive to perform in three dimensions. It is, however, desirable to have a limited range of three-dimensional solutions to confirm the results of the two-dimensional analyses.

4.2 Method of Analysis

Figure 4.1 shows the general problem of an internally pressurized thick-walled cylinder containing an array of equispaced and uniform straight-fronted radial cracks at the bore surface. In analyzing this problem, the following aspects were taken into consideration:

-
- Symmetry and boundary conditions,
 - Mesh refinement and accuracy,

The following sections describe these aspects of the analyses.

4.2.1 Symmetry and Boundary Conditions

The analysis of a cylinder containing N equispaced radial cracks of uniform depth is simplified by taking advantage of the symmetry of the problem. As shown in Figure 4.2, only a $180^\circ/N$ segment of the cylinder need be considered. The cylinder is "sliced" such that one edge of the segment is coincident with the plane of the crack. This reduces the number of elements and nodes required as only one half of the crack is modelled and this is the region in which the finite element meshes tend to be most refined. To maintain the required symmetry conditions, the edges of the cylinder segment must be constrained from movement in the circumferential (hoop) direction with, of course, the exception of the crack surface.

The cracked cylinder model is further simplified if superposition is taken into consideration. For a given geometry, the stress intensity factor for Mode I crack displacement is dependent only on the stresses acting normal to the crack surface. For an internally pressurized cylinder containing fully pressurized cracks, the effective stress distribution normal to the crack surface is equal to the sum of the hoop stress in the uncracked cylinder plus the pressure acting on the crack:

$$\sigma_{\theta c} = \sigma_{\theta} + P \quad - 4.1$$

$$\text{or } \sigma_{\theta c} = \frac{P}{w^2 - 1} [1 + (R_2/r)^2] + P \quad - 4.2$$

$$= \frac{P}{w^2 - 1} [w^2 + (R_2/r)^2] \quad - 4.2$$

It can be shown that the latter equation represents the hoop stress in a cylinder subjected to a tensile stress, equal in magnitude to P , acting on the outer wall surface of the cylinder. This loading scheme is advantageous because in general fewer elements are associated with the outer wall surface than along the inner bore and crack surfaces where higher element mesh densities are required. Though this does not reduce the computing time required, it does simplify the task of defining the load data for the finite element models, and therefore reduces the likelihood of introducing erroneous input to the model.

4.2.2 Mesh Refinement and Accuracy

The choice of parameters controlling finite element meshes for use in the cylinder problems was established by performing a series of accuracy assessment studies on a 'benchmark' problem for which exact solutions were available in the literature. The problem selected was that a single edge cracked plate loaded in uniaxial tension, and in four-point or pure bending as shown in Figure 4.3. Values of SIF

for these problems are given by Rooke and Cartwright [65]. A variety of FEM meshes were used to study such effects as:

- crack element size
- number of elements along the crack face
- number of elements between crack tip and outer surface
- mesh refinement far from the crack tip

Typical meshes investigated are shown in Figure 4.4.

The results of these studies are reported fully in [66], and with regard to the present work the following recommendations for mesh design emerged which would ensure accuracy of SIF of approximately 1-2%:

- (i) Four crack tip elements are required.
- (ii) The crack tip element side length should be in the range of 5-10% of the crack length.
- (iii) At least three, and preferably four elements are required along the crack face.
- (iv) At least three, and preferably four elements are required along the line between the crack tip and the outer surface edge.
- (v) The solution accuracy is relatively insensitive to mesh refinement far away from the crack tip.

Using the above recommendations as guidelines, suitable meshes for the cracked cylinder models were designed using the quadratic isoparametric elements described in Chapter 3.

4.2.3 Summary of 2D FEM Models

Typical finite element meshes for cylinders containing 1, 2, 4, 20 and 40 cracks are shown in Figures 4.5 to 4.9 respectively. Note that for the single crack model, an extra bar element of very small stiffness was attached to the outer edge of the model at $\theta = 90^\circ$. By fixing one node of this element in the X direction (r direction, $\theta = 0^\circ$) the rigid body displacement mode of the model was removed. In all seven models corresponding to crack depths of $a/t = 0.05, 0.1, 0.2, 0.25, 0.3, 0.35, 0.4$ and 0.5 were designed for the single crack problems. The multiple crack models were then obtained simply by removing the appropriate elements from the single crack model and applying the required boundary conditions, as shown in Figures 4.6, 4.7, etc. Models for different cylinder diameter ratios, but for the same crack depth to wall thickness ratio were obtained by linearly scaling the nodal coordinates to give the required dimensions. In the final mesh designs there were at least four elements along the crack face, and between the crack tip and outer wall radius.

4.3 Results for 2D Cracks

The results of the two-dimensional finite element analyses of internally pressurized thick-walled cylinders containing single and multiple radial cracks at the bore are summarized in Tables 4.1 to 4.4 and plotted in Figures 4.10 to 4.13 for the four cylinder radius ratios considered. The stress intensity factors were computed using the crack tip element displacements in the manner described in section 3.3 assuming plane stress (ie. open ended) conditions in the cylinder. For plane strain conditions, these values should be multiplied by $1/(1 - \nu^2)$. The simple parameter $P\sqrt{\pi a}$ has been used to normalize the stress intensity factors for generality.

4.3.1 Comparisons With Published Results

The results obtained using ASAS were compared with published data for a cylinder of $W = 2.0$ containing one and 40 equispaced cracks. The results of this comparison study are summarized in Tables 4.5 and 4.6 and plotted in Figures 4.14 and 4.15.

The results obtained for the single crack case shown in Table 4.5 compare very favourably with the results of Bowie and Freese [24] and Hoff [35], being within $\pm 1\%$ for all crack lengths. Less agreement is obtained when comparing to the results of Pu and Hussain [34], although the differences are still within approximately $\pm 2\%$. It is noted, however, that the results of Pu and Hussain oscillate about the results obtained by ASAS, Hoff, and Bowie and Freese and thus would appear to indicate problems with the analyses conducted in [34].

The results obtained for the 40 crack case are compared to those obtained by Pu and Hussain and by Hoff in Table 4.6. It is apparent that the ASAS results lie approximately midway between the published results, being 5 to 10% higher than those of Pu and Hussain for all crack lengths and approximately 5.5% lower than Hoff's result for $a/t = 0.2$.

The good agreement obtained with published data lends confidence in the finite element modelling approach used to conduct the 2D cracked cylinder studies.

4.3.2 Discussion of Results For Single Cracks

The numerical results for the normalized mode I stress intensity factor, $K_I/P\sqrt{\pi a}$, for single straight-fronted cracks in pressurized cylinders having diameter ratios of $W = 1.5$, 1.75, 2.0 and 2.25 are represented in Figure 4.16.

For all relative crack depths, a/t , the stress intensity factor is observed to increase as cylinder radius ratio decreases. This effect is due to the increase in hoop stresses as W decreases for a given internal pressure load (recall equation 2.2). The cylinder radius ratio also influences the variation of $K_I/P\sqrt{\pi a}$ as a/t increases. For the thick cylinder with $W = 2.25$ $K_I/P\sqrt{\pi a}$ varies only slightly with a/t , exhibiting a minimum near $a/t = 0.2$. As W decreases, the degree of variation in $K_I/P\sqrt{\pi a}$ with a/t becomes more pronounced and the value of a/t at which $K_I/P\sqrt{\pi a}$ is minimum decreases. It is apparent then that straight-forward application of thin-walled vessel results to thicker walled vessels can result in substantial errors.

It can be shown theoretically that the value of $K_I/P\sqrt{\pi a}$ approaches $2.24 \cdot W^2/(W^2 - 1)$ as crack length approaches zero. This corresponds to $K_I/P\sqrt{\pi a}$ values of 4.03, 3.33, 2.99 and 2.79 for $W = 1.5, 1.75, 2.0$ and 2.25 respectively. Extrapolation of the finite element results presented in Figure 4.16 indicated excellent agreement with theory and further substantiates the accuracy of the numerical results.

The effect of cylinder geometry on fatigue life and critical crack size is better understood if the parameters $P\sqrt{R_1}$, and $P\sqrt{t}$ are used to normalize K_I :

$$K_I/P\sqrt{R_1} = K_I/P\sqrt{\pi a} \cdot \sqrt{\pi \cdot a/t} \cdot (W - 1) \quad - 4.4$$

$$K_I/P\sqrt{t} = K_I/P\sqrt{\pi a} \cdot \sqrt{\pi \cdot a/t} \quad - 4.5$$

Figure 4.17 shows the variation of $K_I/P\sqrt{R_1}$ with a/t and W . For a given material fracture toughness, K_{IC} , internal pressure load and bore radius, $K_{IC}/P\sqrt{R_1}$ is fixed. Figure 4.16 implies that the relative crack depth, a/t , at the onset of brittle fracture increases with increasing cylinder diameter ratio. Since $t = R_1 \cdot (W - 1)$, it follows that the critical crack size, a_c , also increases. The increased critical crack size coupled with the lower value of K_I for any given crack depth implies that the fatigue lives for constant bore radius cylinders increases as cylinder radius ratio and/or wall thickness increases. Additionally, if one assumes that the trends indicated in Figure 4.17 can be extrapolated to greater values of a/t , then this would suggest that a leak-before-break situation will occur in thicker walled vessels at lower values of $K_I/P\sqrt{R_1}$.

Figure 4.18 shows the variation of $K_I/P\sqrt{t}$ with a/t and W . From this figure it is clear that the effect of increasing diameter ratio for a given material fracture toughness, internal pressure load and wall thickness is to increase the relative crack depth and critical crack size at the onset of brittle fracture and to increase the fatigue life.

An impressive reduction in cylinder radius ratio dependence is obtained if $K_I/P\sqrt{t}$ is plotted against $a/t/(W - 1)$. This is shown in Figure 4.19 and, as can be seen, the correlation is excellent for the cylinder and crack geometries considered. Two important observations can be made from this figure. Firstly, for given value of K_{IC} , P , and t the critical values of a/t and a_c can be shown to vary in direct proportion to $(W - 1)$. For example, if we consider two different cylinders with $W = 1.5$ and 2.0 , but of the same wall thickness, material, and subjected to the same pressure load, then the critical values of a/t and a_c for the cylinder with $W = 2.0$ will be $(2.0 - 1.0)/(1.5 - 1.0) = 2.0$ times greater than the values for the cylinder with $W = 1.5$.

The second important observation is that if K_{IC} , P and W are assumed fixed, then an increase in vessel wall thickness (equivalent to an increase in the bore radius) results in a decrease in the critical value of a/t . However, the value of a_c changes only slightly. This suggests that the fatigue lives of constant radius ratio cylinders is more or less independent of the actual cross-sectional dimensions. This

observation is supported by fatigue tests of cylinders which demonstrated that the size effect associated with an increase in cross-sectional dimensions is not readily apparent in cylinders of constant radius ratio [25].

A least squares curve fit to the data presented in Figure 4.19 resulted in the following expression:

$$K_I/P\sqrt{t} = 5.714(\alpha)^{0.5} - 4.258(\alpha) + 5.651(\alpha)^{1.5} - 4.6$$

where $\alpha = a/t/(W - 1)$

This expression is accurate to within $\pm 1\%$ of the finite element results and should prove useful for preliminary design studies and fatigue life estimates. Users of the above expression are advised to exercise caution when attempting to extrapolate to cracked cylinders geometries beyond $0.0 < a/t < 0.5$ and $1.5 < W < 2.25$. Further investigation is required to determine if these results for varying cylinder radius ratio can be applied to semi-elliptical crack geometries as discussed in Chapter 5.

4.3.3 Discussion of Results For Multiple Cracks

Figures 4.10 to 4.13 illustrate the effect of multiple cracks on the stress intensity factor for the four cylinder sizes considered. It is evident that the multi-crack effect increases as relative crack depth increases. For large arrays of cracks (ie. $N = 20$ or 40 cracks) the stress intensity factors are significantly lower than for the single crack cases. This effect is due to the redistribution of the

hoop stress in the vicinity of the crack and one may use the analogy of how multiple parallel notches mitigate the stress concentration effect compared to that of one notch. This effect decreases as the number of cracks decreases and for the four crack case, only modest reductions in the stress intensity factor are observed. The "two crack" effect noted by Shannon [25] is quite apparent in Figure 4.12. It is postulated that the effect of two diametrically opposed cracks is to result in an additional bending moment term about the crack tips and that this results in the higher values of stress intensity factor.

The effect of cylinder radius ratio on $K_I/P\sqrt{\pi a}$ for the 20 and 40 multiple crack cases is illustrated in Figures 4.20 and 4.21. As was observed for the single cracks, the stress intensity factor increases as cylinder radius ratio decreases. However, unlike the trend observed for single cracks, the variation of $K_I/P\sqrt{\pi a}$ with a/t is very similar for all radius ratios. The normalized stress intensity factor is observed to decrease rapidly until a/t is approximately 0.2, after which it remains nearly constant.

An attempt to plot $K_I/P\sqrt{t}$ with $a/t/(W - 1)$ failed to produce a correlation comparable to that obtained for the single crack cases. However, it was determined that by plotting the load relief factor, R_N described in section 2.4.1, as a function of $a/t \cdot (W - 1)$, the dependence on cylinder radius ratio was virtually eliminated. This is shown in Figure 4.22. The parameter $a/t \cdot (W - 1)$ is in fact equal to a/R_1 and this suggests that R_N is primarily dependant

on the crack depth to bore radius ratio irrespective of the cylinder radius ratio. Note that for short cracks R_N approaches unity. For large arrays of deep cracks, R_N assumes nearly constant values of approximately 0.525 for 20 cracks and 0.375 for 40 cracks. For the four crack case, R_N at first decreases slightly from unity then starts to rise again and could conceivably become greater than unity for deep cracks (ie. $a/t \cdot (W - 1) > 0.55$). The "two crack" effect evidently increases rapidly with increasing crack depth.

For large arrays of cracks, the decrease in R_N for increasing $a/t \cdot (W - 1)$ implies that fatigue life increases with increasing radius ratio assuming constant bore radius and/or wall thickness. This is consistent with the results for the single cracks. For the two and possibly four crack cases, R_N increases with $a/t \cdot (W - 1)$. The effect of radius ratio on critical crack size and fatigue life is more difficult to predict in this case because the increase in R_N associated with an increase in W may or may not compensate for the decrease in the reference stress intensity factor for the single crack.

The form of presentation shown in Figure 4.22, together with Figure 4.19 or equation 4.6, provides a convenient means of estimating stress intensity factors for cracked cylinder geometries not considered in this thesis. Again, users are cautioned when extrapolating these results beyond $0.0 < a/t < 0.5$ and $1.5 < W < 2.25$ and/or attempting to apply these results to semi-elliptical crack geometries.

Table 4.1 $K_I/P\sqrt{\pi a}$ For a Cylinder With $W = 1.5$

a/t	$K_I/P\sqrt{\pi a}$		
	N = 1	N = 20	N = 40
0.05	3.956	3.808	3.442
0.10	3.984	3.483	2.731
0.20	4.225	2.942	2.158
0.25	4.410	2.794	2.072
0.30	4.616	2.702	2.019
0.35	4.831	2.666	2.002
0.40	5.105	2.688	2.025
0.50	5.675	2.819	2.153

Table 4.2: $K_I/P\sqrt{\pi a}$ For a Cylinder With $W = 1.75$

a/t	$K_I/P\sqrt{\pi a}$		
	N = 1	N = 20	N = 40
0.05	3.207	2.982	2.477
0.10	3.190	2.545	1.856
0.20	3.273	2.026	1.494
0.25	3.357	1.932	1.433
0.30	3.457	1.887	1.396
0.35	3.563	1.878	1.388
0.40	3.683	1.892	1.407
0.50	3.954	2.003	1.500

Table 4.3 $K_I/P\sqrt{\pi a}$ For a Cylinder With $W = 2.0$

a/t	$K_I/P\sqrt{\pi a}$				
	N = 1	N = 2	N = 4	N = 20	N = 40
0.05	2.827	2.844	2.813	2.518	1.957
0.10	2.792	2.846	2.768	2.039	1.464
0.20	2.778	2.957	2.714	1.606	1.189
0.25	2.816	3.075	2.740	1.545	1.135
0.30	2.862	3.222	2.772	1.519	1.110
0.35	2.924	3.392	2.819	1.515	1.106
0.40	2.973	3.557	2.861	1.522	1.114
0.50	3.126	3.992	3.074	1.620	1.201

Table 4.4 $K_I/P\sqrt{\pi a}$ For a Cylinder With $W = 2.25$

a/t	$K_I/P\sqrt{\pi a}$		
	N = 1	N = 20	N = 40
0.05	2.594	2.202	1.712
0.10	2.548	1.732	1.240
0.20	2.472	1.373	1.017
0.25	2.485	1.326	0.966
0.30	2.502	1.309	0.950
0.35	2.541	1.311	0.946
0.40	2.551	1.309	0.969
0.50	2.649	1.403	1.034

Table 4.5 Comparison of Results Obtained For One Crack, $W = 2.0$

a/t	ASAS	Bowie & Freese [24]		Pu & Hussain [34]		Hoff [35]	
	$K_I/P\sqrt{\pi a}$	$K_I/P\sqrt{\pi a}$	% Diff.	$K_I/P\sqrt{\pi a}$	% Diff.	$K_I/P\sqrt{\pi a}$	% Diff.
0.1	2.792	2.800	+ 0.3	2.817	- 0.9	2.765	+ 1.0
0.2	2.778	2.774	+ 0.2	2.837	- 2.1	2.776	+ 0.1
0.3	2.862	2.880	- 0.6	2.824	+ 1.3	2.849	+ 0.5
0.4	2.973	2.987	- 0.5	3.023	- 1.7	2.962	+ 0.4
0.5	3.126	3.146	- 0.7	3.188	- 1.9	3.114	+ 0.4

Table 4.6 Comparison of Results Obtained for 40 Cracks, $W = 2.0$

a/t	ASAS	Pu & Hussain [34]		Hoff [35]	
	$K_I/P\sqrt{\pi a}$	$K_I/P\sqrt{\pi a}$	% Diff.	$K_I/P\sqrt{\pi a}$	% Diff.
0.1	1.464	1.397	+ 4.8	1.259	- 5.5
0.2	1.189	1.119	+ 6.2		
0.3	1.110	1.012	+ 9.7		
0.4	1.114	1.058	+ 5.3		
0.5	1.201	1.123	+ 7.0

$$t = R_2 - R_1$$
$$W = R_2 / R_1$$
$$N = 4$$

Figure 4.1 Two-Dimensional Representation of Cracked Thick-Walled Cylinder Problem

Figure 4.2 Symmetry and Boundary Conditions

a) Uniaxial Tension

b) Pure Bending

Figure 4.3 Edge Cracked Plate Benchmark Problems

Figure 4.4 Typical Meshes For Benchmark Accuracy Studies

Figure 4.5 FE MESH For 1 Crack Model; $a/t = 0.2$

Figure 4.6 FE Mesh For 2 Crack Model; $a/t = 0.2$

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Figure 4.7 FE Mesh For 4 Crack Model; $a/t = 0.5$

Figure 4.8 FE Mesh For 20 Crack Model; $a/t = 0.5$

Figure 4.9 FE Mesh For 40 Crack Model; $a/t = 0.2$

Figure 4.10 $K_I / P \sqrt{\pi a}$ vs a/t ($W = 1.5$)

Figure 4.11 $K_I/P/\sqrt{\pi a}$ vs a/t ($W = 1.75$)

Figure 4.12 $K_I / P\sqrt{\pi a}$ vs a/t , ($W = 2.0$)

Figure 4.13 $K_I/P\sqrt{\pi a}$ vs a/t ($W = 2.25$)

Figure 4.14 Comparison With Published Results For A Cylinder Containing One Crack ($W = 2.0$)

Figure 4.15 Comparison With Published Results For A Cylinder Containing 40 Cracks ($W = 2.0$)

Figure 4.16' $K_I / P \sqrt{\pi a}$ vs a/t , W For Single Cracks

Figure 4.17 $K_I / P \sqrt{R_1}$ vs a/t , W For Single Cracks

Figure 4.18 $K_I/P\sqrt{t}$ vs a/t , W For Single Cracks

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Micro-**D**

Figure 4.19 SIF Correlation For 2D Single Cracks

Figure 4.20 $K_I / P\sqrt{\pi a}$ vs a/t , W For 20 Cracks

Figure 4.21 $K_I / P\sqrt{\pi a}$ vs a/t , W For 40 Cracks

Figure 4.22 Load Relief Factor Correlation For 2D Multiple Cracks

CHAPTER 5

ANALYSIS OF SEMI-ELLIPTICAL (3D) CRACKS IN A PRESSURIZED CYLINDER

5.1 Introduction

In the preceding chapter, solutions for straight-fronted radial cracks occurring at the bore of thick-walled cylindrical pressure vessels were presented. In practice however, cracks occurring in vessels of this type are seldom long and straight-fronted. Instead, they tend to be curve-fronted and become increasingly more so as crack penetration increases. Thus a more descriptive (and hence accurate) model of the cracked cylinder problem requires taking into account the shape of the crack profile. A standard method of performing this is to approximate the crack profile by a mathematical curve such as a circle, or more generally as a semi-ellipse. By introducing the crack shape profile into the analysis, the model becomes three-dimensional. The normalized stress intensity factor, $K_I/P\sqrt{\pi a}$, is a function of the crack shape aspect ratio, b/a , the position along the crack front defined by the angle β , the number of cracks, N , the crack depth to wall thickness ratio, a/t , and the cylinder radius ratio, W , as shown in Figure 5.1.

As was mentioned earlier in Chapter 3, a complex three dimensional finite element analysis is extremely difficult and costly to perform both in terms of effort and in computer CPU time. One method of alleviating the difficulties and

costs, at least effortwise, is to employ a computer programmed mesh generator to facilitate the preparation of the meshes. A second method is to break the total problem down into a number of substructures or components that are common in several analyses. Each substructure is solved by itself only once but can later be linked together with other substructures to form several global analyses. By making more efficient use of each component analysis, substantial savings in costs are incurred and the analysis is reduced to a series of relatively small and manageable computer runs. Both these techniques were employed extensively in the analyses of the 3D crack problems conducted in this thesis and are described in detail in later sections of this chapter.

5.2 Method of Analysis

Figure 5.1 shows the general problem of an internally pressurized thick-walled cylinder containing an array of equispaced and uniform semi-elliptical radial cracks at the bore surface. In analyzing this problem, the following aspects were taken into consideration:

- Symmetry and boundary conditions,
- Mesh generation,
- Substructuring,
- Mesh refinement and accuracy.

The following sections describe these aspects of the analyses.

5.2.1 Symmetry and Boundary Conditions

The analysis of the 3D cylinder problem is analogous to that of the 2D cylinder problem described in section 4.2.1 and is summarized in Figure 5.2. By taking advantage of the symmetry of the problem, only one half of the crack profile and a $180^\circ/N$ segment of the cylinder requires modelling. To maintain the required symmetry conditions, the radial edges of the cylinder must be constrained in the hoop direction except on the crack face. Plane strain conditions are imposed by constraining movements in the axial (z) direction at both ends of the cylinder model. The fully pressurized crack, which is considered to be the worst case for practical application, is modelled by applying a uniform pressure load normal to the crack face and cylinder bore surfaces.

5.2.2 Mesh Generation

As discussed earlier, one method of alleviating the difficulties and costs incurred in performing a three-dimensional finite element analysis is to use a computer programmed mesh generator. The mesh generator provides the following advantages:

- (ii) Error free input data for the finite element program;
- (i) Quick and easy production of three dimensional finite element meshes;
- (iii) A regular mesh format which follows a known progression so that the accuracy and behaviour of the crack-tip elements can be monitored in a systematic fashion.

GENMESH, a finite element mesh generation program for producing models of components containing elliptically shaped cracks was produced by this author as part of his undergraduate thesis work [67] in response to a request for a programmed mesh generator. The source program was written in FORTRAN IV code. In its original format, GENMESH was able to produce models for semi-elliptical cracks occurring in thick-plate and thick-walled cylinder geometries, complete with required boundary conditions. The initial success gained from the analyses of these problems (both of which are covered in this thesis work) prompted further additions, and it has been employed in the analyses of several other problems; notably the problem of surface cracks at weld toes and surface cracks occurring in rotating disks [68, 69].

The mesh generation program has been documented fully [67, 70], but because of its direct relevance to the work covered in this thesis, a short description of GENMESH is included here for completeness.

5.2.2.1 Description of GENMESH

The basic function of GENMESH is to produce three-dimensional finite element meshes that model elliptically shaped cracks in structural components, thus aiding in the analysis of these problems. The program was designed exclusively for the ASAS finite element programs requirements, but could be suitably modified to perform for other programs requirements.

The general procedure for producing the three-dimensional models is shown schematically in Figure 5.3. GENMESH begins by taking information from an ASAS input data file for a two-dimensional mesh, along with mesh parameters describing

the crack shape, overall mesh dimensions and boundary conditions. This information is easily prepared by hand. The three-dimensional mesh formulation then begins with the conversion of the 2-D elements present in the input data file, specifically the QUM8, TRM6 and CTM6 elements, to equivalent 3-D brick element types, BR20, BR15 and CB15 respectively as shown in Figure 5.4. The 2-D input mesh represents the form of the 3-D mesh defined on the $Z = 0$ plane.

The 3-D mesh node coordinate and connectivity generation begins by rotating an 'image' of the 2-D mesh out of the $Z = 0$ plane along an elliptical path defining the crack profile. By specifying the crack shape parameter, b/a , and the number of elements desired along the crack front, the program performs a numerical integration of the elliptic integral, E_2 , defined:

$$E_2 = \int_0^{\pi/2} [1 - (a/b)^2 \sin^2 \phi]^{1/2} d\phi \quad - 5.1$$

from which the values of the 'rotation' angles, ϕ , required to divide the crack front into segments of equal length are determined. The mid-side nodes for the first layer of 3-D elements are then determined via a series of rotation equations. These equations were developed such that the image of the 2D mesh is maintained perpendicular to the crack front with constant element sizing. By rotating the image mesh through the next angle of rotation, the nodes and coordinates for the end-faces of the first layer of elements are determined using the same equations. This sequence of rotations continues in a step-wise fashion until the image mesh has been rotated a full 90 degrees. The three-dimensional model is completed by adjusting positions of the nodes at the boundaries (and adding elements if required) to give the mesh

its required final shape. Boundary conditions data, including displacement constraints, substructuring data and loads are determined automatically by specifying a boundary condition codeword in the input file (PCYL, for an internally pressurized cylinder as an example). All of this data is then written to a data file in formats compatible with the ASAS programs requirements. Boundary condition data are presented in a format that is comprehensive to the user (especially when using substructuring analyses) so that data checks are reasonably systematic. An additional data file for crack-meshes is provided and contains information regarding crack element sizes and positions required for the computation of the stress intensity factors from the ASAS displacement recovery analysis.

5.2.3 Substructuring

Figure 5.5 illustrates the substructuring scheme employed for the 3D cylinder models. For each crack geometry considered (ie. a/t and b/a), a total of six component meshes were developed using the mesh generator GENMESH. The crack mesh, designated SUB1, had an included angle of 5.625° . The remaining components, SUB2 to SUB6, consisted entirely of conventional isoparametric brick elements and had included angles as shown in Figure 5.5.

Using the above substructuring scheme it was possible to analyze a cylinder containing 1, 2, 4, 8, 16 and 32 cracks using the same basic components by combining them in new global analyses. In this manner, the increment in cost per multiple crack analysis is the cost of a new global analysis and the subsequent displacement recovery analysis. This results in substantial savings compared to the case where each multiple crack case is analyzed as a complete structure handled in a single computer run.

For the case of a cylinder with a single crack, the substructures SUB1 to SUB6 are linked together to form the global structure. For a cylinder with two cracks, substructures SUB1 to SUB5 are linked together and the plane at $\theta = 90^\circ$ is suppressed in the hoop direction. Similar globals are analyzed for cylinders with 4, 8, 16 and 32 cracks.

5.2.4 Mesh Refinement and Accuracy

The choice of parameters controlling finite element meshes for use in the analysis 3D cylinder problems was established from the results of a previous study conducted at Carleton University [66]. This study considered two benchmark problems for which accurate solutions were available in the literature. These were the problems of a circular flaw embedded in an infinite medium for which the exact solution is known, and the problem of a semi-circular surface crack in a plate loaded in tension for which the finite element results of Raju and Newman [70] were used in the assessment. A variety of meshes were used to study effects such as:

- the number of crack-tip elements required along the crack front,
- the length of the crack-tip element perpendicular to the crack front relative to the crack depth, ie. a/L ,
- the length of crack-tip element along the crack front relative to the length perpendicular to the crack front, ie. M/L ,

as illustrated in Figure 5.6. A typical mesh for the benchmark problems is shown in Figure 5.7.

The results of this study are reported fully in [72] and, with regard to the present work, the following recommendations for mesh design emerged which would ensure accuracy of approximately 2 - 5% without incurring excessive computing costs:

- (i) A minimum of four crack-tip elements are required along the crack front.
- (ii) The dimensions of the crack-tip elements should be less than $M/L = 2$ and greater than $a/L = 5$.

5.2.5 Summary of 3D Cylinder Models

A typical finite element crack mesh for the 3D cylinder models is shown in Figure 5.8. The length of the cylinder modelled was 3.0 times the wall thickness and plane strain end conditions were applied at the end of the cylinder remote from the crack. Poisson's ratio was taken to be 0.3. The guidelines for crack mesh design described in the previous section were followed as closely as possible. Each model included five crack-tip elements of equal length along the crack front, however it was not possible to keep M/L less than 2 for the semi-elliptical cracks. Consequently it is expected that the accuracy of the results obtained for the higher aspect ratio cracks will be somewhat less than results for the semi-circular cracks, in the area of 5 to 8% [72]. In all, a total of 20 crack meshes were prepared for the analyses which in themselves covered 120 different crack configurations. Table 5.1 summarizes the crack mesh design parameters for the 3D cylinder models.

5.3 Results For Semi - Elliptical Cracks

The results of the three-dimensional finite element analyses of internally pressurized cylinders containing single and multiple semi-elliptical cracks at the bore are summarized in Figures 5.9 to 5.12. The simple parameter $P\sqrt{a}$ has been used to normalize the stress intensity factors for generality. Tabular listings of the 3D analyses results may be obtained by written requests to Professor R. Bell at Carleton University.

Because of the large amount of data involved, a short computer program was written which could read the required crack tip element model displacements from the ASAS recovery files together with files containing crack mesh geometry data (ie. L, β , node numbers) obtained from GENMESH. This program would then compute the stress intensity factors using equations 3.1, 3.2 and 3.3 and save the results in files for subsequent processing and analysis. The calculations were performed assuming plane strain conditions at all points along the crack front except at the free surface which was treated as being under plane stress. The author notes that the use of GENMESH and the post processing program described provided a fully automated procedure which was systematic, error-free and relatively easy to use. The reduction in time and effort was such that a complete analysis (one crack geometry for 1, 2, 4, 8, 16, 32 cracks), starting with the preparation of 2D mesh input files for GENMESH to the plotting of results, could be performed in a single 24 hour period!

5.3.1 General Observations

Plots of $K_I/P\sqrt{\pi a}$ vs β , N and a/t are shown in Figures 5.9 to 5.12 for $b/a = 1.0, 2.0, 3.0$ and 4.0 respectively. For the semi-circular cracks, $K_I/P\sqrt{\pi a}$ is observed to increase steadily from the point of maximum crack penetration (ie. $\beta = 0^\circ$) until $\beta = 81^\circ$ after which it decreases to the free surface value at $\beta = 90^\circ$. The latter observed effect is due to the plane stress assumption used to compute K_I at the free surface. The validity of the K_I calculations at the free surfaces is, however, uncertain and deserves a separate and more detailed investigation. Indeed the meaning of K_I at the point of intersection of the crack front and a free surface has been the subject of debate for some time now [45, 71].

Results for the semi-elliptical cracks show a remarkable contrast to those obtained for the semi-circular cracks. Instead of increasing, the values of $K_I/P\sqrt{\pi a}$ are observed to decrease continuously as β increases for 0° to 90° . The free surface effect is also less apparent for the semi-elliptical cracks. It is interesting to note that $K_I/P\sqrt{\pi a}$ at the free surface for a given crack shape is observed to vary with a/t .

The effects of multiple cracks are, qualitatively, similar for all flaw shapes. For short cracks (ie. $a/t < 0.2$), the results for 1, 2, 4, and 8 cracks are quite similar, but noticeable reductions in $K_I/P\sqrt{\pi a}$ are apparent for the 16 and 32 crack cases. As the relative crack depth increases, the multiple crack effects become more marked and it is apparent that the 2 and 4 crack cases are more severe than the single crack case. It is interesting to note that the 2D results failed to predict the 4 crack case as being more severe than the single crack case. This may be attrib-

uted to the increasing influence of three-dimensional effects on the net section of the cylinder.

5.3.2 Comparisons With Published Results

The results obtained for a single semi-circular crack with $a/t = 0.5$ are compared to those published by Atluri et al [47] for the same crack and cylinder geometry in Table 5.2 and are plotted in Figure 5.13. Good agreement (less than 3% difference) is obtained at the point of maximum crack penetration ($\beta = 0^\circ$) and at the free surface, while the ASAS results are 10% higher at the mid points along the crack. The ASAS results fail to exhibit the trend of a reduction in $K_I/P\sqrt{\pi a}$ as β increases from 0° to a minimum at $\beta = 20^\circ$ and then an increase until $\beta = 90^\circ$ as shown by the results of Atluri. The results of Atluri also do not exhibit the free-surface effect noticeable in the ASAS results between $\beta = 80^\circ$ and 90° .

One comment concerning the published results shown in Figure 5.13 concerns the slope of $K_I/P\sqrt{\pi a}$ vs β at the point of maximum crack penetration. Since this represents a plane of symmetry, one should expect that the slope at $\beta = 0^\circ$ to be zero (ie. the curve should flatten out). This is observed for the ASAS results but is not evident for the results in [47] and may indicate improper application of boundary conditions and/or modelling of the crack geometry by the other authors.

As a further assessment of the present results, a comparison was made with the results of Tan et al [48] who studied the problem of a semi-elliptical surface flaw with

$b/a = 1.25$ in a cylinder with $W = 2.0$ for various crack depth ratios. The results of this comparison are summarized in Tables 5.3 and 5.4 and in Figures 5.14 and 5.15 for $a/t = 0.2$ and 0.5 respectively. Note that this particular flaw shape was not modelled in the present analyses. The ASAS results presented have been estimated by linear interpolation of the results obtained for $b/a = 1.0$ and 2.0 for the same angle β and crack depth a/t . As can be seen, the results agree quite well with a difference of less than 1% at $\beta = 0^\circ$ and 4% at $\beta = 90^\circ$, the maximum difference being about 8% for β between approximately 45° and 75° . The ASAS results do not exhibit a dip in $K_I/P\sqrt{\pi a}$ at approximately 60° which is present in the results of Tan et al.. Also the free surface effect is not readily noticeable in the results of [48]. It is conceivable that some of the differences may be attributed to interpolation errors. In addition one should expect the FEM results to be higher than the boundary integral equations results of Tan et al; that is the FEM always tends to err conservatively.

Despite these differences, the results obtained using ASAS are generally in good agreement with the published results. This lends confidence in the modelling and analysis procedure, particularly for the multiple crack cases for which the author notes there are essentially no known solutions to compare with.

5.3.3 Discussion of Results For Single Cracks

The effect of crack shape on the variation of $K_I/P\sqrt{\pi a}$ with β is clearly illustrated in Figure 5.16 for a single crack with $a/t = 0.1$. Similar trends are observed for other

values of a/t and for multiple cracks and for this reason have not been presented here. As discussed previously, with the exception of the semi-circular cracks, the maximum stress intensity factor occurs at the point of maximum crack penetration (ie. $\beta = 0^\circ$). The values of $K_I/P\sqrt{\pi a}$ increase with b/a at $\beta = 0^\circ$ while the opposite effect occurs at the free surface. The ratio of K_I at $\beta = 90^\circ$ to K_I at $\beta = 0^\circ$ also increases with increasing values of b/a and is less than 1.0 for semi-elliptical cracks and greater than 1.0 for semi-circular cracks. The significance of this latter point is that it suggests semi-circular cracks will tend to become more elliptical in shape (ie. b increases more rapidly than a) as the crack grows. Conversely, semi-elliptical cracks will tend to become more circular in shape as crack growth progresses. Presumably there exists an "equilibrium" value of b/a for which K_I is essentially uniform with angle β around the crack front. Evidence of this phenomena is reported in [73] in which experiments conducted on thick-walled cylinders demonstrated that a crack, once established, tends to a b/a aspect ratio close to 1.25. Inspection of Figures 5.14 and 5.15 do indeed indicate that the variation of $K_I/P\sqrt{\pi a}$ with β for $b/a = 1.25$ is less appreciable than for $b/a = 1.0$, or 2.0. Based on the present results, it is estimated that the equilibrium crack shape for a cylinder with $W = 2.0$ occurs in the vicinity of $b/a = 1.25$ to 1.75.

The variation of $K_I/P\sqrt{\pi a}$ at $\beta = 0^\circ$ with a/t is presented in Figure 5.17 for the single crack cases. Also shown are the results obtained from the 2D analyses for a cylinder with $W = 2.0$ modified for plane strain conditions. These curves demonstrate that the crack depth at which $K_I/P\sqrt{\pi a}$ is minimum is dependent on the crack shape. For semi-circular cracks,

this occurs at an a/t value greater than 0.5, however this value decreases to approximately 0.4, 0.3 and 0.25 for $b/a = 2.0, 3.0, 4.0$ respectively. This figure also illustrates the overly conservativeness estimates of the 2D analyses. It is notable that the 2D predictions are in good agreement with the trends indicated by the 3D results for increasing b/a and this lends further confidence in the finite element models for the 2D and 3D crack studies.

The influence of crack shape is further demonstrated in Figure 5.18 which shows $K_I/P\sqrt{t}$ plotted against a/t and b/a for $\beta = 0^\circ$. The 2D results are also presented for comparison purposes. It is clear from this figure that for any value of a/t , the stress intensity factor increases with increasing b/a . This figure also suggests that the critical flaw size increases with decreasing b/a .

It was determined that the influence of crack shape b/a could be reasonably accounted for if $K_I/P\sqrt{t}$ is plotted against α , where:

$$\alpha = a/t/[3(a/b)^{1.5} + 1] \quad - 5.2$$

This is shown in Figure 5.19 for $\beta = 0^\circ$ and as can be seen, the correlation obtained is excellent. Note that the 2D results are in reasonably close agreement with the 3D results. A least squares curve fit about this correlation resulted in the following expression:

$$\left. K_I/P\sqrt{t} \right|_{\beta = 0^\circ} = 6.942(\alpha)^{0.5} - 9.682(\alpha) + 11.622(\alpha)^{1.5} \quad - 5.3$$

Equation 5.3 is within $\pm 2\%$ of the 3D FEM results.

Figure 5.20 shows the variation of $K_I/P\sqrt{t}$ evaluated at $\beta = 90^\circ$ with a/t and b/a . This figure suggests that crack shape b/a has only a minor influence on the stress intensity factor at the free surface. Thus the data presented in Figure 5.20 were least squares curve fit to the following expression:

$$\left. K_I/P\sqrt{t} \right|_{\beta = 90^\circ} = 3.391(a/t)^{0.5} - 2.140(a/t) + 2.503(a/t)^{1.5} \quad - 5.4$$

Equation 5.4 is generally within $\pm 10\%$ of the numerical results for a/t less than 0.2, and within $\pm 5\%$ for a/t greater than 0.2. Since the numerical results obtained at the free surface are expected to exhibit some uncertainty, the above expression is likely to be adequate for design studies and preliminary fatigue life estimates. The use of equations 5.3 and 5.4 provides a convenient closed-form and reasonably accurate means of estimating stress intensity factors of crack geometries not considered in this thesis and they can easily be incorporated into a fatigue crack growth analysis program. Because the semi-elliptical crack studies have all been performed for a cylinder with $W = 2.0$, it is difficult to comment on the influence of cylinder radius ratio on the stress intensity factors for the semi-elliptical cracks. However, because the 2D results correlate quite well with the 3D results presented in Figure 5.19, it is speculated that the effect of W may reasonably be accounted for by replacing α with $\alpha/(W - 1)$. Further 3D investigations are, however, necessary to substantiate this.

5.3.4 Discussion of Results For Multiple Cracks

Figures 5.9 to 5.12 indicate that the variation of $K_I/P\sqrt{\pi a}$ with β for a given a/t and b/a is similar for single and multiple cracks. It was determined that when computing the load relief factor R_N along the crack front, the variation with crack angle β turned out to be minimal. Figure 5.21, for example, shows the variation of R_N with β for $b/a = 4.0$, $a/t = 0.35$ for 1, 2, 4, 8, 16 and 32 cracks. For each number of cracks, R_N remains nearly constant along the crack front except at the intersection with the bore surface. Since the results at $\beta = 90^\circ$ are somewhat doubtful, we can for discussion purposes assume R_N to be constant.

To illustrate the effect of increasing numbers of cracks, R_N is plotted against number of cracks N and crack shape b/a in Figures 5.22 to 5.26 for $a/t = 0.05$, 0.1, 0.2, 0.35 and 0.5 respectively. Also included in these figures are the 2D results corresponding to a cylinder with $W = 2.0$. The multiple crack effect is observed to increase with both increasing a/t and b/a . For relatively short cracks (ie. $a/t < 0.1$), R_N stays close to 1.0 for small numbers of cracks and gradually dips below 1.0 for larger arrays. The magnitude of R_N also decreases with increasing b/a . For deeper cracks, R_N is greater than 1.0 for the 2 and 4 crack cases and becomes significantly less than 1.0 for large arrays of cracks. It is interesting to note that the 2D results overestimate R_N for two cracks and underestimate R_N for larger arrays of cracks. However the 2D results do follow the trend of increasing b/a for the 3D results and presumably represent the limiting extreme. Again, the good agreement obtained between the 2D and 3D results increases confidence in the modelling and analysis procedures.

It was determined that the correlating parameter $\alpha = a/t/(3(a/b)^{1.5} + 1)$ used for the single crack studies also produced an impressive reduction in crack shape dependence for multiple cracks as shown in Figure 5.27. Note the similarity to Figure 4.22 for the 2D crack cases. It would be interesting to determine if using the parameter $\alpha \cdot (W - 1)$ would correlate the effect of varying cylinder radius ratio for 3D cracks as was determined for the 2D cracks.

The form of presentation shown in Figure 5.27, together with equations 5.3 and 5.4 makes it possible to estimate stress intensity factors for a wide range of crack geometries and configurations. Users are cautioned when attempting to extrapolate these results beyond the ranges of W , b/a , a/t and N considered in this thesis.

Table 5.1

Summary of Crack Mesh Design Parameters
For 3D Cracked Cylinder Models

b/a	a/t	No. * Elements	m/L	a/L	b/L
1.0	0.05	5	1.57	5.00	5.00
	0.10	5	1.57	5.00	5.00
	0.20	5	1.57	5.00	5.00
	0.35	5	1.66	5.28	5.28
	0.50	5	2.00	6.36	6.36
2.0	0.05	5	2.42	5.00	10.00
	0.10	5	2.42	5.00	10.00
	0.20	5	2.42	5.00	10.00
	0.35	5	2.56	5.28	10.56
	0.50	5	3.29	6.36	12.72
3.0	0.05	5	3.35	5.00	15.00
	0.10	5	3.35	5.00	15.00
	0.20	5	3.35	5.00	15.00
	0.35	5	3.54	5.28	15.84
	0.50	5	4.56	6.36	19.08
4.0	0.05	5	4.30	5.00	20.00
	0.10	5	4.30	5.00	20.00
	0.20	5	4.31	5.00	20.00
	0.35	5	4.54	5.28	21.12
	0.50	5	5.84	6.36	25.44

* Number of elements along the crack front.

Table 5.2

Comparison of $K_I/P\sqrt{\pi a}$ vs β

For $W = 2.0$, $b/a = 1.0$, $a/t = 0.5$, $N = 1$ crack

β	ASAS	Atluri & Kathireson [47]	% Difference
0.	1.373	1.336	+ 2.8
15.	1.382	1.298	+ 6.5
30.	1.406	1.296	+ 8.5
45.	1.448	1.308	+ 10.7
60.	1.520	1.355	+ 12.2
75.	1.644	1.451	+ 13.3
90.	1.713	1.685	+ 1.7

Table 5.3

Comparison of $K_I/P\sqrt{\pi a}$ vs β

For $W = 2.0$, $b/a = 1.25$, $a/t = 0.20$, $N = 1$ crack

β	* ASAS	Tan & Fenner [48]	% Difference
0.	1.624	1.617	+ 0.4
15.	1.628	1.604	+ 1.5
30.	1.638	1.588	+ 3.1
45.	1.654	1.565	+ 5.7
60.	1.675	1.565	+ 7.0
75.	1.717	1.604	+ 7.0
90.	1.727	1.782	- 3.1

* ASAS values obtained by linear interpolation of results for $b/a = 1.0$ and $b/a = 2.0$.

Table 5.4

Comparison of $K_I/P\sqrt{\pi a}$ vs β

For $W = 2.0$, $b/a = 1.25$, $a/t = 0.50$, $N = 1$ crack

β	* ASAS	Tan & Fenner [48]	% Difference
0.	1.486	1.477	+ 0.6
15.	1.493	1.450	+ 2.9
30.	1.509	1.447	+ 4.3
45.	1.536	1.432	+ 7.3
60.	1.582	1.468	+ 7.8
75.	1.668	1.551	+ 7.6
90.	1.734	1.677	+ 3.4

* ASAS values obtained by linear interpolation of results for $b/a = 1.0$ and $b/a = 2.0$.

Figure 5.1 Three-Dimensional Representation of Cracked Thick-Walled Cylinder Problem

Figure 5.2 Symmetry and Boundary Conditions for 3D Models

Figure 5.3 Schematic of 3D Mesh Generation Procedure Using GENMESH

Figure 5.4 2D to 3D Element Conversion in GENMESH

Figure 5.5 Substructuring Scheme Employed for 3D Cylinder Models

Figure 5.6 Crack Mesh Design Parameters

Figure 5.7 Typical Mesh for 3D Benchmark Accuracy Studies

Figure 5.8 Typical 3D Crack Mesh

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Figure 5.9a $K_I / P\sqrt{\pi a}$ vs. β , N ($b/a = 1.0$, $a/t = 0.05$)

Figure 5.9b $K_I / P \sqrt{\pi a}$ vs β , N ($b/a = 1.0$, $a/c = 0.1$)

Figure 5.9c $K_I / P \sqrt{\pi a}$ vs β, N ($b/a = 1.0, a/t = 0.2$)

Figure 5.9d $K_I / P \sqrt{\pi a}$ vs β , N ($b/a = 1.0$, $a/t = 0.35$)

Figure 5.9e $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 1.0$, $a/t = 0.5$)

Figure 5.16a $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 2.0$, $a/t = 0.05$)

Figure 5.10b $K_I/P\sqrt{\pi a}$ vs β , $N(b/a = 2.0, a/t = 0.1)$

Figure 5.10c $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 2.0$, $a/t = 0.2$)

Figure 5.10d $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 2.0$, $a/t = 0.35$)

Figure 5.10e $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 2.0$, $a/t = 0.5$)

Figure 5.11a $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 3.0$, $a/t = 0.05$)

Figure 5.11b $K_I / P \sqrt{\pi a}$ vs β , N ($b/a = 3.0$, $a/t = 0.1$)

Figure 5.11c $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 3.0$, $a/t = 0.2$)

Figure 5.11d $K_I / P / \sqrt{\pi a}$ vs β , N ($b/a = 3.0$, $a/t = 0.35$)

Figure 5.11e $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 3.0$, $a/t = 0.5$)

Figure 5.12a $K_I / P / \sqrt{\pi a}$ vs β , N ($b/a = 4.0$, $a/t = 0.05$)

Figure 5.12b $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 4.0$, $a/t = 0.1$)

Figure 5.12c $K_I/P\sqrt{\pi a}$ vs β , N ($b/a = 4.0$, $a/t = 0.2$)

Figure 5.12d $K_I/P/\sqrt{\pi a}$ vs β , N ($b/a = 4.0$, $a/t = 0.35$)

Figure 5.12e $K_I/P/\sqrt{\pi a}$ vs β , N ($b/a = 4.0$, $a/t = 0.5$)

Figure 5.12 Comparison with Published Results for a Cylinder Containing One Crack
($b/a = 1.0$, $a/t = 0.5$)

Figure 5.1: Comparison with Published Results for a Cylinder Containing One Crack
($b/a = 1.25$, $a/t = 0.2$)

Figure 5.15 Comparison with Published Results for a Cylinder Containing One Crack
($b/a = 1.25$, $a/t = 0.5$)

Figure 5.16 Effect of Crack Shape on the Variation of $K_I/P\sqrt{\pi a}$ vs β ($a/t = 0.1$, 1 crack)

Figure 5.17. $K_I/P/\sqrt{\pi a}$ vs a/t , b/a ($\beta = 0^\circ$, 1 crack)

Figure 5.18 $K_I/P\sqrt{t}$ vs a/t , b/a ($\beta = 0^\circ$, 1 crack)

Figure 5.19 SIF Correlation For 3D Single Cracks ($\beta = 0^\circ$)

Figure 5.20 SIF Correlation For 3D Single Cracks ($\beta = 90^\circ$)

Figure 5.21 Variation of Load Relief Factor with β ($b/a = 4.0$, $a/t = 0.35$)

Figure 5.22 Multiple Crack Load Relief Factors ($a/t = 0.05$)

Figure 5.23 Multiple Crack Load Relief Factors ($a/t = 0.1$)

Figure 5.24 Multiple Crack Load Relief Factors ($a/t = 0.2$)

Figure 5.25 Multiple Crack Load Relief Factors ($a/t = 0.35$)

Figure 5.26 Multiple Crack Load Relief Factors ($a/t = 0.5$)

Figure 5.27 Load Relief Factor Correlation For 3D Multiple Cracks

CHAPTER 6

CONCLUSIONS

The finite element method has been used to predict mode I stress intensity factors for single and multiple radial surface cracks located at the bore of internally pressurized thick-walled cylinders. This information is of considerable practical importance and is necessary for the application of linear elastic fracture mechanics to the design and maintenance of items such as isostatic compaction chambers, high pressure piping, and gun barrels. The important new findings and results of the work conducted in this thesis are summarized below.

Straight-Fronted (2D) Crack Studies

1. Stress intensity factors for straight-fronted cracks in thick-walled cylinders have been presented for $W = 1.5$ to 2.25 , $a/t = 0.05$ to 0.5 , and for 1 to 40 equispaced cracks.
2. Comparison of the present results with published solutions demonstrated excellent agreement for the single crack geometries. For the 40 crack case, the present results were determined to lie approximately midway between the results of Pu and Hussain [34] and Hoff [35].
3. The cylinder radius ratio, W , exerts a considerable influence on the stress intensity factors for both single and multiple cracks. The stress intensity factor for any crack depth a/t increases with decreasing W . For short cracks ($a/t < 0.1$), this effect can be accounted for by simply scaling by the hoop stress at

the bore. For deeper cracks, the effects of W are more complex and simple scaling will result in substantial errors. This serves to illustrate the danger of applying thin-walled cylinder results to thicker-walled geometries.

4. For single cracks it has been demonstrated that for a given material, internal pressure load, and wall thickness or bore radius, the effect of increasing W is to increase critical crack depth and size and increase fatigue life. In addition, it has been demonstrated that varying cylinder cross-sectional dimensions while maintaining W constant has minimal effect on the critical crack size and fatigue life. Correlations have been presented in closed-form expressions which will allow designers to quantify such effects for cylinder radius ratios and crack geometries not considered in this thesis.
5. The effect of multiple cracks is to cause a considerable reduction in the stress intensity factor compared to the single crack case as the number of cracks increase from 4. For the 4 crack case, only modest reductions in K_I occur for $a/t < 0.5$, although the trend indicates that the 4 crack case may eventually exceed the single crack values for deeper cracks. The present results indicate the 2 crack configuration represents the worst case. For all crack depths, K_I for 2 cracks exceeds the value of K_I for single cracks and the difference increases with increasing a/t .
6. For large arrays of cracks, the effect of increasing W is to increase fatigue life assuming constant bore radius or wall thickness. This is consistent with the results for single cracks. The effect of W on fatigue

life and critical crack size is less certain for 2 cracks.

7. The multiple crack results have been presented in terms of the load relief factor, R_N . It has been determined that the effect of W can be reasonably accounted for by plotting R_N as a function of $a/t \cdot (W - 1)$.

Semi-Elliptical (3D) Crack Studies

1. Stress intensity factors have been presented for semi-elliptical cracks with aspect ratios varying between 1.0 and 4.0 in a cylinder with $W = 2.0$, $a/t = 0.05$ to 0.5 and for 1 to 32 cracks. This represents a substantial increase in the range of crack geometries and configurations over previously known or published solutions. In particular, this is the first time that results have been presented for cracks of large aspect ratios and for arrays of multiple semi-elliptical cracks in a thick-walled cylinder.
2. The method by which the 3D finite element models were formulated has been discussed in detail. The procedure involved extensive use of the multi-level substructuring capabilities of ASAS in conjunction with an automated mesh generation program, and resulted in accurate, economic, and computationally efficient 3D finite element models.
3. Comparison of the present results to those of Atluri et al. [47] and Tan et al. [48] indicated very good agreement. It was noted some of the discrepancies for $b/a = 1.0$, $a/t = 0.5$ may possibly have resulted from the improper application of boundary conditions and/or modelling of the crack geometry in [47].

4. The point at which the stress intensity is maximum along the crack front occurs at the free surface for semi-circular cracks, and at the point of maximum crack depth for semi-elliptical cracks. This suggests that semi-circular cracks will tend to grow more elliptical in shape, while semi-elliptical cracks will tend to grow more circular. It is estimated that the crack shape for which K_I is essentially constant along the crack occurs in the vicinity of $b/a = 1.25$ to 1.75 (for $W = 2.0$). These observations are consistent with experimental results.
5. Comparisons of the 3D results with the 2D results indicated that the latter are overly conservative for both single and multiple crack configurations. It was noted that the 2D predictions are in good agreement with the trend for increasing b/a indicated by the 3D results.
6. It was determined that the influence of b/a on K_I at the point of maximum crack penetration for single cracks could be accounted for by plotting $K_I/P\sqrt{t}$ vs $a/t/(3 \cdot (a/b)^{1.5} + 1)$. This correlation shows good agreement with the 2D results.
7. It was determined that the stress intensity factor values at the bore surface are only slightly dependent on b/a , but quite dependent on a/t . A reasonable correlation of the results was obtained by plotting $K_I/P\sqrt{t}$ vs a/t .
8. Closed-form expressions for the correlations described in 6 and 7 above have been presented. These expressions can be easily implemented in a fatigue crack growth analysis program and are useful for estimating stress

intensity factors for crack geometries not considered in this thesis.

9. The variation of K_I along the crack front is similar for single and multiple cracks. It has been demonstrated that it is reasonable to assume the load relief factor R_N to be constant along the crack front.
10. The effect of multiple cracks on the stress intensity factor is similar for both 3D and 2D cracks with the exception of the 4 crack case. For this case, the 3D results predict a load relief factor greater than unity whereas the 2D results predict R_N to be less than 1.0.
11. The multiple crack effect increases with increasing b/a . For 2 and 4 cracks, R_N increases with increasing b/a . For more than 4 cracks, R_N decreases with increasing b/a . The 2D results follow the trend of the 3D results for increasing b/a , and presumably represent the limiting extreme.
12. A good correlation of the multiple crack results was obtained by plotting R_N vs $a/t/(3(a/b)^{1.5} + 1)$. This correlation can be used in conjunction with the closed form expressions derived for the single cracks to obtain estimates of the stress intensity factors for crack geometries and configurations not considered in this thesis.

Recommendations for Future Work

1. The current uncertainty in the calculation of the stress intensity factor at the point where the crack front intersects a free surface requires further comprehensive investigation. This is of fundamental importance and,

of course, has application to all surface crack problems. It is quite likely that both numerical and experimental approaches would be required to resolve this problem.

2. With more particular reference to the work in this thesis, further 3D studies are required to determine the effect of varying cylinder radius ratio on the stress intensity factors. It would be interesting to determine if combining the 2D and 3D results is possible. If this is so (and preliminary indications seem to suggest this), then it should be possible to redefine the correlating parameter, α , in equations 5.2 and 5.3 as:

$$\alpha = a/t / (3(a/b)^{1.5} + 1) / (W - 1)$$

Similarly, a/t in equation 5.4 should be replaced by $a/t/(W - 1)$, and α for the multiple crack correlation shown in Figure 5.27 should be replaced by $a/t/(3(a/b)^{1.5} + 1) \cdot (W - 1)$.

3. Finally, it may be desirable to extend the ranges of a/t to values greater than 0.5. If so, the methodologies presented in this thesis should provide considerable guidance in conducting these analyses.

Chapter.7

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