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Degree Sequence Polyhedra and Separation Algorithms

by

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**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy**

**Department of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics**

**Carleton University
Ottawa, Ontario
July, 1992**

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ISBN 0-315-79844-0

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Abstract

The subject of this thesis is polyhedra related to degree sequences of graphs. The degree sequence of a subgraph H of a graph G is the vector having a component for each vertex, which is the number of edges incident to that vertex. A degree sequence polyhedron is the convex hull of all degree sequences determined by a family of subgraphs. In the case where the family consists of all subgraphs of G , we give a linear description of the polyhedron, describe a separation algorithm, and also characterize extreme points and extreme point adjacency. For the more difficult case where the family consists of all subgraphs satisfying certain degree constraints (namely, b -matchings), a linear description was already known. We characterize the facets, and give an efficient separation algorithm. This algorithm generalizes both the maximum cardinality b -matching algorithm of Edmonds and Johnson, and the separation algorithm of Cunningham and Green-Krótki for matching degree sequence polyhedra. We also give a linear description for b -matching a -covering degree sequence polyhedra for bidirected graphs. Finally, we study degree sequence polyhedra of forests and Eulerian subgraphs, and we show that these polyhedra must have complicated linear descriptions.

Acknowledgements

I would like to express my sincere thanks to Professor W. H. Cunningham. To me, he is an ideal supervisor. He introduced me to the beautiful field of polyhedral combinatorics, and brought the topic of this thesis to me. I am deeply grateful for his guidance, kindness, and much more.

I would like to thank Carleton University. The financial aid from the Faculty of Graduate Studies made it possible for me to finish my studies here at Carleton. I enjoyed the friendly environment of the Department of Mathematics and Statistics.

I would like to thank the Institute of Operations Research at the University of Bonn for their hospitality during my stay as a visiting student during the summer of 1989.

I appreciate the referees for their comments and suggestions.

I give special thanks to my wife Qiong for her constant love and support. I dedicate this work to her and our very beloved son Eric.

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Chapter 1

Introduction

1.1 Introduction

Let $G = (V, E)$ be a graph, and let $H = (V, E')$ be a spanning subgraph of G . For $v \in V$, let $d_H(v)$ denote the number of edges of H incident with v . The degree sequence of H is $d_H = (d_H(v) : v \in V)$. Many of the classical results of graph theory concern degree sequences. For example, Tutte's famous factor theorem [25] gives necessary and sufficient conditions for a given vector to be the degree sequence of some subgraph of G .

A special case of Tutte's result is equivalent to a theorem of Erdős and Gallai [13]. It gives necessary and sufficient conditions for a given vector to be the degree sequence of a simple graph, that is, to be the degree sequence of a subgraph of a complete graph. These conditions consist of a system of linear inequalities and some trivial conditions. Koren [18] proved that a vector is a convex combination of some degree sequences if and only if it satisfies those linear inequalities. That is, Koren gave a linear description of the convex hull of the degree sequences of all subgraphs of the complete graph.

Koren's theorem can be thought of as the starting point for this thesis. There are several ways to generalize his result. One is to consider arbitrary graphs (not just complete graphs). Another is to consider degree sequences of a special family of

subgraphs. For example, one might consider the family of matchings of G . In fact, the second degree sequence polyhedron theorem, found by Balas and Pulleyblank [2, 3], did just this. Other directions of generalization are to allow a limited number of multiple copies of edges of G to appear in the subgraphs, (a linear description for this b -matching polyhedron was found by Cunningham and Green-Krótki [5]), or to allow G to be “bidirected”.

In addition to giving linear descriptions for several of the resulting polyhedra, we investigate these polyhedra further. For example, in several cases, we give minimal linear descriptions, and characterize extreme points and extreme point adjacency. We also describe efficient separation algorithms for some of these polyhedra.

A “separation algorithm” for a polyhedron P determines whether a given vector $\bar{x}$ is in P , and if not, finds an inequality $cx \leq c_0$ violated by $\bar{x}$ but not by any point of P . Separation algorithms are fundamental in the subject of polyhedral combinatorics. A theorem of Grötschel, Lovász, and Schrijver [17] implies that there exists an efficient separation algorithm for P if and only if there is an efficient algorithm to optimize any linear function over P . Moreover, the connection between separation algorithm and optimization algorithm is not only theoretical. There are practical “cutting plane” algorithms that iteratively use the simplex method and a separation subroutine to solve the optimization problem.

The main result of this thesis is a good separation algorithm for b -matching degree sequence polyhedra. This algorithm maximizes the sum of the coordinates of a vector of a b -matching polyhedron subject to (fractional) degree constraints. The algorithm generalizes both the separation algorithm of Cunningham and Green-Krótki [6] for matching degree sequence polyhedra and the maximum cardinality b -matching algorithm of Edmonds and Johnson [11]. By using the consequences of this algorithm and theorems of Cunningham and Green-Krótki [5], we derive a linear description for the b -matching α -covering degree sequence polyhedron of a bidirected graph.

Finally, we give some “negative” results. Although for some polyhedra, theoretical considerations suggested the existence of simple linear descriptions, we show that any

linear description must be complicated, in the sense that the coefficients on some of its essential (facet-inducing) linear inequalities are large.

The thesis is organized as follows.

In Chapter 1, we introduce the definitions and background.

In Chapter 2, we study the convex hull of degree sequences of an arbitrary graph G . We give a complete linear description for the polyhedron, characterize the facets, and describe a separation algorithm. We also characterize the vertices and adjacency.

In Chapter 3, we characterize the facets of the convex hull of degree sequences of capacitated b -matchings.

In Chapter 4, we solve the separation problem for the b -matching degree sequence polyhedron by giving an efficient algorithm.

In Chapter 5, we give a linear description for the convex hull of degree sequences of capacitated b -matchings of an arbitrary bidirected graph, which is a generalization of directed graph, undirected graph, and more. We also give a linear description for the convex hull of degree sequences of b -matching α -coverings of an arbitrary bidirected graph. We also describe separation algorithms.

In Chapter 6, we give a general way to make bad facet-inducing inequalities from simple ones. We show that there are bad facet inequalities for polyhedra of degree sequences of Eulerian subgraphs. On the other hand, for directed graphs, we give linear descriptions of the polyhedra of outdegree sequences of Eulerian subgraphs.

In Chapter 7, we discuss polyhedra of degree sequences of forests and spanning trees, and we show that there are nasty facet-inducing inequalities for these polyhedra. We also give general ways to determine valid equalities and to justify whether a valid inequality is a facet-inducing inequality for a polyhedron obtained through linear transformation from another polyhedron.

1.2 Terminology and notation

General notation

Let $\mathbf{R}$, $\mathbf{Z}$ denote the set of real, integer numbers respectively. Where S is a finite set, $\mathbf{R}^S$ denotes the set of all vectors $(x_s \in \mathbf{R} : s \in S)$, $\mathbf{R}_+^S$ denotes $\{x \in \mathbf{R}^S : x_s \geq 0, \forall s \in S\}$, $\mathbf{Z}^S$ denotes the set of all vectors $(x_s \in \mathbf{Z} : s \in S)$, and $\mathbf{Z}_+^S$ denotes $\{x \in \mathbf{Z}^S : x_s \geq 0, \forall s \in S\}$. For $x, y \in \mathbf{R}^S$, $x \geq y$ means $x_s \geq y_s$ for all $s \in S$. For $r \in \mathbf{R}$, $\lfloor r \rfloor$ means the maximum integer not greater than r , and $\lceil r \rceil$ means the minimum integer not less than r . For $a \in \mathbf{Z}^S$, let $\gcd(a)$ denote the greatest common divisor of the components of A .

Let $Q \subseteq S$. We use $\chi^Q \in \mathbf{R}^S$ to denote the incidence vector of Q , namely, $\chi_s^Q = 1$ for $s \in Q$, $\chi_s^Q = 0$ for $s \in S \setminus Q$. For $x \in \mathbf{R}^S$, let $x(Q)$, or xQ , denote $\sum(x_s : s \in Q)$. The projection, or restriction, of $x \in \mathbf{R}^S$ on $\mathbf{R}^Q$ is a vector $x' \in \mathbf{R}^Q$ such that $x'_s = x_s$ for all $s \in Q$. We use $x|_Q$, or x , to denote the projection.

For $X, Y \subseteq \mathbf{R}^S$, let $X + Y$ denote $\{x + y : x \in X, y \in Y\}$. For two disjoint finite sets S, T , and $X \subseteq \mathbf{R}^S, Y \subseteq \mathbf{R}^T$, let $X \times Y$ denote $\{(x, y) : x \in X, y \in Y\}$.

For two sets S and T , let $S \Delta T$ denote the symmetric difference of S and T , that is, $S \Delta T = (S \setminus T) \cup (T \setminus S)$.

Graphs

We introduce some basic terminology of graph theory.

An (undirected) graph $G = (V, E)$ consists of a finite set V of vertices, a finite set E of edges, and an assignment of each $e \in E$ to an unordered pair u, v of V , which are called ends of e . G is called simple if there are no multiple edges and loops. If $e \in E$ has ends u, v , then e is said to be incident with u, v , and u, v are said to be adjacent. In this case, e is often written as $[u, v]$ or uv . An edge-vertex incidence vector of $uv \in E$ is $\chi^{uv} \in \mathbf{Z}^V$ such that $\chi_w^{uv} = 1$ if $w = u$ or v , $\chi_w^{uv} = 0$ if $w \in V \setminus \{u, v\}$. The (edge-vertex) incidence matrix of G is formed by all edge-

vertex incidence vectors as columns. We shall use M_G , or M to mean the incidence matrix of G throughout this thesis.

A subgraph $H = (V_1, E_1)$ of G is a graph such that $V_1 \subseteq V$ and $E_1 \subseteq E$. Let $S \subseteq V$. $G[S]$ is the subgraph induced by S , that is, $G[S] = (S, \gamma(S))$. $G - S$ is $G[V \setminus S]$. For $E' \subseteq E$, $G - E'$ is the subgraph obtained by deleting E' from G . For $A, B \subseteq V$, $[A, B]$ denotes $\{e \in E : e \text{ has one end in } A \text{ and the other end in } B\}$. Let $\delta(S)$ denote $[S, V \setminus S]$, and let $\gamma(S)$ denote $[S, S]$. The neighbour set of S , $N(S)$, is the set $\{v \in V : v \text{ is adjacent to a vertex of } S\}$. If $S = \{v\}$, we simply use $\delta(v)$, $N(v)$ to mean $\delta(\{v\})$, $N(\{v\})$.

$G = (V, E)$ is bipartite if V can be partitioned into two disjoint non-empty sets, $V = V_1 \cup V_2$, such that $E = \delta(V_1)$.

The number of edges in graph G incident with a vertex v is called the degree of v and denoted by $d_G(v)$, or simply $d(v)$ if the graph is understood. The degree sequence of a subgraph H is a vector $d \in \mathbb{Z}^V$ such that $d_v = d_H(v)$ for all $v \in V$.

A walk in G is a sequence $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ such that $v_0, \dots, v_k$ are vertices and $e_i = v_{i-1}v_i$ for $i = 1, \dots, k$. The walk is closed if $v_0 = v_k$ and $k > 0$. The length of the walk is k . The walk is simple if $v_0, \dots, v_k$ are distinct. A simple walk is called a path. A closed walk is called a cycle if $v_0, \dots, v_{k-1}$ are distinct. A graph is connected if any two vertices are connected by a path. A forest F of G is a subgraph that does not contain a cycle. A tree T of G is a connected forest. The tree is a spanning tree if the vertex set of T is V . A vertex $v \in V$ is a leaf of the tree if $d_T(v) = 1$. A rooted tree is a tree T with a root vertex r of T . For each vertex v of T , there is a unique path from r to v . Every vertex u ($\neq v$) of the path is called an ancestor of v and v is called a descendant of u , especially, if edge uv is in the path, then u is called the parent of v , and v is called a child of u .

A matching of G is a set of edges of G such that no two edges have a common end. Given a vector $b \in \mathbb{Z}^V$ and $u \in \mathbb{Z}_+^E$, a $(u$ -capacitated) b -matching is a vector $z \in \mathbb{Z}^E$ such that $z(\delta(v)) \leq b_v$ for all $v \in V$ and $z_e \leq u_e$ for all $e \in E$. If every component of u is $+\infty$, z is called an uncapacitated b -matching. Note that a

1-matching is the incidence vector of a matching.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union, $G_1 \cup G_2$, is a graph $(V_1 \cup V_2, E_1 \cup E_2)$. The intersection, $G_1 \cap G_2$, is a graph $(V_1 \cap V_2, E_1 \cap E_2)$.

A subgraph H of G is **Eulerian** if $d_H(v)$ is even for all $v \in V$, namely, H is a union of edge disjoint cycles and isolated vertices.

A **directed graph**, or **digraph**, G consists of a finite set V of vertices, a finite set E of edges, and an assignment of each $e \in E$ to an ordered pair (u, v) of V . If e is assigned to (u, v) , u is called the **tail** (end) of e , v is called the **head** (end) of e . An **edge-vertex incidence vector** is $\chi^{(u,v)} \in \mathbb{Z}^V$ such that $\chi_w^{(u,v)} = 1$ if $w = v$, $\chi_w^{(u,v)} = -1$ if $w = u$, and $\chi_w^{(u,v)} = 0$ if $w \in V \setminus \{u, v\}$. The (edge-vertex) **incidence matrix** of G is defined as for undirected graphs. A **directed walk** in G is a sequence $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ such that $v_0, \dots, v_k$ are vertices and $e_i = (v_{i-1}, v_i)$ for $i = 1, \dots, k$. We define **directed path** and **directed cycle** similarly.

Let $d_G^+(v)$, or $d^+(v)$, denote the number of edges having tail v and $d_G^-(v)$, or $d^-(v)$, denote the number of edges having head v . $d^+(v)$ and $d^-(v)$ are called **outdegree** and **indegree** of v respectively.

For $A, B \subseteq V$, $(A, B) := \{e \in E : e \text{ has tail in } A, \text{ and head in } B\}$. For any $S \subseteq V$, $\delta^+(S) := (S, V \setminus S)$, $\delta^-(S) := (V \setminus S, S)$.

A subgraph H of G is **independent** if the incidence vectors of H are linearly independent. The following relation is well known. If G is undirected, then H is independent if and only if H contains no even cycles and each component of H contains at most one odd cycle. If G is directed, then H is independent if and only if H is a forest. This relation provides a way to compute the number of edges of a maximal independent subgraph of G .

Polyhedra

A convex **polyhedron** $\mathbf{P}$ is the solution set of a finite set of linear inequalities. This set of linear inequalities is called a **linear defining system**, or **linear description**,

of the polyhedron. We note that a polyhedron has infinite linear defining systems. A linear inequality is called **valid** for P if every vector of P satisfies this inequality. In this thesis, all polyhedra are convex. A **polytope** is a bounded polyhedron. A **polyhedral cone**, or **cone**, is a polyhedron P such that the nonnegative scalar product of any $x \in P$ is in P . The **dominant** of a polyhedron P , $\text{dom}(P)$, is $\{x \in \mathbb{R}^n : \text{there is } y \in P \text{ such that } x \geq y\}$. An **affine space** is the solution set of a finite set of linear equalities.

Let $x^1, x^2, \dots, x^k \in \mathbb{R}^n$. A vector $x \in \mathbb{R}^n$ is a **linear combination** of $x^1, x^2, \dots, x^k$ if $x = \sum_{i=1}^k \lambda_i x^i$ for numbers $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$. x is an **affine combination** of $x^1, x^2, \dots, x^k$ if it is a linear combination such that $\sum_{i=1}^k \lambda_i = 1$. x is an **integral combination** if it is a linear combination such that $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{Z}$. x is a **cone combination** of $x^1, x^2, \dots, x^k$ if it is a linear combination such that $\lambda_i \geq 0$ for $i = 1, \dots, k$. x is a **convex combination** of $x^1, x^2, \dots, x^k$ if x is an affine combination and a cone combination.

Let $S \subseteq \mathbb{R}^n$. The **linear [affine, cone, convex] hull** of S is the set of all linear [affine, cone, convex] combinations of finite subsets of S . They are denoted by $\text{lin}(S)$, $\text{aff}(S)$, $\text{cone}(S)$, $\text{conv}(S)$ respectively. We sometimes omit the brackets.

The **rank [affine rank]** of a set $S \subseteq \mathbb{R}^n$ is the cardinality of the largest linearly [affinely] independent subset of S . The **dimension** of S is the affine rank of S minus 1. The rank of S is denoted as $\text{rank}(S)$, and the dimension of S is denoted as $\text{dim}(S)$. S is **full dimensional** if $\text{dim}(S) = n$. Note that if S contains vector 0 , then $\text{dim}(S) = \text{rank}(S)$.

Let P be a polyhedron. A **face** of P is the set of feasible points of P satisfying a valid inequality with equality. This inequality is said to **induce**, or **define**, the face. A face F of P is a **facet** if it is a maximal proper face, equivalently, $\text{dim}(F) = \text{dim}(P) - 1$. A **facet-inducing inequality** for P is a valid inequality inducing a facet. A **vertex** (or an **extreme point**) of P is a vector x such that $\{x\}$ is a face of P . Two vertices of P are **adjacent** if there is a face containing them and there is no other vertex in the face.

A polyhedron is **rational [integral]** if each face contains a rational [integral] vector. Every face of a rational polyhedron is defined by an inequality $ax \leq b$ such that a, b are integral.

Complexity and input size

The input length of an integer k is $\lceil \log_2(|k|+1) \rceil$, denoted as $l(k)$. For every rational number r , there are two unique integers p, q with $\gcd(p, q) = 1$ and $q > 0$ such that $r = p/q$. The input length of r , $l(r)$, is $l(p) + l(q)$. If a is a vector or a matrix, then the length of a , $l(a)$, is the sum of input lengths of the entries. For any sequence $a_1, \dots, a_k$ of vectors and matrices, its input length $l(a_1, \dots, a_k)$ is $l(a_1) + \dots + l(a_k)$. We define $|V| + |E|$ to be the size of a graph $G = (V, E)$.

Given a class of problems and an algorithm, the time complexity function $f : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ of the algorithm is the maximum time $f(n)$ needed to solve any problem instance of input size $n \in \mathbb{Z}_+$. The algorithm is a **polynomial-time algorithm** if there exists a polynomial $p(x)$ such that $f(n) = O(p(n))$, namely, there exists $C \in \mathbb{R}_+$ such that $f(n) \leq Cp(n)$ for all $n \in \mathbb{Z}_+$.

1.3 Background

In this section, we introduce some basic results. For those well known theorems, we just state them. For more details, see [23] and [19].

Network flow optimization

Let $D = (V, E)$ be a digraph with distinguished vertices s (the source) and t (the sink). Let $u \in \mathbb{R}_+^E$ be the capacity function of E . Then (D, u, s, t) is called a **network**.

A vector $z \in \mathbb{R}_+^E$ is called an $s - t$ flow, or flow, in D if it satisfies the following properties:

1. $z(\delta^+(v)) - z(\delta^-(v)) = 0$ for all $v \in V \setminus \{s, t\}$;

2. $z_e \leq u_e$ for all $e \in E$.

$z(\delta^+(s)) - z(\delta^-(s))$ is called the **amount** of z .

An $s - t$ cut, or cut, in D is $\delta^+(S)$ for some $S \subset V$ such that $s \in S$ and $t \notin S$. $u(\delta^+(S))$ is called its **capacity**. The famous theorem of Ford and Fulkerson [14] tells the relation of the flows and cuts of D .

Theorem 1.3.1 (Max-Flow Min-Cut Theorem) $\max\{z(\delta(s)) : \text{for all flow } z \text{ in } D\} = \min\{u(\delta^+(S)) : \text{for all cut } \delta^+(S) \text{ in } D\}$

Ford and Fulkerson developed an algorithm to solve the maximum flow problem, and thus proved the theorem and its integer version. Later, Edmonds and Karp [12], Dinic [8] found polynomial-time algorithms.

Theorem 1.3.2 *If $u \in \mathbb{Z}_+^E$, then there is a maximum flow z of D such that $z \in \mathbb{Z}^E$.*

In fact, the following result is true.

Proposition 1.3.3 *Let $G = (V, E)$ be a digraph. If $b \in \mathbb{Z}^V$, and $u \in \mathbb{Z}_+^E$, then the polyhedron $P = \{z \in \mathbb{R}^E : M_G z \leq b, 0 \leq z \leq u\}$ is integral.*

Linear programming duality theorem

A linear programming, or LP, problem is to minimize or maximize a linear objective function subject to a finite number of linear constraints. A typical form is

$$(1.1) \quad \begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x \geq 0, \end{array}$$

where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

The dual program of (1.1) is

$$(1.2) \quad \begin{array}{ll} \text{minimize} & yb \\ \text{subject to} & yA \geq c \\ & y \geq 0, \end{array}$$

The fundamental result of linear programming is the following theorem.

Theorem 1.3.4 (LP Duality Theorem) *If one of the programs (1.1) and (1.2) has a finite optimal solution, then so does the other and the two optimal values are equal.*

A weaker result is also useful.

Theorem 1.3.5 *Suppose that x and y are feasible solutions of (1.1) and (1.2) respectively. Then x and y optimize (1.1) and (1.2) if and only if the following conditions are satisfied:*

$$(a) \quad (yA - c)x = 0;$$

$$(b) \quad y(Ax - b) = 0.$$

Conditions 1.3.5a, 1.3.5b are called the **complementary slackness conditions**.

Since there are many different forms of constraints of LP problems, the LP duality theorem and complementary slackness condition appear in different forms. In this thesis, we shall use variants of the theorems without stating them.

Polyhedral theory

Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $Ax \leq b$, $Cx = d$ be a linear description for P . Let m be the number of rows of C .

Theorem 1.3.6 *There exist $x^1, \dots, x^p, g^1, \dots, g^q \in \mathbb{R}^n$ such that $P = \text{conv}\{x^1, \dots, x^p\} + \text{cone}\{g^1, \dots, g^q\}$.*

$Ax \leq b$, $Cx = d$ is a minimal linear description if for any inequality $a^i x \leq b_i$ of $Ax \leq b$, there is an $x^* \in P$ such that $a^i x^* < b_i$, and deleting any inequality or equality changes the polyhedron.

Theorem 1.3.7 *If $Ax \leq b$, $Cx = d$ is a minimal linear description for P , then*

$$(a) \quad \text{aff}(P) = \{x \in \mathbb{R}^n : Cx = d\} \text{ and } m = \text{rank}(C);$$

- (b) $\text{aff}(P)$ and P have dimension $n - m$;
- (c) every inequality $a^i x \leq b_i$ of $Ax \leq b$ defines a facet F of P ;
- (d) if $c'x = d'$ is valid for P , then $c' = yC$ and $d' = yd$ for some $y \in \mathbb{R}^m$; if $a'x \leq b'$ defines a facet of P , then $a' = la + yC$ and $b' = lb + yC$ for exactly one inequality $ax \leq b$ of $Ax \leq b$, and for some $y \in \mathbb{R}^m$ and $l > 0$.

Theorem 1.3.8 $x^0 \in \mathbb{R}^n$ is a vertex of P if and only if there is $c \in \mathbb{Z}^n$ such that $\min\{cx : x \in P\}$ has a unique solution x^0 .

Theorem 1.3.9 Two vertices x^1, x^2 of P are adjacent if and only if there is $c \in \mathbb{R}^n$ such that x^1 and x^2 are the only two vertices optimizing $\min\{cx : x \in P\}$. In fact, vector c can be chosen to be integral if the polyhedron P is rational.

We point out that all polyhedra considered in this thesis are rational.

The following result about the linear description of the product of two polyhedra P_1 and P_2 is especially useful. It tells that instead of considering $P_1 \times P_2$, it is enough to consider P_1 and P_2 .

Theorem 1.3.10 Let $P_1 = \{x \in \mathbb{R}^S : a^i x = b_i, i \in I_1, a^i x \leq b_i, i \in I_2\}$, and $P_2 = \{y \in \mathbb{R}^T : c^j y = d_j, j \in J_1, c^j y \leq d_j, j \in J_2\}$. Then $P_1 \times P_2 = \{(x, y) \in \mathbb{R}^S \times \mathbb{R}^T : a^i x = b_i, i \in I_1, a^i x \leq b_i, i \in I_2, c^j y = d_j, j \in J_1, c^j y \leq d_j, j \in J_2\}$. Furthermore, if the systems defining P_1 and P_2 are minimal, then the system defining $P_1 \times P_2$ is minimal.

From linear algebra, we know that for every affine space X , there exist a vector x^0 and a linear space S such that $X = \{x + x^0 : x \in S\}$.

Proposition 1.3.11 Let $X, Y \subseteq \mathbb{R}^n$, and $x^0 \in \mathbb{R}^n$. If $X = \{x + x^0 : x \in Y\}$, then $\dim(X) = \dim(Y)$.

Proposition 1.3.12 Suppose that $x^1, \dots, x^k$ are affinely independent and satisfy equation $ax = a_0$. If $a_0 \neq 0$, then $x^1, \dots, x^k$ are linearly independent.

Proof: $x^1, \dots, x^k$ are affinely independent, so

$$\begin{pmatrix} x^1 \\ 1 \end{pmatrix}, \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x^n \\ 1 \end{pmatrix}$$

are linearly independent. We do the row operation of the matrix by multiplying each vector with $(a, -a_0)$ and replacing the last row with 0's. The rank of the matrix remains the same. So

$$\begin{pmatrix} x^1 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} x^n \\ 0 \end{pmatrix}$$

are linearly independent. Then $x^1, \dots, x^k$ are linearly independent. ■

Now we mention some basic results on operations on two polyhedra.

Proposition 1.3.13 *Let $X, Y \subseteq \mathbb{R}^n$ be affine spaces, and let $x^0 \in X, y^0 \in Y$. Let S_1, S_2 be linear spaces such that $X = \{x^0 + x : x \in S_1\}, Y = \{y^0 + x : x \in S_2\}$. Then $X + Y$ is an affine space and $\dim(X + Y) = \dim(X) + \dim(Y) - \text{rank}(S_1 \cap S_2)$.*

Proof:

$$\begin{aligned} X + Y &= \{x^0 + x + y^0 + y : x \in S_1, y \in S_2\} \\ &= \{x^0 + y^0 + z : z \in S_1 + S_2\}. \end{aligned}$$

So $X + Y$ is an affine space. Moreover, $\text{rank}(S_1 + S_2) = \text{rank}(S_1) + \text{rank}(S_2) - \text{rank}(S_1 \cap S_2)$. So $\dim(X + Y) = \dim(X) + \dim(Y) - \text{rank}(S_1 \cap S_2)$. ■

In Proposition 1.3.13, we notice that if $S_1 \cap S_2 = \{0\}$, then $\dim(X + Y) = \dim(X) + \dim(Y)$.

Proposition 1.3.14 *Let $X, Y \subseteq \mathbb{R}^n$. Then $\text{aff}(X + Y) = \text{aff}(X) + \text{aff}(Y)$.*

Proof: Since $X + Y \subseteq \text{aff}(X) + \text{aff}(Y)$, $\text{aff}(X + Y) \subseteq \text{aff}(X) + \text{aff}(Y)$ by Proposition 1.3.13.

Let $\{x^0, x^0 + x^1, \dots, x^0 + x^p\}, \{y^0, y^0 + y^1, \dots, y^0 + y^q\}$ be maximal affinely independent subsets of X, Y respectively. Then $\text{aff}(X) = \{x^0 + x : x \in \text{lin}(x^1, \dots, x^p)\}$,

$\text{aff}(Y) = \{y^0 + y : y \in \text{lin}(y^1, \dots, y^q)\}$. $\text{aff}(X) + \text{aff}(Y) = \{x^0 + y^0 + z : z \in \text{lin}(x^1, \dots, x^p) + \text{lin}(y^1, \dots, y^q)\}$. So each $x^0 + y^0 + \alpha_1 x^1 + \dots + \alpha_p x^p + \beta_1 y^1 + \dots + \beta_q y^q \in \text{aff}(X + Y)$. So $\text{aff}(X) + \text{aff}(Y) \subseteq \text{aff}(X + Y)$.

Therefore $\text{aff}(X) + \text{aff}(Y) = \text{aff}(X + Y)$. ■

We end the first chapter by proving the following result.

Proposition 1.3.15 *Let $z^1, \dots, z^k \in \mathbb{R}^n$, and let D be a $m \times n$ matrix. If $Q = \text{conv}\{z^1, z^2, \dots, z^k\}$, and $P = \text{conv}\{Dz^1, Dz^2, \dots, Dz^k\}$, then $P = \{Dz : z \in Q\}$.*

Proof: For any $x \in P$, $x = \sum_1^k \lambda_i Dz^i$ for some $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$ such that $\sum_1^k \lambda_i = 1$. So $x = D(\sum_1^k \lambda_i z^i) = Dz$, where $z \in Q$. So $P \subseteq \{Dz : z \in Q\}$.

For any $z \in Q$, then $z = \sum_1^k \lambda_i z^i$ for some $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$ such that $\sum_1^k \lambda_i = 1$. So $Dz = \sum_1^k \lambda_i Dz^i \in P$. So $\{Dz : z \in Q\} \subseteq P$. Therefore, $P = \{Dz : z \in Q\}$. ■

Proposition 1.3.15 provides a general way to understand degree sequence polyhedra. A degree sequence polyhedron is a linear transformation of another polyhedron related to edge-sets, where the transformation matrix is the edge-vertex incidence matrix.

Chapter 2

Polyhedron of degree sequences of subgraphs

2.1 Introduction

A natural question in graph theory is, given a vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, whether it is the degree sequence of a simple graph. Erdős and Gallai [13] answered the question by giving a necessary and sufficient condition.

Theorem 2.1.1 *$(x_v : v \in V)$ is the degree sequence of a simple graph with vertex set V if and only if*

- (a) $x(A) - x(B) \leq |A|(|A| - 1) + |A|(|V| - |A| - |B|)$ for all disjoint sets $A, B \subseteq V$;
- (b) $x_v \geq 0$ for all $v \in V$;
- (c) $x \in \mathbb{Z}^V$ and $\sum_{v \in V} x_v$ is even.

Koren [18] proved the following theorem, which provides a simple linear description for the convex hull of the degree sequences of simple graphs.

Theorem 2.1.2 *The convex hull of degree sequence of simple graphs with vertex set V is the set of $x \in \mathbb{R}^V$ satisfying*

- (a) $x(A) - x(B) \leq |A|(|A| - 1) + |A|(|V| - |A| - |B|)$ for all disjoint sets $A, B \subseteq V$;

(b) $x_v \geq 0$ for all $v \in V$.

In this chapter, we study the convex hull, $P(G)$, of degree sequences of subgraphs of graph $G = (V, E)$. Obviously, the above polyhedron is $P(G)$ for a complete graph G . In Section 2, we shall give a linear description of $P(G)$ and characterize its facets. In Section 3, we describe the extreme points of $P(G)$ and their adjacency. In Section 4, we describe a separation algorithm for $P(G)$. In Section 5, we discuss a further generalization.

2.2 A linear description and facets

First we make an easy observation.

Proposition 2.2.1 *The convex hull $Q(G)$ of incidence vectors of edge-sets of subgraphs is $\{z \in \mathbb{R}^E : 0 \leq z_e \leq 1, \text{ for every } e \in E\}$. $P(G) = \{x = Mz \in \mathbb{R}^V : 0 \leq z \leq 1\}$.*

Proof: It is clear that $\{z \in \mathbb{R}^E : 0 \leq z \leq 1\}$ is an integral polytope whose vertices are incidence vectors of subgraphs of G . It is also clear that every subgraph incidence vector is in this polytope. So $Q(G) = \{z \in \mathbb{R}^E : 0 \leq z \leq 1\}$.

For every subgraph degree sequence x , there is a subgraph incidence vector z such that $x = Mz$. From Proposition 1.3.15, $P(G) = \{Mz : z \in Q(G)\}$. The result follows. ■

Proposition 2.2.2 $\dim(P(G)) = |V| - (\text{number of bipartite components})$. In particular,

(a) if G is nonbipartite and connected, then $\dim(P(G)) = |V|$;

(b) if G is connected and bipartite with bipartition $\{V_1, V_2\}$, then $\dim(P(G)) = |V| - 1$ and $x(V_1) - x(V_2) = 0$ for all $x \in P(G)$.

Proof: It is clear that we only need to prove (a) and (b). (a) If G is nonbipartite and connected, then the incidence matrix M of G has $|V|$ linearly independent column vectors, each of which is in $P(G)$. Note that 0 is also in $P(G)$. So $\dim(P(G)) = |V|$.

(b) Let G be bipartite with bipartition $\{V_1, V_2\}$. Then the incidence matrix of G has $|V| - 1$ linearly independent vectors, each of which is in $P(G)$. Since 0 is also in $P(G)$, $\dim(P(G)) \geq |V| - 1$. On the other hand, it is clear that $x(V_1) - x(V_2) = 0$ holds for every x in $P(G)$. So $\dim(P(G)) = |V| - 1$. ■

Theorem 2.2.3 $P(G)$ is the set of all $x \in \mathbb{R}_+^V$ satisfying

$x(A) - x(B) \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$, for all $A, B \subseteq V$ such that $A \cap B = \emptyset$.

Proof: It is easy to see that it is enough to prove the result in the case where G is connected. Let F be a facet of $P(G)$, and let $ax \leq b$ be an integral valid inequality of $P(G)$ inducing F . Let $\max = \max\{a_v : v \in V\}$, $\min = \min\{a_v : v \in V\}$. If $\max = \min > 0$, then it is easy to see that $ax \leq b$ is implied by the inequality $x(V) \leq 2|E|$, from which the result follows. If $\max = \min < 0$, then it is implied by the inequalities $-x_v \leq 0$, $v \in V$, and again the result follows. So we may assume that $\min < \max$. Let $A = \{v \in V : a_v = \max\}$, $B = \{v \in V : a_v = \min\}$. Then $A \cap B = \emptyset$. Let $E_+ = \{vw \in E : a_v + a_w \geq 0\}$. Then $b = \max\{ax : x \in P(G)\} = \sum(a_v + a_w : vw \in E_+)$.

Case 1. $\max + \min = 0$. Then $\max > 0$, $\min < 0$. Let

$$a'_v = \begin{cases} a_v - 1, & \text{if } v \in A; \\ a_v + 1, & \text{if } v \in B; \\ a_v, & \text{else.} \end{cases}$$

Let $b' = \max\{a'x : x \in P(G)\}$, then

$$\begin{aligned} b' &= \sum(a'_v + a'_w : vw \in E_+) \\ &= \sum(a_v + a_w : vw \in \gamma(A)) - 2|\gamma(A)| + \sum(a_v + a_w : vw \in \delta(A) \setminus \delta(B)) \\ &\quad - |\delta(A) \setminus \delta(B)| + \sum(a_v + a_w : vw \in E_+ \setminus (\gamma(A) \cup \delta(A))) \\ &= b - 2|\gamma(A)| - |\delta(A) \setminus \delta(B)|. \end{aligned}$$

So $ax \leq b$ is the sum of $a'x \leq b'$ and $x(A) - x(B) \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$, which are valid for $P(G)$. It follows that $F \subseteq F' := \{x \in P(G) : x(A) - x(B) = 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|\}$.

We claim that $F' \neq P(G)$. If G is nonbipartite, then the claim is true, since $\dim(P(G)) = |V|$. Suppose G is bipartite with partition V_1, V_2 . If the claim is not true, then either $A = V_1, B = V_2$, or $B = V_1, A = V_2$. But each of these implies that F is induced by $x(V_1) - x(V_2) = 0$, a contradiction. So $F' \neq P(G)$, and hence $F = F'$. Therefore F is induced by $x(A) - x(B) \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$.

Case 2. $\max + \min > 0$. Then $\max > 0$. Let

$$a'_v = \begin{cases} a_v - 1, & \text{if } v \in A; \\ a_v, & \text{else.} \end{cases}$$

An argument similar to the one for Case 1, shows that in this case F is induced by $x(A) \leq 2|\gamma(A)| + |\delta(A)|$.

Case 3. $\max + \min < 0$. Then $\min < 0$. Let

$$a'_v = \begin{cases} a_v + 1, & \text{if } v \in B; \\ a_v, & \text{else.} \end{cases}$$

In this case one shows that F is induced by $-x(B) \leq 0$.

The proof is complete. ■

We remark here that Theorem 2.2.3 can also be derived from a deeper result, Theorem 3.1.3 of Chapter 3, of Cunningham and Green-Krótki [5]. But that theorem is considerably harder to prove. Recently, Goemans [16] found a shorter proof of Theorem 2.2.3.

Let $x \in P(G)$ such that x is integral and $\sum(v_v : v \in V(G))$ is even. In the case when G is a complete graph, x is the degree sequence of a subgraph of G . We remark that this is not true in general. More conditions are needed for x to be the degree sequence of a subgraph of G , see [25].

When G is bipartite, $P(G)$ has a simpler linear description.

Theorem 2.2.4 *If G is bipartite with bipartition $\{V_1, V_2\}$, then $P(G)$ is the set of $x \in \mathbb{R}^V$ satisfying*

- (a) $x(A) - x(B) \leq |\delta(A) \setminus \delta(B)|$ for $A \subseteq V_1, B \subseteq V_2$;
- (b) $x(V_1) - x(V_2) = 0$.

Proof: It is clear that the inequalities and the equality are valid.

We only need to prove that any other inequality of the defining system is redundant. We first note that $x(B) - x(A) \leq |\delta(B) \setminus \delta(A)|$ is implied by $x(V_1 \setminus A) - x(V_2 \setminus B) \leq |\delta(B) \setminus \delta(A)|$ and $x(V_1) - x(V_2) = 0$ for $A \subseteq V_1, B \subseteq V_2$. For any inequality $x(A) - x(B) \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$, let $A_1 = A \cap V_1, A_2 = A \cap V_2, B_1 = B \cap V_1$, and $B_2 = B \cap V_2$. Then $\delta(A) = (\delta(A_1) \cup \delta(A_2)) \setminus \gamma(A)$, $\delta(B) = (\delta(B_1) \cup \delta(B_2)) \setminus \gamma(B)$. So $\delta(A) \setminus \delta(B) = (\delta(A_1) \setminus \delta(B_2)) \cup (\delta(A_2) \setminus \delta(B_1)) \setminus \gamma(A)$. Thus $x(A) - x(B) = (x(A_1) - x(B_2)) + (x(A_2) - x(B_1)) \leq |\delta(A_1) \setminus \delta(B_2)| + |\delta(A_2) \setminus \delta(B_1)| \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$, as required. ■

Now we characterize the facets of $P(G)$. The results generalize a theorem for complete graph G , due to Peled and Srinivasan [21].

Theorem 2.2.5 *Suppose that G is nonbipartite and connected. Then $x(A) - x(B) \leq 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|$ defines a facet if and only if*

- (a) $G[V \setminus (A \cup B)]$ has no bipartite components;
- (b) $G[A \cup B] - \gamma(A) - \gamma(B)$ is connected.

Proof: Let $F = \{x \in P(G) : x(A) - x(B) = 2|\gamma(A)| + |\delta(A) \setminus \delta(B)|\}$. Let $E_1 = \gamma(A) \cup (\delta(A) \setminus \delta(B))$, and let χ^{E_1} be the incidence vector of E_1 . Let $E_0 = \gamma(B) \cup (\delta(B) \setminus \delta(A))$. Then for every $x = Mz \in F$, $z_e = 1$ for all $e \in E_1$ and $z_e = 0$ for all $e \in E_0$. So $F = \{x + M\chi^{E_1} : x \in P(G - \gamma(A) - \gamma(B) - \delta(V \setminus (A \cup B)))\}$.

$$\begin{aligned} \dim(F) &= \dim(P(G - \gamma(A) - \gamma(B) - \delta(V \setminus (A \cup B)))) \\ &= \dim(P(G[V \setminus (A \cup B)])) + \dim(P(G[A \cup B] - \gamma(A) - \gamma(B))) \end{aligned}$$

$$\begin{aligned}
&= |V \setminus (A \cup B)| - (\text{number of bipartite components of } G[V \setminus (A \cup B)]) + \\
&\quad |A \cup B| - (\text{number of bipartite components of } G[A \cup B] - \gamma(A) - \gamma(B)).
\end{aligned}$$

Since $G[A \cup B] - \gamma(A) - \gamma(B)$ is bipartite, $\dim(F) = |V| - 1$ if and only if (2.2.5a) and (2.2.5b) hold. ■

Theorem 2.2.6 *Suppose that G is connected and bipartite with bipartition $\{V_1, V_2\}$. Then $x(A) - x(B) \leq |\delta(A) \setminus \delta(B)|$, where $A \subseteq V_1, B \subseteq V_2$, defines a facet if and only if*

- (a) $V \neq A \cup B$;
- (b) $G[V \setminus (A \cup B)]$ is connected;
- (c) $G[A \cup B]$ is connected.

Proof: Let $F = \{x \in P(G) : x(A) - x(B) = |\delta(A) \setminus \delta(B)|\}$. Let $E_1 = \delta(A) \setminus \delta(B)$, and let χ^{E_1} be the incidence vector of E_1 . Then $F = \{x + M\chi^{E_1} : x \in P(G - \delta(V \setminus (A \cup B)))\}$, as in the proof of Theorem 2.2.5.

$$\begin{aligned}
\dim(F) &= \dim(P(G - \delta(V \setminus (A \cup B)))) \\
&= \dim(P(G[V \setminus (A \cup B)])) + \dim(P(G[A \cup B])) \\
&= |V \setminus (A \cup B)| - (\text{number of components of } G[V \setminus (A \cup B)]) \\
&\quad + |A \cup B| - (\text{number of components of } G[A \cup B]).
\end{aligned}$$

Since $G[A \cup B]$ is bipartite, $\dim(F) = |V| - 2$ if and only if (2.2.6a), (2.2.6b), and (2.2.6c) hold. ■

2.3 Vertices and adjacency

In this section, we give characterizations of vertices of $P(G)$ and of adjacency of pairs of such vertices. Several of these results generalize earlier ones due to Koren [18] and Peled and Srinivasan [21] for the case where G is a complete graph.

Vertices

For any $c \in \mathbf{R}^V$, let $E_c^+ = \{uv \in E : c_u + c_v > 0\}$, $E_c^0 = \{uv \in E : c_u + c_v = 0\}$, and $E_c^- = \{uv \in E : c_u + c_v < 0\}$.

Proposition 2.3.1 (a) *If there exists $c \in \mathbf{R}^V$ such that $f = M\chi^{E_c^+}$ and $E_c^0 = \emptyset$, then f is a vertex of $P(G)$.*

(b) *For any $c \in \mathbf{R}^V$, $f = M\chi^{E_c^+}$ is a vertex of $P(G)$.*

(c) *For any $c \in \mathbf{R}^V$, $f = M\chi^{E \setminus E_c^-}$ is a vertex of $P(G)$.*

Proof: (a) Clearly, $\chi^{E_c^+}$ is the unique optimal solution of $\max(cMz : 0 \leq z \leq 1)$, so f is the unique solution of $\max(cx : x \in P(G))$, so f is a vertex of $P(G)$.

(b) Choose $\varepsilon > 0$ such that $\varepsilon < \min\{|c_u + c_v| : uv \in E \setminus E_c^0\}$. Let

$$c'_v = \begin{cases} c_v & \text{if } c_v > 0; \\ c_v - \varepsilon & \text{if } c_v \leq 0. \end{cases}$$

Then $f = M\chi^{E_{c'}^+}$ and $E_{c'}^0 = \emptyset$. So f is a vertex of $P(G)$ by (a).

(c) The proof is similar to that of (b). ■

Theorem 2.3.2 *f is a vertex of $P(G)$ if and only if there exists $E' \subseteq E$ such that $\chi^{E'}$ is the unique solution of*

$$(2.1) \quad \begin{cases} Mz = f, \\ 0 \leq z_e \leq 1, \forall e \in E. \end{cases}$$

Proof: First suppose that $\chi^{E'}$ is the unique solution of (2.1). Consider the linear programming problem

$$\begin{aligned} & \min(z(E') - z(E \setminus E') : \text{subject to (2.1)}) \\ &= \max\left(\sum_{v \in V} f_v y_v - z'(E) : y_u + y_v - z'_{uv} \leq 1, \forall uv \in E'; \right. \\ & \quad \left. y_u + y_v - z'_{uv} \leq -1, \forall uv \in E \setminus E'; z'_{uv} \geq 0, \forall uv \in E\right) \end{aligned}$$

Let (y^*, z^*) be an optimal solution of the maximum problem. By the complementary slackness conditions,

for every $uv \in E'$, $y_u^* + y_v^* - z_{uv}^* = 1$, and so $y_u^* + y_v^* \geq 1$;

for every $uv \notin E'$, $z_{uv}^* = 0$, and so $y_u^* + y_v^* \leq -1$.

Let $c = y^*$, then $E_c^+ = E'$ and $E_c^0 = \emptyset$. By Proposition 2.3.1, $f = M\chi^{E'}$ is a vertex of $P(G)$.

Now suppose that f is a vertex of $P(G)$. Then $f = M\chi^{E'}$ for some $E' \subseteq E$. If there exists a different solution z of (2.1), then there exists $e' \in E'$ such that $z_{e'} < 1$, for otherwise $z(\delta(v)) > f_v$ for some $v \in V$, a contradiction. Let $\varepsilon = 1 - z_{e'}$, let

$$z'_e = \begin{cases} 1 - \varepsilon & \text{if } e = e'; \\ \chi_e^{E'} & \text{if } e \neq e'; \end{cases}$$

and let

$$z''_e = \begin{cases} z_e + \varepsilon & \text{if } e = e'; \\ z_e & \text{if } e \neq e'. \end{cases}$$

Let $f' = Mz'$ and $f'' = Mz''$. Then $f' \neq f''$ since $f'(V) = f(V) - 2\varepsilon$ and $f''(V) = f(V) + 2\varepsilon$. Now $f = (f' + f'')/2$, so f is not a vertex, a contradiction. So $\chi^{E'}$ is the unique solution of (2.1). ■

Let E_1 be a subset of E , and let $\overline{E_1}$ denote $E \setminus E_1$. An E_1 -alternating walk W is a sequence $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ such that $e_i = v_{i-1}v_i$ for $i = 1, \dots, k$, $\{e_{2i-1} : 1 \leq i \leq \lceil k/2 \rceil\} \subseteq E_1$, and $\{e_{2i} : 1 \leq i \leq \lfloor k/2 \rfloor\} \subseteq \overline{E_1}$. It allows repeated edges. We call a walk $W: v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ alternating if it is either an E_1 -alternating walk or a $\overline{E_1}$ -alternating walk. W is even [odd] if k is even [odd], and W is closed if $v_0 = v_k$. For example, in Figure 2.1 there are two even closed E_1 -alternating walks, where the wiggly edges are in E_1 and the straight edges are in $\overline{E_1}$. The first one is $W_1 = v_0, e_1, v_1, \dots, v_5, e_6, v_6$, where $v_0 = v_6$. The second one is $W_2 = v_0, e_1, v_1, \dots, v_9, e_{10}, v_{10}$, where $v_0 = v_{10}$, $v_2 = v_9$, $v_3 = v_8$, $v_4 = v_7$, $e_3 = e_9$, and $e_4 = e_8$.

Now we prove a result for the existence of an even closed E_1 -alternating walk. The argument is quite standard.

There is another way to prove the result above. We sketch the proof as follows. Construct a bipartite directed graph $G' = (V_1 \cup V_2, E')$ from G . For each $v \in V$, make two copies v_1, v_2 in V_1, V_2 respectively; for each $vw \in E_1$ make two edges (v_1, w_2) and (w_1, v_2) in E' , and for each $vw \in E \setminus E_1$ make two edges (v_2, w_1) and (w_2, v_1) in E' . Then the new graph G' has the property that $\delta_{G'}^+(v) \geq 1$ and $\delta_{G'}^-(v) \geq 1$ for every $v \in V_1 \cup V_2$. It is well known that a graph with such a property has a directed cycle. So G' has a directed cycle. This cycle exactly corresponds to an even closed E_1 -alternating walk in G .

Proposition 2.3.4 *Let d_v , for $v \in V$, denote the degree of v in G . For $f \in \mathbb{R}_+^V$, let $S_0 = \{v \in V : f_v = 0\}$, $S_1 = \{v \in V : f_v = d_v\}$, and $S = V \setminus (S_0 \cup S_1)$. For $v \in S$, let d'_v denote $d_v - |\{\{v\}, S_1\}|$ and f'_v denote $f_v - |\{\{v\}, S_1\}|$. Then f is a vertex of $P(G)$ if and only if f' is a vertex of $P(G[S])$.*

Proof: For any solution of (2.1), it is clear that for every $v \in S_0$, $z_e = 0$ for every $e \in \delta(v)$, and for every $v \in S_1$, $z_e = 1$ for every $e \in \delta(v)$. So (2.1) has a unique solution χ^{E_1} if and only if $\chi^{E_1 \cap \gamma(S)}$ is the unique solution of

$$(2.2) \quad \begin{cases} M'z = f' \\ 0 \leq z_e \leq 1 \quad \forall e \in \gamma(S), \end{cases}$$

where M' is the incidence matrix of $G[S]$. By Theorem 2.3.2, f is a vertex of $P(G)$ if and only if f' is a vertex of $P(G[S])$. ■

Theorem 2.3.5 *$f = M\chi^{E_1}$ is a vertex of $P(G)$ if and only if there is no even closed E_1 -alternating walk.*

Proof: Suppose that there exists an even closed E_1 -alternating walk W : $v_0, e_1, v_1, \dots, v_{2k-1}, e_{2k}, v_{2k}$. Let $e(W)$ denote the number of times that e appears in W . Let

$$z'_e = \begin{cases} 1 - e(W) & \text{if } e \in \{e_{2i-1} : i = 1, \dots, k\}; \\ e(W) & \text{if } e \in \{e_{2i} : i = 1, \dots, k\}; \\ \chi_e^{E_1} & \text{otherwise,} \end{cases}$$

for a sufficiently small positive number ε . Then z' is another solution of (2.1), so f is not a vertex of $P(G)$ by Theorem 2.3.2.

Now suppose that there exists no even closed E_1 -alternating walk. Then by Lemma 2.3.3, we have $S \neq V$ in Proposition 2.3.4. Therefore, f is a vertex of $P(G)$ if and only if f' is a vertex of $P(G[S])$. But clearly, there is no $(E_1 \cap \gamma(S))$ -alternating walk in $G[S]$, so f is a vertex of $P(G)$ by induction. ■

Now we derive related results for complete graphs. A useful observation is: if G is a complete graph, then G has an even closed E_1 -alternating walk if and only if it has a closed E_1 -alternating walk (cycle) of length 4. This is proved in Proposition 2.3.6.

Figure 2.2: Forbidden induced subgraphs

A graph $G_1 = (V, E_1)$ is **threshold** if G_1 has none of the induced subgraphs shown in Figure 2.2. If G is the complete graph with vertex set V , then G_1 is threshold if and only if there is no closed E_1 -alternating cycle of length 4 in G .

Proposition 2.3.6 *Suppose that G is the complete graph with vertex set V . $G_1 = (V, E_1)$ is a threshold graph if and only if G has no even closed E_1 -alternating walk.*

Proof: If G has no even closed E_1 -alternating walk, then G_1 has no closed E_1 -alternating cycle of length 4 in G . So G_1 is a threshold subgraph.

Suppose that G has an even closed E_1 -alternating walk. We claim that G has a closed E_1 -alternating walk with length 4. If the claim is false, then the shortest even closed E_1 -alternating walk $W = v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ has length greater than 4. Then there is r such that $v_r \neq v_{r+3}$. Without loss of generality, we assume $r = 0$. If the edge $e = v_0v_3 \in E_1$, then we have a shorter walk W' by replacing $v_0, e_1, v_1, e_2, v_2, e_3, v_3$ with v_0, e, v_3 in W , a contradiction. If the edge $e = v_0v_3 \notin E_1$, then we have a shorter

walk $W' = v_0, e_1, v_1, e_2, v_2, e_3, v_3, e, v_0$, a contradiction. So the claim is true. Thus G_1 is not a threshold subgraph. ■

Theorem 2.3.7 *Suppose that G is complete. $f = M\chi^{E_1}$ is a vertex of $P(G)$ if and only if $G_1 = (V, E_1)$ is a threshold subgraph.*

Proof: $f = M\chi^{E_1}$ is a vertex of $P(G)$ if and only if G has no even closed E_1 -alternating walk from Theorem 2.3.5. G is a complete graph. So G has no even closed E_1 -alternating walk if and only if (V, E_1) is a threshold subgraph by Proposition 2.3.6. So the theorem is true. ■

Theorem 2.3.8 *Suppose that G is complete. $f = M\chi^{E_1}$ is a vertex of $P(G)$ if and only if χ^{E_1} is the unique integral solution of (2.1).*

Proof: The “only if” part easily follows from Theorem 2.3.2.

If $f = M\chi^{E_1}$ is not a vertex of $P(G)$, then G_1 is not a threshold graph. So there is a E_1 -alternating cycle of length 4 in G . Then χ^{E_1} is not a unique integral solution of (2.1) since switching the edges of the cycle will result in a different integral solution. This proves the “if” part. ■

Theorem 2.3.7 is due to Peled and Srinivasan [21], and Theorem 2.3.8 is due to Koren [18].

Adjacency

Now we discuss the adjacency of the polyhedron. The results generalize two theorems of Peled and Srinivasan [21] for complete graphs.

Theorem 2.3.9 *Two vertices $f = M\chi^{E_1}, g = M\chi^{E_2}$ of $P(G)$ are adjacent if and only if $|E_1 \Delta E_2| = 1$.*

Proof: Suppose that f and g are adjacent. Then there exists $c \in \mathbb{R}^V$ such that f and g are the only two vertices of $P(G)$ maximizing cx over $P(G)$. For every $uv \in E_1 \cup E_2$, $c_u + c_v \geq 0$, for otherwise one of f and g is not optimal. For every $uv \in E_1 \Delta E_2$, $c_u + c_v = 0$, for otherwise uv is in both E_1 and E_2 . For every $uv \notin E_1 \cup E_2$, $c_u + c_v < 0$, for otherwise $E_1 \cup E_2 \subset E \setminus E_c^-$, and by Proposition 2.3.1, $M_{\chi^{E \setminus E_c^-}}$ is a vertex of $P(G)$ that also maximizes cx .

Therefore, $E_1 \cup E_2$ must be either E_1 or E_2 , since $E \setminus E_c^- = E_1 \cup E_2$ and $M_{\chi^{E \setminus E_c^-}}$ is a vertex of $P(G)$ maximizing cx . Without loss of generality, we assume $E_1 = E_1 \cup E_2$. Let $uv \in E_1 \setminus E_2$, and let $E' = E_1 \setminus \{uv\}$. If $M_{\chi^{E'}}$ is not a vertex of $P(G)$, then there exists an even closed E' -alternating walk $W : v_0, e_1, v_1, \dots, v_{2k-1}, e_{2k}, v_{2k}$ by Theorem 2.3.5, where $\{e_{2i-1} : i = 1, \dots, k\} \subseteq E'$, $\{e_{2i} : i = 1, \dots, k\} \subseteq E \setminus E'$. So for $i = 1, \dots, k$, $c_{v_{2i-2}} + c_{v_{2i-1}} \geq 0$; $c_{v_{2i-1}} + c_{v_{2i}} < 0$, if $e_{2i} \neq uv$; $c_{v_{2i-1}} + c_{v_{2i}} = 0$, if $e_{2i} = uv$. But the first relation implies that $\sum_{i=0}^{2k} c_{v_i}$ is non-negative and the latter two relations imply that it is negative, a contradiction. So $M_{\chi^{E'}}$ is a vertex of $P(G)$. Since it maximizes cx , and is different from f , it follows that $M_{\chi^{E'}} = g$. So $E_2 = E'$ and $|E_1 \Delta E_2| = 1$, as required.

Now suppose that $|E_1 \Delta E_2| = 1$. Say $E_2 = E_1 \setminus \{uv\}$. Since f and g are vertices of $P(G)$, there exist $c, c' \in \mathbb{R}^V$ such that $E_1 = E_c^+$, $E_c^0 = \emptyset$ and $E_2 = E_{c'}^+$, $E_{c'}^0 = \emptyset$. Let $k = -(c_u + c_v)/(c'_u + c'_v)$. Notice that $k > 0$. Let $\hat{c} = c + kc'$. Then $E_{\hat{c}}^+ = E_2$, $E_{\hat{c}}^- = E \setminus E_1$. This implies that f and g are the only two vertices of $P(G)$ maximizing $\hat{c}x$. So f and g are adjacent. ■

Our second criterion for two vertices of $P(G)$ to be adjacent has the advantage that it does not depend on the realization of the vertices. For any $v \in V$, let $\chi^v \in \mathbb{R}^V$ denote the incidence vector of $\{v\}$.

Theorem 2.3.10 *Vertices f, g of $P(G)$ are adjacent if and only if $f - g = \pm(\chi^u + \chi^w)$ for some $uw \in E$.*

Proof: Since f and g are vertices of $P(G)$, $f = M_{\chi^{E_1}}$, $g = M_{\chi^{E_2}}$ for some $E_1, E_2 \subseteq E$.

Suppose f and g are adjacent. Then $|E_1 \Delta E_2| = 1$ by Theorem 2.3.9. So $f - g = \pm(\chi^u + \chi^w)$ for some $uw \in E_1 \Delta E_2 \subseteq E$.

Now suppose $f - g = (\chi^u + \chi^w)$ for some $uw \in E$. If $uw \notin E_1$, let $S' = \{v \in V : |\delta(v) \cap (E_1 \Delta E_2)| \geq 1\}$. $S' \neq \emptyset$ since $u, w \in S'$. Consider $G[S']$. Notice that for every $v \in S'$, $|\delta(v) \cap (E_1 \cap \gamma(S'))| \geq 1$, and $|\delta(v) \cap (\gamma(S') \setminus E_1)| \geq 1$. By Lemma 2.3.3, there exists an even closed E_1 -alternating walk in $G[S']$. So f is not a vertex of $P(G)$ by Theorem 2.3.5, a contradiction. Thus $uw \in E_1$.

If $uw \in E_1 \cap E_2$, let $S' = \{v \in V : |\delta(v) \cap (E_1 \Delta E_2)| \geq 1\}$. Then $u, w \in S'$. Consider $G[S']$. For every $v \in S'$, $|\delta(v) \cap (E_2 \cap \gamma(S'))| \geq 1$, and $|\delta(v) \cap (\gamma(S') \setminus E_2)| \geq 1$. So there is an even closed E_2 -alternating walk and thus g is not a vertex of $P(G)$, a contradiction. So $uw \in E_1 \setminus E_2$.

Now let $E' = E_1 \setminus \{uw\}$ and let $S' = \{v \in V : |\delta(v) \cap (E' \Delta E_2)| \geq 1\}$. If $|S'| \neq 0$, then for the same reason as in the preceding paragraph, g is not a vertex of $P(G)$, a contradiction. So $|S'| = 0$, which implies $|E' \Delta E_2| = 0$ and thus $|E_1 \Delta E_2| = 1$. Therefore by Theorem 2.3.9, f and g are adjacent. ■

2.4 The separation problem

The separation problem for $P(G)$ is: given $x \in \mathbb{R}^V$, either prove that $x \in P(G)$, or find a violated inequality which is valid for $P(G)$. We describe an algorithm to solve this problem.

We begin by repeating the observation of [21] on the separation problem for the polytope $P(G)$, when G is complete. It is easy to see that for a most-violated non-trivial inequality (Theorem 2.1.2), A can be chosen to consist of the vertices having the $|A|$ largest values of x_v , and B can be chosen to consist of the vertices having the $|B|$ smallest values of x_v . Hence, once the vertices are sorted in order of their x_v values, there is a linear-time algorithm for the separation problem.

In the more general situation where G is arbitrary, we can solve the separation problem for $P(G)$ by a network flow method. We construct a flow network G' with

vertex-set $V^1 \cup V^2 \cup \{r, s\}$, where $V^i = \{v^i : v \in V\}$, for $i = 1, 2$. That is, we make two copies of each vertex of G and add a source and a sink. For each edge uv of G , there are directed edges (u^1, v^2) and (v^1, u^2) of G' , each having capacity 1. For each vertex v of G there are directed edges (r, v^1) and (v^2, s) of G' , each having capacity x_v . For example, see Figure 2.3. For a subset A of V and $i = 1$ or 2 , we use A^i to

Figure 2.3: Graph G and G'

denote $\{v^i : v \in A\}$. Suppose that we compute a minimum-capacity $r - s$ cut $\delta'(C)$ in G' . Then $C = \{r\} \cup A^1 \cup B^2$ for sets $A, B \subseteq V$. The capacity of this cut is

$$x(V \setminus A) + x(B) + 2|\gamma(A \setminus B)| + |(A \cap B, V \setminus B)| + |(A \setminus B, V \setminus (A \cup B))|.$$

If we delete $A \cap B$ from both A and B , then this quantity does not increase; in fact, it decreases by precisely $|(A \cap B, V \setminus (A \cup B))|$. That is, we can compute such a cut for which A and B are disjoint, and the cut capacity is then

$$x(V \setminus A) + x(B) + 2|\gamma(A)| + |(A, V \setminus (A \cup B))|.$$

But this means that we can minimize $2|\gamma(A)| + |(A, V \setminus (A \cup B))| - x(A) + x(B)$ over disjoint subsets A, B of V , and hence find an inequality most-violated by x , if one exists, by solving a single minimum-cut problem. We remark that this construction leads to another proof of Theorem 2.2.3. Namely, if no inequality is violated, then by the max-flow min-cut theorem, there is a feasible flow of amount $x(V)$ in G' . If we define z_{uv} to be the average of the flow values on (u^1, v^2) and (v^1, u^2) , it easy to check that $Mz = x$ and that $0 \leq z \leq 1$, so $x \in P(G)$.

2.5 A further generalization

We consider a mixed graph $G = (V, E \cup \vec{E})$, where V is the vertex set, E is the undirected edge set, and $\vec{E}$ is the directed edge set. For every edge $e \in E \cup \vec{E}$, there is a positive number, u_e , as its upper bound. Let M denote the edge-vertex incidence matrix of G . Let U denote the diagonal matrix with diagonal u . Let $P(G, u) = \{x \in \mathbb{R}^V : \text{there is a } z \in \mathbb{R}^{E \cup \vec{E}} \text{ such that } x = Mz \text{ and } 0 \leq z \leq u\}$. It is clear that in the special case where G is undirected and $u = 1$, $P(G, u)$ is just $P(G)$ of previous sections. Moreover, if G is directed, then $x \in P(G, u)$ if and only if there is a feasible flow satisfying x as demand vector and u as capacity vector. In this section, we give a linear description for $P(G, u)$, and characterize its facets.

For the rest of this subsection, we define some notations for mixed graphs.

For $S_1, S_2 \subseteq V$, let $[S_1, S_2] = \{e \in E : e \text{ has one end in } S_1 \text{ and the other end in } S_2\}$, and let $(S_1, S_2) = \{e \in \vec{E} : e \text{ has tail end in } S_1 \text{ and the head end in } S_2\}$. For subset $S \subseteq V$, let $\delta(S) = \{e \in E \cup \vec{E} : e \text{ has exactly one end in } S\}$. Let $\delta^-(S) = (V \setminus S, S) \cup [V \setminus S, S]$, and let $\delta^+(S) = (S, V \setminus S)$. Let $\gamma(S) = (S, S) \cup [S, S]$, and let $\gamma^+(S) = [S, S]$. The underlying graph of G is the undirected graph obtained by replacing every directed edge (v, w) with an edge vw . G is connected if its underlying graph is connected. A graph $G = (V, E \cup \vec{E})$ is weakly bipartite if $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$, $E = [V_1, V_2]$ and $\vec{E} = \gamma(V_1) \cup \gamma(V_2)$. Both directed graphs and undirected bipartite graphs are weakly bipartite graphs.

Linear description

Proposition 2.5.1 *G is weakly bipartite if and only if G has no cycle containing an odd number of edges of E .*

Proof: It is clear that a weakly bipartite mixed graph G has no such cycle. Now suppose that G is a mixed graph having no such cycle. We define the relation “ $\sim$ ” on V by:

$v \sim w$ if there is a simple path, L_{vw} , from v to w such that L_{vw} contains an even number of edges of E . It is clear that " $\sim$ " is an equivalence relation. V is divided into two equivalence classes V_1, V_2 . It is clear that $\vec{E} \subseteq \gamma(V_1) \cup \gamma(V_2)$. If there exists an edge $vw \in E \cap (\gamma(V_1) \cup \gamma(V_2))$, say $vw \in E \cap \gamma(V_1)$, then there is a path L_{vw} containing an even number of edges of E . But this path together with vw is a cycle with an odd number of edges of E , a contradiction. So $[V_1, V_2] = E$ and $\vec{E} = (\gamma(V_1) \cup \gamma(V_2))$. ■

Proposition 2.5.2 $\dim(P(G, u)) = |V| - (\text{number of weakly bipartite components})$.
In particular,

- (a) if G is connected and not weakly bipartite, then $\dim(P(G, u)) = |V|$;
- (b) if G is connected and weakly bipartite with partition $\{V_1, V_2\}$, then $\dim(P(G, u)) = |V| - 1$ and $x(V_1) - x(V_2) = 0$ for all $x \in P(G, u)$.

Proof: We only need to prove (a) and (b). (a) Suppose that G is not weakly bipartite. Then by Proposition 2.5.1, G has a cycle containing an odd number of edges of E . Since G is connected, there is a connected spanning subgraph G_1 such that it contains that cycle and has $|V|$ edges. It is easy to check that the corresponding column vectors of M are linearly independent. So MU has $|V|$ linearly independent column vectors, each of which is in $P(G, u)$. Note that 0 is also in $P(G, u)$. So $P(G, u)$ has $|V| + 1$ affinely independent vectors, so $\dim(P(G, u)) = |V|$.

(b) Suppose that G is weakly bipartite with bipartition $\{V_1, V_2\}$. Since G is connected, the incidence matrix M of G , thus MU , has $|V| - 1$ linearly independent vectors. Since 0 is also in $P(G, u)$, $\dim(P(G, u)) \geq |V| - 1$. On the other hand, it is clear that $x(V_1) - x(V_2) = 0$ holds for every x in $P(G, u)$. So $\dim(P(G, u)) = |V| - 1$. ■

Here is the main theorem. The proof is similar to the proof of Theorem 2.2.3.

Theorem 2.5.3 $P(G, u) = \{x \in \mathbb{R}^V : x(A) - x(B) \leq 2u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B)), \text{ for every } A, B \subseteq V \text{ such that } A \cap B = \emptyset\}$.

Proof: It is clear we only need to prove the result in the case where G is connected. Let F be a facet of $P(G, u)$, and let $ax \leq b$ be an integral valid inequality of $P(G, u)$ inducing F . Let $\max = \max\{a_v : v \in V\}$, $\min = \min\{a_v : v \in V\}$. Let $A = \{v \in V : a_v = \max\}$, $B = \{v \in V : a_v = \min\}$. If $A \cap B \neq \emptyset$, then $A = B = V$. In this case, it is easy to see that F is induced by an inequality in the theorem. Suppose $A \cap B = \emptyset$. Let $E_+ = \{vw \in \gamma^+(V) : a_v + a_w \geq 0\}$, and let $\vec{E}_+ = \{(v, w) \in \vec{E} : a_w - a_v \geq 0\}$. Then $b = \max\{ax : x \in P(G, u)\} = \sum((a_v + a_w)u_{vw} : vw \in E_+) + \sum((a_w - a_v)u_{(v,w)} : (v, w) \in \vec{E}_+)$.

Case 1. $\max + \min = 0$. Then $\max > 0$, $\min < 0$. Let

$$a'_v = \begin{cases} a_v - 1, & \text{if } v \in A, \\ a_v + 1, & \text{if } v \in B, \\ a_v, & \text{if } v \notin A \cup B, \end{cases}$$

and let $b' = \max\{a'x : x \in P(G, u)\}$. Then

$$\begin{aligned} b' &= \sum((a'_v + a'_w)u_{vw} : vw \in E_+) + \sum((a'_w - a'_v)u_{(v,w)} : (v, w) \in \vec{E}_+) \\ &= \sum((a_v + a_w)u_{vw} : vw \in \gamma^+(A)) - 2u(\gamma^+(A)) \\ &\quad + \sum((a_v + a_w)u_{vw} : vw \in \delta^-(A) \setminus \delta^-(B)) \\ &\quad + \sum((a_w - a_v)u_{(v,w)} : (v, w) \in \delta^-(A) \setminus \delta^-(B)) - u(\delta^-(A) \setminus \delta^-(B)) \\ &\quad + \sum((a_w - a_v)u_{(v,w)} : (v, w) \in \delta^+(B)) - u(\delta^+(B)) \\ &\quad + \sum(a_v + a_w : vw \in E_+ \setminus (\gamma^+(A) \cup \delta^-(A))) \\ &\quad + \sum((a_w - a_v)u_{(v,w)} : (v, w) \in \vec{E}_+ \setminus (\delta^-(A) \cup \delta^-(B))) \\ &= b - 2u(\gamma^+(A)) - u(\delta^-(A) \setminus \delta^-(B)) - u(\delta^+(B)). \end{aligned}$$

So $ax \leq b$ is the sum of $a'x \leq b'$ and $x(A) - x(B) \leq 2u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))$, which are valid for $P(G, u)$. So $F \subseteq F' := \{x \in P(G, u) : x(A) - x(B) = 2u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))\}$.

We claim that $F' \neq P(G, u)$. If G is not weakly bipartite, then the claim is true since $\dim(P(G, u)) = |V|$. Suppose G is weakly bipartite with associated partition $\{V_1, V_2\}$. If the claim is not true, then either $A = V_1, B = V_2$, or $B = V_1, A = V_2$. But each of these implies that F is induced by $x(V_1) - x(V_2) = 0$, a contradiction. So $F' \neq P(G, u)$, and hence $F = F'$. Therefore, F is induced by the inequality $x(A) - x(B) \leq 2u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))$.

Case 2. $\max + \min > 0$. Then $\max > 0$. Let

$$a'_v = \begin{cases} a_v - 1, & \text{if } v \in A; \\ a_v, & \text{else.} \end{cases}$$

By a similar argument as in Case 1, we can show that F is induced by the inequality $x(A) \leq 2u(\gamma^+(A)) + u(\delta^-(A))$.

Case 3. $\max + \min < 0$. Then $\min < 0$. Let

$$a'_v = \begin{cases} a_v + 1, & \text{if } v \in B; \\ a_v, & \text{else.} \end{cases}$$

By a similar argument as in Case 1, we can show that F is induced by the inequality $-x(B) \leq u(\delta^+(B))$.

This completes the proof. ■

Now we use Theorem 2.5.3 to prove Gale's Theorem [15], which is equivalent to the max-flow min-cut theorem.

Theorem 2.5.4 *If G is a directed graph, then $P(G, u) = \{x \in \mathbb{R}^V : -x(B) \leq u(\delta^+(B)), \text{ for all } B \subseteq V; x(V) = 0\}$.*

Proof: Since $E = \emptyset$, a facet-inducing inequality has the form $x(A) - x(B) \leq u(\delta^-(A)) + u(\delta^+(B))$, which is the sum of two valid inequalities: $x(A) \leq u(\delta^-(A))$ and $-x(B) \leq u(\delta^+(B))$. Note that $x(V) = 0$ is implied by these two inequalities: let $A = V$, then we have $x(V) \leq 0$; let $B = V$, then we have $x(V) \geq 0$. And $x(A) \leq u(\delta^-(A))$ is equivalent to $-x(V \setminus A) \leq u(\delta^+(V \setminus A))$. The theorem is proved.

■

Now we describe a relation between a weakly bipartite graph G and an associated directed graph G' . For each vertex $v \in V_1$, multiply its corresponding row of M by -1 . Then the resulting matrix M' is the incidence matrix of a directed graph G' . G' can be obtained from G as follows: reverse the direction of each edge of $\gamma(V_1)$ and assign each undirected edge a direction with tail end in V_1 . For example, see Figure 2.4. For each $x \in \mathbb{R}^V$, let

$$x'_v = \begin{cases} -x_v, & \text{if } v \in V_1; \\ x_v, & \text{if } v \in V_2. \end{cases}$$

Figure 2.4: A weakly bipartite graph G and the directed graph G'

Proposition 2.5.5 x is in $P(G, u)$ if and only if x' is in $P(G', u)$.

Theorem 2.5.6 If G is weakly bipartite with bipartition $\{V_1, V_2\}$, then $P(G, u) = \{x \in \mathbb{R}^V : x(A) - x(B) \leq u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B)) \text{ for all } A \subseteq V_1, B \subseteq V_2, x(V_1) - x(V_2) = 0\}$

Proof: We note that the inequalities are valid.

From Gale's theorem, $x' \in P(G', u)$ if and only if $-x'(S) \leq u(\delta_{G'}^-(S))$ for every $S \subseteq V$, that is, $x(S \cup V_1) - x(S \cup V_2) \leq u(\delta^-(S \cup V_1) + u(\delta^+(S \cup V_2))$ for every $S \subseteq V$. The latter inequality is a special case of the general inequality, where $A = S \cup V_1$, $B = S \cup V_2$. By Proposition 2.5.5, the theorem is true. ■

Facets

Theorem 2.5.7 *Suppose that G is connected and not weakly bipartite. Then $x(A) - x(B) \leq u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))$ defines a facet of $P(G, u)$ if and only if*

- (a) $G[V \setminus (A \cup B)]$ has no weakly bipartite components;
- (b) $G[A \cup B] - \gamma^+(A) - \gamma^+(B) - \delta^+(A) - \delta^+(B)$ is connected.

Proof: Let $C = V \setminus (A \cup B)$. Let $E_1 = \gamma^+(A) \cup \delta^+(B) \cup (\delta^-(A) \setminus \delta^-(B))$, and let χ^{E_1} be the incidence vector of E_1 . Let $E_0 = \gamma^+(B) \cup \delta^+(A) \cup (\delta^-(B) \setminus \delta^-(A))$. Let $F = \{x \in P(G, u) : x(A) - x(B) = 2u(\gamma^+(A)) + u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))\}$. Then for every $x = Mz \in F$, $z_e = u_e$ for all $e \in E_1$ and $z_e = 0$ for all $e \in E_0$. So $F = \{x + MU\chi^{E_1} : x \in P(G - \gamma^+(A) - \gamma^+(B) - \delta(C) - \delta^+(A) - \delta^+(B))\}$.

$$\begin{aligned}
 \dim(F) &= \dim(P(G - \gamma^+(A) - \gamma^+(B) - \delta(C) - \delta^+(A) - \delta^+(B))) \\
 &= \dim(P(G[C])) + \dim(P(G[A \cup B] - \gamma^+(A) - \gamma^+(B) - \delta^+(A) - \delta^+(B))) \\
 &= |V \setminus (A \cup B)| - (\text{number of weakly bipartite components of } G[C]) \\
 &\quad + |A \cup B| - (\text{number of weakly bipartite components of } \\
 &\quad G[A \cup B] - \gamma^+(A) - \gamma^+(B) - \delta^+(A) - \delta^+(B)).
 \end{aligned}$$

Since $G[A \cup B] - \gamma^+(A) - \gamma^+(B) - \delta^+(A) - \delta^+(B)$ is weakly bipartite, $\dim(F) = |V| - 1$ if and only if (2.5.7a) and (2.5.7b) hold. ■

Theorem 2.5.8 *Suppose that G is connected and weakly bipartite with bipartition $\{V_1, V_2\}$. Then $x(A) - x(B) \leq u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))$, where $A \subseteq V_1$ and $B \subseteq V_2$, defines a facet if and only if*

- (a) $A \cup B \neq V$;
- (b) $G[V \setminus (A \cup B)]$ is connected;

(c) $G[A \cup B]$ is connected.

Proof: Let $C = V \setminus (A \cup B)$. Let $E_1 = \delta^+(B) \cup (\delta^-(A) \setminus \delta^-(B))$, and let χ^{E_1} be the incidence vector of E_1 . Let $E_0 = \delta^+(A) \cup (\delta^-(B) \setminus \delta^-(A))$. Let $F = \{x \in P(G, u) : x(A) - x(B) = u(\delta^-(A) \setminus \delta^-(B)) + u(\delta^+(B))\}$. Then for every $x = Mz \in F$, $z_e = u_e$ for all $e \in E_1$ and $z_e = 0$ for all $e \in E_0$. So $F = \{x + MU\chi^{E_1} : x \in P(G - \delta(C) - \delta^+(A) - \delta^+(B))\}$.

$$\begin{aligned}
 \dim(F) &= \dim(P(G - \delta(C) - \delta^+(A) - \delta^+(B))) \\
 &= \dim(P(G[C]) + \dim(P(G[A \cup B] - \delta^+(A) - \delta^+(B))) \\
 &= |V \setminus (A \cup B)| - (\text{number of components of } G[C]) + |A \cup B| \\
 &\quad - (\text{number of components of } G[A \cup B] - \delta^+(A) - \delta^+(B)).
 \end{aligned}$$

Since $G[A \cup B] - \delta^+(A) - \delta^+(B)$ is weakly bipartite, $\dim(F) = |V| - 2$ if and only if (2.5.8a) and (2.5.7b) hold. ■

Chapter 3

Facets of capacitated b -matching degree sequence polyhedra

In this chapter, we characterize the facets of the convex hull of degree sequences of u -capacitated b -matchings of a graph.

3.1 Introduction

Let $G = (V, E)$ be an undirected graph. Given a set $T \subseteq V$, T is a **matchable set** if there is a matching in $G[T]$ incident to each vertex of T . Namely, T is a matchable set if $\chi^T \in \mathbb{R}^V$ is a degree sequence of a subgraph of G having maximum degree 1. An equivalent version of the famous theorem of Tutte [24] gives a necessary and sufficient condition for $T \subseteq V$ to be a matchable set.

Theorem 3.1.1 *$T \subseteq V$ is a matchable set if and only if the number of odd components of $G[T] - S$ is at most $|S|$, for all $S \subseteq T$.*

Balas and Pulleyblank [2, 3] first studied matching degree sequence polyhedra. They proved the following nice result.

Theorem 3.1.2 *The convex hull of degree sequences of matchings of G , $P(G, 1)$, is the set of all $x \in \mathbb{R}^V$ satisfying*

- (a) $0 \leq x \leq 1$;
- (b) $\sum_{i=1}^k x(A_i) - x(B) \leq \sum_{i=1}^k (|A_i| - 1)$, for all $B \subseteq V$, and vertex-sets $A_1, \dots, A_k$ of components of $G - B$ such that each $|A_i|$ is odd.

This theorem implies Tutte's Theorem since T is a matchable set if and only if $\chi^T \in P(G, 1)$.

Let z be a (u -capacitated) b -matching. The degree sequence of z is Mz . We say that z is perfect if $Mz = b$. Cunningham and Green-Krótki [5] studied the convex hull of degree sequences of u -capacitated b -matchings of G , $P(G, b, u)$. They proved Theorem 3.1.3, which generalizes Theorem 3.1.2.

Theorem 3.1.3 $P(G, b, u)$ is the set of all $x \in \mathbb{R}^V$ satisfying

- (a) $0 \leq x \leq b$;
- (b) $\sum_{i=0}^k x(A_i) - x(B) \leq \sum_{i=1}^k (b(A_i) - 1) + u(\gamma(A_0)) + u[A_0, V \setminus B]$, for all disjoint subsets B, A_0 of V , and vertex-sets $A_1, \dots, A_k$ of components of $G - (B \cup A_0)$ such that each $b(A_i) + u[A_0, A_i]$ is odd, $i = 1, \dots, k$.

The above result implies Theorem 2.2.3 of Chapter 2 as follows. If $u = 1$ and $b_v = \infty$ for all $v \in V$, then $P(G, u, b)$ is the convex hull of degree sequences of subgraphs of G . Note that in this case the inequality $\sum_{i=0}^k x(A_i) - x(B) \leq \sum_{i=1}^k (b(A_i) - 1) + u(\gamma(A_0)) + u[A_0, V \setminus B]$ is not redundant only if $k = 0$. This gives another proof of Theorem 2.2.3.

The main question in this chapter is: Which inequality in Theorem 3.1.3 defines a facet of $P(G, b, u)$? Balas and Pulleyblank [2, 3] characterized the facet-inducing inequalities for $P(G, 1)$. We shall answer the above question, namely, we characterize the facet-inducing inequalities for $P(G, b, u)$. In Section 2, we consider the bipartite case and in Section 3, we consider the nonbipartite case.

One closely related polyhedron is the convex hull, $Q(G, b, u)$, of u -capacitated b -matchings. A classical theorem of Edmonds and Johnson [11] gives a linear description for $Q(G, b, u)$.

Theorem 3.1.4 $Q(G, b, u)$ is the set of all $z \in \mathbb{R}_+^E$ satisfying

(a) $z(\delta(v)) \leq b_v$, for all $v \in V$;

(b) $2z(\gamma(S)) + z(F) \leq b(S) + u(F) - 1$, for all $S \subseteq V$, $F \subseteq \delta(S)$ with $b(S) + u(F)$ odd.

From Theorem 1.3.15, we have $P(G, b, u) = \{Mz : z \in Q(G, b, u)\}$. We use $x = Mz \in P(G, b, u)$ to mean $z \in Q(G, b, u)$. Cook and Pulleyblank [4] characterized the facets of $Q(G, b, u)$.

In this chapter, we always assume that $u > 0$. For a vector $x \in \mathbb{R}^S$ and $Q \subseteq S$, we often use x to mean the projection $x|_Q$. It is obvious that $P(G, b, u) \neq \emptyset$ if and only if $b \geq 0$. To end this section, we mention a basic property of $P(G, b, u)$. We assume $b \geq 0$. Let $V_0 = \{v \in V : b_v = 0\}$.

Proposition 3.1.5 $\dim(P(G, b, u)) = |V| - |V_0| - (\text{number of bipartite components of } G - V_0)$.

Proof: Suppose that G is connected and $V_0 = \emptyset$. Since every column vector of M_G is in $P(G, b, u)$, we have the following. If G is bipartite, then $\dim(P(G, b, u)) = |V| - 1$. If G is nonbipartite, then $\dim(P(G, b, u)) = |V|$.

For the general case, consider $G - V_0$. It is clear that $\dim(P(G, b, u)) = \dim(P(G - V_0, b, u))$. But $\dim(P(G - V_0, b, u)) = |V| - |V_0| - (\text{number of bipartite components of } G - V_0)$ from the above argument. So the theorem is true. ■

For the remainder of this chapter, we assume that $b_v > 0$ for all $v \in V$, and we assume that G is connected.

3.2 Facets of $P(G, b, u)$ for bipartite graphs

For bipartite graphs, versions of both Theorem 3.1.2 and Theorem 3.1.3 are much simpler.

Theorem 3.2.1 *If G is bipartite with bipartition $\{V_1, V_2\}$, then $P(G, 1)$ is the set of all $x \in \mathbb{R}^V$ satisfying*

- (a) $0 \leq x_v \leq 1$ for all $v \in V$;
- (b) $x(A) - x(N(A)) \leq 0$, for all $A \subseteq V_1$;
- (c) $x(V_1) - x(V_2) = 0$.

Theorem 3.2.2 *If G is bipartite with bipartition $\{V_1, V_2\}$, then $P(G, b, u)$ is the set of all $x \in \mathbb{R}^V$ satisfying*

- (a) $0 \leq x_v \leq b_v$ for all $v \in V$;
- (b) $x(A) - x(B) \leq u[A, N(A) \setminus B]$ for all $A \subseteq V_1, B \subseteq V_2$;
- (c) $x(V_1) - x(V_2) = 0$.

Theorem 3.2.1 is due to Balas and Pulleyblank [2], and Theorem 3.2.2 is due to Cunningham and Green-Krótki [5] in Section 2.

Theorem 3.2.1 and Theorem 3.2.2 can be derived from Theorem 3.1.2 and Theorem 3.1.3. For general graphs, Theorem 3.1.2 and Theorem 3.1.3 are considerably harder to prove than for subgraphs without degree constraints. Matching theory is essential in the proofs. Actually, there are short proofs for Theorem 3.2.1 and Theorem 3.2.2. In fact, Ning [20] showed that the separation problem for $P(G, 1)$ in Theorem 3.2.1 can be solved by a maximum flow algorithm, and Cunningham and Green-Krótki indicated [5] that Theorem 3.2.2 is an easy consequence of a well-known network result, namely, Gale's Theorem, and the separation problem can also be solved by solving a maximum flow problem.

In the following, we characterize the facets of $P(G, b, u)$.

We note that inequality $x_w \geq 0$ is a special form of (3.1.2b) by taking $B = \{w\}$ and $A = \emptyset$. But in this case, it is much easier to decide if it is facet-inducing. We state and prove it separately.

Theorem 3.2.3 *If G is bipartite, $x_w \geq 0$ is facet-inducing for $P(G, b, u)$ if and only if $G - \{w\}$ is connected.*

Proof: Let F be the face induced by $x_w \geq 0$. Note that for any $x = Mz \in F$, $z(\delta(w)) = 0$. So $\dim(F) = \dim(P(G - \{w\}, b, u))$. $\dim(P(G - \{w\}, b, u)) = |V| - 2$ if and only if $G - \{w\}$ is connected. So F is a facet if and only if $G - \{w\}$ is connected.

■

Now we discuss valid inequality $x_w \leq b_w$.

Theorem 3.2.4 *For $w \in V$, let $N' = \{v \in N(w) : b_v \leq u_{vw}\}$, and let $N'' = \{v \in N(w) : b_v > u_{vw}\}$. If G is bipartite, $x_w \leq b_w$ is facet-inducing for $P(G, b, u)$ if and only if either (i) $b(N') + u[w, N''] - b_w = 0$, $N' = \emptyset$, and $G - \{w\}$ is connected, or (ii) $b(N') + u[w, N''] - b_w = 0$, $|N'| = 1$, and $|V| = 2$, or (iii) $b(N') + u[w, N''] - b_w \geq 1$.*

Proof: Let F be the face induced by $x_w \leq b_w$.

If $b(N') + u[w, N''] - b_w = 0$, Let $b'_v \in \mathbb{R}^{V \setminus (N' \cup \{w\})}$ such that $b'_v = b_v - u_{vw}$ for $v \in N''$, $b'_v = b_v$ for $v \notin N''$. Then $\dim(F) = \dim(P(G - (N' \cup \{w\}), b', u)) = |V| - |N'| - 1 - (\text{number of components of } G - (N' \cup \{w\}))$. F is a facet if and only if $\dim(F) = |V| - 2$. So F is a facet if and only if $N' = \emptyset$ and $G - (N' \cup \{w\})$ is connected, or $|N'| = 1$ and $V = N' \cup \{w\}$.

Suppose $b(N') + u[w, N''] - b_w \geq 1$. Let R denote $b(N') + u[w, N'']$. Let $\hat{x} \in \mathbb{R}^V$ be defined by

$$\hat{x}_v = \begin{cases} b_w + 1, & \text{if } v = w, \\ (b_w + 1)b_v/R, & \text{if } v \in N', \\ (b_w + 1)u_{vw}/R, & \text{if } v \in N'', \\ 0, & \text{else.} \end{cases}$$

Now we show that $x_w \leq b_w$ is not implied by any other inequalities by showing that $\hat{x}$ satisfies all inequalities except $x_w \leq b_w$. It is easy to see that $0 \leq x_v \leq b_v$ is satisfied for all $v \in V \setminus \{w\}$ and $\hat{x}(V_1) - \hat{x}(V_2) = 0$. We check it for $x(A) - x(B) \leq u[A, N(A) \setminus B]$. If $w \notin A$, then the inequality holds. Suppose that $w \in A$. Let $K' = N' \cap B$, and let

$K'' = N'' \cap B$. Then

$$\begin{aligned}
 \hat{x}(A) - \hat{x}(B) &= (b_w + 1) - (b_w + 1)(b(K') + u[w, K''])/R \\
 &= (b_w + 1)(b(N' \setminus K') + u[w, N'' \setminus K''])/R \\
 &\leq b(N' \setminus K') + u[w, N'' \setminus K''] \\
 &\leq u[w, N' \setminus K'] + u[w, N'' \setminus K''] = u[w, N(w) \setminus B] \\
 &\leq u[A, N(A) \setminus B].
 \end{aligned}$$

So $x_w \leq b_w$ is facet-inducing. ■

Corollary 3.2.5 [2] *If G is bipartite, $x_w \leq 1$ is facet-inducing for $P(G, 1)$ if and only if either (i) $|V| = 2$, or (ii) $|N(w)| \geq 2$.*

Proof: Condition (i) and (ii) correspond to condition (ii) and (iii) of Theorem 3.2.4. Condition (i) of Theorem 3.2.4 does not occur for $P(G, 1)$ since $u = \infty$. So the claim follows. ■

Theorem 3.2.6 *If G is bipartite, then inequality (3.2.2b), $x(A) - x(B) \leq u[A, N(A) \setminus B]$, is facet-inducing for $P(G, b, u)$ if and only if*

- (a) $u(\delta(v) \cap [A, N(A) \setminus B]) < b_v$, for every $v \in A \cup (N(A) \setminus B)$;
- (b) $G - [A, N(A) \setminus B] - [B, N(B) \setminus A]$ has exactly two components.

Proof: Let $F = \{x \in P(G, b, u) : x(A) - x(B) = u[A, N(A) \setminus B]\}$. Let $G' = G - [A, N(A) \setminus B] - [B, N(B) \setminus A]$, and let $b' \in \mathbb{R}^V$ such that

$$b'_v = \begin{cases} b_v - u(\delta(v) \cap [A, N(A) \setminus B]) & v \in A \cup (N(A) \setminus B), \\ b_v & v \notin A \cup (N(A) \setminus B). \end{cases}$$

Claim: for any $x = Mz \in P(G, b, u)$, $x \in F$ if and only if $z[A, N(A) \setminus B] = u[A, N(A) \setminus B]$, and $z[B, N(B) \setminus A] = 0$. (Proof: For any $x = Mz \in P(G, b, u)$, we have $x(A) =$

$z[A, B] + z[A, N(A) \setminus B]$ and $x(B) = z[A, B] + z[B, N(B) \setminus A]$. So $x(A) - x(B) = z[A, N(A) \setminus B] - z[B, N(B) \setminus A] \leq u[A, N(A) \setminus B]$. So $x \in F$ if and only if $z[A, N(A) \setminus B] = u[A, N(A) \setminus B]$, and $z[B, N(B) \setminus A] = 0$.)

From the claim, we have $F = P(G', b', u) + (b - b')$, and $\dim(F) = \dim(P(G', b', u))$. $P(G', b', u)$ is not empty if and only if $b' \geq 0$. By Proposition 3.1.5, $\dim(P(G', b', u)) = |V| - |V_0| - (\text{number of bipartite components of } G' - V_0)$, where $V_0 = \{v \in V : b'_v = 0\}$. So $\dim(F) = |V| - 2$ if and only if $b' > 0$ and G' has only two components. This proves the theorem. ■

Corollary 3.2.7 [2] *If G is bipartite, then inequality (3.2.1b), $x(A) - x(N(A)) \leq 0$ is facet-inducing for $P(G, 1)$ if and only if $G - [B, N(B) \setminus A]$ has exactly two components.*

Proof: Since in this case $B = N(A)$, $[A, N(A) \setminus B] = \emptyset$. So Condition (a) of Theorem 3.2.6 holds automatically. Condition (b) of Theorem 3.2.6 is equivalent to the condition of the claim. ■

3.3 Facets of $P(G, b, u)$ for nonbipartite graphs

Results

A graph H is 1-critical if $H - \{v\}$ has a perfect matching for every $v \in V(H)$. Balas and Pulleyblank [3] proved Theorem 3.3.1.

Theorem 3.3.1 *If G is nonbipartite, then inequality (3.1.2b) is facet-inducing for $P(G, 1)$ if and only if*

- (a) *every component of $G - B - \cup_1^k A_i$ is nonbipartite;*
- (b) *every component $G[A_i]$ is 1-critical;*
- (c) *the graph $G[B \cup \cup_1^k A_i] - \gamma(\cup_1^k A_i)$ is connected.*

We describe essential inequalities of $P(G, b, u)$ in the following.

Theorem 3.3.2 *If G is nonbipartite, $x_w \geq 0$ is facet-inducing for $P(G, u, b)$ if and only if $G - \{w\}$ is nonbipartite.*

Proof: Let F be the face induced by $x_w \geq 0$. Note that for any $x = Mz \in F$, $z(\delta(w)) = 0$. So $\dim(F) = \dim(P(G - \{w\}, b, u))$. $\dim(F) = |V| - 1$ if and only if $G - \{w\}$ is nonbipartite. ■

We consider $x_w \leq b_w$.

Theorem 3.3.3 *Let $N' = \{v \in N(w) : b_v \leq u_{vw}\}$, and let $N'' = \{v \in N(w) : b_v > u_{vw}\}$. If G is nonbipartite, $x_w \leq b_w$ is facet-inducing for $P(G, b, u)$ if and only if either (i) $b(N') + u[w, N''] - b_w = 0$, $N' = \emptyset$, and $G - \{w\}$ is nonbipartite, or (ii) $b(N') + u[w, N''] - b_w \geq 2$, or (iii) $b(N') + u[w, N''] - b_w = 1$ and $G - \gamma(N')$ is nonbipartite.*

Proof: Let F be the face induced by $x_w \leq b_w$.

(i) $b(N') + u[w, N''] - b_w = 0$. Let $b' \in \mathbb{R}^{V \setminus (N' \cup \{w\})}$ such that $b'_v = b_v - u_{vw}$ for $v \in N''$, $b'_v = b_v$ for $v \notin N''$. Then $\dim(F) = \dim(P(G - (N' \cup \{w\}), b', u)) = |V| - |N'| - 1 - (\text{number of bipartite components of } G - (N' \cup \{w\}))$. So F is a facet if and only if $N' = \emptyset$ and $G - \{w\}$ is nonbipartite.

Suppose $b(N') + u[w, N''] - b_w \geq 1$. Consider $G_1 := (N(w) \cup \{w\}, \delta(w))$, and let F' be the face of $P(G_1, b, u)$ induced by $x \leq b_w$. From Theorem 3.2.4, F' is a facet. So we can find $d := |\delta(w)|$ affinely independent vectors $x^1, \dots, x^d \in F'$. These vectors are linearly independent by Proposition 1.3.12. We can simply extend these vectors to $\mathbb{R}^V$ with 0 entries for new coordinates. We still use $x^1, \dots, x^d$ to denote the extended vectors. It is clear that $x^1, \dots, x^d \in F$. Let X denote matrix $[x^1, \dots, x^d]$. Let W denote the matrix of incidence vectors of $\delta(w)$. Note that each x^i is a linear combination of these incidence vectors. Since $x^1, \dots, x^d$ are linearly independent, there exists an invertible matrix P such that $X = WP$.

(ii) $b(N') + u[w, N''] - b_w \geq 2$. Let $E' \subseteq E \setminus \delta(w)$ such that $E' \cup \delta(w)$ is the edge set of a maximal independent subgraph. Then $p := |E'| = |V| - d$. Let $y^1, \dots, y^p$ be the

incidence vectors of E' . Each y^i can be extended to a vector of F by adding a $z^i \in F'$ for $i = 1, \dots, p$. Let Y, Z denote matrices $[y^1, \dots, y^p], [z^1, \dots, z^p]$ respectively. Note that each z^i is a linear combination of the incidence vectors of $\delta(w)$. So $Z = WQ$ for some matrix Q . $[X, Y+Z] = [X, Y+WQ] = [WP, Y+WQ] = [W, Y] \begin{bmatrix} P & Q \\ 0 & I \end{bmatrix}$. Note that the matrix on the right hand side is invertible. So $x^1, \dots, x^d, y^1 + z^1, \dots, y^p + z^p$ are linearly independent, hence affinely independent, so $\dim(F) = d + p - 1 = |V| - 1$, and so $x_w \leq b_w$ is facet-inducing.

(iii) $b(N') + u[w, N''] - b_w = 1$, and $G' := G - \gamma(N')$ is nonbipartite. Let $E' \subseteq E \setminus \delta(w)$ such that $E' \cup \delta(w)$ is the edge set of a maximal independent subgraph of G' . Then $p := |E'| = |V| - d$. With a similar argument as in (ii), we have that $x_w \leq b_w$ is facet-inducing. Now suppose $b(N') + u[w, N''] - b_w = 1$, and $G' := G - \gamma(N')$ is bipartite with bipartition $\{V_1, V_2\}$. Without loss of generality, we assume $w \in V_1$. Let $B = V_2$ and $A_0 = V_1$. Then $x(A_0) - x(B) \leq 0$ is valid, and it is tight for any $x \in F$. Note that 0 is on the face induced by $x(A_0) - x(B) \leq 0$ but not on F . So F is not a maximal face, and $x \leq b_w$ is not facet-inducing. ■

Now we apply the above theorem to valid inequality $x_w \leq 1$ for $P(G, 1)$. The result corrects the claim made in [3].

Corollary 3.3.4 *If G is nonbipartite, $x_w \leq 1$ is facet-inducing for $P(G, 1)$ if and only if either (i) $d_w \geq 3$, or (ii) $d_w = 2$ and $G - \gamma(N(w))$ is nonbipartite.*

Proof: It follows from Theorem 3.3.3. ■

Now we discuss valid inequality (3.1.3b), $\sum_{i=0}^k x(A_i) - x(B) \leq \sum_{i=1}^k (b(A_i) - 1) + u(\gamma(A_0)) + u[A_0, V \setminus B]$. Let $E_1 = \cup_1^k \{e \in [A_0, A_i] : \text{there is no } b\text{-matching } z \in \mathbb{R}^E \text{ such that } 2z(\gamma(A_i)) + 2z(\delta(A_i)) = b(A_i) + u[A_0, A_i] - 1 \text{ and } z_e = u_e - 1\}$. Let $D = V \setminus (B \cup \cup_0^k A_i)$.

Theorem 3.3.5 *Suppose G is nonbipartite. Then inequality (3.1.3b) is facet-inducing for $P(G, b, u)$ if and only if*

- (a) each component of $G[D]$ is nonbipartite;
- (b) for each $v \in \cup_1^k A_i$, there exists a b -matching z such that $z(\delta(v)) = b_v - 1$, and $2z(\gamma(A_i)) + 2z[A_0, A_i] = b(A_i) + u[A_0, A_i] - 1$ for $i = 1, \dots, k$;
- (c) $G'[V \setminus D]$ is connected, where $G' = G - [B, D] - [A_0, D] - \gamma(B) - \gamma(A_0) - E_1$;
- (d) $b_v > u(\delta(v) \cap E_1) \forall v \in \cup_1^k A_i$, $b_v > u[v, A_0] \forall v \in D$, $b_v > u(\delta(v) \cap E_1) + u[v, A_0] + u[v, D] \forall v \in A_0$.

We note that the above theorem implies Theorem 3.3.1, where $A_0 = \emptyset$. It is clear that $E_1 = \emptyset$ and condition (3.3.5d) holds automatically. Conditions (a), (b), (c) in Theorem 3.3.5 correspond to conditions (a), (b), (c) in Theorem 3.3.1. We remark that for uncapacitated b -matching, the corresponding results for facets can be derived in the similar way as for 1-matching. But for capacitated b -matching, the situation is more complicated.

Proof of Theorem 3.3.5

Let

$$F = \{x \in P(G, b, u) : \sum_0^k x(A_i) - x(B) = \sum_1^k (b(A_i) - 1) + u(\gamma(A_0)) + u[A_0, V \setminus B]\}$$

We first prove a useful property.

Proposition 3.3.6 *Let $x = Mz \in P(G, b, u)$. Then $x \in F$ if and only if $z[B, D] = 0$, $z(\gamma(B)) = 0$, $z(\gamma(A_0)) = u(\gamma(A_0))$, $z[A_0, D] = u[A, D]$, and $2z(\gamma(A_i)) + 2z[A_0, A_i] = b(A_i) + u[A_0, A_i] - 1$, $i = 1, \dots, k$.*

Proof: We have

$$x(A_i) = 2z(\gamma(A_i)) + z[A_i, A_0] + z[A_i, B] \text{ for all } i = 1, \dots, k$$

$$x(A_0) = 2z(\gamma(A_0)) + z[A_0, B] + \sum_1^k z[A_i, A_0] + z[A_0, D]$$

$$x(B) = 2z(\gamma(B)) + z[A_0, B] + \sum_1^k z[A_i, B] + z[B, D]$$

So

$$\begin{aligned} \sum_0^k x(A_i) - x(B) &= 2 \sum_1^k z(\gamma(A_i)) + 2 \sum_1^k z[A_i, A_0] + z[A_0, D] + 2z(\gamma(A_0)) \\ &\quad - z[B, D] - 2z(\gamma(B)) \\ &\leq \sum_1^k (b(A_i) + u[A_0, A_i] - 1) + u[A_0, D] + 2u(\gamma(A_0)) \\ &= \sum_1^k (b(A_i) - 1) + u[A_0, V \setminus B] + u(\gamma(A_0)) \end{aligned}$$

So $x \in F$ if and only if the equation conditions hold. ■

It is also clear from the above that $z(E_1) = u(E_1)$ for any $Mz \in P(G, b, u)$.

Define $b' \in \mathbb{R}^V$ by

$$b' = \begin{cases} b_v & v \in B, \\ b_v - u(\delta(v) \cap E_1) & v \in \cup_1^k A_i, \\ b_v - u[v, A_0] & v \in D, \\ b_v - u[v, A_0] - u[v, D] - u(\delta(v) \cap E_1) & v \in A_0. \end{cases}$$

Let F' denote

$$\{x \in P(G', b', u) : \sum_1^k x(A_i) - x(B) = \sum_1^k (b'(A_i) - 1) + u[A_0, \cup_1^k A_i]\}.$$

From Proposition 3.3.6, it is easy to see that F is empty if and only if F' is empty, and $F = F' + (b - b')$.

Let

$$F'_1 = \{x \in P(G' - D, b', u) : \sum_1^k x(A_i) - x(B) = \sum_1^k (b'(A_i) - 1) + u[A_0, \cup_1^k A_i]\}.$$

It is clear that $\dim(F') = \dim(F'_1) + \dim(P(G[D], b, u))$. So $\dim(F) = \dim(P(G[D], b, u)) + \dim(F'_1)$. And so F is a facet if and only if $\dim(P(G[D], b, u)) = |D|$ and $\dim(F'_1) = |V \setminus D| - 1$.

We note that $\dim(P(G[D], b, u)) = |D|$ if and only if each component of $G[D]$ is nonbipartite.

Proof of necessity of Theorem 3.3.5: If $\dim(F'_1) = |V \setminus D| - 1$, then $b' > 0$, G' is connected and for any $v \in \cup_1^k A_i$, there exists a b -matching z such that $z(\delta(v)) = b_v - 1$, $2z(\gamma(A_i)) + 2z[A_0, A_i] = b(A_i) + u[A_0, A_i] - 1$ for $i = 1, \dots, k$. So the conditions are necessary. ■

To prove sufficiency, we need to prove two special cases of the theorem, Proposition 3.3.7 and Proposition 3.3.8.

Proposition 3.3.7 *If $V = \cup_0^k A_i$, $E_1 = \emptyset$ and $\gamma(A_0) = \emptyset$, then the statement in Theorem 3.3.5 is true.*

Proof: We only need to prove sufficiency.

Let

$$F = \{x \in P(G, b, u) : \sum_0^k x(A_i) = \sum_1^k (b(A_i) - 1) + u[A_0, \cup_1^k A_i]\}.$$

Then $F = \text{conv}\{Mz : \text{for all } b\text{-matching } z \text{ such that } 2z(\gamma(A_i)) + 2z[A_0, A_i] = b(A_i) + u[A_0, A_i] - 1, \text{ for } i = 1, \dots, k\}.$

We show that there exist $|V| = |A_0| + \sum_1^k |A_i|$ affinely independent vectors in F .

We first show that there exist $|A_0|$ affinely independent vectors in F satisfying $z(\delta(v)) = b_v$ for all $v \in \cup_1^k A_i$.

Figure 3.1: An example of G and G_1

Shrinking each A_i to v_i , $i = 1, \dots, k$, let $A = \{v_i : i = 1, \dots, k\}$. Let G_1 be the resulting bipartite graph. For example, see Figure 3.1. We define $u[v_i, w] = u[A_i, w]$ for $i = 1, \dots, k$, $w \in A_0$ and let $a_i = u[v_i, A_0] - 1$ for $i = 1, \dots, k$.

Given a vector z such that $x = Mz \in F$, there exists a reduced vector z^1 in G_1 such that $z^1[v_i, w] = z[A_i, w]$ for all $v_i \in A$ and $w \in A_0$. Especially, if $z(\delta(v)) = b_v$ for all $v \in \cup_1^k A_i$, then $z^1(\delta_{G_1}(v_i)) = a_i$ for $i = 1, \dots, k$. Notice that the reverse is also true: any such vector z^1 can be extended to a vector $z \in F$ such that $z(\delta(v)) = b_v$ for all $v \in \cup_1^k A_i$.

This leads us consider the polyhedron $P := \text{conv}\{x \in \mathbb{R}^{A_0} : x_w = z(\delta(w)) \forall w \in A_0, x = M_{G_1}z, z(\delta(v)) = a_v \text{ for all } v \in A, \text{ for some } b\text{-matching } z \text{ of } G_1\}$.

We claim that $\dim(P) = |A_0| - 1$, namely there are $|A_0|$ affinely independent vectors.

Proof of the claim: Suppose $\sum(c_w x_w : w \in A_0) = d$ is satisfied for all $x \in P$. Let $A'_0 = \{w' : c_{w'} = \max\{c_w : w \in A_0\}\}$. We show that $A'_0 = A_0$. This implies that $\sum(c_w x_w : w \in A_0) = d$ must be the same as $\sum(x_w : w \in A_0) = \sum_1^k a_i$ up to a scalar.

Suppose $A_0 \setminus A'_0 \neq \emptyset$. Let z be an integral vector such that $x = M_{G_1}z \in P$, and let $A_z = \{v' : \text{there exists } w' \in A'_0 \text{ such that } z[v', w'] = u[v', w'] - 1\}$. Since G_1 is connected by (3.3.5c), $[A_z, A_0 \setminus A'_0] \cup [A \setminus A_z, A'_0] \neq \emptyset$. Because of the choice of A_z, A'_0 and vector D , we have $z[A_z, A_0 \setminus A'_0] = u[A_z, A_0 \setminus A'_0]$, $z[A \setminus A_z, A'_0] = u[A \setminus A_z, A'_0]$. Without loss of generality, say $e_0 = [v_1, w_1] \in [A_z, A_0 \setminus A'_0]$. From (3.3.5b), it is clear that for $D_i := (a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_k)$, there exists a perfect D_i -matching z^i , for $i = 1, \dots, k$. So there is a perfect D_1 -matching, and so from the network flow theory, we can find an z -augmenting path from v_1 to a vertex $w_{2q} \in A_0$: $v_1, e_1, w_2, e_2, \dots, v_{2q-1}, e_{2q-1}, w_{2q}$, namely, $z_{e_{2i-1}} < u_{e_{2i-1}}$ and $z_{e_{2i}} > 0$ for $i = 1, \dots, q$. We do an augmentation on this path as follows: add 1 to $z_{e_{2i-1}}$ and subtract 1 from $z_{e_{2i}} > 0$ for $i = 1, \dots, q$.

If $w_{2q} \in A'_0$, then subtract 1 from $z[v_1, w_1]$. And we have a new vector $x' \in P$, where

$$x'_w = \begin{cases} x_w - 1 & \text{if } w = w_1, \\ x_w + 1 & \text{if } w = w_{2q}, \\ x_w & \text{else.} \end{cases}$$

So $d = \sum_{w \in A_0} c_w x_w = \sum_{w \in A_0} c_w x'_w + c_{w_1} - c_{w_{2q}} = d + c_{w_1} - c_{w_{2q}}$. This implies $c_{w_{2q}} = c_{w_1}$, and thus $w_1 \in A'_0$, a contradiction.

If $w_{2q} \in A_0 \setminus A'_0$, then we change $z[v_1, w_2]$ back to $u[v_1, w_2] - 1$. And we have a new vector $x' \in P$, where

$$x'_w = \begin{cases} x_w - 1 & \text{if } w = w_2, \\ x_w + 1 & \text{if } w = w_{2q}, \\ x_w & \text{else.} \end{cases}$$

By an argument similar to the one above, we have $c_{w_{2q}} = c_{w_2}$. But $w_2 \in A'_0$, so $w_{2q} \in A'_0$, a contradiction.

So the only valid equality for P is $\sum(x_w : w \in A_0) = \sum(a_v : v \in A)$. Hence $\dim(P) = |A_0| - 1$, and the claim is proved.

Let $\{x^w = M_{G_1} z^w : w \in A_0\}$ be a set of affinely independent vectors of P . As we indicated earlier, each x^w can be extended to

$$\begin{pmatrix} b'' \\ x^w \end{pmatrix} \in F,$$

where $b'' \in \mathbb{R}^{U_1^k A_i}$ such that $b''_v = b_v$, for every $v \in U_1^k A_i$.

From condition (3.3.5a), we have that for each $v \in U_1^k A_i$, there is a b -matching z^v such that $z^v(\delta(w)) = b_w$ for each $w \in U_1^k A_i \setminus \{v\}$, and $z^v(\delta(w)) = b_w - 1$ for $w = v$. Let $\{x^v : v \in U_1^k A_i\}$ be their degree sequences.

We claim that $\{x^w : w \in A_0\} \cup \{x^v : v \in U_1^k A_i\}$ is a set of affinely independent vectors.

Let w' be a vertex of A_0 . Then the above claim is equivalent to the claim that $\{x^w - x^{w'} : w \in A_0 \setminus \{w'\}\} \cup \{x^v - x^{w'} : v \in U_1^k A_i\}$ is a linearly independent set.

Let

$$\sum(\alpha_w(x^w - x^{w'}) : w \in A_0 \setminus \{w'\}) + \sum(\beta_v(x^v - x^{w'}) : v \in U_1^k A_i) = 0.$$

Obviously $\beta_v = 0$ for all $v \in U_1^k A_i$. So we have that

$$\sum(\alpha_w(x^w - x^{w'}) : w \in A_0 \setminus \{w'\}) = 0.$$

But $\{x^w - x^{w'} : w \in A_0 \setminus \{w'\}\}$ is a set of linearly independent vectors, so $\alpha_w = 0$ for all $w \in A_0 \setminus \{w'\}$.

So $\{x^w - x^{w'} : w \in A_0 \setminus \{w'\}\} \cup \{x^v - x^{w'} : v \in \cup_1^k A_i\}$ is a set of linearly independent vectors. Hence the claim is true, and $\dim(F) = |V| - 1$. ■

Proposition 3.3.8 Suppose $V = B \cup \cup_0^k A_i$, $E_1 = \emptyset$, $\gamma(A_0) = \emptyset$, and $\gamma(B) = \emptyset$. Then the statement in Theorem 3.3.5 is true.

Proof: Again we just need to prove sufficiency.

Let $G_j = (V_j, E_j)$, $j = 1, \dots, r$, be all components of $G - B$. Let $I_j = \{i : A_i \in V_j\}$, $A_j^0 = A_0 \cap V_j$ for $j = 1, \dots, r$. Let $Z_j = \{x \in P(G_j, b, u) : \sum_{i \in I_j} x(A_i) + x(A_j^0) = \sum_{i \in I_j} (b(A_i) + u[A_i, A_j^0] - 1)\}$. Then $\dim(Z_j) = |V_j| - 1$ by Proposition 3.3.7. We extend Z_j to a subspace of $\mathbb{R}^V$, as follows: for any $x \in Z_j$, we define a vector $x' \in \mathbb{R}^V$ such that

$$x' = \begin{cases} x_v & \text{if } v \in Z_j, \\ 0 & \text{if } v \in V \setminus Z_j. \end{cases}$$

We still use Z_j to denote the new subspace.

It is clear that $Z_i \cap Z_j = \emptyset$ for any $i \neq j$. So $\dim(\sum_1^r Z_j) = \sum_1^r \dim(Z_j) = \sum_1^r (|V_j| - 1) = \sum_1^k |A_i| - r$. Also it is clear that $\sum_1^r Z_j \subseteq F$.

Figure 3.2: An example of G and G'

Now we shrink each V_j for $j = 1, \dots, r$ in G , and so we have a connected bipartite graph G' . For example, see Figure 3.2. For a tree T of G' , let the edge set $\{e_1, \dots, e_t\}$. For each e_i of T , let $e'_i \in \mathbb{R}^V$ be the incidence vector of e_i in G , namely,

$$e'_i(v) = \begin{cases} 1 & \text{if } e_i \in \delta_G(v), \\ 0 & \text{otherwise.} \end{cases}$$

Then for each e'_i , we can find a vector $x'_i \in \sum_1^r Z_j$ such that $e'_i + x'_i \in F$, for $i = 1, \dots, t$.

Now take the maximum affinely independent set $\{x_0, x_1, \dots, x_s\}$ of $\sum_1^r Z_j$, where $s = |V| - |B| - r = \sum_1^k |A_i| - r$. We claim that $x_0, x_1, \dots, x_s, e'_1 + x'_1, \dots, e'_t + x'_t$ are affinely independent, namely, $x_1 - x_0, \dots, x_s - x_0, e'_1 + x'_1 - x_0, \dots, e'_t + x'_t - x_0$ are linearly independent.

Let

$$\sum_1^s \alpha_i (x_i - x_0) + \sum_1^t \beta_j (e'_j + x'_j - x_0) = 0.$$

Without loss of generality, assume e_1 is incident to a leaf v of the tree T . Let $a \in \mathbb{R}^V$ such that if $v \in B$, then $a_w = 1$ for $w = v$, $a_w = 0$ for all $w \in V \setminus \{v\}$; if $v \notin B$, v is shrunk from V_j for some j , so $a_w = 1$ for $w \in V_j$, $a_w = 0$ for all $w \in V \setminus V_j$. Then the inner products

$$a \cdot (x_i - x_0) = 0 \quad \text{for } i = 1, \dots, s;$$

$$a \cdot (x'_i - x_0) = 0 \quad \text{for } i = 1, \dots, t;$$

$$a \cdot e_i = 0 \quad \text{for } i = 2, \dots, t;$$

$$a \cdot e_1 = 1.$$

So we have $\beta_1 = 0$. Since $\{e_s, \dots, e_t\}$ is again the edge set of a tree of G' , by induction, we have $\beta_2 = \dots = \beta_t = 0$. But $x_1 - x_0, \dots, x_s - x_0$ are linearly independent. So $\alpha_1 = \dots = \alpha_s = 0$, and so $x_1 - x_0, \dots, x_s - x_0, e'_1 + x'_1 - x_0, \dots, e'_t + x'_t - x_0$ are linearly independent.

If we choose the tree T as a spanning tree of G' , then $t = |B| + r - 1$. So $\dim(F) \geq s + t = (|V| - |B| - r) + (|B| + r - 1) = |V| - 1$. But it is clear that $\dim(F) \leq |V| - 1$. So $\dim(F) = |V| - 1$, and F is a facet. ■

Proof of sufficiency for Theorem 3.3.5: We know that $\dim(P(G[D], b, u)) = |D|$. Since G', b, u satisfy the conditions for Proposition 3.3.8, so $\dim(F'_1) = |V \setminus D| - 1$. The proof is complete. ■

Chapter 4

A separation algorithm for b -matching degree sequence polyhedra

In this chapter, we present an efficient separation algorithm for b -matching degree sequence polyhedra.

4.1 Introduction

An important case of u -capacitated b -matching is uncapacitated b -matching. Let $P(G, b)$ denote the convex hull of degree sequences of (uncapacitated) b -matchings of a graph $G = (V, E)$. Cunningham and Green-Krótki [5] proved the following two results, which give a linear description and a min-max theorem for $P(G, b)$.

Theorem 4.1.1 *$P(G, b)$ is the set of $x \in \mathbb{R}^V$ satisfying*

- (a) $x_v \leq b_v$, for all $v \in V$;
- (b) $x(\cup_0^k A_i) - x(B) \leq \sum_1^k (b(A_i) - 1)$, for all disjoint subsets $B, A_0, A_1, \dots, A_k$, of V such that A_0 is a set of isolated vertices of $G - B$, and $A_1, \dots, A_k$ are vertex-sets of components of $G - B$ with $b(A_i)$ odd, $1 \leq i \leq k$.

Theorem 4.1.2 *Let $x \in \mathbb{R}_+^V$, $x \leq b$. Then*

$$(4.1) \quad \max\{y(V) : y \in P(G, b), y \leq x\} =$$

$$(4.2) \quad \min\left(\sum_1^k (b(A_i) - 1) + x(B) + x(V \setminus \cup_0^k A_i)\right),$$

where the minimum is over all disjoint subsets $B, A_0, A_1, \dots, A_k$, of V such that A_0 is a set of isolated vertices of $G - B$, and $A_1, \dots, A_k$ are vertex-sets of components of $G - B$ with $b(A_i)$ odd, $1 \leq i \leq k$.

The separation problem for $P(G, b)$ obviously reduces to: given $x \in R^V$, $0 \leq x \leq b$, either determine that $x \in P(G, b)$, or find a valid inequality (4.1.1b) that is violated by x . Here we present a combinatorial separation algorithm for $P(G, b)$ that runs in polynomial time. Our algorithm actually solves both (4.1) and (4.2), and thus provides a direct proof of Theorem 4.1.2. It also solves the separation problem for $P(G, b)$ as follows. If x maximizes (4.1), then $x \in P(G, b)$. Otherwise we have minimizing $B, A_0, A_1, \dots, A_k$ of (4.2) such that $x(V) > \sum_1^k (b(A_i) - 1) + x(B) + x(V \setminus \cup_0^k A_i)$, that is, $\sum_0^k x(A_i) - x(B) > \sum_1^k (b(A_i) - 1)$. Therefore, a valid inequality violated by x is found. (Notice that this gives a “strong” separation algorithm, that is, the inequality is a “most-violated” one.) The above argument proves the sufficiency of Theorem 4.1.1 from Theorem 4.1.2. In order to give complete proofs for Theorem 4.1.1 and Theorem 4.1.2, we have the following.

Proof of the easy parts of Theorem 4.1.1 and Theorem 4.1.2: For any $y \in P(G, b)$, there is $z \in Q(G, b)$ such that $y = Mz$. For any $B, A_0, A_1, \dots, A_k$ as in Theorem 4.1.1, $y(\cup_0^k A_i) - y(B) = \sum_1^k 2z(\gamma(A_i)) - z[B, V \setminus (B \cup \cup_0^k A_i)] \leq \sum_1^k (b(A_i) - 1)$. This proves the validity of inequality (4.1.1b) in Theorem 4.1.1.

Furthermore, if $y \leq x$, then $y(V) = y(\cup_0^k A_i) - y(B) + y(B) + y(V \setminus \cup_0^k A_i) \leq \sum_1^k 2z(\gamma(A_i)) - z[B, V \setminus (B \cup \cup_0^k A_i)] + x(B) + x(V \setminus \cup_0^k A_i) \leq \sum_1^k (b(A_i) - 1) + x(B) + x(V \setminus \cup_0^k A_i)$. This proves that the maximum is less than or equal to the minimum of Theorem 4.1.2. ■

Let $Q(G, b)$ denote the convex hull of b -matchings of G . From Theorem 3.1.4, a linear description for $Q(G, b)$ is as follows; see also Pulleyblank [22].

Theorem 4.1.3 $Q(G, b)$ is the set of all $z \in R_+^E$ satisfying

- (a) $z(\delta(v)) \leq b_v$, for all $v \in V$;
- (b) $z(\gamma(S)) \leq (b(S) - 1)/2$, for each $S \subseteq V$ such that $b(S)$ is odd, and $G[S]$ is nonbipartite.

Note that $P(G, b) = \{Mz : z \in Q(G, b)\}$ from Proposition 1.3.15. This relation implies that optimizing any linear function over $P(G, b)$ can be reduced to a weighted b -matching problem, which is solvable in polynomial time; see [11] and [7]. From a theorem of Grötschel, Lovász, and Schrijver [17], there is a polynomial-time separation algorithm for $P(G, b)$. But that algorithm is far from practical. On the other hand, our separation algorithm is combinatorial and efficient.

The following two results describing discreteness of a maximizing $y = Mz$ of (4.1) are consequences of our algorithm.

Theorem 4.1.4 *There is a maximizing y in (4.1) such that each y_v is an integral combination of elements of $\{x_v : v \in V\} \cup \{b_v : v \in V\}$.*

Theorem 4.1.5 *If $y \in P(G, b)$, then there exists $z \in Q(G, b)$ such that y is the degree sequence of z and for each $e \in E$, $2z_e$ is an integral combination of elements of $\{y_v : v \in V\} \cup \{b_v : v \in V\}$.*

Also we can use the algorithm to show the following additional result to Theorem 4.1.2.

Theorem 4.1.6 *Let $x \in R_+^V$, $x \leq b$. If $y^0 \in P(G, b)$ and $y^0 \leq x$, then*

$$(4.3) \quad \begin{aligned} \max\{y(V) : y \in P(G, b), y^0 \leq y \leq x\} = \\ \max\{y(V) : y \in P(G, b), y \leq x\} \end{aligned}$$

The separation algorithm uses ideas from Edmonds' matching algorithm [10] and from the separation algorithm of Cunningham and Green-Krótki [6]. The general approach of the algorithm is as follows. We keep $z \in Q(G, b)$ and $y = Mz \in P(G, b)$ with $y \leq x$, and try to increase $y(V) = 2z(E)$ by augmenting path techniques used

in the maximum cardinality matching algorithm. Here there are two problems to deal with due to the extra constraint $y \leq x$. Since z may be fractional, we must prevent any violation of inequality $2z(S) \leq b(S) - 1$ for every $S \subseteq V$ such that $b(S)$ is odd. This is the main difficulty to overcome. The second problem is that in each augmentation, the increase in $y(V)$ can be very small. This problem was solved in [6] by using scaling techniques and discreteness properties of y and z . We apply the same idea.

4.2 Shrinking families and membership

The key idea of Edmonds' matching algorithm is that a subset $S \subseteq V$ with certain properties can be shrunk into a (pseudo) vertex, so that a matching found in the shrunk graph can be expanded to a matching in the original graph. A more general form of this idea is applied in our algorithm.

We introduce some terminology. Most of it was first presented in Pulleyblank [22]; also see [6]. A family $\mathcal{S}$ of subsets of V is nested if there do not exist $S_1, S_2 \in \mathcal{S}$ with $S_1 \setminus S_2, S_2 \setminus S_1, S_2 \cap S_1$ all non-empty. Given a nested family $\mathcal{S}$, the graph $G \times \mathcal{S}$ is obtained by shrinking each of the maximal members S of $\mathcal{S}$ to a single vertex, which we call S , and deleting all edges in $\gamma(S)$. Notice that each vertex of $G \times \mathcal{S}$ is either an element or a subset of V , and that every element of V is either a vertex of $G \times \mathcal{S}$ or an element of a (unique) vertex of $G \times \mathcal{S}$. The first kind of vertex is called *real*, and the second is called *pseudo*. For a nested family $\mathcal{S}$ of G and $G' = G \times \mathcal{S} = (V', E')$, a vector $x \in \mathbb{R}^V$ induces a vector $x' \in \mathbb{R}^{V'}$. Namely, $x'_v = x_v$ for each real vertex v of G' and $x'_S = x(S) - (b(S) - 1)$ for each pseudo vertex of G' . For $z \in \mathbb{R}^E$, it also induces a vector $z' \in \mathbb{R}^{E'}$. By a slight abuse of notation, we also use x to denote x' , and z to denote z' . We note that as a consequence of this convention, we have $b_S = 1$ for any pseudo vertex S of G' . For a nested family $\mathcal{S}$, let $\mathcal{S}[S]$ denote the family $\{S' \in \mathcal{S} : S' \subset S\}$. A graph $G_1 = (V_1, E_1)$ is *b-critical* if there is a perfect b^w -matching for each $w \in V_1$, where $b^w \in \mathbb{R}^{V_1}$ such that $b^w_v = b_v - 1$

if $v = w$ and $b_v^w = b_v$ if $v \in V_1 \setminus \{w\}$. A b -critical graph is simple if it has a unique (odd) cycle. A shrinking family of G is a nested family $\mathcal{S}$ such that for all $S \in \mathcal{S}$, $G[S] \times \mathcal{S}[S]$ is spanned by a simple b -critical graph.

Proposition 4.2.1 *Let $z \in Q(G, b)$, and let $S \subseteq V$ such that $2z(\gamma(S)) = b(S) - 1$. If $G' = G \times \{S\}$, then $z' \in Q(G', b)$.*

Proof: For any $S' \subseteq V(G')$, if the pseudo vertex $S \notin S'$, then $z'(\gamma_{G'}(S')) = z(\gamma_G(S')) \leq b(S') - 1$; if $S \in S'$, then $2z'(\gamma_{G'}(S')) = 2z(\gamma_G((S' \setminus \{S\}) \cup S)) - 2z(\gamma_G(S)) \leq b(S' \setminus \{S\}) + b(S) - 1 - (b(S) - 1) = b(S') - 1$. So $z' \in Q(G', b)$. ■

Lemma 4.2.2 gives a condition for a vector $y \in \mathbb{R}^V$ to be in $P(G, b)$ when G is a b -critical graph. Proposition 4.2.3 ensures that z' can be expanded to $z \in Q(G, b)$. The latter is an analog of Lemma 2.2 in [6].

Lemma 4.2.2 *Let $G_1 = (S, E)$ be a b -critical graph. If $y \in \mathbb{R}_+^S$ such that $y \leq b$ and $y(S) = b(S) - 1$, then $y \in P(G_1, b)$.*

Proof: For every $v \in S$, there is a b -matching z^v such that

$$z^v(\delta(u)) = \begin{cases} b_u - 1 & \text{if } u = v, \\ b_u & \forall u \in S \setminus \{v\}. \end{cases}$$

Let $z = \sum_{v \in S} (b_v - y_v) z^v$. Since $b_v - y_v \geq 0$ for every $v \in S$ and $\sum_{v \in S} (b_v - y_v) = 1$, $z \in Q(G_1, b)$. Note for every $u \in S$,

$$\begin{aligned} z(\delta(u)) &= \sum_{v \in S} (b_v - y_v) z^v(\delta(u)) \\ &= \sum_{v \in S \setminus \{u\}} (b_v - y_v) b_u + (b_u - y_u)(b_u - 1) \\ &= b_u(b(S) - y(S)) - (b_u - y_u) = y_u. \end{aligned}$$

So $y \in P(G_1, b)$. ■

Proposition 4.2.3 *Let $S \subseteq V$ such that $G[S]$ is b -critical. Suppose $z' \in P(G', b)$ where $G' = G \times \{S\} = (V', E')$. Then z' can be extended to $z \in \mathbb{R}^E$ such that*

$z \in Q(G, b)$ and $z(\gamma(S)) = (b(S) - 1)/2$. Moreover, if $x \in \mathbb{R}^V$ and x, z' satisfy $0 \leq x \leq b$, $z'(\delta_{G'}(v)) \leq x_v$ for $v \in V' \setminus \{S\}$, and $z'(\delta(S)) \leq x(S) - b(S) + 1$, then z can be chosen also to satisfy $z(\delta(v)) \leq x_v$ for all $v \in V$.

Proof: Take a vertex $w \in S$, define y_w to be $x_w - z'(\delta(w) \cap E') + z(\delta(S)) + b(S) - 1 - x(S)$, and define y_v , for $v \in S \setminus \{w\}$, to be $x_v - z'(\delta(v) \cap E')$. It is clear that y satisfies the conditions of Lemma 4.2.2, so $y \in P(G[S], b)$. Let $(z_e : e \in \gamma(S))$ be a vector of $Q(G[S], b)$ such that y is its degree sequence. We extend this vector to $z \in \mathbb{R}^E$ such that $z_e = z'_e$ for all $e \in E'$. We show that z is what we need.

We need to show that $z(\gamma(S')) \leq (b(S') - 1)/2$ for all odd subsets S' of V . If $S' \subseteq S$, it is clearly true. If $S \subseteq S'$, it follows from

$$\begin{aligned} z(\gamma(S')) &= z'(\gamma_{G'}((S' \setminus S) \cup \{S\})) + z(\gamma(S)) \\ &\leq b(S' \setminus S)/2 + (b(S) - 1)/2 = (b(S') - 1)/2. \end{aligned}$$

Suppose that $S \setminus S'$, $S' \setminus S$, $S \cap S'$ are all non-empty. If $b(S \cap S')$ is odd, then

$$\begin{aligned} z(\gamma(S')) + z(\gamma(S)) &\leq z'(\gamma((S \cup S'))) + z(\gamma(S \cap S')) \\ &\leq b(S \cup S')/2 + (b(S \cap S') - 1)/2 \\ &= (b(S) - 1)/2 + (b(S') - 1)/2 \\ &= (b(S') - 1)/2 + z(\gamma(S)), \end{aligned}$$

so $z(\gamma(S')) \leq (b(S') - 1)/2$. If $b(S \cap S')$ is even, then

$$\begin{aligned} z(\gamma(S')) + (b(S) - 1)/2 &= z(\gamma(S')) + z(\gamma(S)) \\ &\leq z'(\gamma((S \setminus S'))) + z(\gamma(S' \setminus S)) + \sum (z(\delta(v)) : v \in S \cap S') \\ &\leq b(S \setminus S')/2 + (b(S' \setminus S) - 1)/2 + b(S \cap S') \\ &= (b(S) - 1)/2 + (b(S') - 1)/2, \end{aligned}$$

so $z(\gamma(S')) \leq (b(S') - 1)/2$. ■

Given $y \in P(G, b)$, a vertex v is deficient (with respect to x) if $x_v - y_v > 0$. The deficiency of v is $x_v - y_v > 0$. The deficiency of y is $x(V) - y(V)$. We point out two important facts:

1. After shrinking or expanding, the deficiency remains the same.
2. If $G[S]$ is spanned by a simple b -critical graph $H[S]$ in Proposition 4.2.3, then extending z needs time $O(|\gamma(S)|)$.

The reason for the first remark is that for any pseudo vertex S of G' , $x(S) - y(S) = x(S) - 2z(\gamma(S)) - z(\delta(S)) = x(S) - (b(S) - 1) - z(\delta(S)) = x_S - z(\delta(S)) = x_S - y_S$. So $x(V) - y(V) = x(V') - y(V')$. As to the second one, for each $S \in \mathcal{S}$, let $z_e = 0$ for all $e \in \gamma(S)$ not in $H[S]$. Finding z_e in $H[S]$ takes time $O(|S|)$. The calculation for z carries on edges not on the cycle, then edges on the cycle. It takes time $O(|S|)$ to find z_e for the first cycle edge e . It takes time $O(|S|)$ to work out z_e for the remaining edges of the cycle. So the complexity is $O(|\gamma(S)|)$.

For $z \in Q(G', b)$, let $G'_+(z)$ denote the spanning subgraph of G' with edge set $\{e \in E : z_e > 0\}$. In the algorithm described later, we will keep $G'_+(z)$ independent by shrinking some subsets of vertices. Proposition 4.2.1 will be used to ensure membership for the current z, y . Recall that a graph is independent if it has no even cycle and each of the components contains at most one odd cycle. It is much easier to find an augmenting path in G' if $G'_+(z)$ is independent.

4.3 Augmenting walks and blossoms

In a walk $L = v_0, e_1, v_1, e_2, \dots, e_k, v_k$, a vertex or an edge is even or odd according to the parity of its subscript in its first occurrence in the sequence. A walk is even or odd according to the parity of its last edge. Let $G(L) = (V(L), E(L))$ denote the graph such that $v \in V(L)$ if v appears on L and $e \in E(L)$ if e appears on L . In a tree T with a distinguished root vertex r , a vertex or an edge is even or odd according to its parity in a simple path in T beginning at r . Similarly, we define even and odd

for vertices and edges in a forest of rooted trees. Given $z \in Q(G, b)$, a z -alternating walk is a walk in G such that $z_e > 0$ for any even edge e of the walk. For $z \in Q(G, b)$ and $Mz \leq x$, a walk is z -augmenting if it is an odd z -alternating walk, and its end vertices are deficient.

Type 1 augmentation

Figure 4.1: A Type 1 z -augmenting walk

A **Type 1 z -augmenting walk** is an augmenting path $L = v_0, e_1, v_1, \dots, v_{2k}, e_{2k+1}, v_{2k+1}$. See an example in Figure 4.1. Augmenting by amount ϵ means to replace z by z^ϵ , where

$$z^\epsilon_e = \begin{cases} z_e + \epsilon & \text{if } e \text{ is an odd edge of } L, \\ z_e - \epsilon & \text{if } e \text{ is an even edge of } L, \\ z_e & \text{if } e \text{ is not on } L. \end{cases}$$

ϵ is chosen to be $\min\{x_{v_0} - z(\delta(v_0)), x_{v_{2k+1}} - z(\delta(v_{2k+1})), z_{e_i}, i = 1, \dots, k\}$.

Proposition 4.3.1 *Let $z \in Q(G, b)$, $G_+(z)$ be independent, and L be a Type 1 z -augmenting path. If every component of $G_+(z)$ with an odd cycle has no vertex in common with L , then after a Type 1 augmentation on L , z is still in $Q(G, b)$, and $G_+(z)$ is independent.*

Proof: The component of $G(L) \cup G_+(z)$ containing L is bipartite. So the new z is in $Q(G, b)$, by Theorem 4.1.3. It is also clear that the new $G_+(z)$ is independent. ■

Type 2 augmentation

A **Type 2 z -augmenting walk** is a closed walk $W = v_0, e_1, v_1, \dots, e_{2k+2l+1}, v_{2k+2l+1}$, for some $k \geq 0$, $l \geq 1$, where $v_0, \dots, v_{k+2l}$ are distinct vertices, $v_{2k+2l+1-i} = v_i$ for $0 \leq i \leq k$, and $e_{2k+2l+1-i} = e_i$ for $0 \leq i \leq k$; that is, it consists of a path LW ,

Figure 4.2: A Type 2 z -augmenting walk

traversed once in each direction, and an odd cycle C_W . See an example in Figure 4.2.

Augmenting z by ε on W means to replace z by z^ε , where

$$z^\varepsilon = \begin{cases} z_e + 2\varepsilon & \text{if } e \text{ is an odd edge on the path of } W, \\ z_e - 2\varepsilon & \text{if } e \text{ is an even edge on the path of } W, \\ z_e + \varepsilon & \text{if } e \text{ is an odd edge on the cycle of } W, \\ z_e - \varepsilon & \text{if } e \text{ is an even edge on the cycle of } W, \\ z_e & \text{if } e \text{ is not on } W. \end{cases}$$

The Type 2 augmenting walks we considered have the property that the component H of $G(W) \cup G_+(z)$ containing $G(W)$ is independent. There are two problems to be solved.

1. Find the maximum ε such that $z^\varepsilon \in Q(G, b)$.

Let

$$\varepsilon_2 = \min\{z_e : e \text{ is an even edge on the cycle of } W\},$$

$$\varepsilon_3 = \min\{z_e/2 : e \text{ is an even edge on the path of } W\},$$

$$\varepsilon_4 = (x_{v_0} - z(\delta(v_0)))/2.$$

It is obvious that $\varepsilon \leq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$.

2. If $\varepsilon < \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$, find a subset S_1 of vertices such that $H[S_1]$ is b -critical, and $2z^{\varepsilon'}(\gamma(S_1)) > b(S_1) - 1$ for any $\varepsilon' > \varepsilon$. Such $H[S_1]$ is called a **blossom**, which is to be shrunk.

Let S_1 be the set of all $S \subseteq V(H)$ such that

$$(S1) \quad C_W \subseteq S;$$

(S2) $b(S)$ is odd;

(S3) $H[S]$ is connected;

(S4) $\delta(S)$ does not contain any odd edge of L_W .

Let

$$(4.4) \quad \varepsilon_1 = \min\{b(S) - 1 - 2z(\gamma_H(S)) : S \in \mathcal{S}_1\}/2.$$

Proposition 4.3.2 *Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. Then after a Type 2 augmentation, z^* is in $Q(G, b)$.*

Proof: Suppose that the statement is not true. Let S be a minimal subset of V such that $b(S)$ is odd, and $2z^*(\delta(S)) > b(S) - 1$. Then 1) $H[S]$ is connected, for otherwise z^* must violate a valid inequality on one of its odd components; 2) $S \subseteq V(H)$ because of 1); 3) $C_W \subseteq S$, since otherwise $H[S]$ is bipartite, a contradiction; 4) $\delta(S)$ contains at most one edge of L_W from 1). If it is an odd edge, then $z^*(g(S)) \leq z(g(S))$, a contradiction. So $S \in \mathcal{S}_1$. But $b(S) - 1 - 2z(\gamma(S)) \geq 2\varepsilon_1$, and $z^*(\gamma(S)) = z(\gamma(S)) + 2\varepsilon \leq z(\gamma(S)) + 2\varepsilon_1 \leq b(S) - 1$, a contradiction. ■

Now we show how to find ε_1 and an optimizing $S_1 \in \mathcal{S}_1$ such that S_1 is minimal. The reason for S_1 to be minimal is that in this case $H[S_1]$ is b -critical, and therefore S_1 can be shrunk. This implication is proved in Theorem 4.3.4. For any $S \subseteq V(H)$, $2z(\gamma_H(S)) + z(\delta_H(S)) = y(S)$. So

$$\begin{aligned} b(S) - 1 - 2z(\gamma_H(S)) &= b(S) - 1 - y(S) + z(\delta_H(S)) \\ &= (b - y)(S) - 1 + z(\delta_H(S)) \\ &= (b - y)(V(H)) - 1 + (y - b)(V(H) \setminus S) + z(\delta_H(S)). \end{aligned}$$

Since $(b - y)(V(H)) - 1$ is a constant, we only need to minimize $(y - b)(V(H) \setminus S) + z(\delta_H(S))$.

Figure 4.3: A subgraph H and the tree T obtained from H

Let T be the rooted tree obtained by contracting odd edges of L_W , and shrinking the cycle C_W to a root vertex r . See an example in Figure 4.3. Let $S \subseteq V(T)$. Then there is a unique maximum vertex set $S' \subseteq V(H)$ such that S is obtained from S' as above. We use $b(S)$ and $y(S)$ to mean $b(S')$ and $y(S')$ respectively, (and we only use these notations in this section). Let T_v be the vertex-set that includes v and all its descendants. We also use T_v to mean the subtree of T induced by this vertex set, having root v . For edge $uv \in E(T)$, where u is the parent of v , we use (u, v) to mean edge uv and this relation. Let $c_{(u,v)} = z_{uv} + (y - b)(T_v)$. Then

$$(y - b)(V(H) \setminus S) + z(\delta_H(S)) = c(\delta_T(S)) = c(\delta_T(V(T) \setminus S)).$$

So (4.4) is equivalent to the following,

$$(4.5) \quad \min\{c(\delta_T(S)) : S \subseteq V(T) \text{ such that (a) } r \notin S, (b) b(V(H)) - b(S) \text{ is odd,} \\ (c) T - S \text{ is connected}\}.$$

Note that without parity condition (b), (4.5) is easy to solve. To overcome this difficulty, we solve the problem recursively. Let

$$(4.6) \quad oc(v) := \min\{c(\delta_T(S)) : S \subseteq T_v \text{ such that (a) } v \notin S, (b) b(S) \text{ is odd,} \\ (c) T - S \text{ is connected}\},$$

$$(4.7) \quad ec(v) := \min\{c(\delta_T(S)) : S \subseteq T_v \text{ such that (a) } v \notin S, (b) b(S) \text{ is even,} \\ (c) T - S \text{ is connected}\}.$$

Suppose that we have maximal sets $S_o(v)$, $S_e(v)$ that optimize (4.6) and (4.7) respectively such that one contains the another. (We will show that we can require

that.) Then

$$\epsilon_1 = \begin{cases} oc(r) + (b - y)(V(H)) - 1 & \text{if } b(V(H)) \text{ is even,} \\ ec(r) + (b - y)(V(H)) - 1 & \text{if } b(V(H)) \text{ is odd,} \end{cases}$$

and

$$S_1 = \begin{cases} V(H) \setminus S_o(r) & \text{if } b(V(H)) \text{ is even,} \\ V(H) \setminus S_e(r) & \text{if } b(V(H)) \text{ is odd.} \end{cases}$$

We give a way to solve (4.6) and (4.7) and find a pair of $S_o(v)$, $S_e(v)$ from the leaves of the tree. We remark it is possible that for some vertex v , especially a leaf, of the tree, $S_o(v)$ does not exist. There are two ways to handle this problem. One way is to introduce a notation *nil* to mean “not exist” with a relation that *nil* is contained in any set including the empty set. Another way is to add, for each leaf of the tree, a new vertex and an edge to connect them. Let $b_v = 1$ for the new vertex, and $c_e = \infty$ for the new edge. We use the first in the algorithm.

For $v \in V(T)$, let u be the parent of v , and let $T'_v = T_v \cup \{u\}$. We also use T'_v to mean the subtree of T induced by T'_v , having u as root. See an example in Figure 4.4.

Figure 4.4: A subtree T_v and a subtree T'_v

Let

$$(4.8) \quad p_v := \min\{c(\delta_T(S)) : S \subseteq T'_v \text{ such that (a) } u \notin S, (b) b(S) \text{ is odd,} \\ (c) \ T - S \text{ is connected}\},$$

$$(4.9) \quad q_v := \min\{c(\delta_T(S)) : S \subseteq T'_v \text{ such that (a) } u \notin S, (b) b(S) \text{ is even,} \\ (c) \ T - S \text{ is connected}\}.$$

Then

$$p_v = \begin{cases} \min\{oc(v), c_{(u,v)}\} & \text{if } b(T_v) \text{ is odd,} \\ oc(v) & \text{if } b(T_v) \text{ is even;} \end{cases}$$

$$q_v = \begin{cases} \min\{ec(v), c_{(u,v)}\} & \text{if } b(T_v) \text{ is even,} \\ ec(v) & \text{if } b(T_v) \text{ is odd.} \end{cases}$$

Let

$$S'_o(v) := \begin{cases} T_v & \text{if } oc(v) \geq c_{(u,v)} \text{ and } b(T_v) \text{ is odd,} \\ S_o(v) & \text{otherwise;} \end{cases}$$

$$S'_e(v) := \begin{cases} T_v & \text{if } oc(v) \geq c_{(u,v)} \text{ and } b(T_v) \text{ is even,} \\ S_e(v) & \text{otherwise.} \end{cases}$$

Then $S'_o(v)$, $S'_e(v)$ are maximal sets optimizing (4.8) and (4.9) respectively, and one of them contains the other.

Let $w \in V(T)$ such that $S'_o(v)$ and $S'_e(v)$ are known for the set $X := \{v_1, \dots, v_k\}$ of its children . Then

$$\begin{aligned} oc(w) &= \min\{p(X') + q(X \setminus X') : X' \subseteq X, |X'| \text{ is odd.}\} \\ (4.10) \quad &= \min\{(p - q)(X') : X' \subseteq X, |X'| \text{ is odd.}\} + q(X). \end{aligned}$$

$$\begin{aligned} ec(w) &= \min\{p(X') + q(X \setminus X') : X' \subseteq X, |X'| \text{ is even.}\} \\ (4.11) \quad &= \min\{(p - q)(X') : X' \subseteq X, |X'| \text{ is even.}\} + q(X). \end{aligned}$$

To find $oc(w)$ and $ec(w)$ of (4.10) and (4.11) is easy, but to find a pair of $S_o(w)$ and $S_e(w)$ is less obvious. We give a way to solve these problems.

Let π be the vector of $\mathbb{Z}^X$ such that $\pi_v = 1$ if $S'_o(v) \subset S'_e(v)$, $\pi_v = -1$ if $S'_e(v) \subset S'_o(v)$. π is well defined. For every $v \in X$, let

$$\psi_v = \begin{pmatrix} p_v - q_v \\ \pi_v \end{pmatrix}.$$

For any $v, u \in X$, $\psi_v \prec \psi_u$ if either $p_v - q_v < p_u - q_u$, or $p_v - q_v = p_u - q_u$ and $\pi_v < \pi_u$. $\psi_v = \psi_u$ if all the components all equal. It is easy to check that this defines a total

order on the vector set. It is called lexicographical order. Let $X_- = \{v \in X : \psi_v < 0\}$, $X_+ = \{v \in X : \psi_v > 0\}$. If $|X_-|$ is odd, then $X_o := X_-$ optimizes (4.10). Let $v \in X_-$ such that ψ_v is the maximum of X_- , and let $u \in X_+$ such that ψ_u is the minimum of X_+ . If $\psi_v + \psi_u < 0$, then $X_e := X_- \cup \{u\}$ optimizes (4.11), otherwise $X_e := X_- \setminus \{v\}$ optimizes (4.11).

If $|X_-|$ is even, then $X_e := X_-$ optimizes (4.11). In this case, (4.10) is solvable by a method similar to that for solving (4.11) in the case when $|X_-|$ is odd.

It is clear that after the above procedure is applied, the conditions (X1), (X2), and (X3) for X_o, X_e are satisfied.

(X1) for any $u, v \in X_o \cup X_e$, $p_u + p_v \leq q_u + q_v$, and if it is tight, then at least one of $S'_e(u) \subset S'_o(u)$, $S'_e(v) \subset S'_o(v)$ holds;

(X2) for any $u, v \in X \setminus X_o \cup X_e$, $p_u + p_v \geq q_u + q_v$, and if it is tight, then at least one of $S'_o(u) \subset S'_e(u)$, $S'_o(v) \subset S'_e(v)$ holds;

(X3) for any $u \in X_o \cup X_e$ and $v \in X \setminus X_o \cup X_e$, $p_u + q_v \leq q_u + p_v$, and if it is tight, then at least one of $S'_e(u) \subset S'_o(u)$, $S'_o(v) \subset S'_e(v)$ holds.

Proposition 4.3.3 $\cup_{v \in X_o} S'_o(v) \cup \cup_{v \in X \setminus X_o} S'_e(v)$ and $\cup_{v \in X_e} S'_e(v) \cup \cup_{v \in X \setminus X_e} S'_o(v)$ are maximal sets optimizing (4.6) and (4.7) respectively and one of them contains the other.

Proof: It is clear that they optimize (4.6) and (4.7), and one of them contains the other. We only need to prove that they are maximal.

If $S_o(w) := \cup_{v \in X_o} S'_o(v) \cup \cup_{v \in X \setminus X_o} S'_e(v)$ found in the algorithm is not maximal, then there is an optimizing $S''_o(w) = \cup_{v \in X''} S''_o(v) \cup \cup_{v \in X \setminus X''} S''_e(v)$ such that $S_o(w) \subset S''_o(w)$, where $b(S''_o(v))$ and $b(S''_e(v))$ are odd and even respectively. Then $X'' \Delta X_o$, the symmetric difference, is not empty, and so there must be at least two vertices $u, v \in X'' \Delta X_o$ since both $|X''|$ and $|X_o|$ are odd.

Case 1. $u, v \in X'' \setminus X_o$. Note that $c(\delta(S''_o(v))) + c(\delta(S''_o(u))) \leq q_u + q_v$, $p_u \leq c(\delta(S''_o(v)))$, and $p_u \leq c(\delta(S''_o(u)))$. $q_u + q_v \leq p_u + p_v$ from (X2). So $q_u + q_v = p_u + p_v$,

whence at least one of $S'_o(u) \subset S'_e(u)$, $S'_o(v) \subset S'_e(v)$ holds. This implies that one of $S'_o(u) \subset S''_o(u)$, $S'_o(v) \subset S''_o(v)$ holds, contradicting the maximality of $S'_o(v)$ and $S'_o(u)$.

Case 2. $u, v \in X_o \setminus X''$. It is similar to Case 1.

Case 3. $u \in X_o \setminus X''$ and $v \in X'' \setminus X_o$. Note that $c(\delta(S''_e(u))) + c(\delta(S''_o(v))) \leq p_u + q_v$, $q_u \leq c(\delta(S''_e(u)))$, and $p_v \leq c(\delta(S''_e(v)))$. $p_u + q_v \leq q_u + p_v$ from (X3). Thus all the above inequalities must be tight, especially, $p_u + q_v = q_u + p_v$. This implies that at least one of $S'_e(u) \subset S'_o(u)$, $S'_o(v) \subset S'_e(v)$ holds. So one of $S'_e(u) \subset S''_e(u)$, $S'_o(v) \subset S''_o(v)$ holds, contradicting the maximality of $S'_e(u)$ and $S'_o(v)$.

For $S_e(w)$, the proof is similar to that of $S_o(w)$. ■

Algorithm C

Input: W, H .

Output: $\varepsilon_1, S_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon$.

Step 0 Shrink the odd cycle C_W and the odd edges of the path L_W . For each $v \in T$, $S_o(v) := \text{nil}$, $S_e(v) := \emptyset$, $oc(v) := \infty$, $ec(v) := 0$. Label the leaves of T . The rest are unlabelled.

Step 1 If the root vertex is labelled, let

$$\varepsilon_1 = \begin{cases} oc(r) + (b - y)(V(H)) - 1 & \text{if } b(V(H)) \text{ is even,} \\ ec(r) + (b - y)(V(H)) - 1 & \text{if } b(V(H)) \text{ is odd,} \end{cases}$$

$$S_1 = \begin{cases} V(H) \setminus S_o(r) & \text{if } b(V(H)) \text{ is even,} \\ V(H) \setminus S_e(r) & \text{if } b(V(H)) \text{ is odd,} \end{cases}$$

$$\varepsilon_2 = \min\{z_e : e \text{ is an even edge on } C_W\},$$

$$\varepsilon_3 = \min\{z_e/2 : e \text{ is an even edge on } L_W\},$$

$$\varepsilon_4 = (x_{v_0} - z(\delta(v_0)))/2,$$

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}. \text{ Stop.}$$

Otherwise take an unlabelled w with all children $v_1, \dots, v_k$ labelled. Let X denote $\{v_1, \dots, v_k\}$. For each $v \in X$, let

$$p_v = \begin{cases} \min\{oc(v), c_{(w,v)}\} & \text{if } b(T_v) \text{ is odd,} \\ oc(v) & \text{if } b(T_v) \text{ is even;} \end{cases}$$

$$q_v = \begin{cases} \min\{ec(v), c_{(w,v)}\} & \text{if } b(T_v) \text{ is even,} \\ ec(v) & \text{if } b(T_v) \text{ is odd;} \end{cases}$$

$$S'_o(v) = \begin{cases} T_v & \text{if } oc(v) \geq c_{(w,v)} \text{ and } b(T_v) \text{ is odd,} \\ S_o(v) & \text{otherwise;} \end{cases}$$

$$S'_e(v) = \begin{cases} T_v & \text{if } oc(v) \geq c_{(w,v)} \text{ and } b(T_v) \text{ is even,} \\ S_e(v) & \text{otherwise;} \end{cases}$$

$$\pi_v = \begin{cases} 1 & \text{if } S'_o(v) \subset S'_e(v), \\ -1 & \text{if } S'_e(v) \subset S'_o(v); \end{cases}$$

$$\psi_v = \begin{pmatrix} p_v - q_v \\ \pi_v \end{pmatrix}.$$

Let $X_- = \{v \in X : \psi_v \prec 0\}$, $X_+ = \{v \in X : \psi_v \succ 0\}$. Let $v \in X_-$ such that ψ_v is maximum over X_- , take $u \in X_+$ such that ψ_u is minimum over X_+ . If $|X_-|$ is odd, let $X_o := X_-$; if $\psi_v + \psi_u \prec 0$, let $X_e := X_- \cup \{u\}$, otherwise let $X_e := X_- \setminus \{v\}$. If $|X_-|$ is even, let $X_e := X_-$; if $\psi_v + \psi_u \prec 0$, let $X_o := X_- \cup \{u\}$, otherwise let $X_o := X_- \setminus \{v\}$. Let

$$S_o(w) = \cup_{v \in X_o} S'_o(v) \cup \cup_{v \in X \setminus X_o} S'_e(v),$$

$$S_e(w) = \cup_{v \in X_e} S'_e(v) \cup \cup_{v \in X \setminus X_e} S'_o(v).$$

Label w . Go to Step 1.

Theorem 4.3.4 *Algorithm C terminates in time $O(|V(H)|)$ with ε_1 and an optimizing S_1 of (4.4) such that S_1 is minimal. If $\varepsilon_1 < \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$, then $H[S_1]$ is b -critical.*

Proof: It is clear that the algorithm terminates in time $O(|V(H)|)$. Proposition 4.3.3 shows S_1 is minimal.

For the second part, after we do Type 2 augmentation, $2z(\gamma(S_1)) = b(S_1) - 1$. For any $S_2 \in \mathcal{S}_1$ such that $S_2 \subset S_1$, $2z(\gamma(S_2)) < b(S_2) - 1$.

If $v_0 \in S_1$, let $v_r := v_0$. If $v_0 \notin S_1$, let v_r be the first vertex of L_W in S_1 . Note that $z(\delta_{H[S_1]}(v_r)) < b_w$, so there is a b -matching z^r of $H[S_1]$ such that $2z^r(\gamma(S_1)) = b(S_1) - 1$ and $z^r(\delta_{H[S_1]}(v_r)) = b_w - 1$.

Let ϵ' be a positive number such that $\epsilon' \leq \min\{\epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_1, \epsilon_4 - \epsilon_1\}$, $\epsilon' \leq \min\{b(S) - 1 - 2z(\gamma(S)) : S \in \mathcal{S}_1, S \in S_1\}$, and $\epsilon' \leq b_w - z(\delta_{H[S_1]}(v_r))$.

For any $w \in S_1 \setminus \{v_r\}$, there is a z -alternating walk from v_r to w in $H[S_1]$. After we do ϵ' alternating change on the walk, i.e. $z_e^{\epsilon'} = z_e + \epsilon'$ for any odd edge of the walk, $z_e^{\epsilon'} = z_e - \epsilon'$ for any even edge of the walk, the vector $z^{\epsilon'}$ induced on $\gamma_H(S_1)$ is in $Q(H[S_1], b)$. So there must be a b -matching z in $H[S_1]$ such that $2z(\gamma(S_1)) = b(S_1) - 1$ and $z(\delta(w)) = b_w - 1$. Since w is arbitrary, $H[S_1]$ must be b -critical. ■

4.4 Finding an augmenting walk

In this section, we design an algorithm to find either a positive augmentation, or an optimizing $y = Mz$ of (4.1), and optimizing $B, A_0, A_1, \dots, A_k$ of (4.2).

Throughout the algorithm, we keep $\mathcal{S}$, $G' = G \times \mathcal{S}$, $z \in Q(G', b)$ such that the current graph $G'_+ := G'_+(z)$ satisfies the following condition.

Condition I

- (1) G'_+ is independent.
- (2) Each component of G'_+ has at most one deficient vertex, and if it has one, then it is a tree.

The algorithm grows a z -alternating rooted forest such that the roots are exactly the deficient vertices. The vertices of the forest are labelled hollow or solid. A vertex

$v \in V$ is **hollow (solid)** if it is a hollow (solid) vertex or contained in a hollow (solid) pseudo vertex of the forest. A labelling is **proper** if each even vertex of the forest is labelled hollow, and each odd vertex v of the forest is labelled solid. One fact is that the labelling in the algorithm is always proper. It will be proved later. The condition below is the termination condition of the algorithm.

Condition T

- (1) No pseudo vertex is solid.
- (2) Every edge of G' incident with a hollow vertex of F is incident with a solid vertex of F .

Theorem 4.4.1 *Suppose that G' , z , S meet Condition T. Let B be the set of solid vertices of the forest F , let A_0 be the set of all real hollow vertices of F , and let $A_1, \dots, A_k$ be the maximal members of S in F , i.e. the pseudo vertices of F . Then a maximizing $y = Mz$ of (4.1), and minimizing $B, A_0, A_1, \dots, A_k$ of (4.2) are found.*

Proof: Note that A_0 is a set of isolated vertices of $G - B$, and each A_i is the vertex-set of a component of $G - B$ such that $b(A_i)$ is odd for $i = 1, \dots, k$.

Suppose that z, y are extended to the original graph. Then $x(V) - y(V) = x(V') - y(V') = \sum_1^k (x_{A_i} - y_{A_i}) + x(A_0) - y(A_0) = \sum_1^k x_{A_i} + x(A_0) - y(B) = \sum_1^k (x(A_i) - b(A_i) + 1) + x(A_0) - x(B) = \sum_0^k x(A_i) - x(B) - \sum_1^k (b(A_i) + 1)$. So $y(V) = \sum_1^k (b(A_i) + 1) + x(B) + x(V \setminus \cup_0^k A_i)$. Thus $y = Mz$ maximizes (4.1) and $B, A_0, A_1, \dots, A_k$ minimize (4.2). ■

Algorithm Augment

Input: a shrinking family S of G , $G' = G \times S$, $z \in Q(G', b)$ such that $G'_+ := G'_+(z)$ is independent.

Output: either new such S of G , $G' = G \times S$, $z \in Q(G', b)$, $G'_+(z)$, such that z has bigger component-sum, or an optimizing $y = Mz$ of (4.1) and optimizing $B, A_0, A_1, \dots, A_k$ of (4.2).

Step 0 Let F be the union of all deficient vertices as roots. Label them hollow. If there is an edge $uv \in E(G') \setminus E(F)$ such that v is labelled hollow, then go to Step 1. If no such edge exists and there is a solid pseudo vertex S of a tree $T = H \times S$ of F ($H[S]$ is simple b -critical), then go to Step 2. If Condition T is satisfied, then go to Step 3.

Step 1 Let T be the tree of F containing v .

1. (Type 1 augmentation) u is hollow and u, v are in different trees of F . A simple augmenting path is found. Do a Type 1 augmentation. Stop.
2. u is hollow and u, v are in the same tree. Let H be the subgraph obtained by adding uv to T . Apply Algorithm C to find S_1, ϵ .
 - (a) If $\epsilon > 0$, go to Step 4.
 - (b) If $\epsilon = 0$, shrink S_1 , and label S_1 hollow. Replace T with $T' := H \times S_1$, S with $S \cup \{S_1\}$. Go to Step 1.
3. (Simple forest extension) $u \notin V(F)$ and the component K of G'_+ containing u is a tree. Add K and uv to F , and label $w \in V(K)$ hollow [solid] if the distance from w to u is odd [even]. Go to Step 1.
4. (Non-simple forest extension) $u \notin V(F)$ and the component K of G'_+ containing u is not a tree. Add K and uv to T , and let H be the resulting graph. Apply Algorithm C to get S_1, ϵ .
 - (a) If $\epsilon > 0$, go to Step 4.
 - (b) If $\epsilon = 0$, shrink S_1 , and label S_1 hollow. Let $T' := H \times \{S_1\}$. Label $w \in V(K)$ hollow [solid] if the distance from w to S_1 is even [odd]. Replace T with T' , S with $S \cup \{S_1\}$. Go to Step 1.

Step 2 (Expanding a pseudo vertex) Expand S such that $z_e = 0$ for every $e \in \gamma(S) \setminus E_H(S)$.

1. $E_0 := \{e \in E(W) : z_e = 0, e \text{ is even}\} \neq \emptyset$. Let T' be the component of $H - E_0$ containing u_1 , the first vertex of W in S . Label u_1 solid, and label $v \in V(T') \setminus V(T)$ hollow [solid] if the distance from v to u_1 in T' is odd [even]. Replace T with T' , S with $S \setminus \{S\}$. Go to Step 1.
2. $E_0 = \emptyset$. Apply Algorithm C to find S_1, ϵ .
 - (a) If $\epsilon > 0$, go to Step 4.
 - (b) If $\epsilon = 0$, shrink S_1 , and label $v \in S \setminus S_1$ hollow [solid] if the distance from v to S in $H \times S_1$ is even [odd]. Delete $\{e : z_e = 0, e \text{ is an edge of } H \times S_1\}$ from $H \times S_1$. Let T' be the component of the remaining graph containing S_1 . Replace T with T' , S with $S \cup \{S_1\} \setminus \{S\}$. Go to Step 1.

Step 3 (Termination) Extend z to $\mathbf{R}_+^E$ and let $y = Mz$. Return $z, y, B, A_0, A_1, \dots, A_k$. Stop.

Step 4 (Type 2 augmentation and updating G') Do a Type 2 augmentation.

1. If $\epsilon = \min\{\epsilon_2, \epsilon_3, \epsilon_4\}$, stop.
2. If $\epsilon = \epsilon_1 < \min\{\epsilon_2, \epsilon_3, \epsilon_4\}$, shrink S_1 . Stop.

Theorem 4.4.2 *Algorithm Augment terminates in time $O(|V||E|)$, with either satisfying termination conditions, or having a positive augmentation.*

Remark The complexity in Theorem 4.4.2 can be sharpened to $O(|E|)$ if we implement Algorithm Augment carefully. The idea is similar to that in the maximum cardinality matching algorithm [19]: each edge is considered only a constant number of times. But the details are more complicated. Also we can change the algorithm slightly so that it outputs a properly labelled forest for later use.

In order to prove Theorem 4.4.2, we claim that the following condition holds after just finishing any step neither resulting in a positive augmentation (i.e. $\epsilon > 0$), nor achieving the termination condition.

Condition A

- (1) The labelling of T' is proper.
- (2) T' is z -alternating.
- (3) If v is a hollow vertex of T , then it is hollow in T' .

Suppose that the claim is true. We prove the theorem.

Proof of Theorem 4.4.2: Let $p(F) = |\{v \in V : v \text{ is hollow in } F\}| - |\{S \in \mathcal{S} : S \text{ is not hollow in } F\}|$. It is clear that $-2|V| \leq p(F) \leq |V|$. It is easy to check that $p(F)$ increases at least by 1 after each step other than a positive augmentation. Performing such a step needs $O(|E|)$ basic operations. So the statement is true. ■

What we will do in the rest of the section is to prove the claim is true.

Proposition 4.4.3 *Let $V_1 \subset S$, $V_2 = S \setminus V_1$. Suppose that $2z(\gamma(S)) = b(S) - 1$.*

- 1. *If $z[V_1, V_2] = 0$, then either $2z(\gamma(V_1)) = b(V_1)$, or $2z(\gamma(V_2)) = b(V_2)$.*
- 2. *If $z(\gamma(S)) \leq b(V_1)$ and $z(\gamma(S)) \leq b(V_2)$, then either $z(\gamma(S)) = b(V_1)$, or $z(\gamma(S)) = b(V_2)$.*

Proof: 1.

$$\begin{aligned} \frac{b(S) - 1}{2} = z(\gamma(S)) &= z(\gamma(V_1)) + z(\gamma(V_2)) \\ &\leq \left\lfloor \frac{b(V_1)}{2} \right\rfloor + \left\lfloor \frac{b(V_2)}{2} \right\rfloor. \end{aligned}$$

So $z(\gamma(V_1)) = \lfloor b(V_1)/2 \rfloor$, $z(\gamma(V_2)) = \lfloor b(V_2)/2 \rfloor$. Since $b(S)$ is odd, one of $b(V_1)$ and $b(V_2)$ is odd. So either $2z(\gamma(V_1)) = b(V_1)$, or $2z(\gamma(V_2)) = b(V_2)$.

2. $b(S) - 1 = 2z(\gamma(S)) \leq b(V_1) + b(V_2) = b(S)$. Since $z(\gamma(S))$ is an integer, either $z(\gamma(S)) = b(V_1)$, or $z(\gamma(S)) = b(V_2)$. ■

Proposition 4.4.4 *For Step 1.2.b, Condition A holds.*

Proof: We only need to prove Condition (A1) holds, since Condition (A2), (A3) follow automatically.

Let $V_1 = \text{hollow}(S_1) := \{w \in S_1 : w \text{ is hollow in } T\}$, $V_2 = \text{solid}(S_1) := \{w \in S_1 : w \text{ is solid in } T\}$. Since the first edge e entering S_1 is even in T and $z_e > 0$, $z(\gamma(S_1)) < b(V_1)$. So by Proposition 4.4.3(2), $z(\gamma(S_1)) = b(\text{solid}(S_1))$. This implies that there is no hollow vertex adjacent to S_1 in T' . So the labelling of T' is proper. Condition (A1) holds. ■

Proposition 4.4.5 *For Step 1.4.b, Condition A holds.*

Proof: We only need to prove Condition (A1) holds, since Condition (A2), (A3) follow automatically.

Let $V_1 = \text{hollow}(S_1) := \{w \in S_1 : w \text{ is hollow in } T\}$, $V_2 = \text{solid}(S_1) := \{w \in S_1 : w \text{ is solid in } T\}$. Let $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$ be components of $H[S_1] - vu$ containing v, u respectively. Then $z(E_1) = \lfloor b(V_1)/2 \rfloor$, $z(E_1) \leq b(\text{hollow}(V_1)) = b(\text{hollow}(S_1))$, $z(E_1) \leq b(\text{solid}(V_1)) = b(\text{solid}(S_1))$.

Since the first edge e entering S_1 is even in T and $z_e > 0$, $z(\gamma(S_1)) < b(V_1)$. So by Proposition 4.4.3(2), $z(\gamma(S_1)) = b(\text{solid}(S_1))$. Hence Condition (A1) holds. ■

Proposition 4.4.6 *For Step 2.1, Condition A holds.*

Proof: For any $v' \in S$ such that $z(\delta_{H[S]}(v')) < b_{v'}$, and any $u' \in S$ such that $z(\delta_{H[S]}(u')) > b_{u'} - 1$, let $y \in \mathbb{R}^S$ such that

$$y_v = \begin{cases} z(\delta_{H[S]}(v)) + \varepsilon & \text{if } v = v', \\ z(\delta_{H[S]}(v)) - \varepsilon & \text{if } v = u', \\ z(\delta_{H[S]}(v)) & \text{else,} \end{cases}$$

where $0 < \varepsilon < \min\{b_{v'} - z(\delta_{H[S]}(v')), z(\delta_{H[S]}(u')) - b_{u'} + 1\}$. Then $0 \leq y \leq b$, and $y(S) = b(S) - 1$. So $y \in P(H[S], b)$ by Lemma 4.2.2.

We choose different kind of v', u' in each of the following cases.

Let $H_1 = (V_1, E_1)$ denote $H[S] \cap T'$. Let $\text{hollow}(V_1) = \{w \in S_1 : w \text{ is hollow in } T'\}$, $\text{solid}(V_1) = \{w \in S_1 : w \text{ is solid in } T'\}$.

1. If $z(E_1) < b(\text{hollow}(V_1))$, then $b(V_1)$ is odd and $z(\gamma(S)) = b(\text{solid}(V_1))$ by Proposition 4.4.3(2). Let $B = \text{solid}(V_1)$, $A_0 = \text{hollow}(V_1)$ in $H[S]$. We choose $v' \in \text{hollow}(V_1)$, $u' \in \text{solid}(V_1)$. Then $y(A_0) - y(B) = z(E_1) - z(E_1) + 2\varepsilon = 2\varepsilon > 0$. So $y \notin P(H[S], b)$, a contradiction. Thus $z(E_1) = b(\text{hollow}(V_1))$, and Condition (A1) holds.

2. Suppose $z(\gamma(S) \setminus E_1) < b(S \setminus V_1)/2$. Then $\lfloor b(V_1)/2 \rfloor = z(E_1) = b(S \setminus V_1)/2$. $b(V_1)$ is even. Choose $u' \in \text{solid}(V_1)$, $v' \in S \setminus V_1$ such that $z(\delta_{H[S]}(v')) < b_v$. Let $B = \text{solid}(V_1)$, $A_0 = \text{hollow}(V_1)$, $A_1 = \{S \setminus V_1\}$ in $H[S]$. Then $y(A_0) + y(A_1) - y(B) = b(A_1) - 1 + 2\varepsilon > b(A_1) - 1$. So $y \notin P(H[S], b)$, a contradiction. Thus $z(\gamma(S) \setminus E_1) = b(S \setminus V_1)/2$, and Condition (A2) holds.

3. Suppose $z_{w_1 w_2} = 0$ for certain even edge $w_1 w_2$ of H_1 , where w_1 is solid and w_2 is hollow in T' . Let $H' = (V', E')$, $H'' = (V'', E'')$ denote the components of $H_1 - w_1 w_2$ containing w_1, w_2 respectively. From (2) and Proposition 4.4.3(2), we have two cases:

a) $z(E') = b(V')/2$, $z(E'') < b(V'')/2$. Choose $u' = w_1$, $v' \in S \setminus V''$ such that $z(\delta_{H[S]}(v')) < b_v$. Let $B = \text{solid}(V') := V' \cap \text{solid}(V_1)$, $A_0 = V' \setminus B$, $A_1 = \{V''\}$ in $H[S]$. Then $y(A_0) + y(A_1) - y(B) = b(A_1) - 1 + 2\varepsilon > b(A_1) - 1$. So $y \notin P(H[S], b)$, a contradiction.

b) $z(E') < b(V')/2$, $z(E'') = b(V'')/2$. Choose $u' = w_2$, $v' \in S \setminus V'$ such that $z(\delta_{H[S]}(v')) < b_v$. Let $B = \text{hollow}(V'') := V'' \cap \text{hollow}(V_1)$, $A_0 = V'' \setminus B$, $A_1 = \{V'\}$ in $H[S]$. Then $y(A_0) + y(A_1) - y(B) = b(A_1) - 1 + 2\varepsilon > b(A_1) - 1$. So $y \notin P(H[S], b)$, a contradiction.

So $z_{w_1 w_2} > 0$ for any even edge $w_1 w_2$ of H_1 , Condition (A3) holds. ■

Proposition 4.4.7 *For Step 2.2.b, Condition A holds.*

Proof: We use $y \in \mathbb{R}^S$ as in the proof of Proposition 4.4.6. It is clear that (A3)

holds. We only need to prove that (A1) and (A2) hold. Since both S and S_1 contain C_W , $S_1 \cap S_2 \neq \emptyset$.

Case 1. $b(S_1 \cap S)$ is even.

1.

$$\begin{aligned}
 \frac{b(S) - 1}{2} + \frac{b(S_1) - 1}{2} &= z(\gamma(S_1)) + z(\gamma(S)) \\
 &\leq z(\gamma(S_1 \setminus S)) + z(\gamma(S \setminus S_1)) + \sum(z(\delta_{H[S_1 \cup S]}(v)) : v \in S_1 \cap S) \\
 &\leq \frac{b(S_1 \setminus S) - 1}{2} + \frac{b(S \setminus S_1) - 1}{2} + b(S_1 \cap S) \\
 &= \frac{b(S) - 1}{2} + \frac{b(S_1) - 1}{2}.
 \end{aligned}$$

So

$$\begin{aligned}
 (4.12) \quad z(\delta_{H[S_1 \cup S]}(v)) &= b_v \quad \forall v \in S_1 \cap S, \\
 z(\gamma(S_1 \setminus S)) &= \frac{b(S_1 \setminus S) - 1}{2}, \\
 z(\gamma(S \setminus S_1)) &= \frac{b(S \setminus S_1) - 1}{2}.
 \end{aligned}$$

Let

$$hollow(S \setminus S_1) = \{v \in S \setminus S_1 : v \text{ is hollow in } H \times S_1\},$$

$$solid(S \setminus S_1) = \{v \in S \setminus S_1 : v \text{ is solid in } H \times S_1\},$$

$$hollow(S_1 \setminus S) = \{v \in S_1 \setminus S : v \text{ is hollow in } T\},$$

$$solid(S_1 \setminus S) = \{v \in S_1 \setminus S : v \text{ is solid in } T\}.$$

If $z(\gamma(S \setminus S_1)) < b(hollow(S \setminus S_1))$, then $z(\gamma(S \setminus S_1)) < b(solid(S \setminus S_1))$ from (2) and Proposition 4.4.3.

Let $B = solid(S \setminus S_1)$, $A_0 = hollow(S \setminus S_1)$ in $H[S]$. We choose $v' \in A_0$, $u' \in B$. Then $y(A_0) - y(B) = z(\gamma(S \setminus S_1)) - z(\gamma(S \setminus S_1)) + 2\varepsilon = 2\varepsilon > 0$. So $y \notin P(H[S], b)$, a contradiction. Thus

$$(4.13) \quad z(\gamma(S \setminus S_1)) = b(hollow(S \setminus S_1)).$$

Let uv be the first edge of $E(T)$ entering $S_1 \setminus S$, where $v \in S_1 \setminus S$. Then v is hollow in T . So $z_{uv} > 0$, $z(\gamma(S_1 \setminus S)) < b(\text{hollow}(S_1 \setminus S))$. From Proposition 4.4.3,

$$(4.14) \quad z(\gamma(S_1 \setminus S)) = b(\text{solid}(S_1 \setminus S)).$$

Condition (A1) follows from (4.12), (4.13), and (4.14).

2. Suppose that there is uv in $H \times S_1$ such that $z_{uv} = 0$, where $u \in \text{solid}(S \setminus S_1)$, $v \notin V(T')$. Let $H'' = (V'', E'')$ denote the component of $H[S] - uv$ containing v . If $2z(E'') < b(V'')$, let $B = \text{solid}(S \setminus S_1)$, $A_0 = \text{hollow}(S \setminus S_1)$, $A_1 = \{V'\}$ in $H[S]$. We choose $v' \in V''$, $u' \in B$. Then $y(A_0) + y(A_1) - y(B) = b(A_1) - 1 + 2\epsilon > b(A_1) - 1$. So $y \notin P(H[S], b)$, a contradiction. So

$$(4.15) \quad z(E'') = b(V'').$$

Since uv is arbitrary, Condition (A2) follows.

Case 2. $b(S_1 \cap S)$ is odd.

$$\begin{aligned} \frac{b(S) - 1}{2} + \frac{b(S_1) - 1}{2} &= z(\gamma(S_1)) + z(\gamma(S)) \\ &\leq z(\gamma(S_1 \cup S)) + z(\gamma(S \cap S_1)) \\ &\leq \frac{b(S_1 \cup S) - 1}{2} + \frac{b(S \cap S_1) - 1}{2} \\ &= \frac{b(S) - 1}{2} + \frac{b(S_1) - 1}{2}. \end{aligned}$$

So

$$\begin{aligned} z(\gamma(S_1 \cup S)) &= \frac{b(S_1 \cup S) - 1}{2}, \\ z(\gamma(S \cap S_1)) &= \frac{b(S \cap S_1) - 1}{2}. \end{aligned}$$

Let $S'_1 = S_1 \times (S_1 \cap S) := (S_1 \setminus S) \cup \{S_1 \cap S_2\}$, $S' = S \times (S_1 \cap S)$. Let

$$\text{hollow}(S') = \{v \in S \setminus S_1 : v \text{ is hollow in } T'\} \cup \{S_1 \cap S\},$$

$$\text{solid}(S') = \{v \in S \setminus S_1 : v \text{ is solid in } T'\},$$

$$\text{hollow}(S'_1) = \{v \in S_1 \setminus S : v \text{ is hollow in } T\},$$

$$\text{solid}(S'_1) = \{v \in S_1 \setminus S : v \text{ is solid in } T\} \cup \{S_1 \cap S\}.$$

1. Since for any edge uv of $E(T)$ entering S_1 , $z_{uv} > 0$. So $z(\gamma(S'_1)) < b(\text{hollow}(S'_1))$, and Thus

$$(4.16) \quad z(\gamma(S'_1)) = b(\text{solid}(S'_1)).$$

If $z(\gamma(S')) < b(\text{hollow}(S'))$, then $z(\gamma(S'_1)) < b(\text{solid}(S'_1))$. Let $B = \text{solid}(S')$, $A_0 = \text{hollow}(S \setminus S_1)$, $A_1 = \{S \cap S_1\}$. We choose $v' \in A_0 \cup A_1$, $u' \in B$. Then

$$\begin{aligned} y(A_0) + y(A_1) - y(B) &= 2\varepsilon + \sum_{v \in A_0} z(\delta_{H[S]}(v)) + \sum_{v \in A_1} z(\delta_{H[S]}(v)) - \sum_{v \in B} z(\delta_{H[S]}(v)) \\ &= 2\varepsilon + \sum_{v \in A_0} z(\delta_{H[S]}(v)) + z(\delta_{H[S]}(A_1)) + 2z(\gamma(A_1)) - z(\gamma(S')) \\ &= 2\varepsilon + b(A_1) - 1, \end{aligned}$$

a contradiction. So

$$(4.17) \quad z(\gamma(S')) = b(\text{hollow}(S)).$$

If $S_1 \setminus S \neq \emptyset$ and $S \setminus S_1 \neq \emptyset$, then $z(\delta_{S_1}(S_1 \cap S)) = 1$, $z(\delta_S(S_1 \cap S)) = 1$ from (4.16) and (4.17). But this is impossible since $1 \geq z(\delta(S_1 \cap S)) \geq 2$. So either $S_1 \subset S$ or $S \subset S_1$. In each of these, we have

$$(4.18) \quad z(\delta_{H[S_1 \cup S]}(S_1 \cap S)) = 1.$$

Condition (A₁) follows from (4.16), (4.17) and (4.18).

2. If $S \in S_1$, Condition (A₂) is obviously true. Suppose $S_1 \in S$. Let $uv \in \gamma(S')$ such that $z_{uv} = 0$ and uv is even in $H \times S$. Let H_1, H_2 be the components of $T'[S'] - uv$ containing u, v respectively. If $2z(\gamma(V'')) < b(V'')$, then $2z(\gamma(V')) = b(\text{hollow}(V''))$, from (4.17), and Proposition 4.4.3. Let $B = \text{hollow}(V'')$, $A_0 = \text{solid}(V'')$. Let $v' \in \text{solid}(V'')$, $u' \in B$. Then $y(A_0) - y(B) = 2\varepsilon > 0$. So $y \notin P(H[S], b)$, a contradiction. So

$$(4.19) \quad 2z(\gamma(V'')) = b(V'').$$

Thus Condition (A₂) hold. ■

4.5 The separation algorithm

Now we give two separation algorithms, which are similar to those of [6]. The first one is not efficient, but it is simple and can be used to prove the theorems in Section 4.1. The second one uses scaling techniques. It works for rational input, and runs in polynomial time.

Separation algorithm 1

Step 0 Let $\mathcal{S} = \emptyset$, $G' = G$, $z = 0$.

Step 1 Apply Algorithm Augment. If it meets (termination) Condition T, stop; otherwise update $\mathcal{S}$, G' , and go to Step 1.

Proposition 4.5.1 *Separation algorithm 1 terminates in a finite number of steps.*

Proof: In the algorithm, z is uniquely determined by $\mathcal{S}$, G'_+ , and the deficient vertices of with respect to x' . Since there are finitely many such structures, the algorithm must terminate in a finite number of steps. ■

We notice that in the algorithm, y , z maintain the following properties:

- (D1) y_v is an integral combination of $\{x_w : w \in V\} \cup \{b_w : w \in V\}$ for each $v \in V$;
- (D2) $2z_e$ is an integral combination of $\{y_v : v \in V\} \cup \{b_v : v \in V\}$ for each e . In fact, z_e is not an integral combination of $\{y_v : v \in V\} \cup \{b_v : v \in V\}$ only if e is on a cycle of G'_+ , or on the cycle of the spanning simple b -critical subgraph of $G[\mathcal{S}] \times \mathcal{S}[\mathcal{S}]$ for $\mathcal{S} \in \mathcal{S}$.

Proof of Theorem 4.1.2: It follows from Proposition 4.5.1, Theorem 4.4.2, and Theorem 4.4.1. ■

Proof of Theorem 4.1.4 and Theorem 4.1.5: The results follow from (D1), (D2), Proposition 4.5.1, Theorem 4.4.2, and Theorem 4.4.1. ■

Suppose that we solved (4.1) and (4.2) for $x'' \in \mathbb{R}^V$, and $x' \in \mathbb{R}^V$ is obtained by increasing some x''_v by a number α . The problems for x' can be solved by using the solution information for x'' as follows.

Let $z, y, B, A_0, A_1, \dots, A_k$ optimize the problems for x'' . If $v \in \cup_0^k A_i$, then the above also optimize the problem for x' . If $v \notin \cup_0^k A_i$, one of the following will happen:

(B1) a tree component of G_+ has two deficient vertices;

(B2) a non-tree component of G_+ has a deficient vertex.

We need to adjust z, y to make (I2) hold. In case (B1), find the path L_W in the tree from v to the other deficient vertex. Let ϵ be the minimum of deficiencies and z_e for all even edge e on L_W . Add ϵ to z_e for each odd edge of L_W , and subtract ϵ from z_e for each even edge of L_W . Then the resulting z is still in $Q(G', b)$, and either one of the deficient vertices of L_W is no longer deficient, or two deficient vertices of L_W are in different components of G'_+ . In case (B2), we apply Algorithm C to find ϵ and S_1 . Do Type 2 augmentation. If $\epsilon < \min\{\epsilon_2, \epsilon_3, \epsilon_4, \dots\}$, shrink S_1 , and update G', S . Then either v is no longer deficient, or the component containing v in new G'_+ is a tree. Now Algorithm Augment is used to optimize problem (4.1) and (4.2) for x' as in Separation Algorithm 1. The above is called **re-optimizing procedure**. It terminates in a finite number of steps for the same reason as does Separation Algorithm 1.

Remark If each x''_v is an integral combination of α , then the re-optimizing procedure terminates in time $O(|V||E|)$. The reason is that the optimizing $y = Mz$ has property (D2), i.e. each $2z_e$ is α times an integer. So the procedure calls Algorithm Augment at most twice.

Proof of Theorem 4.1.6: We apply Separation Algorithm 1 to get optimizing z , $y^0 = Mz$, and a structure S , $G' = G \times S$ of (4.1) for y^0 . Let $V = \{v_1, \dots, v_n\}$. For $i = 1, \dots, n$, let $y^i \in \mathbb{R}^V$ such that

$$y_v^i = \begin{cases} y_v^{i-1} + (x_v - y_v^0) & \text{if } v = v_i, \\ y_v^{i-1} & \text{if } v \neq v_i. \end{cases}$$

We then apply the re-optimization procedure to solve the problem for y^i from the solution of that of y^{i-1} . It takes a finite number of steps to solve the problem for $y^n = x$, which is (4.1). We note that the optimizing $y = Mz$ obtained this way satisfies $y^0 \leq y$, and hence optimizes both (4.1) and (4.3). ■

With a similar argument as for Theorem 4.1.4, we have the following discreteness result for problem (4.3).

Theorem 4.5.2 *There is a maximizing y in (4.3) such that each y_v is an integral combination of elements of $\{x_v : v \in V\} \cup \{b_v : v \in V\} \cup \{y_v^0 : v \in V\}$.*

Now we describe the scaling algorithm. The algorithm applies to rational vectors x . Let m_1 be the least common multiple of the denominators of x , and let m_2 be the maximum value of the numerators of x . Let $l = \lceil \log(m_1 + m_2 + 1) \rceil$, which is bounded by the input length of x . Let $x^i = 2^i \lfloor m_1 x / 2^i \rfloor / m_1$, $i = l, l-1, \dots, 1, 0$. The algorithm solves problems (4.1) and (4.2) for the sequence x^i , $i = l, l-1, \dots, 1, 0$. Since $x^l = 0$, the initial problems are trivial. Since $x^0 = x$, the last problems are the ones we want to solve.

If x_v^i and x_v^{i+1} are different, then $x_v^i = x_v^{i+1} + 2^i / m_1$. x^i is an integral combination of $2^i / m_1$, and so the solution for x^i is an integral combination of $2^i / m_1$. After applying at most $|V|$ times the re-optimizing procedure, which runs in time $O(|V||E|)$, (4.1) and (4.2) for x^i are solved from solution information for x^{i+1} . So the original problem is solved in time $O(|V|^2|E|l)$.

Theorem 4.5.3 *The scaling algorithm solves problems 4.1, 4.2, and thus the separation problem for $P(G, b)$ in time $O(|V|^2|E|l)$.*

Now we apply the scaling algorithm instead of Separation Algorithm 1 in the proof of Theorem 4.1.6, we have the following result.

Theorem 4.5.4 *The scaling algorithm solves problem (4.3) in time $O(|V|^2|E|l(x, y^0))$, where $l(x, y^0)$ is the input length of x, y^0 .*

Chapter 5

Polyhedra of b -matching degree sequences of bidirected graphs

5.1 Introduction

A **bidirected graph** $G = (V, E)$ is a graph that may have undirected edges E^+ , directed edges $\vec{E}$, and edges E^- with two tail ends. An undirected edge is said to have two head ends. An edge-vertex incidence vector $\chi^e \in \mathbb{R}^V$ is defined as: $\chi_v^e = 1$, if v is a head end, $\chi_v^e = -1$, if v is a tail end, $\chi_v^e = 0$, otherwise. Again let M_G (or M) denote the edge-vertex incidence matrix of G . We note that by the definition a column of M may have two -1 entries. For $b \in \mathbb{Z}^V$ and $u \in \mathbb{Z}_+^E$, a (u -capacitated) b -matching of G is a vector $z \in \mathbb{Z}_+^E$ such that $Mz \leq b$ and $z \leq u$. This more general class of graphs was introduced by Edmonds and Johnson [11]. They generalized Edmonds' matching algorithm to bidirected graphs, and they also gave a linear description for the convex hull, $Q(G, b, u)$, of b -matchings of G .

One aim of this chapter is to find a linear description for the convex hull of degree sequences of b -matchings for a bidirected graph G . This is discussed in Section 2.

Given an integral vector $a \in \mathbb{Z}^V$, an a -covering is an integral vector $z \in \mathbb{Z}^E$ such that $Mz \geq a$. The covering problem is very similar to the matching problem. One can establish a series of theorems for coverings parallel to matchings. So can one do it for the degree sequences. Instead of doing that, we consider the more common

generalization. A b -matching a -covering (or a matching-covering) is an integral vector $z \in \mathbb{Z}^E$ satisfying $a \leq Mz \leq b$.

In Section 3, we study the convex hull of degree sequences of b -matching a -coverings. We shall give a linear description and a min-max theorem.

Here we define further notation for bidirected graphs. Let (v, w) denote the edge with tail end v and head end w ; let vw (or $[v, w]$) denote the edge with head end v and head end w ; let $\overline{vw}$ denote the edge with tail end v and tail end w . For any subsets A, B of V , let $(A, B) = \{(v, w) \in E : v \in A, w \in B\}$; $[A, B] = \{vw \in E : v \in A, w \in B\}$; $\overline{[A, B]} = \{\overline{vw} \in E : v \in A, w \in B\}$; $\gamma^+(A) = [A, A]$; $\gamma^-(A) = \overline{[A, A]}$. $\delta^-(A) = (V \setminus A, A) \cup [V \setminus A, A]$; $\delta^+(A) = (A, V \setminus A) \cup \overline{[V \setminus A, A]}$. If $A = \{v\}$, we use $\delta^+(v)$ and $\delta^-(v)$ to mean $\delta^+(\{v\})$ and $\delta^-(\{v\})$ respectively.

5.2 Reduction for bidirected graphs

The results

Let $P(G, b, u)$ denote the convex hull of degree sequences of u -capacitated b -matchings of G , namely, $P(G, b, u) = \text{conv}\{Mz : z \in \mathbb{Z}_+^E, Mz \leq b, z \leq u\}$.

In this section, we prove the following theorems, which give a linear description of $P(G, b, u)$ and a min-max theorem.

Theorem 5.2.1 $P(G, b, u)$ is the set of all $x \in \mathbb{R}^V$ satisfying

- (a) $0 \leq x \leq b$;
- (b) $\sum_{i=0}^k x(A_i) - x(B) \leq \sum_{i=1}^k (b(A_i) - 1) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B))$, for all disjoint subsets B, A_0 of V and vertex-sets $A_1, \dots, A_k$ of components of $G - (B \cup A_0)$ such that $b(A_i) + u(A_i, A_0) + u(B, A_i) + u[\overline{B, A_i}] + u[A_i, A_0]$ is odd, $i = 1, \dots, k$.

Theorem 5.2.2 *Let $x \in \mathbf{R}^V$ and $y^0 \in P(G, b, u)$ such that $y^0 \leq x$. Then*

$$(5.1) \quad \max\{y(V) : y^0 \leq y \leq x, y \in P(G, b, u)\} =$$

$$(5.2) \quad \min\left\{\sum_{i=1}^k (b(A_i) - 1) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] \right. \\ \left. + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B)) + x(B) + x(V \setminus \cup_{i=0}^k A_i)\right\},$$

where the minimum is over all disjoint vertex subsets B, A_0 , and vertex-sets $A_1, \dots, A_k$ of components of $G - (B \cup A_0)$ such that $b(A_i) + u(A_i, A_0) + u(B, A_i) + u[\overline{B, A_i}] + u[A_i, A_0]$ is odd, $i = 1, \dots, k$.

Theorem 5.2.1 and Theorem 5.2.2 generalize the corresponding results for undirected graphs [5]. We indicate that they also generalize the matching theorem of Tutte [25] to bidirected graphs. A u -capacitated b -matching z is **perfect** if $Mz = b$. A component $G[A]$ of $G - (B \cup A_0)$ is called an **odd component** if $b(A) + u(A, A_0) + u(B, A) + u[\overline{B, A}] + u[A, A_0]$ is odd. Let $k(A_0, B)$ denote the number of odd components of $G - (B \cup A_0)$.

Theorem 5.2.3 *There is a perfect u -capacitated b -matching if and only if $k(A_0, B) + b(A_0) \leq b(B) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B))$, for all disjoint subsets B, A_0 of V .*

Proof: Note that there is a perfect u -capacitated b -matching if and only if $b \in P(G, b, u)$. By Theorem 5.2.1, $b \in P(G, b, u)$ if and only if the condition in this theorem holds. ■

We shall also prove the following result.

Theorem 5.2.4 *There exists optimizing $y^* = Mz^*$ of (5.1) with $z^* \in Q(G, b, u)$ such that y_v^* is an integral combination of $\{y_v^0, x_v, b_v : v \in V\} \cup \{u_e : e \in E\}$ for each $v \in V$, and $2z_e^*$ is an integral combination of $\{y_v^*, b_v : v \in V\} \cup \{u_e : e \in E\}$ for each $e \in E(G)$.*

For an undirected graph G , whether $P(G, b, u) = \emptyset$ is a trivial question. For a bidirected graph, we have the following theorem to answer this question.

Theorem 5.2.5 $\text{dom}(P(G, b, u)) = \{x \in \mathbb{R}^V : x(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \geq 0 \ \forall B \subseteq V\}$. Namely, there exists $y \in P(G, b, u)$ such that $y \leq x$ if and only if $x(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \geq 0$ for every $B \subseteq V$.

Proof: If there exists $y = Mz \in P(G, b, u)$ such that $y \leq x$, then

$$\begin{aligned}
 & x(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \\
 \geq & y(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \\
 = & z(\delta^-(B)) + 2z(\gamma^+(B)) - z(B, V \setminus B) - z[\overline{B, V \setminus B}] \\
 & - 2z(\gamma^-(B)) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \\
 \geq & 0,
 \end{aligned}$$

for every $B \subseteq V$.

Suppose that $x(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \geq 0$ for every $B \subseteq V$. Let U denote the diagonal matrix with diagonal u . Let $x' = x - MU\chi^{E^-}$ and let $b' = b - MU\chi^{E^-}$. Let $\vec{G} = (V, \vec{E})$. Then $x'(B) + u(B, V \setminus B) \geq 0$. By Gale's Theorem (page 32), there exists $z' \in \mathbb{R}_+^{\vec{E}}$ such that $M_{\vec{G}}z' \leq x' \leq b'$ and $z' \leq u$. From Proposition 1.3.3, $P = \{z \in \mathbb{R}_+^{\vec{E}} : M_{\vec{G}}z \leq b', z \leq u\}$ is integral. So $z' = \sum_{i=1}^r \lambda_i z^i$ for some integral $z^1, \dots, z^r$ of P and some $\lambda_1, \dots, \lambda_r \in \mathbb{R}$. We extend each z^i to $P(G, b, u)$ as follows, $z_e^i = z_e^i$ if $e \in \vec{E}$, $z_e^i = 1$ if $e \in E^-$, and $z_e^i = 0$ if $e \in E^+$. Let $y = M(\sum_{i=1}^r \lambda_i z^i)$, then $y \leq x$ and $y \in P(G, b, u)$. ■

Now we use Theorem 5.2.2 and Theorem 5.2.5 to prove Theorem 5.2.1.

Proof of Theorem 5.2.1: The validity of the inequalities of Theorem 5.2.1 is shown in the proof of Theorem 5.2.2. Suppose $x \leq b$ and $x \notin P(G, b, u)$. Then either $x \notin \text{dom}(P(G, b, u))$ or $x \in \text{dom}(P(G, b, u))$. In the first case, there is a violated inequality of $\text{dom}(P(G, b, u))$, $x(B) + u(B, V \setminus B) + u[\overline{B, V \setminus B}] + 2u(\gamma^-(B)) \geq 0$,

which is in the system of linear inequalities of Theorem 5.2.1. In the second case, by Theorem 5.2.2,

$$\begin{aligned} x(V) &> \max\{y(V) : y \leq x, y \in P(G, b, u)\} \\ &= \min\left\{\sum_{i=1}^k (b(A_i) - 1) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] \right. \\ &\quad \left. + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B)) + x(B) + x(V \setminus \cup_{i=0}^k A_i)\right\}. \end{aligned}$$

Again an inequality in Theorem 5.2.1 is violated. Thus we proved the other part of Theorem 5.2.1. ■

In the two following subsections, we shall prove the theorems.

The undirected graph case

Cunningham and Green-Krótki [5] proved the following theorem for undirected graphs.

Theorem 5.2.6 *Let $x \in \mathbb{R}^V$. Then*

$$(5.3) \quad \max\{y(V) : y \leq x, y \in P(G, b, u)\} =$$

$$(5.4) \quad \min\left\{\sum_{i=1}^k (b(A_i) - 1) + u[V \setminus B, A_0] + u(\gamma(A_0)) + x(B) + x(V \setminus \cup_{i=0}^k A_i)\right\},$$

where the minimum is over all disjoint vertex subsets B, A_0 , and vertex-sets $A_1, \dots, A_k$ of components of $G - (B \cup A_0)$ such that $b(A_i) + u[A_i, A_0]$ is odd, $i = 1, \dots, k$.

This theorem is slightly weaker than the version of Theorem 5.2.2 for undirected graphs, which we need to use to prove Theorem 5.2.2 and Theorem 5.2.4. However, the proof of Theorem 5.2.6 in [5] can be applied to the undirected versions of Theorem 5.2.2 and Theorem 5.2.4. Furthermore, as we will see, the reduction in the proof gives a way to solve (5.3) and (5.4) in polynomial time by applying the algorithm presented in Chapter 4.

Theorem 5.2.7 *If the graph G is undirected, then the statements in Theorem 5.2.1 and Theorem 5.2.4 are true.*

Proof: “ $\max \leq \min$ ”. It is in the proof of Theorem 5.2.2.

“ $\min \leq \max$ ”. We use a standard method to construct a new graph $G' = (V', E')$ from G . For each edge $vw \in E$ with bound u_{vw} , add to new vertices $v(e)$, $w(e)$, replace the edge by three edges $vv(e)$, $v(e)w(e)$, $w(e)w$. See an example in Figure 5.1.

Figure 5.1: An edge $vw \in E$ and the new edges in E'

We define $b', x' \in \mathbf{R}^{V'}$ such that $b'_v = b_v$ and $x'_v = x_v$ for $v \in V$, and $b'_{v(e)} = x'_{v(e)} = u_e$ for $v \in V$ incident in G to $e \in E$. Then $x' \leq b'$. Let $y'' \in \mathbf{R}^{V'}$ such that $y''_v = y_v^0$ for $v \in V$, and $y''_{v(e)} = u_e$ for $v \in V$ incident in G to $e \in E$. Then $y'' \in P(G', b')$ and $y'' \leq x' \leq b'$. By Theorem 4.1.2 and Theorem 4.1.6, there is $y' \in P(G', b')$ such that $y'' \leq y' \leq x'$ and

$$(5.5) \quad y'(V') = \sum_{i=1}^k (b'(A'_i) - 1) + x'(B') + x'(V' \setminus \cup_{i=0}^k A'_i).$$

It is clear that $y'(V') = y(V) + 2u(E)$. We quote the following from Cunningham and Green-Krótki [5].

Therefore to complete the proof, we need to define $B, A_0, A_1, \dots, A_k \subseteq V$ as in the theorem with

$$\begin{aligned} & \sum_{i=1}^k (b(A_i) - 1) + u(\gamma(A_0)) + u[V \setminus B, A_0] + x(B) + x(V \setminus \cup_{i=0}^k A_i) + 2u(E) \\ &= \sum_{i=1}^k (b'(A'_i) - 1) + x'(B') + x'(V' \setminus \cup_{i=0}^k A'_i). \end{aligned}$$

We denote the last expression by f .

Again we show that $B', A'_0, A'_1, \dots, A'_m$ can be assumed to take a special form. In the following we use repeatedly the observation that for $1 \leq i \leq m$, $x'(A'_i) \geq b'(A'_i) - 1$, since otherwise discarding A'_i would decrease f .

Claim. If $v(e) \in B'$, we may assume that $v \in A'_0$. Suppose the claim is false. We shall delete $v(e)$ from B' . Suppose first that $|\{v, w(e)\} \cap \cup_{i=0}^m A'_i| \leq 1$. If it is 0, obviously f decreases. If it is 1, we remove the corresponding A'_p or element of A'_0 . In the former case, f goes down by at

least $x'_{v(e)} - 1 = u_e - 1 \geq 0$. In the latter case, we have $w(e) \in A'_0$, and f goes down by exactly $x'_{v(e)} - x'_{w(e)} = 0$. Now suppose that $w(e) \in A'_0$ and $v \in A'_p$ for some $p > 0$. Adding both $v(e)$ and $w(e)$ to A'_p , f goes down by

$$2x'_{v(e)} - b'_{v(e)} - b'_{w(e)} = 2u_e - 2u_e = 0.$$

Finally, suppose that $v(e) \in A'_p$, $w(e) \in A'_q$ with $p, q > 0$ and possibly $p = q$. If we discard both A'_p, A'_q , then f goes down by at least $x'_{v(e)} - |\{p, q\}| = u_e - |\{p, q\}|$. So we need only deal with the case when $u_e = 1$ and $p \neq q$. But then we can replace A'_p, A'_q by $A'_p \cup A'_q \cup \{v(e)\}$, and f goes down by at least $2x'_{v(e)} - 1 = 2u_e - 1 > 0$. This completes the proof of the claim.

Now define $k = m$, $B = B' \cap V$, and $A = A' \cap V$ for $0 \leq i \leq k$. It follows from the claim that if $vw \in E$ and $v \in A_p$, $p \geq 1$, then $w \in A_p \cup A_0 \cup B$. Hence A_p is the vertex-set of a component of $G - (B \cup A_0)$, provided that $A_p \neq \emptyset$. But if A_p is empty, then $A'_p \subseteq \{v(e), w(e)\}$ for some $e = uv \in E$. Equality cannot hold, because $b'(A'_p)$ is odd. Moreover, if $A'_p = \{v(e)\}$, then discarding A'_p and adding $v(e)$ to A'_0 does not increase f . Now $v, v(e) \in A'_p$ implies $w(e) \in A'_p$ and $w \in B' \cup A'_p$, or $w(e) \in B'$ and $w \in A'_0$. Hence $b'(A'_p) = b(A_p) + 2u(A_p, B) + 2u(\gamma(A_p)) + u(A_p, A_0)$, so $b(A_p) + u(A_p, A_0)$ is an odd integer. Using the claim, we can easily classify all of the edges vw of G as follows. Here we denote $(V' \setminus B') \setminus \cup_0^k A_i$ by C' and $C' \cap V$ by C . If

$v, w \in A_p, p \geq 1,$	then $v(e), w(e) \in A'_p;$
$v \in A_p, p \geq 1, w \in A_0,$	then $v(e) \in A'_p, w(e) \in B';$
$v \in A_p, p \geq 1, w \in B,$	then $v(e), w(e) \in A'_p;$
$v, w \in A_0,$	then $v(e), w(e) \in B';$
$v \in A_0, w \in B,$	then $v(e) \in B', w(e) \in A'_0;$
$v \in A_0, w \in C,$	then $v(e) \in B', w(e) \in C';$
$v, w \in B,$	then $v(e), w(e) \in C';$
$v \in B, w \in C,$	then $v(e), w(e) \in C';$
$v, w \in C,$	then $v(e), w(e) \in C'.$

We can now compute $x'(B')$ to be $x(B) + u(A_0, \cup_1^k A_i) + 2u(\gamma(A_0)) + u(A_0, B) + u(A_0, C)$. Similarly, $x'(V' \setminus \cup_0^k A'_i) = x'(B') + x'(C')$, and $x'(C') = x(C) + u(A_0, C) + 2u(\gamma(B)) + 2u(B, C) + 2u(\gamma(C))$. Now a calculation shows that

$$\sum_{i=1}^k (b'(A'_i) - 1) + x'(B') + x'(V' \setminus \cup_0^k A'_i) - 2u(E)$$

$$= \sum_{i=1}^k (b(A_i) - 1) + u(\gamma(A_0)) + u(A_0, V \setminus B) + x(B) + x(V \setminus \cup_0^k A_i),$$

as required.

By Theorem 4.5.2 and Theorem 4.1.5, y'_v is an integral combination of $\{y''_v, x''_v, b'_v : v \in V'\} \cup \{u_e : e \in E\}$, so y_v is an integral combination of $\{y''_v, x''_v, b'_v : v \in V\} \cup \{u_e : e \in E\}$ for $v \in V$; $2z'_e$ is an integral combination of $\{y'_v, b'_v : v \in V'\}$, so z_e is an integral combination of $\{y_v, b_v : v \in V\} \cup \{u_e : e \in E\}$ for $e \in E$. This completes the proof. ■

The separation problem. Since $y', z', B', A'_0, A'_1, \dots, A'_m$ of (5.5) can be found in time $O(|V'|^2|E'|l(y'', x'')) = O(|V|^2|E|l(y^0, x, u))$ by the algorithm for (4.3) in Chapter 4, and the special form of $B', A'_0, A'_1, \dots, A'_m$ can be found in time $O(|V||E|)$. So problems (5.3) and (5.4) can be solved in time $O(|V|^2|E|l(y^0, x, u))$, and therefore the separation problem can be solved in time $O(|V|^2|E|l(y^0, x, u))$.

Proofs

Proof of Theorem 5.2.2 and Theorem 5.2.4: Let f denote the expression of (5.2). We first show that “max $\leq$ min”. Let y be the degree sequence of a b -matching z . Then for each $i \geq 1$, one of the following valid inequalities must be strict: $y(A_i) \leq b(A_i)$, $z(A_i, A_0) \leq u(A_i, A_0)$, $z(B, A_i) \leq u(B, A_i)$, $z[\overline{B, A_i}] \leq u[\overline{B, A_i}]$, $z[A_i, A_0] \leq u[A_i, A_0]$, $-z(A_0, A_i) \leq 0$, $-z(A_i, B) \leq 0$, $-z[\overline{A_0, A_i}] \leq 0$, $-z[B, A_i] \leq 0$, since the sum of the right hand sides is odd and the sum of the left hand sides is

$$2z(\gamma^+(A_i)) - 2z(\gamma^-(A_i)) + 2z[A_0, A_i] - 2z[\overline{A_0, A_i}] + 2z(B, A_i) - 2z(A_i, B).$$

So

$$\begin{aligned} (5.6) \quad \sum_{i=1}^k y(A_i) &\leq \sum_{i=1}^k (b(A_i) - 1 - z(A_i, A_0) + u(A_i, A_0) - z(B, A_i) \\ &\quad + u(B, A_i) - z[\overline{B, A_i}] + u[\overline{B, A_i}] - z[A_i, A_0] + u[A_i, A_0] \\ &\quad + z(A_0, A_i) + z(A_i, B) + z[\overline{A_0, A_i}] + z[B, A_i]). \end{aligned}$$

Let C denote $V \setminus (B \cup \cup_{i=0}^k A_i)$. Note that

$$(5.7) \quad \begin{aligned} y(A_0) \leq & \sum_{i=1}^k (z(A_i, A_0) + z[A_i, A_0] - z[\overline{A_0, A_i}] - z(A_0, A_i)) \\ & + 2u(\gamma^+(A_0)) - z[\overline{A_0, B}] + z[A_0, B] + u(B, A_0) \\ & + u(C, A_0) + u[C, A_0] \end{aligned}$$

and

$$(5.8) \quad \begin{aligned} -y(B) \leq & -\sum_{i=1}^k (z(A_i, B) + z[A_i, B] - z[\overline{B, A_i}] - z(B, A_i)) \\ & + 2u(\gamma^-(B)) - z[A_0, B] + u(B, A_0) + z[\overline{B, A_0}] \\ & + u(B, C) + u[\overline{B, C}]. \end{aligned}$$

Adding (5.6), (5.7), and (5.8), we have

$$\begin{aligned} \sum_{i=0}^k y(A_i) - y(B) \leq & \sum_{i=1}^k (b(A_i) - 1) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] \\ & + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B)). \end{aligned}$$

So

$$\begin{aligned} y(V) &= \sum_{i=0}^k y(A_i) - y(B) + y(B) + y(V \setminus \cup_{i=0}^k A_i) \\ &\leq \sum_{i=0}^k y(A_i) - y(B) + x(B) + x(V \setminus \cup_{i=0}^k A_i) \\ &\leq f. \end{aligned}$$

Figure 5.2: A graph G and the graph G'

21 of/de 2

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Now we show that there exist y^* , and $B, A_0, A_1, \dots, A_k$ that make (5.1) and (5.2) equal. To do this, we use a standard method to reduce the bidirected graph G to an undirected graph $G' = (V', E')$. For each vertex $v \in V$, make two copies v_1, v_2 . Let $V_1 = \{v_1 : v \in V\}$, $V_2 = \{v_2 : v \in V\}$, and $V' = V_1 \cup V_2$. For each edge $e \in E$, let

$$e' = \begin{cases} v_1 w_1, & \text{if } e = vw; \\ v_2 w_2, & \text{if } e = \overline{vw}; \\ v_1 w_2, & \text{if } e = (w, v). \end{cases}$$

Let $E' = \{v_1 v_2 : v \in V\} \cup \{e' : e \in E\}$. See an example in Figure 5.2. Define $u' \in \mathbb{R}^{E'}$ by $u'_{e'} = u_e$ for all $e \in E$ and $u'_{v_1 v_2} = +\infty$ for all $v \in V$. Let

$$D = \sum(|b_v| : v \in V, b_v \text{ is finite}) + \sum(|u_e| : e \in E, u_e \text{ is finite}) + \sum(|x_v| + 1 : v \in V).$$

Let $b'_{v_1} = b_v + D$, $b'_{v_2} = D$ for every $v \in V$, and let $x'_{v_1} = x_v + D$, $x'_{v_2} = D$ for every $v \in V$.

Let $y''_{v_1} = y^0_v + D$, $y''_{v_2} = D$ for every $v \in V$. Then $y'' \in P(G', b', u')$ and $y'' \leq x'$. (Since if z^0 is a convex combination of b -matchings such that $y^0 = Mz^0$, letting $z''_e = z^0_e$, for every $e \in E$, $z''_{v_1 v_2} = D - z^0(\delta^+(v))$, for every $v \in V$, then z'' is the same convex combination of the b' -matchings extended from the b -matchings as for z^0 , and $y'' = M_{G'} z''$.)

By Theorem 5.2.7, there exist $y' \in P(G', b', u')$, and subsets $B', A'_0, A'_1, \dots, A'_k$ of V' such that $y'' \leq y' \leq x'$ and

$$(5.9) \quad y'(V') = \sum_{i=1}^k (b(A'_i) - 1) + u[V' \setminus B', A'_0] + u(\gamma^+(A'_0)) + x'(B') + x'(V' \setminus \cup_{i=0}^k A'_i),$$

where B', A'_0 are disjoint, and $A'_1, \dots, A'_k$ are vertex-sets of components of $G' - (B' \cup A'_0)$ such that $y'(A'_i) + u[A'_i, A'_0]$ is odd, $i = 1, \dots, k$.

Let $y^*_v = y'_{v_1} - D$ for every $v \in V$. Then $y^* \in P(G, b, u)$ and $y^0 \leq y^* \leq x$. (Let z' be a convex combination of b' -matchings. Let $z^*_e = z'_{e'}$, for all $e \in E$. Then

z^* is a convex combination of b -matchings with the same coefficients as for z' , and $y^* = Mz^*$.)

$$(5.10) \quad y^*(V) + 2D|V| = y'(V').$$

Let

$$A_0 = \{v \in V : v_1 \in A'_0\},$$

$$B = \{v \in V : v_1 \in B'\},$$

$$A_i = \{v \in V : v_1 \in A'_i\}, \quad i = 1, \dots, k.$$

Then

$$\begin{aligned} & \sum_{i=1}^k (b'(A'_i) - 1) + u'[V' \setminus B', A'_0] + u'(\gamma^+(A'_0)) + x'(B') + x'(V' \setminus \bigcup_{i=0}^k A'_i) \\ &= \sum_{i=1}^k (b(A_i) - 1) + u'[V' \setminus B', A'_0] + u'(\gamma^+(A'_0)) + x(B) + x(V \setminus \bigcup_{i=0}^k A_i) \\ & \quad + 2D|V| + D|B'| - D|A'_0|. \end{aligned}$$

Let C' denote $V' \setminus (B' \cup \bigcup_{i=0}^k A'_i)$. Since D is sufficiently large, we must have $|B'| = |A'_0|$. Since $u'[C', A'_0]$, $u(\gamma^+(A_0))$ and $b'(A_i) + u'[A'_0, A'_i]$ for $i = 1, \dots, k$ are finite, we have $|V_1 \cap A'_0| \leq |V_2 \cap B'|$ and $|V_2 \cap A'_0| \leq |V_1 \cap B'|$. So $|V_1 \cap A'_0| = |V_2 \cap B'|$ and $|V_2 \cap A'_0| = |V_1 \cap B'|$. So for any $v \in V$, one of the following is true:

1. $v_1 \in A'_0, v_2 \in B'$;
2. $v_1 \in B', v_2 \in A'_0$;
3. $v_1, v_2 \in A'_i$, for some i , $1 \leq i \leq k$;
4. $v_1, v_2 \in C'$.

So

$$u'[A'_0, A'_i] = u(A_i, A_0) + u(B, A_i) + u[\overline{B}, A_i] + u[A_i, A_0];$$

$$u'[C', A'_0] = u(A_0, C) + u(B, A_0) + u[\overline{B}, A_0] + u[A_0, C];$$

$$u'(\gamma^+(A'_0)) = u(\gamma^+(A_0)) + u(\gamma^-(B)) + u(B, A_0);$$

$$\begin{aligned} & u'[V' \setminus B', A'_0] + u'(\gamma^+(A'_0)) \\ &= 2u'(\gamma^+(A'_0)) + \sum_{i=1}^k u'[A'_i, A'_0] + u'[C', A'_0] \\ &= 2u(\gamma^+(A_0)) + u(\gamma^-(B)) + u(B, A_0) + u(A_i, A_0) + u(B, A_i) \\ &\quad + u[\overline{B, A_i}] + u[A_i, A_0] + u(A_0, C) + u(B, A_0) + u[\overline{B, A_0}] + u[A_0, C] \\ &= u(B, A_0) + u(C, A_0) + \sum_{i=1}^k (A_i, A_0) + u(B, A_0) + u(B, C) \\ &\quad + \sum_{i=1}^k (B, A_i) + u[\overline{B, C}] + u(\gamma^-(B)) + \sum_{i=1}^k u[\overline{B, A_i}] + u(\gamma^+(A_0)) \\ &\quad + u(C, A_0) + \sum_{i=1}^k u[A_i, A_0] + u(\gamma^-(B)) + u(\gamma^+(A_0)) \\ &= u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] + u[V \setminus B, A_0] \\ &\quad + u(\gamma^+(A_0)) + u(\gamma^-(B)). \end{aligned}$$

$$\begin{aligned} y^*(V) &= \sum_{i=1}^k (b(A_i) - 1) + u'[V' \setminus B', A'_0] + u'(\gamma^+(A'_0)) + x(B) + x(V \setminus \cup_{i=0}^k A_i) \\ &= \sum_{i=1}^k (b(A_i) - 1) + u(V \setminus A_0, A_0) + u(B, V \setminus B) + u[\overline{B, V \setminus A_0}] \\ &\quad + u[V \setminus B, A_0] + u(\gamma^+(A_0)) + u(\gamma^-(B)) + x(B) + x(V \setminus \cup_{i=0}^k A_i). \end{aligned}$$

$b(A'_i) + u'[A'_0, A'_i]$ is odd if and only if $b(A_i) + u(A_i, A_0) + u(B, A_i) + u[\overline{B, A_i}] + u[A_i, A_0]$ is odd, for $i = 1, \dots, k$. So y^* and the subsets $B, A_0, A_1, \dots, A_k$ are what we wanted.

Now we prove Theorem 5.2.4. By Theorem 5.2.7, y'_v is an integral combination of $\{y_v^0, x_v, b_v : v \in V\} \cup \{u_e : e \in E\}$ for $v \in V$; $2x'_e$ is an integral combination of $\{y_v^0, b_v : v \in V\}$, so $2x_e$ is an integral combination of $\{y_v^0, b_v : v \in V\} \cup \{u_e : e \in E\}$ for $e \in E$. Theorem 5.2.4 is proved. ■

The separation problem. Since $y', z', B', A'_0, A'_1, \dots, A'_m$ of (5.9) can be found in time $O(|V'|^2|E'|l(y'', x'')) = O(|V|^2|E|l(y^0, x, u))$, and the special form of $B', A'_0, A'_1, \dots, A'_m$ can be found in time $O(|V||E|)$, problems (5.1) and (5.2) can be solved in time $O(|V|^2|E|l(y^0, x, u))$, and therefore the separation problem can be solved in time $O(|V|^2|E|l(y^0, x, u))$.

5.3 Matching-covering degree sequence polyhedra

The results

Let $P(a, G, b)$ denote the convex hull of degree sequences of b -matching a -coverings of G , namely, $P(a, G, b) = \text{conv}\{Mz \in \mathbb{Z}^V : z \in \mathbb{Z}_+^E, a \leq Mz \leq b\}$. In this section we prove the following theorems, which give a linear description of $P(a, G, b)$ and a min-max theorem.

Theorem 5.3.1 $P(a, G, b)$ is the set of all $x \in \mathbb{R}^V$ satisfying

- (a) $a \leq x \leq b$;
- (b) $\sum_{i=0}^k (x(A_i) - x(S_i)) \leq \sum_{i=1}^k (b(A_i) - a(S_i) - 1)$ for all disjoint subsets S_0, A_0 of V , and vertex-sets $A_1 \cup S_1, \dots, A_k \cup S_k$ of components of $G - (S_0 \cup A_0)$ such that

$$\begin{cases} \delta^+(S_0) = \overline{[S_0, A_0]}, \gamma^-(S_0) = \emptyset; \\ \delta^-(A_0) = [A_0, S_0], \gamma^+(A_0) = \emptyset; \\ A_i \cap S_i = \emptyset, b(A_i) - a(S_i) \text{ is odd, } i = 1, \dots, k. \end{cases}$$

Theorem 5.3.2 Let $x \in \mathbb{R}^V$ and $g \in P(a, G, b)$ such that $g \leq x$. Then

$$(5.11) \quad \max\{y(V) : g \leq y \leq x, y \in P(a, G, b)\} =$$

$$(5.12) \quad \min\left\{\sum_{i=1}^k (b(A_i) - a(S_i) - 1) + \sum_{i=0}^k x(S_i) + x(V \setminus \cup_{i=0}^k A_i)\right\},$$

where the minimum is over all disjoint subsets S_0, A_0 of V , and vertex-sets $A_1 \cup S_1, \dots, A_k \cup S_k$ of components of $G - (S_0 \cup A_0)$ such that

$$\begin{cases} \delta^+(S_0) = \overline{[S_0, A_0]}, \gamma^-(S_0) = \emptyset; \\ \delta^-(A_0) = [A_0, S_0], \gamma^+(A_0) = \emptyset; \\ A_i \cap S_i = \emptyset, b(A_i) - a(S_i) \text{ is odd, } i = 1, \dots, k. \end{cases}$$

We also prove the following result. Let $Q(a, G, b)$ denote the convex hull of b -matching a -coverings of G .

Theorem 5.3.3 *There exist optimizing $y^* = Mz^*$ of (5.11) with $z^* \in Q(a, G, b)$ such that y_v^* is an integral combination of $\{g_v, x_v, a_v, b_v : v \in V\}$ for each $v \in V$, and $2z_e^*$ is an integral combination of $\{a_v, b_v, y_v^* : v \in V\}$ for each $e \in E(G)$.*

Proofs

Proof of Theorem 5.3.1, 5.3.2, and 5.3.3: Let $V_1 = \{v \in V : a_v \text{ is finite}\}$. For each $v \in V_1$, make a copy v_1 and add a directed edge $e_v = (v, v_1)$. Let $G' = (V', E')$ denote the new graph. Let $\theta' \in \mathbb{R}^{V'}$ such that $\theta'_v = a_v$, $\theta'_{v_1} = b_v - a_v$, for all $v \in V_1$, and $\theta'_v = b_v$ for all $v \in V \setminus V_1$. See an example in Figure 5.3.

Figure 5.3: A graph G and the graph G'

Then there is an one-to-one correspondence between

$$\text{conv}\{z \in \mathbb{Z}_+^E : a_v \leq z(\delta^-(v)) - z(\delta^+(v)) \leq b_v \forall v \in V\}$$

and

$$\text{conv}\{z' \in \mathbb{Z}_+^{E'} : z'(\delta^-(v)) - z'(\delta^+(v)) = \theta'_v, \forall v \in V_1; z'(\delta^-(v)) - z'(\delta^+(v)) \leq \theta'_v, \forall v \in V' \setminus V_1\}$$

through the linear transformation

$$\begin{cases} z'_e = z_e, & \text{for all } e \in E; \\ z'_{\delta^-(v)} = z(\delta^-(v)) - z(\delta^+(v)) - a_v, & \text{for all } v \in V_1. \end{cases}$$

The correctness of this observation can be easily checked for the matching-coverings and then their convex combinations.

So there is an one-to-one correspondence between $P(a, G, b)$ and $P(G', V') \cap \{y' \in \mathbb{R}^{V'} : y'_v = a_v, \forall v \in V_1\}$ through the linear transformation

$$(5.13) \quad y'_v = a_v, y'_{v_1} = y_v - a_v, \forall v \in V_1; y'_v = y_v, \forall v \in V \setminus V_1.$$

Let $x', g' \in \mathbb{R}^{V'}$ such that

$$x'_v = g'_v = a_v, g'_{v_1} = g_v - a_v, x'_{v_1} = x_v - a_v, \forall v \in V_1, x'_v = x_v, g'_v = g_v, \forall v \in V \setminus V_1.$$

Then it is easy to see that (5.13) gives an one-to-one correspondence between

$$\{y \in \mathbb{R}^V : y \in P(a, G, b), g \leq y \leq x\}$$

and

$$\{y' \in \mathbb{R}^{V'} : y' \in P(G', V'), g'_v \leq y'_v \leq x'_v, \forall v \in V'\}.$$

So

$$(5.14) \quad \begin{aligned} & \max\{y(V) : y \in P(a, G, b), g \leq y \leq x\} = \\ & \max\{y'(V') : y' \in P(G', V'), g'_v \leq y'_v \leq x'_v, \forall v \in V'\}. \end{aligned}$$

Note that $g' \in P(G', V')$. By Theorem 5.2.2, we have

$$(5.15) \quad \begin{aligned} & \max\{y'(V') : y' \in P(G', V'), g'_v \leq y'_v \leq x'_v, \forall v \in V'\} \\ & = \min\left\{\sum_{i=1}^k (b'(A'_i) - 1) + x'(B') + x'(V' \setminus \cup_{i=0}^k A'_i)\right\}, \end{aligned}$$

where the minimum is over all disjoint subsets B', A'_0 of V' , and vertex-sets $A'_1, \dots, A'_k$ of components of $G' - (B' \cup A'_0)$ such that

$$\begin{cases} (B', V' \setminus B') = \emptyset, \overline{B', V' \setminus B'} = \overline{B', A'_0}, \gamma^-(B') = \emptyset; \\ (V' \setminus A'_0, A'_0) = \emptyset, [A'_0, V' \setminus A'_0] = [A'_0, B'], \gamma^+(A'_0) = \emptyset; \\ b(A'_i) \text{ is odd, } i = 1, \dots, k. \end{cases}$$

Let f denote the expression of (5.12), and let f' denote the expression of (5.15). We shall prove that $\min f = \min f'$.

First we show that $\min f \leq \min f'$. Let $f'_1, B', A'_0, A'_1, \dots, A'_k$ be an optimal solution for (5.15). Let C' denote $V' \setminus (B' \cup \bigcup_{i=0}^k A'_i)$. We claim that $B', A'_0, A'_1, \dots, A'_k$ can be chosen such that for every $v \in V_1$,

1. if $v \in A'_0$, then $v_1 \in A'_0$;
2. if $v \in B'$, then $v_1 \in B'_0$;
3. if $v \in A'_i$, then either $v_1 \in A_i$ or $v_1 \in B'$, for certain $i \geq 1, i \leq k$;
4. if $v \in C'$, then $v_1 \in C'$.

If the claim is true, let

$$S_0 = \{v \in V : v \in B'\};$$

$$A_0 = \{v \in V : v \in A'_0\};$$

$$A_i = \{v \in V : v, v_1 \in A'_i, \text{ or } v \in A_i \text{ and } v \notin V_1\}, i = 1, \dots, k;$$

$$S_i = \{v \in V : v \in A'_i, v_1 \in B'\}, i = 1, \dots, k.$$

Let f_1 denote the resulting value of f . Then

$$b'(A'_i) = b(A_i) + a(S_i), i = 1, \dots, k;$$

$$x'(B') = \sum_{i=1}^k (x - a)(S_i) + x(S_0);$$

$$x'(V' \setminus \bigcup_{i=0}^k A'_i) = \sum_{i=1}^k (x - a)(S_i) + x(S_0) + x(V \setminus (\bigcup_{i=0}^k A_i \cup S_i)).$$

An easy computation from the above equalities shows $f'_1 = f_1$. The vertex-sets $S_0, A_0, A_1, S_1, \dots, A_k, S_k$ satisfy the conditions for (5.12) since $B', A'_0, A'_1, \dots, A'_k$ satisfy the conditions for (5.15). So $\min f \leq \min f'$.

Now we prove the claim.

If $v \in A'_0$, and $v_1 \notin A'_0$, we have two cases.

1) If $v_1 \notin \bigcup_{i=1}^k A'_i$, replace A'_0 with $A'_0 \cup \{v_1\}$, B' with $B' \setminus \{v_1\}$, and C' with $C' \setminus \{v_1\}$. f'_1 is decreased by at least $x_v - a_v \geq 0$.

2) If $v_1 \in A'_i$ for some $i \geq 1$, then $A'_i = \{v_1\}$. Replace A'_0 with $A'_0 \cup \{v_1\}$ and delete A'_i . f'_1 is decreased by $b(v) - a(v) - 1 \geq 0$. We do the above adjustment for every such $v \in B'$; then we have the required A'_0 . We still let $A'_1, \dots, A'_k$ denote the remaining odd sets.

From the condition for B' , it is clear that if $v \in B'$, then $v_1 \in B'_0$. Also it is clear that if $v \in A'_i$, for some i , $1 \leq i \leq k$, then either $v_1 \in A'_i$ or $v_1 \in B'$.

If $v \in C'$, and $v_1 \notin C'$, then $v_1 \in B'$. Replace B' with $B' \setminus \{v_1\}$ and replace C' with $C' \cup \{v_1\}$. f'_1 is decreased by $x_v - a_v \geq 0$. We do the above adjustment for all such $v \in C'$. Then we have the required vertex subsets as stated in the claim.

The claim is proved.

To show that $\min f' \leq \min f$, let $f_1, S_0, A_0, A_1, S_1, \dots, A_k, S_k$ be an optimal solution of (5.12). Let

$$B' = S_0 \cup \{v_1 : v \in S_0\} \cup \{v_1 : v \in \bigcup_{i=1}^k S_i\};$$

$$A'_0 = A_0 \cup \{v_1 : v \in A_0\} \cup \{v : v \in \bigcup_{i=1}^k S_i\};$$

$$A'_i = A_i \cup \{v_1 : v \in A_i\}, \quad i = 1, \dots, k.$$

Then $B', A'_0, A'_1, \dots, A'_k$ satisfy the conditions for (5.15). Let f'_1 be the resulting value of f' . Then $f'_1 = f_1$. So $\min f' \leq \min f$. Thus $\min f' = \min f$.

To prove Theorem 5.3.1, note that $x \in P(a, G, b)$ if and only if $x' \in P(G', b') \cap \{y' \in \mathbb{R}^{V'} : y'_v = a_v \forall v \in V_1\}$. By Theorem 5.2.1, $x' \in P(G', b')$ if and only if $x' \leq b'$ and $\min f' = x'(V')$. Note that $\min f' = x'(V')$ if and only if $\min f = x(V)$, and $x' \in \{y' \in \mathbb{R}^{V'} : y' \leq b'\} \cap \{y' \in \mathbb{R}^{V'} : y'_v = a_v \forall v \in V_1\}$ if and only if $x \in \{y \in \mathbb{R}^V : a \leq y \leq b\}$. So $x \in P(a, G, b)$ if and only if $a \leq x \leq b$ and $\min f \geq x(V)$, which are equivalent to $a \leq x \leq b$ and $f \geq x(V)$ for all f . This proves Theorem 5.3.1.

Now we prove Theorem 5.3.3. By Theorem 5.2.4, y'_v is an integral combination

of $\{y_v'', x_v'', b_v' : v \in V'\}$, so y_v is an integral combination of $\{g_v, x_v, a_v, b_v : v \in V\}$ for $v \in V$; $2z_e'$ is an integral combination of $\{b_v', y_v' : v \in V'\}$, so $2z_e$ is an integral combination of $\{a_v, b_v, y_v : v \in V\}$ for $e \in E$. The proof is complete. ■

Chapter 6

Eulerian subgraph degree sequence polyhedra and a polyhedral composition

In this chapter, we consider the convex hull, $P(G, \mathcal{E})$, of degree sequences of spanning Eulerian subgraphs of a graph G . Recall that a subgraph H of G is Eulerian if H is the union of edge disjoint cycles and isolated vertices. This is a well known class of subgraphs with a close relation with b -matchings. There is a nice linear description for the convex hull, $Q(G, \mathcal{E})$, of incidence vectors of edge-sets of Eulerian graphs of G . It is reasonable to conjecture that there exist nice linear descriptions for $P(G, \mathcal{E})$. But we prove that this is not true. We shall show that there exist facet-inducing inequalities with large coefficients in Section 2. This is done by using some general results for composition of two polyhedra, which we present in Section 1. For directed graphs, we show a positive result related to this polyhedron. In section 3, we consider the convex hull of outdegree sequences of Eulerian subgraphs of a directed graph G . We give a simple linear description based on network flow theory.

6.1 A polyhedral composition

For $X \subseteq \mathbb{R}^{V_0} \times \mathbb{R}^{V_1}$ and $Y \subseteq \mathbb{R}^{V_0} \times \mathbb{R}^{V_2}$, where $V_1 \cap V_2 = \emptyset$, let $X \uplus Y$ denote $\{(u + v, x, y) \in \mathbb{R}^{V_0} \times \mathbb{R}^{V_1} \times \mathbb{R}^{V_2} : (u, x) \in X, (v, y) \in Y\}$. Notice that if V_1, V_2 are empty, then $X \uplus Y = X + Y$, and if V_0 is empty, then $X \uplus Y = X \times Y$. We give the

motivation for the definition in the following.

Suppose that G is the union of graph G_1 and G_2 , where $V_0 := V(G_1) \cap V(G_2) = \{v_0\}$. Then every Eulerian subgraph H of G is the union of a Eulerian subgraph H_1 of G_1 and a Eulerian subgraph H_2 of G_2 . Let x^1, x^2, x denote the degree sequences of H_1, H_2, H respectively. Then $x_{v_0} = x_{v_0}^1 + x_{v_0}^2$, $x_v = x_v^1$ if $v \in V(G_1) \setminus \{v_0\}$, and $x_v = x_v^2$ if $v \in V(G_2) \setminus \{v_0\}$. From the above argument, we can easily see that

$$P(G, \mathcal{E}) = P(G_1, \mathcal{E}) \uplus P(G_2, \mathcal{E}).$$

We remark that a similar fact holds for the convex hull of forest degree sequences. This is why we define $X \uplus Y$.

Proposition 6.1.1 *If X and Y are affine spaces, then $X \uplus Y$ is an affine space.*

Proof: X and Y are affine spaces. So $X = \left\{ \begin{pmatrix} u_0 \\ x_0 \end{pmatrix} + \sum_1^p \lambda_i \begin{pmatrix} u_i \\ x_i \end{pmatrix} : \text{for all } \lambda_1, \dots, \lambda_p \right\}$, and $Y = \left\{ \begin{pmatrix} v_0 \\ y_0 \end{pmatrix} + \sum_1^q \mu_i \begin{pmatrix} v_i \\ y_i \end{pmatrix} : \text{for all } \mu_1, \dots, \mu_q \right\}$. $X \uplus Y = \left\{ \begin{pmatrix} u_0 + v_0 \\ x_0 \\ y_0 \end{pmatrix} + \sum_1^p \lambda_i \begin{pmatrix} u_i \\ x_i \\ 0 \end{pmatrix} + \sum_1^q \mu_i \begin{pmatrix} v_i \\ 0 \\ y_i \end{pmatrix} : \text{for all } \lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q \right\}$. So $X \uplus Y$ is an affine space. $\blacksquare$

Proposition 6.1.2 *If P_1 and P_2 are polyhedra, then $P_1 \uplus P_2$ is a polyhedron.*

Proof: We have $P_1 = \text{conv}\left\{ \begin{pmatrix} u_i \\ x_i \end{pmatrix} : i = 1, \dots, p \right\} + \text{cone}\left\{ \begin{pmatrix} u'_i \\ x'_i \end{pmatrix} : i = 1, \dots, p' \right\}$, and $P_2 = \text{conv}\left\{ \begin{pmatrix} v_i \\ y_i \end{pmatrix} : i = 1, \dots, q \right\} + \text{cone}\left\{ \begin{pmatrix} v'_i \\ y'_i \end{pmatrix} : i = 1, \dots, q' \right\}$. Then $P_1 \uplus P_2 = \text{conv}\left\{ \begin{pmatrix} u_1/2 \\ x_1/2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_p/2 \\ x_p/2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1/2 \\ 0 \\ y_1/2 \end{pmatrix}, \dots, \begin{pmatrix} v_q/2 \\ 0 \\ y_q/2 \end{pmatrix} \right\} + \text{cone}\left\{ \begin{pmatrix} u'_1 \\ x'_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u'_{p'} \\ x'_{p'} \\ 0 \end{pmatrix}, \begin{pmatrix} v'_1 \\ 0 \\ y'_1 \end{pmatrix}, \dots, \begin{pmatrix} v'_{q'} \\ 0 \\ y'_{q'} \end{pmatrix} \right\}$.

$\left(\begin{array}{c} v'_1 \\ 0 \\ y'_1 \end{array} \right), \dots, \left(\begin{array}{c} v'_{q'} \\ 0 \\ y'_{q'} \end{array} \right) \}$. So $P_1 \uplus P_2$ is a polyhedron. ■

It is in general hard to get linear description for $P_1 \uplus P_2$ from P_1 and P_2 , but it is possible to do so for some special cases. Here we discuss the case $|V_0| = 1$.

Theorem 6.1.3 *Let P_1 and P_2 be two polyhedra defined by*

$$(6.1) \quad \begin{cases} u + a^i x \leq b_i & i \in I_1, \\ -u + a^i x \leq b_i & i \in I_2, \\ a^i x \leq b_i & i \in I_3, \end{cases}$$

and

$$(6.2) \quad \begin{cases} v + c^j y \leq d_j & j \in J_1, \\ -v + c^j y \leq d_j & j \in J_2, \\ c^j y \leq d_j & j \in J_3, \end{cases}$$

respectively. Then $P_1 \uplus P_2$ is the set of all $(u, x, y) \in \mathbb{R}^{V_0} \times \mathbb{R}^{V_1} \times \mathbb{R}^{V_2}$ satisfying

$$(6.3) \quad \begin{cases} u + a^i x + c^j y \leq b_i + d_j & i \in I_1, j \in J_1, \\ -u + a^i x + c^j y \leq b_i + d_j & i \in I_2, j \in J_2, \\ a^i x \leq b_i & i \in I_3, \\ c^j y \leq d_j & j \in J_3, \\ (a^i + a^{i'})x \leq b_i + b_{i'} & i \in I_1, i' \in I_2, \\ (c^j + c^{j'})y \leq d_j + d_{j'} & j \in J_1, j' \in J_2. \end{cases}$$

Proof: For any $(w, x, y) \in P_1 \uplus P_2$, there exist u, v such that $w = u + v$, $(u, x) \in P_1$, and $(v, y) \in P_2$. Since (u, x) and (v, y) satisfy (6.1) and (6.2) respectively, therefore (w, x, y) satisfies (6.3).

Suppose (w, x, y) satisfies (6.3). Then $\max(a^i x - b_i, w + c^j y - d_j : i \in I_2, j \in J_1) \leq \min(b_i - a^i x, w + d_j - c^j y : i \in I_1, j \in J_2)$. Pick a number u between these two values, and let $v = w - u$. Then $u \leq b_i - a^i x$ for $i \in I_1$ and $a^i x - b_i \leq u$ for $i \in I_2$. These are equivalent to $u + a^i x \leq b_i$ for $i \in I_1$ and $-u + a^i x \leq b_i$ for $i \in I_2$. Similarly,

we have $v + c^j y \leq d_j$ for $j \in J_1$, and $-v + c^j y \leq d_j$ for $j \in J_2$. So $(u, x) \in P_1$ and $(v, y) \in P_2$, which imply $(w, x, y) \in P_1 \uplus P_2$. ■

Theorem 6.1.4 Suppose that P_1 and P_2 are full dimensional and $|V_0| = 1$. Then

- (a) $P_1 \uplus P_2$ is full dimensional;
- (b) If $\epsilon u + ax \leq a_0$ is facet-inducing for P_1 , and $\epsilon u + cy \leq c_0$ is facet-inducing for P_2 , where $\epsilon \neq 0$, then $\epsilon u + ax + cy \leq a_0 + c_0$ is facet-inducing for $P_1 \uplus P_2$.

Proof: Let p, q denote the dimension of P_1 and P_2 respectively. There are $p + 1$ affinely independent vectors

$$\begin{pmatrix} u^0 \\ x^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ x^1 \end{pmatrix}, \dots, \begin{pmatrix} u^p \\ x^p \end{pmatrix}$$

of P_1 , and $q + 1$ affinely independent vectors

$$\begin{pmatrix} v^0 \\ y^0 \end{pmatrix}, \begin{pmatrix} v^1 \\ y^1 \end{pmatrix}, \dots, \begin{pmatrix} v^q \\ y^q \end{pmatrix}$$

of P_2 . Consider the following vectors

$$\begin{pmatrix} u^0 + v^0 \\ x^0 \\ y^0 \end{pmatrix}, \dots, \begin{pmatrix} u^p + v^0 \\ x^p \\ y^0 \end{pmatrix}, \begin{pmatrix} u^0 + v^1 \\ x^0 \\ y^1 \end{pmatrix}, \dots, \begin{pmatrix} u^0 + v^q \\ x^0 \\ y^q \end{pmatrix}$$

of $P_1 \uplus P_2$. We show that there are $p + q$ affinely independent vectors among them. It is equivalent to showing that there are $p + q - 1$ linearly independent vectors among

$$\begin{pmatrix} u^1 - u^0 \\ x^1 - x^0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u^p - u^0 \\ x^p - x^0 \\ 0 \end{pmatrix}, \begin{pmatrix} v^1 - v^0 \\ 0 \\ y^1 - y^0 \end{pmatrix}, \dots, \begin{pmatrix} v^q - v^0 \\ 0 \\ y^q - y^0 \end{pmatrix}.$$

But this is true since the first p vectors are linearly independent and the remaining q vectors are linearly independent. So $\dim(P_1 \uplus P_2) = p + q - 1$.

Let F_1, F_2 be facets defined by $\epsilon u + ax \leq a_0$ and $\epsilon u + cy \leq c_0$ respectively. Let F be the face of $P_1 \oplus P_2$ defined by $\epsilon u + ax + cy \leq a_0 + c_0$. (It is easy to see that this inequality is indeed valid for $P_1 \oplus P_2$.) There are $p' + 1 = p$ affinely independent vectors

$$\begin{pmatrix} u^0 \\ x^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ x^1 \end{pmatrix}, \dots, \begin{pmatrix} u^{p'} \\ x^{p'} \end{pmatrix}$$

in F_1 , and $q' + 1 = q$ affinely independent vectors

$$\begin{pmatrix} v^0 \\ y^0 \end{pmatrix}, \begin{pmatrix} v^1 \\ y^1 \end{pmatrix}, \dots, \begin{pmatrix} v^{q'} \\ y^{q'} \end{pmatrix}$$

in F_2 . Consider the following vectors of F

$$\begin{pmatrix} u^0 + v^0 \\ x^0 \\ y^0 \end{pmatrix}, \dots, \begin{pmatrix} u^{p'} + v^0 \\ x^{p'} \\ y^0 \end{pmatrix}, \begin{pmatrix} u^0 + v^1 \\ x^0 \\ y^1 \end{pmatrix}, \dots, \begin{pmatrix} u^0 + v^{q'} \\ x^0 \\ y^{q'} \end{pmatrix}.$$

We show that they are affinely independent. This is equivalent to showing that the following vectors

$$\begin{pmatrix} u^1 - u^0 \\ x^1 - x^0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u^{p'} - u^0 \\ x^{p'} - x^0 \\ 0 \end{pmatrix}, \begin{pmatrix} v^1 - v^0 \\ 0 \\ y^1 - y^0 \end{pmatrix}, \dots, \begin{pmatrix} v^{q'} - v^0 \\ 0 \\ y^{q'} - y^0 \end{pmatrix}$$

are linearly independent. Note that the first p' vectors are linearly independent and the remaining q' vectors are linearly independent. We replace the first component of each of them by its inner product with (ϵ, a, c) . This new set of vectors is linearly independent if and only if the old one is, but since all their first components are zero, it is linearly independent. Thus $\dim(F) = p' + q' = (p + q - 1) - 1$, and F is a facet, as required. ■

Proposition 6.1.4 can be used to generate complicated facet-inducing inequalities from simple ones.

6.2 Bad facets of Eulerian subgraph degree sequence polyhedra

In this section, we consider the convex hull, $P(G, \mathcal{E})$, of degree sequences of Eulerian subgraphs of graph $G = (V, E)$. We prove the following theorem.

Theorem 6.2.1 *Let $n \in \mathbb{Z}_+$. There exists a graph $G = (V, E)$ and a facet-inducing inequality $ax \leq 0$ of $P(G, \mathcal{E})$ such that $|V| \leq 7n + 1$, $\{-1, 0, 1, 2, 2^2, \dots, 2^n\} \subseteq \{a_v : v \in V\}$, and $P(G, \mathcal{E})$ is full dimensional.*

Figure 6.1: A facet for degree sequences of Eulerian subgraphs

Proposition 6.2.2 *Let G be the graph of Figure 6.1, and let $a = (-1, -1, -1, 2)$. Then $P(G, \mathcal{E})$ is full dimensional, and $ax \leq 0$ is facet-inducing for $P(G, \mathcal{E})$.*

Proof: It is easy to check that the following degree sequences of cycles,

$$\begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix},$$

are linearly independent. Note that 0 is in $P(G, \mathcal{E})$. So $\dim(P(G, \mathcal{E})) = 4$. It is full dimensional.

Let F be the face induced by $ax \leq 0$. Then F contains $\mathbf{0}$ and

$$\begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix},$$

which are affinely independent. So F is a facet. ■

Figure 6.2: A second facet for Eulerian subgraphs

Proposition 6.2.3 Let G be the graph of Figure 6.2, and let $a = (-1, 0, -1, 0, 1)$. Then $P(G, \mathcal{E})$ is full dimensional, and $ax \leq 0$ is facet-inducing for $P(G, \mathcal{E})$.

Proof: It is easy to check that the following degree sequences of cycles,

$$\begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \end{pmatrix},$$

are linearly independent. Note that $\mathbf{0}$ is in $P(G, \mathcal{E})$. So $\dim(P(G, \mathcal{E})) = 5$. It is full dimensional.

Let F be the face induced by $ax \leq 0$. Then F contains $\mathbf{0}$ and

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix},$$

which are affinely independent. So F is a facet. ■

Figure 6.3: A composed facet for Eulerian subgraph degree sequences

Proposition 6.2.4 Let G , a be as in Figure 6.3. Then $P(G, \mathcal{E})$ is full dimensional, and $ax \leq 0$ is facet-inducing for $P(G, \mathcal{E})$.

Proof: This graph and inequality are obtained from those of Propositions 6.2.2 and 6.2.3 by identifying a vertex with coefficient -1 from each. Therefore the result follows from Theorem 6.1.4. ■

Proof of Theorem 6.2.1: We construct a series of graphs G^i and facet-inducing inequalities $a^i x \leq a_0^i$, $i = 1, \dots, n$, such that $P(G^i, \mathcal{E})$ is full dimensional, $\{-1, 0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$ and $|V(G^i)| \leq 7i + 1$ for $i = 1, \dots, n$.

For $i = 1$, we take G , $ax \leq 0$ of Proposition 6.2.4 as G^1 and $a^1 x \leq a_0^1$. The statement is true by Proposition 6.2.4. For $i > 1$, by the inductive hypothesis, there is a graph G^{i-1} and a facet-inducing inequality $a^{i-1} x \leq a_0^{i-1}$ such that $P(G^{i-1}, \mathcal{E})$ is full dimensional, $\{-1, 0, 1, 2, \dots, 2^{i-1}\} \subseteq \{a_v^{i-1} : v \in V(G^{i-1})\}$ and $|V(G^{i-1})| \leq 7(i-1) + 1$. Note that $2^{i-1}ax \leq 2^{i-1}a_0$ is facet-inducing for $P(G_1)$. We pick a vertex w of G^{i-1}

such that $a_w^{i-1} = 2^{i-1}$ and a vertex w' of G_1 such that $2^{i-1}a_{w'} = 2^{i-1}$, and identify these two vertices. Let G^i be the new graph and a^i be the new vector of $\mathbb{R}^{V(G)}$. Then, by Proposition 6.1.3, $P(G^i, \mathcal{E})$ is full dimensional and $a^i x \leq a_0^i$ is a facet-inducing inequality of $P(G^i, \mathcal{E})$, where $\{-1, 0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$ and $|V(G^i)| \leq |V(G^{i-1})| + |V(G_1)| - 1 = 7i + 1$. ■

6.3 Eulerian subgraphs of a directed graph

For a directed graph $G = (V, E)$, a spanning subgraph H is **Eulerian** if $d_H^-(v) = d_H^+(v)$ for all v in $V(H)$. Equivalently, H is the union of edge disjoint directed cycles and isolated vertices. The outdegree sequence of H is more interesting than the degree sequence of H , which is a vector of 0's. We consider the convex hull of outdegree sequences of spanning Eulerian subgraphs of G . Let $P^+(G, \mathcal{E})$ denote this polyhedron. In this section, we give a linear description for $P^+(G, \mathcal{E})$.

Theorem 6.3.1 $P^+(G, \mathcal{E})$ is the set of $x \in \mathbb{R}_+^V$ satisfying $x(A) - x(B) \leq |(A, N^+(A) \setminus B)|$ for all $A, B \subseteq V$.

We note that in Theorem 6.3.1 the subsets A, B need not be disjoint.

A closely related polyhedron is the assignable subgraph polytope of G , studied by Balas [1]. H is **assignable** if it has a spanning subgraph that is the union of vertex disjoint directed cycles. The assignable subgraph polytope, $P^+(G, AS)$, is the convex hull of incidence vectors of assignable subgraphs of G . Balas gave a linear description for this polyhedron.

Theorem 6.3.2 $P^+(G, AS)$ is the set of $x \in \mathbb{R}_+^V$ satisfying $x \leq 1$, and $x(A \setminus N^+(A)) - x(N^+(A) \setminus A) \leq 0$ for all $A \subseteq V$.

Here we consider a common generalization. Let $M^+ = (a_{vj})$ denote the edge-vertex matrix such that $a_{vj} = 1$ if v is the tail of edge j and $a_{vj} = 0$ otherwise. Let $Q^+(G, u, b) = \{z \in \mathbb{R}^E : Mz = 0, 0 \leq z \leq u, M^+z \leq b\}$, and let $P^+(G, u, b) =$

$\{M^+z : z \in Q^+(G, u, b)\}$. Note that if $u = \infty$ and $b = 1$, then $P^+(G, u, b) = P^+(G, AS)$; if $u = 1$ and $b = \infty$, then $P^+(G, u, b) = P^+(G, \mathcal{E})$. We give a linear description for $P^+(G, u, b)$.

Theorem 6.3.3 $P^+(G, u, b)$ is the set of $x \in \mathbb{R}_+^V$ satisfying $x \leq b$, and $x(A) - x(B) \leq u(A, N^+(A) \setminus B)$ for all $A, B \subseteq V$.

We use the way to construct a bipartite graph $G' = (V_1 \cup V_2, E)$ from G as in [1]. For each vertex $v \in V$, make two copies v_1, v_2 ; for each edge (v, w) , there is an edge $[v_1, w_2]$.

Given $x \in \mathbb{R}_+^V$, we construct a network from G' by adding two vertices s (source vertex), t (terminal vertex), adding edge (s, v_1) and assigning capacity x_v for each $v \in V$, adding edge (v_2, t) and assigning capacity x_v for each $v \in V$, and assigning capacity u_e for each edge $e \in E$.

Proof of Theorem 6.3.3: We first make the following observation. Suppose that $x \in \mathbb{R}_+^V$ and $x \leq b$. Then $x \in P^+(G, u, b)$ if and only if there is a flow with amount $x(V)$.

From the max-flow min-cut Theorem, there is a flow with amount $x(V)$ if and only if $\delta^+(s)$ is a minimum cut. So $x \in P^+(G, u, b)$ if and only if $\delta^+(s)$ is a minimum cut.

For any cut $\delta^+(S)$, where $S \subseteq V_1 \cup V_2 \cup \{s\}$ and $s \in S$, let $A = \{v \in V : v_1 \in S\}$, $B = \{v \in V : v_2 \in S\}$. (This is a 1-1 correspondence. The inverse is, for any $A, B \subseteq V$, $S = \{v_1 : v \in A\} \cup \{v_2 : v \in B\} \cup \{s\}$) Then the capacity of $\delta^+(S)$ is $x(V_1 \setminus S) + x(V_2 \cap S) + u(V_1 \cap S, V_2 \setminus S) = x(V_1) - x(A) + x(B) + u(A, N^+(A) \setminus B)$. So $\delta^+(s)$ is a minimum cut if and only if $x(V_1) - x(A) + x(B) + u(A, N^+(A) \setminus B) \geq x(\delta^+(s))$, that is, $x(A) - x(B) \leq u(A, N^+(A) \setminus B)$, for every $A, B \subseteq V$.

So $x \in P^+(G, u, b)$ if and only if $x(A) - x(B) \leq u(A, N^+(A) \setminus B)$, for every $A, B \subseteq V$. ■

If $x \notin P^+(G, u, b)$, then a cut $\delta^+(S)$ with capacity less than $x(V)$ is found. So a pair $A, B \subseteq V$, violating the valid inequality $x(A) - x(B) \leq u(A, N^+(A) \setminus B)$, is found.

This solves the separation problem for $P^+(G, u, b)$.

Proof of Theorem 6.3.1: It follows from Theorem 6.3.3. ■

Proof of Theorem 6.3.2: Since $u = \infty$, the inequality $x(A) - x(B) \leq u(A, N^+(A) \setminus B)$ is redundant if $N^+(A) \setminus B \neq \emptyset$. So the polyhedron is defined by $x(A) - x(B) \leq 0$, where $N^+(A) \subseteq B$ for all $A, B \subseteq V$. But this inequality is the sum of $x(A) - x(N^+(A)) \leq 0$ and $-x(B \setminus N^+(A)) \leq 0$. So the polyhedron is defined by $x(A) - x(N^+(A)) \leq 0$, that is, $x(A \setminus N^+(A)) - x(N^+(A) \setminus A) \leq 0$, for all $A \subseteq V$. ■

One easy and interesting consequence of Proposition 6.3.2 and Proposition 6.3.3 is as follows.

Corollary 6.3.4 *Suppose that u, b are integral vectors. If $x \in P^+(G, u, b)$ is an integral vector, then there is an integral vector $z \in Q^+(G, u, b)$ such that $x = M^+z$.*

Chapter 7

Forest degree sequence polyhedra

7.1 Introduction

In this chapter, we study the convex hulls of degree sequences of forests and of spanning trees of a graph $G = (V, E)$. We assume G is connected. Let $P(G, \mathcal{F})$ denote the convex hull of degree sequences of forests of G , and let $P(G, \mathcal{T})$ denote the convex hull of degree sequences of spanning trees of G . Let $Q(G, \mathcal{F})$ denote the convex hull of the incidence vectors of forests of G , and let $Q(G, \mathcal{T})$ denote the convex hull of the incidence vectors of spanning trees of G . Then $P(G, \mathcal{F}) = \{Mz : z \in Q(G, \mathcal{F})\}$ and $P(G, \mathcal{T}) = \{Mz : z \in Q(G, \mathcal{T})\}$.

The following two results are due to Edmonds[9]. Let $\mathcal{S} = \{S : \emptyset \neq S \subseteq V\}$.

Theorem 7.1.1 $Q(G, \mathcal{F}) = \{z \in \mathbb{R}_+^E : z(\gamma(S)) \leq |S| - 1 \text{ for every } S \in \mathcal{S}\}$.

Theorem 7.1.2 $Q(G, \mathcal{T}) = \{z \in \mathbb{R}_+^E : z(\gamma(S)) \leq |S| - 1 \text{ for every } S \in \mathcal{S}, z(\gamma(V)) = |V| - 1\}$.

One may think that there are simple linear descriptions for $P(G, \mathcal{F})$ and $P(G, \mathcal{T})$. But the actual situation is quite different. As we shall prove in this chapter, these polyhedra have nasty facet-inducing inequalities, in the sense that some of their coefficients have to be very big. In Section 2, we discuss polyhedra for forests. In Section 3, we deal with polyhedra for spanning trees.

In the rest of this section, we prove two general polyhedral results.

Two linear transformation results

Let $Q = \{z \in \mathbb{R}_+^m : c^1 z \leq b^1, i \in I_1; c^i z = b_i, i \in I_2\} = \{z \in \mathbb{R}_+^m : C^1 z \leq b^1; C^2 z = b^2\}$. Let $I = I_1 \cup I_2$ and $J = \{j : 1 \leq j \leq m\}$. Let D be a $n \times m$ matrix, and let P denote $\{Dz \in \mathbb{R}^n : z \in Q\}$. We need to study polyhedral properties for P . For $a \in \mathbb{R}^n$, let a_0 be $\max\{ax : x \in P\}$.

Lemma 7.1.3 *If $\text{aff}(Q) = \{z \in \mathbb{R}^m : C^2 z = b^2\}$, then $ax = a_0$ is valid for every $x \in P$ if and only if aD is in the row space of C^2 . Furthermore, if $\{a^i : i = 1, \dots, k\}$ is a maximal set of linearly independent solutions of $\alpha D = yC^2$ for α , then $\text{aff}(P) = \{x \in \mathbb{R}^n : a^i x = a_0^i, i = 1, \dots, k\}$ and $\dim(P) = n - k$.*

Proof: $ax = a_0$ is valid for every $x \in P$ if and only if $aDz = a_0$ is valid for every $z \in Q$. Since $\text{aff}(Q) = \{z \in \mathbb{R}^m : C^2 z = b^2\}$, $aDz = a_0$ is valid for every $z \in Q$ if and only if (aD, a_0) is a linear combination of the rows of $[C^2, b^2]$. The remaining claims are direct consequences of the first claim. ■

The linear programming problem

$$(7.1) \quad \max\{aDz : C^1 z \leq b^1, C^2 z = b^2, z \geq 0\}$$

has dual

$$(7.2) \quad \min\{y^1 b^1 + y^2 b^2 : y^1 C^1 + y^2 C^2 \geq aD, y^1 \geq 0\}.$$

It is clear that a_0 is equal to the above value. Let $J_a = \{j \in J : \text{there is an optimal } z \text{ of (7.1) such that } z_j > 0\}$. Let $I_a = \{i \in I : \text{for every optimal } z \text{ of (7.1), } c^i z = b_i\}$.

Lemma 7.1.4 *Let $\dim(P) = n - k$ and $\text{aff}(P) = \{x \in \mathbb{R}^n : a^i x = a_0^i, i = 1, \dots, k\}$. Then $ax \leq a_0$ is facet-inducing for P if and only if a is not a linear combination of $a^1, \dots, a^k$, and $\alpha = la + l_1 a^1 + \dots + l_k a^k$, $l \geq 0$, $l_i \in \mathbb{R}$, $i = 1, \dots, k$ is the general solution of*

$$(7.3) \quad \begin{cases} \left\{ \sum (y_i c_j^i : i \in I_a) \right\} \begin{cases} = \alpha D^j & \forall j \in J_a \\ \geq \alpha D^j & \forall j \in J \end{cases} \\ y_i \geq 0 & \forall i \in I_a \setminus I_2 \\ y_i = 0 & \forall i \in I \setminus I_a \end{cases}$$

Proof: Suppose $ax \leq a_0$ is a facet-inducing inequality. Let y be an optimal solution of (7.2). By the complementary slackness conditions, (y, a) satisfies (7.3). Let (y', a') be a solution of (7.3), and let $F_a, F_{a'}$ be the faces induced by $ax \leq a_0, a'x \leq a'_0$ respectively. For any $x \in F_a$, there is a $z \in Q$ such that $x = Dz$. z is an optimal solution of (7.1). So (z, y') satisfies the complementary slackness conditions for the dual pair

$$(7.4) \quad \max\{a'Dz : C^1z \leq b^1, C^2z = b^2, z \leq 0\} =$$

$$(7.5) \quad \min\{y^1b^1 + y^2b^2 : y^1C^1 + y^2C^2 \geq a'D, y^1 \geq 0\}.$$

Namely, $c^iz < b_i$ (so $i \in I \setminus I_a$) implies $y'_i = 0$; $z_j > 0$ (so $j \in J_a$) implies $\sum(y'_i a^i_j : i \in I_a) = a'D^j$. So $x = Dz \in F_{a'}$; thus $F_a \subseteq F_{a'}$.

But F_a is a facet, so either $F_a = F_{a'}$, whence $a' = la + l_1a^1 + \dots + l_ka^k$ for some $l > 0, l_1, \dots, l_k$, or $F_{a'} = P$, whence $a' = l_1a^1 + \dots + l_ka^k$ for some $l_1, \dots, l_k$. On the other hand, for any $la + l_1a^1 + \dots + l_ka^k$, let $y^i \in \mathbb{R}^I$ such that (y^i, a^i) is a solution of (7.3), for $i = 1, \dots, k$. Then $(ly + l_1y^1 + \dots + l_ky^k, la + l_1a^1 + \dots + l_ka^k)$ is a solution of (7.3). So $la + l_1a^1 + \dots + l_ka^k, l \geq 0, l_i \in \mathbb{R}, i = 1, \dots, k$ is the general solution of (7.3).

Now suppose that $\{la + l_1a^1 + \dots + l_ka^k\}$ is the general solution of (7.3), and $ax \leq a_0$ is not facet-inducing. Then there exists a facet-inducing inequality $a'x \leq a'_0$ such that $F_a \subset F_{a'}$ and $F_a \neq F_{a'}$. Let $y_{a'}$ be an optimal solution of (7.5). Then $(y_{a'}, a')$ satisfies (7.3) since for every $i \in I \setminus I_a$, we have $i \in I \setminus I_{a'}$, and for every $j \in J_a$, we have $j \in J_{a'}$. So $a' = la + l_1a^1 + \dots + l_ka^k$ for certain $l > 0$, and hence $F_{a'} \subseteq F_a$, a contradiction. So $ax \leq a_0$ is a facet-inducing inequality. ■

7.2 Bad facets for forest degree sequence polyhedra

Results

In this section, we discuss the degree sequence polyhedra of forests.

Proposition 7.2.1 $\dim(P(G, \mathcal{F})) = |V| - (\text{number of bipartite components})$. In particular,

- (a) if G is nonbipartite and connected, then $\dim(P(G, \mathcal{F})) = |V|$;
- (b) if G is connected and bipartite with bipartition $\{V_1, V_2\}$, then $\dim(P(G, \mathcal{F})) = |V| - 1$ and $x(V_1) - x(V_2) = 0$ for all $x \in P(G, \mathcal{F})$.

The proof is standard, see the proof for Proposition 2.2.2.

We shall prove the following theorems.

Theorem 7.2.2 Let $n \in \mathbb{Z}_+$. There exist a nonbipartite graph $G = (V, E)$ and a facet-inducing inequality $ax \leq a_0$ of $P(G, \mathcal{F})$ such that $|V| \leq 7n+1$, and $\{0, 1, 2, 2^2, \dots, 2^n\} \subseteq \{a_v : v \in V\}$.

Theorem 7.2.3 Let $n \in \mathbb{Z}_+$. There exist a 2-connected nonbipartite graph $G = (V, E)$ and a facet-inducing inequality $ax \leq a_0$ of $P(G, \mathcal{F})$ such that $|V| \leq 10n - 2$, and $\{0, 1, 2, \dots, n+1\} \subseteq \{a_v : v \in V\}$.

Theorem 7.2.4 Let $n \in \mathbb{Z}_+$. There exist a bipartite graph $G = (V, E)$ and a facet-inducing inequality $ax \leq a_0$ of $P(G, \mathcal{F})$ such that $|V| \leq 14n+1$, and $\{0, 1, 2, 2^2, \dots, 2^n\} \subseteq \{a_v : v \in V\}$.

The rest of this section is to prove the theorems. In the proofs of Theorem 7.2.2, Theorem 7.2.3, and Theorem 7.2.4, we use Lemma 7.1.4 again and again. We remark that for each $a \in \mathbb{R}^V$, S_a is defined as I_a , E_a is defined as J_a . For a vector a^i , we also use S_i , E_i to mean S_{a^i} , E_{a^i} . We need the following decomposition result.

Proposition 7.2.5 Let $G = G^1 \cup G^2$, $V(G^1) \cap V(G^2) = W$, and $E(G^1) \cap E(G^2) = \emptyset$. Let $a \in \mathbb{R}^V$, and let a^1, a^2 be the projection of a on $\mathbb{R}^{V(G^1)}, \mathbb{R}^{V(G^2)}$ respectively. Suppose $S_a = S_1 \cup S_2$. If $a^1 x \leq a_0^1$ is facet-inducing, and a^2 is the unique solution of

$$(7.6) \quad \begin{cases} \alpha_w = a_w & \forall w \in W, \\ \sum_{S \in S_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 & \forall S \in S_2, \end{cases}$$

then

(a) $ax \leq a_0$ is facet-inducing, if G^1 is nonbipartite;

(b) $ax \leq a_0$ is facet-inducing, if G is bipartite.

Proof: If G^1 is nonbipartite, we need to prove that $\{la : l \geq 0\}$ is the general solution of (7.7) and (7.8).

$$(7.7) \quad \begin{cases} \sum_{S \in S_1} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_1, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^1), \end{cases} \\ y_S \geq 0 & \forall S \in S_1, \end{cases}$$

$$(7.8) \quad \begin{cases} \sum_{S \in S_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 & \forall S \in S_2, \end{cases}$$

For any solution $(\hat{a}, \hat{y})$ of (7.7) and (7.8), $(\hat{a}^1, \hat{y}^1)$, induced by $(\hat{a}, \hat{y})$, is a solution of (7.7). Since $a^1 x \leq a_0^1$ defines a facet, $\{la^1 : l \geq 0\}$ is the general solution of (7.7). So $\hat{a}^1 = la^1$ for some $l \geq 0$.

$(\hat{a}^2, \hat{y}^2)$, induced by $(\hat{a}, \hat{y})$, satisfies

$$\begin{cases} \alpha_w = a_w & \forall w \in W, \\ \sum_{S \in S_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 & \forall S \in S_2. \end{cases}$$

If $l > 0$, then $(\hat{a}^2, \hat{y}^2)/l$ satisfies (7.6). So $\hat{a}^2/l = a^2$ since a^2 is the unique solution of (7.6), and thus $\hat{a} = la$. If $l = 0$, then $\hat{a}^2 = 0$. Otherwise $a^2 + \hat{a}^2$ is a different solution of (7.6). So $\hat{a} = 0 \in \{la : l \geq 0\}$. In both cases we have $\hat{a} = la$ for some $l \geq 0$. Since $\hat{a}$ is arbitrary, $\{la : l \geq 0\}$ is the general solution of (7.7) and (7.8). Therefore, $ax \leq a_0$ is facet-inducing by Lemma 7.1.4.

If G is bipartite with bipartition $\{V_1, V_2\}$, we need to prove that $\{la + l_1(\chi^{V_1} - \chi^{V_2})\}$ is the general solution of (7.7) and (7.8). For any solution $(\hat{a}, \hat{y})$ of (7.7) and (7.8), $(\hat{a}^1, \hat{y}^1)$, induced by $(\hat{a}, \hat{y})$, is a solution of the first system of (7.7). So $\hat{a}^1 = la^1 + l_1(\chi^{V_1(G^1)} - \chi^{V_2(G^1)})$ for some $l \geq 0$ and l_1 , by Lemma 7.1.4.

If $l > 0$, then $(\hat{a}^2 - l_1(\chi^{V_1(G^1)} - \chi^{V_2(G^1)}))/l$ is a solution of (7.6) for α . So $(\hat{a}^2 - l_1(\chi^{V_1(G^1)} - \chi^{V_2(G^1)}))/l = a^2$ since a^2 is the unique solution of (7.6), that is, $\hat{a}^2 = la^2 + l_1(\chi^{V_1(G^2)} - \chi^{V_2(G^2)})$. Hence $\hat{a} = la + l_1(\chi^{V_1(G)} - \chi^{V_2(G)})$. If $l = 0$, then $\hat{a}^2 = l_1(\chi^{V_1(G)} - \chi^{V_2(G)})$. Otherwise $a^2 + \hat{a}^2 - l_1(\chi^{V_1(G^2)} - \chi^{V_2(G^2)})$ is a different solution of (7.6). So $\hat{a} = l_1(\chi^{V_1(G)} - \chi^{V_2(G)})$. ■

The following corollary of Proposition 7.2.5 is also a direct consequence of Theorem 6.1.4 in Chapter 6. Here we prove it in a different way.

Corollary 7.2.6 Suppose G^1, G^2 are nonbipartite, and $W := V(G^1) \cap V(G^2) = \{w\}$. Let $a^1 x^1 \leq a_0^1$, $a^2 x^2 \leq a_0^2$ be facet-inducing for $P(G^1, \mathcal{F})$, $P(G^2, \mathcal{F})$ respectively. If $a_w \neq 0$, then $ax \leq a_0$ is facet-inducing.

Proof: $a^2 x^2 \leq a_0^2$ is facet-inducing and G^2 is nonbipartite, so $\{ka^2 : k \geq 0\}$ is the general solution of

$$\begin{cases} \sum_{S \in \mathcal{S}_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 \end{cases} \quad \forall S \in \mathcal{S}_2,$$

by Lemma 7.1.4. Let ka^2 be a solution of (7.6). Then $ka_w^2 = a_w^2$, $k = 1$ since $a_w \neq 0$. So a^2 is the unique solution of (7.6). Hence $ax \leq a_0$ is facet-inducing by Proposition 7.2.5. ■

Nonbipartite graphs

The aim of this subsection is to give the proofs for Theorem 7.2.2 and Theorem 7.2.3.

Figure 7.1: A bad facet for degree sequences of forests

Proposition 7.2.7 *Let G be the graph of Figure 7.1, and let $a = (2, 1, 2, 1, 1, 0, 0, 0)$ and $a_0 = 12$. Then $ax \leq a_0$ is a facet-inducing inequality of $P(G, \mathcal{F})$.*

Proof: It is easy to check that $E_a = E$ and $S_a = \{S_1 = \{1, 2\}, S_2 = \{3, 4\}, S_3 = \{1, 2, 3, 4, 5, 6\}\}$. Let y_i denote y_{S_i} . A subsystem of the corresponding system of (7.3) is

$$\left\{ \begin{array}{l} \alpha_2 + \alpha_4 = y_3 \\ \alpha_2 + \alpha_5 = y_3 \\ \alpha_5 + \alpha_4 = y_3 \\ \alpha_6 + \alpha_7 = 0 \\ \alpha_7 + \alpha_8 = 0 \\ \alpha_6 + \alpha_8 = 0 \\ \alpha_1 + \alpha_6 = y_3 \\ \alpha_3 + \alpha_6 = y_3 \\ y_3 \geq 0 \end{array} \right.$$

The first three equalities give $\alpha_2 = \alpha_4 = \alpha_5 = y_3$, and the next three equalities give $\alpha_6 = \alpha_7 = \alpha_8 = 0$. The last two equalities give $\alpha_1 = \alpha_3 = y_3$. It is easy to check that the above solution satisfies the corresponding system of (7.3). So, we

have the general solution $\alpha = l(2, 1, 2, 1, 1, 0, 0, 0)$, $l = y_3/2 \geq 0$. Thus $ax \leq a_0$ is a facet-inducing inequality by Lemma 7.1.4. ■

Proposition 7.2.5 allows us to compose complicated graph G and facet-inducing inequality $ax \leq a_0$ from simple ones.

Proof of Theorem 7.2.2: We construct a sequence of graphs G^i and facet-inducing inequalities $a^i x \leq a_0^i$, $i = 1, \dots, n$, such that $\{0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$ and $|V(G^i)| \leq 7i + 1$ for $i = 1, \dots, n$.

For $i = 1$, we take G , $ax \leq a_0$ of Proposition 7.2.7 as G^1 and $a^1 x \leq a_0^1$. The statement is true by Proposition 7.2.7. For $i > 1$, by the inductive hypothesis, there is a graph G^{i-1} and a facet-inducing inequality $a^{i-1} x \leq a_0^{i-1}$ such that $\{0, 1, 2, \dots, 2^{i-1}\} \subseteq \{a_v^{i-1} : v \in V(G^{i-1})\}$ and $|V(G^{i-1})| \leq 6(i-1) + 2$. Note that $2^{i-1} ax \leq 2^{i-1} a_0$ is facet-inducing for $P(G_1, \mathcal{F})$. We pick a vertex w of G^{i-1} such that $a_w^{i-1} = 2^{i-1}$ and a vertex w' of G^1 such that $2^{i-1} a_{w'} = 2^{i-1}$, and identify these two vertices. Let G^i be the new graph and a^i be the new vector of $\mathbb{R}^{V(G)}$. Then $a^i x \leq a_0^i$ is a facet-inducing inequality of $P(G^i, \mathcal{F})$ by Corollary 7.2.6, where $\{0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$ and $|V(G^i)| \leq |V(G^{i-1})| + |V(G_1)| - 1 = 7i + 1$. ■

Here we give another composition operation. We assume that there exist a graph G^1 and a vector $a \in \mathbb{R}^{V(G^1)}$ with $w_1, w_2 \in V(G^1)$ and $0 \leq a_{w_1} = \beta < a_{w_2} = \mu$.

Operation A

Let G^2, a^2 be as in Figure 7.2, $V(G^1) \cap V(G^2) = \{w_1, w_2\}$, and $E(G^1) \cap E(G^2) = \emptyset$. Let $G = G^1 \cup G^2$, $a_0 = a_0^1 + a_0^2$ and $a \in \mathbb{R}^{V(G)}$ such that $a_v = a_v^1$ for $v \in V(G^1)$, $a_v = a_v^2$ for $v \in V(G^2)$.

Proposition 7.2.8 *If G^1 is nonbipartite and $a^1 x \leq a_0^1$ is a facet-inducing inequality for $P(G^1, \mathcal{F})$, then Operation A gives a facet-inducing inequality for $P(G, \mathcal{F})$.*

Figure 7.2: Operation A graph

Proof: It is clear that $E_2 = E(G^2)$, and $S_a = S_1 \cup S_2$, where S_2 is the same as that of Proposition 7.2.7. Solving

$$\begin{cases} \alpha_{w_1} = \beta, & \alpha_{w_2} = \mu, \\ \sum_{S \in S_2} (y_S : uv \in \gamma(S)) = \alpha_u + \alpha_v & \forall uv \in E(G^2), \\ y_S \geq 0 & \forall S \in S_2, \end{cases}$$

in the similar way as in the proof of Proposition 7.2.7, we have a unique solution a^2 . So $ax \leq a_0$ is a facet-inducing inequality for $P(G, \mathcal{F})$ by Proposition 7.2.5. ■

Proof of Theorem 7.2.3: We construct inductively a sequence of graphs G^i and facet-inducing inequalities $a^i x \leq a_0^i$ of $P(G^i, \mathcal{F})$, $i = 1, \dots, n$, such that $\{0, 1, 2, \dots, i+1\} \subseteq \{a_v^i : v \in V(G^i)\}$ and $|V(G^i)| \leq 10i - 2$ for $i = 1, \dots, n$. For $i \geq 2$, the graph G^i is 2-connected.

For $i = 1$, we take G , a in Proposition 7.2.7. For $i = 2$, we apply Operation A to G^1 with $\beta = 1$, $\mu = 2$. Then the new graph G^2 and vector a^2 satisfy the required conditions. For $i \geq 3$, consider the graph G^{i-1} and a^{i-1} given by the inductive hypothesis. If i is odd [even], then $k := (i+1)/2$ [$k := i/2$] is an integer. We apply Operation A on G^{i-1} with $\beta = 1$, $\mu = k$. Let G^i be the new graph and $a^i x \leq a_0^i$ be the new facet-inducing inequality. Then $|V(G^i)| = |V(G^{i-1})| + 10 = 10i - 2$, and $\{0, 1, 2, \dots, i+1\} \subseteq \{a_v : v \in V(G^i)\}$. ■

Bipartite graphs

Figure 7.3: A bad facet for bipartite graph

Proposition 7.2.9 *Let G be the graph of Figure 7.3, and let $a = (1, 1, 0, 0, 1, 1, -1, -1, 1, 1, 0, 0, 2, 2, 0)$. Then $ax \leq a_0$ is a facet-inducing inequality of $P(G, \mathcal{F})$.*

Proof: It is easy to check that $E_a = E$ and $\mathcal{S}_a = \{S_1 = \{1, 9\}, S_2 = \{2, 10\}, S_3 = \{1, 2, 3, 9, 10, 11\}, S_4 = \{5, 13\}, S_5 = \{6, 14\}, S_6 = \{5, 6, 7, 13, 14, 15\}\}$. A subsystem of the corresponding system of (7.3) is

$$\left\{ \begin{array}{l} \alpha_3 + \alpha_9 = y_3, \alpha_3 + \alpha_{10} = y_3, \\ \alpha_{11} + \alpha_1 = y_3, \alpha_{11} + \alpha_2 = y_3, \\ \alpha_3 + \alpha_{12} = 0, \alpha_{11} + \alpha_4 = 0, \alpha_4 + \alpha_{12} = 0, \\ \alpha_5 + \alpha_{15} = y_6, \alpha_6 + \alpha_{15} = y_6, \\ \alpha_7 + \alpha_{13} = y_6, \alpha_7 + \alpha_{14} = y_6, \\ \alpha_1 + \alpha_8 = 0, \alpha_5 + \alpha_8 = 0, \\ \alpha_7 + \alpha_{10} = 0, \alpha_4 + \alpha_{15} = 0, \\ \alpha_1 + \alpha_9 = y_1 + y_3, \\ y_1 \geq 0, y_2 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0. \end{array} \right.$$

Let $l = (\alpha_1 + \alpha_9)/2$, then $l = (y_1 + y_3)/2 \geq 0$ from the equality in the eighth line. Let $l_1 = (\alpha_1 - \alpha_9)/2$. Then $\alpha_1 = l + l_1$ and $\alpha_9 = l - l_1$. The equalities in the first and the second lines give $\alpha_{10} = \alpha_9 = l - l_1$, $\alpha_2 = \alpha_1 = l + l_1$, and $\alpha_3 - \alpha_{11} = 2l_1$. The equalities in the third line imply $\alpha_{11} = \alpha_{12}$, $\alpha_3 = \alpha_4$, and $\alpha_3 + \alpha_{11} = 0$. These together with the previous one give $\alpha_3 = l_1$, $\alpha_4 = l_1$, $\alpha_{11} = \alpha_{12} = -l_1$. Similarly, by considering the rest equalities, we have $\alpha_5 = \alpha_6 = l + l_1$, $\alpha_{13} = \alpha_{14} = 2l - l_1$, $\alpha_7 = l_1 - l$, $\alpha_8 = -l - l_1$,

and $\alpha_{15} = -l_1$. It is easy to check that the above solution satisfies the corresponding system of (7.3). So, we have the general solution $\alpha = l a + l_1(\chi^{V_1} - \chi^{V_2})$, where $l \geq 0$. So $\alpha x \leq a_0$ is a facet-inducing inequality by Lemma 7.1.4. ■

Here we give a composition operation. As in Operation A, we assume that there are a graph G^1 and a vector $a \in \mathbb{R}^{V(G^1)}$ with $w_1, w_2 \in V(G^1)$ and $0 \leq a_{w_1} = \beta < a_{w_2} = \mu$.

Operation B

Let G^2, a^2 be in Figure 7.4, $V(G^1) \cap V(G^2) = \{w_1, w_2\}$, and $E(G^1) \cap E(G^2) = \emptyset$. Let $G = G^1 \cup G^2$, $a_0 = a_0^1 + a_0^2$ and $a \in \mathbb{R}^{V(G)}$ such that $a_v = a_v^1$ for $v \in V(G^1)$, $a_v = a_v^2$ for $v \in V(G^2)$.

Figure 7.4: Operation B graph

Proposition 7.2.10 *If G^1 is bipartite with bipartition $\{V_1, V_2\}$ and $a^1 x \leq a_0^1$ is a facet-inducing inequality for $P(G^1, \mathcal{F})$, then Operation B gives a bipartite graph G and a facet-inducing inequality for $P(G, \mathcal{F})$.*

Proof: It is clear that $E_2 = E$, and $S_a = S_1 \cup S_2$, where S_2 is the same as that in Proposition 7.2.9. Solving

$$\begin{cases} \alpha_{w_1} = \beta, \quad \alpha_{w_2} = \mu, \\ \sum_{S \in S_2} (y_S : uv \in \gamma(S)) = \alpha_u + \alpha_v \quad \forall uv \in E(G^2), \\ y_S \geq 0 \end{cases} \quad \forall S \in S_2,$$

in the similar way as that in the proof of Proposition 7.2.9, we have a unique solution a^2 . So $\alpha x \leq a_0$ is a facet-inducing inequality for $P(G, \mathcal{F})$ by Proposition 7.2.5. ■

Proof of Theorem 7.2.4: We construct inductively a sequence of graphs G^i and facet-inducing inequalities $a^i x \leq a_0^i$, $i = 1, \dots, n$, such that $\{0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$, $|V(G^i)| \leq 14i + 1$ for $i = 1, \dots, n$, and the graph G^i is 2-connected.

For $i = 1$, we take G , a in Proposition 7.2.9 as G^1 and $a^1 x \leq a_0^1$. For $i \geq 2$, consider the graph G^{i-1} and a^{i-1} given by the inductive hypothesis.

We apply Operation B on G^{i-1} with $\beta = 0$, $\mu = 2^{i-1}$. Let G^i be the new graph and $a^i x \leq a_0^i$ be the new facet-inducing inequality. Then $|V(G^i)| = |V(G^{i-1})| + 14 = 14i + 1$, and $\{0, 1, 2, \dots, 2^i\} \subseteq \{a_v^i : v \in V(G^i)\}$. ■

We note that determining whether a vector is a convex combination of the degree sequences of some forests of G is solvable in polynomial-time. However, determining whether a vector is the degree sequence of a forest of G is NP-hard. The reason is that determining whether vector $(1, 1, 2, 2, \dots, 2)$ is the degree sequence of a forest of G amounts to determining whether there is a Hamiltonian path, which is NP-hard.

To end this section, we mention two open problems. The first problem is to characterize the extreme points of $P(G, \mathcal{F})$. The second one is to find an efficient combinatorial algorithm for the separation problem for $P(G, \mathcal{F})$.

7.3 Bad facets for spanning tree degree sequence polyhedra

In this section, we study the convex hull of degree sequences of spanning trees of graph $G = (V, E)$. We use the same notation as in Section 7.2. Again we prove the following “negative” result.

Theorem 7.3.1 *Let $n \in \mathbb{Z}_+$. There exists a 2-connected nonbipartite graph $G = (V, E)$ and a facet-inducing inequality $ax \leq a_0$ of $P(G, \mathcal{T})$ such that $|V| \leq 10n - 4$, and $\{0, 1, 2, \dots, n + 1\} \subseteq \{a_v : v \in V\}$.*

In the rest of this section, we study basic properties of $P(G, \mathcal{T})$ and prove Theorem 7.3.1.

Proposition 7.3.2 *Let p denote the number of bipartite blocks of G . Then $\dim(P(G, \mathcal{T})) = |V| - p - 1$.*

Proof: Let $\mathcal{B} = \{B_1, \dots, B_p\}$ denote the set of all bipartite blocks, and let $\mathcal{B}' = \{B'_1, \dots, B'_q\}$ denote the set of all nonbipartite blocks. Then $\text{aff}(Q(G, \mathcal{T})) = \{z \in \mathbb{R}^E : z(\gamma(B_i)) = |B_i| - 1, i = 1, \dots, p; z(\gamma(B'_i)) = |B'_i| - 1, i = 1, \dots, q\}$. Consider

$$\sum_1^p y_i \chi^{\gamma(B_i)} + \sum_1^q y'_i \chi^{\gamma(B'_i)} = \alpha M,$$

namely,

$$(7.9) \quad \begin{cases} y_i = \alpha_u + \alpha_v & \forall uv \in \cup_1^p \gamma(B_i), \\ y'_i = \alpha_u + \alpha_v & \forall uv \in \cup_1^q \gamma(B'_i). \end{cases}$$

We have the solution $\alpha_v = y'_i/2$ for all $v \in B'_i, i = 1, \dots, q$ and $\alpha_v = \alpha_u$ for all u, v in the same side (partition) of $B_i, i = 1, \dots, p$. Now we define a relation “ $\sim$ ” on V as follows. $u \sim v$ if u, v are connected with some vertices u', v' in $G - \gamma(\cup_1^p B_i)$, where u', v' are in the same side of B_i for some $1 \leq i \leq p$.

It is easy to check that “ $\sim$ ” is an equivalent relation. For any solution α of (7.9), if $u \sim v$, then we have $\alpha_v = \alpha_u$. Let $S_1, S_2, \dots, S_k$ be the equivalence classes of V determined by this relation. Then $k = p + 1$ since after shrinking each S_i and replacing $\gamma(B)$ by an edge $e_i, i = 1, \dots, p$, the resulting graph is a tree with p edges. So the general solution of (7.9) is $\alpha = \sum_1^k y_i \chi^{S_i}$. Note that $\chi^{S_1}, \dots, \chi^{S_k}$ are linearly independent, so $\dim(P(G, \mathcal{T})) = |V| - p - 1$ by Lemma 7.1.3. ■

Proposition 7.3.3 *Suppose that G^1 has no bipartite block and $a^1 x \leq a_0^1$ is a facet-inducing inequality for $P(G^1, \mathcal{T})$. Then Operation A creates a facet-inducing inequality $ax \leq a_0$ for $P(G, \mathcal{T})$.*

Proof: We need to prove that $la + l_1 \chi^V, l \geq 0$ is the general solution of (7.10), (7.11), and (7.12).

$$(7.10) \quad \begin{cases} \sum_{S \in \mathcal{S}_1} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_1, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^1), \end{cases} \\ y_S \geq 0 & \forall S \in \mathcal{S}_1 \setminus \{V_1\}, \end{cases}$$

$$(7.11) \quad \begin{cases} \sum_{S \in \mathcal{S}_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 & \forall S \in \mathcal{S}_2 \setminus \{V_2\}, \end{cases}$$

$$(7.12) \quad y_{v_1} = y_{v_2} = y_v$$

For any solution $(\hat{a}, \hat{y})$ of (7.10), (7.11), and (7.12), $(\hat{a}^1, \hat{y}^1)$, induced by $(\hat{a}, \hat{y})$, is a solution of the first system of (7.10). So $\hat{a}^1 = la^1 + l_1 \chi^{V_1}$ for some $l \geq 0$ and l_1 by Lemma 7.1.4.

We note that a^2 is the unique solution of

$$(7.13) \quad \begin{cases} \alpha_{w_1} = \beta, \quad \alpha_{w_2} = \mu, \\ \sum_{S \in \mathcal{S}_2} (y_S : uv \in \gamma(S)) \begin{cases} = \alpha_u + \alpha_v & \forall uv \in E_2, \\ \geq \alpha_u + \alpha_v & \forall uv \in E(G^2), \end{cases} \\ y_S \geq 0 & \forall S \in \mathcal{S}_2 \setminus \{V_2\}, \end{cases}$$

If $l > 0$, then $(\hat{a}^2 - l_1 \chi^{V_2})/l$ is a solution of (7.13) for α . So $\hat{a}^2 = la^2 + l_1 \chi^{V_2}$, and hence $\hat{a} = la + l_1 \chi^V$. If $l = 0$, then $\hat{a}^2 = l_1 \chi^{V_2}$. Otherwise $a^2 + \hat{a}^2 - l_1 \chi^{V_2}$ is a different solution of (7.13). So $\hat{a} = l_1 \chi^V$. So $la + l_1 \chi^V$, $l \geq 0$ is the general solution of (7.10), (7.11), and (7.12), hence $ax \leq a_0$ is facet-inducing by Lemma 7.1.4. ■

Figure 7.5: A bad facet for spanning tree degree sequences

Proposition 7.3.4 Let $G = (V, E)$, $a \in \mathbb{R}^V$, be as in Figure 7.5, and let $a_0 = 12$. Then $ax \leq a_0$ is facet-inducing for $P(G, T)$.

Proof: From Proposition 7.3.2, $\dim(P(G, \mathcal{T})) = |V| - 1 = 5$. Let F be the face induced by $ax \leq a_0$. The following degree sequences of spanning trees are in F :

$$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

It is also easy to check that they are affinely independent. So $\dim(F) = 4$ and $ax \leq a_0$ is facet-inducing. ■

Proof of Theorem 7.3.1: We construct a sequence of graphs G^i and facet-inducing inequalities $a^i x \leq a_0^i$, $i = 1, \dots, n$, such that $\{0, 1, 2, \dots, i+1\} \subseteq \{a_v^i : v \in V(G^i)\}$, $|V(G^i)| \leq 10i - 4$, and G^i is 2-connected for $i = 1, \dots, n$.

For $i = 1$, we take G , a in Proposition 7.3.4 as G^1 and $a^1 x \leq a_0^1$. For $i \geq 2$, by the inductive hypothesis, there is a graph G^{i-1} and a^{i-1} . If i is odd [even], then $k := (i+1)/2$ [$k := i/2$] is an integer. We apply Operation A on G^{i-1} with $\beta = 1$, $\mu = k$. Let G^i be the new graph and $a^i x \leq a_0^i$ be the new facet-inducing inequality. Then $|V(G^i)| = |V(G^{i-1})| + 10 = 10i - 4$, and $\{0, 1, 2, \dots, i+1\} \subseteq \{a_v : v \in V(G^i)\}$. ■

Chapter 8

Concluding remarks

In this thesis, we discussed several degree sequence polyhedra. We gave a linear description of the subgraph degree sequence polyhedron, described a separation algorithm, and also characterized extreme points and extreme point adjacency. For the u -capacitated b -matching degree sequence polyhedron of an undirected graph, we gave an efficient separation algorithm, and characterized the facets. We gave a complete linear description for the polyhedron of u -capacitated b -matching degree sequences of a bidirected graph. We also proved that there are nasty facet-inducing inequalities for forest degree sequence polyhedra and Eulerian subgraph degree sequence polyhedra.

Finally, we end this thesis by raising two open problems.

The separation algorithm for the b -matching degree sequence polyhedron presented in this thesis is polynomial-time but not strongly polynomial-time. A strongly polynomial-time separation algorithm is desired.

For the forest degree sequence polyhedron, find a combinatorial separation algorithm and characterize the extreme points of the polyhedron.

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