Asymptotic Efficiency of Weighted Tests for Symmetry

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Abstract

In 1945 Wilcoxon proposed a nonparametric statistical hypothesis test for the case of two related samples or repeated measurements on a single sample. The Wilcoxon signed rank test is distribution-free in the sense that its use does not require assumptions about the form of the distribution of measurements. It can be used for testing the hypothesis of symmetry. In 2008 Stute introduced a large class of weighted U -statistics of order statistics that, with a specific choice of weight and kernel functions, lead to weighted Wilcoxon-type tests for symmetry. A class of weighted Wilcoxon-type tests includes the classical Wilcoxon test as a particular case. In this thesis, efficiency properties of weighted Wilcoxon-type tests for large sample sizes are investigated. The Pitman efficiency of weighted Wilcoxon-type tests is calculated. Some illustrative examples are included. The main results of the thesis are new.

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Chapter 1

Introduction

Testing for symmetry is an important statistical problem. This topic has been dealt with in many textbooks on Nonparametric Statistics. When testing symmetry of the underlying distribution about a specified point, say zero, we might wish to test the following hypotheses

$$H_0 : F(x) = 1 - F(-x), \quad x \in \mathbf{R},$$

$$H_A : F(x) \leq 1 - F(-x), \quad x \in \mathbf{R},$$

with inequality being strict on a set of positive length. The alternative H_A says that random variable X with cumulative distribution function $F(x)$ is stochastically greater than $-X$. One-sided alternative with inverse inequality sign, and two-sided alternative $H_A : F(x) \neq 1 - F(-x)$ on a set of positive length, are also possible. For illustration consider two examples.

Example 1. Suppose the pair of random variables (T, C) , representing a

treatment and control response, respectively, have a joint distribution function $F(t, c)$. Assume that the treatment and control are assigned independently and at random to the subjects of the experiment. Then the null hypothesis of no difference between treatment and control implies that $F(t, c) = F(c, t)$. Let $X = T - C$ be the usual form for the data analysis. This reduces the problem to a one-sample location model in which we formulate hypotheses concerning $G(x) = P(X \leq x)$.

Under the null hypothesis that $F(t, c) = F(c, t)$ for all (c, t) it follows that $P(T - C \leq x) = P(C - T \leq x) = P(-(T - C) \leq x)$ and hence X and $-X$ have the same distribution which means that under the null hypothesis, X is symmetrically distributed about 0.

If the alternative hypothesis specifies that the treatment effect adds a constant to the control then we will test $H_0 : \theta = 0$ versus $H_A : \theta > 0$, where θ is the center of the sampled symmetric population of differences.

Example 2. Rosenzweig *et al.* [5] describe experiments to determine the effects of environment on brain anatomy. In the experiments three male rats from each of 12 litters are randomly assigned to a standard laboratory cage, an enriched cage containing a variety of toys, and an impoverished environment in which the rats lived in isolation. Various measures of brain weight and enzymatic activity were taken. For this example, the weight gain of the cortex over a specific period of time was considered. If we compare rats in an impoverished environment to rats in an enriched environment, we have a paired-data experiment. Let X and Y denote the impoverished measurement and enriched measurement, respectively. Then the basic random variable of interest is $D = Y - X$. Under the null hypothesis

of no difference in the effects due to the two environments, D has a distribution symmetric about 0. If we let θ denote the center of the distribution of D , then the experiment yields 12 observations $D_1, \dots, D_{12}$ to test $H_0 : \theta = 0$ versus $H_A : \theta > 0$.

Some well-known test statistics used for testing symmetry include signed rank statistic, Wilcoxon signed rank statistic, Kolmogorov–Smirnov statistic, and Cramér–von Mises statistic. In order to introduce rank statistics for testing symmetry we need the following definition.

Definition 1. The *order statistics* $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ of a set of real-valued observations $X_1, \dots, X_n$ are the values of the observations positioned in increasing order. The *rank* R_i of X_i among $X_1, \dots, X_n$ is its position number in the set of order statistics. More precisely, if $X_1, \dots, X_n$ are all different (which is true with probability one in case of sampling from a continuous distribution), then R_i is defined by the equation

$$X_i = X_{(R_i)}.$$

Let $\text{sign}(x)$ be $-1, 0$, or 1 if $x < 0, x = 0$ or $x > 0$, respectively. Define the absolute rank R_i^+ of an observation X_i in a sample $X_1, \dots, X_n$ as the rank of X_i in the sample of absolute values $|X_1|, \dots, |X_n|$. A simple linear signed rank statistic has the form

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{R_i^+} \text{sign}(X_i),$$

where $a_1, \dots, a_n$ are given numbers called the *scores* of the statistic. The main attraction of signed rank statistic is its simplicity and distribution-free property: under H_0 the distribution of signed rank statistic does not depend on the un-

derlying distribution and is the same over the set of all symmetric distributions. Consequently, the critical value of a signed rank statistic can be set without further knowledge of the “shape” of the underlying distribution. Rank statistics can be used when little or no information about the shape of the distribution is available.

For $a_i = i$, the statistic T_n is known as the *Wilcoxon signed rank statistic*

$$W_n^+ = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i^+ \text{sign}(X_i).$$

Large values of this statistic indicate that more observations with large absolute values are positive. Thus large values of the Wilcoxon statistic suggest that the mean of the X_i is bigger than zero and the null hypothesis should be rejected.

There exists a variety of other nonparametric procedures for testing hypothesis of symmetry, see, for example, [4, Ch. IV].

In this thesis we deal with a family of *weighted* Wilcoxon-type statistics. The weights are chosen to make test statistics based on the “middle” portion of observations by removing smallest and/or largest observations in the sample. The motivation for using such weighted U -statistics of order statistics is that in some experiments smallest and/or largest observations may be outliers, or equipment used by the experimenter may not be able to “catch” the smallest and largest measurements precisely. Outliers (observations that are quite far from the bulk of the data) occur very frequently, even in carefully conducted studies, and may be caused by improperly calibrated equipment or by recording and transcription errors. Also, in lifetime testing, one often puts n items on test and stops sampling

after the $[n(1 - \alpha)]$ th item breaks down. In other words, only the first $[n(1 - \alpha)]$ order statistics are available.

The purpose of this thesis is to study efficiency properties of weighted Wilcoxon-type statistics for large sample sizes and to find out how the disregard of a portion of smallest and/or largest observations affect the efficiency of Wilcoxon-type tests for symmetry.

The problem of calculating efficiency of test procedures (or test statistics) is important, because many test statistics were suggested by their authors for heuristic reasons without thorough study of their properties. Among a variety of tests that are often available for testing a statistical hypothesis, the statistician wants to use the “best” one. The choice of the best test can be done by comparing test procedures with regard to their efficiency. It is also of importance and interest to know the “absolute” quality of a test procedure.

The main results of the thesis are new. They include (a) Lemma 4 on page 27 that establishes locally uniform asymptotic normality of test statistics under consideration and (b) the efficiency results of Chapter 5 that contain formulas for the Pitman efficiency of weighted Wilcoxon-type statistics.

Now we state precisely the problem of interest and introduce a family of weighted test statistics for symmetry.

Chapter 2

Statement of the problem and notation

Let $X_1, \dots, X_n$ be a random sample from a continuous distribution with cumulative distribution function (cdf) $F(x - \theta)$ and probability density function (pdf) $f(x - \theta)$, where f is assumed symmetric about zero, and $\theta \geq 0$ is a shift parameter. Put $\mathbf{X} = (X_1, \dots, X_n)$. Denote the set of order statistics by $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$. For given functions k (kernel) on the plane and J (weight) on the open unit square $I^2 = (0, 1) \times (0, 1)$ that obey certain regularity conditions, Stute [6] defines the following statistic:

$$S_n = \frac{1}{n(n-1)} \sum_{i \neq j} J(i/(n+1), j/(n+1)) k(X_{(i)}, X_{(j)}). \quad (2.1)$$

This is a weighted U -statistic of order statistics of degree 2. For $J \equiv 1$, when there are no ties (which is true in case of sampling from a continuous distribution), S_n becomes a classical U -statistic of degree 2. The advantage of using weighted U -

statistic of order statistics is that they can be effectively used with various weight functions, thus producing more flexible test statistics than classical U -statistics.

For the reader's convenience we give here the definition of U -statistic. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample from an unknown distribution. For a kernel function $h(x_1, \dots, x_r)$, $1 \leq r \leq n$, that is assumed permutation symmetric in its r arguments (a given h could always be replaced by a symmetric one), consider estimation of the parameter

$$\theta = Eh(X_1, \dots, X_r).$$

The statistic $h(X_1, \dots, X_r)$ is an unbiased estimator for θ , but it is unnatural since it uses only the first r observations. The U -statistic overcomes this problem. It is defined as follows:

$$U_n = U_n(h, r) = \binom{n}{r}^{-1} \sum_{\beta} h(X_{\beta_1}, \dots, X_{\beta_r}),$$

where the sum is taken over the set of all unordered subsets β of r different integers chosen from $\{1, \dots, n\}$. Since the observations are independent identically distributed (iid), U_n is also an unbiased estimator for θ . Moreover, U_n is permutation symmetric in $\mathbf{X}$ and has a smaller variance than $h(X_1, \dots, X_r)$. The latter fact follows from the representation (see page 161 of [7])

$$U_n = E \left(h(X_1, \dots, X_r) | X_{(1)}, \dots, X_{(n)} \right),$$

and the fact that conditional expectation is a projection that decreases second

moment.

A U -statistic of degree $r = 1$ is a mean $n^{-1} \sum_{i=1}^n h(X_i)$. For the kernel $h(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$ of degree 2, the parameter $\theta = Eh(X_1, X_2) = \text{Var}X_1$ is the variance of the observations. The corresponding U -statistic is

$$U_n = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{1}{2}(X_i - X_j)^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Thus, the sample variance is a U -statistic of order 2.

The parameter $\theta = P(X_1 + X_2 > 0)$ corresponds to the kernel $h(x_1, x_2) = I(x_1 + x_2 > 0)$. The corresponding U -statistic is

$$U_n = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} I(X_i + X_j > 0).$$

This is the *signed rank statistic*, which is the average number of pairs (X_i, X_j) with positive sum $X_i + X_j > 0$. It can be used as a test statistic for investigating whether the distribution of the observations is located at zero. If a great proportion of the total pairs (X_i, X_j) yield a positive sum, then we believe that the distribution is centered to the right of zero. Signed rank statistic is often used for testing the null hypothesis that the underlying absolutely continuous distribution function is symmetric about zero: $F(x) = 1 - F(-x)$ for every x . Under this hypothesis the parameter θ is equal to $1/2$, and the asymptotic variance of U_n is $1/(3n)$. Moreover, under the null hypothesis of continuity and symmetry, the limit distribution of $\sqrt{n}(U_n - 1/2)$ is normal $N(0, 1/3)$ (see page 164 of [7]) and hence it is independent of the underlying distribution. The last property assures that the

sequence U_n is asymptotically distribution-free under the null hypothesis of symmetry. This fact makes it easy to set critical values. The test of asymptotically level α_0 rejects H_0 if $\sqrt{3n}(U_n - 1/2) \geq z_{\alpha_0}$ for every F in the null hypothesis.

Below the symbol $I_A(x)$ is used for the indicator of the set $A \subseteq \mathbf{R}$, that is, $I_A(x) = 1$ if $x \in A$, and $I_A(x) = 0$ otherwise. Similarly, $I_A(x, y)$ means the indicator of $A \subseteq \mathbf{R}^2$, that is, $I_A(x, y) = 1$ if $(x, y) \in A$, and $I_A(x, y) = 0$ otherwise. Another notation for the indicator function of the set A is $I(A)$.

Return to the statistic S_n defined in (2.1). We want to use S_n , with a specific choice of J and k , to test the hypothesis of symmetry versus one-sided alternative:

$$H_0 : F(x) = 1 - F(-x), \quad x \in \mathbf{R},$$

$$H_A : F(x) \leq 1 - F(-x), \quad x \in \mathbf{R},$$

the inequality being strict on a set of positive length. More specifically, we consider statistic S_n with kernel

$$k(x, y) = I(x + y \leq 0), \quad (x, y) \in \mathbf{R}^2,$$

and weight function

$$J(u, v) = J_0(u)J_0(v), \quad J_0(u) = I_{[\alpha, 1-\alpha]}(u), \quad (u, v) \in I^2,$$

where $0 \leq \alpha < 1/2$ is a parameter that controls the amount of sample used for

the construction of test statistic:

$$S_{n,\alpha} = \frac{1}{n(n-1)} \sum_{i \neq j} I_{[\alpha, 1-\alpha] \times [\alpha, 1-\alpha]}(i/(n+1), j/(n+1)) I(X_{(i)} + X_{(j)} \leq 0). \quad (2.2)$$

For large sample sizes, when $\alpha = 0$ the test statistic $S_{n,\alpha}$ is equivalent to the Wilcoxon signed rank statistic W_n^+ (see Section 4.3 for details). Also, choosing $J_0(u) = I_{[0, 1-\alpha]}(u)$ and $J_0(u) = I_{[\alpha, 1]}(u)$ results in the statistics

$$\begin{aligned} S_{n,\alpha}^{(1)} &= \frac{1}{n(n-1)} \sum_{i \neq j} I_{[0, 1-\alpha] \times [0, 1-\alpha]}(i/(n+1), j/(n+1)) I(X_{(i)} + X_{(j)} \leq 0), \\ S_{n,\alpha}^{(2)} &= \frac{1}{n(n-1)} \sum_{i \neq j} I_{[\alpha, 1] \times [\alpha, 1]}(i/(n+1), j/(n+1)) I(X_{(i)} + X_{(j)} \leq 0), \end{aligned}$$

that ignore the largest and the smallest observations, respectively. The statistic $S_{n,\alpha}$ can be easily adjusted for testing

$$H_0 : F(x) = 1 - F(-x), \quad x \in \mathbf{R},$$

$$H_A : F(x) \geq 1 - F(-x), \quad x \in \mathbf{R}.$$

(see Note 4 in Section 4.3 for details).

In this thesis we deal with a one-sided alternative. There are two main reasons for this. First, in most practical applications where testing for symmetry is of interest, a one-sided alternative is used. Second, in case of two-sided alternative, efficiency study is a more delicate problem. To the best of our knowledge, the analytic results that allow us, in our settings, to find the Pitman efficiency of test statistic $S_{n,\alpha}$ are not available for a similar problem with the two-sided alternative.

In [6] the derivation of asymptotic normality of S_n is performed under the following regularity conditions:

- (A) J is twice differentiable on the open unit square I^2 ;
- (B) Letting $J^{(l)}$ and $J^{(lm)}$, $l, m = 1, 2$ denote the first- and second-order partial derivatives of J , respectively, then for $l, m = 1, 2$
 - (i) $\iint_{I^2} [J^{(l)}(u, v) k(F^{-1}(u), F^{-1}(v))]^2 dudv < \infty$,
 - (ii) $|J^{(lm)}(u, v)| \leq C (u(1-u))^{-\beta_1} (v(1-v))^{-\beta_2} I_{(a_1, b_1)}(u) I_{(a_2, b_2)}(v)$ for some real β_1, β_2 , $0 \leq a_1 \leq b_1 \leq 1$ and $0 \leq a_2 \leq b_2 \leq 1$,
 - (iii) $\int_{a_1-\delta_{11}}^{b_1+\delta_{21}} \int_{a_2-\delta_{12}}^{b_2+\delta_{22}} (u(1-u))^{1-\beta_1} (v(1-v))^{1-\beta_2} |k(F^{-1}(u), F^{-1}(v))| dudv < \infty$,

where the δ 's may be any arbitrarily small positive numbers if the corresponding a and b are positive and less than 1, respectively, and are equal to zero otherwise.

- (C) $\iint_{I^2} [J(u, v) k(F^{-1}(u), F^{-1}(v))]^2 dudv < \infty$.

Note 1. Inspection of the proof of Theorem 1.1 in [6] that establishes asymptotic normality of S_n shows that condition (A) can be relaxed: it is enough to assume that

- (A*) J is twice differentiable almost everywhere on the open unit square I^2 .

Note 2. Condition B(i) is a square integrability condition; condition B(ii) imposes a restriction on the growth of $J^{(lm)}$ on the closed unit square; condition

B(iii) is an integral restriction on the growth of $k(x, y)$ near the boundary of the square $[0, 1] \times [0, 1]$; and condition (C) requires square integrability of the product $J \cdot k$.

Note 3. For the test statistic $S_{n,\alpha}$, the main object of our investigation, conditions (A*), (B), and (C) are trivially satisfied. Therefore we can use the results of Stute [6] on asymptotic normality of S_n .

As shown below, the null hypothesis of symmetry is rejected in favour of alternative H_A for large values of $S_{n,\alpha}$, that is, the critical region is of the form $\{S_{n,\alpha} > c\}$. For the shift family $\{f(x - \theta) : \theta \geq 0\}$, where density $f(x)$ is symmetric about 0, the hypotheses to be tested in terms of parameter θ are

$$H_0 : \theta = 0,$$

$$H_1 : \theta > 0.$$

The alternative H_1 is an important particular case of H_A .

In hypothesis testing, the true statistical difficulty is to distinguish between the null hypothesis and alternative when the hypotheses are close to each other. Therefore we fix the null hypothesis and let the sequences of alternatives converge to the null hypotheses at a rate $O(n^{-1/2})$, that is, $\theta = \theta_n = h/\sqrt{n}$, $h \geq 0$. Then in terms of a new parameter h , called *local parameter*, the hypotheses become

$$H_0 : h = 0,$$

$$H_1 : h > 0.$$

In this thesis, for the shift statistical model $\{f(x - h/\sqrt{n}) : h \geq 0\}$ under consideration, we calculate asymptotic efficiency of the test statistics $S_{n,\alpha}$, $0 \leq \alpha < 1/2$, using the so-called Pitman approach.

Now we introduce some notation used in this thesis. A sequence of random variables X_n is said to *converge in distribution* to a random variable X if $P(X_n \leq x) \rightarrow P(X \leq x)$, for every x at which the limit cumulative distribution function $x \mapsto P(X \leq x)$ is continuous. It is denoted by $X_n \rightsquigarrow X$ or $X_n \xrightarrow{d} X$. The symbols $\overset{P}{\rightsquigarrow}$ and $\overset{P}{=}$ denote convergence in distribution and equality in distribution, respectively, under the probability distribution P . The expectation and the variance with respect to $P_\theta^{(n)}$, $\theta \geq 0$, are denoted by E_θ and Var_θ (with index n omitted), respectively. The symbol $o_P(1)$ is short for a sequence of random variables or vectors that converge to zero in probability P . The symbol $\|\cdot\|$ denotes the total variation distance $\|P - Q\| = \int_{\mathbf{R}} |p(x) - q(x)| dx$, where p and q are densities of P and Q , respectively. The natural logarithm (base e) is denoted $\log$. We write $b_n = O(a_n)$, as $n \rightarrow \infty$, if $\lim_{n \rightarrow \infty} (b_n/a_n) = C$ for some constant $C \neq 0$.

Chapter 3

Statistical model and its local asymptotic normality

The study of efficiency properties of the test statistic $S_{n,\alpha}$ requires some regularity conditions on the underlying family of distributions. To be precise, we assume that the family $\{f(x - \theta) : \theta \geq 0\}$ of pdfs satisfies the following two conditions:

(C1) $f(x)$ is absolutely continuous for every $x \in \mathbf{R}$;

(C2) $E_\theta (\partial \log f(X - \theta) / \partial \theta)^2 |_{\theta=0} < \infty$, where X has pdf $f(x - \theta)$.

Definition 2. Let I be an interval of the real line $\mathbf{R}$. A function $f : I \mapsto \mathbf{R}$ is *absolutely continuous* on I if for every positive number ε , there is a positive number δ such that whenever a finite sequence of pairwise disjoint subintervals $[x_k, y_k]$ of I satisfies $\sum_k |y_k - x_k| < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$.

Let P_θ be the distribution of the X_i s, that is, $P_\theta(A) = \int_A f(x - \theta) dx$. The sample $\mathbf{X}$ is then a single observation from the product $P_\theta^{(n)}$ of n copies of P_θ .

The family of probability distributions $\{P_\theta^{(n)} : \theta \geq 0\}$ is called a *statistical experiment* or *statistical model*. When studying the efficiency properties of $S_{n,\alpha}$ as a test statistic for testing symmetry, we want the experiment $\{P_\theta^{(n)} : \theta \geq 0\}$ to be *locally asymptotically normal (LAN)* at $\theta = 0$. The meaning of local asymptotic normality is convergence of the local statistical experiments indexed by the parameter h , $\{P_{h/\sqrt{n}}^{(n)} : h \geq 0\}$, to a normal experiment whose properties are well studied. The idea behind local asymptotic normality is that under certain regularity conditions a sequence of statistical experiment can be approximated by a Gaussian experiment after a suitable reparametrization.

Let us fix a value θ_0 (when testing symmetry, we have $\theta_0 = 0$) and rewrite $P_\theta^{(n)}$ as $P_{\theta_0+h/\sqrt{n}}^{(n)}$, a distribution depending on the parameter h . It turns out that for “smooth” models $\theta \mapsto P_\theta$ the experiments

$$\left\{P_{\theta_0+h/\sqrt{n}}^{(n)} : h \geq 0\right\} \quad \text{and} \quad \left\{N(h, I_{\theta_0}^{-1}) : h \geq 0\right\},$$

where I_{θ_0} is the Fisher information at $\theta = \theta_0$, have similar properties when n is large. While the original experiment may be difficult to analyze, the limit one is a tractable normal experiment which can give us information about the original one. In the second experiment we observe a single observation from the normal distribution with unknown mean h and known variance equal to the inverse of the Fisher information.

The LAN condition can be used as a tool to study the behavior of statistics under contiguous alternatives. “Contiguity” suggests sequences of probability distributions living “on the top of each other” (in the limit).

Definition 3. The sequence of probability distributions Q_n is *contiguous* with respect to the sequence P_n if $P_n(A_n) \rightarrow 0$ implies $Q_n(A_n) \rightarrow 0$ for every sequence of measurable sets A_n . This is denoted $Q_n \triangleleft P_n$. The sequences P_n and Q_n are *mutually contiguous* if both $P_n \triangleleft Q_n$ and $Q_n \triangleleft P_n$. This is denoted $P_n \triangleleft \triangleright Q_n$.

Now we give a formal definition of LAN in case of a real-valued parameter. A general definition can be found in [3, Sec. II.2] or [7, Sec. 7.6].

Definition 4. The sequence of probability distributions $\{P_\theta^{(n)} : \theta \geq 0\}$ is *locally asymptotically normal (LAN)* at $\theta = \theta_0$ if there exist nonzero numbers r_n and I_{θ_0} and a random variable $\Delta_{n,\theta_0} \overset{P_{\theta_0}^{(n)}}{\rightsquigarrow} N(0, I_{\theta_0})$ such that

$$\log \frac{dP_{\theta_0 + r_n^{-1}h}^{(n)}}{dP_{\theta_0}^{(n)}}(\mathbf{X}) = h\Delta_{n,\theta_0} - \frac{1}{2}h^2 I_{\theta_0} + o_{P_{\theta_0}^{(n)}}(1). \quad (3.1)$$

For a normal experiment with parameters θ and σ^2 the LAN condition is trivially satisfied with

$$I_0 = \frac{1}{\sigma^2} \quad \text{and} \quad \Delta_{n,0} = \frac{1}{\sqrt{n}\sigma^2} \sum_{i=1}^n X_i \overset{P_0^{(n)}}{\rightsquigarrow} N(0, I_0).$$

Indeed, let $X_1, \dots, X_n$ be independent normally distributed random variables, each with pdf

$$\varphi(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}.$$

Then the joint pdf is

$$f(\mathbf{x}; \theta, \sigma^2) = \prod_{i=1}^n \varphi(x_i; \theta, \sigma^2) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp \left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \theta)^2 \right)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and the log of the likelihood ratio is

$$\begin{aligned} \log \frac{f(\mathbf{x}; \theta, \sigma^2)}{f(\mathbf{x}; 0, \sigma^2)} &= -\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (x_i - \theta)^2 - \sum_{i=1}^n x_i^2 \right) \\ &= -\frac{1}{2\sigma^2} \left(\sum_{i=1}^n x_i^2 - 2\theta \sum_{i=1}^n x_i + n\theta^2 - \sum_{i=1}^n x_i^2 \right) \\ &= \frac{\theta}{\sigma^2} \sum_{i=1}^n x_i - \frac{n\theta^2}{2\sigma^2} = \frac{h}{\sqrt{n}\sigma^2} \sum_{i=1}^n x_i - \frac{h^2}{2\sigma^2}. \end{aligned}$$

It is well known (see Theorem 2.1 of [3]) that a regular *shift experiment* $\{P_\theta^{(n)} : \theta \geq 0\}$, for which the Fisher information $I_0 = \int_{\mathbf{R}} (f'(x))^2 / f(x) dx$ is finite, is LAN at $\theta = 0$ if the function $f(x)$ is absolutely continuous:

$$\log \frac{dP_{h/\sqrt{n}}^{(n)}(\mathbf{X})}{dP_0^{(n)}} = h\Delta_{n,0} - \frac{1}{2}h^2I_0 + o_{P_0^{(n)}}(1),$$

where

$$\Delta_{n,0} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\partial}{\partial \theta} \log f(X_i - \theta)|_{\theta=0} = -\frac{1}{\sqrt{n}} \sum_{i=1}^n (f'/f)(X_i).$$

Therefore, under the validity of (C1) and (C2), the sequence of shift experiments $\{P_\theta^{(n)} : \theta \geq 0\}$ is LAN at $\theta = 0$.

Under LAN condition (see (3.1)),

$$\log \frac{dP_{h/\sqrt{n}}^{(n)}(\mathbf{X})}{dP_0^{(n)}} \overset{P_0^{(n)}}{\rightsquigarrow} N \left(-\frac{1}{2}h^2I_0, h^2I_0 \right), \quad (3.2)$$

where

$$I_\theta = E_\theta \left(\frac{\partial}{\partial \theta} \log f(X - \theta) \right)^2 = \int_{\mathbf{R}} \frac{(f'(x - \theta))^2}{f(x - \theta)} dx$$

is the Fisher information, so that $I_0 = \int_{\mathbf{R}} (f'(x))^2 / f(x) dx$. Our goal is to calculate the Pitman efficiency of tests based on $S_{n,\alpha}$, $\alpha \in [0, 1/2)$. For this the knowledge of the limit distributions of $S_{n,\alpha}$ under both, the null hypothesis and the alternative, is important. Note that we cannot simply use the results of [6] because, in our context, the results of [6] only assure the asymptotic normality of $S_{n,\alpha}$ under H_0 and under alternative $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f(x - \theta)$, with f symmetric about zero and $\theta > 0$ being fixed. In order to calculate Pitman efficiency we need more — we need “locally uniform” asymptotic normality of $S_{n,\alpha}$ under the sequence of contiguous alternatives. With the help of Le Cam’s third lemma we can obtain locally uniform asymptotic normality of $S_{n,\alpha}$ under the alternative once the limit distribution of $S_{n,\alpha}$ under H_0 is known.

Chapter 4

Asymptotic properties of $S_{n,\alpha}$

4.1 Le Cam's third lemma

Le Cam's third lemma (see, for example, [7, Sec. 6.2]) tells us that if P_n and Q_n are sequences of probability distributions such that Q_n is contiguous to P_n , and X_n is a sequence of random variables satisfying

$$\left(X_n, \log \frac{dQ_n}{dP_n} \right) \overset{P_n}{\rightsquigarrow} N \left(\begin{pmatrix} \mu \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & \sigma^2 \end{pmatrix} \right), \quad (4.1)$$

then

$$X_n \overset{Q_n}{\rightsquigarrow} N(\mu + \tau, \Sigma), \quad (4.2)$$

so that the asymptotic variance of X_n is the *same* under P_n and Q_n , but the mean differs by the asymptotic covariance τ between X_n and the log likelihood ratio.

For the shift experiment under consideration, that is, for which conditions (C1) and (C2) are satisfied, local asymptotic normality implies that the distributions $P_0^{(n)}$ and $P_{h/\sqrt{n}}^{(n)}$ are mutually contiguous, and therefore Le Cam's third lemma can be used. The mutual contiguity of $P_0^{(n)}$ and $P_{h/\sqrt{n}}^{(n)}$ follows from the following result.

Lemma 1 (Lemma 6.4 of [7]). *Let P_n and Q_n be sequences of probability distributions on measurable spaces (Ω_n, A_n) . Then the following statements are equivalent:*

- (i) $Q_n \triangleleft P_n$.
- (ii) If $dP_n/dQ_n \overset{Q_n}{\rightsquigarrow} U$ along a subsequence, then $P(U > 0) = 1$.
- (iii) If $dQ_n/dP_n \overset{P_n}{\rightsquigarrow} V$ along a subsequence, then $EV = 1$.
- (iv) For any statistics $T_n : \Omega_n \mapsto \mathbf{R}^k$: If $T_n \xrightarrow{P_n} 0$, then $T_n \xrightarrow{Q_n} 0$.

Let P_n and Q_n be probability measures on arbitrary measurable spaces such that

$$\frac{dP_n}{dQ_n} \overset{Q_n}{\rightsquigarrow} e^{N(\mu, \sigma^2)}.$$

Because the lognormal variable on the right is positive, that is, $P(e^{N(\mu, \sigma^2)} > 0) = 1$, it follows that $Q_n \triangleleft P_n$ by part (ii) of Lemma 1. The mean of the lognormal distribution is $Ee^{N(\mu, \sigma^2)} = e^{\mu + \frac{1}{2}\sigma^2}$. It is clear that the mean is equal to 1 if and only if $\mu = -\frac{1}{2}\sigma^2$, and under this condition we have $P_n \triangleleft Q_n$ by part (iii) of Lemma 1. Thus, $Q_n \triangleleft \triangleright P_n$ if and only if $\mu = -\frac{1}{2}\sigma^2$, cf. (3.2).

4.2 Limit distribution under alternative: general scheme

A general scheme for deriving the limit distribution of a test statistic under the alternative is as follows. First, assume the LAN experiment, so that (3.2) holds true. Next, assume that the sequence of statistics $S_{n,\alpha}$ can be approximated by an average of the type

$$\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_{0,\alpha}(X_k) + o_{P_0}(1), \quad (4.3)$$

where $\psi_{0,\alpha}(X_1)$ has mean zero and finite second moment. Many sequences of test statistics allow approximation (4.3). For the statistic $S_{n,\alpha}$ decomposition (4.3) holds true in view of Lemma 3 in Section 4.3. For the sequel we need the following result.

Lemma 2 (Slutsky's lemma). *Let X_n , X , and Y_n be random vectors or variables. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} c$ for a constant c , then*

$$X_n + Y_n \xrightarrow{d} X + c, \quad X_n Y_n \xrightarrow{d} cX.$$

It follows from (4.3) that the normalized statistic $S_{n,\alpha}$ has the form $\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)) = \xi_n + \eta_n$, where ξ_n is the normalized sum of iid random variables, for which the Central Limit Theorem (CLT) holds true:

$$\xi_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_{0,\alpha}(X_k) \xrightarrow{P_0^{(n)}} N(0, E[\psi_{0,\alpha}^2(X_1)]),$$

and η_n converges to zero in $P_0^{(n)}$ probability. Therefore by Slutsky's lemma

$$\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)) \stackrel{P_0^{(n)}}{\rightsquigarrow} N(0, E[\psi_{0,\alpha}^2(X_1)]). \quad (4.4)$$

Put

$$l_\theta(x) = \log f(x - \theta), \quad \dot{l}_\theta(x) = \frac{\partial}{\partial \theta} \log f(x - \theta) = -\frac{f'(x - \theta)}{f(x - \theta)}, \quad \dot{l}_0(x) = -\frac{f'(x)}{f(x)}.$$

Next, in view of Slutsky's lemma and the CLT, it follows from (3.2) that the random variable $\log \frac{dP_{h/\sqrt{n}}^{(n)}}{dP_0^{(n)}}(\mathbf{X})$ can be represented as

$$\log \frac{dP_{h/\sqrt{n}}^{(n)}}{dP_0^{(n)}}(\mathbf{X}) \stackrel{P_0^{(n)}}{=} \zeta_n + \delta_n, \quad (4.5)$$

where $\zeta_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n h \dot{l}_0(X_k) - \frac{1}{2} h^2 I_0$ and δ_n converges to zero in $P_0^{(n)}$ probability.

Indeed,

$$\text{Var} \left(\frac{h}{\sqrt{n}} \sum_{k=1}^n \dot{l}_0(X_k) - \frac{1}{2} h^2 I_0 \right) = h^2 E \left(\dot{l}_0^2(X_1) \right) = h^2 E \left(\frac{f'(X_1)}{f(X_1)} \right)^2 = h^2 I_0,$$

and by the CLT the random variable ζ_n is asymptotically normally $N \left(-\frac{1}{2} h^2 I_0, h^2 I_0 \right)$ distributed. Now applying Slutsky's lemma again and using relations (4.3) and (4.5), we conclude that the limit distribution of the two-dimensional vector

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n \left(\psi_{0,\alpha}(X_k), h \dot{l}_0(X_k) - \frac{1}{2} h^2 I_0 \right),$$

is the same as that of

$$\left(\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)), \log \frac{dP_{h/\sqrt{n}}^{(n)}}{dP_0^{(n)}}(\mathbf{X}) \right).$$

In particular, since $E\psi_{0,\alpha}(X_1) = 0$ the limit covariance of $\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0))$ and $\log \frac{dP_{h/\sqrt{n}}^{(n)}}{dP_0^{(n)}}(\mathbf{X})$ is

$$\begin{aligned} & \text{Cov} \left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_{0,\alpha}(X_k), \frac{1}{\sqrt{n}} \sum_{k=1}^n h\dot{l}_0(X_k) - \frac{1}{2}h^2I_0 \right) \\ &= E \left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_{0,\alpha}(X_k) \frac{1}{\sqrt{n}} \sum_{k=1}^n h\dot{l}_0(X_k) \right) \\ &= hE \left(\psi_{0,\alpha}(X_1)\dot{l}_0(X_1) \right). \end{aligned}$$

Now we need the following result.

Theorem 1 (The Multivariate Central Limit Theorem). *Suppose that $\mathbf{X} = (X_1, \dots, X_n)^\top$ is a random vector with a positive definite covariance matrix Σ . Assume that $EX_i^2 < \infty$. If $\mathbf{X}_1, \mathbf{X}_2, \dots$ is a sequence of iid copies of $\mathbf{X}$, then*

$$T_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mathbf{X}_i - E\mathbf{X}_i) \xrightarrow{d} N(0, \Sigma).$$

It follows from the above that the vector

$$\tau_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n \left(\psi_{0,\alpha}(X_k), h\dot{l}_0(X_k) - \frac{1}{2}h^2I_0 \right)$$

is the normalized sum of iid random vectors with asymptotic covariance matrix

$R_\alpha = \begin{pmatrix} E_0[\psi_{0,\alpha}^2(X_1)] & hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \\ hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] & h^2I_0 \end{pmatrix}$, where both $E_0[\psi_{0,\alpha}^2(X_1)]$ and h^2I_0 are finite. The matrix R_α is positive definite. Indeed, recalling that $\dot{l}_0(x) = -f'(x)/f(x)$ and that under the alternative h is positive, using the Cauchy–Schwarz inequality

$$\begin{aligned} \det(R_\alpha) &= h^2E_0[\psi_{0,\alpha}^2(X_1)]I_0 - h^2 \left(E_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \right)^2 \\ &= h^2E_0[\psi_{0,\alpha}^2(X_1)]E_0[\dot{l}_0^2(X_1)] - h^2 \left(E_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \right)^2 \geq 0, \end{aligned}$$

with equality sign if and only if $\psi_{0,\alpha}(X_1)$ is proportional to $(f'/f)(X_1)$ with probability one.

Therefore, applying the Multivariate CLT to τ_n we get

$$\begin{aligned} \tau_n &= \frac{1}{\sqrt{n}} \sum_{k=1}^n \left(\psi_{0,\alpha}(X_k), h\dot{l}_0(X_k) - \frac{1}{2}h^2I_0 \right) \\ &\stackrel{P_0^{(n)}}{\rightsquigarrow} N \left(\begin{pmatrix} 0 \\ -\frac{1}{2}h^2I_0 \end{pmatrix}, \begin{pmatrix} E_0[\psi_{0,\alpha}^2(X_1)] & hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \\ hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] & h^2I_0 \end{pmatrix} \right), \end{aligned}$$

and, in view of Slutsky's lemma,

$$\begin{aligned} &\begin{pmatrix} \sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)), \log \frac{dP_{h/\sqrt{n}}^{(n)}(\mathbf{X})}{dP_0^{(n)}}(\mathbf{X}) \end{pmatrix} \\ &\stackrel{P_0^{(n)}}{\rightsquigarrow} N \left(\begin{pmatrix} 0 \\ -\frac{1}{2}h^2I_0 \end{pmatrix}, \begin{pmatrix} E_0[\psi_{0,\alpha}^2(X_1)] & hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \\ hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] & h^2I_0 \end{pmatrix} \right). \end{aligned} \tag{4.6}$$

4.3 Limit distribution of $S_{n,\alpha}$ under null and alternative hypotheses

The application of Le Cam's third lemma to $\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0))$ yields (compare (4.1), (4.2), and (4.6))

$$\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0)) \overset{P_{h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N\left(hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)], E_0[\psi_{0,\alpha}^2(X_1)]\right).$$

In order to obtain function $\psi_{0,\alpha}(x)$ in decomposition (4.3) for $S_{n,\alpha}$ we will use the following result. For a random sample $X_1, \dots, X_n$ drawn from a continuous distribution with cdf $F(x)$, denote $U_i = F(X_i)$, $i = 1, \dots, n$. It is clear that $U_1, \dots, U_n$ are independent uniform on $[0, 1]$ random variables.

Lemma 3 (Corollary 1.1 of [6]). *Under assumptions (A), (B), and (C),*

$$\begin{aligned} S_n &= \iint_{I^2} J(u, v) k(F^{-1}(u), F^{-1}(v)) \, dudv \\ &= \frac{1}{n} \sum_{i=1}^n \left(\int_0^1 J(U_i, v) k(F^{-1}(U_i), F^{-1}(v)) \, dv - \iint_{I^2} J(u, v) k(F^{-1}(u), F^{-1}(v)) \, dudv \right) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \left(\int_0^1 J(u, U_i) k(F^{-1}(u), F^{-1}(U_i)) \, du - \iint_{I^2} J(u, v) k(F^{-1}(u), F^{-1}(v)) \, dudv \right) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \left(\int_{U_i}^1 \int_0^1 k_1(u, v) \, dudv - EU_1 k_1(U_1, U_2) \right) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \left(\int_0^1 \int_{U_i}^1 k_2(u, v) \, dudv - EU_2 k_2(U_1, U_2) \right) + R_n, \end{aligned}$$

where

$$|R_n| = O(n^{-1}(\ln n)^\gamma) \quad \text{with probability } 1, \quad \gamma > 2(|\beta_1| + |\beta_2| + 1).$$

$$k_l(u, v) = J^{(l)}(u, v)k(F^{-1}(u), F^{-1}(v)), \quad l = 1, 2.$$

From Lemma 3 the function $\psi_{0,\alpha}$ is equal to

$$\begin{aligned} \psi_{0,\alpha}(x) &= \int_0^1 J(F(x), v)k(x, F^{-1}(v))dv - \iint_{I^2} J(u, v)k(F^{-1}(u), F^{-1}(v))dudv \\ &+ \int_0^1 J(u, F(x))k(F^{-1}(u), x)du - \iint_{I^2} J(u, v)k(F^{-1}(u), F^{-1}(v))dudv, \end{aligned}$$

where for $0 \leq \alpha < 1/2$

$$J(u, v) = J_0(u)J_0(v), \quad J_0(u) = I_{[\alpha, 1-\alpha]}(u), \quad k(x, y) = I(x + y \leq 0).$$

For the above functions $k(x, y)$ and $J(u, v)$, the assumptions (A*), (B), and (C) are satisfied, and hence Lemma 3 can be used (see Note 1). With this choice of $J(u, v)$ the statistic $S_{n,\alpha}$ is based on the middle portion of $(1 - 2\alpha)100\%$ observations, that is, the $\alpha 100\%$ smallest and $\alpha 100\%$ largest observations are neglected.

In terms of the shift experiment $\{P_{h/\sqrt{n}}^{(n)} : h \geq 0\}$ under consideration, the null hypothesis of symmetry introduced in Chapter 2 states that $h = 0$. The alternative hypothesis (sequence of hypotheses) is $H_1 : h > 0$. This alternative H_1 is a particular case of the alternative hypothesis of symmetry H_A in Chapter 1.

A nice property of $S_{n,\alpha}$ is that under H_0 it is asymptotically distribution-free,

that is, as shown below, its limit distribution is normal with asymptotic mean and variance independent of the distribution from which the sample was drawn. Moreover, for a wide class of distributions $F(x - \theta)$, the statistic $S_{n,\alpha}$ is consistent against the alternative $H_1 : \theta > 0$.

Definition 5. Consider testing $H_0 : \theta \in \Theta_0 \subset \mathbf{R}$ versus $H_A : \theta \in \Theta_1 = \Theta \setminus \Theta_0$. A sequence of tests with power functions $\theta \mapsto \pi_n(\theta)$ is *consistent* at level α_0 against the alternative $H_A : \theta \in \Theta_1$ if it is asymptotically of level α_0 and $\pi_n(\theta) \rightarrow 1$ as $n \rightarrow \infty$.

The consistency of the test based on $S_{n,\alpha}$ will be shown below, in Section 4.4, after we get the result on the limit distribution of $S_{n,\alpha}$.

Using Lemma 3 and Le Cam's third Lemma we can now establish "locally uniform" asymptotic normality of $S_{n,\alpha}$.

Lemma 4. Consider a shift model $\{P_{h/\sqrt{n}}^{(n)} : h \geq 0\}$ of continuous distributions with pdf $f(x - h/\sqrt{n})$, where $f(x)$ is symmetric about zero and satisfies conditions (C1) and (C2). Then for $h \geq 0$

$$\sqrt{n} \left(S_{n,\alpha} - \mu_\alpha \left(\frac{h}{\sqrt{n}} \right) \right) \overset{P_{h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N(0, \sigma_\alpha^2(0)), \quad n \rightarrow \infty,$$

where

$$\begin{aligned} \mu_\alpha(\theta) &= \frac{(2\alpha - 1)^2}{2} + 2\theta \left(\int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - (1 - 2\alpha) f(F^{-1}(\alpha)) \right), \\ \sigma_\alpha^2(0) &= 1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4. \end{aligned}$$

Proof. By the symmetry of kernel $k(x, y) = I(x + y \leq 0)$, the function $\psi_{0,\alpha}(x)$ in decomposition (4.3) has the form

$$\begin{aligned}\psi_{0,\alpha}(x) &= 2 \int_0^1 J_0(F(x)) J_0(v) k(x, F^{-1}(v)) dv \\ &\quad - 2 \iint_{I^2} J_0(u) J_0(v) k(F^{-1}(u), F^{-1}(v)) du dv \\ &= 2I_1 - 2I_2.\end{aligned}$$

The first integral is equal to

$$\begin{aligned}I_1 &= \int_{\mathbf{R}} J_0(F(x)) J_0(F(t)) k(x, t) f(t) dt = \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} J_0(F(x)) k(x, t) f(t) dt \\ &= J_0(F(x)) \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} I(x + t \leq 0) f(t) dt \\ &= J_0(F(x)) \int_{[F^{-1}(\alpha), F^{-1}(1-\alpha)] \cap (-\infty, -x]} f(t) dt \\ &= \begin{cases} J_0(F(x)) \int_{F^{-1}(\alpha)}^{-x} dF(x) & , \quad \text{if } F(x) \geq \alpha \\ 0 & , \quad \text{if } F(x) < \alpha \end{cases} \\ &= \begin{cases} J_0(F(x)) (1 - F(x) - \alpha) & , \quad \text{if } F(x) \geq \alpha \\ 0 & , \quad \text{if } F(x) < \alpha \end{cases} \\ &= (1 - F(x) - \alpha) I_{[\alpha, 1-\alpha]}(F(x)),\end{aligned}$$

where the last equality is due to the fact that $J_0(u) = I_{[\alpha, 1-\alpha]}(u)$.

For the second integral, denoting

$$D_\alpha = [F^{-1}(\alpha), F^{-1}(1 - \alpha)] \times [F^{-1}(\alpha), F^{-1}(1 - \alpha)] \cap \{(s, t) \in \mathbf{R}^2 : s + t \leq 0\},$$

we have

$$\begin{aligned}
I_2 &= \iint_{\mathbf{R}^2} J_0(F(s)) J_0(F(t)) k(s, t) f(s) f(t) ds dt = \iint_{D_\alpha} dF(s) dF(t) \\
&= \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} dF(s) \int_{F^{-1}(\alpha)}^{-s} dF(t) = \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(s) - \alpha) dF(s) \\
&= \int_{\alpha}^{1-\alpha} (1 - u - \alpha) du = \frac{(2\alpha - 1)^2}{2}.
\end{aligned}$$

From this,

$$\psi_{0,\alpha}(x) = 2[1 - F(x) - \alpha] I_{[\alpha, 1-\alpha]}(F(x)) - (2\alpha - 1)^2.$$

Observe that, as required in decomposition (4.3), $E\psi_{0,\alpha}(X_1) = 0$ and $E\psi_{0,\alpha}^2(X_1)$ is finite.

Now, using the fact that any density function $f(x)$ must satisfy $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 0$ and that under H_0 the density $f(x)$ is symmetric about zero, we get

$$\begin{aligned}
& hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)] \\
&= hE_0 \left\{ (2(1 - F(x) - \alpha) I_{[\alpha, 1-\alpha]}(F(x)) - (2\alpha - 1)^2) \frac{f'(x)}{f(x)} \right\} \\
&= h \int_{\mathbf{R}} (2(1 - F(x) - \alpha) I_{[\alpha, 1-\alpha]}(F(x)) - (2\alpha - 1)^2) f'(x) dx \\
&= h \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (2(1 - F(x) - \alpha) - (2\alpha - 1)^2) df(x) - h \int_{-\infty}^{\infty} (2\alpha - 1)^2 df(x) \\
&= 2h \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x) - \alpha) df(x) - h(2\alpha - 1)^2 \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} df(x)
\end{aligned}$$

$$\begin{aligned}
&= 2h \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x) - \alpha) df(x) \\
&= 2h \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - 2h(1 - 2\alpha) f(F^{-1}(\alpha)). \tag{4.7}
\end{aligned}$$

Next, denoting

$$\sigma_\alpha^2(0) = E_0[\psi_{0,\alpha}^2(X_1)],$$

we have

$$\begin{aligned}
\sigma_\alpha^2(0) &= \int_{\mathbf{R}} \psi_{0,\alpha}^2(x) f(x) dx \\
&= \int_{\mathbf{R}} (2(1 - F(x) - \alpha) I_{(\alpha, 1-\alpha)}(F(x)) - (2\alpha - 1)^2)^2 dF(x) \\
&= \int_{\mathbf{R}} (2(1 - F(x) - \alpha) I_{(\alpha, 1-\alpha)}(F(x)))^2 dF(x) + (-(2\alpha - 1)^2)^2 \int_{\mathbf{R}} dF(x) \\
&\quad + 4(-(2\alpha - 1)^2) \int_{\mathbf{R}} ((1 - F(x) - \alpha) I_{(\alpha, 1-\alpha)}(F(x))) dF(x) \\
&= 4 \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x) - \alpha)^2 dF(x) + (-(2\alpha - 1)^2)^2 \int_{\mathbf{R}} dF(x) \\
&\quad + 4(-(2\alpha - 1)^2) \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x) - \alpha) dF(x) \\
&= 4 \int_{\alpha}^{1-\alpha} (1 - u - \alpha)^2 du + (-(2\alpha - 1)^2)^2 \int_0^1 du \\
&\quad + 4(-(2\alpha - 1)^2) \int_{\alpha}^{1-\alpha} (1 - u - \alpha) du \\
&= 4 \left(\int_{\alpha}^{1-\alpha} u^2 du + (\alpha - 1)^2 \int_{\alpha}^{1-\alpha} du + (2\alpha - 2) \int_{\alpha}^{1-\alpha} u du \right) \\
&\quad + 4(-(2\alpha - 1)^2) \left((1 - \alpha) \int_{\alpha}^{1-\alpha} du - \int_{\alpha}^{1-\alpha} u du \right) + (-(2\alpha - 1)^2)^2 \\
&= 1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4.
\end{aligned}$$

Note that $\sigma_\alpha^2(0)$ is a decreasing function of α . Indeed, differentiating $\sigma_\alpha^2(0)$ with respect to α we get

$$\frac{\partial}{\partial \alpha} \sigma_\alpha^2(0) = -16\alpha + 64\alpha^2 - 64\alpha^3 = -16\alpha(1 - 2\alpha)^2 < 0,$$

provided $0 < \alpha < 1/2$. Next we need the following lemma.

Lemma 5 (Corollary 1.2 of [6]) *Under assumptions (A), (B), and (C), as $n \rightarrow \infty$*

$$n^{1/2} \left(S_n - \int_0^1 \int_0^1 J(u, v) k(F^{-1}(u), F^{-1}(v)) \, du dv \right) \rightsquigarrow N(0, \sigma^2)$$

where S_n is given by (2.1) and σ^2 is the joint variance of the summands in Lemma 3.

From Lemma 5 we get limiting normality of $S_{n,\alpha}$ under H_0 :

$$\sqrt{n} (S_{n,\alpha} - \mu_\alpha(0)) \overset{P_0^{(n)}}{\rightsquigarrow} N(0, \sigma_\alpha^2(0)), \quad n \rightarrow \infty,$$

where $\sigma_\alpha^2(0)$ as above and

$$\mu_\alpha(0) = E_0 S_{n,\alpha} = I_2 = (2\alpha - 1)^2/2.$$

Thus, under H_0 the statistic $S_{n,\alpha}$ is asymptotically distribution-free.

Next, under the alternative, by Le Cam's third lemma (see Section 4.1)

$$\sqrt{n} (S_{n,\alpha} - \mu_\alpha(0)) \overset{P_{h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N(hE_0[\psi_{0,\alpha}(X_1)\dot{l}_0(X_1)], \sigma_\alpha^2(0)), \quad n \rightarrow \infty,$$

or equivalently,

$$\sqrt{n} \left(S_{n,\alpha} - \left(\mu_\alpha(0) + \frac{h}{\sqrt{n}} E_0[\psi_{0,\alpha}(X_1) \dot{l}_0(X_1)] \right) \right) \stackrel{P_{h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N(0, \sigma_\alpha^2(0)), \quad n \rightarrow \infty,$$

with $hE_0[\psi_{0,\alpha}(X_1) \dot{l}_0(X_1)]$ given by (4.7). In particular, under the alternative H_1 : $h > 0$, the asymptotic mean of $S_{n,\alpha}$, $\mu_\alpha(h/\sqrt{n})$, and the asymptotic variance of $S_{n,\alpha}$ multiplied by n , $\sigma_\alpha^2(0)$, are

$$\begin{aligned} \mu_\alpha \left(\frac{h}{\sqrt{n}} \right) &= \frac{(2\alpha - 1)^2}{2} + 2 \frac{h}{\sqrt{n}} \left(\int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - (1 - 2\alpha) f(F^{-1}(\alpha)) \right), \\ \sigma_\alpha^2(0) &= 1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4. \end{aligned}$$

The proof is completed. $\square$

The following result is well known and can be found, for example, in [2, Ch. 2]. It follows immediately from Lemma 4 for $\alpha = 0$.

Corollary of Lemma 4. *Under the conditions of Lemma 4 as $n \rightarrow \infty$*

$$\sqrt{n} \left(S_{n,0} - \mu_0 \left(\frac{h}{\sqrt{n}} \right) \right) \stackrel{P_{h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N(0, 1/3).$$

where

$$\mu_0(\theta) = 1/2 + 2\theta \int_{\mathbf{R}} f^2(x) dx.$$

Note 4. If we wish to test the hypotheses

$$H_0 : F(x) = 1 - F(-x), \quad x \in \mathbf{R},$$

$$H_2 : F(x) \geq 1 - F(-x), \quad x \in \mathbf{R},$$

with inequality being strict on a set of positive length, we can use statistic

$$\tilde{S}_{n,\alpha} = \frac{1}{n(n-1)} \sum_{i \neq j} I_{[\alpha, 1-\alpha] \times [\alpha, 1-\alpha]}(i/(n+1), j/(n+1)) I(X_{(i)} + X_{(j)} \geq 0),$$

with kernel $\tilde{k}(x, y) = I(X_{(i)} + X_{(j)} \geq 0)$ in place of $k(x, y) = I(X_{(i)} + X_{(j)} \leq 0)$, for which (compare with the above expressions for $\mu_\alpha(\theta)$ and $\sigma_\alpha^2(0)$)

$$\begin{aligned} \tilde{\mu}_\alpha(\theta) &= \frac{(2\alpha - 1)^2}{2} - 2\theta \left(\int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - (1 - 2\alpha)f(F^{-1}(\alpha)) \right), \\ \tilde{\sigma}_\alpha^2(0) &= 1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4. \end{aligned}$$

In this case, H_0 is rejected in favour of H_2 for small values of $\tilde{S}_{n,\alpha}$. When $\alpha = 0$ the statistic $\tilde{S}_{n,\alpha}$ (and hence $S_{n,\alpha}$) is asymptotically equivalent to the U -statistic which, in its turn, is equivalent to the Wilcoxon signed rank statistic. In other words,

$$S_{n,0} = \frac{1}{n(n-1)} \sum_{i \neq j} I(X_{(i)} + X_{(j)} \leq 0) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} I(X_i + X_j \leq 0)$$

and

$$\tilde{S}_{n,0} = \frac{1}{n(n-1)} \sum_{i \neq j} I(X_{(i)} + X_{(j)} \geq 0) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} I(X_i + X_j \geq 0)$$

are asymptotically as efficient as the Wilcoxon signed rank statistic.

Indeed, the Wilcoxon signed rank statistic can be written in the form (see page 125 of [2])

$$W_n = \frac{2}{n(n+1)} \sum_{1 \leq i \leq j \leq n} I(X_i + X_j > 0).$$

For continuous distributions, this is asymptotically, as $n \rightarrow \infty$, equivalent to $\tilde{S}_{n,0}$, because

$$\begin{aligned} & \left| W_n - \frac{n-1}{n+1} \tilde{S}_{n,0} \right| \\ & \stackrel{\text{with prob. 1}}{=} \frac{2}{n(n+1)} \left| \sum_{1 \leq i \leq j \leq n} (I(X_i + X_j > 0)) - \sum_{1 \leq i < j \leq n} (I(X_i + X_j > 0)) \right| \\ & = \frac{2}{n(n+1)} \left| \sum_{i=j} (I(X_i + X_j > 0)) \right| = \frac{2}{n(n+1)} \left| \sum_{i=1}^n (I(X_i > 0)) \right| \\ & \leq \frac{2}{n(n+1)} \frac{n}{2} = \frac{1}{n+1} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

4.4 Consistency of the test based on $S_{n,\alpha}$

Consider testing $H_0 : \theta \in \Theta_0$ versus $H_A : \theta \in \Theta_1$. Suppose that test statistic S_n is used and that H_0 is rejected if S_n falls into a critical region K_n . The power function of a test based on S_n is

$$\pi_n(\theta) = P_\theta(S_n \in K_n).$$

In a local statistical experiment indexed by parameter $h = \theta\sqrt{n}$, the local power function of the test based on S_n is defined as

$$\pi(h) = \lim_{n \rightarrow \infty} \pi_n \left(\frac{h}{\sqrt{n}} \right), \quad h \geq 0.$$

The comparison of local power functions of tests is only of interest if the sequences of tests are consistent at fixed alternatives (see Section 14.2 of [7] for details).

Let us show that under extra assumption on $f(x)$ the test based on $S_{n,\alpha}$ is consistent against $H_1 : \theta > 0$. More precisely, when dealing with the test based on $S_{n,\alpha}$, where $\alpha \in [0, 1/2)$ is a given number, in addition to conditions (C1) and (C2), we assume that

$$(C3) \quad (\log f)'(F^{-1}(\alpha)) > 0.$$

This condition is satisfied, for example, for a *unimodal* symmetric density $f(x)$ with peak at $x = 0$. Indeed, since $F^{-1}(\alpha) \in (-\infty, 0)$ for $\alpha \in [0, 1/2)$ and unimodal $f(x)$ is increasing on $(-\infty, 0)$, it follows that $\log f(x)$ is also increasing on $(-\infty, 0)$ and hence $(\log f)'(F^{-1}(\alpha)) > 0$.

The power function of level α_0 test under consideration is

$$\pi_n(\theta) = P_\theta \left(\frac{\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0))}{\sigma_\alpha(0)} > z_{\alpha_0} \right), \quad z_{\alpha_0} = \Phi^{-1}(1 - \alpha_0).$$

This is a nondecreasing function of θ (see formula (4.9) below). This fact will be used later in Chapter 5.

It follows from Lemma 4 that the level α_0 test based on $S_{n,\alpha}$ is consistent

against $H_1 : \theta > 0$ if the relation

$$\lim_{n \rightarrow \infty} P_\theta \left(\frac{\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0))}{\sigma_\alpha(0)} > z_{\alpha_0} \right) = 1 \quad (4.8)$$

holds true. Using the results of the previous section, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_\theta \left(\frac{\sqrt{n}(S_{n,\alpha} - \mu_\alpha(0))}{\sigma_\alpha(0)} > z_{\alpha_0} \right) \\ &= \lim_{n \rightarrow \infty} P_\theta \left(\frac{\sqrt{n}(S_{n,\alpha} - \mu_\alpha(\theta) + (\mu_\alpha(\theta) - \mu_\alpha(0)))}{\sigma_\alpha(0)} > z_{\alpha_0} \right) \\ &= \lim_{n \rightarrow \infty} P_\theta \left(\frac{\sqrt{n}(S_{n,\alpha} - \mu_\alpha(\theta))}{\sigma_\alpha(0)} > z_{\alpha_0} - \sqrt{n} \frac{\mu_\alpha(\theta) - \mu_\alpha(0)}{\sigma_\alpha(0)} \right) \\ &= \lim_{n \rightarrow \infty} P \left(Z > z_{\alpha_0} - \sqrt{n} \frac{\mu_\alpha(\theta) - \mu_\alpha(0)}{\sigma_\alpha(0)} \right), \end{aligned} \quad (4.9)$$

where $Z \sim N(0, 1)$, and $\mu_\alpha(\theta)$ and $\sigma_\alpha^2(0)$ are given in Lemma 4. Thus, if we show that for $\theta > 0$

$$g(\alpha) := \mu_\alpha(\theta) - \mu_\alpha(0) = 2\theta \left(\int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - (1 - 2\alpha)f(F^{-1}(\alpha)) \right) > 0,$$

the consistency at a fixed alternative will be proved.

Note that $\mu_\alpha(\theta) - \mu_\alpha(0) = 2\theta \int_{\mathbf{R}} f^2(x) dx > 0$ when $\alpha = 0$, and $\mu_\alpha(\theta) - \mu_\alpha(0) = 0$ when $\alpha = 1/2$. So, if $g(\alpha)$ is decreasing in $\alpha \in [0, 1/2)$, then it follows that

$\mu_\alpha(\theta) - \mu_\alpha(0) > 0$ for all $\alpha \in [0, 1/2)$. Applying the Leibniz integral rule, we get

$$\begin{aligned}
\frac{g'(\alpha)}{2\theta} &= \frac{dF^{-1}(1-\alpha)}{d\alpha} f^2(F^{-1}(1-\alpha)) - \frac{dF^{-1}(\alpha)}{d\alpha} f^2(F^{-1}(\alpha)) \\
&\quad + \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} \frac{\partial}{\partial \alpha} f^2(x) dx + 2f(F^{-1}(\alpha)) \\
&\quad - (1-2\alpha)f'(F^{-1}(\alpha)) \frac{dF^{-1}(\alpha)}{d\alpha} \\
&= f^2(F^{-1}(1-\alpha)) \left(\frac{dF^{-1}(1-\alpha)}{d\alpha} - \frac{dF^{-1}(\alpha)}{d\alpha} \right) + 2f(F^{-1}(\alpha)) \\
&\quad - (1-2\alpha)f'(F^{-1}(\alpha)) \frac{dF^{-1}(\alpha)}{d\alpha} \\
&= f^2(F^{-1}(1-\alpha)) \left(\frac{-1}{f(F^{-1}(1-\alpha))} - \frac{1}{f(F^{-1}(\alpha))} \right) + 2f(F^{-1}(\alpha)) \\
&\quad - (1-2\alpha)f'(F^{-1}(\alpha)) \frac{1}{f(F^{-1}(\alpha))} \\
&= -(1-2\alpha)(\log f)'(F^{-1}(\alpha)) < 0.
\end{aligned}$$

where the last inequality is due to condition (C3). Thus, we get that $g'(\alpha) < 0$ and conclude that under (C1)–(C3) the test $S_{n,\alpha}$, where $\alpha \in [0, 1/2)$, is consistent against $H_1 : \theta > 0$.

Chapter 5

Asymptotic efficiency of weighted Wilcoxon-type tests

5.1 Concept of asymptotic relative efficiency

Sequences of tests can be compared by using the concept of relative efficiency. Let $X_1, \dots, X_n$ be a random sample from a distribution indexed by an unknown parameter $\theta \in \Theta \subseteq R$. Consider, for example, testing the hypotheses

$$H_0 : \theta = \theta_0,$$

$$H_A : \theta \neq \theta_0.$$

Definition 6. Let $N_T(\alpha_0, \beta, \theta)$ be the minimal sample size needed for the test of level α_0 based on $\{T_n\}$, where $T_n = T_n(X_1, \dots, X_n)$, to have the power larger or equal to β at point $\theta \neq \theta_0$. Suppose that for testing the above hypotheses two

sequences of test statistics $\{T_n\}$ and $\{S_n\}$ are available. The *relative efficiency* of $\{S_n\}$ with respect to $\{T_n\}$ is defined by

$$e_{S,T}(\alpha_0, \beta, \theta) = \frac{N_T(\alpha_0, \beta, \theta)}{N_S(\alpha_0, \beta, \theta)}.$$

If the value of $e_{S,T}(\alpha_0, \beta, \theta)$ is larger than 1, fewer observations for reaching the power β at the level α_0 and the alternative value θ are needed with the sequence $\{S_n\}$, which is considered the better one.

Relative efficiency has a substantial demerit. The value of $e_{S,T}(\alpha_0, \beta, \theta)$ depends on three arguments (and two sequences of statistics). Therefore it is extremely difficult to calculate this value. This difficulty can be overcome by calculating the limiting values of $e_{S,T}(\alpha_0, \beta, \theta)$ as $\alpha_0 \rightarrow 0$, as $\beta \rightarrow 1$, and as $\theta \rightarrow \theta_0$, keeping fixed the values of the two remaining parameters. There are three fundamental types of the asymptotic relative efficiency (ARE).

1. If for $\beta \in (0, 1)$ and $\theta \neq \theta_0$ there exists a limit

$$e_{S,T}^B(\beta, \theta) = \lim_{\alpha_0 \downarrow 0} e_{S,T}(\alpha_0, \beta, \theta),$$

then this limit is called the *Bahadur ARE* of the sequence $\{S_n\}$ with respect to $\{T_n\}$.

2. If for $\alpha_0 \in (0, 1)$ and $\theta \neq \theta_0$ there exists a limit

$$e_{S,T}^{HL}(\alpha_0, \theta) = \lim_{\beta \uparrow 1} e_{S,T}(\alpha_0, \beta, \theta),$$

then this limit is called the *Hodges-Lehmann ARE* of the sequence $\{S_n\}$ with

respect to $\{T_n\}$.

3. If for $0 < \alpha_0 < \beta < 1$ and $\theta \rightarrow \theta_0$ there exists a limit

$$e_{S,T}^P(\alpha_0, \beta, \theta_0) = \lim_{\theta \rightarrow \theta_0} e_{S,T}(\alpha_0, \beta, \theta),$$

then this limit is called the *Pitman ARE* of the sequence $\{S_n\}$ with respect to $\{T_n\}$.

5.2 Pitman efficiency

The Pitman efficiency is a widely used tool in asymptotic comparison of tests. With the regularity conditions which Pitman provided in the 1940's, the calculation of the Pitman efficiency becomes quite easy in many cases. In order to calculate the Pitman efficiency $e_{S,T}^P(\alpha_0, \beta, \theta_0)$ we essentially need the asymptotic normality of the test statistics T_n and S_n under the null hypothesis and under the alternative.

Now, let $\pi_n(\theta)$ denote the power function of the test based on S_n that rejects the null hypothesis for large values of S_n :

$$\pi_n(\theta) = P_\theta(S_n \geq c_n).$$

A sequence of tests based on S_n for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ is called *asymptotically of level α_0* if

$$\limsup_{n \rightarrow \infty} \pi_n(0) \leq \alpha_0.$$

Assume that $\theta = \theta_n = h/\sqrt{n}$ and consider the local limiting power function

$$\pi(h) = \lim_{n \rightarrow \infty} \pi_n \left(\frac{h}{\sqrt{n}} \right), \quad h \geq 0.$$

Due to Pitman we have the following result which yields an expression for $\pi(h)$.

Theorem 2 (Theorem 14.7 of [7]). *Let μ and σ be functions of θ such that*

$$\frac{\sqrt{n}(S_n - \mu(\theta_n))}{\sigma(\theta_n)} \overset{P_{\theta_n}^{(n)}}{\rightsquigarrow} N(0, 1), \quad n \rightarrow \infty, \quad (5.1)$$

holds for every sequence $\theta_n = h/\sqrt{n}$. Suppose that μ is differentiable and that σ is continuous at $\theta = 0$. Then the power functions π_n of the tests that reject $H_0 : \theta = 0$ for large values of S_n and are asymptotically of level α_0 satisfy

$$\lim_{n \rightarrow \infty} \pi_n \left(\frac{h}{\sqrt{n}} \right) = 1 - \Phi \left(z_{\alpha_0} - h \frac{\mu'(0)}{\sigma(0)} \right), \quad h \geq 0,$$

where Φ is the cdf of a standard normal distribution and $z_{\alpha_0} = \Phi^{-1}(1 - \alpha_0)$.

The convergence (5.1) is referred to as *locally uniform* asymptotic normality. The quantity $\mu'(0)/\sigma(0)$ in Theorem 2 is called the *slope* of the test statistic S_n .

Two sequences of asymptotically normal test statistics can be compared by comparing the sizes of their slopes. The square of the quotient of two slopes is called *asymptotic relative efficiency* (ARE) of one test relative to the other. Next theorem is often used for calculating ARE of two tests.

Theorem 3 (Theorem 14.19 of [7]) *Consider statistical models $\{P_{n,\theta} : \theta \geq 0\}$ such that $\|P_{n,\theta} - P_{n,0}\| \rightarrow 0$ as $\theta \rightarrow 0$, for every n . Let $S_{n,1}$ and $S_{n,2}$ be*

sequences of statistics that satisfy (5.1) for every sequence $\theta_n \downarrow 0$ and functions μ_i and σ_i such that μ_i is differentiable at zero and σ_i is continuous at zero, with $\mu'_i(0) > 0$ and $\sigma_i(0) > 0$. Then the relative efficiency of the tests that reject the null hypothesis $H_0 : \theta = 0$ for large values of $S_{n,i}$ is equal to

$$e_{12} = \frac{c_1^2}{c_2^2} = \left(\frac{\mu'_1(0)/\sigma_1(0)}{\mu'_2(0)/\sigma_2(0)} \right)^2,$$

for every sequence of alternatives $\theta_n \downarrow 0$, independently of significance level $\alpha_0 > 0$ and power $\beta \in (\alpha_0, 1)$, where

$$c_i = \lim_{n \rightarrow \infty} \frac{\mu'_i(\theta_n)}{\sigma_i(\theta_n)} = \frac{\mu'_i(0)}{\sigma_i(0)}, \quad i = 1, 2,$$

is the slope or efficacy of the test based on $S_{n,i}$. If the power functions of the tests based on $S_{n,i}$ are nondecreasing for every n , then the assumption of asymptotic normality of $S_{n,i}$ can be relaxed to asymptotic normality under every sequence $\theta_n = O(n^{-1/2})$ only.

It is noted in Section 4.4 that the power function $\pi_n(\theta)$ of the test based on $S_{n,\alpha}$ is nondecreasing in θ . Therefore we may use Theorem 3 with $\theta_n = h/\sqrt{n}$. Also, thanks to local asymptotic normality of the shift experiment $\{P_{h/\sqrt{n}}^{(n)} : h \geq 0\}$ it is true that $\|P_{h/\sqrt{n}}^{(n)} - P_0^{(n)}\| \rightarrow 0$ as $n \rightarrow \infty$.

For large n , the slope measures the rate of change of the asymptotic mean of S_n at the null hypothesis. A test with larger slope c responds faster to alternatives and therefore have better local power properties.

Since our test statistics are asymptotically normal under the null and alternative hypotheses, we use Pitman approach to calculate their efficiency.

5.3 Pitman slope of $S_{n,\alpha}$

Thanks to Lemma 4, the locally uniform asymptotic normality of $S_{n,\alpha}$ holds true with $\mu_\alpha(\theta)$ and $\sigma_\alpha(\theta)$, where $\mu_\alpha(\theta)$ is a linear function of θ , and $\sigma_\alpha(\theta)$ is a constant equal to $\sigma_\alpha(0)$. So, $\mu_\alpha(\theta)$ is differentiable at zero and $\sigma_\alpha(\theta)$ is continuous at zero. Therefore in order to calculate the slope $\mu'_\alpha(0)/\sigma_\alpha(0)$ we can use Theorem 2. The bigger the size of the slope, $|\mu'_\alpha(0)/\sigma_\alpha(0)|$, the better the test for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. From Lemma 4 and Theorem 2 the slope of $S_{n,\alpha}$ is equal to

$$\mu'_\alpha(0)/\sigma_\alpha(0) = \frac{2 \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x)dx - 2(1-2\alpha)f(F^{-1}(\alpha))}{\sqrt{1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4}}.$$

When $\alpha = 0$ the slope of $S_{n,\alpha}$ becomes

$$\mu'_0(0)/\sigma_0(0) = 2\sqrt{3} \int_{\mathbf{R}} f^2(x)dx.$$

5.4 Upper bound on the power function. Absolute Pitman efficiency

In addition to the knowledge of efficiency of one test relative to another, the knowledge of how test procedure is good (or bad) in an absolute scale is very important. In this section we obtain the quantity that characterizes the absolute quality of the test statistic $S_{n,\alpha}$.

Let $X_1, \dots, X_n$ be a random sample from a family of continuous distributions

$\{P_\theta : \theta \geq 0\}$ with pdf $f(x; \theta)$. Consider a local statistical experiment

$$(X_1, \dots, X_n) \sim \{P_{h/\sqrt{n}}^{(n)} : h \geq 0\},$$

indexed by a local parameter $h = \sqrt{n}\theta$. Suppose we wish to test the hypotheses

$$H_0 : h = 0 \quad \text{versus} \quad H_1 : h > 0,$$

using a sequence of test statistics $T_n = T_n(X_1, \dots, X_n)$ such that H_0 is rejected in favour of H_1 for large values of T_n . Let $\pi_n(\theta)$ be the power function of the test based on T_n :

$$\pi_n(\theta) = P_\theta(T_n \geq c_n).$$

Then if the following holds:

$$\lim_{n \rightarrow \infty} \pi_n(h/\sqrt{n}) = 1 - \Phi(z_{\alpha_0} - hc), \quad h \geq 0, \quad (5.2)$$

the asymptotically level α_0 test based on T_n is said to have slope c (see Theorem 2). Next result establishes upper bound on the limiting power function.

Theorem 4 (Theorem 15.4 of [7]). *Let $\Theta \subset \mathbf{R}^k$ be open and let $\psi : \Theta \mapsto \mathbf{R}$ be differentiable at θ_0 , with nonzero gradient $\dot{\psi}_{\theta_0}$ and such that $\psi(\theta_0) = 0$. Let the sequence of experiments $\{P_{n,\theta} : \theta \in \Theta\}$ be locally asymptotically normal at θ_0 with nonsingular Fisher information matrix I_{θ_0} , for constants $r_n \rightarrow \infty$. Then the power functions $\theta \mapsto \pi_n(\theta)$ of any sequence of level α_0 tests for testing $H_0 : \psi(\theta) \leq 0$*

versus $H_1 : \psi(\theta) > 0$ satisfy, for every h such that $\dot{\psi}_{\theta_0} h > 0$,

$$\limsup_{n \rightarrow \infty} \pi_n \left(\theta_0 + \frac{h}{r_n} \right) \leq 1 - \Phi \left(z_{\alpha_0} - \frac{\dot{\psi}_{\theta_0} h}{\sqrt{\dot{\psi}_{\theta_0} I_{\theta_0}^{-1} \dot{\psi}_{\theta_0}^\top}} \right).$$

We will apply this theorem with $k = 1$, $\theta_0 = 0$, $\psi(\theta) = \theta$, and $r_n = n^{1/2}$. A sequence of power functions $\theta \mapsto \pi_n(\theta)$ in the original experiments induces the sequence of power functions $h \mapsto \pi_n(\theta_0 + h/\sqrt{n})$ in the local experiments. Suppose that $\pi_n(\theta_0 + h/\sqrt{n}) \rightarrow \pi(h)$ for every h and some function π . Suppose also that the sequence π_n is of asymptotic level α_0 for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Then $\pi(0) = \lim \pi_n(\theta_0) \leq \alpha_0$ and hence π corresponds to a level α_0 test for $H_0 : h = 0$ versus $H_1 : h > 0$ in the limit experiment.

By Theorem 4, under the LAN condition on the statistical experiment,

$$\lim_{n \rightarrow \infty} \pi(h/\sqrt{n}) \leq 1 - \Phi(z_{\alpha_0} - h\sqrt{I_0}), \quad h \geq 0, \quad (5.3)$$

where for any $\theta \geq 0$

$$I_\theta = E_\theta \left(\frac{\partial}{\partial \theta} \log f(X_1; \theta) \right)^2,$$

is the Fisher information. It follows from (5.2) and (5.3) that the square root of the Fisher information $\sqrt{I_0}$ is the largest possible slope:

$$c \leq \sqrt{I_0}. \quad (5.4)$$

The quantity

$$e = \frac{c^2}{I_0},$$

is the relative efficiency of the test with slope c and the best test. It measures the *absolute* quality of the test with slope c . For this reason, we call this quantity the *absolute* Pitman efficiency of the test with slope c .

5.5 Pitman efficiency of test statistic $S_{n,\alpha}$

Return to the test statistic $S_{n,\alpha}$ defined in (2.2). Using (5.4), we obtain the following upper bound on the slope squared:

$$(\mu'_\alpha(0)/\sigma_\alpha(0))^2 \leq I_0, \quad (5.5)$$

where

$$I_0 = \int_{\mathbf{R}} (f'(x))^2 / f(x) dx,$$

is the Fisher information at $\theta = 0$ for the shift experiment under consideration. Therefore, using the results of the previous sections, the absolute Pitman efficiency of the test based on $S_{n,\alpha}$ is equal to

$$e(S_{n,\alpha}) = \frac{4 \left((1 - 2\alpha) f(F^{-1}(\alpha)) - \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx \right)^2}{(1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4) \int_{\mathbf{R}} (f'(x))^2 / f(x) dx}. \quad (5.6)$$

In particular, for unweighted test statistic for testing symmetry, for which $\alpha = 0$,

$$e(S_{n,0}) = \frac{12 \left(\int_{\mathbf{R}} f^2(x) dx \right)^2}{\int_{\mathbf{R}} (f'(x))^2 / f(x) dx}. \quad (5.7)$$

In the examples below, the underlying density has the form $f(x - \theta)$, where

$$f(x) = f_0(x/\sigma)/\sigma, \quad (5.8)$$

for a known density f_0 that is symmetric about zero, where $\sigma > 0$ is an unknown scale parameter. The corresponding cdf is $F(x) = F_0(x/\sigma)$. Before proceed with examples, let us rewrite formulas (5.6) and (5.7) in terms of f_0 . The numerator on the right-hand side of (5.6) is equal to

$$\begin{aligned} & 4 \left((1 - 2\alpha)(1/\sigma) f_0(\sigma F_0^{-1}(\alpha)/\sigma) - \int_{\sigma F_0^{-1}(\alpha)}^{\sigma F_0^{-1}(1-\alpha)} (1/\sigma^2) f_0^2(x/\sigma) dx \right)^2 \\ &= \frac{4}{\sigma^2} \left((1 - 2\alpha) f_0(F_0^{-1}(\alpha)) - \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(y) dy \right)^2. \end{aligned}$$

Next, $f'(x) = (1/\sigma^2) f_0'(x/\sigma)$, and the Fisher information in the denominator on the right-hand side of (5.6) becomes

$$\int_{\mathbf{R}} [(1/\sigma^2) f_0'(x/\sigma)]^2 / [(1/\sigma) f_0(x/\sigma)] dx = (1/\sigma^2) \int_{\mathbf{R}} (f_0'(y))^2 / f_0(y) dy.$$

Thus we arrive at the following expressions for the Pitman efficiency:

$$e(S_{n,\alpha}) = \frac{4 \left((1 - 2\alpha) f_0(F_0^{-1}(\alpha)) - \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{(1/3 - 8\alpha^2 + (64/3)\alpha^3 - 16\alpha^4) \int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx} \quad (5.9)$$

$$e(S_{n,0}) = \frac{12 \left(\int_{\mathbf{R}} f_0^2(x) dx \right)^2}{\int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx}. \quad (5.10)$$

That is, the absolute Pitman efficiency of $S_{n,\alpha}$ is not affected by the scale parameter σ .

5.6 Pitman efficiency of test statistic $S_{n,\alpha}^{(1)}$

Consider test statistic $S_{n,\alpha}^{(1)}$ with kernel

$$k(x, y) = I(x + y \leq 0),$$

and weight function

$$J(u, v) = J_0(u) J_0(v), \quad J_0(u) = I_{[0, 1-\alpha]}(u), \quad 0 \leq \alpha < 1/2.$$

This test statistic is based on the first $(1 - \alpha)100\%$ ordered observations. Let us find function $\psi_{0,\alpha}^{(1)}(x)$ in the decomposition

$$\sqrt{n} (S_{n,\alpha}^{(1)} - \mu_{\alpha}^{(1)}(0)) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_{0,\alpha}^{(1)}(X_k) + o_{P_0}(1). \quad (5.11)$$

Acting as in the case of statistic $S_{n,\alpha}$ we get

$$\psi_{0,\alpha}^{(1)}(x) = 2I_1^{(1)} - 2I_2^{(1)},$$

where $I_1^{(1)}$ and $I_2^{(1)}$ are given and calculated below.

$$\begin{aligned} I_1^{(1)} &= \int_{\mathbf{R}} J_0(F(x)) J_0(F(t)) k(x, t) f(t) dt = \int_{-\infty}^{F^{-1}(1-\alpha)} J_0(F(x)) k(x, t) f(t) dt \\ &= J_0(F(x)) \int_{-\infty}^{F^{-1}(1-\alpha)} I(x + t \leq 0) f(t) dt \\ &= J_0(F(x)) \int_{(-\infty, F^{-1}(1-\alpha)] \cap (-\infty, -x]} f(t) dt \\ &= \begin{cases} J_0(F(x)) \int_{-\infty}^{F^{-1}(1-\alpha)} dF(x) & , \quad \text{if } F(x) \leq \alpha \\ J_0(F(x)) \int_{-\infty}^{-x} dF(x) & , \quad \text{if } F(x) > \alpha \end{cases} \\ &= \begin{cases} J_0(F(x)) (1 - \alpha) & , \quad \text{if } F(x) \leq \alpha \\ J_0(F(x)) (1 - F(x)) & , \quad \text{if } F(x) > \alpha \end{cases} \\ &= \begin{cases} 1 - \alpha & , \quad \text{if } F(x) \leq \alpha \\ (1 - F(x)) I_{[0, 1-\alpha]}(F(x)) & , \quad \text{if } F(x) > \alpha \end{cases} \\ &= (1 - \max(F(x), \alpha)) I_{[0, 1-\alpha]}(F(x)). \end{aligned}$$

Denoting

$$D_{\alpha}^{(1)} = (-\infty, F^{-1}(1 - \alpha)] \times (-\infty, F^{-1}(1 - \alpha)] \cap \{(s, t) \in \mathbf{R}^2 : s + t \leq 0\},$$

Then

$$\begin{aligned}
I_2^{(1)} &= \iint_{\mathbf{R}^2} J_0(F(s))J_0(F(t))k(s,t)f(s)f(t)dsdt = \iint_{D_\alpha^{(1)}} dF(s)dF(t) \\
&= \int_{-\infty}^{F^{-1}(1-\alpha)} dF(s) \int_{-\infty}^{-s} dF(t) = \int_{-\infty}^{F^{-1}(1-\alpha)} (1-F(s)) dF(s) \\
&= \int_0^{1-\alpha} (1-u)du = \frac{1-\alpha^2}{2}.
\end{aligned}$$

From this, the function $\psi_{0,\alpha}^{(1)}(x)$ in decomposition (5.11) is as follows

$$\psi_{0,\alpha}^{(1)}(x) = 2I_1^{(1)} - 2I_2^{(1)} = 2(1 - \max(F(x), \alpha)) I_{[0,1-\alpha]}(F(x)) - (1 - \alpha^2).$$

Next, cf. (4.7),

$$\begin{aligned}
&hE_0[\psi_{0,\alpha}^{(1)}(X_1)\dot{l}_0(X_1)] \\
&= hE_0 \left\{ 2 \left((1 - \max(F(x), \alpha)) I_{[0,1-\alpha]}(F(x)) - (1 - \alpha^2) \right) \frac{f'(x)}{f(x)} \right\} \\
&= 2h \int_{\mathbf{R}} (1 - \max(F(x), \alpha)) I_{[0,1-\alpha]}(F(x)) df(x) - h(1 - \alpha^2) \int_{\mathbf{R}} df(x) \\
&\quad = 2h \int_{-\infty}^{F^{-1}(1-\alpha)} (1 - \max(F(x), \alpha)) df(x) \\
&= 2h \left\{ \int_{-\infty}^{F^{-1}(\alpha)} (1 - \alpha) df(x) + \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x)) df(x) \right\} \\
&\quad = 2h \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx - 2h\alpha f(F^{-1}(\alpha)).
\end{aligned}$$

Also

$$\begin{aligned}
(\sigma_\alpha^{(1)}(0))^2 &= E_0[\psi_{0,\alpha}^{(1)}(X_1)]^2 = \int_{\mathbf{R}} \left(\psi_{0,\alpha}^{(1)}(x) \right)^2 f(x) dx \\
&= \int_{\mathbf{R}} \left(2(1 - \max(F(x), \alpha)) I_{(0,1-\alpha)}(F(x)) - (1 - \alpha^2) \right)^2 dF(x) \\
&= \int_{\mathbf{R}} \left(2(1 - \max(F(x), \alpha)) I_{(0,1-\alpha)}(F(x)) \right)^2 dF(x) \\
&\quad + (1 - \alpha^2)^2 \int_{\mathbf{R}} dF(x) \\
&\quad - 4(1 - \alpha^2) \int_{\mathbf{R}} \left((1 - \max(F(x), \alpha)) I_{(\alpha,1)}(F(x)) \right) dF(x) \\
&= 4 \int_{-\infty}^{F^{-1}(1-\alpha)} (1 - \max(F(x), \alpha))^2 dF(x) + (1 - \alpha^2)^2 \int_{\mathbf{R}} dF(x) \\
&\quad - 4(1 - \alpha^2) \int_{-\infty}^{F^{-1}(1-\alpha)} (1 - \max(F(x), \alpha)) dF(x) \\
&= 4 \left(\int_{-\infty}^{F^{-1}(\alpha)} (1 - \alpha)^2 dF(x) + \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x))^2 dF(x) \right) \\
&\quad - 4(1 - \alpha^2) \left(\int_{-\infty}^{F^{-1}(\alpha)} (1 - \alpha) dF(x) + \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} (1 - F(x)) dF(x) \right) \\
&\quad + (1 - \alpha^2)^2 \int_{\mathbf{R}} dF(x) \\
&= 4 \left(\int_0^\alpha (1 - \alpha)^2 du + \int_\alpha^{1-\alpha} (1 - u)^2 du \right) \\
&\quad - 4(1 - \alpha^2) \left(\int_0^\alpha (1 - \alpha) du + \int_\alpha^{1-\alpha} (1 - u) du \right) + (1 - \alpha^2)^2 \int_0^1 du \\
&= 1/3 + (4/3)\alpha^3 - 3\alpha^4.
\end{aligned}$$

Then, using the results of Section 5.4, we conclude that the absolute Pitman

efficiency of the test based on $S_{n,\alpha}^{(1)}$ is equal to

$$e(S_{n,\alpha}^{(1)}) = \frac{4 \left(\alpha f(F^{-1}(\alpha)) - \int_{F^{-1}(\alpha)}^{F^{-1}(1-\alpha)} f^2(x) dx \right)^2}{(1/3 + (4/3)\alpha^3 - 3\alpha^4) \int_{\mathbf{R}} (f'(x))^2 / f(x) dx}. \quad (5.12)$$

Rewriting formula (5.12) in terms of f_0 , where $f(x) = f_0(x/\sigma)/\sigma$, we have

$$e(S_{n,\alpha}^{(1)}) = \frac{4 \left(\alpha f_0(F_0^{-1}(\alpha)) - \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{(1/3 + (4/3)\alpha^3 - 3\alpha^4) \int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx}. \quad (5.13)$$

Using the same technique as above, similar results for $S_{n,\alpha}^{(2)}$ with $J_0(u) = I_{[\alpha,1]}(u)$ can be obtained.

5.7 Examples

Using (5.9) and (5.10), we find the values of the absolute Pitman efficiency $e(S_{n,\alpha})$ for $\alpha = 0; 0.05, 0.1; 0.2; 0.25$ for the following shift families: (a) normal, (b) Laplace, (c) Cauchy. In the examples below, we are dealing with a shift experiment, so in each case, the family is $\{f(x - \theta) : \theta \geq 0\}$.

(a) Normal distribution. The pdf of a normal distribution with shift parameter θ and scale parameter σ is $f(x - \theta)$, where $f(x)$ is of the form (5.8) with

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and

$$f'_0(x) = -\frac{x}{\sqrt{2\pi}}e^{-x^2/2}.$$

Below for the calculation of $e(S_{n,\alpha})$ we use formulas (3.321.3) and (3.326.2) from [1]:

$$\begin{aligned} \int_0^\infty \exp(-q^2 x^2) dx &= \frac{\sqrt{\pi}}{2q}, \quad q > 0, \\ \int_0^\infty x^m \exp(-\beta x^n) dx &= \frac{\Gamma(\gamma)}{n\beta^\gamma}, \quad \beta, m, n > 0, \quad \gamma = \frac{m+1}{n}, \end{aligned}$$

and the fact that the gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ satisfies $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(x+1) = x\Gamma(x)$. In view of (5.10) we have for $\alpha = 0$

$$\begin{aligned} e(S_{n,0}) &= \frac{12 \left(\int_{\mathbf{R}} f_0^2(x) dx \right)^2}{\int_{\mathbf{R}} (f'_0(x))^2 / f_0(x) dx} = \frac{12 \left(\frac{1}{2\pi} \int_{\mathbf{R}} e^{-x^2} dx \right)^2}{\int_{\mathbf{R}} \frac{(1/2\pi)e^{-x^2}(x^2)}{(1/\sqrt{2\pi})e^{-x^2/2}} dx} \\ &= \frac{\frac{12}{4\pi^2} \left(\int_{\mathbf{R}} e^{-x^2} dx \right)^2}{(2\pi)^{-1/2} \int_{\mathbf{R}} x^2 e^{-x^2/2} dx} = \frac{3\sqrt{2} \left(\int_{\mathbf{R}} e^{-x^2} dx \right)^2}{\pi^{3/2} \int_{\mathbf{R}} x^2 e^{-x^2/2} dx} \\ &= \frac{3\sqrt{2}}{\pi^{3/2}} \frac{\pi}{2^{3/2}\Gamma(3/2)} = 0.9549. \end{aligned}$$

Next, for $\alpha \in (0, 1/2)$ by formula (5.9)

$$\begin{aligned} e(S_{n,\alpha}) &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2 \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{\sigma_\alpha^2(0) \int_{\mathbf{R}} (f'_0(x))^2 / f_0(x) dx} \\ &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2(2\pi)^{-1} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx \right)^2}{(2\pi)^{-1/2} \sigma_\alpha^2(0) \int_{\mathbf{R}} x^2 e^{-x^2/2} dx} \end{aligned}$$

$$\begin{aligned}
&= \frac{\left(2(1-2\alpha)(2\pi)^{-1/2}e^{-(F_0^{-1}(\alpha))^2/2} - \pi^{-1} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx\right)^2}{(2\pi)^{-1/2}\sigma_\alpha^2(0) \int_{\mathbf{R}} x^2 e^{-x^2/2} dx} \\
&= \frac{\left(2(1-2\alpha)(2\pi)^{-1/2}e^{-(F_0^{-1}(\alpha))^2/2} - \pi^{-1} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx\right)^2}{-16\alpha^4 + (64/3)\alpha^3 - 8\alpha^2 + 1/3}.
\end{aligned}$$

Using statistical software R , we obtain quantiles of normal distribution and values of $e(S_{n,\alpha})$ for different values of α . These results are given in Table 1.

Table 1: *Absolute Pitman efficiency $e(S_{n,\alpha})$ for normal distribution*

α	0	0.05	0.1	0.2	0.25
$F_0^{-1}(\alpha)$	$-\infty$	-1.6449	-1.2816	-0.8416	-0.6745
$F_0^{-1}(1-\alpha)$	∞	1.6449	1.2816	0.8416	0.6745
$\int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx$	1.7725	1.7370	1.6485	1.3578	1.1696
$e(S_{n,\alpha})$	0.9549	0.4270	0.2179	0.0585	0.0285

Clearly, when the value of α increases the value of $e(S_{n,\alpha})$ decreases very fast. Every time the proportion of the smallest and largest observations that are neglected increases by 5% the efficiency $e(S_{n,\alpha})$ decreases by more than 50%. When 10% of the observations (5% smallest and 5% largest) are removed, a huge loss of efficiency occurs. Therefore the weighted Wilcoxon-type statistic $S_{n,\alpha}$ with positive α should not be used when the observations come from a normal distribution.

(b) Laplace distribution. The pdf of Laplace distribution with shift parameter θ and scale parameter σ is $f(x - \theta)$, where $f(x)$ has the form (5.8) with

$$f_0(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{-x} & , \text{ if } x \geq 0, \\ \frac{1}{2}e^x & , \text{ if } x < 0, \end{cases}$$

and

$$f'_0(x) = \begin{cases} -\frac{1}{2}e^{-x} & , \text{ if } x > 0, \\ \text{first derivative doesn't exist} & , \text{ if } x = 0, \\ \frac{1}{2}e^x & , \text{ if } x < 0. \end{cases}$$

By the symmetry of f_0 the Fisher information at $\theta = 0$ is equal to

$$\begin{aligned} I_0 &= E_\theta \left(\frac{\partial}{\partial \theta} \log f_0(\mathbf{X}_1 - \theta) \right)^2 \Big|_{\theta=0} = \int_{\{x \neq 0\}} (f'_0(x))^2 / f_0(x) dx \\ &= 2 \int_0^\infty (f'_0(x))^2 / f_0(x) dx. \end{aligned}$$

Therefore by (5.10)

$$\begin{aligned} e(S_{n,0}) &= \frac{4 \left(\int_{-\infty}^\infty f_0^2(x) dx \right)^2}{(2/3) \int_0^\infty (f'_0(x))^2 / f_0(x) dx} = \frac{24 \left(\int_0^\infty f_0^2(x) dx \right)^2}{\int_0^\infty (f'_0(x))^2 / f_0(x) dx} \\ &= \frac{3 \left(\int_0^\infty e^{-2x} dx \right)^2}{\int_0^\infty e^{-x} dx} = \frac{3}{4} = 0.75. \end{aligned}$$

Also, for $\alpha \in (0, 1/2)$ thanks to (5.9) we get

$$\begin{aligned} e(S_{n,\alpha}) &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 4 \int_0^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{2\sigma_\alpha^2(0) \int_0^\infty (f'_0(x))^2 / f_0(x) dx} \\ &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 4 \int_0^{F_0^{-1}(1-\alpha)} \frac{1}{4} e^{-2x} dx \right)^2}{2\sigma_\alpha^2(0) \int_0^\infty \frac{(1/4)e^{(-2x)}}{(1/2)e^{(-x)}} dx} \\ &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - \int_0^{F_0^{-1}(1-\alpha)} e^{-2x} dx \right)^2}{\sigma_\alpha^2(0) \int_0^\infty e^{-x} dx} \end{aligned}$$

$$= \frac{\left((1 - 2\alpha)e^{F_0^{-1}(\alpha)} + \frac{1}{2}e^{2F_0^{-1}(\alpha)} - \frac{1}{2}\right)^2}{-16\alpha^4 + (64/3)\alpha^3 - 8\alpha^2 + 1/3}.$$

Quantiles of Laplace distribution and values of $e(S_{n,\alpha})$ for several choices of α are summarized in Table 2.

Table 2: *Absolute Pitman efficiency $e(S_{n,\alpha})$ for Laplace distribution*

α	0	0.05	0.1	0.2	0.25
$F_0^{-1}(\alpha)$	$-\infty$	-2.3026	-1.6094	-0.9163	-0.6931
$e(S_{n,\alpha})$	0.75	0.5192	0.3750	0.2046	0.1499

First, note that for Laplace distribution the classical (unweighted) test is not as efficient as in the normal case. The quantity $e(S_{n,\alpha})$ is decreasing in α . When the number of smallest and largest observations that are removed increases, the efficiency $e(S_{n,\alpha})$ decreases at a decreasing rate. When 10% of the observations (5% smallest and 5% largest) are neglected, the loss of efficiency is considerable but smaller than in Example 1. The weighted Wilcoxon-type statistics $S_{n,\alpha}$ with positive α are also not recommended to use for Laplace distribution.

(c) Cauchy distribution. The pdf of Cauchy distribution with shift parameter θ and scale parameter σ is $f(x - \theta)$ with $f(x)$ satisfying (5.8) and

$$f_0(x) = \frac{1}{\pi(x^2 + 1)}.$$

We have

$$f'_0(x) = -\frac{2x}{\pi(x^2 + 1)^2}.$$

For the calculation of $e(S_{n,\alpha})$ we use formulas (2.172) and (2.173.1) from [1]:

$$\begin{aligned}\int \frac{dx}{a+bx+cx^2} &= \frac{2}{\sqrt{\Delta}} \arctan \frac{b+2cx}{\sqrt{\Delta}}, \quad \Delta = 4ac - b^2, \\ \int \frac{dx}{(a+bx+cx^2)^2} &= \frac{b+2cx}{\Delta(a+bx+cx^2)} + \frac{2c}{\Delta} \int \frac{dx}{a+bx+cx^2}.\end{aligned}$$

We apply these formulas with $a = 1$, $b = 0$, $c = 1$ and get

$$\int \frac{dt}{(1+t^2)^2} = \frac{t}{2(1+t^2)} + \frac{1}{2} \arctan t. \quad (5.14)$$

In particular,

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2} = \left[\frac{t}{2(1+t^2)} + \frac{1}{2} \arctan t \right]_{t=-\infty}^{t=\infty} = \frac{\pi}{2}. \quad (5.15)$$

We also use formula (2.147.2) from [1]:

$$\int \frac{x^m dx}{(1+x^2)^n} = -\frac{1}{2n-m-1} \frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1+x^2)^n},$$

where m and n are natural numbers, to get

$$\int \frac{x^2 dx}{(1+x^2)^3} = -\frac{x}{3(1+x^2)^2} + \frac{1}{3} \int \frac{dx}{(1+x^2)^3},$$

and formula (2.148.4) from [1] that gives

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = \left[\frac{3!!}{4 \cdot 2!} \arctan x \right]_{x=-\infty}^{x=\infty} = \frac{\pi}{8}.$$

Therefore by (5.10)

$$\begin{aligned}
 e(S_{n,0}) &= \frac{12 \left(\int_{\mathbf{R}} f_0^2(x) dx \right)^2}{\int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx} = \frac{12 \left(\pi^{-2} \int_{\mathbf{R}} \frac{1}{(x^2+1)^2} dx \right)^2}{\int_{\mathbf{R}} \frac{(\pi)^{-2} (4x^2/(x^2+1)^4)}{(\pi)^{-1} (1/(x^2+1))} dx} \\
 &= \frac{3 \left(\int_{\mathbf{R}} \frac{1}{(x^2+1)^2} dx \right)^2}{\pi^3 \int_{\mathbf{R}} \frac{x^2}{(x^2+1)^3} dx} = \frac{3 (\pi/2)^2}{\pi^3 (\pi/8)} = \frac{6}{\pi^2} = 0.6079.
 \end{aligned}$$

For $\alpha \in (0, 1/2)$

$$\begin{aligned}
 e(S_{n,\alpha}) &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2 \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{\sigma_\alpha^2(0) \int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx} \\
 &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2\pi^{-2} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} \frac{1}{(x^2+1)^2} dx \right)^2}{\sigma_\alpha^2(0) \int_{\mathbf{R}} \frac{\pi^{-2} (4x^2/(x^2+1)^4)}{\pi^{-1} (1/(x^2+1))} dx} \\
 &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2\pi^{-2} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} \frac{1}{(x^2+1)^2} dx \right)^2}{\sigma_\alpha^2(0) 4\pi^{-1} \int_{\mathbf{R}} \frac{x^2}{(x^2+1)^3} dx} \\
 &= \frac{8\pi^{-2} \left(\frac{1-2\alpha}{(F_0^{-1}(\alpha))^2+1} - \pi^{-1} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} \frac{1}{(x^2+1)^2} dx \right)^2}{\sigma_\alpha^2(0)} \\
 &= \frac{8\pi^{-2} \left(\frac{1-2\alpha}{(F_0^{-1}(\alpha))^2+1} - \pi^{-1} \left(\frac{F_0^{-1}(1-\alpha)}{(F_0^{-1}(1-\alpha))^2+1} + \arctan(F_0^{-1}(1-\alpha)) \right) \right)^2}{-16\alpha^4 + (64/3)\alpha^3 - 8\alpha^2 + 1/3}.
 \end{aligned}$$

Quantiles and values of $e(S_{n,\alpha})$ for different values of α are presented in Table 3.

Table 3: *Absolute Pitman efficiency $e(S_{n,\alpha})$ for Cauchy distribution*

α	0	0.05	0.1	0.2	0.25
$F_0^{-1}(\alpha)$	$-\infty$	-6.3138	-3.0777	-1.3764	-1
$F_0^{-1}(1 - \alpha)$	∞	6.3138	3.0777	1.3764	1
$e(S_{n,\alpha})$	0.6079	0.5842	0.5166	0.3048	0.1971

We see from Table 3 that for Cauchy distribution the efficiency of test statistic $S_{n,\alpha}$ is not very high. The quantity $e(S_{n,\alpha})$ is decreasing in α slowly. When 5% smallest and 5% largest observations are removed, there is a small loss of efficiency compared to the classical (unweighted) test with $\alpha = 0$. Therefore, weighted tests with small α are almost as efficient as unweighted test, but the efficiency in both cases (weighted and unweighted) is not high.

Now we consider the test statistic $S_{n,\alpha}^{(1)}$. Using (5.13), we find the absolute Pitman efficiency $e(S_{n,\alpha}^{(1)})$ for $\alpha = 0; 0.05, 0.1; 0.2$ for the same distributions as above. Again, in each case, the family is $\{f(x - \theta) : \theta \geq 0\}$.

(a1) Normal distribution. Applying (5.13), we get

$$\begin{aligned}
e(S_{n,\alpha}^{(1)}) &= \frac{4 \left(\alpha f_0(F_0^{-1}(\alpha)) - \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} f_0^2(x) dx \right)^2}{\left(\sigma_0^{(1)}(\alpha) \right)^2 \int_{\mathbf{R}} (f_0'(x))^2 / f_0(x) dx} \\
&= \frac{\left(2\alpha(2\pi)^{-1/2} e^{-(F_0^{-1}(\alpha))^2/2} - \pi^{-1} \int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx \right)^2}{1/3 + (4/3)\alpha^3 - 3\alpha^4}.
\end{aligned}$$

This formula leads to the following table.

Table 4: *Absolute Pitman efficiency $e(S_{n,\alpha}^{(1)})$ for normal distribution*

α	0	0.05	0.1	0.2
$F_0^{-1}(\alpha)$	$-\infty$	-1.6449	-1.2816	-0.8416
$\int_{F_0^{-1}(\alpha)}^{F_0^{-1}(1-\alpha)} e^{-x^2} dx$	1.7725	1.7370	1.6485	1.3578
$e(S_{n,\alpha})$	0.9549	0.8828	0.7170	0.3023

We see from Table 4 that when α increases the efficiency $e(S_{n,\alpha}^{(1)})$ decreases quite slowly. When 5% largest observations are neglected, the loss of efficiency is small. Therefore, in this case, the test statistic $S_{n,\alpha}^{(1)}$ can be safely used for testing symmetry. When 10% largest observations are neglected, the loss of efficiency is bigger but much smaller compared to statistic $S_{n,\alpha}$ (see Table 1). Therefore statistic $S_{n,\alpha}^{(1)}$ is more efficient than $S_{n,\alpha}$ under normal distribution. In the presence of large outliers, we can use statistic $S_{n,\alpha}^{(1)}$ with not very big α when the observations follow normal distribution.

(b1) Laplace distribution. In view of (5.13) we have

$$\begin{aligned}
e(S_{n,\alpha}^{(1)}) &= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2\int_0^{F_0^{-1}(1-\alpha)} f_0^2(x)dx - 2\int_0^\infty f_0^2(x)dx\right)^2}{2\left(\sigma_0^{(1)}(\alpha)\right)^2 \int_0^\infty (f_0'(x))^2/f_0(x)dx} \\
&= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - 2\int_0^{F_0^{-1}(1-\alpha)} \frac{1}{4}e^{-2x}dx - 2\int_0^\infty \frac{1}{4}e^{-2x}dx\right)^2}{2\left(\sigma_0^{(1)}(\alpha)\right)^2 \int_0^\infty \frac{(1/4)e^{(-2x)}}{(1/2)e^{(-x)}}dx} \\
&= \frac{\left(2(1-2\alpha)f_0(F_0^{-1}(\alpha)) - (1/2)\int_0^{F_0^{-1}(1-\alpha)} e^{-2x}dx - (1/2)\int_0^\infty e^{-2x}dx\right)^2}{\left(\sigma_0^{(1)}(\alpha)\right)^2 \int_0^\infty e^{-x}dx} \\
&= \frac{\left((1-2\alpha)e^{F_0^{-1}(\alpha)} + \frac{1}{4}e^{2F_0^{-1}(\alpha)} - \frac{1}{2}\right)^2}{1/3 - 6\alpha^2 + (44/3)\alpha^3 - 9\alpha^4}.
\end{aligned}$$

This formula gives the following table.

Table 5: *Absolute Pitman efficiency $e(S_{n,\alpha}^{(1)})$ for Laplace distribution*

α	0	0.05	0.1	0.2
$F_0^{-1}(\alpha)$	$-\infty$	-2.3026	-1.6094	-0.9163
$e(S_{n,\alpha})$	0.75	0.7200	0.6328	0.3408

Again, the efficiency $e(S_{n,\alpha}^{(1)})$ is decreasing in α . When the proportion of large observations that are removed increases the efficiency $e(S_{n,\alpha}^{(1)})$ decreases at an increasing rate. When 5% largest observations are neglected, the loss of efficiency is small. As in the previous example, if 10% of the observations are neglected the efficiency $e(S_{n,\alpha}^{(1)})$ is larger compared to $e(S_{n,\alpha})$ (see Table 2). Therefore in the case of Laplace distribution, we can use weighted Wilcoxon-type test statistics $S_{n,\alpha}^{(1)}$.

(c1) Cauchy distribution. Using the same formulas for integrals as we used for the calculation of $e(S_{n,\alpha})$ in case of Cauchy distribution, we get

$$\begin{aligned}
 e(S_{n,\alpha}^{(1)}) &= \frac{8\pi^{-2} \left(\frac{1-2\alpha}{(F_0^{-1}(\alpha))^2+1} - \pi^{-1} \int_{F_0^{-1}(\alpha)}^{\infty} \frac{1}{(x^2+1)^2} dx \right)^2}{\left(\sigma_{\alpha}^{(1)}(0) \right)^2} \\
 &= \frac{8\pi^{-2} \left(\frac{1-2\alpha}{(F_0^{-1}(\alpha))^2+1} - \pi^{-1} \left(\frac{\pi}{4} - \frac{F_0^{-1}(\alpha)}{2(1+(F_0^{-1}(\alpha))^2)} - \frac{1}{2} \arctan(F_0^{-1}(\alpha)) \right) \right)^2}{1/3 - 6\alpha^2 + (44/3)\alpha^3 - 9\alpha^4}.
 \end{aligned}$$

This formula leads to the following table.

Table 6: *Absolute Pitman efficiency $e(S_{n,\alpha}^{(1)})$ for Cauchy distribution*

α	0	0.05	0.1	0.2
$F_0^{-1}(\alpha)$	$-\infty$	-6.3138	-3.0777	-1.3764
$e(S_{n,\alpha})$	0.6079	0.6027	0.5679	0.3492

From Table 6, when α increases the efficiency $e(S_{n,\alpha}^{(1)})$ decreases. When 5% largest observations are neglected, the loss of efficiency is very small. When 10% largest observations are removed, the loss of efficiency is bigger compare to $S_{n,\alpha}$ with 5% smallest and 5% largest observations removed (see Table 3), but still small. Therefore for Cauchy distribution and small $\alpha > 0$ the test statistics $S_{n,\alpha}^{(1)}$ are close to unweighted statistic $S_{n,0}$, from the efficiency point of view. However, in both case (weighted and unweighted), the Pitman efficiency is not high.

Chapter 6

Conclusion

In this thesis, we calculate the Pitman efficiency of weighted Wilcoxon-type test statistics with smallest and/or largest observations removed, $S_{n,\alpha}$ and $S_{n,\alpha}^{(1)}$, and compare them with the classical statistic for testing symmetry that is equivalent to the Wilcoxon signed rank statistic. The parameter $\alpha \in (0, 1/2)$ controls the amount of observations removed from the sample when we construct $S_{n,\alpha}$ and $S_{n,\alpha}^{(1)}$. In Chapter 5 we obtain formulas for finding Pitman slope and for calculating the absolute Pitman efficiency of $S_{n,\alpha}$ and $S_{n,\alpha}^{(1)}$. These formulas are our main results and they are new. The formulas can be used for comparing the weighted Wilcoxon-type tests with other tests for symmetry existing in the literature that are based on asymptotically normal test statistics. Unlike many tests for symmetry, the tests based on $S_{n,\alpha}$ and $S_{n,\alpha}^{(1)}$ with $\alpha \in (0, 1/2)$ are resistant to the presence of outliers. The results are illustrated by examples with normal, Laplace, and Cauchy distributions. From the efficiency point of view, for normal and Laplace distributions, the Wilcoxon-type statistic $S_{n,\alpha}^{(1)}$ is better than $S_{n,\alpha}$

when $\alpha \in (0, 1/2)$. Therefore, in these two cases, in the absence of small outliers, $S_{n,\alpha}^{(1)}$ is preferred to $S_{n,\alpha}$. For Cauchy distribution the loss of efficiency for $S_{n,\alpha}$ is smaller than for $S_{n,\alpha}^{(1)}$. However, in this case, even the Wilcoxon signed rank statistic whose efficiency is higher, is not recommended for use. Using the efficiency formulas of Chapter 5, the Pitman efficiency of test statistics $S_{n,\alpha}$ and $S_{n,\alpha}^{(1)}$ for other distributions can be found, and the comparison of the existing tests of symmetry with tests under consideration can be made.

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