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**Sensitivity Analysis and Optimization
of
High-Speed VLSI Interconnects**

by

Stephen Lum, B.A.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Master of Engineering

Ottawa-Carleton Institute for Electrical Engineering
Department of Electronics
Faculty of Engineering
Carleton University
Ottawa, Canada
September, 1991

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The undersigned recommend to the Faculty of Graduate Studies
and Research acceptance of the thesis,

“SENSITIVITY ANALYSIS AND OPTIMIZATION
OF
HIGH-SPEED VLSI INTERCONNECTS”

submitted by Stephen Lum, B.A.Sc.
in partial fulfillment of the requirements for
the degree of Master of Engineering

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3.19	Sensitivity of voltage V_{out} with respect to parameter $R_{1,1}$ of transmission line #2 in Fig. 3.15.	57
3.20	Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #2 in Fig. 3.15.	58
3.21	Two conductor transmission line near a ground plane.	61
3.22	Transient response of the circuit shown in Fig. 3.15 at node V_{cross} . . .	62
3.23	Sensitivity of voltage V_{cross} with respect to the ratio w/d of transmission line #2 in Fig. 3.15.	63
4.1	Nonlinear network used to illustrate formulation.	68
4.2	Replacement of the nonlinear elements of circuit in Fig. 4.1 with time dependent sources.	70
4.3	Evaluation of $\mathbf{F}(t)$ for circuit in Fig. 4.1.	73
4.4	Evaluation of first column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.	73
4.5	Evaluation of second column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.	74
4.6	Evaluation of first column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.	74
4.7	Evaluation of second column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.	75
4.8	Example circuit containing a transmission line and a nonlinear termination.	78
4.9	Applied input voltage waveform used for example circuit.	79
4.10	Transient response of the circuit shown in Fig. 4.8 at node V_{out} . . .	80
5.1	Circuit for example 1, transmission lines with nonlinear terminations. . .	90
5.2	Applied input voltage waveform used for examples #1 and #2.	91
5.3	Transient response of the circuit shown in Fig. 5.1 at node V_{out}	92
5.4	Sensitivity of voltage V_{out} with respect to L_1 of Fig. 5.1.	93
5.5	Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #7 in Fig. 5.1.	94
5.6	Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #6 in Fig. 5.1.	95
5.7	Circuit for example 2, interconnect model with lossy coupled transmission lines and nonlinear terminations.	97
5.8	Transient response of the circuit shown in Fig. 5.7 at node V_{out}	98
5.9	Sensitivity of voltage V_{out} with respect to R_1 of Fig. 5.7.	99
5.10	Sensitivity of voltage V_{out} with respect to parameter $G'_{1,1}$ of transmission line #2 in Fig. 5.7.	100
5.11	Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #1 in Fig. 5.7.	101
5.12	Circuit for example 3, interconnect network with nonlinear terminations. . .	103
5.13	Applied input voltage waveform used for example #3.	104
5.14	Transient response of the circuit shown in Fig. 5.12 at node V_3	105
5.15	Sensitivity of voltage V_3 with respect to R_1 of Fig. 5.12.	106

Acknowledgments

I wish to thank my supervisor Dr. Michel Nakhla for his invaluable advice and guidance during the course of my research and studies. I am also grateful for his patience and care in reviewing this thesis.

I would like to thank Dr. Q. J. Zhang for his guidance and assistance in the development of the optimization examples and for the use of his minimax optimization package.

In addition, I would like to express my deep appreciation for the encouragement and support my parents have provided me through the years.

I am grateful to Karen Kung for her understanding and patience.

Finally, I must also acknowledge the assistance Richard Griffith provided in the development of the software necessary to test the algorithms and Tak Tang for his contribution to the preparation of the thesis.

Table of Contents

Acceptance Form	ii
Abstract	iii
Acknowledgements	iv
Table of Contents	v
List of Tables	vii
List of Figures	viii
List of Symbols	xi
1 Introduction	1
1.1 Problem Description	1
1.2 Thesis Objectives	2
1.3 Thesis Contributions	3
1.4 Outline of the Thesis	5
2 Time Domain Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks	7
2.1 Introduction	7
2.2 Formulation of the Network Equations	9
2.3 MNA Representation of a Lossy Multiconductor Transmission Line	11
2.4 Time Domain Analysis of the Linear Network	15
2.4.1 Numerical Inverse Laplace Transform Algorithm	15
2.4.2 Application of the NILT to Linear Networks	17
2.4.3 Truncation Error	18
2.5 Example of Analysis using NILT	19
2.6 Summary	23
3 Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks	24
3.1 Introduction	24
3.2 Frequency Domain Sensitivity	25
3.2.1 Adjoint Method for Sensitivity Analysis	26
3.2.2 Sensitivity wrt Lumped Components	28

3.2.3	Sensitivity wrt Distributed Components	30
3.2.4	Sensitivity wrt Physical Parameters	33
3.3	Time Domain Sensitivity using NILT	34
3.4	Summary of the Computational Steps	34
3.5	Examples Comparing the Proposed Method with Perturbation	35
3.6	Summary	64
4	Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations	65
4.1	Introduction	65
4.2	Formulation of the Network Equations	66
4.3	Solution of the Network Equations	69
4.3.1	The Piecewise Linear Time Domain Representation	70
4.3.2	Solution using Newton-Raphson Iterations	75
4.4	Example of Analysis using the PDT	77
5	Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations	81
5.1	Introduction	81
5.2	Sensitivity Analysis of Nonlinear Networks using the PDT	82
5.3	Solution of the Sensitivity of the Linear Subnetwork	84
5.4	Improving the Efficiency of the Method	85
5.5	Summary of Computational Steps	87
5.6	Examples Comparing the Proposed Method with Perturbation	88
5.7	Summary	109
6	Optimization of Time Domain Responses of Multiconductor Transmission Lines	110
6.1	Introduction	110
6.2	Formulation of Error Functions	111
6.3	Formulation of Optimization	113
6.4	Examples of Performance Optimization	114
6.5	Summary	135
7	Conclusions and Further Research	136
7.1	Summary	136
7.2	Suggestions for Further Research	137
	References	138

List of Tables

3.1	CPU comparison between HSPICE and Proposed Method.	36
5.1	CPU comparison between perturbation and the Proposed Method. . .	89
6.1	Parameter values for example #1.	116
6.2	Parameter values for example #2.	125

List of Figures

2.1	Linear network used to illustrate formulation.	11
2.2	Voltages and currents used in the admittance representation of a multiconductor transmission line.	12
2.3	Example circuit containing transmission lines with terminal networks.	20
2.4	Applied input voltage waveform used for example circuit.	21
2.5	Transient response of the circuit shown in Fig. 2.3 at node V_{out}	22
3.1	Circuit for example 1, obtained by cascading circuit shown in Fig. 2.3.	37
3.2	Applied input voltage waveform used for example #1.	38
3.3	Transient response of the circuit shown in Fig. 3.1 at node V_{out}	39
3.4	Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.1.	40
3.5	Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.1.	41
3.6	Sensitivity of voltage V_{out} with respect to the inductance per unit length of transmission line #7 in Fig. 3.1.	42
3.7	Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #35 in Fig. 3.1.	43
3.8	Circuit for example 2, two multiconductor transmission lines with terminal networks.	45
3.9	Applied input voltage waveform used for examples #2 and #3.	46
3.10	Transient response of the circuit shown in Fig. 3.8 at node V_{out}	47
3.11	Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.8.	48
3.12	Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.8.	49
3.13	Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #1 in Fig. 3.8.	50
3.14	Sensitivity of voltage V_{out} with respect to parameter $C_{1,2}$ of transmission line #1 in Fig. 3.8.	51
3.15	Circuit for examples 3 and 4, interconnect model with lossy coupled transmission lines.	53
3.16	Transient response of the circuit shown in Fig. 3.15 at node V_{out}	54
3.17	Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.15.	55
3.18	Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.15.	56

3.19	Sensitivity of voltage V_{out} with respect to parameter $R_{1,1}$ of transmission line #2 in Fig. 3.15.	57
3.20	Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #2 in Fig. 3.15.	58
3.21	Two conductor transmission line near a ground plane.	61
3.22	Transient response of the circuit shown in Fig. 3.15 at node V_{cross} . . .	62
3.23	Sensitivity of voltage V_{cross} with respect to the ratio w/d of transmission line #2 in Fig. 3.15.	63
4.1	Nonlinear network used to illustrate formulation.	68
4.2	Replacement of the nonlinear elements of circuit in Fig. 4.1 with time dependent sources.	70
4.3	Evaluation of $\mathbf{F}(t)$ for circuit in Fig. 4.1.	73
4.4	Evaluation of first column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.	73
4.5	Evaluation of second column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.	71
4.6	Evaluation of first column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.	74
4.7	Evaluation of second column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.	75
4.8	Example circuit containing a transmission line and a nonlinear termination.	78
4.9	Applied input voltage waveform used for example circuit.	79
4.10	Transient response of the circuit shown in Fig. 4.8 at node V_{out}	80
5.1	Circuit for example 1, transmission lines with nonlinear terminations. . .	90
5.2	Applied input voltage waveform used for examples #1 and #2.	91
5.3	Transient response of the circuit shown in Fig. 5.1 at node V_{out}	92
5.4	Sensitivity of voltage V_{out} with respect to L_1 of Fig. 5.1.	93
5.5	Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #7 in Fig. 5.1.	94
5.6	Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #6 in Fig. 5.1.	95
5.7	Circuit for example 2, interconnect model with lossy coupled transmission lines and nonlinear terminations.	97
5.8	Transient response of the circuit shown in Fig. 5.7 at node V_{out}	98
5.9	Sensitivity of voltage V_{out} with respect to R_1 of Fig. 5.7.	99
5.10	Sensitivity of voltage V_{out} with respect to parameter $G_{1,1}$ of transmission line #2 in Fig. 5.7.	100
5.11	Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #1 in Fig. 5.7.	101
5.12	Circuit for example 3, interconnect network with nonlinear terminations. .	103
5.13	Applied input voltage waveform used for example #3.	104
5.14	Transient response of the circuit shown in Fig. 5.12 at node V_3	105
5.15	Sensitivity of voltage V_3 with respect to R_1 of Fig. 5.12.	106

5.16	Sensitivity of voltage V_3 with respect to parameter $C_{3,3}$ of transmission line #3 in Fig. 5.12.	107
5.17	Sensitivity of voltage V_3 with respect to parameter $L_{3,2}$ of transmission line #3 in Fig. 5.12.	108
6.1	Circuit for example 1, 13 4-conductor transmission lines.	117
6.2	Applied input voltage waveform used for examples #1 and #2.	118
6.3	Transient response of the circuit shown in Fig. 6.1 at node V_d before optimization.	119
6.4	Crosstalk response of the circuit shown in Fig. 6.1 at node V_c before optimization.	120
6.5	Transient response of the circuit shown in Fig. 6.1 at node V_d after optimization.	121
6.6	Crosstalk response of the circuit shown in Fig. 6.1 at node V_c after optimization.	122
6.7	Circuit for example 2, 13 4-conductor transmission lines and 2 sources.	126
6.8	Transient response of the circuit shown in Fig. 6.7 at node V_b before optimization.	127
6.9	Transient response of the circuit shown in Fig. 6.7 at node V_d before optimization.	128
6.10	Crosstalk response of the circuit shown in Fig. 6.7 at node V_a before optimization.	129
6.11	Crosstalk response of the circuit shown in Fig. 6.7 at node V_c before optimization.	130
6.12	Transient response of the circuit shown in Fig. 6.7 at node V_b after optimization.	131
6.13	Transient response of the circuit shown in Fig. 6.7 at node V_d after optimization.	132
6.14	Crosstalk response of the circuit shown in Fig. 6.7 at node V_a after optimization.	133
6.15	Crosstalk response of the circuit shown in Fig. 6.7 at node V_c after optimization.	134

List of Symbols

$\mathbf{A}_k$	modified nodal admittance matrix for the linear subnetwork k
$\mathbf{C}$	symmetric matrix representing the capacitance per unit length of the transmission line
$\mathbf{C}_\pi$	constant matrix with entries determined by the lumped linear components in the network π
D	length of the transmission line
$\mathbf{D}_f$	selector matrix that maps the vector $\mathbf{x}(t)$ into the node space $\mathbb{R}^{N_\pi}$
$\mathbf{D}_k$	selector matrix that maps $\mathbf{i}_k(t)$ into the node space $\mathbb{R}^{N_\pi}$
$\mathbf{E}_\pi, \mathbf{e}_\pi(t)$	vector of source currents in the network π
$\mathcal{F}$	vector of functions describing the nonlinear elements
$\mathbf{G}$	symmetric matrix representing the conductance per unit length of the transmission line
$\mathbf{G}_\pi$	constant matrix with entries determined by the lumped linear components in the network π
$\mathbf{\Gamma}$	diagonal matrix containing the square-roots of the eigenvalues of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
γ_i^2	eigenvalues of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
$\mathbf{I}_k$	vector of currents entering the linear subnetwork k
$\mathbf{J}_x$	Jacobian matrix of the reduced equation set
$\mathbf{J}_z$	Jacobian matrix
I	number of piecewise linear segments
$\mathbf{L}$	symmetric matrix representing the inductance per unit length of the transmission line
$\mathcal{L}$	Laplace transform
λ	parameter which the sensitivity is found with respect to

N_f	number of nonzero entries in $\mathcal{F}$
N_k	number of elements in the vector $\mathbf{V}_k$
N_π	number of elements in the vector $\mathbf{V}_\pi$
N_s	number of linear subnetworks
ϕ	system output
$\mathbf{R}$	symmetric matrix representing the resistance per unit length of the transmission line
s	Laplace operator
$\mathbf{U}$	identity matrix
$u(t)$	unit step function
$\mathbf{V}_k$	vector of node voltages and appended branch currents of the linear subnetwork k
$\mathbf{V}_\pi, \mathbf{v}_\pi(t)$	vector of node voltages appended by independent voltage source currents and inductor currents of the network π
$\mathbf{V}_\pi^a$	vector containing the solution of the adjoint network
$\mathbf{v}_\pi(0)$	initial conditions of the network π
$\mathbf{v}_\pi^*$	solution vector $\mathbf{v}_\pi(t)$ at the discrete time point $t = t_r^*$
$\mathbf{x}_i$	eigenvectors of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
$\mathbf{x}(t)$	vector of piecewise linear time dependent sources
$\mathbf{Y}_L$	complex admittance per unit length of the transmission line
$\mathbf{Y}_\pi$	modified nodal admittance matrix for the network π
$\mathbf{Z}_L$	complex impedance per unit length of the transmission line

Chapter 1

Introduction

1.1 Problem Description

Due to recent advances in VLSI technologies and packaging techniques, device sizes have become smaller, ICs larger and interconnections finer. As feature sizes have decreased, the density of devices on a chip has increased leading to a greater number of interconnections both on the IC and between ICs. As well, switching times are shorter while at the same time chip area has increased which can lead to the electrical length of interconnects becoming a significant fraction of a wavelength. Finer and longer interconnections means an increase in losses which can affect signal integrity. All of these factors have made the analysis and design of interconnections a primary concern. Improperly designed interconnects can result in increased signal delay, ringing, inadvertent and false switching [1]–[5]. This phenomena can be observed at the chip level as well as the system level and the interconnected blocks can be analog, digital or mixed.

The conventional lumped impedance interconnect models are not adequate for high-speed applications where the analysis of delay and crosstalk between coupled lines is required. In addition, due to small line geometries, losses in the interconnections can be significant. As a result, a distributed lossy coupled transmission line model should be used.

Currently, there are a number of methods available for the analysis of transmission line networks. These analysis or simulation tools allow a circuit designer to determine the response of a given network. Human expertise is still required to study the effect of variance of network parameters on system performance. Critical components which can affect rise/fall times, overshoot/undershoot, crosstalk, delay, etc. have to be identified through repeated circuit simulations which is, in general, a computationally expensive process. On the other hand, an important step in the design process is the use of optimization techniques to improve the circuit response. Transmission line effects such as crosstalk, delay and reflection can be minimized at vital connection ports. Application of powerful gradient based optimizers depends on the knowledge of sensitivities of the output responses. The direct perturbation approach of sensitivity analysis reduces the efficiency of optimization and can be very expensive if the number of optimization variables is large.

1.2 Thesis Objectives

The first objective of this thesis is to develop a method to determine network sensitivities with respect to parameters of a lossy coupled transmission line [6]–[7].

The proposed method is based on the numerical inversion of Laplace transform [8] [9] and is more efficient and reliable compared to other methods based on FFT or the brute force perturbation technique.

In addition, to precisely analyze sensitivities of high-speed networks, the nonlinearities of the terminations have to be taken into account. The difficulty in this case is the fact that the quasi-transverse electromagnetic propagation modes in lossy transmission lines are defined only in the frequency domain. Whereas the nonlinear terminations can, in the general case, only be analyzed in the time domain. The second objective of this thesis is to utilize the Piecewise Decomposition Technique (PDT) [3] to efficiently analyze the sensitivities of network interconnects with nonlinear terminations [10].

Finally, a practical design example is presented to illustrate the usefulness of sensitivity analysis in combination with optimization [11]. In this example, the output response of a typical interconnect network is optimized to minimize transmission line effects such as crosstalk, delay and reflection. The design variables used for optimization include both component values and physical parameters.

1.3 Thesis Contributions

In this thesis, several contributions are presented to address the design of interconnections in high-speed networks:

1. A method based on the numerical inversion of Laplace transform is developed to evaluate the time domain sensitivity of networks which contain lossy coupled

transmission lines. The sensitivity can be calculated with respect to network components and parameters of transmission lines (Chapter 3).

2. A new method is developed to determine the frequency domain sensitivity of the eigenvalues and eigenvectors of the propagation matrix to transmission line electrical parameters (Section 3.2.3).
3. The method is extended to include sensitivity to physical parameters of the transmission line (Section 3.2.4).
4. Numerical examples comparing the proposed method with the perturbation method demonstrates the accuracy in evaluating network sensitivities. In addition, the sensitivities with respect to all parameters can be computed more efficiently than the sensitivity to one parameter using the perturbation method. An example for a medium sized circuit is given showing up to 2 orders of magnitude in CPU time savings (Section 3.5).
5. A method to evaluate the sensitivity of nonlinear networks with frequency dependent parameters is presented. This method combines the method for the time domain solution of linear network sensitivities with the Piecewise Decomposition Technique. Numerical examples are provided to demonstrate the accuracy of the combined method compared to the perturbation method (Chapter 5).
6. The method for time domain solution of linear network sensitivities is combined with a minimax optimizer to provide an automated design package for network

interconnects. Design examples illustrating the usefulness of this combination are presented (Chapter 6).

1.4 Outline of the Thesis

Chapter 2. Time Domain Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks outlines the time domain solution of a linear network response using numerical inversion of Laplace transform.

Chapter 3. Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks presents the method to determine the sensitivity of the network response to network parameters which include both lumped components and parameters of lossy coupled transmission lines. In addition, the method is extended to cover the physical parameters of the transmission line. Numerical examples are provided to illustrate the accuracy and efficiency of the proposed method compared to the perturbation method.

Chapter 4. Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations provides an outline of the Piecewise Decomposition Technique for the analysis of the time domain solution of nonlinear networks which contain frequency dependent components.

Chapter 5. Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations describes the combination of the method discussed in chapter 3 with the Piecewise Decomposition Technique to evaluate the

sensitivity of networks containing lossy coupled transmission lines with nonlinear terminations. Numerical examples illustrating the results are presented.

Chapter 6. Optimization of Time Domain Responses of Multiconductor Transmission Lines presents a practical application of sensitivity analysis. A minimax optimizer is used to optimize the output response of an interconnect network.

Chapter 7. Conclusions and Further Research summarizes the results of the thesis and suggestions for further research.

Chapter 2

Time Domain Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks

2.1 Introduction

A number of methods [12]–[16] have been proposed for the time domain analysis of networks which contain lossy coupled transmission lines. The common method found in the literature is based on separate equations formulated to describe the transmission line system and the terminal and interconnecting networks in the frequency domain. These equations are combined at the analysis stage. The time domain response is obtained by applying the inverse fast Fourier transform (FFT). The major difficulty with this approach is when the analysis has to span a time interval of several line transient times. For example, the response of a lossless line with short-circuited termination is of infinite duration which makes it impossible to compute the response using FFT. Even for moderately lossy lines, the duration of the response can exceed

many line transient times. This requires that a large number of points be added to the analysis to avoid aliasing problems and as a result greatly reduces the efficiency of the time response computation. In addition, the absence of a simple error criterion for the inverse FFT makes it difficult to establish confidence in the results.

Recently, an alternative method [17] for the analysis of lossy coupled transmission lines with arbitrary linear terminal and interconnecting networks has been proposed. This method is based on a unified formulation technique for the equations describing the transmission line system and the equations describing the terminal and interconnecting networks. The time domain response is obtained using numerical inversion of Laplace transform (NILT) [8]–[9]. This method is more reliable than the FFT-based methods since it does not suffer from the usual aliasing problems.

In the following sections, a review of the method presented in [17] is given. This will serve as the basis for the sensitivity analysis algorithm to be presented in the next chapter. The formulation of the network equations is described in section 2.2. In section 2.3, the modified nodal admittance (MNA) representation of a lossy multiconductor transmission line is derived. The time domain solution can then be obtained from the frequency domain solution using the NILT algorithm presented in section 2.4. A numerical example of a linear network containing lossless single transmission lines comparing NILT with HSPICE is provided in section 2.5.

2.2 Formulation of the Network Equations

Consider a linear network π which contains linear lumped components and arbitrary linear subnetworks. The linear lumped components can be described by equations in either the time or frequency domain. The arbitrary linear subnetworks may contain frequency dependent or distributed elements that are best described in the frequency domain. Without loss of generality the modified nodal admittance matrix [18] equation of the network π can be written as

$$\mathbf{C}_\pi \frac{d\mathbf{v}_\pi(t)}{dt} + \mathbf{G}_\pi \mathbf{v}_\pi(t) + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{i}_k(t) - \mathbf{e}_\pi(t) = 0 \quad (2.1)$$

where

$\mathbf{C}_\pi(t) \in \mathbb{R}^{N_\pi \times N_\pi}$, $\mathbf{G}_\pi(t) \in \mathbb{R}^{N_\pi \times N_\pi}$ are constant matrices with entries determined by the lumped linear components,

$\mathbf{v}_\pi(t) \in \mathbb{R}^{N_\pi}$ is the vector of node voltage waveforms appended by independent voltage source current and inductor current waveforms,

$\mathbf{D}_k = [d_{i,j}], d_{i,j} \in \{0,1\}, i \in \{1,2,\dots,N_\pi\}, j \in \{1,2,\dots,N_k\}$ with a maximum of one nonzero in each row or column is a selector matrix that maps $\mathbf{i}_k(t) \in \mathbb{R}^{N_k}$, the vector of currents entering the linear subnetwork k , into the node space $\mathbb{R}^{N_\pi}$ of the network π ,

N_s is the number of linear subnetworks and

$\mathbf{e}_\pi(t) \in \mathbb{R}^{N_\pi}$ is the vector of source waveforms.

The frequency domain representation can be obtained by taking the Laplace transform of (2.1):

$$[s\mathbf{C}_\pi + \mathbf{G}_\pi]\mathbf{V}_\pi(s) + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{I}_k(s) = \mathbf{E}_\pi(s) + \mathbf{C}_\pi \mathbf{v}_\pi(0) \quad (2.2)$$

Let the frequency domain equations of the linear subnetwork k to be in the form

$$\mathbf{A}_k \mathbf{V}_k(s) = \mathbf{I}_k(s) \quad (2.3)$$

where $\mathbf{V}_k(s)$ and $\mathbf{I}_k(s)$ represent the frequency domain terminal voltages and currents of the subnetwork k and $\mathbf{A}_k$ represents the MNA matrix of the subnetwork.

The node voltages of the linear subnetwork k are related to the node voltages of the linear network π by the equation

$$\mathbf{V}_k = \mathbf{D}_k^t \mathbf{V}_\pi \quad (2.4)$$

Substituting (2.3) and (2.4) into (2.2):

$$\mathbf{Y}_\pi \mathbf{V}_\pi = \mathbf{E}_\pi + \mathbf{C}_\pi \mathbf{v}_\pi(0) \quad (2.5)$$

where

$$\mathbf{Y}_\pi = s\mathbf{C}_\pi + \mathbf{G}_\pi + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{A}_k \mathbf{D}_k^t$$

Example The following matrices illustrate the formulation for the network shown in Fig. 2.1.

$$\mathbf{G}_\pi = \begin{bmatrix} G_1 & -G_1 & 0 & 0 & 0 & 1 \\ -G_1 & G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{C}_\pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V}_\pi = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ I_1 \end{bmatrix} \quad \mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{I}_1 = \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \end{bmatrix} \quad \mathbf{E}_\pi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \end{bmatrix}$$

Figure 2.1: Linear network used to illustrate formulation.

2.3 MNA Representation of a Lossy Multiconductor Transmission Line

In this section the MNA representation of a lossy multiconductor transmission line is derived. The resulting admittance matrix is used in the formulation of the frequency domain equations of the linear subnetworks described by (2.3).

The transmission line is assumed to be uniform along its length with an arbitrary cross-section. The reference directions for the voltage and current signals on a transmission line are shown in Fig. 2.2. The cross-section of a transmission line

with N signal conductors can be represented by the following $N \times N$ matrices of line parameters: the resistance per unit length $\mathbf{R}$, the inductance per unit length $\mathbf{L}$, the conductance per unit length $\mathbf{G}$, and the capacitance per unit length $\mathbf{C}$. Let x represent the length along the transmission line with $x = 0$ corresponding to the near end and $x = D$ corresponding to the far end as shown in Fig. 2.2.

Figure 2.2: Voltages and currents used in the admittance representation of a multi-conductor transmission line.

In the frequency domain the transmission line can be described by the differential equations

$$\frac{d\mathbf{V}(x)}{dx} = -\mathbf{Z}_L \mathbf{I}(x) \quad (2.6)$$

$$\frac{d\mathbf{I}(x)}{dx} = -\mathbf{Y}_L \mathbf{V}(x) \quad (2.7)$$

where $\mathbf{V}(x), \mathbf{I}(x)$ are the line voltages and currents,

$$\mathbf{Z}_L = \mathbf{R} + s\mathbf{L} \quad (2.8)$$

$$\mathbf{Y}_L = \mathbf{G} + s\mathbf{C} \quad (2.9)$$

and s is the Laplace operator.

The differential equations can be combined into the single wave equation

$$\frac{d^2 \mathbf{V}(x)}{dx^2} = \mathbf{Z}_L \mathbf{Y}_L \mathbf{V}(x) \quad (2.10)$$

which will have solutions of the form

$$\mathbf{V}_i(x) = \mathbf{V}_i(0) e^{\pm \gamma_i x} \quad (2.11)$$

$$\mathbf{I}_i(x) = \mathbf{V}_i(0) e^{\pm \gamma_i x} \quad (2.12)$$

where γ_i describes the wave propagating along the conductors with the amplitude and phase on a particular conductor set by the complex vectors $\mathbf{V}_i(0)$ and $\mathbf{I}_i(0)$.

Substituting the solution into the wave equation (2.10) yields

$$(\gamma_i^2 \mathbf{U} - \mathbf{Z}_L \mathbf{Y}_L) \mathbf{V}_i(0) = 0 \quad (2.13)$$

where $\mathbf{U}$ is the identity matrix. This equation has nontrivial solution when γ_i^2 satisfies the eigenvalue problem

$$\det\{\gamma_i^2 \mathbf{U} - \mathbf{Z}_L \mathbf{Y}_L\} = 0 \quad (2.14)$$

There will be N solutions $\gamma_i^2, i = 1 \dots N$ to the eigenvalue problem with associated eigenvectors $\mathbf{x}_i, i = 1 \dots N$. Let $\mathbf{\Gamma}$ be a diagonal matrix $\{\gamma_1, \gamma_2, \dots, \gamma_N\}$ obtained by taking the square roots of the eigenvalues and $\mathbf{S}_v$ be a matrix with the eigenvectors $\mathbf{x}_i$ in the columns. Solutions for the voltages can now be written as

$$\mathbf{V}(x) = \mathbf{S}_v \mathbf{E}(x) \mathbf{C}_1 + \mathbf{S}_v \mathbf{E}^{-1}(x) \mathbf{C}_2 \quad (2.15)$$

where

$\mathbf{E}(x)$ is the diagonal matrix $\{e^{-\gamma_1 x}, e^{-\gamma_2 x}, \dots, e^{-\gamma_N x}\}$ and

$\mathbf{C}_1, \mathbf{C}_2$ are constant vectors.

Substituting (2.15) into (2.10) produces

$$\mathbf{I}(x) = \mathbf{S}_i \mathbf{E}(x) \mathbf{C}_1 + \mathbf{S}_i \mathbf{E}^{-1}(x) \mathbf{C}_2 \quad (2.16)$$

where $\mathbf{S}_i = \mathbf{Z}_L^{-1} \mathbf{S}_v \mathbf{\Gamma}$.

From (2.15) the relationship between the near and far end voltages can be written as

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{V}(D) \end{bmatrix} = \begin{bmatrix} \mathbf{S}_v & \mathbf{S}_v \\ \mathbf{S}_v \mathbf{E}(D) & \mathbf{S}_v \mathbf{E}(D)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \quad (2.17)$$

Similarly, from (2.16) the relationship between the near and far end currents can be written as

$$\begin{bmatrix} \mathbf{I}(0) \\ \mathbf{I}(D) \end{bmatrix} = \begin{bmatrix} \mathbf{S}_i & \mathbf{S}_i \\ \mathbf{S}_i \mathbf{E}(D) & \mathbf{S}_i \mathbf{E}(D)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \quad (2.18)$$

Solving (2.17) for the constants and substituting into (2.18):

$$\begin{aligned} \begin{bmatrix} \mathbf{I}(0) \\ \mathbf{I}(D) \end{bmatrix} &= \begin{bmatrix} \mathbf{S}_i & \mathbf{S}_i \\ \mathbf{S}_i \mathbf{E}(D) & \mathbf{S}_i \mathbf{E}(D)^{-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{S}_v & \mathbf{S}_v \\ \mathbf{S}_v \mathbf{E}(D) & \mathbf{S}_v \mathbf{E}(D)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{V}(D) \end{bmatrix} \end{aligned} \quad (2.19)$$

This equation represents the transmission line in admittance form $\mathbf{A}\mathbf{V} = \mathbf{I}$ where

$$\mathbf{I} = \begin{bmatrix} \mathbf{I}(0) \\ \mathbf{I}(D) \end{bmatrix} \quad (2.20)$$

is a vector of the terminal currents into the transmission line,

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{V}(D) \end{bmatrix} \quad (2.21)$$

is a vector of the terminal voltages of the transmission line and

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{S}_i & -\mathbf{S}_i \\ -\mathbf{S}_i \mathbf{E}(D) & \mathbf{S}_i \mathbf{E}(D)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{S}_v & \mathbf{S}_v \\ \mathbf{S}_v \mathbf{E}(D) & \mathbf{S}_v \mathbf{E}(D)^{-1} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} \\ \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} \end{bmatrix} \end{aligned} \quad (2.22)$$

is an admittance matrix for the transmission line, where $\mathbf{E}_1$ and $\mathbf{E}_2$ are diagonal matrices:

$$\mathbf{E}_1 = \text{diag} \left\{ \frac{e^{-2\gamma_i D} + 1}{1 - e^{-2\gamma_i D}} ; i = 1 \dots N \right\} \quad (2.23)$$

$$\mathbf{E}_2 = \text{diag} \left\{ \frac{2}{e^{-\gamma_i D} - e^{\gamma_i D}} ; i = 1 \dots N \right\} \quad (2.24)$$

2.4 Time Domain Analysis of the Linear Network

2.4.1 Numerical Inverse Laplace Transform Algorithm

The method for the NILT is based on the numerical evaluation of the Laplace transform inversion integral

$$v(t) = \frac{1}{2\pi j} \int_{e^{-j\infty}}^{e^{+j\infty}} V(s) e^{st} ds \quad (2.25)$$

$$= \frac{1}{2\pi j t} \int_{e'^{-j\infty}}^{e'^{+j\infty}} V(z/t) e^z dz \quad (2.26)$$

where $z = st$.

Approximate the function e^z in (2.25) by a Padé rational function

$$\xi_{N,M}(z) = \frac{P_N(z)}{Q_M(z)} = \frac{\sum_{i=0}^N (M+N-i)! \binom{N}{i} z^i}{\sum_{i=0}^M (-1)^i (M+N-i)! \binom{M}{i} z^i} \quad (2.27)$$

which has the first $M+N+1$ terms of its Taylor expansion equal to those of the Taylor expansion of $e^z = \sum_{i=0}^{\infty} z^i/i!$. By inserting (2.27) into (2.25) an approximation $\hat{v}(t)$ to $v(t)$ is obtained:

$$\hat{v}(t) = \frac{1}{2\pi j t} \int_{\epsilon'-j\infty}^{\epsilon'+j\infty} V(z/t) \xi_{N,M}(z) dz \quad (2.28)$$

The integral can be evaluated by residue calculus by closing the path of integration along an infinite arc either to the right or to the left. In order that the path along the infinite arc not contribute to the integral, take N, M such that the function

$$F(z) = V(z/t) \xi_{N,M}(z) \quad (2.29)$$

has at least two more finite poles than zeros. Then

$$\oint F(z) dz = \pm 2\pi j \left(\sum \text{residue at poles inside the closed path} \right) \quad (2.30)$$

where the sign depends on the direction in which the path is closed.

Select any $N < M$. Then

$$\xi_{N,M}(z) = \sum_{i=1}^M K_i / (z - z_i) \quad (2.31)$$

where K_i are the residues and z_i are the poles of $\xi_{N,M}(z)$. Closing the path of integration around the poles in the right half plane produces an equation of the form

$$\hat{v}(t) = -(1/t) \sum_{i=1}^M K_i V(z_i/t); \quad t > 0 \quad (2.32)$$

which is the formula for the calculation of the time response. The formula can be seen as a quadrature formula with complex points or as a weighted sum of function values, evaluated at the complex points z_i/t . A collection of tables for the poles z_i and residues K_i can be found in [8] for different values of M and N . In [9] a proof is given which shows that the inversion method given by (2.32) is equivalent to the integration of differential equations with the order of integration equal to $M + N$.

Real time functions can be evaluated using only the poles z_i in the upper half plane. When M is even

$$\hat{v}(t) = -(1/t) \sum_{i=1}^{M'} \text{Re} [K'_i V(z_i/t)] ; t > 0 \quad (2.33)$$

where $M' = M/2$, $K'_i = 2K_i$. When M is odd, $M' = (M + 1)/2$ and $K'_i = K_i$ for the residue corresponding to the real pole.

2.4.2 Application of the NILT to Linear Networks

Given a Laplace domain MNA representation for a linear network π containing lossy multiconductor transmission lines, $\mathbf{Y}_\pi \mathbf{V}_\pi = \mathbf{E}_\pi + \mathbf{C}_\pi \mathbf{v}_\pi(0)$ where $\mathbf{V}_\pi$ is the circuit response, (2.33) can be used to calculate the time domain response as

$$\hat{\mathbf{v}}_\pi(t) = -(1/t) \sum_{i=1}^{M'} \text{Re} \left\{ K'_i \mathbf{Y}_\pi(z_i/t)^{-1} [\mathbf{E}_\pi(z_i/t) + \mathbf{C}_\pi \mathbf{v}_\pi(0)] \right\} \quad (2.34)$$

The response at each time point is obtained from the M' solutions to the network equation evaluated at the complex frequencies $s = z_i/t$.

Several points of interest should be mentioned when comparing (2.34) with inverse FFT methods. First, the numerical inversion of the Laplace transform does not suffer from aliasing effects. Second, the number of frequency points defined by $M' < 15$ is

usually much less than the number of frequency points required by the inverse FFT algorithm for the same accuracy in the output waveforms. This leads to considerable savings in the CPU requirements. Third, from (2.34), it is obvious that the solution at time t is completely independent of solutions at all other time points. If the circuit response for time t_0 is all that is required this response can be calculated efficiently without calculating the response for any other values of time. Using inverse FFT techniques the entire waveform must be calculated.

2.4.3 Truncation Error

In [8] an expression for the truncation error associated with (2.34) has been derived. From this expression it is easy to prove that (2.34) inverts exactly the first $M + N + 1$ terms of the Taylor series of any time response. In addition, the truncation error rapidly decreases with increasing M, N . Consequently, the size of the error can be approximated by solving the problem twice, first with a given M, N and the second time with the same M but with N smaller by one. The difference in the two solutions will approximate the $(M + N + 1)$ st term of the Taylor expansion. As long as this term is small, the truncation error in the result is small. The accuracy of NILT decreases as time increases. For lumped networks, a stepping algorithm can be used, first the network is solved for the interval $t_0 \leq t \leq t_1$, then the result at $t = t_1$ is considered as the initial point for the next interval. Unfortunately there is no easy way to extend this technique to the case of distributed networks. So as t increases the size of the error increases. If the size of the error becomes unacceptable higher values of M, N have to be used.

2.5 Example of Analysis using NILT

The circuit in Fig. 2.3 contains single lossless transmission lines. The applied voltage is a pulse shown in Fig. 2.4. The time domain response at the output node V_{out} is compared with the response obtained using HSPICE in Fig. 2.5.

Figure 2.3: Example circuit containing transmission lines with terminal networks.

Figure 2.4: Applied input voltage waveform used for example circuit.

Figure 2.5: Transient response of the circuit shown in Fig. 2.3 at node V_{out} .

2.6 Summary

In this chapter the formulation of the network equations and the derivation of the MNA stamp for a multiconductor transmission line was presented. This enables the computation of the solution of the network in the frequency domain. The time domain solution could then be obtained through the use of the NILT algorithm. The NILT was chosen as the basis for the sensitivity analysis algorithm to be presented later for a number of reasons:

1. The NILT algorithm is more reliable since it does not suffer from usual aliasing problems associated with the inverse FFT-based method.
2. The number of points required by the NILT is less than the number required by the inverse FFT for the same accuracy.
3. The solution at any time t can be obtained without calculating the entire waveform which is useful when a circuit designer is interested in a specific time window.

Chapter 3

Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Linear Terminal and Interconnecting Networks

3.1 Introduction

Time domain analysis of the network provides a measure of how a given circuit will perform. Sensitivity analysis is useful as a tool to determine the critical components in a network which can affect circuit performance such as rise/fall times, overshoot/undershoot, crosstalk, etc. The quality of different circuits having the same nominal response could be compared. In addition, the response gradients generated by the sensitivity analysis can be coupled with minimization routines to optimize circuit performance.

The sensitivity of the network response with respect to a parameter λ could be computed using brute force by perturbing the parameter by $\Delta\lambda$ but this method requires too many computations to make it practical. For each parameter λ an

additional formulation and solution of the network equations are required. As well, the accuracy of the sensitivity calculation requires that $\Delta\lambda \rightarrow 0$ which could lead to roundoff errors.

In this chapter a method for the time domain sensitivity analysis of a network with lossy coupled transmission lines is presented. The method works in parallel with the time domain analysis technique described in chapter 2. This allows the sensitivity to be computed in an efficient manner. Consequently, the sensitivity with respect to *many* parameters in the circuit could be calculated in fewer computations than a sensitivity analysis with respect to a *single* parameter using the perturbation method. In section 3.2 the method used for the frequency domain sensitivity analysis is described. This section contains an overview of the adjoint system method. The sensitivity with respect to transmission line parameters is derived in terms of the sensitivity of the eigenvalues and eigenvectors of the propagation matrix. Sensitivities to physical parameters of the transmission line are also taken into account. In section 3.3 the time domain response of sensitivity of the network is obtained from the frequency domain solution using NILT. Section 3.4 provides a summary of the computational steps and a number of examples comparing the sensitivity calculated using the proposed method and perturbation is presented in section 3.5.

3.2 Frequency Domain Sensitivity

The time domain sensitivity of a linear network requires the computation of the frequency domain solution at prespecified complex frequency points. In this section

the method for evaluating the frequency domain sensitivities of a linear network containing lossy coupled transmission lines is described. Sensitivity calculations can be divided into two categories: either sensitivity of all system variables with respect to (wrt) one component or sensitivity of one system output wrt all components. The latter case is more useful in practice and will be studied in detail [19].

3.2.1 Adjoint Method for Sensitivity Analysis

The adjoint method for sensitivity analysis [20] provides an efficient method to calculate the frequency domain sensitivity of a linear circuit with respect to a parameter λ . Consider the system of linear equations given by (2.5)

$$\mathbf{Y}_\pi \mathbf{V}_\pi = \mathbf{E}_\pi + \mathbf{C}_\pi \mathbf{v}_\pi(0) \quad (3.1)$$

where $\mathbf{Y}_\pi$, $\mathbf{E}_\pi$ and $\mathbf{C}_\pi$ are dependent on some parameter λ .

The sensitivity of $\mathbf{V}_\pi$ is evaluated by differentiating (3.1) wrt λ :

$$\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} \mathbf{V}_\pi + \mathbf{Y}_\pi \frac{\partial \mathbf{V}_\pi}{\partial \lambda} = \frac{\partial \mathbf{E}_\pi}{\partial \lambda} + \frac{\partial \mathbf{C}_\pi}{\partial \lambda} \mathbf{v}_\pi(0) \quad (3.2)$$

(3.2) can be rewritten in the form

$$\frac{\partial \mathbf{V}_\pi}{\partial \lambda} = -\mathbf{Y}_\pi^{-1} \left[\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} \mathbf{V}_\pi - \frac{\partial \mathbf{E}_\pi}{\partial \lambda} - \frac{\partial \mathbf{C}_\pi}{\partial \lambda} \mathbf{v}_\pi(0) \right] \quad (3.3)$$

(3.3) represents the sensitivity of all elements of the output vector $\mathbf{V}_\pi$ wrt the parameter λ . In most cases, there is only a single system output consisting of a linear combination of elements in $\mathbf{V}_\pi$. Therefore let the system output be defined as the variable ϕ such that

$$\phi = \mathbf{d}' \mathbf{V}_\pi \quad (3.4)$$

where $\mathbf{d}$ is a constant vector.

Differentiating (3.4) with respect to λ :

$$\frac{\partial \phi}{\partial \lambda} = \mathbf{d}^t \frac{\partial \mathbf{V}_\pi}{\partial \lambda} \quad (3.5)$$

Substituting (3.3) into (3.5):

$$\frac{\partial \phi}{\partial \lambda} = -\mathbf{d}^t \mathbf{Y}_\pi^{-1} \left[\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} \mathbf{V}_\pi - \frac{\partial \mathbf{E}_\pi}{\partial \lambda} - \frac{\partial \mathbf{C}_\pi}{\partial \lambda} \mathbf{v}_\pi(0) \right] \quad (3.6)$$

Define the adjoint vector $\mathbf{V}_\pi^a$ as

$$(\mathbf{V}_\pi^a)^t = -\mathbf{d}^t \mathbf{Y}_\pi^{-1} \quad (3.7)$$

or

$$\mathbf{Y}_\pi^t \mathbf{V}_\pi^a = -\mathbf{d} \quad (3.8)$$

By combining (3.6) and (3.7), the final form of the sensitivity equation becomes

$$\frac{\partial \phi}{\partial \lambda} = (\mathbf{V}_\pi^a)^t \left[\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} \mathbf{V}_\pi - \frac{\partial \mathbf{E}_\pi}{\partial \lambda} - \frac{\partial \mathbf{C}_\pi}{\partial \lambda} \mathbf{v}_\pi(0) \right] \quad (3.9)$$

Note that both the vectors $\mathbf{V}_\pi$ and $\mathbf{V}_\pi^a$ are calculated before the sensitivity calculations are performed. The vector $\mathbf{V}_\pi$ is obtained when the solution of the system is calculated. The vector $\mathbf{V}_\pi^a$ can be evaluated from the LU factors of $\mathbf{Y}_\pi$ produced in solving the system. After performing LU decomposition the matrix $\mathbf{Y}_\pi$ become:

$$\mathbf{Y}_\pi = \mathbf{L}\mathbf{U} \quad (3.10)$$

The adjoint solution $\mathbf{V}_\pi^a$ requires the LU decomposition of the transpose of $\mathbf{Y}_\pi$.

However

$$\mathbf{Y}_\pi^t = \mathbf{U}^t \mathbf{L}^t \quad (3.11)$$

Using (3.8):

$$\mathbf{U}'\mathbf{L}'\mathbf{V}_\pi^a = -\mathbf{d} \quad (3.12)$$

Therefore, there is no need to perform an LU decomposition on $\mathbf{Y}_\pi^t$. Only one forward and one backward substitution are required. As a result, the sensitivity of the system can be obtained at a relatively low additional cost. The final step in obtaining the sensitivity of the linear network wrt to a parameter λ requires the calculation of $\partial\mathbf{Y}_\pi/\partial\lambda$ and $\partial\mathbf{E}_\pi/\partial\lambda$ which is described in the following subsections.

3.2.2 Sensitivity wrt Lumped Components

In general, λ is either a source or a component within the network. When finding the sensitivity to a source $\partial\mathbf{Y}_\pi/\partial\lambda$ is zero. Therefore the sensitivity of the output is

$$\frac{\partial\phi}{\partial\lambda} = -(\mathbf{V}_\pi^a)' \frac{\partial\mathbf{E}_\pi}{\partial\lambda} \quad (3.13)$$

Example 1 Let λ be the current source J_1 connected between nodes i and j . The vector $\mathbf{E}_\pi$ will contain the following elements to represent J_1 :

$$\mathbf{E}_\pi = \begin{matrix} i \\ j \end{matrix} \begin{bmatrix} J_1 \\ -J_1 \end{bmatrix} \quad (3.14)$$

Differentiating (3.14) with respect to J_1 results in

$$\frac{\partial\mathbf{E}_\pi}{\partial\lambda} = (\mathbf{e}_i - \mathbf{e}_j) \quad (3.15)$$

where $\mathbf{e}_i$ is a vector containing the value one in location i and zero elsewhere.

Network sensitivities to a specific component requires the evaluation of $\partial\mathbf{Y}_\pi/\partial\lambda$. In the case of lumped components such as resistors, capacitors and inductors, $\partial\mathbf{E}_\pi/\partial\lambda$

is zero. From (2.5)

$$\mathbf{Y}_\pi = s\mathbf{C}_\pi + \mathbf{G}_\pi + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{A}_k \mathbf{D}_k^t \quad (3.16)$$

If λ represents a lumped component then

$$\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} = s \frac{\partial \mathbf{C}_\pi}{\partial \lambda} + \frac{\partial \mathbf{G}_\pi}{\partial \lambda} \quad (3.17)$$

$\mathbf{C}_\pi$ and $\mathbf{G}_\pi$ are the complex and real parts of the MNA matrix containing the lumped components. The derivative of these matrices with respect to λ is simply the derivative the MNA stamp for which λ represents.

Example 2 Let λ be a conductance G_1 connected between nodes i and j . The MNA matrix will contain entries for G_1 as follows:

$$\mathbf{G}_\pi = \begin{matrix} & \begin{matrix} i & j \end{matrix} \\ \begin{matrix} i \\ j \end{matrix} & \begin{bmatrix} G_1 & -G_1 \\ -G_1 & G_1 \end{bmatrix} \end{matrix} \quad (3.18)$$

Differentiating (3.18) with respect to G_1 and substituting into (3.17) yields

$$\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} = (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^t \quad (3.19)$$

Example 3 Let λ be an inductor L_1 .

$$\mathbf{C}_\pi = \begin{matrix} & \begin{matrix} i & j & I \end{matrix} \\ \begin{matrix} i \\ j \\ m+1 \end{matrix} & \begin{bmatrix} & & \\ & & \\ & & -L_1 \end{bmatrix} \end{matrix} \quad \mathbf{G}_\pi = \begin{matrix} & \begin{matrix} i & j & I \end{matrix} \\ \begin{matrix} i \\ j \\ m+1 \end{matrix} & \begin{bmatrix} & & 1 \\ & & -1 \\ 1 & -1 & \end{bmatrix} \end{matrix} \quad (3.20)$$

where $(m + 1)$ denotes the location of the inductor current in the vector $\mathbf{V}_\pi$.

Differentiating (3.20) with respect to L_1 results in

$$\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} = -s \mathbf{e}_{m+1} \mathbf{e}_{m+1}^t \quad (3.21)$$

The derivatives of the other lumped components can be determined in a similar manner. If λ is a transmission line parameter the evaluation of $\partial \mathbf{Y}_\pi / \partial \lambda$ is relatively more complex and will be described in the next section.

3.2.3 Sensitivity wrt Distributed Components

If λ is a parameter of the subnetwork k differentiating (2.5) with respect to λ results in

$$\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} = \mathbf{D}_k \frac{\partial \mathbf{A}_k}{\partial \lambda} \mathbf{D}_k^t \quad (3.22)$$

Combining (3.22) and (3.9) produces

$$\frac{\partial \phi}{\partial \lambda} = (\mathbf{V}_\pi^a)^t \left[\mathbf{D}_k \frac{\partial \mathbf{A}_k}{\partial \lambda} \mathbf{D}_k^t \right] \mathbf{V}_\pi \quad (3.23)$$

If the subnetwork is a lossy coupled transmission line $\mathbf{A}_k$ is given by (2.22). Rewriting (2.22) as

$$\mathbf{A}_k \begin{bmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{S}_i & 0 \\ 0 & \mathbf{S}_i \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix} \quad (3.24)$$

Differentiating (3.24) results in

$$\begin{aligned} \frac{\partial \mathbf{A}_k}{\partial \lambda} \begin{bmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{bmatrix} &= \begin{bmatrix} \frac{\partial \mathbf{S}_i}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_i}{\partial \lambda} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{S}_i & 0 \\ 0 & \mathbf{S}_i \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda} & \frac{\partial \mathbf{E}_2}{\partial \lambda} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda} & \frac{\partial \mathbf{E}_1}{\partial \lambda} \end{bmatrix} \\ &\quad - \mathbf{A}_k \begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda} \end{bmatrix} \end{aligned} \quad (3.25)$$

In order to find the solution to (3.25) the partial derivatives of $\mathbf{S}_i$, $\mathbf{S}_v$, $\mathbf{E}_1$ and $\mathbf{E}_2$ need to be determined. These partial derivatives can be evaluated as follows:

a) From (2.16), $\mathbf{S}_i$ is defined as

$$\begin{aligned}\mathbf{S}_i &= \mathbf{Z}_L^{-1} \mathbf{S}_v \Gamma \\ \mathbf{Z}_L \mathbf{S}_i &= \mathbf{S}_v \Gamma\end{aligned}\quad (3.26)$$

Differentiating (3.26) results in

$$\mathbf{Z}_L \frac{\partial \mathbf{S}_i}{\partial \lambda} = \frac{\partial \mathbf{S}_v}{\partial \lambda} \Gamma + \mathbf{S}_v \frac{\partial \Gamma}{\partial \lambda} - \frac{\partial \mathbf{Z}_L}{\partial \lambda} \mathbf{S}_i \quad (3.27)$$

b) The matrix Γ is defined as a matrix containing the square-root of the eigenvalues of $\mathbf{Z}_L \mathbf{Y}_L$. The sensitivity of this matrix to λ in relation to the sensitivity of the eigenvalues of $\mathbf{Z}_L \mathbf{Y}_L$ is as follows:

$$\frac{\partial \Gamma}{\partial \lambda} = \text{diag} \left\{ \frac{\partial \gamma_i}{\partial \lambda} ; i = 1 \dots N_k \right\} \quad (3.28)$$

where

$$\frac{\partial \gamma_i}{\partial \lambda} = \frac{1}{2\gamma_i} \frac{\partial \gamma_i^2}{\partial \lambda} \quad (3.29)$$

c) Now the problem is reduced to the evaluation of the sensitivity of the eigenvalues γ_i^2 and eigenvectors $\mathbf{x}_i$ of the matrix $\mathbf{Z}_L \mathbf{Y}_L$. Let

$$(\gamma_i^2 \mathbf{U} - \mathbf{Z}_L \mathbf{Y}_L) \mathbf{x}_i = 0 \quad (3.30)$$

where $\mathbf{U}$ is the identity matrix.

Differentiating (3.30) with respect to λ produces

$$\frac{\partial \gamma_i^2}{\partial \lambda} \mathbf{x}_i + \gamma_i^2 \frac{\partial \mathbf{x}_i}{\partial \lambda} - \frac{\partial \mathbf{Z}_L \mathbf{Y}_L}{\partial \lambda} \mathbf{x}_i - \mathbf{Z}_L \mathbf{Y}_L \frac{\partial \mathbf{x}_i}{\partial \lambda} = 0 \quad (3.31)$$

This is a system of N_k equations with $N_k + 1$ unknowns. To solve the system one more equation has to be defined. An eigenvector can be normalized as follows:

$$\mathbf{x}_i^t \mathbf{x}_i = 1 \quad (3.32)$$

Differentiating (3.32):

$$\frac{\partial \mathbf{x}_i^t}{\partial \lambda} \mathbf{x}_i + \mathbf{x}_i^t \frac{\partial \mathbf{x}_i}{\partial \lambda} = 0 \quad (3.33)$$

Since each term of (3.33) is a scalar quantity it can easily be shown that

$$\frac{\partial \mathbf{x}_i^t}{\partial \lambda} \mathbf{x}_i = -\mathbf{x}_i^t \frac{\partial \mathbf{x}_i}{\partial \lambda} \quad (3.34)$$

Therefore

$$\mathbf{x}_i^t \frac{\partial \mathbf{x}_i}{\partial \lambda} = 0 \quad (3.35)$$

Now with (3.35) inserted into the matrix equation (3.31) a solution for the partial derivatives of the eigenvalues and eigenvectors can be found.

$$\begin{bmatrix} \gamma_i^2 \mathbf{U} - \mathbf{Z}_L \mathbf{Y}_L & \mathbf{x}_i \\ \mathbf{x}_i^t & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_i}{\partial \lambda} \\ \frac{\partial \gamma_i^2}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{Z}_L \mathbf{Y}_L}{\partial \lambda} \mathbf{x}_i \\ 0 \end{bmatrix} \quad (3.36)$$

d) To complete the solution, $\partial \mathbf{E}_1 / \partial \lambda$ and $\partial \mathbf{E}_2 / \partial \lambda$ have to be evaluated. Differentiating $\mathbf{E}_1$ and $\mathbf{E}_2$ produces

$$\frac{\partial \mathbf{E}_1}{\partial \lambda} = \text{diag} \left\{ \frac{-4 \left(\frac{\partial \gamma_i}{\partial \lambda} D + \gamma_i \frac{\partial D}{\partial \lambda} \right) e^{-2\gamma_i D}}{(1 - e^{-2\gamma_i D})^2} \right\} \quad (3.37)$$

$$\frac{\partial \mathbf{E}_2}{\partial \lambda} = \text{diag} \left\{ \frac{2 \left(\frac{\partial \gamma_i}{\partial \lambda} D + \frac{\partial D}{\partial \lambda} \gamma_i \right) (e^{-\gamma_i D} + e^{\gamma_i D})}{(e^{-\gamma_i D} - e^{\gamma_i D})^2} \right\} \quad (3.38)$$

Once all the separate partial derivatives have been calculated and combined, the resultant $\partial \mathbf{A}_k / \partial \lambda$ can be inserted into (3.23) to produce the frequency domain sensitivity $\partial \phi / \partial \lambda$. It should be noted that λ could be an electrical parameter of the

transmission line or a physical parameter such as separation or width of the conductor.

3.2.4 Sensitivity wrt Physical Parameters

Sensitivity with respect to the physical parameters of the transmission line can be determined by extending the algorithm described above. Physical parameters include length of the transmission line, width of the conductors, distance separating conductors, etc.

Determining the sensitivity to physical parameters requires the evaluation of the sensitivity of the electrical parameters of the transmission line to the physical parameter. The electrical parameters can be obtained numerically [21]–[23] or, in some special cases, by using explicit empirical formulae [24]. In either case, using chain rule expansion, the sensitivity of the output can be expressed as

$$\frac{\partial \phi}{\partial \lambda} = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} \left(\frac{\partial \phi}{\partial R_{i,j}} \frac{\partial R_{i,j}}{\partial \lambda} + \frac{\partial \phi}{\partial L_{i,j}} \frac{\partial L_{i,j}}{\partial \lambda} + \frac{\partial \phi}{\partial G_{i,j}} \frac{\partial G_{i,j}}{\partial \lambda} + \frac{\partial \phi}{\partial C_{i,j}} \frac{\partial C_{i,j}}{\partial \lambda} \right) \quad (3.39)$$

When determining sensitivity to the length D of transmission line k , the calculation of $\partial \mathbf{A}_k / \partial D$ can be simplified. From (3.36), the sensitivity of the eigenvalues and eigenvectors to D is zero. Therefore

$$\frac{\partial \mathbf{A}_k}{\partial D} = \begin{bmatrix} \mathbf{S}_i \frac{\partial \mathbf{E}_1}{\partial D} \mathbf{S}_v^{-1} & \mathbf{S}_i \frac{\partial \mathbf{E}_2}{\partial D} \mathbf{S}_v^{-1} \\ \mathbf{S}_i \frac{\partial \mathbf{E}_2}{\partial D} \mathbf{S}_v^{-1} & \mathbf{S}_i \frac{\partial \mathbf{E}_1}{\partial D} \mathbf{S}_v^{-1} \end{bmatrix} \quad (3.40)$$

where

$$\frac{\partial \mathbf{E}_1}{\partial D} = \text{diag} \left\{ \frac{-4\gamma_i e^{-2\gamma_i D}}{(1 - e^{-2\gamma_i D})^2} \right\} \quad (3.41)$$

$$\frac{\partial \mathbf{E}_2}{\partial D} = \text{diag} \left\{ \frac{2\gamma_i (e^{-\gamma_i D} + e^{\gamma_i D})}{(e^{-\gamma_i D} - e^{\gamma_i D})^2} \right\} \quad (3.42)$$

3.3 Time Domain Sensitivity using NILT

The time domain sensitivity of a linear network can be obtained by applying the NILT using the frequency domain sensitivity solution. The sensitivity solution in the frequency domain is obtained using (3.9).

$$\frac{\partial \phi(s)}{\partial \lambda} = (\mathbf{V}_\pi^a)^t \left[\frac{\partial \mathbf{Y}_\pi}{\partial \lambda} \mathbf{V}_\pi - \frac{\partial \mathbf{E}_\pi}{\partial \lambda} - \frac{\partial \mathbf{C}_\pi}{\partial \lambda} \mathbf{v}_\pi(0) \right] \quad (3.43)$$

Using NILT (2.33) the time domain solution can be calculated as follows:

$$\frac{\partial \hat{\phi}(t)}{\partial \lambda} = -(1/t) \sum_{i=1}^{M'} K_i' \frac{\partial \phi(z_i/t)}{\partial \lambda}; \quad t > 0 \quad (3.44)$$

where $\hat{\phi}(t)$ is an approximation of $\phi(t)$.

The sensitivity at each time point is calculated from the weighted sum of M' frequency domain sensitivities evaluated at the complex frequencies $s = z_i/t$.

3.4 Summary of the Computational Steps

The following is a summary of the sensitivity analysis technique proposed in this chapter:

Formulate $\mathbf{Y}_\pi$;

For $j = 1$ to number of time points

 For $i = 1$ to M'

Solve for $\mathbf{V}_\pi$ and $\mathbf{V}_\pi^a$ at $s = z_i/t_j$;

if λ is a transmission line parameter

 Calculate $\partial \mathbf{A}_k / \partial \lambda$ using (3.25);

 Calculate $\partial \phi / \partial \lambda$ using (3.23);

else

 Calculate $\partial \mathbf{Y}_\pi / \partial \lambda$ using (3.17);

 Calculate $\partial \phi / \partial \lambda$ using (3.9);

end;

 Add $\mathbf{V}_\pi$ and $\partial \phi / \partial \lambda$ to the weighted sums of (2.34) and (3.44);

end;

end;

3.5 Examples Comparing the Proposed Method with Perturbation

In this section a number of examples are presented. To show the accuracy of the proposed method, each example has been verified by perturbing the parameter λ . The sensitivity plots have been normalized to represent mV change per 1 percent change in λ .

Example 1 The circuit in Fig. 3.1 contains 35 single lossless transmission lines. The circuit is obtained by cascading the network in Fig. 2.3. The applied voltage is a pulse shown in Fig. 3.2 and the time response of the voltage V_{out} is shown in Fig. 3.3. The time domain sensitivities of the output voltage V_{out} evaluated using the proposed

method are shown in Fig. 3.4–Fig. 3.7 which demonstrate excellent agreement with the results obtained by perturbation using HSPICE. The CPU time comparison is listed in Table 3.1. The comparison was made on a Sun SPARCstation IPC.

Number of Parameters	CPU Seconds		Speedup Ratio
	HSPICE	Proposed Method	
10	382.8	34.3	11.2
All	7516.8	60.3	124.7

Table 3.1: CPU comparison between HSPICE and Proposed Method.

Figure 3.1: Circuit for example 1, obtained by cascading circuit shown in Fig. 2.3.

Figure 3.2: Applied input voltage waveform used for example #1.

Figure 3.3: Transient response of the circuit shown in Fig. 3.1 at node V_{out} .

Figure 3.4: Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.1.

Figure 3.5: Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.1.

Figure 3.6: Sensitivity of voltage V_{out} with respect to the inductance per unit length of transmission line #7 in Fig. 3.1.

Figure 3.7: Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #35 in Fig. 3.1.

Example 2 The circuit shown in Fig. 3.8 contains two coupled lines highly interconnected by a terminal network. The first transmission line has four conductors, is 0.3 meters in length, and is described by the following parameters:

$$\mathbf{L} = \begin{bmatrix} 1.5 & 0.18 & 0 & 0 \\ 0.18 & 1.5 & 0.18 & 0 \\ 0 & 0.18 & 1.5 & 0.18 \\ 0 & 0 & 0.18 & 1.5 \end{bmatrix} \text{ uH/m,}$$

$$\mathbf{C} = \begin{bmatrix} 0.266 & -0.02 & 0 & 0 \\ -0.02 & 0.266 & -0.02 & 0 \\ 0 & -0.02 & 0.266 & -0.02 \\ 0 & 0 & -0.02 & 0.266 \end{bmatrix} \text{ nF/m}$$

and $\mathbf{R} = 0$, $\mathbf{G} = 0$. The second transmission line has two conductors, is 0.4 meters in length and is described by the following parameters:

$$\mathbf{L} = \begin{bmatrix} 750 & 95 \\ 95 & 750 \end{bmatrix} \text{ nH/m,} \quad \mathbf{C} = \begin{bmatrix} 0.133 & -0.009 \\ -0.009 & 0.133 \end{bmatrix} \text{ nF/m}$$

and $\mathbf{R} = 0$, $\mathbf{G} = 0$.

Fig. 3.9 is a plot of the applied voltage. The circuit response of the voltage V_{out} using NILT is shown in Fig. 3.10. The time domain sensitivities of the output voltage V_{out} are compared with the perturbation method in Fig. 3.11–Fig. 3.14.

Figure 3.8: Circuit for example 2, two multiconductor transmission lines with terminal networks.

Figure 3.9: Applied input voltage waveform used for examples #2 and #3.

Figure 3.10: Transient response of the circuit shown in Fig. 3.8 at node V_{out} .

Figure 3.11: Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.8.

Figure 3.12: Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.8.

Figure 3.13: Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #1 in Fig. 3.8.

Figure 3.14: Sensitivity of voltage V_{out} with respect to parameter $C_{1,2}$ of transmission line #1 in Fig. 3.8.

Example 3 The circuit in Fig. 3.15 contains three lossy coupled transmission lines. The lengths of transmission lines 1, 2 and 3 are 0.05m, 0.04m and 0.03m respectively. The transmission line parameters for the three lines are identical:

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{bmatrix} \text{ nH/m}, & \mathbf{C} &= \begin{bmatrix} 62.8 & -4.9 \\ -4.9 & 62.8 \end{bmatrix} \text{ pF/m}, \\ \mathbf{R} &= \begin{bmatrix} 75 & 15 \\ 15 & 75 \end{bmatrix} \Omega/\text{m}, & \mathbf{G} &= \begin{bmatrix} 0.1 & -0.01 \\ -0.01 & 0.1 \end{bmatrix} \text{ S/m}. \end{aligned}$$

The applied voltage for this example is shown in Fig. 3.9 and the circuit response is shown in Fig. 3.16. The time domain sensitivities of the output voltage V_{out} are compared with perturbation in Fig. 3.17–Fig. 3.20.

Figure 3.15: Circuit for examples 3 and 4, interconnect model with lossy coupled transmission lines.

Figure 3.16: Transient response of the circuit shown in Fig. 3.15 at node V_{out} .

Figure 3.17: Sensitivity of voltage V_{out} with respect to C_1 of Fig. 3.15.

Figure 3.18: Sensitivity of voltage V_{out} with respect to R_1 of Fig. 3.15.

Figure 3.19: Sensitivity of voltage V_{out} with respect to parameter $R_{1,1}$ of transmission line #2 in Fig. 3.15.

Figure 3.20: Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #2 in Fig. 3.15.

Example 4 In [24], [25] the formulae for calculating the inductance and capacitance of the transmission line shown in Fig. 3.21 are provided. The transmission line consists of two flat conductors near a ground plane. Assuming a lossless transmission line, the transmission line parameters can be determined with the following equations:

$$C_{1,1} = C_{2,2} \approx \varepsilon_r \varepsilon_o K_{C1} \left(\frac{w}{h} \right) + \frac{\varepsilon_r \varepsilon_o}{4\pi} K_{L1} K_{C1} \left(\frac{w}{h} \right)^2 \ln \left[1 + \left(\frac{2h}{d} \right)^2 \right] \text{ F/m} \quad (3.45)$$

$$C_{1,2} = C_{2,1} \approx -\frac{\varepsilon_r \varepsilon_o}{4\pi} K_{L1} K_{C1} \left(\frac{w}{h} \right)^2 \ln \left[1 + \left(\frac{2h}{d} \right)^2 \right] \text{ F/m} \quad (3.46)$$

$$L_{1,1} = L_{2,2} \approx \frac{\mu_r \mu_o}{K_{L1}} \left(\frac{h}{w} \right) - \frac{\mu_r \mu_o}{4\pi} \ln \left[1 + \left(\frac{2h}{d} \right)^2 \right] \text{ H/m} \quad (3.47)$$

$$L_{1,2} = L_{2,1} \approx \frac{\mu_r \mu_o}{4\pi} \ln \left[1 + \left(\frac{2h}{d} \right)^2 \right] \text{ H/m} \quad (3.48)$$

$$K_{L1} = \frac{120\pi}{Z_{o(\varepsilon_r=1)}} \left(\frac{h}{w} \right) \quad (3.49)$$

$$K_{C1} = \left[\frac{120\pi}{Z_{o(\varepsilon_r=1)}} \sqrt{\frac{\varepsilon_{r(\text{eff})}}{K_{L1} \varepsilon_r}} \left(\frac{h}{w} \right) \right]^2 \quad (3.50)$$

$$Z_{o(\varepsilon_r=1)} \approx 60 \ln \left(\frac{8h}{w} + \frac{w}{4h} \right) \Omega \text{ for } \frac{w}{h} \leq 1 \quad (3.51)$$

$$Z_{o(\varepsilon_r=1)} \approx \frac{120\pi}{\frac{w}{h} + 2.42 - 0.44 \left(\frac{h}{w} \right) + \left(1 - \frac{h}{w} \right)^6} \Omega \text{ for } \frac{w}{h} \geq 1 \quad (3.52)$$

where

ε_r is the relative dielectric constant,

$\varepsilon_o = 10^{-9}/36\pi \text{ F/m}$ is the dielectric constant in vacuum,

$\varepsilon_{r(\text{eff})}$ is a function of h and w described in [25],

μ_r is the relative permeability,

$\mu_o = 4\pi \times 10^{-7}$ H/m is the permeability in vacuum.

Using $w = 0.58$ mm, $h = 1.17$ mm and $d = 2.49$ mm, the following matrices of transmission line parameters are obtained:

$$\mathbf{L} = \begin{bmatrix} 493.11 & 63.04 \\ 63.04 & 493.11 \end{bmatrix} \text{ nH/m}, \quad \mathbf{C} = \begin{bmatrix} 69.62 & -7.09 \\ -7.09 & 69.62 \end{bmatrix} \text{ pF/m}$$

The circuit in Fig. 3.15 is used for this example. The transmission lines have the parameters listed above with the length of the lines being 0.05m, 0.04m and 0.03m for transmission lines 1, 2 and 3, respectively. Fig. 3.23 is a plot of the sensitivity of V_{cross} to the ratio w/d of transmission line # 2 keeping the ratio of h/w constant.

Figure 3.21: Two conductor transmission line near a ground plane.

Figure 3.22: Transient response of the circuit shown in Fig. 3.15 at node V_{cross} .

Figure 3.23: Sensitivity of voltage V_{cross} with respect to the ratio w/d of transmission line #2 in Fig. 3.15.

3.6 Summary

In this chapter the method used for the time domain sensitivity analysis of a linear network was presented. The method encompasses networks which contain lossy coupled transmission lines. The sensitivity of the circuit can be calculated with respect to the electrical parameters of the transmission lines as well as the physical parameters. The advantages of the proposed method using NILT over sensitivity obtained by perturbation are:

1. The method is more reliable than methods based on FFT.
2. The perturbation method requires that $\Delta\lambda \rightarrow 0$ which could lead to roundoff errors.
3. The sensitivity of the network response with respect to many variables could be determined using the proposed method with fewer calculations than the sensitivity with respect to a single variable using perturbation.
4. The sensitivity at any time t can be determined independently.

Chapter 4

Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations

4.1 Introduction

In chapters 2 and 3, the analysis was performed on networks which contained linear elements only. Linear network analysis is acceptable if the nonlinear effects of the components are insignificant. However, with higher signal levels and speeds nonlinearities begin to affect circuit performance to a much greater degree. The problem in the analysis in this case is the fact that the quasi-transverse electromagnetic propagation modes in lossy coupled transmission lines are defined only in the frequency domain whereas the nonlinear terminations can only be analyzed in the time domain. Several methods [13]–[16], [26]–[29] have been proposed for the analysis of networks which contain transmission lines but for the most part the transmission line has been analyzed either by using a series of lumped models in the time domain [15] or by using the frequency domain transformation (FFT) [16], [29].

Recently, a method has been proposed [3],[30]–[31] which is general and is not limited to the analysis of transmission lines. The method is based on a piecewise decomposition technique (PDT) where the network is partitioned into linear and nonlinear subnetworks. The nonlinear ports are excited by piecewise linear time dependent sources. The linear subnetwork network is analyzed in the frequency domain and the transient response is obtained using numerical inversion of Laplace transform. The nonlinear subnetwork is analyzed in the time domain. Using the Newton-Raphson iteration technique, the parameters of the time dependent sources are determined such that the topological and constitutive relations of the subnetworks are satisfied. This method can be easily implemented using the general circuit formulation technique described in chapter 2. In this chapter an overview of the piecewise decomposition technique will be presented. Details of the method can be found in [3].

4.2 Formulation of the Network Equations

Consider the nonlinear network π which contains linear lumped components, arbitrary linear subnetworks and nonlinear components. The modified nodal admittance matrix equation (2.1) can be expanded to include the equations describing the nonlinear components as follows:

$$\mathbf{C}_\pi \frac{d\mathbf{v}_\pi(t)}{dt} + \mathbf{G}_\pi \mathbf{v}_\pi(t) + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{i}_k(t) - \mathcal{F}(\mathbf{v}_\pi(t)) - \mathbf{e}_\pi(t) = 0 \quad (4.1)$$

where $\mathcal{F}(\mathbf{v}_\pi) \in \mathbb{R}^{N_\pi}$ is the vector of functions describing the nonlinear elements of the circuit.

Example The following matrices illustrate the formulation for the network shown in Fig. 4.1.

$$\begin{aligned}
 \mathbf{G}_\pi &= \begin{bmatrix} G_1 & -G_1 & 0 & 0 & 0 & 1 \\ -G_1 & G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \mathbf{C}_\pi &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 \mathbf{v}_\pi &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i_1 \end{bmatrix} & \mathcal{F}(\mathbf{v}_\pi(t)) &= \begin{bmatrix} 0 \\ 0 \\ f_1(v_3, v_4) \\ 0 \\ f_2(v_5) \\ 0 \end{bmatrix} & \mathbf{D}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 \mathbf{i}_1 &= \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_d \end{bmatrix} & \mathbf{e}_\pi &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e \end{bmatrix}
 \end{aligned}$$

The difficulty in finding the solution to the network equation is due to the fact that it implicitly contains a mixture of frequency and time domain representations. To overcome this difficulty the nonzero entries in $\mathcal{F}(\mathbf{v}_\pi)$ are replaced with a set of independent waveforms $\mathbf{x}(t)$. Let

$$\mathcal{F}(\mathbf{v}_\pi(t)) = \mathbf{D}_f \mathbf{x}(t) \quad (4.2)$$

where

$$\mathbf{x}(t) \in \mathfrak{R}^{N_f},$$

$\mathbf{D}_f = [d_{i,j}], d_{i,j} \in \{0,1\}, i \in \{1,2,\dots,N_\pi\}, j \in \{1,2,\dots,N_f\}$ is a selector matrix

and

Figure 4.1: Nonlinear network used to illustrate formulation.

N_f is the number of nonzero entries in $\mathcal{F}(\mathbf{v}_\pi(t))$.

Substituting equation (4.2) into (4.1) reduces the the nonlinear differential equations to a set of linear differential equations of the form:

$$\mathbf{C}_\pi \frac{d\mathbf{v}_\pi(t)}{dt} + \mathbf{G}_\pi \mathbf{v}_\pi(t) + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{i}_k(t) = \mathbf{D}_f \mathbf{x}(t) + \mathbf{e}_\pi(t) \quad (4.3)$$

which can be represented in the complex frequency domain as

$$[s\mathbf{C}_\pi + \mathbf{G}_\pi] \mathbf{V}_\pi(s) + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{i}_k(s) = \mathcal{L}[\mathbf{D}_f \mathbf{x}(t) + \mathbf{e}_\pi(t)] + \mathbf{C}_\pi \mathbf{v}_\pi(0) \quad (4.4)$$

where $\mathcal{L}$ denotes the Laplace transform.

Substituting the relation of the node voltages of the linear subnetwork described by (2.3) into (4.4) yields:

$$\mathbf{Y}_\pi \mathbf{V}_\pi = \mathcal{L}[\mathbf{D}_f \mathbf{x}(t) + \mathbf{e}_\pi(t)] + \mathbf{C}_\pi \mathbf{v}_\pi(0) \quad (4.5)$$

where

$$\mathbf{Y}_\pi = s\mathbf{C}_\pi + \mathbf{G}_\pi + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{A}_k \mathbf{D}_k^t$$

The solution for the time domain node voltages is an equation of the form

$$\mathbf{v}_\pi(t) = \mathcal{L}^{-1} \left[\mathbf{Y}_\pi^{-1} \{ \mathcal{L}[\mathbf{D}_f \mathbf{x}(t) + \mathbf{e}_\pi(t)] + \mathbf{C}_\pi \mathbf{v}_\pi(0) \} \right] \quad (4.6)$$

The nonlinear network has been effectively partitioned into a nonlinear part represented by equation (4.2) and a linear one represented by equation (4.6). This can be interpreted as replacing the nonlinear branches in the network π by a set of time dependent augmenting sources represented by the waveforms $\mathbf{x}(t)$. As a result, the network π can be solved completely in the frequency domain. The waveforms $\mathbf{x}(t)$ must be chosen such that $\mathbf{v}_\pi(t)$ meets the constraints imposed by the equation (4.2). Fig. 4.2 illustrates the replacement of the nonlinear elements in Fig. 4.1 with time dependent sources. The selector matrix $\mathbf{D}_f$ for this example is

$$\mathbf{D}_f^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4.3 Solution of the Network Equations

In order to efficiently solve the network equations a piecewise linear time domain representation of $\mathbf{x}(t)$ is used. Newton-Raphson iterations are then used to satisfy the relations between the linear and nonlinear subnetworks. This method requires only one evaluation of the time domain response of the linear subnetwork.

Figure 4.2: Replacement of the nonlinear elements of circuit in Fig. 4.1 with time dependent sources.

4.3.1 The Piecewise Linear Time Domain Representation

Let $\mathbf{x}(t) = [x_m(t)]$; $m \in \{1, 2, \dots, N_f\}$ be a piecewise linear function described by

$$\mathbf{x}(t) = \sum_{r=1}^L (\mathbf{B}_r + \mathbf{A}_r t) [u(t - t_{r-1}) - u(t - t_r)] \quad (4.7)$$

where

L is the number of linear segments,

$$\mathbf{A}_r = [a_{1r} \ a_{2r} \ a_{3r} \ \dots \ a_{N_f r}]^t,$$

$$\mathbf{B}_r = [b_{1r} \ b_{2r} \ b_{3r} \ \dots \ b_{N_f r}]^t,$$

a_{mr}, b_{mr} are parameters defining the segment r of waveform $x_m(t)$ in the interval

$$t_{r-1} \leq t \leq t_r \text{ such that}$$

$$a_{mr} = \frac{x_m(t_r) - x_m(t_{r-1})}{t_r - t_{r-1}}$$

$$b_{mr} = \frac{x_{m(r-1)}t_r - x_{mr}t_{r-1}}{t_r - t_{r-1}}$$

and $u(t)$ is the unit step function.

Let the interval of analysis $t \in [0, T]$ be divided into L equal length intervals such that $t_r - t_{r-1} = \Delta$ and $L\Delta = T$. (4.7) can be re-written in the form

$$\mathbf{x}(t) = \sum_{r=1}^L \mathbf{u}_h(t - t_{r-1}) (\mathbf{B}_r + \mathbf{A}_r t_{r-1}) + \sum_{r=1}^L \mathbf{u}_g(t - t_{r-1}) \mathbf{A}_r \quad (4.8)$$

where

$$\mathbf{u}_h(t) = \text{diag} \{u(t) - u(t - \Delta)\} \in \mathbb{R}^{N_f \times N_f}$$

$$\mathbf{u}_g(t) = \text{diag} \{t(u(t) - u(t - \Delta))\} \in \mathbb{R}^{N_f \times N_f}$$

Using (4.8) the piecewise linear representation for $\mathbf{x}(t)$ is substituted into (4.6) yielding

$$\mathbf{v}_\pi(t) = \sum_{r=1}^L \mathbf{H}(t - t_{r-1}) (\mathbf{B}_r + \mathbf{A}_r t_{r-1}) + \sum_{r=1}^L \mathbf{G}(t - t_{r-1}) \mathbf{A}_r + \mathbf{F}(t) \quad (4.9)$$

where

$$\mathbf{F}(t) = \mathcal{L}^{-1} \left\{ \mathbf{Y}_\pi^{-1}(s) [\mathcal{L} \{ \mathbf{e}_\pi(t) \} + \mathbf{C}_\pi \mathbf{v}_\pi(0)] \right\} \quad (4.10)$$

$$\mathbf{G}(t) = \mathcal{L}^{-1} \left\{ \mathbf{Y}_\pi^{-1}(s) \mathcal{L} \{ \mathbf{D}_f \mathbf{u}_g(t) \} \right\} \quad (4.11)$$

$$\mathbf{H}(t) = \mathcal{L}^{-1} \left\{ \mathbf{Y}_\pi^{-1}(s) \mathcal{L} \{ \mathbf{D}_f \mathbf{u}_h(t) \} \right\} \quad (4.12)$$

The vector $\mathbf{F}(t)$ represents the output $\mathbf{v}_\pi(t)$ corresponding to the independent sources $\mathbf{e}_\pi(t)$ when the augmenting sources are set to zero. $\mathbf{G}(t)$ and $\mathbf{H}(t)$ are time domain response matrices

$$\mathbf{G}(t) = [g_{i,j}]; \quad i \in \{1, 2, \dots, N_\pi\}, \quad j \in \{1, 2, \dots, N_f\} \quad (4.13)$$

$$g_{i,j} = (\mathbf{v}_\pi(t))_i \begin{cases} x_j(t) = t[u(t) - u(t - \Delta)] \\ x_k(t) = 0; \quad k \neq j, \quad k \in \{1, 2, \dots, N_f\} \\ \mathbf{e}_\pi(t) = 0 \end{cases} \quad (4.14)$$

$$\mathbf{H}(t) = [h_{i,j}]; \quad i \in \{1, 2, \dots, N_\pi\}, \quad j \in \{1, 2, \dots, N_f\} \quad (4.15)$$

$$h_{i,j} = (\mathbf{v}_\pi(t))_i \begin{cases} x_j(t) = u(t) - u(t - \Delta) \\ x_k(t) = 0; \quad k \neq j, \quad k \in \{1, 2, \dots, N_f\} \\ \mathbf{e}_\pi(t) = 0 \end{cases} \quad (4.16)$$

corresponding to the response of the network π when a pulse of the specified form, $x_j(t)$, is applied in place of augmenting source j and all other sources are set to zero.

$\mathbf{F}(t)$, $\mathbf{G}(t)$ and $\mathbf{H}(t)$ can be obtained by analyzing the linear network in the frequency domain and using the inverse Laplace transform to convert the results to the time domain as described in chapter 2. Note that $\mathbf{F}(t)$, $\mathbf{G}(t)$ and $\mathbf{H}(t)$ need to be calculated only once for a given network.

Substituting for $\mathbf{A}_r$ and $\mathbf{B}_r$ in (4.9) yields

$$\mathbf{v}_\pi(t) = \sum_{r=1}^L \mathbf{H}(t - t_{r-1}) \mathbf{x}_{r-1} + \sum_{r=1}^L \mathbf{G}(t - t_{r-1}) \frac{\mathbf{x}_r - \mathbf{x}_{r-1}}{\Delta} + \mathbf{F}(t) \quad (4.17)$$

where $\mathbf{x}_r = \mathbf{x}(t_r)$.

An approximate solution to the network equations is obtained by solving (4.17) with (4.2) at a finite number of points L . The approximate solution may be made arbitrarily accurate by increasing L .

Example Fig. 4.3 – Fig. 4.7 illustrate the application of the piecewise decomposition technique to the circuit in Fig. 4.1. In Fig. 4.3 the nonlinear elements are replaced by open circuits to determine the elements of $\mathbf{F}(t)$. The independent voltage source is

replaced by a short circuit and a current source is added to the appropriate nonlinear port to evaluate $\mathbf{G}(t)$ and $\mathbf{H}(t)$ as shown in Fig. 4.4 – Fig. 4.7.

Figure 4.3: Evaluation of $\mathbf{F}(t)$ for circuit in Fig. 4.1.

Figure 4.4: Evaluation of first column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.

Figure 4.5: Evaluation of second column of $\mathbf{G}(t)$ for circuit in Fig. 4.1.

Figure 4.6: Evaluation of first column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.

Figure 4.7: Evaluation of second column of $\mathbf{H}(t)$ for circuit in Fig. 4.1.

4.3.2 Solution using Newton-Raphson Iterations

To use Newton Raphson iterations the nonlinear equations (4.2) are discretized at L time points resulting in

$$\mathcal{F}(\mathbf{v}_\pi(t_r^*)) - \mathbf{D}_f \mathbf{x}(t_r^*) = 0 \quad (4.18)$$

where $t_{r-1} < t_r^* \leq t_r$ and $r \in \{1, 2, \dots, L\}$. Let $\mathbf{x}_r = \mathbf{x}(t_r)$ and $\mathbf{v}_r^* = \mathbf{v}_\pi(t_r^*)$. At each time point, (4.18) represents a set of N_f nonlinear equations in the N_f unknowns $\mathbf{x}_r = [x_{1r} \ x_{2r} \ \dots \ x_{N_f r}]^t$. Using (4.17), $\mathbf{v}_\pi(t_r^*)$ can be represented as an explicit function of $\mathbf{x}_r$:

$$\mathbf{v}_r^* = \mathbf{R}_r \mathbf{x}_r + \mathbf{S}_r \quad (4.19)$$

where

$$\mathbf{R}_r = \frac{\mathbf{G}(t_r^* - t_{r-1})}{\Delta}$$

$$\mathbf{S}_r = \sum_{k=1}^r \left[\mathbf{H}(t_r^* - t_{k-1}) - \frac{\mathbf{G}(t_r^* - t_{k-1})}{\Delta} \right] \mathbf{x}_{k-1} + \sum_{k=1}^{r-1} \left[\frac{\mathbf{G}(t_r^* - t_{k-1})}{\Delta} \right] \mathbf{x}_k + \mathbf{F}(t_r^*)$$

To determine the solution to (4.18) and (4.19) define the vector

$$\mathbf{z}_r = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{v}_r^* \end{bmatrix} \quad (4.20)$$

Solution of $\mathbf{z}_r$ can be found through Newtons iterations [32] in the form

$$\mathbf{J}_z^{(i)} [\mathbf{z}_r^{(i+1)} - \mathbf{z}_r^{(i)}] = -\psi(\mathbf{z}_r^{(i)}) \quad (4.21)$$

where

$$\psi(\mathbf{z}_r) = \begin{bmatrix} \mathbf{D}_f^t \mathcal{F}(\mathbf{v}_r^*) - \mathbf{x}_r^* \\ \mathbf{v}_r^* - \mathbf{R}_r \mathbf{x}_r - \mathbf{S}_r \end{bmatrix} \quad (4.22)$$

and the Jacobian $\mathbf{J}_z$ is

$$\mathbf{J}_z = \begin{bmatrix} -\frac{t_r^* - t_{r-1}}{\Delta} \mathbf{U} & \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \\ -\mathbf{R}_r & \mathbf{U} \end{bmatrix} \quad (4.23)$$

The iterations are repeated until $\|\mathbf{z}_r^{(i+1)} - \mathbf{z}_r^{(i)}\|$ is less than a predefined tolerance value. The entire time domain solution is calculated by carrying out Newtons iterations until convergence for each time point $r \in \{1, 2, \dots, L\}$.

The equation set can be reduced by eliminating the variable $\mathbf{v}_r^*$ from the iterations such that

$$\mathbf{J}_x^{(i)} [\mathbf{x}_r^{(i+1)} - \mathbf{x}_r^{(i)}] = -[\mathbf{D}_f^t \mathcal{F}(\mathbf{v}_r^{*(i)}) - \mathbf{x}_r^{*(i)}] \quad (4.24)$$

where

$$\mathbf{J}_x = \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \mathbf{R}_r - \frac{t_r^* - t_{r-1}}{\Delta} \mathbf{U} \quad (4.25)$$

and $\mathbf{v}_r^{*(i)}$ can be obtained using (4.19).

4.4 Example of Analysis using the PDT

The circuit in Fig. 4.8 contains a single lossless transmission line. The applied voltage is a pulse shown in Fig. 4.9. The time domain response at the output node V_{out} is compared with the response obtained using HSPICE in Fig. 4.10.

Figure 4.8: Example circuit containing a transmission line and a nonlinear termination.

Figure 4.9: Applied input voltage waveform used for example circuit.

Figure 4.10: Transient response of the circuit shown in Fig. 4.8 at node V_{out} .

Chapter 5

Sensitivity Analysis of Lossy Multiconductor Transmission Lines with Nonlinear Terminations

5.1 Introduction

In this chapter a method for evaluating the time domain sensitivities of a nonlinear network is presented. The method, as in the linear case, works in parallel with the analysis of the time domain response of the circuit. The outline of the nonlinear sensitivity analysis technique can be summarized as follows:

1. Use the piecewise decomposition technique to express the sensitivity of the nonlinear network in terms of the sensitivity of the linear subnetwork.
2. Utilize LU factors of the Jacobian to solve the sensitivity equations for the nonlinear network to reduce computational cost.

3. Use the linear sensitivity analysis technique described in chapter 3 to produce the sensitivity of the linear subnetwork efficiently.
4. Improve the efficiency of the method by evaluating only sensitivities of the linear network which are required.

5.2 Sensitivity Analysis of Nonlinear Networks using the PDT

Using the PDT method, the nonlinear network is decomposed into linear and nonlinear subnetworks. The sensitivity of the nonlinear network is obtained from the derivatives of equations used in the decomposition method presented in the preceding chapter. The resulting equations express the sensitivity of the nonlinear network in terms of the sensitivity of the linear subnetwork. Referring to (4.21)

$$\psi(\mathbf{z}_r, t_r^*) = 0 \quad (5.1)$$

Differentiating (5.1) with respect to a parameter λ , where λ is a parameter of the network:

$$\frac{\partial \psi}{\partial \mathbf{z}_r} \frac{\partial \mathbf{z}_r}{\partial \lambda} + \frac{\partial \psi}{\partial \lambda} = 0 \quad (5.2)$$

Using (4.22):

$$\frac{\partial \psi}{\partial \mathbf{z}_r} = \begin{bmatrix} -\frac{t_r^* - t_{r-1}}{\Delta} \mathbf{U} & \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \\ -\mathbf{R}_r & \mathbf{U} \end{bmatrix} \quad (5.3)$$

Note that this matrix is the Jacobian defined in (4.23) evaluated at the solution point of (5.1).

Using (4.20) and (5.3), (5.2) can be rewritten in the form

$$\begin{bmatrix} -\frac{t_r - t_{r-1}}{\Delta} \mathbf{U} & \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \\ -\mathbf{R}_r & \mathbf{U} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_r}{\partial \lambda} \\ \frac{\partial \mathbf{v}_r^*}{\partial \lambda} \end{bmatrix} = - \begin{bmatrix} \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \lambda} \\ -\frac{\partial \mathbf{R}_r}{\partial \lambda} \mathbf{x}_r - \frac{\partial \mathbf{S}_r}{\partial \lambda} \end{bmatrix} \quad (5.4)$$

Using (4.19)

$$\frac{\partial \mathbf{R}_r}{\partial \lambda} = \frac{1}{\Delta} \frac{\partial \mathbf{G}(t_r^* - t_{r-1})}{\partial \lambda} \quad (5.5)$$

and

$$\begin{aligned} \frac{\partial \mathbf{S}_r}{\partial \lambda} = & \sum_{k=1}^r \left[\frac{\partial \mathbf{H}(t_r^* - t_{r-1})}{\partial \lambda} - \frac{1}{\Delta} \frac{\partial \mathbf{G}(t_r^* - t_{r-1})}{\partial \lambda} \right] \mathbf{x}_{k-1} \\ & + \sum_{k=1}^r \left[\mathbf{H}(t_r^* - t_{k-1}) - \frac{1}{\Delta} \mathbf{G}(t_r^* - t_{k-1}) \right] \frac{\partial \mathbf{x}_{k-1}}{\partial \lambda} \\ & + \sum_{k=1}^{r-1} \frac{1}{\Delta} \frac{\partial \mathbf{G}(t_r^* - t_{r-1})}{\partial \lambda} \mathbf{x}_k \\ & + \sum_{k=1}^{r-1} \frac{1}{\Delta} \mathbf{G}(t_r^* - t_{k-1}) \frac{\partial \mathbf{x}_k}{\partial \lambda} \\ & + \frac{\partial \mathbf{F}(t_r^*)}{\partial \lambda} \end{aligned} \quad (5.6)$$

which are the equations defining the sensitivity of the linear subnetwork.

The set of equations can be reduced by eliminating the variable $\partial \mathbf{v}_r^* / \partial \lambda$. Differentiating (4.19) with respect to λ produces

$$\frac{\partial \mathbf{v}_r^*}{\partial \lambda} = \frac{\partial \mathbf{R}_r}{\partial \lambda} \mathbf{x}_r + \mathbf{R}_r \frac{\partial \mathbf{x}_r}{\partial \lambda} + \frac{\partial \mathbf{S}_r}{\partial \lambda} \quad (5.7)$$

Differentiating (4.18) yields

$$\frac{t_r^* - t_{r-1}}{\Delta} \frac{\partial \mathbf{x}_r}{\partial \lambda} = \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \frac{\partial \mathbf{v}_r^*}{\partial \lambda} - \frac{t_r - t_r^*}{\Delta} \frac{\partial \mathbf{x}_{r-1}}{\partial \lambda} \quad (5.8)$$

and combining with (5.7) results in

$$-\mathbf{J}_x \frac{\partial \mathbf{x}_r}{\partial \lambda} = \mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \left[\frac{\partial \mathbf{R}_r}{\partial \lambda} \mathbf{x}_r + \frac{\partial \mathbf{S}_r}{\partial \lambda} - \frac{t_r - t_r^*}{\Delta} \frac{\partial \mathbf{x}_{r-1}}{\partial \lambda} \right] \quad (5.9)$$

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where $\mathbf{J}_x$ is the Jacobian described by (4.25) evaluated at the solution point of (5.1). Since the LU factors of the Jacobian are already available from the evaluation response of the network, the solution to $\partial \mathbf{x}_r / \partial \lambda$ requires only a forward/backward substitution operation after the elements of the right hand side have been determined.

Once $\partial \mathbf{x}_r / \lambda$ is known, the sensitivity of any output $(\mathbf{v}_r^*)_i$ can be computed as

$$\left(\frac{\partial \mathbf{v}_r^*}{\partial \lambda} \right)_i = \sum_{j=1}^{N_f} \left\{ \left(\frac{\partial \mathbf{R}_r}{\partial \lambda} \right)_{i,j} (\mathbf{x}_r)_j + (\mathbf{R}_r)_{i,j} \left(\frac{\partial \mathbf{x}_r}{\partial \lambda} \right)_j \right\} + \left(\frac{\partial \mathbf{S}_r}{\partial \lambda} \right)_i \quad (5.10)$$

where $i \in \{1, 2, \dots, N_\pi\}$.

(5.9) and (5.10) expresses the sensitivity of the nonlinear network in terms of the time domain sensitivity of the linear subnetwork. The solution to these sensitivities will be described in the next section.

5.3 Solution of the Sensitivity of the Linear Subnetwork

The solution to the sensitivity of the nonlinear network requires the evaluation of $\partial \mathbf{F}(t) / \partial \lambda$, $\partial \mathbf{G}(t) / \partial \lambda$ and $\partial \mathbf{H}(t) / \partial \lambda$ which are the sensitivities of the linear subnetworks defined to solve for $\mathbf{F}(t)$, $\mathbf{G}(t)$ and $\mathbf{H}(t)$ respectively. These are obtained using the linear sensitivity analysis method presented in chapter 3. Following the notations used in chapter 3, the sensitivities are defined by the following equations:

$$\frac{\partial \mathbf{F}(t)}{\partial \lambda} = \left[\frac{\partial f_i}{\partial \lambda} \right] ; i \in \{1, 2, \dots, N_\pi\} \quad (5.11)$$

$$\frac{\partial f_i}{\partial \lambda} = \frac{\partial \phi(t)}{\partial \lambda} \begin{cases} \mathbf{x}(t) = 0 \\ \mathbf{d}^t = \mathbf{e}_i \end{cases} \quad (5.12)$$

$$\frac{\partial \mathbf{G}(t)}{\partial \lambda} = \left[\frac{\partial g_{i,j}}{\partial \lambda} \right] ; i \in \{1, 2, \dots, N_\pi\}, j \in \{1, 2, \dots, N_f\} \quad (5.13)$$

$$\frac{\partial g_{i,j}}{\partial \lambda} = \frac{\partial \phi(t)}{\partial \lambda} \begin{cases} x_j(t) = t[u(t) - u(t - \Delta)] \\ x_k(t) = 0; \quad k \neq j, \quad k \in \{1, 2, \dots, N_f\} \\ \mathbf{e}_\pi(t) = 0 \\ \mathbf{d}^t = \mathbf{e}_i \end{cases} \quad (5.14)$$

$$\frac{\partial \mathbf{H}(t)}{\partial \lambda} = \left[\frac{\partial h_{i,j}}{\partial \lambda} \right]; \quad i \in \{1, 2, \dots, N_\pi\}, \quad j \in \{1, 2, \dots, N_f\} \quad (5.15)$$

$$\frac{\partial h_{i,j}}{\partial \lambda} = \frac{\partial \phi(t)}{\partial \lambda} \begin{cases} x_j(t) = u(t) - u(t - \Delta) \\ x_k(t) = 0; \quad k \neq j, \quad k \in \{1, 2, \dots, N_f\} \\ \mathbf{e}_\pi(t) = 0 \\ \mathbf{d}^t = \mathbf{e}_i \end{cases} \quad (5.16)$$

where $\mathbf{e}_i$ is a vector containing the value one in location i and zero elsewhere.

Note that as in the analysis of the time domain response, the sensitivities of $\mathbf{F}(t)$, $\mathbf{G}(t)$ and $\mathbf{H}(t)$ need to be calculated only once for a given network. When the solution to the sensitivities of the linear subnetwork are determined, the sensitivity of the nonlinear network can be calculated using (5.9) and (5.10).

5.4 Improving the Efficiency of the Method

The efficiency of the sensitivity analysis method for nonlinear networks described so far can be improved by reducing the number of sensitivities of the linear subnetwork that are computed. This is important because each sensitivity requires a forward/backward substitution operation which can be computationally expensive if there is a large number of variables in the system. The reduction can be achieved because the number of sensitivities of the linear subnetwork that is actually required

is in general less than N_π . Referring to (5.9), the matrix $[\mathbf{D}_f^t \partial \mathcal{F} / \partial \mathbf{v}_r^*]$ selects the elements of $\partial \mathbf{R}_r / \partial \lambda$ and $\partial \mathbf{S}_r / \partial \lambda$ which are required to evaluate $\partial \mathbf{x}_r / \partial \lambda$.

Let $\mathcal{F}(\mathbf{v}_r^*)$ be dependent on the set of n variables

$$\mathbf{v}_z \subset \mathbf{v}_r^* ; \quad \mathbf{v}_z = [(\mathbf{v}_r^*)_{z_1} \ (\mathbf{v}_r^*)_{z_2} \ \dots \ (\mathbf{v}_r^*)_{z_n}]^t \quad (5.17)$$

where $\{z_1, z_2, \dots, z_n\}$ are indices into the vector $\mathbf{v}_r^*$.

$[\mathbf{D}_f^t \partial \mathcal{F} / \partial \mathbf{v}_r^*]$ is an $N_f \times N_\pi$ matrix with nonzero elements only in the columns defined by $\{z_1, z_2, \dots, z_n\}$

$$\text{i.e.} \quad \left(\mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} \right)_{i,j} = 0 \quad (5.18)$$

where $i \in \{1, 2, \dots, N_f\}$ and $j \notin \{z_1, z_2, \dots, z_n\}$.

Example The nonzero elements of $[\mathbf{D}_f^t \partial \mathcal{F} / \partial \mathbf{v}_r^*]$ for the circuit shown in Fig. 4.1 are

$$\mathbf{D}_f^t \frac{\partial \mathcal{F}}{\partial \mathbf{v}_r^*} = \begin{bmatrix} 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 \end{bmatrix} \quad (5.19)$$

where x denotes a nonzero element. The number of dependent variables in this example is $n = 3$ and the set of indices $\{z_1, z_2, z_3\}$ is $\{3, 4, 5\}$.

Hence only the elements $(\partial \mathbf{R}_r / \partial \lambda)_{i,j}$ and $(\partial \mathbf{S}_r / \partial \lambda)_i$ where $i \in \{z_1, z_2, \dots, z_n\}$, $j \in \{1, 2, \dots, N_f\}$ are needed to compute (5.9).

Referring to (5.5) and (5.6) the elements $\partial f_i / \partial \lambda$, $\partial g_{i,j} / \partial \lambda$ and $\partial h_{i,j} / \partial \lambda$ where $i \in \{z_1, z_2, \dots, z_n\}$, $j \in \{1, 2, \dots, N_f\}$ are the only sensitivities of the linear subnetworks defined by $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that are required to evaluate the sensitivity of the nonlinear network. Consequently the number of variables for which the sensitivity is needed is reduced from N_π to n .

The sensitivity of any of the dependent variables can be determined using (5.10):

$$\left(\frac{\partial \mathbf{v}_r^*}{\partial \lambda}\right)_i = \sum_{j=1}^{N_f} \left\{ \left(\frac{\partial \mathbf{R}_r}{\partial \lambda}\right)_{i,j} (\mathbf{x}_r)_j + (\mathbf{R}_r)_{i,j} \left(\frac{\partial \mathbf{x}_r}{\partial \lambda}\right)_j \right\} + \left(\frac{\partial \mathbf{S}_r}{\partial \lambda}\right)_i \quad (5.20)$$

where $i \in \{z_1, z_2, \dots, z_n\}$. The sensitivity of any other output in the network can be computed by expanding the set of dependent variables to include the output of interest.

5.5 Summary of Computational Steps

The following is a summary of the sensitivity analysis technique proposed in this chapter:

Formulate $\mathbf{Y}_\pi$;

Solve for $\mathbf{F}(t)$, $\mathbf{G}(t)$ and $\mathbf{H}(t)$ using (4.10)–(4.12) and

solve for $\partial \mathbf{F}(t)/\partial \lambda$, $\partial \mathbf{G}(t)/\partial \lambda$ and $\partial \mathbf{H}(t)/\partial \lambda$ using (5.11)–(5.16);

For $r = 1$ to number of time points

Calculate $\mathbf{R}_r$ and $\mathbf{S}_r$ using (4.19);

Calculate $\partial \mathbf{R}_r/\partial \lambda$ and $\partial \mathbf{S}_r/\partial \lambda$ using (5.5) and (5.6);

Set $i = 0$, $\mathbf{x}_r^{(i)} = \mathbf{x}_{r-1}$;

Repeat

Calculate $\mathbf{v}_r^{*(i)}$ using (4.19);

Calculate $\mathcal{F}(\mathbf{v}_r^{*(i)})$ and $\partial \mathcal{F}/\partial \mathbf{v}_r^{*(i)}$;

Calculate $\mathbf{J}_r^{(i)}$ using (4.25);

Calculate $\mathbf{x}_r^{(i+1)}$ using (4.24);

```

    Increment  $i$ ;
    Until  $\| \mathbf{x}_r^{(i)} - \mathbf{x}_r^{(i-1)} \| \leq \epsilon$ ;
    Calculate  $\partial \mathbf{x}_r / \partial \lambda$  using (5.9);
    Calculate  $\partial \mathbf{v}_r^* / \partial \lambda$  using (5.10);
end;
```

5.6 Examples Comparing the Proposed Method with Perturbation

In this section a number of examples are presented. To show the accuracy of the proposed method, each example has been verified by perturbing the parameter λ . The sensitivity plots have been normalized to represent mV change per 1 percent change in λ .

Example 1 The circuit in Fig. 5.1 contains single lossless transmission lines. The nonlinear elements are defined by the following equations: $I_1 = 0.001V_1^3$, $I_2 = V_2^3/500 + V_2/750$ and $I_{out} = 0.001V_{out}^3$. The applied voltage for this example is the pulse shown in Fig. 5.2. The time domain response at the output node V_{out} is compared with the response obtained using HSPICE in Fig. 5.3. The time domain sensitivities of the output voltage V_{out} evaluated using the proposed method are shown in Fig. 5.4–Fig. 5.6 which demonstrate excellent agreement with the results obtained by perturbation using HSPICE. The approximate CPU time measurements comparing the perturbation method using HSPICE and PDT with the proposed method are listed in Table 5.1. The comparisons were made on a Sun SPARCstation IPC. The

CPU time advantage of the proposed method increases dramatically as the size of the circuit and the number of parameters increase.

Number of Parameters	CPU Seconds		
	HSPICE	PDT	Proposed Method
5	62.8	35.4	9.5
All	554.4	312.7	39.1

Table 5.1: CPU comparison between perturbation and the Proposed Method.

Figure 5.1: Circuit for example 1, transmission lines with nonlinear terminations.

Figure 5.2: Applied input voltage waveform used for examples #1 and #2.

Figure 5.3: Transient response of the circuit shown in Fig. 5.1 at node V_{out} .

Figure 5.4: Sensitivity of voltage V_{out} with respect to L_1 of Fig. 5.1.

Figure 5.5: Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #7 in Fig. 5.1.

Figure 5.6: Sensitivity of voltage V_{out} with respect to the capacitance per unit length of transmission line #6 in Fig. 5.1.

Example 2 The circuit in Fig. 5.7 contains three lossy coupled transmission lines. The lengths of transmission lines 1, 2 and 3 are 0.05m, 0.04m and 0.03m respectively. The transmission line parameters for the three lines are identical:

$$\mathbf{L} = \begin{bmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{bmatrix} \text{ nH/m}, \quad \mathbf{C} = \begin{bmatrix} 62.8 & -4.9 \\ -4.9 & 62.8 \end{bmatrix} \text{ pF/m},$$

$$\mathbf{R} = \begin{bmatrix} 75 & 15 \\ 15 & 75 \end{bmatrix} \Omega/\text{m}, \quad \mathbf{G} = \begin{bmatrix} 0.1 & -0.01 \\ -0.01 & 0.1 \end{bmatrix} \text{ S/m}.$$

The nonlinear element is defined by the equation: $I_{\text{out}} = V_{\text{out}}^3$. The applied input voltage is the pulse shown in Fig. 5.2 and the circuit response at node V_{out} is shown in Fig. 5.8. The time domain sensitivities of the output voltage V_{out} are compared with perturbation in Fig. 5.9–Fig. 5.11.

Figure 5.7: Circuit for example 2, interconnect model with lossy coupled transmission lines and nonlinear terminations.

Figure 5.8: Transient response of the circuit shown in Fig. 5.7 at node V_{out} .

Figure 5.9: Sensitivity of voltage V_{out} with respect to R_1 of Fig. 5.7.

Figure 5.10: Sensitivity of voltage V_{out} with respect to parameter $G_{1,1}$ of transmission line #2 in Fig. 5.7.

Figure 5.11: Sensitivity of voltage V_{out} with respect to parameter $L_{1,1}$ of transmission line #1 in Fig. 5.7.

Example 3 The circuit in Fig. 5.12 contains three lossy transmission lines. Transmission line 1 is made up of 2 coupled conductors, transmission line 2 is a single conductor and transmission line 3 consists of 4 coupled conductors. The lengths of the transmission lines are 0.5m, 0.2m and 0.3m respectively. The parameters of the single conductor transmission line are $L = 600\text{nH/m}$, $C = 1\text{nF/m}$, $R = 1\Omega/\text{m}$ and $G = 5\text{mS/m}$. The parameters for the transmission line 2 are:

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} 600 & 50 \\ 50 & 600 \end{bmatrix} \text{nH/m}, & \mathbf{C} &= \begin{bmatrix} 1.2 & -0.11 \\ -0.11 & 1.2 \end{bmatrix} \text{nF/m}, \\ \mathbf{R} &= \begin{bmatrix} 2.25 & 0.225 \\ 0.225 & 2.25 \end{bmatrix} \Omega/\text{m}, & \mathbf{G} &= \begin{bmatrix} 7.5 & 0 \\ 0 & 7.5 \end{bmatrix} \text{mS/m}. \end{aligned}$$

The parameters of transmission line 3 are:

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} 1 & 0.11 & 0.03 & 0 \\ 0.11 & 1 & 0.11 & 0.03 \\ 0.03 & 0.11 & 1 & 0.11 \\ 0 & 0.03 & 0.11 & 1 \end{bmatrix} \mu\text{H/m}, \\ \mathbf{C} &= \begin{bmatrix} 1.5 & -0.17 & -0.03 & 0 \\ -0.17 & 1.5 & -0.17 & -0.03 \\ -0.03 & -0.17 & 1.5 & -0.17 \\ 0 & -0.03 & -0.17 & 1.5 \end{bmatrix} \text{nF/m}, \\ \mathbf{R} &= \begin{bmatrix} 3.5 & 0.35 & 0.035 & 0 \\ 0.35 & 3.5 & 0.35 & 0.035 \\ 0.035 & 0.35 & 3.5 & 0.35 \\ 0 & 0.035 & 0.35 & 3.5 \end{bmatrix} \Omega/\text{m}, \\ \mathbf{G} &= \begin{bmatrix} 10 & 1 & 0.1 & 0 \\ 1 & 10 & 1 & 0.1 \\ 0.1 & 1 & 10 & 1 \\ 0 & 0.1 & 1 & 10 \end{bmatrix} \text{mS/m}. \end{aligned}$$

The nonlinear elements are all described by the equation: $I = V^3$. The applied voltage waveform is shown in Fig. 5.13 and the circuit response at node V_3 is shown in Fig. 5.14. The time domain sensitivities of the output voltage V_3 are compared with perturbation in Fig. 5.15–Fig. 5.17.

Figure 5.12: Circuit for example 3, interconnect network with nonlinear terminations.

Figure 5.13: Applied input voltage waveform used for example #3.

Figure 5.14: Transient response of the circuit shown in Fig. 5.12 at node V_3 .

Figure 5.15: Sensitivity of voltage V_3 with respect to R_1 of Fig. 5.12.

Figure 5.16: Sensitivity of voltage V_3 with respect to parameter $C_{3,3}$ of transmission line #3 in Fig. 5.12.

Figure 5.17: Sensitivity of voltage V_3 with respect to parameter $L_{3,2}$ of transmission line #3 in Fig. 5.12.

5.7 Summary

In this chapter the method used for the time domain sensitivity analysis of a nonlinear network was presented. The method encompasses networks which contain lossy coupled transmission lines or in the general case, networks which contain frequency dependent components. The sensitivity of the linear subnetworks containing the frequency dependent components is evaluated using the method described in chapter 3 for linear networks. The efficiency of the method can be improved by evaluating only the sensitivities required by the dependent variables of the nonlinear components in the network. The sensitivity of the nonlinear network can be determined after Newtons iterations are performed to satisfy the relations between the linear and nonlinear subnetworks.

Chapter 6

Optimization of Time Domain Responses of Multiconductor Transmission Lines

6.1 Introduction

In high speed digital circuits, transmission line effects such as delay, reflection and crosstalk limit the performance of the overall system. In order to reduce these limiting factors, a circuit designer can use various simulation techniques to obtain circuit responses and from these responses adjust circuit parameters to meet specific requirements. This method can be very time consuming particularly for large networks and the results rely on the circuit designer's experience through trial and error. Instead, an automated design technique is required to optimize the time responses to meet specifications given a set of design constraints.

In this chapter, simultaneous minimization of delay, crosstalk and reflections is described as a minimax optimization problem. Examples of time domain optimization of a network containing multiconductor transmission lines are presented. These

examples link the time domain analysis method and sensitivity analysis method of linear transmission line networks presented in the previous chapters with an efficient two-stage minimax optimization package. Transmission line effects such as crosstalk, delay and reflection are minimized at all the vital connection ports. In addition, optimization is carried out with respect to the physical parameters of the transmission lines.

6.2 Formulation of Error Functions

In this section a set of error functions are formulated for optimization [33]–[34]. Consider a linear network which contains a multiconductor transmission line. Let one of the conductors be excited by a trapezoidal signal. Define Λ as the vector containing the circuit design parameters and $\Phi(\Lambda, t)$, as the vector containing the system output responses.

If $\phi_i(\Lambda, t)$ is on a conductor not connected to the excitation then $\phi_i(\Lambda, t)$ is due to crosstalk. By imposing an upper specification S_c on the magnitude of the response, the weighted error function can be written as

$$e = w_c(t)(\phi_i(\Lambda, t) - S_c(t)) ; 0 < t < \infty \quad (6.1)$$

and

$$e = -w_c(t)(\phi_i(\Lambda, t) + S_c(t)) ; 0 < t < \infty \quad (6.2)$$

If the maximum allowable time for signal to reach a threshold value S_d is τ then the weighted error function for waveform propagation delay with an upper specification

τ can then be written as

$$e = -w_d(\phi_i(\Lambda, \tau) - S_d) \quad (6.3)$$

Due to transmission line effects and terminating values, a signal response may have a rise/fall time which does not meet design specifications. If the signal rise time is required to be t_{rs} during which the signal should rise from below a threshold value of S_l to above S_h then the weighted error function can be written as

$$e = w_{rs}(\phi_i(\Lambda, \tau - \alpha t_{rs}) - S_l) \quad (6.4)$$

and

$$e = -w_{rs}(\phi_i(\Lambda, \tau + \beta t_{rs}) - S_h) \quad (6.5)$$

where $\alpha + \beta = 1$; $\alpha > 0, \beta > 0$.

If the duration of the trapezoid waveform is T then the output response $\phi_i(\Lambda, t)$ should ideally decay to zero after a time $\tau + T$. However due to reflections the signal will continue to exist for an extended period of time. The reflected signal can be minimized by imposing an upper specification $S_r(t)$ on the magnitude of the signal with the weighted error function being written as

$$e = w_r(t)(\phi_i(\Lambda, t) - S_r(t)); \quad t > \tau + T \quad (6.6)$$

and

$$e = -w_r(t)(\phi_i(\Lambda, t) + S_r(t)); \quad t > \tau + T \quad (6.7)$$

6.3 Formulation of Optimization

A set of sample points in the time range of interest is selected. The error functions (6.1)–(6.7) are formulated for all the responses of interest at the required ports and at the sample time points. A vector $\mathbf{e}(\Lambda)$ containing all the weighted error functions is then assembled. The overall objective function for design optimization is

$$u(\Lambda) = \max\{e_1(\Lambda), e_2(\Lambda), \dots, e_m(\Lambda)\} \quad (6.8)$$

where m is the number of error functions.

The overall optimization problem can now be stated as

$$\underset{\Lambda}{\text{minimize}} \quad u(\Lambda) \quad (6.9)$$

subject to

$$\mathbf{g}(\Lambda) \geq 0 \quad (6.10)$$

and

$$\mathbf{h}(\Lambda) = 0 \quad (6.11)$$

where the functions $\mathbf{g}(\Lambda)$ and $\mathbf{h}(\Lambda)$ represent the constraints on the design parameters Λ .

This formulation allows the application of a powerful two-stage minimax optimization algorithm [35]. This optimization algorithm combined with the exact sensitivities produced using the method described in Chapters 3 and 5 provides a powerful and efficient tool for automated design of large transmission line networks. Since the sample time points can be arbitrarily chosen by the circuit designer, the use of the NILT algorithm is particularly well suited to this application. This is due to the fact that the response and sensitivity at each time point can be independently computed.

6.4 Examples of Performance Optimization

In this section, time domain optimization is performed on a large transmission line network. The network represents a typical interconnection configuration and the goal is to improve the response at the output ports. The design parameters include the termination elements as well as the physical dimensions of the transmission lines. A set of linear constraints are imposed on the design parameters.

Example 1 The circuit in Fig. 6.1 contains 13 4-conductor transmission lines representing a 4-bit bus with single conductor transmission lines connecting the terminating elements to the bus. The terminating elements consist of resistors, inductors and capacitors. The output ports of interest are the nodes V_c and V_d . In order to obtain the electrical parameters from the physical parameters, the formulae provided by Walker [24] are used.

The applied voltage is the pulse shown in Fig. 6.2. The time response of the output node V_d is shown in Fig. 6.3. The delay time for the output V_d is approximately 4.1ns based on a 3V threshold criterion. The crosstalk voltage V_c is shown in Fig. 6.4 with the peak voltage being 0.36V.

The specifications for optimization include a 4.0ns upper specification on the delay time of V_d and an upper specification on the magnitude of the crosstalk V_c of 0.2V. The weighting factors were chosen to be $w_c = 1$ and $w_d = 1$. A specification of 0.2V is also placed on V_d between the time range of 13.5ns and 14.5ns to reduce the reflections in the waveform with the weighting factor $w_r = 1$. The total number of error functions is $m = 15$.

The vector of optimization variables $\mathbf{A}$ consists of the lengths of the 13 4-conductor transmission lines, the distance between the conductors, the terminating resistances and the terminating capacitances as listed in Table 6.1.

A constraint was placed on the distance separating the conductors such that $\{d_1 = d_{1,1} = d_{2,1} \dots = d_{13,1}\}, \{d_2 = d_{1,2} = d_{2,2} \dots = d_{13,2}\}$ and $\{d_3 = d_{1,3} = d_{2,3} \dots = d_{13,3}\}$. The minimum distance separating the conductors was set at 0.8mm while the maximum total separation was set at 3mm. The total length of the 13 transmission lines was fixed at 40cm.

The initial value of the overall objective function defined in (6.9) was 0.632. After 7 iterations of optimization, the value was reduced to -0.182. The negative function value indicates that all specifications were satisfied. The parameters before and after optimization are listed in Table 6.1. The output response V_d and the crosstalk response V_c after optimization are compared with the responses before optimization in Fig. 6.5 and Fig 6.6 respectively. The propagation delay was reduced to 3.3ns and the crosstalk voltage was reduced to less than 0.01V. In addition, the reflections in the output response was also significantly reduced.

Variable	Description	Before Optimization	After Optimization
D1	length of transmission line #1	0.025m	0.02915m
D2	length of transmission line #2	0.035m	0.03915m
D3	length of transmission line #3	0.025m	0.02914m
D4	length of transmission line #4	0.04m	0.04454m
D5	length of transmission line #5	0.04m	0.03896m
D6	length of transmission line #6	0.025m	0.01962m
D7	length of transmission line #7	0.035m	0.02945m
D8	length of transmission line #8	0.035m	0.02295m
D9	length of transmission line #9	0.025m	0.01295m
D10	length of transmission line #10	0.04m	0.03592m
D11	length of transmission line #11	0.025m	0.02877m
D12	length of transmission line #12	0.025m	0.03394m
D13	length of transmission line #13	0.025m	0.03545m
d1	distance between conductor 1 & 2	1.0mm	0.8mm
d2	distance between conductor 2 & 3	1.0mm	0.8mm
d3	distance between conductor 3 & 4	1.0mm	3.4mm
R4	resistor	25 Ω	16.12 Ω
R7	resistor	200 Ω	198.76 Ω
R10	resistor	200 Ω	106.86 Ω
R20	resistor	100 Ω	88.16 Ω
C2	capacitor	5pF	0.95pF
C5	capacitor	10pF	5.22pF

Table 6.1: Parameter values for example #1.

Figure 6.2: Applied input voltage waveform used for examples #1 and #2.

Figure 6.3: Transient response of the circuit shown in Fig. 6.1 at node V_d before optimization.

Figure 6.4: Crosstalk response of the circuit shown in Fig. 6.1 at node V_c before optimization.

Figure 6.5: Transient response of the circuit shown in Fig. 6.1 at node V_d after optimization.

Figure 6.6: Crosstalk response of the circuit shown in Fig. 6.1 at node V_c after optimization.

Example 2 The circuit in Fig. 6.7 contains 2 excitation sources. The output ports of interest are the nodes V_a , V_b , V_c and V_d . The applied voltage of both sources is the pulse shown in Fig. 6.2. The time response of the output nodes V_b and V_d are shown in Fig. 6.8 and Fig 6.9. The delay time for the outputs V_b and V_d is approximately 4.1ns and 3.1ns respectively based on a 3V threshold criterion. The crosstalk voltages V_a and V_c are shown in Fig. 6.10 and Fig 6.11 with the peak voltage being 0.74V and 0.59V respectively.

The specification for optimization is a 2ns upper specification on the delay time of V_b and a 4ns upper specification on the delay time of V_d . Upper specifications were placed on the magnitude of the crosstalk voltages V_a and V_c at 0.2V. The weighting factors used were $w_c = 1$ and $w_d = 1$. A specification of 0.2V is also placed on V_d between the time range of 12.5ns and 14.5ns to reduce the reflections in the waveform. The weighting factor $w_r = 1$. The total number of error functions is $m = 61$.

The vector of optimization variables Λ consists of the lengths of the 13 4-conductor transmission lines, the distance between the conductors, the terminating resistances and the terminating capacitances as listed in Table 6.2.

The distance separating the conductors for the 13 4-conductor transmission lines were constrained to be identical between each of the transmission lines. The minimum distance separating the conductors was set at 0.8mm while the maximum total separation between conductors 1 and 4 was set at 5mm. The total length of the 13 transmission lines was fixed at 40cm.

The initial value of the overall objective function defined in (6.9) was 1.713. After 27 iterations of optimization, the value was reduced to -0.143 indicating that all

the specifications were satisfied. The parameters before and after optimization are listed in Table 6.2. The output responses V_b and V_d are shown in Fig. 6.12 and Fig. 6.13. The crosstalk responses V_a and V_c are shown in Fig. 6.14 and Fig 6.15. The propagation delay of V_b and V_d was reduced to 2ns and 3.5ns respectively. The crosstalk voltages V_a and V_c were reduced to 0.07V and 0.16V respectively. In addition, the reflections in the output response V_d was significantly reduced and the rise/fall times of V_b was also improved.

Variable	Description	Before Optimization	After Optimization
D1	length of transmission line #1	0.025m	0.05828m
D2	length of transmission line #2	0.035m	0.03429m
D3	length of transmission line #3	0.025m	0.03392m
D4	length of transmission line #4	0.04m	0.01m
D5	length of transmission line #5	0.04m	0.01m
D6	length of transmission line #6	0.025m	0.01m
D7	length of transmission line #7	0.035m	0.03863m
D8	length of transmission line #8	0.035m	0.01764m
D9	length of transmission line #9	0.025m	0.01227m
D10	length of transmission line #10	0.04m	0.07m
D11	length of transmission line #11	0.025m	0.01681m
D12	length of transmission line #12	0.025m	0.03632m
D13	length of transmission line #13	0.025m	0.05184m
d1	distance between conductor 1 & 2	1.0mm	3.4mm
d2	distance between conductor 2 & 3	1.0mm	0.8mm
d3	distance between conductor 3 & 4	1.0mm	0.8mm
R2	resistor	25 Ω	28.09 Ω
R4	resistor	25 Ω	18.11 Ω
R7	resistor	200 Ω	119.86 Ω
R10	resistor	200 Ω	21.18 Ω
R14	resistor	200 Ω	311.14 Ω
R18	resistor	200 Ω	80.70 Ω
R20	resistor	100 Ω	75.32 Ω
C1	capacitor	5pF	0.21pF
C2	capacitor	5pF	0.19pF
C3	capacitor	5pF	0.35pF
C5	capacitor	10pF	8.91pF

Table 6.2: Parameter values for example #2.

Figure 6.7: Circuit for example 2, 13 4-conductor transmission lines and 2 sources.

Figure 6.8: Transient response of the circuit shown in Fig. 6.7 at node V_b before optimization.

Figure 6.9: Transient response of the circuit shown in Fig. 6.7 at node V_d before optimization.

Figure 6.10: Crosstalk response of the circuit shown in Fig. 6.7 at node V_a before optimization.

Figure 6.11: Crosstalk response of the circuit shown in Fig. 6.7 at node V_c before optimization.

Figure 6.12: Transient response of the circuit shown in Fig. 6.7 at node V_b after optimization.

Figure 6.13: Transient response of the circuit shown in Fig. 6.7 at node V_d after optimization.

Figure 6.14: Crosstalk response of the circuit shown in Fig. 6.7 at node V_a after optimization.

Figure 6.15: Crosstalk response of the circuit shown in Fig. 6.7 at node V_c after optimization.

From the results shown in Table 6.1 and Table 6.2 it can be seen that the optimizer tends to reduce capacitances, maximize conductor spacing and reduce transmission line lengths. These are logical changes a circuit designer can make in a simple design such as the one presented in example 1. The problem becomes much more difficult when there are conflicting specifications such as in the design presented in example 2. In this design there were 2 different crosstalks which needed to be minimized. As well, the delay time for V_b and V_c had to be minimized while maintaining the total length length constraint. These conflicting specifications are best resolved using optimization techniques which will provide an optimal solution. As the number of specifications increase and as the design size and complexity increases the more difficult it is to manually adjust the circuit parameters.

6.5 Summary

In this chapter the method developed for time domain sensitivity analysis of multiconductor transmission lines was used in conjunction with a minimax optimization package. The result is an automated design tool useful for reducing transmission line effects such as crosstalk, delay and reflections at the vital connection ports. Examples of time domain performance optimization on networks containing multiconductor transmission lines with linear terminations was given. Design parameters included the lumped terminations as well as the physical parameters of the distributed elements. The obtained results show the effectiveness of the developed technique. Extension to the nonlinear case is straightforward.

Chapter 7

Conclusions and Further Research

7.1 Summary

This research was motivated by the need for an automated design tool to solve the interconnect problems in high-speed VLSI systems where the interconnects are modeled as lossy coupled transmission lines. The method proposed in this thesis combined with the Piecewise Decomposition Technique provides an efficient and accurate method to determine the sensitivities of high-speed circuits. The sensitivities can then be fed to an optimization package to minimize transmission line effects such as crosstalk, delay and reflection.

The proposed method can evaluate the sensitivities of networks which contain general RLC lumped elements, lossy coupled transmission lines and nonlinear terminations. In addition, the sensitivity can be determined with respect to the electrical and physical parameters of the transmission lines. A unified formulation based on the modified nodal admittance matrix is used to describe the network equations. The

time domain solution of the network response and sensitivities is obtained by using numerical inversion of Laplace transform.

The sensitivities with respect to all network parameters can be determined with less additional cost than an additional evaluation of the time domain response. The numerical examples in the thesis demonstrates excellent agreement with sensitivities produced using the perturbation technique while at the same time providing significant CPU savings. When the proposed method is combined with an optimization package the result is an automated design tool which allows a circuit designer to efficiently solve the interconnection problem.

7.2 Suggestions for Further Research

1. Implementation of optimization of nonlinear circuits. This can be achieved by combining the method for computing the time domain sensitivities of nonlinear networks described in Chapter 5 with the minimax optimization package described in Chapter 6.
2. Computation of sensitivities of networks containing nonuniform transmission lines.
3. Use of the new methods for numerical inversion of Laplace transform [36]–[37] to improve the accuracy and efficiency of the sensitivity analysis methods described in Chapters 3 and 5.

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