Graph Connectivity Augmentation Problems

Master's Thesis

by
Romy Varga, B. Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Science

Department of Mathematics
Carleton University
Ottawa, Ontario

January 19, 1996

ISBN 0-612-13873-9

Name Romy Varga

Dissertation Abstracts International is arranged by broad, general subject categories. Please select the one subject which most nearly describes the content of your dissertation. Enter the corresponding four-digit code in the spaces provided.

MATHEMATICS — SUBJECT TERM

~~0280~~ — SUBJECT CODE 0405

U·M·I

Subject Categories

THE HUMANITIES AND SOCIAL SCIENCES

COMMUNICATIONS AND THE ARTS
Architecture 0729
Art History 0377
Cinema 0900
Dance 0378
Fine Arts 0357
Information Science 0723
Journalism 0391
Library Science 0399
Mass Communications 0708
Music 0413
Speech Communication 0459
Theater 0465

EDUCATION
General 0515
Administration 0514
Adult and Continuing 0516
Agricultural 0517
Art 0273
Bilingual and Multicultural 0282
Business 0688
Community College 0275
Curriculum and Instruction 0727
Early Childhood 0518
Elementary 0524
Finance 0277
Guidance and Counseling 0519
Health 0680
Higher 0745
History of 0520
Home Economics 0278
Industrial 0521
Language and Literature 0279
Mathematics ..... ~~0280~~
Music 0522
Philosophy of 0998
Physical 0523
Psychology 0525
Reading 0535
Religious 0527
Sciences 0714
Secondary 0533
Social Sciences 0534
Sociology of 0370
Special 0529
Teacher Training 0530
Technology 0710
Tests and Measurements 0288
Vocational 0747

LANGUAGE, LITERATURE AND LINGUISTICS
Language
- General 0679
- Ancient 0289
- Linguistics 0290
- Modern 0291

Literature
- General 0401
- Classical 0294
- Comparative 0295
- Medieval 0297
- Modern 0298
- African 0316
- American 0591
- Asian 0305
- Canadian (English) 0352
- Canadian (French) 0355
- English 0593
- Germanic 0311
- Latin American 0312
- Middle Eastern 0315
- Romance 0313
- Slavic and East European 0314

PHILOSOPHY, RELIGION AND THEOLOGY
Philosophy 0422
Religion
- General 0318
- Biblical Studies 0321
- Clergy 0319
- History of 0320
- Philosophy of 0322

Theology 0469

SOCIAL SCIENCES
American Studies 0323
Anthropology
- Archaeology 0324
- Cultural 0326
- Physical 0327

Business Administration
- General 0310
- Accounting 0272
- Banking 0770
- Management 0454
- Marketing 0338

Canadian Studies 0385
Economics
- General 0501
- Agricultural 0503
- Commerce-Business 0505
- Finance 0508
- History 0509
- Labor 0510
- Theory 0511

Folklore 0358
Geography 0366
Gerontology 0351
History
- General 0578
- Ancient 0579
- Medieval 0581
- Modern 0582
- Black 0328
- African 0331
- Asia, Australia and Oceania 0332
- Canadian 0334
- European 0335
- Latin American 0336
- Middle Eastern 0333
- United States 0337

History of Science 0585
Law 0398
Political Science
- General 0615
- International Law and Relations 0616
- Public Administration 0617

Recreation 0814
Social Work 0452
Sociology
- General 0626
- Criminology and Penology 0627
- Demography 0938
- Ethnic and Racial Studies 0631
- Individual and Family Studies 0628
- Industrial and Labor Relations 0629
- Public and Social Welfare 0630
- Social Structure and Development 0700
- Theory and Methods 0344

Transportation 0709
Urban and Regional Planning 0999
Women's Studies 0453

THE SCIENCES AND ENGINEERING

BIOLOGICAL SCIENCES
Agriculture
- General 0473
- Agronomy 0285
- Animal Culture and Nutrition 0475
- Animal Pathology 0476
- Food Science and Technology 0359
- Forestry and Wildlife 0478
- Plant Culture 0479
- Plant Pathology 0480
- Plant Physiology 0817
- Range Management 0777
- Wood Technology 0746

Biology
- General 0306
- Anatomy 0287
- Biostatistics 0308
- Botany 0309
- Cell 0379
- Ecology 0329
- Entomology 0353
- Genetics 0369
- Limnology 0793
- Microbiology 0410
- Molecular 0307
- Neuroscience 0317
- Oceanography 0416
- Physiology 0433
- Radiation 0821
- Veterinary Science 0778
- Zoology 0472

Biophysics
- General 0786
- Medical 0760

EARTH SCIENCES
Biogeochemistry 0425
Geochemistry 0996
Geodesy 0370
Geology 0372
Geophysics 0373
Hydrology 0388
Mineralogy 0411
Paleobotany 0345
Paleoecology 0426
Paleontology 0418
Paleozoology 0985
Palynology 0427
Physical Geography 0368
Physical Oceanography 0415

HEALTH AND ENVIRONMENTAL SCIENCES
Environmental Sciences 0768
Health Sciences
- General 0566
- Audiology 0300
- Chemotherapy 0992
- Dentistry 0567
- Education 0350
- Hospital Management 0769
- Human Development 0758
- Immunology 0982
- Medicine and Surgery 0564
- Mental Health 0347
- Nursing 0569
- Nutrition 0570
- Obstetrics and Gynecology 0380
- Occupational Health and Therapy 0354
- Ophthalmology 0381
- Pathology 0571
- Pharmacology 0419
- Pharmacy 0572
- Physical Therapy 0382
- Public Health 0573
- Radiology 0574
- Recreation 0575
- Speech Pathology 0460
- Toxicology 0383

Home Economics 0386

PHYSICAL SCIENCES

Pure Sciences

Chemistry
- General 0485
- Agricultural 0749
- Analytical 0486
- Biochemistry 0487
- Inorganic 0488
- Nuclear 0738
- Organic 0490
- Pharmaceutical 0491
- Physical 0494
- Polymer 0495
- Radiation 0754

Mathematics 0405
Physics
- General 0605
- Acoustics 0986
- Astronomy and Astrophysics 0606
- Atmospheric Science 0608
- Atomic 0748
- Electronics and Electricity 0607
- Elementary Particles and High Energy 0798
- Fluid and Plasma 0759
- Molecular 0609
- Nuclear 0610
- Optics 0752
- Radiation 0756
- Solid State 0611

Statistics 0463

Applied Sciences

Applied Mechanics 0346
Computer Science 0984
Engineering
- General 0537
- Aerospace 0538
- Agricultural 0539
- Automotive 0540
- Biomedical 0541
- Chemical 0542
- Civil 0543
- Electronics and Electrical 0544
- Heat and Thermodynamics 0348
- Hydraulic 0545
- Industrial 0546
- Marine 0547
- Materials Science 0794
- Mechanical 0548
- Metallurgy 0743
- Mining 0551
- Nuclear 0552
- Packaging 0549
- Petroleum 0765
- Sanitary and Municipal 0554
- System Science 0790

Geotechnology 0428
Operations Research 0796
Plastics Technology 0795
Textile Technology 0994

PSYCHOLOGY
General 0621
Behavioral 0384
Clinical 0622
Developmental 0620
Experimental 0623
Industrial 0624
Personality 0625
Physiological 0989
Psychobiology 0349
Psychometrics 0632
Social 0451

The undersigned recommended to the Faculty of Graduate Studies
and Research acceptance of the thesis

"Graph Connectivity Augmentation Problems"
submitted by Romy Varga, B.Sc.
in partial fulfilment of the requirements for
the degree of Master of Science

Thesis Supervisor

for Chair, Department of Mathematics

Carleton University
January 19, 1996

Contents

1 Abstract

In this thesis I will present some graph connectivity augmentation problems that I have made contributions to while on an academic exchange [1] in Budapest, Hungary visiting Professor ANDRÁS FRANK at Eötvös Loránd University.

Edge–connectivity augmentations will be discussed, both for directed, undirected graphs, as well as hypergraphs. We will introduce the splitting off technique and Frank's Algorithm and show how these techniques solve edge–connectivity augmentation problems. The successive edge–connectivity problem will also be solved using these techniques. Hypergraph connectivity will be k–edge–augmented with edges of size two by applying splitting off techniques. We also solve the k–(S,T)–edge–connectivity problem in digraphs for $k = 1$.

Throughout this thesis I will show the exact places of my own contributions to the above discussed problems.

[1] This scholarship was funded by Carleton International, Carleton University, Ottawa, Canada.

2 Introduction

Research for this Master's Thesis was done while I spent a year on an Academic Exchange in Budapest, Hungary, visiting Professor ANDRÁS FRANK, one of the leading researchers in Graph Theory today. The academic exchange was organised between Carleton International and the Hungarian Ministry of Education, and I was awarded a scholarship to participate in this programme during the 1994/95 academic year.

The area of research I focused on was Connectivity Augmentation Problems, an area in which ANDRÁS FRANK is very active. He is in touch with many other researchers around the world, who consult with him, share their ideas with Professor FRANK, and appreciate his comments and advice.

I got involved in this research process, first by reading and understanding the pre-publications sent to ANDRÁS FRANK, then commenting on them and checking their validity. During this process I discovered some errors in proofs, holes in arguments, suggested corrections to them, found problems with examples, indicated possible fixes, helped with the style of writing and typing mistakes. I kept in touch with both ANDRÁS FRANK and the authors of these articles, working together with them on their research. I feel that during this process I made worthwhile contributions to mathematical research in the area of graph connectivity augmentation problems.

The papers to which I made contributions all deal with edge–connectivity augmentations, both for directed, undirected and hypergraphs. These pre–publications are a direct result of a workshop on hypergraph connectivity augmentations, organised by Professor ANDRÁS FRANK, in November 1994 at Eötvös Loránd University, Budapest.

Professor FRANK asked me to write a summary of the research resulting from that workshop, detailing the major ideas, theorems and current advancements in this area and tieing them together. This paper is due to be published in one of the local journals edited by ANDRÁS FRANK in Budapest. This thesis is an elaboration of that summary.

The list of attendees at that week–long workshop included TIBOR JORDÁN, EDDIE CHENG, JØRG BANG–JENSEN, BILL JACKSON and STEFFEN ENNI, amongst others. However, these were the mathematicians who were ready to publish their results and sent their materials to ANDRÁS FRANK before the end of my exchange programme in Budapest.

In the sections that follow, I will present these papers, pointing out the exact locations of my own contributions.

Connectivity augmentation problems are a subclass of questions in graph theory which arise when one is asked to construct graphs (or subgraphs of a

graph) satisfying certain conditions. usually also worrying about it being of minimal cost.

These may result from some other areas of mathematics (operations research, say), computer science (designing communications networks; increasing network reliability of existing networks either by adding new communication links or new nodes; providing model for studying routing and control algorithms; also in learning theory), electrical engineering (electrical networks are generally viewed as flowgraphs), statistical mechanics (in continuum statistical mechanics there exists a one–to–one correspondence between the so called "configuration integral" of a substance of N molecules and the labelled graphs on N vertices. lattice models of real life substances can be studied in discrete statistical mechanics; especially of crystals, that form spatially–periodic lattices with occasional vacancies), chemistry (molecules can be naturally represented by graphs with vertices representing atoms while the edges are the valence bonds holding the molecule together). geography (the seven bridges of Königsberg; map colourings that led to the Four–Colour Problem; transportation networks being classical representatives of this category). architecture (the process of synthesis of floor plans can be solved by graph theoretic means, regions of a plan [rooms, halls, and other areas of a house, say] being represented by the vertices of a graph, and edges expressing the adjacency relations between regions). social sciences (graphs mostly digraphs — used to models problems of theoretical sociology, psychology, economics and political science, for instance, and many others) and linguistics

(in the study of natural languages, graphs of directed edges connecting parts of the syntax represent dependencies of sentence structures), just to name a few (see [2] for details).

Shortest paths between two specified nodes in a graph, or finding minimum cost spanning trees are special cases of these problems. One may also need to find graphs satisfying planarity requirements, or graphs that are 3–colorable, or perhaps graphs that have a Hamiltonian cycle. Connectivity requirements also fall in this category, and play an important role in the above mentioned applied cases.

One of the simplest connectivity questions to ask is, when given a graph which is not connected, how many new edges do we need to add to make it *strongly connected*? This question was answered by K.P. ESWARAN and R.E. TARJAN [6] for directed graphs, and for undirected graphs by W. CHOU and H. FRANK [4], as we will see in the following section.

However, one can also ask the question, how about if instead of requiring the new graph to be just connected, we want it to be *k–edge–connected*? That is, between any two vertices, we want to have at least k–edge–disjoint paths? Then the problem gets a bit trickier, but is still solvable. There are polynomial time algorithms for the above problems, but other connection problems may be NP–complete (see [7] for some such problems). Also, some

cases have combinatorial solutions, and yet others are solvable only by using non–combinatorial methods, like the *ellipsoid method*.

For defining what is formally called an augmentation problem, let us denote the *number of edge–disjoint paths* between any two vertices x and y of a graph (or digraph) $G = (V, E)$ by $\lambda(x, y)$. Problems which require us to increase the connectivity of a graph from one value (which could be zero) to a higher one, usually k, are called *connectivity augmentation* problems. The new graph with connectivity k and the smallest number of new edges is called an *optimal k–connected augmentation of G*. When we require paths between vertices to be edge–disjoint (or vertex–disjoint) we talk about *edge–connectivity augmentation* (or respectively *vertex–connectivity augmentation*). In this thesis we only talk about problems of edge–connectivity augmentations, so from now on, we may suppress the "edge–connectivity" and speak only of "augmentation problems" while talking about edge–connectivity augmentation problems. Also from time to time we will say "disjoint paths" and mean "edge–disjoint paths"; similarly "k–edge–connected" may be shortened to "k–connected".

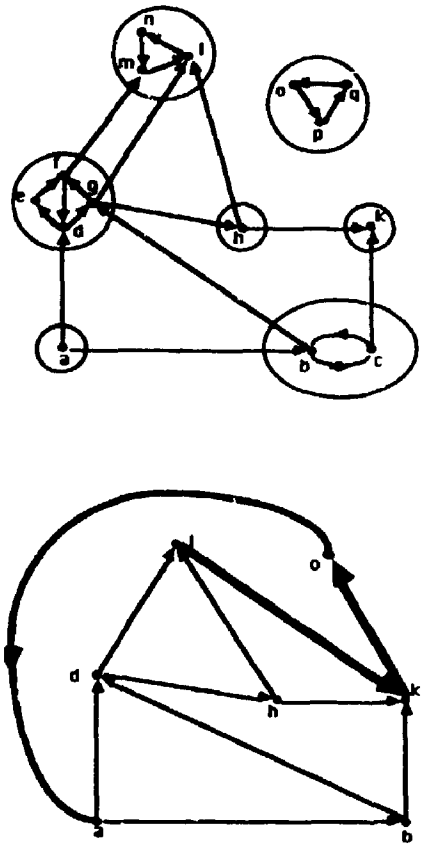
The most general form of an augmentation problem is: given a graph $G = (V, E)$ and a *demand function* $r(x, y)$ on the set of pairs of vertices, what is the minimal number of new edges needed, whose addition to E yields a new graph $G' = (V, E')$ satisfying $\lambda(x, y) \geq r(x, y)$ for all pairs of vertices

x and y in G'?

If the demand function $r(x, y)$ has the same value for all $x, y \in V$, then we talk about a *uniform augmentation problem*, and if it takes different values for different pairs of vertices, then the problem is *non–uniform*. Most problems we consider in this thesis are of the first case, but there will be problems of the latter type.

As mentioned earlier, the minimal number of edges needed to make a directed graph D strongly connected has been determined in [6], and it equals the maximum of the number of minimal *sink–sets* (with respect to containment) and *source–sets*.

A set X of vertices is said to be a sink–set when $\delta(X) = 0$ (the number of edges leaving it is zero). It is called a source–set when $\varrho(X) = 0$ (the number of edges entering it is zero).



ESWARAN and TARJAN developed a linear–time algorithm which identifies all strongly connected components of a digraph, shrinks them all to one new vertex representing each component (*condensation*, as the authors have called it, see the above picture), and then another algorithm which adds the necessary new edges to make the condensed graph strongly connected, still in linear time (these are the ones that are bolded in the second part of the above picture). It is easy to see that the edges added to make the condensed graph strongly connected, will make the original graph strongly connected,

as well.

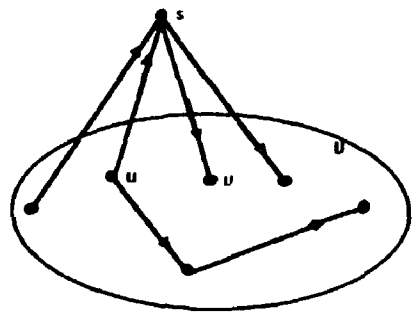
It is not hard to convince ourselves that at least this many new edges (maximum of the number of sink–sets and the number of source–sets) are needed to make the graph strongly connected. This is because each sink–set will need to have at least one outgoing edge, and each source–set will have to have at least one incoming edge. The proof that these many edges suffice is constructive, and gives rise to the above mentioned linear time algorithm.

The minimal number of edges required to make D k–connected has been determined by A. FRANK in [8], using *Mader's Splitting off Theorem.* The idea of *splitting off* pairs of edges was introduced by L. LOVÁSZ [11].

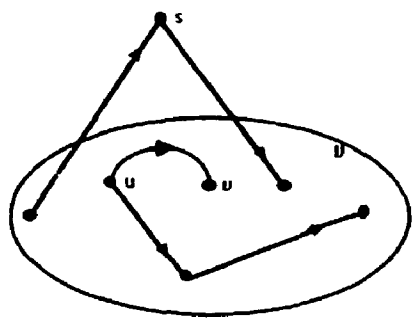
Suppose we have a digraph $D = (V \cup \{s\}, E)$ with a distinguished vertex s and directed edges satisfying

$$\varrho(s) = \delta(s) \quad \text{and} \quad \lambda(x, y) \geq k \quad \forall x, y \in V, \tag{2.1}$$

where $\varrho(x)$ is the number of directed edges with head x, and $\delta(x)$ is the number of directed edges with tail x for any $x \in V \cup \{s\}$.



Then splitting off a pair of edges us and sv (for $u, v \in V'$) means replacing the pair (us, sv) by a new edge uv.



Mader's Theorem states that the edges entering and leaving s can be paired into $\varrho(s)$ pairs so that splitting off these pairs and deleting s leaves a k–edge–connected digraph (for $k \geq 2$).

A. FRANK used MADER's splitting off theorem in [8] and developed a strongly polynomial [2] time algorithm for making a digraph k-edge connected.

[2] A polynomial time algorithm is called strongly polynomial if it uses, besides ordinary data manipulation, only the basic operations like comparing, adding, subtracting, multiplying and dividing numbers, and if the number of these operations is independent of the

This algorithm is being called by many authors *Frank's Algorithm*. It consists of adding a new vertex s to a given digraph $D = (V, E)$ and adding k copies of a new edge $sv \quad \forall v \in V$ and k copies of $us \quad \forall u \in V$, then deleting as many new edges as possible such that Equation (2.1) is still satisfied.

Note that this deleting process can be done greedily and that if needed, one extra edge has to be added back to satisfy the $\varrho(s) = \delta(s)$ part of Equation (2.1). Clearly this new graph is k–connected. Then *Mader's Theorem* guarantees that the edges to and from s can be split off and that the new graph will also be k–connected.

The idea of splitting off and FRANK's Algorithm played an important role in recent work in this area. Most of the authors attending the workshop in Budapest found themselves using them over and over again. Some of the papers we will discuss in the following, will use them as basic tools.

At this point let me introduce the topics of the projects I worked on while under the supervision of Professor FRANK, and made some contributions to these pre–publications. The names of the authors will be mentioned, too, most of whom I was in direct contact with and had discussions about their research with in person (or through email), and others through ANDRÁS FRANK.

numbers occurring in the input.

The *successive edge-augmentation* problem for undirected graphs has been previously solved by WATANABE and NAKAMURA as a byproduct of their algorithm in [16], but using FRANK's Algorithm E. CHENG realised that a better solution is possible.

The basic form of the successive edge–augmentation problem is as follows. Given a graph (digraph) $G = (V, E)$ and parameter k, find a graph $G_i = (V, E_i)$ for all $1 \leq i \leq k$ such that

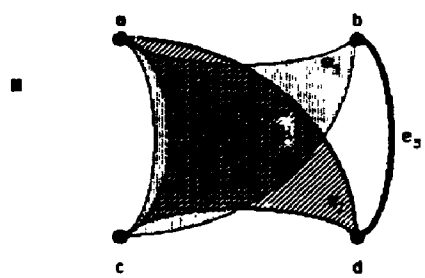
$$\lambda(x, y; G_i) \geq i \qquad \forall x, y \in V$$

(where $\lambda(x, y; G_i)$ is the number of edge–disjoint paths from x to y in G_i) and that

$$G \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k.$$

However, the successive augmentation problem in the directed case had not been solved prior to the end of 1994. Seeing that FRANK's Algorithm can be used in the directed case, T. JORDAN worked out a solution for the directed successive edge–augmentation problem using the same technique. This will be the topic of our Section 3. We will tie in here the work of E. CHENG, who worked out a different solution to this problem for undirected graphs than WATANABE and NAKAMURA's solution. As a matter of fact, T. JORDAN and E. CHENG plan to publish their results in a joint paper.

The splitting off technique also inspired J. BANG–JENSEN and B. JACKSON in their quest for solving hypergraph augmentation. They have been trying to edge–augment hypergraphs by adding new edges of size two to accomplish k–connectedness. A *hypergraph* $H = (V, E)$ can be viewed as a usual graph with vertices $v \in V$ and edges $e \in E$ with an incidence relation that associates a subset of V to each edge $e \in E$. For instance, on the vertices $V = \{a, b, c, d\}$ we could have $E = \{e_1, e_2, e_3\} = \{\{a, b, c\}, \{a, c, d\}, \{b, d\}\}$ as a set of hyperedges (see graph below). Note that, if all hyperedges have size two, then the hypergraph is a graph we are all used to.



The number of edges of size two needed to augment a hypergraph to be k–edge–connected if it was already $(k-1)$–connected has been previously known. In Section 4 we will give a more general result, which is the work of J. BANG–JENSEN and B. JACKSON. They also used the splitting off idea, and developed a strongly polynomial algorithm, similar to FRANK's Algorithm, that finds an optimal augmentation to this problem.

So far we talked about only edge–connectivity problems between any two vertices x and y of a graph (digraph or hypergraph) G. We need to mention a special type of connectivity problem, too, the so-called (S,T)–*connectivity problem*. This one, with parameter k, consists of requiring the number of edge–disjoint paths between any two vertices $s \in S$ and $t \in T$, for two specified non–empty subsets S and T of V (that may or may not be disjoint), to be at least k. It is easy to see that when $S = T = V$ we are back at the regular k–edge–connectivity augmentation problem.

This problem has been previously solved by A. FRANK and T. JORDÁN in [9], but their proof is a consequence of a general result on *crossing bi–supermodular functions* and does not yield a polynomial–time algorithm. In Section 5 we will show how S. ENNI solves the special case when $k = 1$. Unfortunately, at this time, it seems like there is little hope to generalize this approach to higher connectivity requirements. Even the case $k = 2$ looks unsolvable with a similar technique to S. ENNI's.

Having introduced the main new ideas in the area of edge–connectivity augmentation let us turn our heads back for a moment to mention some earlier results and clear up some definitions.

3 Definitions and previous results

In this section we list a few definitions and previous results that will be used in later parts of this thesis.

In graph theory it is customary to denote the *degree of a vertex* x in a graph [3] $G = (V, E)$ by $d(x)$. Similarly, for a subset X of V, $d(X)$ is the number of edges leaving X. Also when $X, Y \subseteq V$, we let $d(X, Y)$ be the number of edges between $X - Y$ and $Y - X$, and $\bar{d}(X, Y)$ the number of edges between $X \cap Y$ and $V - (X \cup Y)$.

In a digraph $D = (V, E)$ $\delta(x)$ is the number of edges with tail x, while $\varrho(x)$ is the number of edges with head x. As we have already defined $d(X)$ for undirected graphs, we now define $\delta(X)$ to be the number of edges with tails in X and heads not in X (outdegree of X) and $\varrho(X)$ to be the number of edges with heads in X and tails not in X (indegree of X). Now $d(X, Y)$ is the number of edges between $X - Y$ and $Y - X$ in any direction. The same definition holds for $\bar{d}(X, Y)$ as in the undirected case, again counting edges going in both directions.

In following sections, proofs will often refer to the following equalities.

[3] In our definition of graphs, we allow parallel edges, but not loops.

For undirected graphs we have:

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \tag{3.1}$$

while for directed graphs:

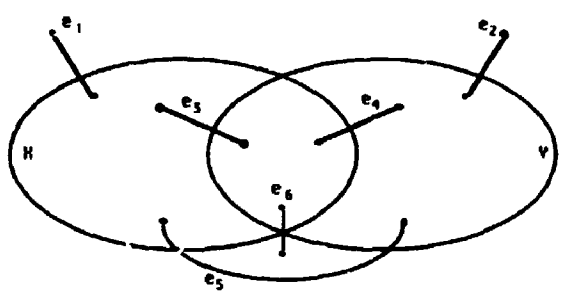
$$\delta(X) + \delta(Y) = \delta(X \cap Y) + \delta(X \cup Y) + d(X, Y), \tag{3.2}$$

$$\varrho(X) + \varrho(Y) = \varrho(X \cap Y) + \varrho(X \cup Y) + d(X, Y). \tag{3.3}$$

For undirected graphs the following also holds:

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y). \tag{3.4}$$

Each of these equations can easily be proved by counting the contributions of each edge on both sides of the equations.



In this picture it is easy to see the reason why Equations (3.1) and (3.4) hold. Note that edges e_1, e_4, e_5 and e_6 all contribute one to $d(X)$. e_3 does not

But on the other hand, e_3 increases $d(X - Y)$ by one, and it also contributes to $d(X \cap Y)$. Edge e_5 only contributes to $d(X - Y)$ and to $d(Y - X)$ on the right hand side, while e_6 is the only edge that increments $\overline{d}(X, Y)$ by one. Similarly, e_5 is the only edge that contributes to $d(X, Y)$. This way, Equations (3.1) and (3.4) for this example look like:

$$4 + 4 = 3 + 3 + 2 \cdot 1,$$

$$4 + 4 = 3 + 3 + 2 \cdot 1,$$

respectively.

Similarly, by drawing a simple example of all possible edges between two sets, one can derive Equations (3.2) and (3.3) for directed graphs by counting edges on both sides of the equations.

Note that some of the time, we will use these equalities in an easier form, by dropping their last terms, and writing them as inequalities. That is, Equation (3.1) will be written as:

$$d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y).$$

Similarly, its directed "relatives" will be shortened, too.

If $\delta(X \cap Y)$ happens to equal $\varrho(X \cap Y)$, then for directed graphs we also have:

$$\delta(X) + \delta(Y) = \delta(X - Y) + \delta(Y - X) + \overline{d}(X, Y) \tag{3.5}$$

$$\varrho(X) + \varrho(Y) = \varrho(X - Y) + \varrho(Y - X) + \overline{d}(X, Y) \tag{3.6}$$

We already talked about $\lambda(x, y)$, the number of edge–disjoint paths in G between x and y. When we want to specify which graph we count the number of edge–disjoint paths in (for instance, we may have a series of graphs G_i with different numbers of edges for different i's, hence different numbers of disjoint paths between different vertices, and we need this number in the i-th graph), we write the following: $\lambda(x, y; G_i)$.

Similarly, we write $d(x; G_i)$ to emphasize the graph in which we count the degree of x (in this case, it would be G_i).

A graph G is called k–edge–connected when the number of edge–disjoint paths between any two vertices x and y of G is at least k.

One form of *Menger's Theorem* states:

Theorem 3.1 *An undirected graph $G = (V, E)$ is k-edge-connected if and only if*

$$d(X) \geq k$$

for every proper subset X of V.

A similar theorem holds for directed graphs.

Theorem 3.2 *A directed graph $D = (V, E)$ is k-edge-connected if and only if*

$$\varrho(X) \geq k \quad \textit{and} \quad \delta(X) \geq k$$

for every proper subset X of V.

As discussed in the Introduction, *splitting off* a pair of edges su and sv in an undirected graph $(s, u, v \in V)$ is equivalent to deleting this pair (su, sv) and replacing it by a new edge uv. Splitting off a pair of edges (su, sv) is called a *k-splitting* when $\lambda(x, y; G') \geq k \quad \forall x, y \in V - \{s\}$, where G' is the new graph we get after splitting off.

A pair of edges is called *k-splittable*, if a k-splitting is possible on that pair of edges in the current graph.

A proper subset X of V is called *k-dangerous* if $d(X) \leq k + 1$, and *k-critical* when $d(X) = k$.

The following theorem is a building block for solving problems where splitting off is used. [4]

Theorem 3.3 *Given a graph $G = (V \cup \{s\}, E)$. A pair of edges (su, sv) is k-splittable in G if and only if there is no k-dangerous set $X \subset V$ such that*

[4] I did not find whom this theorem is due to, however it is well known amongst people working in this area. The proof given here is one provided by me.

$u, v \in X$.

Proof: Assume that (su, sv) is k–splittable in G. By definition, this means that $\lambda(x, y; G') \geq k \quad \forall x, y \in V$, where G' again denotes the graph after the splitting off has occurred. Hence by MENGER's Theorem (3.1) $d(X; G') \geq k \quad \forall X \subset V$. This holds in particular for X such that $u, v \in X$. But for such an X, $d(X; G) = d(X; G') + 2 \geq k + 2$. That is, u and v do not belong to a k–dangerous subset X of V.

Conversely, assume that u and v do not belong to a k–dangerous subset X of V. Then $\forall X \subset V$ with $u, v \in X$, $d(X; G) > k+1$, i.e. $d(X; G) \geq k+2$. But two of those more than $k+2$ edges connect s to u and v. Splitting those edges off still leaves at least k edges exiting from any subset X of V that contain u and v. Splitting off (su, sv) did not affect the degree of any other subsets that do not have both u and v in them, hence their degree is still greater than or equal to k. Overall, $d(X; G') \geq k \quad \forall X \subset V$, which again is equivalent to $\lambda(x, y; G') \geq k \quad \forall x, y \in V$. Hence (su, sv) is k–splittable in G. •

Here we state the exact form of *Mader's Theorem*, already referred to in the Introduction.

Theorem 3.4 *Let $D = (V, E)$ be a digraph with $s \in V$ and*

$$\delta(s) = \varrho(s) \quad \text{and} \quad \lambda(x, y) \geq k \quad \forall x, y \in V - \{s\}.$$

Then the dges entering and leaving s can be partitioned into $\varrho(s)$ pairs such that splitting off all these pairs and deleting s leaves a k-edge-connected digraph.

This theorem is proved by $\varrho(s)$ repetitions of the following theorem, also due to MADER.[5]

Theorem 3.5 *Suppose that, for a node s in a digraph $D = (V, E)$, $\varrho(s) = \delta(s)$ holds, and that*

$$\lambda(x, y) \geq k \quad \forall x, y \in V - \{s\}. \tag{3.7}$$

Then for any edge us there exists another edge sv, such that the pair (us, sv) can be split off without violating (3.7).

Here we provide a proof to this theorem that is due to A. FRANK, given in [8].

Proof: Recall that a pair of edges (us, sv) is not splittable precisely when there is a critical [6] set containing both u and v. Therefore, if there is no

[5] MADER's first formulation of the theorem looked like:
Let $D = (V, E)$ be a digraph with $s \in V$, $0 < d(s) \neq 3$ and there is no cut-edge incident to s. Then there exists a pair of edges (su, sv) such that $\lambda(x, y; D) = \lambda(x, y; D')$ $\forall x, y \in V$, where D' is the digraph resulting after splitting off the pair of edges (su, sv).

[6] Recall that any set in a digraph is called critical if it is either in-critical, or out-critical.

critical set containing u, then any pair of edges (us, sv) is splittable.

We now need the following three propositions, for proving Theorem (3.5).

Proposition 3.1 *If X and Y are intersecting subsets of nodes for which $\{s\} = X \cap Y$ and $\delta(X) = \delta(Y) = k$, then $\delta(X - Y) = \delta(Y - X) = k$, and also $\overline{d}(X, Y) = 0$.*

To prove this proposition, we apply (3.5) to obtain:

$$k + k = \delta(X) + \delta(Y) = \delta(X - Y) + \delta(Y - X) + \overline{d}(X, Y) \geq k + k + \overline{d}(X, Y),$$

from which $\delta(X - Y) = \delta(Y - X) = k$ follows, and that $\overline{d}(X, Y) = 0$.

Proposition 3.2 *If $A, B \subseteq V$ and*

$$\varrho(A) = \varrho(B) = k \leq \min(\varrho(A \cap B), \varrho(A \cup B)),$$

then

$$\varrho(A \cap B) = \varrho(A \cup B) = k \quad \textit{and} \quad d(A, B) = 0.$$

This proposition is proved by applying Equation (3.3), to get:

$$k + k = \varrho(A) + \varrho(B) = \varrho(A \cap B) + \varrho(A \cup B)) + d(A, B) \geq k + k + d(A, B),$$

from which $\varrho(A \cap B) = \varrho(A \cup B) = k$ and $d(A, B) = 0$ follows.

Proposition 3.3 *If A and B are two intersecting critical subsets, then either:*

(i) *$A \cup B$ is critical, or*

(ii) *$B - A$ is critical and $\overline{d}(A, B) = 0$.*

If both A and B are in–critical and $A \cup B \subset V - \{s\}$, then the previous proposition implies that *(i)* will hold. If $A \cup B = V - \{s\}$ (which is never critical), then Proposition (3.1) applied to $X := V - A$ and $Y := V - B$ implies that *(ii)* should hold. The situation is analogous if both A and B are out–critical.

When A is in–critical and B is out–critical, the previous proposition applied to $X := A$ and $Y := V - B$ implies *(ii)* again.

For two intersecting critical sets A and B containing u, only the case *(i)* from above may hold, since the existence of the edge us implies $\overline{d}(A, B) > 0$. Therefore, the union M of all critical sets containing u is critical again.

Returning to the proof of the theorem, we claim that there exists an edge sv with $v \in (V - \{s\}) - M$. We indirectly suppose that no such an edge exists.

If M is in–critical, then $\delta(s) = \varrho(s)$, which implies that $\varrho((V - \{s\}) - M) = \delta(M \cup \{s\}) < \delta(M) = k$, contradicting the requirement that $\varrho(X) \geq k$ for

all subsets X of $V - \{s\}$.

If M is out–critical, then $\delta((V - \{s\}) - M) < \varrho(M) = k$, contradicting the other condition, that $\delta(X) \geq k$ for all $X \subset V - \{s\}$ (by MENGER's Theorem, (3.1)).

By the choice of M, no critical set contains both u and v, and therefore us and sv are splittable. •

In 1974, LOVÁSZ announced at the "Conference on Graph Theory" in Prague [12] the following theorem, analogous to the previous theorem (Theorem (3.5)).

Theorem 3.6 *In a graph $G = (V, E)$, with $s \in V$, $d(s)$ even, $d(s) > 0$, $k \geq 2$ and*

$$d(X) \geq k \quad \forall X \subset V - \{s\} \quad (X \neq \emptyset). \tag{3.8}$$

for every edge su there exists another edge sv such that the pair (su, sv) can be split off without violating (3.8).

This theorem, by applying it repeatedly $d(s)/2$ times can easily be extended to the following, being an analogue to Theorem (3.4) for undirected graphs.

Theorem 3.7 *Let $G = (V, E)$ be a graph with $s \in V$, $d(s)$ even, $d(s) > 0$ and*

$$\lambda(x, y) \geq k \quad \forall x \ y \in V.$$

Then the edges incident with s can be partitioned into $d(s)/2$ pairs such that splitting off all these pairs and deleting s leaves a k-edge-connected graph.

For completeness, we provide a proof of Theorem (3.6) here, that is also due to A. FRANK.

Proof: Let us denote the set of vertices incident with $s \in V$ by S; i.e. $S := \{v \in V - \{s\} | sv \in E\}$. Then we will need to show that the following two claims hold.

CLAIM A. *Let A and B be two intersecting k-dangerous*[7] *subsets of $V - \{s\}$ with $u \in A \cap B$. Then we have*

(i) $\overline{d}(A, B) = 1$, and

(ii) $S \not\subseteq A \cup B$ (in particular, $A \cup B \neq V - \{s\}$).

To prove this claim, we write Equation (3.4):

$$(k+1) + (k+1) \geq d(A) + d(B) = d(A - B) + d(B - A) + 2\overline{d}(A, B) \geq$$

$$\geq k + k + 2\overline{d}(A, B),$$

[7]Remember that a set X is called to be k-dangerous when $d(X) \leq k + 1$.

since we assumed that $d(X) \geq k \ \forall X \subset V - \{s\}$. From this, it follows that $\overline{d}(A, B) \leq 1$. But since $u \in A \cap B$, the existence of the edge su gives us $d(A \cap B, \overline{A \cup B}) \geq 1$, i.e. $\overline{d}(A, B) \geq 1$.[8] Therefore, we have that $\overline{d}(A, B) = 1$.

For part *(ii)*, suppose that $S \subseteq A \cup B$. Let us define $\alpha := d(s, A - B)$ and $\beta := d(s, B - A)$. By symmetry, we can assume that $\alpha \geq \beta$. Since $d(A, B) = 1$ we have $k \leq d((V - \{s\}) - A) = d(A \cup \{s\}) = d(A) - \alpha + \beta - 1 \leq d(A) - 1 \leq k$; from which $\alpha \leq \beta$ follows, and hence $\alpha = \beta$. But this is impossible, since if $S \subseteq A \cup B$, then $d(s) = \alpha + \beta + 1 = 2\alpha + 1$, an odd number, contradicting the requirement of $d(s)$ to be even.

CLAIM B. *If A and B are intersecting dangerous sets with $u \in A \cap B$, and A is maximal dangerous, then $d(A) = d(B) = k + 1$ and $d(A \cap B) = k$.*

We use CLAIM A to prove our current claim. Hence $A \cup B \neq V - \{s\}$, and by the maximality of A, $d(A \cup B) \geq k + 2$. From (3.1) we have:

$$k + 1 + k + 1 \geq d(A) + d(B) \geq d(A \cup B) + d(A \cap B) \geq k + 2 + k,$$

from which CLAIM B follows.

If there is at most one maximal dangerous set X with $u \in X$, then for any edge sv with $v \notin X$, the pair (su, sv) is splittable. Such an edge exists, since otherwise $d((V - \{s\}) - X) = d(X \cup \{s\}) = d(X) - d(s) \leq k + 1 - 2 = k - 1$, contradicting $d(X) \geq k \ \forall X \subset V$.

[8] Recall the definition of $\overline{d}(A, B) := d(A \cap B, \overline{A \cup B})$.

Suppose that X and Y are two distinct maximal dangerous sets with $u \in X \cap Y$ for which $M := X \cap Y$ is maximal. Then X and Y are intersecting, and CLAIM A implies that there is an edge sv with $v \notin X \cup Y$.

Now we claim that the pair (su, sv) is splittable.

Suppose that, indirectly, there is a maximal dangerous set Z with $u, v \in Z$ (by Theorem (3.3) this would mean that (su, sv) is not k-splittable). Applying CLAIM B to $A := X$ and $B := Y$, we have that $d(M) = k$. Hence Z and M must not be intersecting (i.e. disjoint), otherwise CLAIM B could be applied to $A := X$ and $B := M$ implying that $d(M) = k + 1$. Therefore $M \subseteq Z$ and by the maximal choice of M we have $X \cap Z = Y \cap Z = M$. By CLAIM A, $\bar{d}(X, Y) = \bar{d}(Z, Y) = \bar{d}(Z, X) = 1$, and therefore no edge other than su can leave M, contradicting the fact that $k \geq 2$. •

A set of subsets $\{X_1, X_2, \ldots, X_t\}$ of V is called a *sub-partition* of V when the X_i's are pairwise disjoint, and they are all non-empty. Note that it is not required that the union of the X_i's is the whole V.

The following theorem is due to A. FRANK [8], and it determines the minimum number of edges needed to make a graph k-connected:

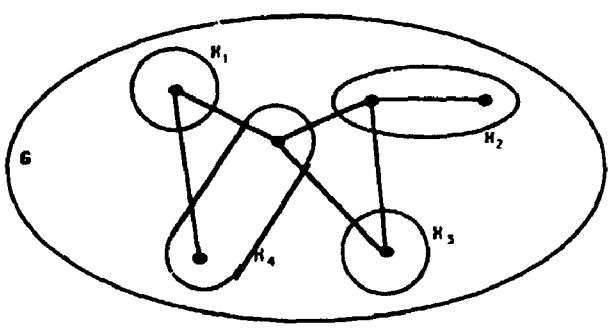
Theorem 3.8 *The minimum number of new edges γ whose addition to an undirected graph $G = (V, E)$ makes the supergraph k-edge-connected ($k \geq 2$)*

is

$$\gamma = \max\left\lceil \frac{\sum_i (k - d(X_i))}{2} \right\rceil,$$

where the maximum is taken over all sub-partitions $\{X_1, \ldots, X_t\}$ *of* V.

Proof: It is not hard to see that at least $\max\lceil(\sum_i(k - d(X_i)))/2\rceil$ new edges are needed. This is because in order to have edge–connectivity equal to k, we need at least k edges exiting any set of vertices. But the current outdegree $d(X_i)$ of a set may be less than k, so to increase it to k clearly $k - d(X_i)$ new edges are needed. And since each edge we add connects two such subsets of vertices, we need to divide by two. Hence the need of at least this many new edges.



[a]

[a]**Note:** Here the current edge–connectivity is $\lambda = 1$. If $k = 3$, then with the current partition of G, the sets X_1, X_2 and X_3 need at least one more edge entering them to hope for achieving $\lambda = k$.

For proving that $\max\lceil(\sum_i(k - d(X_i)))/2\rceil$ new edges are enough, we rephrase the theorem as:

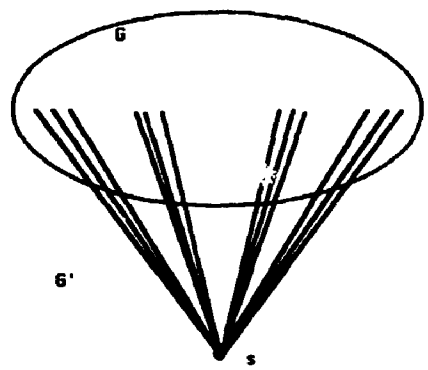
> *A graph G can be made k-edge-connected ($k \geq 2$) by the addition of γ edges if and only if*
>
> $$\sum_i (k - d(X_i)) \leq 2\gamma$$
>
> *for all sub-partitions $\{X_1, \ldots, X_t\}$ of V.*

Hence for proving sufficiency, we need to show that if $\sum_i(k - d(X_i)) \leq 2\gamma$ for all sub-partitions $\{X_1, \ldots, X_t\}$ of V, then G can be made k-connected by adding at most γ new edges.

To do this, we form a new graph by adding a new vertex s to G, and k new edges from s to every vertex of G. This graph (call it G') is clearly k-edge-connected. Then we delete as many new edges as possible, one by one, such that we keep $\lambda(x, y; G') \geq k \ \forall x, y \in V$ satisfied. When no more new edges can be deleted and the degree of s is odd, we add back the last edge deleted. [9] in order to keep $d(s; G')$ even. (Later on, we will need $d(s; G')$ to be even.)

[9] Originally the algorithm calls for adding any deleted edge back. Adding back the "last" deleted edge is my suggestion to improve the algorithm. Simply by looking at the implementation point of view, (this proof to the theorem gives rise to a polynomial time algorithm) one only needs to save the last deleted edge in memory, and not all deleted edges, from which at the end — if needed — choosing one at random to add back.

By MENGER's Theorem, (3.1), this inequality on $\lambda(x, y; G')$ is equivalent to having $d(X; G') \geq k \; \forall X \subset V, X \neq \emptyset$.



Now we claim that the number of new edges left is at most 2γ, i.e.

$$d(s; G') \leq 2\gamma \qquad (3.9).$$

The above deleting process stops when we can not delete any more edges, i.e. when for every edge su, u will be an element of a k–critical set.

Let M_u denote a minimal critical set containing $u \in V$. Let $\mathcal{F}$ be the family of minimal critical sets for each u adjacent to s; that is, $\mathcal{F} := \{M_u | u \text{ adjacent to } s\}$. Let $\{X_1, X_2, \ldots, X_t\}$ be the maximal members of $\mathcal{F}$.

It can be shown, using Equation (3.1), that if $X, Y \subset V$ are critical, $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$, then both $X \cup Y$ and $X \cap Y$ are also critical

(Proposition (3.2)). Similarly, using Equation (3.4), we can show that if $X - Y \neq \emptyset$ and $Y - X \neq \emptyset$, then $X - Y$ and $Y - X$ are also critical.

Proposition 3.4 *If X and Y are intersecting and $d(X) = d(Y) = k$, then*

$$d(X - Y) = d(Y - X) = k \quad \textit{and} \quad \overline{d}(X, Y) = 0.$$

For proving this statement, we write:

$$k + k = d(X) + d(Y) = d(X - Y) + d(Y - X) + \overline{d}(X, Y) \geq k + k,$$

with the last inequality holding by assumption. Since we have $2k \geq 2k$, equality must hold, in particular, $d(X-Y) = d(Y-X) = k$ and $\overline{d}(X, Y) = 0$.

Using this last proposition we can show that the sets X_i are pairwise disjoint.

This is since, assuming that M_u and M_v would be intersecting would lead to $M_v - M_u$ being critical, together with $\overline{d}(X, Y)$ being 0. But then $v \in M_v - M_u$, which is critical, a subset of M_v, a fact that contradicts the minimality of M_v.

Therefore, for the sub–partition $\{X_1, X_2, \ldots, X_t\}$ (since they are pairwise disjoint now) and s we can write:

$$d(s; G') = \sum_{i=1}^{t} (d(X_i; G') - d(X_i; G)) = \sum_{i=1}^{t} (k - d(X_i; G)) \leq 2\gamma,$$

where the last inequality holds by assumption.

Hence, proving the claim (3.9), we have showed that G can be made k-connected by adding at most γ new edges, so $\max\lceil(\sum_i(k - d(X_i)))/2\rceil$ new edges are enough to be added.

Then we have $d(s; G') = 2\gamma$. Using LOVÁSZ's Theorem, (3.7), we finish completing Professor FRANK's proof to Theorem (3.8) by splitting off all the 2γ edges. •

A similar theorem to the above holds for directed graphs as well, and was proved by A. FRANK in [8]:

Theorem 3.9 *A digraph $D = (V, E)$ can be made k-edge-connected by adding at most γ new edges if and only if*

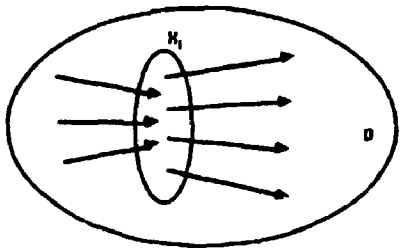
$$\sum_i (k - \varrho(X_i)) \leq \gamma \quad \textit{and} \quad \sum_i (k - \delta(X_i)) \leq \gamma \tag{3.10}$$

hold for every sub-partition $\{X_1, X_2, \ldots, X_t\}$ of V.

Proof: This proof is analogous to the proof of Theorem (3.8), for undirected graphs.

Necessity is again easy to see. Note that in a k-edge-connected graph every subset X_i has at least k edges entering it, and also at least k edges

leaving it. Without this, there would be no way of having k–edge–disjoint paths from (or to) a vertex in X_i to some other vertex outside X_i.



[a]

[a]If for the above picture k would equal 5, then no vertex in X_i would be able to be reached with 5 edge–disjoint paths, the most we can have is 3.

To prove sufficiency, we add a new vertex s to the graph, and γ new edges for every vertex v of V pointing from v to s, and γ more new edges from s to each v. This new graph D' will clearly be k–edge–connected. Then we delete as many of these new edges as possible, one by one, while keeping

$$\varrho(X; D') \geq k \quad \text{and} \quad \delta(X; D') \geq k$$

satisfied for every nonempty subset X of V. This is equivalent to keeping D' k–connected (by Theorem (3.2) of MENGER).

Now we add back any necessary edges that have been deleted in order to make $\varrho(s) = \delta(s)$.

Then by using the equivalent of Theorem (3.3) for directed graphs, we can not delete more new edges when the tail of every edge us (that is, u itself) belongs to an out–critical subset X_u of V, and the head of every edge vs (v) belongs to an in–critical subset X_v of V.[10]

Therefore, by the minimality of D', there is a family $\mathcal{F} = \{X_1, X_2, \ldots, X_t\}$ of in–critical subsets X_i of V covering neighbours of s, and with t being minimal.

Now we need to prove that $\delta(s; D') \leq \gamma$. Once that is done , by symmetry $\varrho(s; D') \leq \gamma$ will also follow.

One case could be that the family $\mathcal{F}$ of in–critical subsets of V consists only of disjoint sets. Then we have:

$$kt = \sum_{i=1}^{t} \varrho(X_i; D') = \delta(s; D') + \sum_{i=1}^{t} \varrho(X_i),$$

and therefore

$$\delta(s; D') = \sum_{i=1}^{t} (k - \varrho(X_i)) \leq \gamma.$$

The last inequality follows, since for showing sufficiency, we assume that equation (3.10) holds, and show that D can be made k- connected by adding at most γ new edges.

Another case could be that there are at least two intersecting members

[10]Recall that a set $X \subset V$ is out–critical (in–critical) for some k in question, when $\delta(X) = k$ ($\varrho(X) = k$).

of $\mathcal{F}$, say A and B. We show that $A \cup B = V$, and hence there could be only two such members of $\mathcal{F}$. For this, we assume that $A \cup B \neq V$.

It is possible to show that the union and the intersection of two in–critical sets is also critical, further that $d(A, B; D') = 0$ for two such sets A and B. Since A and B are in–critical (members of the family of all in–critical sets), $A \cup B$ is also in critical, hence replacing A and B by $A \cup B$ contradicts the minimal choice of t.

Now since $A \cup B = V$, we define $Y_1 := V - A$ and $Y_2 := V - B$. It follows that $\delta(Y_1) = \varrho(A)$ and that $\delta(Y_2) = \varrho(B)$. Since (3.10) holds, we have:

$$\gamma \geq k - \delta(Y_1) + k - \delta(Y_2) = k - \varrho(A) + k - \varrho(B) \geq$$

$$\geq k - \varrho(A; D') + k - \varrho(B; D') = \delta(s; D').$$

This proves the claim that $\delta(s; D') \leq \gamma$.

So we have showed that $\delta(s; D') = \gamma$, and now by applying MADER's Theorem (3.4) we complete the proof of theorem (3.9). •

The above two theorems give rise to combinatorial algorithms for constructing the γ new edges required to k–edge–connect a graph (digraph), called *Frank's Algorithms*. I have described this algorithm for the directed case in the Introduction section of this thesis. It is easy to see from the above proofs where the algorithm originates from. For undirected graphs

the algorithm is analogous, and follows the line of thought for the proof of Theorem (3.8).

Now we will describe FRANK's Algorithm to find a k-edge-connected augmentation of a digraph.

Given $D = (V, E)$.

- Add a new node s to the digraph.
- Add k new parallel edges from s to v for every $v \in V$.
- Order the new edges $f_1, f_2, \ldots, f_t$ so that all parallel edges from s to the same v_i get consecutive labels. [11] (Clearly $t = k \cdot n$, where n is the number of vertices in V.)
- Define $D' := (V \cup \{s\}, E \cup \{f_1, f_2, \ldots, f_t\})$.
- Go through the new edges in the given order, and discard an f_i if this can be done without destroying $\varrho(X; D') \geq k$ for every nonempty proper subset X of V.

[11] In [8] A. Frank describes his algorithm for a more general case, when there are costs c_{in} and c_{out} associated with each vertex v_i. Then the algorithm finds a minimum node-cost k-edge-connected augmentation of D. Hence the edges $f_1, f_2, \ldots, f_t$ are ordered according to the decreasing order of the c_{in} costs of their end-nodes, v_i (while the order of the parallel edges from s to v_i do not matter). In our case, that we have talked about in this paper, we have each $c_{in} \equiv c_{out} \equiv 1$.

- Let γ_1 be the number of the remaining new edges.
- Add k new parallel edges from each u in V to s.
- Order now these new edges $l_1, l_2, \ldots, l_t$ so that all parallel edges from the same u_i to s get consecutive labels. [12] (Again, $t = k \cdot n$.)
- Define $D' := (V \cup \{s\}, E \cup \{l_1, l_2, \ldots, l_t\})$.
- Go through the new edges in the given order, and discard an l_i if this can be done without destroying $\delta(X; D') \geq k$ for every nonempty proper subset X of V.
- Let γ_2 be the number of the remaining new l_i edges.
- Let $\gamma := max(\gamma_1, \gamma_2)$.
- If $\gamma_2 < \gamma_1$, then add $\gamma_1 - \gamma_2$ parallel edges from any u to s. [13]
- If $\gamma_1 < \gamma_2$, then add $\gamma_2 - \gamma_1$ parallel edges from s to any v. [14]
- Let D' denote the new digraph we have now. Note that $\delta(s) = \varrho(s) = \gamma$.
- Apply MADER's Theorem. 3.5, γ times to get an optimal k–edge–augmentation of D.

[12] In the more general case, $l_1, l_2, \ldots, l_t$ are ordered according to the decreasing order of the c_{out} costs of their end-nodes, u_i (while the order of the parallel edges from u_i to s do not matter).

[13] In the general case, add these edges back from the vertex u where the c_{out} is minimum to s.

[14] Now of course, in the general case, one would add the edges from s to the v where c_{in} is minimum.

FRANK's Algorithm to find a k–edge–connected augmentation of an undirected graph.

Given $G = (V, E)$.

- Add a new node s to the graph.
- Add k new parallel edges from s to v for every $v \in V$.
- Order the new edges $f_1, f_2, \ldots, f_t$ so that all parallel edges from s to the same v_i get consecutive labels. (Clearly $t = k \cdot n$, where n is the number of vertices in V.)
- Define $G' := (V \cup \{s\}, E \cup \{f_1, f_2, \ldots, f_t\})$.
- Go through the new edges in the order defined above, and discard an f_i if this can be done without destroying $d(X; D') \geq k$ for every nonempty proper subset X of V.
- If at the end of this procedure there are an odd number of new edges left, then add back the last deleted edge.
- Let γ be the number of the remaining new edges.
- Now we can apply Theorem 3.6 to split off all edges from s. This way we get an optimal k–edge–augmentation of G.

ANDRÁS FRANK proves in Section 9 of [8] that the above described algorithms run in polynomial time. He uses the max flow–min cut problem as subroutine, and techniques developed in the past for this family of problems. (See [10] for a recent survey of these algorithms.)

There Professor FRANK proves that in the algorithms, the deletion process can be done in time $O(n)$ for undirected graphs, and in time $O(n^2)$ for directed ones.

The splitting part is also proved to be of time-complexity $O(n^2)$ in the directed case. The directed case is similar.

Altogether, we have that this algorithm finishes in polynomial time in k and n. However, one might think that it would be polynomial in $\log k$ and n to be a truly polynomial algorithm. It is beyond the scope of this paper to detail these proofs, so we just refer the reader to [8] for the full details.

Having cleared up these problems and clarified definitions, we can turn to the first new topic of my work, the successive edge–augmentation problem.

4 The successive edge–connectivity augmentation problem

We have already introduced the problem of successive edge–connectivity in the Introduction. Here we will prove two theorems that solve this problem, one in both of the directed and undirected cases. This work is due to EDDIE CHENG and TIBOR JORDÁN [3].

4.1 The case of undirected graphs

EDDIE CHENG worked on this part of the problem following the workshop held in Budapest. A preliminary version of the paper was sent to ANDRÁS FRANK a few months after the workshop. Professor FRANK handed the work over to me for checking the details of proofs and arguments, checking its correctness and validity, reporting these back to him.

I have discovered some unclear parts in the procedure used to prove the main theorem, inconsistent notations and a really tedious language trying to express what is going on. I supplied Professor FRANK with a different argument, keeping the original ideas but clarifying the writeup so that it was easier to understand. Also included in this, I supplied arguments to fix the flaws discovered.

First we will prove a theorem that solves the successive edge–connectivity augmentation problem for undirected graphs in a different way than it has previous been solved in [16]. Here we will use the splitting off technique and *Frank's Algorithm.*

Recall that the successive augmentation problem is stated as follows.

Given a graph $G = (V, E)$ find a sequence of graphs $\{G_1, G_2, \ldots, G_k\}$, where each $G_i = (V, E_i)$ for all $1 \leq i \leq k$, such that

$$\lambda(x, y; G_i) \geq i \qquad \forall x, y \in V \tag{4.1.1}$$

$$G \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k, \tag{4.1.2}$$

and that each G_i is an optimal i-edge-connected augmentation of G (uses a minimal number of new edges).

Note that without the restriction of each G_i being an optimal i–augmentation of G, it is easy to construct a sequence of subgraphs that satisfies (4.1.2). However, usually it is not the case that every G_i will have the minimum number of new edges over G.

Also, one can determine the minimum number of new edges needed to make G i–edge–connected (and construct them) using Theorem (3.8) of Section 3. But just adding them for each i independently of the others and

constructing the sequence (4.1.2) will not necessarily be a solution to the successive edge–connectivity augmentation problem. Therefore some care needs to be taken here. [15]

WATANABE and NAKAMURA have proved in [16] that such a sequence exists and gave a polynomial time algorithm for constructing it.

E. CHENG proves something stronger, the *successive non–uniform edge–connectivity augmentation problem.* In this problem we suppose (in addition to conditions of the successive augmentation problem, stated above) that we are also given a sequence of demand functions $\{r_1, r_2, \ldots, r_k\}$, where each of the demand functions is defined on the ordered pairs of vertices, $r_i : (V \times V) \mapsto Z^+$. We are required to find the same sequence of graphs $\{G_i\}$, but instead of Equation (4.1.1), we now are asked to fulfill:

$$\lambda(x, y; G_i) \geq r_i(x, y) \quad \forall x, y \in V \quad \text{and} \quad 1 \leq i \leq k, \tag{4.1.3}$$

together with (4.1.2).

It is not hard to see that the uniform augmentation problem is just the special case when $r_i \equiv i$ for all $1 \leq i \leq k$.

For simplicity and easier understanding, here we prove the uniform problem and note that proving the non-uniform case is parallel to this line of

[15]This paragraph and the previous one was my own discussion, that I asked to be included in the final version of the paper to clarify the difference between this case and the trivial problem.

thought.

The proof of the uniform successive edge–connectivity problem uses the splitting off technique, which makes it different from the WATANABE and NAKAMURA proof.

The following theorem asserts that there exists a solution to the above stated — uniform — problem.

Theorem 4.1 *Let $k > 0$ be a positive integer and let $G = (V, E)$ be an undirected graph. Then there exist k sets of edges $H_1, H_2, \ldots, H_k$ on the same set of vertices V, with*

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k$$

such that $G_i = (V, E \cup H_i)$ is an optimal i-edge-connected augmentation of G for any $i = 1, \ldots, k$. [16]

Proof: We prove the theorem by giving a polynomial–time algorithm that will construct the sequence of edges $\{H_i\}$.

First we use *Frank's Algorithm* to find H_k.

Add a new vertex s and k copies of an edge from s to ever vertex v of V.

[16] In the original version, the author mixed the H_i's and the G_i's, sometimes meaning sets of subgraphs, sometimes sets of edges.

Clearly this new graph is k–edge–connected. Now delete greedily as many new edges as possible so that we keep $\lambda(x, y) \geq k$ for all $x, y \in V$. If $d(s)$ is odd, then add back the last edge deleted. [17] Call the set of new edges we now have F_k.

We can split off the edges in F_k by Theorem (3.8). The set of $d(s)/2$ new edges we get is the set H_k we are looking for. Also, G_k is $(V, E_k) = (V, E \cup H_k)$. This is an optimal k–augmentation of G by *Frank's Algorithm* (proof of Theorem (3.8)).

Now delete as many edges from F_k as possible, such that we keep $\lambda(x, y) \geq k-1$ for all $x, y \in V$. Again, if $d(s)$ is odd, then we add back the last deleted edge. This way we get the set F_{k-1} (note that $F_{k-1} \subseteq F_k$). With this we proceed in the same way as with F_k. That will yield the set H_{k-1} and the graph $G_{k-1} = (V, E \cup H_{k-1})$.

Observe that since $F_{k-1} \subseteq F_k$, and the way H_k and H_{k-1} are derived, we also have that $H_{k-1} \subseteq H_k$.

Similarly we can get the sets F_i, H_i and the graphs G_i for all $1 \leq i \leq k$.

[17] The author adds back any previously deleted edge, which is still correct, but this change makes it easier to implement in practice. That is because one needs to keep in memory only the last edge deleted which — if needed — can be added back, and not the set of all previously deleted edges, so that at the end choosing one at random of these to be added back. This is the same suggestion that I discussed in the previous section, in the proof of Theorem (3.8).

Note that now we have:

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k,$$

and

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k.$$

Also note that, it is enough to find F_i and H_i for i greater than what the edge–connectivity of G is to start with. If $\lambda(x, y; G) \geq \ell \quad \forall x, y \in V$ (that is, ℓ is the edge–connectivity of the original G), then the sets H_i for $i \leq \ell$ are all empty. [18] ($H_i = \emptyset \; \forall i \leq \ell$.)

The claim is that each G_i is an optimal i–edge–augmentation of G for any $i = 1, \ldots, k$.

In order to prove this last statement, it is enough to prove the following theorem.

Theorem 4.2 *If a pair of edges (su, sv) can be split off in some F_j $(1 \leq j \leq k)$, then the same pair (su, sv) can be split off in all F_i for all $j \leq i \leq k$.*

Proof: Theorem (3.3) gives us the condition when a pair of edges is splittable. To prove Theorem (4.2), we assume that (su, sv) is splittable in

[18] This note was not part of the original paper. I added it having read TIBOR JORDÁN's work on the analogous case for directed graphs.

$G_j^* := (V \cup \{s\}, E \cup F_j)$, but not splittable in G_{j+1}^*. Once we have disproved this possibility, then we proceed using induction to prove that (su, sv) is splittable in F_i, for all $j \leq i \leq k$.

If (su, sv) is splittable in G_j^*, but not splittable in G_{j+1}^*, then this means that there exists a $(j+1)$–dangerous subset A of V in G_{j+1}^* with $u, v \in A$; i.e.

$$d_{j+1}(A) \leq j + 2, \text{ [19]}$$

while A is not j–dangerous in G_j^*, so

$$d_j(A) > j + 1.$$

But knowing Theorem (3.1), this is only possible if

$$d_j(A) = d_{j+1}(A) = j + 2.$$

At this point we have two cases. One is when there was no deleted edge that we had to add back in order to keep $d_j(s)$ even in the process of calculating F_j from F_{j+1}.

The other case is when there was such an "added back" edge. Because if there was one — call it sw for some $w \in V$ — then it could happen that w

[19]Here by the subscript on the degree function we emphasize in which graph G_j^* we count the number of edges leaving A, i.e. $d_j(A) := d(A, G_j)$. The author did not make this clear; he used subscripts and superscripts, tracking different things, that made the argument hard to follow.

is in the set A. That is, the set A was $(j+1)$–dangerous in G^*_{j+1}, say. Then we deleted some edge (or edges) from A in the process of calculating F_j. It could be that the set A (now in G^*_j) is dangerous again, and hence we would not be able to split off any pairs of edges that would have an end in A. But since we added this edge sw back, we increased $d_j(A)$ by one, and now by our definition, A is not j–dangerous anymore. [20]

We could try to split off edges that enter this set A, but then those edges will not be splittable in G^*_{j+1}, since in that, A was $(j+1)$–dangerous. So we need to make sure that w does not belong to the set A).

Indeed, E. CHENG proves that w can not be in A. This is being done by carefully examining all possibilities of minimal critical sets containing u and v, their intersections with each other and their union with A (that it turns out that can not equal the whole V). At the end he concludes that all cases lead to contradictions, leaving no room for w. Therefore w can not belong to A.

Then if w is outside of A, it does not play a role in the computations, so we can assume that there is no edge sw that was added back at the end of the deleting process from F_{j+1}.

[20] I asked Professor FRANK to supply this argument to the author, as I thought that it was important to clarify why such a w is a key player in the proof.

Now let M_u and M_v be minimal critical sets in G_j^* containing u and v respectively. That is, $d_j(M_u) = d_j(M_v) = j$.

We can easily show that neither $v \in M_u$ nor $u \in M_v$ hold, otherwise we would have u and v belong to the same critical set in G_j^*, hence the pair (su, sv) would not be splittable in G_j^*. Also M_u and M_v can be taken to be nonintersecting. It is not hard to see that each of M_u and M_v contains a vertex not in A which is adjacent to an edge in $F_{j+1} - F_j$.

That is, $\exists sx, sy \in F_{j+1} - F_j \quad s.t.\ x, y \notin A$ but $x \in M_u$, say, and $y \in M_v$.

This is because in G_{j+1}^* the pair (su, sv) is not splittable. But we know that we can split off all edges in F_{j+1}, hence also su. So by Theorem (3.6), su must have a 'pair' in F_{j+1}, call this sx, with x as above. Similarly, we can proceed to find y in M_v. Therefore, it is clear that $A \cup M_u \neq V$ and $A \cup M_v \neq V$.

Now using Equation (2.1), the following inequality must hold:

$$(j+2) + j = d_j(A) + d_j(M_u) \geq d_j(A \cup M_u) + d_j(A \cap M_u) \geq (j+2) + j,$$

where the last inequality follows since $u, v \in A \cup M_u$ and (su, sv) are splittable in G_j^*. Hence $A \cup M_u$ is not dangerous, therefore $d_j(A \cup M_u) \geq j + 2$ (Theorem (3.3)). The other part of the right hand side, $d_j(A \cap M_u) \geq j$, follows since $d_i(X) \geq i$ for all $1 \leq i \leq k$ (by MENGER'S Theorem (3.1)).

Since in the above inequality we have $2j+2$ on both sides, equality must hold everywhere. In particular, $d_j(A \cap M_u) = j$ must hold.

Recall that the degree of A does not increase as we go from G_j^* to G_{j+1}^* $(d_j(A) = d_{j+1}(A) = j+2)$. Therefore, the degree of every subset of A stays the same also, in going from G_j^* to G_{j+1}^*. Hence $d_{j+1}(A \cap M_u) = d_j(A \cap M_u) = j$. But this is a contradiction to MENGER'S Theorem. Therefore, such a set A does not exist, and we have proved Theorem (4.2). •

This also completes the proof that the algorithm described above constructs a set of edges H_i for $i = 1, 2, \ldots, k$ satisfying the conditions of Theorem (4.1). •

4.2 The case of directed graphs

Now we turn our attention to directed graphs. T. JORDÁN proves a similar theorem to (4.1) for digraphs. The proof is done by constructing a polynomial time algorithm, much like the one described in that section.

Theorem 4.3 *Let $k > 0$ be a positive integer and let $G = (V, E)$ be a directed graph.* [21] *Then there exist k sets of edges $H_1, H_2, \ldots, H_k$ on the same set of*

[21] Note that in previous sections we denoted digraphs by $D = (V, E)$. Here for simplicity and easier comparison with the previous subsection, we will stay with the $G = (V, E)$ notation while talking about directed graphs.

vertices V, with

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k$$

such that $G_i = (V, E \cup H_i)$ is an optimal i-edge-connected augmentation of G for any $i = 1, \ldots, k$.

Proof: We start the algorithm by adding a new vertex s. Then we add k copies of edges from s to every vertex $v \in V$, and k copies of edges from all v's to s. In this new graph $\delta(s) = \varrho(s)$ and it is clearly k–edge–connected.

We delete as many edges as we can, such that we keep

$$\delta(s) = \varrho(s), \quad \delta(X) \geq k \quad \text{and} \quad \varrho(X) \geq k \tag{4.2.1}$$

satisfied for all non–empty proper subsets X of V.

Call the set of new edges we now have F_k. Splitting off these $\varrho(s)$ pairs of edges we get the set of edges H_k whose addition to G optimally k-edge-augments it. [22] Then $G_k := (V, E \cup H_k)$.

Similarly, we define F_i for all $1 \leq i \leq k-1$ by deleting edges from F_{i+1} while keeping the following satisfied:

$$\delta_i(s) = \varrho_i(s), \quad \delta_i(X) \geq i \quad \text{and} \quad \varrho_i(X) \geq i \tag{4.2.2}$$

[22] Note that so far this algorithm is the same as *Frank's Algorithm*.

for all nonempty $X \subset V$. Here δ_i and ϱ_i are the degree-functions in the graphs $G_i := (V, E \cup H_i)$, with H_i being the set of edges we get after splitting off the $\varrho_i(s)$ pairs (i.e. $\delta_i(X) := \delta(X; G_i)$).

Note that we delete edges from F_{i+1} until we can not delete more, that is, with the deletion of any one more edge, we would not have Equation (4.2.2) satisfied. Then to obtain F_i we do have to delete more edges. The size of the sets decrease, so we have $|F_i| < |F_{i+1}|$. But it may happen that we can not delete more than one edge. In which case $\delta_i(s) \neq \varrho_i(s)$, so we need to add back this one single deleted edge. Therefore we always have $|F_i| \leq |F_{i+1}|$.

Let us call such a monotone decreasing sequence of edges $\{F_1, F_2, \ldots, F_k\}$ *proper*.

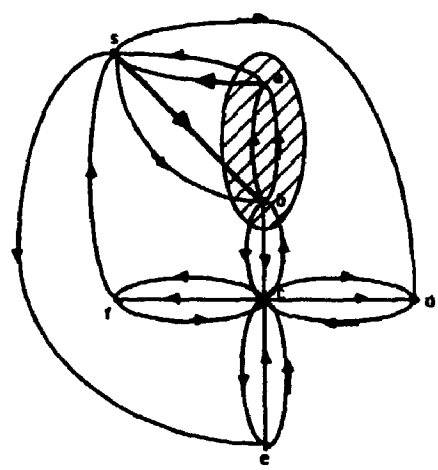
If we would have a theorem like Theorem (4.2), that is

> "*If a pair of edges is splittable in G_j then the same pair is splittable in every F_i for $1 \leq j < i \leq k$ where the F_i's form a proper sequence,*"

then that would prove Theorem (4.3). Let us call this our "Wish Theorem" for now.

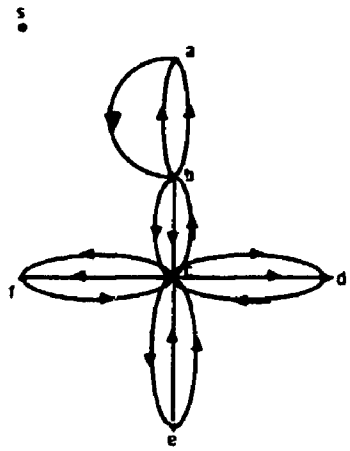
But in general this "wish theorem" does not hold for every proper sequence $\{F_i\}$. For instance consider the following counterexample.

Let $G = (V, E) = (\{a, b, c, d, e, f\}, \{ba, ba, cb, bc, bc, cd, dc, dc, ce, ec, ec, cf, cf, fc\})$ be a directed graph, with edge connectivity 0.



Then after adding the new vertex s, F_2 could be the set to $\{as, as, fs, sb, sd, se\}$. Splitting these off gives G_2, a 2-connected augmentation of G. Deleting edges from F_2 until possible will give us F_1. One such F_1 would be $\{as, sb\}$. Note that $F_1 \subset F_2$. In the picture, the two edges in F_1 are drawn in bold. Also note that F_2 and F_1 satisfy all conditions of Equations (4.2.1) and (4.2.2). Therefore they form a proper sequence. Then G_1 is the graph we get after splitting off the pair of edges (as, sb) (the only

pair in F_1), see the next picture.



However, observe that the pair of edges (as, sb) is *not* splittable in G_2. This is since the set $\{a, b\}$ in G_2 (the set that is shaded in the picture on the previous page) is 3–in–dangerous (i.e. $\varrho_2(\{a, b\}) = 3$). Therefore, even though the edges entering and leaving s can be split off to satisfy our conditions, the resulting set of edges (by splitting off F_2) will not be a superset of the set resulting from splitting off the edges in F_1

But the problem we face is not hopeless. We can modify an existing proper sequence, and form another proper sequence, such that our "wish theorem" above will hold true for the new proper sequence.

The idea is that starting from $G_{\ell+1}$, [23] we find a pair of edges that are splittable in $G_{\ell+1}$, that will be called (us, sv). Then check whether this pair is splittable in $G_{\ell+2}$, $G_{\ell+3}$, and so on .... All the way up to G_k. If (us, sv) is splittable in all G_i for $i \leq k$, then we split it off in every F_i. Then start over with the reduced graph $G^r := (V, E \cup \{uv\})$ and the reduced proper sequence

$$F_i^r := F_i - \{us, sv\} \quad \forall i = \ell + 1, \ldots, k.$$

However, if this pair (us, sv) is splittable all the way up to some G_t, but not splittable in G_{t+1}, then we construct a new proper sequence $\{F'_{\ell+1}, \ldots, F'_k\}$ and a new pair of edges that will be splittable in $G'_{\ell+1}$, $G'_{\ell+2}$, ..., all the way up to G'_t, and also splittable in G'_{t+1}. This way we move up by induction, until we reach F_k. Then we split off this new pair of edges, and start over again.

We construct the new proper sequence $\{F'_i\}$ by modifying the sets F_i for $\ell + 1 \leq i \leq t$ and letting $F'_i := F_i$ for $t + 1 \leq i \leq k$.

Recall, that the pair of edges (us, sv) is not splittable in G_i for some i, if there exists a proper subset X of V such that $\delta(X) = i$ or $\varrho(X) = i$.

[23] Recall that ℓ was the edge-connectivity of G, and that the sets F_i and H_i are all empty for $i \leq \ell$.

We will also need the following two inequalities. If $X \subseteq Y \subset V$ and $\ell + 1 \leq i < j \leq k$ then

$$\delta_j(Y) - \delta_i(Y) \geq \delta_j(X) - \delta_i(X) \tag{4.2.3}$$

and

$$\varrho_j(Y) - \varrho_i(Y) \geq \varrho_j(X) - \varrho_i(X) \tag{4.2.4}$$

must hold. These can be easily proved by using the fact explained above that $|F_i| \leq |F_{i+1}| \quad \forall i$, hence $\delta_i(X) \leq \delta_j(X) \quad \forall X \subset V$ and $\ell + 1 \leq i < j \leq k$.

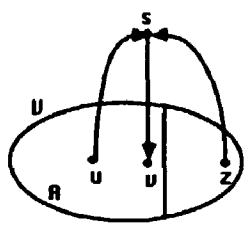
We assume that the pair (us, sv) is splittable in G_t but not splittable in G_{t+1}. This means that there exists a proper subset A^* of V such that $\delta_{t+1}(A^*) = t + 1$ or $\varrho_{t+1}(A^*) = t + 1$.

Let A_u and A_v be maximal critical subsets of V containing u and v respectively. It is easy to show that A_u (and A_v) are the union of all critical subsets containing u (and v, respectively), and also that $A_u = A_v$. [24] Call this set A. Obviously $A^* \subseteq A$, so for maximality we use A instead of A^* in the following.

Two cases are possible. One is when A is out–critical in G_{t+1}, $\delta_{t+1}(A) = t + 1$, or in–critical, $\varrho_{t+1}(A) = t + 1$. The two cases are analogous, and here we work out only the first case. So assume that A is out–critical.

[24] The author proves that $A_u = A_v$ in his paper, but this is not needed, since both this claim and that A_u is the union of all critical subsets containing u has been proved in FRANK's [8] publication.

Since us and sv are not splittable in G_{t+1}, but all $2\varrho_{t+1}(s)$ edges can be paired up so that we can split off all of them, there must exist an edge, say zs in $F_{t+1} - F_t$, with $z \notin A$ such that the pair (sv, zs) is splittable in G_{t+1}. Since $zs \in F_{t+1} - F_t$ we also have $zs \notin F_j$ for all $j < t$. If z were in A, then this pair again would not be splittable in G_{t+1} since A is critical. [25]



We have now formed the new pair of edges (sv, zs), and we formed them in such a way that they are splittable in G_{t+1}. All we have left to do is to modify the sets F_i for $\ell + 1 \leq i \leq t$ and show that the new pair is splittable in these, and show that the new sequence $\{F'_{\ell+1}, \ldots, F'_k\}$ is proper.

Define

$$F'_i := (F_i - \{us\}) \cup \{zs\} \quad \forall \ell + 1 \leq i \leq t,$$

$$F'_i := F_i \quad \forall t < i \leq k,$$

[25]This paragraph was supplied by me, to clear up a circular argument of the author to prove the existence of such a vertex z, hence the existence of the edge zs.

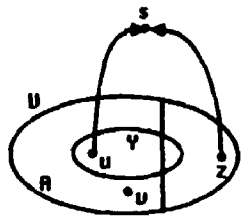
and

$$G_i' := (G_i - F_i) \cup F_i' \quad \forall i.$$

Let us first show that this new sequence is proper. [26]

In order to do this, we need to show that Equation (4.2.2) is satisfied for all $i = \ell + 1, \ldots, t$. Assume that (4.2.2) is not satisfied for some j, $\ell + 1 \leq j \leq t$. This implies that there exists a subset Y of V, such that $u \in Y$, $z \notin Y$ and $\delta_j(Y) < j$. This is because both $\delta_j(s)$ and $\varrho_j(s)$ stayed the same as before, they were not affected by deleting the edge us and adding zs to F_j. Similarly, the indegree of any subset of V has not been changed either, hence $\varrho_j(Y)$ stayed the same as well. Also, the case when $z \in Y$ is not interesting, because then by deleting the edge us from F_j we decreased the out–degree of Y by one, and adding zs we increased its out–degree back to its original value, which satisfied Equation (4.2.2). Therefore, the only possible case when Equation (4.2.2) is not satisfied is when $\delta_j(Y) < j$ and $u \in Y$, $z \notin Y$. Note that, we also have $Y \cup A \neq V$.

[26]The author did prove that the new sequence is proper and that the pair of edges (sv, zs) is splittable in all F_i''s, but in the argument the cases and subcases ran together, it was hard to tell one subcase from the other. The following paragraphs are the cleared up arguments that I provided the author with. They separate the bits and pieces of the proof, making it more readable and understandable.



We write Equation (4.2.3) with subscripts j and $t+1$ for the sets A and $A \cap Y$:

$$\delta_{t+1}(A) - \delta_j(A) \geq \delta_{t+1}(A \cap Y) - \delta_j(A \cap Y).$$

In here $\delta_{t+1}(A \cap Y) \geq t+1$ since Equation (4.2.2) does hold for $i = t+1$, and $\delta_{t+1}(A) = t+1$, as this is what we assumed. This way we get that:

$$\delta_j(A \cap Y) \geq \delta_j(A).$$

Substituting this into Equation (2.2) for A and Y

$$\delta_j(A) + \delta_j(Y) \geq \delta_j(A \cap Y) + \delta_j(A \cup Y), \quad \text{since} \quad d(A, Y) \geq 0$$

we get:

$$\delta_j(A) + j > \delta_j(A) + (j+1),$$

clearly a contradiction. [27]

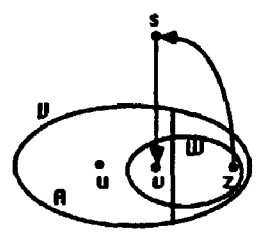
[27] Writing the above four equations were expanded by me to explain what the author expressed in the following single line:

$$k + 1 - c + j = \delta_j(A) + \delta_j(Y) \geq \delta_j(A \cap Y) + \delta_j(A \cup Y) \geq k + 1 - c + j + 1,$$

where $c := k + 1 - \delta_j(A)$, and concluding that this is a contradiction.

Therefore Equation (4.2.2) is satisfied for all $i = \ell + 1, \ldots, t$. The same equation still holds for $i > t$ since we have not modified the sets F_i $i = t + 1, \ldots, k$, therefore the sequence $\{F'_{\ell+1}, \ldots, F'_k\}$ is proper.

Now all we are left to prove is that the pair (sv, zs) is splittable in all G'_i for $i \leq t$. To do this, we assume that it is not splittable in some G'_j, $\ell + 1 \leq j \leq t$. That would happen when there would exist a critical set W in G'_j with v and z in W. Also $u \notin W$ would have to hold, since then the pair (us, sv) would not have been splittable in G_j.



We said that W is critical. Again, there are two cases to consider. First say that W is out critical in G'_j, that is $\delta_j(W; G'_j) = j$.

Above we concluded that $u \notin W$. Then we get that $\delta_j(W) < \delta_j(W; G'_j) = j$, since by adding zs to F_j, we have increased the out-degree of W by one. This is in contradiction with Equation (4.2.2).

Therefore we must have that W is in–critical, $\varrho_j(W; G'_j) = j$. We still have that $u \notin W$. Define W^* to be $V \cup \{s\} - W$. [28] Write Equation (4.2.3) for subscripts $t+1$, j and sets A, $A \cap W^*$:

$$\delta_{t+1}(A) - \delta_j(A) \geq \delta_{t+1}(A \cap W^*) - \delta_j(A \cap W^*).$$

Here $\delta_{t+1}(A \cap W^*) \geq t+1$ and $\delta_{t+1}(A) = t+1$ for reasons mentioned before. Then this yields that $\delta_j(A \cap W^*) \geq \delta_j(A)$.

Write Equation (2.2) for A and W^*:

$$\delta_j(A) + \delta_j(W^*) \geq \delta_j(A \cap W^*) + \delta_j(A \cup W^*) + 1,$$

since $d(A, W^*) \geq 1$ (because of the edges sv, zs).

Substituting $\delta_j(W^*) = \varrho_j(W) = j$, $\delta_j(A \cap W^*) \geq \delta_j(A)$ and $\delta_j(A \cup W^*) = \varrho_j(W - A) \geq j$ (this one follows since Equation (4.2.2) must also hold for $W - A \subset V$) into the above inequality, we get the following contradiction: [29]

$$j \geq j + 1.$$

Therefore we have constructed a new pair of edges (sv, zs) that is splittable in all G_i's for $i \leq t$ and it is also splittable in G_{t+1}. We have deleted

[28] The author denotes this by W'. I've suggested to use some other notation, like W^*, so that this is not confused with the the symbol "prime" used for G' and F'.

[29] The above equations are again my expansions of the author's one line argument, that $k+1-c+j = \delta_j(A) + \delta_j(W^*) = \delta_j(A \cap W^*) + \delta(A \cup W^*) + d_j(A, W^*) \geq k+1-c+j+1$, with c as in the second previous footnote, (21), and concluding that this is a contradiction.

the pair (sv, us) from all F_i, $i \leq t$, (that was only splittable in the G_i's for $i \leq t$ but not splittable in G_{t+1}), and replaced it with the new pair of edges (sv, zs). Using this as the induction step, we can proceed ahead and split off all edges entering and leaving s to get the sets $F_1, F_2, \ldots, F_k$, and hence the sets $H_1, H_2, \ldots, H_k$ in the way Theorem (4.3) claimed it.

This finishes the proof of Theorem (4.3). •

We have also provided a polynomial time algorithm to solve the successive edge–augmentation problem, just by following the steps of the proof.

We now turn our attention to edge–augmenting hypergraphs by edges of size two.

5 Augmenting hypergraphs by edges of size two

In this section we give the minimum number of edges of size two whose addition to a given hypergraph H makes it k–edge–connected. This work is due to JØRG BANG–JENSEN and BILL JACKSON, two other participants at the workshop organised by Professor FRANK in Budapest. [30]

We need to introduce the notion of the *component of a graph* G, that is defined as a maximal connected subgraph of G. We shall denote the *number of components* of G by $\omega(G)$.

The number of edges of size two whose addition to H makes H k-connected when it is already $(k-1)$–connected has previously been determined. In this section we give a solution to the general problem by proving the following theorem:

Theorem 5.1 *Let $H = (V, E)$ be a hypergraph and k a natural number.*

[30]This paper was the one that had the least amount of problematic sections in it, amongst the papers I've checked for Professor FRANK. All of the proofs worked as written, all the examples fit the ideas they were to illustrate, the algorithm corresponded to the main theorem of the section and accounted for all possible cases. There have been only minor typos, a few mismatched subscripts and misplaced words (used "Backtracking Step" at one point instead of "Forward Step"), but in general a very thorough work. I've reported these little mistakes to ANDRÁS FRANK, who in turn forwarded them to the authors. I have somewhat rewritten this paper, added the examples that will follow, reorganised the order of theorems so that they fit into this thesis in a logical way with the other problems discussed.

Then the minimal number of edges of size two whose addition to H results in a k-edge-connected hypergraph is

$$\gamma_k(H) = \max\left(\left\lceil\frac{\sum(k - d(X_i))}{2}\right\rceil, c_k(H) - 1\right),$$

where the maximum is taken over all subpartitions $\{V_1, V_2, \ldots, V_k\}$ of V, and $c_k(H)$ is the maximum value of $\omega(H - A)$ taken over all $A \subseteq E$ with $|A| \leq k - 1$. [31]

Before proving this theorem, we need a few more definitions and results specific only to this section.

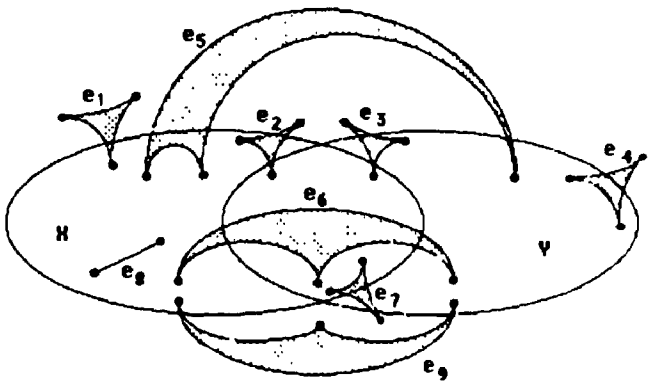
We call a hypergraph *simple* if no two edges correspond to the same subset of vertices. Similar equations to (2.1) and (2.4) also hold for hypergraphs. For this we define the following (with $X, Y \subseteq V$):

- $d_1(X, Y)$ to be the number of edges which intersect precisely [32] $X - Y$ and $Y - X$ (see picture below for example);
- $d_2(X, Y)$ to be the number of edges which intersect precisely $X - Y$, $Y - X$ and exactly one of $X \cap Y$ or $\overline{X \cup Y}$;
- $d_3(X, Y)$ to be the number of edges intersecting precisely $X \cap Y$ and $\overline{X \cup Y}$;

[31] This version of the theorem is my reformulation of the original, so that it fits in with the previous sections of this thesis.

[32] In this context "precisely" means that we are counting only those hyper-edges — viewing them as subsets of the vertex set — whose vertices are either in $X - Y$ or in $Y - X$.

- $d_4(X, Y)$ to be the number of edges intersecting precisely $X \cap Y$, $\overline{X \cup Y}$ and exactly one of $X - Y$ or $Y - X$.



[a]

[a]In this picture, e_5 would add one to $d_1(X, Y)$, and also to $d(X)$ and to $d(Y)$. However, it does not contribute neither to $d_2(X, Y), d_3(X, Y)$ nor $d_4(X, Y)$. On the other hand, both e_6 and e_9 add one to $d_2(X, Y)$ (and naturally to $d(X)$ and $d(Y)$). To $d_3(X, Y)$ the edge e_7 adds one, while both e_2 and e_3 increment $d_4(X, Y)$ by one.
Also note that edges e_1 and e_4 increment only $d(X)$ and $d(Y)$ respectively, and e_8 does not play a role here.
For the above example, Equation (5.1) (see below) looks like:

$$7 + 7 = 4 + 6 + 2 \cdot 1 + 2,$$

while Equation (5.2) has the form:

$$7 + 7 = 5 + 5 + 2 \cdot 1 + 2.$$

Then we have:

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d_1(X, Y) + d_2(X, Y). \qquad (5.1)$$

and

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d_3(X, Y) + d_4(X, Y). \qquad (5.2)$$

Deriving these equations is analogous to proving Equations (3.1), (3.2), (3.3) and (3.4). Consider $d(X)+d(Y)$ and compare it with $d(X \cap Y)+d(X \cup Y)$. Now observe that everything counted on the right hand side is counted on the left hand side as well. Now there are some other edges that have not been counted on the right hand side, and it's not hard to see that those exactly add up to $2d_1(X, Y) + d_2(X, Y)$.

For hypergraphs the notion of a path is slightly different, the rest of the definitions are all the same. A *path in a hypergraph* H is an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, v_3, \ldots, e_p, v_p$, such that the set $\{v_i, v_{i+1}\} \subseteq e_i$ for all $i = 1, 2, \ldots, p-1$.

We now introduce the notion of contracting parts of a hypergraph. When $T \subseteq V$, we denote by H/T the hypergraph obtained from $H - T$ by adding a new vertex t and edges $(e - T) \cup \{t\}$ for all $e \in E$, such that $e \cap T \neq \emptyset$. H/T is the hypergraph obtained from H by *contracting* T.

Splitting in hypergraphs is the same as for graphs, but here we only

split pairs of edges, each of size two. We have one more new notion to define regarding splittings. A *complete k-splitting* of s (s defined just as in the section with undirected graphs) is a sequence of hypergraphs $H = G_0, G_1, G_2, \ldots, G_m$, where $m = \frac{1}{2}d(s)$, such that G_{i+1} is obtained from G_i by performing a k–splitting in G_i.

In general the existence of one k–splitting at s does not imply the existence of a complete k–splitting at s. The following theorem determines when this is possible.

Theorem 5.2 *Let $G = (V \cup \{s\}, E)$ be a hypergraph such that $\lambda(x, y) \geq k, \quad \forall x, y \in V$ and $d(s) = 2m$. Then either*

- *there is a complete k-splitting of s, or*
- *there exists $A \subseteq E$ with $|A| = k - 1$ and $\omega(G - \{s\} - A) \geq m + 2$.*

Now we are ready to prove Theorem (5.1). [33]

Proof: It can be easily seen that

$$\gamma_k(H) \geq \max\left(\left\lceil \sum \frac{(k - d(X_i))}{2} \right\rceil, c_k(H) - 1\right). \tag{5.3}$$

hence it only remains to show that:

$$\gamma_k(H) \leq \max\left(\lceil \sum (k - d(X_i))/2 \rceil, c_k(H) - 1\right).$$

[33] I have somewhat rewritten the proof, leaving the original ideas intact, but changing the flow of the argument, so that it is easier to understand.

Start with the hypergraph H. Add a new vertex s and k copies of an edge from s to every vertex v of V. Then delete as many edges from s as possible until either

(1) we can not delete more edges without violating $\lambda(x,y) \geq k \quad \forall x, y \in V$, or

(2) there are exactly $2(c_k(H) - 1)$ edges incident with s.

If $d(s)$ is odd, then we add back an edge from s to an arbitrary $v \in V$. [34] Call the hypergraph we now have G. Note that by Theorem (5.2) there is a complete k–splitting of s in G, and hence $\gamma_k(H) \leq \frac{1}{2}d(s)$.

If (2) occurs first in the above deletion process, then $\gamma_k(H) \leq c_k(H) - 1$ and hence combining this with Equation (5) we get that $\gamma_k(H) = c_k(H) - 1$.

If (1) occurs first, then we try to follow the same procedure as above, by first proving that $d(s)$ at the end of the deletion process, before adding back the necessary extra edge, is less than or equal to $\max \sum(k - d(X_i))$ over all subpartitions $X_1, X_2, \ldots, X_r$ of V. This would imply that $d(s) \leq \lceil \max \sum(k - d(X_i))/2 \rceil$. Hence using Equation (5) we would get that $d(s) = \lceil \max \sum(k - d(X_i))/2 \rceil$.

Since at the end of the deletion process (before adding back a possible extra edge), s is such that every edge incident with it enters a k–critical set

[34] Again here we should add back the last deleted edges, just as in previous sections, which makes a difference at the implementation of the algorithm on a computer.

X in V. Let S be the set of all vertices incident with s at this stage. Let W_v denote a minimal (to inclusion) k–critical set containing v.

Define $\mathcal{F} = \{W_v | v \in S\}$ and $X_1, X_2, \ldots, X_q$ be the maximal members of $\mathcal{F}$. Using the fact that if X and Y are k–critical subsets of V, then each of $X - Y$ and $Y - X$ are k–critical and $d_3(X, Y) = d_4(X, Y) = 0$, we can prove that the sets $X_1, X_2, \ldots, X_q$ form a subpartition of V.

Now we have:

$$d(s) = \sum_{i=1} t(d'(X_i) - d(X_i)) = \sum_{i=1} t(k - d(X_i)) \leq$$
$$\leq \max \sum (k - d(X_i))$$

over all subpartitions $X_1, X_2, \ldots, X_r$ of V, where $d'(X_i)$ is the degree of the set X_i with s and all the edges incident with it present, and $d(X_i)$ is just the degree function in H without s and incident edges.

This completes the proof of Theorem (5.1). •

In the following we present a strongly polynomial algorithm that finds an optimal k–augmentation of a hypergraph H by edges of size two. [35]

The *initialization step* of the algorithm consists of adding a new vertex s to H along with k copies of an edge from s to each $v \in V$. Let H' denote the

[35]By strongly polynomial, here we mean that the complexity of the algorithm depends on only the number of vertices and edges in the starting hypergraph, provided we count all basic arithmetic operations as constant time procedures.

resulting graph, which is clearly k–edge–connected. Label the vertices of V by $v_1, v_2, \ldots, v_n$. Remove as many edges from s to v_1 as possible so that the resulting hypergraph H'' satisfies $\lambda(x, y; H'') \geq k \quad \forall x, y \in V$. Then do the same for $v_2, \ldots, v_n$ in that order. Call the resulting hypergraph H_1. Now in H_1 no edge incident with s can be removed without violating the condition that $\lambda(x, y; \overline{H}) \quad \forall x, y \in V$ in the resulting hypergraph $\overline{H}$. If $d(s; H_1)$ is odd, then we add back an edge from s to v_n. Let G denote the hypergraph obtained this way from H_1. Define $m = d(s; G)/2$.

The following step is the *splitting step*. This constitutes of greedily performing a k–splitting on pairs of edges incident with s, always splitting away as many parallel edges as possible, until we either have a complete k–splitting at s (and hence obtain an augmentation of H), or we find an edge st which is not k–splittable.

If a complete k–splitting is possible, then we split off the $\max \sum(k - d(X_i))$ pairs of edges and start over with the reduced hypergraph. [36]

If a complete k–splitting is not possible, i.e. we have an edge st that is not splittable, then $d(s; G_r) \geq 4$ in the resulting hypergraph G_r. Then we apply the following lemma and theorem to find $2(m - r)$ components $H_1, \ldots, H_{2(m-r)}$ of $G - \{s\} - A$ for some $A \in E$ with $|A| = k - 1$ and $\omega((G_r -$

[36] What we mean here, is that perform a k–splitting for one pair of vertices, if possible. Then perform another one on another pair of vertices, until there are such pairs left. If we split off all of the pairs, then we are done. If not, then we go to the following case.

$\{s\}) - A) = d(s; G_r) = 2(m - r)$, as needed.

Lemma 5.1 *If $H = (V \cup \{s\}, E)$ is a hypergraph such that $\lambda(x, y; H) \geq k \quad \forall x, y \in V, d(s) = 2m$ and $st \in E$ with st not k-splittable in H, then we can find $2m$ components $H_1, \ldots, H_{2m}$ of $H - \{s\} - A$, where $A \subseteq E$, $|A| = k - 1$ and $\omega(H - \{s\} - A) = 2m$.*

The proof of this lemma gives rise to a sub-algorithm to find the components $H_1, \ldots, H_{2m}$, and uses the following theorem.

Theorem 5.3 *If $H = (V \cup \{s\}, E)$ is a hypergraph such that $\lambda(x, y; H) \geq k \quad \forall x, y \in V, d(s) = 2m$ and $st \in E$, then either*

- *there is a k-splitting at s using the edge st, or*
- *$m \geq 2$ and there exists a set A in E with $|A| = k - 1$ and*
  - *$\omega(H - \{s\} - A) = 2m$*
  - *s has one neighbor in each component of $H - \{s\} - A$*
  - *each $e \in A$ intersects every component of $H - \{s\} - A$.*

To find component H_1 we denote the neighbors of s in V by $u_1, \ldots, u_{2m}$ and delete the edge su_1. Then let $W_1 = \{v \in V | \lambda(v, u_2; H - \{su_1\}) = k - 1 = \lambda(v, u_3; H - \{su_1\})\}$. Then the subgraph of H spanned by W_1 is H_1. Similarly we can find $H_2, \ldots, H_{2m}$.

Continuing the *splitting step*, we let $G = G_0, G_1, \ldots, G_r$ be the sequence of hypergraphs obtained when we made the k–splittings above. Here going from G_j to G_{j+1} corresponds to k–splitting one pair of edges (su, sv). (Thus in the case where we split a set of ℓ parallel edges su with ℓ parallel edges sv, then we may think of this as going from G_j to $G_{j+\ell}$. The sequence $G_j, \ldots, G_{j+\ell}$ is then taken to be any arbitrary ordering of the hypergraphs which can be obtained from G_j by adding one edge at a time between u and v and deleting one pair of edges (su, sv), until we have added ℓ new edges between u and v.)

Now we perform the *backtracking step*. Go backwards through the sequence of hypergraphs $G_r, \ldots, G_1, G_0$. For each $0 \leq j < r$ replace the edge uv that we added to go from G_j to G_{j+1} by two edges su and sv. We stop either when we reach G_0, or we find the largest i such that $\omega(G_i - \{s\} - A) = \omega(G_{i+1} - \{s\} - A)$. Note that since we have already found the components $H_1, \ldots, H_{2(m-r)}$ we can identify this i.

First suppose that we stop before we reach G_0, i.e. we have found such an i. In G_i we have $d(s; G_i) = 2(m-r) + 2(r-i)$ and $\omega(G_i - \{s\} - A) = 2(m-r) + (r-i) - 1$. Then we can show that at least two of the sets $W_1, \ldots, W_{2(m-r)}$ (with the W_i's as defined above) have exactly one edge to s in G_i. Without loss of generality, we may assume that these two sets are H_1 and H_2.

Now we perform the *forward step*, that consists of performing a sequence of single k–splittings to obtain a sequence of graphs $G_1 = G'_1, G'_{1+1}, \ldots, G'_q$ such that G'_{j+1} is obtained from G'_j by performing a k–splitting using the edge sw, where w is the unique neighbor of s in some component of $G'_j - \{s\} - A$, $\quad 1 \leq j < q$.

We assume that q is as large as possible. Then we get from the above that

$$\omega(G'_{j+1} - \{s\} - A) = \omega(G'_j - \{s\} - A) - 1.$$

Then using a technique that was used in the proof of Theorem (5.2), we get that $q \geq r + 1$. Thus we either obtain a complete k–splitting this way (suppressing s when it becomes of degree two), or we reach a situation where $d(s; G_q) \geq 4$. If the latter case is what we reach, then using Lemma (5.1) we find the $2(m-q)$ components $H'_1, \ldots, H'_{2(m-q)}$ of $G_q - \{s\} - A'$ for some $A' \subseteq E$ with $|A'| = k-1$ in G_q. Also $\omega(G_q - \{s\} - A') = d(s; G_q) = 2(m-q)$. Now we repeat the *backtracking step* starting from G_q.

Continue this way until we have either found a complete k splitting of s, or we have backtracked all the way up to G_0 in some *backtracking step* Every time we start a new *backtracking step* the degree of s in the starting hypergraph has decreased by at least two.

Suppose that this process has stopped because we backtracked all the way

up to $G_0 = G$. Let G_q be the last hypergraph from which we have started the last *backtracking step*. Then $d(s;G) = 2m$ and using Theorem (5.2) we can prove that there does not exist a complete k–splitting of s in G. However we claim that $c_k(H) = c_k(G - \{s\}) = 2(m - q') + q'$. So we have either constructed a complete k–splitting of s, or have constructed a set of $c_k(H)$ components with $d(s;G) \leq 2c_k(H) - 4$.

Lastly we perform an *adding step*. Let G^* be the hypergraph obtained from G by adding $2((m - q') - 1)$ parallel edges from s to v_n. Using Theorem (5.2) again, there exists a complete k–splitting of s in G^*.

To finish the description of the algorithm, it remains to show how to find this last complete k–splitting of s. We perform a sequence of *forward steps* and *backtracking steps* on G^* instead of on G. From the above it follows that this will lead to a complete k–splitting of s and hence to a k–edge–augmentation of H with $c_k(H) - 1$ edges.

Therefore we have provided an algorithm that determines the number of edges of size 2 that k–edge–connects a given hypergraph H.

In the following section we turn our attention to a special kind of connectedness augmentation problem, the so–called (S,T)–connectedness problem.

6 A 1–(S,T)–edge–connectivity augmentation algorithm

In this section I present STEFFEN ENNI's work ([5]), who devised a combinatorial time algorithm for the 1–(S,T)–edge–connectivity augmentation problem in digraphs, a special case of the general (S,T)–connectivity problem. [37]

In the following, we will often use the union of a set and a one-element set. For this reason we will abbreviate $X \cup \{v\}$ to $X + v$ and similarly $f(\{v\})$ to $f(v)$. A vertex v is called isolated if there are no edges incident with v. For a subset of vertices $S \subset V$ we let $G[S]$ denote the graph induced by the vertices in S. Recall that the *component of directed graph* G is a maximal (with respect to inclusion) set of vertices which are connected using the edges of G in the undirected sense.

A directed graph $G = (V, E)$ with two specified non-empty subsets of vertices S and T (which need not be disjoint) is said to be k–(S,T)–*edge connected* if $\lambda(s,t;G) \geq k$ for all choices of $s \in S$, $t \in T$.

[37] STEFFEN ENNI was another participant at the November '94 workshop in Budapest referred to in the Introduction. In the months that followed, I have received transcripts of his work through Professor FRANK. From then on, we kept in touch through email while working on the ideas in the paper, and even met in person, when STEFFEN came to Budapest for a short visit. We finalised the proofs then, in person. On the following pages I will indicate more closely the places where my ideas were used, but I have had comments and suggestions to every paragraph of his paper.

A family $\mathcal{F}$ of subsets of vertices is (S,T)-*independent* if $S-X$ and $X \cap T$ are both nonempty for every $X \in \mathcal{F}$ and no two members X, Y of $\mathcal{F}$ separates the same pair of vertices $s \in S, \quad t \in T$. This definition is equivalent to the following: for every two members X, Y of $\mathcal{F}$ we either have $X \cap Y \cap T = \emptyset$ or $(S-X) \cap (S-Y) = \emptyset$. This says that, either any two sets are disjoint in T — do not have a common element in T — or they cover S. It is this later definition that we will use in what follows.

Given a family of subsets of vertices $\mathcal{F}$ and a function p defined on the power-set of V, we define $p(\mathcal{F}) := \sum(p(X)|X \in \mathcal{F})$.

The following theorem was proved in [9] as a consequence of coverings of crossing bi-supermodular functions using the ellipsoid method and does not yield a polynomial time algorithm for finding the γ set of edges needed to augment a graph with.

Theorem 6.1 *A digraph $G = (V, E)$ with two non-empty subsets $S, T \subseteq V$ can be augmented to a k-(S,T)-edge-connected graph with a set F of γ edges with tails in S and heads in T if and only if*

$$\sum_{X \in \mathcal{F}} p_{def}(X) \leq \gamma$$

for all (S,T)-independent families $\mathcal{F}$ of subsets of V, where $p_{def}(X) :=$ $\max(0, k - \varrho(X))$.

However, note that, when in the above theorem we let $S = T = V$, then we are back at the k-edge-augmentation problem already discussed before for digraphs. There are good combinatorial algorithms to solve that.

In the above theorem, when $k = 1$ (the case we will be dealing with in the remainder of this section), the definition of $p_{def}(X)$ reduces to the maximum of 0 or 1, whether $\varrho(X) > 0$ or $\varrho(X) = 0$, in that order. That is:

$$p_{def}(X) = \begin{cases} 0, & \varrho(X) > 0, \\ 1, & \varrho(X) = 0 \end{cases}$$

So the maximum value the sum $\sum_{X \in \mathcal{F}} p_{def}(X)$ can reach occurs when all $p_{def}(X)$'s are 1 for all X (S,T)-independent subsets of V. That is, when $\varrho(X) = 0 \ \forall X \in \mathcal{F}$. Now the sum equals the maximum number of (S,T)-independent subsets of V. Note that, therefore, when we are given a maximal (S,T)-independent set of subsets of V, we know that for each subset we must have $\varrho(X) = 0$ holding. Otherwise, if there were an incoming edge into a set X, that would make $\varrho(X) > 0$, hence reduce the value of the above sum. Then the sum will not be maximal, hence the set of subsets is not a maximal (S,T)-independent family. [38]

Hence (by the above observation and Theorem (6.1)) we get that the minimum number of edges γ whose addition to a given graph G makes it 1 (S,T)-connected is greater than or equal to the maximum number of (S,T)

[38]This observation was cleared up by me, and sent to STEFFEN to be included in the final version.

independent subsets of V, i.e.

$$\gamma \geq \nu, \tag{6.1}$$

where by ν we denoted the maximum number or (S,T)-independent sets of V. We may write ν_G to emphasize which graph we count the maximum number of (S,T)-independent subsets in.

Proof: We solve the problem of 1-(S,T)-connectedness in two major steps.

First we will derive an algorithm that reduces any starting graph $G = (V, E)$ (with two subsets S and T of V) to a special class of bipartite directed graphs. At every step of the reduction algorithm we will show that the maximal number of (S,T)-independent sets does not change. Therefore, the minimal number of edges needed to 1-(S,T)-connect the original graph is the same as the number of edges needed to 1-(S,T)-connect the reduced graph of the special class.

Once our starting graph is transformed so that it belongs to this class, we derive a formula for the number of edges needed to 1-(S,T)-connect it. We will prove that the transformation process does not increase, nor decrease, the number of edges required to do so. We will do this by showing that the maximum number of (S,T)-independent subsets of V, ν, stays the same at every step of the transformation. This is because if ν were to change, by Equation (6.1) γ would change as well.

6.1 Reduction part

In the following we may assume that S and T are disjoint, since if they aren't, then we can replace a vertex $v \in S \cap T$ in T by a new vertex v_T, and connect v (now only in S) and v_T (in T) by two edges, one directed from v to v_T, the other backwards. [39]

The special class of bipartite graphs we would like any starting graph to reduce to is of the form $G = (S \cup T, E)$, a simple directed bipartite graph with bipartitions S and T consisting of components of the following three types and possibly some isolated vertices:

(1) A pair of vertices $s \in S, t \in T$ which are connected with a pair of oppositely directed edges,

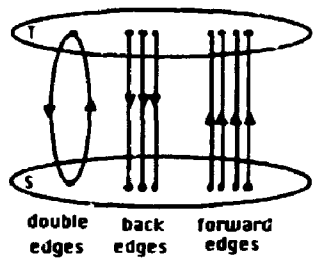
(2) all edges are directed from S to T and

(3) all edges are directed from T to S.

We call two oppositely directed edges between a pair of vertices a *double edge*, edges from T to S *back edges*, and edges from S to T *forward edges*. [40]

[39] I added this argument to account for the vertices that belong both to S and T

[40] Originally STEFFEN denoted these types of edges by "Type 1", "Type 2", and "Type 3". I found that very confusing, because in the arguments to come, we kept mixing their types, so we came up with the names "double", "back" and "forward", which is much clearer. One does not have to go back to the beginning of the paper to remind her/himself of what "Type 1" was.



The transformation process is done in four steps.

Step 1. Compute the transitive closure G_{tr} of G. It is easy to see that $\nu_{G_{tr}} = \nu_G$, because we may assume that no member X of an (S,T)-independent family has $p_{def}(X) = 0$ and thus have no entering edge. In other words, for every set X of a maximal (S,T)-independent family of G, we have $p_{def}(X) = 0$ (see discussion just before Equation (6.1)), i.e. no set has an incoming edge. Then when computing the transitive closure of G, any subset X of vertices will not have an incoming edge, unless the original set had incoming edges. Therefore, by computing the transitive closure, we are preserving the maximal number of (S,T)-independent sets.

Step 2. Let $G_{ST} := G_{tr}[S \cup T]$ be the graph spanned by $S \cup T$ in G_{tr}. The inequality $\nu_{G_{ST}} \geq \nu_{G_{tr}}$ follows immediately, since given an (S,T)-independent family $\mathcal{F}$ in G_{tr}, the family $\mathcal{F}' := \{X \cap (S \cup T) | X \in \mathcal{F}\}$ is (S,T)-independent.

To prove the opposite inequality, consider an (S,T)-independent family $\mathcal{F}$ in G_{ST}. For every member X define $X' := X \cup \{v \in V - (S \cup T) | \exists vx \in$

$E(G_{tr}), x \in X\}$. Using the fact that G_{tr} is a transitively closed graph, it is easy to see that $p_{def}(X') = p_{def}(X) = 1\ \forall X \in \mathcal{F}$ (because in a maximal (S,T)-independent family $\mathcal{F}$ all sets X have $p_{def}(X) = 1$, i.e. no incoming edges, and by the definition of X', we have not added any incoming edges, hence $p_{def}(X')$ is also 1) and that the family defined this way is (S,T)-independent.

Step 3. Let G' be the result of shrinking those strongly connected components of G_{ST} which are contained entirely in S or in T to single vertices. As no set X with $p_{def}(X) = 1$ can intersect a strongly connected component, this operation clearly preserves the maximum in Equation (6.1). The graphs spanned by S and by T are acyclic and the only cycles in G' consist of pairs of oppositely directed edges between a vertex in S and a vertex in T (recall that G' is transitively closed).

Step 4. Construct a bipartite graph H from G' as follows. Remove all vertices in S which have an out-neighbor $s' \in S$ and all vertices in T with an in-neighbor $t' \in T$. That is, H is defined with vertex-set

$$V(H) := \{s \in S \mid \nexists s s' \in E(G'), s' \in S\} \cup \{t \in T \mid \nexists t' t \in E(G'), t' \in T\}$$

and edge-set induced by $V(H)$ in G'. [41]

To show that $\nu_H \geq \nu_{G'}$, let $\mathcal{F}$ be an (S,T)-independent family in G'

[41] This step contained an error, and the present form was suggested by me and sent to the author.

and let X be a member of $\mathcal{F}$. By assumption both $S - X$ and $X \cap T$ are non-empty and $\varrho(X) = 0$. We show that both $S_H - X$ and $T_H \cap X$ are non-empty (where S_H (T_H)is the set of vertices of S (T) left over in H after the removing process above).

If, say, $S_H - X$ is empty, then by assumption there is a vertex $s \in S - X$ and because $s \notin H$ there is an edge ss' with $s' \in S_H \subseteq X$, this is a contradiction to $\varrho(X) = 0$. (Similar arguments apply if $T_H \cap X = \emptyset$.) Furthermore the restriction of $\mathcal{F}$ to H is (S,T)-independent (if not then we deduce that $\mathcal{F}$ is not (S,T)-independent) and $\nu_H \geq \nu_{G'}$ follows.

To prove $\nu_H \leq \nu_{G'}$ we proceed in a similar fashion. Let $\mathcal{F}$ be an (S,T)-independent family in H. Define a family $\mathcal{F}'$ with members $X' = X \cup \{s \in S | \exists s' \in S_H \cap X, ss' \in E(G')\}$. Using the fact that G' is the transitive closure of G and contains no cycles in S or in T it is easy to conclude that $\varrho(X') = 0$.

We claim that $\mathcal{F}'$ is (S,T)-independent in G'. If not then there are two members X', Y' separating the same pair of vertices $s \in S, t \in T$. If $s \in S_H, t \in T_H$ then the contradiction follows immediately as the corresponding sets $X, Y \in \mathcal{F}$ separates the same pair in H. If $s \notin S_H$ then there is at least one out-neighbor $s' \in S_H$. By the definition of X', Y' we have $s' \notin X \cup Y$ (similar arguments show the existence of a vertex $t' \in X \cap Y \cap T_H$) and then X, Y is dependent in H, a contradiction. We have now shown that $\nu_H \leq \nu_{G'}$

and equality follows.

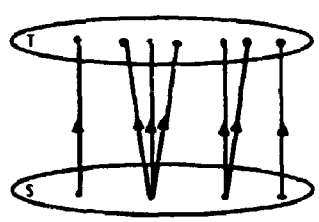
This concludes the reduction part of our algorithm. It is not hard to see that the graph H we derived from any starting graph G is a bipartite graph falling in the class of graphs described above.

6.2 Determining γ

In this second part of the algorithm, we derive the number of edges needed to 1-(S,T)-augment the particular set of bipartite digraphs we discussed earlier. We break up the problem into three parts, and determine γ separately for each part.

Proposition 6.1 *If G has only forward edges and isolated vertices, then a 1-(S,T)-edge-connectivity augmentation with γ edges requires*

$$\gamma \geq |S||T| - |E(G)|.$$

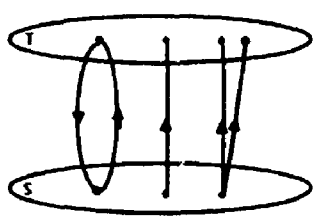


The first proposition is obviously true, and can be proved by an elementary construction of an (S,T)-independent family $\mathcal{F}$ with $p_{def}(\mathcal{F}) = |S||T| - |E(G)|$ on one hand. On the other hand by adding one new edge for every pair of vertices (s,t) with $s \in S, t \in T$ whenever no such edge already exists (the number of which being clearly $|S||T| - |E(G)|$). It is easy to see that adding at least these edges, will make the digraph so that every vertex in T will be reachable from every vertex in S.

Proposition 6.2 *If G has precisely one double edge and no back edges, then a 1-(S,T)-edge-connectivity augmentation with γ edges requires*

$$\gamma \geq |S| + |T| - 2.$$

[42]



[42]In this proposition, originally S. ENNI formulated the inequality incorrectly, it stated that $\gamma \geq |S| + |T| - 2|S||T|$ Clearly this was incorrect, I've sent STEFFEN a note on this, suggesting the current form. It is not hard to see what γ needs to be, we need to reach each $t \in T$ from every $s \in S$, and we only have edges from S to T together with one double edge. Joining the end of the double edge that is in S with every other vertex in T, and every other vertex in S to the end of the double edge in T will form very neatly a digraph where a 1-(S,T)-connectivity is achieved. Seeing this, I realised that the original formulation was wrong, and by counting these edges that I've added, I suggested the present form. Moreover, most of the time, the quantity $|S| + |T| - 2|S||T|$ is negative.

To prove this proposition, we let $s \in S, t \in T$ be the pair of vertices connected by a double edge. For every $u \in T - t$ we let $C_u := S - s + u$ [43] and for every $v \in S - s$ we let $D_v := S + t - v$. This family $\mathcal{F} = \{C_u | u \in T - t\} \cup \{D_v | v \in S - s\}$ of sets is (S,T)-independent and $p_{def}(\mathcal{F}) = |S| + |T| - 2$.

Conversely, augmenting with the set of edges $\{su | u \in T - t\} \cup \{vt | v \in S - s\}$ clearly gives a 1-(S,T)-edge-connected digraph.

Proposition 6.3 *If G does not satisfy the conditions of Proposition (6.1) or (6.2) then a 1-(S,T)-edge-connectivity augmentation with γ edges requires*

$$\gamma \geq |S| + |T| - \min(|S_B|, |T_B|),$$

where S_B (respectively T_B) denotes the set of vertices $v \in S$ ($u \in T$) incident with a back edge uv directed from T to S.

We prove this last proposition by induction on $l := |S| + |T|$. For this, we construct an augmentation using $|S| + |T| - \min(|S_B|, |T_B|)$ edges.

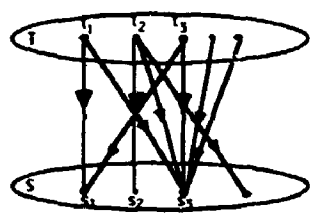
The base case is $l = 2$ in which G must consist of only one edge directed from T to S. (Otherwise either Proposition (6.1) or (6.2) would apply.) [44] In this case the statement is trivial. Suppose now $l > 2$; then there are two different possibilities.

[43] In this section, we are abbreviating things like $(S - \{s\}) \cup \{v\}$ to $S - s + u$

[44] This case was cleared up by me, the original graph G supplied by the author was not of the required form, it did not fit into this category.

Case 1. Every vertex is incident with an edge directed from T to S, i.e. there are no forward edges and no isolated vertices.

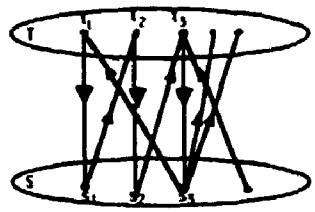
Let $M = \{s_1t_1, \ldots, s_nt_n\}$ be a maximum cardinality matching in the set of edges directed from T to S.



[a]

[a]Here the maximum cardinality matching constitutes of the vertices adjacent to the edges in bold.

Construct an augmentation giving a 1-(S,T)-edge-connected graph as follows: connect s_i to t_{i+1} for $i \in \{1, 2, \ldots, n\}$ (subscripts modulo n) and connect the unmatched vertices $\{s_{n+1}, \ldots, s_{n+k}, t_{n+1}, \ldots, t_{n+l}\}$ pairwise with edges $s_{n+i}t_{n+i}$ for $i \in \{1, 2, \ldots, \min(l, k)\}$. If $k > l$ we connect s_{n+i} to t_{n+l} for $i > l$ and if $l > k$ we connect s_{n+l} to t_{n+i} for $i > k$. The result is a 1-(S,T)-edge-connected digraph and we have used $n + \max(k, l) = \max(|S|, |T|) = |S| + |T| - \min(|S_B|, |T_B|)$ edges.



[a]

[a]Now we show the new edges from S to T (the ones not in bold) needed to 1-(S,T)-connect the maximum cardinality matching of the back edges.

Case 2. There exists either an isolated vertex or a component spanned by forward edges. Let $C = \{s_1, \ldots, s_l, t_1, \ldots, t_k\}$ be a component which is either an isolated vertex or spanned by forward edges. If we delete C then the resulting graph G' satisfies the hypothesis of Proposition (6.3) because G did. By induction there is an augmentation F' of G' using $|S'| + |T'| - \min(|S'_B|, |T'_B|)$ edges (where the primes refer to G'). Note that $|S'_B| = |S_B|$ and $|T'_B| = |T_B|$. Let ts be an edge in G' directed from T to S. Adding the set of edges $\{s_i t\}_{i=1}^{l} \cup \{st_i\}_{i=1}^{k}$ to F' clearly gives an augmenting set of edges which 1-(S,T)-edge-connects G. The total number of edges in this set is $|S'| + |T'| - \min(|S'_B|, |T'_B|) + l + k = |S| + |T| - \min(|S_B|, |T_B|)$.[45]

To finish the proof we have to construct an (S,T)-independent family $\mathcal{F}$ for which $p_{def}(\mathcal{F}) = |S| + |T| - \min(|S_B|, |T_B|)$. We consider only the case when $|T_B| \geq |S_B|$, the construction in the other case is slightly different but

[45]In this paragraph it was unclear what the author meant. The present form of the argument was sent to him by me, and was included in the final version of the paper

similar.

Let us introduce some new notations. Write S and T as disjoint unions $S = S_1 \cup S_2 \cup S_3$ and $T = T_1 \cup T_2 \cup T_3$, where S_1 (T_1) is the set of vertices incident with a double edge, S_2 (T_2) the set of vertices incident with forward edges and S_3 (T_3) the vertices incident with back edges. For further simplicity we will assume that there are no isolated vertices, these shall be treated just as vertices incident with forward edges. Then we give the following sets with indegree zero:

- For each $t_d \in T_1$ we define $X := S_2 + t_d + s_d$ where $\{t_d s_d, s_d t_d\}$ is the pair of double edges.
- For each $t_f \in T_2$, $X := S_2 + t_f$.
- For each $t_b \in T_3$, $X := S_2 + t_b$.
- For each $s_f \in S_2$, $X := S - s_f + T - T_2$.

An easy inspection reveals that the family $\mathcal{F}$ consisting of these sets is (S,T)-independent and $p_{def}(\mathcal{F}) = |S_2| + |T| = |S| + |T| - \min(|S_B|, |T_B|)$.

This finishes the proof of the algorithm that determines the number of edges needed to 1-(S,T)-augment the special class of bipartite digraphs. Since we have an algorithm that reduces any starting graph to this special

kind of graphs, the two algorithms combined determine the number of edges needed to augment any digraph G. •

We can summarize the results of the algorithm in the following theorem.[46]

Theorem 6.2 *Let $G = (S \cup T, E)$ be a simple bipartite directed graph for which every component containing edges in both directions between S and T has precisely two vertices. Let γ denote the number of new edges with tails in S and heads in T in a minimal 1-(S,T)-edge-connected augmentation of G. Then*

$$\gamma = \begin{cases} |S||T| - |E|, & \text{if all edges are directed from } S \text{ to } T. \\ |S| + |T| - 2, & \text{if only one edge } ts \text{ is directed from } T \\ & \text{to } S \text{ and the reverse edge } st \in E. \\ |S| + |T| - min(|S_B|, |T_B|), & \text{otherwise.} \end{cases}$$

Where S_B (respectively T_B) denote the number of vertices in S (T) incident with an edge directed from T to S.

[46] I have suggested STEFFEN to have a single theorem that summarises the results, so that for further reference it is easy to look up. We decided to place this theorem at the end of the paper, rather than at the beginning, since then we were worried that it might confuse the reader.

7 Conclusions

In the previous sections we have demonstrated the power of the splitting off technique, and the wide area of applications of *Frank's Algorithm*. We hope that there will be many more theorems proved and problems solved using these techniques in the future.

All that is written in this thesis was written by me, pulling the work of the authors we discussed in earlier sections into one concise paper, in order to show the unifying ideas behind these problems and techniques common to solving them.

I've showed the exact places where I've made contributions to the original work of the authors presented in this thesis. Most significant places were E. CHENG's paper on undirected successive–edge–connectivity augmentation problem, and S. ENNI's work on 1-(S,T)-augmentation for digraphs.

In the process of evaluating the correctness of CHENG's work, I've discovered places in the proof that did not fit, inconsistent notations and I've cleared up the arguments from the tedious language used in order for better understanding. I have also supplied E. CHENG with arguments to fix the flaws I've discovered in his proofs.

The sections dealing with T. JORDÁN's work and the hypergraph aug-

mentation section have been somewhat rewritten so that they fit with the rest of the thesis, although there have not been any significant changes. I've added a few examples and reorganised the order of the theorems and proofs in the hypergraph augmentation section, but did not need to fix any flaws in proofs or theorems.

On the other hand, the section solving the 1–(S,T)–connectivity problem has been changed quite a bit. There have been propositions that I've discovered to be untrue, suggested what they should read, supplied the author with fixes to proofs of other propositions, corrected examples and helped with the style of the whole paper for easier comprehension.

Besides these problems that we've introduced here, there are still many open problems in this area that we tried to give a brief introduction to. Perhaps the most important open problem in graph augmentation is *vertex connectivity* in directed and undirected graphs. Some special cases have been solved, but the general case is still open.

The general k–(S,T)–edge–connectivity problem also stands unsolvable (at least by a combinatorial algorithm) today. There are still many unanswered questions in this area, as well, left open for the reader to tackle.

We would like to draw the reader's attention to an interesting note on comparing two algorithms described in previous sections. Recall how in Sec-

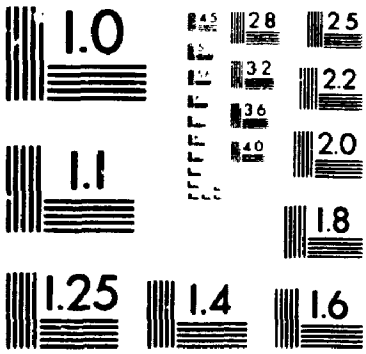
tion 4.2, when we were working on the successive edge–augmentation problem for directed graphs, we needed a kind of "backtracking step" when a pair of edges us, sv was not splittable in some G_j. Very much like in Section 5, when we augmented hypergraphs by edges of size two. There, too, we needed a backtracking step to take care of problematic edges. The similarity between the two unrelated problems is astonishing.

I hope that this thesis has succeeded in introducing the reader to this vast area of edge–connectivity augmentation problems, and showed some of the techniques used in solving problems falling into this category. The thesis could also serve as a good overview of the recent research in graph edge–augmentation. It is our hope that this thesis will help further research in the area.

Index

2 of/de 2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT



PRECISION[SM] RESOLUTION TARGETS

References

[1] J. BANG-JENSEN and B. JACKSON. *Augmenting hypergraphs by edges of size two*, preprint, December 1994.

[2] L. W. BEINEKE and R. J. WILSON. *Applications of graph theory*. Academic Press, London, 1979.

[3] E. CHENG and TIBOR JORDÁN. *The successive edge-connectivity augmentation problem*, preprint, May 1995.

[4] W. CHOU and H. FRANK, *Survivable Communication Networks, and the Terminal Capacity Matrix*, IEEE Trans. Circuit Theory, Vol. CT-17, No. 2, May 1970, pp. 192–197.

[5] S. ENNI, *A 1-(S,T)-edge-connectivity augmentation algorithm*, preprint, January 1995.

[6] K.P. ESWARAN and R.E. TARJAN. *Augmentation Problems*, SIAM J. Computing, Vol. 5, No. 4, December 1976.

[7] A. FRANK. *Preserving and increasing local edge-connectivity in mixed graphs*, SIAM J. Disc. Math., Vol. 8, No. 2, May 1995, pp. 155-178.

[8] A. FRANK, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. on Discrete Mathematics, Vol. 5, No. 1, February 1992, pp. 22-53.

[9] A. FRANK and T. JORDÁN, *Minimal Edge-coverings of Pairs of Sets*, J. of Combinatorial Theory, Ser. B, submitted 1993.

[10] A.V. GOLDBERG, R. TARDOS and R.E. TARJAN, *Network flow algorithms*, in *Paths, Flows and VLSI layouts*, Springer–Verlag, 1990, pp. 101–164.

[11] L. LOVÁSZ, *Combinatorial problems and exercises*, North–Holland, Amsterdam, 1979.

[12] L. LOVÁSZ, *Conference on Graph Theory*, lecture, Prague, 1974.

[13] L. LOVÁSZ, *Connectivity in digraphs*, J. of Combinatorial Theory, Ser. B, No. 15, 1973, pp. 174–177.

[14] W. MADER, *Konstruktion aller n-fach kantenzusammenhängenden Digraphen*, European J. of Combinatorics, No. 3, 1982, pp. 63–67.

[15] W. MADER, *A reduction method for edge-connectivity in graphs*, Annals of Discrete Math, No. 3, 1978, pp. 145–164.

[16] T. WATANABE and A. NAKAMURA, *Edge-connectivity Augmentation Problems*, Computer and System Sciences, Vol. 35, No. 1, 1987, pp. 96–144.

END

21-11-96

FIN