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**ON THE OSCILLATION OF
SECOND ORDER DIFFERENTIAL EQUATIONS**

by

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the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

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Abstract

In this work, we present both oscillation and non-oscillation theorems for second order linear and nonlinear scalar and matrix differential equations. First, we extend a well-known scalar non-oscillation criterion to linear matrix systems through the introduction of any fixed matrix norm (Chapter 1). Next, we generalize very recent Kamenev-type maximum eigenvalue oscillation criteria for linear matrix systems to a whole family of criteria based on general nonlinear functionals (Chapter 2). We also extend an oscillation criterion involving the trace to the case of positive linear functionals (Chapter 3). Finally (Chapter 4), we present a non-oscillation theorem for a nonlinear scalar differential equation in the spirit of Atkinson's [3] classic result.

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Samuel Dubé
December 2003

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Introduction

A fascinating and very fundamental area of research contained within the vast field of differential equations is that branch of qualitative theory known as oscillation theory. The study of oscillation concerns itself with the characteristics and behaviour of the “zeros” of solutions of differential equations without necessarily solving these related differential equations. One could easily argue that oscillation theory is at the root of the ubiquitous phenomena of periodicity, quasi-periodicity, and ultimately chaos theory, so fundamental is the nature of this theory and its consequences.

We will concern ourselves in this work with the oscillation (and non-oscillation) theory of second order scalar and matrix differential equations, with details to follow.

Perhaps the most significant of the early references in this vein is the 1836 paper by Sturm [65], which deals with oscillation and comparison theorems for linear homogeneous second order ordinary differential equations. Around the same time the work of Liouville [44] introduced the “Sturm-Liouville Problem”, a type of boundary problem which involved the study of the asymptotic behaviour of solutions and of eigenvalues of linear second order differential equations using integral equations.

Considering the second order scalar differential equation

$$(A) \quad y'' + q(t)y = 0, \quad t \in [0, \infty),$$

where $q(t) \in C^0[0, \infty)$, Kneser [36] showed in 1893 that $q(t) \leq 1/4t^2$ implies non-oscillation while $q(t) \geq (1 + \delta)/4t^2$, $\delta > 0$ implies oscillation. In 1917, Bôcher [7] extended the oscillation, comparison, and separation theorems of Sturm.

A year later, Fite [23] showed that $q(t) > 0$ and $\int_0^\infty q(s)ds = \infty$ together imply oscillation (see also [73] and [74]). In 1949, Wintner [75] removed the

condition $q(t) > 0$ in Fite's result, proving that

$$(B) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_T^t \left[\int_T^s q(u) du \right] ds = \infty$$

implies oscillation, probably the first criterion using a type of mean or average.

Three years later, Hartman [25] showed that the non-existence of the limit in (B) as a finite number combined with

$$(C) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \left[\int_T^s q(u) du \right] ds > -\infty$$

imply oscillation.

In 1976, Etgen and Pawlowski [19] tackled systems of differential equations by providing an oscillation criterion for the matrix differential equation

$$(D) \quad Y'' + Q(t)Y = 0, \quad t \in [t_0, \infty)$$

where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric, the criterion being

$$(E) \quad \lim_{t \rightarrow \infty} g \left[\int_0^t Q(s) ds \right] = \infty,$$

where g is any positive linear functional on symmetric matrices. (It should be noted that Reid [59] gives a detailed history of the contributions and developments influencing the evolution of systems of differential equations.)

In a conjecture apparently due to Reid, but propagated by Hinton and Lewis [31], the vector differential equation

$$(F) \quad y'' + Q(t)y = 0,$$

when $y \in \mathbb{R}^n$ and $Q(t)$ is as in (D), is claimed to be oscillatory if

$$(G) \quad \lim_{t \rightarrow \infty} \lambda_1 \left[\int_0^t Q(s) ds \right] = \infty,$$

where $\lambda_1[\cdot]$ denotes the largest eigenvalue of the matrix under consideration. Combining Hartman's results (C) with a particular positive linear functional, the "trace" functional, Mingarelli [47][49] proved the conjecture with the added condition

$$(H) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr} \left[\int_0^s Q(u) du \right] ds > -\infty.$$

In addition, others [41] [9] [35] [4] have also given a proof of the conjecture under additional similar mild conditions, including Byers, Harris, and Kwong who involved the use of weighted means [12].

The work of Kamenev [34] precipitated an expansion into the investigation of integral criteria (so called “Kamenev-type theorems”) for oscillation and non-oscillation, culminating so far with the work of Sun [66] which we will deal very directly with later on.

One of our chief motivations is that there has been comparatively little research done extending oscillation and non-oscillation results from scalar second order linear and nonlinear differential equations to systems and their related matrix equations.

This work is divided into four chapters. The first chapter extends a non-oscillation criterion for second order linear scalar differential equations due to Mingarelli [48] to second order linear matrix differential equations through the use of any fixed matrix norm. In Chapter 2, we first generalize the recent Kamenev-type integral oscillation criteria for second order linear matrix differential equations of Sun [66] by considering a more general kernel. Secondly, we extend our new result even further so as not to depend explicitly on the largest eigenvalue of the matrices in question, but instead on very general nonlinear functionals. Chapter 3 concerns itself with second order nonlinear matrix differential equations extending an oscillation theorem of Praharaj [56] involving the trace to the case of positive linear functionals. In Chapter 4 we present a general non-oscillation criterion for a second order nonlinear scalar differential equation in the spirit of Atkinson’s [3] classic result in this field in 1955. Although stronger than Atkinson’s original criterion, our last result unifies sublinear, linear, and superlinear cases under a common theme, eliminating the need for specific techniques in each case.

Terminology and Preliminary Lemmas

We begin our investigation by considering the self-adjoint differential system

$$(I) \quad [P(t)y']' + Q(t)y = 0, \quad t \in J$$

for a vector $y \in \mathbb{R}^n$ where J is a real interval, and P, Q are real, $n \times n$ continuous matrix functions on J such that Q is symmetric and P is symmetric and positive definite. We adopt the notation and conventions used in [49].

Definition I.1: The two distinct points $t_1, t_2 \in J = [a, b]$ will be (*mutually*) *conjugate* relative to (I) if there exists a solution $y(t) \not\equiv 0$ of (I) satisfying $y(t_1) = y(t_2) = 0$ (the zero vector) on J .

Definition I.2: The equation (I) will be called *disconjugate* on J , if there are no conjugate points therein, i.e., if every nontrivial solution of (I) vanishes at most once in J .

The term *conjugate* originates in a fixed point theorem credited to Lagrange in the Calculus of Variations [64, p. 311).

Definition I.3: When $J = [0, \infty)$ the equation (I) will be termed *oscillatory at* ∞ if for every $a > 0$, (I) is not disconjugate on $[a, \infty)$. It will be *non-oscillatory* otherwise.

Associated with equation (I) is the matrix differential equation

$$(J) \quad [P(t)Y']' + Q(t)Y = 0, \quad t \in [0, \infty),$$

where P and Q are the same as in (I) and Y is an $n \times n$ real, continuously differentiable matrix function on $[0, \infty)$. It is clear that equations such as (J) have solutions for $t \in [0, \infty)$ since we can reduce them to a first-order system where Q becomes a $2n \times 2n$ matrix and apply a known existence theorem (see [14], Chapter 1). We may also note that $Y(t)$ is a solution of (J) if and only if its columns are solutions of (I).

Definition I.4(a): A solution $Y(t)$ of equation (J) is called *non-trivial* if $\det Y(t) \neq 0$ for at least one $t \in [0, \infty)$, (where $\det Y(t)$ as usual will be used to denote the determinant of the matrix $Y(t)$).

Definition I.4(b): A solution $Y(t)$ of equation (J) is said to be *non-singular* at $t = a$, $a \in [0, \infty)$, if and only if $\det Y(a) \neq 0$.

We thus have an expanded yet analogous definition for the oscillation of solutions of the matrix equation.

Definition I.5: A non-trivial solution $Y(t)$ of equation (J) will be called *oscillatory at ∞* if and only if $\det Y(t)$ has infinitely many zeros in $[0, \infty)$ having no finite point of accumulation. It will be *non-oscillatory* otherwise.

A simple differentiation shows that whenever $Y(t)$ is a solution of equation (J), we have

$$(K) \quad Y^*(t)[P(t)Y'(t)] - [P(t)Y'(t)]^*Y(t) = C$$

where C is a constant $n \times n$ matrix and the asterisk “*” will be used to denote the conjugate transpose of a matrix, e.g., if A is a Hermitian matrix, $A^* = A$. Equation (K) defines the *Wronkian* of $P(t)Y'(t)$ and $Y(t)$ of equation (J), $W[PY', Y]$ (refer to [5]).

Definition I.6: A solution $Y(t)$ of equation (J) is said to be *prepared, self-conjugate*, or *conjoined* if

$$(L) \quad Y^*(t)[P(t)Y'(t)] = [P(t)Y'(t)]^*Y(t), \quad t \in [0, \infty)$$

or equivalently if $C = 0$ (the zero $n \times n$ matrix) in (K).

Setting the initial condition $Y(0) = 0$ to obtain $C = 0$ guarantees a prepared solution.

We can see clearly that equation (L) is saying that $P(t)Y'(t)Y^{-1}(t)$ is a symmetric matrix, $t \in [0, \infty)$.

In the following example, credited to Noussair and Swanson [52], it can be seen that the “preparedness” of a solution $Y(t)$ of equation (J) is required in order to develop an analog of the classical theory of oscillation in the scalar

($n = 1$) case for linear systems. In other words, it is possible that (J) admits a non-trivial, non-prepared, non-oscillatory solution. Consider the matrix differential equation

$$(M) \quad Y'' + Y = 0, \quad t \in [0, \infty),$$

which is simply equation (J) with $P = I_n = Q$, where I_n denotes the $n \times n$ identity matrix. This is the analog of the scalar equation $y'' + y = 0$ whose solutions are of the form

$$(N) \quad y(t) = c_1 \sin t + c_2 \cos t,$$

where c_1 and c_2 are arbitrary constants. The solutions of (N) are oscillatory. But if we consider the solution

$$Y(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in [0, \infty)$$

of equation (M), we get that

$$\begin{aligned} Y^*(t)[P(t)Y'(t)] &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} [P(t)Y'(t)]^*Y(t) &= [Y'(t)]^*P^*(t)Y(t) \\ &= \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

so that $Y^*(t)[P(t)Y'(t)] - [P(t)Y'(t)]^*Y(t)$

$$\begin{aligned} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\ &\neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

So $Y(t)$ is not prepared. Also, $\det Y \equiv 1$ on $[0, \infty)$ so $Y(t)$ is non-oscillatory as well (and clearly non-trivial).

Equation (M) also admits a non-trivial, symmetric, non-prepared, non-oscillatory solution. Consider

$$Y(t) = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}, \quad t \in [0, \infty)$$

where

$$\begin{aligned} Y^*(t)[P(t)Y'(t)] &= \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} [P(t)Y'(t)]^*Y(t) &= [Y'(t)]^*P^*(t)Y(t) \\ &= \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

so that $Y^*(t)[P(t)Y'(t)] - [P(t)Y'(t)]^*Y(t)$

$$\begin{aligned} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \\ &\neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

So $Y(t)$ is not prepared, and $\det Y \equiv -1$ on $[0, \infty)$ so $Y(t)$ is non-oscillatory as well (and clearly non-trivial).

It is known ([28], p. 238, Theorem 10.2) that whenever equation (I) is disconjugate on $J = [a, \infty)$, $a \geq 0$, there exists a non-trivial prepared solution of the associated matrix differential equation (J) such that $Y(t)$ is non-singular in (a, ∞) . In this case, $Y^{-1}(t)$ exists, and

$$(O) \quad V(t) \equiv P(t)Y'(t)Y^{-1}(t)$$

will be a symmetric matrix, since (suppressing the argument “ t ” for convenience)

$$Y^*PY' - Y'^*PY = 0 \quad \text{from } Y \text{ being prepared}$$

$$Y^*PY' = Y'^*PY$$

$$(Y^*)^{-1}Y^*PY' = (Y^*)^{-1}Y'^*PY \quad (\text{left multiplication by } (Y^*)^{-1})$$

$$\begin{aligned}
PY' &= (Y^*)^{-1}Y'^*PY \\
PY'Y^{-1} &= (Y^*)^{-1}Y'^*PY Y^{-1} \quad (\text{right multiplication by } Y^{-1}) \\
PY'Y^{-1} &= (Y^*)^{-1}Y'^*P \\
\therefore V &= (Y^*)^{-1}Y'^*P \\
&= (Y^{-1})^*Y'^*P^* \quad \text{since } (Y^*)^{-1} = (Y^{-1})^* \text{ and } P = P^* \\
\therefore V &= (Y^{-1})^*(PY')^* \\
&= (PY'Y^{-1})^* \\
&= V^*
\end{aligned}$$

And so $V^2(t)$ will be a non-negative definite matrix.

If $P(t) = I_n$, the $n \times n$ identity matrix, in equation (J), the result is

$$(P) \quad Y'' + Q(t)Y = 0, \quad t \in [0, \infty).$$

We note that the oscillation of equations (J) and (P) must be studied separately since (just as in the scalar case) there is no oscillation-preserving transformation of the independent variable that allows us to relate the two different forms when $P(t) \neq I_n$ (p. 20 [16]).

The equivalence of the oscillation of equations (I) and (J) was shown by Tomastik [69]. While the oscillation of equation (I) is dealt with directly in [41], [4], [26], and [35], other works, including [9], [11], [16], [33], [49], and [71] study the oscillation of equation (I) through the investigation of equation (J). We will follow the tendency of the latter papers in our own work.

We will be using the standard notation of matrix algebra and calculus throughout. Unless otherwise indicated, capital letters $(A, B, P, Q, Y, \dots)$ will be used to represent $n \times n$ matrices, and thus the term “matrix” will be used to refer to square matrices. The symbol I (rather than I_n) will denote the identity matrix regardless of dimension. Similarly, the symbol 0 will denote the zero matrix.

If $A = (a_{ij})$ is a matrix, it is said to be *non-singular* if $\det A \neq 0$, and in this case, the inverse of A , A^{-1} , exists. If A is not non-singular, then it is *singular*. Since all our matrices are real, we use the same notation $A = A^*$ to denote the Hermitian and symmetric property for A . A symmetric matrix A is

said to be *positive semi-definite* (respectively *negative semi-definite*), denoted by $A \geq 0$ (respectively $A \leq 0$), if $\langle A, x \rangle \geq 0$ (respectively $\langle A, x \rangle \leq 0$) for every vector x in $\mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. Similarly, a symmetric matrix A is said to be *positive definite* (respectively *negative definite*), denoted by $A > 0$ (respectively $A < 0$), if $\langle A, x \rangle > 0$ (respectively $\langle A, x \rangle < 0$) for every vector x in $\mathbb{R}^n$. The *trace* of a matrix A is defined by $\sum_{i=1}^n a_{ii}$, the sum of all diagonal elements of A , and is denoted by $\text{tr}A$.

If $A(t)$ is a matrix each of whose entries is a function, then $A(t)$ will be termed continuous, differentiable, etc., when each of its entries has the indicated property.

A complex number λ is said to be an *eigenvalue* or *characteristic value* of a matrix A if there exists a non zero vector x in $\mathbb{R}^n$ such that $Ax = \lambda x$. Here, x is called an *eigenvector* associated with the eigenvalue λ . The *characteristic equation* of a matrix A is defined by setting $\det(A - \lambda I) = 0$.

Let S be the real linear vector space of $n \times n$ symmetric matrices. Whenever A is in S , its eigenvalues (all necessarily real) will be denoted by $\lambda_k(A)$, $1 \leq k \leq n$, where $\lambda_1(A)$ (respectively $\lambda_n(A)$) is the largest (respectively the smallest) eigenvalue of A .

A nonlinear functional (see [26]) $\kappa: S \rightarrow \mathbb{R}$ is called *super-additive* and *(positively) superhomogeneous* if

$$(Q) \quad \kappa(A + B) \geq \kappa(A) + \kappa(B) \quad \text{and}$$

$$(R) \quad \kappa(\alpha A) \geq \alpha \kappa(A) \quad \text{for } \alpha \geq 0.$$

Correspondingly, a functional $\wp: S \rightarrow \mathbb{R}$ is called *sub-additive* and *(positively) subhomogeneous* if

$$(S) \quad \wp(A + B) \leq \wp(A) + \wp(B) \quad \text{and}$$

$$(T) \quad \wp(\alpha A) \leq \alpha \wp(A) \quad \text{for } \alpha \geq 0.$$

We should note that conditions (Q) and (R) (or (S) and (T)) imply that κ (or $\wp$) is concave (or convex) and hence continuous on S . ([26]).

A real linear functional $g: S \rightarrow \mathbb{R}$ (i.e., satisfying

$$(U) \quad g(A + B) = g(A) + g(B), \quad g(\alpha A) = \alpha g(A)$$

for A and B in $\mathcal{S}$, $\alpha \in \mathbb{R}$) is called *positive* (respectively *non-negative*) if

$$g(A) > 0 \quad (\text{respectively } g(A) \geq 0)$$

whenever $A > 0$ (respectively $A \geq 0$).

Since (by (Q) and (R), or (S) and (T)) a positive linear functional is continuous, we have

$$g\left[\int_a^t A(s)ds\right] = \int_a^t g[A(s)]ds, \quad a, t \in [0, \infty),$$

whenever A is integrable, and

$$g[B'(t)] = \{g[B(t)]\}', \quad t \in [0, \infty),$$

whenever B is differentiable, (see [19]).

There are many results which have used positive linear functionals to investigate the oscillation of equation (J), such as [19], [22], [51], [31], [71], and [72]. Etgen and Pawłowski applied the idea to equation (F) [19], while the former and Lewis followed up with [22], formulating a comparison theorem relating the oscillatory behaviour of equation (F) (as a special case) to the oscillatory behaviour of a scalar differential equation. The basic result obtained is that equation (J) is oscillatory on $[0, \infty)$ provided there exists a positive linear functional g such that the scalar equation

$$(V) \quad y'' + g[Q(t)]y = 0$$

is oscillatory, where it is assumed that $g(I) = n$. In [26], Hartman showed that many of the oscillation criteria can be generalized by replacing linear functionals with non-linear functionals (as seen in (Q) and (R), or (S) and (T)). Hinton and Lewis [31] have extended some of the results of [22] to generalized differential systems (essentially “Stieltjes integrodifferential systems”) with new oscillation criteria having been discovered in this more general setting. Butler and Erbe [9] have obtained several oscillation criteria for equation (J), and have shown through example that there exists continuous symmetric matrix functions $Q(t)$ such that for any positive linear functional, equation (V) is non-oscillatory but equation (J) is oscillatory.

Other works ([9], [11], and [12]) have produced criteria for the oscillation of equation (J) in terms of the maximum eigenvalue (λ_1) and trace of the

integrals of the matrix functions P and Q . Butler, Erbe and Mingarelli applied two of the fundamental approaches used to investigate the oscillation of the scalar equation

$$(W) \quad [p(t)y']' + q(t)y = 0, \quad t \in [0, \infty),$$

that is, the Riccati integral equation and the calculus of variations, to produce several oscillation criteria for equation (J). Their work [11] is a generalization of the results of Kwong and Zettl [39] from the scalar equations to systems of equations. These authors believe that greater insight is gained into how the oscillatory nature of the equation is influenced by conditions imposed on the coefficient matrix or its integral by using these two different approaches. Their paper was also a prime example of how scalar oscillating criteria can be generalized to systems, obtaining sharper results using the largest eigenvalue instead of positive linear functionals such as the trace.

Very recently, results have been obtained which generalize work done in linear systems to a more general class of Hamiltonian systems ([38], [46], and [51]). In [38], Kumani and Umamaheswaram generalize earlier results in [19], [18], and [45] for equation (J) or its special cases. In [46], Meng and Mingarelli develop new criteria for oscillation involving general monotone (non-decreasing) functionals instead of the usual largest eigenvalue (λ_1). Finally, in [51], Mingarelli extends and unifies most known results dealing with oscillation criteria and the particular maximum eigenvalue functional by using nonlinear functionals and a more general class of Hamiltonian systems with symmetric coefficients.

Preliminary Lemmas

The following lemmas play a vital role in presenting the work in later chapters. Unless otherwise specified, these results are referred to in [6], [32] and [29].

Lemma I.1

Let A and B be $n \times n$ real matrices.

- (a) *If $A = A^*$ then the eigenvalues of A are real and are ordered $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$.*
- (b) *A is singular if and only if 0 is an eigenvalue of A .*
- (c) *$\text{tr} A = \sum_{i=1}^n \lambda_i$ and $\det A = \prod_{i=1}^n \lambda_i$.*
- (d) *$\text{tr}(AB) = \text{tr}(BA)$. (p. 96, [6]).*
- (e) *Each distinct eigenvalue of A has associated with it an eigenvector determined up to a scalar multiple.*
- (f) *An eigenvector of A cannot be associated with two distinct eigenvalues of A . (p. 36, [6]).*
- (h) *If $A = A^*$, then $A > 0$ (respectively $A \geq 0$) if and only if all the eigenvalues of A are positive (respectively non-negative). (p. 54, [6]).*
- (i) *If $\det A \neq 0$ then $AA^* = A^*A > 0$ (p. 54, [6]), and if $A = A^*$, $A^2 > 0$.*
- (j) *If the eigenvalues of A are all distinct, then their associated eigenvectors are linearly independent. (p. 54, [6]).*
- (k) *If λ is a root of the characteristic equation $\det(A - \lambda I) = 0$ with multiplicity k , then λ has associated with it k linearly independent eigenvectors. Every eigenvector of λ is a linear combination of these k eigenvectors. (p. 55, [6]).*
- (l) *If $A = A^*$,*

$$\lambda_1(A) = \max_x \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \quad \text{and} \quad \lambda_n(A) = \min_x \frac{\langle x, Ax \rangle}{\langle x, x \rangle},$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$. (p. 112, [6]).

- (m) If λ is an eigenvalue of A , the set of all vectors x satisfying the equation $Ax = \lambda x$ is called the **eigenspace** of A corresponding to the eigenvalue λ . Every nonzero element of this eigenspace is an eigenvector of A corresponding to λ . The dimension of the eigenspace of λ is known as the **geometric multiplicity** of the eigenvalue λ . The **algebraic multiplicity** of the eigenvalue λ is the multiplicity of λ considered as a root of the characteristic equation $\det(A - \lambda I) = 0$. (p. 57-58, [32]).
- (n) The algebraic multiplicity of an eigenvalue λ of A is greater than or equal to its geometric multiplicity. (p. 60, [32]).
- (o) If $A \geq 0$ and all the eigenvalues of A are zero, then $A = 0$. (p. 304, [29]).
- (p) If $A = A^*$ and $B = B^*$, then $\text{tr}(AB)$ is real [17].

Lemma I.2

Assume all matrices are in S .

- (a) $A > B$ (respectively $A \geq B$) if and only if $A - B > 0$ (respectively $A - B \geq 0$).
- (b) $AB = (AB)^*$ if and only if $AB = BA$.
- (c) If $A \geq B$ and $C \geq D$, then $A + C \geq B + D$.
- (d) If $A \not\leq B$ and $B \geq C$, then $A \not\leq C$.
- (e) If $A \geq 0$, $A \not\leq B$ and $C > 0$, then $ACA \not\leq BCB$.
- (f) If $C \geq 0$, then $(A + B)C(A + B) \leq 2(ACA + BCB)$.
- (g) If $A \geq B$, then $CAC \geq CBC$.
- (h) Let $A \geq 0$ and $B > 0$. If $ABA > CBC$, then $A \geq C$.
- (i) $A^2 \geq 0$.
- (j) If $A > 0$, then $\det A > 0$, and so A is nonsingular.
- (k) If $A \geq 0$, $B \geq 0$ and $AB = BA$, then $AB \geq 0$.
- (l) $(A + B)^2 \leq 2(A^2 + B^2)$

This is obtained by noting that

$$(A + B)^2 = A^2 + AB + BA + B^2$$

and $(A - B)^2 = A^2 - AB - BA + B^2$ along with $(A - B)^2 \geq 0$ implies $(A + B)^2 \leq A^2 + A^2 + B^2 + B^2 = 2(A^2 + B^2)$ as required.

- (m) If $A \geq 0$ and f is a non-negative function on $[a, b]$, then $f(t)A \geq 0$ for $t \in [a, b]$.
- (n) If $A > 0$ and f is a positive function on $[a, b]$, then $f(t)A > 0$ for $t \in [a, b]$.
- (o) If $A(t) = (a_{ij}(t)) \geq 0$, $t \in [a, b]$, and $|\int_a^b a_{ij}(t)dt| < \infty$, $1 \leq i, j \leq n$, then $\int_a^b A(t)dt \geq 0$, which follows directly from the definition of A being positive semi-definite.

(See [16] for (c)-(g), [12] for (h), and [29] for (k).)

Lemma I.3 (Lemma 1.1 in [49]).

Let A and B be in $\mathcal{S}$. Then both $\lambda_1, \lambda_n: \mathcal{S} \rightarrow \mathbb{R}$ are nonlinear functions with the following properties:

- (a) $\lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B)$ and
 $\lambda_1(\mu A) \leq \mu \lambda_1(A)$ whenever $\mu \geq 0$ is real.
- (b) $\lambda_n(A + B) \geq \lambda_n(A) + \lambda_n(B)$ and
 $\lambda_n(\mu A) \geq \mu \lambda_n(A)$ whenever $\mu \geq 0$ is real.
- (c) Whenever A is integrable on a compact set $J = [a, b]$, then λ_1 and λ_n satisfy the Jensen-type inequalities
 - (i) $\lambda_1\left[\int_a^b A(s)ds\right] \leq \int_a^b \lambda_1[A(s)]ds$, and
 - (ii) $\lambda_n\left[\int_a^b A(s)ds\right] \geq \int_a^b \lambda_n[A(s)]ds$.
- (d) If $A > B$ (i.e., if $A - B > 0$) then

$$\lambda_k(A) \geq \lambda_k(B), \quad \text{for each } k, \quad 1 \leq k \leq n.$$
- (e) $\lambda_1(A^2) \geq |\lambda_1(A)|^2$.

Lemma I.4 (Lemma 1.2 in [1]).

If $A(t) \in \mathcal{S}$ is a non-negative definite matrix ($A \geq 0$) which is locally integrable on $J = [0, \infty)$, then there holds the inequality

$$\int_0^t \lambda_1[A(s)]ds \geq \lambda_1 \left[\int_0^t A(s)ds \right] \geq \frac{1}{n} \int_0^t \lambda_1[A(s)]ds$$

for $t \in J$.

Corollary I.5 (Corollary 1.3 in [1]).

If $A(t) \in \mathcal{S}$ is a non-negative definite matrix ($A \geq 0$) which is locally integrable on $J = [0, \infty)$ and is such that

$$\lim_{t \rightarrow \infty} \lambda_1[A(t)] = +\infty$$

then

$$\lim_{t \rightarrow \infty} \lambda_1 \left[\int_0^t A(s)ds \right] = +\infty.$$

Note that Lemma I.4 implies that if $A(t)$ is a non-negative definite matrix, then $\lim_{t \rightarrow \infty} \lambda_1 \left[\int_0^t A(s)ds \right] = +\infty$ if and only if $\lim_{t \rightarrow \infty} \int_0^t \lambda_1[A(s)]ds = +\infty$.

Lemma I.6. (See [60] and [22] with proof p. 12, [1]).

If g is a positive linear functional on $\mathcal{S}$, then there holds the generalized Cauchy-Schwarz Inequality

$$|g(A)|^2 \leq g(I)g(A^2)$$

for all $A \in \mathcal{S}$.

Corollary I.7. [49]

If $A \in \mathcal{S}$, then the trace of A , $\text{tr}A$, satisfies the Schwarz-type inequality

$$|\text{tr}A|^2 \leq n(\text{tr}A^2).$$

Let $A(t) \in \mathcal{S}$ be a matrix function defined on $J = [0, \infty)$ and suppose that

$$(X) \quad A(t_2) > A(t_1), \quad T \leq t_1 \leq t_2 < \infty,$$

Then $\lambda_k[A(t_2)] > \lambda_k[A(t_1)]$, $1 \leq k \leq n$, i.e., the real valued function $\lambda_k[A(t)]$ is increasing for large t . Hence we obtain

Lemma I.8. (Corollary 1.1 in [49])

Let $A(t) \in \mathcal{S}$ satisfy (X). Then

$$(Y) \quad \lim_{t \rightarrow \infty} \lambda_k[A(t)]$$

exists for each k , $k = 1, 2, \dots, n$ (but may be infinite).

Finally, if $A \in \mathcal{S}$ is non-negative definite ($A \geq 0$), the following bounds hold:

$$(Z) \quad \lambda_1(A) \leq \operatorname{tr} A \leq n\lambda_1(A),$$

and in fact for any matrix $A \in \mathcal{S}$ we always have

$$(AA) \quad \lambda_n(A) \leq \frac{\operatorname{tr} A}{n} \leq \lambda_1(A).$$

Walters [72] actually obtained a similar inequality for positive linear functionals with norm 1. First, he showed that the norm of a positive linear functional g is equal to its value at the identity matrix (Theorem 3.2 in [72]), i.e.,

$$(BB) \quad \|g\| = \sup_{\|A\|_M=1} |g(A)| = g(I)$$

where $\|\cdot\|_M$ is the operator norm of a matrix and $A \in \mathcal{S}$, that is,

$$\|A\|_M = \sup_{\|x\|=1} \|Ax\|_x,$$

where x ranges over all column n -vectors and $\|\cdot\|_x$ is any vector norm which is consistent with the matrix norm. Hence, the inequality he formulated for positive linear functionals reads thus:

Theorem I.9 (Theorem 3.2 in [72])

Let g be a positive linear functional with norm 1 ($\|g\| = 1$). Then for every $A \in \mathcal{S}$, we have

$$(CC) \quad \lambda_n(A) \leq g(A) \leq \lambda_1(A).$$

This result, combined with the theorem of Etgen and Pawlowski [20] stated below suggests that perhaps some eigenvalue criterion might be stronger than the criteria of Etgen and Pawlowski using positive linear functionals:

Theorem I.10 [20]

Let $P(t)$ be positive definite ($P > 0$). If there exists a positive linear functional g such that the scalar equation

$$(DD) \quad \{g[P(t)]y'\}' + g[Q(t)]y = 0, \quad t \in [0, \infty),$$

is oscillatory, then so is

$$(J) \quad [P(t)Y']' + Q(t)Y = 0, \quad t \in [0, \infty).$$

Indeed, Tomastik has some results in this direction [70] [71]. However, it should be noted that St. Mary has shown [63] that the oscillation of

$$(EE) \quad \{\lambda_n[P(x)y']\}' + \lambda_1[Q(x)]y = 0$$

is not sufficient for the oscillation of equation (J).

Lemma I.11 (Lemma 1.8 in [1]).

Let A be a real $n \times n$ matrix and x be a column vector in $\mathbb{R}^n$. Then,

$$(FF) \quad \text{tr}(xx^*A) = \langle Ax, x \rangle,$$

where $\langle, \rangle$ is the usual inner product on $\mathbb{R}^n$.

Chapter 1

A NON-OSCILLATION THEOREM FOR DIFFERENTIAL MATRIX SYSTEMS

Consider the second order linear matrix differential equation

$$(1.1) \quad [P(t)Y']' + Q(t)Y = 0, \quad t \geq a$$

where $P(t)$, $Q(t)$, and $Y(t)$ are real, $n \times n$, continuous matrix functions on $[a, \infty)$ such that $Q(t)$ is symmetric and $P(t)$ is symmetric and positive definite ($P(t) > 0$). When $P(t) = I$, we obtain the matrix differential equation

$$(1.2) \quad Y'' + Q(t)Y = 0, \quad t \geq a.$$

It is clear that solutions of equations (1.1) and (1.2) exist since they can be reduced to a first order system where $Q(t)$ becomes a $2n \times 2n$ matrix and then apply a known existence theorem (see Chapter 1, [14], and p. 257, [59]). Oscillation of equations (1.1) and (1.2) must be studied separately when $P(t) \neq I$ since there is no oscillation-preserving transformation of the independent variables that allows us to relate the two different forms (p. 20, [14]).

Tomastik [69] showed the equivalence of equation (1.1) to the associated vector differential system

$$(1.3) \quad [P(t)y']' + Q(t)y = 0, \quad t \geq a$$

where $P(t)$ and $Q(t)$ are as in equation (1.1) and $y \in \mathbb{R}^n$. $Y(t)$ is a solution of (1.1) if its columns are solutions of (1.3).

As previously stated, there have been many results over the years, among them [19], [22], [26], [31], [71], and [72], dealing with the oscillation of equation (1.1) through the creation (or modification) of criteria using positive linear functionals. One such notable result is a theorem by Etgen and Lewis [21]:

Theorem 1.4: (Theorem 4.3, [22])

If there exists a positive linear functional g such that the scalar equation

$$(1.5) \quad \{g[P(t)]y'\}' + g[Q(t)]y = 0, \quad t \geq a$$

is oscillatory, then all solutions of the associated matrix differential equation (1.1) are oscillatory.

As a consequence of this result, the oscillation of equation (1.1) can be considered and studied in terms of the oscillation of an associated scalar equation of the form

$$(1.6) \quad [p(t)y']' + q(t)y = 0, \quad t \geq a,$$

where $p(t)$ and $q(t)$ are continuous, real-valued functions on $[a, \infty)$ with $p(t) > 0$ (the scalar analog to $P(t) > 0$) for all $t \in [a, \infty)$. Thus, any of the vast number of well-known oscillation criteria for equation (1.6) may be used to determine a corresponding criteria for equation (1.3) (and consequently equation (1.1)), as research on the former equation dates back as far as the works of Sturm [65] and Liouville [44] in the 1830's, and has inspired countless papers.

Perhaps due in part to the fact that earlier works were concerned more with criteria for oscillation rather than for non-oscillation, and that the tendency is to extend existing results, there are markedly fewer criteria dealing with the latter in the literature. One notable non-oscillation result hails from Wintner (p. 371, [76]) in 1951, and concerns the scalar second order differential equation

$$(1.7) \quad y'' + q(t)y = 0, \quad t \geq a,$$

where $q(t)$ is a continuous, positive, real-valued function on $[a, \infty)$.

Theorem 1.8 (p. 371, [76])

Suppose that the integral

$$a(t) \equiv \int_t^\infty q(s)ds$$

exists in the limiting sense, that is

$$a(t) = \lim_{T \rightarrow \infty} \int_t^T q(s)ds$$

exists and is finite.

If for all $t \geq t_0$,

$$a^2(t) \leq \frac{1}{4}q(t),$$

then equation (1.7) is non-oscillatory.

In 1957, Opial (p. 312, [54]) extended Wintner's result to the following theorem.

Theorem 1.9 (p. 312, [54])

Suppose that the integral

$$a(t) \equiv \int_t^\infty q(s)ds$$

converges (possibly conditionally) and $a(t) \geq 0$. If for all $t \geq t_0$

$$\int_t^\infty a^2(s)ds \leq \frac{1}{4}a(t),$$

then equation (1.7) is non-oscillatory.

In 1980, Mingarelli (Theorems 2.1.1–2.1.8, [48]) generalized many non-oscillation results for second order linear scalar differential equations (of the form of equation (1.7)) by considering the more general linear Volterra-Stieltjes integral equation

$$(1.10) \quad y'(t) = c + \int_a^t y(s)d\sigma(s), \quad t \geq a,$$

where $\sigma(t)$ is a right-continuous function locally of bounded variation on $[a, \infty)$, and

$$\lim_{t \rightarrow \infty} \sigma(t)$$

exists and may or may not be infinite. One such result (Theorem 2.1.1, [48]) gives a necessary and sufficient condition for the non-oscillation of equation (1.10) in terms of the existence of solutions of a certain nonlinear integral equation (a Riccati-type equation).

Theorem 1.11 (Theorem 2.1.1, [48]).

Let

$$\lim_{t \rightarrow \infty} \sigma(t) = 0.$$

Then equation (1.10) is non-oscillatory if and only if the integral equation

$$(1.12) \quad v(t) = \sigma(t) + \int_t^\infty v^2(s)ds$$

has a solution, for sufficiently large t , which is square integrable at infinity, i.e., $v \in L^2[t_0, \infty)$, for some $t_0 \in [a, \infty)$.

Another result of Mingarelli (Theorem 2.1.6, [48]) in the same work gives a criteria for $\sigma(t)$ which guarantees the non-oscillation of equation (1.10).

Theorem 1.13 (Theorem 2.1.6, [48])

Let

$$\lim_{t \rightarrow \infty} \sigma(t) = 0.$$

If for all $t \geq t_0$,

$$\int_t^\infty \sigma^2(s)ds \leq \frac{1}{4}|\sigma(t)|$$

then equation (1.10) is non-oscillatory.

By Theorem 1.11, the proof of Theorem 1.13 is only required to show that equation (1.12) has a solution, which is accomplished in [48] by the use of Schauder's Fixed Point Theorem.

If we set

$$\sigma(t) \equiv a(t)$$

into Theorems 1.11 and 1.13, and notice then that equation (1.10) is now equivalent to equation (1.7), we obtain the specific application of these two theorems to differential equations of the form of equation (1.7) (Theorems 2.1.1A and 2.1.6A in [48] respectively.)

Theorem 1.14 (Theorem 2.1.1A, [48])

Equation (1.7) is non-oscillatory if and only if the nonlinear integral equation

$$(1.15) \quad v(t) = \int_t^\infty q(s)ds + \int_t^\infty v^2(s)ds$$

has a solution for sufficiently large t .

This result extends a theorem of Hille (p. 243, [30]) from 1948 where it was assumed that the coefficient function $q(t)$ was positive ($q(t) > 0$). It is also interesting to note that the existence of a function $v(t)$ satisfying equation (1.15) is similar to Wintner's criterion (p. 375, [76]).

Theorem 1.16 (Theorem 2.1.6A, [48])

Suppose that the integral

$$a(t) = \int_t^\infty q(s)ds$$

converges (possibly conditionally).

If for all $t \geq t_0$,

$$(1.17) \quad \int_t^\infty a^2(s)ds \leq \frac{1}{4}|a(t)|$$

then equation (1.7) is non-oscillatory.

This result of Mingarelli is a direct extension of Theorem 1.9 due to Opial (p. 312, [54]), where the integral $a(t)$ need no longer be non-negative.

Our main result extends Theorem 1.16 due to Mingarelli (Theorem 2.1.6A, [48]) to matrix differential equations of the form (1.2) by creating a criterion for the non-oscillation of equation (1.2) essentially by considering

$$a(t) = \int_t^\infty Q(s)ds$$

and applying any fixed matrix norm $\|\cdot\|$.

Theorem 1.18

Consider the second order linear homogeneous matrix differential equation

$$(1.2) \quad Y'' + Q(t)Y = 0, \quad t \geq a$$

where $Y(t)$ and $Q(t)$ are real, $n \times n$ continuous matrix functions and $Q(t)$ is symmetric. Assume that

$$\int_t^\infty Q(s)ds$$

exists in the limiting sense, that is

$$\lim_{T \rightarrow \infty} \int_t^T Q(s)ds$$

exists and is finite for all sufficiently large t .

If, for all $t \geq a$,

$$(1.19) \quad \int_t^\infty \left\| \int_s^\infty Q(u) du \right\|^2 ds \leq \frac{1}{4} \left\| \int_t^\infty Q(s) ds \right\|,$$

then equation (1.2) is non-oscillatory (in the matrix sense) where $\|\cdot\|$ is any fixed matrix norm.

The matrix equivalent of Theorem 1.14, that is, the analog of the theorem applying to equation

$$(1.20) \quad y'' + Q(t)y = 0, \quad t \geq a, \quad y \in \mathbb{R}^n,$$

and thus, through the equivalency demonstrated by Tomastik [69] to equations of the form (1.2), is given by Reid (p. 284, Theorem 6.3, [59]) and by Mingarelli (Corollary 3.4.2, [50]) and will be appealed to in our proof of Theorem 1.18.

Corollary 1.21 (Corollary 3.4.2, [50])

Assume $Q(t)$ has the property that

$$\lim_{T \rightarrow \infty} \int_t^T Q(s) ds$$

exists, for each sufficiently large t , and that the limit matrix has finite norm, for said t , i.e.,

$$(1.21) \quad \left\| \int_t^\infty Q(s) ds \right\| < \infty$$

where $\|\cdot\|$ is any fixed (but unspecified) matrix norm.

Let the Riccati matrix integral system associated with equation (1.20), viz.,

$$(1.22) \quad V(t) = \int_t^\infty Q(s) ds + \int_t^\infty V^2(s) ds$$

have an absolutely continuous and symmetric matrix solution on some closed half-line $[t_0, \infty)$, $t_0 \geq a$, where $\|V\|$ is in $L^2(t_0, \infty)$. Then equation (1.20) is non-oscillatory. Hence equation (1.2) is non-oscillatory.

It should be noted that Corollary 1.21 is actually the matrix differential equation analog (i.e., to equation (1.2)) of Theorem 1.14.

In our proof of Theorem 1.18 we will parallel certain arguments found in the proofs of the following results of Mingarelli: Theorem 1.13 (Theorem 2.1.6, [48]), Theorem 2.1.2 in [48], and Theorem 3.47 in [50], the latter of which is the

generalization to matrix differential equations (of the form (1.2)) of Theorem 2.1.2A in [48], which is itself the application of Theorem 2.1.2 in [48] to the scalar differential equation (1.7). Thus, we will require the use of the Schauder Fixed Point Theorem.

Theorem 1.23 (Schauder Fixed Point Theorem) (p. 57, [62])

Let X be a convex subset of a Banach space and T be a continuous map leaving X invariant, i.e., $TX \subset X$, and such that TX is compact. Then T has a fixed point, i.e., for some $x \in X$, $x = Tx$.

In order to prove the compactness alluded to in Theorem 1.23, we will invoke the following lemma:

Lemma 1.24 (Lemma 3.4.6, [50])

Let F be a family of symmetric $n \times n$ matrices defined on $[a, \infty)$ and such that, for every member $V \in F$,

$$\|V(t)\| = \sup_{i,j} |V_{ij}(t)|$$

and $\|V\| \in L^2[a, \infty)$, i.e.,

$$\left[\int_a^\infty \|V(s)\|^2 ds \right]^{1/2} < \infty.$$

In addition, we assume that the following conditions hold for every $V \in F$:

(a) There is an $M > 0$ such that, for every $V \in F$,

$$\int_a^\infty \|V(s)\|^2 ds \leq M.$$

(b) If $E_A = \{s : A \leq s < \infty\}$, then, for given $\varepsilon > 0$, and for every $V \in F$,

$$\int_{E_A} \|V(s)\|^2 ds < \varepsilon$$

for A sufficiently large.

(c) For $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$, such that, for every $V \in F$,

$$\int_a^\infty \|V(s+h) - V(s)\|^2 ds < \varepsilon$$

whenever $|h| < \delta$.

Then F is compact.

We note that the choice of norm used in Lemma 1.24 is of no particular importance and any matrix norm can be used.

Proof of Theorem 1.18

By Corollary 1.21 we need only show that equation (1.22) has an absolutely continuous and symmetric solution on some closed half line $[t_0, \infty)$, $t_0 \geq a$, whose norm is in $L^2[t_0, \infty)$. We will thus employ Theorem 1.23.

We consider the Banach space

$$L^2[a, \infty) = \{V: [a, \infty) \rightarrow \mathcal{S} \mid \|V\|_2 < \infty\}$$

where $\|\cdot\|$ is any fixed matrix norm and $\|\cdot\|_2$ is the L^2 -norm, given by

$$\|\cdot\|_2 \equiv \left[\int_a^\infty \|\cdot\|^2 ds \right]^{1/2}.$$

Thus, $L^2[a, \infty)$ is the Banach space of real $n \times n$ matrix-valued functions defined on $[a, \infty)$ whose matrix norm is square-integrable at ∞ . We define a subset X of this Banach space by

$$X = \left\{ V \in L^2[a, \infty) : V(t) = V^*(t), \left\| V(t) - \int_t^\infty Q(s) ds \right\| \leq \left\| \int_t^\infty Q(s) ds \right\| \right\}$$

For $V \in X$ we define the map T by

$$(1.25) \quad (TV)(t) = \int_t^\infty Q(s) ds + \int_t^\infty V^2(s) ds.$$

We need to show that X is convex, $TX \subset X$, T is continuous, and TX is compact in order to apply Theorem 1.23.

If $\alpha \in [0, 1]$ and $U, W \in X$,

$$\begin{aligned} & \left\| \alpha U(t) + (1 - \alpha)W(t) - \int_t^\infty Q(s) ds \right\| \\ &= \left\| \alpha \left[U(t) - \int_t^\infty Q(s) ds \right] + (1 - \alpha) \left[W(t) - \int_t^\infty Q(s) ds \right] \right\| \\ &\leq \left\| \alpha \left[U(t) - \int_t^\infty Q(s) ds \right] \right\| + (1 - \alpha) \left\| W(t) - \int_t^\infty Q(s) ds \right\| \\ &\leq \alpha \left\| \int_t^\infty Q(s) ds \right\| + (1 - \alpha) \left\| \int_t^\infty Q(s) ds \right\| \\ &\leq \left\| \int_t^\infty Q(s) ds \right\|. \end{aligned}$$

Thus, X is convex.

Furthermore, TV is symmetric since both Q and V are symmetric. Moreover, for $V \in X$,

$$\begin{aligned}
 (1.26) \quad \|V(t)\| &= \|V(t) - \int_t^\infty Q(s)ds + \int_t^\infty Q(s)ds\| \\
 &\leq \|V(t) - \int_t^\infty Q(s)ds\| + \|\int_t^\infty Q(s)ds\| \\
 &\leq \|\int_t^\infty Q(s)ds\| + \|\int_t^\infty Q(s)ds\| \\
 &= 2\|\int_t^\infty Q(s)ds\|, \quad t \geq a.
 \end{aligned}$$

From equation (1.25) we have

$$(TV)(t) - \int_t^\infty Q(s)ds = \int_t^\infty V^2(s)ds$$

and applying our matrix norm we obtain

$$\begin{aligned}
 \|(TV)(t) - \int_t^\infty Q(s)ds\| &= \|\int_t^\infty V^2(s)ds\| \\
 &\leq \int_t^\infty \|V^2(s)\|ds \\
 &\leq \int_t^\infty \|V(s)\|^2ds \quad \text{from p. 492, [59]} \\
 &\leq \int_t^\infty 4\|\int_s^\infty Q(u)du\|^2ds \quad \text{by equation (1.26)} \\
 &= 4\int_t^\infty \|\int_s^\infty Q(u)du\|^2ds \\
 &\leq 4\left[\frac{1}{4}\|\int_t^\infty Q(s)ds\|\right] \quad \text{by equation (1.19) (hypothesis)} \\
 &= \|\int_t^\infty Q(s)ds\|.
 \end{aligned}$$

Thus, $TX \subset X$.

Let $V_n \in X$ and $V_n \rightarrow V$ in X . Then

$$\begin{aligned}
 (1.27) \quad \|(TV_n)(t) - (TV)(t)\| &= \|\int_t^\infty Q(s)ds + \int_t^\infty V_n^2(s)ds \\
 &\quad - [\int_t^\infty Q(s)ds + \int_t^\infty V^2(s)ds]\| \\
 &= \|\int_t^\infty V_n^2(s)ds - \int_t^\infty V^2(s)ds\| \\
 &\leq \int_t^\infty \|V_n^2(s) - V^2(s)\|ds.
 \end{aligned}$$

Now,

$$\begin{aligned} V_n^2(t) - V^2(t) &= V_n(t)V(t) - V^2(t) + V_n^2(t) - V_n(t)V(t) \\ &= [V_n(t) - V(t)]V(t) + V_n(t)[V_n(t) - V(t)]. \end{aligned}$$

Therefore,

(1.28)

$$\begin{aligned} \|V_n^2(t) - V^2(t)\| &\leq \|[V_n(t) - V(t)]V(t)\| + \|V_n(t)[V_n(t) - V(t)]\| \\ &\leq \|V_n(t) - V(t)\| \|V(t)\| + \|V_n(t)\| \|V_n(t) - V(t)\| \\ &\leq \|V_n(t) - V(t)\| [2\| \int_t^\infty Q(s)ds \|] \\ &\quad + [2\| \int_t^\infty Q(s)ds \|] \|V_n(t) - V(t)\| \text{ by equation (1.26)} \\ &= 4\| \int_t^\infty Q(s)ds \| \|V_n(t) - V(t)\| \end{aligned}$$

So (1.27) becomes

$$(1.29) \quad \|(TV_n)(t) - (TV)(t)\| \leq 4 \int_t^\infty \| \int_s^\infty Q(u)du \| \|V_n(s) - V(s)\| ds$$

We now apply the following lemma to inequality (1.29).

Lemma 1.30 (Cauchy-Schwarz Inequality in L^2)

Suppose f and g are two functions in $L^2[a, \infty)$. Then (fg) is in $L^1[a, \infty)$, and

$$(1.31) \quad \int_a^\infty f(s)g(s)ds \leq \left\{ \int_a^\infty [f(s)]^2 ds \right\}^{1/2} \left\{ \int_a^\infty [g(s)]^2 ds \right\}^{1/2}.$$

Applying inequality (1.31) to inequality (1.29) we obtain

$$\begin{aligned} \|(TV_n)(t) - (TV)(t)\| &\leq 4 \left[\int_t^\infty \| \int_s^\infty Q(u)du \|^2 ds \right]^{1/2} \left[\int_t^\infty \|V_n(s) - V(s)\|^2 ds \right]^{1/2} \\ &\leq 4 \| \int_t^\infty Q(s)ds \|_2 \|V_n(t) - V(t)\|_2. \end{aligned}$$

But by hypothesis,

$$(1.19) \quad \int_t^\infty \| \int_s^\infty Q(u)du \|^2 ds \leq \frac{1}{4} \| \int_t^\infty Q(s)ds \|^2.$$

Therefore,

$$\begin{aligned} \int_t^\infty \|V_n^2(s) - V^2(s)\| ds &\leq 4 \|V_n(t) - V(t)\|_2 \left[\frac{1}{2} \| \int_t^\infty Q(s)ds \|^{1/2} \right] \\ &= 2 \|V_n(t) - V(t)\|_2 \| \int_t^\infty Q(s)ds \|^{1/2}. \end{aligned}$$

But $V_n \rightarrow V$ in X by hypothesis, which in turn, implies that $\|(TV_n)(t) - (TV)(t)\| \rightarrow 0$ point-wise for sufficiently large t , say $t_0 \geq a$.

Now, since $TX \subset X$, we have

$$\begin{aligned}
 \|(TV_n)(t) - (TV)(t)\|^2 &= \|(TV_n)(t) - (TV)(t)\| \|(TV_n)(t) - (TV)(t)\| \\
 &\leq [\|(TV_n)(t)\| + \|(TV)(t)\|] [\|(TV_n)(t)\| + \|(TV)(t)\|] \\
 &\leq \left[2\left\| \int_t^\infty Q(s)ds \right\| + 2\left\| \int_t^\infty Q(s)ds \right\| \right] \left[2\left\| \int_t^\infty Q(s)ds \right\| \right. \\
 &\quad \left. + 2\left\| \int_t^\infty Q(s)ds \right\| \right] \text{ by equation (1.26)} \\
 &= \left[4\left\| \int_t^\infty Q(s)ds \right\| \right] \left[4\left\| \int_t^\infty Q(s)ds \right\| \right] \\
 &= 16\left\| \int_t^\infty Q(s)ds \right\|^2
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 \int_a^\infty \|(TV_n)(s) - (TV)(s)\|^2 ds &\leq 16 \int_a^\infty \left\| \int_s^\infty Q(u)du \right\|^2 ds \\
 \|(TV_n)(t) - (TV)(t)\|_2^2 &\leq 16 \left\| \int_s^\infty Q(t)dt \right\|_2^2
 \end{aligned}$$

and the Lebesgue Dominated Convergence Theorem (p. 255, Theorem 3.5.2, [53]) now gives that $\|(TV)(t) - (TV)(t)\|_2 \rightarrow 0$ so that T is continuous.

In order to show that TX is compact, we use Lemma 1.24. For $V \in X$,

$$\begin{aligned}
 \|(TV)(t)\|_2 &= \left[\int_a^\infty \|(TV)(s)\|^2 ds \right]^{1/2} \\
 &\leq \left[4 \int_a^\infty \left\| \int_s^\infty Q(u)du \right\|^2 ds \right]^{1/2} \\
 &\quad \text{by equation (1.26) and since } TX \subset X \\
 &= 2 \left\| \int_s^\infty Q(u)du \right\|_2
 \end{aligned}$$

which is bounded by a constant matrix, since $\int_s^\infty Q(u)du$ is a matrix of finite norm by hypothesis (see (1.19)). Thus,

$$\left\| \int_t^\infty Q(s)ds \right\| \text{ is in } L^2[a, \infty).$$

Furthermore, (b) is clear since, by (1.19),

$$\lim_{A \rightarrow \infty} \int_{E_A} \left\| \int_s^\infty Q(u)du \right\|^2 ds = 0.$$

Next,

(1.32)

$$\begin{aligned}
& \|(TV)(t+h) - (TV)(t)\| \\
& \leq \left\| \int_t^{t+h} Q(s)ds \right\| + \left\| \int_t^{t+h} V^2(s)ds \right\| \\
& \leq \left\| \int_t^{t+h} Q(s)ds \right\| + \int_t^{t+h} \|V(s)\|^2 ds, \text{ by (p. 492, [59])} \\
& \leq \left\| \int_t^{t+h} Q(s)ds \right\| + \left| \int_t^{t+h} 4 \left\| \int_s^\infty Q(u)du \right\|^2 ds \right|, \text{ by equation (1.26)} \\
& = \|\mathcal{Q}(t+h) - \mathcal{Q}(t)\| + \left| \int_t^{t+h} 4 \left\| \int_s^\infty Q(u)du \right\|^2 ds \right|
\end{aligned}$$

where

$$\mathcal{Q}(t) \equiv \int_t^\infty Q(s)ds$$

By hypothesis,

$$\|\mathcal{Q}(t)\| = \left\| \int_t^\infty Q(s)ds \right\|$$

is finite, and so $\|\mathcal{Q}(t)\|$ is in $L^2[a, \infty)$. Thus, by one of Lebesgue's theorems (see p. 397, Exercise 19, [68]), if $\varepsilon > 0$,

$$(1.33) \quad \int_{t_0}^\infty \|\mathcal{Q}(s+h) - \mathcal{Q}(s)\|^2 ds < \frac{\varepsilon}{4}$$

whenever $|h| < \delta_1$, where $t_0 \geq a$ is sufficiently large but fixed. Similarly, writing

$$(1.34) \quad \mathcal{V}(t) \equiv \int_t^\infty 4 \left\| \int_s^\infty Q(u)du \right\|^2 ds$$

we have $\mathcal{V}(t) \in L^2[a, \infty)$ by (1.19), and so

$$(1.35) \quad \int_{t_1}^\infty |\mathcal{V}(s+h) - \mathcal{V}(s)|^2 ds < \frac{\varepsilon}{4}$$

for $|h| < \delta_2$, where $t_1 \geq a$ is sufficiently large and fixed. By equation (1.34), inequality (1.32) becomes

$$\|(TV)(t+h) - (TV)(t)\| \leq \|\mathcal{Q}(t+h) - \mathcal{Q}(t)\| + |\mathcal{V}(t+h) - \mathcal{V}(t)|$$

and thus

$$(1.36) \quad \|(TV)(t+h) - (TV)(t)\|^2 \leq [\|\mathcal{Q}(t+h) - \mathcal{Q}(t)\| + |\mathcal{V}(t+h) - \mathcal{V}(t)|]^2$$

We now apply the following lemma.

Lemma 1.37

If $a, b \in \mathbb{R}$, then $(a \pm b)^2 \leq 2a^2 + 2b^2$.

Proof

For $a, b \in \mathbb{R}$,

$$\begin{aligned} (a \pm b)^2 &= a^2 \pm 2ab + b^2 \\ &\leq 2a^2 + 2b^2 \end{aligned}$$

since

$$(a \mp b)^2 = a^2 + b^2 \mp 2ab \geq 0.$$

Applying Lemma 1.37 to the right side of inequality (1.36) we obtain

$$\|(TV)(t+h) - (TV)(t)\|^2 \leq 2\|\mathcal{Q}(t+h) - \mathcal{Q}(t)\|^2 + 2|\mathcal{V}(t+h) - \mathcal{V}(t)|^2$$

Hence,

$$\begin{aligned} \int_{t_2}^{\infty} \|(TV)(s+h) - (TV)(s)\|^2 ds &\leq 2 \int_{t_2}^{\infty} \|\mathcal{Q}(s+h) - \mathcal{Q}(s)\|^2 ds + 2 \int_{t_2}^{\infty} |\mathcal{V}(s+h) - \mathcal{V}(s)|^2 ds \\ &< 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) \text{ by (1.33) and (1.35)} \\ &= \varepsilon, \end{aligned}$$

provided $|h| < \delta = \min\{\delta_1, \delta_2\}$ and $t_2 = \max\{t_0, t_1\}$. This shows that TX is an “ L^2 -equicontinuous” family (p. 72, [50]) and consequently by Lemma 1.24, TX is compact.

Applying Theorem 1.23 now shows the existence of a fixed point for T in X , i.e., $V = TV$ for some $V \in X$, which completes the proof.

Chapter 2

NON-LINEAR FUNCTIONALS IN MATRIX OSCILLATION THEORY

Consider the second order linear matrix differential equation

$$(2.1) \quad [P(t)Y']' + Q(t)Y = 0, \quad t \geq t_0$$

where $P(t)$, $Q(t)$ and $Y(t)$ are real, $n \times n$, continuous matrix functions on $[t_0, \infty)$ such that $Q(t)$ and $P(t)$ are symmetric and $P(t)$ is positive definite ($P(t) > 0$). When $P(t) \equiv I$, we obtain the matrix differential equation

$$(2.2) \quad Y'' + Q(t)Y = 0, \quad t \geq t_0.$$

As mentioned at the beginning of Chapter 1, many authors, including Allegretto and Erbe [2], Etgen and Lewis [22][21], Hartman [26], Hinton and Lewis [31], Tomastik [71], and Walters [72] have shown that equation (2.1) (or (2.2)) is oscillatory if a corresponding scalar equation obtained by applying a positive linear functional is oscillatory. Hence, we can study the oscillation of the matrix equations (2.1) and (2.2) in terms of a vast number of well-known oscillation criteria of the associated second order linear scalar equations

$$(2.3) \quad [p(t)y']' + q(t)y = 0, \quad t \geq t_0$$

where $p(t)$ and $q(t)$ are continuous real-valued functions on $[t_0, \infty)$, and when $p(t) \equiv 1$,

$$(2.4) \quad y'' + q(t)y = 0, \quad t \geq t_0.$$

Other oscillation criteria for equations (2.1) and (2.2) have involved conditions on the eigenvalues of $P(t)$ and $Q(t)$ or their integrals, which also have their origins in analogous conditions on equations (2.3) or (2.4). Let us briefly trace these origins.

In 1918, Fite [23] determined that for $q(t) > 0$ the condition

$$(2.5) \quad \lim_{T \rightarrow \infty} \int_{t_0}^T q(s)ds = \infty$$

implies that equation (2.4) is oscillatory. (Later, this condition became known as the Fite-Wintner-Leighton criterion [42], but Leighton's work dealt with the more general equation (2.3)). In 1949, Wintner [75] allowed $q(t)$ to assume negative values for arbitrarily large values of t , and proved that the weaker condition

$$(2.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^T \left[\int_{t_0}^s q(u) du \right] ds = \infty$$

implies the oscillation of equation (2.4). This was likely the first result based on an integral average.

In 1952, Hartman [25] proved that the limit cannot be replaced by the upper limit in (2.6), and that the condition

$$(2.7) \quad \begin{aligned} -\infty &< \liminf_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^T \left[\int_{t_0}^s q(u) du \right] ds \\ &< \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^T \left[\int_{t_0}^s q(u) du \right] ds \leq \infty \end{aligned}$$

suffices for the oscillation of equation (2.4). Coles [15], Willett [73], and Kwong and Zettl [40] extended this result by considering weighted averages of the integral of $q(t)$.

In 1978, Kamenev [34] established a new integral criterion for the oscillation of equation (2.4) with the result of Wintner as a special case. More precisely, Kamenev proved that if $\alpha > 1$ is any real number, then

$$(2.8) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_{t_0}^t (t-s)^\alpha q(s) ds = \infty,$$

implies that equation (2.4) is oscillatory. This result was later extended by Yan [78] and Yeh [79] to the damped linear equation

$$(2.9) \quad y'' + k(t)y' + q(t)y = 0, \quad t \geq t_0 > 0$$

where $k(t)$ is a real-valued continuous function allowed to assume negative values for arbitrarily large values of $t \in [t_0, \infty)$. Chen [13] extended one of Yan's results to the general nonlinear second order damped differential equation

$$(2.10) \quad [p(t)y']' + k(t)y' + q(t)f(y) = 0,$$

which was then extended in 2000 for the case $p(t) = 1$ by Wong [77] using general means.

Meanwhile, in 1980 Hinton and Lewis [31] proposed a conjecture originally attributed to Reid, that the matrix differential equation (2.2) is oscillatory if

$$(2.11) \quad \lim_{t \rightarrow \infty} \lambda_1 \left[\int_{t_0}^t Q(s) ds \right] = \infty$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ denotes the usual ordering of the eigenvalues of the symmetric matrix A . The conjecture was settled with additional assumptions on the rate of growth of the trace of $\int_{t_0}^t Q(s) ds$ by Mingarelli [49], Kwong et al. [41], and Butler and Erbe [9], [10]. The conjecture was finally proved for the case $n = 2$ by Kwong and Kaper [35] and for arbitrary n by Byers, Harris, and Kwong [12], in 1985 and 1986 respectively. A year later, Butler, Erbe, and Mingarelli [11] extended the scalar criteria (2.6) and (2.7) to the matrix equation (2.2) using Riccati techniques and variational principles.

In 1989, Kamenev's theorem was extended by Philos [55], still in the scalar case of equation (2.4), to the following theorem:

Theorem 2.12 (Theorem 1, [55]).

Let $H(t, s)$ and $h(t, s)$ be continuous on $D = \{(t, s) : t \geq s \geq t_0\}$ such that $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ for $t > s \geq t_0$. Assume further that the partial derivative $\frac{\partial H}{\partial s}(t, s)$ is non-positive and continuous for $t \geq s \geq t_0$, and $h(t, s)$ is defined by

$$\frac{\partial H}{\partial s}(t, s) = -h(t, s)[H(t, s)]^{1/2}, \quad (t, s) \in D.$$

Then equation (2.4) is oscillatory if

$$(2.13) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s)q(s) - \frac{1}{4}h^2(t, s) \right] ds = \infty.$$

However, Philos' result could not be applied to the Euler differential equation

$$(2.14) \quad y'' + \frac{\gamma}{t^2}y = 0$$

where $\gamma > 0$ is a constant. It is well known that equation (2.14) is oscillatory if $\gamma > \frac{1}{4}$ and non-oscillatory if $\gamma \leq \frac{1}{4}$ (see p.45, [67]), but

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, 1)} \int_1^t \left[H(t, \xi) \frac{\gamma}{\xi^2} - \frac{1}{4}h^2(t, \xi) \right] d\xi \\ & \leq \limsup_{t \rightarrow \infty} \int_1^t \frac{\gamma}{\xi^2} d\xi = \gamma < \infty, \end{aligned}$$

i.e., (2.13) does not apply to equation (2.14).

In 1995, Li [43] produced oscillation criteria which improved the result of Philos by using a generalized Riccati transformation due to Yu [80], with the new result being applicable to the question of the oscillation of equation (2.14).

Meanwhile, two years before, a prominent generalization of Philos' theorem occurred when Erbe, Kong, and Ruan [18] extended the result to matrix differential systems of the form of equation (2.1). We state the result for $P(t) \equiv I$, i.e., for equation (2.2).

Theorem 2.15 (Theorem 1, [18])

Let $H(t, s)$ and $h(t, s)$ be defined as in Theorem 2.12. System (2.2) is oscillatory if

$$(2.16) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left\{ \int_{t_0}^t \left[H(t, s)Q(s) - \frac{1}{4}h^2(t, s)I \right] ds \right\} = +\infty,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \lambda_n(A)$ denotes the usual ordering of the eigenvalues of the symmetric matrix A , and I is the $n \times n$ identity matrix.

However, consider the Euler differential system

$$(2.17) \quad Y'' + \text{diag} \left(\frac{\gamma}{t^2}, \frac{\beta}{t^2} \right) Y = 0, \quad t \geq 1$$

where $Y(t)$ and $Q(t) = \text{diag} \left(\frac{\gamma}{t^2}, \frac{\beta}{t^2} \right)$ are 2×2 matrices, and $\gamma \geq \beta > 0$ are constants. Since

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, 1)} \lambda_1 \left\{ \int_1^t \left[H(t, s)Q(s) - \frac{1}{4}h^2(t, s)I \right] ds \right\} \\ & \leq \limsup_{t \rightarrow \infty} \int_1^t \frac{\gamma}{s^2} ds = \gamma < \infty, \end{aligned}$$

Theorem 2.15 cannot be applied to system (2.17).

In 1998, Meng, Wang, and Zheng [45] improved on Theorem 2.15, proving that system (2.17) is oscillatory for $\gamma > \frac{1}{4}$ by using the following theorem:

Theorem 2.18 (Theorem 1, [45])

Let $H(t, s)$ and $h(t, s)$ be defined as in Theorem 2.12. If there exists a function $f \in C^1[t_0, \infty)$ such that

$$(2.19) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left\{ \int_{t_0}^t \left[H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right] ds \right\} = +\infty,$$

where $a(t) = \exp \left\{ -2 \int^t f(s) ds \right\}$, $R(t) = a(t)[Q(t) + f^2(t)I - f'(t)I]$, then system (2.2) is oscillatory.

Since this condition necessitates a very smart choice of f it is relatively difficult to apply to the general case. In addition, both [45] and [18] give criteria of the form $\limsup_{t \rightarrow \infty} \lambda_1[\cdot] = +\infty$, which is not very sharp.

Most recently, and following the underlying thread developed in [18] and [45] of using a matrix Riccati transformation coupled with an appropriate kernel, Sun [66] has produced two new sharper conditions of Kamenev-type of the form $\limsup_{t \rightarrow \infty} \lambda_1[\cdot]$ is greater than some constant for system (2.2) to be oscillatory.

Theorem 2.20 (Theorem 1, [66])

System (2.2) is oscillatory provided that for each $r \geq t_0$ and for some $\alpha > \frac{1}{2}$

$$(2.21) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \left[\int_r^t (s-r)^{2\alpha} (t-s)^2 Q(s) ds \right] > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Theorem 2.22 (Theorem 2, [66])

System (2.2) is oscillatory provided that for each $r \geq t_0$ and for some $\alpha > \frac{1}{2}$

$$(2.23) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \left[\int_r^t (t-s)^{2\alpha} (s-r)^2 Q(s) ds \right] > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Applying Theorem 2.20 to the Euler differential system (2.17), we note that for each $r \geq t_0$

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \left[\int_r^t (s-r)^{2\alpha} (t-s)^2 Q(s) ds \right] \\ &= \lim_{t \rightarrow \infty} \frac{1}{t^{2\alpha+1}} \int_r^t (s-r)^{2\alpha} (t-s)^2 \frac{\gamma}{s^2} ds \\ &= \frac{\gamma}{\alpha(2\alpha-1)(2\alpha+1)} \lim_{t \rightarrow \infty} \frac{(t-r)^{2\alpha}}{t^{2\alpha}} \\ &= \frac{\gamma}{\alpha(2\alpha-1)(2\alpha+1)}. \end{aligned}$$

For any $\gamma > \frac{1}{4}$, there exists an $\alpha > \frac{1}{2}$ such that

$$\frac{\gamma}{\alpha(2\alpha-1)(2\alpha+1)} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

i.e., $\gamma > \alpha^2$. This means that condition (2.21) holds. So by Theorem 2.20, system (2.17) is oscillatory for $\gamma > \frac{1}{4}$.

Our first key result in this chapter will be to extend the result of Sun [66] by considering a more general kernel than the ones used in conditions (2.21) and (2.23). The end result is that Sun's two theorems become corollaries of our new result.

Our second key result draws some inspiration from a very recent work of Mingarelli [51] which itself follows through on an idea first introduced by Meng and Mingarelli [46], and Kong [37]. Originally inspired by Hartman [26] [27], the idea is to produce criteria that do not depend explicitly on the largest eigenvalue of the matrices under consideration. We will show that very general (and possibly discontinuous) nonlinear functionals can be used to produce a more general oscillation criterion when applied to our first main result.

Theorem 2.24

System (2.2) is oscillatory provided that for each $r \geq t_0$ and for some $p, q \geq 2$

$$(2.25) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^{p+q-1}} \lambda_1 \left[\int_r^t (s-r)^p (t-s)^q Q(s) ds \right] \\ > \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)}$$

where Γ is the gamma function defined by (p. 251, [61])

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx \\ = (z-1)\Gamma(z-1)$$

(and $\Gamma(n) = (n-1)!$ for $n \geq 1$, n an integer).

Proof.

Suppose to the contrary that there exists a prepared solution $Y(t)$ of equation (2.2) which is not oscillatory. So $Y^{-1}(t)$ exists for $t \geq T \geq t_0$, and thus $\det Y(t) \neq 0$ for $t \geq T \geq t_0$. We define

$$(2.26) \quad V(s) = -Y'(s)Y^{-1}(s), \quad s \geq T$$

which is essentially equation (O) with $P(t) \equiv I$ from the Introduction. Following the development of equation (O), $V(s) = V^*(s)$, and so $V^2(s)$ is a non-negative definite matrix.

From equation (2.26) we take derivatives to obtain

$$(2.27) \quad \begin{aligned} V'(s) &= -[Y'(s)]'Y^{-1}(s) - Y'(s)[Y^{-1}(s)]' \\ &= -Y''(s)Y^{-1}(s) - Y'(s)[Y^{-1}(s)]'. \end{aligned}$$

But

$$I = Y(s)Y^{-1}(s)$$

and taking derivatives we obtain

$$\begin{aligned} [Y(s)Y^{-1}(s)]' &= 0 \\ Y'(s)Y^{-1}(s) + Y(s)[Y^{-1}(s)]' &= 0 \\ Y(s)[Y^{-1}(s)]' &= -Y'(s)Y^{-1}(s) \\ [Y^{-1}(s)]' &= -Y^{-1}(s)Y'(s)Y^{-1}(s), \end{aligned}$$

and so equation (2.27) becomes

$$(2.28) \quad \begin{aligned} V'(s) &= -Y''(s)Y^{-1}(s) + Y'(s)Y^{-1}(s)Y'(s)Y^{-1}(s) \\ &= -Y''(s)Y^{-1}(s) + [Y'(s)Y^{-1}(s)]^2. \end{aligned}$$

Finally, from equations (2.26) and (2.2), equation (2.28) becomes

$$(2.29) \quad \begin{aligned} V'(s) &= Q(s) + V^2(s), \text{ or} \\ Q(s) &= V'(s) - V^2(s), \quad s \geq T. \end{aligned}$$

Multiplying equation (2.29) by $(s - T)^p(t - s)^q$ and integrating with respect to s from T to t , we obtain

$$\begin{aligned} &\int_T^t (s - T)^p(t - s)^q Q(s) ds \\ &= \int_T^t (s - T)^p(t - s)^q V'(s) ds - \int_T^t (s - T)^p(t - s)^q V^2(s) ds. \end{aligned}$$

Using integration by parts on the first integral,

$$\begin{aligned} &\int_T^t (s - T)^p(t - s)^q Q(s) ds \\ &= [(s - T)^p(t - s)^q V(s)] \Big|_T^t - \int_T^t \frac{d}{ds} [(s - T)^p(t - s)^q] V(s) ds \\ &\quad - \int_T^t (s - T)^p(t - s)^q V^2(s) ds \\ &= 0 - \int_T^t [p(s - T)^{p-1}(t - s)^q - q(s - T)^p(t - s)^{q-1}] V(s) ds \end{aligned}$$

$$\begin{aligned}
& - \int_T^t (s-T)^p (t-s)^q V^2(s) ds \\
& = - \int_T^t (s-T)^{p/2} (t-s)^{q/2} [p(s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - q(s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1}] V(s) ds \\
& \quad - \int_T^t (s-T)^p (t-s)^q V^2(s) ds.
\end{aligned}
\tag{2.30}$$

Now, setting

$$\begin{aligned}
(2.31) \quad a &= \left[\frac{p}{2} (s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - \frac{q}{2} (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1} \right] I, \text{ and} \\
b &= (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}} V(s)
\end{aligned}$$

for simplicity, we see that

$$\begin{aligned}
(2.32) \quad & \int_T^t (s-T)^p (t-s)^q Q(s) ds \\
& = -2 \int_T^t (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}} \left[\frac{p}{2} (s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - \frac{q}{2} (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1} \right] V(s) ds \\
& \quad - \int_T^t (s-T)^p (t-s)^q V^2(s) ds \\
& = -2 \int_T^t ab ds - \int_T^t b^2 ds \text{ by equations (2.31).}
\end{aligned}$$

But since $-2ab - b^2 = a^2 - (b+a)^2$, equation (2.32) gives

$$\begin{aligned}
(2.33) \quad & \int_T^t (s-T)^p (t-s)^q Q(s) ds \\
& = \int_T^t a^2 ds - \int_T^t (b+a)^2 ds \\
& = \int_T^t \left[\frac{p}{2} (s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - \frac{q}{2} (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1} \right]^2 I^2 ds \\
& \quad - \int_T^t \left\{ (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}} V(s) \right. \\
& \quad \left. + \left[\frac{p}{2} (s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - \frac{q}{2} (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1} \right] I \right\}^2 ds.
\end{aligned}$$

Therefore, we have that

$$(2.34) \quad \int_T^t (s-T)^p (t-s)^q Q(s) ds$$

$$\leq \int_T^t \left[\frac{p}{2}(s-T)^{\frac{p}{2}-1}(t-s)^{\frac{q}{2}} - \frac{q}{2}(s-T)^{\frac{p}{2}}(t-s)^{\frac{q}{2}-1} \right]^2 I \, ds.$$

Since the last integral of equation (2.33) is clearly non-negative, it follows from inequality (2.34) that

$$\begin{aligned} & \lambda_1 \left[\int_T^t (s-T)^p (t-s)^q Q(s) \, ds \right] \\ & \leq \int_T^t \left[\frac{p}{2}(s-T)^{\frac{p}{2}-1}(t-s)^{\frac{q}{2}} - \frac{q}{2}(s-T)^{\frac{p}{2}}(t-s)^{\frac{q}{2}-1} \right]^2 ds \\ & = \int_T^t \left\{ \left[\frac{p}{2}(s-T)^{\frac{p}{2}-1}(t-s)^{\frac{q}{2}} \right]^2 - 2 \left[\frac{pq}{4}(s-T)^{\frac{p}{2}-1+\frac{p}{2}}(t-s)^{\frac{q}{2}+\frac{q}{2}-1} \right] \right. \\ & \quad \left. + \left[\frac{q}{2}(s-T)^{\frac{p}{2}}(t-s)^{\frac{q}{2}-1} \right]^2 \right\} ds \\ & = \int_T^t \left[\frac{p^2}{4}(s-T)^{p-2}(t-s)^q - \frac{pq}{2}(s-T)^{p-1}(t-s)^{q-1} + \frac{q^2}{4}(s-T)^p(t-s)^{q-2} \right] ds \\ & = \int_T^t (s-T)^{p-2}(t-s)^{q-2} \left[\frac{p^2}{4}(t-s)^2 - \frac{pq}{2}(s-T)(t-s) + \frac{q^2}{4}(s-T)^2 \right] ds \\ & = \int_T^t (s-T)^{p-2}(t-s)^{q-2} \left[\frac{p}{2}(t-s) - \frac{q}{2}(s-T) \right]^2 ds \\ & = \frac{1}{4} \int_T^t (s-T)^{p-2}(t-s)^{q-2} [p(t-s) - q(s-T)]^2 ds, \\ & \text{(and setting } u = s - T) \\ & = \frac{1}{4} \int_0^{t-T} u^{p-2}(t-T-u)^{q-2} [p(t-T-u) - qu]^2 du \\ & \text{(and setting } w = t - T) \\ & = \frac{1}{4} \int_0^w u^{p-2}(w-u)^{q-2} [p(w-u) - qu]^2 du. \end{aligned}$$

We will evaluate this integral using Euler's beta function [61],

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \quad \operatorname{Re}(\mu, \nu) > 0.$$

Thus,

$$\begin{aligned} & (2.35) \\ & \frac{1}{4} \int_0^w u^{p-2}(w-u)^{q-2} [p(w-u) - qu]^2 du \\ & = \frac{1}{4} \int_0^w u^{p-2}(w-u)^{q-2} [p^2(w-u)^2 - 2pqu(w-u) + q^2u^2] du \\ & = \frac{1}{4} \int_0^w \left[p^2u^{p-2}(w-u)q - 2pqu^{p-1}(w-u)^{q-1} + q^2u^p(w-u)^{q-2} \right] du \\ & = \frac{1}{4} \left[p^2 \int_0^w u^{p-2}(w-u)^q du - 2pq \int_0^w u^{p-1}(w-u)^{q-1} du + q^2 \int_0^w u^p(w-u)^{q-2} du \right] \end{aligned}$$

Evaluating the first integral using Euler's beta function, we obtain

$$\begin{aligned}
 (2.36) \quad & \int_0^w u^{p-2}(w-u)^q du \\
 & \text{(and setting } x = \frac{u}{w} \text{)} \\
 & = \int_0^1 (wx)^{p-2}(w-wx)^q w dx \\
 & = w^{p+q-1} \frac{\Gamma(p-1)\Gamma(q+1)}{\Gamma(p+q)}
 \end{aligned}$$

Evaluating the second and third integrals in (2.35) in a similar fashion, we obtain

$$\begin{aligned}
 (2.37) \quad & \int_0^w u^{p-1}(w-u)^{q-1} du \\
 & = w^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.38) \quad & \int_0^w u^p(w-u)^{q-2} du \\
 & = w^{p+q-1} \frac{\Gamma(p+1)\Gamma(q-1)}{\Gamma(p+q)}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \frac{1}{4} \int_0^w u^{p-2}(w-u)^{q-2} [p(w-u) - qu]^2 du \\
 & = \frac{1}{4} \left[p^2 \frac{\Gamma(p-1)\Gamma(q+1)}{\Gamma(p+q)} w^{p+q-1} - 2pq \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} w^{p+q-1} \right. \\
 & \quad \left. + q^2 \frac{\Gamma(p+1)\Gamma(q-1)}{\Gamma(p+q)} w^{p+q-1} \right] \\
 & = \frac{w^{p+q-1}}{4\Gamma(p+q)} \left[p^2 \Gamma(p-1)\Gamma(q+1) - 2pq\Gamma(p)\Gamma(q) + q^2 \Gamma(p+1)\Gamma(q-1) \right] \\
 & = \frac{w^{p+q-1}}{4\Gamma(p+q)} \left[p^2 q(q-1)\Gamma(p-1)\Gamma(q-1) - 2p(p-1)q(q-1)\Gamma(p-1)\Gamma(q-1) \right. \\
 & \quad \left. + p(p-1)q^2 \Gamma(p-1)\Gamma(q-1) \right] \\
 & = \frac{w^{p+q-1}\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} pq \left[p(q-1) - 2(p-1)(q-1) + (p-1)q \right] \\
 & = pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} w^{p+q-1}
 \end{aligned}$$

(2.39)

Substituting back in for $w = t - T$, equation (2.39) gives

$$(2.40) \quad pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (t-T)^{p+q-1}.$$

So we have that

$$\lambda_1 \left[\int_T^t (s-T)^p (t-s)^q Q(s) ds \right] \leq pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (t-T)^{p+q-1}, \quad t \geq T \geq t_0.$$

Thus,

$$(2.41) \quad \frac{1}{t^{p+q-1}} \lambda_1 \left[\int_T^t (s-T)^p (t-s)^q Q(s) ds \right] \leq pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \frac{(t-T)^{p+q-1}}{t^{p+q-1}},$$

Taking the lim sup on both sides of inequality (2.41) we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{p+q-1}} \lambda_1 \left[\int_T^t (s-T)^p (t-s)^q Q(s) ds \right] \\ & \leq pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \limsup_{t \rightarrow \infty} \left(\frac{t-T}{t} \right)^{p+q-1} \\ & = pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \limsup_{t \rightarrow \infty} \left(1 - \frac{T}{t} \right)^{p+q-1} \\ & = pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (1^{p+q-1}) \\ & = pq(p+q-2) \frac{\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \end{aligned}$$

which contradicts condition (2.25).

Remarks

1. The values $p = q = 1$ are prohibited as a simple evaluation of the integrals in equation (2.35) with these values shows. Hence, the restriction on p and q being greater than 1. The values $1 < p, q < 2$ are admissible into the Γ functions of the general result.
2. We can plainly see that Sun's two main theorems are corollaries to Theorem 2.24.

Corollary 2.42

See our Theorem 2.20 (Theorem 1, [66]).

Proof.

Setting $p = 2\alpha$, $\alpha > \frac{1}{2}$ and $q = 2$ in condition (2.25), we obtain

$$\begin{aligned}
 & \limsup_{t \rightarrow \infty} \frac{1}{2^{2\alpha+1}} \lambda_1 \left[\int_r^t (s-r)^{2\alpha} (t-s)^2 Q(s) ds \right] \\
 & > \frac{2\alpha(2)(2\alpha+2-2)\Gamma(2\alpha-1)\Gamma(2-1)}{4\Gamma(2\alpha+1)} \\
 & = \frac{4\alpha(2\alpha)\Gamma(2\alpha-1)\Gamma(1)}{4(2\alpha+1)(2\alpha)(2\alpha-1)\Gamma(2\alpha-1)} \\
 & = \frac{\alpha}{(2\alpha+1)(2\alpha-1)},
 \end{aligned}$$

which is exactly Sun's condition (2.21).

Corollary 2.43

See our Theorem 2.22 (Theorem 2, [66]).

Proof.

Setting $p = 2$, and $q = 2\alpha$, $\alpha > \frac{1}{2}$ in condition (2.25), we obtain

$$\begin{aligned}
 & \limsup_{t \rightarrow \infty} \frac{1}{2^{2\alpha+1}} \lambda_1 \left[\int_r^t (s-r)^2 (t-s)^{2\alpha} Q(s) ds \right] \\
 & > \frac{2(2\alpha)(2+2\alpha-2)\Gamma(2-1)\Gamma(2\alpha-1)}{4\Gamma(2\alpha+1)} \\
 & = \frac{4\alpha(2\alpha)\Gamma(1)\Gamma(2\alpha-1)}{4(2\alpha+1)(2\alpha)(2\alpha-1)\Gamma(2\alpha-1)} \\
 & = \frac{\alpha}{(2\alpha+1)(2\alpha-1)},
 \end{aligned}$$

which is exactly Sun's condition (2.23).

For our second key result, we will generalize Sun's results even further by replacing λ_1 in our first key result with a general negativity-preserving nonlinear functional, thus producing a criterion for oscillation that does not depend explicitly on the largest eigenvalue of the matrices under consideration.

Definition 2.44

Let $\mathcal{S}$ be the real linear vector space of $n \times n$ symmetric matrices.

A nonlinear (and possibly discontinuous) functional $\kappa : \mathcal{S} \rightarrow \mathbb{R}$ with $\kappa(A) \leq 0$ whenever $A \leq 0$ is called *negativity-preserving*, the class of all such negativity-preserving functionals on $\mathcal{S}$ being denoted by $\mathcal{N}(\mathcal{S})$.

Examples of negativity-preserving functionals include $\kappa(A) = \lambda_1(A)$; $\kappa(A) = \text{tr}(A - P)$ where P is positive semi-definite and fixed; $\kappa(A) = \lambda_1[p(A)]$ where $p(A)$ is any fixed polynomial in A with positive coefficients and odd powers only; $\kappa(A) = \frac{\lambda_1(A)}{1 - \lambda_1(A)}$; $q(A) = a_{ii}$, $1 \leq i \leq n$.

We also note that any positive linear functional on $\mathcal{S}$ is also negativity-preserving. Thus, functionals in the class $\mathcal{N}(\mathcal{S})$ make up all the functionals being used in the current study of matrix oscillation theory [51]. Note that “ κ ” here is the q for nonlinear negativity-preserving functionals in Mingarelli’s paper [51].

Theorem 2.45

System (2.2) is oscillatory provided that there exists a κ in $\mathcal{N}(\mathcal{S})$ such that for each $r \geq t_0$ and for some $p, q > 1$,

$$(2.46) \quad \limsup_{t \rightarrow \infty} \kappa \left[\frac{1}{t^{p+q-1}} \int_r^t (s-r)^p (t-s)^q Q(s) ds - \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \left(1 - \frac{r}{t}\right)^{p+q-1} I \right] > 0$$

Proof.

The proof follows the proof of Theorem 2.24 exactly up to and including inequality (2.34). At this point, we move both integrals to the left-hand side of the inequality to obtain

$$\int_T^t (s-T)^p (t-s)^q Q(s) ds - \int_T^t \left[\frac{p}{2} (s-T)^{\frac{p}{2}-1} (t-s)^{\frac{q}{2}} - \frac{q}{2} (s-T)^{\frac{p}{2}} (t-s)^{\frac{q}{2}-1} \right]^2 I ds \leq 0$$

The second integral is the diagonal matrix whose entries are evaluated in the proof of Theorem 2.24, and given by equation (2.40). Thus,

$$\int_T^t (s-T)^p (t-s)^q Q(s) ds - \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (t-T)^{p+q-1} I \leq 0,$$

and applying κ , we obtain

$$\begin{aligned}
& \kappa \left[\int_T^t (s-T)^p (t-s)^q Q(s) ds \right. \\
& \quad \left. - \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (t-T)^{p+q-1} I \right] \leq 0 \\
& \kappa \left\{ \frac{1}{t^{p+q-1}} \int_T^t (s-T)^p (t-s)^q Q(s) ds \right. \\
& \quad \left. - \frac{1}{t^{p+q-1}} \left[\frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} (t-T)^{p+q-1} I \right] \right\} \leq 0 \\
& \kappa \left[\frac{1}{t^{p+q-1}} \int_T^t (s-T)^p (t-s)^q Q(s) ds \right. \\
& \quad \left. - \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \left(1 - \frac{T}{t}\right)^{p+q-1} I \right] \leq 0
\end{aligned}$$

Noting that κ is not necessarily continuous, we find

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \kappa \left[\frac{1}{t^{p+q-1}} \int_T^t (s-T)^p (t-s)^q Q(s) ds \right. \\
& \quad \left. - \frac{pq(p+q-2)\Gamma(p-1)\Gamma(q-1)}{4\Gamma(p+q)} \left(1 - \frac{T}{t}\right)^{p+q-1} I \right] \leq 0,
\end{aligned}$$

which contradicts condition (2.46) in the statement of the theorem.

Chapter 3

AN OSCILLATION THEOREM FOR DIFFERENTIAL MATRIX SYSTEMS

Consider the nonlinear second order matrix differential equation

$$(3.1) \quad [P(t)Y']' + Q(t)F(Y) = 0, \quad t \geq a,$$

where $a \in \mathbb{R}$, $P(t)$ and $Q(t)$ are real $n \times n$ continuous symmetric matrix functions on $[a, \infty)$ with $P(t)$ positive definite ($P(t) > 0$), and $F: \mathcal{S} \rightarrow \mathcal{S}$ where $\mathcal{S}$ is the real linear vector space of $n \times n$ symmetric matrices. If $P(t) = I$, equation (3.1) assumes the form

$$(3.2) \quad Y'' + Q(t)F(Y) = 0, \quad t \geq a,$$

and we note that as in the linear case, when $P(t) \neq I$ the oscillatory behaviour of equations (3.1) and (3.2) must be studied separately since there is no oscillation-preserving transformation of the independent variable that allows us to relate the two different forms (p. 20, [16]).

There are relatively few results available concerning the oscillation and non-oscillation of nonlinear second order matrix differential systems compared to the linear case. In the 1950's a few results appeared (such as [57], [58], and [64]) stating sufficient conditions for the oscillation of solutions of equations (3.1) and (3.2). It appears that Howard (Theorem 7, [33]) in 1967 was among the first to look at this topic using non-variational methods, in contrast to the efforts of his predecessors.

Since matrix system results may often be obtained through the extension of their scalar equation analogs, we consider the work done concerning the oscillation of the scalar nonlinear second order differential equation

$$(3.3) \quad y'' + q(t)f(y) = 0, \quad t \geq 0,$$

where $q(t)$ is continuous.

In 1955, Atkinson [3] gave the first characterization of oscillating solutions for equation (3.3) in the special case where $f(y)$ has the form $f(y) = y^{2n+1}$, $n = 1, 2, \dots$, i.e., for solutions of the differential equation

$$(3.4) \quad y'' + q(t)y^{2n+1} = 0, \quad n = 1, 2, \dots, \quad t \geq 0,$$

where $q(t) > 0$ and continuous, the criterion for oscillation of $y(t)$ being an integral condition on the coefficient $q(t)$:

$$(3.5) \quad \int_0^\infty tq(t)dt = \infty.$$

Twenty years later, Butler [8] generalized Atkinson's result to equation (3.3), where $q(t)$ is unrestricted as to its sign, $f(y)$ is absolutely continuous, has the same sign as y , is monotone increasing, and satisfies the "superlinearity condition"

$$(3.6) \quad \pm \int_{\pm 1}^{\pm \infty} \frac{dy}{f(y)} < \infty,$$

thus turning equation (3.3) into a "superlinear equation", the simplest such case occurring when $f(y)$ behaves like y^3 . One of our main results (see Chapter 4) concerns itself with an extension of Atkinson's result.

In 1980, Mingarelli [48] extended the work of Butler to Volterra-Stieltjes integral equations of the form

$$(3.7) \quad y'(t) = c - \int_a^t f[y(s)]d\sigma(s), \quad t \in [a, \infty),$$

where σ is a right-continuous function, locally of bounded variation, and such that

$$(3.8) \quad Q(t) \equiv \lim_{T \rightarrow \infty} \int_t^T d\sigma(s)$$

exists and may be infinite.

More recently, Praharaj [56] establishes criteria for the oscillation of equation (3.1) using the trace linear functional, and through the imposition of the property of symmetry on the solution matrix $Y(t)$.

Main Results

In this chapter we will be extending a theorem of Praharaj (Theorem 4.3, [56]), generalizing the result using the trace to a result using positive linear

functionals. Our proof draws its inspiration from the techniques of Howard [33], and Etgen and Pawlowski [19].

We know that the trace is a particular example of a positive linear functional [19], where for a matrix $A = (a_{ij})$ we have

$$\text{tr } A = \sum_{i=1}^n g_{e_i}(A)$$

where the e_i are the canonical basis vectors, and $g_{e_i}(A) = a_{ii}$. We also know that every positive linear functional g on $\mathcal{S}$ can be written

$$g_c(A) = \text{tr } (AC)$$

where $C > 0$ is a symmetric matrix which depends on g (p. 32, [1]). We will see that the proof of our result does not involve a simple application of these facts.

Lemma 3.9

Suppose $u(t)$ is a continuous, real-valued function on $[a, \infty)$ such that

$$(3.10) \quad \lim_{t \rightarrow \infty} u'(t) = \infty.$$

Then $u(t) > 0$ for all sufficiently large $t \in [a, \infty)$.

Proof

Let $u(t)$ be a continuous, real-valued function on $[a, \infty)$ such that (3.10) holds. Then eventually $u'(t)$ must be greater than 1, i.e., there exists a $t_1 \in [a, \infty)$ such that

$$u'(t) > 1, \quad \text{for all } t \geq t_1.$$

Integrating both sides, we obtain

$$\begin{aligned} \int_{t_1}^T u'(s) ds &> \int_{t_1}^T ds \\ u(T) - u(t_1) &> T - t_1 \\ u(T) &> u(t_1) + T - t_1. \end{aligned}$$

Since $t_1 \in [a, \infty)$ is fixed, so is $u(t_1)$, and thus

$$\liminf_{T \rightarrow \infty} u(T) = \infty.$$

So

$$\lim_{T \rightarrow \infty} u(T) \text{ exists, and equals } \infty,$$

i.e.,

$$\lim_{t \rightarrow \infty} u(t) = \infty.$$

So it follows that $u(t) > 0$ for all sufficiently large $t \in [a, \infty)$.

Theorem 3.11 (Theorem 4.3, [56])

Consider the nonlinear second order matrix differential equation (3.1), and let the following assumptions hold:

$P^{-1}(t) \geq I$, $Q(t)$ is positive semi-definite ($Q(t) \geq 0$), and $F(X)X^{-1} \geq I$ for every nonsingular matrix $X \in \mathcal{S}$. For a continuous matrix function $X(t) \in \mathcal{S}$, we assume that $F[X(t)]$ is a continuous matrix function.

If

$$(3.12) \quad \lim_{t \rightarrow \infty} \int_a^t \text{tr } Q(s) ds = \infty$$

then every nontrivial prepared symmetric solution $Y(t)$ of equation (3.3) oscillates.

We are actually going to extend this theorem of Praharaj to equation (3.2) for positive linear functionals. We will deal with the case $P(t) = I$ and generalize the oscillation criterion (3.12) to equation (3.2) in the following theorem:

Theorem 3.13

Consider the nonlinear second order matrix differential equation (3.2), and let the following assumptions hold:

$Q(t)$ is positive semi-definite ($Q(t) \geq 0$), and $F(X)X^{-1} \geq kI$, where k is real and $k > 0$, for every nonsingular matrix $X \in \mathcal{S}$. For a continuous matrix function $X(t) \in \mathcal{S}$, we assume that $F[X(t)]$ is a continuous matrix function such that $F(X)$ has a power series expansion in powers of X with an infinite radius of convergence, so that

$$F(Y(t)) = \sum_{i=0}^{\infty} a_i [Y(t)]^i \text{ with } [Y(t)]^0 = I, \quad a_i \in \mathbb{R}.$$

If there exists a positive linear functional g such that

$$(3.14) \quad \lim_{t \rightarrow \infty} g \left[\int_a^t Q(s) ds \right] = \infty$$

then every nontrivial, prepared, symmetric solution $Y(t)$ of equation (3.2) oscillates.

Proof:

Let us assume the contrary, i.e., that there exists a positive linear functional g such that (3.14) holds, and let $Y(t)$ be a nontrivial, prepared, symmetric solution of (3.2) which is not oscillatory. We shall arrive at a contradiction.

Since $Y(t)$ is not oscillatory, there exists a $t_0 > a$ such that $\det Y(t) \neq 0$ for all $t \geq t_0 > a$. Hence $Y^{-1}(t)$ exists for all $t \geq t_0 > a$. Let us define the matrix

$$(3.15) \quad S(t) \equiv -Y'(t)Y^{-1}(t).$$

Differentiating with respect to t we obtain

$$(3.16) \quad \begin{aligned} S'(t) &= -[Y'(t)]'Y^{-1}(t) - Y'(t)[Y^{-1}(t)]' \\ &= -Y''(t)Y^{-1}(t) - Y'(t)[Y^{-1}(t)]'. \end{aligned}$$

But (3.2) gives $Y''(t) = -Q(t)F[Y(t)]$, and since $I = Y(t)Y^{-1}(t)$, taking the derivative with respect to t gives

$$(3.17) \quad \begin{aligned} 0 &= Y(t)[Y^{-1}(t)]' + Y'(t)Y^{-1}(t) \\ Y(t)[Y^{-1}(t)]' &= -Y'(t)Y^{-1}(t) \\ [Y^{-1}(t)]' &= -Y^{-1}(t)Y'(t)Y^{-1}(t) \end{aligned}$$

and thus

$$(3.18) \quad \begin{aligned} S'(t) &= Q(t)F[Y(t)]Y^{-1}(t) + Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t) \\ &= Q(t)R(t) + [Y'(t)Y^{-1}(t)]^2 \\ &= Q(t)R(t) + S^2(t) \end{aligned}$$

where $R(t) \equiv F[Y(t)]Y^{-1}(t)$. Note that $Y(t)$ symmetric implies that

$$\begin{aligned} [R(t)]^* &= \{F[Y(t)]Y^{-1}(t)\}^* \\ &= [Y^{-1}(t)]^*\{F[Y(t)]\}^* \\ &= [Y^*(t)]^{-1}F[Y^*(t)] \\ &= Y^{-1}(t)F[Y(t)] \quad \text{since } Y(t) = Y^*(t) \\ &= F[Y(t)]Y^{-1}(t) \quad \text{since } F[Y(t)]Y(t) = Y(t)F[Y(t)] \\ &= R(t). \end{aligned}$$

So $R(t)$ is symmetric for $t \geq t_0$.

Now,

$$\begin{aligned}
 S^*(t) &= -[Y'(t)Y^{-1}(t)]^* \\
 &= -\{[Y^{-1}(t)]^*[Y'(t)]^*\} \\
 &= -\{[Y^*(t)]^{-1}[Y^*(t)]'\} \\
 &= -Y^{-1}(t)Y'(t) \quad \text{since } Y(t) = Y^*(t).
 \end{aligned}$$

But since $Y(t)$ is prepared,

$$\begin{aligned}
 [Y'(t)]^*Y(t) &= Y^*(t)Y'(t) \\
 [Y'(t)]^*Y(t)Y^{-1}(t) &= Y^*(t)Y'(t)Y^{-1}(t) \\
 [Y'(t)]^* &= Y^*(t)Y'(t)Y^{-1}(t) \\
 (3.19) \quad Y'(t) &= Y(t)Y'(t)Y^{-1}(t) \quad \text{since } Y(t) = Y^*(t)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 S^*(t) &= -Y^{-1}(t)Y(t)Y'(t)Y^{-1}(t) \\
 &= -Y'(t)Y^{-1}(t) \\
 &= S(t)
 \end{aligned}$$

So $S(t)$ is also symmetric for $t \geq t_0$. Thus $SS^* = S^2$ is positive semi-definite ($S^2 \geq 0$).

We need to show that $Q(t)R(t)$ is symmetric. Since $Q(t)$ and $R(t)$ are symmetric, we have that

$$\begin{aligned}
 [Q(t)R(t)]^* &= R^*(t)Q^*(t) \\
 &= R(t)Q(t).
 \end{aligned}$$

So we need to show that $R(t)Q(t) = Q(t)R(t)$ to obtain $Q(t)R(t)$ symmetric.

From (3.19) we have

$$Y'(t) = Y(t)Y'(t)Y^{-1}(t)$$

and so

$$\begin{aligned}
 [Y'(t)]' &= [Y(t)Y'(t)Y^{-1}(t)]' \\
 &= Y'(t)[Y'(t)Y^{-1}(t)] + Y(t)[Y'(t)Y^{-1}(t)]' \\
 &= Y'(t)Y'(t)Y^{-1}(t) + Y(t)\{Y''(t)Y^{-1}(t) + Y'(t)[Y^{-1}(t)]'\} \\
 &= Y'(t)Y'(t)Y^{-1}(t) + Y(t)Y''(t)Y^{-1}(t) + Y(t)Y'(t)[Y^{-1}(t)]'.
 \end{aligned}$$

So by equations (3.17) and (3.2) we obtain

$$\begin{aligned}
 (3.20) \quad Y''(t) &= Y'(t)Y'(t)Y^{-1}(t) - Y(t)Q(t)F[Y(t)]Y^{-1}(t) - Y(t)Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t) \\
 &= Y'(t)Y'(t)Y^{-1}(t) - Y(t)Q(t)F[Y(t)]Y^{-1}(t) - Y'(t)Y'(t)Y^{-1}(t) \text{ by (3.19)} \\
 &= -Y(t)Q(t)F[Y(t)]Y^{-1}(t).
 \end{aligned}$$

In addition, equation (3.2) implies that

$$\begin{aligned}
 (3.21) \quad [Y''(t)]^* &= -\{Q(t)F[Y(t)]\}^* \\
 &= -F[Y^*(t)]Q^*(t) \\
 &= -F[Y(t)]Q(t)
 \end{aligned}$$

Equating equations (3.20) and (3.21) (since $Y''(t) = [Y''(t)]^*$) we obtain

$$\begin{aligned}
 Y(t)Q(t)F[Y(t)]Y^{-1}(t) &= F[Y(t)]Q(t) \\
 Y^{-1}(t)Y(t)Q(t)F[Y(t)]Y^{-1}(t) &= Y^{-1}(t)F[Y(t)]Q(t) \\
 Q(t)F[Y(t)]Y^{-1}(t) &= Y^{-1}(t)F[Y(t)]Q(t) \\
 Q(t)F[Y(t)]Y^{-1}(t) &= F[Y(t)]Y^{-1}(t)Q(t) \quad \text{since } F[Y(t)]Y^{-1}(t) = Y^{-1}(t)F[Y(t)]
 \end{aligned}$$

and since $R(t) = F[Y(t)]Y^{-1}(t)$ we have finally

$$(3.22) \quad Q(t)R(t) = R(t)Q(t).$$

Thus,

$$\begin{aligned}
 [Q(t)R(t)]^* &= R^*(t)Q^*(t) \\
 &= R(t)Q(t) \\
 &= Q(t)R(t) \text{ from equation (3.22)}
 \end{aligned}$$

as required, i.e. $Q(t)R(t)$ is symmetric.

Now, let us define the $\mathcal{S}$ -valued function T by

$$(3.23) \quad T(t) \equiv S(t) - \int_{t_0}^t Q(s)R(s)ds$$

where $T(t)$ is also symmetric (since $S(t)$ is symmetric) and $Q(t)R(t)$ being symmetric implies $\int_{t_0}^t Q(s)R(s)ds$ is symmetric. Then,

$$\begin{aligned}
 (3.24) \quad T'(t) &= S'(t) - Q(t)R(t) \\
 &= Q(t)R(t) + S^2(t) - Q(t)R(t) \text{ from (3.18)} \\
 &= S^2(t)
 \end{aligned}$$

Hence,

$$(3.25) \quad T'(t) = \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right]^2 \text{ from (3.23)}$$

and so $T'(t) = S^2(t)$ is positive semi-definite ($T'(t) \geq 0$ since $T'(t)$ is symmetric ($[T'(t)]^* = T'(t)$)).

By hypothesis we have that $F(X)X^{-1} \geq kI$, where $k > 0$, for every $X \in \mathcal{S}$, and so

$$R(t) = F[Y(t)]Y^{-1}(t) \geq kI \quad \text{by hypothesis.}$$

Thus,

$$(3.26) \quad Q(t)R(t) = Q(t)F[Y(t)]Y^{-1}(t) \geq kQ(t).$$

Integrating (3.26) we obtain the inequality

$$(3.27) \quad \int_{t_0}^t Q(s)R(s)ds \geq k \int_{t_0}^t Q(s)ds.$$

Now since $T'(t)$ is positive semi-definite ($T'(t) \geq 0$), we have $T(t) \geq T(t_0)$ whenever $t \geq t_0$. Thus,

$$(3.28) \quad T(t) + \int_{t_0}^t Q(s)R(s)ds \geq T(t_0) + k \int_{t_0}^t Q(s)ds.$$

Applying g to both sides of inequality (3.28) we obtain

$$(3.29) \quad \begin{aligned} g\left[T(t) + \int_{t_0}^t Q(s)R(s)ds\right] &\geq g\left[T(t_0) + k \int_{t_0}^t Q(s)ds\right] \\ &= g\left[T(t_0)\right] + kg\left[\int_{t_0}^t Q(s)ds\right]. \end{aligned}$$

But by hypothesis (3.14),

$$\lim_{t \rightarrow \infty} g\left[\int_{t_0}^t Q(s)ds\right] = \infty.$$

Taking limits as $t \rightarrow \infty$ of inequality (3.29) gives

$$(3.30) \quad \lim_{t \rightarrow \infty} g\left[T(t) + \int_{t_0}^t Q(s)R(s)ds\right] = \infty.$$

By Lemma I.6 (the Cauchy-Schwarz Inequality) we know that

$$|g(A)|^2 \leq g(I)g(A^2) \text{ for all } A \in \mathcal{S}.$$

Setting $A = T(t) + \int_{t_0}^t Q(s)R(s)ds$ we obtain

$$(3.31) \quad \left| g\left[T(t) + \int_{t_0}^t Q(s)R(s)ds\right] \right|^2 \leq \frac{1}{\rho} g\left\{ \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right]^2 \right\}$$

where we have set $\rho = \frac{1}{g(T)}$ and $\rho > 0$ since g is positive (p. 100, [19]).

However, equation (3.25) gives us

$$(3.32) \quad \left| g \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right] \right|^2 \leq \frac{1}{\rho} g[T'(t)].$$

By equation (3.30) we must have

$$(3.33) \quad \lim_{t \rightarrow \infty} \left| g \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right] \right|^2 = \infty$$

as well, and so by inequality (3.32) we also have

$$(3.34) \quad \lim_{t \rightarrow \infty} g[T'(t)] = \infty.$$

Since g is linear and continuous, $g[T'(t)] = \{g[T(t)]\}'$ and so

$$(3.35) \quad \lim_{t \rightarrow \infty} \{g[T(t)]\}' = \infty.$$

We now claim that equation (3.35) implies

$$(3.36) \quad \lim_{t \rightarrow \infty} g[T(t)] = \infty.$$

Now, setting $u(t) = g[T(t)]$ and applying the proof of Lemma 3.9 we obtain that equation (3.35) implies equation (3.36), i.e., that

$$(3.36) \quad \lim_{t \rightarrow \infty} g[T(t)] = \infty$$

so that $g[T(t)] > 0$ for all sufficiently large $t \in [a, \infty)$.

We have

$$\begin{aligned} \{g[T(t)]\}' &= g[T'(t)] \text{ since } g \text{ is continuous} \\ &= g \left\{ \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right]^2 \right\} \text{ from (3.25)} \\ &\geq \rho \left\{ g \left[T(t) + \int_{t_0}^t Q(s)R(s)ds \right] \right\}^2 \text{ from (3.31)} \\ &= \rho \left\{ g[T(t)] + g \left[\int_{t_0}^t Q(s)R(s)ds \right] \right\}^2. \end{aligned}$$

By equation (3.27) we have

$$(3.37) \quad g \left[\int_{t_0}^t Q(s)R(s)ds \right] \geq kg \left[\int_{t_0}^t Q(s)ds \right]$$

and so

$$(3.38) \quad \{g[T(t)]\}' \geq \rho \left\{ g[T(t)] + kg \left[\int_{t_0}^t Q(s)ds \right] \right\}^2.$$

Choosing $t_1 \in [a, \infty)$ large enough such that $g[T(t)] > 0$ for all $t \geq t_1$ and such that (3.14) is positive as well, equation (3.38) gives us

$$\begin{aligned} \{g[T(t)]\}' &\geq \rho\{g[T(t)]\}^2 \text{ for all } t \geq t_1 \\ \frac{\{g[T(t)]\}'}{\{g[T(t)]\}^2} &\geq \rho \text{ for all } t \geq t_1. \end{aligned}$$

Integrating we obtain

$$\begin{aligned} \int_{t_1}^t \frac{\{g[T(s)]\}'}{\{g[T(s)]\}^2} ds &\geq \rho \int_{t_1}^t ds \\ \left. \frac{-1}{g[T(s)]} \right|_{t_1}^t &\geq \rho s \Big|_{t_1}^t \\ (3.39) \quad \frac{-1}{g[T(t)]} + \frac{1}{g[T(t_1)]} &\geq \rho(t - t_1). \end{aligned}$$

Noting that $g[T(t_1)]$ is fixed and

$$\lim_{t \rightarrow \infty} \rho(t - t_1) = \infty,$$

we must have

$$\lim_{t \rightarrow \infty} \frac{-1}{g[T(t)]} = \infty.$$

But equation (3.36) tells us that $g[T(t)] \rightarrow \infty$ as $t \rightarrow \infty$. So

$$\lim_{t \rightarrow \infty} \frac{-1}{g[T(t)]} = 0,$$

and so we have a contradiction.

Hence $Y(t)$ must be oscillatory. So Theorem 3.11 (Theorem 4.3, [56]) is thus extended using positive linear functionals instead of the trace.

Chapter 4

A NON-OSCILLATION THEOREM IN THE SCALAR CASE

In this chapter, we present a general non-oscillation criterion for the second order nonlinear scalar differential equation of the form

$$(4.1) \quad y'' + f(x)g(y) = 0, \quad x \in [0, \infty),$$

where $f(x)$ is non-negative and continuous on $[0, \infty)$, and we will introduce a special condition on g . We are inspired by the classic result of Atkinson [3] from 1955, known as “Atkinson’s Oscillation Theorem”.

Theorem 4.2 (Atkinson’s Oscillation Theorem, Theorem 1, [3])

Let $f = f(x)$ be positive and continuous for $x \geq 0$, and let n be an integer greater than 1. Then a necessary and sufficient condition for all solutions of the second order nonlinear scalar differential equation

$$(4.2) \quad y'' + f(x)y^{2n-1} = 0, \quad x \in [0, \infty)$$

to be oscillatory is

$$\int_0^\infty t f(t) dt = \infty.$$

(We remark that in the linear case the criterion is sufficient but not necessary.)

Our new result makes it possible to consider the classical cases (sublinear, linear, and superlinear) together, unifying them under one common theme. We will be using a fixed point theorem to prove this result.

Definition 4.3 (p. 374 [53])

Let (X, d) be a metric space and T a mapping of X into itself ($TX \subseteq X$). The function T is said to be a *contraction* or a *contraction mapping* if there is a real number k , $0 < k < 1$, such that

$$d[T(u), T(v)] \leq kd(u, v),$$

for every $u, v \in X$.

Every contraction mapping T is uniformly continuous.

Definition 4.4 (p. 8, [14])

A function $g: [a, b] \rightarrow \mathbb{R}$ is said to be of class (C, Lip) in $[0, \infty)$ if there exists a $K > 0$ such that for every $y, z \in [a, b]$

$$|g(y) - g(z)| \leq K|y - z|.$$

The quantity K is called a *Lipschitz constant*.

Clearly, if T is a contraction map, then T is of class (C, Lip) with Lipschitz constant $K = k$.

We will also require the following lemma.

Lemma 4.5

Let $f(x)$ be non-negative and continuous on $[0, \infty)$. If

$$\int_0^\infty tf(t)dt < \infty$$

then

$$F(x) \equiv \int_x^\infty (t - x)f(t)dt$$

satisfies

- (a) $F(x)$ is non-increasing on $[0, \infty)$, and
- (b) $\lim_{x \rightarrow \infty} F(x) = 0$.

Proof.

Now, we note that for any given $x \geq 0$, we have

$$\int_0^\infty tf(t)dt \geq \int_x^\infty tf(t)dt \geq x \int_x^\infty f(t)dt$$

and so

$$\int_x^\infty f(t)dt \text{ exists (since } f(t) \geq 0 \text{) and is finite}$$

for every $x \geq 0$.

Next,

$$\begin{aligned} F(x) &= \int_x^\infty (t - x)f(t)dt \\ &= \int_x^\infty tf(t)dt - x \int_x^\infty f(t)dt \end{aligned} \tag{4.6}$$

is necessarily of class $C^1[0, \infty)$ and so

$$\begin{aligned} F'(x) &= -xf(x) - \int_x^\infty f(t)dt - x[-f(x)] \\ &= - \int_x^\infty f(t)dt \\ &\leq 0 \quad \text{since } f(x) \geq 0. \end{aligned}$$

Hence (a) is proven.

Condition (b) is clear since both improper integrals in equation (4.6) converge absolutely.

Theorem 4.7 (Banach-Cacciopoli Fixed Point Theorem [24]). (See also p. 374, Theorem 4.10.1, [53].)

Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a contraction mapping. Then T has a unique fixed point.

Our main result is as follows:

Theorem 4.8

Let g be of class (C, Lip) with Lipschitz constant $K > 0$. Assume that for some $M > 0$ we have

$$0 \leq g(t) \leq M \quad \text{whenever } t \in [0, M].$$

In addition, let $f = f(t)$ be continuous and non-negative on $[0, \infty)$, and satisfy

$$(4.9) \quad \max \left\{ K \int_0^\infty tf(t)dt, \int_0^\infty tf(t)dt \right\} < 1.$$

Then, the second order nonlinear differential equation (4.1) has a non-trivial, non-oscillatory solution.

Proof.

We base ourselves on the original argument of Atkinson [3], noting that if the nonlinear integral equation

$$(4.10) \quad y(x) = M - \int_x^\infty (t-x)f(t)g[y(t)]dt, \quad x \in [0, \infty),$$

has a continuous solution that is uniformly bounded as $x \rightarrow \infty$, then it must be a non-oscillatory solution of equation (4.1). The existence of such a solution implies that $y(x) \rightarrow M$ and $y'(x) \rightarrow 0$ as $x \rightarrow \infty$. Once such a solution is found in X we may conclude that $y(x) \geq 0$ implies that $y(x) > 0$ for all

$x \in (0, \infty)$ because of the basic assumption of uniqueness of solutions of initial value problems associated with equation (4.1) and thus equation (4.10).

To find our solution to equation (4.10), we will invoke Theorem 4.7, the Banach-Cacciopoli Fixed Point Theorem. We start by defining the space

$$X \equiv \{y: \mathbb{R}^+ \rightarrow \mathbb{R} \mid y \in C[0, \infty), \text{ and } 0 \leq y(t) \leq M \text{ for all } t \in [0, \infty)\}.$$

Since X is a closed subset of $C[0, \infty)$, it follows that $(X, |\cdot|_X)$ is a complete metric space where the metric is induced by the uniform norm

$$|y|_X = \sup_{t \in [0, \infty)} |y(t)|.$$

Next, we define a mapping T based on equation (4.10) and show that it satisfies the conditions of Theorem 4.7. To this end, for $y \in X$ we define

$$(4.11) \quad (Ty)(x) = M - \int_x^\infty (t-x)f(t)g[y(t)]dt, \quad x \in [0, \infty).$$

However, for $g \in X$, we have

$$0 \leq g[y(t)] \leq M$$

by definition of the space X and by our original hypothesis on g . Thus, for $x \in [0, \infty)$,

$$\begin{aligned} 0 &\leq \int_x^\infty (t-x)f(t)g[y(t)]dt \\ &\leq M \int_x^\infty (t-x)f(t)dt \quad \text{since } (t-x)f(t) \geq 0 \\ &\leq M \int_0^\infty tf(t)dt \quad \text{by Lemma 4.5} \\ &\leq M \quad \text{by condition (4.9).} \end{aligned}$$

Thus, it follows from equation (4.11) that $(Ty)(x) \geq 0$ and $(Ty)(x) \leq M$ for all $x \in [0, \infty)$, since the improper integral in equations (4.10) and (4.11) is non-negative. Thus $TX \subset X$.

Now, for $u, v \in X$,

$$\begin{aligned}
 |(Tu)(x) - (Tv)(x)| &= \left| \int_x^\infty (t-x)f(t)\{g[u(t)] - g[v(t)]\}dt \right| \\
 &\leq \int_x^\infty (t-x)f(t)|g[u(t)] - g[v(t)]|dt \\
 &< K \sup_{t \in [0, \infty)} |u(t) - v(t)| \int_x^\infty (t-x)f(t)dt \quad \text{since } g \in (C, Lip) \\
 &\leq K|u - v|_X \int_0^\infty tf(t)dt \\
 &\leq \alpha|u - v|_X
 \end{aligned}$$

where $\alpha = K \int_0^\infty tf(t)dt < 1$, by condition (4.9).

Thus T is a contraction mapping on X (so, in fact, T is continuous there), and so applying Theorem 4.7, T has a unique fixed point $(Ty)(x) = y(x)$ in X such that equation (4.10) is satisfied, i.e. such that equation (4.1) has a solution $y(x)$ with

$$\lim_{x \rightarrow \infty} y(x) = M > 0$$

and so is necessarily non-oscillatory. This concludes our proof.

Remarks.

1. Some may argue that $y(x) \equiv 0$ is the contemplated solution to equation (4.10). In order to avoid this peculiar state of affairs we may simply assume without too much loss of generality that M is chosen so that the constant

$$M - g(0) \int_0^\infty tf(t)dt \neq 0.$$

In this case $y(x) \equiv 0$ cannot be a solution of equation (4.10), and thus, as stated, any solution of equation (4.10) must be non-oscillatory (since it converges to $M > 0$ as $x \rightarrow \infty$).

2. If $g(0) \neq 0$, and if g is a self-map on $[0, M]$ with $M > 0$ it follows that there is an $x_0 \in (0, M]$ such that $g(x_0) = 0$. Thus, there is always the trivial constant non-oscillatory solution $y(x) = x_0$ on $[0, \infty)$, $x_0 \neq 0$.
3. If $g(0) = 0$, $y(x) \equiv 0$ is never a solution of equation (4.10) and so equation (4.1) is non-oscillatory (see Remark 1, above).

4. Although our theorem is stronger than Atkinson's [3], it applies to sub-linear, linear, and superlinear cases simultaneously. In this sense it is of some interest.

Basically, the smaller M is in the statement of the theorem, the "larger" we may choose $f(t)$. Similarly, the smaller $f(t)$ is, the larger we may choose M to be (in compliance with condition (4.9)).

Applications

Example 1: A linear case.

Let $g(y) = y$ and choose f so that condition (4.9) is satisfied. Then $K = 1$, $M > 0$ is arbitrary and so

$$y'' + f(x)y = 0, \quad x \in [0, \infty)$$

has unique non-oscillatory solutions converging to M for each $M > 0$ whenever

$$\int_0^\infty tf(t)dt < 1.$$

(Refer to p. 151, Theorem 3.5.2, [48])

Example 2: A superlinear case.

Let $g(y) = y^3$ so that $K = 3$ in the theorem if we choose $M = 1$. If f is any non-negative continuous function such that

$$\int_0^\infty tf(t)dt < \frac{1}{3},$$

then

$$y'' + f(x)y^3 = 0, \quad x \in [0, \infty)$$

has a unique non-oscillatory solution converging to 1 as $x \rightarrow \infty$.

Example 3: A transcendental case (i.e., $\int \frac{dt}{g(t)}$ does not exist on $[0, \infty)$).

Let $g(y) = \sin\left(\frac{\pi}{2}y\right)$ on $[0, 1]$. Then we can choose $M = 1$ in the theorem. Also, $K = \frac{\pi}{2}$ here. Thus, if f is any non-negative continuous function satisfying

$$\int_0^\infty tf(t)dt < \frac{2}{\pi},$$

then

$$y'' + f(x)\sin\left[\frac{\pi}{2}y(x)\right] = 0$$

will have a unique non-oscillatory solution satisfying $y(x) \rightarrow 1$ as $x \rightarrow \infty$.

We note that such a transcendental case is rarely covered in the literature by such general theorems.

Example 4: A sublinear case.

Let $g(y) = \sqrt{|y+1|}$. Now $g: [0, 2] \rightarrow [0, 2]$, so $M = 2$. Actually $g([0, 2]) = [1, \sqrt{3}] \subset [0, 2]$. $K = \frac{1}{2}$ since $g'(y) = \frac{1}{2\sqrt{y+1}}$, for $y > 0$. Thus, $g'(0) = \frac{1}{2}$. We need to choose $f(t)$ so that

$$\frac{1}{2} \int_0^\infty tf(t)dt < 1$$

and, actually,

$$\int_0^\infty tf(t)dt < 1$$

by condition (4.9). Then

$$y'' + f(x)\sqrt{|y(x)+1|} = 0, \quad x \in [0, \infty)$$

has a unique non-oscillatory solution with $y(x) \rightarrow 2$ as $x \rightarrow \infty$.

Conclusion

We have presented here theorems concerning the oscillation (in Chapters 2 and 3) and non-oscillation (in Chapters 1 and 4) of linear and non-linear scalar and matrix differential equations.

In Chapter 1 we formulated a matrix equation oscillation criterion from a pre-existing scalar condition in the second order linear case, while in Chapter 2 we presented a whole new family of oscillation criteria of Kamenev-type for matrix equations. In the unorthodox world of nonlinear differential equations, Chapter 3, inspired by the extensive work done over the years in linear systems, generalized a trace-based oscillation theorem to positive linear functionals. In Chapter 4 we introduced a new perspective on a classic result concerning the non-oscillation of a scalar equation, thus unifying three cases in a novel way.

We hope that our work will add another layer to the fertile ground upon which new insights and results can take root, in the important and dynamic field that is the oscillation theory of differential equations.

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