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CONVERGENCE AS A NEARNESS CONCEPT

by

Wallace Andrew Robertson, B.Sc., M.Sc.

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

Department of Mathematics

Carleton University

Ottawa, Ontario

January, 1975

The undersigned recommend to the Faculty
of Graduate Studies and Research acceptance of the thesis
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ABSTRACT

New approaches to topological problems, usually generalizing the notion of topology, proximity or uniformity, have proved fruitful in recent years. Among these, SNear, the category of semi-nearness spaces and near maps, and its subcategory Near of near spaces (both due to H. Herrlich) are particularly rich in structure. However these categories, besides being too general for many purposes, lack natural mapping space structures, and spaces in SNear do not usually have structures suitable to the construction of extensions. For these reasons we introduce two bicoreflective, cartesian closed subcategories of SNear.

The first is Grill, determined by those semi-nearness spaces which can be characterized by a family of grills, called near grills (grills are the nearness analogue to filters; in fact there is an anti-isomorphism between the family of grills and the family of filters on a set, both ordered by inclusions). For such spaces, grills act as a basis for the semi-nearness structure: maps are near if and only if near grills are carried to near grills. Limits (such as products) are elegantly described via grills. Bun, the category of bunch-determined spaces and near maps of H. L. Bentley, is found to be determined by those spaces which are in both Grill and Near.

The second is R_Con, a subcategory of Grill, which inherits its products as well as its colimits. R_Con is isomorphic to a subcategory of Con, the category of convergence spaces and continuous maps of D. Kent. In a convergence space a notion of convergent

filter, generalizing the topological space definition, gives the structure. The anti-isomorphism between grills and filters may be used to associate a family of grills, and hence a grill-determined space, with any convergence space. However, some structural information may be lost in translation. We introduce a weak symmetry axiom which overcomes this difficulty, and in topological spaces is equivalent to the R_0 symmetry axiom. The subcategory of Con determined by those spaces satisfying the axiom is bireflective in Con and is isomorphic to R_0 Con in Grill.

R_0 Con has some important bireflective subcategories. Among those we consider are R_0 Lim (from the limit spaces of H. R. Fischer), which is also cartesian closed, and TNear, which is isomorphic to the category of R_0 topological spaces and continuous maps. We show TNear is determined by the near spaces in R_0 Con, and that R_0 Con is the coreflective hull in SNear of TNear.

Finally, we show that every space in Grill has an R_0 Con extension, and that the grill-determined spaces are precisely the SNear subspaces of spaces in R_0 Con.

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NOTATION

SET THEORY: We use the usual notation $a, b, x, y, \dots$ for elements of sets $A, B, X, Y, \dots$. Perhaps less standard are:

$\underline{A}, \underline{B}, \underline{F}, \underline{G}, \dots$ for families of sets. Usually $\underline{A} \subseteq PX$ for some set X . $\underline{E}$ and $\underline{F}$ in particular are usually filters on a set; $\underline{G}$ and $\underline{H}$ are usually grills (see 3.1).

$\underline{a}, \underline{b}, \underline{x}, \underline{y}, \dots$ for families of families of sets. Usually $\underline{x} \subseteq P^2X$ for some set X , representing a structure on the set.

Structures which may be defined in a different way (for instance, the families in some $\underline{x} \subseteq P^2X$ may also be assigned to particular elements of X). In such a case, it is not sufficient to give only $\underline{x}$ are denoted by $\underline{g}, \underline{f}, \underline{x}, \underline{y}, \dots$.

Notations for higher orders of families will be introduced as required.

If X is a set and $\underline{A}, \underline{B} \subseteq PX$,

$$\bigcap \underline{A} = \bigcap_{A \in \underline{A}} A$$

$$[\underline{A}] = \{B \subseteq X \mid A \subseteq B \text{ for some } A \in \underline{A}\}$$

$$\underline{A} \vee \underline{B} = \{A \cup B \mid A \in \underline{A}, B \in \underline{B}\}. \text{ Notice that in general}$$

$$\underline{A} \vee \underline{B} \neq \underline{A \cup B} = \{C \subseteq X \mid C \in \underline{A} \text{ or } C \in \underline{B}\}.$$

$\underline{A} < \underline{B}$ means: for every $A \in \underline{A}$ there exists $B \in \underline{B}$, $B \subseteq A$.

Thus $\underline{A} \subseteq \underline{B}$ implies $\underline{A} < \underline{B}$.

$$\text{sec } \underline{A} = \{B \subseteq X \mid B \cap A \neq \emptyset \text{ for every } A \in \underline{A}\}.$$

$$\text{c}\underline{A} = \{B \subseteq X \mid X \setminus B \in \underline{A}\}.$$

If $f: X \rightarrow Y$ is a function, we use f^+ instead of f^{-1} .

$f(\underline{A}) = \{f(A) \mid A \in \underline{A}\}$. If $\underline{F}$ is a filter on X , we do not use $f\underline{F}$ to denote the filter on Y generated by $f(\underline{F})$, as this could

lead to confusion. Instead we use $[f(\underline{F})]$. (see above), the precise description. $f\underline{A}$ means $f(\underline{A})$.

For $x \in X$, $\dot{x} = \{A \subseteq X \mid x \in A\}$, the fixed ultrafilter at x .
 $\dot{f}(\dot{x}) = \{B \subseteq Y \mid f(x) \in B\} = [f(\dot{x})]$.

If $\{f_i: X_i \rightarrow Y_i \mid i \in I\}$ is a set of functions and $\prod X_i, \prod Y_i$ are the cartesian products, then $\pi f_i: \prod X_i \rightarrow \prod Y_i$ is defined by
 $(\pi f_i)(x_i) = (f_i(x_i))$ for $(x_i) \in \prod X_i$.

CATEGORY THEORY. By subcategory we mean full, isomorphism-closed subcategory.

If B and C are subcategories of a category A , by $B \cap C$ we mean the subcategory of A whose objects are those in both B and C . $B \subseteq A$ means B is a subcategory of A .

If X is an object of a category A (A -object) we write $X \in A$.

0. INTRODUCTION

Topologists have often found the usual axioms for a topological space inadequate or awkward for a variety of reasons. Topological spaces do not admit a natural completion as metric spaces do. Mapping spaces may have several possible topologies placed on them; no one seems to be adequate for all purposes. Quotient maps are more complex than subspace maps; for instance, even a finite product of quotient maps need not be a quotient map. Furthermore, some concepts such as boundedness and problems such as generalizing the calculus of normed vector spaces to non-normed spaces [FB] seem to be topological in flavour, but do not successfully relate to the traditional idea of open set.

The lack of completions led to the development of uniform spaces [I], proximities (an idea of 'nearness' between pairs of sets), and to cover a wider class of topological spaces, contiguities [II] (defining a 'nearness' relation for finite families of sets).

At least three approaches have been suggested to broaden the notion of what is 'topological'. One is to look at the 'convergent filter' axiomatization of topology and to weaken the axioms to a lesser or greater degree. Thus we have the category Con of convergent spaces and continuous maps [Ke], the greatest weakening of the axioms, and subcategories successively closer to the topological spaces: Lim, determined by the limit spaces [F], PsT, determined by the pseudo-topological spaces [C] and PrT, determined by the pretopological spaces [Ke]. This approach has the appeal of placing to the fore a basic 'building block', the filter; filters or filter-like structures have been fundamental to almost all

constructions of topological extensions (such as the Stone-Čech compactification, Hewitt realcompactification, Smirnov compactification, and compactifications via contiguities). Further, these categories of convergence spaces are cartesian closed [N] ; that is, they admit unique natural convergence structures on the mapping spaces. Cartesian closedness implies many other desirable properties [H3] ; for instance, a finite product of quotient maps is a quotient map. The limit spaces are also general enough to encompass the two non-topological concepts mentioned at the end of the first paragraph.

The second approach seems to be best represented by the nearness spaces of Herrlich (see [H1] and [H2]), where a 'nearness' relation is defined for families of sets (with no cardinality restriction). This gives rise to Near, the category of nearness spaces and nearness maps or, weakening the axioms somewhat, SNear, the category semi-nearness spaces and near maps. As might be expected, proximities and contiguities fit very well within this more general setting; more surprising is that uniform spaces, topological spaces (with a weak symmetry restriction, since the nearness of a family of sets does not depend on the order in which the sets are presented) and many other structures considered by topologists are neatly described by nearness theory. Despite their intuitive appeal, Near and SNear are not cartesian closed categories, nor is there always a filter-like structure by which semi-nearness structures can be characterized. Moreover, the all-encompassing nature of SNear, and even Near, is not always an advantage: it is often just too general.

The third approach is the categorical one, to give a notion of 'topological category' [H3] ; that is, structures giving rise to

many of the mapping properties which we associate with the category of topological spaces and continuous maps. This approach seems to be useful, since the theorems of topological categories are applicable in many settings such as Con and SNear. Unfortunately it admits such weakly structured categories as Set (the category of all sets and all functions) and excludes some categories we think of as topological, such as Haus (all Hausdorff topological spaces and continuous functions). The second problem has been tackled by weakening the conditions for topological categories slightly to get initially structured categories $[N]$, which admit categories of spaces with separation properties.

Our object here is to reconcile the intuitive and wide-ranging nature of nearness spaces to the useful 'building block' structure of convergence spaces. First we find the analogue to filter in nearness language, and explore the subcategory Grill of SNear, whose spaces are those definable in terms of this analogue, the grill. There is a natural relationship between filters and grills, and using this we are able to show how convergence spaces—with a suitable symmetry restriction, as was needed in the topological space setting—can be considered as semi-nearness spaces. Such spaces then determine a subcategory R_{Con} of SNear. These new subcategories of SNear are very well-behaved: both are bicoreflective in SNear, and both are cartesian closed. Both have structures which are easy to work with. R_{Con} is a minimal well-behaved subcategory of SNear containing the topological spaces; and the spaces of Grill can be 'built up' from the spaces of R_{Con} . They seem, therefore, to combine many of the advantages of convergence spaces and nearness spaces.

The first two chapters summarize known results. They are included to make later results coherent, and because the papers quoted are not readily available at this time.

In chapter 1 we state the categorical results (beyond the basic category theory reference, [HS]) which will be needed later: the definition and fundamental properties of topological categories, and cartesian closedness in such categories. All come from [H2] , [H3] or [N] except Theorem 1.20, which seems to be new.

Chapter 2 gives a brief account of semi-nearness and nearness structures, as found in [H1] and [H2] , and relates SNear to Mer, the category of merotopic spaces and merotopic maps [K1] (which define a notion of families of sets with 'arbitrarily small' members). The characterization of suitable topological spaces in near space language is also reviewed.

In chapter 3 we give the definition of grill (as noted above, our basic structure). The duality between grills and filters is important to chapters 5 and 7. A clear account of grills and the duality is found in [T] (although grills did not originate in that work); needed results are listed in the chapter. Theorem 3.9 is a new result essential to all that follows, since it shows how the duality is carried between sets by functions.

In chapter 4 we introduce grill-determined spaces: those semi-nearness spaces which can be characterized by grills. Grill, the subcategory of SNear determined by these spaces, is nicely embedded in SNear in that it is a bicoreflective subcategory (4.9). Standard constructions such as products, coproducts, subobjects and quotients (in general; initial and final structures; see chapter 1)

are found to be, when not identical to those in SNear, more manageable (4.11, 4.13, 4.14). Furthermore, Grill is cartesian closed (4.15). We close the chapter by exploring the relationship between Grill and Bun, the category of bunch-determined spaces and near maps introduced in [Be].

In chapter 5 we remark upon the possibility of using filters to define grill-determined spaces by utilizing the duality described in chapter 3. This duality also allows us to relate Grill to the category of filter merotopic spaces and merotopic maps, a subcategory of Mer described in [K1].

Chapter 6 begins with a short list of definitions and results on Con, the category of convergence spaces and continuous maps, and some of its important subcategories. These are gathered mainly from [Ke], but include results from [F]. The main purpose of the chapter, however, is to present an adequate 'symmetry' axiom analogous to the R_0 axiom for topological spaces (6.9), which is not too restrictive (6.11) and coincides with the usual R_0 axiom in the topological space setting (6.10). The subcategory $R_0\text{Con}$ of Con, defined by those spaces satisfying the new axiom, is found to be bireflective in Con (6.13).

In chapter 7 we present the main purpose for introducing such a symmetry axiom: restricting attention to $R_0\text{Con}$ allows us, using the remarks of chapter 5, to consider convergence spaces as grill-determined spaces (7.1; see also 7.4). Furthermore the embedding is a useful one in that $R_0\text{Con}$ inherits many of its structures from Grill; for $R_0\text{Con}$ is bicoreflective in Grill (7.8; see also 7.11). The association with Grill is in fact close enough that cartesian

closedness is inherited (7.12). The remainder of the chapter characterizes the subcategories of $R_0\text{Con}$ described in chapter 6 as grill-determined spaces, and briefly studies their interactions with SNear , Grill , and one another. Best known of these subcategories is TNear , the category of R_0 topological spaces considered as nearness spaces (see also 2.13). We find that TNear is determined by those convergence spaces which are also in Near (7.43). Finally we show that $R_0\text{Con}$ is in one sense a minimal well-behaved subcategory of SNear containing the topological spaces: $R_0\text{Con}$ is the coreflective hull in SNear of TNear (7.53).

In chapter 8 we show that $R_0\text{Con}$ is related to Grill in SNear somewhat as the T_2 compact spaces are related to the Tychonoff spaces in the category of topological spaces (but not precisely, of course; $R_0\text{Con}$ is coreflective in Grill , not reflective): every grill-determined space can be densely embedded in an R_0 convergence space (8.9) and the grill-determined spaces are precisely the SNear subspaces of the R_0 convergence spaces (8.10).

1. TOPOLOGICAL CATEGORIES

1.1 DEFINITION [H2] A concrete category A is a category with

(a) objects $(X, \underline{x})$, where X is a set, called the underlying set, and a structure $\underline{x}$ on X , called the underlying A -structure.

(b) Morphisms $f: (X, \underline{x}) \rightarrow (Y, \underline{y})$ are functions $f: X \rightarrow Y$ compatible in some sense with the underlying structures.

$1_X: (X, \underline{x}) \rightarrow (X, \underline{x})$ (where $1_X(x) = x$ for all $x \in X$) is a morphism for any object $(X, \underline{x})$.

1.2 DEFINITION [H2] Let A be a concrete category. Let X be

a set, and $(Y_i, \underline{y}_i)$ an A -object for each $i \in I$ (I may be

a proper class). Let $f_i: X \rightarrow Y_i$ be a function for each $i \in I$.

An A -structure $\underline{x}$ on X is initial with respect to

$(X, (f_i)_{i \in I}, (Y_i, \underline{y}_i)_{i \in I})$ iff the following conditions hold:

(a) $f_i: (X, \underline{x}) \rightarrow (Y_i, \underline{y}_i)$ is an A -morphism for each $i \in I$.

(b) If $(Z, \underline{z})$ is an A -object, and $g: Z \rightarrow X$ is a function such that for each $i \in I$, $f_i \circ g: (Z, \underline{z}) \rightarrow (Y_i, \underline{y}_i)$ is an A -morphism, then $g: (Z, \underline{z}) \rightarrow (X, \underline{x})$ is an A -morphism.

1.3 DEFINITION [H3] A concrete category A is called topological

iff the following conditions are satisfied:

(a) A has initial structures. That is, for any set X , any family $(Y_i, \underline{y}_i)_{i \in I}$ of A -objects and any family

$(f_i: X \rightarrow Y_i)_{i \in I}$ of functions, there exists an A -structure $\underline{x}$ on X which is initial with respect to $(X, (f_i)_{i \in I},$

$(Y_i, \underline{y}_i)_{i \in I})$.

(b) A is properly fibred. That is (i) for every set X , the class of all A -structures on X (called the A -fibre of X) is a set and (ii) if x and y are A -structures on X so that $1_X: (X, x) \rightarrow (X, y)$ and $1_X: (X, y) \rightarrow (X, x)$ are A -morphisms, then $x = y$.

1.4 EXAMPLES (a) Top, the category of all topological spaces and continuous functions, is a topological category. (b) Set, the category of all sets and all functions, is also topological.

1.5 DEFINITION [H2] Let A be a concrete category. Let Y be a set, $(X_i, x_i)_{i \in I}$ a class of A -objects and $(f_i: X_i \rightarrow Y)_{i \in I}$ a class of functions. An A -structure y on Y is final (co-initial) with respect to $((X_i, x_i)_{i \in I}, (f_i)_{i \in I}, Y)$ iff the following conditions hold:

(a) $f_i: (X_i, x_i) \rightarrow (Y, y)$ is an A -morphism for each $i \in I$.

(b) If (Z, z) is an A -object and $g: Y \rightarrow Z$ is a function such that for each $i \in I$, $g \circ f_i: (X_i, x_i) \rightarrow (Z, z)$ is an A -morphism, then $g: (Y, y) \rightarrow (Z, z)$ is an A -morphism.

1.6 DEFINITION [H2] A morphism $f: (X, x) \rightarrow (Y, y)$ in a concrete category A is called

(a) an embedding iff $f: X \rightarrow Y$ is one-to-one and x is initial with respect to $(X, f, (Y, y))$. In this case (X, x) is called an (A) -subobject or subspace of (Y, y) .

(b) f is called a quotient map if $f: X \rightarrow Y$ is onto and y is final with respect to $((X, x), f, Y)$. In this case (Y, y) is called a quotient object of (X, x) , or quotient space.

1.7 NOTE The definitions of embedding and quotient do not necessarily correspond to the categorical notions, but to the more classical structural ones. For example in Top, a topological quotient need not be a categorical-quotient, but will be a quotient in the sense of 1.6 (b).

1.8 THEOREM [H2] Let A be a topological category. Then

- (a) A has final structures.
- (b) A is complete. Limits are constructed by giving the appropriate initial structure to the corresponding limit in Set.
- (c) A is cocomplete. Colimits are constructed by giving the appropriate final structure to the corresponding colimit in Set.
- (d) For every set X there is a discrete A -structure d on X . That is, for any A -object (Y, γ) and any function $f: X \rightarrow Y$, $f: (X, d) \rightarrow (Y, \gamma)$ is an A -morphism.
- (e) For every set X , there is an indiscrete A -structure i on X . That is, for any A -object (Y, γ) and any function $f: Y \rightarrow X$, $f: (Y, \gamma) \rightarrow (X, i)$ is an A -morphism.
- (f) An A -morphism $f: (X, x) \rightarrow (Y, \gamma)$ is
 - (i) a monomorphism iff $f: X \rightarrow Y$ is one-to-one,
 - (ii) an epimorphism iff $f: X \rightarrow Y$ is onto,
 - (iii) an isomorphism iff it is an embedding and a quotient map,
 - (iv) an extremal monomorphism iff it is an embedding, and
 - (v) an extremal epimorphism iff it is a quotient map. |||

1.9 EXAMPLE Let $\{(X_i, x_i) \mid i \in I\}$ be a family of A -objects. The product in A is given by $(p_k: (\prod X_i, \pi_{x_i}) \rightarrow (X_k, x_k))_{k \in I}$ where

$\prod X_i$ is the cartesian product of $\{X_i \mid i \in I\}$, $p_k: \prod X_i \rightarrow X_k$ is the k^{th} projection map and π_{X_i} is initial with respect to $(\prod X_i, (p_i)_{i \in I}, (X_i, \chi_i)_{i \in I})$.

1.10 THEOREM [H2], [N] Let B be a subcategory of a topological category A .

(a) B is epireflective in A iff B is closed under products and subobjects in A .

If B contains all indiscrete A -objects, the following statements are equivalent:

(i) B is epireflective in A .

(ii) B is bireflective in A .

(iii) B is closed under the formation of initial mono-sources in A . That is, if $(Y_i, \chi_i)_{i \in I}$ is a class of objects in B and $(f_i: (X, \chi) \rightarrow (Y_i, \chi_i))_{i \in I}$ is an initial mono-source in A , then $(X, \chi) \in B$.

(iv) Initial structures in B are formed by constructing the initial structures in A (we will say B is closed under initial structures in A).

(b) Suppose B contains at least one object with non-empty underlying set (we say B is non-trivial). Then the following statements are equivalent:

(i) B is coreflective in A .

(ii) B is bicoreflective in A , and contains all discrete A -objects.

(iii) B is closed under coproducts and quotient objects in A .

(iv) B is closed under the formation of final epi-sinks.

(v) Final structures in B are formed by constructing the final structures in A (we will say B is closed under final structures in A).

PROOF. We look only at (b); (a) follows by a dual argument. (i), (ii) and (iii) are equivalent by [H2]. (i) and (iv) are equivalent by [N]. (iv) and (v) are equivalent in view of the fact that if γ is final with respect to $((X_i, x_i)_{i \in I}, (f_i)_{i \in I}, Y)$ ($Y \neq \emptyset$), then by adjoining (Y, d) , where d is the discrete structure on Y , to $(X_i, x_i)_{i \in I}$ and $l_y: Y \rightarrow Y$ to $(f_i)_{i \in I}$, we get a final epi-sink, and γ will not have changed. |||

1.11 THEOREM [H2] Any bireflective or bcoreflective subcategory of a topological category is topological. |||

1.12 THEOREM [N] Let A be a topological category and B a non-trivial subcategory of A . Then an A -object A belongs to the coreflective hull of B in A iff there exists a final epi-sink from B -objects to A . |||

1.13 THEOREM [H2] Let A be a topological category. Let B be a bireflective subcategory of A , and if (X, x) is an A -object, denote the reflection by $l_x: (X, x) \rightarrow (X, x_B)$. Let C be a bcoreflective subcategory of A with coreflection $l_x: (X, x_C) \rightarrow (X, x)$.

(a) For any C -object (X, x) the following conditions are equivalent:

(i) (X, x) is B-structurable; that is, there exists B-object (X, y) with $x_C = x$.

(ii) $x = (x_B)_C$.

(b) For any B-object (X, x) the following conditions are equivalent:

(i) (X, x) is C-structurable; that is, there exists C-object (X, y) with $x_B = x$.

(ii) $x = (x_C)_B$.

(c) The subcategory C_B of C whose objects are the B-structurable objects is bireflective in C . For any C-object (X, x) , the reflection is given by $l_x: (X, x) \rightarrow (X, (x_B)_C)$.

(d) The subcategory B_C of B whose objects are the C-structurable objects is bicoreflective in B . For any B-object (X, x) the coreflection is given by $l_x: (X, (x_C)_B) \rightarrow (X, x)$. |||

1.14 THEOREM [H2] Let A be a topological category, C a bicoreflective subcategory of A , and B a bireflective subcategory of C . Then the subcategory of A consisting of objects whose C -coreflection is in B is bireflective in A . |||

1.15 DEFINITION A category A is said to be cartesian closed iff the following conditions are satisfied:

(a) A has finite products.

(b) For any $A \in A$, the functor $Ax_-: A \rightarrow A$ has a right adjoint. That is, for any $B \in A$ there exists $B^A \in A$ and

an A -morphism $e_{A,B}: A \times B^A \rightarrow B$ such that for any $C \in A$, and A -morphism $f: A \times C \rightarrow B$ there exists a unique A -morphism $g: C \rightarrow B^A$ such that the diagram

$$\begin{array}{ccc} A \times B^A & \xrightarrow{e_{A,B}} & B \\ & \searrow \quad \swarrow & \\ & A \times C & \end{array} \quad \begin{array}{c} \\ f \end{array}$$

commutes.

1.16 THEOREM [H3] If A is a topological category the following conditions are equivalent:

(a) A is cartesian closed.

(b) For any $A \in A$, the functor $A \times _$ preserves final epi-sinks. That is, for any final epi-sink $(f_i: B_i \rightarrow B)_{i \in I}$ in A , $(1_A \times f_i: A \times B_i \rightarrow A \times B)_{i \in I}$ is a final epi-sink.

(c) For any $A, B \in A$, the set $A(A; B)$ of morphisms from A to B can be given the structure of an A -object (which will be B^A) so that (i) $e_{A,B}: A \times B^A \rightarrow B$ defined by $e_{A,B}(a, g) = g(a)$ is an A -morphism and (ii) for any $C \in A$, $\psi: (B^A)^C \rightarrow B^{A \times C}$ defined by $\psi(f) = e_{A,B} \cdot (1_A \times f)$ is onto. |||

Several other characterizations of cartesian closed topological categories appear in [H3]. Many of the consequences of cartesian closedness for a topological category are also given there and in [N]. We present only a small selection.

1.17 THEOREM [H3] If A is a cartesian closed topological category,

and A, B, C, A_i, B_i ($i \in I$, a set) $\in A$, then

(a) $A^{B \times C} = (A^B)^C$

(b) $(\prod A_i)^B = \prod (A_i^B)$

$$(c) \quad A^{\coprod B_i} = \coprod (A^{B_i})$$

$$(d) \quad A \times \coprod B_i = \coprod (A \times B_i)$$

(e) Finite products of quotient maps are quotient maps. |||

1.18 THEOREM [N] If A is a cartesian closed topological category and B a non-trivial coreflective subcategory of A closed under the formation of finite products in A , then B is cartesian closed. |||

1.19 THEOREM [N] If A is a cartesian closed topological category and C is the coreflective hull of a non-trivial class B such that C contains all finite A -products of objects in B then C is cartesian closed. |||

1.20 THEOREM Let A be a cartesian closed topological category and B a bireflective subcategory of A . For $A \in A$, let $1_A: A \rightarrow A_B$ be the B -reflection. If for every $A \in A$ and $B \in B$ $(B \times A)_B = B \times A_B$, then B is cartesian closed.

PROOF. Let $(f_i: X_i \rightarrow Y)_{i \in I}$ be a final epi-sink in B . Suppose $Z \in A$ has the same underlying set as Y , and A -structure making $(f_i: X_i \rightarrow Z)_{i \in I}$ a final epi-sink in A . Now the B -reflection preserves final epi-sinks [N]; so $(f_i: X_i \rightarrow Z_B)_{i \in I}$ is a final epi-sink in B . Since Z and Y have the same underlying set, $Z_B = Y$.

Choose $B' \in B$. Then $(1_B \times f_i: B \times X_i \rightarrow B \times Z)_{i \in I}$ is a final epi-sink in A by 1.16. Since $B \times X_i \in B$ for every $i \in I$, $(1_B \times f_i: B \times X_i \rightarrow (B \times Z)_B)_{i \in I}$ is a final epi-sink in B . But $(B \times Z)_B = B \times Z_B = B \times Y$; so $(1_B \times f_i: B \times X_i \rightarrow B \times Y)_{i \in I}$ is a final epi-sink in B . By 1.16, B is cartesian closed. |||

2. NEARNESS

2.1 DEFINITION [H2] Given a set X , and $\underline{x} \subset P^2 X$. Then $\underline{x}$ is called a pre-nearness structure on X if for $\underline{A}, \underline{B} \subset PX$,

(N1) If $\underline{A} \subset \underline{B}$ and $\underline{B} \in \underline{x}$, then $\underline{A} \in \underline{x}$.

(N2) If $\bigcap \underline{A} \neq \emptyset$, then $\underline{A} \in \underline{x}$.

(N3) If $\emptyset \in \underline{A}$, then $\underline{A} \in \underline{x}$.

If $\underline{x}$ is a pre-nearness structure on X , we call $(X, \underline{x})$ a pre-nearness (P-near) space.

2.2 DEFINITION [H2] If $(X, \underline{x})$ and $(Y, \underline{y})$ are pre-nearness spaces, and $f: X \rightarrow Y$ is a function, then $f: (X, \underline{x}) \rightarrow (Y, \underline{y})$ is called a near map (N-map) if $\underline{A} \in \underline{x}$ implies $f(\underline{A}) \in \underline{y}$.

Clearly $1_X: (X, \underline{x}) \rightarrow (X, \underline{x})$ is a near map for any P-near space $(X, \underline{x})$, and the composition of near maps is a near map. Thus we have a category, denoted PNear, whose objects are the P-near spaces and morphisms are the near maps.

2.3 DEFINITION [H2] If $(X, \underline{x})$ is a P-near space and $\underline{x}$ also satisfies the condition

(N4) If $\underline{A} \vee \underline{B} \in \underline{x}$, then $\underline{A} \in \underline{x}$ or $\underline{B} \in \underline{x}$,

then $\underline{x}$ is called a semi-nearness (S-near) structure on X , and $(X, \underline{x})$ is called a semi-near (S-near) space.

The subcategory of PNear whose objects are the S-near spaces will be denoted by SNear.

2.4 NOTATION. If $(X, \underline{x}) \in \text{PNear}$, we can define a 'closure' operator $cl_{\underline{x}}$ on subsets A of X by

$$cl_{\underline{x}} A = \{x \in X \mid (\{x\}, A) \in \underline{x}\}$$

Then for $A, B \subseteq X$, we always have

$$(i) \quad cl_{\underline{x}} A \cup cl_{\underline{x}} B = cl_{\underline{x}} (A \cup B) \quad \text{and}$$

$$(ii) \quad A \subseteq cl_{\underline{x}} A.$$

However, even for $(X, \underline{x}) \in \underline{SNear}$ it need not be the case that

$$cl_{\underline{x}} cl_{\underline{x}} A = cl_{\underline{x}} A; \quad \text{so the 'closure' need not be topological.}$$

2.5 DEFINITION [H1] If $(X, \underline{x})$ is an S-near space and $\underline{x}$ also satisfies the condition

$$(N5) \quad \text{If } cl_{\underline{x}} A = \{cl_{\underline{x}} A \mid A \in \underline{x}\} \text{ then } A \in \underline{x},$$

$\underline{x}$ is called a near structure on X , and $(X, \underline{x})$ is called a near (N-) space. The subcategory of $\underline{SNear}$ (and hence of $\underline{PNear}$) whose objects are the near spaces will be denoted by $\underline{Near}$.

Notice that if $(X, \underline{x}) \in \underline{Near}$, then we have, for $A \subseteq X$,

$$(iii) \quad cl_{\underline{x}} cl_{\underline{x}} A = cl_{\underline{x}} A$$

and so $cl_{\underline{x}}$ defines a topological closure.

2.6 NOTES. $\underline{Near}$ has been studied extensively in [H1]. $\underline{PNear}$, $\underline{SNear}$, $\underline{Near}$ and many important subcategories of them are studied at length in [H2]. In this dissertation, $\underline{SNear}$ will be the category used primarily. As noted in [H2] at least two other approaches to the problem of finding a more satisfactory setting for topological problems are equivalent to the concept of S-near space. One is to remove the star-refinement condition from the definition of uniformity to get quasi-uniformities [1]. It can be useful to translate from S-near spaces to quasi-uniformities when one wishes to deal with problems analogous to those of uniform space theory (such as an analogue to 'totally bounded'—see [K1]).

The other concept is that of merotopic space, introduced by Katětov in [K1] .

2.7 DEFINITIONS [K1]

(a) Given a set X and $\mathcal{MSP}^2 X$. Then $\underline{m}$ is called a merotopic structure (merotopy) on X if for $\underline{A}, \underline{B} \in \mathcal{P}X$,

(M1) If $\underline{A} \subset \underline{B}$ and $\underline{A} \in \underline{m}$, then $\underline{B} \in \underline{m}$.

(M2) $\{\{x\}\} \in \underline{m}$ for every $x \in X$.

(M3) $\emptyset \notin \underline{m}$.

(M4) If $\underline{A} \cup \underline{B} \in \underline{m}$, then $\underline{A} \in \underline{m}$ or $\underline{B} \in \underline{m}$.

If $\underline{x}$ is a merotopy on X , we call $(X, \underline{m})$ a merotopic space.

(b) If $(X, \underline{m})$ and $(Y, \underline{n})$ are merotopic spaces and $f: X \rightarrow Y$ is a function, then $f: (X, \underline{m}) \rightarrow (Y, \underline{n})$ is called a merotopically continuous (continuous, merotopic) map if for every $\underline{A} \in \underline{m}$;
 $f(\underline{A}) \in \underline{n}$.

(c) The category of all merotopic spaces and merotopic maps will be denoted by $\underline{\text{Mer}}$.

2.8 THEOREM [H2] There is an isomorphism $I: \underline{\text{SNear}} \rightarrow \underline{\text{Mer}}$. It is given by the condition: If $(X, \underline{m}) = I(X, \underline{x})$, $\underline{A} \in \underline{m}$ iff
 $\text{sec } \underline{A} = \{B \subseteq X \mid B \cap A \neq \emptyset \text{ for every } A \in \underline{A}\} \in \underline{x}$. Conversely, $\underline{B} \in \underline{x}$ iff
 $\text{sec } \underline{B} \in \underline{m}$. |||

2.9 THEOREM [H2]

(a) $\underline{\text{SNear}}$ is a bicoreflective subcategory of $\underline{\text{PNear}}$.

If $(X, \underline{x}) \in \underline{\text{PNear}}$ the coreflection is given by $l_X: (X, \underline{x}_S) \rightarrow (X, \underline{x})$, where $\underline{x}_S$ is defined by: $\underline{A} \in \underline{x}_S$ iff there do not exist $\underline{A}_1, \dots, \underline{A}_n$ not in $\underline{x}$ with $\underline{A}_1 \vee \underline{A}_2 \vee \dots \vee \underline{A}_n < \underline{A}$.

(b) Near is a bireflective subcategory of SNear.

If $(X, \underline{x}) \in \underline{SNear}$, the reflection is given by $1_X: (X, \underline{x}) \rightarrow (X, \underline{x}_n)$, where $\underline{x}_n$ is defined as follows:

By transfinite induction define operators cl^α on PX :

$cl^0 A = A$; $cl^{\alpha+1} A = \{x \in X \mid \{cl^\alpha A, cl^\alpha \{x\}\} \in x\}$; $cl^\beta A = \bigcup \{cl^\alpha A \mid \alpha < \beta\}$

if β is a limit ordinal. Let $cl A = \bigcup \{cl^\alpha A \mid \alpha \text{ ordinal}\}$.

Then $\underline{A} \in \underline{x}_n$ iff $cl A \in x$. Note that $cl_{\underline{x}_n} = cl_{\cdot} |||$

2.10 THEOREM [H2] PNear, SNear and Near are topological categories.

(a) Initial structures are described as follows: Let X be a set, $(Y_i, \underline{y}_i)_{i \in I}$ a class of spaces and $(f_i: X \rightarrow Y_i)_{i \in I}$ a class of functions. We wish to define $\underline{x}$, the structure on X

initial with respect to $(X, (f_i)_{i \in I}, (Y_i, \underline{y}_i)_{i \in I})$.

(i) If $(Y_i, \underline{y}_i) \in \underline{PNear}$ for every $i \in I$,

$\underline{x} = \{\underline{A} \in PX \mid f_i(\underline{A}) \in \underline{y}_i \text{ for every } i \in I\}$ is the initial structure in PNear.

(ii) If $(Y_i, \underline{y}_i) \in \underline{SNear}$ for every $i \in I$,

$\underline{x} = \{\underline{A} \in PX \mid f_i(\underline{A}) \in \underline{y}_i \text{ for every } i \in I\}_S$ (see 2.9(a))

is the initial structure in SNear.

(iii) If $(Y_i, \underline{y}_i) \in \underline{Near}$ for every $i \in I$, then $\underline{x}$ as

in (ii) is the initial structure in Near.

(b) Final structures are described as follows: Let $(X_i, \underline{x}_i)_{i \in I}$ be a class of spaces, Y a set, and $(f_i: X_i \rightarrow Y)_{i \in I}$ a class of functions. We wish to define $\underline{y}$, the structure on Y final with respect to $((X_i, \underline{x}_i)_{i \in I}, (f_i)_{i \in I}, Y)$.

(i) If $(X_i, \underline{x}_i) \in \underline{PNear}$ for every $i \in I$,

$\underline{y} = \{\underline{B} \in PY \mid \bigcap \underline{B} \neq \emptyset\} \cup \{\underline{B} \in PY \mid \underline{B} \subseteq [f_i(\underline{A})], \text{ some } i \in I, \text{ some } \underline{A} \in \underline{x}_i\}$

$= \{\underline{B} \in PY \mid \bigcap \underline{B} \neq \emptyset\} \cup \{\underline{B} \in PY \mid f_i^{-1} \underline{B} \in \underline{x}_i, \text{ some } i \in I\}$.

is the final structure in PNear.

(ii) If $(X_i, x_i) \in \underline{\text{SNear}}$ for every $i \in I$, then $\underline{y}$ as in (i) is the final structure in SNear.

(iii) If $(X_i, x_i) \in \underline{\text{Near}}$ for every $i \in I$,

$$\underline{x} = (\{ \underline{B} \subseteq \text{PY} \mid \bigcap \underline{B} \neq \emptyset \} \cup \{ \underline{B} \subseteq \text{PY} \mid f_i^{-1} B \in x_i, \text{ some } i \in I \})_n \quad (\text{see 2.9(b)})$$

is the final structure in Near.

2.11 NOTE. Initial structures in SNear and Near are rather complicated to construct. The special case in which I is a single element can be described somewhat more neatly:

If $f: X \rightarrow Y$ is a function and $(Y, y) \in \underline{\text{SNear}}$, then

$$\underline{x} = \{ \underline{A} \subseteq \text{PX} \mid f(\underline{A}) \in y \} \text{ is initial in } \underline{\text{SNear}} \text{ with respect to } (X, \underline{x}, (Y, y)).$$

In particular if f is one-to-one, this describes sub-objects in SNear.

2.12 NOTE. If $(X, \underline{x}) \in \underline{\text{Near}}$, we have seen that $(X, \text{cl}_{\underline{x}})$ is a topological space. Not every topological space can be described in this way, as there is a symmetry built into the nearness axioms: nearness of a family $\underline{A}$ does not depend on the order in which the sets of $\underline{A}$ are chosen. The class of topological spaces (X, cl) we get are those satisfying the axiom

$$(R_o) \text{ if } x \in \text{cl}\{y\}, \text{ then } y \in \text{cl}\{x\}.$$

The subcategory of Top whose objects are those satisfying

$$(R_o) \text{ will be called } R_o\text{Top}.$$

$R_o\text{Top}$ contains a very wide class of topological spaces: for example, all T_1 -spaces are in $R_o\text{Top}$.

2.13 DEFINITION [H1] Let TNear be the subcategory of Near whose objects are those $(X, \underline{x})$ such that

$$\underline{A} \in \underline{x} \text{ iff } \bigcap_{\underline{x}} \underline{A} \neq \emptyset.$$

2.14 THEOREM [H1] $\underline{TNear}$ is isomorphic to $\underline{R_{0Top}}$, $(X, \underline{x}) \in \underline{TNear}$ being identified with $(X, \underline{cl}_{\underline{x}}) \in \underline{R_{0Top}}$. |||

2.15 THEOREM [H1] $\underline{TNear}$ is a bicoreflective subcategory of $\underline{Near}$.

If $(X, \underline{x}) \in \underline{TNear}$ the coreflection is given by $1_X : (X, \underline{x}_t) \rightarrow (X, \underline{x})$ where

$$\underline{A} \in \underline{x}_t \text{ iff } \bigcap_{\underline{x}} \underline{A} \neq \emptyset.$$

$$\text{Note that } \underline{cl}_{\underline{x}_t} = \underline{cl}_{\underline{x}}. |||$$

2.16 NOTE. In [H1] it is also shown how many other important categories of spaces which have been studied by topologists can be embedded nicely in $\underline{Near}$. For example: the category $\underline{Unif}$ of uniform spaces and uniformly continuous maps is isomorphic to a bireflective subcategory $\underline{UNear}$ of $\underline{Near}$. The category of contiguity spaces and contiguous maps (see [11]) is isomorphic to a bireflective subcategory $\underline{CNear}$ of $\underline{Near}$. Also the category $\underline{Prox}$ of proximity spaces and δ -maps is isomorphic to $\underline{UNear} \cap \underline{CNear}$.

3. GRILLS AND FILTERS

3.1 DEFINITION Let X be a set. $G \subseteq P^2 X$ is called a grill on X if $\emptyset \notin G$, and for $A, B \subseteq X$, $A \cup B \in G$ iff $A \in G$ or $B \in G$.

Notice that if $A \in G$ and $A \subseteq B$, then $B \in G$.

3.2 DEFINITION [T] For $H \subseteq P^2 X$ (not necessarily a grill) let

$$cH = \{A \subseteq X \mid X \setminus A \notin H\}.$$

Then $c: P^2 X \rightarrow P^2 X$ is a function.

3.3 PROPOSITION [T] (a) $ccH = H$. (b) $H \subseteq G$ iff $cG \subseteq cH$. (c) If I is any index set, and $\{H_i \mid i \in I\}$ a family of subsets of $P^2 X$, then (i) $c(\cup \{H_i \mid i \in I\}) = \cap \{cH_i \mid i \in I\}$ and (ii) $c(\cap \{H_i \mid i \in I\}) = \cup \{cH_i \mid i \in I\}$. |||

The concept of grill is dual to that of filter in the following sense:

3.4 THEOREM [T] c defines an anti-isomorphism between the family of all grills on X , ordered by inclusion, and the family of all filters on X , ordered by inclusion. That is, G is a grill on X iff cG is a filter on X , and F is a filter on X iff cF is a grill on X . Furthermore, c reverses inclusions (as shown in 3.3(b)) and $c^2 = \text{identity}$ (as shown in 3.3(a)). |||

Notice that the grills which are also filters are exactly the ultrafilters; if U is an ultrafilter, $cU = U$.

Using 3.4 and the well-known result that filters are the intersections of ultrafilters, we have immediately

3.5 PROPOSITION [T] $\underline{G}$ is a grill iff $\underline{G}$ is a union of ultrafilters.

Thus any union of grills is a grill. |||

3.6 THEOREM [T] If $\underline{F}$ is a filter on a set X , and $\underline{G}$ is a grill on X with $\underline{F} \subseteq \underline{G}$, then there exists an ultrafilter $\underline{U}$ with $\underline{F} \subseteq \underline{U} \subseteq \underline{G}$. |||

3.7 NOTE. It is also worthwhile to note that for any grill or filter $\underline{H}$, $c\underline{H} = \text{sec } \underline{H}$.

Let X and Y be sets, and $f: X \rightarrow Y$ a function. It is well-known that if $\underline{F}$ is a filter on X , then $[f(\underline{F})] = \{B \subseteq Y \mid f(F) \subseteq B, \text{ some } F \in \underline{F}\}$ is a filter on Y . It is straightforward to verify

3.8 LEMMA If $\underline{G}$ is a grill on a set X and $f: X \rightarrow Y$ is a function, then $[f(\underline{G})] = \{B \subseteq Y \mid f(G) \subseteq B, \text{ some } G \in \underline{G}\}$ is a grill on Y . |||

3.9 THEOREM If $\underline{H}$ is a filter (or grill) on a set X , and $f: X \rightarrow Y$ is a function, then $[f(c\underline{H})] = c[f(\underline{H})]$.

PROOF. (i) $f(c\underline{H}) \subseteq c[f(\underline{H})]$. For suppose $A \in c\underline{H}$, $f(A) \not\subseteq c[f(\underline{H})]$. Then $Y \setminus f(A) \in [f(\underline{H})]$; so there exists $F \in \underline{H}$, $f(F) \subseteq Y \setminus f(A)$. Then $F \subseteq X \setminus A$, and $X \setminus A \in \underline{H}$, giving $A \notin \underline{H}$ (contradiction).

(ii) If $B \in c[f(\underline{H})]$, $f^+(B) \in c\underline{H}$. Otherwise, $X \setminus f^+(B) \in \underline{H}$. Then $f(X \setminus f^+(B)) \subseteq Y \setminus B$ implies $Y \setminus B \in [f(\underline{H})]$, and $B \not\subseteq c[f(\underline{H})]$, a contradiction. $f(f^+(B)) \subseteq B$ implies $B \in [f(c\underline{H})]$.

(i) gives $[f(c\underline{H})] \subseteq c[f(\underline{H})]$, and (ii) gives $c[f(\underline{H})] \subseteq [f(c\underline{H})]$.

Notice that this proof uses only the fact that $A \in \underline{H}$, $A \subseteq B$ implies $B \in \underline{H}$, and so works for filters or grills. |||

We give a final result about grills in products which will

be needed later. It also follows via a result about filters.

3.10 THEOREM Let I be a finite index set, and $\{X_i | i \in I\}$ a family of sets. If $\underline{G}_i$ is a grill on X_i for each $i \in I$, then there is a unique maximal grill $\underline{G}$ on $\prod X_i$ satisfying $p_i \underline{G} = \underline{G}_i$ for all $i \in I$ (p_i is the i^{th} projection). That is, if $\underline{H}$ is a grill on $\prod X_i$ with $p_i \underline{H} = \underline{G}_i$ for all $i \in I$, then $\underline{H} \subseteq \underline{G}$.

Furthermore, if $\{Y_i | i \in I\}$ is a family of sets, and $f_i: X_i \rightarrow Y_i$ a function for every $i \in I$, then $[(\pi f_i)(\underline{G})]$ is the unique maximal grill on $\prod Y_i$ with $q_k [(\pi f_i)(\underline{G})] = [f_k(\underline{G}_k)]$ for all $k \in I$ ($q_k: \prod Y_i \rightarrow Y_k$ is the k^{th} projection map).

PROOF. For each i , $c\underline{G}_i$ is a filter. Then it is well-known that $\underline{F} = [\pi c\underline{G}_i] = [(\pi f_i) | f_i \in c\underline{G}_i \text{ for all } i \in I]$ is a filter on $\prod X_i$. Now $p_k \underline{F} = c\underline{G}_k$, and if $\underline{E}$ is a filter on $\prod X_i$ with $p_k \underline{E} = c\underline{G}_k$ for all $k \in I$, then $\underline{F} \subseteq \underline{E}$. Let $\underline{G} = c\underline{F}$. $\underline{G}$ is a grill, and $p_i \underline{G} = p_i c\underline{F} = c p_i \underline{F}$ (3.9). So $p_i \underline{G} = c(c\underline{G}_i) = \underline{G}_i$ for all $i \in I$. $\underline{G}$ is the required maximal grill (see 3.4).

$[(\pi f_i)\underline{F}] = [(\pi f_i)(\pi c\underline{G}_i)] = [\pi [f_i(c\underline{G}_i)]]$, the minimal filter on $\prod Y_i$ with k^{th} projection $[f_k(c\underline{G}_k)]$. Again by the anti-isomorphism of 3.4, $[(\pi f_i)(\underline{G})] = c [(\pi f_i)\underline{F}]$ (3.9) is the maximal grill on $\prod Y_i$ with k^{th} projection $[f_k(\underline{G}_k)]$. |||

4. GRILL-DETERMINED SPACES

4.1 DEFINITIONS Let X be a set.

(a) Let $N(X)$ be the family of all semi-near structures on X . Then $N(X)$ is a partially ordered set under $\leq$ defined by $\underline{x} \leq \underline{y}$ iff $\underline{x} \subseteq \underline{y}$.

(b) Let $\underline{G}(X)$ be the family of all families $\underline{G}$ of grills satisfying the condition

(G) for all $x \in X$, there exists $\underline{G} \in \underline{G}$ with $\dot{x} \in \underline{G}$.

(In particular, any family $\underline{G}$ of grills on X with $\dot{x} \in \underline{G}$ for all $x \in X$ will be in $\underline{G}(X)$). Such a family $\underline{G}$ will be called a grill structure on X , and $(X, \underline{G})$ a grill space.

Define partial order $\leq$ on $\underline{G}(X)$ by $\underline{G} \leq \underline{H}$ iff $\underline{H} \subseteq \underline{G}$ (that is, every grill in $\underline{H}$ is also in $\underline{G}$).

4.2 THEOREM Let $\underline{G}$ be a grill structure on X . Then

$$\Psi(\underline{G}) = \{ \underline{A} \subseteq PX \mid \underline{A} \subseteq \underline{G} \text{ for some } \underline{G} \in \underline{G} \}$$

is a semi-nearness structure on X .

Conversely, if $\underline{x}$ is a semi-nearness structure on X , then

$$\Phi(\underline{x}) = \{ \underline{G} \subseteq PX \mid \underline{G} \text{ is a grill and } \underline{G} \subseteq \underline{x} \}$$

is a grill structure on X .

PROOF. (N1) $\underline{B} \in \Psi(\underline{G})$ implies $\underline{B} \subseteq \underline{G}$, some $\underline{G} \in \underline{G}$. If $\underline{A} < \underline{B}$, then for all $\underline{A} \in \underline{A}$, there exists $\underline{B} \in \underline{B}$, $\underline{B} \in \underline{A}$. But $\underline{B} \in \underline{G}$, $\underline{G}$ a grill, implies $\underline{A} \in \underline{G}$. That is, $\underline{A} \subseteq \underline{G}$, or $\underline{A} \in \Psi(\underline{G})$.

(N2) If $\cap \underline{A} \neq \emptyset$, then $\underline{A} \subseteq \underline{x}$, some $x \in X$. But $\dot{x} \in \underline{G}$, some $\underline{G} \in \underline{G}$; so $\underline{A} \subseteq \underline{G}$, and $\underline{A} \in \Psi(\underline{G})$.

(N3) If $\phi \in A$, then $A \in G$, any grill G , and $A \in \Psi(G)$.

(N4) If $A \in \Psi(G)$ and $B \in \Psi(G)$, then for every $G \in G$, there exists $A \in A$, $B \in B$, $A, B \in G$. Then $A \cup B \in G$, which means that $A \cup B \in G$, and $A \cup B \in \Psi(G)$.

The converse is immediate, since $x \in x$ for every $x \in X$, any S-near structure x (by (N2)). |||

4.3 DEFINITION If $G \in \Psi(x)$ (that is, G is a grill, and $G \in x$) we will call G an x -grill.

4.4 NOTE. Theorem 4.2 defines functions

$$\Psi: (G(X), \leq) \rightarrow (N(X), \leq) \text{ and}$$

$$\Phi: (N(X), \leq) \rightarrow (G(X), \leq)$$

which reverse partial orders and such that $\Psi\Phi(x) \leq x$ for all $x \in N(X)$ and $\Phi\Psi(G) \leq G$ for all $G \in G(X)$. Thus Ψ and Φ are anti-isotone, and we have a Galois correspondence between $(N(X), \leq)$ and $(G(X), \leq)$.

4.5 THEOREM $G = \Phi\Psi(G)$ (that is, G is the family of all $\Psi(G)$ -grills) iff the following conditions hold:

(G1) $x \in G$ for all $x \in X$ and

(G2) If $G \in G$ and H is a grill with $H \leq G$, then $H \in G$.

PROOF. (G1) is necessary by (N2), (G2) by (N1). Sufficiency follows immediately from the definitions of Ψ and Φ . |||

4.6 EXAMPLE In general for $(X, x) \in \text{SNear}$, $\Psi\Phi(x) \neq x$; that is, there may be $A \in x$ such that A is not a subfamily of any x -grill. An example of such a space is given in [Be] where a near space which is not bunch-determined is described (see 4.16).

4.7 DEFINITION If $\underline{x} = \Psi(G)$ for some grill structure G on X (that is, $\underline{x}$ is a closed element under the Galois correspondence) we will say G determines $\underline{x}$ and call $(X, \underline{x})$ a grill-determined space. Then $(X, \underline{x})$ is grill-determined precisely when

$\underline{A} \in \underline{x}$ iff there is an $\underline{x}$ -grill $\underline{G}$ with $\underline{A} \in \underline{G}$. (that is, $\Psi\Phi(\underline{x}) = \underline{x}$).

The subcategory of SNear whose objects are the grill-determined spaces will be called Grill.

If $(X, \underline{x})$ is a grill-determined space with $\underline{x} = \Psi(G)$, then G may be thought of as a base for the semi-nearness structure. For we have

4.8 PROPOSITION Let $(X, \underline{x}) \in \text{Grill}$ with $\underline{x} = \Psi(G)$, and $(Y, \underline{y}) \in \text{SNear}$. A function $f: X \rightarrow Y$ is a near map iff $f(\underline{G}) \in \underline{y}$ for every $\underline{G} \in G$.

PROOF. If f is a near map then the condition is satisfied, since every $\underline{G} \in G$ is an $\underline{x}$ -grill.

Conversely, if the condition holds: $\underline{A} \in \underline{x}$ implies $\underline{A} \in \underline{G}$, some $\underline{G} \in G$. Now $f(\underline{A}) \in f(\underline{G}) \in \underline{y}$; so $f(\underline{A}) \in \underline{y}$ by (N1). |||

4.9 THEOREM Grill is a bicoreflective subcategory of SNear.

PROOF. If $(X, \underline{x}) \in \text{SNear}$, let $\underline{x}_g = \Psi\Phi(\underline{x})$; so that $(X, \underline{x}_g) \in \text{Grill}$. The coreflection is given by $1_X: (X, \underline{x}_g) \rightarrow (X, \underline{x})$:

Since $\Psi\Phi(\underline{x}) \subseteq \underline{x}$, this is a near map.

If $(Y, \underline{y}) \in \text{Grill}$, and $f: (Y, \underline{y}) \rightarrow (X, \underline{x})$ is a near map,

$\underline{H} \in \Phi(\underline{y})$ implies $f(\underline{H}) \in \underline{x}$.

But $f(\underline{H}) \subseteq [f(\underline{H})]$, a grill. $[f(\underline{H})] < f(\underline{H})$; so $[f(\underline{H})] \in \underline{x}$ by (N1).

Thus $[f(\underline{H})] \in \Phi(\underline{x})$ and $f(\underline{H}) \in \Psi\Phi(\underline{x}) = \underline{x}_g$. By 4.8, $f: (Y, \underline{y}) \rightarrow (X, \underline{x}_g)$ is a near map. |||

4.10 COROLLARY Grill is a topological category.

PROOF. The result follows immediately from 4.9, 2.10 (SNear is a topological category) and 1.11. |||

4.11 COROLLARY Grill is closed under final structures in SNear. |||

4.12 NOTE. For any $(X, \underline{x}) \in \text{Near}$, $\text{cl}_{\underline{x}} = \text{cl}_{\underline{x}_g}$: The fact that $\text{cl}_{\underline{x}_g} A \subseteq \text{cl}_{\underline{x}} A$ is immediate, since $\underline{x}_g \subseteq \underline{x}$. If $x \in \text{cl}_{\underline{x}} A$, then $\{x\}$, $A \in \underline{C}(x) = \{B \subseteq X \mid x \in \text{cl}_{\underline{x}} B\}$, an $\underline{x}$ -grill. That is, $x \in \text{cl}_{\underline{x}_g} A$; or $\text{cl}_{\underline{x}} A \subseteq \text{cl}_{\underline{x}_g} A$.

4.13 THEOREM Grill is closed under subobjects in SNear.

PROOF. We should first recall that our definition of subobject (1.6) is not the categorical one.

Let $(X, \underline{x}) \in \text{Grill}$, $(A, \underline{a})$ a subspace of $(X, \underline{x})$ in SNear. That is, $A \subseteq X$, and if $\underline{B} \subseteq \underline{P}A$, $\underline{B} \in \underline{a}$ iff $\underline{B} \in \underline{x}$ (2.11).

If $\underline{B} \in \underline{a}$, $\underline{B} \subseteq \underline{G}$, some $\underline{x}$ -grill $\underline{G}$. Let $\underline{G}_A = \{D \subseteq A \mid D \in \underline{G}\}$. $\underline{G}_A$ is a grill on A : $\phi \in \underline{G}$ implies $\phi \in \underline{G}_A$. If $C, D \subseteq A$, then $C \cup D \in \underline{G}_A \leftrightarrow C \cup D \in \underline{G}$ or $D \in \underline{G} \leftrightarrow C \in \underline{G}_A$ or $D \in \underline{G}_A$. Furthermore, $\underline{G}_A \in \underline{x}$, since $\underline{G} \subseteq \underline{x}$. But $\underline{G}_A \subseteq \underline{P}A$, which means that $\underline{G}_A$ is an $\underline{a}$ -grill. $\underline{B} \subseteq \underline{G}_A$; so $(A, \underline{a}) \in \text{Grill}$. |||

In general, initial structures in Grill are not inherited from SNear (as subobjects are). In fact, they can be described much more simply. Compare 2.10(a)(ii) with

4.14 THEOREM Let I be a class, X a set, $(Y_i, \underline{y}_i) \in \text{Grill}$ and $f_i: X \rightarrow Y_i$ a function for all $i \in I$. Then the structure $\underline{x} \in P^2 X$ given by

$\underline{A} \in \underline{x}$ iff there exists a grill $\underline{G}$ on X with $\underline{A} \subseteq \underline{G}$ and $f_i(\underline{G}) \in \underline{y}_i$ for every $i \in I$

is initial in Grill with respect to $(X, (f_i)_{i \in I}, (Y_i, \underline{y}_i)_{i \in I})$.

PROOF. If $x \in X$, $f_i(x) \in f_i(\underline{x}) \in \underline{y}_i$ for every $i \in I$. Thus $x \in \underline{x}$ for every $x \in X$. But then the family of all grills in $\underline{x}$ satisfies (G) of 4.1. In view of the definition of $\underline{x}$, 4.2 and 4.7 give $(X, \underline{x}) \in \text{Grill}$.

Suppose $(Z, \underline{z}) \in \text{Grill}$ and $g: Z \rightarrow X$ is a function such that $f_i \cdot g: (Z, \underline{z}) \rightarrow (Y_i, \underline{y}_i)$ is a near map for every $i \in I$. If $\underline{J}$ is a $\underline{z}$ -grill, $f_i(g\underline{J}) \in \underline{y}_i$, and thus by (N1) $f_i[g\underline{J}] \in \underline{y}_i$ for every $i \in I$. Since $[g\underline{J}]$ is a grill on X , $[g\underline{J}] \in \underline{x}$. Therefore $g\underline{J} \in \underline{x}$, and by 4.8, $g: (Z, \underline{z}) \rightarrow (X, \underline{x})$ is a near map. |||

4.15 THEOREM Grill is cartesian closed.

PROOF. By 1.16 it suffices to show that if $((X_i, \underline{x}_i) \xrightarrow{f_i} (Y, \underline{y}))_{i \in I}$ is a final epi-sink in Grill and $(A, \underline{a}) \in \text{Grill}$, then $((A \times X_i, \underline{a} \times \underline{x}_i) \xrightarrow{1_A \times f_i} (A \times Y, \underline{a} \times \underline{y}))_{i \in I}$ is a final epi-sink $((A \times Y, \underline{a} \times \underline{y}))$ is the product in Grill of $(A, \underline{a})$ and $(Y, \underline{y})$.

From 2.10(b)(ii) $\underline{B} \in \underline{y}$ iff for some $i \in I$, there exists $\underline{A} \in \underline{x}_i$ so that $\underline{B} \subseteq [f_i(\underline{A})]$. Since all spaces are in Grill, this is

equivalent to: a grill $H \in \mathcal{Y}$ iff for some $i \in I$, there exists x_i -grill J so that $H \subseteq [f_i(J)]$. We must show that if grill $M \in \mathcal{A} \times \mathcal{Y}$, then for some i there exists $(\mathcal{A} \times x_i)$ -grill G so that $M \subseteq [(1_A \times f_i)G]$. Now products in Grill are initial structures; so $M \in \mathcal{A} \times \mathcal{Y}$ iff $p_A M \in \mathcal{A}$, $p_Y M \in \mathcal{Y}$ (p_A and p_Y are the A - and Y - projections respectively). $p_A M$ is an $\mathcal{A}$ -grill and $p_Y M$ is a $\mathcal{Y}$ -grill (3.8). Then for some $i \in I$ there exists an x_i -grill J so that $p_Y M \subseteq [f_i(J)]$. Let G be the maximal grill on $A \times X_i$ with $p_A G = p_A M$ and $p_{X_i} G = J$ (its existence is guaranteed by 3.10). Then $G \in \mathcal{A} \times x_i$. By 3.10, $[(1_A \times f_i)G]$ is the maximal grill on $A \times Y$ with $p_A [(1_A \times f_i)G] = [1_A(p_A M)] = p_A M$ and $p_Y [(1_A \times f_i)G] = [f_i(J)]$. Then $p_Y M \subseteq [f_i(J)]$ implies $M \subseteq [(1_A \times f_i)G]$. Otherwise $M \cup [(1_A \times f_i)G]$ strictly contains $[(1_A \times f_i)G]$ and has the same A - and Y -projections, contradicting the maximality of $[(1_A \times f_i)G]$. |||

4.15 implies that Grill is a very well-behaved category: for instance, finite products of quotients are quotients (see 1.17).

H. L. Bentley introduced in [Be] the subcategory Bun of Near: the bunch-determined near spaces. We now study the relationship between Bun and Grill.

4.16 DEFINITIONS [Be] (a) If $(X, x) \in \text{Near}$, a family $\mathcal{B} \subset \mathcal{P}X$ is called an x -bunch on X iff (i) $\mathcal{B}$ is an x -grill and (ii) $c \downarrow_x \mathcal{B} \in \mathcal{B}$ implies $\mathcal{B} \in \mathcal{B}$.
 (b) $(X, x) \in \text{Near}$ is called bunch-determined (and x is called a bunch-determined structure on X) iff for every $\mathcal{A} \in x$, there exists an x -bunch $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$.

(c) The subcategory of Near whose objects are the bunch-determined spaces will be denoted by Bun.

Clearly Bun $\subset$ Grill. In Chapter 7, we will give several examples of grill-determined spaces which are not bunch-determined.

The $\underline{x}$ -bunches on a space $(X, \underline{x}) \in \underline{\text{Bun}}$ cannot be thought of as forming a 'base' for $\underline{x}$ in the sense that the $\underline{x}$ -grills can when $(X, \underline{x}) \in \underline{\text{Grill}}$ (see 4.8 and preceding remark). For the analogue to 4.8 clearly holds, but in order to define an $\underline{x}$ -bunch, the near structure $\underline{x}$ must already be known. If $(X, \underline{x}) \in \underline{\text{Grill}}$, $\underline{x}$ is determined by a family of grills found purely set-theoretically.

However we do have

4.17 LEMMA Let $(X, \underline{x}) \in \underline{\text{Near}}$. If $\underline{G}$ is a grill on X ,

$\underline{G}_b = \{A \in X \mid \text{cl}_{\underline{x}} A \in \underline{G}\}$ is a grill. In particular, if $\underline{G}$ is an $\underline{x}$ -grill, $\underline{G}_b$ is an $\underline{x}$ -bunch.

PROOF. $\text{cl}_{\underline{x}} \phi = \phi$; so $\phi \notin \underline{G}_b$.

$A \cup B \in \underline{G}_b \leftrightarrow \text{cl}_{\underline{x}} (A \cup B) = \text{cl}_{\underline{x}} A \cup \text{cl}_{\underline{x}} B \in \underline{G} \leftrightarrow \text{cl}_{\underline{x}} A \in \underline{G}$ or

$\text{cl}_{\underline{x}} B \in \underline{G} \leftrightarrow A \in \underline{G}_b$ or $B \in \underline{G}_b$.

If $\text{cl}_{\underline{x}} A \in \underline{G}$, $\text{cl}_{\underline{x}} A = \text{cl}_{\underline{x}} (\text{cl}_{\underline{x}} A) \in \underline{G}$, which means $A \in \underline{G}_b$.

If $\underline{G} \in \underline{x}$, $\underline{G}_b \in \underline{x}$ (by (N5)). |||

4.18 THEOREM Grill $\cap$ Near = Bun.

PROOF. ($\subseteq$) Suppose $(X, \underline{x}) \in \underline{\text{Near}}$ is such that $\underline{x}$ is determined by a family $\underline{G}$ of grills. If $\underline{G} \in \underline{G}$, $\underline{G}_b$ is an $\underline{x}$ -bunch by 4.17.

If $A \in \underline{x}$, $A \in \underline{G}_b$ for some $\underline{G} \in \underline{G}$; so $(X, \underline{x}) \in \underline{\text{Bun}}$.

($\supseteq$) Bun $\subseteq$ Near by definition, and since $\underline{x}$ -bunches are $\underline{x}$ -grills, Bun $\subset$ Grill. |||

4.19 COROLLARY Bun is closed under subobjects in SNear.

PROOF. Near is bireflective in SNear, and hence is closed under SNear subobjects. But Grill is also closed under SNear subobjects (4.13). The result follows from the above theorem. |||

4.20 LEMMA (a) If $(X, \underline{x}) \in \text{Grill}$, $(X, \underline{x}_n) \in \text{Bun}$ (Recall from 2.9(b) that $(X, \underline{x}_n)$ is the Near reflection of $(X, \underline{x})$).

(b) If $(X, \underline{x}) \in \text{Near}$, $(X, \underline{x}_g) \in \text{Bun}$.

PROOF. (a) By definition of $\underline{x}_n$ (2.9(b)), $A \in \underline{x}_n$ iff $\text{cl}_{\underline{x}_n} A \in \underline{x}$. Thus there exists an $\underline{x}$ -grill $\underline{G}$ with $\text{cl}_{\underline{x}_n} A \in \underline{G}$. Now $\underline{G}_b = \{A \in X \mid \text{cl}_{\underline{x}_n} A \in \underline{G}\}$ is a grill by 4.17, and since $\text{cl}_{\underline{x}_n} \underline{G}_b \subseteq \underline{G}$, $\underline{G}_b \in \underline{x}_n$ by (N5). $A \in \underline{G}_b$; so $(X, \underline{x}_n) \in \text{Grill}$. By 4.18, $(X, \underline{x}_n) \in \text{Bun}$.

(b) We show $\underline{x}_g$ satisfies (N5):

By 4.12, $\text{cl}_{\underline{x}} = \text{cl}_{\underline{x}_g}$. Then if $\text{cl}_{\underline{x}_g} A \in \underline{x}_g$, $\text{cl}_{\underline{x}} A \in \underline{x}$. That is, $\text{cl}_{\underline{x}} A \in \underline{G}$ for some $\underline{x}$ -grill $\underline{G}$. By 4.17 $\underline{G}_b = \{A \in X \mid \text{cl}_{\underline{x}} A \in \underline{G}\}$ is an $\underline{x}$ -grill, and $A \in \underline{G}_b$. Therefore $A \in \underline{x}_g$.

This means that $(X, \underline{x}_g) \in \text{Near}$, and by 4.18, $(X, \underline{x}_g) \in \text{Bun}$. |||

4.21 THEOREM (a) The category of Near-structurable Grill spaces is precisely Bun. Thus Bun is a bireflective subcategory of Grill. If $(X, \underline{x}) \in \text{Grill}$, the reflection is given by

$$l_X: (X, \underline{x}) \rightarrow (X, \underline{x}_n).$$

(b) The category of Grill-structurable Near spaces is precisely Bun. Thus Bun is a bicoreflective subcategory of Near. If $(X, \underline{x}) \in \text{Near}$, the coreflection is given by

$$l_X: (X, \underline{x}_g) \rightarrow (X, \underline{x}).$$

PROOF. (a) If $(X, \underline{x}) \in \underline{\text{Bun}}$, 4.18 gives $\underline{x}_{ng} = \underline{x}_g = \underline{x}$, which means that $(X, \underline{x})$ is Near-structurable (1.13(a)).

Conversely, suppose $(X, \underline{x}) \in \underline{\text{Grill}}$ is Near-structurable. Then there exists $(X, \underline{y}) \in \underline{\text{Near}}$ with $\underline{y}_g = \underline{x}$. By 4.20(b), $(X, \underline{x}) \in \underline{\text{Bun}}$.

By 1.13(c), Bun is bireflective in Grill, and if $(X, \underline{x}) \in \underline{\text{Grill}}$ the reflection is given by $1_X: (X, \underline{x}) \rightarrow (\tilde{X}, \underline{x}_{ng})$. But by 4.20(a), $\underline{x}_{ng} = \underline{x}_n$.

(b) If $(X, \underline{x}) \in \underline{\text{Bun}}$, 4.18 gives $\underline{x}_{gn} = \underline{x}_n = \underline{x}$; so $(X, \underline{x})$ is Grill-structurable (1.13(b)).

Conversely, if $(X, \underline{x}) \in \underline{\text{Near}}$ is Grill-structurable, there exists $(X, \underline{y}) \in \underline{\text{Grill}}$ with $\underline{y}_n = \underline{x}$. By 4.20(a), $(X, \underline{x}) \in \underline{\text{Bun}}$.

Then Bun is bireflective in Near (1.13(d)), and if $(X, \underline{x}) \in \underline{\text{Near}}$ the coreflection is given by $1_X: (X, \underline{x}_{gn}) \rightarrow (X, \underline{x})$. But by 4.20(b), $\underline{x}_{gn} = \underline{x}_g$. |||

The fact that Bun is bireflective in Near was proved directly in [Be].

5. FILTERS AND NEARNESS

The duality between filters and grills described in Chapter 3 allows us to construct semi-nearness structures from families of filters. Theorem 4.2 becomes

5.1 THEOREM Let F be a family of filters on a set X satisfying the condition

$$(F) \text{ for all } x \in X, \text{ there exists } \underline{F} \in F \text{ with } \underline{F} \subseteq x.$$

Then

$$\psi c(F) = \{ \underline{A} \subseteq PX \mid \underline{A} \subseteq c \underline{F} \text{ for some } \underline{F} \in F \}$$

is a semi-nearness structure (in fact, a grill-determined structure) on X .

Conversely, if x is a semi-nearness structure on X , then the family of filters

$$c\phi(x) = \{ \underline{F} \subseteq PX \mid \underline{F} \text{ is a filter and } c \underline{F} \subseteq x \}$$

satisfies condition (F). |||

Theorem 4.5 becomes

5.2 THEOREM $F = (c\phi)(\psi c)(F)$ (that is, F is the family of all filters $\underline{F}$ so that $c \underline{F}$ is a $\psi c(F)$ -grill) iff the following conditions hold:

$$(F1) \quad x \in \underline{F} \text{ for all } x \in X \text{ and}$$

$$(F2) \quad \text{If } \underline{F} \in F \text{ and } \underline{E} \text{ is a filter with } \underline{F} \subseteq \underline{E}, \text{ then}$$

$$\underline{E} \in F. |||$$

5.3 NOTE. If $(X, x) \in \text{Near}$, then $B \subseteq PX$ is an x -bunch iff $c B \in c\phi(x)$ is a filter with a base consisting of x -open sets (that is, open in the topology determined by $c \ell_x$):

The filter-grill duality can also be used to show the relationship between Grill and the category of filter merotopic spaces, a subcategory of Mer (see [K1] and 2.7).

5.4 DEFINITION [K1] $(X, \underline{m}) \in \underline{\text{Mer}}$ is said to be filter merotopic ($\underline{m}$ is a filter merotopy on X) if there is a fundamental system F for $\underline{m}$ consisting of filters on X . That is, F is a family of filters with $\underline{F} \subseteq \underline{m}$, and if $\underline{n}$ is a merotopy on X with $\underline{F} \subseteq \underline{n}$, then $\underline{m} \subseteq \underline{n}$.

The subcategory of Mer whose objects are the filter merotopic spaces will be called Fil.

It is not difficult to see that if $(X, \underline{m}) \in \underline{\text{Fil}}$, F is a family of filters which is a fundamental system for $\underline{m}$, and F satisfies condition (F) of 5.1, then $\underline{m} = \{ \underline{A} \subseteq \underline{P}X \mid \underline{F} \subseteq \underline{A} \text{ for some } \underline{F} \in F \}$. Further, any fundamental system F can be replaced by $F \cup \{ \underline{x} \mid x \in X \}$; so all filter merotopic spaces can be constructed from families F satisfying (F).

5.5 THEOREM The isomorphism $I: \underline{\text{SNear}} \rightarrow \underline{\text{Mer}}$ induces an isomorphism $J: \underline{\text{Grill}} \rightarrow \underline{\text{Fil}}$.

If $(X, \underline{x}) \in \underline{\text{Grill}}$, $\underline{x}$ is determined by grill structure G , and $(X, \underline{m}) = J(X, \underline{x})$, then $\underline{c}G = \{ \underline{c}G \mid G \in G \}$ is a fundamental system for $\underline{m}$. In particular, $\underline{c}\phi(\underline{x})$ is the family of all filters in $\underline{m}$.

PROOF. Suppose $(X, \underline{x}) \in \underline{\text{Grill}}$ and G determines $\underline{x}$. $I(X, \underline{x}) = (X, \underline{m})$, where $\underline{B} \in \underline{m}$ iff $\text{sec } \underline{B} \subseteq \underline{x}$ (see 2.8). Thus if $\underline{B} \in \underline{m}$, there exists $\underline{G} \in G$ such that $\text{sec } \underline{B} \subseteq \underline{G}$. Let $\underline{F} = \underline{c}G$. Then $\underline{F}$ is a filter on X , and $\text{sec } \underline{F} = \underline{c}\underline{F} = \underline{G}$; so $\underline{F} \in \underline{m}$. But

$\text{sec } \underline{B} \text{ sec } \underline{F}$ implies $\underline{F} < \underline{B}$. That is, $(X, \underline{m}) \in \underline{\text{Fil}}$, with cG as a fundamental system for $\underline{m}$. If $\underline{F}$ is a filter and $\underline{F} \in \underline{m}$, then $cF = \text{sec } \underline{F} \in \underline{m}$. Therefore $\underline{F} \in c\Phi(\underline{m})$.

Conversely, if $(X, \underline{m}) \in \underline{\text{Fil}}$ and $I(X, \underline{x}) = (X, \underline{m})$, then using a similar argument, we find that $\underline{x}$ is determined by

$\{cF | F \text{ is a filter on } X \text{ and } F \in \underline{m}\}$. Hence $(X, \underline{x}) \in \underline{\text{Grill}}$. |||

6. CONVERGENCE SPACES

Filters appear naturally in the theory of topological spaces: we can describe a topological space by giving all filters convergent to each point in the space. Since $\text{TNear} \subset \text{Grill}$ (For $(X, x) \in \text{T-Near}$, the family of point clusters $\underline{C}(x) = \{A \subseteq X \mid x \in \text{cl}_X A\}$, for $x \in X$, determines $\underline{x}$) an obvious question is: for $(X, x) \in \text{T-Near}$, does $\text{cl}(x)$ bear any relationship to the family of convergent filters? The answer is that these families are the same. But we can develop the theory in a more general setting: convergence spaces. These were first considered (beyond the topological space setting) by Choquet in [C] and later pursued by others, particularly Fischer in [F] and Kent in [Ke].

6.1 DEFINITIONS Let X be a set and for each $x \in X$, let $\mathcal{q}_x$

be a family of filters on X (called the filters $\mathcal{q}$ -convergent to x). Then $\mathcal{q} = \{\mathcal{q}_x \mid x \in X\}$.

(a) [Ke]. $(X, \mathcal{q})$ is called a convergence space ($\mathcal{q}$ is a convergence structure on X) iff the following conditions are satisfied:

(A1) $x \in \mathcal{q}_x$ for every $x \in X$,

(A2) If $\underline{F} \in \mathcal{q}_x$ and $\underline{G}$ is a filter with $\underline{F} \subseteq \underline{G}$, then $\underline{G} \in \mathcal{q}_x$,

(C) $\underline{F} \in \mathcal{q}_x$ implies $\underline{F} \cap \mathcal{q}_x$.

(b) [F]. $(X, \mathcal{q})$ is called a limit space ($\mathcal{q}$ is a limit structure on X) iff (A1), (A2) and

(L) $\underline{F}, \underline{G} \in \mathcal{q}_x$ implies $\underline{F} \cap \underline{G} \in \mathcal{q}_x$

are satisfied.

Clearly every limit space is a convergence space.

(c) [C]. (X, q) is called a pseudo-topological space

(q is a pseudo-topological structure on X) iff (A1), (A2) and

(PS) If $\underline{F}$ is a filter, and $\underline{U} \in qx$ for every ultrafilter $\underline{U}$ containing $\underline{F}$, then $\underline{F} \in qx$ are satisfied.

Every pseudo-topological space is a limit space. For if $\underline{F}, \underline{G} \in qx$, then every ultrafilter $\underline{U}$ containing $\underline{F} \cap \underline{G}$ contains either $\underline{F}$ or $\underline{G}$, and so $\underline{U} \in qx$. Therefore condition (PS) implies condition (L).

(d) [Ke]. (X, q) is called a pretopological space (q is a pretopological structure on X) iff (A1), (A2) and

(PR) $\underline{N}(x) = \bigcap \{ \underline{F} \mid \underline{F} \in qx \} \in qx$ for every $x \in X$ are satisfied.

Every pretopological space is pseudo-topological (it is easy to check that condition (PR) implies condition (PS)).

(e) (X, q) is called a topological space (q is a topological structure on X) iff (A1), (A2), (PR) and

(T) for every $\underline{V} \in \underline{N}(x)$ there exists $\underline{W} \in \underline{N}(x)$ so that $\underline{V} \in \underline{N}(y)$ for every $y \in \underline{W}$ are satisfied.

This is just the "convergent filter" approach to defining topological spaces. For a topological space (X, q) as above, the neighbourhoods of a point $x \in X$ are precisely the members of $\underline{N}(x)$,

the neighbourhood filter at x . Conversely, if (X, τ) is a topological space (where τ is the family of all open sets) then for $x \in X$, $N(x) = \{A \subseteq X \mid \exists G \in \tau \text{ with } x \in G\}$ is a filter and if $q_x = \{F \mid F \text{ a filter with } N(x) \subseteq F\}$ (X, q) is a topological space as defined in (e).

6.2 NOTATION. If (X, q) is a convergence space, then for $A \subseteq X$ define

$$\Gamma_q A = \{x \in X \mid A \in F \text{ for some } F \in q_x\}.$$

In general Γ_q need not be the closure operator defining a topology, but if $(X, q) \in \text{Top}$, Γ_q is the closure for topology τ determined by q as in the remark above.

6.3 DEFINITION If (X, q) and (Y, r) are convergence spaces and $f: X \rightarrow Y$ is a function, $f: (X, q) \rightarrow (Y, r)$ is said to be continuous iff for every $x \in X$, $F \in q_x$ implies $[f(F)] \in r_{f(x)}$.

6.4 NOTATION. (a) We denote the category of all convergence spaces and continuous functions by Con.

(b) Denote the subcategory of Con whose objects are the limit spaces Lim.

(c) Denote the subcategory of Con whose objects are the pseudo-topological spaces PST.

(d) Denote the subcategory of Con whose objects are the pretopological spaces PrT.

(e) Denote the subcategory of Con whose objects are the topological spaces Top. Then Top is the usual category of topological spaces.

We have $\text{Top} \subset \text{PrT} \subset \text{PsT} \subset \text{Lim} \subset \text{Con}$.

6.5 THEOREM [Ke] , [F] Lim , PsT , PrT and Top are bireflective subcategories of Con . If $(X, q) \in \text{Con}$

(a) define (X, q_ℓ) by:

$F \in q_\ell x$ iff there exist $F_1, F_2, \dots, F_n \in qx$ so that $\bigcap \{F_i \mid 1 \leq i \leq n\} \subseteq F$.

Then (X, q_ℓ) is the Lim reflection of (X, q) .

(b) Define (X, q_{ps}) by:

$F \in q_{ps} x$ iff $U \in qx$ for every ultrafilter U , $F \subseteq U$.

Then (X, q_{ps}) is the PsT reflection of (X, q) .

(c) Define (X, q_{pr}) by:

$F \in q_{pr} x$ iff $\bigcap \{G \mid G \in qx\} \subseteq F$.

Then (X, q_{pr}) is the PrT reflection of (X, q) .

(d) If $A \subseteq X$, call A q -open if $x \in A$ implies $A \in F$ for each $F \in qx$. Then the family of all q -open sets defines a topology on X .

If we define (X, q_T) by:

$F \in q_T x$ iff $N_x \subseteq F$, where N_x is the filter generated by all q -open sets containing x ,

(X, q_T) is the Top reflection of (X, q) .

PROOF. These results appear in [Ke]. (c) and (d) follow from results in [F]. |||

6.6 THEOREM Con , Lim and PsT are cartesian closed topological categories. Top is topological but not cartesian closed.

PROOF. The first statement is proved in [N]. The second is well-known. |||

6.7 NOTATION. If $(X, q) \in \text{Con}$, let $F(q)$ be the set of all q -convergent filters. We note that q consists of a family of filters, each assigned to a particular element of X ; in $F(q)$, the assignment to points is omitted.

It follows from (A1) and (A2) that $F(q)$ satisfies conditions (F1) and (F2) of 5.2, and so determines a semi-near structure $\psi_c(F(q)) = q$ on X (in fact, $(X, q) \in \text{Grill}$). However, the information about point-assignment has been lost.

6.8 EXAMPLE Let $X = \{x, y, z\}$. Define q and r on X by:

$$q_x = q_z = \{F \mid F \text{ a filter on } X\}; \quad q_y = \{\dot{y}\}, \quad r_x = \{\dot{x}\}, \\ r_z = \{\dot{z}\}, \quad r_y = \{F \mid F \text{ a filter on } X\}.$$

Then $(X, q), (X, r) \in \text{Top}$ and (X, q) is not isomorphic to (X, r) . But $F(q) = F(r)$; so (X, q) is isomorphic in Grill to (X, r) —both have the indiscrete near structure.

This is a problem similar to the one in topological spaces (see 2.12): in general, convergence spaces have a lacking in symmetry which prevents us from considering them as spaces defined by a notion of "nearness". So we introduce for Con an analogue to the R_0 axiom for topological spaces. As we shall see, our analogue is equivalent to the R_0 axiom for the subcategory Top of Con; thus we will use the term R_0 in this wider sense (notice that in example 6.6 (X, q) and (X, r) considered as topological spaces do not satisfy the R_0 axiom).

6.9 DEFINITION $(X, q) \in \text{Con}$ is said to be R_0 (an R_0 convergence space) iff for a filter F on X the condition

(R_0) $\underline{Fnx} \in F(q)$ implies $\underline{F}eqx$ is satisfied.

The subcategory of Con whose objects are the R_0 convergence spaces will be denoted $R_0\text{Con}$.

Notice that if $(X, q) \in R_0\text{Con}$, then once $F(q)$ is known, q is determined uniquely.

6.10 THEOREM (a) If $(X, q) \in \underline{\text{Lim}}$ the condition (R_0) is equivalent to

$\dot{x}eqy$ implies $qx = qy$.

(b) If $(X, q) \in \underline{\text{Top}}$, the condition (R_0) is equivalent to

$x \in \Gamma_q(y)$ implies $y \in \Gamma_q(x)$,

the usual R_0 axiom for topological spaces.

PROOF. (a) Suppose (R_0) holds and $\dot{x}eqy$. If $\underline{F}eqy$, $\underline{Fnx}eqy$ by (L), which means $\underline{F}eqx$, and $qy \leq qx$. In particular, $y \in qx$ which implies, as above, $qx \leq qy$.

Conversely, suppose $\dot{x}eqy$ implies $qx = qy$. If $\underline{Fnx} \in F(q)$, $\underline{Fnx}eqy$, some $y \in X$. Since $\underline{Fnx} \leq x$, $\dot{x}eqy$ by (A2), and $qx = qy$. So $\underline{Fnx}eqx$, and by (A2), $\underline{F}eqx$.

(b) Suppose (R_0) holds, and $x \in \Gamma_q(y)$. Then $\{y\} \in F$, some $\underline{F}eqx$; so $y \in qx$. By part (a) of this theorem, $qx = qy$. This gives $y \in \Gamma_q(x)$.

Conversely, suppose the usual R_0 axiom holds (see 2.12). $\dot{x}eqy$ implies $x \in N$ for every neighbourhood N of y , and thus $y \in \Gamma_q(x)$. But then $x \in \Gamma_q(y)$, or $y \in N$ for every neighbourhood N of x . Hence $\underline{N}(x) = \underline{N}(y)$, and by definition of q , $qx = qy$. By part (a), (R_0) holds. |||

6.11 NOTES. $(X, q) \in \underline{\text{Con}}$ is said to be a T_1 -space iff.

(T1) $x \in qy$ implies $x = y$

is satisfied (This definition, for limit spaces, is found in [F]).

(T1) is equivalent to: $\Gamma_q\{x\} = \{x\}$ for all $x \in X$.

We denote by $T_1\underline{\text{Con}}$ the subcategory of $\underline{\text{Con}}$ whose objects are the T_1 -spaces. It is easy to check that $T_1\underline{\text{Con}} \subset R_0\underline{\text{Con}}$.

In fact, $R_0\underline{\text{Con}}$ includes most of the convergence spaces studied to date. Most work has been done in $\underline{\text{Lim}}$, which seems to be the natural setting for many problems. In [CF], Cook and Fischer defined uniform convergence structures on limit spaces, the analogue to uniformities on topological spaces. A limit space need not be T_1 in order to have a uniform convergence structure, but it is not difficult to show that it must be R_0 .

Also, in order that a limit space have a compatible group structure as defined in [F], it must be R_0 .

Limit spaces with compatible vector space structure are used in an important way in [FB] (they are called pseudo-topological vector spaces by the authors; in fact the language of the subject is very unsettled: the term "convergence space" is often used to mean what we call limit space). These "limit vector spaces" appear to be the natural structures within which to develop a calculus for vector spaces without norm. Topological spaces have been found to be badly behaved from this point of view. Limit vector spaces are always R_0 spaces, since the vector space structure is a group structure.

6.12 EXAMPLE Theorem 6.10(b) suggests that it might be possible to express our generalized R_0 condition in terms of the operator

Γ_q as $x \in \Gamma_q\{y\}$ implies $y \in \Gamma_q\{x\}$. This is not the case. The R_0 condition is readily seen to imply this condition, but:

Let X be any infinite set. Let $I = \{A \subseteq X \mid X \setminus A \text{ is finite}\}$. Choose $a, b \in X$ and define q by: $qa = \{\underline{F} \mid \underline{F} \text{ a filter, } a \cap b \subseteq \underline{F}\}$; $qb = \{\underline{F} \mid I \cap a \cap b \subseteq \underline{F}\}$; $qx = \{\dot{x}\}$ for $x \neq a, b$. Then $(X, q) \in \text{PrT}$. A straightforward verification shows $x \in \Gamma_q\{y\}$ implies $y \in \Gamma_q\{x\}$ for any $x, y \in X$. But $(I \cap b) \cap a \notin qb$, $I \cap b \notin qa$; so $(X, q) \notin R_0 \text{Con}$.

If we have as an aim the constructing of semi-nearness structures to define convergence spaces, it would not be better to take $x \in \Gamma_q\{y\}$ implies $y \in \Gamma_q\{x\}$ as the symmetry axiom. For returning to the above example, define s on X by:

$sa = sb = qb$; $sx = qx$ for $x \neq a, b$. Then $(X, s) \in \text{PrT}$. (X, s) and (X, q) are not isomorphic in Con , but $F(s) = F(q)$; that is, they determine the same semi-nearness structure (see also 6.7).

6.13 THEOREM $R_0 \text{Con}$ is a bireflective subcategory of Con .

PROOF. If $(X, q) \in \text{Con}$, define q_R on X by:

Filter $\underline{F} \in q_R x$ iff $\underline{F} \cap x \in F(q)$ (that is, $\underline{F} \cap x \in qy$, some $y \in X$).

It is easy to verify that $(X, q_R) \in R_0 \text{Con}$. We claim that the reflection is given by $l_X: (X, q) \rightarrow (X, q_R)$. l_X is continuous, since $\underline{F} \in qx$ implies $\underline{F} \cap x \in qx$. Let $(Y, r) \in R_0 \text{Con}$ and $g: (X, q) \rightarrow (Y, r)$ be continuous. If $\underline{F} \in q_R x$, $\underline{F} \cap x \in qz$, some $z \in X$. By continuity,

$$[g(\underline{F})] \cap \dot{g(x)} = [g(\underline{F} \cap \dot{x})] \in \chi g(z) .$$

By (R_0) , $[g(\underline{F})] \in \chi g(x)$; hence

$g: (X, q_R) \rightarrow (Y, r)$ is continuous. |||

6.14 COROLLARY $R_0 \underline{\text{Lim}} = R_0 \underline{\text{Con}} \cap \underline{\text{Lim}}$, $R_0 \underline{\text{PsT}} = R_0 \underline{\text{Con}} \cap \underline{\text{PsT}}$,

$R_0 \underline{\text{PrT}} = R_0 \underline{\text{Con}} \cap \underline{\text{PrT}}$ and $R_0 \underline{\text{Top}} = R_0 \underline{\text{Con}} \cap \underline{\text{Top}}$ are all bireflective subcategories of $\underline{\text{Con}}$, and hence of $R_0 \underline{\text{Con}}$. |||

6.15 COROLLARY $R_0 \underline{\text{Con}}$, $R_0 \underline{\text{Lim}}$, $R_0 \underline{\text{PsT}}$, $R_0 \underline{\text{PrT}}$ and $R_0 \underline{\text{Top}}$ are topological categories. |||

The fact that $R_0 \underline{\text{Top}}$ is topological is well-known (see 6.10(b), 2.14, 2.15, 2.10 and 1.11).

7. CONVERGENCE AS A NEARNESS CONCEPT

As was indicated in chapter 6,

7.1 THEOREM $R_{\mathcal{O}}\text{Con}$ is isomorphic to a subcategory of Grill .

PROOF. If $(X, \mathcal{q}) \in R_{\mathcal{O}}\text{Con}$, $F(\mathcal{q})$, (the family of all $\mathcal{q}$ -convergent filters) determines $\mathcal{q}$ via: $\underline{F} \in \mathcal{q}x$ iff $F \cap \underline{x} \in F(\mathcal{q})$. As noted in 6.7, if $\underline{q} = \psi c(F(\mathcal{q}))$, $(X, \underline{q}) \in \text{Grill}$ ($\underline{q}$ determined by $cF(\mathcal{q})$). Define $F: R_{\mathcal{O}}\text{Con} \rightarrow \text{Grill}$ by $F(X, \mathcal{q}) = (X, \underline{q})$, and $F(f) = f$ for continuous functions f . By the above remarks, it suffices to show that for every continuous function f , $F(f)$ is a near map, and that every near map $f: F(X, \mathcal{q}) \rightarrow F(Y, \mathcal{r})$ is a continuous map $f: (X, \mathcal{q}) \rightarrow (Y, \mathcal{r})$.

Suppose $f: (X, \mathcal{q}) \rightarrow (Y, \mathcal{r})$ is continuous. $\underline{A} \in \mathcal{q}$ implies $\underline{A} \subseteq c\underline{F}$ for some $\underline{F} \in F(\mathcal{q})$; so $\underline{F} \in \mathcal{q}x$, some $x \in X$. Then $[f\underline{F}] \in \mathcal{r}f(x)$, which means that $[f(c\underline{F})] = c[f(\underline{F})] \in \mathcal{r}$. $f(\underline{A}) \in [f(c\underline{F})]$ implies $f(\underline{A}) \in \mathcal{r}$.

Conversely, if $f: F(X, \mathcal{q}) \rightarrow F(Y, \mathcal{r})$ is a near map: $\underline{F} \in \mathcal{q}x$ implies $\underline{F} \cap \underline{x} \in \mathcal{q}$, and $c(\underline{F} \cap \underline{x}) \in \underline{q}$. f near means $fc(\underline{F} \cap \underline{x}) \in \mathcal{r}$. By (N1) $c[f(\underline{F} \cap \underline{x})] = [fc(\underline{F} \cap \underline{x})] \in \mathcal{r}$. Hence $[f\underline{F}] \cap f(\underline{x}) = [f(\underline{F} \cap \underline{x})] \in \mathcal{r}$. By $(R_{\mathcal{O}})$, $[f\underline{F}] \in \mathcal{r}f(x)$. |||

7.2 NOTATION. The subcategory $F(R_{\mathcal{O}}\text{Con})$ of Grill will also be denoted by $R_{\mathcal{O}}\text{Con}$. Which category is meant should be clear from notation and context.

7.3 NOTE. If $(X, \mathcal{q}) \in R_{\mathcal{O}}\text{Con}$, then the operators $\Gamma_{\mathcal{q}}$ and $c\mathcal{q}$ are the same. For if $\underline{A} \subseteq X$, $x \in \Gamma_{\mathcal{q}} \underline{A}$ means there exists $\underline{F} \in \mathcal{q}x$, $\underline{A} \in \underline{F}$.

Then $X \setminus \{x\}, X \setminus \bigcup_{q \in \mathcal{Q}} F_q$. Thus $\{x\}, \bigcup_{q \in \mathcal{Q}} F_q \in \mathcal{Q}$, and $x \in \bigcup_{q \in \mathcal{Q}} A_q$.

Conversely, $x \in \bigcup_{q \in \mathcal{Q}} A_q$ implies $\{\{x\}, A\} \in \mathcal{Q}$; so $\{x\}, \bigcup_{q \in \mathcal{Q}} F_q \in \mathcal{Q}$, some $F \in \mathcal{Q}$. $X \setminus \{x\} \not\subseteq F$ means that $F = F_q$, and by (R_0) , $F_q \in \mathcal{Q}$. Since $X \setminus \{x\} \not\subseteq F$, there exists a filter E with $F \subseteq E$, $A \in E$. By (A2), $E \in \mathcal{Q}$; so $x \in \bigcup_{q \in \mathcal{Q}} A_q$.

Therefore the T_1 condition given in [F] (see 6.11) is equivalent to the one given in [H1] for nearness spaces $(X, \mathcal{Q})$:

$$(T_1) \quad \text{cl}_{\mathcal{Q}} \{x\} = \{x\} \text{ for all } x \in X.$$

7.4 THEOREM If $(X, \mathcal{Q}) \in \text{Grill}$, then $(X, \mathcal{Q}) \in R_{\text{Con}}$ iff for every $A \in \mathcal{Q}$, there exists $x \in X$ so that $A \cup x \in \mathcal{Q}$.

PROOF. $(\Rightarrow)$ If $A \in \mathcal{Q}$, $A \subseteq F$ for some $F \in \mathcal{Q}$ (where $(X, \mathcal{Q}) = F(X, \mathcal{Q})$). Then $F \in \mathcal{Q}$, some $x \in X$. By (C), $F \cap x \in \mathcal{Q}$; hence $cF \cup x = c(F \cap x) \in cF \cap \mathcal{Q}$. $A \cup x \subseteq cF \cup x$ implies $A \cup x \in \mathcal{Q}$.

$(\Leftarrow)$ Define $\mathcal{Q}$ on X by:

$$F \in \mathcal{Q} \text{ iff } F \cap x \in \mathcal{Q} \text{ for some } x \in X.$$

Clearly (A1) and (A2) are satisfied, since $c\mathcal{Q}$ satisfies (F1) and (F2). Conditions (C) and (R_0) are also immediate. Therefore, $(X, \mathcal{Q}) \in R_{\text{Con}} (= \text{Con})$.

Now if $A \in \mathcal{Q}$, $A \subseteq F$, some $F \in \mathcal{Q}$. Then for some $x \in X$, $c(F \cap x) = cF \cap x \in \mathcal{Q}$; so $F \cap x \in \mathcal{Q}$, and $F \in \mathcal{Q}$. Thus $x \in \bigcup_{q \in \mathcal{Q}} (F \cap q)$.

Conversely, if $A \in \mathcal{Q}$, $A \subseteq F$ for some filter F such that $F \cap x \in \mathcal{Q}$, some $x \in X$. $F \cap x \subseteq F$ implies $F \in \mathcal{Q}$, and thus $cF \in \mathcal{Q}$. So we have $A \in \mathcal{Q}$. By the definition of F in 7.1, $F(X, \mathcal{Q}) = (X, \mathcal{Q})$. |||

7.5 NOTE. If $(X, \mathcal{Q}) \in R_{\text{Con}}$, then for each $A \in \mathcal{Q}$, $\bigcap_{x \in X} A \neq \emptyset$.

As will be seen later, $\bigcap_{x \in A} A \neq \emptyset$ does not imply $A \in \mathcal{X}$, and the condition ' $A \in \mathcal{X}$ implies $\bigcap_{x \in A} A \neq \emptyset$ ' does not characterize $R_{\mathcal{O}} \text{Con}$.

7.6 DEFINITION [K1] A filter-merotopic space $(X, \underline{m})$ is said to be localized if for any filter $\underline{F} \in \underline{m}$ there exists $x \in X$ such that $\{F_U(x) \mid F \in \underline{F}\} \in \underline{m}$.

The subcategory of Fil whose objects are the localized filter spaces will be denoted Loc.

7.7 THEOREM The isomorphism $I : \underline{SNear} \rightarrow \underline{Mer}$ induces an isomorphism $K : R_{\mathcal{O}} \text{Con} \rightarrow \underline{Loc}$.

PROOF. If $(X, \underline{x}) \in R_{\mathcal{O}} \text{Con}$, let $(X, \underline{m}) = I(X, \underline{x})$. By 5.5 $(X, \underline{m}) \in \underline{Fil}$, and if $\underline{F} \in \underline{m}$ is a filter, $\underline{F} \in \mathcal{C}(\underline{x})$. $\underline{cF} \in \underline{x}$ implies $\underline{cF}_U \in \underline{x}$, some $x \in X$; so $\underline{F}_U x = \underline{c}(\underline{cF}_U) \in \underline{m}$. But $\underline{F}_U x = [\{F_U(x) \mid F \in \underline{F}\}]$, and $(X, \underline{m}) \in \underline{Loc}$.

Conversely, if $(X, \underline{m}) \in \underline{Loc}$ with $I(X, \underline{x}) = (X, \underline{m})$, a similar argument shows $(X, \underline{x}) \in R_{\mathcal{O}} \text{Con}$.

In [K2] it is shown that the study of Loc is equivalent to the study of a family of categories of convergence structures defined in terms of nets (rather than filters). A family of categories is introduced because of the need to define admissible classes of directed sets. The axioms of convergence are consequently rather complicated.

7.8 THEOREM $R_{\mathcal{O}} \text{Con}$ is a bicoreflective subcategory of Grill, and hence of SNear. If $(X, \underline{x}) \in \underline{Grill}$, the coreflection is given by $1_X : (X, \underline{x}_c) \rightarrow (X, \underline{x})$, where

$\underline{A} \in \underline{x}_c$ iff $\underline{A} \cup \underline{x} \in \underline{x}$ for some $\underline{x} \in X$.

If $(X, \underline{x}) \in \underline{SNear}$, the coreflection is given by

$1_X: (X, \underline{x}_{gc}) \rightarrow (X, \underline{x})$. $\underline{x}_{gc}$ will be shortened to $\underline{x}_c$. So

$\underline{A} \in \underline{x}_c$ iff $\underline{A} \subseteq G$ for some $\underline{x}$ -grill G such that $\underline{G} \cup \underline{x} \in \underline{x}$, some $\underline{x} \in X$.

PROOF. It is easy to see that if $(X, \underline{x}) \in \underline{Grill}$, $(X, \underline{x}_c) \in \underline{R}_{\underline{OCon}}$, and that $1_X: (X, \underline{x}_c) \rightarrow (X, \underline{x})$ is a near map. Suppose $(Y, \underline{y}) \in \underline{R}_{\underline{OCon}}$,

$f(\underline{B}) \cup f(\underline{y}) \subseteq [f(\underline{B} \cup \underline{y})] \in \underline{x}$ (by nearness of f and (N1)); so by definition of $\underline{x}_c$, $f(\underline{B}) \in \underline{x}_c$. Hence $f: (Y, \underline{y}) \rightarrow (X, \underline{x}_c)$ is a near map. |||

7.9 NOTE. If $(X, \underline{x}) \in \underline{Grill}$, $cl_{\underline{x}} = cl_{\underline{x}_c}$: If $\underline{A} \in \underline{x}$, $cl_{\underline{x}_c} \underline{A} \subseteq cl_{\underline{x}} \underline{A}$, since $\underline{x}_c \subseteq \underline{x}$. Conversely, if $\underline{x} \in cl_{\underline{x}_c} \underline{A}$, $\{\{\underline{x}\}, \underline{A}\} \in \underline{x}$ implies $\{\{\underline{x}\}, \underline{A}\} \cup \underline{x} \in \underline{x}$ by (N1), and so $\{\{\underline{x}\}, \underline{A}\} \in \underline{x}_c$, or $\underline{x} \in cl_{\underline{x}_c} \underline{A}$.

7.10 NOTE. The fact that $\underline{R}_{\underline{OCon}}$ is a topological category was proved in 6.15 (It would also follow as a corollary to 7.8). Hence $\underline{R}_{\underline{OCon}}$ is closed under final structures in $\underline{SNear}$.

Initial structures must be discussed separately. $\underline{R}_{\underline{OCon}}$ is not closed under subobjects in $\underline{Grill}$. This will be seen in the results of chapter 8.

We do have

7.11 THEOREM $\underline{R}_{\underline{OCon}}$ is closed under products in $\underline{Grill}$.

PROOF. Let I be an index set, and $(X_i, \underline{x}_i) \in R_{\underline{O} \text{Con}}$ for all $i \in I$. Then the product in Grill is $((\prod X_i, \pi_{\underline{x}_i}) \xrightarrow{p_k} (X_k, \underline{x}_k))_{k \in I}$, where $\underline{A} \in \pi_{\underline{x}_i}$ iff there exist a grill $\underline{G}$ on $\prod X_i$ with $\underline{A} \subseteq \underline{G}$ and $p_i \underline{G} \in \pi_{\underline{x}_i}$ for all $i \in I$ (4.14). Since $(X_i, \underline{x}_i) \in R_{\underline{O} \text{Con}}$, $p_i \underline{G} \in \pi_{\underline{x}_i}$ for some $\underline{y}_i \in X_i$, every $i \in I$. Let $\underline{y} = (\underline{y}_i)_{i \in I} \in \prod X_i$. Then $p_i(\underline{G} \underline{y}) = p_i \underline{G} \in \pi_{\underline{x}_i}$ for every $i \in I$. That means $\underline{G} \underline{y} \in \pi_{\underline{x}_i}$; so $\underline{A} \underline{y} \in \pi_{\underline{x}_i}$. By 7.4, $(\prod X_i, \pi_{\underline{x}_i}) \in R_{\underline{O} \text{Con}}. |||$

7.12 THEOREM $R_{\underline{O} \text{Con}}$ is cartesian closed.

PROOF. This follows as an immediate consequence of the facts that Grill is cartesian closed, $R_{\underline{O} \text{Con}}$ is bicoreflective in Grill, and the above theorem (see 1.18). |||

7.13 NOTATION. We will also denote the subcategory $F(R_{\underline{O} \text{Lim}})$ ($F: R_{\underline{O} \text{Con}} \rightarrow \text{Grill}$ as defined in 7.1) of $R_{\underline{O} \text{Con}}$ by $R_{\underline{O} \text{Lim}}$.

As indicated in 6.11, $R_{\underline{O} \text{Lim}}$ is the best setting for many problems which need more general structures than topological spaces have.

7.14 THEOREM Let $(X, \underline{x}) \in \text{Grill}$. $(X, \underline{x}) \in R_{\underline{O} \text{Lim}}$ iff the following condition holds:

$\underline{A} \cup \underline{B} \in \underline{x}$ iff there exists $\underline{x} \in X$ such that $\underline{A} \underline{x} \in \underline{x}$ and $\underline{B} \underline{x} \in \underline{x}$.

PROOF. ($\Rightarrow$) If $\underline{A} \cup \underline{B} \in \underline{x}$, there exists $\underline{x} \in X$, $\underline{A} \underline{x} \in \underline{x}$ (by 7.4).

(N1) gives $\underline{A} \underline{x}$ and $\underline{B} \underline{x} \in \underline{x}$.

Conversely, if $\underline{A} \underline{x}$ and $\underline{B} \underline{x} \in \underline{x}$ and $(X, \underline{x}) = F(X, \underline{g})$, then $\underline{A} \underline{x} \subseteq \underline{F}$, $\underline{B} \underline{x} \subseteq \underline{G}$, where $\underline{F}, \underline{G} \in F(\underline{g})$. So $\underline{F} = \underline{F} \underline{x}$ and $\underline{G} = \underline{G} \underline{x}$.

By (R_0) $\underline{F}, \underline{G} \in \underline{q}x$. By (L), $\underline{F} \cap \underline{G} \in \underline{q}x$. $\underline{A} \cup \underline{B} \in c\underline{F} \cup c\underline{G} = c(\underline{F} \cap \underline{G})$ implies $\underline{A} \cup \underline{B} \in x$.

($\leftarrow$) $\underline{A} = \underline{A} \cup \underline{\phi} \in x$ implies that $\underline{A} \cup x \in x$, some $x \in X$. By 7.4, $(X, x) \in R_0 \text{Con}$. Suppose $(X, x) = F(X, q)$. We must show (X, q) satisfies condition (L). If $\underline{F}, \underline{G} \in \underline{q}x$, $\underline{F} \cap x$ and $\underline{G} \cap x \in \underline{q}x$; so $c\underline{F} \cap x = c(\underline{F} \cap x) \in x$ and $c\underline{G} \cap x \in x$. Setting $\underline{A} = c\underline{F} \cap x$, $\underline{B} = c\underline{G} \cap x$ in the above condition we get $c(\underline{F} \cap \underline{G}) \cap x = c\underline{F} \cap c\underline{G} \cap x$. Then $\underline{F} \cap \underline{G} \in F(q)$ and by (R_0) , $\underline{F} \cap \underline{G} \in \underline{q}x$. |||

7.15 EXAMPLE $R_0 \text{Lim} \neq R_0 \text{Con}$.

Let R be the real numbers, and $r \in P^2 R$ defined by:

$\underline{A} \in r$ iff there exists $x \in R$, $d(x, A) < 1$ for every $A \in \underline{A}$ (d is the usual metric).

$(R, r) \in \text{Grill}$: r is determined by the grills

$\underline{M}(x) = \{A \mid d(x, A) < 1\}$, where $x \in R$.

$(R, r) \in R_0 \text{Con}$: If $\underline{A} \in r$, $\underline{A} \subseteq \underline{M}(x)$, some $x \in R$, and $\underline{A} \cup x \in r$.

$(R, r) \notin R_0 \text{Lim}$: Let $\underline{A} = \{\{0\}\}$, $\underline{B} = \{\{4\}\}$. Then $\underline{A} \cup 2 \in r$ and $\underline{B} \cup 2 \in r$, but $\underline{A} \cup \underline{B} \notin r$.

7.16 NOTE. We know that $R_0 \text{Lim}$ is a bireflective subcategory of $R_0 \text{Con}$, but do not have the reflection explicitly. We have the construction of the Lim reflection of spaces in Con (6.5(a)). Unfortunately, for $(X, x) \in R_0 \text{Con}$ it does not suffice to translate this construction via 7.1 in order to get the $R_0 \text{Lim}$ reflection: if $(X, q) \in R_0 \text{Con}$, (X, q_ℓ) need not be in $R_0 \text{Lim}$. In fact, $(X, q_{\ell R})$ (see 6.13) need not be in $R_0 \text{Lim}$. For consider (R, r) from 7.15. If $(R, r) = F(R, x)$, then filter $\underline{F} \in r$ iff $c(\underline{F} \cap x) = c\underline{F} \cap x$. That is, $c\underline{F} \in \underline{M}(x)$, some

$y \in R$ with $d(x, y) \leq 1$. Equivalently, $\underline{F}$ contains

$cM(y) = \{B \subseteq R \mid R \setminus B \notin M(y)\} = \{B \mid d(y, R \setminus B) > 1\} = \{B \mid B \text{ is a neighbourhood of } [y-1, y+1]\}$ (neighbourhood in the usual topology on R).

So $\underline{F} \in r_x$ iff $\underline{F}$ contains every neighbourhood of $[y-1, y+1]$,

some y such that $x \in [y-1, y+1]$. By 6.5(a), $\underline{F} \in r_{\ell} x$ iff there

exist $\underline{F}_1, \dots, \underline{F}_n \in r_x$, $\bigcap \{\underline{F}_i \mid 1 \leq i \leq n\} \subseteq \underline{F}$. This may be seen to be the

condition: $\underline{F} \in r_{\ell} x$ iff $\underline{F}$ contains every neighbourhood of $[x-2, x+2]$.

Comparing with the definition of r_{ℓ} we conclude $(R, r_{\ell}) \notin R_{\text{Lim}}$ (since $(R, r) \notin R_{\text{Lim}}$).

Further, $(R, r_{\ell R}) \notin \text{Lim}$: $\underline{F} \in r_{\ell R} x$ iff $\underline{F} \cap x \in r_{\ell} y$, some

$y \in R$. Let $\underline{F} = \{A \mid A \text{ is a neighbourhood of } [-4, 0]\}$, $\underline{G} = \{B \mid B \text{ is a neighbourhood of } [0, 4]\}$. Then $\underline{F} = \underline{F} \cap 0 \in r_{\ell R} 0$, $\underline{G} \in r_{\ell R} 0$, but

$\underline{F} \cap \underline{G} = \{A \mid A \text{ is a neighbourhood of } [-4, 4]\} \notin r_{\ell R} 0$.

In order to give the R_{Lim} reflection explicitly, we need

7.17 DEFINITION For $(X, \underline{x}) \in \text{SNear}$ define a binary relation $C(\underline{x})$ on X by

$x C(\underline{x}) y$ iff there is a chain $x = z_0, z_1, \dots, z_n = y$ of elements of X such that $\{\{z_i\}, \{z_{i+1}\}\} \in \underline{x}$, $0 \leq i < n$.

It is easily seen that $C(\underline{x})$ is an equivalence relation.

Denote the equivalence class containing $y \in X$ by $\circ C(y)$.

7.18 DEFINITION For $(X, \underline{x}) \in R_{\text{Con}}$ define $\underline{x}_L \subseteq \mathcal{P}^2 X$ by $\underline{A} \in \underline{x}_L$ iff there exist $\underline{A}_1, \dots, \underline{A}_n \subseteq \mathcal{P} X$ and $x_1, \dots, x_n \in X$ such that $C(x_1) = C(x_2) = \dots = C(x_n)$, $\underline{A}_i \cup x_i \in \underline{x}$, $1 \leq i \leq n$, and $\underline{A} \subseteq \bigcup \{\underline{A}_i \mid 1 \leq i \leq n\}$.

7.19 THEOREM If $(X, \underline{x}) \in R_{\text{Con}}$, then $(X, \underline{x}_L) \in R_{\text{Lim}}$ and is the R_{Lim} reflection of $(X, \underline{x})$.

PROOF. $(X, \underline{x}_L) \in \text{Grill}$: Suppose $\underline{A} \in \underline{x}_L$. Then $\underline{A} \subseteq \bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \}$, $\underline{A}_i$ as in 7.18. Then $\underline{A}_i \cup \underline{x}_i \subseteq \underline{G}_i$ for some $\underline{x}$ -grill $\underline{G}_i$, $1 \leq i \leq n$. $\underline{G}_i = \underline{G}_i \cup \underline{x}_i \in \underline{x}$ implies $\bigcup \{ \underline{G}_i \mid 1 \leq i \leq n \} \in \underline{x}_L$. $\bigcup \{ \underline{G}_i \mid 1 \leq i \leq n \}$ is a grill, and $\underline{A} \subseteq \bigcup \{ \underline{G}_i \mid 1 \leq i \leq n \}$. Clearly $\underline{x} \in \underline{x}_L$ for every $\underline{x} \in X$; so by 4.2, $(X, \underline{x}_L) \in \text{Grill}$.

$(X, \underline{x}_L) \in R_{\text{O-Lim}}$: If $\underline{A} \cup \underline{B} \in \underline{x}_L$, $\underline{A} \subseteq \bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \}$, where $\underline{A}_i \cup \underline{x}_i \in \underline{x}$, some $\underline{x}_i$ ($1 \leq i \leq n$) so that $C(\underline{x}_1) = \dots = C(\underline{x}_n)$. Then $\underline{A} \cup \underline{B} \subseteq \bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \} \cup \underline{x}_1$, which satisfies the conditions of 7.18; so $\underline{A} \cup \underline{B} \in \underline{x}_L$. By (N1), $\underline{A} \cup \underline{x}_1, \underline{B} \cup \underline{x}_1 \in \underline{x}_L$.

Conversely, suppose $\underline{A} \cup \underline{x}, \underline{B} \cup \underline{x} \in \underline{x}_L$. Then $\underline{A} \cup \underline{x} \subseteq \bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \}$ where $\underline{A}_i \cup \underline{x}_i \in \underline{x}$, $C(\underline{x}_1) = \dots = C(\underline{x}_n)$; and $\underline{B} \cup \underline{x} \subseteq \bigcup \{ \underline{B}_j \mid 1 \leq j \leq m \}$ where $\underline{B}_j \cup \underline{y}_j \in \underline{x}$, $C(\underline{y}_1) = \dots = C(\underline{y}_m)$. Now $\{ \underline{x} \} \in \underline{A}_k$, some k implies $\{ \{ \underline{x} \}, \{ \underline{x}_k \} \} \in \underline{x}$; and $\{ \underline{x} \} \in \underline{B}_h$, some h implies $\{ \{ \underline{x} \}, \{ \underline{y}_h \} \} \in \underline{x}$. Therefore $C(\underline{x}_k) = C(\underline{x}) = C(\underline{y}_h)$. $\underline{A} \cup \underline{B} \subseteq (\bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \}) \cup (\bigcup \{ \underline{B}_j \mid 1 \leq j \leq m \})$, and $C(\underline{x}_1) = \dots = C(\underline{x}_n) = C(\underline{y}_1) = \dots = C(\underline{y}_m)$ gives $\underline{A} \cup \underline{B} \in \underline{x}_L$.

$1_X : (X, \underline{x}) \rightarrow (X, \underline{x}_L)$ is a near map: If $\underline{A} \in \underline{x}$, $\underline{A} \cup \underline{x} \in \underline{x}$, some $\underline{x} \in X$ (7.4); so $\underline{A} \in \underline{x}_L$.

The reflection property holds: Suppose $(Y, \underline{y}) \in R_{\text{O-Lim}}$ and

$f : (X, \underline{x}) \rightarrow (Y, \underline{y})$ is a near map.

If $\underline{A} \in \underline{x}_L$, $\underline{A} \subseteq \bigcup \{ \underline{A}_i \mid 1 \leq i \leq n \}$, $\underline{A}_i \cup \underline{x}_i \in \underline{x}$

for some $\underline{x}_i$ with $C(\underline{x}_1) = \dots = C(\underline{x}_n)$.

Since f is a near map, $f(\underline{A}_i \cup \underline{x}_i) \in \underline{y}$, and by (N1) $f(\underline{A}_i) \cup f(\underline{x}_i) \in \underline{y}$ for $1 \leq i \leq n$.

$C(\underline{x}_1) = C(\underline{x}_2)$ means that for some chain $\underline{x}_1 = \underline{z}_0, \dots, \underline{z}_m = \underline{x}_2$ in X , $\{ \{ \underline{z}_j \}, \{ \underline{z}_{j+1} \} \} \in \underline{x}$ for $0 \leq j < m$. $\{ \{ f(\underline{z}_j) \}, \{ f(\underline{z}_{j+1}) \} \} \in \underline{y}$, and so $f(\underline{z}_j) \cup f(\underline{z}_{j+1}) \in \underline{y}$, $0 \leq j < m$. We have $f(\underline{A}_1) \cup f(\underline{z}_0) \in \underline{y}$.

$f(\widehat{z_0}) \cup f(\widehat{z_1}) \in \underline{Y}$, and since $(Y, \underline{y}) \in R_{\underline{O}} \underline{Lim}$, $f(\underline{A_1}) \cup f(\widehat{z_1}) \in \underline{Y}$ (7.14).

Proceeding inductively we finally get $f(\underline{A_1}) \cup f(\widehat{x_2}) \in \underline{Y}$. Repeating

the argument, $f(\underline{A_1}) \cup f(\widehat{x_n}) \in \underline{Y}$, and in fact $f(\underline{A_1}) \cup f(\widehat{x_n}) \in \underline{Y}$,

$1 \leq i \leq n$. By 7.14, $\cup \{f(\underline{A_i}) \mid 1 \leq i \leq n\} \in \underline{Y}$. But $f(\underline{A}) \subseteq \cup \{f(\underline{A_i}) \mid 1 \leq i \leq n\}$ gives

$f(\underline{A}) \in \underline{Y}$. |||

7.20 NOTE. If $(X, \underline{x}) \in R_{\underline{O}} \underline{Con}$, $C(x) = \{x\}$ for every $x \in X$ iff

$(X, \underline{x})$ is a T_1 space (see 7.3). So for T_1 space $(X, \underline{x}) \in R_{\underline{O}} \underline{Con}$,

$A \in \underline{X_L}$ iff there exist $\underline{A_1}, \dots, \underline{A_n} \in \underline{X}$ and $x \in X$ so that

$\underline{A_1} \cup x \in \underline{X}$, $1 \leq i \leq n$, and $\underline{A} \subseteq \cup \{\underline{A_i} \mid 1 \leq i \leq n\}$. This is precisely the

translation of the $\underline{Lim}$ reflection of a space in $\underline{Con}$ via F (of 7.1).

$\underline{X_L}$ may be characterized in this simpler way for some

non- T_1 spaces as well: for example, let X be any set with more

than one element and $\underline{x}$ the indiscrete near structure ($A \in \underline{x}$ iff

$\emptyset \neq A$). Then $\underline{cl}_{\underline{x}}\{y\} = X$ for all $y \in X$, but $(X, \underline{x}) \in R_{\underline{O}} \underline{Lim}$; so

$\underline{x_L} = \underline{x}$ and the above characterization holds.

7.21 LEMMA If $(X, \underline{x}) \in R_{\underline{O}} \underline{Lim}$ and $(Y, \underline{y}) \in R_{\underline{O}} \underline{Con}$, $(X \times Y, (\underline{x} \times \underline{y})_L) =$

$(X \times Y, \underline{x} \times \underline{y_L})$.

PROOF. By 7.11, the product in $R_{\underline{O}} \underline{Con}$ is described in 4.14. $R_{\underline{O}} \underline{Lim}$

is bireflective in $R_{\underline{O}} \underline{Con}$ and thus has the same product.

By reflectivity, $(\underline{x} \times \underline{y})_L \subseteq \underline{x} \times \underline{y_L}$.

Suppose $\underline{G}$ is an $(\underline{x} \times \underline{y_L})$ -grill. Then $p_{\underline{x}} \underline{G}$ is an $\underline{x}$ -grill

and $p_{\underline{y}} \underline{G}$ is a $\underline{y_L}$ -grill; that is, there exists $x \in X$ with

$p_{\underline{x}} \underline{G} \cup x \in \underline{X}$ and there exist grills $\underline{H_i}$ on Y and $y_i \in Y$ ($1 \leq i \leq n$) so

that $p_{\underline{y}} \underline{G} \subseteq \cup \{\underline{H_i} \mid 1 \leq i \leq n\}$, $\underline{H_i} \cup y_i \in \underline{Y}$ and $C(y_1) = C(y_2) = \dots = C(y_n)$.

For each i , let $\underline{J_i}$ be the maximal grill on $X \times Y$ so that

$p_{\underline{x}} \underline{J_i} = p_{\underline{x}} \underline{G}$, $p_{\underline{y}} \underline{J_i} = \underline{H_i}$. Then $\underline{J_i} \cup (x, y_i) \in \underline{x} \times \underline{y}$ for each i , and

$C(x, y_1) = C(x, y_2) = \dots = C(x, y_n)$. By the construction in 3.10,
 $U(\underline{J}_1 | 1 \leq i \leq n) = U\{c[cp_{\underline{X}} \underline{G} \times c\underline{H}_1]\} = c\cap([cp_{\underline{X}} \underline{G} \times c\underline{H}_1]) = c[cp_{\underline{X}} \underline{G} \cap (c\underline{H}_1)] =$
 $c[cp_{\underline{X}} \underline{G} \times cU(\underline{H}_1)] = \underline{J}$. $\underline{J}$ is the maximal grill on $X \times Y$ with
 $p_{\underline{X}} \underline{J} = p_{\underline{X}} \underline{G}$, $p_{\underline{Y}} \underline{J} = U(\underline{H}_1 | 1 \leq i \leq n)$. Thus $\underline{G} \subseteq \underline{J} = U(\underline{J}_1 | 1 \leq i \leq n)$, and
 $\underline{G} \in (X \times Y)_L$. |||

7.22 THEOREM $R_{\underline{O}} \underline{Lim}$ is cartesian closed.

PROOF. $R_{\underline{O}} \underline{Con}$ is a cartesian closed topological category (7.12).

In view of the above lemma, the result follows from 1.20. |||

7.23 THEOREM Let $\underline{L}$ be the subcategory of $\underline{SNear}$ whose objects are those $(X, \underline{x})$ such that $(X, \underline{x}) \in R_{\underline{O}} \underline{Lim}$. Then $\underline{L}$ is bireflective in $\underline{SNear}$ and $R_{\underline{O}} \underline{Lim}$ is bicoreflective in $\underline{L}$ (with coreflection $1_X: (X, \underline{x}) \rightarrow (X, \underline{x})$ for $(X, \underline{x}) \in \underline{L}$).

The objects of $\underline{L}$ are the spaces $(X, \underline{x})$ satisfying condition:

If $\underline{G}, \underline{H}$ are $\underline{x}$ -grills, and there exists $x \in X$ with $\underline{G} \dot{x}$, $\underline{H} \dot{x} \in \underline{x}$, then $\underline{G} \dot{H} \dot{x}$.

PROOF. For the first part of the proof we refer to 1.14.

Suppose $(X, \underline{x}) \in \underline{L}$. If $\underline{G}$ and $\underline{H}$ are $\underline{x}$ -grills with $\underline{G} \dot{x}$, $\underline{H} \dot{x} \in \underline{x}$ for some $x \in X$, then $\underline{G} \dot{x}$, $\underline{H} \dot{x} \in \underline{x}_c$ (7.8) and since $(X, \underline{x}_c) \in R_{\underline{O}} \underline{Lim}$, $\underline{G} \dot{H} \dot{x}_c$. Hence $\underline{G} \dot{H} \dot{x}$.

Conversely, if the condition is satisfied: $\underline{A} \dot{x}$, $\underline{B} \dot{x} \in \underline{x}_c$ implies $\underline{A} \dot{x} \subseteq \underline{G}$, $\underline{B} \dot{x} \subseteq \underline{H}$, some $\underline{x}$ -grills $\underline{G} = \underline{G} \dot{x}$, $\underline{H} = \underline{H} \dot{x}$. By the condition, $\underline{G} \dot{H} = \underline{G} \dot{H} \dot{x} \in \underline{x}$. $\underline{G} \dot{H}$ is a grill; so $\underline{G} \dot{H} \dot{x}_c$.

Since $\underline{A} \dot{B} \subseteq \underline{G} \dot{H}$, $\underline{A} \dot{B} \dot{x}_c$, and $(X, \underline{x}_c) \in R_{\underline{O}} \underline{Lim}$. |||

7.24 COROLLARY $\underline{LG} = \underline{L} \cap \underline{Grill}$ is bireflective in $\underline{Grill}$ and bico-reflective in $\underline{L}$. $R_{\underline{O}}\underline{Lim}$ is bicoreflective in $\underline{LG}$. The objects of $\underline{LG}$ are those $(X, \underline{x}) \in \underline{Grill}$ satisfying the condition:

If $\underline{A} \underline{U} \underline{x}$ and $\underline{B} \underline{U} \underline{x} \in \underline{x}$ for some $\underline{x} \in X$; then $\underline{A} \underline{U} \underline{B} \underline{x}$. |||

7.25 NOTE. The $\underline{L}$ bireflection of a space $(X, \underline{x}) \in \underline{SNear}$ is given by $l_X: (X, \underline{x}) \rightarrow (X, \underline{x}_{NL})$ where

$\underline{A} \underline{E} \underline{x}_{NL}$ iff $\underline{A} \underline{E} \underline{x}$ or $\underline{A} \subseteq \bigcup \{G_i \mid 1 \leq i \leq n\}$ where G_i is a grill, $1 \leq i \leq n$ and there exist $x_1, \dots, x_n \in X$ so that $C(x_1) = \dots = C(x_n)$ and $G_i \underline{U} x_i \in \underline{x}$ for $1 \leq i \leq n$.

The proof of this is similar to the proof of 7.19.

7.26 NOTATION. We will denote the subcategory $F(R_{\underline{O}}\underline{PsT})$ by $R_{\underline{O}}\underline{PsT}$ (recalling that F is an isomorphism).

7.27 THEOREM Let $(X, \underline{x}) \in \underline{SNear}$. $(X, \underline{x}) \in R_{\underline{O}}\underline{PsT}$ iff the following condition holds:

$\underline{A} \underline{E} \underline{x}$ iff there exists a grill $\underline{G}$ on X and $\underline{x} \in X$ with $\underline{A} \subseteq \underline{G}$ and such that for every ultrafilter $\underline{U} \subseteq \underline{G}$, $\underline{U} \underline{U} \underline{x} \in \underline{x}$.

PROOF. ($\rightarrow$) If $\underline{A} \underline{E} \underline{x}$ then $\underline{A} \subseteq \underline{G}$ for some $\underline{x}$ -grill $\underline{G}$. $\underline{G}$ clearly satisfies the condition.

Conversely, suppose $\underline{G}$ is a grill and there exists $\underline{x} \in X$ so that $\underline{U} \underline{U} \underline{x} \in \underline{x}$ for every ultrafilter $\underline{U} \subseteq \underline{G}$. If $(X, \underline{q}) = F(X, \underline{g})$ and $\underline{U}$ is an ultrafilter with $\underline{c} \underline{G} \subseteq \underline{U}$, then $\underline{U} = \underline{c} \underline{U} \subseteq \underline{G}$. $\underline{U} \underline{U} \underline{x} \in \underline{x}$ implies $\underline{U} \in \underline{q} \underline{x}$. $(X, \underline{q}) \in R_{\underline{O}}\underline{PsT}$; so $\underline{c} \underline{G} \in \underline{q} \underline{x}$. This means $\underline{G} \underline{E} \underline{x}$.

($\leftarrow$) Clearly $(X; \underline{x}) \in \underline{Grill}$ if the condition is satisfied.

Furthermore, $(X, \underline{x}) \in R_{\underline{O}}\underline{Con}$: $\underline{A} \underline{E} \underline{x}$ implies $\underline{A} \subseteq \underline{G}$, $\underline{G}$ as in the condition.

Then $\underline{G}\underline{U}\dot{x}\in x$, for if $\underline{U}$ is an ultrafilter, $\underline{U}\subseteq \underline{G}\underline{U}\dot{x}$ implies $\underline{U}\subseteq \underline{G}$ or $\underline{U} = \dot{x}$. In either case, $\underline{U}\underline{U}\dot{x}\in x$. Thus $\underline{A}\underline{U}\dot{x}\in x$.

$(X, \underline{x}) \in R_{\underline{O}}\underline{PsT}$: Let $(X, \underline{x}) = F(X, \underline{q})$. If $\underline{F}$ is a filter such that for every ultrafilter $\underline{U}$ containing $\underline{F}$, $\underline{U}\in \underline{q}x$, then $\underline{U} = \underline{c}\underline{U}\subseteq \underline{c}\underline{F}$, and $\underline{U}\underline{U}\dot{x}\in x$. By the condition and arguing as above, $\underline{c}\underline{F}\underline{U}\dot{x}\in x$, which means $\underline{F}\in \underline{q}x$. That is, $(X, \underline{q}) \in \underline{PsT}$. |||

7.28 EXAMPLE $R_{\underline{O}}\underline{Lim} \neq R_{\underline{O}}\underline{PsT}$.

Let X be any infinite set and define $\underline{x} \in P^2X$ by:
 $\underline{A} \in \underline{x}$ iff there exist ultrafilters $\underline{U}_1, \dots, \underline{U}_n$ on X with $\underline{A} \subseteq \underline{U}\{\underline{U}_i \mid 1 \leq i \leq n\}$. It is easy to check $(X, \underline{x}) \in R_{\underline{O}}\underline{Lim}$; but $(X, \underline{x}) \notin R_{\underline{O}}\underline{PsT}$: Since $\underline{U}\underline{U}\dot{x}\in x$ for every ultrafilter $\underline{U}$ on X , $(X, \underline{x}) \in R_{\underline{O}}\underline{PsT}$ would imply $PX \setminus \emptyset \in x$. But $PX \setminus \emptyset$ is clearly not contained in any finite union of ultrafilters.

The complications which arose in constructing the $R_{\underline{O}}\underline{Lim}$ reflection of a space in $R_{\underline{O}}\underline{Con}$ do not appear in finding the $R_{\underline{O}}\underline{PsT}$ reflection of a space in $R_{\underline{O}}\underline{Lim}$: we can simply translate the $\underline{PsT}$ reflection of a convergence space into the language of grill-determined spaces. For we have

7.29 LEMMA If $(X, \underline{q}) \in R_{\underline{O}}\underline{Lim} (\subseteq \underline{Con})$, then $(X, \underline{q}_{\underline{ps}}) \in R_{\underline{O}}\underline{PsT}$.

PROOF. It suffices to show $(X, \underline{q}_{\underline{ps}}) \in R_{\underline{O}}\underline{Con}$. $\underline{F}\dot{x}\in \underline{q}_{\underline{ps}}y$, some $y \in X$ implies that for every ultrafilter $\underline{U}$ such that $\underline{F}\dot{x}\subseteq \underline{U}$, $\underline{U}\in y$. Let $\underline{V}$ be an ultrafilter, $\underline{F}\dot{x}\subseteq \underline{V}$. Then $\underline{F}\dot{x}\subseteq \underline{V}$; so $\underline{V}\in y$. In particular, $\dot{x}\in y$, and by (L) (which $(X, \underline{q})$ satisfies) $\underline{V}\dot{x}\in y$. By $(R_{\underline{O}})$, $\underline{V}\in \underline{q}x$. Hence $\underline{F}\in \underline{q}_{\underline{ps}}x$. |||

7.30 THEOREM If $(X, \underline{x}) \in R_{\underline{O}} \text{Con}$, the $R_{\underline{O}} \text{PsT}$ reflection of $(X, \underline{x})$ is given by $(X, \underline{x}) \xrightarrow{1_X} (X, \underline{x}_L) \xrightarrow{1_X} (X, \underline{x}_{LPS})$ where for $(Y, \underline{y}) \in R_{\underline{O}} \text{Lim}$, $\underline{y}_{PS}$ is defined by:

$\underline{A} \in \underline{y}_{PS}$ iff $\underline{A} \subseteq \underline{G}$ for some grill $\underline{G}$ so that there exists $y \in Y$ with $\underline{U} \underline{y} \in \underline{y}$ for every ultrafilter $\underline{U} \subseteq \underline{G}$.

PROOF. The definition of $\underline{y}_{PS}$ for $(Y, \underline{y}) \in R_{\underline{O}} \text{Lim}$ is simply a translation of the PsT reflection of a convergence space (6.5(b)). In view of 7.29, this gives the $R_{\underline{O}} \text{PsT}$ reflection.

The first part of the theorem follows if we apply reflectivity twice. |||

7.31 LEMMA Let $\{(X_i, \underline{x}_i) \mid i \in I\}$ be a family of spaces in $R_{\underline{O}} \text{Lim}$.

Then $(\prod X_i, (\pi \underline{x}_i)_{PS}) = (\prod X_i, \pi(\underline{x}_{iPS}))$.

PROOF. By reflectivity, $(\pi \underline{x}_i)_{PS} \subseteq \pi(\underline{x}_{iPS})$.

Let $\underline{G}$ be a $\pi(\underline{x}_{iPS})$ -grill. Then $p_{x_i} \underline{G}$ is an $\underline{x}_{iPS}$ -grill for each $i \in I$. That means there exists $x_i \in X_i$ such that if $\underline{U} \subseteq p_{x_i} \underline{G}$ is an ultrafilter, $\underline{U} \underline{x}_i \in \underline{x}_i$. Let $\underline{V} \subseteq \underline{G}$ be an ultrafilter. $p_{x_i} \underline{V} \subseteq p_{x_i} \underline{G}$ is an ultrafilter for each $i \in I$. Thus $p_{x_i} \underline{V} \underline{x}_i \in \underline{x}_i$, and $\underline{V} \underline{(x_i)} \in \pi \underline{x}_i$. Therefore $\underline{G} \in (\pi \underline{x}_i)_{PS}$. |||

7.32 THEOREM $R_{\underline{O}} \text{PsT}$ is cartesian closed.

PROOF. $R_{\underline{O}} \text{PsT}$ is bireflective in $R_{\underline{O}} \text{Lim}$, a cartesian closed topological category (7.22). By the above lemma, 1.20 applies, and the result follows. |||

7.33 THEOREM Let $\underline{PS}$ be the subcategory of $\underline{SNear}$ whose objects are those $(X, \underline{x})$ such that $(X, \underline{x}_c) \in R_{\underline{O}} \text{PsT}$. Then $\underline{PS}$ is

bireflective in $\underline{S}\text{Near}$ and $R_{\underline{O}}\text{PsT}$ is bicoreflective in $\underline{PS}$
(with coreflection $1_X: (X, \underline{x}_c) \rightarrow (X, \underline{x})$ for $(X, \underline{x}) \in \underline{PS}$).

The objects of $\underline{PS}$ are $(X, \underline{x}) \in \underline{S}\text{Near}$ satisfying the condition:

If $\underline{G}$ is a grill on X and there exists $x \in X$ such that $\underline{U} \cup \underline{x}$ for every ultrafilter $\underline{U} \in \underline{G}$, then $\underline{G} \in \underline{x}$. |||

7.34 COROLLARY $\underline{PSG} = \underline{PS} \cap \underline{Grill}$ is bireflective in $\underline{Grill}$ and bicoreflective in $\underline{PS}$. $R_{\underline{O}}\text{PsT}$ is bicoreflective in $\underline{PSG}$. |||

7.35 NOTATION. Following our earlier conventions, we will denote the subcategory $F(R_{\underline{O}}\text{PrT})$ by $R_{\underline{O}}\text{PrT}$.

7.36 THEOREM Let $(X, \underline{x}) \in \underline{S}\text{Near}$. Then $(X, \underline{x}) \in R_{\underline{O}}\text{PrT}$ iff the following condition holds:

$\underline{A} \in \underline{x}$ iff $\bigcap \text{cl}_{\underline{x}} \underline{A} \neq \emptyset$.

PROOF. (*) We have shown in 7.5 that $\underline{A} \in \underline{x}$ implies $\bigcap \text{cl}_{\underline{x}} \underline{A} \neq \emptyset$ for every $(X, \underline{x}) \in R_{\underline{O}}\text{Con}$.

Conversely, suppose $\bigcap \text{cl}_{\underline{x}} \underline{A} \neq \emptyset$ and let $(X, \underline{q}) = F(X, \underline{q})$. By 7.3, $\bigcap \text{cl}_{\underline{q}} \underline{A} \neq \emptyset$. So there exists $x \in X$ so that for every $\underline{A} \in \underline{q}$, $\underline{A} \in \underline{F}_A \in \underline{q}_x$, some filter $\underline{F}_A$. Since $(X, \underline{q}) \in R_{\underline{O}}\text{PrT}$, $\underline{F} = \bigcap \{\underline{F}_A \mid \underline{A} \in \underline{q}\} \in \underline{q}_x$. $\underline{x} \setminus \underline{A} \notin \underline{F}$, or $\underline{A} \in \underline{F}$ for every $\underline{A} \in \underline{q}$. Since $\underline{cF} \in \underline{x}$, $\underline{A} \in \underline{x}$.

(+) $(X, \underline{x}) \in \underline{Grill}$: For $x \in X$, let $\underline{C}(x) = \{\underline{A} \in X \mid x \in \text{cl}_{\underline{x}} \underline{A}\}$. Then $\underline{C}(x)$ is a grill (since $\text{cl}_{\underline{x}} (\underline{A} \cup \underline{B}) = \text{cl}_{\underline{x}} \underline{A} \cup \text{cl}_{\underline{x}} \underline{B}$) and $\bigcap \text{cl}_{\underline{x}} \underline{C}(x) \neq \emptyset$. $\underline{A} \in \underline{x}$ iff $\underline{A} \subseteq \underline{C}(x)$, some $x \in X$; so $\{\underline{C}(x) \mid x \in X\}$ is a grill structure determining $\underline{x}$.

$(X, \underline{x}) \in R_{\underline{O}}\text{Con}$: If $\underline{A} \in \underline{x}$, there exists $x \in \bigcap \text{cl}_{\underline{x}} \underline{A}$; so

$(\cap_{\underline{x}} \text{cl}_X A) \cap (\cap_{\underline{x}} \text{cl}_X \dot{x}) \neq \emptyset$, and $\underline{A} \cup \underline{x} \in \underline{x}$.

$(X, \underline{x}) \in R_{\underline{O}} \text{PrT}$: Let $(X, \underline{x}) = F(X, q)$. For each $x \in X$, $\underline{C}(x) = \underline{C}(x) \cup \underline{x} \in \underline{x}$ which means that $\underline{N}(x) = \text{cl}_{\underline{C}(x)} \underline{x}$. If $\underline{F} \in q_x$, $\underline{cF} \cup \underline{x} = \text{cl}_{\underline{C}(x)} \underline{x}$; so $x \in \cap_{\underline{x}} \text{cl}_X \underline{cF}$. Thus $\underline{cF} \subseteq \underline{C}(x)$, or $\underline{N}(x) \subseteq \underline{F}$. $\cap(\underline{F} | \underline{F} \in q_x) = \underline{N}(x) \in q_x$; so $(X, q) \in \text{PrT}$. |||

7.37 EXAMPLE $R_{\underline{O}} \text{PsT} \neq R_{\underline{O}} \text{PrT}$.

We note that for finite sets X , $(X, q) \in \underline{\text{Lim}}$ implies $(X, q) \in \text{PrT}$, since q_x must be finite for each $x \in X$. So any example must be infinite (thus the infiniteness of X in 7.28 was also necessary).

Let X be any infinite set. Define $\underline{x} \in P^2 X$ by: $\underline{A} \in \underline{x}$ iff $\underline{A} \subseteq \{x_i | 1 \leq i \leq n\}$, where $x_i \in X$, $1 \leq i \leq n$. Clearly $(X, \underline{x}) \in R_{\underline{O}} \text{Con}$.

$(X, \underline{x}) \in R_{\underline{O}} \text{PsT}$: Suppose $\underline{G}$ is a grill, $\underline{G} \notin \underline{x}$. We show that there is an ultrafilter $\underline{U} \subseteq \underline{G}$, $\underline{U} \notin \underline{x}$. By the definition of $\underline{x}$ and 7.27 this will give the result. For any $x_1, \dots, x_n \in X$, $\underline{G} \not\subseteq \{x_i | 1 \leq i \leq n\}$; that is, for every finite $F \subset X$, there exists $\underline{G} \in \underline{G}$, $\underline{G} \cap F = \emptyset$. Let $\underline{L} = \{X \setminus F | F \text{ finite}\}$. $\underline{L}$ is a filter, and $\underline{L} \subseteq \underline{G}$. By 3.6 there exists ultrafilter $\underline{U}$ with $\underline{L} \subseteq \underline{U} \subseteq \underline{G}$. If $\underline{U} \subseteq \{x_i | 1 \leq i \leq n\}$ and $F = \{x_i | 1 \leq i \leq n\}$, then $X \setminus F \notin \underline{U}$, but $X \setminus F \in \underline{L}$, contradicting $\underline{L} \subseteq \underline{U}$. That is, $\underline{U} \notin \underline{x}$.

$(X, \underline{x}) \notin R_{\underline{O}} \text{PrT}$: $\text{cl}_{\underline{x}} \{x\} = X$ for all $x \in X$; so $\cap_{\underline{x}} \text{cl}_X \{x\} \neq \emptyset$, but $\{\{x\} | x \in X\} \notin \underline{x}$, as X is infinite.

As was the case for $R_{\underline{O}} \text{PsT}$ (see 7.29-30) the $R_{\underline{O}} \text{PrT}$ reflection of $(X, \underline{x}) \in \underline{\text{Lim}}$ is found simply by translating the PrT reflection in Con .

7.38 LEMMA If $(X, q) \in R_{\underline{O}} \text{Lim}$ ($\subset \text{Con}$), then $(X, q_{\text{pr}}) \in R_{\underline{O}} \text{PrT}$.

PROOF. It suffices to show $(X, q_{pr}) \in R_{OCon}$. If $\bigcap_{x \in X} q_{pr} x \neq y$ some $y \in X$, $N(y) \subseteq \bigcap_{x \in X} q_{pr} x$ (where $N(y) = \bigcap \{F \mid F \in q_{pr} y\}$). Suppose $x \notin q_{pr} y$. Then $F \not\subseteq x$ for every $F \in q_{pr} y$; so there exists $F \in F$, $x \notin F$. But this means $x \notin \bigcup \{F \mid F \in q_{pr} y\} \in N(y)$ (contradiction). $\therefore x \in q_{pr} y$ and $(X, q) \in R_{OLim}$ imply $q_{pr} x = q_{pr} y$. Then $N(x) \subseteq E$, and $E \in q_{pr} x$. |||

7.39 THEOREM If $(X, x) \in R_{OCon}$, the R_{OPrT} reflection is given by $(X, x) \xrightarrow{1_X} (X, x_L) \xrightarrow{1_X} (X, x_{LPR})$, where for $(Y, y) \in R_{OLim}$, y_{PR} is defined by

$A \in y_{PR}$ iff there exists $y \in Y$ and a family $\{A_i \mid i \in I\}$ with $A_i \cup y \in y$ for all $i \in I$ and $A \subseteq \bigcup \{A_i \mid i \in I\}$.

PROOF. The definition of y_{PR} is a translation from 6.5(c), which may be seen by 7.38 to be the reflection. |||

7.40 THEOREM Let PR be the subcategory of $SNear$ whose objects are those (X, x) such that $(X, x_c) \in R_{OPrT}$. PR is bireflective in $SNear$ and R_{OPrT} is bicoreflective in PR (with coreflection $1_X: (X, x_c) \rightarrow (X, x)$ for $(X, x) \in PR$). The objects of PR are those $(X, x) \in SNear$ satisfying the condition:

If $\{G_i \mid i \in I\}$ is a family of x -grills and there exists $x \in X$ with $G_i \cup x \in x$ for each $i \in I$, then $\bigcup \{G_i \mid i \in I\} \in x$. |||

7.41 COROLLARY $PRG = PR \cap Grill$ is bireflective in $Grill$ and bicoreflective in PR . R_{OPrT} is bicoreflective in PRG . The objects of PRG are those $(X, x) \in Grill$ satisfying:

$\bigcap_{x \in X} A \neq \emptyset$ implies $A \in x$.

PROOF. If $(X, \underline{x}) \in \underline{\text{PRG}}$, $\text{cl}_{\underline{x}} = \text{cl}_{\underline{x}_c}$ (7.9). 7.36 gives: $\cap \text{cl}_{\underline{x}} A \neq \emptyset$ implies $\underline{A} \in \underline{x}_c$.

Conversely, $\underline{A} \in \underline{x}_c$ implies $\cap \text{cl}_{\underline{x}_c} A \neq \emptyset$ (7.5), and if $\cap \text{cl}_{\underline{x}_c} A \neq \emptyset$, $\cap \text{cl}_{\underline{x}} A \neq \emptyset$ (7.9). Choose $x \in \cap \text{cl}_{\underline{x}} A$. $(\cap \text{cl}_{\underline{x}} A) \cap (\cap \text{cl}_{\underline{x}} x) \neq \emptyset$; so by the condition, $\underline{A} \cup x \in \underline{x}$. By 7.8, $\underline{A} \in \underline{x}_c$. Applying 7.36, we have $(X, \underline{x}_c) \in \underline{\text{R}}_{\text{OT}} \text{PrT}$. |||

7.42 NOTE. In view of 6.10(b) and the fact that $\underline{\text{TNear}}$ is isomorphic to $\underline{\text{R}}_{\text{OT}} \text{Top}$ (see 2.14) we have $\underline{\text{TNear}} = F(\underline{\text{R}}_{\text{OT}} \text{Top})$.

7.43 THEOREM $\underline{\text{R}}_{\text{OT}} \text{Con} \cap \underline{\text{TNear}} = \underline{\text{TNear}}$.

PROOF. ($\subseteq$) If $(X, \underline{x}) \in \underline{\text{R}}_{\text{OT}} \text{Con} \cap \underline{\text{TNear}}$, $\underline{A} \in \underline{x}$ implies $\cap \text{cl}_{\underline{x}} A \neq \emptyset$, by 7.5. By (N5), this means $\underline{A} \in \underline{x}$ iff $\cap \text{cl}_{\underline{x}} A \neq \emptyset$; so $(X, \underline{x}) \in \underline{\text{TNear}}$.

($\supseteq$) If $(X, \underline{x}) \in \underline{\text{TNear}}$, $(X, \underline{x}) \in \underline{\text{Near}}$ (by definition). The grills $\underline{C}(x) = \{\underline{A} \in \underline{x} \mid x \in \text{cl}_{\underline{x}} A\}$, $x \in X$, determine $\underline{x}$. If $\underline{A} \in \underline{x}$, there exists $x \in \text{cl}_{\underline{x}} A$, which gives $\underline{A} \cup x \in \underline{x}$ and $(X, \underline{x}) \in \underline{\text{R}}_{\text{OT}} \text{Con}$ (7.4). |||

7.44 EXAMPLE $\underline{\text{TNear}} \neq \underline{\text{R}}_{\text{OT}} \text{PrT}$.

For $A \subseteq \mathbb{R}^2$ ($\mathbb{R}$ the reals) define $\text{cl} A = \{(x, y) \mid \text{there is a sequence } (x_i) \text{ in } A \text{ with } x_i \rightarrow x \text{ (in the usual topology) and } \{(x_i, y) \mid i \in \mathbb{N}\} \subseteq A \text{ or there is a sequence } (y_i), y_i \rightarrow y, \{(x, y_i) \mid i \in \mathbb{N}\} \subseteq A\}$. Define $\underline{\text{sCP}}^2(\mathbb{R}^2)$ by: $\underline{A} \in \underline{s}$ iff $\cap \text{cl}_{\underline{s}} A \neq \emptyset$. It is easy to check $(\mathbb{R}^2, \underline{s}) \in \underline{\text{Grill}}$. Now

$\text{cl}_{\underline{s}} A = \{z \mid \text{cl}\{z\} \cap \text{cl} A \neq \emptyset\} = \text{cl} A$ (since $\text{cl}\{z\} = \{z\}$). By 7.36, $(\mathbb{R}^2, \underline{s}) \in \underline{\text{R}}_{\text{OT}} \text{PrT}$. But $(\mathbb{R}^2, \underline{s}) \notin \underline{\text{TNear}}$, since $\text{clcl} \neq \text{cl}$: let $A = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$. Then $\text{cl} A = \{(x, y) \mid 0 \leq x \leq 1, 0 < y < 1\} \cup \{(x, y) \mid 0 < x < 1, 0 \leq y \leq 1\}$, but $\text{clcl} A = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\} \neq \text{cl} A$.

7.45 LEMMA If $(X, \underline{x}) \in \underline{\text{Near}}$, $\underline{x}_t = \underline{x}_c$; that is, the $\underline{\text{TNear}}$ coreflection is $1_X: (X, \underline{x}_c) \rightarrow (X, \underline{x})$.

PROOF. It suffices (by 7.43) to show $(X, \underline{x}_c) \in \underline{\text{Near}}$. By 4.12,

7.9 and 7.8, $cl_{\underline{x}_c} = cl_{\underline{x}}$. If $cl_{\underline{x}_c} A \in \underline{x}_c$, $cl_{\underline{x}} A \in G$, some $\underline{x}$ -grill G such that $G \cup \underline{x} \in \underline{x}$, some $x \in X$. Let $G_b = \{B \mid cl_{\underline{x}} B \in G\}$. G_b is an $\underline{x}$ -grill (4.17) and $cl_{\underline{x}_b} G \cup cl_{\underline{x}} \underline{x} \in G \cup \underline{x}$ implies $G_b \cup \underline{x} \in \underline{x}$. Since $A \in G_b$, $A \in \underline{x}_c$, and (N5) holds. |||

7.46 LEMMA If $(X, \underline{x}) \in R_{\underline{OCon}}$, $(X, \underline{x}_n) \in \underline{\text{TNear}}$.

PROOF. By 4.20(a) $(X, \underline{x}_n) \in \underline{\text{Grill}}$.

$A \in \underline{x}_n$ implies $cl_{\underline{x}_n} A \in \underline{x}$ (by definition of $\underline{x}_n$). Thus there exists $x \in X$, $cl_{\underline{x}_n} A \cup \underline{x} \in \underline{x}$ (7.4). From (N1), we have $cl_{\underline{x}_n} A \cup cl_{\underline{x}_n} \underline{x} \in \underline{x}$, which means $A \cup \underline{x} \in \underline{x}_n$ (definition of $\underline{x}_n$). 7.4 gives $(X, \underline{x}_n) \in R_{\underline{OCon}}$, and the result follows from 7.43. |||

7.47 THEOREM (a) The category of $\underline{\text{Near}}$ -structurable $R_{\underline{OCon}}$ spaces is precisely $\underline{\text{TNear}}$. Thus the $\underline{\text{TNear}}$ bireflection of $(X, \underline{x}) \in R_{\underline{OCon}}$ is given by $1_X: (X, \underline{x}) \rightarrow (X, \underline{x}_n)$.

(b) The category of $R_{\underline{OCon}}$ -structurable $\underline{\text{Near}}$ spaces is precisely $\underline{\text{TNear}}$.

PROOF. (a) If $(X, \underline{x}) \in \underline{\text{TNear}}$, $\underline{x}_{nc} = \underline{x}_c = \underline{x}$ by 7.43; so $(X, \underline{x})$ is $\underline{\text{Near}}$ -structurable.

Conversely, if $(X, \underline{x}) \in R_{\underline{OCon}}$ is $\underline{\text{Near}}$ -structurable, then for some $(X, \underline{y}) \in \underline{\text{Near}}$, $\underline{x} = \underline{y}_c$. By 7.45, $(X, \underline{x}) \in \underline{\text{TNear}}$.

According to 1.13(c), the reflection is given by

$$l_X: (X, \underline{x}) \rightarrow (X, \underline{x}_{nc}) , \text{ but } \underline{x}_{nc} = \underline{x}_n \quad (7.46).$$

(b) If $(X, \underline{x}) \in \underline{\text{TNear}}$, $\underline{x}_{cn} = \underline{x}_n = \underline{x}$ (7.43); so $(X, \underline{x})$ is R_{Con} -structurable.

Conversely, if $(X, \underline{x}) \in \underline{\text{Near}}$ is R_{Con} -structurable, there is $(X, \underline{y}) \in R_{\text{Con}}$, $\underline{y}_n = \underline{x}$. Then 7.46 gives $(X, \underline{x}) \in \underline{\text{TNear}}$. |||

The property for $(X, \underline{x}) \in \underline{\text{SNear}}$ that $(X, \underline{x}_c) \in \underline{\text{TNear}}$ does not characterize Near:

7.48 EXAMPLE Let R be the reals and cl the usual closure.

Define $\underline{r} \subset P^2 R$ by: $\underline{A} \in \underline{r}$ iff $\cap cl \underline{A} \neq \emptyset$ or $\underline{A} \cap Q$ is infinite for every $\underline{A} \in \underline{r}$ (Q denotes the rationals).

$(R, \underline{r}) \in \underline{\text{Grill}}$: $\underline{C}(x) = \{A \mid x \in cl A\}$, for $x \in R$ and

$\underline{D} = \{A \mid \underline{A} \cap Q \text{ infinite}\}$ determine $\underline{r}$.

$(R, \underline{r}) \notin \underline{\text{Near}}$: $cl \underline{r} A = cl A$ for $A \in R$. Let $B = [0, 1] \setminus Q$, $C = [2, 3] \setminus Q$. $cl A = [0, 1]$, $cl B = [2, 3]$; so $cl A \cap cl B = \emptyset$ and B and C contain no rationals. Thus $\{A, B\} \notin \underline{r}$. But $\{cl A, cl B\} \in \underline{r}$; so (N5) does not hold.

$(R, \underline{r}_c) \in \underline{\text{TNear}}$: $\underline{A} \in \underline{r}_c$ iff $\underline{A} \cup \underline{x} \in \underline{r}$, some $x \in R$. But $\{x\} \cap Q$ cannot be infinite; so this means $\cap cl \underline{A} \neq \emptyset$. Therefore $(R, \underline{r}_c)$ is the space of reals with the usual topology.

7.49 THEOREM Let $\underline{T}$ be the subcategory of SNear whose objects are those $(X, \underline{x})$ with $(X, \underline{x}_c) \in \underline{\text{TNear}}$. Then $\underline{T}$ is bireflective in SNear and TNear is bicoreflective in $\underline{T}$ (with coreflection $l_X: (X, \underline{x}_c) \rightarrow (X, \underline{x})$ for $(X, \underline{x}) \in \underline{T}$).

Furthermore, Near is bireflective in $\underline{T}$ (and Near $\neq \underline{T}$, by 7.48). |||

7.50 COROLLARY $TG = T \cap Grill$ is bireflective in $Grill$ and bico-reflective in T . $TNear$ is bicoreflective in TG . Furthermore, Bun is bireflective in TG (and $Bun \neq TG$, by 7.48).

The objects of TG are those $(X, \underline{x}) \in Grill$ satisfying the conditions (i) $cl_{\underline{x}} cl_{\underline{x}} = cl_{\underline{x}}$ and (ii) $\cap cl_{\underline{x}} A \neq \emptyset$ implies $A \in \underline{x}$.

PROOF. Suppose $(X, \underline{x}) \in TG$. Then $cl_{\underline{x}} = cl_{\underline{x}_c}$, and since $(\bar{X}, \underline{x}_c) \in TNear$, $cl_{\underline{x}} cl_{\underline{x}} = cl_{\underline{x}}$. Condition (ii) follows by 7.41 ($TG \subset PRG$).

Conversely, suppose $(X, \underline{x}) \in Grill$ satisfies (i) and (ii).

If $cl_{\underline{x}_c} A \in \underline{x}_c$, $cl_{\underline{x}_c} A \in \underline{x}$, some $x \in X$. Then $x \in cl_{\underline{x}} cl_{\underline{x}} A = cl_{\underline{x}} A$ for every $A \in \underline{x}$ (using $cl_{\underline{x}} = cl_{\underline{x}_c}$ and (N1)). $\cap (cl_{\underline{x}} A \cup cl_{\underline{x}} x) \neq \emptyset$ implies $A \in \underline{x}$ (ii); so $A \in \underline{x}_c$, and (N5) holds. That means $(X, \underline{x}_c) \in TNear$, or by 7.43, $(X, \underline{x}) \in TNear$. |||

The property for $(X, \underline{x}) \in SNear$ that $(X, \underline{x}_n) \in TNear$ does not characterize $R_0 Con$:

7.51 EXAMPLE Let R be the reals, and define $rCP^2 R$ by: $A \in r$ iff there exists $x \in R$ such that $d(x, A) < 1$ for every $A \in A$ or A is infinite for every $A \in A$. (d is the usual metric on R . Compare this example with 7.15).

$(R, r) \in Grill$, but $(R, r) \notin R_0 Con$, since $I = \{A \mid A \text{ infinite}\} \in r$, but $I \cup x \notin r$, any $x \in R$.

$(R, r_n) \in TNear$: $cl^0 A = A$; $cl^1 A = \{x \mid \{\{x\}, A\} \in r\} = \{x \mid d(x, A) < 2\}$, $cl^2 A = \{x \mid \{cl^1 \{x\}, cl^1 A\} \in r\} = R$ if $A \neq \emptyset$.

Thus, $cl_{r_n} A = R$ for all $\emptyset \neq A \in R$. So $A \in r_n$ iff $\cap cl_{r_n} A \neq \emptyset$.

which means $\phi \notin A$. r_n is thus the indiscrete near structure on R , and $(R, r_n) \in \underline{TNear}$.

7.52 LEMMA [K1] $(X, \underline{x}) \in R_{\underline{O}Con}$ iff there exists a family $\{(X_i, \underline{x}_i) \mid i \in I\}$ of $\underline{TNear}$ spaces and functions $\{f_i: X_i \rightarrow X \mid i \in I\}$ so that $\underline{x}$ is final in $\underline{SNear}$ with respect to $((X_i, \underline{x}_i)_{i \in I}, (f_i)_{i \in I}, X)$.

PROOF. $(\Rightarrow)$ There exists a family $\{G_i \mid i \in I\}$ of $\underline{x}$ -grills determining $\underline{x}$ such that $G_i = G_1 \cup x_i$ for some $x_i \in X$, each $i \in I$. Define $\underline{x}_i \in \mathcal{CP}^2 X$ by: $A \in \underline{x}_i$ iff $A \subseteq G_i$ or $A \subseteq y$, some $y \in X$. Clearly $(X, \underline{x}_i) \in R_{\underline{O}Con}$.

$(X, \underline{x}_i) \in \underline{TNear}$: If $A \subseteq X$, $cl_{\underline{x}_i} A = \{x \mid x \in A \text{ or } \{x\}, A \in G_i\}$. Thus if $A \notin G_i$, $cl_{\underline{x}_i} A = A$, and if $A \in G_i$, $cl_{\underline{x}_i} A = A \cup \{x \mid \{x\} \in G_i\}$. If $cl_{\underline{x}_i} A \in \underline{x}_i$, either (i) there exists $x \in X$, $x \in cl_{\underline{x}_i} A$, or (ii) $cl_{\underline{x}_i} A \in G_i$. (i) If $\{x\} \in G_i$, $A \subseteq G_i$; if $\{x\} \notin G_i$, $A \subseteq x$. (ii) It is easy to check $A \subseteq G_i$. Thus $A \in \underline{x}_i$, and (N5) holds. By 7.43, $(X, \underline{x}_i) \in \underline{TNear}$.

$\underline{x}$ is final with respect to $((X_i, \underline{x}_i)_{i \in I}, (1_{X_i})_{i \in I}, X)$; for $A \in \underline{x} \Leftrightarrow A \subseteq G_i$, some i , $\Leftrightarrow A \subseteq 1_{X_i} G_i$, $G_i \in \underline{x}_i$.

$(\Leftarrow)$ $R_{\underline{O}Con}$ is closed under final structures in $\underline{SNear}$ (1.10). |||

7.53 THEOREM $R_{\underline{O}Con}$ is the coreflective hull in $\underline{SNear}$ (or $\underline{Grill}$) of $\underline{TNear}$.

PROOF. In view of 1.12, this follows immediately from the lemma. |||

7.54 NOTE. By 1.19, the above gives an alternate proof that $R_{\underline{O}Con}$ is cartesian closed, since $\underline{Grill}$ is cartesian closed.

7.55 To clarify the relationships among the categories considered in this chapter, we give an "organization diagram".

R denotes bireflective subcategory

C denotes bicoreflective subcategory

8. EXTENSIONS

8.1 NOTE. Banaschewski's use of families of filters to construct all extensions of topological spaces [Ba] can be translated via the duality between filters and grills into the language of nearness spaces (see [Be]). Given a suitable topological space (X, cl) , where cl is the closure, the Near (in fact Bun) structures $\underline{x}$ on X compatible with the topology ($cl_{\underline{x}} = cl$) give all the strict extensions of (X, cl) , and in fact all extensions can be constructed in this way if we translate [Ba], section 2. The nearness approach is useful because important classes of extensions can be constructed by restricting to particular subcategories of Near (see 2.16). For example, the contigual near spaces give the compact extensions (see [II]). There seems to be scope for further study along these lines: using nearness structures as a tool for finding topological extensions.

A somewhat different extension problem is considered in [H1] : for $(X, \underline{x}) \in \text{Near}$, 'completion' $(X^*, \underline{x}^*)$ is constructed which is analogous to the completion of a uniform space. The completion is always in Near, and $(X, \underline{x})$ is densely embedded in it (denseness being with respect to $cl_{\underline{x}^*}$) if $(X, \underline{x})$ is a T_1 -space (see 7.3).

It is not possible to copy the construction of [H1] to get a 'completion' for SNear (or even Grill) spaces in general. Unless it is in R_{PrT} , a space $(X, \underline{x}) \in \text{Grill}$ need not have the property that every $\underline{x}$ -grill is contained in a unique maximal $\underline{x}$ -grill, or even that $\underline{x}$ is contained in a unique maximal $\underline{x}$ -grill for $\underline{x} \in X$. These maximal grills were needed in [H1] as the elements of the completion. However, we have been able to construct, for $(X, \underline{x}) \in \text{Grill}$,

(non-unique) R_{Con} extensions: spaces in R_{Con} containing $(X, \underline{x})$ as a dense subspace. The method of construction is the standard one: $\underline{x}$ -grills are used as the elements of a new space which is given a structure defined in terms of structure $\underline{x}$.

8.2 NOTATION. Let X be a set. Then $\mathcal{X}$ will denote a non-empty set of grills on X . We denote subsets of $\mathcal{X}$ by $A, B, \dots$, and subsets of PX by $\underline{A}, \underline{B}, \dots$. As usual, if $\underline{G} \in X$, $\underline{G} = \{A \in X \mid \underline{G} \in A\}^\circ$.

8.3 DEFINITION Define a function $g: PX \rightarrow PX$ by

$$g(A) = \{\underline{G} \in X \mid A \in \underline{G}\}.$$

8.4 LEMMA (a) If $\underline{G} \subseteq H$, $g(\underline{G}) \subseteq g(H)$.

$$(b) \quad g(\underline{G} \cup \underline{H}) = g(\underline{G}) \cup g(\underline{H}).$$

8.5 LEMMA (a). If $\underline{G} \in X$, $g^+(\underline{G}) = \underline{G}$.

(b) If $\underline{G}$ is a grill on X , $g^+(\underline{G})$ is a grill on X .

PROOF. (a) If $\underline{G} \in X$, $g(A) \in \underline{G} \leftrightarrow \underline{G} \in g(A) \leftrightarrow A \in \underline{G}$.

(b) $A \cup B \in g^+(\underline{G}) \leftrightarrow g(A \cup B) \in \underline{G}$. This means, by 8.4,

$$g(A) \cup g(B) \in \underline{G} \leftrightarrow g(A) \in \underline{G} \text{ or } g(B) \in \underline{G} \leftrightarrow A \text{ or } B \in g^+(\underline{G}).$$

Suppose $(X, \underline{x}) \in \text{SNear}$ and $X \subseteq \phi(\underline{x})$ (see 4.2). We wish to define an R_{Con} structure $\underline{x}$ on X related in a natural way to the structure $\underline{x}$. That means that $\underline{x}$ must be defined in terms of a family of grills on X , each associated with an element of X .

8.5 allows us to define this structure by going back to $(X, \underline{x})$:

8.6 DEFINITION Let $(X, \underline{x}) \in \text{SNear}$ and $\phi \neq X \subseteq \phi(\underline{x})$. Define $\underline{x} \in P^2 X$ by

$\underline{A} \in \underline{X}$ iff there is a grill $\underline{G}$ on $\underline{X}$ and $\underline{H} \in \underline{X}$ such that $\underline{A} \subseteq \underline{G}$ and $g^+(\underline{G}) \cup \underline{H} \in \underline{X}$.

8.7 THEOREM $(X, \underline{x}) \in R_{\underline{O}} \text{Con}$.

PROOF. If $\underline{G} \in \underline{X}$, $g^+(\underline{G}) = \underline{G} \in \underline{X}$; so $\underline{G} \in \underline{X}$. By 4.2, $(X, \underline{x}) \in \text{Grill}$.

If $\underline{A} \in \underline{X}$ there is a grill $\underline{G}$ on $\underline{X}$ and $\underline{H} \in \underline{X}$ with $\underline{A} \subseteq \underline{G}$, $g^+(\underline{G}) \cup \underline{H} \in \underline{X}$. Then $g^+(\underline{G} \cup \underline{H}) = g^+(\underline{G}) \cup g^+(\underline{H}) = g^+(\underline{G}) \cup \underline{H} \in \underline{X}$. Thus $\underline{G} \cup \underline{H} \in \underline{X}$; so $\underline{A} \cup \underline{H} \in \underline{X}$. By 7.4, $(X, \underline{x}) \in R_{\underline{O}} \text{Con}$. ||||

8.8 NOTE. It is interesting to note that if $\underline{X} = \{\underline{x} \mid x \in X\}$, $(X, \underline{x})$ is isomorphic to $(X, \underline{x}_c)$, the $R_{\underline{O}} \text{Con}$ coreflection of $(X, \underline{x})$.

So we can't improve 8.7 to a subcategory of $R_{\underline{O}} \text{Con}$.

We wish to use this construction to find $R_{\underline{O}} \text{Con}$ extensions.

The most natural way to get $\underline{X}$ as a subset of X (especially in view of 8.5(a)) is to require $\underline{x} \in X$ for all $x \in X$ and define $i: X \rightarrow X$ by $i(x) = \underline{x}$. Obviously i is one-to-one.

8.9 THEOREM. Suppose $(X, \underline{x}) \in \text{Grill}$ and $\underline{X} \subseteq \Phi(\underline{x})$ satisfies (i) $\underline{x} \in X$ for all $x \in X$ and (ii) for all $\underline{G} \in \Phi(\underline{x})$ (all $\underline{x}$ -grills) there exists $\underline{H} \in X$ so that $\underline{G} \cup \underline{H} \in \underline{X}$. Then $i: (X, \underline{x}) \rightarrow (X, \underline{x})$ is an SNear extension.

PROOF. i is a near map: Let $\underline{G}$ be an $\underline{x}$ -grill. Then $[i(\underline{G})]$ is a grill on X (3.8). $g^+[i(\underline{G})] = \{A \subseteq X \mid g(A) \in [i(\underline{G})]\}$. But $g(A) \cap i(X) = \{\underline{x} \mid \underline{x} \in g(A)\} = \{\underline{x} \mid x \in A\} = i(A)$; so $g^+[i(\underline{G})] = \underline{G}$. By (ii), there exists $\underline{H} \in X$, $g^+[i(\underline{G})] \cup \underline{H} \in \underline{X}$; so $[i(\underline{G})] \in \underline{X}$. By 4.8 i is near.

i is an embedding: If $i(A) \in \underline{X}$, there is an $\underline{x}$ -grill $\underline{G}$, $i(A) \subseteq \underline{G}$ (8.7). Since $i(A) = g(A) \cap i(X)$, $g(A) \in \underline{G}$ for every $A \in \underline{A}$.

Thus $A \in g^*(G)$. By definition of $\underline{x}$, $g^*(G) \in \underline{x}$; so $A \in \underline{x}$.

$i(X)$ is dense in $(X, \underline{x})$: If $\bar{G} \in \underline{x}$,

$$g^*([iG] \cup \bar{G}) = g^*[iG] \cup \bar{G} = G \in \underline{x}; \text{ so } [iG] \cup \bar{G} \in \underline{x}.$$

$i(X)$, $\{G\} \in [iG] \cup \bar{G}$ gives by (N1) $G \in \text{cl}_{\underline{x}} i(X)$. |||

Notice that the conditions on X are not too restrictive:

for any $(X, \underline{x}) \in \text{Grill}$, $X = \Phi(\underline{x})$ will satisfy them. So we have

8.10 THEOREM The objects of Grill are precisely the SNear subspaces of the objects of R₀Con.

PROOF. By 8.9 (and the following remark) every $(X, \underline{x}) \in \text{Grill}$ is the SNear subspace of a space in R₀Con. Conversely, every subspace of a space in R₀Con is in Grill (4.13). |||

8.11 NOTE. This theorem cannot be strengthened by replacing R₀Con by any of the subcategories of R₀Con we have considered. For let $(A, \underline{a})$ be any SNear subspace of $(X, \underline{x}) \in \text{R-Lim}$. Suppose $B, C \in \text{SPA}$, and there is an $a \in A$ with $B \cup a \in \underline{a}$, $C \cup a \in \underline{a}$. Then $B \cup a, C \cup a \in \underline{x}$ (now reading $\underline{a}$ as $\{D \in X \mid a \in D\}$). Since $(X, \underline{x}) \in \text{R-Lim}$, $B \cup C \in \underline{x}$ (7.14), and hence $B \cup C \in \underline{a}$. By 7.24, $(A, \underline{a}) \in \text{LG}$. But $\text{LG} \neq \text{Grill}$: any space in R₀Con but not R-Lim is in Grill but not LG.

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An extensive bibliography of the literature of cartesian closedness
is found in [H3] , and of nearness structures in [H2] .

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