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THE BARYON AS A TOPOLOGICAL SOLITON

by

Andrew Lefcoe

A thesis submitted to the Faculty of Graduate Studies
in partial fulfilment of the requirements for the degree
of Master of Science.

Carleton University

Ottawa, Ontario

December, 1985

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ISBN 0-315-29887-1

The undersigned hereby recommend to the Faculty of Graduate
Studies and Research acceptance of the thesis,

"THE BARYON AS A TOPOLOGICAL SOLITON"

submitted by Andrew Lefcoe, in partial fulfilment of the requirements
for the degree of Master of Science.

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ABSTRACT

The interpretation of the baryon as a topological soliton has received much attention and validation in recent years. In this thesis we present a line of reasoning by which the baryons emerge as the topological solitons of QCD. This is accomplished by first establishing an equivalence between QCD and an effective theory of mesons. In the weak coupling limit baryons are shown to behave as solitons in the meson theory. We then examine a model theory of mesons which admits soliton solutions; the Skyrme model for two flavours. It is shown that the solitons can be correctly characterized as fermions with a conserved baryon number. Extending the model to three flavours we find it necessary to add a new term to the Skyrme Lagrangian to cure the axial anomaly. This term (Wess-Zumino term) ensures that the solitons are fermions.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Pierre Mathieu, for his guidance and assistance. I also wish to thank Roselyn O'Driscoll for typing this manuscript and Les Copley for his council. Special thanks to Nancy and Louise.

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INTRODUCTION

The concept of a soliton, or stationary wave, was first formally advanced in 1842 at the British Association for the Advancement of Science, when J. Scott Russell reported the following (Russell, 1845):

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon..."

Though, no doubt, Scott Russell had mastered contemporary fluid dynamical theory to a fare-thee-well, he mistakenly attributed the above phenomena to the properties of the medium. It was not until some time later that physicists realized that certain nonlinear wave equations admit soliton solutions that propagate undistorted with time.

The identification of the soliton as an elementary particle was first proposed by Skyrme in his now famous papers from the early sixties (Skyrme, 1960, 1961). At that time, the interactions of pseudoscalar mesons were being investigated by use of non-linear Lagrangians. Skyrme showed that such Lagrangians admit soliton solutions, which he proposed be interpreted as the nucleon, delta, etc. Skyrme and other physicists (Finkelstein and Misner, 1959; Finkelstein, 1965; Finkelstein and Rubinstein, 1968), realized the importance of the non-trivial topological properties of the field configuration to particle phenomena. Skyrme conjectured that the conserved topological charge associated with a soliton solution is the baryon number. Skyrme and Williams (Williams, 1970), put forward general arguments showing that for two flavours a proper quantization scheme would lead to half odd integer spin.

After a period of neglect the ideas of Skyrme, and others, were revived by Pak and Tze (Pak and Tze, 1979). More recently, the work of the Syracuse group (Balachandran, Nair, Rajeev and Stern, 1983), and Witten (Witten, 1983) have created great interest in the subject.

Witten, in extending the theory to three flavours, found it necessary to add an extra term to Skyrme's Lagrangian, the Wess-Zumino term. Witten's analysis has shown that the conjectures of Skyrme are, in fact, implied by QCD.

The connection between QCD and Skyrme's phenomenological model of mesons can be understood in different ways. In the low energy limit there exists a very obvious connection. In this limit, QCD is equivalent to a theory of Goldstone bosons (the pseudoscalar mesons), whose interactions can be described by a non-linear Lagrangian, such as that used by Skyrme. However this limit, being valid for energies much less than that of a quark bound in a baryon, is inappropriate for baryon physics. It has been possible, though, to find an equivalence between

QCD, at any energy, and a meson theory.

This equivalence, developed by Witten (Witten, 1979; see also Coleman, 1979), was first established by 'tHooft ('tHooft, 1974), who considered the generalization from an $SU(3)$ to an $SU(N)$ gauge group (N is the number of colours). In Chapter 1 we show that this leads us to interpret QCD as an effective theory of mesons with a coupling constant $1/N$. In the limit of large N the coupling is weak and thus the tree approximation is valid. The tree approximation is shown to correspond to solving the classical equations of motion derived from some effective Lagrangian. The idea is then to find an effective Lagrangian of local meson fields, generating baryons by searching for soliton solutions. An example of such a Lagrangian is furnished by the Skyrme model.

Chapter 2 is devoted to a detailed discussion of the Skyrme model for the case of two flavours. In Chapter 3 we consider the case of three flavours, known as the Skyrme-Witten model.

1.1 QCD for Large N

1.1.1 Introduction

QCD is now regarded as the correct theory of the strong interactions. For the theory to be of use we must be able to calculate quantitative predictions to compare with physical observables. In theories as complex as QCD this is accomplished by perturbative techniques, requiring a free parameter to do the expansion. Since the coupling constant g fixes the mass scale it is not really a free parameter. Thus $SU(3)$ QCD appears to be a zero parameter theory. We say "appears to be" since one cannot ignore the possibility of a hidden parameter. A candidate for such a parameter was discovered by 'tHooft ('tHooft, 1974).

'tHooft proposed the parameter N , the number of colours, as an expansion parameter. Although in nature $N = 3$, the idea is to generalize from $SU(3)$ to $SU(N)$ and attempt to solve the theory in the large N limit. One then hopes that results for $N = 3$ will be both qualitatively and quantitatively similar to those for the large N limit.

In this section we discover how QCD simplifies in the large N limit. Without detailed mathematical analysis we can extract the qualitative large N behaviour of mesons and baryons. A quantitative approximation scheme, however, seems to elude physicists at present.

1.1.2 Diagram Combinatorics

The nature of QCD in the large N limit is dictated by the combinatorics of Feynman diagrams. The combinatoric factors arise from the many accessible intermediate states made available by the large number of colours. Our first task, then, is to find a method to enumerate the possible intermediate states. Summing over these will give the desired combinatoric factors. To this end we first define the gluon and quark fields for an $SU(N)$ gauge group, leading us to a

convenient notation for following colour indices in Feynman diagrams.

The gluon field in SU(N) QCD is an $N \times N$ matrix, A_{ij}^i , which has $N^2 - 1$ components. For large N we neglect the difference between $N^2 - 1$ and N^2 and consider A_{ij}^i as having N^2 components. The quark and antiquark fields are

$$q^i, \bar{q}_i \quad i = 1, \dots, N.$$

To follow colour indices, we use the 'tHooft double line notation,

$$q^i \longrightarrow$$

and

$$\bar{q}_i \longleftarrow$$

Since A_{ij}^i has an upper index like a quark field and a lower index like an antiquark field, we can think of the gluon as being a combination of quark-antiquark;

$$A_{ij}^i \longrightarrow \longleftarrow$$

It must be stressed that this identification is for the purpose of following colour indices only.

We can now translate Feynman diagrams into the double line notation. Recalling that colour conservation demands that each colour line entering a vertex leave it, we arrive at the following diagrammatic equivalences:

(a)

(b)

(c)

FIGURE 1.1

Figure 1.1 (c) is the gluon contribution to the one-loop gluon vacuum polarization. The quantum number i and j are fixed by the initial and final states, leaving a closed colour line. The value of k , the colour index contracted with itself over this line, is unspecified. The sum over all possible values of k gives this diagram a combinatoric factor of N .

The order of the diagram should also depend upon the coupling factor at the two interaction vertices. If we wish the one-loop gluon vacuum polarization to have a smooth limit for large N , the coupling constant at each vertex must be $g/\sqrt{N}$, where g does not depend upon N . The coupling constant vanishes for large N , exactly cancelling the combinatoric factor and giving this diagram a smooth N limit. We say that 1.1 (c) is of order $N (\frac{1}{\sqrt{N}})^2 = 1$.

Since every vertex in a Feynman diagram has a factor of order $1/\sqrt{N}$, a diagram will survive as we take the limit $N \rightarrow \infty$ only if it has a combinatoric factor large enough to cancel the vertex couplings. This will select a particular class of surviving diagrams, hence simplifying the theory.

To discover the distinguishing properties of this class, consider the following diagrams:

(a)

(b)

(c)

FIGURE 1.2

While the diagrams of (a) and (b) have a smooth N limit, that of (c) is strongly suppressed. What distinguishes between these diagrams is that while (c) has six interaction vertices it has only one, albeit tangled, closed colour line. That this line crosses itself, rather than forming four closed colour loops, is the reason for this diagram being suppressed; it is not planar.

A planar diagram is one which can be drawn on the plane. If one were to add an extra gluon to a planar diagram, this would contribute a factor $(1/N)^2$ coming from the two interaction vertices. If the diagram remained planar then one could be easily convinced that a new closed colour line would be produced, giving a factor of N to cancel the extra vertex factors. If, on the other hand, the extra gluon destroyed the planarity of the diagram, the number of closed colour lines would decrease by one. The diagram would receive an extra factor of $1/N^2$. Thus our first selection rule is that non-planar diagrams are suppressed by a factor of $1/N^2$.

We can discover more selection rules by adding quarks into the scheme. The one-quark loop contribution to the gluon propagator is shown in Figure 1.3.

FIGURE 1.3

This diagram has a combinatoric factor of one and an N -dependence from the vertices of $(1/\sqrt{N})^2 = \frac{1}{N}$. The reason for this diagram being suppressed is that the quark corresponds to one colour line. An internal quark loop can never form a closed colour line but will always add two vertices. This gives us our second selection rule; internal quark loops are suppressed by a factor of $1/N$.

For the study of mesons we will be more interested in the matrix elements of quark bilinears $q\bar{q}$ than the gluon propagator. Figure 1.4 gives the amplitude for the vacuum to create and annihilate a $q\bar{q}$ pair.

FIGURE 1.4

If we add any arbitrary number of gluons such that Figure 1.4 remains planar, it will remain to be of order N .

Demanding planarity is not enough to ensure the survival of a diagram contributing to the matrix elements of quark bilinears. The process,

FIGURE 1.5

containing four interaction vertices, and only one closed colour line, is of order $(1/N)^4 N = 1/N$. Adding a gluon to the edge gives two extra interaction vertices while creating no new closed colour lines. Thus the dominant diagrams of this type contain only quarks at the edge.

The dominant diagrams of any type of interaction contain, as the second selection rule above tells us, a minimum number of internal quark loops. In considering quark bilinears every diagram must contain at least one $q\bar{q}$ pair. It is not unreasonable to suppose, then, that the dominant diagrams contain only one quark-loop at the edge of the diagram. Figure 1.6 verifies this.

FIGURE 1.6

Thus for large N the dominant contributions to the matrix elements of quark bilinears come from planar diagrams with only a single quark loop at the edge.

1.1.3 Properties of Mesons for Large N

Now that we have simplified QCD, what information can we elicit from our selection rules? We cannot extract any quantitative results since as yet we have no way to sum the planar diagrams. However, with the knowledge that for large N it is the planar diagrams that dominate, and a simple assumption, we can form a qualitative picture of the behaviour of mesons and baryons in the large N limit. The assumption is that, as for $N=3$, QCD is a confining theory for arbitrarily large N . That is to say, the planar diagrams sum to give a confining theory.

To begin the investigation of meson physics for large N we consider quark bilinears, such as $q\bar{q}$ or $\bar{q}_\gamma q_\mu$ generically denoted by $J(x)$, whose matrix elements have the proper quantum numbers to create a meson. By considering the two-point function of J we can derive the following static properties of mesons in the large N limit:

- 1) mesons are stable particles,
- 2) meson masses have a smooth limit for large N , and
- 3) the number of mesons is infinite.

To derive these results from the large N expansion we need first show that the operator J creates only one meson states. Since diagrams with one $q\bar{q}$ pair are dominant, and confinement persists, we know that we have only one meson. What we must rule out is the possibility of an intermediate state of one meson plus a glue state. Then we can write the two-point function as the sum of tree diagrams where J creates a meson, with amplitude $a_n = \langle 0 | J | n \rangle$,

which propagates with a propagator $(k^2 - m_n^2)^{-1}$:

$$\langle J(k)J(-k) \rangle = \sum_n \frac{a_n^2}{k^2 - m_n^2} \quad 1.1-1$$

a_n a_n
 $\sum_n$ $x \text{-----} x$

This can be verified by showing that if one cuts a leading order diagram to 1.1-1, the only intermediate states are one-meson states.

A typical leading order diagram to the two-point function is

FIGURE 1.7

where the vertical line represents a cut, revealing a quark anti-quark pair plus two gluons. Confinement demands that these form a colour singlet hadron. By considering the $SU(N)$ indices we can show that these intermediate states form a perturbative approximation to a single hadron.

The cut in Figure 1.7 reveals the quarks and gluons coupled as

$$\bar{q}_i \quad A_j^i \quad A_k^j \quad q^k$$

1.1-2

This is simply read by inspecting Figure 1.7 in double line notation.

FIGURE 1.8

The four fields in 1.1-2 are coupled to a colour singlet. Can we factorize them into two colour singlets? The answer is that no smaller combination of these fields can form a colour singlet. The intermediate states of planar diagrams are always of the form 1.1-2; they are one-meson states. Thus 1.1-1 is verified.

Now we can deduce the above enumerated results. Since the one-particle poles of 1.1-1 are on the real axis, otherwise violating the Lehman spectral representation, mesons are stable. That mesons have a smooth limit can be seen by considering the N dependence of 1.1-1. Since the left-hand side is the sum of planar diagrams, it has a smooth N limit. The lowest order diagram for 1.1-1 is

FIGURE 1.9

which is of order N . Adding any number of internal gluons will not alter the N dependence, as long as the diagram remains planar. Thus the two-point function is of order N . This implies that the right-hand side of 1.1-1 has a smooth N limit, is of order N , thus giving

$$a_n \sim O(\sqrt{N})$$

$$m_n \sim O(1).$$

Finally, we deduce that the numbers of meson states is infinite. By asymptotic freedom, the left-hand side of 1.1-1 behaves logarithmically for large K^2 . Thus the sum must be infinite, otherwise leaving the right-hand side of 1.1-1 with a $\frac{1}{K^2}$ dependence for large K^2 .

The dynamical properties of mesons in the large N limit can also be deduced by counting powers of N in the leading order diagrams. To derive the amplitude for a two body decay, $A \rightarrow BC$, we consider the three-point function

FIGURE 1.10

where we have assumed that the state revealed by cutting a leading order diagram to $\langle JJJ \rangle$ will reveal a one-meson state. The proof is much as before and so is not reproduced here. A typical leading order

diagram is

FIGURE 1.11

which is of order N . The amplitude for this process contains a piece $a_n^3 \Gamma_{nnn}$. Since a_n is of order $\sqrt{N}$, the trilinear meson-meson-meson vertex is of order $N/N^{3/2} = 1/\sqrt{N}$. Thus the amplitudes for two-body decays $A \rightarrow BC$ are of order $1/\sqrt{N}$.

The amplitudes for two-body meson scattering are similarly derived. If one were to cut a leading order diagram to the four-point function, the intermediate state revealed would be, not unexpectedly, a one-meson state. Thus we can represent the four-point function as

$$\langle JJJJ \rangle = \sum \text{[diagram 1]} + \sum \text{[diagram 2]} + \sum \text{[diagram 3]} + \sum \text{[diagram 4]} + \dots$$

FIGURE 1.12

Thus the leading order diagrams contributing to the two, three and four point functions are tree diagrams with physical mesons exchanged. The typical leading order diagram for $\langle JJJJ \rangle$,

FIGURE 1.13

which is of order N , contains a piece $a_n^4 r_{nnn}^2$; four currents, each creating one meson from the vacuum, and two trilinear vertices. This is consistent with our previous result for r_{nnn} . The four point function also contains a term $a_n^4 r_{nnnn}$; four currents, creating mesons from the vacuum, plus a quartic meson vertex. Thus the quartic meson vertex is of order $N/(N^{1/2})^4 = 1/N$. From this we conclude that at $N=\infty$ mesons are free and non-interacting, and QCD is a free field theory of mesons. This result is crucial to the program of this thesis.

Since mesons are non-interacting, two of them are unlikely to be found bound together into an exotic meson; at least, this would be suppressed. Exotics, while not totally absent from phenomenology, are rare and thus appear to be suppressed. The large N expansion is the only known field theoretic explanation for this suppression.

Another phenomenological rule that has thus far been explained only by the large N expansion is the Zweig rule. Consider the Zweig-violating decay $A \rightarrow BC$, shown below.

FIGURE 1.14

The quark loops contribute a combinatoric factor of N^2 while the vertices give the process a total factor of $N^2 / (N^{1/2})^4 = 1$.

A diagram for the process $A \rightarrow BC$ that is Zweig conserving is

•FIGURE 1.15

which is of order $N^3/(N^{1/2})^4 = N$. Thus a Zweig violating diagram is suppressed by a factor of $1/N$ relative to one obeying Zweigs rule. As $N \rightarrow \infty$ Zweigs rule is seen to be exact.

1.1.4 Properties of Baryons for Large N

The treatment of baryons in the large N limit differs markedly from that of mesons. Considering their structure, this should come as no surprise. A meson is a $q\bar{q}$ pair and remains so for any number of colours. Although the order of a Feynman diagram depends upon N, its shape does not. A baryon, on the other hand, is an antisymmetric state composed of N quarks. Thus, for different values of N one must draw different diagrams. Another consequence of the structural difference between mesons and baryons is the N dependence of their mass. Although baryons are not much heavier than mesons, in the large N limit their mass is calculated to be of order N, whereas mesons have a mass of order unity. To see this consider the following simplistic quark model.

A baryon, composed of N quarks, will have a mass given by the sum of the quark masses, kinetic energies and inter-quark potentials. If the quark masses are m , then these will contribute an amount Nm . If one quark has a kinetic energy T , we assume that N quarks have a kinetic energy NT . Finally, the interaction between a quark pair is of order N^{-1} , contributing an amount $N^{-1}V$, where V is some potential. There are $\frac{1}{2}N(N-1)$ quark pairs, giving a total baryon mass of

$$M = NM + NT + \frac{1}{2} N^2 \frac{V}{N}, \quad 1.1-3$$

where we disregard the difference between $N-1$ and N for large N . Equation 1.1-3 may be written

$$M = N(f(g^2, m) + O(1/N)) \quad 1.1-4$$

for some function, f , of the quark mass and coupling constant.

The consequences of 1.1-4 is rather severe. Consider the amplitude for the propagation of an N quark state for a time t ,

$$e^{-iM_B t}.$$

If we take $f(g^2, m)$ to be of the form

$$f(g, m) = m(1+g^2)$$

and attempt an expansion of the amplitude, we find

$$\begin{aligned} e^{-iM_B t} &= e^{-iNf(g^2, m)t} \\ &= e^{-itNm} (1 - iNmNg^2 - \frac{1}{2} m t^2 N^2 g^4 + \dots). \end{aligned} \quad 1.1-5$$

The higher powers of g are successively more divergent in N ; clearly an undesirable property in a perturbative expansion.

This phenomenon is evident in the simplest of diagrams. The process of one gluon exchange, corresponding to the lowest order correction to the free propagation of a N quark state, has the following diagram:

FIGURE 1.16

Let us count the powers of N in this diagram. The two quark-gluon vertices contribute a factor $(1/\sqrt{N})^2 = 1/N$. The combinatoric factor associated with Fig. 1.16 arises since the gluon may be exchanged between any two quark pairs: There are $\frac{1}{2} N(N-1)$ such pairs which, for large N , give a factor of $\frac{1}{2} N^2$. Thus Fig. 1.16 is of order N and diverges for large N . The higher order process involving the exchange of two gluons receives a factor $(1/\sqrt{N})^4 N^4 = N^2$. As was evident from 1.1-5, the perturbative expansion seems inappropriate.

From 1.1-4 we know that the origin of this apparently divergent behaviour is the order of the baryon mass. This does not, then, preclude the existence of a large N limit for baryons. It does, however, preclude the use of the diagrammatic method used for mesons. The large N limit for baryons does exist and the method used is due, like so much of the material in this thesis, to Witten (Witten, 1979).

The method used is a simple, non-relativistic Hamiltonian analysis. We assume that the quarks are heavy, allowing us to write a non-relativistic Schrodinger equation for an N-quark bound system. For heavy quarks we assume a simple Coulomb potential, arriving at the Hamiltonian,

$$H = Nm + \sum_i \left(-\frac{\nabla_i^2}{2m} \right) - \frac{g^2}{N} \sum_{i < j} \frac{1}{(x_i - x_j)}. \quad 1.1-6$$

Note that, although the potential term appears to be of order $1/N$, we cannot treat this as a perturbation since there are $\frac{1}{2} N^2$ terms in the sum. What we can do is consider the baryon as a bound state of pairwise weakly interacting particles, ($O(1/N)$), with a cumulative potential of order N (order unity experienced by each particle). This is the problem which the Hartree approximation was born for. In the following we treat the baryon to a Hartree analysis, finding reasons to believe the approximation exact for $N \rightarrow \infty$. Although the assumptions required to write 1.1-6 are not realistic, the qualitative results we shall derive are in fact valid in the relativistic case (Witten, 1979).

We first consider the problem of a ground state baryon. Thus each quark must be placed in the ground state of some average potential that it experiences. We must first determine this average potential for each quark.

Since quarks are fermions, the total wave function for the N quarks must be antisymmetric with respect to quark interchange. In a baryon the wave function is antisymmetric with respect to colour, as this is why colour was first introduced. If we ignore flavour for now, we must symmetrize with respect to space and spin. This effectively leaves us with a system of N identical bosons. Thus in the ground state of the baryon we must place all N quarks in the same ground state of the average Hartree potential. Thus each quark experiences the same average potential, greatly simplifying the analysis.

This allows us to write the wave function for the system of N quarks as a product of N identical one particle wave functions:

$$\psi(x_1, \dots, x_N) = \prod_{i=1}^N \phi(x_i) \quad 1.1-7$$

(We shall soon show how this equation becomes exact for large N). The exact wave function $\psi(x_1, \dots, x_N)$ can be obtained by demanding that the following functional be stationary;

$$\begin{aligned} \langle \psi | H - E | \psi \rangle = & N m + N \int d^3 x \frac{\bar{\nabla} \phi^* \bar{\nabla} \phi}{2m} \\ & + \frac{1}{2} N^2 \left(\frac{-g^2}{N} \right) \iint \frac{d^3 x d^3 y \phi^*(x) \phi(x) \phi^*(y) \phi(y)}{|x-y|} - N \epsilon \int d^3 x \phi^* \phi \end{aligned} \quad 1.1-8$$

where

$$E = N \epsilon$$

defines ϵ to be the energy per quark and E the total energy.

The variational equation derived from 1.1-8, obtained by varying with respect to ϕ , is

$$\frac{-\nabla^2}{2m} \phi(x) - g^2 \phi(x) \int d^3 y \frac{\phi^*(y) \phi(y)}{|x-y|} = \epsilon \phi(x) \quad 1.1-9$$

The fact that 1.1-9 is independent of N allows us to deduce two important static properties of baryons. Firstly, 1.1-9 is independent of N precisely because every term of 1.1-8 is of order N . That is to say that for wave functions such as 1.1-7, the expectation value of the Hamiltonian is of order N . This reinforces our earlier simplistic determination of the order of the baryon mass. Secondly, the size and shape of a baryon is independent of N . The charge density $\phi^*(x)\phi(x)$ is independent of N , since 1.1-9, which determines ϕ , has no N dependence. The size and shape are determined by the N independent function $\phi^*(x)\phi(x)$, while the charge density is $N\phi^*\phi$.

That 1.1-7 becomes exact as $N \rightarrow \infty$ can be rigorously proven by path integral methods (Witten, [979]). We give a more qualitative proof here. The argument rests on the fact that while the size of the baryon is of order one, the charge density is of order N . Thus for large N the system becomes dense so that, as for an atom with many electrons, 1.1-7 becomes exact as N (Z for atoms) approaches ∞ .

We are now in a position to calculate the N dependence of various baryon scattering processes. Consider first baryon-baryon scattering, whose dominant interaction results from the exchange of a quark pair.

FIGURE 1.17

In Figure 1.17(a) we have a combinatoric factor of N^2 coming from choosing one quark each from the two baryons. The two quark-gluon vertices give this process to be of order N . In the case of gluonless quark exchange (Figure 1.17b), the exchanged quarks must have the same $SU(N)$ quantum numbers, there being no gluon to ensure colour neutrality. Thus we have a combinatoric factor of N resulting only from choosing the first quark, the second being determined by the first. Thus baryons interact strongly. Since the baryon mass is also of order N , we need not worry that baryon-baryon scattering is a divergent process. For a given velocity the baryon kinetic energy T is of order N as is the interaction energy E . Thus the scattering cross section

$$\sigma \propto \frac{E}{T}$$

1.1-10

has a smooth N limit.

The dominant process for baryon-antibaryon scattering involves the annihilation of a $q\bar{q}$ pair, one each from the baryon and antibaryon, respectively.

FIGURE 1.18

We can choose a q and $\bar{q}$ from any N in each baryon, with the gluon couplings contributing a factor of $1/N$. The amplitude for this process is thus of order N , assuring a smooth N limit for the scattering cross section.

The last scattering channel to be considered is meson-baryon scattering. Here the baryon and meson exchange one quark. We can choose this quark from among N in the baryon, whereas a meson has only one quark. The obligatory gluon exchange (to conserve colour) contributes a factor of $1/\sqrt{N}$ at each vertex. The process is thus of order $N (1/\sqrt{N})^2 = 1$, and is negligible compared to the baryon kinetic energy. Thus, to leading order in $1/N$, the baryon propagates freely, unaffected by the presence of the mesons. Since the meson mass is of order unity, the meson will be scattered by the baryon.

By now the reader should be aware that the diagrammatic method is still applicable to baryons. This only became obvious after we had a mathematical framework (Hartree approximation) which told us how to interpret the results from diagram combinatorics.

As a last example we consider the forbidden process $e^+e^- \rightarrow B\bar{B}$.

FIGURE 1.19

The virtual photon will create a $q\bar{q}$ pair but has a probability, p , say, to create an extra $q\bar{q}$ pair. To create a $B\bar{B}$ pair we need $N-1$ extra $q\bar{q}$ pairs. The probability of these being created is p^{N-1} , so that the amplitude vanishes as e^{-CN} where $C = -\ln p$. Here, then, is a quantity without a smooth N limit.

1.1.5 Summary

What we have learned of the nature of QCD in the large N limit is summarized below:

- 1) (a) Planar diagrams dominate while nonplanar diagrams are suppressed by a factor of order $1/N^2$.
 (b) Internal quark loops are suppressed by a factor of order $1/N$.
- 2) (a) Meson masses are of order unity
 (b) Mesons interact weakly, with decay amplitudes of order $1/\sqrt{N}$ and elastic scattering amplitudes of order $1/N$.
- (c) Interactions are described by tree diagrams with physical mesons exchanged.
- 3) (a) Baryon masses are of order N .
 (b) Baryons interact strongly, with the baryon-baryon interactions being of order N while baryon-meson interactions are of order unity.

From the properties listed in 2) above we can conclude that for large N QCD is a weakly coupled field theory of physical mesons. The mesons are represented as effective local fields with effective local interactions of order $1/N$. How, then, can we understand the baryon in such a picture?

The mass of the baryon is inversely proportional to the coupling constant. The baryon size and shape are independent of the coupling constant. Further, baryon interactions are strong, having non-trivial small coupling limits. In short, the baryon possesses all the properties of a soliton in a weakly coupled field theory (see next section). Baryons behave like solitons in the large N approximation of QCD.

1.2 Solitons in Weakly Coupled Field Theories

1.2.1 Classical Solitons

Consider a model theory of mesons described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{g^2} V(g\phi) \quad 1.2-1$$

where g is a dimensionless coupling constant and the potential V attains its minimum value at $\phi = 0$. If we choose this minimum value to be zero, V will have the power series expansion

$$\frac{1}{g^2} V(g\phi) = \frac{1}{2} m^2 \phi^2 + O(g\phi^3) + O(g^2 \phi^4) + \dots \quad 1.2-2$$

The equations of motion,

$$\partial_\mu \partial^\mu \phi + \frac{1}{g} V'(g\phi) = 0, \quad V'(g\phi) = \frac{dV(g\phi)}{d(g\phi)} \quad 1.2-3$$

can be rewritten, using 1.2-2, as

$$-\partial_\mu \partial^\mu \phi = m^2 \phi + O(g\phi^2) + O(g^2\phi^3) + \dots$$

1.2-4

Thus, the parameter g characterizes the strength of the non-linear couplings. Note that when $g = 0$ 1.2-4 becomes linear, giving only plane wave solutions. Using the loose definition of a soliton as a spatially confined, non-dispersive solution of a classical field theory, we see that soliton solutions are possible only for non-linear couplings. Since g characterizes the strength of these couplings we consider the g -dependence of various soliton properties.

Let ϕ be a classical solution to 1.2-3. Defining

$$\phi \rightarrow \frac{\phi}{g}$$

1.2-5

the coupling constant may be scaled out of the Lagrangian as an overall factor:

$$\mathcal{L} = \frac{1}{g^2} \mathcal{L}_0$$

1.2-6

where

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

1.2-7

Since $\mathcal{L}_0$ is independent of g , so also is its classical solution, σ . As long as g is non-zero, then, the existence of the soliton is independent of g . From 1.2-5 we see that the soliton solution becomes singular as $g \rightarrow 0$. We also learn from 1.2-6 that the soliton mass is of order $1/g^2$. Thus the mass diverges in the limit $g \rightarrow 0$. The story is beginning to sound quite familiar.

The soliton solutions can be classified into two general types, topological and nontopological; we shall be interested in only the former species. In what follows we consider separately the cases

where V has a discrete or continuous set of minima.

Consider 1.2-7 in one space dimension. Static solutions are given by

$$\frac{d^2\sigma}{dx^2} - \frac{dV}{d\sigma} = 0 \quad 1.2-8$$

which, after multiplying by $d\sigma/dx$ and integrating, becomes

$$\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 - V(\sigma) = \text{constant.} \quad 1.2-9$$

The energy functional is given by

$$E[\sigma] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + V(\sigma) \right]. \quad 1.2-10$$

In order that the soliton have finite energy at spatial infinity, both $V(\sigma)$ and $d\sigma/dx$ must vanish at $x = \pm \infty$. Let the minimum value of $V(\sigma)$ occur at n points σ^i , $i = 1, \dots, n$, which are also its zeros:

$$V(\sigma) = 0 \text{ for } \sigma = \sigma^i \quad i = 1, \dots, n.$$

Thus finite energy requires that σ take on one of these values at spatial infinity. Taking the limit $x \rightarrow \infty$ in 1.2-9, we see that the integration constant is zero.

Thus we are interested in finite energy solutions to

$$\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 = V(\sigma) \quad 1.2-11$$

with the above boundary conditions.

There is a useful mechanical analogue to 1.2-11. Considering σ as a particle's position and x the time, the particle can be said to move in a potential $-V(\sigma)$, shown below:

FIGURE 1.20

At time $x = -\infty$ the particle begins along a trajectory starting at one of the boundary values, σ^a . As time evolves the particle moves along the path, reaching the position σ^b at $x = +\infty$. From 1.2-11 it can be shown that all derivatives of σ vanish at $\sigma = \sigma^b$; this is energy conservation. Thus as $x \rightarrow \infty$ the particle will rest at $\sigma = \sigma^b$ rather than, say, go on to the right or back to σ^a . To do otherwise would require infinite energy.

Thus for a non-singular finite-energy solution, the field must approach, at any instant, the value of a minimum of V at every point of spatial infinity. In one space dimension infinity consists of the two points $x = \pm \infty$. If at some time t_0 ,

$$\lim_{x \rightarrow \infty} \phi(x, t_0) = \phi_1,$$

then ϕ_1 is an absolute minimum of V . As time evolves the field ϕ will vary continuously with t and x , according to some differential equation. In particular, the field $\phi(\infty, t)$ will be a continuous function of t . Since $\phi(\infty, t)$ must always be one of the minima of V , a discrete set, it cannot jump to another, $\phi_2(\infty, t)$, say. Thus $\phi(\infty, t)$ remains constant.

As a consequence we may divide the space of non-singular finite energy solutions into topologically unconnected sectors. A field from one sector, characterized by its asymptotic behaviour, cannot be continuously distorted into another without violating the requirement of finite energy. Since time evolution is an example of a continuous distortion, a field from some sector stays within that sector as time evolves. This signals the existence of a conserved quantity, the topological index.

The topological index arises in quite a different manner when the potential has a continuous set of minima. The simplest example of this, $V(\phi)=0$, is provided by the non-linear $O(4)$ model⁺, consisting of four real fields, ϕ_a , subject to the constraint

$$\phi_a \phi_a \equiv \bar{\phi} \cdot \bar{\phi} = 1. \quad 1.2-12$$

The dynamics of the system are derived from the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\partial_\mu \bar{\phi}) \cdot (\partial^\mu \bar{\phi}) \\ &= \frac{1}{2} (\partial_\mu \phi_a) (\partial^\mu \phi_a). \end{aligned} \quad 1.2-13$$

Note that $\bar{\phi}$ can be considered as a vector in internal space, whose components are labelled by the index a , as opposed to vectors in coordinate space, labelled by Lorentz indices such as μ .

The equations of motion are obtained by extremizing the action

$$S[\phi] = \int d\bar{x} dt \left[\frac{1}{2} (\partial_\mu \bar{\phi}) \cdot (\partial^\mu \bar{\phi}) + \lambda(\bar{x}, t) (\bar{\phi} \cdot \bar{\phi} - 1) \right]$$

where $\lambda(\bar{x}, t)$ is the undetermined Lagrange multiplier. For a static field in three space dimensions, these are

+ This is essentially the non-linear sigma model, with $\sigma = \phi_0$, $\vec{\pi} = \frac{1}{F_\pi} (\phi_1, \phi_2, \phi_3)$

$$\nabla^2 \bar{\phi} - (\bar{\phi} \cdot \nabla^2 \bar{\phi}) \bar{\phi} = 0.$$

The energy of a static solution is given by

$$E = \frac{1}{2} \int (\partial_i \bar{\phi}) \cdot (\partial_i \bar{\phi}) d\bar{x} \quad i=1,2,3. \quad 1.2-14$$

We note in passing that the classical vacua corresponding to zero energy, are given by

$$\partial_i \bar{\phi} = 0$$

or

$$\bar{\phi}(\bar{x}) = \bar{\phi}_0.$$

While $\bar{\phi}_0$ is $\bar{x}$ -independent, it may point in any direction in internal space. Thus we have a degenerate continuous family of vacua, related to one another by $O(4)$ rotations in internal space. This is a case of spontaneous symmetry breaking. More on this later.

The finite energy soliton solutions must satisfy

$$\lim_{x \rightarrow \infty} \bar{\phi}(\bar{x}) = \bar{\phi}_0 \quad 1.2-15$$

where $\bar{\phi}_0$ is some unit vector in internal space. As we approach spatial infinity from any direction, the limit 1.2-15 must be satisfied.

Otherwise the angular component of the gradient would not vanish. Thus the physical space R^3 is compactified into the three-sphere, S^3 ; all points of spatial infinity are mapped into one point, the north pole of S^3 . The condition 1.2-12 tells us that the internal space is also a unit three-sphere. Therefore any finite-energy static configuration is a mapping of S^3 into S^3 . All such mappings can be classified into homotopy sectors. We will examine this in detail in the next chapter and for now state the result that these homotopy sectors form a group isomorphic to the group of integers. Furthermore, a mapping from one

sector cannot be continuously deformed into that of another. The integer associated with each sector is the conserved topological index for this example.

This is to be contrasted with the last example, where the topological sectors arose from differing boundary values of the fields at infinity. Here, the boundary condition 1.2-15 only serves to compactify the coordinate space.

The topological charge differs from the usual examples of conserved quantities in that it is not derived from a symmetry of the Lagrangian. What is of interest is that topological indices can be related to spontaneous symmetry breaking. The existence of the topological sectors is related to the existence of a set of degenerate minima of the potential. If both the Lagrangian and potential are invariant under a transformation acting on the fields, the complete set of degenerate minima of the potential must be also. They may not, however, be individually invariant; one may be transformed into another. This will be of some importance to us when we quantize the $O(4)$ model in Chapter 2. First we must consider the question of quantizing about classical solutions.

1.2.2 Quantization

We begin with the example of the relationship between static solutions of Newton's equation of motion and non-relativistic quantum energy levels, whose generalization to field theory forms the basis for quantizing static solutions.

Classically, the motion of a non-relativistic particle of unit mass, under the influence of a potential $V(x)$, is described by Newton's equation

$$\frac{d^2 x(t)}{dt^2} = - \frac{dV(x)}{dx}$$

1.2-12

In quantum theory, the energy eigenstates $\psi_n(x)$ satisfy

$$H_n \psi_n(x) = E_n \psi_n(x)$$

1.2-13

where

$$H = \frac{\hat{p}^2}{2} + V(\hat{x}) \text{ and } \hat{p} = i\hbar \frac{\partial}{\partial x}$$

Consider a potential $V(x)$ with extrema at $x = a, b$ and c , as shown:

FIGURE 1.21

Static solutions of 1.2-12 are then given by these extremes:

$$\frac{dV}{dx} = 0.$$

Since $x = a$ is a global minimum we shall call the solution $x(t)=a$ the classical ground state, whose energy is

$$E_0^{Cl} = V(a).$$

This state, with zero momentum and a fixed position, clearly violates the Heisenberg Uncertainty Principle. Quantum theory demands that the lowest eigenvalue of 1.2-13 reflect the presence of some zero point motion, leading to a ground state

$$\begin{aligned} E &= V(a) + \Lambda_0 \\ &= E_0^{cl} + \Lambda_0, \end{aligned}$$

where Λ_0 represents the energy contribution from this zero point motion. We find Λ_0 by a weak coupling expansion of the potential.

If the potential function is approximately harmonic in a neighbourhood of $x=a$, we can find Λ_0 by Taylor expanding $V(x)$ and terming this a weak coupling expansion:

$$V(x) = V(a) + \frac{w^2}{2}(x-a)^2 + \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (x-a)^r$$

For λ_r sufficiently small the low-lying energy eigenstates, whose spread is localized about $x=a$, will satisfy

$$\lambda_r \langle (x-a)^r \rangle \ll \frac{w^2}{2} \langle (x-a)^2 \rangle, \quad r \geq 3 \quad 1.2-14$$

Treating the anharmonic terms of $V(x)$ as a perturbation to the harmonic oscillator potential, we arrive at the well known solution:

$$E_n = V(a) + (n + \frac{1}{2}) \pi w + o(\lambda_r), \quad 1.2-15$$

with a ground state

$$E_0 = V(a) + \frac{\pi w}{2} + o(\lambda_r). \quad 1.2-16$$

The left-hand side of equation 1.2-16 is the energy of a quantum ground state while all the other terms on the right hand side can be related to the corresponding classical solution. The first term is simply the classical ground state; the second, the classical stability frequency, related to the second derivative of the potential. The higher order quantum corrections involve derivatives of the potential at the classical solution $x(t) = a$. Further, 1.2-15 demonstrates that not only the ground state energy but that of a tower of low-lying states can be similarly related to the classical solution.

For us the most important feature of this method is its failure for the zero frequency modes, when for $w=0$ the weak coupling condition, 1.2-14, cannot be satisfied. Before discussing this we first generalize the above results to field theory.

Consider a scalar field $\phi(x,t)$ governed by a Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad 1.2-17$$

where $V(\phi)$ is any function of ϕ which has a lower bound. The Euler-Lagrange equation derived from 1.2-17 is

$$\frac{\partial^2 \phi}{\partial t^2} = - \frac{\delta U[\phi]}{\delta \phi}$$

where the functional $U[\phi]$ is given by

$$U[\phi] = \int d^3x \left(\frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi + V(\phi) \right). \quad 1.2-18$$

The static solutions are again extrema of the potential $U[\phi]$ in field space. Let ϕ_0 be such a solution. We again attempt a weak coupling expansion, now by use of a functional Taylor expansion of U about ϕ_0 :

$$U[\phi] = U[\phi_0] + \int d^3x \frac{1}{2} \left(\eta(x) \left[-\nabla^2 + \frac{d^2 V}{d\phi^2} \Big|_{\phi_0(x)} \right] \eta(x) \right) + \dots$$

where $\eta(x) \equiv \phi(x) - \phi_0(x)$. The frequencies w_i are obtained by solving the eigenvalue equation for the second derivative operator:

$$\left[-\nabla^2 + \frac{d^2 V}{d\phi^2} \Big|_{\phi_0(x)} \right] \eta_i(x) = w_i^2 \eta_i(x) \quad 1.2-19$$

where $\eta_i(x)$ are the orthonormal normal modes.

If we expand $\eta(x,t)$ in terms of its normal modes

$$\begin{aligned} \eta(x,t) &= \phi(x,t) - \phi_0(x) \\ &= \sum_i C_i(t) \eta_i(x), \end{aligned} \quad 1.2-20$$

then

$$\begin{aligned} \frac{1}{2} \int d^3x \left(\frac{\partial \phi}{\partial t} \right)^2 &= \frac{1}{2} \sum_{ij} \dot{C}_i(t) \dot{C}_j(t) \int \eta_i(x) \eta_j(x) d^3x \\ &= \frac{1}{2} \sum_i [\dot{C}_i(t)]^2 \end{aligned}$$

and

$$U[\phi] = U[\phi_0] + \frac{1}{2} \sum_i [C_i(t)]^2 w_i^2 + \dots$$

thus the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \left(\sum_i [\dot{C}_i(t)]^2 - [C_i(t)]^2 w_i^2 \right) - U[\phi_0] \quad 1.2-21$$

Apart from the constant $U[\phi_0]$, this is the Lagrangian for a set of harmonic oscillators, one for each normal mode. By quantizing the coefficients $C_i(t)$, we derive the energy for these states to be

$$E_{\{n_i\}} = U[\phi_0] + \hbar \sum_i (n_i + \frac{1}{2}) \omega_i. \quad 1.2-22$$

We now turn our attention to quantizing in the presence of zero frequency modes.

1.2.3 Zero Modes and Collective Coordinates

The zero frequency modes warned of earlier are by no means accidental occurrences but manifestations of some invariance of the theory. To understand this consider the case of translational invariance, where the potential, Lagrangian and field equations are all invariant. If $\phi_0(x)$ is a solution to the field equations then so is $\phi_0(x-a)$, for any real a . Furthermore, both solutions have the same potential. The family of solutions $\phi_0(x-a)$, for all a , form an equipotential curve in function space. If we displace the field along this equipotential curve, all functional derivatives of $U[\phi]$ will vanish, in particular the second derivative. This is the origin of the zero frequency eigenvalue for the second derivative operator.

In general, let $U[\phi]$ be invariant under a transformation

$$\phi \rightarrow \phi + \delta\phi$$

Symbolically, the eigenvalue equation for the generalized second derivative operator is

$$\frac{\delta^2 U}{\delta \phi^2} \eta(x) = \omega^2 \eta(x)$$

where

$$\eta(x) = \phi(x) - \phi_0(x)$$

and ϕ_0 is a solution of the field equations. Displacing along the equipotential, $\eta(x) = \delta\phi$, we see that $\delta\phi$ is the eigenmode associated with the zero frequency eigenvalue. Thus the zero frequency eigenmode is the local tangent to the equipotential curve.

If the theory is invariant under many transformations and/or internal symmetries, there will be more than one zero frequency mode. In fact, there will be one for each parameter of the symmetry groups. Thus for translational invariance in three dimensions we would expect three zero frequency modes corresponding to three independent translations of a non-trivial solution. We now determine the effects of these modes on quantum wave functions that are constructed about the classical solutions.

The classical, static solutions, ϕ_c , are local minima of the potential, except along the equipotential curve. Orthogonal to this curve the potential rises, allowing us to construct approximate oscillator energy eigenstates in a weak coupling expansion about ϕ_c . Along the equipotential curve the potential has no confining influence, allowing the wave functions to spread. The quantum states are no longer localized about ϕ_c . Thus the presence of a zero mode demands a different treatment than was outlined above for quantizing classical solutions. We shall examine the method of collective coordinates.

We begin with an example in non-relativistic quantum mechanics in two dimensions. A particle's motion is described by the Lagrangian

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2}), \quad \dot{x} = \frac{dx}{dt}$$

L is invariant under rotations and we would expect one zero frequency mode as a result. A static solution of the equations of motion will be a minimum of V . Let $R = \sqrt{x^2 + y^2}$ be this minimum. Then every point on the circle of radius R about the origin is a minimum.

The oscillator approximation to the energy eigenfunctions is given by

$$E_{n_1, n_2} = V(R) + \sum_{i=1}^2 (n_i + \frac{1}{2}) \hbar \omega + \dots$$

Clearly, the frequency corresponding to angular vibrations will vanish. When we attempt to calculate higher order corrections we find that both ω_i appear in denominators. The assumptions of the semi-classical method have been violated.

To rectify the situation we transform coordinates, in this case, to polar coordinates. In terms of (r, θ) the symmetry becomes explicit:

$$L = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - V(r)$$

The θ dependence of the problem is easily solved. Since θ is cyclic its conjugate momenta, π_θ , is conserved:

$$\pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = 0$$

where

$$\pi_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$= r^2 \dot{\theta}$$

$$= l$$

We now have a problem in one variable, r , and an effective potential

$$V_{\text{eff}} = \frac{\ell^2}{2r^2} + V(r).$$

Since rotations leave r unaffected, we no longer have any continuous symmetry and thus no zero mode. The static classical solutions satisfy

$$\left(\frac{d V_{\text{eff}}}{dr} \right)_R = 0,$$

leading to oscillator energy eigenvalues

$$E_{n,\ell} = V(R) + \frac{1}{2} \left(n + \frac{1}{2} \right) \hbar \omega + \frac{\ell^2}{2R}$$

Here the problem of the zero mode was avoided by transforming to a new coordinate system. In these new coordinates the Lagrangian was found to be independent of the variable θ , the very degree of freedom responsible for the zero mode.

This procedure can be generalized to field theory, where the collective coordinates play the same role as θ in this example.

Consider again the system defined by equations 1.2-17 to 1.2-22, in one space dimension. The system exhibits translational invariance so that we expect one zero mode. Let $\sigma(x-a)$ be a static solution, with a mass

$$\begin{aligned} M &= \int \frac{1}{2} \left(\frac{d\sigma}{dx} \right)^2 + V(\sigma) dx \\ &= \int \left(\frac{d\sigma}{dx} \right)^2 dx \end{aligned}$$

1.2-23

that is independent of a .

The method of collective coordinates here involves omitting the zero eigenmode, $\eta_0(x-a)$ say, from the expansion 1.2-20 and writing instead

$$\phi(x,t) = \sigma(x-a(t)) + \sum_{n=1}^{\infty} q_n(t) \eta_n(x-a(t)) \quad 1.2-24$$

where now a is a dynamical variable, $a(t)$. We call $a(t)$ the collective coordinate associated with translational invariance. The idea is to use $a(t)$ and $q_n(t)$ as the coordinates of the system to be quantized, rather than the $C_n(t)$ of 1.2-20.

The Lagrangian is now given by

$$L = \frac{1}{2} \left\{ \left[\dot{a}(t) \right]^2 + \sum_{n=1}^{\infty} \dot{a}(t) \dot{q}_n(t) B_n + \left[\dot{q}_n(t) \right]^2 \right\} - U[q_n] \quad 1.2-25$$

where

$$\alpha = \int dx \left[\frac{d\sigma}{dx} + \sum_{n=1}^{\infty} q_n \frac{d\eta_n}{dx} \right]^2$$

$$B_n = - \int dx \left[\sum_{m=1}^{\infty} q_m \frac{d\eta_m}{dx} + \frac{d\sigma}{dx} \right] \eta_n$$

and

$$U[q_n] = M + \frac{1}{2} \sum_{n=1}^{\infty} q_n^2 \omega_n^2$$

The Hamiltonian is given by

$$H = \sum_{i=0}^{\infty} \pi_i \dot{u}_i - L$$

where $\pi_i = \frac{\partial L}{\partial \dot{u}_i}$

and $u_i(t) = \begin{cases} a(t) & i=0 \\ q_i(t) & i \geq 1 \end{cases}$

We find

$$H = \frac{1}{2} \sum_{i=0}^{\infty} \pi_i^2 + U[q_n].$$

1.2-26

Note that the collective coordinate $u_0(t) = a(t)$ is cyclic, leading to a conserved conjugate momentum π_0 .

The system is now quantized by the usual canonical method, converting the u_n and π_n into operators. This method can be used to handle any number of continuous symmetries. In cases where this becomes too complex, one searches for a physical argument allowing one to ignore other symmetries as a first approximation. Such is the case for the $O(4)$ model discussed previously. This model, as developed by Skyrme, is the subject of the next chapter, where the question of quantization will be treated in detail. To end this Chapter, we reconsider the large N limit of QCD, discussing the relevance of this limit to the material developed, in this section on classical solutions.

1.2.4 Large N as a Classical Limit

To elicit the relation between classical and quantum solutions, consider again the Lagrangian 1.2-6:

$$\mathcal{L} = \frac{1}{g^2} \mathcal{L}_0.$$

A state vector in the corresponding quantum theory is represented by the superposition of the amplitudes e^{iA} , where A is the action

$$A = \frac{1}{\hbar} \int \mathcal{L} d^4x$$

$$= \frac{1}{\hbar g^2} \int \mathcal{L} d^4x, \quad 1.2-27$$

the superposition being over all possible paths leading from some initial to final configurations. In the classical limit, $\hbar \rightarrow 0$, the integrand of the path integral, e^{iA} , will vary with greater frequency unless A is fairly constant as a functional of ϕ . This occurs when A is stationary and ϕ is the classical trajectory determined by the Euler-Lagrange equations.

Considering $\hbar$ as a weak coupling parameter, one can expand the generating functional in an asymptotic series in $\hbar$. The first term in the series gives the classical contribution to the Green's functions while the $O(\hbar)$ term gives the first quantum corrections. The Feynman rules derived from the first term give us the tree approximation to the theory while those from the $O(\hbar)$ term give the loop corrections. Moreover, if ϕ_0 is the field configuration about which we expand, then

$$\phi_0 = \phi_{cl} + O(\hbar) \quad 1.2-28$$

where ϕ_{cl} is the classical solution. Thus the wave functionals will have large amplitudes for ϕ_0 to be near the classical solutions.

In the large N limit, $g \rightarrow g/\sqrt{N}$, so that in effect the large N limit is a classical limit. From 1.2-27 we see that an expansion in $\hbar$ is equivalent to an expansion in $1/N$, with the first term giving the tree approximation to QCD.

2.1 INTRODUCTION

2.1.1 Chiral Symmetry

The reader is referred to Appendix A on the σ -model for a detailed example of what follows.

The underlying Lagrangian for QCD contains a kinetic term for free, massless quarks. Such a Lagrangian leads to conserved vector and axial vector currents;

$$V_{\mu}^i = \bar{q}_{\alpha} \tau^i \gamma_{\mu} q_{\alpha} \quad 2.1-1$$

$\alpha=1, \dots, N_f$, where N_f is the number of flavours

$$A_{\mu}^i = \bar{q}_{\alpha} \tau^i \gamma_{\mu} \gamma_5 q_{\alpha}$$

The combinations $V \pm A$ describe the weak isospin structure of the left- and right-handed quarks;

$$L_{\mu}^i = \bar{q}_{\alpha} \tau^i \gamma_{\mu} (1 - \gamma_5) q_{\alpha}$$

2.1-2

$$R_{\mu}^i = \bar{q}_{\alpha} \tau^i \gamma_{\mu} (1 + \gamma_5) q_{\alpha}$$

Because of the commutation relations amongst V and A currents, we have two independent representations of $SU(N_f)$; one for left-handed, the other for right-handed particles. The invariance of a theory under separate transformations for left- and right-handed particles is referred to as chiral $SU(N_f)_L \times SU(N_f)_R$ invariance. Thus a theory is chirally invariant if no piece of the Lagrangian mixes left- and right-handed particles.

It is precisely the quark mass terms in the QCD Lagrangian which violate this symmetry. When quarks are taken to be massless (a good approximation for $N_f \leq 3$), QCD exhibits a global symmetry under chiral $SU(N_f)_L \times SU(N_f)_R$ transformations. As is discussed in Appendix A, the fact that nucleons are not massless leads us to require a mechanism to create this mass, related to the confinement of quarks. The generation of this mass breaks the $SU(N_f)_L \times SU(N_f)_R$ symmetry to the diagonal vector subgroup $SU(N_f)$. This in turn implies the existence of $N_f^2 - 1$ Goldstone modes, which we identify with the pseudoscalar mesons $\pi, \eta, k, \dots$. For $SU(2)$ we have three Goldstone particles which are the lightest mesons, the pions.

This symmetry breaking also implies that the vacuum states of the theory, labelled by a unitary matrix U , are in one to one correspondence with points in the $SU(2)$ manifold. To describe low energy dynamics, we assume that everywhere in space we have an approximate vacuum state where the parameters labelling the vacuum are slowly varying functions of space-time. Thus we promote U to a space-time dependent field $U(\vec{x}, t)$ which describes the Goldstone modes. U has the expansion

$$U = U_0 \left(1 + \frac{i}{F_\pi} \sum_{i=1}^3 \tau^i \pi^i + O(\pi^2) \right) \quad 2.1-3$$

where U_0 is the vacuum expectation value of U , τ^i are the generators of $SU(2)$ and π^i are the three Goldstone fields.

Within the context of the σ -model (see Appendix A), it is conventional to express the field U in the form

$$U = \frac{1}{F_\pi} (\sigma + i \vec{\tau} \cdot \vec{\pi}) \\ \equiv \phi_4 + i \tau^j \phi_j \quad 2.1-4$$

where σ is a scalar, isoscalar field.

Since

$$U^\dagger U = 1,$$

2.1-5

$$\phi^\dagger \phi_\alpha = 1$$

Under $SU(2)_L \times SU(2)_R$, U transforms nonlinearly as

$$U \rightarrow AUB^{-1}$$

2.1-6

where $A \in SU(2)_L$

and $B \in SU(2)_R$

We now show how to describe the dynamics of these Goldstone modes by an effective Lagrangian and then proceed to develop such a Lagrangian, leading to a study of the Skyrme model.

2.1.2 Effective Lagrangians

The matrix elements for soft-pion interactions can be obtained by use of the commutation and conservation relations of vector and axial-vector currents of chiral $SU(2) \times SU(2)$; by use of the so-called "current algebras". The same matrix elements may be calculated from an $SU(2) \times SU(2)$ invariant phenomenological Lagrangian which embodies all of current algebra. To find this Lagrangian, we use a "theorem" (unproven) of Weinberg (Weinberg, 1979):

Theorem: If one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles.

Our effective Lagrangian will thus be the most general $\mathcal{L}$ with all the physical meson fields which are consistent with the symmetries of QCD. For low energy, one need only consider the pion fields, described by U . Since U is a unitary matrix ($UU^\dagger = 1$), it is impossible to write a chirally invariant Lagrangian without derivatives of U . For low energy the dominant terms will be those with the least number of derivatives. Thus we write

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \dots \quad 2.1-7$$

where the subscript refers to the number of derivatives.

We will see that $\mathcal{L}_2$ alone cannot support solitonic solutions.

A minimal term of fourth order, ensuring the existence of soliton solutions, has been proposed by Skyrme. The study of this model is the main topic of this thesis.

2.2 The Skyrme Model

2.2.1 Skyrme Lagrangian

The unique choice for the two-derivative term is

$$\mathcal{L}_2 \propto \text{Tr} \partial_\mu U^\dagger \partial_\mu U$$

To coincide with the non-linear sigma model (Appendix A), which embodies the entire content of current algebras, we set

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \text{Tr} \partial_\mu U^\dagger \partial_\mu U \quad 2.2-1$$

where $F_\pi = 93 \text{ MeV}$ and $F_\pi \sim \sqrt{N_c}$.

For convenience we define

$$L_{\mu}^{\alpha} = \frac{1}{2i} \text{Tr} [\tau^{\alpha} U_{\mu}^{\dagger} \partial_{\mu} U]$$

and

$$L_{\mu} = i \tau^{\alpha} L_{\mu}^{\alpha}$$

$$= U^{\dagger} \partial_{\mu} U$$

2.2-2

allowing 2.2-1 to be written

$$\mathcal{L}_2 = \frac{-F^2}{4} \text{Tr} L_{\mu} L_{\mu}$$

$$= \frac{F^2}{2} L_{\mu}^a L_{\mu}^a$$

2.2-3

With only the $\mathcal{L}_2$ term, $\mathcal{L}$ is unable to support a static, energetically stable structure. For a static solution, the energy is given by

$$E = \int \mathcal{L}_2 d^3x$$

$$= \frac{F^2}{2} \int L_i^a L_i^a d^3x$$

$$= V_2 \geq 0.$$

Consider now a one parameter family of field configurations defined by

$$U(\vec{x}; \lambda) = U(\lambda \vec{x}; 1), \text{ where } \lambda > 0.$$

Then

$$\frac{\partial U}{\partial x_i} \rightarrow \lambda \frac{\partial U}{\partial (\lambda x_i)}$$

and

$$\int^3 x \rightarrow \lambda^{-3} \int (\lambda^3 x)$$

As a function of λ , the energy becomes

$$E(\lambda) = \lambda^{-1} V_2$$

2.2-4

Since λ multiplies $\bar{x}$, $1/\lambda$ is a measure of the solitons size.

We see that as we shrink the soliton size to zero the energy vanishes.

Thus we cannot have energetic stability, leading us to consider another term for ;

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$$

2.2-5

The energy is now

$$E(\lambda) = \lambda^{-1} V_2 + \lambda V_4$$

2.2-6

where V_4 is of the form

$$V_4 = \int d^3 x (\vec{\nabla} \theta \cdot \vec{\nabla} \theta)^2$$

As we shrink the soliton, E would have no lower bound unless we demand that $V_4 \geq 0$. It is a unique result of these stability arguments that each term must be positive.

According to Hamilton's principle, E must be stationary against variations of λ at $\lambda = 1$. Thus

$$\left. \frac{\partial E}{\partial \lambda} \right|_{\lambda=1} = -V_2 + V_4 = 0$$

giving

$$V_2 = V_4$$

$$F(\lambda) = (\lambda^{-1} + \lambda) V_2$$

2.2-7

For stability we want the solution to be a local minimum of the energy, requiring

$$\left. \frac{d^2 E}{d\lambda^2} \right|_{\lambda=1} > 0$$

A simple calculation gives

$$\frac{d^2 E}{d\lambda^2} = 2V_2$$

2.2-8,

which is positive, as required.

In n spatial dimensions, 2.2-7 and 2.2-8 would read

$$V_2 = -(n-1)/(n-2) V_4$$

$$\left. \frac{d^2 E}{d\lambda^2} \right|_{\lambda=1} = 2(n-2)V_2 > 0 \text{ for } n \geq 3. \quad 2.2-9$$

Since both V_2 and V_4 are positive, 2.2-9 indicates that

$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$ can support an energetically stable solution for $n = 3$ only. (The reader should be warned that this scaling analysis by no means suffices to ensure the existence of solitons!) Thus we look for a static classical solution of solitonic nature to the

Euler-Lagrange equations of motion obtained from

$$\begin{aligned} \mathcal{L} &= -\frac{F_\pi^2}{4} \text{Tr}[L_\mu L_\mu] + \frac{\epsilon^2}{4} \text{Tr}[L_\mu, L_\nu]^2 \\ &= \frac{F_\pi^2}{2} L_\mu^a L_\mu^a + 2\epsilon^2 [L_\mu^a L_\mu^b L_\nu^a L_\nu^b - L_\mu^a L_\mu^b L_\nu^b L_\nu^a] \end{aligned} \quad 2.2-10$$

where the second term is that first used by Skyrme (Skyrme, 1961 and 1962), and is known as the Skyrme term. The soliton solution which Skyrme discovered from 2.2-10 is known as the Skyrmion or chiral soliton. Since 2.2-10 represents an effective Lagrangian for large N QCD, $\epsilon \sim \sqrt{N}$. The constant ϵ can be constrained by experiment (Balachandran, 1983). In terms of the ϕ_e (see 2.1-3),

$$\begin{aligned} \mathcal{L}_4 &= \frac{\epsilon^2}{4} \text{Tr}[L_\mu, L_\nu]^2 \\ &= \frac{\epsilon^2}{2} (\nabla \phi_e \times \nabla \phi_e)^2 \end{aligned} \quad 2.2-11$$

for static solutions,

so that $V_4 \geq 0$ explicitly.

The Euler-Lagrange equations read

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta L_\mu^a} = 0. \quad 2.2-12$$

We define the canonically conjugate variables to be

$$\begin{aligned} P_\mu^a &= \frac{\delta \mathcal{L}}{\delta \dot{L}_\mu^a} \\ &= F_\pi^2 \dot{L}_\mu^a + 8\epsilon^2 [L_\mu^b L_\nu^b L_\nu^a - L_\mu^a L_\nu^b L_\nu^b] \end{aligned}$$

The equations of motion read

$$\partial_\mu \left[\left(S^{ab} + \frac{2\varepsilon^2}{F_1^2} (L^a L^b) \right) L_\mu^{ab} \right] = 0$$

which for the static case reduces to

$$\partial_\mu \left[\left(S^{ab} + \frac{2\varepsilon^2}{F_1^2} (L^a L^b) \right) L_\mu^{ab} \right] = 0$$

2.2-13

where

$$L^{ab} = L^a_\mu L^b_\mu S^{ab} - L^a_\mu L^b_\mu$$

The energy density for a time dependent solution is

$$= F_1^2 \left[L^0 L^0 + L^1 L^1 \right] + 2\varepsilon^2 \left[2L^0 L^0 L^0 L^0 - 2L^0 L^0 L^1 L^1 + L^0 L^0 L^1 L^1 + L^1 L^1 L^1 L^1 \right]$$

From the inequalities

$$\begin{aligned} L^0 L^0 &\geq 0, \quad L^1 L^1 \geq 0 \\ L^0 L^0 L^0 L^0 &\geq L^0 L^0 L^1 L^1 \\ L^0 L^0 L^1 L^1 &\geq L^1 L^1 L^1 L^1 \end{aligned}$$

we see that

$$\tau_{00} \geq 0.$$

Before proceeding we first give some qualitative justifications for Skyrme's choice of $\mathcal{L}_4$, 2.2-11. This choice is not unique, the most general quartic term being given by:

$$\mathcal{L}_4 = \alpha \text{Tr} [L_a^2 L_b^2] + \beta \text{Tr} [L_a L_b L_a L_b] + \gamma \text{Tr} [L_a L_b L_c L_d] + \delta \text{Tr} [L_a L_b L_c L_d] \quad 2.2-14$$

Any other invariant can be cast into one of the above forms by use of the Maurer-Cartan identities (see Skyrme, 1960),

$$\partial_\mu L_\nu - \partial_\nu L_\mu = -[L_\mu, L_\nu] \quad 2.2-15$$

which expresses the integrability condition for 2.2-2. This expression can be derived directly from the definition of L_μ .

Skyrme's choice is given by $\gamma = \delta = 0$, $\alpha = -\beta$. This is the unique term that leads to a positive definite Hamiltonian, which is very attractive although not necessary for a semi-classical expansion.* This choice also leaves the field equations derived from $\mathcal{L}$ linear in the time derivatives and, as will be shown, leads to a minimal energy.

The above three attractions of the Skyrme term would not be possible if the last two terms in 2.2-14 were non-zero. Thus, we shall prove minimality of the energy disregarding these terms;

$$\mathcal{L}_4 = \alpha \text{Tr} [L_a^2 L_b^2] + \beta \text{Tr} [L_a L_b L_a L_b]$$

We can restrict α, β by the requirement $V_4 \geq 0$ and then eliminate one by the condition of minimal energy.

For a static solution, $V_4 \geq 0$ gives $\alpha \text{Tr} [L_a^2 L_b^2] + \beta \text{Tr} [L_a L_b L_a L_b] \leq 0$, or

$$2L_a^a L_b^a L_c^b L_d^b + \beta [2L_a^a L_b^a L_c^b L_d^b - L_a^a L_b^a L_c^b L_d^b] \leq 0.$$

* The Hamiltonian need only be positive definite in the neighbourhood of the static soliton solution.

Using the inequality

$$3(L_i^a L_i^a)(L_i^b L_i^b) \geq (L_i^a L_i^a)(L_i^b L_i^b) \geq (L_i^a L_i^a)(L_i^b L_i^b) \geq 0$$

we find

$$A \geq B \quad \text{for} \quad A - B \geq 0$$

$$B \geq 3A \quad \text{for} \quad A - B \geq 0$$

giving the minimal choice

$$B = -A$$

Skyrme's choice of λ_4 is motivated by simplicity of the resulting field equations and minimization of the energy.

Returning to our discussion of $\mathcal{L}$, we see that the requirement of finite energy demands that L_μ vanish ($U(\chi)$ become constant) at infinity. Thus we have the boundary condition

$$U(\vec{\chi}, t) \rightarrow U_0 \quad \text{as} \quad |\vec{\chi}| \rightarrow \infty$$

where U_0 is some constant matrix. Since $\mathcal{L}$ is invariant under global chiral rotations, we can rotate U

$$U \rightarrow U_0^{-1} U$$

and hereafter take the boundary condition satisfied by U to be

$$U(\vec{\chi}, t) \rightarrow 1 \quad \text{as} \quad |\vec{\chi}| \rightarrow \infty$$

2.2.2 Topology

Since $U(\vec{x}, t)$ approaches the same constant value at all points of spatial infinity, the physical space R^3 is compactified onto the three-sphere S^3 . Let us call this S^3_{phy} . The internal $SU(2)$ space, the space of the fields ϕ , is topologically equivalent to S^3 , since $\phi \phi^\dagger = 1$. Let us call this S^3_{int} . The field $U(\vec{x}, t)$, for fixed t , maps the physical space R^3 into the internal space $SU(2) \approx S^3_{\text{int}}$. Thus we are interested in the mappings,

$$S^3_{\text{phy}} \rightarrow SU(2) \approx S^3_{\text{int}}$$

which can be classified into homotopy sectors as follows.

Let

$$\phi: R^n \rightarrow Y$$

where Y is some manifold, and let Q be the set of all such mappings ϕ satisfying a boundary condition, such as 2.2-16;

$$\phi(x) \rightarrow y_0 \text{ as } |x| \rightarrow \infty, \quad y_0 \in Y, \quad 2.2-17$$

To avoid confusion we state here that in the following we are interested in the topology of the field manifold (Y , the space of the dependent variables ϕ), not of the space-time manifold (R^n , the space of the independent variables).

Q is the disjoint union of a set of equivalence classes*, or homotopy sectors. Two mappings $\phi_1, \phi_2 \in Q$ are equivalent if and only if they are homotopic. That is to say, if there exists a path $\phi(x, \lambda)$, $0 \leq \lambda \leq 1$, joining ϕ_1 and ϕ_2 such that

* See bottom of next page.

- 1) $\theta(x, \alpha)$ is continuous in the variables (x, α) .
- 2) $\theta(x, \alpha)$, for all $0 \leq \alpha \leq 1$, satisfies the boundary condition 2.2-17
- 3) $\theta(x, 0) = \theta_1(x)$, $\theta(x, 1) = \theta_2(x)$

Homotopic means simply that θ_1 and θ_2 can be joined by a curve in Q . This implies that on the manifold Y , θ_1 and θ_2 may be continuously deformed into one another. If θ_1 and θ_2 are not homotopic, they cannot be continuously deformed into one another. We may use time, t , as a continuous parameter to label a path. Thus a mapping belonging to a certain class Q_a possesses characteristics which in time cannot develop into the characteristics of mappings belonging to another class Q_b . This alerts us to the presence of a conservation law, which is why we are interested in the connectivity of Q . The tool to study the connectivity of Q is homotopy theory.

* The relation $\sim$ on elements of a group G is said to be an equivalence relation on G if for all a, b, c in G :

- 1) $a \sim a$ termed reflexivity
- 2) $a \sim b$ implies $b \sim a$ symmetry
- 3) $a \sim b$ and $b \sim c$ implies $a \sim c$ transitivity

The equivalence class of $g \in G$ is the set $c(g) = \{g' \in G \mid g \sim g'\}$.

The equivalence classes of $\sim$ provide a decomposition of G into mutually disjoint subsets, the collection of which also forms a group. (See Herstein, 1975).

The equivalence classes are the group elements of the n^{th} homotopy group, denoted $\pi_n(Y)$. For the theory considered here $n = 3$, $Y = S^3$, and we say that $U(\vec{x}, t)$, for fixed t , satisfying 2.2-16, is an element of the third homotopy group $\pi_3(SU(2)) \approx \pi_3(S^3_{\text{int}})$. It is known from tables that

$$\pi_n(S^n) = \mathbb{Z}, \quad \forall n$$

where $\mathbb{Z}$ is the additive group of integers. Thus we can label the homotopy sectors by an integer i ;

$$Q = \bigcup_{i \in \mathbb{Z}} Q_i.$$

Thus a mapping is characterized by an integer, we say, which represents the homotopy sector of the map. W is known as the winding number since it is a measure of the number of times S^3_{int} is covered when S^3_{phy} is traversed once; how many times it has been wrapped. Consider, for example, mappings from the real line R^1 into the circle S^1 ,

$$\alpha: R^1 \rightarrow S^1$$

subject to

$$\alpha(x) \rightarrow (0,1) \text{ as } |x| \rightarrow \infty$$

where we have parameterized S^1 by two real variables (ϕ_1, ϕ_2) .

The field α may be visualized as a strip stretching from $x = -\infty$ to $x = +\infty$ along R^1 , with the value of $\alpha(x)$ being determined by the angular twist of the strip at the point x . We then associate $W=1$ to the case where the strip exhibits a twist through 2π (similar to a Mobius strip).

Equivalently, consider the mapping of the physical space S_{phy}^1 parameterized by an angle θ , modulo 2π , into the internal space S_{int}^1 . The mapping is given by the continuous function $\mathcal{L}$; modulo 2π ;

$$\mathcal{L}: S_{\text{phy}}^1 \rightarrow S_{\text{int}}^1$$

or, diagrammatically

$$\mathcal{L}: \bigcirc \rightarrow \bigcirc$$

The field in the internal symmetry space is $\mathcal{L}(x)$, defined over the points $x \in S_{\text{phy}}^1$. Then S_{int}^1 is the manifold of possible values of $\mathcal{L}$.

Consider first a trivial mapping

$$\mathcal{L}(\theta) = \text{constant}$$

Here S_{phy}^1 is mapped into one point of S_{int}^1 , which is not covered once; thus $W=0$. We visualize this as,

$$W = 0$$

FIGURE 2.1 (a)

Consider now the map

$$\mathcal{L}(\theta) = \theta$$

As θ spans $[0, 2\pi]$, S^1_{int} is wrapped about once; thus $W = 1$.

$$W = 1$$

FIGURE 2.1 (b)

Here we attribute a direction to the wrapping so as to distinguish $W = 1$ from $W = -1$. For $\alpha(\theta) = n\theta$, we see that as θ spans $[0, 2\pi]$, S^1_{int} is wrapped about once. As θ spans the entire interval $[0, 2\pi]$, S^1_{int} will be covered n times, giving $W = n$. Here α maps S^1_{phy} onto S^1_{int} . An expression for W can be written down, for the case $\alpha: S^1 \rightarrow S^1$, as

$$W = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{d\theta} d\theta$$

Two mappings with the same winding number may be continuously deformed into one another. To see this, consider

$$\alpha'(\theta) = \begin{cases} s\theta & 0 \leq \theta \leq \pi \\ s(2\pi - \theta) & \pi \leq \theta \leq 2\pi \end{cases} \quad s \in [0, 1]$$

Clearly, as θ ranges over $[0, 2\pi]$, S^1_{int} will not be covered since

at $\theta = \pi$ the map begins to retrace its steps. Thus α' also has $W = 0$.

By varying s continuously to zero, α' can be deformed into α .

Again, since time evolution is an example of a smooth deformation, the field $U(\vec{x}, t)$, and thus a soliton solution, cannot change its value of W . The number W is conserved irrespective of the dynamics of the system; it is a topological charge and is not to be associated with a Noether current. Following Skyrme (Skyrme, 1960), we define the winding number W for the mapping $\Pi_3(S^3)$ from the current

$$W_M(x) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} [L_\mu L_\nu L_\alpha L_\beta]$$

2.2-18

Writing

$$L_\mu = \partial_\mu U$$

we see that

$$\partial_\mu W_M(x) = 0$$

since

$$\epsilon_{\mu\nu\alpha\beta} \partial_\mu L_\nu = \epsilon_{\mu\nu\alpha\beta} \partial_\mu (\partial_\nu U) = 0$$

This shows how the current is conserved independent of any symmetry of the Lagrangian.

The charge associated with W_M is the winding number;

$$W = \int d^3x W_0(x)$$

2.2-19

That 2.2-19 is integral valued is a consequence of the normalization of 2.2-18. Skyrme made the prophetic proposal that a soliton solution of 2.2-13 with a $W=1$ could be interpreted as a baryon, with W identified with the baryon number. The proof of the latter proposition is deferred until Chapter 3. To show that the Skyrmion is a baryon, we must show that it can be a fermion (recall that $\mathcal{L}$ contains only boson fields). This has been proven by several authors (Finkelstein and Rubinstein, 1968; Williams, 1970; for a review see: Pak and Tze, 1979; also Witten, 1983) as is detailed in the next section, showing that the Skyrme model admits spin.

2.2.3 Spin

We wish to identify baryons with the solitons of QCD. A first step in this identification is to show that the Skyrme solitons can be fermions. An important property of a many-fermion state is the double-valuedness of the wave function under exchange. For a model of Skyrmes type, it has been shown (Finkelstein and Rubenstein, 1968) that this implies double-valuedness of the $W = 1$ sector wave functions under 2π rotation. The classical fields considered so far have been single-valued under the rotation group $SO(3)$, and are thus bosons. To show that the Skyrme model can admit spin we must define a wave function $\Psi(U(\vec{x}, t))$ which is double-valued under 2π rotation.

Consider a mapping θ from n -dimensional Euclidian space R^n into the n -sphere, S^n ;

$$\theta: R^n \rightarrow S^n, \quad \theta(x) \rightarrow 1 \text{ as } |x| \rightarrow \infty$$

Let Q be the set of all such mappings and Q_m , $m \in \mathbb{Z}$, be the equivalence classes, the group elements of $\pi_n(S^n) = \mathbb{Z}$. We wish to know the conditions under which wave functions may be defined on Q_1 which are double-valued under 2π rotations (the wave functions will be functionals of the mappings $\theta: R^3 \rightarrow SU(2)$ in Skyrme model notation). First we must define exactly what is meant by a double-valued functional.

Let the functional be labelled by an additional parameter* which describes the path by which the argument of the functional was reached. It is possible to define a double-valued functional on Q_1 if we can follow a path P in Q_1 , beginning and ending at θ_1 , say, such that the value of the functional changes continuously along P and does not return to its original value at the end of the path, θ_1 . If the path

* Here we use the concept of a universal covering space (Finkelstein, 1966; Hocking and Young, 1961).

is traversed twice, then it must return to its original value; P must be of order 2 (Intuitively, one can think of a Riemann sheet of a function). Note that in order to avoid a contradiction, P must not be trivial (deformable to a point in Q_1). The path P^2 , which is equivalent to P traversed twice, must be trivial. Thus we require information about the path structure of Q_1 , and again homotopy theory provides the answer.

We can represent P as a mapping from R^1 into Q_1 ,

$$P: R^1 \rightarrow Q_1$$

by $P(S) = \theta_1$ for $|S| = \infty$

The group $\pi_1(Q_1)$ is the set of equivalence classes of such closed paths in Q_1 . We can now state that we can define double-valued functionals on Q_1 if and only if $\pi_1(Q_1)$ has an element of order two. This is a first condition for the theory to admit spin.

1) $\pi_1(Q_1)$ has an element of order two

This is not enough to guarantee the existence of spin. We must show that this double-valuedness, defined by following closed paths in Q_1 , actually corresponds to rotating the system through 2π ; that the closed loops of Q_1 are generated by actual $SO(3)$ 2π spatial rotations. This, combined with non-triviality, gives us our second condition;

2) closed paths in Q_1 , corresponding to a 2π spatial rotation, are non-trivial.

Note that condition 2) implies condition 1). Condition 1) is useful as a check, since it is easier to verify. If condition 1)

is not satisfied, neither will condition 2).

Let us check condition 1) first. By establishing a homomorphism between any two sectors Q_i, Q_j it can be shown (Whitehead, 1946) that

$$\pi_1(Q_i) \cong \pi_1(Q_j)$$

2.2-20

The problem can now be reduced to investigating the path structure of the trivial sector, since from 2.2-20

$$\pi_1(Q_0) \cong \pi_1(Q_0)$$

Consider a path in $Q_0, \theta(x, s), s \in R^1$, such that

- 1) $\theta(x, s)$ is continuous in x and s
- 2) $\theta(x, s) = \theta_0$ for $|x| = \infty$ where θ_0 is the identity map
- 3) $\theta(x, \infty) = \theta(x, -\infty) = \theta(x)$

$\theta(x, s)$ is equivalent to a mapping of R^4 into S^3 as follows. Let

$$S: R^1 \rightarrow Q_0, S \in \pi_1(Q_0)$$

and

$$S(\infty) = S(-\infty) = \theta_0$$

Then we define

$$\theta(x, s): R^4 \rightarrow S^3$$

by

$$\theta(x, s) = [S(s)](x)$$

Thus $\theta(x, s)$ is an element of one of the homotopy classes of $\pi_4(S^3)$. We have shown a correspondence between paths in Q_0 and

mappings from R^4 into S^3 . The triviality or non-triviality of $\xi(s)$ depends upon that of $\theta(x,s)$; leading to the statement of isomorphism*

$$\pi_1(Q_1) \cong \pi_4(S^3). \quad 2.2-21$$

It is known (Toda, 1962) that

$$\pi_4(S^3) \cong \mathbb{Z}_2. \quad 2.2-22$$

where $\mathbb{Z}_2 = \{1, -1\}$ does indeed have an element of order two. Equations 2.2-20 and 2.2-21 tell us that $\pi_1(Q_1)$ does indeed satisfy condition 1). We can conceptualize 2.2-22 as follows. The field $U(\bar{x}, t)$ is simply the function $\theta(\bar{x}, s)$ above with s the continuous variable representing the time. The two elements of $\pi_4(S^3)$ can be described as follows. We begin at $t \rightarrow -\infty$ with a constant field $U = 1$. As time evolves, create a soliton-anti-soliton pair and separate them; rotate the soliton through a 2π angle and then bring together the soliton-anti-soliton pair to annihilate. This field would be weighted with a factor -1 , while the field without the 2π rotation would be weighted with a 1 ; it is homotopically trivial.

We must now show that a path in Q_1 corresponding to a 2π rotation is non-trivial and belongs to the element of $\pi_1(Q_1)$ which is of second order. Before doing so it is useful to consider a two dimensional theory of mappings from R^2 into S^2 , with a suitable boundary condition at spatial infinity. From previous considerations we can state immediately that

$$\pi_1(Q_1) \cong \pi_3(S^2)$$

where $\pi_3(S^2) = \mathbb{Z}$. Thus we have para-statistics and the theory does not admit spin. This should come as no surprise since the rotation

* See Whitehead, 1966.

group $SO(2)$ is not doubly connected. In fact, as with Q_1 , it is infinitely multiply connected;

$$\pi_1(SO(2)) = \mathbb{Z}.$$

In three dimensions spin wave functions exist because $SO(3)$ is doubly connected;

$$\pi_1(SO(3)) = \mathbb{Z}_2$$

In general,

$$\pi_1(SO(n)) = \mathbb{Z}_2, \quad n \geq 3$$

and

$$\pi_1(Q_n) = \mathbb{Z}_2, \quad n \geq 3, \quad \text{when the range is } S^n.$$

This suggests that the connectivity of Q_1 is directly related to the connectivity of $SO(n)$. It turns out that in three dimensions, the triviality or non-triviality of a path in $SO(3)$ determines the triviality or non-triviality of the corresponding path in Q_1 . Equation 2.2-21 tells us that we must investigate mappings from R^4 into S^3 . First, let us consider lower dimensional mappings from R^3 into S^2 . These correspond to one of the three independent fields ϕ in $U(x)$ (only three are independent by virtue of equation 2.1-5) vanishing identically. The field manifold is thus reduced from S^3 to S^2 . Such solutions, corresponding to the Hopf mapping (Flanders, 1963) also possess a conserved topological charge and represent a class of solutions to the Skyrme model (see section 2.3.4). In what follows we omit mathematical details and simply outline the procedure. The reader is referred to Williams, 1970 and Pak and Tze, 1979 for the details.

Consider a mapping

$$g: R^3 \rightarrow S^2$$

subject to

$g(x) \rightarrow (x, 1)$ a point on the S^2 manifold,
as $|x| \rightarrow \infty$. R^3 is thus compactified into S^3 .

Since

$$\pi_3(S^2) = \mathbb{Z},$$

we can again label the homotopy sectors by an integer i, Q_i say, where Q_0 is the class of constant mappings, taking the whole of R^3 into a point of S^2 . An example of a mapping in the class Q_1 is given by

$$\begin{aligned}\theta_1 &= -2(\gamma_4 \gamma_1 - \gamma_3 \gamma_2), \\ \theta_2 &= -2(\gamma_4 \gamma_2 + \gamma_3 \gamma_1), \\ \theta_3 &= 1 - 2(\gamma_1^2 + \gamma_2^2),\end{aligned}$$

where a point of S^2 is denoted by $(\theta_1, \theta_2, \theta_3)$; $S^3, (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. This is the celebrated Hopf mapping which takes great circles of S^3 into single points of S^2 . We now look for an example of a mapping

$$f: S^4 \rightarrow S^3$$

which also belongs to the first homotopy sector Q_1 . From 2.2-22 we see that all non-trivial examples of f belong to Q_1 . We find this mapping by using the concept of a suspension.

Let

$$\lambda: S^m \rightarrow S^n$$

where we can consider S^m and S^n as being equators of the spheres

S^{m+1} and S^{n+1} . A mapping

$$E(\lambda): S^{m+1} \rightarrow S^{n+1},$$

reducing to λ on the equator, is known as the suspension of λ . It is known (Hilton, 1966) that the suspension of the Hopf mapping is non-trivial and thus an element of Q_1 . Finding such a suspension would satisfy condition 2) and we will have proved that Skyrmions can be fermions. The search for this suspension involves a complex though not very interesting series of mappings (Williams, 1970). That this suspension exists allows us to state that a Skyrmion can be a fermion.

To recap, we have shown that Q_1 admits two different types of paths, trivial and second order. The triviality, or non-triviality, of a 2π rotation path in Q_1 was shown to depend upon that of a mapping in Q_0 , which in turn was related to a mapping from S^4 into S^3 . This mapping, being a suspension of the Hopf mapping, is non-trivial. Thus a 2π rotation path in Q_1 was shown to be non-trivial.

2.3 Simple Solutions

2.3.1 Hedgehog Ansatz

Simple solutions occur when the fields ϕ of 2.1-4 lie in a fixed direction. The hedgehog ansatz corresponds to a radial direction of $SU(2)$ isospin, diagrammatically represented as a hedgehog, with the arrows

FIGURE 2.2

representing the direction of internal isospin and the circle

representing the physical space S^3 . Figure 2.2 suggests a correlation between internal and physical spaces, which will be discussed shortly.

The static field configurations can now be written as

$$U = q + e^{\hat{r} \cdot \hat{A} \theta(r)} \quad 2.3-1$$

Choosing

$$\Phi_0 = \cos \theta(r)$$

$$\Phi_i = \hat{r}_i \sin \theta(r)$$

we have

$$U = e^{e \hat{r} \cdot \hat{A} \theta(r)} \\ = \cos \theta(r) + e \hat{r} \cdot \hat{A} \sin \theta(r) \quad 2.3-2$$

From equation 2.2-16 we can take our boundary conditions to be

$$\theta(\infty) = \pi \\ \theta(0) = 0 \quad 2.3-3$$

The hedgehog is a special case of the general Witten Ansatz (Witten, 1976) which can be written

$$L_c^a = \frac{G_1}{r} (S_{ab} - \frac{x_a x_b}{r^2}) + \frac{G_2}{r} \epsilon_{abc} \frac{x_c}{r} + A_1 \frac{x_a x_b}{r^2} \quad 2.3-4$$

where L_i^a is defined by equation 2.2-2.

We can restrict ϕ_1 , ϕ_2 , and A_1 by use of the integrability condition 2.2-15;

$$\partial_j L_i^c - \partial_i L_j^c = \pm \epsilon_{ijk} L_k^a L_j^b.$$

2.3-5

Using 2.3-4 this gives

$$\begin{aligned} \partial_j L_i^c - \partial_i L_j^c &= \frac{\phi_1'}{r^2} (x_i \delta_{cj} - x_j \delta_{ci}) + (\pm \frac{\phi_2}{r^4} - \frac{\phi_1'}{r^3}) (\delta_{ijk} x_k + \epsilon_{ijk} x_k) \\ &\quad + \frac{A_1}{r^2} (x_i \delta_{cj} - x_j \delta_{ci}) + 2 \frac{\phi_2}{r^2} \epsilon_{cij} c \end{aligned}$$

and

$$\epsilon_{ijk} L_i^a L_j^b = \frac{\phi_1^2}{r^2} [\epsilon_{ijk} + \epsilon_{ikc} \frac{x_j x_c}{r^2} - \epsilon_{jkc} \frac{x_i x_c}{r^2}] + \frac{\phi_2^2}{r^4} \epsilon_{ijk} x_k x_c + \frac{\phi_1 \phi_2}{r^3} [\epsilon_{ikc} x_j x_c + \epsilon_{jkc} x_i x_c]$$

where the prime, ' , denotes differentiation with respect to r .

Multiplying 2.3-5 by x^c and x^i gives

$$\begin{aligned} \phi_1^2 + \phi_2^2 &= \phi_2 \\ 2 \phi_1 A_1 &= \phi_2' \end{aligned}$$

If we set

$$\phi_1 = \frac{1}{2} \sin \alpha(r), \quad \frac{\alpha(r)}{2} = \theta(r).$$

we have

$$\phi_2 = (1 \pm \cos 2\theta(r))/2$$

$$A_1 = \mp \alpha'(r)/2$$

Equation 2.2-16 requires that

$$\lim_{|x| \rightarrow \infty} \phi_2(r) = 0$$

which selects

$$\phi_2 = (1 + \cos \alpha(r))/2$$

$$\phi_1 = \frac{1}{2} \sin \alpha(r)$$

and

$$A = \alpha(r)/2$$

This corresponds to the Hedgehog Ansatz; as can be seen by calculating L_m (2.2-2) with 2.3-2.

The correlation between spin and isospin spaces mentioned earlier will now be developed. The Ansatz 2.3-2 is not invariant under separate spin and isospin rotations but under combined rotations. If we rotate the fields through an angle $\delta\phi$, the fields change by the amount

$$\delta U = i \delta\phi [G, U]$$

where G is the generator of this infinitesimal rotation.

For ordinary spin rotation about the axis i

$$G_i = (\vec{r} \times \vec{p})_i + \sigma_i/2$$

where σ acts on the spin indices. The field U changes by an amount

$$\delta U = i \delta\phi \frac{\sin\theta}{r} \epsilon_{ijk} x_j \tau_k$$

2.3-6

For isospin rotations

$$G_i = \tau_i/2$$

where τ acts on the isospin indices changing U by an amount

$$\delta U = -i \delta\phi \frac{\sin\theta}{r} \epsilon_{ijk} x_j \tau_k$$

2.3-7

Thus we can see explicitly that U is invariant under the operator

$$\begin{aligned}\bar{K} &= \bar{I} + \bar{J} \\ &= \bar{r} \times \bar{p} + \frac{\bar{q}}{2} + \frac{\bar{r}}{2}\end{aligned}\quad 2.3.8$$

The simultaneous correlations in I and J spaces lead us to expect a lowering of the energy, as indeed occurs for nuclear systems (See Bohr and Mottelson, 1975). The lowest energy state will be an eigenvector of 2.3-8 with eigenvalues $K = K_z = 0$. Whether or not 2.3-2 is the minimal energy Ansatz has not, however, been shown formally.

2.3.2 Equations of Motion

We now write the equations of motion, 2.2-13, using the Witten Ansatz.

The quantities needed are

$$L_i^a L_i^a = \frac{2}{r^2} (\Phi_1^2 + \Phi_2^2) + A_1^2$$

$$L_i^a L_i^b = \left(\frac{\Phi_1^2 + \Phi_2^2}{r^2} \right) (\delta_{ab} - \frac{x_a x_b}{r^2}) + A_1^2 \frac{x_a x_b}{r^2}$$

so that

$$\Lambda^{ab} = L_i^a L_i^b - (L_i^a)^2 \delta^{ab}$$

$$= \left[\frac{\Phi_1^2 + \Phi_2^2}{r^2} - A_1^2 \right] (\delta_{ab} - \frac{x_a x_b}{r^2}) - \frac{2(\Phi_1^2 + \Phi_2^2)}{r^2} \delta^{ab} \quad 2.3-9$$

Then, using $x_b L_i^b = A_1 x_i$, we find

$$\begin{aligned}\lambda_i \Lambda^{ab} L_i^b &= \left[\left(\frac{\Phi_1^2 + \Phi_2^2}{r^2} \right) - A_1^2 \right] \left[-\frac{2x_a \Phi_1^2}{r^3} \right] - \frac{2}{r} \left(\frac{\Phi_1^2 + \Phi_2^2}{r^2} \right) A_1 x_a \\ &= \frac{2}{r} \left(\frac{\Phi_1^2 + \Phi_2^2}{r^2} \right) \frac{x_a}{r} \left[A_1^2 + \frac{2A_1}{r} - \frac{2\Phi_1^2}{r^2} \right]\end{aligned}$$

and

2.3-10

Using 2.3-8 and setting $z = \ln r$, the equation of motion reads

$$\ddot{\theta} + \epsilon - \sin 2\theta + \frac{1}{r_0^2} e^{-2z} \left[\dot{\theta} \sin \theta - \theta \sin \theta + \frac{1}{2} \dot{\theta}^2 \sin 2\theta - \frac{1}{2} \sin^2 \theta \sin 2\theta \right] = 0 \quad 2.3-11$$

where

$$\dot{\theta} = d\theta/dz$$

and r_0 is an integration constant.

The reader should be warned that the equations of motion quoted in Pak and Tze (Pak and Tze, 1979) are incorrect.

For $r \rightarrow \infty$, 2.3-11 reduces to

$$\ddot{\theta} + \dot{\theta} - \sin 2\theta = 0$$

We adjust the boundary conditions 2.3-3 such that we have a soliton at the origin,

$$\begin{aligned} \theta(\infty) &= 0 \\ \theta(0) &= \pi \end{aligned}$$

2.3-12

allowing us to write $\sin 2\theta \approx 2\theta$ and

$$\ddot{\theta} + \dot{\theta} - 2\theta = 0$$

2.3-13

with solution

$$\begin{aligned}\theta &= A e^{-2r} \\ &= A r^3 / r^3\end{aligned}\quad A = \text{constant} \quad 2.3-14$$

For $r \rightarrow 0$, 2.3-11 reduces to

$$(\ddot{\theta} - \dot{\theta}) \sin^2 \theta + \dot{\theta}^2 \sin \theta \cos \theta - \dot{\theta} \sin^3 \theta \cos \theta = 0 \quad 2.3-14$$

To lowest order, $\theta = \text{constant}$ is a solution. We set

$$\theta = n\pi + \beta(r),$$

where $\beta(r)$ is small and

$$\beta(0) = 0.$$

Now 2.3-14 reads

$$(\ddot{\beta} - \dot{\beta}) \beta + \frac{1}{2} (\dot{\beta})^2 - \frac{1}{2} \beta^2 = 0 \quad 2.3-15$$

with solution

$$\beta = a e^{-r} \quad a = \text{constant}$$

so that

$$\theta = n\pi + ar/r^2$$

2.3.3 Winding Number of the Skymion

We derive the winding number of the Skymion for the hedgehog ansatz, using 2.2-19. The component of the current (2.2-18) needed is

$$\begin{aligned}W_0 &= \frac{1}{24\pi^2} \epsilon_{\alpha\beta\gamma} \text{Tr} [L_\alpha L_\beta L_\gamma] \\ &= \frac{1}{12\pi^2} \epsilon_{ijk} \epsilon_{abc} L_i^a L_j^b L_k^c \\ &= \frac{1}{12\pi^2} \frac{A}{r^2} [\phi_1^2 + \phi_2^2] \quad \text{in Witten Ansatz notation} \\ &= \frac{1}{4\pi^2 r^2} (1 - \cos 2\theta) \frac{d\theta}{dr}\end{aligned} \quad 2.3-16$$

The winding number is given by

$$W = \int d^3r W_0(r) \\ = \frac{1}{\pi} \int_0^\infty (1 - \cos 2\theta) \frac{d\theta}{dr} dr$$

Since

$$U \rightarrow 1 \quad \text{as } r \rightarrow \infty \text{ we set}$$

$$= \frac{1}{\pi} \left[\theta(\infty) - \theta(0) - \frac{1}{2} \sin 2\theta(\infty) + \frac{1}{2} \sin 2\theta(0) \right]$$

$$\theta(\infty) = l\pi$$

$$\theta(0) = m\pi$$

$$l, m \in \mathbb{Z}$$

so that

$$W = n \quad n \in \mathbb{Z}$$

As discussed previously, n is an integer which measures the number of times S^3 is covered during the map $S^3 \rightarrow S^3$. The $W=1$ sector is physically very important, as it should include the nucleons. The $W=0$ sector solutions describe the usual mesons.

2.3.4 Hopf Mapping

The hedgehog ansatz, while being the most symmetrical, is not the only configuration leading to a conserved topological index. There also exist the so called "smoke rings" (Gibson and Tze, 1981) which correspond to mappings

$$V: S^3 \rightarrow S^2.$$

2.3-17

The field configurations V , constructed from three real fields ϕ_k , can be used in the Skyrme Lagrangian in place of the fields U . The solutions to the Euler-Lagrange equations, known as Hopf solitons, possess a conserved topological index, the Hopf index. As will be seen, the existence of a Skyrme solution guarantees the

existence of a Hopf solution.

The new field configuration is given by

$$V = i \tau_i \phi_i \quad 2.3-18$$

where the fields ϕ_i are defined to be

$$\phi_i = z^\dagger \tau_i z, \quad z z^\dagger = 1 \quad 2.3-19$$

and

$$z = \begin{bmatrix} \cos \theta + \frac{i x_3}{r} \sin \theta \\ \frac{i (x_1 + i x_2)}{r} \sin \theta \end{bmatrix} \quad 2.3-20$$

is the first column of the hedgehog configuration

$$U_0 = e^{i \theta \hat{n} \cdot \vec{\tau}}$$

$$= \begin{bmatrix} \cos \theta + \frac{i x_3}{r} \sin \theta & \frac{i (x_1 - i x_2)}{r} \sin \theta \\ \frac{i (x_1 + i x_2)}{r} \sin \theta & \cos \theta - \frac{i x_3}{r} \sin \theta \end{bmatrix}$$

Inserting V into the Skyrme Lagrangian in favour of U yields the same equations of motion for the variable (2.3-11) up to a scale change $r \simeq \sqrt{2} r'$. Thus the existence of a Skyrme solution guarantees the existence of the Hopf soliton, which has a radius $\sqrt{2}$ times that of the Skyrmion and is $8\sqrt{2/3}$ times as massive. (Fujii, Otsuki and Toyoda, 1985).

Since $\phi(r)$ satisfies the boundary conditions 2.3-12, the physical space R^3 is still compactified into S^3 . The three fields ϕ_i live in the space S^2 since from 2.3-19

$$\phi_i \phi_i = \mathbf{z}^T \mathbf{z} \mathbf{z} \mathbf{z}^T \mathbf{z} = 1$$

Thus the Hopf mapping can be classified by an integer:

$$\pi_3(S^2) = \mathbb{Z}$$

This integer, the Hopf index, is given by

$$H = \frac{1}{4\pi^2} \int d^3x \epsilon^{ijk} A_i F_{jk} \quad (2.3-21)$$

where

$$A_i = \mathbf{z}^T \partial_i \mathbf{z}$$

and

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

The field 2.3-19 can also be written as (Fujii et. al., 1985)

$$V = i U \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} U^\dagger \quad (2.3-22)$$

Any solution to the field equations can be written as a product of two others, according to the group composition laws of $SU(2)$. Since the winding number of a product is given by the sum of the winding numbers of the two fields (Skyrme, 1961), the Hopf soliton is seen to have winding number zero. The Hopf soliton can thus be regarded as a Skyrmion-anti-Skyrmion composite.

2.4 Variational Bounds

2.4.1 Lower Bounds

With the choice of the commutator for the quartic term of our Lagrangian (2.2-10), the energy is bounded by the topological charge. To derive the bound we first write the energy in terms of L_i^a and its dual.

Define the dual of L_i^a , $\star L_i^a$, as

$$\star L_i^a = \epsilon_{ijk} \epsilon^{abc} L_j^b L_k^c$$

2.4-1

Then

$$\begin{aligned} \mathcal{L}_0 &= \frac{F_\pi^2}{2} L_i^a L_i^a + \frac{\epsilon^2}{2} A_i^a \star L_i^a \\ &\geq \frac{\sqrt{2}}{2} \epsilon F_\pi |L_i^a \star L_i^a| \end{aligned}$$

since

$$a^2 + b^2 \geq 2|ab|$$

The energy is given by

$$E = \int \mathcal{L}_0 d^3r$$

$$\geq \frac{\sqrt{2}}{2} \epsilon F_\pi \int |L_i^a \star L_i^a| d^3r$$

$$\geq \frac{\sqrt{2}}{2} \epsilon F_\pi \left| \int L_i^a \star L_i^a d^3r \right|$$

2.4-2

We can write the winding number, 2.2-18, in terms of L_i^a and $\star L_i^a$

$$W = \int \mathbf{L} \cdot d^3r$$

$$= \frac{1}{24\pi^2} \epsilon_{ijk} \int \text{Tr} [L_i^a L_j^a L_k^a] d^3r$$

$$= \frac{1}{12\pi^2} \int L_i^a \star L_i^a d^3r$$

The lower bound for E is thus simply related to W by

$$E \geq \frac{1}{2} \frac{W}{L} \quad (2.4-3)$$

2.4-3

By writing

$$L = \frac{1}{2} \frac{W}{E} \quad (2.4-4)$$

we can see that the bound on E will be saturated for

$$L^2 = \frac{1}{2} \frac{W}{E} L \quad (2.4-4)$$

2.4-4

a sort of self-duality condition.

Then, for a particle with topological number N ,

$$E_N = K N \quad K = \text{constant}$$

so that

$$E_N = \frac{N}{M} E_M \quad (2.4-5)$$

2.4-5

Equation 2.4-5 tells us that the particles are not interacting.
Note that 2.4-4 implies

$$\sum_{j,k} \epsilon^{ijk} L_j^h L_k^c = K L_L^h, \quad K = \frac{F_{\pi}}{f_{\pi}^2}$$

Applied to the integrability condition (2.2-15), this yields

$$K L_i^2 = \epsilon_0 k \partial_k L_i^2$$

or, in vector notation,

$$(\vec{\nabla} \times \vec{L}^2) = -K \vec{L}^2$$

clearly a false statement.

Thus the "self-duality" condition is incompatible with the integrability condition, which establishes the strict inequality

$$E > K |W|,$$

telling us that there exists a short range repulsion.

So far we have been unable to derive a lower bound for the energy without any particular choice for the currents L_i^a . The hedgehog ansatz possesses a high degree of symmetry and we would expect the bound obtained from it to be lower than 2.4-3 above. Using 2.3-9, 2.3-8 and 2.2-14 we obtain,

$$\mathcal{L}_{00} = \frac{F_\pi^2}{2} \left[(\dot{\theta})^2 + 2 \frac{\sin^2 \theta}{r^2} \right] + 4\epsilon \frac{\sin^2 \theta}{r^2} \left[\frac{\sin^2 \theta}{r^2} + 2(\dot{\theta})^2 \right]$$

and

$$E = \int \mathcal{L}_{00} d^3r$$

$$= 4\pi \epsilon F_\pi \int_0^\infty dz \left[(\dot{\theta})^2 \left[\frac{z^2}{2} + 8 \sin^2 \theta \right] + \frac{\sin^2 \theta}{z^2} \left[4 \sin^2 \theta + 1 \right] \right] \quad 2.4-6$$

where z is a dimensionless variable,

$$z = \frac{F_\pi}{\epsilon} r$$

We can find a lower bound using

$$\frac{\dot{\theta}^2 z^2}{2} + 4 \frac{\sin^4 \theta}{z^2} \geq 2\sqrt{2} \sin^2 \theta |\dot{\theta}| \quad 2.4-7$$

Now 2.4-6 becomes the inequality,

$$E \geq 4\pi \epsilon F_\pi \int_0^\infty dz \left[2\sqrt{2} \sin^2 \theta |\dot{\theta}| + 8 \sin^2 \theta |\dot{\theta}|^2 + \sin^2 \theta \right]$$

$$= 4\pi \epsilon F_\pi \int_0^\pi d\theta \sin^2 \theta \left[2\sqrt{2} + 8|\dot{\theta}| + \frac{1}{|\dot{\theta}|} \right] \quad 2.4-8$$

Since $|\theta| \geq 0$,

$$2\sqrt{2} + 8|\theta| + \frac{1}{|\theta|^2} \geq 6\sqrt{2}$$

so that

$$E \geq 24\sqrt{2} \pi F_{\pi} \epsilon \int_0^{\infty} \sin^2 \theta dr$$

$$= 12\sqrt{2} \pi^2 \epsilon F_{\pi} \int_0^{\infty} \sin^2 \theta dr$$

2.4-9

Surprisingly, this is the same bound as that derived in 2.4-3

Similarly, one can derive the bound for a Hopf soliton. The ansatz 2.3-18 gives a bound (Gibson and Tze, 1981).

$$E \leq 17.65 \pi^2 \epsilon F_{\pi} = 4.35 \times 10^4 \epsilon F_{\pi}$$

2.4.2 Upper Bound

For static solutions the Hamiltonian density is

$$\mathcal{H} = -\mathcal{L}$$

Thus an extrema of $\mathcal{L}$ (the Skyrmon solution of Euler-Lagrange equations) minimizes the classical energy. Thus any twice differentiable function $\theta(r)$ which satisfies the boundary conditions will lead to a value of the energy greater than that resulting from a solution of 2.3-11. Following Skyrme (Skyrme, 1961) we choose a function, obeying the boundary conditions, which has some free parameter. Minimizing the energy with respect to this parameter will give an upper bound to the Skyrmon energy.

We use the trial function

$$\theta = \begin{cases} \pi(1 - r/r_0) & r \leq r_0 \\ 0 & r > r_0 \end{cases}$$

2.4-10

Equation 2.4-6 becomes

$$E < 4\pi \epsilon F_{\pi} \left[\frac{1}{2} (4n^2 \pi^2 + n^2 \pi^2) + 2\epsilon \left(\frac{n^2}{2} + \frac{1}{2} \right) \right] \quad 2.4-11$$

where $2\epsilon = F_{\pi} r_0$ and we have used

$$\int_0^{\pi} \frac{\sin^4 y}{y^2} dy < \int_0^{\pi} \frac{\sin^4 y}{y} dy.$$

For $n = 1$, this is minimized to

$$E < 258.58 \epsilon F_{\pi}$$

so that the upper bound for the Skymion energy is

$$E \leq 258.58 \epsilon F_{\pi}.$$

2.4-12

This is to be compared with the lower bound

$$E \geq 167.49 \epsilon F_{\pi}.$$

2.5 Quantization

2.5.1 Method of Adkins, Nappi and Witten

Recall that we proposed baryons to be the solitons of QCD. How is the Hedgehog Ansatz, with $W=1$, relevant to the nucleon? Note that the hedgehog solutions are not eigenstates of $\bar{I}$ and $\bar{J}$ but of $\bar{K} = \bar{I} + \bar{J}$ (see 2.3-8). Good eigenstates of I and J require quantum corrections. We now explore the problem of quantization of classical solutions, treated in Chapter 1.

We begin by noting that if U_0 is a soliton solution then so too is

$$U = A U_0 B^{-1}, \quad A, B \in SU(2).$$

The boundary condition 2.2-16 requires that $A = B$, ensuring that U and U_0 have the same behaviour at spatial infinity. Thus

$$U = A U_0 A^{-1} \quad 2.5-1$$

is a solution with the same configuration. Thus U and U_0 have the same energy, which implies the existence of a zero frequency mode. As discussed earlier, the existence of a zero frequency mode invalidates the use of a harmonic expansion about the classical wave function. The difficulty is overcome by the use of the method of collective coordinates, where we elevate A to a dynamical variable $A(t)$ and proceed with the following assumption (Adkins, Nappi and Witten, 1983): the dominant degrees of freedom are the three associated with internal $SU(2)$ rotation. We then approximate the quantum ground state by retaining only these three modes. Since there are an infinite number of modes (see 1.2-24) this approximation has been much criticized, though it can be somewhat rationalized.

In the absence of degenerate classical ground states, the quantum ground state, Ψ , will be peaked about the classical solution U_0 . In the degenerate case, Ψ will be a superposition of all the ground states which in this case are given by 2.5-1. Since $SU(2)$ rotations are responsible for the degeneracy, we approximate the quantum ground state by retaining only the modes generated by such rotations.

Before proceeding to quantize these modes we must answer the alert readers objection that spatial rotations have been ignored. If

$$U_0 = \cos \theta(r) + i \hat{r} \cdot \hat{\tau} \sin \theta(r)$$

is a classical solution then

$$U = \cos \theta(r) + i \hat{r} \cdot \hat{\tau} \sin \theta(r) \quad \text{RE. 5.0(3)} \quad 2.5-2$$

is also a solution with the same energy. This is, however, equivalent to an $SU(2)$ rotation. This is due to the existence of a two to one

map

$$A \rightarrow R(A)$$

from $SU(2)$ onto $SO(3)$, given by

$$A \in A^\dagger = Z, R_{ij}(A)$$

2.5-3

Using 2.5-3 one sees that 2.5-2 is indeed equivalent to 2.5-1.

We may now proceed to the problem of quantization of collective coordinates.

Inserting U of 2.5-1 in place of U_0 , the Lagrangian 2.2-10 becomes

$$L = -M + \lambda \text{Tr} [\partial_\mu A \partial_\mu A^{-1}]$$

2.5-4

where M is the hedgehog classical energy (2.4-6), evaluated with the solution to 2.3-11, θ_0 , say, and

$$\lambda = \frac{8\pi\epsilon^3}{3F\pi} \int_0^\infty dz z^2 \sin^2 \theta_0 [1 + 5(\dot{\theta}_0^2 + \frac{\sin^2 \theta_0}{z^2})]$$

2.5-5

Since $A \in SU(2)$, it can be written as

$$A = a_0 + i \vec{a} \cdot \vec{\tau}$$

2.5-6

so that

$$AA^{-1} = 1$$

becomes

$$a_0^2 + \vec{a} \cdot \vec{a} = 1$$

2.5-7

Equation 2.5-4, in terms of the a_i , is

$$L = M + \frac{1}{2} \lambda \sum_{i=1}^3 (\dot{a}_i)^2, \quad \dot{a}_i = \frac{da_i}{dt} \quad 2.5-8$$

The Hamiltonian is

$$\begin{aligned} H &= \pi \dot{a}_i - L \\ &= M + \frac{1}{2} \lambda \sum_{i=1}^3 (\dot{a}_i)^2 \\ &= M + \frac{1}{8\lambda} \sum_{i=1}^3 \pi_i^2 \end{aligned} \quad 2.5-9$$

where

$$\pi_i = \frac{\partial L}{\partial \dot{a}_i} = 4\lambda \dot{a}_i$$

are the momenta conjugate to the coordinates a_i .

We quantize by the usual procedure, setting $\pi_i = -i\hbar \frac{\partial}{\partial a_i}$ to obtain

$$H = M + \frac{1}{8\lambda} \sum_{i=1}^3 \frac{\partial^2}{\partial a_i^2} \quad 2.5-10$$

The constant 2.5-5 leads us to interpret the operator

$$\sum_{i=1}^3 \frac{\partial^2}{\partial a_i^2}$$

as the Laplacian on the three-sphere. By analogy with the usual spherical harmonics, the corresponding wave functions are polynomials in the a_i . By applying the spin and isospin operators

$$I_K = \frac{c}{2} \left(a_0 \frac{\partial}{\partial a_K} - a_K \frac{\partial}{\partial a_0} - \sum_{l,m} \epsilon_{Klm} a_l \frac{\partial}{\partial a_m} \right)$$

$$J_K = \frac{c}{2} \left(a_K \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_K} - \sum_{l,m} \epsilon_{Klm} a_l \frac{\partial}{\partial a_m} \right)$$

to a typical wave function $(a_0 + ia_1)^{\lambda}$, we see that such a wave function has spin and isospin equal to $\lambda/2$ (recall $\vec{k} = \vec{I} + \vec{U} = 0$ for ground state);

$$\begin{aligned} T_K \psi &= \frac{i}{2} (a_0 + ia_1)^{1-1} (a_0 - ia_1) \\ &= -\frac{1}{2} (a_0 + ia_1)^2 \end{aligned}$$

Note that for $U = AU_0A^{-1}$, the substitution $A \rightarrow -A$ gives us the same field U . It has been shown (Finklestein and Rubenstein, 1968) that there are two consistent ways to quantize a soliton. For bosons, we require $\psi(A) = \psi(-A)$; for fermions, $\psi(A) = -\psi(-A)$. Following the second requirement restricts the fermion wave functions to be of odd degree in the a_i .

Thus the nucleons with $I=J=\frac{1}{2}$ will be linear in the a_i while the delta, $I=J=3/2$, will be cubic with higher excitations corresponding to fifth order and higher. We can write down wave functions for the nucleons and deltas as follows:

$$|p \uparrow\rangle = \frac{1}{\pi} (a_0 + ia_1) \quad |p \downarrow\rangle = \frac{i}{\pi} (a_0 - ia_1)$$

$$|n \uparrow\rangle = \frac{i}{\pi} (a_0 + ia_3)$$

$$|n \downarrow\rangle = \frac{1}{\pi} (a_1 - ia_2)$$

2.5-11

$$|\Delta^{++} S=3/2\rangle = \frac{\sqrt{2}}{\pi} (a_1 + ia_2)^3$$

etc....

We check that these are indeed the correct wave functions:

$$\langle P | P \rangle = \langle P | P \rangle + \langle P | P \rangle$$

$$= (a_1^2 + a_2^2 + a_3^2 + a_4^2) = 1$$

$$T_H | P \rangle = \left(\frac{1}{2\pi} \left[-\frac{1}{2} \sum_{i,j} a_i a_j (S_{ij} + S_{ji}) \right] \right)$$

$$= \frac{1}{2\pi} (a_1 + a_2)$$

etc....

$$= \frac{1}{2} | P \rangle$$

One can now use these results to compute static properties of nucleons. To obtain expressions for the nucleon mass, we find eigenvalues of the hamiltonian 2.5-10:

$$\sum_{i,j} \left(\frac{-\partial}{\partial a_i} \right) (a_i a_j) = l(l+2) (a_i - a_j)$$

giving the eigenvalues

$$E = M + \frac{1}{8\lambda} l(l+2)$$

2.5-12

The nucleon and delta masses are thus

$$M_N = M + \frac{3}{8\lambda}$$

$$M_\Delta = M + \frac{15}{8\lambda}$$

2.5-13

Considering λ , and the masses, as functions of ξ and F_π ; we can fit the nucleon and delta masses by adjusting ξ and F_π . The results from such an investigation predict a value of F_π 30% lower than the experimental value $F_\pi = 93 \text{ MeV}$ (Adkins et al, 1983). This error is characteristic of calculations in this model.

2.5.2 Second Method

Let us briefly reconsider the method of 2.5.1. In the Lagrangian 2.2-10, $F_\pi \sim \sqrt{N}$ and $\xi \sim \sqrt{N}$, so that N can be taken out as an overall factor in front of the Lagrangian. In the large N limit, the functional integral will be dominated by the stationary points of $\mathcal{L}$; the classical solutions. Thus we expect the wave functions of the low-lying baryons to have large amplitudes for U to be peaked about the low energy classical solutions,

$$U = AU_0 A^{-1}$$

where U_0 is the Hedgehog solution,

$$U_0 = e^{i \vec{\sigma} \cdot \hat{r} \theta_0(r)}$$

2.5-14

and $\theta_0(r)$ is computed numerically from 2.3-11. We can illustrate the different approach of this section by an analogy from nonrelativistic quantum mechanics.

Consider a particle in a central potential, thus rendering the variables θ and ϕ cyclic. The Hamiltonian is given by

$$H = \frac{1}{2} m \dot{r}^2 + V(r) + \frac{l(l+1) \hbar^2}{2mr^2}$$

To lowest order the energy levels, computed by expanding about the minimum of $V(r)$, are

$$E_l = V(r_0) + \frac{1}{2} \hbar \omega_l + \frac{l(l+1)\hbar^2}{2mr_0^2}$$

2.5-15

where

$$\omega_l^2 = V''(r_0)/m$$

Clearly a more accurate method would be to expand about the minimum r_0 of the effective potential

$$V_{\text{eff}} = V(r) + \frac{l(l+1)\hbar^2}{2mr^2}$$

2.5-16

To lowest order, the energy levels are now

$$E_l = V_{\text{eff}}(r_0') + \frac{1}{2} \hbar \omega_l'$$

2.5-17

where

$$(\omega_l')^2 = V_{\text{eff}}''(r_0')/m$$

In this section we will attempt this method, which for the Skyrme model involves minimizing the effective Hamiltonian (Lagrangian) with respect to some static configurations, which for simplicity are taken to be spherical. The resulting solution corresponds to a classical solution uniformly rotating with angular momentum $L^2 = l(l+1)\hbar^2$. This differs markedly from solving the static equations and then rotating the resulting fixed solution in spin and isospin (equation 2.5-1 with $A = A(t)$). We will find that the effective potential for the Skyrme model possesses no minimum, showing that spherically symmetric solitons in this theory are unstable.

The effective Hamiltonian for this problem is given by

$$H[\theta] = M[\theta] + \frac{(1/2)\hbar^2}{8\pi\lambda[\theta]} \quad 2.5-18$$

as in 2.5-10, whereas now M and λ (equations 2.4-6 and 2.5-5) are functionals of θ and not θ_0 , the solution to 2.3-11. The function θ_0 minimizes $M[\theta]$ and not $H[\theta]$, and thus the solutions 2.5-1 obtained in the previous section will not satisfy the Euler-Lagrange equations for the effective Hamiltonian.

Since we will only be interested in the asymptotic behaviour of the solutions to the resulting equations of motion, we disregard the quartic terms in the Skyrme Lagrangian which, as can be seen by a scaling argument, are important only at small distances. To this order, the functionals in 2.5-18 are

$$M[\theta] = 4\pi\varepsilon F_\pi \int_0^\infty z^2 \left[\frac{(\theta')^2}{2} + \frac{\sin^2\theta}{z^2} \right] dz$$

and

$$\lambda[\theta] = \frac{8\pi\varepsilon^3}{3F_\pi} \int_0^\infty z^2 \sin^2\theta dz \quad 2.5-19$$

We can eliminate ε from 2.5-19 by substituting $r = \frac{\varepsilon}{F_\pi} z$

$$M[\theta] = 4\pi F_\pi^2 \int_0^\infty r^2 \left[\frac{(\theta')^2}{2} + \frac{\sin^2\theta}{r^2} \right] dr$$

$$\lambda[\theta] = \frac{8\pi F_\pi^2}{3} \int_0^\infty r^2 \sin^2\theta dr \quad 2.5-20$$

where

$$\theta' = \frac{d\theta}{dr}$$

2 OF/DE 2

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The Euler-Lagrange equation from 2.5-18 (recall $H=-L$ for static solutions) are

$$4\pi F_H^2 \left[2r\theta' + r^2\theta'' - \sin^2\theta \right] + \frac{3\lambda(l+2)}{64\pi F_H^2} \left[\frac{r^2 \sin^2\theta}{(Sdr r^2 \sin^2\theta)^2} \right] = 0 \quad 2.5-21$$

For large r we can linearize the above, taking the soliton at the origin, with $\theta \rightarrow 0$:

$$\theta'' + \frac{2}{r}\theta' - \frac{2}{r^2}\theta + \frac{3}{128} \frac{\lambda(l+2)}{\pi^2 F_H^2} \left[\frac{\theta}{(Sdr r^2 \sin^2\theta)^2} \right] = 0 \quad 2.5-22$$

Equation 2.5-22 is of the form of a free particle Schroedinger equation (Dicke and Witke, 1960, page 170) with angular momentum equal to one and wave number

$$k^2 = \frac{3\lambda(l+2)}{128\pi^2 F_H^2 (Sdr r^2 \sin^2\theta)^2} \quad 2.5-23$$

Discarding the $1/r^2$ term in 2.5-22 and setting

$$\theta = u/r$$

we have simply

$$\frac{\partial^2 u}{\partial r^2} + k^2 u = 0$$

with solutions

$$u = \sin kr, \quad u = \cos kr$$

or

$$\theta = \frac{\sin kr}{r}, \quad \theta = \frac{\cos kr}{r}$$

2.5-24

These asymptotic solutions, however, give a non-convergent value for the integral in the denominator of 2.5-22. We are searching for solutions to 2.5-21 for any finite value of λ . For $k^2=0$, ψ behaves as $1/r^2$ giving a convergent value for the integral in the denominator of 2.5-23 and thus a non-zero value for k^2 . This contradiction shows that no such solution to the effective potential

$$V_{\text{eff}}[\theta] = M[\theta] + \frac{l(l+2)}{8\pi[\theta]}$$

exists. This was first pointed out by Bander and Hayot (Bander and Hayot 1984).

Additional refinements to the methods involved in this section are:

- 1) Addition to $\mathcal{L}$ of a pion mass term (Adkins and Nappi, 1984; Braaten and Ralston, 1985; Rajaraman, Sommermann, Webach and Wyld, 1985.)

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \frac{m_\pi^2 F_\pi^2}{2} (2 - T_{\text{rot}}) + \dots$$

Now the function θ which minimizes $H[\theta]$ develops oscillatory behaviour at large distances. This reflects the fact that a rotating configuration will lower its energy by emitting pions, unless its angular frequency ω is less than the pion mass m_π . A configuration which emits pions is losing energy and thus not uniformly rotating. Thus a uniformly rotating classical solution cannot exist for $m_\pi = 0$.

The addition of the chiral symmetry breaking term proportional to m_π^2 gives a large distance behaviour

$$\theta(r) \rightarrow e^{-\mu r}/r$$

where

$$\mu^2 = m_\pi^2/F_\pi^2 - l(l+2)/6\pi$$

Note that for $\mu^2 < 0$ the solutions would be oscillatory and the functional $\mathcal{L}[\theta]$ would diverge. The condition $\mu^2 > 0$ divides the

parameter space ($\frac{m_\pi}{F_\pi}, \epsilon$) into stable and unstable regions. The empirical nucleon and delta masses are reproduced in the unstable region (Rajaraman et al, 1985). It is possible to have a stable nucleon and unstable delta, however.

- 2) - Include quantum corrections due to vibrational modes (Biedenhorn, Dothan and Tarlini, 1984)..

The energy of the Skyrmion is

$$E = M + \text{Quantum corrections}$$

where

$$M \sim F_\pi^2 \sim N$$

and the quantum corrections due to rotation calculated above are

$$E_{\text{ROT}} \sim O(1/N).$$

The vibrational modes are of order

$$E_{\text{VIB}} \sim O(1)$$

and are thus of considerable interest. The above authors find the rotational-vibrational coupling changes the decoupled spectrum to that more familiar from molecular physics, with rotational bands built on vibrational excitations. The stability of these states has yet to be investigated.

- 3) The effects of quantizing other neglected modes or considering non-spherical solutions could improve the prospects for a quantized Skyrme model. Although much work is being done at present in the literature, a self-consistent quantization of the Skyrme model seems at present to be elusive.

3.1 Witten Analysis

3.1.1 Wess-Zumino Effective Lagrangian

Recall that the chiral Lagrangian

$$\mathcal{L} = -\frac{F_\pi^2}{4} \text{Tr}[L_\mu L_\mu] + \frac{\varepsilon^2}{4} \text{Tr}[L_\mu L_\nu]^2$$

3.1-1

represents a low-energy effective theory of QCD to leading order in $1/N_c$ that is known to incorporate all relevant symmetries of QCD. It has been pointed out (Witten, 1983) that for three flavours the Lagrangian 3.1-1 respects a spurious symmetry which is not present in the case of two flavours. To obtain a Lagrangian which respects only the appropriate symmetries we must add a term to 3.1-1, which breaks the undesired symmetry, known as the Wess-Zumino term.

With $SU(3)_L \times SU(3)_R$ spontaneously broken to $SU(3)$, the low energy dynamics of the theory are described by a field $U(\vec{x})$, with

$$U(\vec{x}) \in SU(3) \quad \forall \vec{x}.$$

The field U has the expansion

$$U = \exp\left(\frac{i}{F_\pi} \vec{\lambda}^a \pi^a\right)$$

$$= 1 + \frac{i}{F_\pi} \vec{\lambda}^a \pi^a + \dots$$

a=1,...,8

3.1-2

where $\vec{\lambda}^a$ are the $SU(3)$ generators and π^a are the Goldstone bosons. Let us consider the symmetries of 3.1-1 and their physical consequences.

The Lagrangian 3.1-1 is invariant under $U \rightarrow U^T$. Considering only the first three SU(3) generators this has the effect,

$$\begin{aligned} \lambda^1 &\rightarrow \lambda^1 \\ \lambda^2 &\rightarrow -\lambda^2 \\ \text{and } \lambda^3 &\rightarrow \lambda^3 \end{aligned}$$

so that in terms of the physical pion fields

$$\begin{aligned} \pi^+ &= \pi^1 + i\pi^2 \\ \pi^- &= \pi^1 - i\pi^2 \\ \pi^0 &= \pi^3 \end{aligned}$$

this symmetry corresponds to ordinary charge conjugation,

$$\begin{aligned} \pi^+ &\leftrightarrow \pi^- \\ \pi^0 &\leftrightarrow \pi^0 \end{aligned}$$

$\mathcal{L}$ is also invariant under a naive parity transformation;

$$\begin{aligned} P_0 : \chi &\rightarrow -\chi \\ t &\rightarrow t \\ u &\rightarrow u \end{aligned}$$

P_0 is termed naive since it does not account for the intrinsic parity of the fields. An additional symmetry of $\mathcal{L}$ is $U \rightarrow U^T$.

From 3.1-2 we see that this corresponds to $\pi^0 \rightarrow -\pi^0$. Acting on a state with N_B bosons this operation yields a factor of $(-1)^{N_B}$.

Note that the simple process

$$K^+ K^- \rightarrow \pi^+ \pi^0 \pi^-$$

respects all symmetries of QCD but violates $(-1)^{N_B}$. Also, since P_0 disregards intrinsic parity, this process violates P_0 . Thus QCD does not respect the symmetries P_0 and $(-1)^{N_B}$ but rather $P = P_0(-1)^{N_B}$;

$$\begin{aligned} \chi &\rightarrow -\chi \\ t &\rightarrow t \\ u &\rightarrow u \end{aligned}$$

In order that our theory be effective we need add a term to 3.1-1, rendering a Lagrangian which obeys only the required symmetries.

Recall that the term with two derivatives was unique, so that the required term need be of higher order. The Euler-Lagrange equation derived from 3.1-1 reads

$$\partial^\mu L_\mu = 0 \quad 3.1-3$$

where we have disregarded the stabilizing quartic term. Note that if the equation were

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu L_\nu = 0,$$

P_0 would be violated. Since this is not the case, we must add a term to 3.1-3 which is Lorentz invariant, violates P_0 and thus

must contain the anti-symmetric pseudotensor $\epsilon_{\mu\nu\alpha\beta}$. The unique P_0 violating term with four derivatives is

$$\epsilon^{\mu\nu\alpha\beta} L_\mu L_\nu L_\alpha L_\beta \quad 3.1-4$$

which also respects $P = P_0(-1)$. However, to derive this term from a Lagrangian would seem impossible since

$$\epsilon^{\mu\nu\alpha\beta} \text{Tr}[L_\mu L_\nu L_\alpha L_\beta] = 0.$$

To see that the problem is soluble, consider the simple example of a particle of unit mass constrained to move on the surface of a two-sphere. The coordinate $x(t)$ is a mapping

$$x(t): S^1 \rightarrow S^2,$$

where the time coordinate has been compactified by boundary conditions imposed at infinity;

$$\chi(\pm\infty) = (a, h), \quad (a, h) \in S^2.$$

The particle moves under a force expressed as the constraint

$$\sum_i \dot{x}_i^2 = 1, \quad 3.1-5$$

its motion dictated by the Lagrangian

$$L = \frac{1}{2} \sum_i \dot{x}_i^2, \quad 3.1-6$$

Since the δx_i are not independent, the equations of motion read

$$\ddot{x}_i = \lambda x_i$$

Multiplying by x_i we find that

$$\begin{aligned} \lambda &= x_i \ddot{x}_i \\ &= -\dot{x}_i \dot{x}_i \end{aligned}$$

where the last equality comes from differentiating 3.1-5 twice.

Thus the equations of motion are

$$\ddot{x}_i + m x_i \dot{x}_k \dot{x}_k = 0.$$

This system separately respects the symmetries

$$t \rightarrow -t$$

$$x_i \rightarrow -x_i$$

To break the two separate symmetries, while preserving their product, we add a simple term to 3.1-7:

$$\ddot{x}_i + \dot{x}_i \dot{x}_K^2 = \lambda \epsilon_{ijk} x_j \dot{x}_K \quad 3.1-8$$

where λ is some constant yet to be determined. Again, since

$$\epsilon_{ijk} x_i x_j \dot{x}_K = 0,$$

there is no obvious term to add to L whose variation will yield the required term. Fortunately, the solution to this problem is well known. By introducing a vector potential such that

$$\nabla \times \vec{A} = \frac{\vec{x}}{|\vec{x}|^3}$$

the term on the right-hand side of 3.1-8 can be understood as the Lorentz force for an electric charge interacting with a magnetic monopole at the spheres center. The action for this problem is

$$I = \int \left(\frac{1}{2} \dot{x}_i^2 + \lambda A_i \dot{x}_i \right) dt \quad 3.1-9$$

We are not yet finished since the potential A_i contains a Dirac string which will not respect the symmetries of the problem. When quantizing this model the Feynman path integral

$$\text{Tr } e^{-\beta H} = \int dx_4 \int dx_i e^{-iI}$$

will contain the term

$$\exp[i\lambda \int A_i dx_i]$$

3.1-10

which is not well defined.

As the integration in 3.1-10 is over the particle-orbit, γ , we can eliminate the vector potential by use of Gauss's law;

$$\exp[i\alpha \int_{\gamma} A_i dx_i] = \exp[i\alpha \int_D F_{ij} d\varepsilon^{ij}], \quad 3.1-11$$

where the integration is now over the disc D bounded by γ .

We can now use the right-hand side of 3.1-11 in our path integral as it does not suffer from a Dirac string. Here we encounter yet another problem; the curve γ bounds the disc D and, in the opposite sense, the disc D' which is the remainder of the sphere exterior to γ . Stated mathematically, since $\pi_1(S^2) = 0$ any curve γ can be continuously deformed into another on S^2 and, in particular, a curve can be looped around the sphere and turned inside out, bounding the exterior disc in the opposite sense.

Since we have no reason to prefer D to D' we must demand that integration over either be equal;

$$\exp[i\alpha \int_D F_{ij} d\varepsilon^{ij}] = \exp[-i\alpha \int_{D'} F_{ij} d\varepsilon^{ij}].$$

or

$$1 = \exp[i\alpha \int_{D+D'} F_{ij} d\varepsilon^{ij}].$$

3.1-12

Since $D+D'$ is the sphere S^2 and

$$\int_{S^2} F_{ij} d\varepsilon^{ij} = 4\pi,$$

3.1-12 restricts α to be integer or half-integer. This is recognized as Dirac's quantization condition for the product of electric and magnetic charge.

We can solve our $SU(3)$ problem in a similar manner. Considering space-time to be a four-dimensional sphere M , the field U maps M into $SU(3)$. Since $\pi_4(SU(3))=0$, the four-sphere in $SU(3)$ defined

by $U(x)$ is the boundary of a five-dimensional disc Q and, in the opposite sense, Q' , where $Q+Q'$ is a closed five-dimensional sphere on the $SU(3)$ manifold. As before, we need an object to be integrated over Q to define an action.

Let

$\Gamma = \int_Q w_{ijklm} d\epsilon^{ijklm}$
 where w_{ijklm} is the unique anti-symmetric tensor on the $SU(3)$ manifold which is $SU(3) \times SU(3)$ invariant (see Witten; 1983). Since Γ could be defined as an integral over Q' , we must demand that

$$1 = \exp[i\Gamma(P - P')], \quad 3.1-13$$

where

$$\Gamma' = - \int_{Q'} w_{ijklm} d\epsilon^{ijklm}$$

Since $Q+Q'$ is a closed five-sphere, 3.1-13 requires

$$\int_S w_{ijklm} d\epsilon^{ijklm} = 2\pi \cdot \text{integer}$$

for any five-sphere S in the $SU(3)$ manifold. Whereas in the two dimensional case we integrated over the domain, S^2 (3.1-12), here we integrate over any five-sphere in $SU(3)$. Since $\pi_5(SU(3)) = \mathbb{Z}$, every five-sphere in $SU(3)$ is topologically a multiple of some basic five-sphere S_0 . We normalize w such that

$$\int_{S_0} w_{ijklm} d\epsilon^{ijklm} = 2\pi$$

and take as our new Lagrangian

$$L = \frac{1}{4} F_{\mu\nu}^2 + \frac{n}{4} \text{Tr}[L_{\mu\nu} L_{\mu\nu}] d^4x + n \Gamma + \dots$$

3.1-14

where n is an arbitrary integer.

To write Γ in terms of the field U and its derivatives, we use a result from the Bott periodicity theorem (Bott and Seeley, 1978):

$$d \sum_{ijklm} \omega_{ijklm} = \frac{-i}{240\pi^2} d \sum_{ijklm} \text{Tr} [L_i L_j L_k L_l L_m]$$

where

$$L_i = U^\dagger \frac{\partial U}{\partial y_i}$$

$y_i (i=1,5)$ being the coordinates of the disc Q .

This allows us to write Γ as

$$\Gamma = \frac{-i\eta}{240\pi^2} \epsilon_{ijklm} \int d^5x \text{Tr} [L_i L_j L_k L_l L_m].$$

3.1-15

Γ is the Wess-Zumino effective Lagrangian (Wess-Zumino, 1971) which, as we shall discuss later, describes the effects of anomalies in current algebra. The new lagrangian, 3.1-14, satisfies only the required symmetries of QCD, as was desired.

From 3.1-2 we have the weak field approximation

$$L_i = U^\dagger \partial_i U = \frac{1}{F_\pi} \partial_i A + O(A^2), \quad A = \lambda^a \pi^a$$

Then

$$\omega_{ijklm} d \sum_{ijklm} = 1/(240\pi^2 F_\pi^5) d \sum_{ijklm} \text{Tr} [A \partial_i A \partial_j A \partial_k A \partial_l A] + O(A^6)$$

Thus Γ is, to order A^5 , the integral of a total divergence. Stokes theorem enables us to write this as an integral over the boundary of Q , which is M , space-time;

$$\Gamma = \frac{1}{240\pi^2 F_\pi^5} \int d^4x \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A \partial_\mu A \partial_\nu A \partial_\alpha A \partial_\beta A] + \dots$$

3.1-16

3.1.2 Baryon Number of Skyrmion

We begin this section with an analysis of the value of the integer n (3.1-14) in QCD. To do this we couple to electromagnetism and investigate the low-energy dynamics of pions and photons, comparing our results with QCD results.

The functional Γ is invariant under global $U(1)$ charge rotations

$$U_i \rightarrow U_i + i\epsilon [Q, U_i]$$

where

$$Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

is the usual charge matrix for three flavours and ϵ is constant. To promote this to a local symmetry one would usually try to gauge via the minimal coupling

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + i\epsilon A_\mu$$

$$A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \epsilon(x)$$

where A_μ is, of course, the photon field. However, as Γ itself (3.1-16) is not manifestly $SU(2)_L \times SU(3)_R$ invariant, this procedure fails. One finds that under local charge rotation

$$\Gamma \rightarrow \Gamma - \int d^4x (\partial_\mu \epsilon(x)) J^\mu(x)$$

where

$$J^\mu(x) = \frac{1}{48\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} [L_\nu (Q + U Q U^\dagger) L_\alpha L_\beta] \quad 3.1-7$$

To build an invariant $\mathcal{L}$ we must add a minimal coupling term

$$\Gamma \rightarrow \Gamma' = \Gamma - e \int d^4x A_\mu J^\mu(x).$$

Now Γ would be gauge invariant if $J^\mu(x)$ were, which it is not. By trial and error, Witten (Witten, 1983) found the following gauge invariant functional;

$$\hat{\Gamma}(U, A_\mu) = \Gamma(U) - e \int d^4x A_\mu J^\mu + \frac{ie^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu) \partial_\alpha A_\beta$$

$$\times \text{Tr} [Q^2 (\partial_\beta U) U^{-1} + Q^2 U^{-1} (\partial_\beta U) + Q U Q U^{-1} (\partial_\beta U) U^{-1}]$$

giving

$$L = -\frac{F_\pi^2}{4} \int d^4x \text{Tr} [A_\mu U \partial_\mu U] + n \hat{\Gamma} \quad 3.1-19$$

as our $U(1)$ gauge invariant Lagrangian, where n is as yet an unspecified integer. If 3.1-19 is to reproduce low energy theorems from QCD then in fact n is not arbitrary but equal to the number of colours, N .

To see this consider the term in 3.1-18 of order $\mathcal{O}_0 = e^2/4\pi$, which contains a piece describing $\pi^0 \rightarrow 2\gamma$ to lowest order;

$$I = \frac{ie^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu) A_\alpha \text{Tr} [Q^2 (\partial_\beta U) U^{-1} + Q^2 U^{-1} \partial_\beta U + Q U Q U^{-1} (\partial_\beta U) U^{-1}]$$

Expanding U by 3.1-2 the trace term is

$$\frac{3i}{F_\pi} \text{Tr} [Q^2 \partial_\beta A + O(A^2)] \quad , \quad A = \pi^a \lambda^a$$

$$= \frac{3i}{F_\pi} \text{Tr} [Q^2 \lambda^3 \partial_\beta \pi^3 + \dots + O(A^2)]$$

$$= \frac{i}{F_\pi} (\partial_\beta \pi^0 + \dots + O(A^2))$$

where π^0 is the physical field $\pi^0 = \pi^3$. We can write I as

$$I = \frac{-ne^2}{24\pi^2 F_\pi} \int d^4x \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu) A_\alpha \partial_\beta \pi^0 + \dots + O(A^2)$$

which, after integrating by parts, contains a term

$$\begin{aligned} & \frac{ne^2}{24\pi^2 F_\pi} \int d^4x \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu \partial_\alpha A_\beta \pi^0 \\ &= \frac{ne^2}{96\pi^2 F_\pi} \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \pi^0, \end{aligned}$$

This agrees with the corresponding result from QCD triangle diagrams (Adler, 1969; Bell and Jackiw, 1969, Bardeen, 1969) if $n = N$, the number of colours. The Noether coupling term $-e A_\mu J^\mu$ contains a piece

$$\frac{-ie n}{12\pi^2 F_\pi^3} \int d^4x \epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\nu \pi^+ \partial_\alpha \pi^- \partial_\beta \pi^0$$

which describes a $\gamma \pi^+ \pi^0 \pi^-$ vertex. Again this result agrees with QCD calculations (Adler and Bardeen, 1969) if $n = N$. Having established this identification we now calculate the baryon number of the soliton.

In Chapter 2 we mentioned that Skyrme proposed that a soliton with a winding number W is a baryon with baryon number $N_B = W$.

With the results above this is simply derived.

The baryon current defined at the quark level is related to

$$J_\mu^3 = \bar{q} \gamma_\mu \gamma_5 q$$

The Noether current corresponding to this, resulting from the n^{Γ} term, is

$$J_{\mu}^2 = \frac{N}{24\pi^2} \epsilon^{\mu\nu\lambda R} \text{Tr} [\lambda^a L_{\nu} L_{\lambda} L_R] \quad 3.1-20$$

Since a baryon is made from N quarks each quark can be defined to have baryon number $1/N$. If we let $\lambda \rightarrow 1/N$ in 3.1-20 then the baryon current is

$$J_{\mu} = \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda R} \text{Tr} [L_{\nu} L_{\lambda} L_R] \quad 3.1-21$$

so that the baryon number is

$$\begin{aligned} N_B &= \int d^3x J_0 \\ &= \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} [L_i L_j L_k] \end{aligned} \quad 3.1-22$$

which, it will be recalled, is the properly normalized expression for the winding number of a soliton (2.2-18). Thus the $W=1$ sector contains the ordinary nucleons, as stated previously, while the trivial sector $W=0$ contains the mesons.

To see that 3.1-21 is the proper current, consider gauging $SU(3)_L \times SU(3)_R \times U(1)$. If B_{μ} is a $U(1)$ gauge field, we add a Noether coupling $-eJ^{\mu} B_{\mu}$ plus whatever higher-order terms are required by gauge invariance. Using J^{μ} defined above one would find a generalization of that properly accounts for anomalous diagrams involving the baryon number current. This current can also be derived by a variation of the full Lagrangian (Chan, 1985) or directly from QCD (Goldstone and Wilczek, 1981).

For the case of $SU(2) \times SU(2)$ the result is unchanged. Although the Wess-Zumino term vanishes (the Lagrangian for two flavours does not exhibit spurious symmetries), the triangle anomaly $\pi^0 \rightarrow 2\gamma$ still exists, as does the current 3.1-21.

3.1.3 Soliton is a Fermion

In section 2.2.3 it was shown how a Skymion for the group $SU(2) \times SU(2)$ can be a fermion. Witten (Witten, 1983) has proven a much stronger result for the group $SU(3) \times SU(3)$, valid also for four or more flavours. To accomplish this Witten compared the amplitude for the two processes used in section 2.2.3 to distinguish between the two elements of $\pi_4(SU(2)) \approx Z_2$.

Firstly, consider a soliton at rest for a long time, T . The amplitude for this process is

$$\begin{aligned} e^{iS} &= \exp[i \int d^4x \mathcal{L}] \\ &= \exp[i \int dt \int d^3x \mathcal{L}] \\ &= \exp[i \int dt (-H)] \\ &= e^{-iMT} \end{aligned}$$

where

$$\begin{aligned} \int d^3x \mathcal{L} &= -H \\ &= -M \end{aligned}$$

since $\vec{\Pi} = 0$ for a soliton at rest (see 3.1-16). In the second process a soliton is adiabatically rotated through an angle of 2π over a long period of time T . The usual terms in $\mathcal{L}$ are at best quartic in time derivatives and cannot distinguish between the two processes:

$$\int dt \text{Tr} \left[\frac{\partial u}{\partial t} \frac{\partial u^{-1}}{\partial t} \right] \rightarrow 0$$

for an adiabatic process. The Wess-Zumino term, however, is linear in the time derivatives and can distinguish between a soliton at rest and one that has been adiabatically rotated. It can be shown (Witten, 1983) that for a soliton adiabatically rotated through a 2π angle, $\Gamma = \pi$. Thus, the amplitude for this process is

$$e^{-iMT} e^{iN\pi} = (-1)^N e^{-iMT}$$

The factor of $(-1)^N$ tells us that for N odd (even) the soliton must be a fermion (boson). Since $N = 3$, we have a much stronger result than previously. We have yet to prove, however, that the quantized soliton corresponds to the ordinary nucleons rather than some topologically excited solitonic baryon. The following result seems to directly establish this. —

The result is a comparison between the expected quantum numbers I, J in the quark model and the soliton model. Assuming that the multi-quark wave function is symmetric in space and antisymmetric in colour, the spin and isospin components must be symmetric. We represent a quark by ϕ_i , where $i = 1, 4$ is a combined spin-isospin index. Since, in the large N limit, a baryon consists of N quarks in a colour singlet, we must form symmetric combinations of N four-vectors ϕ_i . The general symmetric expression for N quarks is

$$(\phi^2)^k A_{i_1 \dots i_{N-2k}} \quad 0 \leq k \leq N/2$$

where

$$\phi^2 = \sum_{i=1}^4 \phi_i^2$$

and

$$A_{i_1 \dots i_P}^{(P)} = (\phi_{i_1} \phi_{i_2} \dots \phi_{i_P} - \text{trace terms})$$

is a traceless tensor of rank P .

If N is even, we can pair spin and isospin to cancel or produce even integral results (recall that for a quark $I=J=1/2$);

N even, $I=J=0,1,2,\dots$

If N is odd we will always have a half-integral result;

N odd, $I=J=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

Since $N=3$, we have the second sequence of the nucleon ($I=J=\frac{1}{2}$) and the delta ($I=J=\frac{3}{2}$). We know from section 2.5.1 that in quantizing the soliton as a fermion we also obtain the second sequence

$$I=J=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

While the above analysis applies to two flavours, quantization of the collective coordinates for three or more flavours will still give quantum numbers in agreement with the quark model (Witten, 1983). The analysis is more complicated, however, as the Wess-Zumino term is present.

The main results of this section were derived from the Wess-Zumino term. This term was found to be necessary to counter a spurious symmetry of the Skyrme Lagrangian, present for three or more flavours. The relation between the Wess-Zumino term and anomalies is made explicit in the next section, where the Wess-Zumino term is rederived by integrating directly the Bardeen anomaly.

3.2 Connection to QCD

3.2.1 Axial Anomaly

Before deriving the Wess-Zumino term we give an elementary description of the axial anomaly.

The axial anomaly arises from a conflict between chiral invariance and gauge invariance. When gauge invariance is enforced one finds that left and right chiral states do not decouple in the massless

limit. The masslessness of a theory is expressed by the invariance of the Lagrangian under $U(1)$ chiral transformations:

$$\psi(x) \rightarrow e^{-i\omega \gamma_5} \psi(x). \quad 3.2-1$$

By Noethers theorem, this implies the existence of a conserved axial-vector current,

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi. \quad 3.2-2$$

From a formal analysis of the equations of motion, one can derive

$$\partial_\mu j_5^\mu = 2m j_5 \quad 3.2-3$$

where j_5 is the chiral density

$$j_5 = i \bar{\psi} \gamma_5 \psi \quad 3.2-4$$

and m is the mass. Thus j_5^μ would appear to be conserved in the massless limit. This is, however, not the case.

The invariance of a Lagrangian under a $U(1)$ gauge transformation

$$\psi \rightarrow e^{-i\omega} \psi \quad 3.2-5$$

gives rise to a conserved current

$$j^\mu = \bar{\psi} \gamma^\mu \psi. \quad 3.2-6$$

What we will find is that when gauge invariance, defined by

$$\partial_\mu j^\mu = 0 \quad 3.2-7$$

is enforced, equation 3.2-3 will read

$$\partial_\mu j_5^\mu = 2m j_5^\mu + \frac{d_0}{4\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad 3.2-8$$

so that chiral invariance will not be satisfied in the massless limit.

The simplest process in which the axial anomaly is manifest is the triangle graph, which corresponds to the lowest order description of processes such as

$$\begin{aligned} \pi^0 &\rightarrow 2\gamma \\ Z &\rightarrow 2\gamma \text{ (virtual)}. \end{aligned} \quad 3.2-9$$

The triangle graph is a closed fermion loop with one axial-vector vertex and two vector vertices:

FIGURE 3.1

The above two diagrams are collectively referred to as the triangle graph, denoted by

$$t_{\alpha\beta\mu}(k_1, k_2) = S_{\mu\beta\mu}(k_1, k_2) + S_{\mu\alpha\mu}(k_1, k_2) \quad 3.2-10$$

where

$$S_{ABM}(k_1, k_2) = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left(\frac{1}{p+k-m} \gamma_A \frac{1}{p-m} \gamma_B \frac{1}{p-k_2-m} \gamma_5 \right) \quad 3.2-11$$

Although 3.2-11 is linearly divergent, the sum 3.2-10 is convergent.

Gauge invariance can be stated as

$$k_1^\alpha \epsilon_{ABM}(k_1, k_2) = k_2^\beta \epsilon_{ABM}(k_1, k_2) = 0 \quad 3.2-12$$

The statements of "naive" chiral invariance, 3.2-3, implies

$$-(k_1 + k_2)^\mu \epsilon_{ABM} = 2m V_{AB}(k_1, k_2) \quad 3.2-13$$

where

$$V_{AB}(k_1, k_2) = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left(\frac{1}{p+k_1-m} \gamma_A \frac{1}{p-m} \gamma_B \frac{1}{p-k_2-m} \gamma_5 \right) + (k_1, k_2 \rightarrow k_2, k_1) \quad 3.2-14$$

We now show the incompatibility of 3.2-12 and 3.2-13.

Equation 3.2-13 can be easily verified since

$$(k_1 + k_2)^\mu S_{ABM}(k_1, k_2) = 2m \text{Tr} \left(\frac{1}{p+k_1-m} \gamma_A \frac{1}{p-m} \gamma_B \frac{1}{p-k_2-m} \gamma_5 \right),$$

the other two terms vanishing due to dependence on only one external momentum. In the limit $m \rightarrow 0$ naive chiral invariance is obeyed.

One also finds

and

$$k_1^\alpha \epsilon_{ABM}(k_1, k_2) = -i/4\pi^2 k_1^\alpha k_2^\nu \epsilon_{ABM\nu}$$

$$k_2^\beta \epsilon_{ABM}(k_1, k_2) = i/4\pi^2 k_1^\alpha k_2^\nu \epsilon_{\alpha B M \nu}$$

so that gauge invariance is violated. We can enforce gauge invariance

by redefining the amplitude to be

$$T_{ABM}(k_1, k_2) = t_{ABM}(k_1, k_2) - \frac{i}{4\pi^2} (k_1 - k_2)^\nu \epsilon_{ABM\nu}.$$

This would yield,

$$(k_1 + k_2)^\mu T_{ABM}(k_1, k_2) = 2i\pi^2 V_{ABM}(k_1, k_2) + \frac{i}{2\pi^2} k_1^\mu k_2^\nu \epsilon_{ABM\nu} \quad 3.2-15$$

so that chiral invariance would not be obeyed in the massless limit. This is how equation 3.2-8 is derived. Thus chiral invariance is violated when gauge invariance is enforced.

From 3.2-15 we see that in the massless limit the contribution from the triangle graph comes from the anomaly alone. It can be shown (Adler and Bardeen, 1969) that higher order corrections to the triangle graph do not contribute to the axial anomaly, so that the anomalous (second) term of 3.2-15 is exact to all orders of perturbation.

3.2.2 Anomalous Ward Identities

The insistence upon the local gauge invariance of QCD imposes restrictions on the Green's functions known as the Ward identities. The effective Lagrangian being used to describe the low energy behaviour of QCD must satisfy these non-trivial constraints. In the presence of anomalies, however, the normal Ward identities are not satisfied, though new anomalous Ward identities are. Below we give these identities in terms of variations of the vacuum functional and derive the consistency relations the anomalous Ward identities satisfy (Wess and Zumino, 1971).

Let $W(U, V_\mu, A_\mu)$ be the one particle vacuum functional in the presence of vector and axial vector fields

$$V_\mu = \frac{1}{2} V_{\mu i} \lambda_i$$

$$A_\mu = \frac{1}{2} A_{\mu i} \lambda_i, \quad i=1,8. \quad 3.2-16$$

The variation of the vacuum functional is described by the infinitesimal gauge operators

$$-X_i = \delta_m \frac{\delta}{\delta V_{mi}} + f^{ijk} V_{mj} \frac{\delta}{\delta V_{ik}} + f^{ijk} A_{mj} \frac{\delta}{\delta A_{ik}} + \dots$$

$$-Y_i = \delta_m \frac{\delta}{\delta A_{mi}} + f^{ijk} A_{mj} \frac{\delta}{\delta V_{ik}} + f^{ijk} V_{mj} \frac{\delta}{\delta A_{ik}} + \dots$$

where f^{ijk} are the SU(3) structure constants and the dots refer to operators acting on the dependence of W upon the chiral field U .

The operators X, Y satisfy the commutation relations

$$[X_i(x), X_j(x')] = f_{ijk} \delta(x-x') X_k(x) \quad (a)$$

$$[X_i(x), Y_j(x')] = f_{ijk} \delta(x-x') Y_k(x) \quad (b)$$

$$[Y_i(x), Y_j(x')] = f_{ijk} \delta(x-x') X_k(x) \quad (c)$$

3.2-17

The anomalous Ward identities, stated in terms of the variation of the functional W , are

$$X_i W = 0$$

$$Y_i W = G_i$$

3.2-18

where $G_i = G_i(A_m, V_m)$ is the axial anomaly derived by Bardeen (Bardeen, 1969; see Wess-Zumino, 1971, for explicit form in this notation).

Note that W has been defined here such that the vector Ward identities are anomaly free with the axial vector anomaly depending only upon the fields A_m and V_m . Since the Ward identities are expressed as the variations of a single functional W we can easily derive strong restrictions on the form of the anomaly. From 3.2-18

$$[X_i(x), Y_j(x')] W = X_i(x) G_j(x')$$

which, using 3.2-17(b), yields the consistency condition

$$X_i(x) G_j(x') = \delta_{ij} + \delta(x-x') G_k(x).$$

3.2-19

Also 3.2-18 yields

$$\begin{aligned} Y_i(x) G_j(x) - Y_j(x') G_i(x) &= [Y_i(x), Y_j(x')] W \\ &= \delta_{ij} \delta(x-x') X_k(x) W \text{ from 3.2-17(c),} \end{aligned}$$

giving the second consistency condition (using 3.2-18)

$$Y_i(x) G_j(x') - Y_j(x') G_i(x) = 0. \quad 3.2-20$$

For a G_i satisfying these consistency conditions (which the Bardeen expression does), equations 3.2-18 are locally integrable. Because of the interest in the topological structure of the solutions, several authors (Rossi, Testa and Yoshida, 1984; for a review see Pak and Rossi, 1984) were led to investigate the global integrability of 3.2-18. The results, given below, are in effect a derivation of NF from QCD. The action functional W , which agrees with Witten's result for $A_A = V_A = 0$, was first derived by Wess and Zumino (Wess and Zumino, 1971), and describes all modifications to low energy theorems due to the presence of anomalies.

It is known from the theory of differential forms that local integrability implies global integrability only if the domain of the solution (the space of mappings $G = \{U: S^1 \times S^3 \rightarrow SU(3) \mid U \rightarrow I\}$, where $S^1 \times S^3$ is the compactified space-time manifold) is simply connected. Since

$$\pi_1(SU(3)) = 0$$

G is connected but because

$$\pi_5(SU(3)) = \mathbb{Z},$$

there exists a covering space of G

$G = \{ \varphi : D \times S^3 \rightarrow SU(3) \mid \varphi \xrightarrow{\sim} I \}$ (D is a two dimensional disc) which is not isomorphic to the trivial embedding of G in $\hat{G}$ (See Choquet-Bruhat, Witt-Morette and Dillard-Bleick, 1977, page 18). Thus G is not simply connected and the global integrability of 3.2-18 is not immediate.

The method of Rossi, Testa and Yoshida is to define a sequence of chiral transformations $\mathcal{L}(x, \tau)$, $\tau \in [0, 1]$, subject to

$$\mathcal{L}(x, \tau) = \begin{cases} I & \tau = 0 \\ \mathcal{L}_1(x) & \tau = 1 \end{cases}$$

3.2-21

where

$$\mathcal{L}_1^2(x) = U(x).$$

The range of $\mathcal{L}$ is a five dimensional manifold $Q = D \times S^3$. The chiral transformations on Q are

$$U^{\mathcal{L}} = \mathcal{L}^{-1} U \mathcal{L}$$

$$(V_{\mu} + A_{\mu})^{\mathcal{L}} = \mathcal{L} (V_{\mu} + A_{\mu}) \mathcal{L}^{-1} + \mathcal{L} \partial_{\mu} \mathcal{L}^{-1}$$

$$(V_{\mu} - A_{\mu})^{\mathcal{L}} = \mathcal{L}^{-1} (V_{\mu} - A_{\mu}) \mathcal{L} + \mathcal{L}^{-1} \partial_{\mu} \mathcal{L}$$

We now construct the functional

$$H(\tau) = W(U^{\mathcal{L}}, V^{\mathcal{L}}, A^{\mathcal{L}})$$

which is a solution to 3.2-18 on Q . Using 3.2-18 we can write the total derivative of H as

$$\frac{dH}{d\tau} = \frac{i}{2} \int d^4x (\mathcal{L} \partial_{\tau} \mathcal{L}^{-1} - \mathcal{L}^{-1} \partial_{\tau} \mathcal{L}) \cdot G_r(V_{\mu}^{\mathcal{L}}, A_{\mu}^{\mathcal{L}}).$$

where

$$(\Omega \partial_\mu \Omega^{-1} - \Omega^{-1} \partial_\mu \Omega)_c \equiv \text{Tr}[\lambda^c (\Omega \partial_\mu \Omega^{-1} - \Omega^{-1} \partial_\mu \Omega)].$$

Using the boundary conditions 3.2-21 this can be integrated to give

$$W(U, V_\mu, A_\mu) = W(1, V_\mu^{(1)}, A_\mu^{(1)}) - \frac{i}{2} \int_0^1 d\tau \int d^4x (\Omega \partial_\mu \Omega^{-1} - \Omega^{-1} \partial_\mu \Omega)_c \text{Tr}(V_\mu^{(2)}, A_\mu^{(2)}).$$

Since the integration constant is $SU(3) \times SU(3)$ invariant it can be set to zero. One now easily recovers Witten's formula, setting $V_\mu = A_\mu = 0$;

$$\begin{aligned} P(U) &= W(1, 00) = \frac{i}{48\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_0^1 d\tau \int d^4x \text{Tr}[L_\tau L_\mu L_\nu L_\rho L_\sigma] \\ &= \frac{i}{240\pi^2} \epsilon_{ijklm} \int d^5x \text{Tr}[L_i L_j L_k L_l L_m] \end{aligned}$$

where the extra factor of five is combinatorial, resulting from replacing a four dimensional Levi-Civita Symbol by a five dimensional one.

CONCLUSION: The Baryon as a Topological Soliton

The objective of this thesis has been to present a line of reasoning by which the baryons emerge as the solitons of QCD. Historically, the argument begins with Skyrme who, in a series of ingenious papers, developed a meson theory which admitted solitonic solutions. Skyrme's greatest contribution was not the theory itself but his identification of the solitons as fermions, with a topological index corresponding to the baryon number. As this development predated the formulation of QCD, the theoretical basis for our argument lies not with Skyrme but with 'tHooft. For without 'tHooft's development of $SU(N)$ QCD, Skyrme's model would remain a mere phenomenological ansatz with some highly speculative insights. To understand this one must understand the link between QCD and, via the $1/N$ expansion, the Skyrme model.

In the low energy limit, QCD is equivalent to a theory of Goldstone bosons. Since one can use a model meson theory, such as Skyrme's, to describe the interactions of the Goldstone bosons, why do we need the $1/N$ expansion? The answer is that this limit, being valid for energies much less than that of a quark bound in a baryon, is inappropriate for baryon physics. One could assume that, for any energy, QCD is equivalent to a local theory of mesons. This may be a useful phenomenological ansatz but hardly represents a theoretical breakthrough. To use Skyrme's model, and his conjectures, to learn something of theoretical importance about QCD, we need a well defined equivalence between QCD and a meson theory. The $1/N$ expansion provides this desired equivalence.

In Chapter one we discovered that QCD, for any N , is equivalent to a meson theory. It must be stressed that Skyrme's is not the meson theory which is equivalent to QCD; it is a model used to solve for the leading order effects in the weak coupling (large N) limit. While the

actual meson theory which is equivalent to QCD remains unknown, we have been able to derive much with only the knowledge of its existence. Having derived many features of the strong interaction phenomenology, such as Zweigs rule, we can confidently claim that QCD is equivalent to some complicated meson theory with a coupling constant $1/N$ and selection rules discovered from large N combinatorics. What does this tell us about the baryons?

In Chapter one the baryons were shown to behave as solitons in a weakly coupled field theory. Thus we assume that, for any N , baryons are solitons in the meson theory that is equivalent to QCD. For large N the coupling is weak and the solitons can be treated semiclassically. This is where the Skyrme model enters into the picture as an example of an effective theory of mesons which yields solitonic solutions. One crucial question that must be answered is whether $N=3$ is large enough to justify a semiclassical treatment.

As Witten once remarked:

$$\frac{e^2}{4\pi} = \frac{1}{137} \Rightarrow e = 0.3$$

This does not ensure that the $1/N$ expansion of QCD will be as successful as the perturbative expansion of QED; it is merely a quick rebuttal to an outright rejection of the method based on the magnitude of N . If, for instance, the expansion parameter of QED were $4\pi e^2$, perturbation theory would not be such a spectacular success. If the characteristic parameter of the non planar diagrams is actually $1/4\pi N^2 = 1/113$, rather than $1/N^2 = 1/9$, the contributions from the nonplanar diagrams could be about as small as those from electromagnetic corrections. We can only know how large N must be when we know the magnitude of these contributions. Given that we cannot sum even the planar diagrams, a theoretical validation of the expansion is at present impossible. The best that we

can do is to rely upon the qualitative predictions derived in Chapter One. While much has been done without an explicit expression for the leading order term, this is surely a problem which must soon be solved.

Assuming that the $1/N$ expansion is valid, the results of quantizing the soliton can only lead to the interpretation of the baryons as solitons in a meson theory, with the semi-classical expansion corresponding to the large N limit of QCD. The most important results leading to this conclusion are the correct characterization of the baryon as a fermion and the existence of a conserved baryon number. The latter is due to the anomaly induced coupling of the baryon current to the Goldstone bosons. With this theoretical edifice we now examine the resulting calculations.

Since the large N effective Lagrangian of QCD is unknown we perform calculations with a simple model, such as Skyrmes. Such calculations, as detailed in section 2.5-1, are generally accurate to within 30%. This error is due to the inadequacy of either the semiclassical expansion or the Skyrme model itself. One can certainly improve upon the Skyrme model by introducing the vector mesons (Adkins and Nappi, 1984) or other degrees of freedom. The semiclassical method also deserves some scrutiny. In particular, we need a self-consistent and accurate method of quantization. A systematic study of the modes neglected during quantization would seem necessary. Most physicists assume, however, that the semiclassical method, applied to the proper meson theory, would be equivalent to the $1/N$ expansion and thus contribute an error of less than 30%. One can only know with certainty after the planar diagrams have been summed.

Until that time effort should be concentrated upon verifying the main prediction of the $1/N$ expansion; that baryon physics can be deduced from meson physics. Apart from some selection rules, which should be present in any model, the $1/N$ expansion has no predictive power for mesons. Thus we can take meson physics from experiment to test the main prediction of the $1/N$ expansion. (See, for example, Donoghue, 1985).

Appendix A: The σ - Model

The σ -model provides a simple example of a system incorporating chiral symmetry and partial conservation of the axial current. The model was first introduced by Gell-Mann and Levy (Gell-Mann and Levy, 1960) to reconcile the large mass of the nucleon with the fact that the axial current for the nucleon was partially conserved.

The Lagrangian density for a free nucleon is given by

$$\mathcal{L}(x) = i \bar{\psi} \gamma \psi - m_N \bar{\psi} \psi \quad \text{A-1}$$

Under the chiral transformation,

$$\begin{aligned} \psi &\rightarrow e^{-i \bar{\psi} \cdot \vec{\alpha} \gamma_5 / 2} \psi \\ \bar{\psi} &\rightarrow e^{-i \bar{\psi} \cdot \vec{\alpha} \gamma_5 / 2} \bar{\psi} \end{aligned} \quad \text{A-2}$$

the two terms of A-1 transform as

$$i \bar{\psi} \gamma \psi \rightarrow i \bar{\psi} \gamma \psi,$$

since

$$\{\gamma_\mu, \gamma_5\} = 0,$$

and

$$m_N \bar{\psi} \psi \rightarrow m_N \bar{\psi} e^{-i \bar{\psi} \cdot \vec{\alpha} \gamma_5} \psi,$$

The second term is not invariant under A-2 and is termed chirally odd.

This problem is avoided by introducing new fields, σ and $\vec{\pi}$, and replacing the second term, of A-1 with

$$g \bar{\psi} (\sigma + i \bar{\psi} \cdot \vec{\pi} \gamma_5) \psi,$$

A-3

The fields σ and π are defined to transform so as to cancel the transformation A-2;

$$(\sigma + i \vec{\pi} \cdot \vec{\gamma}_5) \rightarrow e^{i \vec{\pi} \cdot \vec{\alpha} \gamma_5 / 2} (\sigma + i \vec{\pi} \cdot \vec{\pi} \gamma_5) e^{i \vec{\pi} \cdot \vec{\alpha} \gamma_5 / 2}$$

A-4

or, for α infinitesimal

$$\sigma \rightarrow \sigma - \alpha \cdot \vec{\pi}$$

$$\vec{\pi} \rightarrow \vec{\pi} + \sigma \vec{\alpha}$$

A-5

Note that $\sigma^2 + \pi^2$ is invariant under this transformation.

The most general chirally invariant Lagrangian density involving the nucleon, σ and π fields, is

$$\mathcal{L}(\chi) = i \bar{\psi} \not{\partial} \psi + g \bar{\psi} (\sigma + i \vec{\pi} \cdot \vec{\gamma}_5) \psi + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \vec{\pi})^2 - \frac{1}{4} \lambda^2 [(\sigma^2 + \pi^2) - v^2]^2$$

A-6

The potential term of A-6 is

$$V = \frac{\lambda^2}{4} (\sigma^2 + \pi^2)^2 - \frac{\lambda^2 v^2}{2} (\sigma^2 + \pi^2) + \frac{\lambda^2 v^4}{4}$$

A-7

For $v^2 > 0$, the potential has the well known "Mexican hat" shape, with a minimum on the surface $(\sigma^2 + \pi^2) = v^2$. The vacuum is not invariant under $SU(2) \times SU(2)$ but $SU(2)$. One can reach any vacuum state by acting on $(\sigma, \vec{\pi})$ with an element of $SU(2)$. Thus the vacuum states are in one-to-one correspondence with points of the $SU(2)$ manifold.

Since a nonzero expectation value for $\bar{\pi}$ would violate parity, we expand about one of the minima $\sigma = \pm v$;

$$\begin{aligned}\sigma &\rightarrow \sigma + v \\ \bar{\pi} &\rightarrow \bar{\pi}\end{aligned}$$

A-8

The Lagrangian A-6 becomes, upon substitution of A-8,

$$\begin{aligned}L(\chi) = & \bar{\psi}(i\gamma + gv)\psi + g\bar{\psi}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\lambda^2 v^2)\sigma^2 \\ & + \frac{1}{2}(\partial_\mu\vec{\pi})^2 - \lambda^2\sigma(\sigma^2 + \vec{\pi}^2) - \frac{1}{4}\lambda^2(\sigma^2 + \vec{\pi}^2)^2.\end{aligned}$$

A-9

The nucleon has now acquired a mass gv and the original $SU(2)\times SU(2)$ symmetry has certainly been hidden. The σ has also acquired a mass,

$$m_\sigma^2 = 2\lambda^2 v^2.$$

A-10

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