Isometries and CAT(0) Metric Spaces

by

Naomi Lynne Wolfson, B.Math

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Abstract

Using the CAT(0) inequality, non-positively curvature in the Riemannian sense can be generalized to the much broader setting of geodesic metric spaces. By studying CAT(0) spaces and groups which act on them properly and cocompactly by isometries, a great deal can be learned about the structure of both. This thesis presents an introduction to CAT(0) spaces and explores the consequences of proper cocompact group actions on these spaces, including the Flat Torus Theorem, the Splitting Theorem, and a solution to the word problem for such groups.

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Chapter 0

Introduction

The objective of this thesis is to supply a straightforward introduction to $\text{CAT}(0)$ spaces and the groups which act on them properly and cocompactly by isometries. The classical theory for groups acting on hyperbolic manifolds was very successful and in the latter part of the 20th century attempts were made to encapsulate the properties of non-positively curved manifolds in the setting of geodesic metric spaces. Two of the schemes used to accomplish this were Gromov's δ -hyperbolicity and the $\text{CAT}(0)$ inequality [Gromov]. Although both approaches have been tremendously profitable, this thesis focuses on the latter. Named for E. Cartan, A.D. Alexandrov, and A. Topogonov, the $\text{CAT}(0)$ inequality for geodesic metric spaces successfully generalizes many of the properties of hyperbolic manifolds. Moreover, the Flat Torus Theorem, the Splitting Theorem, and the Solvable Subgroup Theorem, which were proved for the fundamental groups of compact non-positively curved Riemannian man-

ifolds [Gromoll & Wolf], [Lawson & Yau] have analogous theorems for groups which act properly and cocompactly by isometries on a $\text{CAT}(0)$ metric space.

The principle resource available on the topic is the book *Metric Spaces of Non-positive Curvature* written in 1999 by Martin Bridson and André Haefliger [Bridson & Haefliger]. This 700-page text is comprehensive and encyclopedic in nature, containing reference to more or less everything that was known about $\text{CAT}(0)$ spaces at the time it was written. This book is not necessarily suitable as an introductory text for graduate students who wish to learn about $\text{CAT}(0)$ spaces however, as it omits numerous details in its proofs while covering more material than appropriate for an introduction. In order to provide a more effortless introduction, we have chosen to cover selected chapters and sections from this book, presenting all the omitted details in the relevant proofs. By choosing an appropriate route through this material, we build progressively from the basic definitions of geodesic and $\text{CAT}(0)$ metric spaces to attain the ultimate goal of proving the Splitting Theorem and Flat Torus Theorem in the context of $\text{CAT}(0)$ spaces. This thesis provides a snap-shot of the theory of $\text{CAT}(0)$ spaces while still providing sufficient detail to be rigorous.

The thesis can be logically divided into two parts where the first part consists of Chapters 1 and 2 and second part contains the remaining chapters. In Chapter 1 we will introduce geodesic metric spaces, triangles and angles as well as a collection of uniformly curved metric spaces which will serve as a basis upon which to define $\text{CAT}(0)$ or $\text{CAT}(\kappa)$ spaces. Then in Chapter 2 we define $\text{CAT}(0)$ spaces and begin examining their global structure. We pay

special attention to the consequences of convexity in these spaces and define the centre and radius of a bounded set, which will be used later to prove the Bruhat-Tits Fixed Point Theorem for complete $\text{CAT}(0)$ spaces. The chapter will finish with a section which examines the presence of flat subspaces in a $\text{CAT}(0)$ space and a proof of the Product Decomposition Theorem which will be used to study axes of hyperbolic isometries in Chapters 5 and 6.

In Chapter 3 we change direction and introduce group actions on topological spaces. We also prove a useful result due to Macbeath that gives a finite presentation for a group acting properly and cocompactly by isometries on a simply-connected geodesic metric space. The subsequent chapter elaborates on this, showing that any group acting properly and cocompactly by isometries on a $\text{CAT}(0)$ space has a decidable word problem and in fact a quadratic isoperimetric function.

The final two chapters are dedicated to the isometries of $\text{CAT}(0)$ spaces. In Chapter 5 we look at the properties of individual isometries via their displacement functions and translation lengths. In Chapter 6, the final chapter, we will examine the structural consequences of groups which act properly (not necessarily cocompactly) by isometries on a $\text{CAT}(0)$ space. We finish our discussions by presenting proofs for the Splitting Theorem and the Flat Torus Theorem. This thesis assumes basic knowledge of metric spaces, group theory, and topology.

Chapter 1

Geodesics and the Model Spaces

In this chapter, we introduce geodesic metric spaces and explore a class of such metric spaces called the Model Spaces. The notions of curvature we will explore in this thesis are based on the properties of the Model Spaces which means that a solid understanding of their properties are essential in the chapters which follow. The material in this section is presented largely without proof and one may refer to [Bridson & Haefliger pg. 1-26] for more details.

1.1 What are Geodesics?

In a metric space X we define a *path* $c : [a, b] \rightarrow X$ to be a continuous function from a closed bounded interval $[a, b] \subset \mathbb{R}$ to the metric space X . A

path c is said to be *rectifiable* if its length ℓ_c defined by

$$\ell_c = \sup_{a=t_0 \leq t_1 \leq \dots \leq t_n=b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \quad (1.1)$$

exists. We may verify that in the space $\mathbb{R}^n$ this notion of the length of a path is equivalent to the arc length of a curve.

In a similar manner, we define a *ray* in a metric space X to be a continuous map $c : [a, \infty) \rightarrow X$ and a *line* to be a continuous map $c : \mathbb{R} \rightarrow X$. In this thesis however, we deal almost exclusively with paths, rays, or lines in which the distance between any two points on the path is actually equal to the length of the portion of the path which joins them. We call such paths *geodesic*; the precise definition is as follows.

Definition 1.1.1. A *geodesic path* in a metric space X is a continuous function $c : [0, \lambda] \rightarrow X$ satisfying:

$$d(c(t), c(t')) = |t - t'| \quad (1.2)$$

for every $t, t' \in [0, \lambda]$.

Given a geodesic path $c : [0, \lambda] \rightarrow X$ we say that c *originates* at $c(0)$ and *terminates* at $c(\lambda)$; or that c is a geodesic path from $c(0)$ to $c(\lambda)$. As a consequence of how a geodesic path c is defined, its length ℓ_c is equal to the distance between its originating and terminating points; in particular we have $\ell_c = \lambda$. Furthermore, for any $t < t' < t'' \in [0, \lambda]$ the triangle inequality reduces

to an equality.

$$d(c(t), c(t'')) = d(c(t), c(t')) + d(c(t'), c(t'')) \quad (1.3)$$

We also apply the concept of geodesics to lines and rays in a metric space X where a line $c : \mathbb{R} \rightarrow X$ is said to be geodesic if for every $t, t' \in \mathbb{R}$ the equality $d(c(t), c(t')) = |t - t'|$ holds. Geodesic rays are defined similarly.

Each geodesic path $c : [0, \lambda] \rightarrow X$ in a metric space X has as its domain the interval $[0, \lambda]$ where λ is equal to the length ℓ_c of that path. In a collection of several geodesic paths however, it is not necessary that these lengths are all the same and so the geodesics may have different domains. To simplify matters, it is useful to regard a geodesic path as a continuous image of the unit interval.

Definition 1.1.2. A *linearly reparameterized geodesic path* in a metric space X is a continuous function $c : [0, 1] \rightarrow X$ such that for some $\lambda > 0$:

$$d(c(t), c(t')) = \lambda |t - t'| \quad (1.4)$$

for every $t, t' \in [0, 1]$. Note that in this case, it is the *reparameterization factor* λ which is equal to the length ℓ_c of the path.

Let $I \subseteq \mathbb{R}$ be a closed interval. The path, ray, or line $c : I \rightarrow X$ is said to be *locally geodesic* if for every $t \in I$ there is a neighbourhood $J \subset I$ of t such that $c|_J$, after being suitably reparameterized, is a geodesic path.

Given any choice of a geodesic path $c : I \rightarrow X$ originating at x and terminating at y in metric space X , we write $[x, y]$ to denote the image of c in X ; this image $[x, y]$ is called a *geodesic segment*. We often speak of a geodesic segment $[x, y]$ without mention of the specific geodesic path c , whose image is $[x, y]$. Geodesics need not be unique however, so we must take care to avoid ambiguity. For example, given antipodal points a and b on the unit sphere $\mathbb{S}^2$, any great circle through a and b defines two geodesic paths from a to b .

Furthermore, given any geodesic path c which defines a geodesic segment $[x, y]$ there exists a linearly reparameterized geodesic path which defines the same geodesic segment. As such, if we have a geodesic path c and a linearly reparameterized geodesic path c' originating and terminating at the same points such that c and c' have the same image, the two paths are equivalent in most of our discussions.

For any geodesic segment $[a, b]$ defined by the geodesic path $c : [0, \lambda] \rightarrow X$ we define the reverse geodesic segment $[b, a]$ to be the image of $\bar{c} : [0, \lambda] \rightarrow X$ where $\bar{c}(t) = c(\lambda - t)$ for each $t \in [0, \lambda]$.

Definition 1.1.3. Given two geodesic paths $c : [0, \lambda] \rightarrow X$ and $c' : [0, \lambda'] \rightarrow X$ such that the terminating point of c is the originating point of c' we define the concatenation $cc' : [0, \lambda + \lambda'] \rightarrow X$ of c and c' to be given by:

$$cc'(t) = \begin{cases} c(t) & t \in [0, \lambda] \\ c'(t - \lambda) & t \in [\lambda, \lambda + \lambda'] \end{cases} \quad (1.5)$$

If we have two geodesic segments $[x, y]$ and $[y, z]$ we shall write $[x, y] \cup [y, z]$ to represent the image of the concatenation of their corresponding geodesic paths. We must be careful however, for it is not necessarily the case that this union is itself a geodesic segment.

Proposition 1.1.4. Suppose $c_1 : [0, \lambda_1] \rightarrow X$ and $c_2 : [0, \lambda_2] \rightarrow X$ are geodesic paths such that the terminating point of c_1 is the originating point of c_2 . Then the concatenation of these two paths is itself a geodesic path if and only if $d(x, z) = d(x, y) + d(y, z)$ where $x = c_1(0)$, $y = c_1(\lambda_1) = c_2(0)$, $z = c_2(\lambda_2)$.

Proof. ($\Rightarrow$) Suppose that the concatenation $c_1 c_2$ is a geodesic path. Then by (1.3) we have

$$\begin{aligned} d(c_1 c_2(0), c_1 c_2(\lambda_1 + \lambda_2)) &= d(c_1 c_2(0), c_1 c_2(\lambda_1)) + d(c_1 c_2(\lambda_1), c_1 c_2(\lambda_1 + \lambda_2)) \\ &= d(c_1(0), c_1(\lambda_1)) + d(c_2(0), c_2(\lambda_2)) \end{aligned} \tag{1.6}$$

So we have $d(x, z) = d(x, y) + d(y, z)$ as desired.

($\Leftarrow$) Suppose $d(x, z) = d(x, y) + d(y, z) = \lambda_1 + \lambda_2$. To show that $c_1 c_2$ is a geodesic path, we only need to consider $t \in [0, \lambda_1]$ and $t' \in [\lambda_1, \lambda_1 + \lambda_2]$ since

we already know that c_1 and c_2 are geodesic paths. We see that:

$$\begin{aligned}
d(c_1c_2(0), c_1c_2(\lambda_1 + \lambda_2)) &\leq d(c_1c_2(0), c_1c_2(t)) + d(c_1c_2(t), c_1c_2(t')) \\
&\quad + d(c_1c_2(t'), c_1c_2(\lambda_1 + \lambda_2)) \\
&\leq d(c_1c_2(0), c_1c_2(t)) + d(c_1c_2(t), c_1c_2(\lambda_1)) \\
&\quad + d(c_1c_2(\lambda_1), c_1c_2(t')) \\
&\quad + d(c_1c_2(t'), c_1c_2(\lambda_1 + \lambda_2)) \\
&= d(c_1(0), c_1(t)) + d(c_1(t), c_1(\lambda_1)) \\
&\quad + d(c_2(0), c_2(t - \lambda_1)) + d(c_2(t' - \lambda_1), c_2(\lambda_2)) \\
&= t + (\lambda_1 - t) + (t' - \lambda_1) + (\lambda_2 - t' + \lambda_1) \\
&= \lambda_1 + \lambda_2 \\
&= d(c_1c_2(0), c_1c_2(\lambda_1 + \lambda_2))
\end{aligned} \tag{1.7}$$

Thus, equality holds everywhere and in particular this implies that for any $t \in [0, \lambda_1]$ and $t' \in [\lambda_1, \lambda_1 + \lambda_2]$ we have:

$$d(c_1c_2(t), c_1c_2(t')) = (\lambda_1 - t) + (t' - \lambda_1) = t' - t \tag{1.8}$$

We conclude that c_1c_2 is a geodesic path as desired. $\square$

1.2 Geodesic Metric Spaces and Convexity

We are interested in those metric spaces for which there exists a geodesic segment joining each pair of points. Metric spaces which satisfy this condition

are called *geodesic metric spaces*. A slightly less restrictive condition for X is that every ball in X of radius r is a geodesic metric space; such spaces are called r -geodesic. A stronger condition for geodesic metric spaces is that for each pair of points x, y in the space X there is a *unique* geodesic path originating at x and terminating at y . Such spaces are called *uniquely geodesic metric spaces*; similarly, any r -geodesic metric space in which each ball $B(x, r) \subseteq X$ is a uniquely geodesic metric space is called *uniquely r -geodesic*.

Example 1.2.1. The 2-dimensional sphere $\mathbb{S}^2$ is a geodesic metric space when endowed with the metric which assigns to each pair of points $x, y \in \mathbb{S}^2$ a distance $d(x, y)$ which is equal to the length of the shortest curve in $\mathbb{S}^2$ which joins them. The space $\mathbb{S}^2$ with this metric, however, is not uniquely geodesic.

Proposition 1.2.2. Given geodesic segments $[a, b]$ and $[b, c]$ in a uniquely geodesic metric space X . The geodesic segment $[a, c]$ is equal to the union $[a, b] \cup [b, c]$ if and only if $b \in [a, c]$

Proof. $(\Rightarrow)$ This direction is clear since $[a, c] = [a, b] \cup [b, c]$ implies $b \in [a, c]$. $(\Leftarrow)$ If $b \in [a, c]$ then $d(a, c) = d(a, b) + d(b, c)$ so by Proposition 1.1.4, the concatenation $[a, b] \cup [b, c]$ is in fact a geodesic segment, say $[a, b] \cup [b, c] = [a, c]'$. The metric space X is uniquely geodesic however, so $[a, b] \cup [b, c] = [a, c]' = [a, c]$ as desired. $\square$

Given a geodesic metric space X , a subset $C \subset X$ is called *convex* if for every $x, x' \in C$ there exists a geodesic path $c : [0, \lambda] \rightarrow \mathbb{R}$ in X originating at

x and terminating at x' and for every such geodesic path, its corresponding geodesic segment $[x, x']$ is completely contained within C .

Remark 1.2.3. A convex subspace C of any geodesic metric space X is a geodesic metric space. This is because any two points $x, y \in C$ have a geodesic segment $[x, y]$ in X joining them and this segment is contained within C . Similarly, any convex subspace of an r -geodesic space is r -geodesic and any convex subset of a uniquely geodesic space (uniquely r -geodesic space) is uniquely geodesic (uniquely r -geodesic).

Before continuing, we present some examples of geodesic metric spaces. Refer to [Serre] for more details.

Example 1.2.4 (Metric Cayley Graphs). A *combinatorial graph* G consists of a set of vertices V and a set of edges E together with two maps $\partial_0 : E \rightarrow V$ and $\partial_1 : E \rightarrow V$ (the endpoint maps). We say that the edge e originates at $\partial_0(e)$ and terminates at $\partial_1(e)$. Using the combinatorial graph G we define the *metric graph* X_G by taking the set $E \times [0, 1]$ modulo the equivalence relation

$$(e, i) \sim (e', i') \iff \partial_i(e) = \partial_{i'}(e') \quad (1.9)$$

where $(e, i), (e', i') \in E$ and $i, i' \in \{0, 1\}$. We denote this space X_G and take the distance $d(x, y)$ to be the infimum of the lengths of all the piecewise linear paths joining x to y in X_G . This defines a geodesic metric space provided G

is connected. If the corresponding metric graph X_G is connected and simply connected, then it is called a *tree*.

The *Cayley graph* for a group Γ with generating set S is defined to be the combinatorial graph consisting of the vertex set Γ and the edge set E which contains an edge $e_{(\gamma,s)}$ originating at γ and terminating at $\gamma \cdot s$ for each pair $\gamma \in \Gamma$, $s \in S$. The Cayley graph is denoted $C_S(\Gamma)$ but we will also abuse notation slightly and write $C_S(\Gamma)$ to denote the *metric Cayley graph*. The metric Cayley graph is a geodesic metric space. It will be a uniquely geodesic metric space if and only if it is a tree, which occurs if and only if Γ is free and S is a set of free generators.

For example, the free group on two generators $F(a, b)$ has the Cayley graph as illustrated in Figure 1.1

This metric graph is a tree and thus it is a uniquely geodesic metric space.

Example 1.2.5. We define the *Manhattan metric* on $\mathbb{R}^2$ as follows::

$$d((x, y), (x', y')) = \begin{cases} |x - x'| & y = y' \\ |x| + |x'| + |y - y'| & y \neq y' \end{cases} \quad (1.10)$$

Endowed with this metric, $\mathbb{R}^2$ is a uniquely geodesic metric space.

Figure 1.1: Section of the Cayley graph of $F(a, b)$

1.3 Geodesic Triangles and Angles Between Geodesics

When we begin exploring the curvature of geodesic metric spaces, our primary method of characterization shall be by investigating the properties of their triangles. By comparing these triangles to triangles in spaces of known curvature, say the Euclidean plane for instance, we can begin to understand how geodesics in our metric space behave.

Definition 1.3.1. Given three points x , y , and z in a metric space X which have geodesic segments $[x, y], [y, z], [z, x] \in X$ joining them, we define the *geodesic triangle* $\Delta([x, y], [y, z], [z, x])$ to be the union of the geodesic segments $[x, y] \cup [y, z] \cup [z, x] \subset X$. If the geodesic segments $[x, y]$, $[y, z]$, and $[z, x]$ are unique then we may omit these choices in our notation and write $\Delta(x, y, z)$ in the place of $\Delta([x, y], [y, z], [z, x])$.

Figure 1.2: Geodesic Triangle

Geodesic triangles are an powerful tool for understanding the properties

of a metric space. We shall see later that the curvature of a metric space is associated with whether its geodesic triangles are *fat* or *thin*. First though, we need to find ways of describing our geodesic triangles, but of what use are *triangles* if you have no notion of *angle*? So, to develop a meaning for *angle* in our spaces, we shall associate with each geodesic triangle in a space X a triangle in the Euclidean plane $\mathbb{E}^2$, which is a metric space in which we have a good understanding of the relationships between angles and triangles. The idea of using comparison triangles is due to Alexandrov.

Definition 1.3.2. Let X be a metric space and $x, y, z \in X$. A *comparison triangle* in $\mathbb{E}^2$ for the triple (x, y, z) is a geodesic triangle $\Delta(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{E}^2$ such that $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$.

The triangle inequality guarantees that we can always find $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ and furthermore, this choice is unique up to isometry. When a specific choice of points $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ for a comparison triangle is made, it is typically denoted $\bar{\Delta}(x, y, z)$.

Figure 1.3: Comparison Triangle

The *comparison angle* of the points x, y, z at the point x is the interior angle of any comparison triangle $\overline{\Delta}(x, y, z) = \Delta(\overline{x}, \overline{y}, \overline{z})$ at the vertex labelled $\overline{x} \in \mathbb{E}^2$. The comparison angle is denoted $\overline{\angle}_x(y, z)$ and is well defined provided $x \neq y$ and $x \neq z$.

We now have a way of describing the angle between triples of points in a metric space. Typically however, we are used to defining angles at the intersection of two lines, not at triples of points. As such, we would like to find a way of defining the angle between two geodesic paths which originate at the same point. In the Euclidean plane, the angle between two geodesic paths originating at z (which are straight lines in this case) coincides with the comparison angle between x and y at z for *any* choice of x from the first line and y from the second (excluding any choice where $x = z$ or $y = z$). In an arbitrary metric space however, it is not necessarily the case that two different choices of points would yield the same comparison angle, so we introduce the *Alexandrov angle* between two geodesic segments.

Definition 1.3.3. Given two geodesic paths (or linearly reparameterized geodesic paths) $c : [0, a] \rightarrow X$ and $c' : [0, a'] \rightarrow X$ in a metric space X originating from the same point $c(0) = c'(0)$, the *Alexandrov angle* between c and c' is defined to be:

$$\angle(c, c') := \lim_{\varepsilon \rightarrow 0} \sup_{0 < t, t' < \varepsilon} \overline{\angle}_{c(0)}(c(t), c'(t')) \quad (1.11)$$

In this expression, $\overline{\angle}_{c(0)}(c(t), c'(t'))$ is the comparison angle for the triple of points $(c(0), c(t), c'(t'))$ as defined previously. Given geodesic segments $[a, b]$

and $[a, c]$ we write $\angle([a, b], [a, c])$ to denote the Alexandrov angle $\angle(c, c')$ where c and c' are geodesic paths whose corresponding geodesic segments are $[a, b]$ and $[a, c]$ respectively.

Figure 1.4: Comparison Triangle

The Alexandrov angle possesses several properties analogous to the standard angle in Euclidean geometry. For example, the angle between two opposite rays in a geodesic line is π , as outlined in the following proposition.

Proposition 1.3.4. Given geodesic segments $[a, b]$ and $[a, c]$ in a metric space X such that $[b, a] \cup [a, c] = [b, c]$ is a geodesic segment, then the following equality holds

$$\angle([a, b], [a, c]) = \pi \quad (1.12)$$

Proof. Let $c : [0, 1] \rightarrow X$ and $c' : [0, 1] \rightarrow X$ be the linearly reparameterized geodesic paths with images $[a, b]$ and $[a, c]$ respectively. If we pick points $t, t' \in (0, 1]$ then any comparison triangle $\overline{\Delta}(c(0), c(t), c'(t')) = \Delta(\overline{c(0)}, \overline{c(t)}, \overline{c'(t')})$ for

$c(0)$, $c(t)$, and $c'(t')$ is degenerate. This is because by (1.3) we have

$$d(c(t), c'(t')) = d(c(t), c(0)) + d(c(0), c'(t')) \quad (1.13)$$

Thus $\overline{Z}_{c(0)}(c(t), c'(t')) = \pi$ which gives:

$$\angle([a, b], [a, c]) = \angle(c, c') = \lim_{\varepsilon \rightarrow 0} \sup_{0 < t, t' < \varepsilon} \pi = \pi \quad (1.14)$$

as desired □

Example 1.3.5. In the metric graph X for a combinatorial graph G , the angle between two geodesic paths is either 0 or π , depending if they coincide for non-zero time or not.

In Euclidean space, angles are additive in the sense that if we cut an angle into two parts, it will be the sum of the two angles created. The Alexandrov angle is only subadditive.

Proposition 1.3.6. Suppose $c : [0, 1] \rightarrow X$, $c' : [0, 1] \rightarrow X$, and $c'' : [0, 1] \rightarrow X$ are linearly reparameterized geodesic paths in the metric space X all originating from the same point, $c(0) = c'(0) = c''(0)$. Then the following inequality holds:

$$\angle(c, c'') \leq \angle(c, c') + \angle(c', c'') \quad (1.15)$$

Proof. First, if $\angle(c, c'') = 0$ then (1.15) holds trivially. Let us now suppose that $\angle(c, c'') > 0$. The proof proceeds by contradiction. Suppose that the

inequality $\angle(c, c'') > \angle(c, c') + \angle(c', c'')$ holds. This implies that there exists $\delta > 0$ such that $\angle(c, c'') > \angle(c, c') + \angle(c', c'') + 3\delta$; let us assume further that $\delta < \angle(c, c'')$. From the definition of the Alexandrov angle (Definition 1.3.3), we may find $\varepsilon_1 > 0$ such that for all $t, t' < \varepsilon_1$:

$$\overline{Z}_{c(0)}(c(t), c'(t')) < \angle(c, c') + \delta \quad (1.16)$$

Similarly, there exists $\varepsilon_2 > 0$ such that for all $t', t'' < \varepsilon_2$ we have:

$$\overline{Z}_{c(0)}(c'(t'), c''(t'')) < \angle(c', c'') + \delta \quad (1.17)$$

Furthermore, if we let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ then we may find $t, t'' < \varepsilon$ such that:

$$\overline{Z}_{c(0)}(c(t), c''(t'')) > \angle(c, c'') - \delta \quad (1.18)$$

Let us fix t and t'' from (1.18) and consider a triangle, $\Delta(O, x, x'') \in \mathbb{E}^2$ such that $d(O, x) = t$, $d(O, x'') = t''$ and the vertex angle α at O satisfies:

$$\angle(c, c'') - \delta < \alpha < \overline{Z}_{c(0)}(c(t), c''(t'')) \quad (1.19)$$

Since $\overline{Z}_{c(0)}(c(t), c''(t'')) \leq \pi$ we have that $\alpha < \pi$ and since $\overline{Z}_{c(0)}(c(t), c''(t'')) > \angle(c', c'') - \delta > 0$ we can select $\alpha > 0$ which guarantees that Δ is not degenerate. Then (1.19) implies that $d(x, x'') < d(c(t), c''(t''))$.

 Figure 1.5: Split α into the sum of α' and α''

Now, if we combine our initial assumption that

$$\angle(c, c'') > \angle(c, c') + \angle(c', c'') + 3\delta \quad (1.20)$$

with our choice of α such that $\alpha > \angle(c, c'') - \delta$ then we see that

$$\alpha > \angle(c, c') + \angle(c', c'') + 2\delta \quad (1.21)$$

It is now possible to find a point $x' \in [x, x''] \subset \Delta$ such that the geodesic segment $[O, x']$ splits α into α' and α'' satisfying $\alpha' > \angle(c, c') + \delta$ and $\alpha'' > \angle(c', c'') + \delta$. See Figure 1.5.

If we let $r = \max\{d(O, x), d(O, x'')\} = \max\{t, t''\}$ then, by the convexity of balls in $\mathbb{E}^2$ we have $[x, x''] \subseteq B(O, r)$ which implies that $d(O, x') \leq r = \max\{t, t''\}$. Thus, if we let $t' = d(O, x')$ then $t, t' < \varepsilon_1$ which by (1.16) implies that $\overline{Z}_{c(0)}(c(t), c'(t')) < \angle(c, c') + \delta < \alpha'$. The law of cosines now gives

$d(c(t), c'(t')) < d(x, x')$. Similarly, $d(c'(t'), c''(t'')) < d(x', x'')$. Together, these inequalities tell us that

$$\begin{aligned} d(c(t), c''(t'')) &> d(x, x'') \\ &= d(x, x') + d(x', x'') \\ &> d(c(t), c'(t')) + d(c'(t'), c''(t'')) \end{aligned} \tag{1.22}$$

which contradicts the triangle inequality. $\square$

Corollary 1.3.7. Consider two geodesic segments $[a, b]$ and $[c, d]$ in a metric space X , such that $c \in [a, b]$, $c \neq a$, $c \neq b$. Then if $[c, a]$ and $[c, b]$ are geodesic segments such that $[a, c] \cup [c, b] = [a, b]$ then $\angle([a, c], [c, d]) + \angle([c, d], [c, b]) \geq \pi$.

Figure 1.6: Split the geodesic segment $[a, b]$ by α and α'

Proof. By Proposition 1.3.4 we have $\pi = \angle([c, a], [c, b])$ and by Proposition 1.3.6 we also have $\angle([c, a], [c, b]) \leq \angle([c, a], [c, d]) + \angle([c, d], [c, b])$. Thus the desired inequality holds. $\square$

In other words, if the angles α and α' split the geodesic segment $[a, b]$ then $\alpha + \alpha' \geq \pi$. For the spaces we introduce in the following two sections, we shall find that $\alpha + \alpha' = \pi$, a property which is key in proving the major result of this chapter, Alexandrov's Lemma.

1.4 The Euclidean, Spherical, and Hyperbolic Metric Spaces

In this section, we introduce the following metric spaces: Euclidean space $\mathbb{E}^n$, spherical space $\mathbb{S}^n$, and hyperbolic space $\mathbb{H}^n$. Each of these spaces satisfies a variation on the classical law of cosines. Furthermore, we will use these spaces to form a basis upon which to describe what it means to be a *negatively* or *positively* curved space.

Definition 1.4.1. The n -dimensional Euclidean space $\mathbb{E}^n$ is the vector space $\mathbb{R}^n$ equipped with the metric which, given $x, y \in \mathbb{E}^n$, assigns $\|x - y\|$ to be the distance between x and y where $\|\cdot\|$ denotes the ℓ_2 -norm. Euclidean space is a uniquely geodesic metric space where the geodesic segment joining a pair of points $x, y \in \mathbb{E}^n$ is given by the straight line segment:

$$[x, y] = \{ty + (1 - t)x : t \in [0, 1]\} \quad (1.23)$$

The metric space $\mathbb{E}^n$ obeys the classical law of cosines: given any triangle $\Delta(x, y, z)$ in $\mathbb{E}^n$ where the side lengths are given by $d(x, y) = a$, $d(x, z) = b$,

$d(y, z) = c$, and $\angle([x, y], [x, z]) = \theta$, we have:

$$c^2 = a^2 + b^2 - 2ab \cos \theta \quad (1.24)$$

Figure 1.7: Euclidean Law of Cosines

Now we introduce a metric space in which, unlike in Euclidean space, geodesic paths and lines are *curved*.

Definition 1.4.2. We define the n -dimensional spherical space $\mathbb{S}^n$ to be the unit hypersphere, $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ together with the metric which, for each $x, y \in \mathbb{S}^n$, assigns the distance from x to y to be the length of a great arc which joins them. More precisely, the metric is given by the following equation:

$$d(x, y) = \cos^{-1}(x \cdot y) \quad (1.25)$$

where $x \cdot y$ is the standard dot product in $\mathbb{R}^{n+1}$

A geodesic segment in $\mathbb{S}^n$ joining any two points x and y is a great arc

Figure 1.8: Spherical Distances

between them and this geodesic is unique provided $d(x, y) < \pi$. Consequently, $\mathbb{S}^n$ is a geodesic metric space which is uniquely $\pi/2$ -geodesic and any ball of radius less than or equal to $\pi/2$ in $\mathbb{S}^n$ is convex.

Remark 1.4.3. The n -dimensional sphere $\mathbb{S}^n$ is not a geodesic metric space with the induced metric from $\mathbb{E}^{n+1}$.

If we define the angle between any two geodesic paths to be the *Alexandrov angle* from the previous section then the geodesic triangles in $\mathbb{S}^n$ obey a law of cosines analogous to the Euclidean case. This is called the *spherical law of cosines*: given a triangle $\Delta(x, y, z) \in \mathbb{S}^n$ such that $d(x, y) = a$, $d(x, z) = b$, $d(y, z) = c$ and $\angle([x, y], [x, z]) = \theta$, we have:

$$\cos c = \cos a \cos b + \sin a \sin b \cos \theta \quad (1.26)$$

Furthermore, geodesic triangles in $\mathbb{S}^n$ appear *thick* in the sense that their sides seem to swell outwards and their vertex angles have a sum which is

Figure 1.9: Spherical Law of Cosines

greater than or equal to π . This inevitably leads us to the question as to the existence of a metric space in which the opposite occurs. That is, geodesic triangles appear to be *thin* in the sense that the sum of their angles is less than or equal to π . In the following definition, we introduce a metric space in which this does occur; it is a subspace of $\mathbb{E}^{n,1}$, which is the vector space $\mathbb{R}^{n+1}$ together with the symmetric bilinear form:

$$\langle u, v \rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^n u_i v_i \quad (1.27)$$

Definition 1.4.4. Define n -dimensional hyperbolic space $\mathbb{H}^n$ to be the set

$$\{u \in \mathbb{E}^{n,1} \mid \langle u, u \rangle = -1, u_{n+1} > 0\} \quad (1.28)$$

together with the metric which assigns to each $a, b \in \mathbb{H}^n$ the distance $d(a, b)$ satisfying:

$$\cosh(d(a, b)) = -\langle a, b \rangle \quad (1.29)$$

The space $\mathbb{H}^n$ is uniquely geodesic and any ball is convex. Just as in the spherical case, $\mathbb{H}^n$ obeys a *hyperbolic law of cosines* with respect to the Alexandrov angle:

Given a geodesic triangle $\Delta(x, y, z)$ in $\mathbb{H}^n$ such that $d(x, y) = a$, $d(x, z) = b$, $d(y, z) = c$ and $\angle([x, y], [x, z]) = \theta$, we have:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \theta \quad (1.30)$$

Figure 1.10: Hyperbolic Law of Cosines

For a more comprehensive approach to hyperbolic geometry, refer to [Anderson].

In the preceding three metric spaces, if α and α' are angles which split a

geodesic segment, then the following equality holds

$$\alpha + \alpha' = \pi \quad (1.31)$$

The Alexandrov angle between two curves originating at the same point z in one of these spaces, is equal to the Spherical, Hyperbolic, or Euclidean angle respectively [Bridson & Haefliger pg. 173]. Thus, the Alexandrov angle is equal to the angle between the tangent vectors of these curves at the point z in the tangent plane [Bridson & Haefliger pg. 17,20]. Since the additivity of angles holds in the Euclidean plane $\mathbb{E}^2$, (1.31) holds in all three spaces.

1.5 The Model M_κ^n Spaces

The spaces $\mathbb{S}^n$ and $\mathbb{H}^n$, which we defined in the previous section have geodesic triangles which compare to triangles in $\mathbb{E}^2$ in somewhat opposite ways. We say that the n -dimensional spherical metric space is *positively curved* and that the n -dimensional hyperbolic metric space is *negatively curved*; the n -dimensional Euclidean plane, which seems to meet the other two halfway between, is said to be *flat* or have *zero* curvature. A natural question to ask is whether there are spaces possessing uniform curvature which lie somewhere in-between these others. As we will see in the following definition, the answer is to the affirmative.

Definition 1.5.1. The n -dimensional model space of uniform curvature κ ,

denoted M_κ^n is defined as follows:

- if $\kappa = 0$ then $M_0^n = \mathbb{E}^n$
- if $\kappa > 0$ then M_κ^n is obtained from $\mathbb{S}^n$ by assigning to it a new distance function $d_\kappa : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{S}^n$ we have:

$$d_\kappa(x, y) = d(x, y)/\sqrt{\kappa} \quad (1.32)$$

where $d : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ is the standard spherical metric.

- if $\kappa < 0$ then M_κ^n is obtained from $\mathbb{H}^n$ in a similar manner. The new distance function $d_\kappa : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ is given by:

$$d_\kappa(x, y) = d(x, y)/\sqrt{-\kappa} \quad (1.33)$$

for any $x, y \in M_\kappa^n$ where in this case we use the standard hyperbolic metric $d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}$.

For any $\kappa \in \mathbb{R}$ we define the *diameter* of M_κ^n to be $D_\kappa := \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$ and $D_\kappa = \infty$ otherwise.

Remark 1.5.2. The metric spaces of uniform curvature M_κ^n , inherit the geodesic properties of their parent spaces $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$. In particular, M_κ^n is a uniquely geodesic metric space for $\kappa \leq 0$ and if $\kappa > 0$ then M_κ^n is uniquely $\frac{\pi}{2\sqrt{\kappa}}$ -geodesic.

Since the model spaces are obtained by scaling the metrics for $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ their corresponding cosine laws may be similarly rescaled. Thus, for each $\kappa \in \mathbb{R}$ we obtain the following:

Given a geodesic triangle $\Delta(x, y, z) \subset M_\kappa^n$ such that $d(x, y) = a$, $d(x, z) = b$, $d(y, z) = c$ and $\angle([x, y], [x, z]) = \theta$ then one of the following cosine laws will hold:

- if $\kappa = 0$:

$$c^2 = a^2 + b^2 - 2ab \cos \theta \quad (1.34)$$

- if $\kappa < 0$:

$$\cosh(\sqrt{|\kappa|}c) = \cosh(\sqrt{|\kappa|}a) \cosh(\sqrt{|\kappa|}b) - \sinh(\sqrt{|\kappa|}a) \sinh \sqrt{|\kappa|}b \cos \theta \quad (1.35)$$

- if $\kappa > 0$:

$$\cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a) \cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a) \sin(\sqrt{\kappa}b) \cos \theta \quad (1.36)$$

Note that these cosine laws imply that if a and b are held constant then c is a strictly increasing function of θ . This is a property which is used frequently.

Remark 1.5.3. The model spaces also have the property that any two angles α and α' which split a geodesic sum to π .

Just as we had comparison triangles in $\mathbb{E}^2$ for triples of points in any metric space X , we can often define comparison triangles in M_κ^n for these points as well.

Definition 1.5.4. Given $x, y, z \in X$, provided $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, we can find $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ such that $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$. The triangle $\bar{\Delta}(x, y, z) = \Delta(\bar{x}, \bar{y}, \bar{z})$ is called a κ -comparison triangle for the triple (x, y, z) and is unique up to isometry. Given any geodesic triangle $\Delta(x, y, z) \subset X$ we define a κ -comparison triangle for $\Delta(x, y, z)$ to be any κ -comparison triangle $\bar{\Delta}(x, y, z)$ for the triple (x, y, z) , provided it exists.

Figure 1.11: κ -comparison Triangle

If a κ -comparison triangle $\bar{\Delta}(x, y, z) = \Delta(\bar{x}, \bar{y}, \bar{z})$ exists for $x, y, z \in X$ then the κ -comparison angle between y and z at x is defined to be the Alexandrov angle between the geodesic segments $[\bar{x}, \bar{y}]$ and $[\bar{x}, \bar{z}]$. This angle is denoted $\bar{\angle}_x^{(\kappa)}(y, z)$.

Definition 1.5.5. A metric space X has the *geodesic extension property* if for every local geodesic path $c : [a, b] \rightarrow X$ in X can be extended to a local

geodesic line $\tilde{c} : \mathbb{R} \rightarrow X$ such that $\tilde{c}|_{[a,b]} = c$.

Since the Euclidean, Hyperbolic and Spherical spaces have the geodesic extension property, the model spaces M_κ^n have it as well. Furthermore, any local geodesic path with length less than D_κ will be a geodesic path. We are now prepared to state and prove Alexandrov's Lemma; for clarity we have split it into two parts.

Theorem 1.5.6 (Alexandrov's Lemma *the first half*). Suppose we have four distinct points $A, B, B', C \in M_\kappa^2$ such that $d(C, B) + d(C, B') + d(A, B) + d(A, B') < 2D_\kappa$. Suppose further that B and B' lie on opposite sides of the locally geodesic line extending $[A, C]$. Consider $\Delta(A, B, C)$ and $\Delta(A, B', C)$ as shown in Figure 1.12.

Then if $\gamma + \gamma' \geq \pi$ the following inequality will hold:

$$d(B, C) + d(B', C) \leq d(B, A) + d(B', A) \quad (1.37)$$

Proof. In M_κ^2 we can extend the geodesic segment $[B, C]$ to contain a point $\tilde{B}' \in M_\kappa^2$ such that $d(C, B') = d(C, \tilde{B}')$. See Figure 1.13.

Since $[A, C]$ splits the geodesic segment $[B, \tilde{B}']$, Remark 1.5.3 implies that $\gamma + \angle_C(A, \tilde{B}') = \pi$. Combining this with the given condition that $\gamma + \gamma' \geq \pi$, we conclude that $\angle_C(A, \tilde{B}') \leq \gamma'$, which implies that $d(A, B') \geq d(A, \tilde{B}')$ by the appropriate law of cosines. We remark that by the law of cosines $d(A, B') = d(A, \tilde{B})$ if and only if $\gamma' = \angle_C(A, \tilde{B}')$ if and only if $\gamma + \gamma' = \pi$.

Figure 1.12: Quadrilateral with vertices A , B , B' , and C

Figure 1.13: Construction of $\tilde{B}'$

Our work so far implies

$$\begin{aligned} d(A, B) + d(A, B') &\geq d(A, B) + d(A, \tilde{B}') \\ &\geq d(B, \tilde{B}') \end{aligned} \tag{1.38}$$

where the second inequality is a consequence of the triangle inequality. By (1.3) we know that $d(B, \tilde{B}') = d(B, C) + d(C, \tilde{B}')$ because C lies on the geodesic segment $[B, \tilde{B}']$. By construction $d(C, \tilde{B}') = d(C, B')$, so we conclude that $d(A, B) + d(A, B') \geq d(B, \tilde{B}') = d(B, C) + d(B', C)$ as desired. $\square$

Theorem 1.5.7 (Alexandrov's Lemma *the second half*). Let us take the four points, A, B, B' , and $C \in M_\kappa^2$ from the previous lemma together with the requirement that $\gamma + \gamma' \geq \pi$. Consider a geodesic triangle $\Delta(\bar{A}, \bar{B}, \bar{B}') \subseteq M_\kappa^2$ such that $d(A, B) = d(\bar{A}, \bar{B})$, $d(A, B') = d(\bar{A}, \bar{B}')$, and $d(\bar{B}, \bar{B}') = d(B, C) + d(C, B')$. Such a triangle exists by the first part of Alexandrov's Lemma. Let $\bar{C} \in [\bar{B}, \bar{B}']$ be the unique point such that $d(\bar{B}, \bar{C}) = d(B, C)$ and $d(\bar{B}', \bar{C}) = d(B', C)$. We define $\bar{\alpha}, \bar{\beta}, \bar{\beta}'$ as in Figure 1.14.

Then we have $\bar{\alpha} \geq \alpha + \alpha'$, $\bar{\beta} \geq \beta$, $\bar{\beta}' \geq \beta'$ and $d(A, C) \leq d(\bar{A}, \bar{C})$. If any one of these is an equality, then they will all be equalities, which occurs if and only if $\gamma + \gamma' = \pi$.

Proof. Let $\tilde{B}'$ be as in the proof of the first half of Alexandrov's Lemma. Recall that in the proof of Theorem 1.5.6 we showed that

$$d(\bar{A}, \bar{B}') = d(A, B') \geq d(A, \tilde{B}') \tag{1.39}$$

 Figure 1.14: Construction of Triangle $\Delta(\bar{A}, \bar{B}, \bar{B}')$

and remarked that equality holds if and only if $\gamma + \gamma' = \pi$.

Also,

$$d(\bar{B}, \bar{B}') = d(B, C) + d(C, B') \geq d(B, B') \quad (1.40)$$

with equality if and only if $C \in [B, B']$ if and only if $\gamma + \gamma' = \pi$.

The inequality (1.39), combined with the law of cosines gives $\bar{\beta} \geq \beta$ while the inequality (1.40) gives $\bar{\alpha} \geq \alpha + \alpha'$. Applying the law of cosines to $\bar{\beta} \geq \beta$ we obtain $d(A, C) \leq d(\bar{A}, \bar{C})$. Any of these inequalities are equalities if and only if the corresponding inequality (1.39), (1.40) is an equality if and only if $\gamma + \gamma' = \pi$. $\square$

Chapter 2

Introduction to $\text{CAT}(\kappa)$ Spaces

In this chapter we introduce $\text{CAT}(\kappa)$ spaces, which are spaces whose curvature is bounded from above, and then discuss the structural properties of $\text{CAT}(0)$ spaces. We explore the convexity in the context of $\text{CAT}(0)$ spaces and examine the conditions under which flat spaces occur in a $\text{CAT}(0)$ space. The chapter concludes with the Product Decomposition Theorem which will be important in the chapters which follow.

2.1 The $\text{CAT}(\kappa)$ Inequality

In the previous section we examined a spectrum of spaces of uniform curvature. It is possible however, to have a curved space of non-uniform curvature which is still bounded above. That is, its geodesic triangles are *thinner* than the triangles in M_κ^n for some $\kappa \in \mathbb{R}$. With this motivation, we introduce

$CAT(\kappa)$ spaces via the $CAT(\kappa)$ inequality.

The following definition uses the notion of a *comparison point*. Given a geodesic triangle $\Delta([p, q], [q, r], [r, p]) \in X$ which has a comparison triangle $\bar{\Delta} = \Delta(\bar{p}, \bar{q}, \bar{r}) \in M_\kappa^2$ for p, q, r ; we say that $\bar{x}$ is a *comparison point* for $x \in [p, q]$ if it is the unique point $\bar{x} \in [\bar{p}, \bar{q}]$ such that $d(p, x) = d(\bar{p}, \bar{x})$. See Figure 2.1.

Figure 2.1: Comparison Point

Definition 2.1.1. A geodesic triangle $\Delta = \Delta([p, q], [q, r], [r, p])$ in a metric space X is said to satisfy the $CAT(\kappa)$ inequality for $\kappa \in \mathbb{R}$ if the following hold:

1. The perimeter of Δ is less than $2D_\kappa$, so there exists a κ -comparison triangle $\bar{\Delta}(p, q, r) = \Delta(\bar{p}, \bar{q}, \bar{r}) \in M_\kappa^2$ for the triangle Δ .
2. For any points $x, y \in \Delta$ and their comparison points $\bar{x}, \bar{y} \in \bar{\Delta}(p, q, r)$, the following inequality holds:

$$d(x, y) \leq d_\kappa(\bar{x}, \bar{y}) \quad (2.1)$$

Thus we can now define what it means for X to be a $\text{CAT}(\kappa)$ metric space.

- if $\kappa \leq 0$ then X is called a $\text{CAT}(\kappa)$ space if it is a geodesic metric space and all of its geodesic triangles satisfy the $\text{CAT}(\kappa)$ inequality.
- if $\kappa > 0$ then X is called a $\text{CAT}(\kappa)$ space if it is a D_κ -geodesic metric space and all of its geodesic triangles of perimeter less than $2D_\kappa$ satisfy the $\text{CAT}(\kappa)$ inequality.

To define a $\text{CAT}(\kappa)$ space, item (2) in the definition of the $\text{CAT}(\kappa)$ inequality may be equivalently replaced by any of the following: [Bridson & Haefliger pg. 161]

1. For any pair of points $x \in [p, q]$ and $y \in [p, r]$ such that $x \neq p$ and $y \neq p$, the following inequality holds:

$$\overline{Z}_p^{(\kappa)}(x, y) \leq \overline{Z}_p^{(\kappa)}(q, r) \quad (2.2)$$

Note that a κ -comparison triangle exists for p, x, y provided one exists for p, q, r because

$$\begin{aligned} d(p, x) + d(x, y) + d(y, p) &\leq d(p, x) + d(x, q) + d(q, r) + d(r, y) + d(y, p) \\ &= d(p, q) + d(q, r) + d(r, p) \\ &\leq 2D_\kappa \end{aligned} \quad (2.3)$$

2. The Alexandrov angle, $\angle([p, q], [p, r])$ satisfies the inequality:

$$\angle([p, q], [p, r]) \leq \overline{Z}_p^{(\kappa)}(q, r) \quad (2.4)$$

3. If $\Delta(\tilde{p}, \tilde{q}, \tilde{r}) \subset M_\kappa^2$ is a geodesic triangle such that $d(p, q) = d(\tilde{p}, \tilde{q})$, $d(p, r) = d(\tilde{p}, \tilde{r})$ and $\angle_{\tilde{p}}(\tilde{q}, \tilde{r}) = \angle_p(q, r)$ then the following inequality must hold:

$$d(q, r) \geq d(\tilde{q}, \tilde{r}) \quad (2.5)$$

In the proceeding sections, we will use the terms $\text{CAT}(\kappa)$ inequality or $\text{CAT}(\kappa)$ condition to refer to any one of the inequalities just mentioned.

Here are a few of the properties that $\text{CAT}(\kappa)$ spaces satisfy:

Remark 2.1.2. Note that (2.4) implies that in a $\text{CAT}(0)$ space the angles of a geodesic triangle sum to at most π .

Proposition 2.1.3. Suppose X is a $\text{CAT}(\kappa)$ space, then X satisfies the following:

1. For $\kappa \leq 0$, the metric space X is uniquely geodesic and otherwise, X is uniquely D_κ -geodesic. (Note that in the definition of $\text{CAT}(\kappa)$ we only required the existence of geodesics, not the uniqueness)
2. Every locally geodesic path in X of length at most D_κ is a geodesic path.
3. The balls in X of radius less than $D_\kappa/2$ are convex.

Proof. 1. Let $p, q \in X$ be such that $d(p, q) < D_\kappa$ and let $[p, q]$ and $[p, q]'$ be any two geodesic segments joining p and q . Let $x \in [p, q]$ and $x' \in [p, q]'$ be such that $d(p, x) = d(p, x')$. Now, consider the triangle $\Delta([p, q]', [p, x], [x, q])$ where $[p, x] \cup [x, q] = [p, q]$.

Figure 2.2: Comparison Triangle for $\Delta([p, q]', [p, x], [x, q])$

If we take a comparison triangle $\Delta(\bar{p}, \bar{q}, \bar{x})$ in M_κ^2 for $\Delta([p, x], [x, q], [p, q])$ then since $d(\bar{p}, \bar{q}) = d(\bar{p}, \bar{x}) + d(\bar{x}, \bar{q})$ then by Proposition 1.1.4 it must be the case that $[\bar{p}, \bar{x}] \cup [\bar{x}, \bar{q}]$ is a geodesic segment. Since the model spaces M_κ^n are uniquely geodesic for segments of length less than D_κ , we have $[\bar{p}, \bar{q}] = [\bar{p}, \bar{x}] \cup [\bar{x}, \bar{q}]$ and the comparison triangle is degenerate.

Therefore if $\bar{x}' \in [\bar{p}, \bar{q}]$ is the comparison point for $x' \in [p, q]'$, then we may conclude that the equality $\bar{x} = \bar{x}'$ holds; by the $CAT(\kappa)$ inequality, this implies that $x = x'$ and thus $[p, q] = [p, q]'$ as desired.

2. Consider a local geodesic path $c : [0, \lambda] \rightarrow X$ with length $\lambda < D_\kappa$ and let $S = \{t \in [0, \lambda] : c|_{[0, t]} \text{ is a geodesic path}\}$. We wish to show that $S = [0, \lambda]$ which will imply that c is a geodesic path. This will be done by showing that S is open, closed and non-empty in the connected

subspace $[0, \lambda]$ of $\mathbb{R}$.

Since c is a local geodesic, S contains a neighbourhood of zero and is thus non-empty. We now show that S is closed. Consider a sequence $\{t_n\} \subset S \subseteq [0, \lambda]$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$. Note that $t \in [0, \lambda]$ because $[0, \lambda]$ is closed. The convergence of $\{t_n\}$ implies that for all $t', t'' \in [0, t)$ there exists $t_i \in \{t_n\}$ such that $t', t'' \in [0, t_i]$. Since by hypothesis, $c|_{[c(0), c(t_i)]}$ is a geodesic, we know that $d(c(t'), c(t'')) = |t' - t''|$ for any $t', t'' \in [0, t)$. Furthermore, for any $t' \in [0, t)$ have

$$d(t', t) = \lim_{i \rightarrow \infty} d(c(t'), c(t_i)) = \lim_{i \rightarrow \infty} |t' - t_i| = |t' - t| \quad (2.6)$$

thus we have verified that $c|_{[0, t]}$ is a geodesic path and therefore that S is closed.

Now we will show that S is open. Let $t_0 \in S$ be such that $0 < t_0 < \lambda$. There is no need to consider $t_0 = \lambda$ because in that case, $S = [0, \lambda]$ which would conclude the proof. Since c is a local geodesic, there exists $0 < \varepsilon < \lambda - t_0$ such that $c|_{[t_0 - \varepsilon, t_0 + \varepsilon]}$ is a geodesic path. This means that if we consider the geodesic triangle $\Delta(c(0), c(t_0), c(t_0 + \varepsilon))$, then the side $[c(0), c(t_0)]$ is the geodesic segment corresponding to $c|_{[0, t_0]}$ and the side $[c(t_0), c(t_0 + \varepsilon)]$ is the geodesic segment corresponding to $c|_{[t_0, t_0 + \varepsilon]}$ by the uniqueness of geodesic paths with length less than D_κ .

Let $\overline{\Delta}(c(0), c(t_0), c(t_0 + \varepsilon)) = \overline{\Delta(c(0), c(t_0), c(t_0 + \varepsilon))}$ be a κ -comparison

 Figure 2.3: Comparison Triangle For $c(0), c(t_0), c(t_0 + \epsilon)$

triangle for $\Delta(c(0), c(t_0), c(t_0 + \epsilon))$ as per Figure 2.3 and suppose that it is not degenerate. Then we have the following strict inequality

$$d(\overline{c(t_0 - \epsilon)}, \overline{c(t_0 + \epsilon)}) < d(\overline{c(t_0 - \epsilon)}, \overline{c(t_0)}) + d(\overline{c(t_0)}, \overline{c(t_0 + \epsilon)}) \quad (2.7)$$

Taken with the $\text{CAT}(\kappa)$ inequality, this implies that

$$\begin{aligned} d(c(t_0 - \epsilon), c(t_0 + \epsilon)) &\leq d(\overline{c(t_0 - \epsilon)}, \overline{c(t_0 + \epsilon)}) \\ &< d(\overline{c(t_0 - \epsilon)}, \overline{c(t_0)}) + d(\overline{c(t_0)}, \overline{c(t_0 + \epsilon)}) \\ &= d(c(t_0 - \epsilon), c(t_0)) + d(c(t_0), c(t_0 + \epsilon)) \\ &= d(c(t_0 - \epsilon), c(t_0 + \epsilon)) \end{aligned} \quad (2.8)$$

which is a contradiction. Therefore $\overline{\Delta}$ is degenerate, which by the $\text{CAT}(\kappa)$ inequality, implies that Δ must be degenerate as well. Therefore $c|_{[0, t_0 + \epsilon]}$ is a geodesic path and $(t_0 - \epsilon, t_0 + \epsilon) \subseteq S$ implying that S is open. We have seen that S is non-empty, open, and closed so $S = [0, \lambda]$ and we conclude that c is a geodesic path.

3. Let $B(x_0, r)$ be a ball in X such that $r < \frac{D_\kappa}{2}$. Consider $x, y \in B(x_0, r)$ and the geodesic segment $[x, y] \in X$ joining them. This geodesic segment exists because $d(x, y) \leq d(x, x_0) + d(y, x_0) < D_\kappa$. We want to show that $[x, y] \subseteq B(x_0, r)$ which will require that for any $x' \in [x, y]$ we have $x' \in B(x_0, r)$.

Consider $\Delta(x_0, x, y)$ and let $\bar{\Delta}(x_0, x, y) = \Delta(\bar{x}_0, \bar{x}, \bar{y}) \in M_\kappa^2$ be a κ -comparison triangle for $\Delta(x_0, x, y)$. Since balls of radius $r < D_\kappa/2$ in the model space M_κ^2 are convex, the comparison point $\bar{x}' \in [\bar{x}, \bar{y}]$ is in the ball $B(\bar{x}, r)$. Therefore, by the $\text{CAT}(\kappa)$ inequality, $d(x_0, x') < r$ which implies that $x' \in B(x, r)$ and so the ball is convex as desired.

□

Any metric space X which satisfies the $\text{CAT}(\kappa)$ condition will satisfy the $\text{CAT}(\kappa')$ condition for any $\kappa' \geq \kappa$. In particular, the negatively curved model spaces M_κ^n , where $n < 0$ are $\text{CAT}(0)$ and thus n -dimensional hyperbolic space $\mathbb{H}^n$ is also $\text{CAT}(0)$ [Bridson & Haefliger pg. 165].

The following corollary to Proposition 2.1.3 is presented without proof, for details refer to [Bridson & Haefliger pg. 208].

Corollary 2.1.4. A sufficient condition for a $\text{CAT}(0)$ space X to have the geodesic extension property is that for any local geodesic path $c : [a, b] \rightarrow X$ there exists $\varepsilon > 0$ such that we can find a local geodesic path $\tilde{c} : [a, b + \varepsilon] \rightarrow X$ which extends c .

Remark 2.1.5. Any $\text{CAT}(0)$ space is simply connected. The proof of this has been omitted, for details refer to [Bridson & Haefliger pg. 161]

Remark 2.1.6. As a consequence to Remark 1.2.3 any convex subset of a $\text{CAT}(\kappa)$ space will again be a D_κ -geodesic space. Furthermore, its triangles will satisfy the $\text{CAT}(\kappa)$ inequality, so any convex subset of a $\text{CAT}(\kappa)$ space will again be a $\text{CAT}(\kappa)$ space.

Example 2.1.7. A metric graph is a $\text{CAT}(\kappa)$ space provided it has no essential loops with length shorter than $2D_\kappa$. In particular, if $\kappa \leq 0$ then a metric graph is a $\text{CAT}(\kappa)$ space if and only if it is a tree [Bridson & Haefliger pg. 167].

Example 2.1.8. A uniquely geodesic metric space T is called an $\mathbb{R}$ -tree if given any two geodesic segments $[a, b]$ and $[b, c]$ such that $[a, b] \cap [b, c] = \{b\}$, their concatenation $[a, b] \cup [b, c]$ is again a geodesic segment. For example the plane $\mathbb{R}^2$ endowed with the *Manhattan metric* as described in Example 1.2.5 is an $\mathbb{R}$ -tree.

A metric space T is an $\mathbb{R}$ -tree if and only if it is a $\text{CAT}(\kappa)$ space for every $\kappa \in \mathbb{R}$ [Bridson & Haefliger pg. 167].

2.2 Insights into Convexity

Recall the definition of a convex function.

Definition 2.2.1. If I is any interval in $\mathbb{R}$, then a function $f : I \rightarrow \mathbb{R}$, is said

to be *convex* provided that for all $t, t' \in I$ and $s \in [0, 1]$ we have:

$$f((1-s)t + st') \leq (1-s)f(t) + sf(t') \quad (2.9)$$

For a convex function, given three points $x < y < z \in I$, the image of y in the graph of f lies below the secant line joining x and z . See Figure 2.4.

Figure 2.4: Graph of a Convex Function

We can extend this notion of convexity to apply to functions which have a geodesic metric space X as their domain, instead of an interval in $\mathbb{R}$. We say that a function $f : X \rightarrow \mathbb{R}$ is *convex* if, for any linearly reparameterized geodesic path $c : [0, 1] \rightarrow X$, the function $\phi : [0, 1] \rightarrow \mathbb{R}$ defined by $t \mapsto f(c(t))$ is convex as in the previous definition.

Remark 2.2.2. For a geodesic metric space, it is equivalent to define a function $f : X \rightarrow \mathbb{R}$ to be convex if for every linearly reparameterized geodesic

path $c : [0, 1] \rightarrow X$ and every $t \in [0, 1]$, the following inequality holds

$$f(c(t)) \leq (1 - t)f(c(0)) + tf(c(1)) \quad (2.10)$$

This is because for any $t, t' \in [0, 1]$ we can reparameterize the geodesic path $c|_{[t, t']}$ to be from the unit interval.

Similarly, we say that for a geodesic metric space X , the distance function $d : X \times X \rightarrow \mathbb{R}$ is convex if for any two linearly reparameterized geodesic paths, $c : [0, 1] \rightarrow X$ and $c' : [0, 1] \rightarrow X$ and any $t \in [0, 1]$ the following inequality holds:

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)) \quad (2.11)$$

Proposition 2.2.3. If X is a $CAT(0)$ metric space then the distance function, $d : X \times X \rightarrow \mathbb{R}$ is convex.

Proof. Let c and c' be linearly reparameterized geodesic paths in X with reparameterization factors λ and λ' respectively. We first handle the case where c and c' originate from the same point, so $c(0) = c'(0)$.

Let $\overline{\Delta}(c(0), c(1), c'(1)) = \Delta(\overline{c(0)}, \overline{c(1)}, \overline{c'(1)})$ be a comparison triangle in $\mathbb{E}^2$ for the triangle $\Delta(c(0), c(1), c'(1)) \in X$. For any $t \in [0, 1]$ and comparison points $\overline{c(t)}, \overline{c'(t)} \in \overline{\Delta}$, the triangle $\Delta(\overline{c(0)}, \overline{c(t)}, \overline{c'(t)})$ in $\mathbb{E}^2$ is similar to $\overline{\Delta}$. See Figure 2.5.

Figure 2.5: Comparison Triangle for $(c(0), c(1), c'(1))$

This is because:

$$\frac{d(\overline{c(0)}, \overline{c'(t)})}{d(\overline{c(0)}, \overline{c'(1)})} = \frac{d(c(0), c'(t))}{d(c(0), c'(1))} = \frac{\lambda' t}{\lambda'} = t \quad (2.12)$$

and similarly

$$\frac{d(\overline{c(0)}, \overline{c(t)})}{d(\overline{c(0)}, \overline{c(1)})} = \frac{d(c(0), c(t))}{d(c(0), c(1))} = \frac{\lambda t}{\lambda} = t \quad (2.13)$$

Thus, $d(\overline{c(t)}, \overline{c'(t)}) = td(\overline{c(1)}, \overline{c'(1)})$. By the $\text{CAT}(0)$ inequality, we have $d(c(t), c'(t)) \leq d(\overline{c(t)}, \overline{c'(t)})$ which gives us the desired result

$$d(c(t), c'(t)) \leq td(\overline{c(1)}, \overline{c'(1)}) = td(c(1), c'(1)) \quad (2.14)$$

For the case where $c(0) \neq c'(0)$, we define $c'' : [0, 1] \rightarrow X$ to be the linearly reparameterized geodesic path such that $c''(0) = c(0)$ and $c''(1) = c'(1)$.

Now, by the above method applied to the comparison triangle formed by c

Figure 2.6: Geodesic paths c , c' and c''

and c'' , and the comparison triangle obtained from the reverse geodesic paths of c' and c'' , one obtains the following inequalities:

$$d(c(t), c''(t)) \leq td(c(1), c''(1)) \quad (2.15)$$

$$d(c''(t), c'(t)) \leq (1 - t)d(c''(0), c'(0)) \quad (2.16)$$

Combining these results with the triangle inequality yields our desired result,

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)) \quad (2.17)$$

□

The following proposition is used frequently when dealing with convex functions.

Proposition 2.2.4. If a function $f : X \rightarrow \mathbb{R}$ is convex and bounded on a

geodesic metric space X with the geodesic extension property, then it must be constant.

Proof. Let $f : X \rightarrow \mathbb{R}$ be a convex bounded function on a geodesic metric space X , say $|f(x)| \leq B$ for every $x \in X$.

For any $x_1, x_2 \in X$ such that $x_1 \neq x_2$ let $[x_1, x_2]$ be a geodesic segment joining these two points and $c : \mathbb{R} \rightarrow X$ be a geodesic line which extends this geodesic segment. Consider the function $g = f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ and suppose that it is not constant on $\mathbb{R}$. Then there exist $r_1 < r_2 \in \mathbb{R}$ such that $g(r_1) \neq g(r_2)$.

Case 1: If $g(r_1) < g(r_2)$ then for each integer $n > 0$ we have

$$\begin{aligned} g(r_2) &\leq \frac{n}{n+1}g(r_1) + \frac{1}{n+1}g(r_2 + n(r_2 - r_1)) \\ \Rightarrow g(r_2 + n(r_2 - r_1)) &\geq (n+1)g(r_2) - ng(r_1) \\ \Rightarrow g(r_2 + n(r_2 - r_1)) &\geq (n+1)(g(r_2) - g(r_1)) + g(r_1) \end{aligned} \tag{2.18}$$

which means that g gets arbitrarily large as $n \rightarrow \infty$ and g is unbounded.

Case 2: If $g(r_2) < g(r_1)$ then for each integer $n > 0$ we have

$$\begin{aligned} g(r_1) &\leq \frac{n}{n+1}g(r_2) + \frac{1}{n+1}g(r_1 - n(r_2 - r_1)) \\ \Rightarrow g(r_1 - n(r_2 - r_1)) &\geq (n+1)g(r_1) - ng(r_2) \\ \Rightarrow g(r_1 - n(r_2 - r_1)) &\geq (n+1)(g(r_1) - g(r_2)) + g(r_2) \end{aligned} \tag{2.19}$$

so, just as in Case 1, g gets arbitrarily large as $n \rightarrow \infty$ and g is unbounded.

In both cases, we have a contradiction. Thus, we conclude that g must be constant which implies that $f(x_1) = f(x_2)$. Since x_1 and x_2 were arbitrary, f

is constant. $\square$

2.3 Projection onto a Convex Subspace

An *orthogonal projection* to a convex subset C is a map $\pi : X \rightarrow C$ defined as in the following proposition.

Proposition 2.3.1. Let X be a $CAT(0)$ metric space and $C \subset X$ be a convex subset which is complete in the induced metric. Then:

1. For every $x \in X$ there is a unique point $y \in C$ such that $d(x, y) = d(x, C) = \inf_{v \in C} d(x, v)$. We define $\pi : X \rightarrow C$ by $\pi(x) = y$.
2. Whenever $x' \in [x, \pi(x)]$, then the equality $\pi(x') = \pi(x)$ holds.
3. For points $x, y \in X$ such that $x \notin C$ and $y \in C$, if $y \neq \pi(x)$ then $\angle_{\pi(x)}(x, y) \geq \pi/2$.

Proof. 1. First we show the existence of $y = \pi(x)$. Since we have

$$D = d(x, C) = \inf_{y' \in C} d(x, y') \tag{2.20}$$

we can find a sequence $\{y_n\} \subset C$ such that $d(x, y_n) \rightarrow d(x, C)$ as $n \rightarrow \infty$.

We will show that $\{y_n\}$ is a Cauchy sequence:

Let $\varepsilon > 0$ and fix $N \in \mathbb{N}$ such that for all $n > N$ the inequality $d(x, y_n) < D + \varepsilon$ holds. Consider $n, m > N$ and let $\overline{\Delta}(x, y_n, y_m) = \Delta(\overline{x}, \overline{y_n}, \overline{y_m}) \subset \mathbb{E}^2$ be a comparison triangle for $\Delta(x, y_n, y_m) \subset X$.

Consider the ball $B(\bar{x}, D + \varepsilon)$ and the open ball $B(\bar{x}, D)$ and let $A = B(\bar{x}, D + \varepsilon) - B(\bar{x}, D)$ be the annular region between these balls. See Figure 2.7. Clearly $\bar{y}_n$ and $\bar{y}_m$ are contained within the annular region A and by the convexity of balls in $\mathbb{E}^2$, the geodesic segment $[\bar{y}_n, \bar{y}_m] \subset \mathbb{E}^2$ is contained in the larger ball, $B(\bar{x}, D + \varepsilon)$.

Figure 2.7: Comparison Points Relative to the Annular Region

Furthermore, the geodesic segment $[\bar{y}_n, \bar{y}_m] \subset \mathbb{E}^2$ must be contained within the annular region A . Indeed, if it were not then there would exist $\bar{z} \in [\bar{y}_n, \bar{y}_m]$ such that $d(\bar{x}, \bar{z}) < D$. This would imply that the point $z \in [y_n, y_m]$ for which $\bar{z}$ is a comparison point satisfies

$$d(x, z) \leq d(\bar{x}, \bar{z}) < D \quad (2.21)$$

where the first inequality is a consequence of the $CAT(0)$ condition. Since

$z \in C$, this contradicts our choice of $D = d(x, C)$ and so we conclude that $[\overline{y_n}, \overline{y_m}]$ is completely contained within the annular region A .

A straightforward exercise in Euclidean geometry shows that a geodesic path completely contained in this annular region A can have length at most $2\sqrt{2\varepsilon D + \varepsilon^2}$. Therefore $d(y_n, y_m) = d(\overline{y_n}, \overline{y_m}) \leq 2\sqrt{2\varepsilon D + \varepsilon^2}$, and thus the sequence $\{y_n\}$ is Cauchy.

Since C is complete in the induced metric, there exists $y \in C$ such that $\{y_n\} \rightarrow y$ as $n \rightarrow \infty$, giving $d(x, y) = d(x, C)$. To show the uniqueness of y , consider any other sequence $\{y'_n\}$ with the above properties, and its limit y' . The sequence $\{y''_n\}$ obtained by alternating terms from $\{y_n\}$ and $\{y'_n\}$ would still be Cauchy by the above arguments, so $y = y'$ and thus $y = \pi(x)$ is unique and has the desired properties.

2. To show that $\pi(x) = \pi(x')$ for any $x' \in [\pi(x), x]$, we suppose $\pi(x) \neq \pi(x')$ and derive a contradiction. Since $\pi(x')$ is the unique closest point to x' in C , $d(x', \pi(x')) < d(x', \pi(x))$. This implies however, that we have

$$\begin{aligned} d(x, \pi(x)) &= d(x, x') + d(x', \pi(x)) \\ &> d(x, x') + d(x', \pi(x')) \\ &\geq d(x, \pi(x')) \end{aligned} \tag{2.22}$$

which contradicts our choice of $\pi(x)$. Therefore $\pi(x) = \pi(x')$ and we are done.

3. Suppose to the contrary that $\angle_{\pi(x)}(x, y) < \pi/2$. By the definition of the Alexandrov angle, there exists points $x' \in [\pi(x), x]$ and $y' \in [\pi(x), y]$ such that the comparison angle $\bar{\angle}_{\pi(x)}(x', y')$ is less than $\pi/2$. Consider a comparison triangle $\bar{\Delta}(\pi(x), x', y') = \Delta(\overline{\pi(x)}, \bar{x}', \bar{y}') \subset \mathbb{E}^2$ for the triangle $\Delta(\pi(x), x', y')$. We can find a point $\bar{p} \in [\overline{\pi(x)}, \bar{y}']$ sufficiently close to $\overline{\pi(x)}$ to make the angle $\angle_{\bar{x}'}(\bar{p}, \overline{\pi(x)})$ acute. See Figure 2.8. Since $\angle_{\overline{\pi(x)}}(\bar{x}', \bar{y}')$ is also acute, $\angle_{\bar{p}}(\overline{\pi(x)}, \bar{x}')$ is obtuse.

 Figure 2.8: Location of $\bar{p} \in [\overline{\pi(x)}, \bar{y}']$

Thus we have $d(\bar{p}, \bar{x}') < d(\overline{\pi(x)}, \bar{x}')$. Taken with the $CAT(0)$ inequality the corresponding point $p \in [\pi(x), y']$ for which $\bar{p}$ is a comparison point has

$$\begin{aligned} d(p, x') &\leq d(\bar{p}, \bar{x}') \\ &< d(\overline{\pi(x)}, \bar{x}') \\ &= d(\pi(x), x') \end{aligned} \tag{2.23}$$

Since C is convex we know that $[\pi(x), y] \subset C$ which gives $p \in C$ and thus $\pi(x) \neq \pi(x')$. This contradicts (2) which asserts that $\pi(x') = \pi(x)$.

Therefore we have derived a contradiction and it must be the case that

$$\angle_{\pi(x)}(x, y) \geq \pi/2.$$

□

Corollary 2.3.2. If C is a complete convex subset of a $CAT(0)$ metric space X then $d_C : X \rightarrow \mathbb{R}$ given by $x \mapsto d(x, C)$ is a convex function.

Proof. Let π be the projection of X onto C . Let $c : [0, 1] \rightarrow X$ be a linearly parameterized geodesic segment, and $c' : [0, 1] \rightarrow C$ be the linearly reparameterized geodesic path whose corresponding geodesic segment is $[\pi(c(0)), \pi(c(1))]$. Since C is convex, we have the inclusion $[\pi(c(0)), \pi(c(1))] \subseteq C$.

Now, for any $t \in [0, 1]$ we have $d_C(c(t)) = d(c(t), C) \leq d(c(t), c'(t))$ since $c'(t) \in C$. Furthermore, the convexity of the distance function gives us

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)) \quad (2.24)$$

Since $d(c(0), c'(0)) = d_C(c(0))$ and $d(c(1), c'(1)) = d_C(c(1))$ we may conclude that

$$d_C(c(t)) \leq (1 - t)d_C(c(0)) + td_C(c(1)) \quad (2.25)$$

which implies that d_C is a convex function. □

2.4 The Centre of a Bounded Set

In this section, we introduce the centre and radius of a bounded set. The existence and uniqueness of such centres in complete $CAT(0)$ spaces will be used later to prove the Bruhat-Tits Fixed Point Theorem for isometries of complete $CAT(0)$ metric spaces. This approach via centres is due to Serre. Refer to [Brown], [Serre], [Bruhat & Tits],

Definition 2.4.1. If Y is a bounded subset in a metric space X then the *radius* of Y , denoted r_Y , is given by

$$r_Y = \inf\{r \in \mathbb{R} \mid Y \subseteq \overline{B}(x, r) \text{ some } x \in X\} \quad (2.26)$$

Proposition 2.4.2. Let X be a complete $CAT(\kappa)$ metric space. If $Y \subseteq X$ is a bounded subset of radius $r_Y < D_\kappa/2$ then there exists a unique point $c_Y \in X$ such that $Y \subseteq \overline{B}(c_Y, r_Y)$. The point c_Y is called the *centre* of Y .

Proof. The definition of r_Y implies that there exist sequences of points $\{x_n\} \subset X$ with a corresponding sequence $\{r_n\} \subset \mathbb{R}$ satisfying $r_n \rightarrow r_Y$ and $Y \subseteq \overline{B}(x_n, r_n)$ for each $n \in \mathbb{N}$. Our goal is to show that $\{x_n\}$ is a Cauchy sequence.

Let $\Omega \in M_\kappa^2$ be a fixed basepoint. For $\varepsilon > 0$ we may choose $R \in (r_Y, D_\kappa/2)$ and $R' < r_Y$ such that any geodesic segment contained within the annular region $A = B(\Omega, R) \setminus B(\Omega, R')$ has length less than $\varepsilon/2$ (cf. the proof of Proposition 2.3.1 (1)).

Since $r_n \rightarrow r_Y$ and $r_n \geq r_Y$ for all n , there exists $N > 0$ such that

for all $n > N$ we have $r_n \in [r_Y, R]$. Let $m, n > N$. For each $y \in Y$ the geodesic triangle $\Delta_y = \Delta(y, x_n, x_m)$ is unique because $d(y, x_n) < r_n < D_\kappa/2$ and $d(y, x_m) < r_m < D_\kappa/2$, giving Δ_y a perimeter of less than $2D_\kappa$.

Consider a comparison triangle $\overline{\Delta}_y = \Delta(\Omega, \overline{x_n}, \overline{x_m}) \subset M_\kappa^2$ for the triangle $\Delta(y, x_n, x_m)$. Let m be the midpoint of the geodesic segment $[x_n, x_m]$ and suppose its comparison point $\overline{m} \in [\overline{x_n}, \overline{x_m}]$ lies within the ball $B(\Omega, R') \subset M_\kappa^2$. By the CAT(κ) inequality this gives $d(m, y) \leq d(\Omega, \overline{m}) < R' < r_Y$.

Thus, if it were the case that for every $y \in Y$, the comparison triangle $\overline{\Delta}_y$ has the midpoint $\overline{m}$ of $[\overline{x_n}, \overline{x_m}]$ contained within $B(\Omega, R')$ then this would imply that $d(m, y) < R'$ for every $y \in Y$ so $Y \subseteq B(m, R') \subset X$ which contradicts our choice of $R' < r_Y$.

This implies that for some $y \in Y$, the midpoint $\overline{m} \in [\overline{x_n}, \overline{x_m}] \in \overline{\Delta}_y$ is contained within the annular region A . We now show that one of $[\overline{x_n}, \overline{m}]$ or $[\overline{m}, \overline{x_m}]$ lies within the annular region A . Suppose to the contrary that there exists $x' \in [\overline{x_n}, \overline{m}]$ and $x'' \in [\overline{m}, \overline{x_m}]$ such that $x', x'' \in B(\Omega, R')$. Then by the convexity of balls in M_κ^2 this implies that the geodesic segment $[x', x'']$ and consequently $\overline{m}$ are contained within $B(\Omega, R')$. This contradicts our choice of having $\overline{m} \in A$. Therefore, we conclude that either $d(x_n, m) = d(\overline{x_n}, \overline{m}) < \varepsilon/2$ or $d(m, x_m) = d(\overline{m}, \overline{x_m}) < \varepsilon/2$. Since m is the mid-point of $[x_n, x_m]$, we have $d(x_n, x_m) = 2d(x_n, m) = 2d(m, x_m) < \varepsilon$ and the sequence is Cauchy.

Since X is complete, $\lim\{x_n\} = c_Y$ must exist and its uniqueness is proved in the same manner as the uniqueness of the projection in the proof of Proposition 2.3.1. $\square$

Example 2.4.3. The centre of a bounded subset of a $CAT(0)$ space need not be contained in that set. For example in $\mathbb{E}^2$ the set $\{(0, 0), (0, 1)\}$ is bounded and has centre $(0, \frac{1}{2})$ and the set $\{x \in \mathbb{E}^2 \mid d(x, (0, 0)) = 1\}$ has centre $(0, 0)$.

The following lemma will be used to prove the Bruhat-Tits Fixed Point Theorem [Bruhat & Tits].

Lemma 2.4.4. Let X be a complete $CAT(0)$ space and suppose that Y is a bounded subset of X with centre c_Y . If $f : X \rightarrow X$ is an isometry such that $f(Y) = Y$ then $f(c_Y) = c_Y$.

Proof. Since c_Y is the centre of Y we have $Y \subseteq \overline{B}(c_Y, r_Y)$ where r_Y is the radius of Y . Since f acts by isometries, we have $f(Y) \subseteq \overline{B}(f(c_Y), r_Y)$, but $Y = f(Y)$ so $Y \subseteq \overline{B}(f(c_Y), r_Y)$. By the uniqueness of centres, this implies that $f(c_Y) = c_Y$ as desired. $\square$

2.5 Flat Subspaces and the Product Decomposition Theorem

The goal of this section is to prove a theorem that will allow us to split the subspace consisting of all the geodesic lines parallel to a given geodesic line c in a $CAT(0)$ space X into the product of a convex subspace of X and $\mathbb{R}$. This will require us to examine under which conditions subspaces of a $CAT(0)$ are isometric to flat Euclidean space.

Definition 2.5.1. Let X be a geodesic metric space and let A be a subset of X . The *convex hull* of A , denoted $C(A)$, is the intersection of all convex subspaces of X which contain A . This will be the smallest convex subset in X containing A .

The following proposition shows how one can end up with *flat triangles* in a non-positively curved metric space. We will later use this to describe *flat quadrilaterals* in a CAT(0) space and eventually *flat strips*.

Proposition 2.5.2 (Flat Triangles). Let X be a CAT(0) metric space and consider a geodesic triangle, $\Delta = \Delta(p, q, q') \subset X$. If the (Alexandrov) angle $\angle_p(q, q')$ is equal to its comparison angle $\bar{\angle}_p(q, q')$ then the convex hull $C(\Delta)$ of $\Delta \subset X$ is isometric to the convex hull $C(\bar{\Delta})$ where $\bar{\Delta}$ is any comparison triangle in $\mathbb{E}^2$ for $\Delta(p, q, q')$.

Proof. Let $\Delta = \Delta(p, q, q') \subset X$ be any geodesic triangle, such that $\angle_p(q, q') = \bar{\angle}_p(q, q')$. Let $\bar{\Delta}(p, q, q') = \Delta(\bar{p}, \bar{q}, \bar{q}')$ be a comparison triangle for $\Delta(p, q, q')$.

The first step of the proof is to show that the equality $d(p, r) = d(\bar{p}, \bar{r})$ holds for any point $r \in [q, q']$.

Fix $r \in [q, q']$ such that $r \neq q$ and $r \neq q'$. Considering the geodesic triangles $\Delta' = \Delta(p, q, r)$ and $\Delta'' = \Delta(p, r, q')$, we can find points $\tilde{p}, \tilde{q}, \tilde{q}', \tilde{r} \in \mathbb{E}^2$ such that $\tilde{\Delta}' = \Delta(\tilde{p}, \tilde{q}, \tilde{r})$ and $\tilde{\Delta}'' = \Delta(\tilde{p}, \tilde{r}, \tilde{q}')$ are comparison triangles for Δ' and Δ'' respectively. Furthermore, these points may be selected such that $\tilde{q}$ and $\tilde{q}'$ lie on opposite sides of the line which runs through $\tilde{p}$ and $\tilde{r}$. See Figure 2.9.

Let $\tilde{\gamma} = \angle_{\tilde{r}}(\tilde{p}, \tilde{q})$ and $\tilde{\gamma}' = \angle_{\tilde{r}}(\tilde{p}, \tilde{q}')$ as in Figure 2.9. Since $\gamma = \angle_r(p, q)$

 Figure 2.9: Arrangement of $\tilde{\Delta}'$ and $\tilde{\Delta}''$ in $\mathbb{E}^2$

and $\gamma' = \angle_r(p, q')$ split a geodesic segment, we have $\gamma + \gamma' \geq \pi$ by Proposition 1.3.6. By the $\text{CAT}(0)$ inequality we get $\tilde{\gamma} \geq \gamma$ and $\tilde{\gamma}' \geq \gamma'$ which implies $\tilde{\gamma} + \tilde{\gamma}' \geq \pi$.

We can now apply the second part of Alexandrov's Lemma 1.5.7 to $\tilde{p}$, $\tilde{q}$, $\tilde{q}'$, and $\tilde{r}$ to get: $\angle_{\tilde{p}}(\tilde{q}, \tilde{r}) + \angle_{\tilde{p}}(\tilde{q}', \tilde{r}) \leq \angle_{\tilde{p}}(\tilde{q}, \tilde{q}')$. Furthermore, we observe

$$\begin{aligned} \angle_p(q, q') &\leq \angle_p(q, r) + \angle_p(r, q') \\ &\leq \angle_{\tilde{p}}(\tilde{q}, \tilde{r}) + \angle_{\tilde{p}}(\tilde{q}', \tilde{r}) \\ &\leq \angle_{\tilde{p}}(\tilde{q}, \tilde{q}') \end{aligned} \tag{2.27}$$

where the first inequality is a consequence of Proposition 1.3.6, the second inequality is a consequence of the $\text{CAT}(0)$ condition, and the third inequality we just proved.

Our initial assumption that $\angle_p(q, q') = \angle_{\tilde{p}}(\tilde{q}, \tilde{q}')$ implies that equality holds

everywhere in (2.27); in particular $\angle_{\bar{p}}(\tilde{q}, \tilde{r}) + \angle_{\bar{p}}(\tilde{q}', \tilde{r}) = \angle_{\bar{p}}(\bar{q}, \bar{q}')$. At this point, the equality condition in the second part of Alexandrov's Lemma 1.5.7 gives $d(\bar{p}, \bar{r}) = d(\tilde{p}, \tilde{r})$, but $d(\tilde{p}, \tilde{r}) = d(p, r)$ so $d(p, r) = d(\bar{p}, \bar{r})$ as desired.

To complete the proof, we define the map, $j : C(\bar{\Delta}) \longrightarrow X$ as follows:

For every $\bar{r} \in [\bar{q}, \bar{q}']$ we isometrically send the geodesic segment $[\bar{p}, \bar{r}]$ to $[p, r]$ fixing $j(\bar{p}) = p$ and $j(\bar{r}) = r$. This is only possible because $d(p, r) = d(\bar{p}, \bar{r})$. Note that for any $\bar{x} \in C(\bar{\Delta}) - \{\bar{p}\}$ there is a unique point, $\bar{r} \in [\bar{q}, \bar{q}']$ such that $\bar{x} \in [\bar{p}, \bar{r}]$, so j is well-defined.

To show that j is an isometry, let $\bar{x}, \bar{x}' \in C(\bar{\Delta})$ be any two points such that $\bar{x}, \bar{x}' \neq \bar{p}$. Let $\bar{r}, \bar{r}' \in [\bar{q}, \bar{q}']$ be such that $\bar{x} \in [\bar{p}, \bar{r}]$ and $\bar{x}' \in [\bar{p}, \bar{r}']$. Without loss of generality, we may assume $\bar{r} \in [\bar{q}, \bar{r}']$.

Figure 2.10: Location of $\bar{x}$ and $\bar{x}'$ in $\Delta(\bar{p}, \bar{q}, \bar{q}')$

Let $x = j(\bar{x})$, $x' = j(\bar{x}')$, $r = j(\bar{r})$, and $r' = j(\bar{r}')$. Since $d(p, r) = d(\bar{p}, \bar{r})$, the triangle $\Delta(\bar{p}, \bar{q}, \bar{r})$ is a comparison triangle in $\mathbb{E}^2$ for $\Delta(p, q, r)$.

Similarly, $\Delta(\bar{p}, \bar{r}, \bar{r}')$ is a comparison triangle for $\Delta(p, r, r')$ and $\Delta(\bar{p}, \bar{r}', \bar{q}')$ is a comparison triangle for $\Delta(p, r', q')$. Thus, by the $CAT(0)$ inequality, we obtain $\delta_1 = \angle_p(q, r) \leq \angle_{\bar{p}}(\bar{q}, \bar{r}) = \bar{\delta}_1$. Similarly, $\delta_2 = \angle_p(r, r') \leq \angle_{\bar{p}}(\bar{r}, \bar{r}') = \bar{\delta}_2$ and $\delta_3 = \angle_p(r', q') \leq \angle_{\bar{p}}(\bar{r}', \bar{q}') = \bar{\delta}_3$. Combining these results gives:

$$\begin{aligned}
 \angle_{\bar{p}}(\bar{q}, \bar{q}') &= \bar{\delta}_1 + \bar{\delta}_2 + \bar{\delta}_3 \\
 &\geq \delta_1 + \delta_2 + \delta_3 \\
 &= \angle_p(q, r) + \angle_p(r, r') + \angle_p(r', q') \\
 &\geq \angle_p(q, q') \quad (\text{Proposition 1.3.6})
 \end{aligned} \tag{2.28}$$

By hypothesis, we have $\angle_p(q, q') = \bar{\angle}_p(q, q') = \angle_{\bar{p}}(\bar{q}, \bar{q}')$. Thus equality holds everywhere in the above expression, which implies that $\delta_2 = \bar{\delta}_2$. Let us fix a comparison triangle $\bar{\Delta}(p, x, x')$ for $\Delta(p, x, x')$. We wish to show that it is congruent to $\Delta(\bar{p}, \bar{x}, \bar{x}')$. Since $d(p, x) = d(\bar{p}, \bar{x})$ and $d(p, x') = d(\bar{p}, \bar{x}')$ it suffices to show that $\bar{\angle}_p(x, x') = \angle_{\bar{p}}(\bar{x}, \bar{x}')$. We have the following inequalities:

$$\begin{aligned}
 \delta_2 &= \angle_p(r, r') \\
 &= \angle_p(x, x') \quad (\text{since } x \in [p, r] \text{ and } x' \in [p, r']) \\
 &\leq \bar{\angle}_p(x, x') \quad (\text{by the } CAT(0) \text{ condition}) \\
 &\leq \bar{\angle}_p(r, r') \quad (\text{by the } CAT(0) \text{ condition}) \\
 &= \bar{\delta}_2
 \end{aligned} \tag{2.29}$$

We know that $\bar{\delta}_2 = \delta_2$ so in (2.29) equality holds everywhere; in particular $\bar{\angle}_p(x, x') = \bar{\angle}_p(r, r')$ which implies that the comparison triangle $\bar{\Delta}(p, x, x')$ is

congruent to $\Delta(\bar{p}, \bar{x}, \bar{x}')$. So we deduce that $d(\bar{x}, \bar{x}') = d(x, x') = d(j(\bar{x}), j(\bar{x}'))$ and we have an isometry as desired.

All that remains to be shown is that $C(\Delta) = j(C(\bar{\Delta}))$. Since we know that $\Delta \subseteq j(C(\bar{\Delta}))$ and $j(C(\bar{\Delta}))$ is convex (as the isometric image of a convex set), we have $C(\Delta) \subseteq j(C(\bar{\Delta}))$. Since j is injective, we define a left inverse $j^{-1} : j(C(\bar{\Delta})) \rightarrow C(\bar{\Delta})$. By using the same argument as above, we see that $j^{-1}(C(\Delta)) \supseteq C(\bar{\Delta})$ so $C(\Delta) \supseteq j(C(\bar{\Delta}))$. Thus, we may conclude that $C(\Delta)$ is isometric to $C(\bar{\Delta})$ which completes the proof. $\square$

We have seen that one may find *flat triangles* in a $\text{CAT}(0)$ space. The following proposition is similar, but addresses *flat quadrilaterals*.

Proposition 2.5.3 (Flat Quadrilaterals). Let p, q, r, s be four points in a $\text{CAT}(0)$ space X such that $\angle_p(q, s) + \angle_s(p, r) + \angle_r(s, q) + \angle_q(r, p) \geq 2\pi$. Then equality holds in the previous inequality and the convex hull of the four points is isometric to the convex hull of a convex quadrilateral in $\mathbb{E}^2$.

Proof. Let $\Delta_1 = \Delta(p, q, s)$ and $\Delta_2 = \Delta(r, q, s)$ be geodesic triangles. We can pick comparison triangles $\bar{\Delta}_1$ and $\bar{\Delta}_2$ for Δ_1 and Δ_2 respectively, such that the comparison points for q and s in each triangle coincide and the comparison points for p and r lie on opposite sides of the segment $[\bar{q}, \bar{s}]$. See Figure 2.11.

Let $\bar{\alpha}, \bar{\gamma}, \bar{\beta}_1, \bar{\beta}_2, \bar{\delta}_1$, and $\bar{\delta}_2$ be defined as in Figure 2.11. If we let $\alpha = \angle_p(q, s)$, $\beta = \angle_q(p, r)$, $\gamma = \angle_r(q, s)$, and $\delta = \angle_s(r, p)$ then the $\text{CAT}(0)$ inequality gives us $\alpha \leq \bar{\alpha}$ and $\gamma \leq \bar{\gamma}$. Furthermore, by Proposition 1.3.6

Figure 2.11: Placement of $\bar{\Delta}_1$ and $\bar{\Delta}_2$

we have $\delta = \angle_s(r, p) \leq \angle_s(r, q) + \angle_s(q, p)$. By the $CAT(0)$ inequality again, $\angle_s(r, q) \leq \bar{\delta}_2$ and $\angle_s(q, p) \leq \bar{\delta}_1$ so $\delta \leq \bar{\delta}_1 + \bar{\delta}_2$. Similarly, we have $\beta \leq \bar{\beta}_1 + \bar{\beta}_2$.

This implies that

$$\bar{\alpha} + \bar{\beta}_1 + \bar{\beta}_2 + \bar{\gamma} + \bar{\delta}_1 + \bar{\delta}_2 \geq \alpha + \beta + \delta + \gamma \geq 2\pi \quad (2.30)$$

The sum of the angles of a Euclidean quadrilateral is equal to 2π however, so equality holds in the above expression. In particular, $\bar{\alpha} = \alpha$, $\bar{\gamma} = \gamma$, $\bar{\delta}_1 + \bar{\delta}_2 = \delta$, and $\bar{\beta}_1 + \bar{\beta}_2 = \beta$.

By the previous proposition, this implies that $C(\Delta_1)$ and $C(\Delta_2)$ are isometric to $C(\bar{\Delta}_1)$ and $C(\bar{\Delta}_2)$ respectively. Thus, we know that there exist isometries, $j_1 : C(\bar{\Delta}_1) \rightarrow C(\Delta_1)$ and $j_2 : C(\bar{\Delta}_2) \rightarrow C(\Delta_2)$. Additionally $\bar{\beta}_1 + \bar{\beta}_2 = \beta = \angle_q(p, r) \leq \pi$ and similarly $\bar{\delta}_1 + \bar{\delta}_2 \leq \pi$ which implies that the convex hull, $C(\{\bar{p}, \bar{q}, \bar{r}, \bar{s}\})$ is equal to the union of $C(\bar{\Delta}_1)$ and $C(\bar{\Delta}_2)$.

We must show that there is an isometry mapping $C(\{\bar{p}, \bar{q}, \bar{r}, \bar{s}\})$ to $C(\{p, q, r, s\})$. Define $j : C(\{\bar{p}, \bar{q}, \bar{r}, \bar{s}\}) \rightarrow X$ by $j|_{C(\bar{\Delta}_1)} = j_1$ and $j|_{C(\bar{\Delta}_2)} = j_2$. Since $C(\bar{\Delta}_1) \cap C(\bar{\Delta}_2) = [\bar{q}, \bar{s}]$ and the maps j_1 and j_2 agree on the set $[\bar{q}, \bar{s}]$, j is well-defined. All we must now show is that j is an isometry.

For all $\bar{x}_1, \bar{x}_2 \in C(\bar{\Delta}_1)$ we have

$$d(j(\bar{x}_1), j(\bar{x}_2)) = d(j_1(\bar{x}_1), j_1(\bar{x}_2)) = d(\bar{x}_1, \bar{x}_2) \quad (2.31)$$

The same holds for points in $C(\bar{\Delta}_2)$. Thus, in order to show that j is an

isometry, we must verify that j preserves the distance between points $\bar{x}_1 \in C(\bar{\Delta}_1)$ and $\bar{x}_2 \in C(\bar{\Delta}_2)$.

Let $\bar{x}_1 \in C(\bar{\Delta}_1)$ and $\bar{x}_2 \in C(\bar{\Delta}_2)$ with images $j(\bar{x}_1) = x_1$ and $j(\bar{x}_2) = x_2$ respectively, such that $\bar{x}_1, \bar{x}_2 \neq \bar{q}$. See Figure 2.11. In order to show that $d(x_1, x_2) = d(\bar{x}_1, \bar{x}_2)$ it is sufficient to show $\angle_q(x_1, x_2) = \angle_{\bar{q}}(\bar{x}_1, \bar{x}_2)$. Indeed, suppose it is true. Then by the $CAT(0)$ inequality $\bar{Z}_q(x_1, x_2) \geq \bar{Z}_q(r_1, r_2) = \angle_{\bar{q}}(\bar{x}_1, \bar{x}_2)$. So if $\Delta(\tilde{q}, \tilde{x}_1, \tilde{x}_2)$ is a comparison triangle for $\Delta(q, x_1, x_2)$ then $d(x_1, x_2) = d(\tilde{x}_1, \tilde{x}_2) \geq d(\bar{x}_1, \bar{x}_2)$ by the law of cosines. To see the opposite inequality, if we let $\bar{y} \in [\bar{x}_1, \bar{x}_2]$ be the unique point which intersects the geodesic segment $[\bar{q}, \bar{s}]$ and $y = j(\bar{y})$, then

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, y) + d(y, x_2) \\ &= d(\bar{x}_1, \bar{y}) + d(\bar{y}, \bar{x}_2) \\ &= d(\bar{x}_1, \bar{x}_2) \end{aligned} \tag{2.32}$$

which gives $d(x_1, x_2) = d(\bar{x}_1, \bar{x}_2)$ as desired.

So let us prove that $\angle_q(x_1, x_2) = \angle_{\bar{q}}(\bar{x}_1, \bar{x}_2)$. Let $\mu = \angle_q(x_1, x_2)$ and $\bar{\mu} = \angle_{\bar{q}}(\bar{x}_1, \bar{x}_2)$. Note that since j_1 and j_2 are isometries, we have

$$\begin{aligned} \beta &\leq \angle_q(p, x_1) + \mu + \angle_q(x_2, r) \\ &\leq \angle_q(p, x_1) + (\angle_q(x_1, s) + \angle_q(s, x_2)) + \angle_q(x_2, r) \\ &= \angle_{\bar{q}}(\bar{p}, \bar{x}_1) + (\angle_{\bar{q}}(\bar{x}_1, \bar{s}) + \angle_{\bar{q}}(\bar{s}, \bar{x}_2)) + \angle_{\bar{q}}(\bar{x}_2, \bar{r}) \\ &= \bar{\beta}_1 + \bar{\beta}_2 \end{aligned} \tag{2.33}$$

But $\overline{\beta_1} + \overline{\beta_2} = \beta$, so equality holds everywhere in the above expression. Therefore,

$$\begin{aligned}
 \angle_q(p, x_1) + \mu + \angle_q(x_2, r) &= \angle_{\overline{q}}(\overline{p}, \overline{x_1}) + (\angle_{\overline{q}}(\overline{x_1}, \overline{s}) + \angle_{\overline{q}}(\overline{s}, \overline{x_2})) + \angle_{\overline{q}}(\overline{x_2}, \overline{r}) \\
 &= \angle_{\overline{q}}(\overline{p}, \overline{x_1}) + \overline{\mu} + \angle_{\overline{q}}(\overline{x_2}, \overline{r}) \\
 &= \angle_q(p, x_1) + \overline{\mu} + \angle_q(x_2, r)
 \end{aligned} \tag{2.34}$$

so we deduce that $\mu = \overline{\mu}$ as desired. By the same method as in the proof of Proposition 2.5.2 we can show that the image of j is the convex hull of the quadrilateral and formed by p, q, r, s and thus the proof is complete. $\square$

Definition 2.5.4. Let X be a geodesic metric space, and let $c : \mathbb{R} \rightarrow X$, $c' : \mathbb{R} \rightarrow X$ be two geodesic lines in X . We say that the lines c and c' are *asymptotic* geodesic lines if there exists a non-negative constant K such that the inequality $d(c(t), c'(t)) \leq K$ holds for all $t \in \mathbb{R}$. We say that c and c' are *parallel* if $d(c(t), c'(t))$ is constant.

Just as in Euclidean space $\mathbb{E}^n$, in a CAT(0) space, any two *asymptotic* geodesic lines are in fact *parallel*. Furthermore, their convex hull is isometric to a strip in $\mathbb{E}^2$ as outlined in the following proposition.

Proposition 2.5.5 (The Flat Strip Theorem). Let $c : \mathbb{R} \rightarrow X$, $c' : \mathbb{R} \rightarrow X$ be two geodesic lines in a CAT(0) space X . If c and c' are asymptotic, then $C(c(\mathbb{R}) \cup c'(\mathbb{R}))$ is isometric to a flat strip $\mathbb{R} \times [0, D] \subset \mathbb{E}^2$ for some $D > 0$.

Proof. If we consider the subspaces $c(\mathbb{R}) \subset X$ and $c'(\mathbb{R}) \subset X$ then they are

clearly complete and convex, as the isometric images of the complete and convex metric space $\mathbb{R}$.

Let $\pi : X \rightarrow c(\mathbb{R})$ and $\pi' : X \rightarrow c'(\mathbb{R})$ be the projection maps of X onto $c(\mathbb{R})$ and $c'(\mathbb{R})$ respectively, as defined in the previous section.

First observe that if $a, b \in \mathbb{R}$ are fixed then

$$\begin{aligned} d(c(t+a), c'(t+b)) &\leq d(c(t+a), c(t)) + d(c(t), c'(t)) + d(c'(t), c'(t+b)) \\ &\leq a + K + b \end{aligned} \tag{2.35}$$

since $d(c(t), c'(t)) \leq K$ for every $t \in \mathbb{R}$. Hence, $d(c(t+a), c'(t+b))$ is bounded for all $t \in \mathbb{R}$ so without loss of generality we may reparameterize c' so that $\pi(c'(0)) = c(0)$.

Now, by Proposition 2.2.3 the function $t \mapsto d(c(t), c'(t))$ is convex and non-negative. Furthermore, it is bounded by assumption. Therefore by Proposition 2.2.4, it is constant, that is for some $D \in \mathbb{R}$, we have $d(c(t), c'(t)) = D$ for all $t \in \mathbb{R}$. Similarly, by (2.35), we see that the function $t \mapsto d(c(t+a), c'(t))$ is constant for all $a \in \mathbb{R}$. Because $\pi(c'(0)) = c(0)$, we have

$$\begin{aligned} d(c'(t), c(t+a)) &= d(c'(0), c(a)) \\ &\geq d(c'(0), c(0)) \\ &= d(c'(t), c(t)) \end{aligned} \tag{2.36}$$

so $\pi(c'(t)) = c(t)$ for all $t \in \mathbb{R}$.

Conversely, for any $t \in \mathbb{R}$, we have

$$\begin{aligned} d(c(0), c'(t)) &= d(c(-t), c'(0)) \\ &\geq d(c(0), c'(0)) \end{aligned} \tag{2.37}$$

so $\pi'(c(0)) = c'(0)$ where π' is the projection of X onto $c'(\mathbb{R})$. By the same method as in (2.36), we can show that $\pi'(c(t)) = c'(t)$ for all $t \in \mathbb{R}$.

Let us define a map $j : \mathbb{R} \times [0, D] \longrightarrow X$ such that j sends (t, s) to the point on $[c(t), c'(t)]$ a distance s from $c(t)$. We will show that j is in fact an isometry. Let $t < t' \in \mathbb{R}$ and consider the quadrilateral in X with vertices, $c(t), c(t'), c'(t')$, and $c'(t)$. By Proposition 2.3.1 (3) we have the following

$$\begin{aligned} \angle_{c(t)}(c(t'), c'(t)) &\geq \pi/2 \\ \angle_{c(t')}(c(t), c'(t')) &\geq \pi/2 \\ \angle_{c'(t')}(c(t'), c'(t)) &\geq \pi/2 \\ \angle_{c'(t)}(c'(t'), c(t)) &\geq \pi/2 \end{aligned} \tag{2.38}$$

By the Flat Quadrilaterals Theorem 2.5.3, this implies that the convex hull of $\{c(t), c(t'), c'(t'), c'(t)\}$ is isometric to the Euclidean quadrilateral, $[0, D] \times [t, t']$. In particular, we have $d((t, s), (t', s')) = d(j(t, s), j(t', s'))$ as desired. So j is an isometry and by the same argument as in the previous two theorems, its image is the convex hull $C(c(\mathbb{R}) \cup c'(\mathbb{R}))$ as desired. $\square$

This property shows that asymptotic lines are parallel.

The big theorem of the section now requires only the following lemma as

preparation:

Lemma 2.5.6. Let $c_1, c_2, c_3 : \mathbb{R} \rightarrow X$ be three geodesis lines in a metric space X . Suppose that the union of the images of each pair of these lines is isometric to the union of two parallel lines in $\mathbb{E}^2$. Let $p_{i,j}$ be the map which assigns to each point in $c_j(\mathbb{R})$ the unique closest point in $c_i(\mathbb{R})$. Then the equality $p_{1,3} \circ p_{3,2} \circ p_{2,1} = p_{1,1}$ holds, where $p_{1,1}$ is the identity map on $c_1(\mathbb{R})$.

Proof. Since each pair c_i, c_j of lines have images which are isometric to a pair of parallel lines in $\mathbb{E}^2$, the projection $p_{i,j}$ is an isometry. Thus the composition, $p_{1,3} \circ p_{3,2} \circ p_{2,1}$ is an isometry. The only isometries of $\mathbb{R}$ are reflections and translations; we claim that $p_{1,3} \circ p_{3,2} \circ p_{2,1}$ must be a translation.

To derive a contradiction, suppose $p_{1,3} \circ p_{3,2} \circ p_{2,1}$ is a reflection about a point $c_1(a) \in \mathbb{R}$, so we have $p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(a+x)) = c_1(a-x)$. We may reparameterize c_1 so that $c_1(a)$ becomes $c_1(0)$ and then reparameterize c_2 so that $p_{2,1}(c_1(0)) = c_2(0)$. Finally, we reparameterize c_3 so that $p_{3,2}(c_2(0)) = c_3(0)$. Thus, $p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(x)) = c_1(-x)$ for all $x \in \mathbb{R}$.

Set

$$\begin{aligned} d_1 &= d(c_1(0), c_2(0)) \\ d_2 &= d(c_2(0), c_3(0)) \\ d_3 &= d(c_3(0), c_1(0)) \end{aligned} \tag{2.39}$$

If we consider a real number $x \in \mathbb{R}$ such that $x > 1/2(d_1 + d_2 + d_3)$, then

$$\begin{aligned}
& d(c_1(x), p_{2,1}(c_1(x))) + d(p_{2,1}(c_1(x)), p_{3,2} \circ p_{2,1}(c_1(x))) \\
& + d(p_{3,2} \circ p_{2,1}(c_1(x)), p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(x))) \\
& \geq d(c_1(x), p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(x))) \\
& = 2x \\
& > d_1 + d_2 + d_3
\end{aligned} \tag{2.40}$$

This implies that one of the following will hold:

$$\begin{aligned}
d_1 & < d(c(x), p_{2,1}(c(x))) \\
d_2 & < d(p_{2,1}(c(x)), p_{3,2} \circ p_{2,1}(c(x))) \\
d_3 & < d(p_{3,2} \circ p_{2,1}(c(x)), p_{1,3} \circ p_{3,2} \circ p_{2,1}(c(x)))
\end{aligned} \tag{2.41}$$

which contradicts the assumption that each pair of geodesic lines has images which are isometric to a pair of parallel lines in $\mathbb{E}^2$. Thus, we conclude that $p_{1,3} \circ p_{3,2} \circ p_{2,1}$ must be a translation by a constant $b \in \mathbb{R}$. This gives $p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(t)) = c_1(t + b)$ for all $t \in \mathbb{R}$.

By reparameterizing c_2 such that $p_{2,1}(c_1(0)) = c_2(0)$ and then reparameterizing c_3 such that $p_{3,2}(c_2(0)) = c_3(0)$ we get that $p_{1,3} \circ p_{3,2} \circ p_{2,1}(c_1(0)) = c_1(b)$.

Now, let $a_1 = d(c_1(\mathbb{R}), c_2(\mathbb{R}))$, $a_2 = d(c_2(\mathbb{R}), c_3(\mathbb{R}))$, $a_3 = d(c_3(\mathbb{R}), c_1(\mathbb{R}))$. Since we assumed that the union of any two of the given lines is isometric to two parallel lines in $\mathbb{E}^2$, we have $d(c_1(t), c_2(t+s)) = \sqrt{a_1^2 + s^2}$ for any $t \in \mathbb{R}$. Similarly, $d(c_2(t), c_3(t+s)) = \sqrt{a_2^2 + s^2}$ and $d(c_3(t), c_1(s+t)) = \sqrt{a_3^2 + (s-b)^2}$.

holds for any $t \in \mathbb{R}$.

Thus, for all $s \in \mathbb{R}$ we have

$$\begin{aligned}
 & d(c_1(0), c_1((a_1 + a_2 + a_3)s + b)) \\
 & \leq d(c_1(0), c_2(a_1s)) + d(c_2(a_1s), c_3((a_1 + a_2)s)) \\
 & \quad + d(c_3((a_1 + a_2)s), c_1((a_1 + a_2 + a_3)s + b)) \\
 & = \sqrt{a_1^2 + a_1^2 s^2} + \sqrt{a_2^2 + a_2^2 s^2} + \sqrt{a_3^2 + (a_3s + b - b)^2} \\
 & = (a_1 + a_2 + a_3)\sqrt{1 + s^2}
 \end{aligned} \tag{2.42}$$

But c_1 is a geodesic path, so $d(c_1(0), c_1((a_1 + a_2 + a_3)s + b)) = |(a_1 + a_2 + a_3)s + b|$ which tells us that $|(a_1 + a_2 + a_3)s + b| \leq (a_1 + a_2 + a_3)\sqrt{1 + s^2}$ for all $s \in \mathbb{R}$.

If we consider the case where $b \geq 0$ then for every $s > 0$ we have

$$(\sqrt{s^2 + 1} - s)(a_1 + a_2 + a_3) \geq b \geq 0 \tag{2.43}$$

and as $s \rightarrow \infty$ we have $\sqrt{s^2 + 1} - s \rightarrow 0$ and thus $b = 0$. If, on the other hand, $b \leq 0$ then for every $s < 0$ we have

$$0 \geq b \geq -(a_1 + a_2 + a_3)(\sqrt{1 + s^2} + s) \tag{2.44}$$

and as $s \rightarrow -\infty$ we have $s + \sqrt{s^2 + 1} \rightarrow 0$ so $b = 0$ and we have the equality $p_{1,3} \circ p_{3,2} \circ p_{2,1} = p_{1,1}$ as desired. $\square$

We now proceed with the main theorem of this section. It will be used later to find axes of hyperbolic isometries.

Theorem 2.5.7 (Product Decomposition Theorem). Let X be a $CAT(0)$ space and $c : \mathbb{R} \rightarrow X$ be a geodesic line. Then:

1. The union of the collection of all images of geodesic lines $c' : \mathbb{R} \rightarrow X$ which are asymptotic to c forms a convex subspace X_c of X and hence is $CAT(0)$.
2. Let $\pi : X \rightarrow c(\mathbb{R})$ be the orthogonal projection of X onto $c(\mathbb{R})$ and let $p : X_c \rightarrow c(\mathbb{R})$ be its restriction to X_c . Then $X_c^0 = p^{-1}(c(0))$ is a convex subset of X and X_c is canonically isometric to the product $X_c^0 \times \mathbb{R}$.

Proof. 1. We wish to show that for any two points x_1, x_2 in X_c , the geodesic segment $[x_1, x_2]$ is contained in X_c . First observe that the triangle inequality implies that being asymptotic is an equivalence relation on the set of geodesic lines.

Since X_c is the collection of geodesic lines asymptotic to c , there exist geodesic lines c_1 and c_2 such that $x_1 \in c_1(\mathbb{R})$ and $x_2 \in c_2(\mathbb{R})$. Since c_1 and c_2 are asymptotic, the Flat Strip Theorem 2.5.5 tells us that the convex hull, $C(c_1(\mathbb{R}) \cup c_2(\mathbb{R}))$ is isometric to a flat strip, $\mathbb{R} \times [0, D]$. For any point $(a, b) \in \mathbb{R} \times [0, D]$, the geodesic line $\{(p, b) : p \in \mathbb{R}\}$ is asymptotic to $\{(p, 0) : p \in \mathbb{R}\}$. Thus, any point in $C(c_1(\mathbb{R}) \cup c_2(\mathbb{R}))$ is on a geodesic line which is asymptotic to c_1 . So we have $C(c_1(\mathbb{R}) \cup c_2(\mathbb{R})) \subset X_c$ which implies that $[x_1, x_2] \subset C(c_1(\mathbb{R}) \cup c_2(\mathbb{R})) \subset X_c$ as desired. Furthermore, a convex subset of a $CAT(0)$ space is $CAT(0)$ so X_c is a $CAT(0)$ metric space by Remark 2.1.6.

2. For every $x \in X_c^0$ there exists a unique geodesic line $c_x : \mathbb{R} \rightarrow X$ asymptotic to c such that $x = c_x(0)$ and $p(c_x(0)) = c(0)$. Moreover, for every $x' \in X_c$ there exists $x \in X_c^0$ such that $x' \in c_x(\mathbb{R})$. We now define a map $j : X_c^0 \times \mathbb{R} \rightarrow X_c$ by $j(x, t) = c_x(t)$.

By definition j is surjective, so we must show that j is an isometry. Let (x_1, t_1) and (x_2, t_2) be points in $X_c^0 \times \mathbb{R}$. Note that $d((x_1, t_1), (x_2, t_2)) = \sqrt{d(x_1, x_2)^2 + (t_1 - t_2)^2}$. Let us compute $d(c_{x_1}(t_1), c_{x_2}(t_2))$. We know that $C(c_{x_1}(\mathbb{R}) \cup c_{x_2}(\mathbb{R}))$ is isometric to a flat strip, so we obtain Figure 2.12, where p_{x_2, x_1} is the projection of c_{x_1} onto c_{x_2} .

Figure 2.12: Calculating the distance between $c_{x_1}(t_1)$ and $c_{x_2}(t_2)$

Thus we have

$$\begin{aligned} & d(c_{x_1}(t_1), c_{x_2}(t_2)) \\ &= \sqrt{d(c_{x_1}(t_1), p_{x_2, x_1}(c_{x_1}(t_1)))^2 + d(p_{x_2, x_1}(c_{x_1}(t_1)), c_{x_2}(t_2))^2} \end{aligned} \quad (2.45)$$

If we can show that $p_{x_2, x_1}(x_1) = x_2$ then since p_{x_2, x_1} is an isometry it will follow that $p_{x_2, x_1}(c_{x_1}(t)) = c_{x_2}(t)$ for all t and in particular $p_{x_2, x_1}(c_{x_1}(t_1)) = c_{x_2}(t_1)$. Then (2.45) will become $\sqrt{d(x_1, x_2) + (t_2 - t_1)^2}$.

Let p_{0, x_2} be the projection of $c_{x_2}(\mathbb{R})$ onto $c(\mathbb{R})$ and let $p_{x_2, 0}$ be the projection of $c(\mathbb{R})$ onto $c_{x_1}(\mathbb{R})$. By Lemma 2.5.6 we know that $p_{x_1, 0} \circ p_{0, x_2} \circ p_{x_2, x_1}$ is the identity map on $c_{x_1}(\mathbb{R})$. Thus $p_{x_1, 0} \circ p_{0, x_2} \circ p_{x_2, x_1}(x_1) = x_1$. But $p_{x_1, 0}$ and p_{0, x_2} are isometries, so they have inverses p_{0, x_1} and $p_{x_2, 0}$ giving $p_{x_2, x_1}(x_1) = p_{x_2, 0} \circ p_{0, x_1}(x_1)$. Furthermore, since $x_1 \in X_c^0$ we have $p_{0, x_1}(x_1) = c(0)$; similarly, $p_{0, x_2}(x_2) = c(0)$ which gives $p_{x_2, 0}(c(0)) = x_2$ and thus $p_{x_2, x_1}(x_1) = x_2$.

This completes the proof that j is an isometry and that X_c is isometric to $X_c^0 \times \mathbb{R}$.

The convexity of X_c^0 now follows: if $x, y \in X_c^0$ and $[x, y] \subseteq X$ is a geodesic segment, then $[x, y] \subseteq X_c$ and so $j^{-1}([x, y]) = [(x, 0), (y, 0)] \subseteq X_c^0 \times \mathbb{R}$ is a geodesic segment. Clearly, for any $(z, t) \in [(x, 0), (y, 0)]$ we have $t = 0$. This means that $[(x, 0), (y, 0)]$ is a geodesic segment in $X_c^0 \times \{0\}$ from $(x, 0)$ to $(y, 0)$. Hence $[x, y] = j(X_c^0 \times \{0\}) = X_c^0$. Thus X_c^0 is convex as desired.

□

Chapter 3

Group Actions

In this chapter we will introduce the basic definitions of group actions as well as discuss some properties of groups which act properly and cocompactly by isometries on a metric space. The main results of this chapter include Macbeath's theorem [Macbeath], that a group is finitely presented if and only if it acts properly and cocompactly on a simply-connected geodesic metric space and the Bruhat-Tits Fixed Point Theorem. The latter is a generalization of Cartan's theorem from the theory of Lie groups [Brown]. The former will be used in Chapter 4 to solve the word problem for such groups.

3.1 Basics of Group Actions

We begin by describing actions of a group on an arbitrary metric space, paying specific attention to those actions which are proper and cocompact.

First though, we need a few definitions.

Definition 3.1.1. Given a topological space X and a group Γ we define an *action* of Γ on X to be a homomorphism $\Phi : \Gamma \rightarrow \text{Homeo}(X)$. We write $\gamma \cdot x$ in place of $(\Phi(\gamma))(x)$.

Definition 3.1.2. Group actions with the following properties are used extensively throughout the proceeding sections:

1. A group action $\Phi : \Gamma \rightarrow \text{Homeo}(X)$ is called *faithful* if $\ker \Phi = \{1\}$.
2. The action, $\Phi : \Gamma \rightarrow \text{Homeo}(X)$ is said to be *free* if $\gamma \cdot x \neq x$ for each $\gamma \in \Gamma - \{1\}$ and $x \in X$.
3. If there exists a compact subset $K \subseteq X$ such that $\Gamma \cdot K = X$ then Φ is said to be *cocompact*.
4. The *stabilizer* of a point $x \in X$ is the set $\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}$. This set is a subgroup of Γ called the *isotropy subgroup* of x .
5. Given a point $x \in X$ the *orbit* of x is the set $\Gamma \cdot x = \{\gamma \cdot x \mid \gamma \in \Gamma\}$.

We shall be using the following definition for a group acting on a metric space.

Definition 3.1.3. An isometric action of a group Γ on a metric space X is a group action $\Phi : \Gamma \rightarrow \text{Homeo}(X)$ such that each homeomorphism in the image of Φ is an isometry of X , that is $\Phi(\Gamma) \subseteq \text{Isom}(X)$. We say that the action is

proper, or that Γ acts *properly* on X , if for every $x \in X$ there exists $r > 0$ such that the set $\{\gamma \in \Gamma \mid \gamma \cdot B(x, r) \cap B(x, r) \neq \emptyset\}$ is finite.

Remark 3.1.4. If X is a *proper* metric space and Γ acts on X by isometries then Definition 3.1.3 is equivalent to the more general property that for every compact set $K \subseteq X$ the set $\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\}$ is finite. Note that a metric space X is said to be *proper* if every closed ball $\overline{B}(x, r)$ forms a compact subspace of X . In fact, every compact set K has an open neighbourhood U such that the set $\{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$ is finite

Proof. Let us fix a compact subset K of X ; by Definition 3.1.3, for every $x \in K$ we may find $r_x > 0$ such that the following set is finite

$$\{\gamma \in \Gamma \mid \gamma \cdot B(x, r_x) \cap B(x, r_x) \neq \emptyset\} \quad (3.1)$$

Since the set of balls $B(x, \frac{r_x}{2})$, $x \in K$ forms an open cover for the compact set K , there exists a finite subset $x_1, x_2, \dots, x_n \in K$ such that the balls $B(x_i, r_i/2)$ cover K , where $r_i = r_{x_i}$. Let $U = \bigcup_{i=1}^n B(x_i, r_i/2)$ and note that U is an open neighbourhood of K .

To derive a contradiction we will suppose that the following set is infinite

$$A = \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\} \quad (3.2)$$

For each $\gamma \in A$ we may find x_i, x_j such that $\gamma \cdot B(x_i, r_i/2) \cap B(x_j, r_j/2) \neq \emptyset$. Since A is infinite, this implies that we can find a pair x_i, x_j such that the set

$B = \{\gamma \in A \mid \gamma \cdot B(x_i, r_i/2) \cap B(x_j, r_j/2) \neq \emptyset\}$ is infinite. Without loss of generality, we will assume that $r_i \leq r_j$. Let $s_1, s_2 \in B$ be two distinct elements. We know that $s_1 \cdot B(x_i, r_i/2) \cap B(x_j, r_j/2) = B(s_1 \cdot x_i, r_i/2) \cap B(x_j, r_j/2) \neq \emptyset$ and that $s_2 \cdot B(x_i, r_i/2) \cap B(x_j, r_j/2) = B(s_2 \cdot x_i, r_i/2) \cap B(x_j, r_j/2) \neq \emptyset$ which tells us that

$$d(s_1 \cdot x_i, x_j) \leq r_i/2 + r_j/2 \leq r_j \quad (3.3)$$

and similarly

$$d(s_2 \cdot x_i, x_j) \leq r_i/2 + r_j/2 \leq r_j \quad (3.4)$$

From this we see that both $s_1 \cdot x_i$ and $s_2 \cdot x_i$ are contained within $B(x_j, r_j)$ which tells us that $s_2 s_1^{-1} \cdot (s_1 \cdot x_i) = s_2 \cdot x_i$ must be contained in both the ball $(s_2 s_1^{-1}) \cdot B(x_j, r_j)$ and the ball $B(x_j, r_j)$. In particular, we have $(s_2 s_1^{-1}) \cdot B(x_j, r_j) \cap B(x_j, r_j) \neq \emptyset$. Since we have an infinite number of distinct elements $s_i, s_j \in B$, this implies that the set $\{\gamma \in \Gamma \mid \gamma \cdot B(x_j, r_j) \cap B(x_j, r_j) \neq \emptyset\}$ is infinite, which is a contradiction. This completes the proof. $\square$

Proposition 3.1.5. Let Γ act properly and by isometries on a metric space X . Then the following hold:

1. For every $x \in X$, there exists $\varepsilon > 0$ such that $\gamma \in \Gamma_x$ whenever we have $\gamma \cdot B(x, \varepsilon) \cap B(x, \varepsilon) \neq \emptyset$.
2. The space X/Γ of Γ -orbits of X is a metric space with the distance function:

$$d(\Gamma \cdot x, \Gamma \cdot y) = \inf_{\gamma, \gamma' \in \Gamma} d(\gamma \cdot x, \gamma' \cdot y) \quad (3.5)$$

3. Let Y be a subspace of X . If Y is invariant under the action of a subgroup $H \subseteq \Gamma$ then H acts properly on Y .
4. Suppose the action of Γ on X is also cocompact. Then for each $x \in X$, the isotropy subgroup Γ_x is finite and the set $\{\Gamma_x \mid x \in X\}$ is the union of finitely many conjugacy classes of finite subgroups. That is, there are only finitely many conjugacy classes of isotropy subgroups.

Proof. 1. Given $x \in X$, since Γ acts properly on X , we can find $r > 0$ such that the following set is finite

$$S = \{\gamma \in \Gamma \mid \gamma \cdot B(x, r) \cap B(x, r) \neq \emptyset\} \quad (3.6)$$

Now let $T = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \subseteq S$ be the finite subset of S consisting precisely of those members which do not stabilize x . Suppose we let ε be given by the following equation:

$$\varepsilon = \frac{1}{2} \min\{r, d(x, \gamma_1 \cdot x), d(x, \gamma_2 \cdot x), \dots, d(x, \gamma_n \cdot x)\} \quad (3.7)$$

We claim that ε is the desired value. Suppose $\gamma \cdot B(x, \varepsilon) \cap B(x, \varepsilon) \neq \emptyset$; since $\varepsilon \leq r$, we have $\gamma \in S$. Let $x' \in B(x, \varepsilon) \cap \gamma^{-1} \cdot B(x, \varepsilon)$. Since Γ acts by isometries, we have $x' \in B(\gamma \cdot x, \varepsilon)$. So,

$$d(x, \gamma \cdot x) \leq d(x, x') + d(x', \gamma \cdot x) < \varepsilon + \varepsilon = 2\varepsilon \quad (3.8)$$

Since $d(x, \gamma \cdot x) < 2\varepsilon$, we may conclude that γ cannot be in the set T so this implies $\gamma \in \Gamma_x$ as desired.

2. First we show that d is a pseudometric. Clearly d is symmetric. Furthermore, since Γ acts by isometries, we have $d(\Gamma \cdot x, \Gamma \cdot y) = d(x, \Gamma \cdot y)$ for any $x, y \in X$.

To establish the triangle inequality, consider $x, y, z \in X$. For every $\varepsilon > 0$ we can find $x' \in \Gamma \cdot x$ and $y' \in \Gamma \cdot y$ such that

$$d(x', y') \leq d(\Gamma \cdot x, \Gamma \cdot y) + \varepsilon/2 \quad (3.9)$$

We can then find $z' \in \Gamma \cdot z$ such that $d(y', z') \leq d(\Gamma \cdot y, \Gamma \cdot z) + \varepsilon/2$. This gives

$$\begin{aligned} d(\Gamma \cdot x, \Gamma \cdot z) &\leq d(x', z') \\ &\leq d(x', y') + d(y', z') \\ &\leq d(\Gamma \cdot x, \Gamma \cdot y) + d(\Gamma \cdot y, \Gamma \cdot z) + \varepsilon \end{aligned} \quad (3.10)$$

Since this holds for any $\varepsilon > 0$ we have the desired result: for any points $x, y, z \in X$, we have

$$d(\Gamma \cdot x, \Gamma \cdot z) \leq d(\Gamma \cdot x, \Gamma \cdot y) + d(\Gamma \cdot y, \Gamma \cdot z) \quad (3.11)$$

Since it is clear that if $\Gamma \cdot x = \Gamma \cdot y$ then $d(\Gamma \cdot x, \Gamma \cdot y) = 0$, all

that is left to be shown is that if $d(\Gamma \cdot x, \Gamma \cdot y) = 0$ then $\Gamma \cdot x = \Gamma \cdot y$. By (1), there exists $\varepsilon > 0$ such that if $\gamma \cdot B(y, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$ then $\gamma \in \Gamma_y$. Since $d(x, \Gamma \cdot y) = 0$, there exists a sequence, $\{\gamma_n\} \subseteq \Gamma$ such that $d(x, \gamma_n \cdot y) \rightarrow 0$. This means that we may find $N \in \mathbb{N}$ such that for any $n \geq N$ we have $d(x, \gamma_n \cdot y) < \varepsilon/2$. Now, for any $n, m \geq N$ this implies that we have:

$$\begin{aligned} d(\gamma_n \cdot y, \gamma_m \cdot y) &\leq d(\gamma_n \cdot y, x) + d(\gamma_m \cdot y, x) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned} \tag{3.12}$$

Since Γ acts by isometries, $d(y, \gamma_n^{-1} \gamma_m \cdot y) < \varepsilon$ and thus $\gamma_n^{-1} \gamma_m \cdot B(y, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$ which, by our choice of ε , implies that $\gamma_n^{-1} \gamma_m \in \Gamma_y$. This tells us that $\gamma_n^{-1} \gamma_m \cdot y = y$ so we see that $\gamma_n \cdot y = \gamma_m \cdot y$ for all $n, m \geq N$. Thus, for all $n \geq N$ we have $d(x, \gamma_n \cdot y) = d(x, \gamma_N \cdot y)$. Since $d(x, \gamma_n \cdot y) \rightarrow 0$ as $n \rightarrow \infty$ we have $d(x, \gamma_N \cdot y) = 0$ and thus $x = \gamma_N \cdot y$. This implies that $x \in \Gamma \cdot y$ and $\Gamma \cdot y = \Gamma \cdot x$ as required.

3. This is a direct consequence of the definitions.
4. First note that for any $x \in X$ and $\gamma \in \Gamma$ we have $\gamma \Gamma_x \gamma^{-1} = \Gamma_{\gamma \cdot x}$ because

$$\begin{aligned} \alpha \in \Gamma_{\gamma \cdot x} &\Leftrightarrow \alpha \gamma \cdot x = \gamma \cdot x \\ &\Leftrightarrow \gamma^{-1} \alpha \gamma \cdot x = x \\ &\Leftrightarrow \alpha \in \gamma \Gamma_x \gamma^{-1} \end{aligned} \tag{3.13}$$

Suppose that the action of Γ on X is cocompact and fix a compact set $K \subset X$ such that $\Gamma \cdot K = X$. Since Γ acts properly on X , for each $x \in K$ there exists $r_x > 0$ such that $\{\gamma \in \Gamma : \gamma \cdot B(x, r_x) \cap B(x, r_x) \neq \emptyset\}$ is finite. The set of all such balls $B(x, r_x)$ is an open cover for K so by compactness we can find a finite subcover $\{B(x_1, r_1), B(x_2, r_2), \dots, B(x_n, r_n)\}$ of K , where $r_i = r_{x_i}$ for each $1 \leq i \leq n$. Let us define Σ to be:

$$\Sigma = \bigcup_{i=1}^n \{\gamma \in \Gamma \mid \gamma \cdot B(x_i, r_i) \cap B(x_i, r_i) \neq \emptyset\} \quad (3.14)$$

As a finite union of finite sets, Σ is clearly finite as well.

For each $x \in X$ there exists $\gamma \in \Gamma$ such that $\gamma \cdot x \in K$. Then $\gamma \Gamma_x \gamma^{-1} = \Gamma_{\gamma \cdot x}$. Since $\Gamma_{\gamma \cdot x}$ stabilizes $\gamma \cdot x \in K$, for each $\gamma' \in \Gamma_{\gamma \cdot x}$ we have $\gamma' \cdot K \cap K \neq \emptyset$. Thus $\gamma' \in \Sigma$ and $\Gamma_{\gamma \cdot x} \subseteq \Sigma$. So Γ_x is conjugate to a subset of the finite set Σ and thus the isotropy subgroups of Γ are finite. Furthermore, since each conjugacy class of isotropy subgroups has a member which is contained in Σ , there can only be finitely many such classes.

□

Remark 3.1.6. If X is a geodesic metric space such that there exists a group which acts properly and cocompactly on X , then X is complete and locally compact. Thus, by the Hopf-Rinow Theorem, it is a proper geodesic space [Bridson & Haefliger pg. 35, 132].

Example 3.1.7. We can define an action of Γ on the metric Cayley graph $C_A(\Gamma)$ where A is a finite generating set for Γ . For each $\gamma \in \Gamma$ we define the action of γ to take each edge e originating at γ_0 labelled by $a \in A$ isometrically to the edge labelled by a originating at $\gamma\gamma_0$. This action is proper because any ball of radius $1/2$ in $C_A(\Gamma)$ intersects only its trivial translate. Conversely, the translates of any closed ball of radius 1 will cover $C_A(\Gamma)$ so the action is cocompact.

3.2 Group Presentations and Actions

The aim of this section is to prove that a group acts properly and cocompactly on a simply-connected geodesic metric space if and only if it is finitely presented. We will begin by describing group presentations and then proceeding to Macbeath's theorem [Macbeath] which provides a presentation for any group acting properly and cocompactly by homeomorphisms on a simply-connected topological space. Given a set S we write $F(S)$ to denote the free group generated by S .

Definition 3.2.1. A *presentation* for a group G consists of a set S , an epimorphism $\pi : F(S) \rightarrow G$ and a subset $R \subset F(S)$ such that $\ker \pi = \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the *normal closure* of R . A presentation is normally denoted $G = \langle S \mid R \rangle$ where S is called the set of *generators* for the presentation of G and R is the set of *relators*. If the sets S and R are finite then the presentation is said to be finite and if a group G admits a finite presentation, then G is

called *finitely presentable*.

The following lemma is a standard consequence of the Seifert-van Kampen theorem from Algebraic Topology and is stated without proof. For more details, refer to [Bridson & Haefliger pg.135] and [Massey pg. 113-116].

Lemma 3.2.2. Let Γ be a group with presentation $\langle A \mid R \rangle$ and let $C_A(\Gamma)$ be its metric Cayley graph. Let $R' \subseteq \langle\langle R \rangle\rangle$ and consider the 2-complex obtained by gluing a 2-cell to each loop in $C_A(\Gamma)$ whose edges are labelled by a reduced word, $r \in R'$. Then this 2-complex is simply connected if and only if $\langle\langle R' \rangle\rangle = \langle\langle R \rangle\rangle$.

The following theorem is due to Macbeath [Macbeath].

Theorem 3.2.3. Let Γ be a group acting by homeomorphisms on a non-empty connected topological space X . If U is an open set such that $\Gamma \cdot U = X$, then:

1. The set $S = \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$ is a generating set for Γ
2. If we also require that U be path-connected and X be both path-connected and simply connected then Γ has a presentation $\Gamma = \langle A_S \mid R \rangle$ where A_S is a set of symbols indexed by S and the set of relators R is defined as follows:

$$R = \{a_{s_1} a_{s_2} a_{s_3}^{-1} \mid s_i \in S; U \cap s_1 \cdot U \cap s_3 \cdot U \neq \emptyset; s_1 s_2 = s_3\}. \quad (3.15)$$

Proof. 1. Let $H = \langle S \rangle \subseteq \Gamma$ be the subgroup generated by S . We wish to show that $H = \Gamma$ so if we let $I = \Gamma - H$ then it is equivalent to show that I is empty.

Suppose first that $H \cdot U \cap I \cdot U \neq \emptyset$. This implies that there exists elements $u, u' \in U$ and $h \in H, i \in I$ such that $h \cdot u = i \cdot u'$. This gives $h^{-1}i \cdot u' = u \in U$ which implies that $h^{-1}i \in S$ and thus $i \in HS$. Unfortunately, $HS = H$ so we conclude that $i \in H$ which is a contradiction. We now know that $H \cdot U \cap I \cdot U = \emptyset$. The set U is open however, so $H \cdot U = \bigcup \{h \cdot U \mid h \in H\}$ and $I \cdot U = \bigcup \{i \cdot U \mid i \in I\}$ are unions of open sets since Γ acts by homeomorphisms. Thus $H \cdot U$ and $I \cdot U$ are also open. Since $\Gamma = H \cup I$, we have $X = \Gamma \cdot U = H \cdot U \cup I \cdot U$. But X is connected, so we conclude that either $H \cdot U = \emptyset$ or $I \cdot U = \emptyset$.

The set $H \cdot U$ is non-empty since both U and S are non-empty ($1 \in S$). This implies that $I \cdot U = \emptyset$ and since $U \neq \emptyset$, it must be the case that $I = \emptyset$, as desired.

2. Let $C_{A_S}(\Gamma)$ be the metric Cayley graph of Γ with generators in A_S . If K is the 2-complex obtained from $C_{A_S}(\Gamma)$ by attaching a 2-cell to each of the edge loops labelled by a word in R , according to the Lemma 3.2.2, in order to show that R is a set of defining relators of Γ , it is sufficient to show that K is simply-connected. Note that we may extend the action of Γ on $C_{A_S}(\Gamma)$ to an action on K where $\gamma \in \Gamma$ sends the 2-cell glued to the edge loop labelled by r , based at the vertex x , homeomorphically to

the 2-cell glued to the edge loop labelled by r based at the vertex $\gamma \cdot x$.

In order to prove that K is simply connected, we will show that any continuous map $\ell : \partial D \rightarrow C_{A_S}(\Gamma)$ from the boundary of a disk D can be extended to a continuous map $\tilde{\ell} : D \rightarrow K$.

We will first construct a Γ -equivariant map from K to X . Fix a base point $x_0 \in U$. Now, for each generator $s \in S$, we select a point x_s in the non-empty set $U \cap s \cdot U$. Since U is path-connected, we can now find a path $\tilde{c}_s$ in U originating at x_0 and terminating at x_s . Furthermore, since $s \cdot U$ is path-connected, we may find a path $\tilde{c}_s'$ from x_s to $s \cdot x_0 \in s \cdot U$. Let $c_s : I \rightarrow X$ be the concatenation of these two paths (so c_s originates at x_0 and terminates at $s \cdot x_0$). See Figure 3.1.

Figure 3.1: Constructing the path c_s

We can now define a map $p : C_{A_S}(\Gamma) \rightarrow X$ which sends any vertex labelled by $\gamma \in \Gamma$ to the point $\gamma \cdot x_0$ and any edge labelled by a_s originating at γ and terminating at $\gamma \cdot a_s$ to the path $\gamma \cdot c_s$ in X .

To construct a Γ -equivariant extension of p to K , we will extend p to each cell glued to edge loops which are based at 1. Let $r \in R$ and consider $e_r : S^1 \rightarrow C_A(\Gamma)$, the loop based at 1 and labeled by r . Let c_r be the cell which was glued to this loop. Since X is simply-connected, we may extend to map $p \circ e_r : S^1 \rightarrow X$ to a map $\overline{p \circ e_r} : S^1 \cup c_r^o \rightarrow X$ where c_r^o is the interior of the cell c_r , thus we can find a continuous extension $\tilde{p}_{e_r} : C_A(\Gamma) \cup c_r \rightarrow X$ of p . If we find this extension $\tilde{p}_{e_r}$ for each $r \in R$ then we can define $\tilde{p} : K \rightarrow X$ as follows:

$$\tilde{p}(x) = \begin{cases} p(x) & x \in C_{A_S}(\Gamma) \\ \gamma \cdot \tilde{p}_{e_r}(x) & x \in \gamma \cdot c_r \end{cases} \quad (3.16)$$

This map is continuous since it is continuous when restricted to $C_{A_S}(\Gamma)$ as well as when restricted to each open 2-cell in K . Furthermore, it is Γ -equivariant by definition.

Let us now return to our map $\ell : \partial D \rightarrow C_{A_S}(\Gamma)$. We assume without loss of generality that ℓ is a closed edge path labeled by a cyclically reduced word (so the first letter is not the inverse of the last).

Since X is simply-connected, the map $\tilde{p} \circ \ell : \partial D \rightarrow X$ can be continuously extended to a map $\phi : D \rightarrow X$. Our goal is to find a triangulation T for the disk D such that a suitably reparameterized version of our loop, $\ell' : D \rightarrow C_{A_S}(\Gamma)$ can be continuously extended to a map $\tilde{\phi} : T \rightarrow C_{A_S}(\Gamma)$. Furthermore, we shall do this in such a manner that the image of each

triangle in T is an edge loop in $C_{A_s}(\Gamma)$ which is labeled by a word in R , which will allow us to extend this map to a map from the entire disk D . Therefore $\tilde{\phi}$ can be extended to D and we shall be able to conclude that ℓ is null-homotopic.

Let us now examine the image of ℓ in C_{A_s} more carefully. Let γ_0 be a base point for the loop ℓ and let $s_1 s_2 s_3 \cdots s_n$ be its edge labels. Let $\gamma_i = \gamma_0 s_1 s_2 s_3 \cdots s_i$ be the label of the i^{th} vertex through which ℓ runs, $i = 1, 2, \dots, n$. Let $d_0, d_1, \dots, d_{n-1}$ be the points on ∂D which are mapped under ℓ to $\gamma_0 \cdot x_0, \gamma_1 \cdot x_0, \gamma_2 \cdot x_0, \dots, \gamma_{n-1} \cdot x_0$ respectively. Note that $\gamma_n \cdot x_0 = x_0$. See Figure 3.2

Thus, if we consider the loop $\tilde{p} \circ \ell : \partial D \rightarrow X$ we shall see that it is the concatenation of the paths $\gamma_0 \cdot c_{s_1}, \gamma_1 \cdot c_{s_2} \dots \gamma_{n-1} \cdot c_{s_n}$. Moreover, we defined each path c_{s_i} to be the concatenation of a path in U from x_0 to x_{s_i} and a path in $s_i \cdot U$ from x_{s_i} to $s_i \cdot x_0$. For each $0 \leq i \leq n-1$ let $d'_i \in \partial D$ be the point which gets mapped under $\tilde{p} \circ \ell$ to $\gamma_i \cdot x_s$.

Now, this means that image of the segment of ∂D from d'_{i-1} to d'_i for $1 \leq i < n$ (as well as from d'_{n-1} to d'_0) under $\tilde{p} \circ \ell$ is contained within $\gamma_{i+1} \cdot U$ (respectively $\gamma_0 \cdot U$). Since D is compact, there exists ε_i for each $0 \leq i < n$ such that any ball of radius ε_i around a point in the segment from d'_i to d'_{i+1} is mapped into $\gamma_{i+1} \cdot U$ by $\tilde{p} \circ \ell$.

Furthermore, since $\Gamma \cdot U = X$, the set $\{\phi^{-1}(\gamma \cdot U) : \gamma \in \Gamma\}$ is an open cover of D . Thus by the Lebesgue Covering Lemma [Munkres], we can

Figure 3.2: Relationship between D , $C_{As}(\Gamma)$, and X

find $\varepsilon' > 0$ such that any ball of radius ε' in D lies within one of the open sets of the cover. In particular, this means that the image under ϕ of any such ball is completely contained within $\gamma \cdot U$ for some $\gamma \in \Gamma$. Let $\varepsilon = \min\{\varepsilon', \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}\}$ and take a finite triangulation T of the disk D such that the diameter of every triangle is less than ε and each of $d_0, d_1, \dots, d_{n-1}, d'_0, d'_1, \dots, d'_{n-1}$ is included in the vertex set of T . In particular, the ball of radius ε around a vertex v in the triangulation T contains all the triangles in T which are incident at v .

Figure 3.3: Selection of δ_i and δ'_i

We can now define a map from the vertices of T not on ∂D to Γ by $v \mapsto \gamma_v$ where we fix γ_v to be an element of Γ such that ϕ maps all of the triangles incident at v into $\gamma_v \cdot U$. For each $v \in \partial D$ we choose γ_v as follows:

- if $v = d'_i$ for some $0 \leq i < n - 1$ we choose $\gamma_v = \gamma_{i+1}$

- if v lies on the segment of ∂D between d'_i and d'_{i+1} for $0 \leq i \leq n-1$ then we choose $\gamma_v = \gamma_{i+1}$ and if it lies between d'_{n-1} and d'_0 we choose $\gamma_v = \gamma_0$

Let us examine a triangle $t \in T$ with vertices v_1, v_2, v_3 .

$$\begin{aligned} \emptyset \neq \phi(t) &\subseteq \gamma_{v_1} \cdot U \cap \gamma_{v_2} \cdot U \cap \gamma_{v_3} \cdot U \\ &= \gamma_{v_1}(U \cap (\gamma_{v_1}^{-1}\gamma_{v_2} \cdot U) \cap (\gamma_{v_1}^{-1}\gamma_{v_3} \cdot U)) \\ &= \gamma_{v_2}((\gamma_{v_2}^{-1}\gamma_{v_1} \cdot U) \cap U \cap (\gamma_{v_2}^{-1}\gamma_{v_3} \cdot U)) \end{aligned} \quad (3.17)$$

Thus, if we let $b_1 = \gamma_{v_1}^{-1}\gamma_{v_2}$, let $b_2 = \gamma_{v_2}^{-1}\gamma_{v_3}$, and let $b_3 = \gamma_{v_1}^{-1}\gamma_{v_3}$, then $b_1, b_2, b_3 \in S$ and furthermore, $a_{b_1}a_{b_2}a_{b_3}^{-1} \in R$. We can now extend our map $v \mapsto \gamma_v$ to a map $\tilde{\phi} : T \rightarrow C_{A_S}(\Gamma)$ by sending the edge which connects v_1 to v_2 to the edge b_1 as defined above and similarly for the other edges. Note that $\tilde{\phi}|_{\partial D}$ is a reparameterization of ℓ .

Now, since each triangle is mapped to a circuit in $C_{A_S}(\Gamma)$ which is labelled by a word in R , we can extend $\tilde{\phi}$ to a continuous map $D \rightarrow K$. Therefore K is simply connected as desired.

□

Corollary 3.2.4. A group Γ is finitely presented if and only if it acts properly and cocompactly by isometries on a simply-connected geodesic metric space X .

Proof. ($\Leftarrow$) Suppose Γ is a group which acts properly and cocompactly on a

simply-connected geodesic metric space X . Let K be a compact set such that $\Gamma \cdot K = X$ and take an open ball $B(x_0, r)$ which contains K . Then we have $\Gamma \cdot B(x_0, r) = X$ and can apply Theorem 3.2.3 with $U = B(x_0, r)$. By Remark 3.1.6, the space X is proper and the action of Γ is proper, so we know that the set $\{\gamma \in \Gamma : \gamma \cdot \overline{B}(x_0, r) \cap \overline{B}(x_0, r) \neq \emptyset\}$ is finite by Remark 3.1.4. Since as defined in Theorem 3.2.3, S is contained in this set, it is finite and hence R is finite as well.

($\Rightarrow$) Suppose $\Gamma = \langle A | R \rangle$ with A and R finite. If we construct the simply-connected geodesic metric space K by gluing an n -gon to each edge loop labeled by a word $r \in R$ of length n in the Cayley graph $C_A(\Gamma)$ then the action of Γ on $C_A(\Gamma)$ extends naturally to an action on K . In this action, if $\gamma \in \Gamma$ and c is an n -gon labeled by $r \in R$ and based at $\gamma_0 \in \Gamma$, then γ sends c isometrically to the n -gon glued to the edge labeled by r based at $\gamma\gamma_0$. The space K is simply-connected by Lemma 3.2.2.

Let m be the length of the longest word in R . If we take $B = \overline{B}(1, m) \subseteq K$ then B is compact since both A and R are finite. Furthermore, the transitivity of the action of Γ on the vertex set of $C_A(\Gamma)$ guarantees that the translates of B cover K . Thus, the action of Γ on K is cocompact. Furthermore, for any ball $B(x, 1/2)$ in K , if $\gamma \in \Gamma$ such that $\gamma \cdot B(x, 1/2) \cap B(x, 1/2) \neq \emptyset$ then $\gamma = 1$ so the action is proper and so we have defined a proper cocompact action of Γ on a simply-connected geodesic metric space, as desired. $\square$

Corollary 3.2.5. A group Γ which acts properly and cocompactly on a proper

CAT(0) space is finitely presented.

Proof. Any CAT(0) space is a simply connected geodesic space by Remark 2.1.5, so by Corollary 3.2.4, we have that Γ is finitely presented. $\square$

3.3 Bruhat-Tits Fixed Point Theorem

In this section, we present a fixed point theorem for isometries of CAT(0) spaces as well as several properties regarding conjugates of finite subgroups.

Theorem 3.3.1. Let Γ be a group acting by isometries on a CAT(0) space X .

1. **[Bruhat-Tits Fixed Point Theorem]** If X is a complete CAT(0) space and Γ has a bounded orbit then the fixed-point set of Γ is a non-empty convex subspace of X .
2. If Γ acts properly and cocompactly on X then each finite subgroup is contained in an isotropy subgroup and hence there are only finitely many conjugacy classes of finite subgroups in Γ .

Proof. 1. Let Y be a bounded orbit of Γ and let $\gamma \in \Gamma$ be any element. Then by Proposition 2.4.4 the centre c_Y of Y is fixed under the action of γ . Thus c_Y is in the fixed-point set of Γ .

To show that the fixed point set is convex, let x, y be fixed points of Γ and consider the (unique) geodesic segment $[x, y]$ joining them. For any

$\gamma \in \Gamma$, the set $\gamma \cdot [x, y]$ is a geodesic segment from $\gamma \cdot x$ to $\gamma \cdot y$ because γ is an isometry. Since x and y are fixed by γ and X is uniquely geodesic, we conclude that $\gamma \cdot [x, y] = [x, y]$ and $[x, y]$ is in the fixed set of Γ as desired.

2. Suppose H is a finite subgroup of Γ . Let $x_0 \in X$. Then $H \cdot x_0$ is a bounded orbit and thus by (1) we know that H has a non-empty fixed point set. Suppose $x \in X$ is in the fixed point set of H , then $H \subseteq \Gamma_x$. Since by Proposition 3.1.5 we also know that there are only finitely many conjugacy classes of isotropy subgroups in Γ and hence finitely many conjugacy classes of subgroups of isotropy subgroups, we conclude that Γ contains only finitely many conjugacy classes of finite subgroups.

□

Corollary 3.3.2. Suppose Γ acts properly and cocompactly by isometries on a CAT(0) space. Then Γ has only finitely many conjugacy classes of elements of finite order and in particular finitely many central elements of finite order.

Proof. Choose a representative for each of the finitely many conjugacy classes of finite subgroups. Each $\gamma \in \Gamma$ of finite order is conjugate to one of the finitely many elements of subgroups from this finite list of subgroups. The second remark follows since central elements are their own conjugacy class. □

An important application of the Bruhat-Tits Fixed Point Theorem is to the case of a group acting on a tree. Serre [Serre] uses the result to prove if

$\Gamma_1 *_H \Gamma_2$ is an amalgamated product, then any finite subgroup is conjugate to a subgroup of Γ_1 or Γ_2 .

Similarly to the case of a finite subgroup, if a topological group Γ acts by isometries on a CAT(0) space and $K \subseteq \Gamma$ is compact, then K has a bounded orbit and hence a fixed point. Cartan's fixed point theorem is a special case of this.

Chapter 4

Decision Problems

In this chapter we shall prove that any group which acts properly and cocompactly on a $\text{CAT}(0)$ metric space has a decidable word problem.

4.1 The Word Problem

We begin with a definition.

Definition 4.1.1. A group Γ with a finite generating set A has a *decidable word problem* if there exists a Turing machine halting on each input that decides whether or not a given word $w \in F(A)$ represents the identity in Γ .

It must be noted that we only require the existence of such a Turing machine, not an effective method by which to construct it.

Since the aim of this section is to prove that such a Turing machine exists for groups which act properly and cocompactly by isometries on a $\text{CAT}(0)$

metric space, let us fix such a group Γ and CAT(0) metric space X for the entire section. Note that by Remark 3.1.6, X is a proper space. Since the action of Γ on X is cocompact, there exists a compact subset $K \subseteq X$ such that $\Gamma \cdot K = X$. Furthermore, since K is bounded, we can find a radius $D/3$ and basepoint $x_0 \in X$ such that $K \subseteq B(x_0, D/3)$. This tells us that $\Gamma \cdot B(x_0, D/3) = X$; let us also fix this constant D for the remainder of the section.

We begin our discussions with a lemma which identifies a generating set for our group Γ .

Lemma 4.1.2. The set $A = \{a \in \Gamma : d(a \cdot x_0, x_0) \leq D+1\}$ is a finite generating set for Γ . Moreover, for any $\gamma \in \Gamma$ such that $d(\gamma \cdot x_0, x_0) \leq 2D+1$ there exists a four-letter word $a_1 a_2 a_3 a_4 \in F(A)$ which represents γ in the group Γ .

Proof. If we let $U = B(x_0, D/3)$ then we have $\Gamma \cdot U = X$ which by Theorem 3.2.3 implies that the set $S = \{\gamma \in \Gamma : \gamma \cdot B(x_0, D/3) \cap B(x_0, D/3) \neq \emptyset\}$ is a generating set for Γ .

Pick any element $s \in S$; we wish to show that s is an element of A . There exists $x \in B(x_0, D/3)$ such that x is in $s \cdot B(x_0, D/3)$ where $s \cdot B(x_0, D/3) = B(s \cdot x_0, D/3)$ since Γ acts by isometries. From this, we can compute:

$$\begin{aligned} d(x_0, s \cdot x_0) &\leq d(x_0, x) + d(x, s \cdot x_0) \\ &\leq 2D/3 \\ &\leq D+1 \end{aligned} \tag{4.1}$$

Consequently, we may conclude that $s \in A$ and thus $S \subseteq A$. Since we know that $\langle S \rangle = \Gamma$ we see that $\langle A \rangle = \Gamma$ as desired.

Now we shall show that A is finite. Consider the set:

$$T = \{\gamma \in \Gamma : \gamma \cdot \overline{B}(x_0, D+1) \cap \overline{B}(x_0, D+1) \neq \emptyset\} \quad (4.2)$$

The metric space X is proper, so the closed ball $\overline{B}(x_0, D+1)$ is a compact subspace of X . Since Γ acts properly on X , by Remark 3.1.4 we see that the set T must be finite.

Given an element $a \in A$, by definition we have $d(a \cdot x_0, x_0) \leq D+1$ which implies that $a \cdot x_0 \in \overline{B}(x_0, D+1)$. From this we deduce that $a \cdot \overline{B}(x_0, D+1) \cap \overline{B}(x_0, D+1) \neq \emptyset$ which gives $a \in T$. Thus, A is a subset of T and the set A is a finite generating set for Γ as desired.

To prove the second half of the lemma, let $\gamma \in \Gamma$ be an element such that $d(\gamma \cdot x_0, x_0) \leq 2D+1$ and consider the geodesic path $c_\gamma : [0, \lambda] \rightarrow X$ which originates at x_0 and terminates at $\gamma \cdot x_0$. Note that the length of the path λ is equal to $d(x_0, \gamma \cdot x_0)$ and so $\lambda \leq 2D+1$.

We fix the following points in the image of c_γ :

$$\begin{aligned} x_1 &= c_\gamma(\lambda/3) \\ x_2 &= c_\gamma(\lambda/2) \\ x_3 &= c_\gamma(2\lambda/3) \end{aligned} \quad (4.3)$$

Since $\Gamma \cdot B(x_0, D/3) = X$ we can find elements $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that:

$$\begin{aligned} x_1 &\in B(\gamma_1 \cdot x_0, D/3) \\ x_2 &\in B(\gamma_2 \cdot x_0, D/3) \\ x_3 &\in B(\gamma_3 \cdot x_0, D/3) \end{aligned} \tag{4.4}$$

Figure 4.1: Dividing the geodesic segment $[x_0, \gamma \cdot x_0]$ into four parts

For the element γ_1 we have:

$$\begin{aligned} d(x_0, \gamma_1 \cdot x_0) &\leq d(x_0, x_1) + d(x_1, \gamma_1 \cdot x_0) \\ &\leq \lambda/3 + D/3 \\ &\leq (2D + 1)/3 + D/3 \\ &< D + 1 \end{aligned} \tag{4.5}$$

So we may conclude that $\gamma_1 \in A$. Let us assign $a_1 = \gamma_1$. Similarly, we find that $d(x_0, \gamma_1^{-1}\gamma_2 \cdot x_0) < D + 1$ and $d(x_0, \gamma_2^{-1}\gamma_3 \cdot x_0) < D + 1$. See Figure 4.1. Thus we may assign $a_2 = \gamma_1^{-1}\gamma_2 \in A$ and $a_3 = \gamma_2^{-1}\gamma_3 \in A$. Finally, we note that $d(x_0, \gamma_3^{-1}\gamma \cdot x_0) < D + 1$ so we have $a_4 = \gamma_3^{-1}\gamma \in A$. We conclude by pointing out that the word $a_1a_2a_3a_4$ in $F(A)$ represents the element $\gamma_1(\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3)(\gamma_3^{-1}\gamma) = \gamma$ in Γ and is therefore our desired

representation. $\square$

Definition 4.1.3. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is an isoperimetric function for a group Γ given by a finite presentation $\langle A \mid R \rangle$ if all words w in $F(A)$ of length at most n representing the identity in Γ can be written as a product of at most $f(n)$ conjugates of relators and their inverses.

The following theorem says that Γ has a quadratic isoperimetric function and a decidable word problem [Epstein et. al.].

Theorem 4.1.4. Let $R \subset F(A)$ be the set of reduced words of length at most 10 which represent the identity in Γ . Then a word $w \in F(A)$ represents the identity in Γ if and only if the following equality in $F(A)$ holds:

$$w = \prod_{i=1}^N x_i r_i x_i^{-1} \quad (4.6)$$

where $N \leq (D+1)|w|^2$ and $|x_i| \leq (D+1)|w|$ with each $r_i \in R$.

Proof. For each $\gamma \in \Gamma$ we associate with it a word σ_γ in $F(A)$ as follows. Let $c_\gamma : [0, \lambda] \rightarrow X$ be the unique geodesic path which originates at x_0 and terminates at $\gamma \cdot x_0$. We extend c_γ to a ray by taking $c_\gamma(t) = \gamma \cdot x_0$ for each $t \geq \lambda$.

For each integer $i \geq 0$, since $\Gamma \cdot B(x_0, D/3) = X$, we can find $\sigma_\gamma(i) \in \Gamma$ such that $c_\gamma(i) \in B(\sigma_\gamma(i) \cdot x_0, D/3)$. For $i = 0$, let us take $\sigma_\gamma(i) = 1$ and for each $i \geq d(x_0, \gamma \cdot x_0) = \lambda$ we take $\sigma_\gamma(i) = \gamma$.

For each integer $i > 1$ we make the following observation:

$$\begin{aligned}
 d(\sigma_\gamma(i-1)^{-1}\sigma_\gamma(i) \cdot x_0, x_0) &= d(\sigma_\gamma(i) \cdot x_0, \sigma_\gamma(i-1) \cdot x_0) \\
 &\leq d(\sigma_\gamma(i) \cdot x_0, c_\gamma(i)) + d(c_\gamma(i), c_\gamma(i-1)) \\
 &\quad + d(c_\gamma(i-1), \sigma_\gamma(i-1) \cdot x_0) \\
 &\leq D/3 + 1 + D/3 \\
 &< D + 1
 \end{aligned} \tag{4.7}$$

From this we conclude that $\sigma_\gamma(i-1)^{-1}\sigma_\gamma(i) \in A$ so set $a_i = \sigma_\gamma(i-1)^{-1}\sigma_\gamma(i)$. We can now define the word σ_γ to be $a_1a_2 \cdots a_n$ where $d(\gamma \cdot x_0, x_0) \leq n < d(\gamma \cdot x_0, x_0) + 1$. It is easily verified that

$$\gamma = \prod_{i=1}^n \sigma(i-1)^{-1}\sigma(i) \tag{4.8}$$

and so $\sigma_\gamma \in F(A)$ represents γ in Γ .

Let us now fix an element $\gamma \in \Gamma$ and a letter $b \in A$; let $\gamma' = \gamma b$. We shall be considering the words σ_γ and $\sigma_{\gamma'}$ in $F(A)$ representing γ and γ' respectively, as defined previously. By adding sufficiently many empty characters to the shorter word, we may assume that the two words have the same length, that is $|\sigma_\gamma| = |\sigma_{\gamma'}| = n$ where $\sigma_\gamma = a_1a_2 \cdots a_n$ and $\sigma_{\gamma'} = a'_1a'_2 \cdots a'_n$. Our goal is to show that the distance from $\sigma_\gamma(i) \cdot x_0$ to $\sigma_{\gamma'}(i) \cdot x_0$ is bounded above by $2D + 1$. Note that in adding empty letters to either the word σ_γ or $\sigma_{\gamma'}$ we remain consistent with our choice of $a_i = \sigma_\gamma(i-1)^{-1}\sigma_\gamma(i)$ and $a'_i = \sigma_{\gamma'}(i-1)^{-1}\sigma_{\gamma'}(i)$ since $\sigma_\gamma(i) = \gamma$ for every $i \geq d(\gamma \cdot x_0, x_0)$.

Our goal is to show that the distance between $\sigma_\gamma(i) \cdot x_0$ and $\sigma_{\gamma'}(i) \cdot x_0$ is bounded above by $2D + 1$. This is an analogue of the fellow traveler property [Epstein et. al.].

We begin by making the observation that

$$\begin{aligned} d(\gamma \cdot x_0, \gamma' \cdot x_0) &= d(\gamma \cdot x_0, \gamma b \cdot x_0) \\ &= d(x_0, b \cdot x_0) \\ &\leq D + 1 \end{aligned} \tag{4.9}$$

The last inequality arises due to the fact that b is an element of A .

Let c_γ and $c_{\gamma'}$ be the unique geodesic paths originating at x_0 and terminating at $\gamma \cdot x_0$ and $\gamma' \cdot x_0$ respectively. Just as before, we define $c_\gamma(t) = \gamma \cdot x_0$ for every $t > d(\gamma \cdot x_0, x_0)$ and $c_{\gamma'}(t) = \gamma' \cdot x_0$ for every $t > d(\gamma' \cdot x_0, x_0)$.

The next step is to show that the inequality:

$$d(c_\gamma(t), c_{\gamma'}(t)) \leq D + 1 \tag{4.10}$$

holds for every $t > 0$. Consider the geodesic triangle $\Delta(x_0, \gamma \cdot x_0, \gamma' \cdot x_0)$ and its comparison triangle in $\mathbb{E}^2$, given by $\overline{\Delta}(x_0, \gamma \cdot x_0, \gamma' \cdot x_0) = \Delta(\overline{x_0}, \overline{\gamma \cdot x_0}, \overline{\gamma' \cdot x_0})$. Without loss of generality, we assume that $d(\gamma \cdot x_0, x_0) \leq d(\gamma' \cdot x_0, x_0)$. We further assume that the triangle $\overline{\Delta}$ is not degenerate, for if it were then the CAT(0) inequality would guarantee that the inequality (4.9) holds for every $t > 0$.

Let $a \in [\overline{x_0}, \overline{\gamma' \cdot x_0}]$ be the unique point such that $d(\overline{x_0}, a) = d(\overline{x_0}, \overline{\gamma \cdot x_0})$

and let b be any point on the geodesic segment $[a, \overline{\gamma' \cdot x_0}]$ such that $b \neq \overline{\gamma' \cdot x_0}$.

Figure 4.2: $\Delta(\overline{x_0}, \overline{\gamma \cdot x_0}, \overline{\gamma' \cdot x_0})$ with a and b illustrated

Let α , α' , β , and β' be angles as shown in Figure 4.2. Since the triangle $\Delta(\overline{x_0}, a, \overline{\gamma \cdot x_0})$ is isosceles and non-degenerate, we know that $\alpha < \pi/2$. This implies that $\beta > \pi/2$ and since $\beta' = \beta + \alpha'$ we know that $\beta' > \pi/2$ (note that if the points a and b coincide, then $\beta' = \beta > \pi/2$). Since the angle at b in the non-degenerate triangle $\Delta(b, \overline{\gamma' \cdot x_0}, \overline{\gamma \cdot x_0})$ is obtuse, we conclude that

$$d(b, \overline{\gamma \cdot x_0}) < d(\overline{\gamma' \cdot x_0}, \overline{\gamma \cdot x_0}) \leq D + 1 \quad (4.11)$$

Also, (4.11) holds trivially if $b = \overline{\gamma' \cdot x_0}$. Taking $a = b$, the inequality $d(\overline{\gamma \cdot x_0}, a) \leq D + 1$ holds

Consider any $t > 0$ and let $\overline{c_\gamma(t)}$ and $\overline{c_{\gamma'}(t)}$ be comparison points in $\overline{\Delta}$ for $c_\gamma(t)$ and $c_{\gamma'}(t)$ respectively. We shall split our discussion into two cases:

1. If $t \leq d(\gamma \cdot x_0, x_0)$ then we find that $\overline{c_{\gamma'}(t)}$ lies on the geodesic segment $[\overline{x_0}, a]$ in $\overline{\Delta}$. Furthermore, the geodesic triangle $\Delta(\overline{x_0}, \overline{c_{\gamma}(t)}, \overline{c_{\gamma'}(t)})$ is similar to $\Delta(\overline{x_0}, \overline{\gamma \cdot x_0}, a)$.

Figure 4.3: Comparing isosceles triangles

We now have the following inequalities:

$$\begin{aligned}
 d(c_{\gamma}(t), c_{\gamma'}(t)) &\leq d(\overline{c_{\gamma}(t)}, \overline{c_{\gamma'}(t)}) \\
 &\leq d(\overline{\gamma \cdot x_0}, a) \\
 &\leq D + 1
 \end{aligned} \tag{4.12}$$

The first inequality is a consequence of the CAT(0) condition.

2. The other possibility is that $t > d(\gamma \cdot x_0, x_0)$ in which case $c_{\gamma}(t) = \gamma \cdot x_0$. This means that if we consider the comparison points $\overline{c_{\gamma}(t)}$ and $\overline{c_{\gamma'}(t)}$ for $c_{\gamma}(t)$ and $c_{\gamma'}(t)$ respectively, then $\overline{c_{\gamma}(t)} = \overline{\gamma \cdot x_0}$ and $\overline{c_{\gamma'}(t)} \in [a, \overline{\gamma' \cdot x_0}]$. We now have the following inequality:

$$\begin{aligned}
d(c_\gamma(t), c_{\gamma'}(t)) &\leq d(\overline{c_\gamma(t)}, \overline{c_{\gamma'}(t)}) \\
&= d(\overline{\gamma \cdot x_0}, \overline{c_{\gamma'}(t)}) \\
&\leq D + 1
\end{aligned} \tag{4.13}$$

The first inequality is a consequence of the CAT(0) condition and the last inequality is a consequence of (4.11)

To conclude this part of the proof, for every integer $i \geq 0$ we have the following inequalities:

$$\begin{aligned}
d(\sigma_\gamma(i) \cdot x_0, \sigma_{\gamma'}(i) \cdot x_0) &\leq d(\sigma_\gamma(i) \cdot x_0, c_\gamma(i)) + d(c_\gamma(i), c_{\gamma'}(i)) \\
&\quad + d(c_{\gamma'}(i), \sigma_{\gamma'}(i) \cdot x_0) \\
&\leq D/3 + (D + 1) + D/3 \\
&< 2D + 1
\end{aligned} \tag{4.14}$$

This gives us the inequality $d(\sigma_\gamma(i)^{-1}\sigma_{\gamma'}(i) \cdot x_0, x_0) \leq 2D + 1$ which by Lemma 4.1.2 implies that we can find a four-letter word $\alpha(i)$ in $F(A)$ which represents $\sigma_\gamma(i)^{-1}\sigma_{\gamma'}(i)$ in the group Γ for each integer $1 \leq i \leq n$. For $i = 0$ we take $\alpha(0)$ to be the empty word and for $i = n$ we set $\alpha(n) = b$ which is possible since $\sigma_\gamma(n)^{-1}\sigma_{\gamma'}(n) = \gamma^{-1}\gamma' = b$ in Γ .

To split σ_γ and $\sigma_{\gamma'}$ into subwords in $F(A)$ we let $p_\gamma(i) = a_1a_2 \cdots a_i$ and $p_{\gamma'}(i) = a'_1a'_2 \cdots a'_i$ for each integer $0 \leq i \leq n$. Note that we assign $p_\gamma(0)$ and $p_{\gamma'}(0)$ to both be the empty word. It is now trivial to verify that the following

 Figure 4.4: γ and γ' in the metric Cayley graph of Γ .

telescoping equality holds in the free group $F(A)$:

$$\sigma_\gamma b \sigma_{\gamma'}^{-1} = \prod_{i=0}^{n-1} p_{\gamma'}(i) [\alpha(i)^{-1} a_{i+1} \alpha(i+1) (a'_{i+1})^{-1}] p_{\gamma'}(i)^{-1} \quad (4.15)$$

The word $\alpha(i)^{-1} a_{i+1} \alpha(i+1) (a'_{i+1})^{-1}$ in $F(A)$ represents the following element in Γ :

$$(\sigma_\gamma(i)^{-1} \sigma_{\gamma'}(i))^{-1} (\sigma_\gamma(i)^{-1} \sigma_\gamma(i+1)) (\sigma_\gamma(i+1)^{-1} \sigma_{\gamma'}(i+1)) (\sigma_{\gamma'}(i)^{-1} \sigma_{\gamma'}(i+1))^{-1} \quad (4.16)$$

which reduces to the identity. Furthermore, it has no more than 10 letters so it is a word in our set R .

We can now proceed with the proof of the isoperimetric function. Let $w = b_1 b_2 \cdots b_m \in F(A)$ be a word which represents the identity in Γ and let γ_i be the element in Γ represented by $b_1 b_2 \cdots b_i$ for each $0 \leq i \leq m$, so $\gamma_0 = 1 = \gamma_m$. Denote by $|w|$ the length of the word w . If $|w| \leq 10$ then w can clearly be written in the desired form in $F(A)$ since $w \in R$; this means

we may restrict our attention to $|w| > 10$. We may trivially equate w to the following telescoping product in $F(A)$:

$$w = \prod_{j=1}^m \sigma_{\gamma_{j-1}} b_j \sigma_{\gamma_j}^{-1} \quad (4.17)$$

For each $1 \leq j \leq m$, if we let $\gamma = \gamma_{j-1}$ and $\gamma' = \gamma_j$ then we have by (4.15), the following equality in $F(A)$:

$$\sigma_{\gamma_{j-1}} b_j \sigma_{\gamma_j} = \prod_{i=0}^{n(\gamma_{j-1}, \gamma_j)-1} p_{\gamma_j}(i) R_{ij} p_{\gamma_j}(i)^{-1} \quad (4.18)$$

where $n(\gamma_{j-1}, \gamma_j)$ is the length of the longer word $\sigma_{\gamma_{j-1}}$, or σ_{γ_j} and each R_{ij} is a word in the set R . We have now written w as a product in $F(A)$ of the correct form:

$$w = \prod_{j=0}^m \prod_{i=0}^{n(\gamma_{j-1}, \gamma_j)-1} p_{\gamma_j}(i) R_{ij} p_{\gamma_j}(i)^{-1} \quad (4.19)$$

This means that all that remains to be shown is that $|p_{\gamma_j}(i)| \leq (D+1)|w|$ for each pair i, j in the product, and that the following bound holds:

$$\sum_{j=0}^m n(\gamma_{j-1}, \gamma_j) \leq (D+1)|w|^2 \quad (4.20)$$

To prove that these bounds hold, we first show that the length of each word σ_{γ_j} is less than $\frac{1}{2}(D+1)|w| + 1$. Consider γ_j and note the following, recalling

that $\gamma_0 \cdot x_0 = \gamma_m \cdot x_0 = x_0$

$$\begin{aligned}
 d(\gamma_j \cdot x_0, x_0) &\leq \min \left\{ \begin{array}{l} \sum_{k=1}^j d(\gamma_{k-1} \cdot x_0, \gamma_k \cdot x_0) \\ \sum_{k=j+1}^m d(\gamma_{k-1} \cdot x_0, \gamma_k \cdot x_0) \end{array} \right. \\
 &\leq \min \left\{ \begin{array}{l} j(D+1) \\ (m-j)(D+1) \end{array} \right. \\
 &\leq \frac{1}{2}(j(D+1) + (m-j)(D+1)) \\
 &= \frac{D+1}{2}m \\
 &= \frac{D+1}{2}|w|
 \end{aligned} \tag{4.21}$$

Since $d(\gamma_j \cdot x_0, x_0) < |\sigma_{\gamma_j}| \leq d(\gamma_j \cdot x_0, x_0) + 1$ we may conclude that $|\sigma_{\gamma_j}| < \frac{D+1}{2}|w| + 1$ as desired.

The word $p_{\gamma_j}(i)$ is a subword of σ_{γ_j} and thus satisfies $|p_{\gamma_j}(i)| \leq \frac{(D+1)}{2}|w| + 1 \leq (D+1)|w|$ since $|w| > 10$.

Furthermore, $n(\gamma_j, \gamma_{j-1}) < \frac{1}{2}(D+1)|w| + 1 \leq (D+1)|w|$ so

$$\sum_{j=0}^m n(\gamma_{j-1}, \gamma_j) \leq (D+1)|w|m = (D+1)|w|^2 \tag{4.22}$$

This completes the proof of Theorem 4.1.4. $\square$

Corollary 4.1.5. If a group Γ acts properly and cocompactly on a CAT(0) metric X , then the group Γ has a decidable word problem

Proof. Given the sets A , R and a freely reduced word $w \in F(A)$ as in Theorem 4.1.4, a Turing machine can enumerate the finite list of words of the form (4.6)

within the given upper bounds and compute their freely reduced forms. The Turing machine then checks if w is in the list of reduced words. By Theorem 4.1.4, w represents the identity in Γ if and only if it can be written in the form of (4.6), within the given bounds. Thus, the Turing machine answers yes if w is in the list and no otherwise. $\square$

Chapter 5

Investigations of Isometries

Having presented the basic properties of $\text{CAT}(0)$ spaces in Chapter 2, as well as introduced the fundamental concept of group actions in Chapter 3, we may begin to characterize the isometries of $\text{CAT}(0)$ spaces. In this chapter, we shall be paying special attention to the properties of individual isometries and how they interact with the structure of our metric space. We will always present our isometries as group elements *acting* on our space.

5.1 Displacement Functions and Translation Length

In Section 5.2 we shall divide isometries into three classes, depending on the properties of their *displacement function* and their *translation length*.

Definition 5.1.1. Given a metric space X and an isometry γ of X we define the *displacement function* of γ to be $d_\gamma : X \rightarrow \mathbb{R}$ where for any $x \in X$ the displacement is given by $d_\gamma(x) = d(\gamma \cdot x, x)$.

To get a better impression of how γ affects the entire space X , however, we are more interested in the *translation length* of γ which is given by

$$|\gamma| = \inf\{d_\gamma(x) \mid x \in X\} \quad (5.1)$$

The set of elements in X for which d_γ actually attains this infimum is called $\text{Min}(\gamma)$ and if we are investigating a group Γ of isometries, then we set $\text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma)$.

We can now separate the isometries of a $\text{CAT}(0)$ space into two types, those for which $\text{Min}(\gamma)$ is empty and those for which it is not. In the case that $\text{Min}(\gamma)$ is non-empty we say that γ is a *semi-simple* isometry of X . Later, we shall see that two of our to-be-defined types of isometries are semi-simple while the other is not. Before proceeding however, we shall collect several useful facts about the displacement function, the translation length, and the set $\text{Min}(\gamma)$.

Proposition 5.1.2. Let X be a metric space and Γ a group acting by isometries on X . If γ is an element of Γ , then the following hold:

1. If $\alpha \in \Gamma$ is another isometry of X then the translation length $|\gamma|$ is equal to the translation length $|\alpha\gamma\alpha^{-1}|$ and $\alpha \cdot \text{Min}(\gamma) = \text{Min}(\alpha\gamma\alpha^{-1})$. Thus, if

α and γ commute then $\text{Min}(\gamma)$ is an α -invariant subspace of X and if $N \trianglelefteq \Gamma$ is a normal subgroup, then $\text{Min}(N) = \bigcap_{n \in N} \text{Min}(n)$ is Γ -invariant.

2. The sets $\text{Min}(\gamma)$ and $\text{Min}(\Gamma)$ are γ -invariant.
3. If X is a CAT(0) space, then the displacement function d_γ is convex and $\text{Min}(\gamma)$ is a closed convex subset of X .
4. Given a non-empty, complete, convex, γ -invariant subset C of a CAT(0) space X then the translation length of γ is equal to the translation length of γ when restricted to C . Furthermore, γ is semi-simple if and only if $\gamma|_C$ is semi-simple. In particular, $\text{Min}(\gamma)$ is non-empty if and only if $C \cap \text{Min}(\gamma)$ is non-empty.

Proof. 1. First observe that for all $x \in X$ we have

$$d(\alpha\gamma\alpha^{-1} \cdot (\alpha \cdot x), \alpha \cdot x) = d(\gamma \cdot x, x) \quad (5.2)$$

For any $\varepsilon > 0$ there exists $x \in X$ such that $d(\gamma \cdot x, x) \leq |\gamma| + \varepsilon$. But this implies that $d(\alpha\gamma\alpha^{-1} \cdot (\alpha \cdot x), \alpha \cdot x) \leq |\gamma| + \varepsilon$ by (5.2), so we see that $|\alpha\gamma\alpha^{-1}| \leq |\gamma| + \varepsilon$ for every $\varepsilon > 0$. This tells us that $|\alpha\gamma\alpha^{-1}| \leq |\gamma|$. Using a similar argument, we derive that $|\gamma| \leq |\alpha\gamma\alpha^{-1}|$ and thus the desired equality holds.

To show that $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$, consider $x \in \text{Min}(\gamma)$. We know that the equality, $d(\gamma \cdot x, x) = |\gamma|$ holds and we wish to show that $\alpha \cdot x \in$

$\text{Min}(\alpha\gamma\alpha^{-1})$. By (5.2) we have

$$d(\alpha\gamma\alpha^{-1} \cdot (\alpha \cdot x), \alpha \cdot x) = |\gamma| = |\alpha\gamma\alpha^{-1}| \quad (5.3)$$

so $\alpha \cdot x \in \text{Min}(\alpha\gamma\alpha^{-1})$ and $\alpha \cdot \text{Min}(\gamma) \subseteq \text{Min}(\alpha\gamma\alpha^{-1})$. Dually, we have $\alpha^{-1} \cdot \text{Min}(\alpha\gamma\alpha^{-1}) \subseteq \text{Min}(\gamma)$ so $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$ as desired.

The above equality clearly implies that if α and γ commute, then $\text{Min}(\gamma)$ is α -invariant.

Now, suppose N is a normal subgroup of Γ . Then

$$\begin{aligned} \gamma \cdot \text{Min}(N) &= \bigcap_{n \in N} \gamma \cdot \text{Min}(n) \\ &= \bigcap_{n \in N} \text{Min}(\gamma n \gamma^{-1}) \\ &= \bigcap_{n \in N} \text{Min}(n) \\ &= \text{Min}(N) \end{aligned} \quad (5.4)$$

and $\text{Min}(N)$ is Γ -invariant as desired.

2. Since $\Gamma \trianglelefteq \Gamma$ we may conclude by (1) that $\text{Min}(\Gamma)$ is Γ -invariant and thus γ -invariant. Similarly, since γ commutes with itself, we apply (1) to conclude that $\text{Min}(\gamma)$ is γ -invariant.
3. Fix $\gamma \in \Gamma$; to show that d_γ is convex, let $c : I \rightarrow X$ be a linearly reparameterized geodesic path. We wish to show that for all $t \in I$ the

following inequality holds:

$$d_\gamma(c(t)) \leq (1-t)d_\gamma(c(0)) + td_\gamma(c(1)) \quad (5.5)$$

In a CAT(0) space however, the distance function is convex by Proposition 2.2.3. If we consider the linearly reparameterized geodesic path c along with the linearly reparameterized geodesic path $\gamma \cdot c$ we have the following:

$$\begin{aligned} d_\gamma(c(t)) &= d(\gamma \cdot c(t), c(t)) \\ &\leq (1-t)d((\gamma \cdot c)(0), c(0)) + td((\gamma \cdot c)(1), c(1)) \\ &= (1-t)d(\gamma \cdot c(0), c(0)) + td(\gamma \cdot c(1), c(1)) \\ &= (1-t)d_\gamma(c(0)) + td_\gamma(c(1)) \end{aligned} \quad (5.6)$$

so d_γ is convex as desired.

To show that $\text{Min}(\gamma)$ is convex, consider $x, y \in \text{Min}(\gamma)$ and $z \in [x, y]$. Let $c : [0, 1] \rightarrow X$ be a linearly reparameterized geodesic path with corresponding geodesic segment $[x, y]$ and suppose $z = c(t)$. Then by (5.6) we have

$$\begin{aligned} d_\gamma(z) &\leq (1-t)d_\gamma(x) + td_\gamma(y) \\ &\leq (1-t)|\gamma| + t|\gamma| \\ &= |\gamma| \end{aligned} \quad (5.7)$$

so $z \in \text{Min}(\gamma)$ as desired.

4. We first wish to show that the translation length of γ when restricted to C is equal to $|\gamma|$. Let $\pi : X \rightarrow C$ be the orthogonal projection of X onto C . We begin by showing that π is γ -equivariant. For any $x \in X$ and any $c \in C$, since $\gamma^{-1} \cdot c \in C$ we have that $d(\gamma^{-1} \cdot c, x) \geq d(\pi(x), x)$. Moreover, since γ is an isometry, this now implies that $d(c, \gamma \cdot x) \geq d(\gamma \cdot \pi(x), \gamma \cdot x)$ for every $c \in C$. As C is γ -invariant, we have that $\gamma \cdot \pi(x) \in C$ and hence $\pi(\gamma \cdot x) = \gamma \cdot \pi(x)$.

Next, we shall establish that

$$d(\gamma \cdot x, x) \geq d(\pi(\gamma \cdot x), \pi(x)) = d(\gamma \cdot \pi(x), \pi(x)) \quad (5.8)$$

Consider comparison triangles $\overline{\Delta} = \Delta(\overline{x}, \overline{\pi(x)}, \overline{\pi(\gamma \cdot x)})$ for the triangle $\Delta(x, \pi(x), \pi(\gamma \cdot x))$ and $\overline{\Delta}' = \Delta(\overline{x}, \overline{\gamma \cdot x}, \overline{\pi(\gamma \cdot x)})$ for $\Delta(x, \gamma \cdot x, \pi(\gamma \cdot x))$ such that the segments $[\overline{x}, \overline{\pi(\gamma \cdot x)}]$ coincide in each triangle and $\overline{\gamma \cdot x}$ and $\overline{\pi(x)}$ lie on opposite sides of the geodesic line which runs through $\overline{x}$ and $\overline{\pi(\gamma \cdot x)}$. See Figure 5.1.

Figure 5.1: Constructing a quadrilateral in $\mathbb{E}^2$ for comparison

By the CAT(0) condition, we know that

$$\angle_{\overline{\pi(x)}}(\overline{x}, \overline{\pi(\gamma \cdot x)}) \geq \angle_{\pi(x)}(x, \pi(\gamma \cdot x)) \geq \pi/2 \quad (5.9)$$

where the last inequality is a consequence of Proposition 2.3.1 (3). Furthermore, we have

$$\begin{aligned} \angle_{\overline{\pi(\gamma \cdot x)}}(\overline{\gamma \cdot x}, \overline{\pi(x)}) &= \angle_{\overline{\pi(\gamma \cdot x)}}(\overline{\gamma \cdot x}, \overline{x}) + \angle_{\overline{\pi(\gamma \cdot x)}}(\overline{x}, \overline{\pi(x)}) \\ &\geq \angle_{\pi(\gamma \cdot x)}(\gamma \cdot x, x) + \angle_{\pi(\gamma \cdot x)}(x, \pi(x)) \\ &\geq \angle_{\pi(\gamma \cdot x)}(\gamma \cdot x, \pi(x)) \\ &\geq \pi/2 \end{aligned} \quad (5.10)$$

again by the CAT(0) inequality and Proposition 2.3.1 (3). Thus we have $d(\overline{x}, \overline{\gamma \cdot x}) \geq d(\overline{\pi(x)}, \overline{\pi(\gamma \cdot x)})$ which implies that we have the inequality $d(x, \gamma \cdot x) \geq d(\pi(x), \pi(\gamma \cdot x))$ as desired.

Clearly $|\gamma| \leq |\gamma|_C$. To show the opposite inequality, let $\varepsilon > 0$, then there exists $x \in X$ such that $d(\gamma \cdot x, x) \leq |\gamma| + \varepsilon$. Then by (5.8) we have

$$\begin{aligned} d(\gamma \cdot \pi(x), \pi(x)) &= d(\pi(\gamma \cdot x), \pi(x)) \\ &\leq d(\gamma \cdot x, x) \\ &\leq |\gamma| + \varepsilon \end{aligned} \quad (5.11)$$

Thus $|\gamma|_C \leq d(\gamma \cdot \pi(x), \pi(x)) \leq |\gamma| + \varepsilon$. Since ε was arbitrary, $|\gamma|_C \leq |\gamma|$ and equality holds as desired.

It is now immediate that $\text{Min}(\gamma) \cap C = \text{Min}(\gamma|_C)$.

□

We are now prepared to introduce the three classes of isometries.

5.2 Three Classes of Isometries

We now proceed with dividing the individual isometries of a $\text{CAT}(0)$ space into three different types based upon the properties of their translation lengths.

Definition 5.2.1. Consider a $\text{CAT}(0)$ metric space X and a group Γ which acts on X by isometries. If $\gamma \in \Gamma$ is an individual isometry, then γ may be classified as one of the following:

1. We say that γ is *elliptic* if it has a fixed point. This means that $|\gamma| = 0$ and $\text{Min}(\gamma) \neq \emptyset$.
2. If $\text{Min}(\gamma) \neq \emptyset$ but $|\gamma| > 0$ then we say that γ is *hyperbolic*.
3. Finally, in the case that $\text{Min}(\gamma) = \emptyset$ then γ is said to be *parabolic*.

Note: elliptic and hyperbolic isometries are semi-simple and by the previous proposition, conjugate isometries are of the same type. A semi-simple isometry γ such that $\text{Min}(\gamma) = X$ is called a *Clifford Translations*.

Proposition 5.2.2. Let X be a complete $\text{CAT}(0)$ metric space and let γ be an isometry of X . Then γ is elliptic if and only if γ has a bounded orbit. Furthermore, if γ^n is elliptic for some $n > 0$ then γ is elliptic as well.

Proof. Suppose that γ is elliptic. Therefore we can find an element $x \in X$ such that $\gamma \cdot x = x$. This implies that the orbit of x under γ is the singleton $\{x\}$ which is clearly bounded.

Suppose that γ has a bounded orbit Y . The set Y is Γ -invariant so by the Bruhat-Tits Fixed Point Theorem 3.3.1 (1), γ has a non-empty fixed point set, so γ is elliptic.

To prove the second part of the theorem let γ^n be elliptic. By definition, there exists a point $x \in X$ such that $\gamma^n \cdot x = x$. In particular, this implies that the orbit of x under $\langle \gamma \rangle$ is finite and thus bounded. Applying the first half of the proposition we see that γ is elliptic, as desired. $\square$

Example 5.2.3. Every isometry of $\mathbb{E}^n$ is semi-simple.

Before proceeding, we state a small lemma, which will be used in the proof of the theorem which follows. The proof of this lemma may be found in [Bridson & Haefliger pg. 239].

Lemma 5.2.4. Suppose X is a product of the metric spaces X_1 and X_2 . An isometry $\gamma \in \text{Isom}(X)$ decomposes as a product (γ_1, γ_2) where γ_1 is an isometry of X_1 and γ_2 is an isometry of X_2 if and only if, for every $x_1 \in X_1$, there exists a point $\gamma_1(x_1)$ such that $\gamma_1(\{x_1\} \times X_2) = \{\gamma_1(x_1)\} \times X_2$.

A semi-simple isometry γ induces a splitting on $\text{Min}(\gamma)$ as we shall see in the following theorem.

Theorem 5.2.5. Let X be a complete CAT(0) metric space and consider an isometry γ of X .

1. The isometry γ is hyperbolic if and only if there exists a geodesic line $c : \mathbb{R} \rightarrow X$ whose image is a γ -invariant subspace of X and the action of γ when restricted to $c(\mathbb{R})$ is a non-trivial translation $c(t) \mapsto c(t + |\gamma|)$. Such a geodesic line is called an *axis* of γ .
2. If γ is a hyperbolic isometry of X then the axes of γ are parallel to each other and their union is equal to $\text{Min}(\gamma)$
3. If γ is hyperbolic then $\text{Min}(\gamma)$ splits as a product $Y \times \mathbb{R}$ and the restriction of γ to $\text{Min}(\gamma)$ acts as $(y, t) \mapsto (y, t + |\gamma|)$. Moreover, Y is a convex subset of X and hence $Y \times \mathbb{R}$ is a CAT(0) space.
4. If α is an isometry which commutes with γ then it leaves $\text{Min}(\gamma)$ invariant. Furthermore, its restriction of to $\text{Min}(\gamma) = Y \times \mathbb{R}$ splits as (α', α'') where α' is an isometry of Y and α'' is a translation of $\mathbb{R}$.
5. If we take X to be a complete metric space and γ^m is a hyperbolic isometry for some $m > 0$ then γ is hyperbolic.

Proof. 1. ($\Rightarrow$) Suppose γ is hyperbolic. This means that we can find an element $x \in X$ such that $d(\gamma \cdot x, x) = |\gamma|$. Let $c : [0, |\gamma|] \rightarrow X$ be the geodesic path whose corresponding geodesic segment is $[x, \gamma \cdot x]$. Consider the set of paths $c_i : [0, |\gamma|] \rightarrow X$ defined by $c_i(t) = \gamma^i \cdot c(t)$ for every integer i . Since γ is an isometry, these are geodesic paths and

furthermore, for every integer i we have

$$\begin{aligned}
 c_{i+1}(0) &= \gamma^{i+1}c(0) \\
 &= \gamma^i(\gamma \cdot c(0)) \\
 &= \gamma^i \cdot c(|\gamma|) \\
 &= c_i(|\gamma|)
 \end{aligned} \tag{5.12}$$

This means that we can define a line $\tilde{c} : \mathbb{R} \rightarrow X$ to be the concatenation of the paths c_i . More precisely, if $t \in \mathbb{R}$ and $t = a + b|\gamma|$ where $b \in \mathbb{N}$ and $a \in [0, |\gamma|)$ then $\tilde{c}(t) = c_b(a)$. Our goal is to show that this line is locally a geodesic. Since for every integer i we know $\tilde{c}|_{[i|\gamma|, (i+1)|\gamma|]}$ is a geodesic, it is sufficient to show that for every integer i we have $\tilde{c}|_{[(i-1/2)|\gamma|, (i+1/2)|\gamma|]}$ is a geodesic path. In light of Proposition 1.1.4 it is sufficient to show that

$$\begin{aligned}
 &d(\tilde{c}((i-1/2)|\gamma|), \tilde{c}((i+1/2)|\gamma|)) \\
 &= d(\tilde{c}((i-1/2)|\gamma|), \tilde{c}(i|\gamma|)) + d(\tilde{c}(i|\gamma|), \tilde{c}((i+1/2)|\gamma|))
 \end{aligned} \tag{5.13}$$

From the triangle inequality we know that

$$\begin{aligned}
 &d(\tilde{c}((i-1/2)|\gamma|), \tilde{c}((i+1/2)|\gamma|)) \\
 &\leq d(\tilde{c}((i-1/2)|\gamma|), \tilde{c}(i|\gamma|)) + d(\tilde{c}(i|\gamma|), \tilde{c}((i+1/2)|\gamma|))
 \end{aligned} \tag{5.14}$$

but we also know that

$$\begin{aligned}
 & d(\tilde{c}((i - 1/2)|\gamma|), \tilde{c}(i|\gamma|)) + d(\tilde{c}(i|\gamma|), \tilde{c}((i + 1/2)|\gamma|)) \\
 = & 1/2|\gamma| + 1/2|\gamma| \\
 = & |\gamma| \\
 \leq & d(\tilde{c}((i - 1/2)|\gamma|), \gamma \cdot \tilde{c}((i - 1/2)|\gamma|)) \\
 = & d(\tilde{c}((i - 1/2)|\gamma|), \tilde{c}((i + 1/2)|\gamma|))
 \end{aligned} \tag{5.15}$$

and thus equality holds and we may conclude that $\tilde{c}|_{[(i-1/2)|\gamma|, (i+1/2)|\gamma|]}$ is a geodesic path and $\tilde{c}$ is a locally geodesic line. In CAT(0) spaces however, local geodesics are geodesic which means that $\tilde{c}$ is a geodesic line. Furthermore, from the way we defined $\tilde{c}$ it is clear that the action of γ on the image of $\tilde{c}$ is by translations of length $|\gamma|$ as desired.

($\Leftarrow$) Suppose there exists a geodesic line $c : \mathbb{R} \rightarrow X$ which is γ -invariant and such that γ acts on c by translation. Then $c(\mathbb{R})$ is complete, convex, and γ -invariant and so by Proposition 5.1.2 (4) we have $|\gamma|_{c(\mathbb{R})} = |\gamma|$ and thus γ is hyperbolic. Furthermore, this implies that the translation length of γ is the amount by which γ translates $c(\mathbb{R})$.

2. Let $c : \mathbb{R} \rightarrow X$ and $c' : \mathbb{R} \rightarrow X$ be axes of γ . For any $t \in \mathbb{R}$ we have

$$\begin{aligned}
 d(c(t), c'(t)) &= d(\gamma \cdot c(t), \gamma \cdot c'(t)) \\
 &= d(c(t + |\gamma|), c'(t + |\gamma|))
 \end{aligned} \tag{5.16}$$

This shows that the function $t \mapsto d(c(t), c'(t))$ is periodic and thus

bounded. Furthermore, it is convex so it must be constant by Proposition 2.2.4, implying that c and c' are parallel. Any axis of γ is contained in $\text{Min}(\gamma)$ so to show that $\text{Min}(\gamma)$ is actually equal to the union of all the axes of γ consider a point $x \in \text{Min}(\gamma)$. Using the construction in the proof of (1) we find an axis of γ which contains x . Thus $\text{Min}(\gamma)$ is equal to the union of all the axes of γ as desired.

3. We proved in Proposition 5.1.2 (3) that $\text{Min}(\gamma)$ is closed and convex for any $\gamma \in \Gamma$ so by Remark 2.1.6, it is a $\text{CAT}(0)$ metric space in the induced metric. By applying the Product Decomposition Theorem 2.5.7 to $\text{Min}(\gamma)$ we have $\text{Min}(\gamma) = Y \times \mathbb{R}$ where each $\{y\} \times \mathbb{R}$ is an axis of γ . Furthermore, for any $(y, t) \in Y \times \mathbb{R}$ we have $\gamma \cdot (y, t) = (y, t + |\gamma|)$ as desired.
4. Suppose that α is an isometry of X which commutes with γ . The invariance of $\text{Min}(\gamma)$ is a consequence of Proposition 5.1.2 (1). We now show that α splits as a product (α_1, α_2) .

For each $y \in Y$ we claim that the set $\alpha \cdot (\{y\} \times \mathbb{R})$ is an axis of γ . If we let $c : \mathbb{R} \rightarrow X$ be the map $c(t) = (y, t)$ then c is an axis of γ and $\alpha \cdot c(\mathbb{R}) = \alpha \cdot (\{y\} \times \mathbb{R})$. Thus we need to show that $\alpha \cdot c$ is also an axis of γ . For any $t \in \mathbb{R}$ we have

$$\begin{aligned} \gamma \cdot (\alpha \cdot c(t)) &= \alpha \cdot (\gamma \cdot c(t)) \\ &= \alpha \cdot c(t + |\gamma|) \end{aligned} \tag{5.17}$$

so $\gamma \cdot (\alpha \cdot c(t)) \in \alpha \cdot c(\mathbb{R})$ so $\alpha \cdot c(\mathbb{R})$ is γ -invariant and $\alpha \cdot c$ is an axis of γ and thus $\alpha \cdot (\{y\} \times \mathbb{R}) = \{y'\} \times \mathbb{R}$ for some $y' \in Y$. Thus, if we define $\alpha_1(y)$ for each $y \in Y$ to be given by $\alpha_1(y) = y'$ then we can apply Lemma 5.2.4 to deduce that α splits as (α_1, α_2) where α_1 is an isometry of Y and α_2 is an isometry of $\mathbb{R}$.

Since γ which splits as (id, γ') where γ' is a translation of $\mathbb{R}$ and α_2 commutes with γ' we conclude that α_2 is also a translation of $\mathbb{R}$. This is because the only isometries of $\mathbb{R}$ which commute with a translation are translations.

5. Suppose γ^m is hyperbolic for some $m \in \mathbb{N}$. By (3) $\text{Min}(\gamma^m)$ splits as a product $Y \times \mathbb{R}$ and γ^m splits as (id, γ') where γ' is a translation of $\mathbb{R}$. Since γ^m commutes with γ , by (4) we see that γ splits as (γ_1, γ_2) where γ_1^m is the identity on Y and γ_2 is a translation of $\mathbb{R}$.

Note that $\text{Min}(\gamma)$ is a complete CAT(0) metric space, as it is a closed and convex subspace of X and thus by the Product Decomposition Theorem 2.5.7, Y is a complete CAT(0) space. Since γ_1^m is elliptic, by Proposition 5.2.2, we know that γ_1 is elliptic and hence has a fixed point $c_y \in Y$. Thus γ acts by translations on $\{c_y\} \times \mathbb{R}$ by some constant $k \geq |\gamma|$. The slice $\{c_y\} \times \mathbb{R}$ is an axis of γ^m , however, so $|\gamma^m| = mk \geq m|\gamma|$. By the triangle inequality, we have $mk = |\gamma^m| \leq m|\gamma| = mk$ and thus the equality $|\gamma| = k$ holds and γ is semi-simple. Since $mk = |\gamma^m| > 0$ we deduce that $k > 0$ and so γ is hyperbolic.

□

Proposition 5.2.6. Let X be a metric space which splits as the product $X_1 \times X_2$ and suppose γ is an isometry which splits as a pair of isometries (γ_1, γ_2) preserving this splitting. Then the equality $\text{Min}(\gamma) = \text{Min}(\gamma_1) \times \text{Min}(\gamma_2)$ holds. In particular, this implies that γ is semi-simple if and only if both γ_1 and γ_2 are semi-simple.

Proof. Let $x_1, y_1 \in X_1$ and let $x_2, y_2 \in X_2$ then the following two properties hold:

$$\begin{aligned} d(\gamma_1(x_1), x_1) &\leq d(\gamma_1(y_1), y_1) \\ \iff d(\gamma(x_1, x_2), (x_1, x_2)) &\leq d(\gamma(y_1, x_2), (y_1, x_2)) \end{aligned} \quad (5.18)$$

and

$$\begin{aligned} d(\gamma_2(x_2), x_2) &\leq d(\gamma_2(y_2), y_2) \\ \iff d(\gamma(x_1, x_2), (x_1, x_2)) &\leq d(\gamma(x_1, y_2), (x_1, y_2)) \end{aligned} \quad (5.19)$$

Take $(x_1, x_2) \in \text{Min}(\gamma)$, then for any $y_1 \in X_1$ we have

$$d(\gamma(x_1, x_2), (x_1, x_2)) \leq d(\gamma(x_1, y_2), (x_1, y_2)) \quad (5.20)$$

which using (5.19) implies that

$$d(\gamma_2(x_2), x_2) \leq d(\gamma_2(y_2), y_2) \quad (5.21)$$

so $x_2 \in \text{Min}(\gamma_2)$. Similarly, using (5.18) one shows that $x_1 \in \text{Min}(\gamma_1)$.

Conversely, let us take $(x_1, x_2) \in \text{Min}(\gamma_1) \times \text{Min}(\gamma_2)$ and consider any pair

$(y_1, y_2) \in X_1 \times X_2$. We have

$$\begin{aligned}
 d(\gamma(x_1, x_2), (x_1, x_2))^2 &= d(\gamma_1(x_1), x_1)^2 + d(\gamma_2(x_2), x_2)^2 \\
 &\leq d(\gamma_1(y_1), y_1)^2 + d(\gamma_2(y_2), y_2)^2 \\
 &= d(\gamma(y_1, y_2), (y_1, y_2))^2
 \end{aligned} \tag{5.22}$$

Thus, $(x_1, x_2) \in \text{Min}(\gamma)$ and we conclude that $\text{Min}(\gamma) = \text{Min}(\gamma_1) \times \text{Min}(\gamma_2)$ as desired.

□

Lemma 5.2.7. Let X be a CAT(0) space with the geodesic extension property. Let Γ be a group which acts cocompactly by isometries on X and $\alpha \in \text{Isom}(X)$ be an isometry which commutes with every isometry in Γ ; then $\text{Min}(\alpha) = X$ (so α is a Clifford Translation).

Proof. Consider the displacement function d_α of α . For any $x \in X$ and any $\gamma \in \Gamma$ we have

$$\begin{aligned}
 d_\alpha(\gamma \cdot x) &= d(\alpha \cdot (\gamma \cdot x), \gamma \cdot x) \\
 &= d(\gamma \alpha \cdot x, \gamma \cdot x) \\
 &= d(\alpha \cdot x, x) \\
 &= d_\alpha(x)
 \end{aligned} \tag{5.23}$$

which implies that the displacement function d_α is Γ -equivariant. Since the action of Γ is cocompact, there exists a compact subset K such that $\Gamma \cdot K = X$. In particular, we can find a ball $B(x_0, r) \supseteq K$ which gives $\Gamma \cdot B(x_0, r) = X$. Let $d = d(\alpha \cdot x_0, x_0)$. For each $x \in X$ there exists $\gamma \in \Gamma$ such that $\gamma \cdot x = x' \in$

$B(x_0, r)$. This gives

$$\begin{aligned}
 d_\alpha(x) &= d_\alpha(\gamma \cdot x) \\
 &= d(x', \alpha \cdot x') \\
 &\leq d(x', x_0) + d(x_0, \alpha \cdot x_0) + d(\alpha \cdot x_0, \alpha \cdot x') \\
 &\leq r + d + r
 \end{aligned} \tag{5.24}$$

so d_α is bounded.

The function d_α is convex and so, since X has the geodesic extension property, it must be constant by Proposition 2.2.4. This gives us the desired result that $d_\alpha(x) = d_\alpha(y)$ for every $x, y \in X$, so $\text{Min}(\alpha) = X$. $\square$

Chapter 6

Groups of Isometries

In the last chapter we examined the properties of individual isometries of a $\text{CAT}(0)$ space X and what they reveal about the structure of this space. In this chapter we will look more broadly at entire groups of isometries which act properly and cocompactly on a $\text{CAT}(0)$ space X to find some relationships between the structure of the group and the structure of the space.

6.1 More on the Structure of Isometries

We shall first collect some properties of proper and cocompact actions of a group on a metric space X .

Proposition 6.1.1. Let X be a metric space and suppose that Γ is a group which acts properly by isometries on X .

1. If the action of Γ on X is cocompact then any $\gamma \in \Gamma$ is a semi-simple

isometry of X

2. Assume that X splits isometrically as a product $X' \times X''$ and that each element γ of Γ splits as (γ', γ'') . Suppose that N is a normal subgroup of Γ formed by elements of the form $(id_{X'}, n)$ for which there exists a compact subset $K \subseteq X''$ such that $\bigcup_{(id_{X'}, n) \in N} n \cdot K = X''$. Then the induced action of Γ/N on X' is proper (where in the induced action, $(\gamma', \gamma'')N$ acts as γ' on X').

Proof. 1. Let K be a compact set such that $\Gamma \cdot K = X$ and take a sequence of points $\{x_n\} \in X$ such that $d_\gamma(x_n) \rightarrow |\gamma|$ as $n \rightarrow \infty$. Since $\Gamma \cdot K = X$, for each x_n we can find $\gamma_n \in \Gamma$ such that $\gamma_n \cdot x_n \in K$; let $y_n = \gamma_n \cdot x_n$ for each n . We note that

$$\begin{aligned} d(\gamma_n \gamma \gamma_n^{-1} \cdot y_n, y_n) &= d(\gamma \gamma_n^{-1} \cdot y_n, \gamma_n^{-1} \cdot y_n) \\ &= d(\gamma \cdot x_n, x_n) \\ &\rightarrow |\gamma| \end{aligned} \tag{6.1}$$

as $n \rightarrow \infty$. Furthermore, for every $x \in K$ we have

$$\begin{aligned} &d(\gamma_n \gamma \gamma_n^{-1} \cdot x, x) \\ &\leq d(\gamma_n \gamma \gamma_n^{-1} \cdot x, \gamma_n \gamma \gamma_n^{-1} \cdot y_n) + d(\gamma_n \gamma \gamma_n^{-1} \cdot y_n, y_n) + d(y_n, x) \\ &\leq 2\text{diam}(K) + d(\gamma \cdot x_n, x_n) \end{aligned} \tag{6.2}$$

so we see that the sequence $\{d(\gamma_n \gamma \gamma_n^{-1} \cdot x, x)\}$ is bounded by a constant M for each independent $x \in K$. Let $x_0 \in K$ such that $K \subseteq B(x_0, R)$.

Then for every n , we have $\gamma_n \gamma \gamma_n^{-1} \overline{B}(x_0, R + M) \cap \overline{B}(x_0, R + M) \neq \emptyset$ by (6.2). By Remark 3.1.6 X is proper so by the properness of the action, this means that the set $\{\gamma_n \gamma \gamma_n^{-1} \mid n > 0\}$ is finite. By passing to a subsequence, we may assume that $\gamma_n \gamma \gamma_n^{-1} = \overline{\gamma}$ for every $n > 0$ and some $\overline{\gamma} \in \Gamma$. Since K is compact, by passing to another subsequence we may assume that the sequence $\{\gamma_n \cdot x_n\}$ converges to $y \in K$. This gives

$$\begin{aligned}
 d(\gamma_1^{-1} \cdot y, \gamma \gamma_1^{-1} \cdot y) &= d(\gamma_1 \gamma \gamma_1^{-1} \cdot y, y) \\
 &= \lim_{n \rightarrow \infty} d(\overline{\gamma} \cdot y_n, y_n) \\
 &= \lim_{n \rightarrow \infty} d(\gamma \gamma_n^{-1} \cdot y_n, \gamma_n^{-1} \cdot y_n) \\
 &= \lim_{n \rightarrow \infty} d(\gamma \cdot x_n, x_n) \\
 &= |\gamma|
 \end{aligned} \tag{6.3}$$

Thus γ is semi-simple as desired.

2. Let $x' \in X'$ be any point. We wish to show that there exists $\varepsilon > 0$ such that the following set is finite:

$$S = \{(\gamma', \gamma'')N \in \Gamma/N \mid \gamma' \cdot B(x', \varepsilon) \cap B(x', \varepsilon) \neq \emptyset\} \tag{6.4}$$

Since the action of Γ is proper by Remark 3.1.4, there exists $\varepsilon > 0$ such that the set $A = \{\gamma \in \Gamma \mid \gamma \cdot (B(x', \varepsilon) \times K) \cap (B(x', \varepsilon) \times K) \neq \emptyset\}$ is finite.

Suppose $(\gamma', \gamma'')N \in \Gamma/N$ is such that $\gamma' \cdot B(x', \varepsilon) \cap B(x', \varepsilon) \neq \emptyset$. Let

$x'' \in K$. Since $\bigcup_{(id_{X'}, n) \in N} n \cdot K = X''$, there exists $(id_{X'}, n) \in N$ such that $n\gamma'' \cdot x'' \in K$. Thus we have both $\gamma' \cdot B(x', \varepsilon) \cap B(x', \varepsilon) \neq \emptyset$ and $n\gamma'' \cdot K \cap K \neq \emptyset$ which means that $(\gamma', n\gamma'')$ is in the set A . Since each coset in the set S has a representative in the finite set A , we can only have finitely many elements $(\gamma', \gamma'')N$ in S and can conclude that the action of Γ/N on X' is proper.

□

Theorem 6.1.2. Let X be a CAT(0) metric space and let Γ be a finitely generated group acting faithfully on X by hyperbolic isometries (aside from the identity). Suppose Γ has a central subgroup $A \cong \mathbb{Z}^n$. Then there exists a finite index subgroup $H \subseteq \Gamma$ which contains A as a direct factor.

Proof. We shall prove the result by induction on n , the rank of A . If the rank of A is zero then A is trivial and we may take $H = \Gamma$. Now assume that for any subgroup $A' \cong \mathbb{Z}^j$ of any group Γ' acting faithfully by hyperbolic isometries on a CAT(0) space, with $j \leq n - 1$, the claim holds.

Suppose that A has rank n and fix a non-trivial element $\alpha \in A$. Since α is in the centre of Γ , $\text{Min}(\alpha)$ is Γ -invariant. Furthermore, $\text{Min}(\alpha)$ splits as a product $Y \times \mathbb{R}$ and the restriction of each $\gamma \in \Gamma$ to $Y \times \mathbb{R}$ splits as (γ', γ'') where γ'' is a translation of $\mathbb{R}$ by Theorem 5.2.5. In particular, α acts as a non-trivial translation on $\{y\} \times \mathbb{R}$ for each $y \in Y$.

We can now map Γ onto a finitely generated group of translations of $\mathbb{R}$ via the homomorphism $\gamma \mapsto \gamma''$. Since Γ is finitely generated, this group

of translations is isomorphic to $\mathbb{Z}^m$ for some integer $m > 0$, so we define $\psi : \Gamma \rightarrow \mathbb{Z}^m$ in the obvious manner. The image $\psi(\Gamma)$ is non-trivial because $\psi(\alpha)$ is non-trivial.

Since the image $\psi(\alpha)$ is non-trivial, we may find a suitable direct factor in $\mathbb{Z}^m$ such that if $\phi : \Gamma \rightarrow \mathbb{Z}$ is the composition of ψ with the projection onto that factor, then the image $\phi(A)$ is non-trivial. We may now find an element $a \in A$ such that $\phi(a)$ generates $\phi(A)$. Let us say $\phi(a) = i$ and $\phi(A) = i\mathbb{Z}$, $i \neq 0$.

Let $H_0 = \phi^{-1}(\phi(A)) = \phi^{-1}(i\mathbb{Z})$. Then $[\Gamma : H_0] = [\mathbb{Z} : i\mathbb{Z}] = i$ and so H_0 has finite index in Γ .

Let $f : \phi(A) \rightarrow H_0$ be given by $\phi(a) \mapsto a$. This function induces a homomorphism as $\phi(a)$ freely generates $\phi(A)$. We then have the following split short exact sequence where $K = \ker \phi$.

$$1 \rightarrow K \rightarrow H_0 \rightarrow \phi(A) \rightarrow 1 \quad (6.5)$$

Clearly $\langle a \rangle \cap K$ is trivial. Thus we have that H_0 is the semidirect product of K and $\langle a \rangle$. But as a is central, this reduces to the direct product, $H_0 = K \times \langle a \rangle$.

Thus $A = (K \cap A) \times \langle a \rangle$, so if we let $A' = K \cap A$ then since $A = A' \times \langle a \rangle$ we know that A' has rank $n - 1$. So by the inductive hypothesis applied to A' and K there exists a finite index subgroup H' of K such that H' splits as $H' = H'' \times A'$. This gives us $H = H' \times \langle a \rangle = H'' \times A' \times \langle a \rangle = H'' \times A$ so all that is left to show is that H has finite index in Γ .

We know however, that H' has finite index in $\ker \phi$ so $H' \times \langle a \rangle$ has finite index in $K \times \langle a \rangle$. But $H = H' \times \langle a \rangle$ and $H_0 = K \times \langle a \rangle$, so H has finite index in H_0 . Since H_0 has finite index in Γ we may conclude that H has finite index in Γ to complete the proof. $\square$

As a corollary to the first part of the proof to Theorem 6.1.2 we have the following:

Corollary 6.1.3. If Γ is a finitely generated group acting by isometries on a CAT(0) space X and the centre of Γ contains an element acting by a hyperbolic isometry, then X contains a subspace isometric to $\mathbb{R}$ and there is a non-trivial homomorphism $\psi : \Gamma \rightarrow \mathbb{R}$.

The above corollary applies in particular when Γ acts properly and cocompactly on a CAT(0) space and Γ has a central element of infinite order by Proposition 6.1.1

6.2 Splitting CAT(0) Spaces

The goal of this section is to prove a splitting theorem for CAT(0) metric spaces. We begin this section by presenting a lemma which is used in the proof of the splitting theorem. The reader can refer to [Bridson & Haefliger pg. 239] for the proof.

Lemma 6.2.1. Suppose that X is a CAT(0) space with the geodesic extension property and that Γ is a group which acts properly and cocompactly by

isometries on X . If Γ has a torsion-free subgroup of finite index and trivial centre, then the centralizer of Γ in $\text{Isom}(X)$ is trivial.

The following lemma outlines another property which will be used later.

Lemma 6.2.2. Suppose X is a complete CAT(0) metric space which has the geodesic extension property. If Γ is a group which acts cocompactly on X by isometries then the only closed convex, Γ -invariant subsets of X are the empty set and the entire space X .

Proof. Let C be any non-empty, closed, convex, Γ -invariant subset of X . We wish to show that C is equal to the entire space X . Let $\pi : X \rightarrow C$ be the orthogonal projection onto C .

Since the action is cocompact there exists a ball $B(x_0, r/2) \subseteq X$ such that $\Gamma \cdot B(x_0, r/2) = X$. In particular, as C is Γ -invariant, $B(x_0, r/2) \cap C \neq \emptyset$. For every $x \in X$ we can find an element $\gamma \in \Gamma$ such that $\gamma \cdot x \in B(x_0, r/2)$. Thus $d(\gamma \cdot x, C) \leq r$ so by the Γ -invariance of C , we have that $d(x, C) \leq r$ for every $x \in X$.

Now suppose that there exists $x \in X - C$ and consider the non-constant geodesic segment $[x, \pi(x)]$. Since X has the geodesic extension property, we can extend this geodesic segment to a geodesic ray $c : [0, \infty] \rightarrow X$ with $c(0) = \pi(x)$. For any $t > 0$ we have $\pi(c(t)) = \pi(x)$ by Proposition 2.3.1(2). This implies that $d(c(r+1), C) = r+1$ contradicting our conclusion that $d(z, C) \leq r$ for every $z \in X$. This means that no such $x \in X - C$ exists and $C = X$ as desired. $\square$

Lemma 6.2.3. Let Γ be a group acting by isometries on a CAT(0) space X . Let C be a Γ -invariant, closed, convex subspace of X . Then the projection $p : X \rightarrow C$ is Γ -equivariant.

Proof. Let $x \in X$, then we have

$$d(\gamma \cdot x, p(\gamma \cdot x)) \leq d(\gamma \cdot x, \gamma \cdot p(x)) \quad (6.6)$$

Similarly,

$$d(\gamma \cdot x, \gamma \cdot p(x)) = d(x, p(x)) \leq d(x, \gamma^{-1} \cdot p(\gamma \cdot x)) = d(\gamma \cdot x, p(\gamma \cdot x)) \quad (6.7)$$

Thus we combine (6.6) and (6.7) to deduce that $d(\gamma \cdot x, p(\gamma \cdot x)) = d(\gamma \cdot x, \gamma \cdot p(x))$ and since projections are unique, we conclude that $p(\gamma \cdot x) = \gamma \cdot p(x)$. $\square$

Before presenting the splitting theorem, we need two technical results that will be used in the proof.

Proposition 6.2.4. Let X be a complete CAT(0) space with the geodesic extension property and let Γ be a group which acts properly and cocompactly on X . If the group Γ splits as $\Gamma = \Gamma_1 \times \Gamma_2$ and there exists a closed convex hull C of a Γ_1 -orbit in X such that the action of Γ_1 on C is cocompact, then X splits as a product of metric spaces $X_1 \times X_2$ and Γ preserves this splitting. Furthermore, for this splitting the closed convex hulls of the Γ_1 -orbits are precisely the sets of the form $X_1 \times \{x_2\}$.

Proof. Let Σ be the set of all non-empty closed, convex, Γ_1 -invariant subspaces of X and let N be the subset of Σ containing all those sets which are minimal in Σ with respect to inclusion.

If C_1 and C_2 are in Σ then they are both closed, convex, and Γ_1 -invariant so their intersection $C_1 \cap C_2$ is also closed, convex, and Γ_1 -invariant and thus in Σ . This implies that since the intersection of any two elements of Σ is again in Σ , we have that the elements of N are disjoint.

Additionally, for any $C' \in N$ if $x \in C'$ is an element then the closed convex hull of the Γ_1 -orbit of x , $C(\Gamma_1 \cdot x)$ is contained in Σ and is also a subset of C' by definition. Since C' is minimal under inclusion however, we conclude that the equality $C' = C(\Gamma_1 \cdot x)$ holds and thus all the elements in N are the closed convex hulls of some Γ_1 -orbit in X .

Claim 1. The set N is non-empty.

Let us take C to be the closed, convex hull of a Γ_1 -orbit as in the proposition and let K be a compact subset of X such that $\Gamma_1 \cdot K = C$. Let M be the set of subspaces in Σ which are contained in C and consider a decreasing sequence of such subspaces in M :

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq C_3 \dots \quad (6.8)$$

For each C_n in the sequence we have $C_n \subseteq C = \Gamma_1 \cdot K$. Since C_n is non-empty, we pick $x_n \in C_n$ and find $\gamma_n \in \Gamma_1$ such that $\gamma_n \cdot x_n \in K$. Since C_n is Γ_1 -invariant however, we have $\gamma_n \cdot x_n \in C_n$ so $C_n \cap K \neq \emptyset$.

It follows that

$$C_0 \cap K \supseteq C_1 \cap K \supseteq C_2 \cap K \supseteq \dots \quad (6.9)$$

is a decreasing chain of closed subsets of K and hence has a non-empty intersection by the compactness of K . Thus $C' = \bigcap_{i=0}^{\infty} C_n \neq \emptyset$.

As an intersection of closed sets, C' is itself closed. To show that C' is convex, consider $x, y \in C'$. We have $x, y \in C_n$ for every $n \geq 0$ and since C_n is convex for each n , we have $[x, y] \in C_n$. So $[x, y] \in C'$ and C' is convex. Similarly, for every $x \in C'$ and $\gamma \in \Gamma_1$ we know that $x \in C_n$, which by the Γ -invariance of C_n implies that $\gamma \cdot x \in C_n$ for each $n \geq 0$. Thus $\gamma \cdot x \in C'$, so C' is Γ_1 -invariant. This tells us that C' is an element of Σ but $C' \subseteq C$ so $C' \in M$. By Zorn's Lemma, we can now conclude that M has a minimal element. This element will also be minimal in Σ so we have that N is non-empty as desired. This completes the proof of Claim 1.

Consider two elements C_1, C_2 of N and let $p_i : X \rightarrow C_i$ be the orthogonal projection. Let $d = d(C_1, C_2)$ be the distance between the two sets.

Claim 2. There is a unique isometry $j : C_1 \times [0, d] \rightarrow C(C_1 \cup C_2)$ such that $j(x, 0) = x$ and $j(x, 1) = p_2(x)$.

Consider the function $d_{C_1} : C_2 \rightarrow \mathbb{R}$ given by $d_{C_1}(x) = d(x, C_1)$ for each $x \in C_2$. We claim that this function is constant. If it were not constant, then there would exist elements $x, y \in C_2$ such that $d_{C_1}(x) < d_{C_1}(y)$. Let us consider the set $A = \{z \in C_2 : d_{C_1}(z) \leq d_{C_1}(x)\}$, note that $x \in A$ and

$y \notin A$. Since the function d_{C_1} is continuous, this set is clearly closed in C_2 and thus closed in X . Furthermore, the convexity of d_{C_1} implies that A is convex. Indeed, if $z_1, z_2 \in A$ and $c : [0, 1] \rightarrow C_2$ is a linearly reparamaterized geodesic path originating at z_1 and terminating at z_2 , then by the convexity of d_{C_1} , for any $t \in [0, 1]$ we have

$$d_{C_1}(c(t)) \leq (1-t)d_{C_1}(z_1) + td_{C_1}(z_2) \leq (1-t)d_{C_1}(x) + td_{C_1}(x) = d_{C_1}(x) \quad (6.10)$$

thus $c(t) \in A$ and $[z_1, z_2] \subseteq A$ so we conclude that A is convex.

Moreover, for any $\gamma \in \Gamma_1$ we have

$$\begin{aligned} d_{C_1}(\gamma \cdot x) &= d(\gamma \cdot x, C_1) \\ &= d(x, C_1) \\ &= d_{C_1}(x) \end{aligned} \quad (6.11)$$

so A is Γ_1 -invariant. Thus A is a non-empty, closed, convex, Γ_1 -invariant subspace strictly contained in C_2 which contradicts the minimality of C_2 . Thus we may conclude that d_{C_1} is constant on C_2 , so $d_{C_1}(x) = d$ for all $x \in C_2$.

We shall now define an isometry $j : C_1 \times [0, d] \rightarrow C(C_1 \cup C_2)$ as follows. For any $(x, t) \in C_1 \times [0, d]$ we let $j(x, t) = c_x(t)$ where we define the path $c_x : [0, d] \rightarrow C(C_1 \cup C_2)$ to be the unique geodesic path which originates at x and terminates at $p_2(x)$. Note that we are only able to define this map because $d(x, p_2(x)) = d$.

To show that this map is an isometry, consider any two points $(x_1, t_1),$

$(x_2, t_2) \in C_1 \times [0, d]$. If we consider the quadrilateral formed by the geodesic segments $[x_1, x_2]$, $[x_1, p_2(x_1)]$, $[p_2(x_1), p_2(x_2)]$, and $[p_2(x_2), x_2]$ then by Proposition (2.3.1)(3) we see that each angle in this quadrilateral is greater than or equal to $\pi/2$ which guarantees that the sum of these angles is greater than or equal to 2π . We can now apply the Flat Quadrilaterals Theorem 2.5.3 to see that j , when restricted to $[x_1, x_2] \times [0, d]$ is in fact an isometry, so $d(j((x_1, t_1)), j((x_2, t_2))) = d((x_1, t_1), (x_2, t_2))$ and $C(C_1 \cup C_2)$ is isometric to $C_1 \times [0, d]$. This completes the proof of Claim 2.

Claim 3. For every $x \in X$ there exists a unique element of N , denoted C_x which contains x . The set C_x is equal to the closed convex hull of the Γ_1 -orbit of x , $C(\Gamma_1 \cdot x)$.

If such a C_x exists for each $x \in X$, its uniqueness is clear as the elements of N are disjoint. To show that such a C_x exists for each $x \in X$ we let $B = \bigcup \{C_i : C_i \in N\}$. It is sufficient to show that $B = X$; this follows from Lemma 6.2.2 once we show that B is closed, convex, non-empty and Γ -invariant.

Clearly B is non-empty. To show that B is Γ -invariant, note that for each $x \in B$ the set $C_x = C(\Gamma_1 \cdot x)$ is contained in N . Then for each $\gamma \in \Gamma$, it is sufficient to show that $\gamma \cdot C_x \in N$. We know that $\gamma = \alpha\beta$ where $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$ so $\gamma \cdot C_x = \beta \cdot C_x$ since C_x is Γ_1 -invariant. Since β is an isometry, we know that $\beta \cdot C_x = C(\beta \cdot \Gamma_1 \cdot x) = C(\Gamma_1 \cdot (\beta \cdot x))$ is a closed convex Γ_1 -invariant subspace of X . This set is also minimal in Σ for if it were not, then there would exist a non-empty closed convex Γ_1 -invariant subspace C' of X

such that $C' \subset C(\Gamma_1 \cdot \beta \cdot x)$. This would give $\beta^{-1} \cdot C' \subset C$ contradicting the minimality of C . Thus B is Γ -invariant.

To show that B is convex, consider $x, y \in B$, so $C_x, C_y \in N$. By Claim 2 we know that $C(C_1 \cup C_2)$ is isometric to $C_1 \times [0, d]$ via an isometry $j : C_1 \times [0, d] \rightarrow C(C_1 \cup C_2)$. Furthermore, any subspace of the form $j(C_1 \times \{t\})$ is Γ_1 -invariant because each element $\gamma \in \Gamma_1$ takes the geodesic segment $[x, p_2(x)]$ isometrically to the geodesic segment $[\gamma \cdot x, \gamma \cdot p_2(x)] = [\gamma \cdot x, p_2(\gamma \cdot x)]$. Moreover, $j(C_1 \times \{t\})$ is closed, convex, and non-empty. If it were not minimal then its projection onto C_1 would contradict the minimality of C_1 . Thus $C(C_x \cup C_y) \subseteq B$ and B is convex.

Now we prove that B is closed. Let $\{x_n\}$ be a sequence of points in B which converges to $x \in X$. We wish to show that $x \in B$. By passing to a subsequence, we may assume that $d(x_n, x_{n+1}) \leq 1/2^n$ for each integer $n > 0$. For each element x_n in our sequence, let C_n be the element of N containing x_n and let $p_n : X \rightarrow C_n$ be the orthogonal projection. Consider the composition, $p_n p_{n-1} \cdots p_3 p_2 : X \rightarrow C_n$ and let $P_n : C_1 \rightarrow X$ be the restriction of this function to C_1 .

For each $z \in C_1$ the sequence $P_n(z)$ satisfies the following inequality:

$$\begin{aligned}
 d(P_n(z), P_{n+k}(z)) &\leq d(P_n(z), P_{n+1}(z)) + d(P_{n+1}(z), P_{n+2}(z)) \\
 &\quad d(P_{n+2}(z), P_{n+3}(z)) + \cdots + d(P_{n+k-1}(z), P_{n+k}(z)) \\
 &= d(C_n, C_{n+1}) + \cdots + d(C_{n+k-1}, C_{n+k}) \\
 &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\
 &\leq 1/2^{n-1}
 \end{aligned} \tag{6.12}$$

Thus $\{P_n\}$ is a Cauchy sequence in the space of continuous functions from C_1 to X with the uniform metric. Since X is complete, so is the function space [Munkres] and thus P_n converges to a continuous function $P : C_1 \rightarrow X$. Moreover, P is Γ_1 -equivariant since for every $y \in C_1$ and $\gamma \in \Gamma_1$ we have the following:

$$\begin{aligned}
 P(\gamma \cdot y) &= \lim_{n \rightarrow \infty} P_n(\gamma \cdot y) \\
 &= \lim_{n \rightarrow \infty} \gamma \cdot P_n(y) \\
 &= \gamma \cdot \lim_{n \rightarrow \infty} P_n(y) \\
 &= \gamma \cdot P(y)
 \end{aligned} \tag{6.13}$$

where the second equality is because P_n is Γ_1 -equivariant and the third equality is because γ acts as a continuous map. Now we shall show that P is an

isometry. For any $y, z \in C_1$ we have

$$\begin{aligned} d(P(y), P(z)) &= d(\lim_{n \rightarrow \infty} (P_n(y), P_n(z))) \\ &= \lim_{n \rightarrow \infty} d(P_n(y), P_n(z)) \\ &= d(y, z) \end{aligned} \tag{6.14}$$

This implies that $P(C_1)$ is not only Γ_1 -invariant, but also closed, convex and non-empty. Furthermore, it is minimal in Σ as P^{-1} would take any closed convex Γ_1 -invariant subset of $P(C_1)$ to an analogous subset of C_1 . Thus we conclude that $P(C_1) \in N$. The final step is to show that $x \in P(C_1)$, so we shall construct a sequence in the closed set $P(C_1)$ converging to x .

For each x_n in our original sequence, let $z_n \in C_1$ be the unique element such that $P_n(z_n) = x_n$. Then we find that

$$\begin{aligned} d(P(z_n), x) &\leq d(P(z_n), x_n) + d(x_n, x) \\ &= d(P(z_n), P_n(z_n)) + d(x_n, x) \\ &\leq 1/2^{n-1} + d(x_n, x) \end{aligned} \tag{6.15}$$

which decreases to zero as $n \rightarrow \infty$ so we conclude that $x \in P(C_1)$ and thus B is closed. By Proposition 6.2.2 we now know that $X = B$ so for all $x \in X$ we may find $C_x \in N$ such that $x \in C_x$.

Claim 4. For any three sets $C_1, C_2, C_3 \in N$ if we consider the projection maps $p_i : X \rightarrow C_i$ then the equality $p_1 = p_1 p_2$ holds when restricted to C_3

If each $C_i \in N$ is a single point then the claim holds trivially. Otherwise,

we may assume that some element of N , and hence each element of N by Claim 2, has at least two points. This means that for each $x \in C_3$ we can find a distinct element $y \in C_3$ and thus extend the geodesic segment $[x, y]$ to a non-trivial geodesic line $c : \mathbb{R} \rightarrow X$. For any $C \in N$ such that $C \neq C_3$, we know by Claim 2 that $C(C_3 \cup C)$ is isometric to $C_3 \times [0, d]$ where $d = d(C_3, C)$. This implies that $c(\mathbb{R})$ cannot intersect C and thus the line c is completely contained within C_3 as $X = \bigcap N$. Claim 2 also guarantees that the lines p_1c and p_2c are asymptotic to c and thus, by Proposition 2.5.6, we know that the following equality holds

$$p_{3,1}|_{p_1c(\mathbb{R})} \circ p_{1,2}|_{p_2c(\mathbb{R})} \circ p_{2,3}|_{c(\mathbb{R})} = p_{3,3}|_{c(\mathbb{R})} \quad (6.16)$$

and so

$$p_{1,2}|_{p_2c(\mathbb{R})} \circ p_{2,3}|_{c(\mathbb{R})} = p_{1,3}|_{c(\mathbb{R})} \quad (6.17)$$

Therefore $p_1(x) = p_1p_2(x)$ for every $x \in C_3$ and thus $p_1 = p_1p_2$ when restricted to C_3 , as desired. This completes the proof of Claim 4.

We are now prepared to define the splitting of our CAT(0) space X . Fix any $X_1 \in N$ and let $p : X \rightarrow X_1$ be the orthogonal projection. Then p is Γ_1 -equivariant by Lemma 6.2.3.

Let X_2 be the metric space (N, d') where the distance function is given by

$$d'(C, C') = \inf\{d(x, x') \mid x \in C, x' \in C'\} \quad (6.18)$$

Note that Claim 2 implies that for any $x \in C$ we have $d'(C, C') = d(x, p'(x))$ where $p' : X \rightarrow C'$ is the orthogonal projection. To prove that X_2 is a metric space:

1. For any $C, C' \in N$ the inequality $d'(C, C') \geq 0$ holds trivially, as does symmetry.
2. Clearly, for any $C \in N$ we have $d'(C, C) = 0$. Furthermore, if for some $C, C' \in N$ the equality $d'(C, C') = 0$ holds, then by Claim 2, for any $x \in C$ we have $d(x, p'(x)) = 0$ which implies that $x \in C'$. Hence $C = C'$ since elements of N are disjoint.
3. To show that the triangle inequality holds, consider $C, C', C'' \in N$ and let $x \in C$. If $p' : X \rightarrow C'$ and $p'' : X \rightarrow C''$ are the orthogonal projections onto C' and C'' respectively, then we have the following

$$\begin{aligned}
 d'(C, C'') &= d(x, p''(x)) \\
 &\leq d(x, p'(x)) + d(p'(x), p''(x)) \\
 &= d(x, p'(x)) + d(p'(x), p''(p'(x))) \text{ (Claim 4)} \\
 &= d'(C, C') + d'(C', C'')
 \end{aligned} \tag{6.19}$$

so the triangle inequality holds, as desired.

We have shown that (N, d') is a metric space. The next step is to define an isometric action of $\Gamma = \Gamma_1 \times \Gamma_2$ on metric space $X_1 \times X_2$. We wish for the action to preserve the splitting, so we need to do this by defining an action $\odot$

of Γ_1 and Γ_2 on each of the factors separately and showing that each of these actions are by isometries.

Since X_1 is Γ_1 -invariant, we can define the action of Γ_1 on X_1 to be the restriction to X_1 of the old isometric action, so for every $\gamma \in \Gamma_1$ and every $x \in X_1$ we define the action $\odot : \Gamma_1 \times X_1 \rightarrow X_1$ to be $\gamma \odot x = \gamma \cdot x$. Furthermore, each $C \in N$ is Γ_1 -invariant, so we define the action of Γ_1 on X_2 to be trivial, which is clearly isometric.

The action of Γ_2 on X_2 is defined by $\gamma \odot C_x = C_{\gamma \cdot x}$ for each $\gamma \in \Gamma_2$ and $C_x \in N$. Note that as sets, we have $\gamma \cdot C_x = \gamma \odot C_x$ for any $\gamma \in \Gamma_2$. This is because $\gamma \cdot C_x$ is minimal, closed, convex, Γ_1 -invariant (since Γ_1 and Γ_2 commute and C_x is Γ_1 -invariant) and thus in N . Furthermore, $\gamma \cdot x \in \gamma \cdot C_x$ so by the disjointness of elements in N we have $\gamma \odot C_x = C_{\gamma \cdot x} = \gamma \cdot C_x$. This implies that the action is well-defined.

Let C_x, C_y be any two elements of N and consider $\gamma \in \Gamma_2$. If we define $p_{\gamma \cdot y} : X \rightarrow C_{\gamma \cdot y}$ to be the orthogonal projection onto $C_{\gamma \cdot y}$ then we have $d(C_{\gamma \cdot x}, C_{\gamma \cdot y}) = d(\gamma \cdot x, p_{\gamma \cdot y}(\gamma \cdot x))$. Since γ maps C_y surjectively to $C_{\gamma \cdot y}$ we can

find $z \in C_y$ such that $\gamma \cdot z = p_{\gamma \cdot y}(\gamma \cdot x)$. This give us

$$\begin{aligned}
 d(\gamma \odot C_x, \gamma \odot C_y) &= d(C_{\gamma \cdot x}, C_{\gamma \cdot y}) \\
 &= d(\gamma \cdot x, p_{\gamma \cdot y}(\gamma \cdot x)) \\
 &= d(\gamma \cdot x, \gamma \cdot z) \\
 &= d(x, z) \\
 &\geq d(C_x, C_z) \\
 &= d(C_x, C_y)
 \end{aligned} \tag{6.20}$$

By replacing C_x by $\gamma \odot C_x$, C_y by $\gamma \odot C_y$ and γ by γ^{-1} we obtain the opposite inequality and can thus conclude that the action is by isometries and since the action of Γ_1 on X_2 is trivial, the actions commute.

We define the action of Γ_2 on X_1 to be given by $\gamma \odot x = p(\gamma \cdot x)$ for every $\gamma \in \Gamma_2$ and $x \in X_1$. Let $\gamma_1, \gamma_2 \in \Gamma_2$. Consider Figure 6.1 where $p : \gamma_2 \cdot X_1 \rightarrow X_1$ and $p' : \gamma_1 \gamma_2 \cdot C_1 \rightarrow \gamma_1 \cdot C_1$ are the restrictions to $\gamma_2 \cdot X_1$ and $\gamma_1 \gamma_2 \cdot X_1$ respectively of the orthogonal projections onto X_1 and $\gamma_1 \cdot X_1$ respectively. Since γ_1 is an isometry, this diagram commutes. Thus, for any $x_1 \in X_1$ we have

$$\begin{aligned}
 \gamma_1 \cdot p(\gamma_2 \cdot x_1) &= p'(\gamma_1 \gamma_2 \cdot x_1) \\
 \Rightarrow p(\gamma_1 \cdot p(\gamma_2 \cdot x_1)) &= p(p'(\gamma_1 \gamma_2 \cdot x_1)) \\
 \Rightarrow p(\gamma_1 \cdot p(\gamma_2 \cdot x_1)) &= p(\gamma_1 \gamma_2 \cdot x_1) \text{ (by Claim 4)} \\
 \Rightarrow \gamma_1 \odot (\gamma_2 \odot x_1) &= (\gamma_1 \gamma_2) \odot x_1
 \end{aligned} \tag{6.21}$$

and the action is well-defined. Furthermore, this action clearly commutes with

Figure 6.1: Commutative Diagram for γ_2 , p , and p'

the action of Γ_1 on X_1 since p is Γ_1 -equivariant.

To show that the action is by isometries, consider $x, y \in X_1$ and $\gamma \in \Gamma_2$. We already established that $\gamma \cdot x$ and $\gamma \cdot y$ lie in the same element $\gamma \cdot X_1$ of N . Furthermore, by Claim 2, the projection p when restricted to $\gamma \cdot X_1$ is an isometry, so we have

$$\begin{aligned}
 d(\gamma \odot x, \gamma \odot y) &= d(p(\gamma \cdot x), p(\gamma \cdot y)) \\
 &= d(\gamma \cdot x, \gamma \cdot y) \\
 &= d(x, y)
 \end{aligned} \tag{6.22}$$

so the action is by isometries, as desired.

The last step of the proof is to show that there exists a Γ -equivariant isometry $f : X \rightarrow X_1 \times X_2$.

We define f to be the map $x \mapsto (p(x), C_x)$ which is clearly Γ -equivariant since for every $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ we have

$$\begin{aligned}
 f(\gamma_1\gamma_2 \cdot x) &= (p(\gamma_1\gamma_2 \cdot x), C_{\gamma_1\gamma_2 \cdot x}) \\
 &= \gamma_1\gamma_2 \odot x, \gamma_1\gamma_2 \odot C_x \\
 &= (\gamma_1\gamma_2) \odot (p(x), C_x) \\
 &= (\gamma_1\gamma_2) \odot f(x)
 \end{aligned} \tag{6.23}$$

Now to show that f is an isometry, let $x, x' \in X$ and let $p' : X \rightarrow C_{x'}$ be the orthogonal projection of X onto $C_{x'}$. The by Claim 2 we have

$$\begin{aligned}
 d(x, x')^2 &= d(x, p'(x))^2 + d(p'(x), x')^2 \\
 &= d(C_x, C_{x'})^2 + d(pp'(x), p(x'))^2 \\
 &= d(C_x, C_{x'})^2 + d(p(x), p(x'))^2 \text{ (by Claim 4)} \\
 &= d((p(x), C_x), (p(x'), C_{x'})) \\
 &= d(f(x), f(x'))
 \end{aligned} \tag{6.24}$$

which means that f is a Γ -equivariant isometry as desired. Finally we show that f is surjective. If $(x, C_y) \in X_1 \times X_2$ then $p : C_y \rightarrow X_1$ is an isometry. Thus, there exists $z \in C_y$ such that $p(z) = x$. Then $f(z) = (p(z), C_z) = (x, C_y)$ and f is surjective. This means that the splitting $X = X_1 \times X_2$ has the desired properties and the proof is complete. $\square$

We now present a second technical lemma. It will be used to satisfy the requirement in the previous proposition that the action of Γ_1 on the closed

convex hull of some Γ_1 -orbit be cocompact.

Lemma 6.2.5. Let X be a proper CAT(0) space and $\Gamma = \Gamma_1 \times \Gamma_2$ be a group acting properly on X by isometries. Suppose moreover that the centre of Γ_2 is finite. Let $C \subseteq X$ be a closed, convex, Γ_1 -invariant subspace for which there exists a compact subset $K \subseteq X$ such that $C \subseteq \Gamma \cdot K$. Then the action of Γ_1 on C is cocompact.

Proof. The proof shall proceed by contradiction; suppose that no compact set $K' \subseteq C$ such that $\Gamma_1 \cdot K' = C$ exists. Since K is compact we can find $x \in K$ and $d > 0$ such that $K \subseteq B(x, d)$. Construct a sequence of compact subsets of C as follows. For each $n \geq 0$ set

$$K_n = \overline{B}(x, d + n) \cap C \tag{6.25}$$

Since X is proper, the closed ball $\overline{B}(x, d + n)$ is compact for each $n \geq 0$ which means that since K_n is the intersection of a closed set and a compact set, it is also compact.

Since $K_n \subseteq C$, the Γ_1 -invariance of C implies that $\Gamma_1 \cdot K_n \subseteq C$. Our initial assumption then implies that $C - (\Gamma_1 \cdot K_n) \neq \emptyset$. Now we can construct a sequence $\{x_n\}$ such that for each $n \geq 0$ we have $x_n \in C$ and $x_n \notin \Gamma_1 \cdot K_n$.

Since $x_n \notin \Gamma_1 \cdot K_n$ for each $n \geq 0$, we have $\gamma \cdot x_n \notin K_n$ for any $\gamma \in \Gamma_1$. Since $\gamma \cdot x_n \in C$ this implies that $\gamma \cdot x_n \notin \overline{B}(x, d + n)$. So if $k \in K$ and $\gamma \in \Gamma_1$

we have

$$\begin{aligned}
 d(\gamma \cdot x_n, k) + d &\geq d(\gamma_n \cdot x_n, k) + d(k, x) \\
 &\geq d(\gamma \cdot x_n, x) \\
 &\geq d + n
 \end{aligned} \tag{6.26}$$

and thus $d(x_n \Gamma_1 \cdot K) \geq n$ which implies that $d(x_n, \Gamma_1 \cdot K) \rightarrow \infty$ as $n \rightarrow \infty$.

Since $C \subseteq \Gamma \cdot K$, for each $n \geq 0$ we can find $\alpha_n \in \Gamma$ such that $\alpha_n \cdot x_n \in K$. As $\Gamma = \Gamma_1 \times \Gamma_2$, we can write $\alpha_n = \gamma_n \beta_n$ where $\gamma_n \in \Gamma_1$ and $\beta_n \in \Gamma_2$ and thus $\gamma_n \beta_n \cdot x_n \in K$ which implies that $\beta_n \cdot x_n \in \Gamma_1 \cdot K$. We fix this sequence $\{\beta_n\}$.

Fix x_n and $\varepsilon > 0$. We can find $z \in \Gamma_1 \cdot K$ and $y \in \beta_n^{-1} \cdot K$ such that $d(x, y) \leq d(\Gamma_1 \cdot K, \beta_n^{-1} \cdot K) + \varepsilon$. This gives:

$$\begin{aligned}
 d(x_n, \Gamma_1 \cdot K) &= d(\gamma_n \cdot x_n, \Gamma_1 \cdot K) \\
 &\leq d(\gamma_n \cdot x_n, z) \\
 &\leq d(\gamma_n \cdot x_n, y) + d(y, z) \\
 &\leq d(\beta_n \gamma_n \cdot x_n, \beta_n \cdot y) + d(\Gamma_1 \cdot K, \beta_n^{-1} \cdot K) + \varepsilon \\
 &\leq \text{diam}(K) + d(\Gamma_1 \cdot K, \beta_n^{-1} \cdot K) + \varepsilon
 \end{aligned} \tag{6.27}$$

Since $d(\beta_n^{-1} \cdot K, \Gamma_1 \cdot K) \leq d(\beta_n^{-1} \cdot K, K)$ the above inequality implies that $d(x_n, \Gamma_1 \cdot K) - \text{diam}(K) \leq d(\beta_n^{-1} \cdot K, K)$. Since $d(x_n, \Gamma_1 \cdot K) \rightarrow \infty$ as $n \rightarrow \infty$, we conclude that $d(\beta_n^{-1} \cdot K, K) \rightarrow \infty$ as $n \rightarrow \infty$. This means that by passing to a subsequence, we may assume that the elements of $\{\beta_n\}$ are distinct.

For each $\beta \in \Gamma_2$ consider the displacement function d_β . For any $\gamma \in \Gamma_1$

and $y \in X$ we have

$$\begin{aligned}
 d_\beta(\gamma \cdot y) &= d(\beta\gamma \cdot y, \gamma \cdot y) \\
 &= d(\gamma\beta \cdot y, \gamma \cdot y) \\
 &= d(\beta \cdot y, y) \\
 &= d_\beta(y)
 \end{aligned} \tag{6.28}$$

This implies that d_β is constant on any Γ_1 -orbit in X .

Let $r = \max\{d_\beta(y) \mid y \in K\}$. The compactness of K guarantees that r is defined as $d_\beta|_K : K \rightarrow \mathbb{R}$ is continuous.

Let us define the set $U \subseteq X$ as follows

$$U = \{y \in X \mid d_\beta(y) \leq r\} \tag{6.29}$$

Since d_β is continuous, this set is closed. Furthermore, it was shown in (6.28) that d_β is constant on Γ_1 -orbits, so U is Γ_1 -invariant. Finally, we note that for any $u, v \in X$ and any linearly reparameterized geodesic path $c : [0, 1] \rightarrow X$ originating at u and terminating at v , if $z = c(t)$ then by the convexity of d_β we have

$$d_\beta(z) \leq (t - 1)d_\beta(u) + td_\beta(v) \leq r \tag{6.30}$$

so $z \in U$ and U is convex. By definition we have $K \subseteq U$ so since U is closed, convex, and Γ_1 -invariant, we have $C(\Gamma_1 \cdot K) \subseteq U$. Thus, we conclude that d_β is bounded by r on $C(\Gamma_1 \cdot K)$.

Now we have for any $y \in K$

$$\begin{aligned}
 d(\beta_n \beta \beta_n^{-1} \cdot y, y) &\leq d(\beta_n \beta \beta_n^{-1} \cdot y, \beta_n \beta \beta_n^{-1} \cdot (\beta_n \gamma_n \cdot x_n)) \\
 &\quad + d(\beta_n \beta \beta_n^{-1} \cdot (\beta_n \gamma_n \cdot x_n), \beta_n \gamma_n \cdot x_n) + d(\beta_n \gamma_n \cdot x_n, y) \\
 &= 2d(\beta_n \gamma_n \cdot x_n, y) + d(\gamma_n \beta_n \beta \cdot x_n, \gamma_n \beta_n \cdot x_n) \\
 &\leq 2\text{diam}(K) + d_\beta(\gamma_n \beta_n \cdot x_n) \\
 &\leq 2\text{diam}(K) + r
 \end{aligned} \tag{6.31}$$

Let $K'' = \overline{B}(x, 2\text{diam}(K) + r)$. Since X is proper, K'' is compact and since Γ_2 acts properly on X , the following set is finite

$$A = \{\gamma \in \Gamma_2 \mid \gamma \cdot K'' \cap K'' \neq \emptyset\} \tag{6.32}$$

Since for each β_n we have that $d_{\beta_n \beta \beta_n^{-1}}$ is bounded by $2\text{diam}(K) + r$ on K , we have $d(x, \beta_n \beta \beta_n^{-1} \cdot x) \leq 2\text{diam}K + r$ so each element $\beta_n \beta \beta_n^{-1}$ is contained in A . Since A is finite, by again passing to a subsequence, we may assume that $\beta_n \beta \beta_n^{-1} = \beta_m \beta \beta_m^{-1}$ for each $n, m \geq 0$. Thus, $\beta_m^{-1} \beta_n \beta = \beta \beta_m^{-1} \beta_n$ and we have found an infinite number of distinct elements which commute with β . We repeat this process for each member of the finite generating set for Γ_2 restricting to a further subsequence $\{\beta_n\}$ each time. In the end, we are left with infinitely many distinct elements of the form $\beta_m^{-1} \beta_n \in Z(\Gamma_2)$. This contradicts our assumption that Γ_2 has a finite centre. Thus our initial assumption that no such K' exists was false and the proof is complete. $\square$

We have now come to the statement and proof of the major result of this section.

Theorem 6.2.6 (Splitting Theorem). Let X be a complete CAT(0) space with the geodesic extension property and suppose $\Gamma = \Gamma_1 \times \Gamma_2$ is a group which acts properly and cocompactly on X by isometries. If one of the following holds

1. Γ has a finite centre and Γ_1 is torsion-free
2. the abelianization of Γ_1 is finite

then X splits as a product $X = X_1 \times X_2$ and Γ preserves this splitting: the action of $\Gamma = \Gamma_1 \times \Gamma_2$ is the product action on $X = X_1 \times X_2$ and both the actions of Γ_1 on X_2 and Γ_2 on X_1 are trivial.

Proof. (1) Note that by Remark 3.1.6, X is a proper metric space. Suppose Γ has a finite centre and Γ_1 is torsion-free. Since Γ_2 commutes with Γ_1 , we have $Z(\Gamma_2) \subseteq Z(\Gamma)$ and thus the centre of Γ_2 is finite. If we pick any point $x \in X$ and let $C = C(\Gamma_1 \cdot x)$ then since Γ acts cocompactly there exists a compact set $K \subseteq X$ such that $\Gamma \cdot K = X$. In particular, $C \subseteq \Gamma \cdot K$ and we can apply Lemma 6.2.5 to conclude that the action of Γ_1 on C is cocompact. Since C is the closed convex hull of a Γ_1 -orbit we may now apply Proposition 6.2.4 to obtain a splitting $X = X_1 \times X_2$ which is preserved by Γ and such that the subspaces of the $X_1 \times \{x_2\}$ are the closed convex hulls of Γ_1 -orbits. Thus the action of Γ_1 on X_2 is trivial. To complete the proof, we only need to show that the action of Γ_2 on X_1 is also trivial.

Since the centre of Γ_1 is finite and Γ_1 is torsion-free, its centre must be trivial. By Proposition 6.2.4, the action of Γ_1 on X_1 is cocompact since Γ_1 acts cocompactly on the closed convex hulls of Γ_1 -orbits. We have thus satisfied the requirements of Lemma 6.2.1 and can conclude that the centralizer of Γ_1 in $\text{Isom}(X_1)$ is trivial.

Moreover, since Γ_2 commutes with Γ_1 we can conclude that its image in $\text{Isom}(X_1)$ is in the centralizer of the image of Γ_1 in $\text{Isom}(X_1)$ and thus by Lemma 6.2.1, Γ_2 acts trivially on X_1 as desired. This completes the proof of the first part of the splitting theorem.

(2) Suppose that the centre of Γ is infinite. Since the abelianization of Γ_1 is finite, the centre of Γ_1 cannot have any elements of infinite order. Indeed, by Proposition 6.1.1 such an element γ must act as a semi-simple isometry of X . Since γ has infinite order and the action is proper, by Proposition 3.1.5 it cannot stabilize a point. Hence γ acts as a hyperbolic isometry so by Corollary 6.1.3 there is a non-trivial homomorphism $\Gamma_1 \rightarrow \mathbb{R}$, which contradicts the assumption that the abelianization of Γ_1 is finite. By Corollary 3.3.2 $Z(\Gamma)$ has only finitely many torsion elements. As the centre of Γ is infinite, it must therefore contain an element (α_1, α_2) with infinite order. Since $\alpha_1 \in Z(\Gamma_1)$ has finite order, $\alpha_2 \in Z(\Gamma_2)$ has infinite order.

Since α_2 has infinite order it cannot be elliptic since by Proposition 3.1.5, the isotropy subgroups of Γ are finite. By Proposition 6.1.1 α_2 is semi-simple so it must be hyperbolic and thus by Theorem 5.2.5 $\text{Min}(\alpha_2)$ splits as $Y \times \mathbb{R}$ where Y is a convex subspace of X . The element α_2 commutes with Γ so by

Lemma 5.2.7, α_2 is a Clifford translation and $\text{Min}(\alpha_2) = X$. In particular $X = Y \times \mathbb{R}$ and since each $\gamma \in \Gamma$ commutes with α_2 , the element γ splits as $\gamma = (\gamma', \gamma'')$ where γ'' is a translation of $\mathbb{R}$ by Theorem 5.2.5.

Since the abelianization of Γ_1 is finite, the homomorphism $\gamma \mapsto \gamma''$ taking each element of Γ_1 to a translation of $\mathbb{R}$ must be trivial. In particular, the closed convex hulls of Γ_1 -orbits are contained in slices of the form $Y \times \{t\}$.

An *embedded flat* in a metric space X is a subset of X which is isometric to $\mathbb{R}^n$ for some integer n . We shall now prove by induction on the maximum dimension of embedded flats in X , that the action of Γ_1 on the closed convex hulls of Γ_1 -orbits is cocompact. To see that there is in fact a maximum dimension of embedded flats, refer to [Bridson & Haefliger pg. 247]. If there are no embedded flats in X then the centre of Γ must be finite as we have just shown that if $Z(\Gamma)$ is infinite then $X = Y \times \mathbb{R}$. Then we may apply Lemma 6.2.5, as in the previous case, to conclude that the action of Γ_1 on any closed convex hull of a Γ_1 -orbit is cocompact.

Suppose that the assertion holds for every group $\Gamma = \Gamma_1 \times \Gamma_2$ such that the abelianization of Γ is finite, where Γ acts properly and cocompactly on a $\text{CAT}(0)$ space whose maximum dimension of embedded flats is $n - 1$. Now suppose that the maximum dimension of embedded flats in X is n . If the centre of Γ is finite, we can use the same argument as the base case to show that the action of Γ_1 on the closed convex hull of a Γ_1 -orbit is cocompact. If the centre is infinite however, we can find an element $\alpha_2 \in Z(\Gamma_2)$ of infinite order, as described above to get the splitting of X as a product $X = Y \times \mathbb{R}$.

Since α_2 acts trivially on Y we can define an action of $\Gamma/\langle\alpha_2\rangle = \Gamma_1 \times \Gamma_2/\langle\alpha_2\rangle$ on Y by $\gamma'\langle\alpha_2\rangle \cdot y = \gamma' \cdot y$ for all $y \in Y$. By Proposition 6.1.1 (2) this action is proper. Furthermore, we can show that the action is cocompact. Indeed, there exists a compact set $K \subset X$ such that $\Gamma \cdot K = X$. The projection K' of K to Y is compact and for every $(x, t) \in Y \times \mathbb{R}$ there exists $\gamma = (\gamma', \gamma'') \in \Gamma$ such that $\gamma \cdot (x, t) = (\gamma' \cdot x, \gamma'' \cdot t) \in K$. Therefore, for every $x' \in Y$ there exists $(\gamma', \gamma'')N \in \Gamma/N$ such that $\gamma' \cdot x = (\gamma', \gamma'')N \cdot x \in K'$. We have thus defined a proper cocompact action of $\Gamma/\langle\alpha_2\rangle = \Gamma_1 \times \Gamma_2/\langle\alpha_2\rangle$ on the proper and complete CAT(0) space Y and clearly the abelianization of Γ_1 is still finite. Since X has the geodesic extension property, given any distinct elements $y_1, y_2 \in Y$ the geodesic segment $[(y_1, 0), (y_2, 0)]$ can be extended to a geodesic line and since $X = Y \times \mathbb{R}$, this line will be contained in the slice $Y \times \{0\}$. Thus Y has the geodesic extension property and so does Y .

Since $X = Y \times \mathbb{R}$, the space Y has a maximum dimension $n-1$ of embedded flats and so we have satisfied all the requirements in the inductive hypothesis. Therefore, we can apply the inductive hypothesis to deduce that the action of Γ_1 on the closed convex hull of a Γ_1 -orbit in Y is cocompact. Since each Γ_1 orbit in X is contained in a slice $Y \times \{t\}$ of $Y \times \mathbb{R}$ we conclude that the action of Γ_1 on the closed convex hull of some Γ_1 -orbit in X is cocompact.

We can now apply Proposition 6.2.4 to obtain a splitting of $X = X_1 \times X_2$ such that Γ preserves this splitting and the action of Γ_1 on X_2 is trivial. To complete the proof, we must show that the action of Γ_2 on X_1 , as defined in the proof of Proposition 6.2.4, is trivial. Let $p : X \rightarrow X_1$ be the orthogonal

projection; under our splitting, p is the projection to the first coordinate. For each $x \in X$ we need $p(\gamma_2 \cdot x) = x$. If we can find a closed convex hull of a Γ_1 -orbit, $C = X_1 \times \{x_2\}$ such that $p\gamma_2$ restricts to the identity on C then for every $(x_1, x_2) \in C$ we shall have $\gamma_2 \cdot (x_1, x_2) = (p(\gamma_2 \cdot x_1), \gamma_2 \cdot x_2) = (x_1, \gamma_2 \cdot x_2)$ and thus $\gamma_2 \odot x_1 = x_1$ for every $x_1 \in X_1$.

By Proposition 6.1.1 (1), γ_2 is a semi-simple isometry of X . If γ_2 is elliptic, then take $x = (x_1, x_2) \in \text{Min}(\gamma_2)$. Let C be the closed, convex hull of $\Gamma_1 \cdot x_2$. Then $\gamma_2 \cdot x = x$ and thus $\gamma_2 \cdot C = C$ since γ_2 commutes with Γ_1 and hence takes closed, convex hulls of Γ_1 -orbits to closed convex hulls of Γ_1 -orbits. Thus, $p\gamma_2 = p$ on C .

If γ_2 is hyperbolic then $\text{Min}(\gamma_2)$ splits as $Y' \times \mathbb{R}$. As we saw earlier in the proof, Γ_1 preserves this splitting, and its action on the second factor is trivial. Thus each Γ_1 -orbit in $\text{Min}(\gamma_2)$ is contained in a slice $Y' \times \{t\}$. If we take $x \in \text{Min}(\gamma_2)$ and the set $C = C(\Gamma_1 \cdot x)$, then the action of γ_2 on C is trivial and thus $p\gamma_2$ when restricted to C is p .

We have shown that the action of Γ_2 on X_1 is trivial and this completes the proof of the splitting theorem. $\square$

6.3 The Flat Torus Theorem

Theorem 6.3.1 (The Flat Torus Theorem). Let A be a free abelian group of rank n which acts properly by semi-simple isometries on a $\text{CAT}(0)$ space X . Then

1. $\text{Min}(A) = \bigcap_{\alpha \in A} \text{Min}(\alpha)$ is non-empty and splits as a product $Y \times \mathbb{E}^n$.
2. Every $\alpha \in A$ leaves $\text{Min}(A)$ invariant and respects this decomposition.
The action of α on the first factor Y is trivial and the action on $\mathbb{E}^n$ is a translation.
3. The quotient of each n -flat $\{y\} \times \mathbb{E}^n$ by the action of A is an n -torus.
4. Any isometry of X which normalizes A leaves $\text{Min}(A)$ invariant and preserves the product decomposition.
5. If a subgroup $\Gamma \subseteq \text{Isom} X$ normalizes A then a subgroup of finite index in Γ centralizes A . Moreover, if Γ is finitely generated and contains A , then Γ has a subgroup of finite index that contains A as a direct factor.

Proof. The proof of (1), (2), and (3) will proceed by induction on the rank of A . If the rank of A is zero then $A = \{1\}$ and the claims hold trivially. Suppose (1), (2), and (3) hold for any free abelian group of rank $n - 1$.

Suppose $\alpha \in A$ is elliptic. Then α stabilizes a point and by Proposition 3.1.5 (4) the isotropy groups of points are finite and thus α has finite order. Hence $\alpha = 1$ as A is torsion-free. Since α is semi-simple, we conclude that every non-trivial element in A is a hyperbolic isometry.

Since A is a free abelian group of rank n we can choose a set of free generators $\alpha_1, \alpha_2, \dots, \alpha_n$ for A . Now, α_1 is hyperbolic so $\text{Min}(\alpha_1)$ splits as $Z \times \mathbb{E}^1$ where α_1 acts trivially on Z and as a translation on $\mathbb{E}^1$ by Theorem 5.2.5 (3). Moreover, every $\alpha \in A$ commutes with α_1 so each α splits as (α', α'')

where α' is an isometry of Z and α'' is a translation of $\mathbb{E}^1$ by Theorem 5.2.5 (3) again.

Consider the subgroup $N \subseteq A$ consisting of precisely those elements which act trivially on the first factor Z . By Proposition 6.1.1 (2), N acts properly on $\mathbb{E}^1$. The subgroup N is a free abelian group with rank less than or equal to n . Suppose N has rank of at least 2 and suppose n_1 and n_2 are free generators; they will be translations of $\mathbb{E}$ by η_1 and η_2 respectively. Since η_1 and η_2 are $\mathbb{Z}$ -independent, they are $\mathbb{Q}$ -independent so by Kronecker's Theorem [Hardy & Wright] the set $\mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$ is dense in $\mathbb{R}$. This means that for every $\varepsilon > 0$ there exist non-zero integers a, b such that $n_1^a n_2^b$ acts on $\mathbb{E}^1$ as a translation of length less than ε . This contradicts the properness of the action and thus N is cyclic. Since α_1 is primitive in A it is a generator for N . Moreover, the action of $N = \langle \alpha_1 \rangle$ on $\mathbb{E}$ is cocompact with quotient a circle.

The free abelian group $A_0 = A/N$ has rank $n - 1$. By Proposition 6.1.1 (2), its induced action on Z is proper and by Proposition 5.2.6 this action is by semi-simple isometries. Finally, as a convex subspace of X , the space $Z \times \{0\} \cong Z$ is CAT(0). By induction, $\text{Min}(A_0)$ splits as $Y \times \mathbb{E}^{n-1}$ where A_0 acts trivially on Y and by translations on $\mathbb{E}^{n-1}$ and the quotient of $\mathbb{E}^{n-1}$ by the action of A_0 is an $(n - 1)$ -torus.

Thus, using Proposition 5.2.5 we have $\text{Min}(A) = Y \times \mathbb{E}^{n-1} \times \mathbb{E} = Y \times \mathbb{E}^n$. Since for each $\alpha \in A$, the element $\alpha N \in A/N$ acts trivially on Y and by translations on $\mathbb{E}^{n-1}$, and N acts trivially on $Y \times \mathbb{E}^{n-1}$ and by translations on $\mathbb{E}^1$, we see that α acts trivially on Y and by translations on $\mathbb{E}^n$. The quotient

$\mathbb{E}^1/N$ is a circle and by the inductive hypothesis, the quotient $\mathbb{E}^{n-1}/A_0$ is an $(n-1)$ -torus. Furthermore, the action of A is compatible with this splitting. Thus the quotient of $\mathbb{E}^n$ under the action of A is an n -torus as desired.

(4) If an isometry γ normalizes $\text{Min}(A)$ then by Proposition 5.1.2 (1) $\text{Min}(A)$ is γ -invariant. Furthermore, for any A -orbit $A \cdot x$ in X we have $\gamma A \cdot x = A \cdot (\gamma \cdot x)$, so γ takes A -orbits to A -orbits. From (3) we have that each A -orbit, $A \cdot (y, t)$ is a lattice in $\{y\} \times \mathbb{E}^n$ and so the convex hull of each A -orbit is precisely an n -flat $\{y\} \times \mathbb{E}^n$. Therefore $\gamma \cdot (\{y\} \times \mathbb{E}^n) = \{\gamma_1 \cdot y\} \times \mathbb{E}^n$ and thus by Proposition 5.2.4, the isometry γ respects this splitting.

(5) For any number r there can only be finitely many $\alpha \in A$ such that $|\alpha| = r$. Indeed, the set $K = \overline{B}((y, 0), r)$ is compact so by the properness of the action, the set $\{\gamma \in A \mid \gamma \cdot K \cap K \neq \emptyset\}$ is finite. Thus, only finitely many elements of A have a translation length less than or equal to r . Consider the homomorphism $f : \Gamma \rightarrow \text{Aut}(A)$ where $f(\gamma)(\alpha) = \gamma\alpha\gamma^{-1}$ for each $\alpha \in A$ and each $\gamma \in \Gamma$. We have that the image of f is finite since each generator α_i for A can only be conjugated to a finite number of distinct elements $\gamma\alpha_i\gamma^{-1}$ since $|\gamma\alpha_i\gamma^{-1}| = |\alpha_i|$. Thus, $\ker f$ is a finite index subgroup of Γ which centralizes A .

The action of Γ on X is faithful because $\Gamma \subseteq \text{Isom}X$, so if Γ is finitely generated and $A \subseteq \Gamma$, then $\ker f$ is a finitely generated subgroup containing A to which Theorem 6.1.2 applies. Thus $\ker f$ and hence Γ contains a finite index subgroup containing A as a direct factor. $\square$

We can generalize the flat torus theorem to groups which contain a finite index subgroup which is free abelian of rank n .

Corollary 6.3.2. Let Γ be a finitely generated group which acts properly by semi-simple isometries on a CAT(0) space X . Suppose that Γ contains a finite index subgroup that is free abelian of rank n . Then

1. X contains a Γ -invariant closed convex subspace isometric to a product $Y \times \mathbb{E}^n$.
2. The action of Γ preserves the product structure on $Y \times \mathbb{E}^n$, acting as the identity on the first factor and cocompactly on the second.
3. Any isometry of X which normalizes Γ preserves $Y \times \mathbb{E}^n$ and its splitting.

Proof. Let $A_0 \cong \mathbb{Z}^n$ be a subgroup of finite index in Γ . Because Γ is finitely generated, there are only finitely many subgroups in Γ of index $|\Gamma/A_0|$. Let A be the intersection of these finitely many subgroups; note that A has finite index in Γ . Moreover, A is free abelian of rank n since A has finite index in A_0 .

As automorphisms preserve the index of subgroups, any automorphism of Γ will send a subgroup of index $|\Gamma/A_0|$ to a subgroup of index $|\Gamma/A_0|$ which implies that A is characteristic in Γ . In particular, any isometry of X which normalizes Γ will also normalize A .

As a subgroup of Γ , A acts properly and by semi-simple isometries on X so by the Flat Torus Theorem 6.3.1, $\text{Min}(A)$ splits as a product $Z \times \mathbb{E}^n$ where A

acts trivially on the first factor and cocompactly on the second. Furthermore, since A is normal in Γ , the action of Γ on $\text{Min}(A)$ preserves this splitting.

Thus, we have an induced action of the finite group Γ/A on Z where γA acts as γ on each $z \in Z$. According to the Bruhat-Tits Fixed Point Theorem 3.3.1 (1) the fixed point set Y of Γ/A on Z is non-empty and by Proposition 5.1.2 (3) this set Y is a closed convex subset of X (actually, it is a closed convex subset of Z but this means that it is a closed convex subset of X as well).

The set $Y \times \mathbb{E}^n$ is our desired subspace. The group Γ acts trivially on the first factor and since the action of A on the second is cocompact, the action of Γ will be cocompact as well. Any isometry γ of X which normalizes Γ also normalizes A and hence leaves $Z \times \mathbb{E}^n$ invariant and preserves the structure by the Flat Torus Theorem 6.3.1. Thus, in order to show that γ preserves the splitting $Y \times \mathbb{E}^n$, it suffices to show that if $y \in Y$, then $\gamma \cdot y \in Y$. We prove that $\gamma \cdot y \in Y$ by showing that it is fixed by the action of Γ/A . Suppose $\gamma' A \in \Gamma/A$. Since γ normalizes Γ in $\text{Isom}(X)$ there exists $\gamma'' \in \Gamma$ such that $\gamma' \gamma \cdot x = \gamma \gamma'' \cdot x$ for every $x \in X$. Thus

$$\begin{aligned} \gamma' A \cdot (\gamma \cdot y) &= \gamma' \gamma \cdot y \\ &= \gamma \gamma'' \cdot y \\ &= \gamma \cdot y \end{aligned} \tag{6.33}$$

Hence, $\gamma \cdot y \in Y$ and γ preserves the structure. $\square$

Bibliography

- [1] J. W. Anderson. Hyperbolic Geometry. Springer-Verlag, Berlin, 1999.
- [2] M. R. Bridson and A. Haefliger. Metric Spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen [Fundamental Principles of Mathematical Sciences] Springer-Verlag, Berlin, 1999.
- [3] K. S. Brown, Buildings. Springer-Verlag, Berlin, 1988.
- [4] F. Bruhat and J. Tits. *Groups Réductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5-251.
- [5] I. Chiswell., Introduction to Λ -trees, World Scientific Publishing Co., Inc., River Edge, NJ, 2001
- [6] N. J. Cutland. Computability: An Introduction to Recursive Function Theory, Cambridge University Press, 1980.
- [7] A. Dummit and T. Foote, Abstract Algebra, Third Edition, John Wiley and Sons, New York, 2004.

- [8] H. B. Enderton, *A Mathematical Introduction to Logic*. 2nd ed., Harcourt Academic Press, 2001, Toronto.
- [9] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Patterson, W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett, Boston MA, 1992.
- [10] D. Gromoll and J. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of non-positive curvature*, *Bulletins of the American Mathematical Society* **77** (1971), 545-552.
- [11] M. Gromov *Asymptotic invariants of infinite groups*, *Geometric Group Theory* (G. A. Niblo and M.A. Roller, ed), *LMS Lecture Note Series* **182** Cambridge Univ. Press, Cambridge, 1993, 75-263.
- [12] M. Gromov, *Hyperbolic Groups*, *Essays in group theory* (S. M. Gersten, ed), Springer Verlag, MSRI Publ. **8** (1987), 75-263.
- [13] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, New York, 1979.
- [14] H. B Lawson and S. T. Yau, *Compact manifolds of non-positive curvature*, *J. Differential Geometry* **7** (1972), 211-228.
- [15] A. M. Macbeath, *Groups of homomorphisms of a simply connected space*, *Annals of Mathematics*, (2) **79** (1964), 473-488.

- [16] W. S. Massey Algebraic Topology: An Introduction, Graduate texts in mathematics bf 56. Springer-Verlag, Berlin, 1967.
- [17] J. R. Munkres Topology: a first course, Prentice-Hall, 1975, Englewood Cliffs, NJ.
- [18] J. P. Serre Trees, Springer-Verlag, 1980, Translation of "Arbres, Amalgames, SL_2 ", Astérisque **46**, 1977.
- [19] Group theory from a geometrical viewpoint. Proceedings of the workshop held in Trieste, March 26–April 6, 1990. Edited by É. I. Ghys, A. Haefliger and A. Verjovsky. World Scientific Publishing Co., Inc., River Edge, NJ, 1991.