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FELLEGI, Ivan Peter, 1935-
SAMPLING WITHOUT REPLACEMENT WITH
PROBABILITIES PROPORTIONAL TO SIZE.

Carleton University, Ph.D., 1961
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

SAMPLING WITHOUT REPLACEMENT
WITH PROBABILITIES PROPORTIONAL TO SIZE

by

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A Thesis Submitted to the Faculty of Carleton
University in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Ottawa

1961

SAMPLING WITHOUT REPLACEMENT WITH PROBABILITIES PROPORTIONAL TO SIZE

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Carleton University, Ottawa, 1961.

Advisory Committee: Dr. M. S. Macphail, Mr. D. K. Dale, Dr. J. W. Hopkins.

The study develops a new approach to the determining of the selection probabilities associated with sampling without replacement from a finite population. As the basis of this study a sampling method was defined, based on the following criteria:

- 1) The sample is selected from a population of N units.
- 2) Two measurements Y and $M > 0$ are associated with each unit. Y_i is the value of the i -th unit and is unknown before the sample is selected. The M_i value is a size measure which is known for every unit of the population. There is a high positive correlation between Y_i and M_i .
- 3) Utilizing the M_i values in a way described below, positive quantities p_i are calculated where $\sum_{i=1}^N p_i = 1$. Then n units are selected one by one without replacement and in such a way that the conditional probability of the i_j -th unit being selected as the j -th unit in the sample (given that the i_1 -th, i_2 -th, ..., i_{j-1} -th units are already selected) is equal to

$$\frac{p_{i_j}}{1 - p_{i_1} - p_{i_2} - \dots - p_{i_{j-1}}}$$

- 4) Denoting by $\pi_i(p_1, p_2, \dots, p_N)$ the probability of the event that the i -th unit is in the sample, the estimator proposed by Horvitz and Thompson

$$\frac{Y_{i_1}}{\pi_{i_1}} + \frac{Y_{i_2}}{\pi_{i_2}} + \dots + \frac{Y_{i_n}}{\pi_{i_n}} \quad (1)$$

is used to estimate the population total $\sum_{i=1}^N Y_i$.

For the choice of p_i in criterion (3), the study considers two alternatives, the first of which is to set

$$p_i = \frac{M_i}{\sum_{i=1}^N M_i} \quad (i=1, 2, \dots, N) \quad (2)$$

In this case the arithmetically involved operation of calculating the corresponding π_i is referred to as the "direct problem". In the second alternative the p_i are determined from the equations

$$\pi_i(p_1, p_2, \dots, p_N) = n \frac{M_i}{\sum_{i=1}^N M_i} \quad (i=1, 2, \dots, N) \quad (3)$$

and the solution of these equations is referred to as the "reverse problem". The direct problem does not present any theoretical difficulties, but does present serious practical difficulties due to the heavy arithmetic involved. The "reverse problem" presents difficulties that are both theoretical and practical.

In this thesis a set of functions $\delta_i(p_1, p_2, \dots, p_N)$ are developed to serve as approximations of the functions $\pi_i(p_1, p_2, \dots, p_N)$. These functions are of a sufficiently simple nature so that the substitution implied by (2) is not too cumbersome and the solution for p_i of the equations

$$\delta_i(p_1, p_2, \dots, p_N) = n \frac{M_i}{\sum_{i=1}^N M_i} \quad (i=1, 2, \dots, N)$$

is feasible.

The thesis provides a comprehensive review of the pertinent literature before developing the functions δ_i , first for the special case where $n = 2$. The general case, where n is any finite size, is also considered. For this case a set of equations in the functions δ_i are evolved in several stages. Some empirical observations are utilized in this development. The problem of the solution of the system of equations in δ_i is reduced to the solution of a single irrational equation in one unknown and an iterative process is suggested for the solution of this latter equation.

The investigation of the case where N tends to infinity according to the specified conditions indicates that, if N is large enough and $\sum p_i^2 < \frac{1}{2}$, the system of equations in δ_i is solvable. The solution in δ_i is such that $\delta_i - \pi_i = O(\frac{1}{N})$, $0 \leq \delta_i < 1$, and that the use of δ_i in place of π_i gives a bias in the estimates and variance estimates that tends to zero.

Numerical examples are given where the values of N are small, ranging from four to fifteen. The examples are not all favourable, yet even under unfavourable circumstances the values δ_i and π_i appear to be remarkably close. The two alternatives for the choice of p_i are compared, that is the "direct" approach of (2) and the "reverse" approach of (3).

ACKNOWLEDGEMENT

The subject of this thesis is pertinent to the Dominion Bureau of Statistics and arose in connection with the Canadian Labour Force Survey. I am greatly indebted to the Dominion Bureau of Statistics for generously granting an educational leave and thus providing for me the opportunity of completing this thesis.

I also wish to express my gratitude to Dr. M. S. Macphail, Mr. D. K. Dale and Dr. J. Hopkins for their careful scrutiny of this study and for their many helpful comments.

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CHAPTER I
REVIEW OF BACKGROUND MATERIAL

1 - Introduction

The theory of sampling from a finite population with probability proportional to a measure which is known for every unit of the population (this measure is usually referred to as "size") originated with an article by Hansen and Hurwitz in 1943 [4]. This method proved to be most useful in reducing the variance of estimates and it was adopted widely, although it was limited to sampling with replacement.

It is recognized that in sampling with equal probabilities there is no distinct advantage in sampling with replacement, which permits a unit to be selected more than once. Sampling without replacement is intuitively appealing, as at every draw a new unit is obtained and new information is secured regarding the population. Aside from this consideration, often it is impossible or impractical to sample with replacement. Such would be the case where the sample units, in the course of measuring, are destroyed or effected. This occurs, for example, in some quality control procedures or in sampling from human populations for enumerative surveys where repeated selection of the same individuals may result in a resistance to answer questions or where the answer at subsequent interviews is conditioned in an undesirable manner by previous interviews.

The most widely used estimator for the total of a characteristic when sampling with replacement and equal probabilities is the arithmetic mean of the sample observations (each taken as many times as it occurs in the sample) multiplied by N , the number of units in the population. Another

unbiased estimator is the sum of the sample observations (each observation taken only once) divided by $1 - \left(\frac{N-1}{N}\right)^n$ (where n is the sample size). This estimator is less efficient than the previously mentioned estimator. This is intuitively apparent, because the number of terms in it is a random variable, that is the number of different units selected. The denominator $1 - \left(\frac{N-1}{N}\right)^n$ is merely the probability that a specified unit is in the sample. When sampling with replacement with unequal probabilities, the estimator widely used is the sum of the sample observations Y_i divided by np_i (that is $\sum \frac{Y_i}{np_i}$) where p_i is the probability that any one of the n selections unit numbered i is sampled. This is a generalization of the estimator noted above in connection with sampling with equal probabilities. Another unbiased estimator is defined as the weighted sum of the measurements, where the weight for unit numbered i , if it is selected, is $\frac{1}{1 - (1-p_i)^n}$. Again $1 - (1-p_i)^n$ is the probability that the unit numbered i is in the sample. This latter estimator can be generalized to sampling without replacement, in which case the number of different units selected is exactly n .

In effect two different sets of probabilities are involved. One set is employed in the procedure of selection, the other in the procedure of estimation. In the above example for sampling with replacement, the two sets are $[p_i]$ and $[1 - (1-p_i)^n]$, while in sampling without replacement the second set of probabilities can be denoted by $[\pi_i]$ which are functions of $[p_i]$. Unfortunately, these functions $\pi_i(p_1, p_2, \dots, p_N)$ are not as easy to evaluate when sampling without replacement as $1 - (1-p_i)^n$ would be. In fact, the arithmetic involved is prohibitive if n is at all large. At the present, following the suggestion of Yates and Grundy [9], np_i is used as an approximation of π_i .

One of the objectives of this thesis is to obtain an improved approximation of π_i . This is referred to as the "direct problem". The second objective is to solve approximately for p_i in the system $\pi_i(p_1, p_2, \dots, p_N) = C_i (i=1, \dots, N)$ where C_i are predetermined constants. This second problem, referred to as the "reverse problem", arises in connection with the efficiency of the estimate.

2 - The Estimator of Horvitz and Thompson

The first paper to provide unbiased estimators of the total measurement of the population and its variance when sampling without replacement, was prepared by Horvitz and Thompson [5]. They assumed that a sample of n units was selected without replacement from a population of N units. Their estimator of the total is reproduced below.

Suppose $Y_1, Y_2, \dots, Y_N$ are the measurements of the units in the population numbered serially from 1 to N and a sample of n units is drawn, say those numbered $i_1, i_2, \dots, i_n$. As an unbiased estimator of the total, $\sum_{i=1}^N Y_i$, Horvitz and Thompson proposed the statistic

$$\hat{Y} = \sum_{i \in s} \frac{Y_i}{\pi_i} \quad (1)$$

where the summation extends over all i contained in the set of integers $s: \{i_1, i_2, \dots, i_n\}$ and π_i is the probability that Y_i is in the sample. It is easy to show, that this estimator is unbiased and is the only possible unbiased linear estimator with the property that the weight of Y_i depends on i only and is independent of the other units in the sample and of the order of selection. In fact if

$$\sum_{i \in s} w_i Y_i$$

is an unbiased estimate of the total and π_s is the probability of obtaining sample s , then

$$E \left(\sum_{i \in s} w_i Y_i \right) = \sum_{i=1}^N Y_i$$

that is

$$\sum_s \pi_s \left(\sum_{i \in s} w_i Y_i \right) = \sum_{i=1}^N w_i Y_i \left(\sum_{s: i \in s} \pi_s \right) = \sum_{i=1}^N \pi_i w_i Y_i$$

therefore $w_i = \frac{1}{\pi_i}$ where $\sum_{s: i \in s}$ designates summation over all samples s containing i .

Note: In succeeding formulae, unless otherwise stated, all indexes of summation run from 1 to N ; so $\sum_{i < j}$ stands for summation over all pairs of integers i, j such that $i < j$ and $\sum_{j(\neq i)}$ indicates summation with respect to j from 1 to N excepting $j=i$.

The variance of Y is obtained by the author as

$$\text{Var}(\hat{Y}) = \sum \frac{1-\pi_i}{\pi_i} Y_i^2 - 2 \sum_{i < j} \frac{\pi_i \pi_j - \pi_{ij}}{\pi_i \pi_j} Y_i Y_j$$

where π_{ij} denotes the probability that both Y_i and Y_j are in the sample. It is assumed that π_i and π_{ij} are non-zero ($i \neq j$). To estimate this variance they recommend

$$\sum_{i \in s} \frac{1-\pi_i}{\pi_i} Y_i^2 - 2 \sum_{\substack{i, j \in s \\ i < j}} \frac{\pi_i \pi_j - \pi_{ij}}{\pi_i \pi_j \pi_{ij}} Y_i Y_j \quad (2)$$

It is easily shown that this is indeed an unbiased estimate of the variance. Its main deficiency is, however, that it is frequently negative and, partly for this reason, its variance is observed to be large.

3 - The Variance Estimator of Yates and Grundy

Yates and Grundy [10] were led to further research by the unfavourable properties of the variance estimator (2). Making use of the identities

$$\sum \pi_i = n \quad \sum_{j \neq i} \pi_{ij} = (n-1)\pi_i \quad \sum_{i < j} \pi_{ij} = \frac{n(n-1)}{2} \quad (3)$$

they were able to obtain $\text{Var}(\hat{Y})$ in another form:

$$\text{Var}(\hat{Y}) = \sum_{i < j} (\pi_i \pi_j - \pi_{ij}) \left(\frac{Y_i}{\pi_i} - \frac{Y_j}{\pi_j} \right)^2 \quad (4)$$

It follows from (4) that an unbiased estimate of $\text{Var}(\hat{Y})$ is given by:

$$\text{Est.Var}(\hat{Y}) = \sum_{\substack{i, j \in s \\ i < j}} \frac{\pi_i \pi_j - \pi_{ij}}{\pi_{ij}} \left(\frac{Y_i}{\pi_i} - \frac{Y_j}{\pi_j} \right)^2 \quad (5)$$

They advocated the use of $\hat{Y}$ as given by (1) with its variance estimated by (5) instead of (2). In order to derive explicit formulae for π_i and π_{ij} , one has to specify the sampling method. Probably the most commonly used method, apart from sampling with equal probabilities, is sampling with probabilities proportional to a measure of size, abbreviated by pps. It is assumed that there is a measurement M_i (referred to as "measure of size") that is determinable for each unit in the population before the sample is selected. The pps sampling method utilizes these M_i values in the process of selecting the sample by first determining the quantities

$$p_i = \frac{M_i}{\sum M_i}$$

and then selecting the first unit in such a way, that the probability that the i_1 -th unit is selected at this step equals p_{i_1} . Having selected the i_1 -th unit, the next unit is then chosen among the remaining ones with probabilities proportional to the remaining M_i . The probability of

selecting the i_2 -th unit at this step is then equal to

$$\frac{M_{i_2}}{\sum M_i - M_{i_1}} = \frac{p_{i_2}}{1 - p_{i_1}}$$

In general the probability of selecting the i_j -th unit at the j -th step, having already selected units $i_1, i_2, \dots, i_{j-1}$, equals $\frac{p_{i_j}}{1 - p_{i_1} - \dots - p_{i_{j-1}}}$.

The probability of obtaining the units numbered $i_1, i_2, \dots, i_n$ as our sample, and in this order, equals $p_{i_1} \frac{p_{i_2}}{1 - p_{i_1}} \dots \frac{p_{i_n}}{1 - p_{i_1} - \dots - p_{i_{n-1}}}$. The probability π_s of selecting these units in any order is given by

$$\pi_s = \sum p_{i_1} \frac{p_{i_2}}{1 - p_{i_1}} \dots \frac{p_{i_n}}{1 - p_{i_1} - \dots - p_{i_{n-1}}} \quad (6)$$

the $\sum$ indicating summation over all permutations of $i_1, i_2, \dots, i_n$.

Finally

$$\pi_i = \sum_{s \ni i} \pi_s \text{ and } \pi_{ij} = \sum_{s \ni i, j} \pi_s$$

where, as before, s denotes the set of integers $i_1, \dots, i_n$. The identities (1) are easily confirmed.

It has not yet been shown that with this sampling scheme (5) is a positive quantity. Des Raj [2] has proved it for the special case $n=2$. On the other hand an example is yet to be produced such that (5) is not positive under (pps).

It can be seen by substituting $\pi_i = \frac{n}{N}$ and $\pi_{ij} = \frac{n(n-1)}{N(N-1)}$ in (2) that $\hat{Y}$ reduces to the so-called best unbiased linear estimator in the special case when the probabilities p_i are all equal.

4 - The Estimators of Des Raj and Murthy

The uncertainty as to the positive value of the variance estimate (5) led to further research. In 1956 Des Raj proposed a new estimator for the total ΣY_i , and an estimator of its variance [2]. He succeeded in producing an estimate such that the corresponding variance estimate is always positive. His estimator of ΣY_i , however, seems to be inferior to $\hat{Y}$ for other reasons. First, it is not only dependent on the sample itself, but also on the order in which the sample units are drawn. Roy and Chakravarti [9] demonstrated, that it is always possible to improve the efficiency of such an estimator. It may be mentioned that this estimator fails to reduce to the so-called best unbiased linear estimator in the special case of equal probabilities ($\frac{1}{n} \Sigma_{i \in s} Y_i$).

Clearly there was room for improvement on Des Raj's estimator. Once the sample is drawn and the values $Y_{i_1}, Y_{i_2}, \dots, Y_{i_n}$ are available one can produce Des Raj's estimates corresponding to every possible order of drawing this sample. The weighted average of these estimates, (the weights being the conditional probabilities of obtaining the specific orders, given that the sample consists of the units numbered $i_1, i_2, \dots, i_n$) is also an unbiased estimate of ΣY_i . This was shown by Murthy in 1957 [8] following up a conjecture of Lahiri. This estimator overcame some of the shortcomings of Des Raj's estimator. In particular the weight of Y_{i_1} in the estimator is independent of the order in which the sample units were drawn and the estimator reduces to the so-called best unbiased linear estimator in the special case of equal probabilities. On the other hand, though the weight of Y_{i_1} is independent of the order in which the sample units were drawn, it is still dependent on the sample as a whole, not just on Y_{i_1} alone. In addition, the variance estimate is not

shown as necessarily being positive.

In comparing the relative merits of $\hat{Y}$ with the Yates-Grundy variance estimator and Murthy's estimator of the total and its variance estimator, the following differences are noted. $\hat{Y}$ with the Yates-Grundy variance estimator is of a very simple structure. The factor $\frac{\pi_{i_1} \pi_{i_2} \dots \pi_{i_n}}{\pi_{i_1 i_2 \dots i_n}}$ in (5) is a generalization of the well known variance factor for sampling with equal probabilities. Murthy's estimator in contrast is rather complicated and has little intuitive appeal; it involves two types of conditional probabilities. With respect to the calculations, both estimating methods are laborious, although Murthy's appears to be more so than Yates-Grundy's. In practice, both variance estimators seem to be positive, but this is only proved in the special case when $n=2$. In light of expediency, therefore, it would be preferable to use the Yates-Grundy estimate.

5 - Godambe's Generalization of the Notion of Efficiency

Another important consideration is the efficiency of the estimator. Godambe [6] has shown that there is no best estimate among the unbiased linear estimates of the total or mean. In so doing, he defines a linear estimate in the following way: consider all possible samples (s) admitted by the sampling procedure and define in advance for all conceivable samples of size n_s , a set of constants $(B_{si_1}, B_{si_2}, \dots, B_{si_{n_s}})$. Then the set of $\bar{B}_s$ vectors corresponding to the set of logically possible samples is a linear estimator, B. If sample s, consisting of units $i_1, i_2, \dots, i_{n_s}$ was realized, then e_s the linear estimate is defined as

$$e_s = B_{si_1} Y_{i_1} + B_{si_2} Y_{i_2} + \dots + B_{si_{n_s}} Y_{i_{n_s}} = (\bar{B}_s, \bar{Y}_s)$$

where $\bar{Y}_s$ is the vector $(Y_{i_1}, Y_{i_2}, \dots, Y_{i_{n_s}})$ and $(\bar{B}_s, \bar{Y}_s)$ is the scalar product of the two vectors $\bar{B}_s$ and $\bar{Y}_s$.

Godambe has shown, that there is no sampling procedure resulting in an unbiased linear estimator B_0 in which the corresponding e_{s_0} would have the optimum property:

$$\text{Variance } e_{s_0} \leq \text{Variance } e_s$$

regardless of which unbiased linear estimator has produced e_s . In this sense neither Yates-Grundy's nor Murthy's estimator can be the uniformly best unbiased linear estimator.

No estimator being the best, there might, however, be some that are better than others. An estimator is called inadmissible by Roy and Chakravarti if there is another linear unbiased estimator uniformly better than it, that is, having smaller variance regardless of the values $Y_1, Y_2, \dots, Y_N$. An admissible estimator is such that no other estimator is uniformly better than it. Roy and Chakravarti were able to show that the Horvitz - Thompson estimator is admissible. Even this much is not shown about the estimator of Murthy and because of its rather complicated nature it would seem to be a difficult task to attempt this proof. The estimator of Des Raj is inadmissible since Murthy's estimator is uniformly better.

It is a reasonable criterion to prefer an estimator that is better than any other estimator on the average. This criterion was accepted by Godambe. An intuitive interpretation of what he did might be stated as follows: given the measurements M_i for every unit of the population,

Godambe postulates the existence of an a priori conditional probability distribution for Y_i . This idea is not unique to Godambe. It can be found in Cochran's earlier works [1]. The given finite population is regarded as a sample of one from a super-population, consisting of classes of finite populations, a class being the collection of all finite populations of N units having a common set of related measurements M_i . Denoting the expected value under this a priori conditional probability distribution by ξ , then $\xi(Y_1), \xi(Y_2), \dots, \xi(Y_n)$ are assumed to be well-defined quantities.

A few simplifying assumptions are made:

- 1) $\xi(Y_i)$ is assumed to be unaffected by the drawing of units other than Y_i that is $\xi(Y_i | Y_j) = \xi(Y_i)$ or $\xi(Y_i Y_j) = \xi(Y_i) \xi(Y_j)$
- 2) $\xi[Y_i - \xi(Y_i)]^2 = v(Y_i)$ is assumed to exist.

Minimizing $\xi(\text{Var } e_s)$ under conditions which will ensure the unbiasedness of e_s Godambe obtains a system of equations for B_{si_j} . For a general sampling method he does not succeed in solving the equations. However, he solves the equations for a method satisfying three conditions:

- 1) the design specified that the sample is of size n
- 2) sampling is done without replacement
- 3) the probability of the i -th unit being included in the sample is proportional to $\xi(Y_i)$, that is

$$\pi_i = \frac{\xi(Y_i)}{\sum \xi(Y_i)}.$$

Under these conditions Godambe proves that the estimator with $B_{si} = \frac{1}{\pi_i}$ is the best in the broader sense that $\xi(\text{Var } e_s)$ is minimized. Since sampling with pps is a design satisfying the above conditions,

provided $\pi_i = n \frac{\xi(Y_i)}{\sum \xi(Y_i)}$, it follows that the Horvitz-Thompson estimator is the best unbiased linear estimator in this broader sense. If the further assumption is made, that $\xi(Y_i)$ is proportional to M_i , then condition 3) takes the form $\pi_i = n \frac{M_i}{\sum M_i} = np_i$. That is, in order to obtain the best estimator in the sense of Godambe, one would use the Horvitz-Thompson estimator, but instead of using the p_i proportions as the sampling probabilities, the system of equations $n \frac{M_i}{\sum M_i} = \pi_i(p_1, p_2, \dots, p_N)$ should be solved for $p_1, p_2, \dots, p_N$ where $p_1, p_2, \dots, p_N$ are constants which when used as the sampling probabilities (selecting the i_1 -th unit at first with probability p_{i_1} ; the i_2 -th, once i_1 was selected, with probability $\frac{p_{i_2}}{1-p_{i_1}}$; etc.) the corresponding π_i equal $n \frac{M_i}{\sum M_i}$.

Godambe also shows, that in fact, more is true if also $v(Y_i) \propto \xi^2(Y_i)$ holds. With this condition among all possible designs entailing the drawing of n units without replacement, the one satisfying the condition $\pi_i = n \frac{\xi(Y_i)}{\sum \xi(Y_i)}$ is the best design. However, if the design is such that this condition is satisfied, the Horvitz-Thompson estimator is the best estimator, the adjective "best" being interpreted in the broader sense as given by Godambe. Thus pps sampling with the Horvitz-Thompson estimator is the best design with the best estimator under one condition, $\xi(Y_i) = n \frac{M_i}{\sum M_i} = \pi_i$. To satisfy this condition, one has to solve the equations $\pi_i = n \frac{M_i}{\sum M_i}$ for p_i , as indicated in the previous paragraph.

It might be mentioned that Murthy also considered an inadmissible estimator, quoted as one obtained by Das, and applied to it the procedure

(later investigated in its generality by Roy and Chakravarti) of un-ordering the sample. Though he obtained a more efficient estimator, it was so hopelessly complicated that he didn't even go so far as to express the estimate explicitly in terms of Y_i . Mickey [7] in an article, apparently written years ago but published just recently, reproduced the estimates of Des Raj and Murthy.

6 - Statement of the Problem

In light of the foregoing considerations, it seems to be a reasonable decision to favor the Horvitz-Thompson estimator of the total coupled with the Yates-Grundy estimate of its variance. One has, however, two reservations. First of all, the amount of arithmetic looms high even for relatively small values of N and n . For $N=100$, $n=10$, the number of possible ordered samples is of the order of 10^{26} with a probability associated with each. To calculate any of these probabilities, quantities

like $p_{i_1} \frac{p_{i_2}}{1-p_{i_1}} \dots, \frac{p_{i_{10}}}{1-p_{i_1} - \dots - p_{i_9}}$ have to be calculated. To get a π_{ij}

value all samples involving i and j have to be considered and the corresponding probabilities added. The number of additions to get π_{ij} is of the order of 10^{26} . The total number of operations to get π_{ij} is of the order of 10^{27} . This number might be reduced using some shortcuts, but even so, the number of operations is rather prohibitive.

As was mentioned, Yates and Grundy proposed to circumvent this difficulty by using in place of π_i the value np_i . Thereby the biased estimator

$$\hat{Y}_B = \sum_{i \in s} \frac{Y_i}{np_i}$$

would be used in which the bias equals

$$\text{Bias} = \sum (\pi_i - np_i) \frac{Y_i}{np_i}$$

If $\frac{Y_i}{np_i}$ is close to a constant, the bias is insignificant. However, this may be an artificial assumption.

$$\text{Var} (\hat{Y}_B) = \sum_{i,j} (\pi_i \pi_j - \pi_{ij}) \left(\frac{Y_i}{np_i} - \frac{Y_j}{np_j} \right)^2$$

In order to estimate this variance, the π_i, π_j, π_{ij} quantities are needed. A biased estimate of the variance would be obtained should these quantities be replaced arbitrarily by the corresponding values had the sampling been with equal probabilities.

A further concern arises in obtaining the p_i values so that they result in π_i proportionate to M_i (the measure of size), as in the previous discussion. The need for this was already intuitively felt by Horvitz and Thompson. They produced an approximate solution of the problem for the special case $n=2$. Another solution was obtained by Yates and Grundy again for $n=2$. Both problems would be solved if a reasonably simple relationship between p_i and π_i could be obtained which would hold at least approximately. The way Horvitz and Thompson obtained this relationship is described below because the logic of it (though not the specific detail) lends itself to generalization.

Horvitz and Thompson start by noting that "if a sample of size 2 is to be drawn using one set of selection probabilities and with replacement, then the probability that element Y_i will be selected only once in the two draws is $2p_i(1-p_i)$. If the conditions are such that sampling without replacement is not much different from sampling with replacement, this probability will be approximately equal to π_i ". On account of this

observation they set $\pi_i = 2p_i (1-p_i) = 0$, then substituting $\pi_i = n \frac{M_i}{\sum M_i}$, they solve for p_i and adjust the results so that $\sum p_i = 1$.

This concludes the introductory remarks and the critical review of the relevant literature. In later Chapters, attention will be focused on the two problems referred to as the "direct problem" and the "reverse problem". The direct problem is that of finding an approximation of π_i when p_i are known. The reverse problem is that of solving $\pi_i(p_1, \dots, p_N) = n \frac{M_i}{\sum M_i}$ for p_i . Both problems can be considered as solved if a reasonably simple relationship is found which connects two sets of variables δ_i and p_i and is such that

- a) when substituting known constants for p_i and solving for δ_i , we get approximations to $\pi_i(p_1, \dots, p_N)$.
- b) when substituting the known constants $n \frac{M_i}{\sum M_i}$ for δ_i and solving for p_i , we get p_i values corresponding to which $\pi_i(p_1, \dots, p_N)$ are approximately equal to $n \frac{M_i}{\sum M_i}$.

Note: We will use in subsequent chapters, the quantities defined below:

$$V_K(m) = \frac{\sum_{i < j} \pi_{ij} (m_i - m_j)^2}{n-1}$$

$$= \sum \pi_i m_i^2 - \frac{2}{n-1} \sum_{i < j} \pi_{ij} m_i m_j \quad (7)$$

and

$$V(m) = \sum_{i < j} (\pi_i \pi_j - \pi_{ij}) (m_i - m_j)^2$$

$$= \sum \pi_i (1-\pi_i) m_i^2 - 2 \sum_{i < j} (\pi_i \pi_j - \pi_{ij}) m_i m_j \quad (8)$$

where m_i are variables. It may be mentioned, that if $m_i = \frac{Y_i}{\pi_i}$, $V(m)$ is equal to $\text{Var}(\hat{Y})$ while $V_k(m)$ is the expected value of

$$\frac{n \sum_{i \in s} (m_i - \bar{m})^2}{n-1} \quad (9)$$

where

$$\bar{m} = \frac{1}{n} \sum_{i \in s} m_i$$

(9) is an unbiased variance estimate of $\hat{Y}$ if the sampling method is that of (pps) with replacement [1].

CHAPTER II

THE CASE $n=2$

Chapter II is devoted to the case, when the sample size, n , is 2, drawn from a finite population of size N . The main results are collected in a single theorem. Later in the thesis a procedure will be introduced whereby we will have $N \rightarrow \infty$. Most of the results of the theorem below concern the error committed by estimating π_i and π_{ij} in a certain manner. The upper bounds of errors are of the form $\frac{G}{N^p}$ where p is a positive integer and G a quantity depending on N . It will turn out later, when the limiting process will be defined, that the constants G will all be bounded as $N \rightarrow \infty$ and so the inequalities in the theorem will take the form of order relations.

Theorem 1:

If $\frac{G_1}{N} < p_i < \frac{G_2}{N}$ and $\sum p_i^2 < \frac{1}{3}$ ($i = 1, \dots, N$) then among the solutions of the system of equations

$$\delta_i^2 - 4\delta_i + \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} p_i(1 - p_i) = 0 \quad i = 1, \dots, N$$

there is one and only one which satisfies the conditions

$$0 \leq \delta_i < 1$$

These values of δ_i are approximations of π_i with an accuracy given by

$$|\delta_i - \pi_i| < \frac{G_3}{N^3}$$

The set of equations in δ_{ij}

$$\delta_i \delta_j + 2\delta_{ij} - \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} p_i p_j = 0$$

have solutions such that

$$|\delta_{ij} - \pi_{ij}| < \frac{G_4}{N^4}$$

The relationships

$$\sum_{j(\neq i)} \delta_{ij} = \delta_i \quad \text{and} \quad \sum \delta_i = 2$$

hold. If $|Y_i| < \frac{G_5}{N}$ then

$$|E[\sum_{i \in s} \frac{Y_i}{\delta_i} - \hat{Y}]| < \frac{G_6}{N^2}$$

$$\text{Var} [\sum_{i \in s} \frac{Y_i}{\delta_i}]$$

is estimated by

$$\sum_{\substack{i, j \in s \\ i < j}} \frac{\delta_i \delta_j - \delta_{ij}}{\delta_{ij}} \left(\frac{Y_i}{\delta_i} - \frac{Y_j}{\delta_j} \right)^2$$

with a bias whose absolute value is less than $\frac{G_7}{N^2}$

Proof:

$$\pi_i = p_i \left(1+A - \frac{p_i}{1-p_i} \right) \quad \pi_{ij} = p_i p_j \left(\frac{1}{1-p_i} + \frac{1}{1-p_j} \right)$$

where $A = \sum_{i=1}^N \frac{p_i}{1-p_i}$. The formula for π_{ij} is obvious, and the one for π_i can be obtained from it by summing with respect to j . The following notation will be used:

$$B = \sum_{i=1}^N \frac{p_i^2}{1-p_i} \quad B(m) = \sum_{i=1}^N \frac{p_i^2}{1-p_i} m_i^2 \quad B(m^2) = \sum_{i=1}^N \frac{p_i^2}{1-p_i} m_i^2$$

$$A(m) = \sum_{i=1}^N \frac{p_i}{1-p_i} m_i \quad A(m^2) = \sum_{i=1}^N \frac{p_i}{1-p_i} m_i^2$$

$$P(m) = \sum_{i=1}^N p_i m_i \quad P(m^2) = \sum_{i=1}^N p_i m_i$$

Then

$$A = B + 1$$

$$A(m) = B(m) + P(m)$$

From the Special Note at the end of Chapter I:

$$\begin{aligned} V_k(m) &= \sum p_i (1+A-\frac{p_i}{1-p_i}) m_i^2 - 2 \sum_{i < j} p_i p_j (\frac{1}{1-p_i} + \frac{1}{1-p_j}) m_i m_j \\ &= (B+2)P(m^2) - B(m^2) - \sum_{i < j} \frac{p_i}{1-p_i} p_j m_i m_j - 2 \sum_{i < j} \frac{p_j}{1-p_j} p_i m_i m_j \\ &= (B+2)P(m^2) - B(m^2) - \sum \frac{p_i}{1-p_i} m_i [P(m) - p_i m_i] - \sum p_j m_j [A(m) - \frac{p_j}{1-p_j} m_j] \\ &= (B+2)P(m^2) + B(m^2) - 2P(m)A(m) \\ &= (B+2)P(m^2) + B(m^2) - 2B(m)P(m) - 2P^2(m) \\ &= (B+2)[P(m^2) - P^2(m)] + B[\frac{B(m^2)}{B} - \frac{B^2(m)}{B^2}] + B[\frac{B(m)}{B} - P(m)]^2 \quad (1) \end{aligned}$$

Similarly

$$\begin{aligned} V(m) &= \sum \pi_i (1-\pi_i) m_i^2 - 2 \sum_{i < j} \pi_i \pi_j m_i m_j + 2 \sum_{i < j} \pi_i \pi_j m_i m_j \\ &= \sum \pi_i (1-\pi_i) m_i^2 - \sum \pi_i m_i (\sum \pi_j m_j - \pi_i m_i) + 2 \sum_{i < j} \pi_i \pi_j m_i m_j \\ &= P(m^2)(A+1) - B(m^2) - [(A+1)P(m) - B(m)]^2 + 2A(m)P(m) - 2B(m^2) \\ &= (B+2)[P(m^2) - P^2(m)] - 3B[\frac{B(m^2)}{B} - \frac{B^2(m)}{B^2}] - B(B+3)[P(m) - \frac{B(m)}{B}]^2 \quad (2) \end{aligned}$$

Hence

$$V(m) + 3V_k(m) = 4(B+2)[P(m^2) - P^2(m)] - B^2[P(m) - \frac{B(m)}{B}]^2$$

From (7) and (8) of Chapter I follows that:

$$\begin{aligned} V(m) + 3V_K(m) &= \sum_{i < j} (\pi_i \pi_j - \pi_{ij})(m_i - m_j)^2 + 3 \sum_{i < j} \pi_{ij}(m_i - m_j)^2 \\ &= \sum_{i < j} (\pi_i \pi_j + 2\pi_{ij})(m_i - m_j)^2 \end{aligned}$$

Also:

$$P(m^2) - P^2(m) = \sum_{i < j} p_i p_j (m_i - m_j)^2$$

as is easily verified.

So

$$\sum_{i \neq j} (\pi_i \pi_j + 2\pi_{ij})(m_i - m_j)^2 = 4(B+2) \sum_{i \neq j} p_i p_j (m_i - m_j)^2 - 2B^2 \left[P(m) - \frac{B(m)}{B} \right]^2 \quad (3)$$

Setting $m_j = 0$ for every j except $j = i$ and $m_i = 1$:

$$\sum_{j(\neq i)} (\pi_i \pi_j + 2\pi_{ij}) = 4(B+2) \sum_{j(\neq i)} p_i p_j - 2B^2 \left[p_i - \frac{p_i^2}{B(1-p_i)} \right]^2$$

or summing with respect to $j(\neq i)$

$$4\pi_i - \pi_i^2 = 4(B+2)p_i(1-p_i) - 2B^2 \left[p_i - \frac{p_i^2}{B(1-p_i)} \right]^2 \quad (4)$$

Again in (3) setting $m_i = 1$, $m_j = 1$, $m_k = 0$ for every k except $k = i$

and $k = j$:

$$\begin{aligned} \sum_{k(\neq i, j)} (\pi_i \pi_k + 2\pi_{ik} + \pi_j \pi_k + 2\pi_{jk}) &= 4(B+2) \sum_{k(\neq i, j)} (p_i + p_j) p_k - 2B^2 \left[p_i + p_j - \frac{p_i^2}{B(1-p_i)} \right. \\ &\quad \left. - \frac{p_j^2}{B(1-p_j)} \right]^2 \end{aligned}$$

Using (4) we obtain (5) and (6) below.

$$\pi_i \pi_j + 2\pi_{ij} = 4(B+2)p_i p_j + 2B^2 \left[p_i - \frac{p_i^2}{B(1-p_i)} \right] \left[p_j - \frac{p_j^2}{B(1-p_j)} \right] \quad (5)$$

$$8 - \sum \pi_i^2 = 4(B+2)(1 - \sum p_i^2) - 2B^2 \sum p_i^2 \left[1 - \frac{p_i}{B(1-p_i)} \right]^2 \quad (6)$$

Now if $p_i < \frac{G_1}{N}$ ($i=1, \dots, N$) then

$$0 < B < \frac{H_2}{N} \quad \text{where} \quad H_2 = \frac{G_1^2}{1 - \frac{1}{N}}$$

and

$$0 < 2B^2 \sum p_i^2 \left[1 - \frac{p_i}{B(1-p_i)}\right]^2 < \frac{H_3}{N^3} \quad \text{where} \quad H_3 = 2H_2^2 G_1^2$$

From (6) then

$$0 < 4(B+2) - \frac{8 - \sum \pi_i^2}{1 - \sum p_i^2} < \frac{H_4}{N^3} \quad \text{where} \quad H_4 = \frac{H_3}{1 - \frac{1}{N}}$$

$$0 < 4(B+2)p_i p_j - \frac{8 - \sum \pi_i^2}{1 - \sum p_i^2} p_i p_j < \frac{H_4 G_1^2}{N^5}$$

Now from (5) it can be shown similarly, that

$$|\pi_i \pi_j + 2\pi_{ij} - \frac{8 - \sum \pi_i^2}{1 - \sum p_i^2} p_i p_j| < \frac{H_5}{N^4} \quad (7)$$

Summing with respect to $j (\neq i)$ we obtain

$$|\pi_i^2 - 4\pi_i + \frac{8 - \sum \pi_i^2}{1 - \sum p_i^2} p_i (1-p_i)| < \frac{H_5}{N^3} \quad (8)$$

where H_5 is again a bounded function of N .

Now if we consider the system of equations

$$\delta_i^2 - 4\delta_i + \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} p_i (1-p_i) = 0 \quad (9)$$

and having solved it we also solve for δ_{ij} (10) below:

$$\delta_i \delta_j + 2\delta_{ij} - \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} p_i p_j = 0 \quad (10)$$

we will have arrived at approximate values of π_i and π_{ij} , satisfying

$$\sum \delta_i = 2 \quad \sum_{j(\neq i)} \delta_{ij} = \delta_i$$

as follows by summing (9) and (10) with respect to i and j . To investigate the order of magnitude of the error involved in the approximation we put

$$\pi_i - \delta_i = \varepsilon_i \quad \max |\varepsilon_i| = \varepsilon$$

$$\pi_i = \delta_i + \varepsilon_i$$

$$\pi_i^2 = \delta_i^2 + \varepsilon_i(\pi_i + \delta_i)$$

Obviously

$$0 < \pi_i \leq 1$$

and we will assume for a moment that the same inequality holds for δ_i .

Then

$$\begin{aligned} |\sum \pi_i^2 - \sum \delta_i^2| &\leq \varepsilon \sum (\pi_i + \delta_i) = 4\varepsilon \\ \left| \frac{8 - \sum \pi_i^2}{1 - \sum p_i^2} - \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} \right| &\leq \frac{4}{1 - \frac{G_1}{N}} \varepsilon = H_6 \varepsilon \end{aligned} \quad (11)$$

From (8) and (9)

$$\begin{aligned} -\frac{H_5}{N^3} &< \pi_i^2 - 4\pi_i - \delta_i^2 + 4\delta_i + \frac{\sum \delta_i^2 - \sum \pi_i^2}{1 - \sum p_i^2} p_i(1-p_i) < \frac{H_5}{N^3} \\ -\frac{H_5}{N^3} - G_6 p_i(1-p_i)\varepsilon &< (\pi_i + \delta_i - 4)(\pi_i - \delta_i) < \frac{H_5}{N^3} + H_6 p_i(1-p_i)\varepsilon \end{aligned}$$

Hence

$$2|\varepsilon_i| < \frac{H_5}{N^3} + H_6 p_i(1-p_i)\varepsilon$$

$$\leq \frac{H_5}{N^3} + \frac{H_6}{4} \varepsilon$$

for every i .

Then

$$\frac{H_5}{N^3} > (2 - \frac{H_6}{4})\epsilon = \frac{1 - 2\frac{G_1^2}{N}}{1 - \frac{G_1^2}{N}} \epsilon = H_7 \epsilon$$

As is demonstrated later, the limiting process to be introduced is such, that the values G_1 will be bounded and hence H_7 will be positive for a sufficiently large value of N . Then

$$\epsilon < \frac{H_5}{H_7} \frac{1}{N^3} = \frac{H_8}{N^3}$$

and so

$$|\pi_i - \delta_i| < \frac{H_3}{N^3} = \frac{G_3}{N^3}$$

Since

$$\pi_i = p_i (1 + A \frac{p_i}{1-p_i}) < \frac{G_1}{N} (1 + \frac{G_1}{1 - \frac{G_1}{N}}) = \frac{H_9}{N}$$

it also follows that

$$\delta_i < \frac{H_{10}}{N} \quad \text{and similarly} \quad \frac{H_{11}}{N} < \delta_i$$

and then without difficulty (from 10))

$$|\delta_{ij} - \pi_{ij}| < \frac{G_4}{N^4}$$

It should be noted, that if in (9) we put $\delta_i = n \frac{M_i}{\sum M_i}$ we obtain a system of second degree equations in p_i whose solution will approximate the solution of the reverse problem. The order of the error is not quite so important since it does not contribute to bias.

We will estimate the bias introduced by using $Y_\delta = \sum_{i \in s} \frac{Y_i}{\delta_i}$ instead of $\hat{Y} = \sum_{i \in s} \frac{Y_i}{\pi_i}$

$$E[\hat{Y}_\delta] = \sum \frac{\pi_i}{\delta_i} Y_i$$

$$\text{Bias} = \sum \frac{\pi_i - \delta_i}{\delta_i} Y_i$$

If now

$$|Y_i| < \frac{G_5}{N}$$

$$|\text{Bias}| < \frac{H_8 G_5}{H_{11}} \frac{1}{N^2} = \frac{G_6}{N^2}$$

Similarly, it can be seen, that

$$\text{Var}[\hat{Y}_\delta] = \sum_{i < j} (\pi_i \pi_j - \pi_{ij}) \left(\frac{Y_i}{\delta_i} - \frac{Y_j}{\delta_j} \right)^2$$

while

$$E \left\{ \sum_{i < j} \frac{\delta_i \delta_j - \delta_{ij}}{\delta_{ij}} \left(\frac{Y_i}{\delta_i} - \frac{Y_j}{\delta_j} \right)^2 \right\} = \sum_{i < j} \frac{\pi_{ij}}{\delta_{ij}} (\delta_i \delta_j - \delta_{ij}) \left(\frac{Y_i}{\delta_i} - \frac{Y_j}{\delta_j} \right)^2$$

and it can be shown that these latter quantities differ from each other by a quantity less than a constant over N^2 .

Finally whenever $\sum p_i^2 \leq \frac{1}{3}$ we have $0 < \delta_i < 1$

Let the p_i be ordered by their magnitude from p_1 to p_N and denote

$$p_i(1-p_i) = r_i$$

If δ_N is so determined, that it satisfies the equation

$$\sum \sqrt{4 + \frac{r_i}{r_N} (\delta_N^2 - 4\delta_N)} = 2(N-1) \quad 0 < \delta_N < 1 \quad (12)$$

the solution of (9) is

$$\delta_i = 2 - \sqrt{4 + \frac{r_i}{r_N} (\delta_N^2 - 4\delta_N)}$$

In fact, substituting

$$4 - 4\delta_i + \delta_i^2 = (2 - \delta_i)^2 = 4 + \frac{r_i}{r_N}(\delta_N^2 - 4\delta_N)$$

$$\delta_i^2 - 4\delta_i = \frac{r_i}{r_N}(\delta_N^2 - 4\delta_N) \quad (13)$$

If (12) is satisfied, then $\delta_i = 2N - 2(N-1) = 2$, so summing (13)

$$\sum \delta_i^2 - 8 = \frac{1 - \sum p_i^2}{r_N}(\delta_N^2 - 4\delta_N)$$

and so

$$\delta_i^2 - 4\delta_i + \frac{8 - \sum \delta_i^2}{1 - \sum p_i^2} p_i(1 - p_i) = 0$$

(12) can be satisfied since $\sum \sqrt{4 + \frac{r_i}{r_N}(\delta_N^2 - 4\delta_N)} = F(\delta_N)$ is a continuous function of δ_N in the domain where all the square roots are real.

$$F(0) = 2N$$

$$F(1) = \sum \sqrt{4 - 3 \frac{r_i}{r_N}} < \sum (2 - \frac{3}{4} \frac{r_i}{r_N}) = 2N - \frac{3}{4} \frac{1 - \sum p_i^2}{p_N(1 - p_N)}$$

$$\leq 2(N-1)$$

Hence there is a value of δ_N such that $0 < \delta_N < 1$ and

$$F(\delta_N) = 2(N-1)$$

This value is even unique since in the interval $[0,1]$ $\frac{dF(\delta_N)}{d\delta_N} < 0$.

Hence

$$0 = 2 - \sqrt{4} \leq \delta_i < 2 - \sqrt{4 + \frac{r_i}{r_N}(\delta_N^2 - 4\delta_N)} < 2 - \sqrt{4 - 3} = 1$$

since $\frac{r_i}{r_N} \leq 1$. Further $\delta_i = 0$ only if $r_i = 0$ that is if $p_i = 0$.

An interesting corollary may be derived from the previous results.

Consider the method of sampling without replacement. Suppose that the probability of the event that the i -th unit is selected as the k -th sampling unit is independent of k and is equal to $\frac{M_i}{\sum M_i}$. Compare this method with that of sampling with pps without replacement and such that

$$\pi_i = n \frac{M_i}{\sum M_i}$$

Then the estimator

$$\sum_{i \in s} \frac{y_i}{\pi_i} \quad (14)$$

is an unbiased estimator under both sampling schemes.

Corollary:

The estimator (14) has a smaller variance under sampling without replacement than under sampling with replacement.

Proof:

From (1) and (2)

$$V_k(m) - V(m) = \sum_{i < j} (2\pi_{ij} - \pi_i \pi_j) (m_i - m_j)^2 \geq 0 \quad (15)$$

The quadratic form (15) in the variables m_i is positive semidefinite. According to Durbin [3] this is equivalent to the statement of the Corollary.

In other words, when $n=2$ and the reverse problem is solved, sampling without replacement is preferable to sampling with replacement.

In this Chapter, dealing with the case where $n=2$ the solution of

the reverse problem is of primary importance. The solution of the direct problem is of significance in establishing the method which can be adopted to the more general case where n is an arbitrary integer smaller than N . As for $n=2$, it would not be unduly difficult to calculate the exact values of π .

CHAPTER III

THE CASE OF GENERAL n

1 - The Recurrence Formula

When trying to imitate the procedure that worked for $n=2$ for the case $n>2$, we find the basic difficulty is the lack of a tractable formula for π_i and π_{ij} in terms of p_i ; or thinking of p_i as equaling π_i , when $n=1$, the lack of a formula connecting π_i^n and π_{ij}^n with π_i^{n-1} and π_{ij}^{n-1} . * In order to develop an approximate formula of this kind, we proceed as follows:

The event of including Y_i and Y_j in the sample of n units is made up of the following mutually exclusive events:

- a) Y_i and Y_j are both selected in the first $n-1$ draws.
- b) Y_i is selected in the first $(n-1)$ draws and Y_j in the n -th draw.
- c) Y_j is selected in the first $(n-1)$ draws and Y_i in the n -th draw.

The total probability equals π_{ij}^n . Case a) has probability π_{ij}^{n-1} . The probability of case b) is the product of two probabilities: $\Pr(Y_i, \text{ but not } Y_j, \text{ was selected in a sample of } n-1)$ and $\Pr(Y_j \text{ was selected in the } n\text{-th draw} \mid \text{a sample of } n-1 \text{ containing } Y_i \text{ but not } Y_j \text{ was selected before})$. The estimation of the probabilities involved will be dealt with in some length. Case c), of course, is similar to b).

*

Superscripts are introduced to refer to the sample size.

The probability of the event " Y_i was selected in a sample of $n-1$, but not Y_j " (in case no one p dominates the distribution) is slightly overestimated by π_i^{n-1} . Invoking the logic of Horvitz and Thompson, if "conditions are such that sampling without replacement is not much different from sampling with replacement" the probability in question would be $(n-1)p_i (1-p_i-p_j)^{n-2}$ while π_i^{n-1} would be $(n-1)p_i (1-p_i)^{n-2}$. If all p_i are of the order of $\frac{1}{N}$, the ratio of the two expressions is of the order of $(\frac{N-2}{N-1})^{n-2}$.

The second probability is estimated, in the same spirit, by its value, had the design been that of "sampling with partial replacement", replacing at each draw everything but Y_i . Then the second probability is estimated by $\frac{p_j}{1-p_i}$.

One can be more rigorous in the interpretation of the statement "the situation is very much like sampling with replacement, or sampling from an infinite population". When looking for a process whereby in the limit the finite population approaches an infinite population, one has several choices. In the limit, however, we want to have a population with N classes of elements, the probability of selecting a single element from the i -th class being p_i . It seems to be a natural process to duplicate, triplicate, quadruplicate, etc., every element in the population while reducing the corresponding probabilities by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. At the K -th step, we will have N classes with K distinguishable elements in each class, the probability of selecting Y_{ik} , the k -th element in the i -th class being $p_{ik} = \frac{p_i}{K}$ ($i=1, \dots, N$; $k=1, \dots, K$). The total number of elements is NK .

Notation:

$\Pr(Y_{ik} \text{ was selected in a sample of } n \text{ but not } Y_{jm}) = \pi_{ik|jm}^n$

$\Pr(Y_{ik} \text{ was selected in a sample of } n) = \pi_{(ik)}^n$. $\pi_{(ik)}^n$ is different from π_{ik}^n .

$\Pr(\text{a sample of } n-1 \text{ was selected containing } Y_{ik}, \text{ but not } Y_{jm} \text{ and } Y_{jm} \text{ was selected at the } n\text{-th draw}) = \pi_{(ik)jm}^n$

Theorem 2:

If the number of elements tends to infinity as described above:

$$\lim_{K \rightarrow \infty} \frac{\pi_{ik|jm}^n}{\pi_{(ik)}^n} = 1, \quad \lim_{K \rightarrow \infty} \frac{\pi_{(ik)}^n}{n p_{ik}} = 1, \quad \lim_{K \rightarrow \infty} \frac{\pi_{(ik)jm}^n}{\pi_{ik|jm}^{n-1}} \frac{1-p_{ik}}{p_{jm}} = 1$$

Proof:

$$\pi_{ik|jm}^n = \sum \frac{p_{i_1 k_1} p_{i_2 k_2} \dots p_{i_{n-1} k_{n-1}}}{(1-p_{i_1 k_1})(1-p_{i_1 k_1} - p_{i_2 k_2}) \dots (1-p_{i_1 k_1} - \dots - p_{i_{n-2} k_{n-2}})}$$

the summation being taken over possible selections of ordered samples of elements from NK elements always including Y_{ik} and excluding Y_{jm} . Now $p_{i_r k_r} = \frac{p_i}{K}$ and hence we can rewrite the sum by adding first all the terms corresponding to a particular selection of an ordered sample of size n selected with replacement from among the N classes, always including the i -th class. If $K \geq n$, there are $N^n - (N-1)^n$ such ordered samples of classes, each contributing n_s terms to the summation above $s=1, 2, \dots, N^n - (N-1)^n$.

Then

$$\pi_{ik|jm}^n = \sum_s n_s \frac{\frac{p_{i_1}}{K} \frac{p_{i_2}}{K} \dots \frac{p_{i_{n-1}}}{K}}{(1 - \frac{p_{i_1}}{K})(1 - \frac{p_{i_1}}{K} - \frac{p_{i_2}}{K}) \dots (1 - \frac{p_{i_1}}{K} - \dots - \frac{p_{i_{n-2}}}{K})}$$

The number n_s depends on the number of different classes selected for sample s . If $i_1, i_2, \dots, i_{n-1}$ are all different from one another and from i and j , then $n_s = K^{n-1}$. If they are all equal but different from i and j , $n_s = \binom{K}{n-1} (n-1)! = K(K-1) \dots (K-n+2)$. For other cases if there are t classes represented in the sample, each $n_1, n_2, \dots, n_t$ times ($n_1 + n_2 + \dots + n_t = n-1$) and if none of these classes is the i -th or j -th then

$$\begin{aligned} n_s &= \binom{K}{n_1} \binom{K}{n_2} \dots \binom{K}{n_t} n_1! n_2! \dots n_t! \\ &= K(K-1) \dots (K-n_1+1) K(K-1) \dots (K-n_2+1) \dots K(K-1) \dots (K-n_t+1) \end{aligned}$$

If any of the numbers $i_1, i_2, \dots, i_{n-1}$ happens to be i or j , the corresponding binomial coefficient is modified by the replacement of K by $K-1$. In any case $\frac{n_s}{K^{n-1}} \rightarrow 1$ as $K \rightarrow \infty$. Hence

$$\lim_{K \rightarrow \infty} K \pi_{ik|jm}^n = \sum_s p_{i_1} p_{i_2} \dots p_{i_{n-1}}$$

Dividing through by p_i , we will get every term $p_{i_1} p_{i_2} \dots p_{i_{n-1}}$ (corresponding to the selection of $s'(i_1, i_2, \dots, i_{n-1})$ with replacement from among $1, 2, \dots, N$) exactly n times, once for each position of p_i before the division. Hence

$$\lim_{K \rightarrow \infty} \frac{K \pi_{ik|jm}^n}{n p_i} = \lim_{K \rightarrow \infty} \frac{\pi_{ik|jm}^n}{n p_{ik}} = \sum_{s'} p_{i_1} p_{i_2} \dots p_{i_{n-1}} = 1$$

The last equality is easily obtained by summing on the subscripts one by one.

A similar argument shows that $\frac{\pi_{(ik)}^n}{n p_{ik}} \rightarrow 1$ as $K \rightarrow \infty$. Hence

$$\lim_{K \rightarrow \infty} \frac{\pi_{ik|jm}^n}{\pi_{(ik)}^n} = \lim_{K \rightarrow \infty} \frac{\pi_{ik|jm}^n}{n p_{ik}} \cdot \frac{n p_{ik}}{\pi_{(ik)}^n} = 1$$

It remains to be shown that

$$\lim_{K \rightarrow \infty} \frac{\pi_{(ik)jm}^n}{\pi_{ik|jm}^{n-1}} \cdot \frac{1-p_{ik}}{p_{jm}} = 1$$

or, since $p_{ik} \rightarrow 0$, in light of the previous considerations regarding $\pi_{ik|jm}^{n-1}$ equivalently

$$\lim_{K \rightarrow \infty} \frac{\pi_{(ik)jm}^n}{(n-1)p_{ik}p_{jm}} = \lim_{K \rightarrow \infty} K^2 \frac{\pi_{(ik)jm}^n}{(n-1)p_i p_j} = 1$$

$$\begin{aligned} \pi_{(ik)jm}^n &= \sum \frac{p_{i_1 k_1} p_{i_2 k_2} \dots p_{ik} \dots p_{i_{n-2} k_{n-2}} p_{jm}}{(1-p_{i_1 k_1}) \dots (1-p_{i_1 k_1} \dots p_{ik} \dots p_{i_{n-2} k_{n-2}})} \\ &= \sum n_s \frac{\frac{p_{i_1}}{K} \frac{p_{i_2}}{K} \dots \frac{p_i}{K} \dots \frac{p_{i_{n-2}}}{K} \frac{p_j}{K}}{(1 - \frac{p_{i_1}}{K}) \dots (1 - \frac{p_{i_1}}{K} \dots \frac{p_i}{K} \dots \frac{p_{i_{n-2}}}{K})} \end{aligned}$$

The first summation is extended over all ordered samples of n elements that can be selected from among the NK units, such that p_{jm} is the n -th selected unit and p_{ik} is selected always. The second summation extends over all possible ordered samples of n classes out of N , selected with replacement, and such that the j -th class is the n -th selected class and the i -th class is selected. As before, it can be seen that we now have

$$\frac{n_s}{K^{n-2}} \rightarrow 1, \text{ hence}$$

$$\lim_{K \rightarrow \infty} K^2 \pi_{(ik)jm}^n = p_j \sum p_{i_1} p_{i_2} \dots p_i \dots p_{i_{n-2}}$$

the summation being taken over all possible ordered samples of $n-1$ classes out of N , i always being selected. Again, as before, when

dividing each term of the summation by p_i , we will get exactly $(n-1)$ times each term $p_{i_1} \dots p_{i_{n-2}}$ corresponding to the selection of $i_1, i_2, \dots, i_{n-2}$ with replacement out of $1, 2, \dots, N$. So

$$\lim_{K \rightarrow \infty} K^2 \pi_{(ik)jm}^n = (n-1)p_i p_j$$

which was to be shown.

Theorem 2 may be generalized.

Definition: We say that sampling without replacement approaches sampling with replacement as the size of the population $N \rightarrow \infty$ if $p_1, \dots, p_N$ can be divided into N_0 classes with $K_i(N)$ elements in each ($i=1, \dots, N_0$) such that for each class there are numbers $m_i(N)$, $M_i(N)$ and

$$a) m_i(N) \leq p_{ik}(N) \leq M_i(N) \quad (i=1, \dots, N_0, \quad k=1, \dots, K_i(N))$$

$$b) \lim_{N \rightarrow \infty} K_i(N)m_i(N) = \lim_{N \rightarrow \infty} K_i(N)M_i(N) = t_i > 0 \quad \text{and} \quad t_{i+1} \geq t_i$$

and

$$c) \lim_{N \rightarrow \infty} \frac{K_i(N)}{K_j(N)} = 1$$

Theorem 3:

If sampling without replacement approaches sampling with replacement, then

$$\lim_{N \rightarrow \infty} \frac{\pi_{ik|jm}^n}{\pi_{(ik)}^n} = 1 \quad \lim_{N \rightarrow \infty} \frac{\pi_{ik|jm}^n}{n p_{ik}} = 1 \quad \lim_{N \rightarrow \infty} \frac{\pi_{(ik)jm}^{n-1}}{\pi_{ik|jm}^{n-1}} \frac{1-p_{ik}}{p_{jm}} = 1$$

Proof:

The sum representing $\pi_{ik|jm}^n$ can be rearranged as

$$\pi_{ik|jm}^n = \sum_s \sum_{k_s} \frac{p_{i_1 k_1} \dots p_{i_k k} \dots p_{i_{n-1} k_{n-1}}}{(1-p_{i_1 k_1}) \dots (1-p_{i_1 k_1} - \dots - p_{i_k k} - \dots - p_{i_{n-1} k_{n-1}})}$$

the first sum extending over all ordered samples $s (i_1, \dots, i_{n-1})$ containing i and drawn with replacement out of the N classes, the second over all samples k_s drawn from the classes, such that the number of units selected from each class equals the number of times the class was selected in s , always selecting Y_{ik} however, but never Y_{jm} . The number of terms, n_s under $\sum_{k_s}$ (in analogy to the proof of Theorem 2) can be seen to be dependent on the sample s . If there are t different among the units $i_1, \dots, i_{n-1}$, say $j_1, \dots, j_t$, and they are selected $n_{j_1}, \dots, n_{j_t}$ times ($n_{j_1} + \dots + n_{j_t} = n-1$) then

$$n_s = \binom{K_{j_1}}{n_{j_1}} \dots \binom{K_{j_t}}{n_{j_t}} n_{j_1}! \dots n_{j_t}!$$

provided none of $j_1, \dots, j_t$ is i or j . If any of them is i or j , the corresponding binomial coefficient becomes $\binom{K_i-1}{n_i}$ or $\binom{K_j-1}{n_j}$. In any case,

it can be seen as in the proof of Theorem 2, that $\lim_{N \rightarrow \infty} \frac{n_s}{(K_{j_1})^{n_{j_1}} \dots (K_{j_t})^{n_{j_t}}} = 1$

On the other hand by condition a)

$$\sum_s n_s \frac{m_{i_1} \dots m_{i_{n-1}}}{(1-m_{i_1}) \dots (1-m_{i_1} - m_{i_1} - \dots - m_{i_{n-1}})} \leq \pi_{ik|jm}^n \leq \sum_s n_s \frac{M_{i_1} \dots M_{i_{n-1}}}{(1-M_{i_1}) \dots (1-M_{i_1} - M_{i_1} - \dots - M_{i_{n-1}})}$$

Then clearly

$$\lim_{N \rightarrow \infty} n_s \frac{m_{i_1} \dots m_{i_{n-1}}}{1} = \lim_{N \rightarrow \infty} n_s \frac{M_{i_1} \dots M_{i_{n-1}}}{1} = t_{i_1} \dots t_{i_{n-1}}$$

so

$$\lim_{N \rightarrow \infty} K_i \pi_{ik|jm}^n = \sum_s t_{i_1} \dots t_{i_{n-1}}$$

or dividing by t_i

$$\lim_{N \rightarrow \infty} \frac{K_i \pi_{ik|jm}^n}{t_i} = \sum t_{i_1} \dots t_{i_{n-1}}$$

the summation extending over all samples $(i_1, \dots, i_{n-1})$ drawn with replacement from the N_0 units $1, \dots, N_0$. The sum is equal to one if

$$\sum_{i=1}^{N_0} t_i = 1$$

It is easy to see that this is in fact the case, since from a) summing with respect to $i (i=1, \dots, N_0)$ and $k (k=1, \dots, K_i)$

$$\sum_{i=1}^{N_0} K_i m_i \leq 1 \leq \sum_{i=1}^{N_0} K_i M_i$$

Letting $N \rightarrow \infty$ we obtain

$$\sum_{i=1}^{N_0} t_i \leq 1 \leq \sum_{i=1}^{N_0} t_i$$

and so

$$\sum_{i=1}^{N_0} t_i = 1$$

Since

$$\lim_{N \rightarrow \infty} \frac{t_i}{K_i p_{ik}} = 1$$

it follows that

$$\lim_{N \rightarrow \infty} \frac{\pi_{ik|jm}^n}{n p_{ik}} = 1$$

The rest of the statements are proved similarly to the corresponding statements in Theorem 2.

Remark: It may be mentioned, that due to condition c)

$$\lim_{N \rightarrow \infty} \frac{K_i}{N} = \text{finite}$$

so $O(\frac{1}{K_i}) = O(\frac{1}{N})$ and so the probabilities $\pi_{ik|jm}$ and $\pi_{(ik)}$ are of $O(\frac{1}{N})$.

We will adopt again a single subscript for the elements of the population, writing for example $\pi_{(i)j}^n$ for $\pi_{(ik)jm}^n$ and π_i^n for $\pi_{(ik)}^n$. Throughout this chapter we shall use $\alpha_{ij}, \alpha'_{ij}, \dots$ to denote quantities which are $O(\frac{1}{N^2})$ and $\beta_i, \beta'_i, \dots$ to denote quantities which are $O(\frac{1}{N})$.

We may now formulate the following corollary.

Corollary:

If sampling without replacement approaches sampling with replacement, then

$$\pi_{ij}^n = \pi_{ij}^{n-1} + \pi_i^{n-1} \frac{p_j}{1-p_i} + \pi_j^{n-1} \frac{p_i}{1-p_j} + \alpha_{ij}$$

and

$$\pi_i^{(n)} = \pi_i^{(n-1)} + \frac{p_i}{n-1} \left(\sum \frac{\pi_j^{n-1}}{1-p_j} - \frac{\pi_i^{n-1}}{(1-p_i)} \right) + \beta_i$$

where

$$\sum_{j(\neq i)} \alpha_{ij} = \beta_i, \quad \sum \beta_i = 0$$

Proof:

We write

$$\pi_{(i)j}^{n-1} = \frac{\pi_{(i)j}^n}{\pi_i^{(n-1)}} \pi_i^{n-1}$$

then since

$$\lim_{N \rightarrow \infty} \frac{\frac{\pi_{(i)j}^n}{\pi_i^{n-1}} - \frac{p_j}{1-p_i}}{\frac{p_j}{1-p_i}} = 0$$

and $p_j = O(\frac{1}{N})$, it follows that

$$\frac{\pi_{(i)j}^n}{\pi_i^{n-1}} = \frac{p_j}{1-p_i} + o(\frac{1}{N})$$

Since by the above remark also $\pi_i^n = O(\frac{1}{N})$ it follows that

$$\begin{aligned} \pi_{(i)j}^n &= \pi_{ij}^{n-1} + \pi_{(i)j}^{n-1} + \pi_{(j)i}^{n-1} \\ &= \pi_i^{n-1} \frac{p_j}{1-p_i} + o(\frac{1}{N^2}) \end{aligned}$$

and hence

$$\pi_{ij}^n = \pi_{ij}^{n-1} + \pi_{(i)j}^{n-1} + \pi_{(j)i}^{n-1} = \pi_{ij}^{n-1} + \pi_i^{n-1} \frac{p_j}{1-p_i} + \pi_j^{n-1} \frac{p_i}{1-p_j} + \alpha_{ij} \quad (1)$$

Using (1)

$$\begin{aligned} \pi_i^n &= \frac{1}{n-1} \left[\sum_{j(\neq i)} \pi_{ij}^{n-1} + \frac{\pi_i^{n-1}}{1-p_i} \sum_{j(\neq i)} p_j + p_i \sum_{j(\neq i)} \frac{\pi_j^{n-1}}{1-p_j} + \sum_{j(\neq i)} \alpha_{ij} \right] \\ &= \pi_i^{n-1} + \frac{p_i}{n-1} \left(\sum_{j(\neq i)} \frac{\pi_j^{n-1}}{1-p_j} - \frac{\pi_i^{n-1}}{1-p_i} \right) + \beta_i \quad (2) \end{aligned}$$

It is easy to see, that since (3) of Chapter I is satisfied, the sum of the β_i terms is zero. In fact

$$\sum \left[\pi_i^{n-1} + \frac{p_i}{n-1} \left(\sum_{j(\neq i)} \frac{\pi_j^{n-1}}{1-p_j} - \frac{\pi_i^{n-1}}{1-p_i} \right) \right] = n-1 + \frac{1}{n-1} \sum \frac{\pi_j^{n-1}}{1-p_j} - \frac{1}{n-1} \sum \frac{\pi_i^{n-1} p_i}{1-p_i} =$$

$$= n-1 + \frac{1}{n-1} \sum \pi_i^{n-1} = n$$

So

$$\sum \beta_i = 0$$

One more remark: the fact that $\frac{\pi_{(ik)}^n}{n p_{ik}} \rightarrow 1$ suggests, that as a rough approximation to π_i^n , $n p_i$ can be used.

2 - The Initial Approximation

In the course of trying to duplicate the method used for $n=2$, the next step would be to consider $V_k^n(m)$ and $V^n(m)$ as given in Chapter I, Section 7. Some notation will be necessary. First of all approximate values of π_i^n and π_{ij}^n as obtained in this Chapter will be denoted by δ_i and δ_{ij} respectively.

Theorem 4:

If sampling without replacement approaches sampling with replacement, then the system of equations

$$\delta_i^2 - 2n \delta_i + \frac{2n^2 - \sum \delta_i^2}{1 - \sum p_i^2} p_i(1-p_i) = 0 \quad (i=1,2,\dots,N)$$

is satisfied by a set of values δ_i such that $\delta_i - \pi_i^n = o(\frac{1}{N})$ whenever $\sum p_i^2 \leq \frac{1}{2}$. In this case

$$\delta_i \delta_j + \frac{n}{n-1} \delta_{ij} - \frac{2n^2 - \sum \delta_i^2}{1 - \sum p_i^2} p_i p_j = 0 \quad (i,j = 1,2,\dots,N; i \neq j)$$

as equations in δ_{ij} have solutions such that $\delta_{ij} - \pi_{ij}^n = o(\frac{1}{N^2})$

The identities

$$\sum \delta_i = n$$

$$\sum_{j \neq i} \delta_{ij} = (n-1) \delta_i$$

hold. If such a solution exist, then $\delta_i \neq 0$.

Proof:

Let

$$A^n = \sum \frac{\pi_i^n}{1-p_i}$$

$$B^n = \sum \frac{\pi_i^n p_i}{1-p_i}$$

$$A^n(m) = \sum \frac{\pi_i^n}{1-p_i} m_i$$

$$B^n(m) = \sum \frac{\pi_i^n p_i}{1-p_i} m_i$$

$$A^n(m^2) = \sum \frac{\pi_i^n}{1-p_i} m_i^2$$

$$B^n(m^2) = \sum \frac{\pi_i^n p_i}{1-p_i} m_i^2$$

$$D^n(m) = \sum \pi_i^n m_i$$

$$P(m) = \sum p_i m_i$$

$$D^n(m^2) = \sum \pi_i^n m_i^2$$

$$P(m^2) = \sum p_i m_i^2$$

where $m_1, m_2, \dots, m_N$ are arbitrary variables.

The identities $A^{n-1} = (n-1) + B^{n-1}$; $A^{n-1}(m) = D^{n-1}(m) + B^{n-1}(m)$ are easy to check. Now using (1) and (2) we obtain

$$\begin{aligned} V^n(m) + (2n-1) V_k^n(m) &= 2n \sum \pi_i^n m_i^2 - (\sum \pi_i^n m_i)^2 - \frac{2n}{n-1} \sum_{i < j} \pi_{ij}^n m_i m_j \quad (3) \\ &= 2n D^{n-1}(m^2) + \frac{2n A^{n-1}}{n-1} P(m^2) - \frac{2n}{n-1} B^{n-1}(m^2) \\ &= [D^{n-1}(m) + \frac{A^{n-1}}{n-1} P(m) - \frac{1}{n-1} B^{n-1}(m)]^2 \\ &\quad - \frac{2n}{n-1} \sum_{i < j} \pi_{ij}^{n-1} m_i m_j - \frac{2n}{n-1} \sum_{i < j} \left(\frac{\pi_i^{n-1} p_i}{1-p_i} + \frac{\pi_j^{n-1} p_j}{1-p_j} \right) m_i m_j \\ &\quad + \sum \beta_i m_i^2 + \sum_{i < j} \alpha_{ij} m_i m_j \end{aligned}$$

the last equality being a consequence of (1) and (2). Here β_i and α_{ij} are not necessarily the same as the constants before.

Now from (3)

$$\sum_{i < j} \pi_{ij}^{n-1} m_i m_j = - \frac{(n-2)}{2(n-1)} [V^{n-1}(m) + (2n-3) V_k^{n-1}(m)] + (n-2) \sum \pi_i^{n-1} m_i^2$$

$$= \frac{n-2}{2(n-1)} (\sum \pi_i^{n-1} m_i)^2$$

so

$$V^n(m) + (2n-1) V_k^n(m) = \frac{n(n-2)}{(n-1)^2} [V^{n-1}(m) + (2n-3) V_k^{n-1}(m)] - \frac{2n(n-2)}{n-1} D^{n-1}(m^2)$$

$$+ \frac{n(n-2)}{(n-1)^2} [D^{n-1}(m)]^2 + 2n D^{n-1}(m^2) + \frac{2nA^{n-1}}{n-1} P(m^2)$$

$$- \frac{2n}{n-1} B^{n-1}(m^2) - (D^{n-1}(m) + \frac{A^{n-1}}{n-1} P(m) - \frac{1}{n-1} B^{n-1}(m))^2$$

$$= \frac{2n}{n-1} A^{n-1}(m) P(m) + \frac{2n}{n-1} B^{n-1}(m^2) + \sum \beta_i m_i^2 + \sum_{i < j} \alpha_{ij} m_i m_j$$

As in the case of $n=2$, $B^{n-1}(m^2)$ cancels out, and after some algebra, we obtain

$$V^n(m) + (2n-1) V_k^n(m) = \frac{(n-2)}{(n-1)^2} [V^{n-1}(m) + (2n-3) V_k^{n-1}(m)] + \frac{2n}{n-1} A^{n-1} \sum p_i (m_i - P(m))^2$$

$$+ 2n \sum \frac{\pi_i^{n-1}}{n-1} (m_i - \frac{D^{n-1}(m)}{n-1})^2 + (2n-1) (\frac{D^{n-1}(m)}{n-1} - P(m))^2$$

$$+ 2B^{n-1}(P(m) - \frac{D^{n-1}(m)}{n-1}) (P(m) - \frac{B^{n-1}(m)}{B^{n-1}}) - \frac{(B^{n-1})^2}{(n-1)^2} (P(m) -$$

$$- \frac{B^{n-1}(m)}{B^{n-1}})^2 + \sum \beta_i m_i^2 + \sum_{i < j} \alpha_{ij} m_i m_j$$

Now setting $m_j = 0$ for every $j \neq i$ and $m_i = 1$:

$$\begin{aligned}
 2n \pi_i^n - (\pi_i^n)^2 &= \frac{n(n-2)}{(n-1)^2} [2(n-1) \pi_i^{n-1} - (\pi_i^{n-1})^2] + \frac{2n}{n-1} A^{n-1} p_i (1-p_i) \\
 &+ 2n \frac{\pi_i^{n-1}}{n-1} (1 - \frac{\pi_i^{n-1}}{n-1} + (2n-1)(\frac{\pi_i^{n-1}}{n-1} - p_i)^2 \\
 &+ 2B^{n-1} (p_i - \frac{\pi_i^{n-1}}{n-1})(p_i - \frac{p_i \pi_i^{n-1}}{B^{n-1}(1-p_i)}) \\
 &- \frac{(B^{n-1})^2}{(n-1)^2} (p_i - \frac{p_i \pi_i^{n-1}}{B^{n-1}(1-p_i)})^2 + \beta_i
 \end{aligned} \tag{4}$$

Setting $m_i = 1$, $m_j = 1$, zero for the other m values and making use of (4):

$$\begin{aligned}
 -2\pi_i^n \pi_j^n - \frac{2n}{n-1} \pi_{ij}^n &= \frac{n(n-2)}{(n-1)^2} [-2\pi_i^{n-1} \pi_j^{n-1} - \frac{2(n-1)}{n-2} \pi_{ij}^{n-1}] - 2 \frac{2n}{n-1} A^{n-1} p_i p_j \\
 &- 4n \frac{\pi_i^{n-1}}{n-1} \frac{\pi_j^{n-1}}{n-1} + 2(2n-1)(\frac{\pi_i^{n-1}}{n-1} - p_i)(\frac{\pi_j^{n-1}}{n-1} - p_j) \\
 &+ 2B^{n-1}(p_i - \frac{\pi_i^{n-1}}{n-1})(p_j - \frac{p_j \pi_j^{n-1}}{B^{n-1}(1-p_j)}) \\
 &+ 2B^{n-1}(p_j - \frac{\pi_j^{n-1}}{n-1})(p_i - \frac{p_i \pi_i^{n-1}}{B^{n-1}(1-p_i)}) \\
 &- \frac{(B^{n-1})^2}{(n-1)^2} (p_i - \frac{p_i \pi_i^{n-1}}{B^{n-1}(1-p_i)})(p_j - \frac{p_j \pi_j^{n-1}}{B^{n-1}(1-p_j)}) + \alpha_{ij}
 \end{aligned} \tag{5}$$

(5) when summed with respect to $j (\neq i)$ yields (4). Now let us observe, that since

$$\pi_i^{n-1} = (n-1)p_i + o(\frac{1}{N})$$

we can in (4) and (5) make this substitution in the third term, correspondingly modifying β_i and α_{ij} to β_i' and α_{ij}' respectively, and still have (5) add up to (4). Also, since $\sum \pi_i^{n-1} = (n-1) \sum p_i$, and

$\sum \beta_i = 0$, and $p_i = O(\frac{1}{N})$ we have from (4)

$$\sum \beta_i' = o(\frac{1}{N})$$

The fourth to sixth terms of (4) and (5) are all of $o(\frac{1}{N^2})$. We can neglect them and correspondingly modify α_{ij}' and β_i' to α_{ij}'' and β_i'' . Then

$$\sum \beta_i'' = o(\frac{1}{N}) \quad (6)$$

It should be emphasized that this is true although β_i'' itself is $o(\frac{1}{N})$.

Then

$$\pi_i^n \pi_j^n + \frac{n}{n-1} \pi_{ij}^n = \frac{n(n-2)}{(n-1)^2} [\pi_i^{n-1} \pi_j^{n-1} + \frac{n-1}{n-2} \pi_{ij}^{n-1}] + 2n(\frac{A^{n-1}}{n-1} + 1)p_i p_j + \alpha_{ij}'' \quad (7)$$

and we can obtain from (4) or equivalently from (5), summing with respect to $(j \neq i)$:

$$2n \pi_i^n - (\pi_i^n)^2 = \frac{n(n-2)}{(n-1)^2} [2(n-1) \pi_i^{n-1} - (\pi_i^{n-1})^2] + 2n(\frac{A^{n-1}}{n-1} + 1)p_i(1-p_i) + \beta_i'' \quad (8)$$

Using the recurrence formulae (7) and (8) simultaneously over and over again until on the right hand side $\pi_i^{(2)}$, $\pi_{ij}^{(2)}$ appear:

$$\begin{aligned} \pi_i^n \pi_j^n + \frac{n}{n-1} \pi_{ij}^n &= \frac{n(n-2)}{(n-1)^2} [\pi_i^{n-1} \pi_j^{n-1} + \frac{n-1}{n-2} \pi_{ij}^{n-1}] \\ &\quad + \frac{2n}{n-1} (A^{n-1} + n-1)p_i p_j + \alpha_{ij}'' (\frac{1}{N^2}) \\ &= \frac{n}{n-1} \frac{n-3}{n-2} [\pi_i^{n-2} \pi_j^{n-2} + \frac{n-2}{n-3} \pi_{ij}^{n-2}] + \frac{2n}{n-1} [A^{n-1} + A^{n-2} + (n-1) + \\ &\quad + (n-2)] p_i p_j + \alpha_{ij}''' = \dots = \end{aligned}$$

$$= \frac{n}{n-1} \frac{1}{2} [\pi_i^2 \pi_j^2 + 2\pi_{ij}^2] + \frac{2n}{n-1} [A^{n-1} + \dots + A^2 + (n-1) + \dots + 2] p_i p_j + \alpha_{ij}^{n-1}$$

Since from Chapter II:

$$\pi_i^2 \pi_j^2 + 2\pi_{ij}^2 = 4(A^1 + 1)p_i p_j + o(\frac{1}{N^4})$$

we altogether obtain

$$\pi_i^n \pi_j^n + \frac{n}{n-1} \pi_{ij}^n = \frac{2n}{n-1} [A^{n-1} + \dots + A^1 + \frac{n(n-1)}{2}] p_i p_j + \alpha_{ij}^n \quad (9)$$

and similarly from (8)

$$2n \pi_i^n - (\pi_i^n)^2 = \frac{2n}{n-1} [A^{n-1} + \dots + A^1 + \frac{n(n-1)}{2}] p_i (1-p_i) + \beta_i^n \quad (10)$$

where

$$\sum \beta_i^n = o(\frac{1}{N})$$

Now omitting the superscripts for π_i^n and π_{ij}^n and summing (10) we obtain, using (6)

$$2n^2 - \sum \pi_i^2 = \frac{2n}{n-1} [A^{n-1} + \dots + A^1 + \frac{n(n-1)}{2}] (1 - \sum p_i^2) + o(\frac{1}{N})$$

Substituting this last equation in (9) and (10) we obtain

$$\pi_i \pi_j + \frac{n}{n-1} \pi_{ij} = \frac{2n^2 - \sum \pi_i^2}{1 - \sum p_i^2} p_i p_j + \alpha_{ij}^{n+1} \quad (11)$$

$$\pi_i^2 - 2n \pi_i + \frac{2n^2 - \sum \pi_i^2}{1 - \sum p_i^2} p_i (1-p_i) + \beta_i^{n+1} = 0 \quad (12)$$

Alternatively the following equations in δ_i and δ_{ij}

$$\delta_i \delta_j + \frac{n}{n-1} \delta_{ij} = \frac{2n^2 - \sum \delta_i^2}{1 - \sum p_i^2} p_i p_j \quad (13)$$

$$\delta_i^2 - 2n \delta_i + \frac{2n^2 - \sum \delta_i^2}{1 - \sum p_i^2} p_i (1-p_i) = 0 \quad (14)$$

yield approximations to π_i and π_{ij} . It can be shown that

$$\sum \delta_i = n,$$

$$\sum_{j(\neq i)} \delta_{ij} = (n-1)\delta_i.$$

It can be verified as in Chapter II that a solution of (12) is

$$\delta_i = n - \sqrt{n^2 - \frac{r_i}{r_N}(2n\delta_N - \delta_N^2)} \quad (15)$$

where

$$r_i = p_i(1-p_i)$$

and δ_N has to be determined so that

$$\sum \sqrt{n^2 - \frac{r_i}{r_N}(2n\delta_N - \delta_N^2)} = n(N-1) \quad (16)$$

If p_i are so numbered that

$$\max r_i = r_N$$

then, viewing the left hand side of (16) as a function of δ_N it is a continuous real valued function in the interval $[0, n]$. At $\delta_N = 0$ its value equals Nn . At $\delta_N = n$

$$\begin{aligned} \sum \sqrt{n^2 - \frac{r_i}{r_N}(2n^2 - n^2)} &= n \sum \sqrt{1 - \frac{r_i}{r_N}} \leq n \sum (1 - \frac{r_i}{2r_N}) \\ &= n(N - \frac{1 - \sum p_i^2}{2r_N}) \leq n(N-1) \end{aligned}$$

the last inequality being a consequence of the fact that $r_N = p_N(1-p_N) \leq \frac{1}{4}$ and $\sum p_i^2 \leq \frac{1}{2}$.

Hence there is a value of δ_N , between 0 and n, which satisfies (16)
Taking this value of δ_N

$$\delta_i = n - \sqrt{n - \frac{r_i}{r_N} (2n \delta_N - \delta_N^2)} \geq n - n = 0$$

Also it is easily seen that if $p_i = \frac{1}{N}$ for every i, then $\delta_i = \frac{n}{N}$.
This is exactly the value of π_i in this case.

We know that $\pi_i = O(\frac{1}{N})$ and now it follows from (15) that the same
holds for δ_i as well.

If, as in the case $n=2$ we let

$$\pi_i - \delta_i = \varepsilon_i \quad \max \varepsilon_i = \varepsilon$$

we obtain from (12) in a way similar to what was done in Chapter II that

$$\pi_i - \delta_i = o(\frac{1}{N})$$

and

$$\pi_{ij} - \delta_{ij} = o(\frac{1}{N^2})$$

This completes the proof of the theorem.

3 - A Further Refinement

It is to be noted that the equations (13) and (14) can as approximate
identities be improved upon. The first approximation that was used for the
fourth term in (4) and (5) was of the form $\pi_i^n = np_i + o(\frac{1}{N})$, or
 $\frac{\pi_i^n}{n} = p_i + o(\frac{1}{N})$. We write

$$\frac{\pi_i^n}{n} = d_i + \beta_i \quad (17)$$

where d_i' is to be determined as a function of p_i such that

$$\sum d_i' = 1,$$

$$\sum \beta_i = 0$$

and try to extract some further terms of $O(\frac{1}{N^2})$ from the error term. This, as will be seen presently will not increase the order in $\frac{1}{N}$ of the error term, but for moderately small values of N is found in practice to yield an improved approximation. Let us substitute from (15) into the third terms of (4) and (5)

$$\pi_i^n \pi_j^n + \frac{n}{n-1} \pi_{ij}^n = \frac{n(n-2)}{(n-1)^2} [\pi_i^{n-1} \pi_j^{n-1} + \frac{n-1}{n-2} \pi_{ij}^{n-1}] + \frac{2n}{n-1} A^{n-1} p_i p_j + 2n d_i' d_j' + \alpha_{ij}$$

and

$$2n \pi_i^n - (\pi_i^n)^2 = \frac{n(n-2)}{(n-1)^2} [2(n-1) \pi_i^{n-1} - (\pi_i^{n-1})^2] + \frac{2n}{n-1} A p_i (1-p_i) + 2n d_i' (1-d_i') + \beta_i$$

As before, it follows that

$$\pi_i^n \pi_j^n + \frac{n}{n-1} \pi_{ij}^n = \frac{2n}{n-1} [A^{n-1} + \dots + A^1] p_i p_j + \frac{2n}{n-1} \frac{n(n-1)}{2} d_i' d_j' + \alpha_{ij}^n$$

$$2n \pi_i^n - (\pi_i^n)^2 = \frac{2n}{n-1} [A^{n-1} + \dots + A^1] p_i (1-p_i) + \frac{2n}{n-1} \frac{n(n-1)}{2} d_i' (1-d_i') + \beta_i^n$$

Since

$$\frac{\pi_i^n}{n} = p_i + o(\frac{1}{N})$$

d_i' must be equal to p_i up to $o(\frac{1}{N})$ and $d_i' d_j'$ must be equal to $p_i p_j$ up to $o(\frac{1}{N^2})$. Let us put

$$d_i' = c p_i (1 - k p_i)$$

and try to determine the value of c and k .

Summing on i

$$1 = c(1-k \sum p_i^2)$$

For the rest of the thesis we shall use the notation

$$s = \sum p_i^2$$

Then

$$d_i = \frac{p_i(1-kp_i)}{1-ks} \quad (18)$$

Substituting from (17) we get without difficulty, that

$$\frac{A_i}{i} = 1 + s + c\left(\frac{1}{N}\right)$$

$$d_i d_j = p_i p_j (1-kp_i - kp_j + ks) + O\left(\frac{1}{N^3}\right)$$

Then

$$\pi_i \pi_j + \frac{n}{n-1} \pi_{ij} = \frac{2n}{n-1} \frac{n(n-1)}{2} p_i p_j [2+s-kp_i - kp_j + ks] + \alpha_{ij}^{n+1} \quad (19)$$

Summing on j (≠ i)

$$-n_i^2 + 2n \pi_i = n^2 p_i [2 - (k+2)p_i + (k+1)s] + \beta_i^{n+1}$$

Including now terms of $O\left(\frac{1}{N^2}\right)$ (except the one that is of $O\left(\frac{1}{N^2}\right)$ due to a factor of p_i^2) in β_i^{n+1} we obtain

$$\pi_i = n p_i \left[1 - \frac{k+2}{2} p_i\right] + \beta_i^{n+2}$$

and adjusting so that the error terms add up to zero

$$\frac{\pi_i}{n} = \frac{p_i(1-k_2 p_i)}{1-k_2 s} + \beta_i^{n+3}$$

where

$$k_2 = \frac{k+2}{2}$$

Comparing this with (17) and (18) we see that the process just described can be repeated to yield

$$k_3 = \frac{k_2 + 2}{2}$$

and in general

$$k_{n+1} = \frac{k_n + 2}{2}$$

It is easily shown, that $k_n \rightarrow 2$ no matter what k was.

Substituting in (19) and factoring on the right hand side $2+3s$ we obtain

$$\pi_i \pi_j + \frac{n}{n-1} \pi_{ij} = (2+3s) n^2 p_i p_j [1-p_i-p_j] + \alpha_{ij}^{n+4}$$

Summing with respect to $j (\neq i)$

$$- \pi_i^2 + 2n \pi_i = (2+3s) n^2 p_i (1-2p_i+2p_i^2-s) + \beta_i^{n+4}$$

Setting

$$p_i' = p_i (1-2p_i+2p_i^2-s) = p_i [p_i^2 + (1-p_i)^2 - s] \quad (20)$$

we have

$$(2+3s)n^2 = \frac{2n^2 - \sum_i \pi_i^2}{\sum_i p_i'} + o(1)$$

and so we obtain the modified versions of (13) and (14):

$$\delta_i' \delta_j' + \frac{n}{n-1} \delta_{ij}' = \frac{2n^2 - \sum_i \delta_i'^2}{\sum_i p_i'} p_i' p_j' (1 - p_i' - p_j') \quad (21)$$

$$\delta_i'^2 - 2n \delta_i' + \frac{2n^2 - \sum_i \delta_i'^2}{\sum_i p_i'} p_i' = 0 \quad (22)$$

These equations are satisfied by π_i, π_{ij} up to $o(\frac{1}{N^2})$ and $o(\frac{1}{N})$ respectively.

The identities

$$\sum \delta_i' = n$$

$$\sum_{j \neq i} \delta_{ij}' = (n-1) \delta_i'$$

can be established directly from (21) and (22)

4 = Empirical Corrections

In Section 4 some empirical corrections are recommended. In Chapter V it will be shown, that they don't disturb the order of approximation, as established in Section 2, however, for small values of N or large values of n they were found in practice to provide very much worthwhile improvements.

Equations (21) and (22) represent corrections of (13) and (14) in terms of p_i . But δ_i^n is also a function of n. If $\frac{n}{N}$ is not negligible, another correction is needed. Obviously, when $n=N$, we want $\delta_i = 1$ to be the solution of (14). Summing over i we obtain

$$K = \frac{2n^2 - \sum \delta_i^2}{1 - s}$$

Then if $N=n$, $\delta_i=1$, we want

$$1 - 2N + \frac{2N^2 - N}{1 - s} p_i (1-p_i)$$

to be corrected so that it equals zero for every p_i . Clearly K needs to be corrected by adding a term which for $n=N$ reduces to $\frac{N(2N-1)}{1-s}$ and at the same time a constant has to be added reducing in this case to $2N-1$. This could be achieved by adding to (14) $\frac{(2n-1)n}{1-s} [\frac{1-s}{N} - p_i(1-p_i)]$.

However for values of n smaller than N this correction is found to be much too large. So it needs to be reduced by a factor that is less than one for $n < N$ but it equals 1 for $n = N$. After much experimentation, a suitable factor was found in the form of $\frac{n(n-2)}{N(N-2)}$. The whole correction then reduces to zero for $n=2$, which is in line with the fact, that (14) is better for $n=2$ than for any other value of n . Modified this way (14) and correspondingly (13) will be changed to:

$$\delta_i^2 - 2n \delta_i + K p_i (1-p_i) + \frac{a}{1-s} \left(\frac{1-s}{N} - p_i (1-p_i) \right) = 0 \quad (23)$$

$$\delta_i \delta_j + \frac{n}{n-1} \delta_{ij} = K p_i p_j + \frac{a}{1-s} \left(\frac{1-s}{N(N-1)} - p_i p_j \right) \quad (24)$$

where

$$a = \frac{n^2(n-2)(2n-1)}{N(N-2)} \quad (25)$$

It should be noted that the correction in (23) is $O(\frac{1}{N^3})$, in (24) $O(\frac{1}{N^4})$.

Summing (23) K is found to have the same form it had before:

$$K = \frac{2n^2 - \sum \delta_i^2}{1-s}$$

To obtain an empirical correction of (22) the same kind of argument which was used for the correction of (21) could be applied. It seems though, that better results can be obtained in a different way. If we order p_i from p_1 to p_N by their magnitude, p_1 being the smallest, it is found that for a given distribution of p_i we can obtain better approximations to π_1 from (23) than from (22). So we correct (22) in such a way, that its solution for δ_1 should equal that of (23). If we do so, then in the case $n=N$, δ_1 as obtained from the corrected form of (22) will be equal to 1. It will be shown later, that in this case $\delta_i=1$ for all i .

We will, therefore, modify (22) and get the new equations

$$\delta_i'^2 - 2n \delta_i' + \frac{2n^2 - \sum \delta_i'^2}{\sum p_i'} p_i' + C\left(\frac{\sum p_i'}{N} - p_i'\right) = 0$$

where for the new unknown, C we have the extra condition

$$\frac{2n^2 - \sum \delta_i'^2}{1-s} r_1 + a\left(\frac{1-s}{N} - r_1\right) = \frac{2n^2 - \sum \delta_i'^2}{\sum p_i'} p_1' + C\left(\frac{\sum p_i'}{N} - p_1'\right)$$

where a is given by (25). To avoid the necessity of solving (22) in order to get the value of $\sum \delta_i'^2$ to be substituted into the left hand side of the equation above, we approximate $\sum \delta_i'^2$ by $\sum \delta_i'^2$ and writing simply δ_i' for δ_i' everywhere:

$$\frac{2n^2 - \sum \delta_i'^2}{1-s} r_1 + a\left(\frac{1-s}{N} - r_1\right) = \frac{2n^2 - \sum \delta_i'^2}{\sum p_i'} p_1' + C\left(\frac{\sum p_i'}{N} - p_1'\right) \quad (26)$$

Another empirical observation can be made. For small values of p_i the corresponding true value of π_i is greater than np_i and for large values of p_i , π_i is smaller than np_i . Also the rate at which π_i increases with p_i is smaller for large values of p_i . This makes good sense since both π_i and np_i add to n and it is reasonable to expect that π_i increases with p_i , but $\pi_i \leq 1$, so whenever $p_i > \frac{1}{n}$ we have to have $np_i > \pi_i$ and hence for small values of p_i we should get $np_i < \pi_i$. Plotting the points (p_i, π_i) they should therefore be on a concave curve. The point of the p axis at which this curve intersects the straight line $np = \pi$ in all of the large number of examples considered seems to be coinciding with the point $p=s$. (See graphs in the Appendix). We can utilize this observation for another possible correction of (22). We can write

$$\delta_i'^2 - 2n \delta_i' + \frac{2n^2 - \sum \delta_i'^2}{\sum p_i'} p_i' + D\left(\frac{\sum p_i'}{N} - p_i'\right) = 0$$

where for the unknown D we have the extra equation:

$$n^2 s^2 - 2n^2 s + \frac{2n^2 - \sum \delta_i^2}{\sum p_i} s[(1-s)^2 + s^2 - s] + D\{-s[(1-s)^2 + s^2 - s] + \frac{\sum p_i'}{N}\} = 0 \quad (27)$$

It is recommended, that the weighted average of these corrections be used. Set

$$E = \frac{(\frac{N-n}{N} + s)C + \frac{N-n}{N} D}{2 \frac{N-n}{N} + s} \quad (28)$$

and modify (22) and (21)

$$\delta_i^2 - 2n \delta_i + \frac{2n^2 - \sum \delta_i^2}{\sum p_i} p_i' + E(\frac{\sum p_i'}{N} - p_i') = 0 \quad (29)$$

$$\delta_i \delta_j + \frac{n}{n-1} \delta_{ij} = \frac{2n^2 - \sum \delta_i^2}{\sum p_i} p_i p_j (1-p_i-p_j) + E[\frac{\sum p_i'}{N(N-1)} - p_i p_j (1-p_i-p_j)] \quad (30)$$

where C and D are to be found from (26) and (27) respectively. The joint solution of all these equations is treated in the next Chapter.

Note: The weights in (28) were once again empirically found. If $\frac{n}{N}$ is small and s is of order $\frac{1}{N}$ then it is almost an unweighted average. For $n=N$ however $E=C$.

CHAPTER IV

The Solution

1 - The Solution of the Direct Problem

The first step is to solve the equations (29) in the previous Chapter for δ_i when p_i are given, (the direct problem), then the reverse problem will be considered as well. In applications the order would be reversed. Our aim is to reduce the system of equations given by (29) of Chapter III to a linear system of equations. This is possible according to the next theorem.

Theorem 5

The system of second degree equations (29) can be reduced to a system of linear equations by the change of variables:

$$\delta_i = n - X_i$$

Proof:

Denote

$$X_i = n - \delta_i \tag{1}$$

Then

$$\begin{aligned} \delta_i^2 - 2n \delta_i &= X_i^2 - n^2 \\ \frac{2n^2 - \sum \delta_i^2}{\sum p_i} &= \frac{Nn^2 - \sum X_i^2}{\sum p_i} \end{aligned}$$

Therefore (29) becomes

$$X_i^2 - n^2 + \frac{Nn^2 - \sum X_i^2}{\sum p_i} + E\left(\frac{\sum p_i}{N} - p_i\right) = 0$$

Using the relations in (26) of Chapter III, C becomes

$$C = Nn^2 c_1 + \frac{Nn^2}{\sum p_i} + (c_2 - \frac{1}{\sum p_i}) \sum x_i^2$$

where

$$c_2 = \frac{1 - N \frac{p_1(1-p_1)}{1-s}}{\sum p_i - Np_1} \quad (2)$$

$$c_1 = [\frac{(2n-1)(n-2)}{N^2(N-2)} - 1]c_2 \quad (3)$$

Similarly, expressing D from (27) of Chapter III

$$D = Nn^2 d_1 + \frac{Nn^2}{\sum p_i} + (d_2 - \frac{1}{\sum p_i}) \sum x_i^2 \quad (4)$$

where

$$d_2 = \frac{1}{\sum p_i - Ns(1-s)(1-2s)} \quad (5)$$

$$d_1 = - (1-s)^2 d_2 \quad (6)$$

Finally

$$E = Nn^2 e_1 + \frac{Nn^2}{\sum p_i} + (e_2 - \frac{1}{\sum p_i}) \sum x_i^2 \quad (7)$$

where

$$e_1 = \frac{(\frac{N-n}{N} + s)c_1 + \frac{N-n}{N} d_1}{2 \frac{N-n}{N} + s} \quad (8)$$

$$e_2 = \frac{(\frac{N-n}{N} + s)c_2 + \frac{N-n}{N} d_2}{2 \frac{N-n}{N} + s} \quad (9)$$

With this notation (29) of Chapter III becomes

$$X_i^2 - (Nn^2 e_1 + e_2 \sum X_i^2) p_i' + (Nn^2 e_1 + e_2 \sum X_i^2 - \frac{\sum X_i^2}{\sum p_i'}) \frac{\sum p_i'}{N} = 0$$

Denoting

$$a_i = e_2 (\sum p_i' - N p_i') - 1 \quad \sum a_i = -N$$

$$b_i = Nn^2 e_1 (N p_i' - \sum p_i') \quad \sum b_i = 0$$

we obtain the system of linear equations in X_i^2

$$a_i X_1^2 + a_i X_2^2 + \dots + (N+a_i) X_i^2 + \dots + a_i X_N^2 = b_i \quad i=1,2,\dots,N \quad (10)$$

This concludes the proof.

Theorem 6

The system of linear equations (10) can be solved in terms of $X_1^2 = (n - \delta_1)^2$ or $X_N^2 = (n - \delta_N)^2$. The solution in terms of X_1^2 is

$$X_i^2 = \frac{b_i a_1 - b_1 a_i}{N a_1} + \frac{a_i}{a_1} X_1^2$$

and similarly for X_N .

In order to complete the solution X_1 (or X_N) is to be evaluated in such a way that the equation be satisfied:

$$\sum \delta_i = n$$

or equivalently

$$\sum X_i = \sum \sqrt{\frac{b_i a_1 - b_1 a_i}{N a_1} + \frac{a_i}{a_1} X_1^2} = (N-1)n$$

Proof:

The determinant of the system is zero, the sum of the row vectors being zero. Now consider the $(N-1) \times (N-1)$ subdeterminant in the lower right corner.

$$\begin{aligned}
 & \begin{vmatrix} N+a_2 & a_2 & \dots & a_2 & \dots & a_2 \\ a_3 & N+a_3 & \dots & a_3 & \dots & a_3 \\ & & \dots & & & \\ a_N & a_N & \dots & a_N & \dots & N+a_N \end{vmatrix} \\
 & = \begin{vmatrix} -a_1 & -a_1 & \dots & -a_1 & \dots & -a_1 \\ a_3 & N+a_3 & \dots & a_3 & \dots & a_3 \\ & & \dots & & & \\ a_N & a_N & \dots & a_N & \dots & N+a_N \end{vmatrix} \\
 & = \begin{vmatrix} -a_1 & 0 & & 0 & \dots & 0 \\ 0 & N & & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & & 0 & \dots & N \end{vmatrix} \\
 & = -N^{N-2} a_1
 \end{aligned}$$

Similarly the upper right $(N-1) \times (N-1)$ subdeterminant equals $-N^{N-2} a_N$. At least, one of these has to be different from zero since the numbers a_i form a monotone sequence with sum equal to $-N$.

Suppose $a_1 \neq 0$. Then the system for X_i^2 can be solved in terms of X_1^2 .

$$D_1 = \begin{vmatrix} N+a_2 & a_2 & \dots & a_2 & \dots & b_2 - a_2 X_1^2 & \dots & a_2 \\ a_3 & N+a_3 & \dots & a_3 & \dots & b_3 - a_3 X_1^2 & \dots & a_3 \\ & & & \dots & & & & \\ a_i & a_i & \dots & a_i & \dots & b_i - a_i X_1^2 & \dots & a_i \\ & & & \dots & & & & \\ a_N & a_N & \dots & a_N & \dots & b_N - a_N X_1^2 & \dots & N+a_N \end{vmatrix}$$

$$= \begin{vmatrix} -a_1 & -a_1 & \dots & -a_1 & \dots & -b_1 + NX_1^2 + a_1 X_1^2 & \dots & -a_1 \\ a_3 & N+a_3 & \dots & a_3 & \dots & b_3 - a_3 X_1^2 & \dots & a_3 \\ & & & \dots & & & & \\ a_i & a_i & \dots & a_i & \dots & b_i - a_i X_1^2 & \dots & a_i \\ & & & \dots & & & & \\ a_N & a_N & \dots & a_N & \dots & b_N - a_N X_1^2 & \dots & N+a_N \end{vmatrix}$$

$$= \begin{vmatrix} -a_1 & 0 & \dots & 0 & \dots & -b_1 + NX_1^2 & \dots & 0 \\ a_3 & N & \dots & 0 & \dots & b_3 & \dots & 0 \\ & & & \dots & & & & \\ a_i & 0 & \dots & 0 & \dots & b_i & \dots & 0 \\ & & & \dots & & & & \\ a_N & 0 & \dots & 0 & \dots & b_N & \dots & N \end{vmatrix}$$

$$= \begin{vmatrix} -a_1 & 0 & \dots & 0 & \dots & -b_1 + NX_1^2 & \dots & 0 \\ 0 & N & \dots & 0 & \dots & 0 & \dots & 0 \\ & & & \dots & & & & \\ a_i & 0 & \dots & 0 & \dots & b_i & \dots & 0 \\ & & & \dots & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & N \end{vmatrix}$$

$$= N^{N-3} [a_i b_1 - a_1 b_i - N a_i X_1^2]$$

Hence

$$X_i^2 = \frac{b_i a_1 - b_1 a_i}{N a_1} + \frac{a_i}{a_1} X_1^2 \quad i=2, \dots, N \quad (11)$$

This can be seen to hold identically also for $i=1$

The additional equation for X_1 is provided by

$$\sum X_i^2 = (N-1)n \quad (12)$$

It is convenient to solve the system in terms of $\sum X_i^2$ instead of X_1^2 .

From (11)

$$\sum X_i^2 = \frac{b_1 - N X_1^2}{a_1}$$

Substituting $X_1^2 = \frac{b_1}{N} - a_1 \frac{\sum X_i^2}{N}$ in (11), gives

$$X_i^2 = \frac{b_i}{N} - a_i \frac{\sum X_i^2}{N} \quad i=1, 2, \dots, N \quad (13)$$

The solution of (12) and (13) can adopt the method of successive approximations. It appears that in so doing it is more convenient to work with equations (13) rather than (11). An initial approximation of

$$\sum X_i^2 = \sum (n - \delta_i)^2 = Nn^2 - 2n^2 + \sum \delta_i^2$$

is provided by

$$(N-2) n^2 + n^2 \sum p_i^2$$

Other approximation are given by

$$n^2 \left[\frac{\sum p_i^2 (1-p_i)^2}{(1-s)^2} + N-2 \right]$$

or

$$n^2 \left[\frac{\sum p_i'^2}{(\sum p_i')^2} + N-2 \right]$$

Denoting any one of these values by $N\sigma$ and $\frac{\sum x_i^2}{N}$ by $\sigma + y$, it is required to solve equation (14) for y :

$$\sum \sqrt{\frac{b_i}{N} - a_i (\sigma + y)} = (N-1)n \quad (14)$$

The square root may be approximated by the first three terms of the Maclaurin series:

$$\sqrt{\frac{b_i}{N} - a_i (\sigma + y)} = \frac{b_i}{N} - a_i \sigma - \frac{a_i}{2 \sqrt{\frac{b_i}{N} - a_i \sigma}} - y^2 \sum \frac{a_i^2}{8(\frac{b_i}{N} - a_i \sigma)^{3/2}} y^2$$

and then solving for y in the following equation

$$(N-1)n = \sum \sqrt{\frac{b_i}{N} - a_i \sigma} - y \sum \frac{a_i}{2 \sqrt{\frac{b_i}{N} - a_i \sigma}} - y^2 \sum \frac{a_i^2}{8(\frac{b_i}{N} - a_i \sigma)^{3/2}} \quad (15)$$

Having solved for y , $\sigma + y = \sigma_2$ will be used in place of σ , the procedure ending when $\sum \sqrt{\frac{b_i}{N} - a_i \sigma} = (N-1)n$ is obtained up to the required number of decimals. The entire calculation is easily established in a form facilitating the use of a desk calculator or an electric computer.

TABLE I

Calculations to obtain δ_i

	[1]	[2]	[3]
	$p_i' = p_i[(1-p_i)^2 + p_i^2 + s]$	$Np_i' - \sum p_i'$	$a_i = -e_2 \times [2] - 1$
1			
2			
.			
.			
.			
N			
Total	$\sum p_i'$	0	- N
	[4]	[5]	[6]
	$\frac{b_i}{N} = n^2 e_1 \times [2]$	$[4] - \sigma \times [3]$	$\sqrt{[5]}$
1			
2			
.			
.			
.			
N			
Total	0	$N\sigma$	

	[7]	[8]	[9]	[10]
	[3] / [6]	[7] ² / [6]	[5]-y ₁ *[3]	✓ [9]
1				
2				
.				
.				
.				
N				
Total				

The instruction (or programme) may be given as follows:

Start by calculating $s = \sum p_i^2$. Complete column [1], obtain the total, complete column [2]. Check that the total of column [2] equals zero. Obtain $c_2, c_1, d_2, d_1, e_2, e_1$ using formulae (2) to (9). Also

$$\sigma_1 = n^2 \left[\frac{N-2}{N} + \frac{\sum p_i'^2}{N(\sum p_i')^2} \right]$$

Next complete columns [3], [4], [5] and [6]. At every step check the totals. Every entry in column [5] should be positive, otherwise the initial value of σ has to be modified. This never happened in the examples. If the total of column [6] equals $(N-1)n$ to the required decimal places, the entries in this column provide the solution of the problem in terms of X_i and an extra column $n-X_i$ yields the solution in terms of δ_i . Otherwise obtain columns [7] and [8].^{*} The total of [7]

^{*} If it is more convenient, the term containing y^2 in (15) may be omitted, taking only the first two terms of the expansion. Then correspondingly column [8] may be dropped.

divided by -2 gives the coefficient of y^2 in (15). Solve (15) for y , obtain y_1 . Complete column [9], whose total should be equal the total of [5] plus Ny_1 . If y_1 is sufficiently small, the square root in column [10] is obtained easily by $\frac{[9]+[5]}{2 \times [6]}$. If the total of column [10] equals $(N-1)n$ the extra column $n-[10]$ yields δ_i . If not, repeat the operations analogous to the completion of columns [7], [8], [9] and [10]. At the last step, the total of column [5] equals $\sum X_i^2$. Next, to evaluate δ_{ij} , obtain the following quantities:

$$[a] = \frac{\sum X_i^2}{N}$$

$$[b] = -n^2 e_1 - e_2 \frac{\sum X_i^2}{N} = -n^2 e_1 - [a] e_2$$

$$[c] = N[b]$$

$$[d] = \frac{1}{N-1} (-[b] \times \sum p_i' + n^2 - [a])$$

Then expressing δ_{ij} from equation (30) of Chapter III, we obtain

$$\delta_{ij} = \frac{n-1}{n} [[c] \times p_i p_j (1-p_i-p_j) + [d] - \delta_i \delta_j]$$

2 - Solution of the Reverse Problem

The reverse problem can be solved using similar methods. Now π_i are given ($n \frac{M_i}{\sum M_i}$) and it is required to solve for the approximate values of p_i which would give rise to these π_i . From (23) of Chapter III

$$\pi_i^2 - 2n\pi_i + \left(\frac{2n^2 - \sum \pi_i^2}{1-s} - \frac{n^2(n-2)(2n-1)}{N(N-2)(1-s)} \right) p_i(1-p_i) + \frac{n^2(n-2)(2n-1)}{N(N-2)} = 0$$

which, using the notation of (25) in Chapter III is

$$p_i^2 - p_i + \frac{1 - \sum p_i^2}{2n^2 - \sum \pi_i^2 - a} [2n\pi_i - \pi_i^2 - \frac{a}{n}] = 0$$

Substituting

$$y_i = \frac{1}{2} - p_i$$

and reordering we obtain

$$a'_1 y_1^2 + a'_2 y_2^2 + \dots + (1+a'_i) y_i^2 + \dots + a'_N y_N^2 = b'_i \quad i=1, \dots, N$$

where

$$a'_i = \frac{\frac{a}{N} - 2n\pi_i + \pi_i^2}{2n^2 - \sum \pi_i^2 - a} \quad \sum a'_i = -1$$

$$b'_i = \frac{Na'_i + 1}{4} \quad \sum b'_i = 0$$

This is essentially the same kind of system of equations as in (10), and it would be exactly the same had we multiplied through by N. The additional equation, to the analogue of (12) is given by

$$\sum y_i = \frac{N}{2} - 1$$

Then the solution of this system, using the analogy, can be obtained as

$$y_i^2 = b'_i - a'_i \sum y_i^2$$

As before, an approximation of $\sum y_i^2$ is given by

$$\frac{N-4}{4} + \frac{\sum \pi_i^2 (2n-\pi_i)^2}{(2n^2 - \sum \pi_i^2)^2} = \sigma$$

The successive stages of solution are now similar to those described above for the solution of (10) and (12).

3 - Summary of Recommended Procedures

The following procedure is recommended:

Given the values M_1 , substitute $n \frac{M_1}{\sum M_1}$ in place of δ_1 in formula (23) of Chapter III. Solve for p_1 to obtain an approximate solution of the reverse problem. The values $\pi_1(p_1, p_2, \dots, p_N)$ will not be exactly equal to $n \frac{M_1}{\sum M_1}$ since $p_1, p_2, \dots, p_N$ is only an approximate solution of the reverse problem. If now (23) of Chapter III is used to approximate $\pi_1(p_1, p_2, \dots, p_N)$, we would obtain $n \frac{M_1}{\sum M_1}$ as the approximate values of $\pi_1(p_1, p_2, \dots, p_N)$ and hence these values may be used in the estimator (1) of Chapter I. It is, however, recommended that an improved approximation of $\pi_1(p_1, p_2, \dots, p_N)$ be obtained from (29) of Chapter III.

It may appear that a simpler procedure would be to use only (29) of Chapter III for both the reverse and the direct problem. In this case one would substitute $n \frac{M_1}{\sum M_1}$ in place of δ_1 and solve this latter equation for p_1 . Then $\pi_1(p_1, p_2, \dots, p_N)$ may be approximated by $n \frac{M_1}{\sum M_1}$. It should be noted, however, that (29) is in the third degree in p_1 while (23) is in the second degree. Therefore, it is more convenient to use (23) for the reverse problem while keeping (29) for the direct problem. The fact that the improved approximation is used for the direct problem is in line with the observation made by Horvitz and Thompson, that we may afford more laxity in the approximate solution of the reverse problem than in the direct problem. This observation may be made by noting that the values

M_i are only guessed values of Y_i or values of a related variable and hence even if the exact solution of the reverse problem was effected, this solution would be subject to some uncertainty due to the inaccuracy of M_i . However, having solved the reverse problem we have the p_i values which are used in the sampling scheme. Correspondingly, we ought to use $\pi_i(p_1, p_2, \dots, p_N)$ in the estimator (1) of Chapter I. Our failure to do so increases the bias. Therefore, if π_i is not used, at least its best available approximation should be entered into (1).

Further it may be noted that if the δ_i , as obtained from (29) of Chapter III, are found to differ appreciably from $n \frac{M_i}{\sum M_i}$, one may consider the possibility of using (29) for the reverse problem, attempting to solve (29) numerically for p_i . Again successive approximations may be recommended, starting with $p_i^1 = p_i$ as obtained from (23) and $\delta_i^1 = \delta_i(p_1^1, p_2^1, \dots, p_N^1)$ as obtained from (29). Then approximating

$$\frac{\partial \delta_i(p_1, p_2, \dots, p_N)}{\partial p_i} \approx n \text{ solve the equation for } p_i^2:$$

$$\frac{n \frac{M_i}{\sum M_i} - \delta_i^1}{p_i^2 - p_i^1} = n$$

Then

$$p_i^2 = p_i^1 + \frac{M_i}{\sum M_i} - \frac{1}{n} \delta_i^1$$

and

$$\delta_i^2 = \delta_i(p_1^2, p_2^2, \dots, p_N^2)$$

using (29).

Next approximate $\frac{\partial \delta_i}{\partial p_i}$ by $\frac{\delta_i^2 - \delta_i^1}{p_i^2 - p_i^1}$ and so obtain the next equation for p_i^3 :

$$\frac{n \frac{M_i}{\sum M_i} - \delta_i^2}{p_i^3 - p_i^2} = \frac{\delta_i^2 - \delta_i^1}{p_i^2 - p_i^1}$$

then

$$\delta_i^3 = \delta_i(p_1^3, p_2^3, \dots, p_N^3)$$

The procedure may be continued until $\delta_i = n \frac{M_i}{\sum M_i}$ holds with the required precision.

CHAPTER V

LIMITING PROPERTIES

1 - Limiting Properties of δ_i

In this Chapter we shall investigate some of the properties of the approximation δ_i for large values of N , more precisely under the condition (as set out in Chapter III) that sampling without replacement approaches sampling with replacement. Once again we shall have to resort to the notation of Chapter III, writing p_{ik} in place of p_i , $\pi_{(ik)}$ in place of π_i , $\delta_{(ik)}$ in place of δ_i , Y_{ik} in place of Y_i , $\pi_{ik,jm}$ in place of π_{ij} and $\delta_{ik,jm}$ in place of δ_{ij} .

In the course of the Chapter, the following lemma will be used.

Lemma:

If u and v are real numbers such that $u > 0$ and $u^2 + v \geq 0$ then

$$\sqrt{u^2 + v} \geq u + \frac{uv}{2u^2 + v} \quad (1)$$

Proof:

The right hand side is obviously positive, so it is sufficient to show, that the square of the left hand side is not smaller than the right hand side:

$$u^2 + v \geq u^2 + \frac{2u^2v}{2u^2 + v} + \frac{u^2v^2}{(2u^2 + v)^2}$$

u^2 cancels out, and multiplying through by $2u^2 + v > 0$ the inequality above holds if and only if

$$2u^2v + v^2 \geq 2u^2v + \frac{u^2v^2}{2u^2 + v}$$

which is easily seen to hold, since $u^2 + v > 0$. Equality occurs only if $v=0$ or $v = -u^2$.

Now we shall formulate one of the main theorems of this Chapter.

Theorem 7

- If 1) sampling without replacement approaches sampling with replacement
2) the limiting distribution $t_1, t_2, \dots, t_{N_0}$ is such that

$$\frac{1}{N_0} < \sum_{i=1}^{N_0} t_i^2 < \frac{2}{N_0} \quad *$$

- 3) N is larger than some number H ,
then the system of equations (29)
of Chapter III can be solved.

Proof:

The constants, $s, c_1, d_1, c_2, d_2, a_1, b_1$, etc., of Chapter III will be all functions of N and should therefore be denoted by $s(N), c_1(N), d_1(N), c_2(N), d_2(N), a_{ik}(N), b_{ik}(N)$, etc. For convenience the argument N will be omitted. All the limits are understood to refer to $N \rightarrow \infty$.

* The left hand side of the inequality simply means that the t_i are not all equal. The right hand side is easily shown to be equivalent to the condition, that the coefficient of variation of the distribution of t_i is less than 100%. In most cases this is not a very much restricting condition, i.e., in a distribution such that $t_i - t_{i-1} = \text{constant}$ this condition is satisfied no matter how much larger is t_{N_0} than t_1 .

Denote

$$\sum_{i=1}^{N_0} t_i^2 = s_0$$

Then

$$\sum_{i=1}^{N_0} \frac{N}{K_i} K_{iM_i}^2 \leq N \sum p_{ik}^2 = Ns = \sum_{i=1}^{N_0} \frac{N}{K_i} K_{iM_i}^2$$

Since

$$\lim K_{iM_i}^2 = \lim K_{iM_i}^2 = t_i^2$$

and

$$\lim \frac{N}{K_i} = N_0$$

we get

$$\lim Ns = N_0 s_0$$

$$\lim s = 0$$

Also

$$\lim \frac{p_{ik}}{p_{ik}} = \lim [p_{ik}^2 + (1-p_{ik})^2 - s] = 1 \quad (2)$$

$$\lim \sum p_{ik}^3 = \lim (1 - 3s + 2 \sum p_{ik}^3) = 1 \quad (3)$$

$$\lim c_2 = \lim \frac{1 - N \frac{p_{11}(1-p_{11})}{1-s}}{\sum p_{ik} - Np_{11}} = 1$$

$$\lim c_1 = \lim \left[\frac{(2n-1)(n-2)}{N^2(N-2)} - 1 \right] c_2 = -1$$

$$\lim d_2 = \lim \frac{1}{\sum p_{ik} - Ns(1-s)(1-2s)} = \frac{1}{1-N_0 s_0} < 0$$

$$\lim d_1 = \lim [- (1-s)^2 d_2] = \frac{1}{N_0 s_0 - 1} > 0$$

$$\lim e_1 = \frac{\lim c_1 + \lim d_1}{2} = \frac{2-N_0 s_0}{2(N_0 s_0 - 1)} > 0 \quad (4)$$

$$\lim e_2 = \frac{\lim c_2 + \lim d_2}{2} = \frac{N_0 s_0 - 2}{2(N_0 s_0 - 1)} < 0 \quad (5)$$

$$\lim N(c_1 + c_2) = \lim N c_2 \frac{(2n-1)(n-2)}{N^2 (Ns-2)} = 0$$

$$\lim N(d_1 + d_2) = \lim N d_2 (2s - s^2) = \frac{2N_0 s_0}{1-Ns_0} < 0$$

$$\begin{aligned} \lim N(e_1 + e_2) &= \lim N \frac{(\frac{N-n}{N} + s)(c_1 + c_2) + \frac{N-n}{N} (d_1 + d_2)}{2 \frac{N-n}{N} + s} \\ &= \frac{N_0 s_0}{1-N_0 s_0} < 0 \end{aligned} \quad (6)$$

$$\lim a_{ik} = \lim [e_2 (\sum p'_{ik} - N p'_{ik}) - 1] = \frac{N_0 s_0 - 2}{2(N_0 s_0 - 1)} (1 - N_0 t_1) - 1 \quad (7)$$

$$\lim \frac{b_{ik}}{N} = \lim n^2 e_1 (N p'_{ik} - \sum p'_{ik}) = n^2 \frac{2-N_0 s_0}{2(N_0 s_0 - 1)} (N_0 t_1 - 1) \quad (8)$$

Let us choose N large enough so that

$$e_1 > 0 \quad e_2 < 0 \quad e_1 + e_2 < 0 \quad (9)$$

It follows from (4), (5) and (6) that this is always possible. Let us order p_{ik} in an increasing sequence, starting with p_{11} . Then, we shall have an increasing sequence for p'_{ik} as well, if only N is large enough.

In fact

$$\begin{aligned} \frac{dp_{ik}}{dp_{ik}} &= 1 - 4p_{ik} + 6p_{ik}^2 - s \geq 1 - 4p_{N_0} K_{N_0} - s \\ &\geq 1 - 4M_{N_0} - s > 0 \end{aligned} \quad (10)$$

since $M_{N_0} \rightarrow 0$ and $s \rightarrow 0$ as $N \rightarrow \infty$. Then

$$a_{11} = e_2 (\sum p_{ik} - Np_{11}) - 1 < 0 \quad (11)$$

In as much as $a_{11} \neq 0$, we can express X_{ik}^2 in terms of X_{11}^2 according to (11) of Chapter IV.

$$\begin{aligned} X_{ik}^2 &= \frac{b_{ik}a_{11} - a_{ik}b_{11}}{N a_{11}} + \frac{a_{ik}}{a_{11}} X_{11}^2 \\ &= \frac{n^2 e_1 (Np_{11} - Np_{ik})}{a_{11}} + \frac{a_{ik}}{a_{11}} X_{11}^2 \end{aligned} \quad (12)$$

Then, according to Theorem 6, we have to show, that if N is large enough, there is a number X_{11}^2 such that

$$F_N(X_{11}^2) = \sum X_{ik}^2 = (N-1)n \quad (13)$$

has a solution. Let us replace X_{11}^2 in (12) by an independent variable X , obtaining the function $X_{ik}^2(X)$. Now let us consider the functions

$$G_N(X) = F_N(X) - (N-1)n = \sum X_{ik}^2(X) = \sum \sqrt{X_{ik}^2(X)}$$

$G_N(X)$ is a continuous function of X in any closed interval in which $X_{ik}^2(X) \geq 0$. But $X_{ik}^2(X)$, being monotone functions of X , are non-negative in any closed interval as long as they are non-negative at the end-points. Therefore, if we can find two points such that $G_N(X)$ is non-negative at one of these points and non-positive at the other and at the same time all the

functions $X_{ik}^2(X)$ are non-negative at both of these points, then between these two points there is a point at which $G_N(X) = 0$. This will prove (13). We shall show that the closed interval whose end-points are $X = n^2$ and $X = -n \frac{e_1}{e_2}$ is satisfying all the requirements. In fact

$$\begin{aligned}
 G_N(n^2) &= \sum \sqrt{\frac{n^2 e_1 (Np_{11}' - Np_{ik}')}{a_{11}} + \frac{a_{ik}}{a_{11}} n^2 - (N-1)n} \\
 &= n \sum \sqrt{\frac{e_1 (Np_{11}' - Np_{ik}') + a_{ik} - a_{11}}{a_{11}} + 1 - (N-1)n} \\
 &= n \sum \sqrt{\frac{(e_1 + e_2)(Np_{11}' - Np_{ik}')}{a_{11}} + 1 - (N-1)n} \\
 &= n \sum \sqrt{\frac{(e_1 + e_2)(Np_{11}' - Np_{ik}') + a_{11} \frac{2N-1}{N^2}}{a_{11}} + \frac{(N-1)^2}{N^2} - (N-1)n} \quad (14)
 \end{aligned}$$

To estimate (14) we shall apply the lemma to every one of the square roots above. For the values u and v we shall choose

$$\begin{aligned}
 u &= \frac{N-1}{N} \\
 v &= \frac{(e_1 + e_2)(Np_{11}' - Np_{ik}') + a_{11} \frac{2N-1}{N^2}}{a_{11}}
 \end{aligned}$$

Before we can apply the lemma, we have to show that

$$u^2 + v \geq 0 \quad (15)$$

Since $u^2 + v = X_{ik}^2(n^2)$, it will also follow from (15) that

$$X_{ik}^2(n^2) \geq 0 \quad i=1,2,\dots,N_0 \quad k=1,2,\dots,K_i$$

We have

$$p'_{ik} = p_{ik} [(1-p_{ik})^2 + p_{ik}^2 - s] < p_{ik} \leq M_i \quad (16)$$

and then due to (9) and (11)

$$\begin{aligned} v + u^2 &= \frac{N(e_1 + e_2)(p'_{11} - p'_{ik}) + a_{11} \frac{2N-1}{N^2}}{a_{11}} + \left(\frac{N-1}{N}\right)^2 \\ &> \frac{N(e_1 + e_2)(p'_{11} - M_i) + a_{11} \frac{2N-1}{N^2}}{a_{11}} + \left(\frac{N-1}{N}\right)^2 \\ &\quad i=1, \dots, N_0 \quad k=1, 2, \dots, K_i \end{aligned}$$

The right hand side has limit 1 as $N \rightarrow \infty$. Since N_0 is a fixed finite number, we can choose N large enough so that for every i at the same time the right hand side is positive.

So we can apply the lemma and obtain

$$\begin{aligned} G_N(n^2) &\geq n \frac{N-1}{N} \sum \frac{N(e_1 + e_2)(p'_{11} - p'_{ik}) + a_{11} \frac{2N-1}{N^2}}{a_{11} [2\left(\frac{N-1}{N}\right)^2 + \frac{N(e_1 + e_2)(p'_{11} - p'_{ik}) + a_{11} \frac{2N-1}{N^2}}{a_{11}} + \frac{2N-1}{N^2}]} \\ &= n \frac{N-1}{N} \sum \frac{N(e_1 + e_2)(p'_{11} - p'_{ik}) + a_{11} \frac{2N-1}{N^2}}{a_{11} [2\left(\frac{N-1}{N}\right)^2 + \frac{2N-1}{N^2}] + N(e_1 + e_2)(p'_{11} - p'_{ik})} \quad (17) \end{aligned}$$

The general term of the sum above is a ratio, therefore its derivative with respect to p'_{ik} is also a ratio. The denominator of the derivative is a complete square and so it is positive. The numerator

is equal to

$$- 2N(e_1 + e_2) a_{11} \left(\frac{N-1}{N}\right)^2$$

and so, in light of (9) and (11) is a negative quantity. Therefore according to (16) we decrease (17) if we substitute in place of p'_{ik} .

$$G_N(n^2) \geq n \frac{N-1}{N} \sum_{i=1}^{N_0} K_i \frac{N(e_1+e_2)(p'_{11}-M_i) + a_{11} \frac{2N-1}{N^2}}{a_{11} [2\left(\frac{N-1}{N}\right)^2 + \frac{2N-1}{N^2}] + N(e_1+e_2)(p'_{11}-M_i)}$$

But

$$\lim \left\{ a_{11} \left[2\left(\frac{N-1}{N}\right)^2 + \frac{2N-1}{N^2} \right] + N(e_1+e_2)(p'_{11}-M_i) \right\} = 2 \lim a_{11}$$

$$\lim G_N(n^2) = \frac{n}{2 \lim a_{11}} \left[\frac{N_0 s_0}{1-N_0 s_0} (N_0 t_1 - 1) + \frac{N_0 s_0 - 2}{N_0 s_0 - 1} (1 - N_0 t_1) - 2 \right]$$

$$= \frac{-nN_0 t_1}{\lim a_{11}} > 0$$

Therefore if N is large enough, we shall have

$$G_N(n^2) > 0$$

Next we shall show that whenever N is large enough

$$x_{ik}^2 \left(-n \frac{e_1}{e_2} \right) \geq 0$$

and

$$G_N \left(-n^2 \frac{e_1}{e_2} \right) < 0$$

$$\begin{aligned}
 G_N(-n^2 \frac{e_1}{e_2}) &= \sum \sqrt{\frac{n^2 e_1 N(p'_{11} - p'_{ik}) - a_{ik} n^2 \frac{e_1}{e_2}}{a_{11}}} - (N-1)n \\
 &= N \sqrt{-n^2 \frac{e_1}{e_2}} - (N-1)n
 \end{aligned} \tag{18}$$

Since

$$\lim \frac{e_1}{e_2} = -1$$

the expression under the square root (which is the common value of all the $X_{ik}^2(-n \frac{e_1}{e_2})$) is positive if N is large enough. The expression under (18) is negative whenever

$$N \sqrt{-n^2 \frac{e_1}{e_2}} < (N-1)n$$

or, equivalently,

$$e_1 + (\frac{N-1}{N})^2 e_2 < 0 \tag{19}$$

But since

$$\lim [e_1 + (\frac{N-1}{N})^2 e_2] = \lim (e_1 + e_2) < 0$$

it follows that, the expression (18) is negative whenever N is sufficiently large and hence

$$G_N(-n \frac{e_1}{e_2}) < 0$$

Accordingly, if N is large enough, there exists a number X_{11}^2 such that

$$F_N(X_{11}^2) = (N-1)n$$

This completes the proof of Theorem 7. The statement as well as the proof of the theorem implies that

$$X_{ik}^2 \geq 0 \tag{20}$$

Theorem 8

If the conditions of Theorem 7 are satisfied then, whenever N is large enough we will have for every i, k at the same time

$$0 \leq \delta_{(ik)} < 1 \quad i=1, \dots, N_0 \quad k=1, \dots, K_i$$

Proof:

It follows from the argument in the proof of Theorem 7 that X_{11}^2 is between $-n^2 \frac{e_1}{e_2}$ and n^2 . According to (19) however

$$-\frac{e_1}{e_2} < 1$$

So we may write

$$0 < -n^2 \frac{e_1}{e_2} \leq X_{11}^2 \leq n^2 \quad (21)$$

or equivalently

$$\delta_{(11)} = n - X_{11} \geq 0 \quad (22)$$

We shall show that for every i and k

$$\delta_{(ik)} \geq 0 \quad (23)$$

In fact

$$\frac{dX_{ik}^2}{dp_{ik}} = \frac{dX_{ik}^2}{dp_{ik}'} \frac{dp_{ik}'}{dp_{ik}}$$

But according to (12)

$$\frac{dX_{ik}^2}{dp_{ik}'} = \frac{-n^2 Ne_1 - Ne_2 X_{11}^2}{a_{11}}$$

$$\begin{aligned}
 &= - \frac{N}{a_{11}} (n^2 e_1 + x_{11}^2 e_2) \\
 &< - \frac{N}{a_{11}} (n^2 e_1 - n^2 \frac{e_1}{e_2} e_2) \\
 &= 0
 \end{aligned}$$

This means, in light of (10), that

$$\frac{dx_{ik}^2}{dp_{ik}} < 0 \quad (24)$$

Since $p_{11} \leq p_{ik}$ for every i and k , it follows that

$$0 \leq x_{ik}^2 \leq x_{11}^2 \leq n^2$$

hence

$$0 \leq x_{ik} \leq x_{11} \leq n$$

and finally

$$0 \leq \delta_{(11)} \leq \delta_{(ik)}$$

Another consequence of (24) is that since p_{ik} are ordered in an increasing sequence, starting with p_{11} and ending with $p_{N_0 K_{N_0}}$ $\delta_{(ik)}$ are ordered similarly.

Since

$$\lim \frac{e_1}{e_2} = -1$$

it follows from (21) that

$$\lim x_{11}^2 = n^2$$

Then

$$\begin{aligned}
 0 &\leq 2n \delta_{(ik)} = n^2 + \delta_{(ik)}^2 - x_{ik}^2 \\
 &\leq n^2 + \delta_{(ik)} - x_{ik}^2 \\
 &< n^2 + \delta_{(ik)} - \frac{n^2 e_1 N(p'_{11} - p'_{ik}) + a_{ik} x_{11}^2}{a_{11}} \\
 &= \delta_{(ik)} - n^2 \frac{e_1 N(p'_{11} - p'_{ik}) + a_{ik} - a_{11}}{a_{11}} + \frac{a_{ik}}{a_{11}} (n^2 - x_{11}^2) \\
 &= \delta_{(ik)} - n^2 \frac{N(e_1 + e_2)(p'_{11} - p'_{ik})}{a_{11}} + \frac{e_2 (\sum p'_{ik} - N p'_{ik}) - 1}{a_{11}} (n^2 - x_{11}^2)
 \end{aligned}$$

According to (9) the coefficient of p'_{ik} is positive in the second term and negative in the third term above. Therefore (16) implies that

$$0 \leq (2n-1) \delta_{(ik)} < -n^2 \frac{N(e_1 + e_2)(p'_{11} - M_{N_0})}{a_{11}} + \frac{e_2 \sum p'_{ik} - 1}{a_{11}} (n^2 - x_{11}^2)$$

The right hand side is independent of i and k and tends to zero as N tends to infinity. Hence

$$\delta_{(ik)} \rightarrow 0$$

uniformly, in the sense, that if N is large enough all the $\delta_{(ik)}$ will be smaller than an arbitrarily small positive quantity. In particular, they will all be smaller than 1.

$$0 \leq \delta_{(ik)} < 1 \quad i=1, \dots, N_0 \quad k=1, \dots, K_i$$

Theorem 9

If the conditions of Theorem 7 are satisfied then

$$\pi_{(ik)} - \delta_{(ik)} = o\left(\frac{1}{N}\right)$$

Proof:

According to the proof of the previous theorem, corresponding to an arbitrary positive quantity, $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$0 \leq \delta_{(ik)} < \varepsilon$$

for every i and k , whenever $N > N(\varepsilon)$. In this case

$$\sum \delta_{(ik)}^2 < \varepsilon \sum \delta_{(ik)} = n\varepsilon$$

In other words

$$\lim \sum \delta_{(ik)}^2 = 0 \quad (25)$$

Now from (29) of Chapter III

$$\delta_{(ik)} = n - \sqrt{n^2 - \frac{2n^2 - \sum \delta_{(ik)}^2}{\sum p_{ik}'} p_{ik}' - E \left(\frac{\sum p_{ik}'}{N} - p_{ik}' \right)} \quad (26)$$

where E is given by (28) of Chapter III.

It will be shown, that $E \rightarrow 0$ as $N \rightarrow \infty$. According to the notation of Chapter IV

$$C = N n^2 c_1 + \frac{N n^2}{\sum p_{ik}'} + \left(c_2 - \frac{1}{\sum p_{ik}'} \right) \sum x_{ik}^2$$

$$\begin{aligned}
 &= N n^2 c_1 + \frac{N n^2}{\sum p_{ik}} + (c_2 - \frac{1}{\sum p_{ik}}) [(N-2)n^2 + \sum \delta_{(ik)}^2] \\
 &= n^2 N (c_1 + c_2) - 2n^2 (c_2 - \frac{1}{\sum p_{ik}}) + (c_2 - \frac{1}{\sum p_{ik}}) \sum \delta_{(ik)}^2
 \end{aligned}$$

Also similarly

$$D = n^2 N (d_1 + d_2) - 2n^2 (d_2 - \frac{1}{\sum p_{ik}}) + (d_2 - \frac{1}{\sum p_{ik}}) \sum \delta_{(ik)}^2$$

so

$$E = n^2 N (e_1 + e_2) - 2n^2 (e_2 - \frac{1}{\sum p_{ik}}) + (e_2 - \frac{1}{\sum p_{ik}}) \sum \delta_{(ik)}^2 \quad (27)$$

Now from (3), (5), (6) and (25)

$$\lim E = \frac{N n^2 s}{1 - N s} - 2n^2 \left(\frac{N s - 2}{2(N s - 1)} - 1 \right) = 0 \quad (28)$$

Finally consider

$$\Delta_{ik} = \frac{\delta_{(ik)} - n p_{ik}}{p_{ik}} = \frac{1}{p_{ik}} [n(1 - p_{ik}) - \sqrt{n^2 - \frac{2n^2 - \sum \delta_{(ik)}^2}{\sum p_{ik}} p_{ik} - E(\frac{\sum p_{ik}}{N} - p_{ik})}]$$

It will be shown that $\Delta_{ik} \rightarrow 0$ uniformly

Since $\delta_{(ik)}$ tend to zero uniformly as N tends to infinity, given any $\epsilon_1 > 0$, we have

$$|p_{ik} \Delta_{ik}| \leq \delta_{(ik)} + n p_{ik} \leq \frac{\epsilon_1}{2} + n M_1 \leq \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1$$

as soon as all the $\delta_{(ik)}$ are less than $\frac{\xi_1}{2}$ and all the finite number (N_i) of M_i are less than $\frac{\xi_1}{2}$.

But

$$[p_{ik}\Delta_{ik} - n(1-p_{ik})]^2 = n^2 - \frac{2n^2 - \sum \delta_{(ik)}^2}{\sum p_{ik}} p_{ik} - E\left(\frac{\sum p_{ik}}{N} - p_{ik}\right)$$

$$p_{ik}^2 \Delta_{ik}^2 + n^2 - 2n^2 p_{ik} + n^2 p_{ik}^2 - 2np_{ik}(1-p_{ik})\Delta_{ik} = n^2 - \frac{2n^2 - \sum \delta_{(ik)}^2}{\sum p_{ik}} p_{ik} - E\left(\frac{\sum p_{ik}}{N} - p_{ik}\right)$$

$$\Delta_{ik} = \frac{\frac{2n^2 - \sum \delta_{(ik)}^2}{\sum p_{ik}} \frac{p_{ik}}{p_{ik}} + E\left(\frac{\sum p_{ik}}{N} \frac{p_{ik}}{p_{ik}} - \frac{p_{ik}}{p_{ik}}\right) - 2n^2 + n^2 p_{ik}}{2n(1-p_{ik}) - \Delta_{ik} p_{ik}}$$

Hence

$$|\Delta_{ik}| \leq \frac{1}{2n(1-M_i) - \xi_1} \left[2n^2 \frac{|1 - \sum p_{ik}|}{\sum p_{ik}} + n^2 \frac{\xi_1}{2} + |E| \left| \frac{\sum p_{ik}}{N} - \frac{p_{ik}}{p_{ik}} \right| \right]$$

Since

$$\frac{\sum p_{ik}}{N} - \frac{p_{ik}}{p_{ik}}$$

tends to $\frac{1}{t_i} - 1$ as $N \rightarrow \infty$ it is certainly bounded and so whatever $\xi_2 > 0$ is given, we shall have

$$|\Delta_{ik}| < \xi_2$$

whenever N is large enough.

Since according to Theorem 3 also

$$\lim \frac{\pi_{(ik)} - np_{ik}}{p_{ik}} = 0 \quad (31)$$

it follows that

$$\lim \frac{\delta_{(ik)} - \pi_{(ik)}}{p_{ik}} = 0$$

Finally since

$$p_{ik} = O\left(\frac{1}{N}\right)$$

$$\delta_{(ik)} = \pi_{(ik)} + o\left(\frac{1}{N}\right)$$

This proves the theorem.

The result concerning the uniform convergence of Δ_{ik} is more than is needed here, but this result will be used later.

2 - Limiting Properties of the Estimators (1) and (5) of Chapter I.

Next the limiting properties of the estimators (1) and (5) of Chapter I are to be investigated as sampling without replacement approaches sampling with replacement. However, in order to do so, certain conditions have to be imposed on the values Y_i (or rather Y_{ik} , using the double subscripts which have been adopted to deal with the limiting process). These conditions are similar to those imposed on p_{ik} in Chapter III.

According to the definition given in Chapter III, when sampling without replacement approaches sampling with replacement as $N \rightarrow \infty$, the population consists of a fixed number of classes (N_0), each class

containing $K_i(N)$ units. The statistic to be measured on the k -th unit of the i -th class is $Y_{ik}(N)$ ($i=1,2,\dots,N_0$; $k=1,2,\dots,K_i(N)$). The following conditions are considered to be extensions of the definition of the process whereby "sampling without replacement approaches sampling with replacement":

- a) within each class there are numbers $y_i'(N)$ and $Y_i'(N)$ such that

$$y_i'(N) \leq Y_{ik}(N) \leq Y_i'(N) \quad i=1,2,\dots,N_0; \quad k=1,2,\dots,K_i(N)$$

b) $\lim K_i(N) y_i'(N) = \lim K_i(N) Y_i'(N) = Y_{i0}$

Again, the argument N will be dropped, as before.

Theorem 10

If the conditions of Theorem 7 as well as the two conditions above are satisfied then the bias of the estimator

$$\hat{Y}_\delta = \sum_{i,k \in s} \frac{Y_{ik}}{\delta(ik)}$$

tends to zero.

Proof:

$$\sum_{i=1}^{N_0} K_i y_i' \leq \sum Y_{ik} \leq \sum_{i=1}^{N_0} K_i Y_i'$$

Since both the left hand side and the right hand side tends to $\sum_{i=1}^{N_0} Y_{i0}$ as $N \rightarrow \infty$

$$\lim \sum Y_{ik} = \sum_{i=1}^{N_0} Y_{i0}$$

It is to be shown that

$$E(\hat{Y}_{\delta}) - \sum Y_{ik} = \sum \frac{\pi_{(ik)} - \delta_{(ik)}}{\delta_{(ik)}} Y_{ik} \rightarrow 0 \text{ as } N \rightarrow \infty$$

A lemma is needed.

Lemma:

If the conditions of Theorem 5 are satisfied then there are numbers q_i , Q_i , w_i , W_i in each of the N_0 classes of the population such that

$$q_i \leq \delta_{(ik)} \leq Q_i ; \quad N(Q_i - q_i) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (32)$$

$$W_i \leq \pi_{(ik)} \leq W_i ; \quad N(W_i - w_i) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (33)$$

From (12)

$$\begin{aligned} 2n\delta_{(ik)} - \delta_{(ik)}^2 &= n^2 - x_{ik}^2 = \frac{n^2 e_1 (p'_{ik} - p'_{11}) - a_{ik} x_{11}^2}{a_{11}} + n^2 \\ &= \frac{Nn^2 e_1 (p'_{ik} - p'_{11}) - x_{11}^2 [e_2 (\sum p'_{ik} - Np'_{ik}) - 1]}{a_{11}} + n^2 \end{aligned}$$

If now we put in succession m_i and M_i in place of p_{ik} to obtain q'_i and Q'_i it follows from (24) that

$$q'_i \leq n^2 - x_{ik}^2 = 2n\delta_{(ik)} - \delta_{(ik)}^2 \leq Q'_i$$

for every i and k if only N is large enough.

Then

$$q_i = \frac{q'_i}{2n} \leq \delta_{(ik)} \leq \frac{Q'_i + \delta_{(ik)}}{2n} = Q_i$$

It can be shown without difficulty that

$$\lim N(Q_i' - q_i') = 0$$

In order to show that also

$$\lim N(Q_i - q_i) = 0$$

it is sufficient to demonstrate that $N\delta_{(ik)}^2$ tends to zero uniformly with respect to k . But it is known already that

$$\Delta_{ik} = \frac{\delta_{(ik)} - np_{ik}}{p_{ik}}$$

as well as $\delta_{(ik)}$ tend to zero uniformly as $N \rightarrow \infty$. Hence given any $\epsilon_2 > 0$ we shall have

$$|\delta_{(ik)} - np_{ik}| < \epsilon_2 p_{ik}$$

$$\delta_{(ik)} < \epsilon_2$$

if only N is large enough. Hence

$$\delta_{(ik)}^2 + n^2 p_{ik}^2 - 2np_{ik} \delta_{(ik)} < \epsilon_2^2 p_{ik}^2$$

$$\delta_{(ik)}^2 < \epsilon_2^2 M_i^2 + 2nM_i \epsilon_2$$

$$N \delta_{(ik)}^2 < \epsilon_2^2 (NM_i^2 + 2nNM_i)$$

Since NM_i has a finite limit, it is certainly bounded and hence $N \delta_{(ik)}^2$ can be made as small as required for every k at the same time, by choosing ϵ_2 sufficiently small.

In order to demonstrate (33) the following logic is used.

Consider the expression for $\pi_{(il)}$

$$\pi_{(il)} = \sum \frac{p_{i_1 k_1} \cdots p_{i_l} \cdots p_{i_n k_n}}{(1-p_{i_1 k_1}) \cdots (1-p_{i_1 k_1} - p_{i_2 k_2} - \cdots - p_{i_l} - \cdots - p_{i_{n-1} k_{n-1}})} \quad (34)$$

the sum extending over possible samples containing unit (il). Replace p_{i_1} by M_i and M_i to obtain w_i and W_i respectively. It is to be shown that

$$w_i \leq \pi_{(ik)} \leq W_i \quad k=1,2,\dots,K_i \quad (35)$$

(35) is evident for $k=1$. To show this relationship for any other value of k split the sum obtained for w_1 from (34) into two parts

$$w_1 = \Sigma_1 + \Sigma_2$$

where Σ_1 contains those terms in which p_{ik} occurs and hence is a sum over all samples containing both (ik) and (il); Σ_2 contains those terms in which p_{ik} does not occur and hence Σ_2 is a sum over all samples containing (il) but not (ik). Next consider $\pi_{(ik)}$

$$\pi_{(ik)} = \sum \frac{p_{i_1 k_1} \cdots p_{ik} \cdots p_{i_n k_n}}{(1-p_{i_1 k_1}) \cdots (1-p_{i_1 k_1} - \cdots - p_{ik} - \cdots - p_{i_{n-1} k_{n-1}})}$$

Split this latter sum again into two parts:

$$\pi_{(ik)} = \Sigma'_1 + \Sigma'_2$$

Σ'_1 containing the terms in which both (ik) and (il) occur; Σ'_2 the terms in which (ik) occurs but not (il). If now in Σ'_1 p_{i_1} is replaced by m_i , every term of the sum is decreased and exactly Σ_1 is obtained.

$$\Sigma_1 \leq \Sigma'_1$$

If in $\Sigma_2' p_{ik}$ is replaced by m_i , every term of the sum is decreased and exactly Σ_2 is obtained. So

$$\Sigma_2 \leq \Sigma_2'$$

and hence the left hand part of (35) follows by adding the last two inequalities. The right hand part of it follows similarly.

A proof very similar to that of Theorem 2 can be constructed to verify the statement below:

$$\lim N(W_i - w_i) = 0 \quad (36)$$

This completes the proof of the lemma. Now the proof of Theorem 10 will be considered.

Suppose for the present that $Y_{ik} \geq 0$ for every i and k and that none of the q_i and Q_i values are zero.

$$\sum_{i=1}^{N_0} \left(\frac{w_i}{Q_i} - 1 \right) \sum_{k=1}^{K_i} Y_{ik} \leq \sum Y_{ik} \frac{\pi_{(ik)} - \delta_{(ik)}}{\delta_{(ik)}} \leq \sum_{i=1}^{N_0} \left(\frac{w_i}{q_i} - 1 \right) \sum_{k=1}^{K_i} Y_{ik} \quad (37)$$

Since

$$\lim N \delta_{(ik)} = \lim N n p_{ik} = n t_i$$

it follows that also

$$\lim N Q_i = n t_i > 0$$

hence

$$\begin{aligned} \lim \frac{w_i}{Q_i} &= \lim \frac{N(w_i - \pi_{(i1)})}{N Q_i} + \lim \frac{N(\pi_{(i1)} - \delta_{(i1)})}{N Q_i} + \lim \frac{\delta_{(i1)}}{Q_i} \\ &= 1 \end{aligned}$$

Similarly

$$\lim \frac{W_i}{g_i} = 1$$

But

$$\lim \sum_{k=1}^{K_i} Y_{ik} = Y_{i0} = \text{finite}$$

so both left hand side and right hand side of (37) tend to zero and this proves the theorem. The restriction that $Y_{ik} \geq 0$ can easily be overcome by separating the terms involving negative values of Y_{ik} . The restriction regarding the values of q_i and Q_i can also be removed by adding or subtracting $\frac{1}{N^2}$.

It should be noted that $\text{Var}(\hat{Y}_G)$ may be estimated by replacing in (5) of Chapter I π_i by δ_i and π_{ij} by δ_{ij} . The bias so committed may be shown to tend to zero if the conditions of Theorem 10 are satisfied.

Having investigated one extreme case, namely where $n \ll N$ it may be of some interest to consider the other extreme case, namely where $n=N$.

Theorem 11

If $n=N$ then $\delta_i = \pi_i = 1$ for every i .

Proof:

In this case it follows from (8) and (9) of Chapter IV that

$$e_1 = c_1$$

$$e_2 = c_2$$

$$c_1 = - \left(\frac{N-1}{N}\right)^2 c_2$$

$$a_i = c_2 (\sum p_i' - N p_i') - 1$$

$$b_i = - N^3 \left(\frac{N-1}{N}\right)^2 c_2 (N p_i' - \sum p_i') = N^3 \left(\frac{N-1}{N}\right)^2 (1 + a_i)$$

$$x_i^2 = N^2 \left(\frac{N-1}{N}\right)^2 (1 + a_i) - a_i \frac{\sum x_i^2}{N} = (N-1)^2 + a_i [(N-1)^2 - \frac{\sum x_i^2}{N}]$$

Clearly

$$x_i^2 = (N-1)^2$$

So

$$\delta_i = 1 \quad i=1, \dots, N$$

is the only solution satisfying

$$0 \leq \delta_i \leq 1$$

CHAPTER VI

NUMERICAL EXAMPLES

1 - The Direct Problem and The Reverse Problem

The use of δ_i rather than np_i , suggested by Yates and Grundy as an approximation to $\pi_i(p_1, p_2, \dots, p_N)$ is illustrated in the following examples. It is noted in the examples given, that the δ_i values are invariably closer to π_i than are the np_i values. The estimator, $\hat{Y}_B$, obtained from (1) of Chapter I when using np_i instead of π_i and the estimator $\hat{Y}_\delta$, arising from the use of δ_i instead of π_i are recorded below:

$$\hat{Y}_B = \sum_{i \in S} \frac{Y_i}{np_i} \quad (1)$$

$$\hat{Y}_\delta = \sum_{i \in S} \frac{Y_i}{\delta_i} \quad (2)$$

The biases of the respective estimates are

$$\text{Bias } (\hat{Y}_B) = \sum (np_i - \pi_i) \frac{Y_i}{np_i} \quad (3)$$

$$\text{Bias } (\hat{Y}_\delta) = \sum (\delta_i - \pi_i) \frac{Y_i}{\delta_i} \quad (4)$$

It is clear from (3) and (4) that the respective biases depend on the magnitude of the differences $|np_i - \pi_i|$ and $|\delta_i - \pi_i|$. In fact, when all the Y_i values, except Y_j , are zero, the ratio of the biases (3) and (4) is equal to the ratio of $np_j - \pi_j$ and $\delta_j - \pi_j$. In the following examples this ratio may be noted to be as high as 100 or even 1000 in a number of instances. If the Y_i values increase faster with i than the p_i values or if a few extreme values of Y_i occur at the upper end of its

distribution, the reduction of the bias (4) in comparison with (3) may be particularly impressive, since the greatest improvement of the approximation δ_i over np_i occurs at the extremes of the distribution.

The first two examples are given a more complete analysis. Here, apart from the "direct problem", mentioned above, the "reverse problem" is considered as well.

All the examples are restricted to relatively small values of N (ranging from 4 to 15) partly because this shows the closeness of the approximation under unfavourable circumstances, partly because (for the sake of comparison) the values π_i had to be calculated and the amount of arithmetic involved barred the consideration of examples with larger values of N .

Example 1.

The first example is that considered by Yates and Grundy in which a sample of 2 is selected from a population of 4 units. The variation in p_i as well as the magnitude of the sampling rate $n/N = \frac{1}{2}$ may be considered quite extreme. The values of p_i , Y_i , δ_i , π_i and np_i for this example are given by Table I below.

<u>TABLE I</u>				
Values of p_i , Y_i , δ_i , π_i and np_i				
i	1	2	3	4
p_i	.1	.2	.3	.4
Y_i	.5	1.2	2.1	3.2
δ_i	.234481	.441051	.607233	.717235
π_i	.234524	.441270	.608333	.715873
np_i	.200000	.400000	.600000	.800000
$n=2$		$\Sigma Y_i = 7.0$		

The expected value and bias of the estimates $\hat{Y}_\delta$ and $\hat{Y}_B$ are exhibited in Table II.

TABLE II		
Comparison of $\hat{Y}_\delta$ and $\hat{Y}_B$		
Estimate	$\hat{Y}_\delta$	$\hat{Y}_B$
Expected Value	6.998415	6.902778
Bias	0.001585	0.097222
Percentage Bias	0.2%	1.39%
Bias Index	100	695

The bias of the estimate $\hat{Y}_B$ is almost 70 times as much as the bias of $\hat{Y}_\delta$. However, it may be mentioned here that if the population values, Y_i , were 0.1, 0.2, 0.9, 4.0, then the bias resulting from the use of np_i would be equal to 7.12%, while with this same population, the bias due to using δ_i in place of π_i would be only 0.11%; still an insignificant proportion.

Continuing this example, the bias of the variance estimate is investigated when δ_i, δ_{ij} are used instead of π_i, π_{ij} in expression (5) of Chapter I. We shall denote the resulting expression by $\text{var}(\hat{Y}_\delta)$

$$\text{var}(\hat{Y}_\delta) = \sum_{\substack{i,j \in s \\ i < j}} \frac{\delta_i \delta_j - \delta_{ij}}{\delta_{ij}} \left(\frac{Y_i}{\delta_i} - \frac{Y_j}{\delta_j} \right)^2 \quad (5)$$

$$\sqrt{E(\text{var} \hat{Y}_\delta)} = 0.9090$$

The square root of the mean square error of $\hat{Y}_\delta$

$$\sqrt{\text{MSE}(\hat{Y}_\delta)} = 0.9138$$

Comparing these figures it is seen that the bias of the error estimate is 0.0048, or 0.07% of the population total to be estimated.

For the sake of later comparisons the standard deviation of the Horvitz-Thompson estimate, $\hat{Y}$, and that of Murthy's estimate, $\hat{Y}_M$, are also recorded here:

$$\sqrt{\text{Var}(\hat{Y})} = 0.9071 \quad (6)$$

$$\sqrt{\text{Var}(\hat{Y}_M)} = 0.5582 \quad (7)$$

When Yates and Grundy constructed the above example, they assumed that the measures of size, M_i , are utilized in the sampling method by setting the p_i values proportional to M_i :

$$p_i = \frac{M_i}{\sum M_i}$$

Alternatively, M_i may be used more gainfully if the reverse problem is solved, that is, if the values p_{iT} are calculated from the equations

$$n_i(p_{1T}, p_{2T}, \dots, p_{NT}) = n \frac{M_i}{\sum M_i} \quad i=1, 2, \dots, N$$

In the present example the proportions $\frac{M_i}{\sum M_i}$ are 0.1, 0.2, 0.3, 0.4. The corresponding values of p_{iT} have been found by the method of trial and error and are shown in Table III.

TABLE III

Solution of the reverse problem

i	1	2	3	4
p_{iT}	.0799	.1674	.2703	.4824

In the case when $n=2$ Yates and Grundy suggest a method of successive approximations for the solution of this problem. The results of the first five steps of this process appear in Table IV.

TABLE IV

Successive approximations of Yates-Grundy to the solution of the reverse problem

$i \backslash j$ -th step	1	2	3	4	5
1	.084479	.081985	.081032	.080529	.080248
2	.179594	.172991	.170359	.169045	.168327
3	.293115	.283606	.277675	.274361	.272538
4	.442813	.461419	.470934	.476066	.478887

The corresponding π_i values corresponding to the solution p_{iT} are, of course, 0.2, 0.4, 0.6, 0.8. The π_i corresponding to the five approximations above appear in Table V.

TABLE V

The π_i values corresponding to the p_i values of TABLE IV

$i \backslash j$ -th step	1	2	3	4	5
1	.205140	.201829	.200950	.200530	.200299
2	.413364	.405130	.402510	.401365	.400763
3	.617274	.611231	.606343	.603498	.601932
4	.764223	.781811	.790197	.794607	.797006

The convergence of π_4 appears to be particularly slow.

The approximate solution (p_{iR}) of the reverse problem, based on the method outlined in Section 3 of Chapter IV yields the results contained in Table VI. This Table also contains the values of the corresponding

δ_i , π_i values (denoted by δ_{iR} and π_{iR} respectively) as well as the values np_{iR} .

TABLE VI

Approximate solution of reverse problem
with corresponding δ_{iR} , π_{iR} and np_{iR} values

i	1	2	3	4
p_{iR}	.080551	.168828	.273710	.476912
δ_{iR}	.2	.4	.6	.8
π_{iR}	.200709	.401167	.602832	.795292
np_{iR}	.161102	.337656	.547420	.953824

The solution of the reverse problem is undertaken in order to reduce the standard deviation of the estimate $\hat{Y}$ (or its approximation $\hat{Y}_\delta$).

$$\sqrt{\text{Var}(\hat{Y})} = 0.5471$$

Comparing this result with (6) it is readily seen that the approximate solution of the reverse problem reduces the standard deviation of $\hat{Y}$ considerably, in fact it is now smaller than the standard deviation of Murthy's estimator (7).

We may compare again the approximations δ_{iR} and np_{iR} by comparing the estimates obtained from the Horvitz-Thompson estimate by replacing π_{iR} by δ_{iR} and np_{iR} respectively.

TABLE VII

Comparison of $\hat{Y}_\delta$ and $\hat{Y}_B$ after the approximate solution of the reverse problem

Estimate	$\hat{Y}_\delta$	$\hat{Y}_B$
Expected Value	6.996364	7.029345
Bias	0.003636	0.029345
Percentage Bias	0.05%	0.42%
Bias Index	100	84.0

Both biases are insignificant, although that of $\hat{Y}_B$ is considerably larger than that of $\hat{Y}_\delta$. However, with population values (Y_i) 0.1, 0.2, 0.9, 4.0 the bias of $\hat{Y}_B$ increases to 9.84% while for the same set of Y's the bias of $\hat{Y}_\delta$ is only 0.35%.

Finally the error estimate of $\hat{Y}_\delta$, as given by (5), is compared with its mean square error

$$\sqrt{E(\text{var } \hat{Y}_\delta)} = 0.5353$$

$$\sqrt{\text{MSE } (\hat{Y}_\delta)} = 0.5389$$

The bias of the error estimate is 0.0036, or 0.05% of the population total, ΣY_i , to be estimated.

Example 2

In the second example a population of 6 units is considered from which a sample of 3 is selected. The Table below contains the values of p_i , Y_i , δ_i , π_i and np_i .

TABLE VIII

$p_i, Y_i, \delta_i, \pi_i$ and np_i for the second example considered						
i	1	2	3	4	5	6
p_i	0.09	0.12	0.15	0.18	0.21	0.25
Y_i	2.655	3.480	4.425	5.220	6.195	7.250
δ_i	.312664	.402008	.478939	.544535	.600164	.661693
π_i	.310595	.397745	.475740	.544209	.603164	.668547
np_i	.270000	.360000	.450000	.540000	.630000	.750000

$$n=3 \quad \Sigma Y_i = 29.225$$

Comparing np_i and δ_i with π_i we can see the improvement of the approximation δ_i over np_i . However, in the example chosen, the Y_i values are very nearly proportional to p_i : $Y_1/p_1, Y_3/p_3, Y_5/p_5$ are equal to 29.5; $Y_2/p_2, Y_4/p_4, Y_6/p_6$ are equal to 29.0. In light of (3) and (4) it may be expected, that the bias of $\hat{Y}_B$ will be negligible. In fact if for every i

$$\frac{Y_i}{p_i} \approx c$$

then

$$\sum (np_i - \pi_i) \frac{Y_i}{np_i} \approx \frac{c}{n} \sum (np_i - \pi_i) = 0$$

We can see in the following Table, that in fact the bias of $\hat{Y}_B$ is even smaller than that of $\hat{Y}_\delta$. This occurs due to the special character of the values of Y_i and is explainable by the cancellation of terms in (3). The individual terms of (3) are, however, considerably larger than those of (4), as can be seen from the comparison of the differences $|np_i - \pi_i|$ and $|\delta_i - \pi_i|$.

TABLE IX

Estimate	Comparison of $\hat{Y}_\delta$ and $\hat{Y}_B$	
	$\hat{Y}_\delta$	$\hat{Y}_B$
Expected Value	29.2439	29.2316
Bias	0.0189	0.0034
Percentage Bias	0.06%	0.02%
Bias Index	100	33

As in Example 1, the square root of the expected value of the estimate $\text{var}(\hat{Y}_\delta)$ as defined by (5) is compared with the mean square error of $\hat{Y}_\delta$

$$\sqrt{E(\text{var } \hat{Y}_\delta)} = 1.0982$$

$$\sqrt{\text{MSE } (\hat{Y}_\delta)} = 1.1217$$

The bias of the error estimate is, therefore, 0.0235 or 0.08% of the population total, $\sum Y_i$, to be estimated. This quantity may safely be judged as negligible.

Again, for later comparisons, the standard deviations of the Horvitz-Thompson estimate, $\hat{Y}$, and Murthy's estimate $\hat{Y}_M$ are also recorded.

$$\sqrt{\text{Var } (\hat{Y})} = 1.0323 \quad (8)$$

$$\sqrt{\text{Var } (\hat{Y}_M)} = 0.1551 \quad (9)$$

The approximate solution (p_{iR}) of the reverse problem is contained in the Table below. The values p_{iR} were obtained by solving the equations below for p_i

$$\delta_i(p_1, p_2, \dots, p_N) = n \frac{M_i}{\sum M_i} \quad i=1, 2, \dots, N$$

The values of $\frac{M_i}{\sum M_i}$ are 0.09, 0.12, 0.15, 0.18, 0.21, 0.25. For the solution the numerical method explained at the end of Section 3, Chapter IV was used. Since $n=3$, the method of Yates and Grundy is not applicable.

TABLE X

Approximate solution of the reverse problem for Example 2

with corresponding δ_{iR} , π_{iR} and np_{iR} values

i	1	2	3	4	5	6
p_{iR}	.073959	.100473	.131603	.169216	.220587	.304162
δ_{iR}	.269374	.358573	.449402	.540544	.636692	.745411
π_{iR}	.270018	.354470	.444252	.538232	.639755	.753272
np_{iR}	.221877	.301219	.394809	.507648	.661761	.912486

The next Table contains the comparison of $\hat{Y}_\delta$ and $\hat{Y}_B$, when these estimates are prepared after having approximately solved the reverse problem.

TABLE XI
Comparison of $\hat{Y}_\delta$ and $\hat{Y}_B$ after the approximate solution
of the reverse problem

Estimate	$\hat{Y}_\delta$	$\hat{Y}_B$
Expected Value	29.2248	29.8139
Bias	0.0002	0.5889
Percentage Bias	0.00%	2.02%
Bias Index	100	294,400

There seems to be a considerable reduction in the bias if $\hat{Y}_\delta$ is used instead of $\hat{Y}_B$. As Table X indicates, if all the Y_i would be small in comparison with Y_6 , $\hat{Y}_B$ could be biased by as large a quantity as 20% of the total to be estimated.

Let us consider the bias of the estimated error of $\hat{Y}_\delta$

$$\sqrt{E(\text{var } \hat{Y}_\delta)} = 0.0940$$

$$\sqrt{\text{MSE } (\hat{Y}_\delta)} = 0.0991$$

The bias of the error estimate equals 0.0051 or about 0.02% of the population total, ΣY_i , to be estimated.

The standard deviation of the Horvitz-Thompson estimator, $\hat{Y}$, is

$$\sqrt{\text{Var } (\hat{Y})} = 0.1457$$

The comparison of this figure with (8) and (9) will be sufficient to demonstrate the gain in the precision of estimates, due to the approximate

solution of the reverse problem.

2 - Further Numerical Examples

In what follows only the closeness of the approximation δ_i will be demonstrated in contrast with the approximation accorded by np_i . The values p_i , δ_i , π_i and np_i will be exhibited. It is felt that there are few examples of the values $\pi_i(p_1, p_2, \dots, p_N)$ recorded in the literature and so there may be some interest in constructing further examples even beyond the context of the above mentioned comparison.

Example 3

Again, as in Example 2, a population of 6 units is considered and a sample size of 3. The variation in the p_i values is somewhat smaller than in Example 2 and hence this example may be judged more favourable for the approximation δ_i .

TABLE I

p_i, δ_i, π_i and np_i values for Example 3; $n=3$						
i	1	2	3	4	5	6
p_i	.11	.13	.15	.18	.20	.23
δ_i	.3622	.4192	.4708	.5388	.5784	.6306
π_i	.3649	.4192	.4695	.5377	.5779	.6308
np_i	.3300	.4900	.4500	.5400	.6000	.6900

Example 4

As a most extreme example, the Yates-Grundy population of p_i values is considered once again, this time, however, a sample size of 3 is assumed. The sampling ratio, n/N , is therefore $3/4$. Altogether then

this example may be judged as most unfavourable for the approximation δ .

TABLE II

p_i, δ_i, π_i and np_i values for the Yates-Grundy population; $n=3$

i	1	2	3	4
p_i	0.1	0.2	0.3	0.4
δ_i	.4551	.7561	.8659	.9229
π_i	.4488	.7588	.8702	.9222
np_i	.3000	.6000	.9000	1.2000

Example 5

The p_i values of Example 2 are considered again, but instead the sample size now is assumed to be 4, instead of 3, as in Example 2. This makes the conditions for the approximation δ_i more unfavourable than it was in Example 2.

TABLE III

p_i, δ_i, π_i and np_i values for Example 5; $n=4$

i	1	2	3	4	5	6
p_i	.09	.12	.15	.18	.21	.25
δ_i	.4527	.5552	.6431	.7178	.7809	.8504
π_i	.4562	.5661	.6541	.7221	.7746	.8269
np_i	.3600	.4800	.6000	.7200	.8400	1.0000

Example 6

The π_i values for Examples 6 and 7 were calculated at the Central Bureau of Statistics, Israel, and are borrowed from an unpublished paper by Mr. Hertzstein.

In Example 6 n/N is not as high as in the previous examples, but now the variation in the p_i values is considerably larger, in fact, it

may be said to be extreme, relative to the size of the population, $N=15$.

The sample size is $n=4$.

TABLE IV

p_i, δ_i, π_i and np_i values for Example 6; $n=4$								
i	1-2	3	4-7	8-11	12	13	14	15
p_i	1/77	2/77	3/77	5/77	7/77	9/77	10/77	15/77
δ_i	.0605	.1181	.1731	.2757	.3686	.4522	.4907	.6544
π_i	.0608	.1184	.1730	.2756	.3683	.4521	.4909	.6542
np_i	.0519	.1039	.1558	.2597	.3636	.4675	.5195	.7792

Example 7

In Example 7 the variation in the p_i values is small, so from this aspect, it is more favourable than the previous example. However, the sample size is larger, $n=5$, while the population size is smaller, $N=13$ and so n/N is larger and thus is unfavourable.

TABLE V

p_i, δ_i, π_i and np_i values for Example 7; $n=5$				
i	1-3	4-6	7-8	9-13
p_i	17/243	18/243	19/243	20/243
δ_i	.3543	.3724	.3903	.4079
π_i	.3564	.3734	.3900	.4061
np_i	.3498	.3704	.3908	.4115

In Conclusion, the following may be noted. In the examples considered it appeared that:

- 1) δ_i is an appreciably better approximation than np_i
- 2) if there is some good ground for the assumption that

Y_i/p_i is close to being constant, $\hat{Y}_B$ may be gainfully employed

- 3) the approximate solution of the reverse problem yields a decrease of the sampling error of either $\hat{Y}$ or $\hat{Y}_\delta$.

A P P E N D I X

Graphical Representation of the
points (p_i, π_i) and the line $\pi = \pi p$
for Examples 1-7 of Chapter VI.

EXAMPLE 1.

The points (p_i, π_i) for example and the line $\pi = 2p$

$$s = \sum p_i^2 = 0.3$$

EXAMPLE 2.

The points (p_i, π_i) and the line $\pi = 3p$

$$s = \sum p_i^2 = 0.1840$$

EXAMPLE 3.

The points (p_i, π_i) and the line $\pi = 3p$

$$s = \sum p_i^2 = 0.1768$$

EXAMPLE 4.

The points (p_i, π_i) and the line $\pi = 3p$

$$s = \sum p_i^2 = 0.3$$

EXAMPLE 5.

The points (p_i, π_i) and the line $\pi = 4p$

$$s = \sum p_i^2 = 0.1840$$

EXAMPLE 6.

The points (p_i, π_i) and the line $\pi = 4p$

$$s = \sum p_i^2 = 0.1007$$

EXAMPLE 7.

The points (p_i, π_i) and the line $\pi = 5p$

$$s = \sum p_i^2 = 0.0772$$

LIST OF REFERENCES

- [1] W.G. Cochran: "Sampling Techniques" New York. John Wylie and Sons, (1953) page 175.
- [2] Des Raj: Some estimators in sampling for varying probabilities without replacement. "Journal of American Statistics Association", Volume 51 (1956). pp. 269-284.
- [3] J. Durbin: Some results in sampling theory when the units are selected with unequal probabilities. "Journal of the Royal Statistical Society." Volume 15 (1953) pp. 262-269.
- [4] M.H. Hansen - W.N. Hurwitz; On the theory of sampling from finite populations. "Annals of Mathematical Statistics", Volume 14 (1943). pp. 333-362.
- [5] D.G. Horvitz - D.J. Thompson: A generalization of sampling without replacement from a finite universe. "Journal of American Statistical Association", Volume 47 (1952). pp. 663-685.
- [6] V.P. Godambe: A unified theory of sampling from finite populations. "Journal of the Royal Statistical Society", Series B, Volume 17 (1955) pp. 269-278.
- [7] M.R. Mickey: Some finite population unbiased ratio and regression estimators. "Journal of the American Statistical Association", Volume 54 (1959) pp. 594-613.
- [8] M.N. Murthy: Ordered and unordered estimators in sampling without replacement. "Sankhya", Volume 18, (1957) pp. 379-390.
- [9] J. Roy - I.M. Chakravarti: Estimating the meaning of a finite population. "Annals of Mathematical Statistics", Volume 31 (1960) pp. 392-398.
- [10] F. Yates - P.M. Grundy: Selection without replacement from within strata with probability proportional to size. "Journal of the Royal Statistical Society", Series B, Volume 15 (1953) pp. 253-261.