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**SERIES SOLUTIONS OF NONLINEAR ORDINARY
DIFFERENTIAL EQUATIONS**

by

MOHAMED ALI HEMMI, B.Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Master of Science

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

April, 1994

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**SERIES SOLUTIONS OF NONLINEAR ORDINARY
DIFFERENTIAL EQUATIONS**

submitted by
MOHAMED ALI HEMMI, B.Eng.
in partial fulfilment of the requirements for
the degree of Master of Science

Thesis Supervisor

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Abstract

This thesis deals with series solutions of second- and third-order nonlinear ordinary differential equations. General solutions are expressed in terms of series expansions about arbitrary ordinary points, movable poles (Laurent series), and movable branch points (psi-series). Convergence proofs are discussed, and carried out in detail for various examples. Estimates of the radius of convergence for each type of series are made.

Series solutions about ordinary points, with various initial conditions, are calculated and plotted for the Kuramoto-Sivashinsky equation.

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INTRODUCTION

A nonlinear ordinary differential equation (ODE) (or system) is generally said to be *integrable* if it has the *Painlevé property*, i.e., its general solution has no movable singularities other than poles. A singularity is movable if its location in the complex plane of the independent variable depends upon the arbitrary constants in the solution, or, equivalently, upon the initial data. For example, the (linear) equation

$$z \frac{d^2 u}{dz^2} + \frac{du}{dz} + u = 0$$

has a fixed singularity at $z = 0$, whereas the (nonlinear) equation

$$\left[\frac{d}{dz} \left(\frac{1}{u} \frac{du}{dz} \right) \right]^2 + 4 \left(\frac{1}{u} \frac{du}{dz} \right)^2 = 0 ,$$

with general solution

$$u(z) = a e^{\frac{1}{z-b}} ,$$

has a movable essential singularity at $z = b$.

In the late 19th century, Painlevé (see, e.g., Ince [4]) determined all first-order ODEs of the form

$$\frac{du}{dz} = f(z, u) ,$$

where f is rational in u and analytic in z , which have the Painlevé property. Painlevé and his collaborators subsequently performed a singularity analysis of all second-order ODEs of the form

$$\frac{d^2 u}{dz^2} = F(z, u, u') ,$$

where F is analytic in z , algebraic in u and rational in u' (see [4]). They found that fifty such equations possess the Painlevé property, forty-four of which were solvable in terms of elementary functions or quadratures, and the remaining six require new transcendental functions for their solutions, the so-called Painlevé transcendents. A similar analysis for third-order equations has been carried out by Bureau [22].

Interest in the Painlevé property of nonlinear ODEs was revived in the 1970s following the conjecture of Ablowitz, Ramani and Segur (ARS) [1, 2] that every nonlinear partial differential equation (PDE) that is either directly linearizable or solvable via an inverse scattering transform (see, e.g., Ablowitz and Segur [5]) has the property that every ODE obtained from the PDE by symmetry reduction has the Painlevé property. Ablowitz *et al.* [1, 2] (see also Ramani *et al.* [6]) developed an algorithm, generally known as the *ARS algorithm* (and also called the *Painlevé test*), consisting of necessary (but not sufficient) conditions for a given ODE to have the Painlevé property.

For equations passing the ARS test, general solutions may be expressed in (Laurent) series about arbitrary movable poles. For equations failing the test, series solutions may be possible, albeit about arbitrary movable branch points, and such series are generally referred to as *psi-series* (see, e.g., Hille [3] and Tabor [10]). A convergence proof for psi-series solutions of quadratic systems has been given by Hille [12]. The main thrust of this thesis is the provision of convergence proofs for various other types of psi-series, as well as for analytic and for Laurent series.

In Chapter 1, we discuss the ordinary series method (i.e., series solutions about ordinary points) for a class of nonlinear ODEs, where our main concern is the convergence proofs and the estimates of the radii of convergence. In Chapter 2, a detailed description of the ARS algorithm is presented, with applications to particular equations. For ODEs passing the Painlevé test, it is shown that the appropriate Laurent series converges absolutely on an interval. Chapter 3 deals with the various different forms that psi-series must take in order to constitute general solutions about

arbitrary movable branch points, at least formally. Proofs of absolute convergence for the various types of psi-series are given in Chapter 4. Finally, in Chapter 5, we present a study of the nonlinear ODE governing traveling-wave solutions of the Kuramoto-Sivashinsky equation (a fourth-order nonlinear PDE). A proof of convergence of the appropriate psi-series is given, as well as that of the ordinary series solutions for various initial conditions.

Chapter 1

SERIES SOLUTIONS ABOUT ORDINARY POINTS

Most differential equations (DEs), especially nonlinear ones, do not admit solutions which can be expressed explicitly or implicitly in terms of elementary functions. However, the problem of finding solutions is not entirely hopeless. In fact, there are available graphical methods, numerical methods and series methods for solving DEs. For certain differential equations, implicit solutions in terms of elementary functions can be found. However, it is frequently true that such implicit solutions are less useful than series or numerical ones. Implicit solutions are, for most differential equations, such complicated expressions that it is extremely difficult to find values of the dependent variable for given values of the independent variable.

The subject of this chapter is the employment of series as a method for deriving solutions to nonlinear ordinary differential equations. The solutions are expanded about arbitrary ordinary points and convergence is proved.

We begin by giving a brief review of power series in the next section.

1.1 Review of Power Series

A series of the form

$$\sum_{k=0}^{\infty} a_k(t - t_0)^k = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots, \quad (1.1)$$

where $a_0, a_1, a_2, \dots, t_0$ are constants and t is the independent variable is called a **power series**. The following are pertinent results regarding power series :

1. A power series (1.1) is said to converge at a point t if

$$\lim_{m \rightarrow \infty} \sum_{k=0}^m a_k(t - t_0)^k \equiv \sum_{k=0}^{\infty} a_k(t - t_0)^k$$

exists. The series certainly converges for $t = t_0$; it may converge for all t , or it may converge for some values of t and not for others.

2. The series (1.1) is said to converge *absolutely* at a point t if the series

$$\sum_{k=0}^{\infty} |a_k(t - t_0)^k|$$

converges. If a series converges absolutely, then the series also converges; however, the converse is not true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test. It states that if for a fixed value of t

$$|t - t_0| \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = l,$$

then the power series converges absolutely at that value of t if $l < 1$, and diverges if $l > 1$. If $l = 1$ the test is inconclusive, i.e, the series may either converge or diverge.

4. If a power series converges at $t = t_1$, then it converges absolutely for all t such that $|t - t_0| < |t_1 - t_0|$; and if it diverges at $t = t_1$, then it diverges for all t such that $|t - t_0| > |t_1 - t_0|$.

5. The radius of convergence of a power series is defined to be the positive constant ρ such that the power series converges absolutely for $|t - t_0| < \rho$ and diverges for $|t - t_0| > \rho$. The interval $I = \{t : |t - t_0| < \rho\}$ is called the interval of convergence of the series.

6. Power series can be added or subtracted termwise, and

$$\sum_{k=0}^{\infty} a_k(t - t_0)^k \pm \sum_{k=0}^{\infty} b_k(t - t_0)^k = \sum_{k=0}^{\infty} (a_k \pm b_k)(t - t_0)^k .$$

7. Series can also be formally multiplied, and

$$\left(\sum_{k=0}^{\infty} a_k(t - t_0)^k \right) \left(\sum_{k=0}^{\infty} b_k(t - t_0)^k \right) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^k a_l b_{k-l} \right) (t - t_0)^k .$$

8. A function $f(t)$ is said to be analytic at $t = t_0$ if it has a Taylor series expansion about $t = t_0$,

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} (t - t_0)^k , \quad (1.2)$$

with a radius of convergence $\rho > 0$.

1.2 Series Solutions About Ordinary Points

Consider the n th-order nonlinear ordinary differential equation

$$u^{(n)} = F(u, u', u'', \dots, u^{(n-1)}) . \quad (1.3)$$

Definition 1.1 A point t_0 is called an ordinary point of a particular solution $u_p(t)$ of (1.3) if $u_p(t)$ is analytic at $t = t_0$, i.e., $u_p(t)$ has a Taylor series expansion (1.2) about $t = t_0$ with a positive radius of convergence. The point t_0 is called a singular point if $u_p(t)$ is not analytic at $t = t_0$.

In this chapter we shall deal with ordinary points. Singular points will be discussed in later chapters.

Since every ODE of type (1.3) can be cast into the form of an equivalent first-order system, the existence of analytic series solutions is guaranteed by the “existence and uniqueness” theorem (Tennenbaum and Pollard [7]). In this section we shall illustrate, by means of two examples, that appropriate series expansions about ordinary points contain the required number of free constants.

Example 1.2.1: Let us begin by considering the second-order ODE

$$u'' = -2uu' - u^2 . \quad (1.4)$$

We seek solutions to (1.4) of the form

$$u = \sum_{k=0}^{\infty} a_k \tau^k , \quad (1.5)$$

where $\tau = t - t_0$, t_0 is the arbitrary ordinary point and the a_k are constants. We shall show that the series (1.5) contains a_0 and a_1 as free constants, and hence *formally* constitutes a general solution. From (1.5) we have

$$u' = \sum_{k=0}^{\infty} k a_k \tau^{k-1} ,$$

$$u'' = \sum_{k=0}^{\infty} k(k-1) \tau^{k-2} , \quad (1.6)$$

$$uu' = \sum_{k=0}^{\infty} \sum_{m=0}^k m a_m a_{k-m} \tau^{k-1} , \quad (1.7)$$

$$u^2 = \sum_{k=0}^{\infty} \sum_{m=0}^k a_m a_{k-m} \tau^k . \quad (1.8)$$

Substituting (1.6), (1.7) and (1.8) into the ODE (1.4), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} k(k-1) a_k \tau^{k-2} &= -2 \sum_{k=0}^{\infty} \sum_{m=0}^k m a_m a_{k-m} \tau^{k-1} - \sum_{k=0}^{\infty} \sum_{m=0}^k a_m a_{k-m} \tau^k \\ &= \sum_{k=0}^{\infty} \left\{ -2 \sum_{m=0}^{k-1} m a_m a_{k-m-1} - \sum_{m=0}^{k-2} a_m a_{k-m-2} \right\} \tau^{k-2} . \end{aligned}$$

The coefficients of τ^{k-2} must balance, hence we obtain the recursion relation for a_k in terms of a_m , $m < k$, $\forall k \geq 0$, namely,

$$k(k-1)a_k = -2 \sum_{m=0}^{k-1} m a_m a_{k-m-1} - \sum_{m=0}^{k-2} a_m a_{k-m-2} .$$

When

$$k = 0, \text{ we have } 0 \cdot a_0 = 0 \Rightarrow a_0 \text{ is a free constant,}$$

and when

$$k = 1, \text{ we have } 0 \cdot a_1 = 0 \Rightarrow a_1 \text{ is a free constant.}$$

Therefore, as claimed, we have two free constants a_0 and a_1 that can be used to satisfy any initial conditions. For example, if the initial conditions are $u(t_0) = u_0$, $u'(t_0) = u_1$, then $a_0 = u_0$ and $a_1 = u_1$. For $k \geq 2$, the constants a_k are given by

$$a_k = -\frac{2}{k(k-1)} \sum_{m=0}^{k-1} m a_m a_{k-m-1} - \frac{1}{k(k-1)} \sum_{m=0}^{k-2} a_m a_{k-m-2} , \quad (1.9)$$

and are uniquely determined. Therefore, the series solution (1.5) is a formal general solution of (1.4).

Example 1.2.2: As a second example, let us consider the third-order nonlinear partial differential equation (NLPDE) known as the Korteweg-de Vries-Burgers (KdV-B) equation

$$v_t + vv_x + \gamma v_{xx} + v_{xxx} = 0 , \quad (1.10)$$

where γ is a nonzero constant. This equation governs waves on thin films [19], the undular bore in an open channel flow [20] and the weak shock profile in a plasma [21]. Traveling-wave solutions of (1.10) satisfy a third-order ODE, obtained by making the transformation $\xi = x - ct$, $u(\xi) = v(x, t)$, where c is a constant (the wave speed).

We obtain

$$\begin{aligned} v_t &= -cu' , \quad v_x = u' , \\ v_{xx} &= u'' , \quad v_{xxx} = u''' , \end{aligned}$$

which upon substitution into (1.10), give the third-order ODE

$$-cu' + uu' + \gamma u'' + u''' = 0 .$$

Integrating the latter with respect to ξ , we obtain the second-order ODE

$$u'' + \gamma u' + \frac{1}{2}u^2 - cu = A , \quad (1.11)$$

where A is a constant of integration. We now seek a solution of (1.11) in the form

$$u(\xi) = \sum_{k=0}^{\infty} a_k \tau^k , \quad (1.12)$$

where $\tau = \xi - \xi_0$ and ξ_0 is the arbitrary ordinary point. Again, this series expansion contains a_0 and a_1 as free constants. To see this, we proceed as in the previous example, differentiating the series term by term and substituting into (1.11) to obtain

$$\sum_{k=0}^{\infty} k(k-1)a_k \tau^{k-2} + \gamma \sum_{k=0}^{\infty} k a_k \tau^{k-1} + \frac{1}{2} \sum_{k=0}^{\infty} \sum_{m=0}^k a_m a_{k-m} \tau^k - c \sum_{k=0}^{\infty} a_k \tau^k = A .$$

Changing indices to make τ^{k-2} a common factor, requiring the coefficients to vanish, and defining $a_k = 0$ for $k < 0$, we obtain the recursion relation $\forall k \geq 0$, namely,

$$k(k-1)a_k = -\gamma(k-1)a_{k-1} - \frac{1}{2} \sum_{m=0}^{k-2} a_m a_{k-m-2} + c a_{k-2} + A \delta_{k,2} , \quad (1.13)$$

where

$$\delta_{k,2} = \begin{cases} 1 & \text{if } k = 2 , \\ 0 & \text{otherwise.} \end{cases}$$

From the above recursion relation, we have

$$\begin{aligned} 0 \cdot a_0 &= 0 , & \text{when } k = 0 , \\ 0 \cdot a_1 &= 0 , & \text{when } k = 1 . \end{aligned}$$

This implies that the two constants a_0 and a_1 are free. Therefore, the series expansion (1.12) is a formal general solution of the ODE (1.11).

Remark: We have seen, in both examples, that a_0 and a_1 are free constants, where the ODEs are of the second order. We also have t_0 (or ξ_0 in the second example) as an arbitrary point about which the solution is expanded. It seems, therefore, that we have three free constants, which is impossible. However, t_0 , a_0 , and a_1 are not independently arbitrary since $a_0 = u(t_0)$ and $a_1 = u'(t_0)$. That the coefficients do not depend explicitly upon t_0 merely reflects the fact that the equation is autonomous and therefore invariant under time translations. A choice of (a_0, a_1) uniquely determines a trajectory T , for any value of t_0 .

1.3 Convergence of Series Expansions About Ordinary Points

In the previous section, we have shown that series expansions (1.5) and (1.12) *formally* constitute general solutions to the equations considered in that they contain the required number of free constants. In this section we prove that the series solutions have positive radii of convergence, so that they are actual solutions.

Let us consider the first example (1.4). The recursion relation defining a_k in terms of a_m , $m < k$, for $k \geq 2$ is

$$a_k = -\frac{2}{k(k-1)} \sum_{m=0}^{k-1} m a_m a_{k-m-1} - \frac{1}{k(k-1)} \sum_{m=0}^{k-2} a_m a_{k-m-2}. \quad (1.14)$$

To prove that the series solution $u = \sum_{k=0}^{\infty} a_k \tau^k$ of (1.4), where $a_0 = u(t_0)$, $a_1 = u'(t_0)$ and a_k , $k \geq 2$, are uniquely determined by (1.14), has a positive radius of convergence for any initial conditions, we proceed as follows. We shall prove that we can find positive numbers M and K such that

$$|a_k| \leq M K^k \quad \forall k \geq 0. \quad (1.15)$$

Since a_0 and a_1 are determined by the initial conditions, for any given initial

conditions, let

$$M = \max \{ |a_0|, |a_1| \}. \quad (1.16)$$

With M as in (1.16) and given a positive integer $N \geq 2$, we can certainly find a positive constant K such that (1.15) holds for $0 \leq k \leq N-1$. Taking this to be our induction hypothesis we shall show that (1.15) holds for $k = N$, with the same value of K , and hence, by induction, holds for all k .

From the recursion relation (1.14) we have

$$\begin{aligned} |a_N| &\leq \frac{2}{N(N-1)} \sum_{m=1}^{N-1} m |a_m| |a_{N-m-1}| + \frac{1}{N(N-1)} \sum_{m=0}^{N-2} |a_m| |a_{N-m-2}| \\ &\leq \frac{2}{N(N-1)} M^2 K^{N-1} \sum_{m=1}^{N-1} m + \frac{1}{N(N-1)} M^2 K^{N-2} \sum_{m=0}^{N-2} 1 \\ &= M^2 K^{N-1} + \frac{M^2 K^{N-2}}{N} \\ &\leq M^2 K^{N-1} + \frac{M^2}{2} K^{N-2} \quad \text{for } N \geq 2. \end{aligned}$$

If we impose the conditions that

$$K \geq \frac{M + \sqrt{M^2 + 2M}}{2} \equiv K_{\min} \quad \text{and} \quad N \geq 2,$$

we get

$$|a_N| \leq M K^N,$$

which shows that (1.15) holds for $k = N$, assuming it holds for $0 \leq k \leq N-1$ with $N \geq 2$ and $K \geq K_{\min}$. Therefore, by induction (1.15) holds for all $k \geq N$ with the same value of K . Hence we have

$$\begin{aligned} |u(t)| &\leq \sum_{k=0}^{\infty} |a_k| |\tau|^k \\ &\leq \sum_{k=0}^{\infty} M K^k |\tau|^k, \end{aligned}$$

and by the ratio test, the series on the right-hand side converges if

$$|\tau| < \frac{1}{K}.$$

Therefore the series $u = \sum_{k=0}^{\infty} a_k \tau^k$ converges at least for t such that $|t - t_0| < \frac{1}{K} = R$. The number R is a lower bound for the radius of convergence and is found as follows. By (1.15), we have

$$K \geq \max_{1 \leq k \leq N-1} \left[\left(\frac{|a_k|}{M} \right)^{\frac{1}{k}} \right].$$

From the above proof we have the conditions on K and N ,

$$K \geq K_{\min} \text{ and } N \geq 2.$$

Hence K is given by

$$K = \max \left[\frac{|a_1|}{M}, \frac{M + \sqrt{M^2 + 2M}}{2} \right],$$

which gives

$$R = \frac{1}{\max \left[\frac{|a_1|}{M}, \frac{M + \sqrt{M^2 + 2M}}{2} \right]}.$$

For the second example (1.11) (the KdV-B equation), the convergence proof for its series solution is the same, but with different conditions on K . First, recall that the recursion relation for $k \geq 2$ is

$$a_k = -\frac{\gamma}{k} a_{k-1} - \frac{1}{2k(k-1)} \sum_{m=0}^{k-2} a_m a_{k-m-2} + \frac{c}{k(k-1)} a_{k-2}. \quad (1.17)$$

Let $M = \max \{|a_0|, |a_1|\}$ and $K > 0$ such that (1.15) holds for $0 \leq k \leq N-1$ and prove that it holds for $k = N$.

From (1.17) we have

$$|a_N| = \frac{|\gamma|}{N} |a_{N-1}| + \frac{1}{2N(N-1)} \sum_{m=0}^{N-2} |a_m| |a_{N-m-2}| + \frac{c}{N(N-1)} |a_{N-2}|$$

$$\begin{aligned}
&\leq \frac{|\gamma|}{N}MK^{N-1} + \frac{1}{2N(N-1)}M^2K^{N-2} \sum_{m=0}^{N-2} 1 + \frac{c}{N(N-1)}MK^{N-2} \\
&\leq \frac{|\gamma|}{N}MK^{N-1} + \frac{1}{2N}M^2K^{N-2} + \frac{c}{N(N-1)}MK^{N-2} \\
&\leq \frac{|\gamma|}{2}MK^{N-1} + \frac{1}{4}M^2K^{N-2} + \frac{c}{2}MK^{N-2} \quad \text{for } N \geq 2.
\end{aligned}$$

If we impose the conditions

$$K \geq \frac{|\gamma| + \sqrt{\gamma^2 + 8|c| + 4M}}{4} \quad \text{and} \quad N \geq 2,$$

we get

$$|a_N| \leq MK^N,$$

and hence (1.15) holds for all $k \geq 0$. Therefore, as in the previous example, the series solution to (1.11) converges at least for ξ such that $|\xi - \xi_0| < \frac{1}{K} = R$, where R is a lower bound of the radius of convergence and is given by

$$R = \frac{1}{\max \left[\frac{|a_1|}{M}, \frac{|\gamma| + \sqrt{\gamma^2 + 8|c| + 4M}}{4} \right]}.$$

1.4 Continuation Beyond the Radius of Convergence

In this section, we shall describe a procedure by means of which extension of the series solution of (1.3) beyond the radius of convergence, say up to a point t_{end} , becomes possible. The procedure consists of a sequence of re-expansions about ordinary points within the appropriate radii of convergence.

Let the initial conditions for (1.3) be given by

$$u(t_0) = u_0, \quad u^{(k)}(t_0) = u_k, \quad k = 1, \dots, n-1, \quad (1.18)$$

where $u^{(k)}(t_0)$ denotes the k th derivative of $u(t)$ evaluated at $t = t_0$. In order to obtain an analytic solution over the interval $[t_0, t_{\text{end}}]$ satisfying (1.18), we proceed as follows:

1. Expand about the initial ordinary point t_0 with the first n coefficients being

$$a_0 = u_0 \text{ and}$$

$$a_k = \frac{y_k}{k!}, \quad k = 1, 2, \dots, n-1,$$

and compute the radius of convergence R_0 of the initial expansion.

2. Pick any point t_1 within the radius of convergence R_0 and compute, using the series expansion about t_0 , the values of the solution and its first $n-1$ derivatives at $t = t_1$ to any desired accuracy by taking a sufficient number of terms in the series. Repeat step 1 using t_1 as a new initial point.
3. Repeat step 2 until the end point t_{end} is reached.

1.5 Conclusion

In this chapter, we have seen that series expansions provide a means of obtaining analytic solutions for nonlinear ODEs. The existence of a positive radius of convergence was proved for two examples. These two examples show that nonlinear ODEs of type (1.3) admit series solutions with positive radii of convergence. The idea of the convergence proof is the same; that is, find an appropriate K for which the induction hypothesis (1.15) holds for $n = N$ (N is the order of the ODE) and for any larger one, and then apply the ratio test to obtain convergence. The proof not only shows convergence, but also gives a lower estimate of the radius of convergence, which is easily computed, once the value of K is found.

In Chapter 5, we shall apply this series method to a well-known equation and see how we can extend the solution beyond the radius of convergence. This is done

by re-expanding about another point within the radius of convergence. The process is repeated as desired or until the subsequent radii of convergence approach zero.

Chapter 2

THE ARS ALGORITHM AND THE PAINLEVÉ PROPERTY

In the case of simple nonlinear differential equations such as

$$\frac{d}{dt}u(t) + u^2(t) = 0, \quad (2.1)$$

whose general solution is easily obtained as

$$u(t) = \frac{1}{t + t_0}, \quad (2.2)$$

where $t_0 = 1/u(0)$, the nature of their movable singularities is revealed by the form of the exact solutions (e.g. eq. (2.2)). However, for most nonlinear differential equations, the exact solutions are not obtainable. This may seem to be a setback in the analysis of such ODEs. It turns out, however, that the nature of the movable singularities can be determined by studying the “local” properties of the solutions, that is, solutions in a neighbourhood of an arbitrary movable singularity. This study reveals many properties of dynamical systems governed by nonlinear differential equations. In Section 2.1, we present an algorithm known as the *ARS algorithm* [1, 2, 3] as a method for determining the nature of the movable singularities of nonlinear ODEs (or systems of ODEs), without knowing the exact general solution. In Section 2.2, we apply the algorithm to some examples.

2.1 The ARS Algorithm

The ARS algorithm was originally developed to determine whether a nonlinear ODE (or a system of ODEs) admits movable branch points, either algebraic or logarithmic [2, 3]. It is important to mention that movable essential singularities [3] cannot be detected by this algorithm. Thus the algorithm provides a method of determining necessary (but not sufficient) conditions for a given nonlinear ODE to possess the *Painlevé property* (i.e., to have no movable singularities other than poles). The focus of this thesis is the application of this algorithm and its consequences to a class of autonomous, semilinear ODEs.

Let us consider an ODE of the form

$$u^{(n)} = F(u, u', \dots, u^{(n-1)}) , \quad (2.3)$$

where the superscripts denote differentiations with respect to the independent variable t , and F is a function of $u, u', \dots, u^{(n-1)}$. The main assumption upon which this algorithm is based is that the dominant behaviour of the solutions in a neighbourhood of an arbitrary movable singularity t_0 is of the form

$$u \sim a_0(t - t_0)^p \quad t \rightarrow t_0 , \quad (2.4)$$

where a_0 and p are referred to as the *leading coefficient* and the *leading order*, respectively.

The ARS algorithm consists of three steps, dealing with the dominant behaviour (i.e., the leading order p), the resonances and the arbitrary constants.

STEP 1: The Dominant Behaviours

In this step we determine all possible leading orders by substituting (2.4) into (2.3) and finding all possible negative values of p for which two or more terms in the equation balance each other and produce the most singular behaviour. For each such choice of p , the balance of these so-called leading terms also determines the

corresponding values of a_0 . Let us illustrate this by means of an example. For the equation

$$u'' = -2uu' - u^2, \quad (2.5)$$

substituting (2.4) into (2.5), we obtain

$$a_0 p(p-1)(t-t_0)^{p-2} = -2pa_0^2(t-t_0)^{2p-1} - a_0^2(t-t_0)^{2p}. \quad (2.6)$$

Balancing the most singular terms ($p-2 = 2p-1$) gives the leading order $p = -1$. The leading coefficient a_0 is determined by equating the coefficients of the balanced terms, that is,

$$a_0 p(p-1) = -2pa_0^2. \quad (2.7)$$

For a nontrivial solution we get $a_0 = 1$. This leads to the leading-order behaviour $u = (t-t_0)^{-1}$.

In this example we have seen that the leading order turned out to be an integer. The leading order not being an integer indicates that t_0 is an algebraic branch point, which is a strong indication that the ODE does not possess the Painlevé property. If all possible leading orders are integers, then for each one of them the leading behaviour can be viewed as the first term of a Laurent series expansion about a movable pole, i.e., eq. (2.4) generalizes to

$$u = \sum_{k=0}^{\infty} a_k(t-t_0)^{k+r}. \quad (2.8)$$

For an n -th order ODE, its general solution must contain n arbitrary constants, including the arbitrary point t_0 . Thus there remain $n-1$ arbitrary constants to be sought among the a_k in (2.8). The values of k at which such constants arise are termed "*resonances*" of the series (2.8), and it is to their determination that we now turn.

STEP 2: The Resonances

The resonances are the values of k for which the recursion relation, resulting from substituting (2.8) into the ODE (2.3), has the form

$$0 \cdot a_k = G(a_0, a_1, \dots, a_{k-1}) ,$$

where G is a function in its arguments. Having determined the leading order p and the leading coefficient a_0 , a quick way of finding the resonances is by substituting into the ODE the expression

$$u = a_0 \tau^p + \gamma \tau^{r+p} , \quad r > 0, \quad (2.9)$$

where $\tau = t - t_0$, and retain only the terms linear in γ , to obtain an expression of the form

$$Q(r)\gamma = 0 , \quad (2.10)$$

where $Q(r)$ is a polynomial in r of degree $\leq n$. The values r for which $Q(r) = 0$ are the *resonances* of the expansion (2.8). Assuming that $Q(r)$ has degree n , it admits the factorization

$$Q(r) = (r + 1)(r^{n-1} + c_2 r^{n-2} + \dots + c_n) . \quad (2.11)$$

We note that the root $r = -1$ of (2.11) is related to the arbitrariness of the location of the singularity t_0 .

Let us illustrate this step by means of the previous example (2.5). Substituting (2.9) (with $p = -1$ and $a_0 = 1$) into (2.5) and retaining only the terms linear in γ , we obtain

$$(r + 1)(r - 2)\gamma = 0 , \quad (2.12)$$

yielding the resonances $r = -1$ and $r = 2$.

If for every leading-order behaviour of step 1, all of the resonances are positive integers, then we may proceed to the next step.

STEP 3: The Arbitrary Constants

In this step, we check whether the expansion (2.8) contains the required number of free constants. To this end, we substitute (2.8) into the full equation (2.3). We then identify the terms order by order in powers of $\tau = t - t_0$, and after (possibly) a shifting of indices, we obtain a recursion relation (RR) of the form

$$Q(k)a_k = R^{(k)}(a_0, a_1, \dots, a_{k-1}), \quad k \geq 1, \quad (2.13)$$

where $Q(k) = (k+1)(k^{n-1} + c_2 k^{n-2} + \dots + c_n)$ (assuming $\deg(Q) = n$). $Q(k)$ then factorizes as $Q(k) = (k+1)(k-r_1)(k-r_2)\dots(k-r_{n-1})$, where $r_1 < r_2 < \dots < r_{n-1}$ are the resonances. Equation (2.13) thus takes the form

$$(k+1)(k-r_1)(k-r_2)\dots(k-r_{n-1})a_k = R^{(k)}(a_0, a_1, \dots, a_{k-1}), \quad (2.14)$$

leading to the following observations :

1. For $k < r_1$ (r_1 being the smallest positive resonance) eq. (2.4) determines a_k .
  2. At $k = r_1$, since $Q(r_1) = 0$, the following compatibility condition must be satisfied
- $$R^{(r_1)}(a_0, a_1, \dots, a_{r_1-1}) = 0. \quad (2.15)$$
3. If (2.15) is satisfied (a_{r_1} is free) then for $r_1 < k < r_2$, eq. (2.14) again determines a_k .
  4. The same procedure is repeated successively at each higher resonance up to the largest one.

After executing the above steps, we obtain $n - 1$ compatibility conditions that must be satisfied, i.e.,

$$R^{(r_i)}(a_0, a_1, \dots, a_{r_i-1}) = 0, \quad i = 1, \dots, n-1. \quad (2.16)$$

If a compatibility condition is satisfied, the corresponding resonance is said to be *compatible*, otherwise it is *incompatible*. The compatibility of a resonance, say r_1 , implies that a_{r_1} is free. Hence, having all $n - 1$ compatible resonances implies that the $n - 1$ constants $a_{r_1}, a_{r_2}, \dots, a_{r_{n-1}}$, together with t_0 , comprise n free parameters (where the differential equation has order n), and the expansion (2.8) formally constitutes a general solution of the differential equation.

An n th-order ODE (or system of ODEs) is said to pass the Painlevé test if the leading order is a negative integer and all of the resonances are positive integers and compatible. In other words, if its general solution can be expanded in a Laurent series (2.8) near every one of its movable singularities $t = t_0$. ODEs with the Painlevé property are (usually) called *integrable*.

A differential equation may be *nonintegrable* for any of the following reasons:

1. The leading order p is not a negative integer.
2. All of the resonances are positive integers (except $r = -1$) but at least one is incompatible, i.e., the expansion (2.8) is inconsistent.
3. At least one of the resonances (other than $r = -1$) is not a positive integer, in which case there are fewer than $n - 1$ free constants.

In the first case, the solution may be multivalued, but this is not always a problem, as one may be able to transform the analysis into one with negative integer leading order as we shall see by an example in the next section.

We conclude the presentation of the ARS algorithm by making some important remarks:

1. As we have already seen, $r = -1$ is always a resonance (i.e., a root of (2.11)). This is related to the arbitrariness of the singularity location t_0 [5].
2. Any resonance r with $\text{Re}(r) < 0$ (except $r = -1$) must be ignored since it violates the assumption that p is the power of the leading term in the expansion

(2.8). The occurrence of negative and/or non-integer resonances implies that the singular expansion (2.8) cannot be a general solution of the equation, and this is again an indication that the ODE does not possess the Painlevé property.

3. If in step 2 any of the resonances (other than $r = -1$) turn out not to be a positive integer, then the algorithm terminates at that step. If a resonance r is positive and rational, then it must be checked whether the algebraic branch point can be removed by a coordinate transformation.
4. In the event that p is rational, the appearance of a rational resonance r indicates finite branching with multiplicity determined by the leading singularity. In this case, making the change of variable $z = \tau^{1/[d_p, d_r]}$, and expanding in powers of z , transforms the expansion into one with negative leading order and positive integer resonance. The quantity $[d_p, d_r]$ is defined to be the least common multiple of d_p and d_r , these numbers being the denominators of the leading order and the resonance, respectively. This is the so-called “*weak Painlevé*” concept [6]. This special case, in which the ODE may pass the Painlevé test in the new variable z , is illustrated by an example in the next section.

2.2 Applications of the ARS Algorithm

In this section we apply the ARS algorithm, outlined in Section 2.1, to a second-order example. Also, by means of a second-order example, we see how in the case of negative rational leading order and positive rational resonance, a simple transformation can convert the expansion into one with negative leading order and positive integer resonance.

Let us begin by the application of the ARS algorithm to the ODE

$$u'' = -2uu' + Bu , \quad (2.17)$$

where B is a constant. To the leading order p and the leading coefficient a_0 , we substitute Substituting

$$u = a_0 \tau^p$$

into (2.17) to obtain

$$p(p-1)a_0\tau^{p-2} = -2pa_0^2\tau^{2p-1} + Ba_0\tau^p = 0 .$$

Balancing the first two terms ($p-2 = 2p-1$), which produce the most singular behaviour, we get the leading order $p = -1$. The leading coefficient a_0 is determined by equating the coefficients of the balanced terms, that is

$$p(p-1)a_0 = -2pa_0^2 .$$

For a nontrivial solution, we get $a_0 = 1$. Next, we find the resonances by substituting the expression

$$u = \tau^{-1} + \gamma\tau^{r-1}$$

into the ODE, retaining only the terms linear in γ , to obtain the resonance equation

$$(r+1)(r-2)\gamma = 0 ,$$

which gives the resonances $r = -1$ and $r = 2$. The singular expansion [cf. eq. (2.8)] then takes the form

$$u = \sum_{k=0}^{\infty} a_k \tau^{k-1} . \quad (2.18)$$

The next step is to check for the required free constants (i.e., check the compatibility of the positive resonance $r = 2$). From (2.18) we have

$$u' = \sum_{k=0}^{\infty} (k-1)a_k \tau^{k-2} ,$$

$$u'' = \sum_{k=0}^{\infty} (k-1)(k-2)a_k \tau^{k-3},$$

$$uu' = \sum_{k=0}^{\infty} \sum_{m=0}^k (m-1)a_m a_{k-m} \tau^{k-3}.$$

Substituting these into the full equation (2.17), we obtain the recursion relation $\forall k \geq 1$:

$$(k+1)(k-2)a_k = -2 \sum_{m=1}^{k-1} (m-1)a_m a_{k-m} - Ba_{k-2}. \quad (2.19)$$

From the above recursion relation, we have

$$k = 1 : -2a_1 = 0 \Rightarrow a_1 = 0.$$

$$k = 2 : 0 \cdot a_2 = -Ba_0 = -B.$$

We see that the compatibility condition at $k = 2$ is satisfied only if $B = 0$. Therefore, if $B = 0$, the resonance $r = 2$ is compatible, and a_2 is a free constant, in addition to the singular point t_0 . Thus the Laurent series (2.18) contains two free constants and hence constitutes a general solution of the ODE (2.17) (if $B = 0$), and the ODE passes the Painlevé test. On the other hand if $B \neq 0$, the compatibility condition at $k = 2$ is not satisfied, leading to an inconsistent recursion relation. Thus the Laurent series expansion cannot be a solution. In this case, the ODE (2.17) is said to fail the Painlevé test and is nonintegrable.

In the case that a nonlinear ODE passes the Painlevé test, a Laurent series is the appropriate expansion that formally constitutes a general solution. Even though such an expansion is singular at $t = t_0$, it has a nonzero radius of convergence. Hence it represents the behaviour of the solutions of the ODE in a neighbourhood of an arbitrary movable pole. We now prove the convergence of the Laurent series for the example (2.17) with $B = 0$. Thus the ODE is

$$u'' = -2uu', \quad (2.20)$$

and by the above analysis, this ODE passes the Painlevé test. The Laurent series

expansion

$$u(t) = \sum_{k=0}^{\infty} a_k \tau^{k-1}, \quad (2.21)$$

where $\tau = t - t_0$, constitutes a general solution of (2.20).

Theorem 2.2.1 *There exists $R > 0$ such that $\forall t$, $0 < |t - t_0| < R$, the series (2.21) converges absolutely for any choice of the arbitrary constant a_2 .*

Proof:

The proof is based on the following claim:

Claim: *Given $N \geq 2$, there exists $K > 0$ such that*

$$|a_k| \leq K^k, \quad \forall k \text{ with } 1 \leq k \leq N-1. \quad (2.22)$$

To prove the claim, we simply note that for any choice of a_2 , (2.22) is satisfied with

$$K = \max_{1 \leq k \leq N-1} [|a_k|^{1/k}].$$

To prove the theorem, let (2.22) be our induction hypothesis and let us prove that, with the same K , the result holds for $k = N$. Hence by induction it will hold for all $k \geq 1$. From the recursion relation (2.19), with $B = 0$, we have

$$\begin{aligned} |a_N| &\leq \frac{2}{(N+1)(N-2)} \sum_{m=2}^{N-1} (m-1) |a_m| |a_{N-m}| \\ &\leq \frac{2}{(N+1)(N-2)} \sum_{m=2}^{N-1} (m-1) K^m K^{N-m} \\ &= \frac{2}{(N+1)(N-2)} K^N \sum_{m=1}^{N-1} (m-1) \\ &= \frac{N-1}{N+1} K^N \\ &\leq K^N \end{aligned}$$

for all $N \geq 2$. Thus the result holds for $k = N$ and hence holds $\forall k \geq 1$. From the expansion (2.21), we have

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k \tau^{k-1}| &= |a_0| |\tau|^{-1} + \sum_{k=1}^{\infty} |a_k| |\tau|^{k-1} \\ &\leq |a_0| |\tau|^{-1} + \sum_{k=1}^{\infty} K^k |\tau|^{k-1}. \end{aligned}$$

By the ratio test, the series on the right-hand side converges if

$$|\tau| < \frac{1}{K}.$$

This shows that $\forall t$ such that $0 < |\tau| = |t - t_0| < \frac{1}{K} \equiv R$, the Laurent series expansion (2.21) converges absolutely, and the theorem is proved. $\square$

A lower bound for the radius of convergence R for the present example (2.20) is given by

$$\frac{1}{\max[|a_1|, \sqrt{|a_2|}]} = \frac{1}{\sqrt{|a_2|}}, \quad \text{since } a_1 = 0.$$

This example shows that for a nonlinear ODE of type (2.3) which passes the Painlevé test, a Laurent series expansion represents its general solution in a neighbourhood of a movable pole.

The next example deals with the case of negative rational leading order and positive rational resonance. This case, as mentioned earlier, is related to the weak Painlevé concept, that is, the Painlevé test in a new variable.

Let us consider the ODE

$$u'' = -\frac{8}{3}u'u^2 - \frac{7}{12}u^5. \quad (2.23)$$

Applying steps 1 and 2 of the ARS algorithm, we find that the leading order is $p = -\frac{1}{2}$ and the resonances are $r_1 = -1$ and $r_2 = \frac{1}{3}$. Since the positive resonance r_2 is not an integer, a Laurent series expansion (2.8) in powers of $\tau = t - t_0$ will not

contain the required number of free constants. Hence by definition of the Painlevé test, equation (2.23) fails the test. However, since both the leading order and the positive resonance are rational, we shall see that making the transformation

$$z = \tau^{1/6} \quad (2.24)$$

converts the expansion into one with negative integer leading order and positive integer resonance, in the new variable z . From (2.24), using the chain rule, we have

$$\frac{d}{dt} = \frac{1}{6} z^{-5} \frac{d}{dz} .$$

To find the new leading order $\tilde{p}$, we substitute the expression

$$u = a_0 z^{\tilde{p}}$$

into the ODE (2.23) to get

$$\tilde{p}(\tilde{p} - 6)a_0 z^{\tilde{p}-12} = -16\tilde{p}a_0^3 z^{3\tilde{p}-6} - 21a_0^5 z^{5\tilde{p}} .$$

Balancing the most singular terms ($\tilde{p} - 12 = 3\tilde{p} - 6 = 5\tilde{p}$) gives the new leading order $\tilde{p} = -3$. We note, here, that the new leading order is the old one ($-1/2$) multiplied by 6, which is the least common multiple of 2 and 3. The new leading order, $\tilde{p} = -3$, indicates that $z = 0$ is a pole. The new leading coefficient, a_0 , is obtained by equating the coefficients of the balanced terms, that is,

$$\tilde{p}(\tilde{p} - 6)a_0 = -16a_0^3\tilde{p} - 21a_0^5 .$$

For a nontrivial solution, with $\tilde{p} = -3$, a_0 satisfies

$$7a_0^4 - 16a_0^2 + 9 = 0 . \quad (2.25)$$

One solution of (2.25) is $a_0 = 1$.

Using $a_0 = 1$, the new resonances are found by substituting

$$u = z^{-3} + \gamma z^{r-3}$$

into the ODE and retaining the terms that are linear in γ . After some manipulation, we obtain

$$(\tau^2 + 4\tau - 12)\gamma = 0 ,$$

which gives the new resonances $r_1 = -6$ and $r_2 = 2$. We note, again, that the new resonances are the old ones multiplied by 6.

The above example leads to the following important remarks:

1. When expanding in powers of $\tau = t - t_0$, the ODE (2.23) had leading order $p = -1/2$ and resonances $r_1 = -1$, $r_2 = 1/3$, and when re-expanding in powers of $z = \tau^{1/6}$, it had integer leading order $\tilde{p} = -3$ and integer resonances $r_1 = -6$, $r_2 = 2$. This is an interesting observation, because both the new leading order and the new resonances are the old ones multiplied by 6, the number which occurs in (2.24).
2. The above observation suggests that for second-order ODEs of type (2.3), failing the Painlevé test because of finite branching at the arbitrary singularity t_0 (i.e., both the leading order and the positive resonance are rational), one can make the change of variable $z = \tau^{1/d}$, where d is the least common multiple of the denominators of the leading order and the resonance, and proceed with the Painlevé test in the new variable.
3. The Painlevé test, when applied in the new independent variable z , is called the weak Painlevé test.
4. If the ODE passes the weak Painlevé test, a Laurent series expansion in powers of $z = \tau^{1/d}$ will form a general solution.

2.3 Conclusion

In summary, we say that an ODE (2.3) satisfies the necessary conditions for the *Painlevé property* (i.e., having no movable singularities other than poles) if its general

solution can be expanded in a Laurent series (2.8) near every one of its movable singularities $t = t_0$. In other words, following the ARS algorithm as outlined in Section 2.1, we should come across no branch points.

But even for those ODEs where all of the necessary conditions for the Painlevé property are fulfilled, the ARS algorithm provides only an indication that they actually do possess the Painlevé property, as movable essential singularities, which cannot be detected by this algorithm, may still exist.

A second-order ODE failing the Painlevé test because of finite branching may still pass the weak Painlevé test, as was illustrated by example (2.23). The weak Painlevé test is the Painlevé test in a new variable.

For ODEs that are nonintegrable, for whatever reason, new types of series expansions are necessary for the representations of solutions near arbitrary singularities. Such series are called *psi-series* [3, 10, 11], whose forms are discussed in the next chapter.

Chapter 3

FORMS OF PSI-SERIES

In the previous chapter, we presented the ARS algorithm as a method of possibly identifying ODEs having the Painlevé property. For such ODEs, general solutions are obtained by means of Laurent series expansions, near their movable singularities (poles). We have also seen that for ODEs failing the Painlevé test, series of type (2.8) do not constitute general solutions. Such ODEs were termed nonintegrable. In this chapter, we shall be concerned with nonintegrable ODEs and the generalization of Laurent series for the construction of general solutions of certain second- and third-order ODEs of type (2.3).

We have seen that the resonance equation is usually of the form

$$(r + 1)P(r) = 0 ,$$

where P is a polynomial of degree $n - 1$ (where the DE is of order n). (we remark that such is not always the case.) Thus the polynomial P has $n - 1$ roots, but some may not be positive integers. Some may even be complex. In the case of non-integer resonances, and also in the case of incompatible positive integer resonances, the Laurent series must be generalized into a more complicated one. Such series are called *psi-series*. They contain the required number of free constants, and hence constitute general solutions.

3.1 The Second-Order Case

We shall be concerned with second-order ODEs of the form

$$u'' = F(u, u') , \quad (3.1)$$

where F is a polynomial in u and u' . For ODEs of this type, there are two cases of nonintegrability:

- The resonance is a positive integer and incompatible, or
- The resonance is not a positive integer.

These two cases are treated separately.

CASE 1: Incompatible positive integer resonance

First, recall that for a linear ordinary differential equation of the form

$$y'' + f(t)y' + g(t)y = 0 \quad (3.2)$$

the Frobenious method (Boyce and DiPrima [9]) states that if the indicial roots r_1, r_2 differ by an integer, then one solution is

$$y_1(t) = |t|^{r_1} \left(1 + \sum_{k=1}^{\infty} a_k t^k \right) \quad (3.3)$$

and another solution is

$$y_2(t) = A y_1(t) \log |t| + |t|^{r_2} \left(1 + \sum_{k=1}^{\infty} b_k t^k \right) , \quad (3.4)$$

where A is a constant and the series converges for $0 < |t| < R$, $R > 0$. It is precisely due to the presence of an incompatible resonance that $A \neq 0$ and the general solution must contain a logarithm.

Motivated by this observation, the same idea is applied to nonlinear ODEs, and it has been found [10] that if a series solution of type (2.8) contains one incompatible

positive integer resonance, say r , then the Laurent series (2.8) must be generalized to the series

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r \log \tau)^l, \quad (3.5)$$

where τ is assumed to be positive (i.e., this series is valid for $t > t_0$, $\tau = t - t_0$). The series in (3.5) is known as a *logarithmic psi-series*.

This logarithmic series expansion will result in a recursion relation containing two resonances at $(k, l) = (0, 1)$ and $(k, l) = (r, 0)$. The resonance at $(k, l) = (0, 1)$ is compatible, leading to a free constant $a_{0,1}$ which can be chosen to make the resonance at $(k, l) = (r, 0)$ compatible and hence $a_{r,0}$ free. Thus (3.5) constitutes a general solution. This will be illustrated in detail in the next two examples.

Example 3.1.1: Incompatible integer resonance:

Consider the ODE

$$u'' + 2uu' + u^2 = 0. \quad (3.6)$$

where $u' \equiv \frac{du}{dt}$. We start by seeking a general solution in the form (2.8), namely

$$u = \sum_{k=0}^{\infty} a_k \tau^{k+p}. \quad (3.7)$$

where, again, p is the leading order to be determined. $\tau = t - t_0$. t_0 is the arbitrary singularity, and the a_k 's are constants. Substituting

$$u = a_0 \tau^p$$

into the DE (3.6) we get

$$p(p-1)a_0\tau^{p-2} + 2pa_0^2\tau^{2p-1} + a_0^2\tau^{2p} = 0.$$

The leading terms (least powers of τ) are τ^{p-2} and τ^{2p-1} , and they must be balanced in order to satisfy the DE. Thus $p-2 = 2p-1$ gives the leading order $p = -1$. Note

that if at least two leading terms do not “*balance*,” then all of the a_k ’s are forced to be zero, as a_0 is forced to be zero. With the leading order $p = -1$, equating the coefficients of the balanced terms gives

$$\begin{aligned} 2a_0 - 2a_0^2 &= 0 , \\ \Rightarrow a_0 &= 1 . \end{aligned}$$

For the resonances, substituting

$$u = \tau^{-1} + \gamma \tau^{r-1}$$

into the DE with only the leading terms, retaining only linear terms in γ , gives the resonance equation

$$(r+1)(r-2)\gamma = 0 ,$$

and hence the resonances $r_1 = -1$, $r_2 = 2$. Next, we proceed to derive the recursion relation to check the compatibility of the integer resonance $r = 2$.

With $p = -1$, from the series (3.7), we have

$$u' = \sum_{k=0}^{\infty} (k-1)a_k \tau^{k-2} ,$$

$$u'' = \sum_{k=0}^{\infty} (k-1)(k-2)a_k \tau^{k-3} , \quad (3.8)$$

$$uu' = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (m-1)a_m a_{k-m} \tau^{k-3} , \quad (3.9)$$

$$u^2 = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_m a_{k-m} \tau^{k-2} . \quad (3.10)$$

Substituting (3.8), (3.9), (3.10) into the DE gives

$$\sum_{k=0}^{\infty} (k-1)(k-2)a_k \tau^{k-3} + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} 2(m-1)a_m a_{k-m} \tau^{k-3} + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_m a_{k-m} \tau^{k-2} = 0 .$$

Replace k by $k-1$ in the last sum, define $a_n = 0$ if $n < 0$, and combine all sums to get:

$$\sum_{k=0}^{\infty} \left\{ (k-1)(k-2)a_k + \sum_{m=0}^k 2(m-1)a_m a_{k-m} + \sum_{m=0}^{k-1} a_m a_{k-m-1} \right\} \tau^{k-3} = 0 .$$

For all $k \geq 0$, the coefficient of τ^{k-3} must vanish. Thus we have $\forall k \geq 0$,

$$(k-1)(k-2)a_k + \sum_{m=0}^k 2(m-1)a_m a_{k-m} + \sum_{m=0}^{k-1} a_m a_{k-m-1} = 0. \quad (3.11)$$

When $k = 0$, (3.11) gives

$$2a_0 - 2a_0^2 = 0,$$

and hence $a_0 = 1$ (as before).

The first sum in (3.11) may be expressed as

$$\sum_{m=0}^k 2(m-1)a_m a_{k-m} = 2(k-2)a_0 a_k + \sum_{m=1}^{k-1} 2(m-1)a_m a_{k-m},$$

and substitution into (3.11), using $a_0 = 1$, keeping terms involving a_k to one side and terms involving a_m , $m < k$, to the other side, yields the recursion relation $\forall k \geq 1$:

$$(k+1)(k-2)a_k = - \sum_{m=1}^{k-1} 2(m-1)a_m a_{k-m} - \sum_{m=0}^{k-1} a_m a_{k-m-1}. \quad (3.12)$$

Note that the resonances are given by roots of the coefficient of a_k in (3.12).

When $k = 1$, we have

$$\begin{aligned} 2a_1 &= -a_0^2, \\ \Rightarrow a_1 &= -\frac{1}{2}. \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} 0 \cdot a_2 &= 2a_0 a_1 \\ &\neq 0. \end{aligned}$$

This shows that the resonance $r = 2$ is incompatible (i.e., the recursion relation (3.12) is inconsistent), and the series expansion (3.7) cannot constitute a solution. Hence the appropriate series takes the form [cf. eq. (3.5)]

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-1} (\tau^2 \log \tau)^l. \quad (3.13)$$

We shall show that this expansion has a second free constant, in addition to t_0 , and hence formally constitutes a general solution of (3.6).

From (3.13), we have the following :

$$\begin{aligned}
u' &= \sum_{k=0}^{\infty} \left\{ (k+2l-1) \log^l \tau + l \log^{l-1} \tau \right\} a_{kl} \tau^{k+2l-2}, \\
u'' &= \sum_{k=0}^{\infty} \left\{ (k+2l-1)(k+2l-2) \log^l \tau + (2k+4l-3)l \log^{l-1} \tau \right. \\
&\quad \left. + l(l-1) \log^{l-2} \tau \right\} a_{kl} \tau^{k+2l-3}, \\
u^2 &= \sum_{k,l=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^l a_{mn} a_{k-m,l-n} \tau^{k+2l-2} \log^l \tau, \\
uu' &= \sum_{k,l=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^l (m+2n-1) a_{mn} a_{k-m,l-n} \tau^{k+2l-3} \log^l \tau,
\end{aligned}$$

which upon substitution into the DE give

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left\{ (k+2l-1)(k+2l-2) a_{kl} \tau^{k+2l-3} \log^l \tau + (2k+4l-3)l a_{kl} \tau^{k+2l-3} \log^{l-1} \tau \right. \\
&\quad \left. + l(l-1) a_{kl} \tau^{k+2l-3} \log^{l-2} \tau + \sum_{m=0}^k \sum_{n=0}^l \left\{ 2(m+2n-1) a_{mn} a_{k-m,l-n} \tau^{k+2l-3} \log^l \tau \right. \right. \\
&\quad \left. \left. + 2n a_{mn} a_{k-m,l-n} \tau^{k+2l-3} \log^{l-1} \tau + a_{mn} a_{k-m,l-n} \tau^{k+2l-2} \log^l \tau \right\} \right\} = 0.
\end{aligned}$$

When $k = l = 0$, we have

$$2a_{0,0} - 2a_{0,0}^2 = 0 \Rightarrow a_{0,0} = 1.$$

When $k = 1, l = 0$, we have

$$-2a_{0,0}a_{1,0} + a_{0,0}^2 = 0 \Rightarrow a_{1,0} = \frac{1}{2}.$$

Next, adjusting terms (changing indices l, k so that $\tau^{k+2l-3} \log^l \tau$ is a common factor), defining $a_{kl} = 0$ for $k < 0$ or $l < 0$, we obtain

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left\{ (k+2l-1)(k+2l-2) a_{kl} + (l+1)(2k+4l-3) a_{k-2,l+1} \right.$$

$$\begin{aligned}
& + (l+2)(l+1)a_{k-4,l+2} + \sum_{m=0}^k \sum_{n=0}^l 2(m+2n-1)a_{mn}a_{k-m,l-n} \\
& + \sum_{m=0}^{k-2} \sum_{n=0}^{l+1} 2na_{mn}a_{k-m-2,l-n+1} + \sum_{m=0}^{k-1} \sum_{n=0}^l a_{mn}a_{k-m-1,l-n} \} \tau^{k-2l-3} \log^l \tau = 0 .
\end{aligned}$$

Requiring the coefficients of $\tau^{k-2l-3} \log^l \tau$ to vanish $\forall (k, l)$ and combining all terms involving a_{kl} , we obtain the doubly-indexed recursion relation:

$$\begin{aligned}
(k+2l-1)(k+2l-2)a_{kl} = & -(l+1)(2k+4l-3)a_{k-2,l+1} - (l+2)(l+1)a_{k-4,l+2} \\
& - \sum_{m=0}^k \sum_{n=0}^l 2(m+2n-1)a_{mn}a_{k-m,l-n} \\
& - \sum_{m=0}^{k-2} \sum_{n=0}^{l+1} 2na_{mn}a_{k-m-2,l-n+1} \\
& - \sum_{m=0}^{k-1} \sum_{n=0}^l a_{mn}a_{k-m-1,l-n} .
\end{aligned} \tag{3.14}$$

When $(k, l) = (0, 1)$, we have

$$0 \cdot a_{0,1} = -2a_{0,0}a_{0,1} + 2a_{0,1}a_{0,0} = 0 ,$$

which implies that $a_{0,1}$ is a free constant.

When $(k, l) = (2, 0)$, we have

$$\begin{aligned}
0 \cdot a_{2,0} = & -a_{0,1} - \sum_{m=0}^2 2(m-1)a_{m,0}a_{2-m,0} - \sum_{n=0}^1 2na_{0,n}a_{0,1-n} - \sum_{m=0}^1 a_{m,0}a_{1-m,0} \\
= & -a_{0,1} + 2a_{0,0}a_{2,0} - 2a_{2,0}a_{0,0} - 2a_{0,1}a_{0,0} - 2a_{0,0}a_{1,0} \\
= & -3a_{0,1} - 1 .
\end{aligned}$$

Since $a_{0,1}$ is free, it can be chosen so that the recursion relation is consistent at $(k, l) = (2, 0)$, thus making the resonance at $(k, l) = (2, 0)$ compatible. Therefore, by choosing $a_{0,1} = -\frac{1}{3}$ we have $a_{2,0}$ as a free constant, in addition to t_0 , and the

logarithmic psi-series (3.13) formally constitutes a general solution of the ODE (3.6).

Example 3.1.2: The Korteweg-de Vries-Burgers (KdV-B) equation

Consider the partial differential equation

$$v_t + vv_x + \gamma v_{xx} + v_{xxx} = 0 , \quad (3.15)$$

where γ is a nonzero constant.

We seek wave solutions by letting $\xi = x - ct$ and $u(\xi) = v(x, t)$, where c is the wave speed. Then $v_t = -cu'$, $v_x = u'$, $v_{xx} = u''$, $v_{xxx} = u'''$. Equation (3.15) transforms into the third-order ODE

$$-cu' + uu' + \gamma u'' + u''' = 0 . \quad (3.16)$$

Integrating (3.16) once with respect to ξ we get the second-order ODE

$$u'' + \gamma u' + \frac{1}{2}u^2 - cu = A , \quad (3.17)$$

where A is a constant of integration.

We now seek a solution of the form

$$u = \sum_{k=0}^{\infty} a_k \tau^{k+p}$$

where p is the leading order to be determined and $\tau = \xi - \xi_0$, ξ_0 an arbitrary singularity. Following step 1 of the ARS algorithm, we find the leading order $p = -2$. Then our solution becomes

$$u = \sum_{k=0}^{\infty} a_k \tau^{k-2} , \quad (3.18)$$

from which we obtain

$$u' = \sum_{k=0}^{\infty} (k-2)a_k \tau^{k-3} ,$$

$$\begin{aligned}
u'' &= \sum_{k=0}^{\infty} (k-2)(k-3)a_k \tau^{k-4}, \\
uu' &= \sum_{k=0}^{\infty} \sum_{m=0}^k (m-2)a_m a_{k-m} \tau^{k-5}, \\
u^2 &= \sum_{k=0}^{\infty} \sum_{m=0}^k a_m a_{k-m} \tau^{k-4}.
\end{aligned}$$

Substitution into (3.17) gives

$$\sum_{k=0}^{\infty} \left\{ (k-3)(k-2)a_k \tau^{k-4} + \gamma(k-2)a_k \tau^{k-3} + \sum_{m=0}^k \frac{1}{2} a_m a_{k-m} \tau^{k-4} - ca_k \tau^{k-2} \right\} = A.$$

When $k = 0$, we have

$$\begin{aligned}
6a_0 + \frac{1}{2}a_0^2 &= 0, \\
\Rightarrow a_0 &= -12.
\end{aligned}$$

Changing indices to make τ^{k-4} a common factor, defining $a_n = 0$ for $n < 0$, we obtain

$$(k-3)(k-2)a_k + \gamma(k-3)a_{k-1} + \sum_{m=0}^k \frac{1}{2} a_m a_{k-m} - ca_{k-2} = A\delta_{k,4}, \quad (3.19)$$

where

$$\delta_{k,4} = \begin{cases} 1 & \text{if } k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

The sum in (3.19) is expressed as

$$\sum_{m=0}^k \frac{1}{2} a_m a_{k-m} = 2a_0 a_k + \sum_{m=1}^{k-1} \frac{1}{2} a_m a_{k-m} = -24a_k + \sum_{m=1}^{k-1} \frac{1}{2} a_m a_{k-m},$$

and substitution into (3.19) and a rearrangement of terms give the recursion relation $\forall k \geq 1$:

$$(k+1)(k-6)a_k = ca_{k-2} - \gamma(k-3)a_{k-1} - \sum_{m=0}^{k-1} \frac{1}{2} a_m a_{k-m} + A\delta_{k,4}. \quad (3.20)$$

From (3.20), we see that the resonance occurs at $k = 6$.

For $1 \leq k \leq 5$, (3.20) gives

$$\begin{aligned} a_1 &= 12\frac{\gamma}{5}, & a_2 &= c + \left(\frac{\gamma}{5}\right)^2, \\ a_3 &= \left(\frac{\gamma}{5}\right)^3, & a_4 &= \frac{35}{20}\left(\frac{\gamma}{5}\right)^4 - \frac{1}{20}(c^2 + 2A), \\ a_5 &= \frac{385}{60}\left(\frac{\gamma}{5}\right)^5 - \frac{11}{60}(c^2 + 2A)\left(\frac{\gamma}{5}\right). \end{aligned}$$

At the resonance $k = 6$, using the values of a_1, a_2, a_3, a_4, a_5 above, we have the compatibility condition

$$0 \cdot a_6 = 5 \left(\frac{\gamma}{5}\right)^2 \left[c^2 + 2A - 36 \left(\frac{\gamma}{5}\right)^4 \right], \quad (3.21)$$

which is satisfied iff A (the constant of integration) is chosen to satisfy

$$A = 18 \left(\frac{\gamma}{5}\right)^4 - \frac{1}{2}c^2. \quad (3.22)$$

Hence if A is chosen as in (3.22), the resonance at $k = 6$ is compatible and a_6 is a second free constant, in addition to ξ_0 . This implies that the expansion (3.18) constitutes a general solution of the ODE (3.17).

However, we are interested in the nonintegrable case of (3.17), so we consider the case where $A \neq [18(\frac{\gamma}{5})^4 - \frac{1}{2}c^2]$. This makes the resonance at $k = 6$ incompatible, and (3.17) nonintegrable. In this case, the appropriate series expansion takes the form [cf. eq. (3.5)]

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-2} (\tau^6 \log \tau)^l. \quad (3.23)$$

In the following, we will take $A = 0$ and assume that $18(\frac{\gamma}{5})^4 - \frac{1}{2}c^2 \neq 0$. With this choice of A our ODE (3.17) becomes

$$u'' + \gamma u' + \frac{1}{2}u^2 - cu = 0. \quad (3.24)$$

Following the same technique as in Example 3.1.1, we find that

$$a_{00} = -12$$

and $\forall (k, l), (k, l) \neq (0, 0)$, the recursion relation is

$$\begin{aligned}
 (k + 6l - 3)(k + 6l - 2)a_{kl} &= -(l + 1)(2k + 12l - 5)a_{k-6, l+1} \\
 &\quad -(l + 1)(l + 2)a_{k-12, l+2} \\
 &\quad -\gamma(k + 6l - 3)a_{k-1, l} - \gamma(l + 1)a_{k-7, l+1} \\
 &\quad -\frac{1}{2} \sum_{m=0}^k \sum_{n=0}^l a_{mn} a_{k-m, l-n} + ca_{k-2, l}. \quad (3.25)
 \end{aligned}$$

When $(k, l) = (0, 1)$, from (3.25), we have

$$\begin{aligned}
 12a_{0,1} &= -\frac{1}{2}a_{0,0}a_{0,1} - \frac{1}{2}a_{0,1}a_{0,0} \\
 &= -a_{0,0}a_{0,1} \\
 &= 12a_{0,1},
 \end{aligned}$$

and as claimed we have $a_{0,1}$ free. For $l = 0$, and $0 \leq k \leq 5$, we find that

$$\begin{aligned}
 a_{1,0} &= 12\frac{\gamma}{5}, \\
 a_{2,0} &= c + \left(\frac{\gamma}{5}\right)^2, \\
 a_{3,0} &= \left(\frac{\gamma}{5}\right)^2, \\
 a_{4,0} &= \frac{35}{20}\left(\frac{\gamma}{5}\right)^4 - \frac{1}{20}c^2, \\
 a_{5,0} &= \frac{395}{60}\left(\frac{\gamma}{5}\right)^5 - \frac{11}{60}\left(\frac{\gamma}{5}\right)c^2.
 \end{aligned}$$

When $(k, l) = (6, 0)$, (3.25) gives the compatibility condition

$$\begin{aligned}
 0 \cdot a_{6,0} &= ca_{4,0} - \frac{1}{2}a_{3,0}^2 - 3\gamma a_{5,0} - a_{1,0}a_{5,0} - a_{2,0}a_{4,0} - 7a_{0,1} \\
 &= 5\left(\frac{\gamma}{5}\right)^2 \left[c^2 - 36\left(\frac{\gamma}{5}\right)^4 \right] - 7a_{0,1}. \quad (3.26)
 \end{aligned}$$

Note that this compatibility condition is similar to the one in (3.21) except for the additional free constant $a_{0,1}$. It is due to the presence of this free constant that the compatibility condition at $(k, l) = (6, 0)$ will be satisfied. Hence, by choosing $a_{0,1}$ as

$$a_{0,1} = \frac{5}{7} \left(\frac{\gamma}{5}\right)^2 \left[c^2 - 36\left(\frac{\gamma}{5}\right)^4 \right].$$

we have $a_{e,0}$ as a second free constant, in addition to ξ_0 , and the logarithmic psi-series (3.23) constitutes a formal general solution of (3.24).

In conclusion, if a Laurent series of type (2.8) results in an incompatible positive integer resonance, say at $k = r$, then the logarithmic psi-series (3.5) contains $a_{0,1}$ as a free constant which can be chosen in such a way that the compatibility condition at $(r, 0)$ is satisfied. This makes $a_{r,0}$ a second free constant, in addition to the arbitrary singularity about which the solution is expanded, and thus a series of type (3.5) formally constitutes a general solution of the ODE.

CASE 2: Noninteger positive resonance

There are two possible types of noninteger positive resonances, namely, rational and irrational. In the case where the resonance is rational, we have seen in Chapter 2 that the problem can be transformed into one with a positive integer resonance via the weak Painlevé test. In the case where the resonance is irrational (and positive), the appropriate psi-series [11] takes the form

$$\begin{aligned} u &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r)^l \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+rl+p}, \end{aligned} \quad (3.27)$$

where $\tau = t - t_0$ (t_0 is the arbitrary singularity), r is the resonance and p is the leading order. The series in (3.27) contains $a_{0,1}$ as a free constant, and therefore formally constitutes a general solution. The next example will illustrate this.

Example 3.1.3 : Noninteger positive resonance

Consider the ODE

$$u'' + 4uu' + u^3 = 0. \quad (3.28)$$

We start by seeking a solution in the form of a Laurent series

$$u = \sum_{k=0}^{\infty} a_k \tau^{k+p}. \quad (3.29)$$

The leading order p and the leading coefficient a_0 are easily found to be $p = -1$ and $a_0 = 2 \pm \sqrt{2}$ (note that a_0 is irrational). Substituting (3.29) into (3.28) and rearranging terms, we get the recursion relation $\forall k \geq 1$:

$$(k+1)[k - (4 - 4a_0)]a_k = - \sum_{m=1}^{k-1} 4(m-1)a_m a_{k-m} - \sum_{m=1}^{k-1} (a_0 a_m + b_m) a_{k-m} = 0,$$

where

$$b_m = \sum_{n=0}^m a_n a_{m-n}.$$

The resonances are those values of k for which the coefficient of a_k in the recursion relation vanishes. They are at $k_1 = -1$ and $k_2 = 4 - 4a_0$. Since a_0 is irrational, the positive resonance is irrational, and by the ARS algorithm the Laurent series (3.29) cannot be a general solution. Note that a_0 may have two possible values, $2 - \sqrt{2}$ and $2 + \sqrt{2}$. We consider only the value $2 - \sqrt{2}$, because if the other is chosen, the second resonance k_2 will be negative and this will violate the assumption that the most singular behaviour that the ODE can support is $p = -1$. Choosing $a_0 = 2 - \sqrt{2}$, the positive resonance is $k_2 = \beta = 4\sqrt{2} - 4$ and the appropriate psi-series (3.27) takes the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+\beta l-1}. \quad (3.30)$$

We shall now show that this series expansion contains $a_{0,1}$ as a free constant.

From (3.30), we have

$$u' = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k + \beta l - 1) a_{kl} \tau^{k+\beta l-2}.$$

$$\begin{aligned}
u'' &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k + \beta l - 1)(k + \beta l - 2) a_{kl} \tau^{k+\beta l-3}, \\
uu' &= \sum_{k,l=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^l (m + \beta n - 1) a_{mn} a_{k-m,l-n} \tau^{k+\beta l-3}, \\
u^3 &= \sum_{k,l=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^l b_{mn} a_{k-m,l-n} \tau^{k+\beta l-3},
\end{aligned}$$

where

$$b_{mn} = \sum_{i=0}^m \sum_{j=0}^n a_{ij} a_{m-i,n-j}.$$

Substitution into the ODE, requiring the coefficient of $\tau^{k+\beta l-3}$ to vanish $\forall k, l \geq 0$ gives the recursion relation

$$(k + \beta l - 1)(k + \beta l - 2) a_{kl} + \sum_{m=0}^k \sum_{n=0}^l \{4(m + \beta n - 1) a_{mn} + b_{mn}\} a_{k-m,l-n} = 0.$$

When $(k, l) = (0, 0)$, using $b_{0,0} = a_{0,0}^2$, we have

$$\begin{aligned}
2a_{0,0} - 4a_{0,0}^2 + a_{0,0}^3 &= 0, \\
\Rightarrow a_{0,0} &= 2 \pm \sqrt{2}.
\end{aligned}$$

Note that $a_{0,0}$ has the same values as a_0 in the Laurent series.

When $(k, l) = (0, l)$, we obtain the recursion relation for a_{0l} , namely,

$$(\beta l - 1)(\beta l - 2) a_{0l} + \sum_{n=0}^l \{4(\beta n - 1) a_{0n} + b_{0n}\} a_{0,l-n} = 0,$$

and when $l = 1$, we have

$$(\beta - 1)(\beta - 2) a_{0,1} + 4(\beta - 2) a_{0,1} a_{0,0} + b_{0,0} a_{0,1} + b_{0,1} a_{0,0} = 0.$$

Using the fact that $b_{0,0} = a_{0,0}^2$, $b_{0,1} = 2a_{0,0}a_{0,1}$ and $a_{0,0}^2 = 4a_{0,0} - 2$, we have the compatibility condition

$$(\beta + 1)[\beta - (4 - 4a_{0,0})] a_{0,1} \equiv 0 \cdot a_{0,1} = 0,$$

which is trivially satisfied (the resonance $\beta = 4 - 4a_{0,0}$). This shows that $a_{0,1}$ is a free constant. Hence, including the arbitrariness of t_0 , the movable singularity, we have two free constants and the psi-series (3.30) formally constitutes a general solution of our ODE (3.28).

3.2 The Third-Order Case

In this section, we consider ODEs of the form

$$u''' = F(u, u', u'') \quad (3.31)$$

with negative integer leading orders in their Laurent series expansions. For third-order ODEs, the polynomial P in the resonance equation is of degree at most 2. If it has degree 2, then it admits 2 roots (the resonances). Such ODEs may be nonintegrable for the following reasons:

- The resonances are positive integers, but at least one is incompatible.
- The resonances are complex conjugates.
- The resonances are irrational conjugates.
- The resonances are either both rational or one is a positive integer and the other is a positive rational, but the ODE fails the weak Painlevé test.

In this thesis, we shall consider only the first three cases.

CASE 1: At least one integer resonance is incompatible

In this case, where r_1 and r_2 ($r_1 < r_2$) are the positive integer resonances, it has been found [11] that a logarithmic psi-series expansion (3.5), with r being the smallest incompatible resonance, suffices to obtain a consistent expansion containing

the required number of free constants. As in the second-order case, one of these free constants is $a_{0,1}$ which, together with another free constant, can be defined in such a way as to make the resonances at $(r_i, 0)$ compatible, and thus $a_{r_1,0}$ and $a_{r_2,0}$ free. Hence the logarithmic psi-series formally constitutes a general solution.

The variations (the smaller or larger resonance being incompatible, or both) in this case are explored by means of the example

$$u''' + Au'' + A(u')^2 + Bu^2 + Cuu' = 0, \quad (3.32)$$

where A, B and C are constants. The leading order p and the leading coefficient a_0 for this example are easily found to be $p = -1$ and $a_0 = 2/A$. The recursion relation for the a_k in the Laurent series is also easily found to be, $\forall k \geq 1$,

$$\begin{aligned} (k+1)(k-2)(k-3)a_k = & -A(k-3) \sum_{m=2}^{k-1} (m-1)a_m a_{k-m} - B \sum_{m=0}^{k-2} a_m a_{k-m} \\ & - C \sum_{m=0}^{k-1} (m-1)a_m a_{k-m-1}, \end{aligned}$$

from which we note that the resonances are positive integers at $k = 2$ and $k = 3$.

When $k = 1$, we have

$$\begin{aligned} 4a_1 = Ca_0^2 &= 4\frac{C}{A^2}, \\ \Rightarrow a_1 &= \frac{C}{A^2}. \end{aligned}$$

When $k = 2$, the first resonance, we get the first compatibility condition

$$0 \cdot a_2 = \left(\frac{C}{A}\right)^2 - 2\frac{B}{A},$$

and when $k = 3$, the second resonance, we get the second compatibility condition

$$0 \cdot a_3 = -2Ba_0a_1 = \frac{BC}{A^4}.$$

These two compatibility conditions lead to the following possibilities:

Conditions	$r_1 = 2$	$r_2 = 3$
$C = 0$ and $B = 0$	Compatible	Compatible
$BC \neq 0$ and $A = \frac{C^2}{2B}$	Compatible	Incompatible
$B = 0, C \neq 0$ or $C = 0, B \neq 0$	Incompatible	Compatible
$BC \neq 0$ and $A \neq \frac{C^2}{2B}$	Incompatible	Incompatible

We shall consider the last case where both resonances are incompatible. The appropriate series expansion is then

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-1} (\tau^2 \log \tau)^l. \quad (3.33)$$

Differentiating (3.33), substituting into (3.32), and rearranging indices, we get the rather long recursion relation for a_{kl} ($\alpha = k + 2l$ and $\beta = m + 2n$):

$$\begin{aligned}
(\alpha - 1)(\alpha - 2)(\alpha - 3)a_{kl} = & -(l + 1)(l + 2)(l + 3)a_{k-6,l+3} \\
& -3(l + 1)(l + 2)(\alpha - 6)a_{k-4,l+2} \\
& -(l + 1) \left[3(\alpha - 2)^2 - 23 \right] a_{k-2,l+1} \\
& -A \sum_{m=0}^k \sum_{n=0}^l (\alpha - 3)(\beta - 1)a_{mn}a_{k-m,l-n} \\
& -A \sum_{m=0}^{k-2} \sum_{n=0}^{l+1} [2(\beta - 1)(ml + 1) - n] a_{mn}a_{k-m-2,l-n+1} \\
& -A \sum_{m=0}^{k-4} \sum_{n=0}^{l+2} n(l + 1)a_{mn}a_{k-m-4,l-n+2} \\
& -B \sum_{m=0}^{k-2} \sum_{n=0}^l a_{mn}a_{k-m-2,l-n} - C \sum_{m=0}^{k-3} \sum_{n=0}^{l+1} n a_{mn}a_{k-m-3,l-n+1} \\
& -C \sum_{m=0}^{k-1} \sum_{n=0}^l (\beta - 1)a_{mn}a_{k-m-1,l-n}.
\end{aligned}$$

When $(k, l) = (0, 0)$, the RR gives

$$\begin{aligned} -6a_{0,0} &= -3Aa_{0,0}^2, \\ \Rightarrow a_{0,0} &= \frac{2}{A}. \end{aligned}$$

When $(k, l) = (0, 1)$, we have

$$0 \cdot a_{0,1} = -A\{a_{0,0}a_{0,1} - a_{0,1}a_{0,0}\} = 0$$

$\Rightarrow a_{0,1}$ is the first free constant, as expected.

When $(k, l) = (1, 0)$, we have

$$0 \cdot a_{1,0} = -2Aa_{0,0}a_{1,0} + Ca_{0,0}^2 \Rightarrow a_{1,0} = \frac{C}{A^2}$$

When $(k, l) = (1, 1)$, we have

$$0 \cdot a_{1,1} = -C\{-a_{0,0}a_{0,1} + a_{0,1}a_{0,0}\} = 0$$

$\Rightarrow a_{1,1}$ is the second free constant.

When $(k, l) = (2, 0)$ (where $r_1 = 2$ is the first resonance), we have the compatibility condition

$$\begin{aligned} 0 \cdot a_{2,0} &= 25a_{0,1} + Aa_{0,0}a_{0,1} - Ba_{0,0}^2 + Ca_{0,0}a_{1,0} \\ &= 25a_{0,1} - 4\frac{B}{A^2} + 2\frac{C^2}{A^3}. \end{aligned}$$

Since $a_{0,1}$ is a free constant, it can be chosen to be

$$a_{0,1} = \frac{4AB - 2C^2}{25A^3},$$

thus making $a_{2,0}$ free.

Similarly for $(k, l) = (3, 0)$ (where $r_2 = 3$ is the second resonance), the second free constant can be used in such a way that the compatibility condition at $(k, l) = (3, 0)$ is satisfied, and $a_{3,0}$ is free.

CASE 2: Complex or irrational conjugate resonances

It has been found [11] that if the resonances r_1, r_2 are either complex conjugates (i.e. $r_2 = \bar{r}_1$) or irrational conjugates, then the appropriate psi-series expansion is of the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^{r_1})^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+p} (\tau^{r_1})^l, \quad (3.34)$$

where in the complex case, $\bar{a}_{kl}$ is the complex conjugate of a_{kl} . Two irrational numbers are said to be irrational conjugates if they are the two roots of a quadratic polynomial.

The psi-series (3.34) contains $a_{0,1}$ and $\bar{a}_{0,1}$ as free constants (as well as t_0 , where $\tau = t - t_0$) and hence formally constitutes a general solution of the third-order ODE (3.31), with non-integer resonances.

As an illustrative example, consider the ODE

$$u''' + (u')^2 - 5u'u^2 = 0. \quad (3.35)$$

By inspection, the leading order is $p = -1$. The leading coefficient a_0 in the Laurent series is found to satisfy

$$\begin{aligned} 5a_0^2 + a_0 - 6 &= 0, \\ \Rightarrow a_0 &= \frac{-1 \pm \sqrt{121}}{10} = -\frac{6}{5} \text{ or } 1. \end{aligned}$$

Following the Laurent series analysis as outlined in Step 2 of the ARS algorithm, the resonance equation is found to be

$$(r+1)(r^2 - 7r + 12 - a_0) = 0. \quad (3.36)$$

The two resonances are therefore

$$r, \bar{r} = \frac{7 \pm \sqrt{4a_0 + 1}}{2} = \begin{cases} \frac{7 \pm \sqrt{5}}{2} & \text{if } a_0 = 1, \\ \frac{7 \pm i\sqrt{\frac{49}{5}}}{2} & \text{if } a_0 = -\frac{6}{5}. \end{cases}$$

We see that this example can exhibit nonintegrability in two different ways, depending upon the value of the leading coefficient a_0 . If $a_0 = 1$, we have real but irrational resonances, and if $a_0 = -\frac{6}{5}$, we have complex resonances.

Before we verify that the psi-series in (3.34) contains $a_{0,1}$ and $\bar{a}_{0,1}$ as free constants, we state and prove the following lemma.

Lemma 3.2.1 *If s_1 and s_2 are given by*

$$s_1 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+lr+p} \quad \text{and} \quad s_2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+l\bar{r}+q},$$

where $r + \bar{r} = \alpha$, α a positive integer, then

$$\begin{aligned} s_1 \cdot s_2 = & \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{\alpha} \rfloor} \sum_{n=0}^{k-m\alpha} \left\{ \delta_l \bar{a}_{nm} a_{k-m\alpha-n, l+m} \tau^{k+lr+p+q} \right\} \\ & + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{\alpha} \rfloor} \sum_{n=0}^{k-m\alpha} \left\{ \delta_l a_{nm} \bar{a}_{k-m\alpha-n, l+m} \tau^{k+l\bar{r}+p+q} \right\}, \end{aligned}$$

where

$$\delta_l = \begin{cases} \frac{1}{2} & \text{if } l = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Proof: Write s_1 and s_2 as

$$s_1 = \tau^p \sum_{l=0}^{\infty} A_l \tau^{lr} \quad \text{and} \quad s_2 = \tau^q \sum_{l=0}^{\infty} \bar{A}_l \tau^{l\bar{r}},$$

where $A_l = \sum_{k=0}^{\infty} a_{kl} \tau^k$ and $\bar{A}_l = \sum_{k=0}^{\infty} \bar{a}_{kl} \tau^k$, then

$$s_1 \cdot s_2 = \tau^{(p+q)} \left\{ \begin{aligned} & A_0 \bar{A}_0 \tau^0 + A_0 \bar{A}_1 \tau^{\bar{r}} + A_0 \bar{A}_2 \tau^{2\bar{r}} + A_0 \bar{A}_3 \tau^{3\bar{r}} + \dots \\ & + A_1 \bar{A}_0 \tau^r + A_1 \bar{A}_1 \tau^{\alpha} + A_1 \bar{A}_2 \tau^{r+\alpha} + A_1 \bar{A}_3 \tau^{2\bar{r}+\alpha} + \dots \\ & + A_2 \bar{A}_0 \tau^{2r} + A_2 \bar{A}_1 \tau^{r+\alpha} + A_2 \bar{A}_2 \tau^{2\alpha} + A_2 \bar{A}_3 \tau^{r+2\alpha} + \dots \\ & + A_3 \bar{A}_0 \tau^{3r} + A_3 \bar{A}_1 \tau^{2r+\alpha} + A_3 \bar{A}_2 \tau^{r+2\alpha} + A_3 \bar{A}_3 \tau^{3\alpha} + \dots \\ & \vdots \end{aligned} \right\}.$$

Summing along the diagonals, we obtain

$$s_1 \cdot s_2 = \tau^{(p+q)} \left\{ \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \delta_l A_{l+m} \bar{A}_m \tau^{lr+m\alpha} + \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \delta_l A_m \bar{A}_{l+m} \tau^{l\bar{r}+m\alpha} \right\}.$$

Using

$$A_{l+m} \bar{A}_m = \sum_{k=0}^{\infty} \left(\sum_{n=0}^k \bar{a}_{n,m} a_{k-n,l+m} \right) \tau^k \equiv \sum_{k=0}^{\infty} B_{l,m,k} \tau^k$$

and

$$A_m \bar{A}_{l+m} = \sum_{k=0}^{\infty} \left(\sum_{n=0}^k a_{n,m} \bar{a}_{k-n,l+m} \right) \tau^k \equiv \sum_{k=0}^{\infty} \bar{B}_{l,m,k} \tau^k,$$

we obtain

$$s_1 \cdot s_2 = \tau^{(p+q)} \left\{ \sum_{l=0}^{\infty} \delta_l \tau^{lr} \underbrace{\left(\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} B_{l,m,k} \tau^{k+m\alpha} \right)}_I + \sum_{l=0}^{\infty} \delta_l \tau^{l\bar{r}} \underbrace{\left(\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \bar{B}_{l,m,k} \tau^{k+m\alpha} \right)}_{II} \right\}.$$

It is easily verified that

$$I = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\lfloor \frac{m}{\alpha} \rfloor} B_{l,k,m-k\alpha} \right) \tau^m \quad \text{and} \quad II = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\lfloor \frac{m}{\alpha} \rfloor} \bar{B}_{l,k,m-k\alpha} \right) \tau^m,$$

$\Rightarrow$

$$s_1 \cdot s_2 = \tau^{p+q} \left\{ \sum_{l=0}^{\infty} \tau^{lr} \delta_l \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{m}{\alpha} \rfloor} B_{l,k,m-k\alpha} \tau^m + \sum_{l=0}^{\infty} \tau^{l\bar{r}} \delta_l \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{m}{\alpha} \rfloor} \bar{B}_{l,k,m-k\alpha} \tau^m \right\}.$$

Finally, substitute back the expressions for $B_{l,k,m-k\alpha}$ and $\bar{B}_{l,k,m-k\alpha}$, to get

$$\begin{aligned} s_1 \cdot s_2 &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{\alpha} \rfloor} \sum_{n=0}^{k-m\alpha} \left\{ \delta_l \bar{a}_{nm} a_{k-m\alpha-n,l+m} \tau^{k+lr+p+q} \right\} \\ &\quad + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{\alpha} \rfloor} \sum_{n=0}^{k-m\alpha} \left\{ \delta_l a_{nm} \bar{a}_{k-m\alpha-n,l+m} \tau^{k+l\bar{r}+p+q} \right\}, \end{aligned}$$

and this completes the proof of the lemma. $\square$

Returning to our example (3.35), we note that in both the complex and the irrational resonance cases, $r + \bar{r} = 7 = \alpha$ in the lemma. We now verify that $a_{0,1}$ and $\bar{a}_{0,1}$

in the psi-series are free constants. The following procedure applies to both cases, with no reference as to whether r and $\bar{r}$ are complex or irrational conjugates. Substituting the psi-series (3.34) into the ODE (3.35), using the lemma, and performing a lengthy calculation, we obtain the simultaneous recursion relations for the a_{kl} and the $\bar{a}_{kl}$:

$$\begin{cases} (k + lr - 1)(k + lr - 2)(k + lr - 3)a_{kl} + C_{kl} - 5E_{kl} = 0 , \\ (k + l\bar{r} - 1)(k + l\bar{r} - 2)(k + l\bar{r} - 3)\bar{a}_{kl} + \bar{C}_{kl} - 5\bar{E}_{kl} = 0 , \end{cases}$$

where

$$\begin{aligned} C_{kl} &= \sum_{m=0}^k \sum_{n=0}^l A_{mn} A_{k-m, l-n} + 2\delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} \bar{A}_{nm} A_{k-7m-n, l+m} , \\ \bar{C}_{kl} &= \sum_{m=0}^k \sum_{n=0}^l \bar{A}_{mn} \bar{A}_{k-m, l-n} + 2\delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} A_{nm} \bar{A}_{k-7m-n, l+m} , \\ E_{kl} &= \sum_{m=0}^k \sum_{n=0}^l A_{mn} D_{k-m, l-n} + \delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} \{ \bar{D}_{nm} A_{k-7m-n, l+m} + \bar{A}_{nm} D_{k-7m-n, l+m} \} , \\ \bar{E}_{kl} &= \sum_{m=0}^k \sum_{n=0}^l \bar{A}_{mn} \bar{D}_{k-m, l-n} + \delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} \{ A_{nm} \bar{D}_{k-7m-n, l+m} + D_{nm} \bar{A}_{k-7m-n, l+m} \} , \\ D_{kl} &= \sum_{m=0}^k \sum_{n=0}^l a_{mn} a_{k-m, l-n} + 2\delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} \bar{a}_{nm} a_{k-7m-n, l+m} , \\ \bar{D}_{kl} &= \sum_{m=0}^k \sum_{n=0}^l \bar{a}_{mn} \bar{a}_{k-m, l-n} + 2\delta_l \sum_{m=0}^{\lfloor \frac{k}{7} \rfloor} \sum_{n=0}^{k-7m} a_{nm} \bar{a}_{k-7m-n, l+m} , \\ A_{kl} &= (k + lr - 1)a_{kl} , \\ \bar{A}_{kl} &= (k + l\bar{r} - 1)\bar{a}_{kl} . \end{aligned}$$

When $(k, l) = (0, 0)$, the RR gives

$$\begin{cases} -6a_{0,0} + C_{0,0} - 5E_{0,0} = 0 , \\ -6\bar{a}_{0,0} + \bar{C}_{0,0} - 5\bar{E}_{0,0} = 0 . \end{cases}$$

Simple calculations give

$$\begin{aligned} C_{0,0} &= a_{0,0}(a_{0,0} + \bar{a}_{0,0}) , & \bar{C}_{0,0} &= \bar{a}_{0,0}(a_{0,0} + \bar{a}_{0,0}) , \\ E_{0,0} &= -a_{0,0}(a_{0,0} + \bar{a}_{0,0})^2 , & \bar{E}_{0,0} &= -\bar{a}_{0,0}(a_{0,0} + \bar{a}_{0,0})^2 , \end{aligned}$$

$\Rightarrow a_{0,0}$ and $\bar{a}_{0,0}$ satisfy

$$\begin{aligned} 5(a_{0,0} + \bar{a}_{0,0})^2 + (a_{0,0} + \bar{a}_{0,0}) - 6 &= 0, \\ \Rightarrow a_{0,0} + \bar{a}_{0,0} &= -\frac{6}{5} \text{ or } 1. \end{aligned}$$

Because $(a_{0,0} + \bar{a}_{0,0})$ satisfies the same condition as a_0 (the leading coefficient of the single series), $a_{0,0}$ and $\bar{a}_{0,0}$ can be assigned any convenient values such that the sum is equal to a_0 .

When $(k, l) = (0, 1)$, the RR gives

$$\begin{cases} (r-1)(r-2)(r-3)a_{0,1} + C_{0,1} - 5E_{0,1} = 0, \\ (\bar{r}-1)(\bar{r}-2)(\bar{r}-3)\bar{a}_{0,1} + \bar{C}_{0,1} - 5\bar{E}_{0,1} = 0. \end{cases}$$

Again simple calculations give

$$\begin{aligned} C_{0,1} &= -2(r-1)(a_{0,0} + \bar{a}_{0,0})a_{0,1}, \quad E_{0,1} = (r-3)(a_{0,0} + \bar{a}_{0,0})^2 a_{0,1}, \\ \bar{C}_{0,1} &= -2(\bar{r}-1)(a_{0,0} + \bar{a}_{0,0})\bar{a}_{0,1}, \quad \bar{E}_{0,1} = (\bar{r}-3)(a_{0,0} + \bar{a}_{0,0})^2 \bar{a}_{0,1}, \end{aligned}$$

$\Rightarrow a_{0,1}$ and $\bar{a}_{0,1}$ satisfy

$$\begin{cases} (r-1)(r-2)(r-3)a_{0,1} - 2(r-1)(a_{0,0} + \bar{a}_{0,0})a_{0,1} - 5(r-1)(a_{0,0} + \bar{a}_{0,0})^2 a_{0,1} = 0, \\ (\bar{r}-1)(\bar{r}-2)(\bar{r}-3)\bar{a}_{0,1} - 2(\bar{r}-1)(a_{0,0} + \bar{a}_{0,0})\bar{a}_{0,1} - 5(\bar{r}-1)(a_{0,0} + \bar{a}_{0,0})^2 \bar{a}_{0,1} = 0. \end{cases}$$

The above equations factorize to

$$\begin{cases} (r+1)(r^2 - 7r + 12 - (a_{0,0} + \bar{a}_{0,0}))a_{0,1} = 0 \\ (\bar{r}+1)(\bar{r}^2 - 7\bar{r} + 12 - (a_{0,0} + \bar{a}_{0,0}))\bar{a}_{0,1} = 0, \end{cases}$$

and in view of the fact that $(a_{0,0} + \bar{a}_{0,0}) = a_0$ and that r and $\bar{r}$ are the roots of the quadratic in (3.36), we obtain $0 \cdot a_{0,1} = 0$ and $0 \cdot \bar{a}_{0,1} = 0$. Thus both $a_{0,1}$ and $\bar{a}_{0,1}$ are free constants in addition to the arbitrary singularity, and this proves that the psi-series in (3.34) constitutes a formal general solution.

3.3 Conclusion

For nonintegrable ODEs, a Laurent series was found to be either inconsistent or short of the necessary degrees of freedom (the arbitrary constants). The various psi-series were found to be the appropriate series expansions in that they introduce the additional degrees of freedom needed. Such psi-series were said to be formal general solutions. Until convergence of such series is proved, they cannot be regarded as actual solutions. The convergence is dealt with in the next chapter.

Chapter 4

CONVERGENCE OF PSI-SERIES

In the previous chapter, various forms of psi-series, logarithmic and nonlogarithmic, were shown to constitute formal general solutions of nonintegrable second- and third-order ODEs of type (3.1) and (3.31). Until such series are shown to admit positive radii of convergence, they are merely symbols and not actual solutions.

This chapter is devoted to the convergence proofs of the various types of psi-series encountered in Chapter 3. In the proof, we shall be assuming that $\tau = t - t_0 > 0$. Also, we shall consider only those cases where the resonances have real parts greater than 1.

All of the proofs will proceed according to the outline described in Section 4.1. In Section 4.2, we shall see how psi-series can be converted into single series with coefficients that are functions. In Section 4.3, certain lemmas are stated and proved. In Section 4.4, we employ the examples from Chapter 3 and prove convergence of their psi-series solutions, as well as giving an estimation formula for their radii of convergence.

4.1 Outline of the Proof

As was seen in Chapter 3, the recursion relations for the a_{kl} in the psi-series are very complicated. A much better way of dealing with such double series is by converting them into single series with coefficients that are functions. The single series are of the form

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha+p}, \quad (4.1)$$

where p is the leading order, $z = \log \tau$ and $f_{\alpha}(z)$ is some function of z . The conversion from a double series into a single one, termed "resummation of psi-series," is described in the next section. This is the first step in the proof of convergence.

Any n th-order differential equation of type (2.3) can be put into the form of a system

$$(S) = \begin{cases} u'_1 &= u_2 \\ u'_2 &= u_3 \\ \vdots &\vdots \\ u'_{n-1} &= u_n \\ u'_n &= R(u_1, u_2, \dots, u_{n-1}), \end{cases}$$

where $u_1 = u$, $u_i = u^{(i-1)}$ for $2 \leq i \leq n$ ($u^{(i)}$ denoting the i -th derivative of u), and R is a polynomial in its arguments. Substitution of the series (4.1) into the above system (S) yields a sequence of linear, nonhomogeneous, first-order vector ODEs

$$\vec{f}'_{\alpha}(z) + A_{\alpha} \vec{f}_{\alpha}(z) = \vec{F}_{\alpha}(z), \quad (4.2)$$

where

$$\vec{f}_\alpha = \begin{pmatrix} f_{\alpha,1} \\ f_{\alpha,2} \\ \vdots \\ f_{\alpha,n} \end{pmatrix}, \quad \vec{F}_\alpha = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F_\alpha \end{pmatrix},$$

$$f_{\alpha,1} = f_\alpha,$$

$$f_{\alpha,i} = f'_{\alpha,i-1} + (\alpha - p - i + 2)f_{\alpha,i-1}, \quad i = 2, 3, \dots, n,$$

$$F_\alpha = F(f_{\alpha,1}, f_{\alpha,2}, \dots, f_{\alpha-1,n}) \text{ is some polynomial function of its arguments}$$

and A_α is an $n \times n$ matrix. For $\alpha > \text{Re}(r)$ (where r is any resonance), the vector equation (4.2) has the solution

$$\vec{f}_\alpha(z) = \int_{-\infty}^z \vec{F}_\alpha(x) e^{(x-z)A_\alpha} dx, \quad (4.3)$$

where the exponential of a square matrix is given by :

Definition 4.1 Let $A = (a_{ij})$ be a real or complex $n \times n$ matrix. Then

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \dots$$

This is step two in the convergence proof, namely, obtaining and solving a sequence of ODEs for the coefficient functions $f_\alpha(z)$.

The norm of the vector solution (4.3) can be shown to satisfy the inequality

$$\|\vec{f}_\alpha(z)\| \leq M \frac{(a + bw(z))^{\alpha/q}}{\sqrt{\alpha+1}} \quad \forall \alpha \geq 1, \quad (4.4)$$

where the norm of a matrix that we employ here is defined by :

Definition 4.2 Let $A = (a_{ij})$ be a real or complex $m \times n$ matrix. Then we define $\|A\|$ by

$$\|A\| = \max_{1 \leq i \leq m} \left(\sum_{j=1}^n |a_{ij}| \right).$$

In (4.4), a and b are appropriately chosen constants, $w(z)$ is some function of z , $0 < M \leq 1$, and $q = r$ (the smallest incompatible positive integer resonance) in the logarithmic psi-series case, and $q = 1$ in the nonlogarithmic psi-series case. Since $|f_\alpha(z)| \leq \|\vec{f}_\alpha(z)\|$, $|f_\alpha(z)|$ is also dominated as in (4.4). This is step three in the convergence proof, and the most delicate one. It is achieved by means of induction and is best illustrated by means of examples (Section 4.4).

In view of (4.4), we have

$$\sum_{\alpha=0}^{\infty} |f_\alpha(z)| |\tau^{\alpha+p}| \leq M \sum_{\alpha=0}^{\infty} \frac{[a + bw(z)]^{\alpha/q}}{\sqrt{\alpha+1}} |\tau^{\alpha+p}| ,$$

and by the ratio test, the series on the right converges if

$$[a + bw(z)]^{1/q} |\tau| < 1 , \text{ or } [a + bw(z)] < e^{-zq} \quad (4.5)$$

is satisfied in an interval. In all of the examples that we shall be considering, the inequality (4.5) is valid in an interval.

We now summarize the 4 steps comprising the convergence proof :

1. Convert the double series into the single series (4.1) with function coefficients.
2. Employ (4.1) in the given ODE to obtain a sequence of linear vector ODEs. Solve the sequence of vector ODEs and prove by induction that the solutions are bounded as in (4.4), and hence deduce that the function coefficients in (4.1) are also bounded by the same bound.
3. Use the ratio test to find the condition for absolute convergence, and hence the interval of convergence.

4.2 Resummation of Psi-Series

In this section, we shall see how the psi-series discussed in Chapter 3 are converted into single series (4.1). In all of the following, the resonances are assumed to have real parts greater than 1.

4.2.1 Logarithmic Psi-Series

Recall that logarithmic psi-series are of the form.

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r \log \tau)^l = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+lr+p} \log^l \tau, \quad (4.6)$$

where r is the smallest incompatible positive integer resonance and p is the leading order. Let $\alpha = k + lr$ and rewrite the series as

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+lr+p} \log^l \tau = \sum_{\alpha=0}^{\infty} \left(\sum_{k+lr=\alpha} a_{kl} \log^l \tau \right) \tau^{\alpha+p}. \quad (4.7)$$

Thus summation over all (k, l) with $k, l \geq 0$ is achieved by summing over all (k, l) with $k + rl = \alpha$, for all $\alpha \geq 0$. The inner sum in (4.7) is a finite sum, over all points $(k, l) \in \mathbb{Z}^2$ on the line segment $k + rl = \alpha$ in the (k, l) -plane. This is illustrated graphically in Fig. 4.1. To help visualize the transition from (k, l) to α , the following table shows, for each value of α , the corresponding (k, l) pairs.

α	(k, l)
0	(0,0)
1	(1,0)
2	(2,0)
$\vdots$	$\vdots$
r	($r, 0$) ($0, 1$)
$r + 1$	($r + 1, 0$) ($1, 1$)
$\vdots$	$\vdots$
$2r$	($2r, 0$) ($r, 1$) ($0, 2$)
$\vdots$	$\vdots$

Table 4.1. The single index α versus the double index (k, l)

We can see from the table that for each α there correspond $([\frac{\alpha}{r}] + 1)$ (k, l) pairs. This means that the inner sum in (4.7) can be written as

$$\sum_{k+l=r=\alpha} a_{kl} \log^l \tau = \sum_{n=0}^{[\frac{\alpha}{r}]} a_n^{(\alpha)} \log^n \tau, \quad (4.8)$$

where $a_n^{(\alpha)} = a_{\alpha - nr, n}$. The sum in (4.8) is simply a polynomial in $z = \log \tau$ of degree $[\frac{\alpha}{r}]$, where $[\frac{\alpha}{r}]$ is the greatest integer less than or equal to $(\frac{\alpha}{r})$. Hence, defining the function $f_\alpha(z)$ by

$$f_\alpha(z) = \sum_{n=0}^{[\frac{\alpha}{r}]} a_n^{(\alpha)} z^n,$$

the double series (4.6) is converted into the single series (4.1), i.e.,

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-p} (\tau^r \log \tau)^l = \sum_{\alpha=0}^{\infty} f_\alpha(z) \tau^{\alpha+p}, \quad (4.9)$$

where $f_\alpha(z)$ is a polynomial in z of degree $[\frac{\alpha}{r}]$.

4.2.2 Nonlogarithmic Psi-Series

In the second-order case where the positive resonance r is not an integer, we have seen that the psi-series takes the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r)^l = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l+r+p}. \quad (4.10)$$

In order to convert such a series into the single series (4.1), we write it as

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l+r+p} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l+p} \tau^{(r-1)l}$$

and let $\alpha = k + l$ to get

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l+p} \tau^{(r-1)l} = \sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} a_{kl} \tau^{(r-1)l} \right) \tau^{\alpha+p}. \quad (4.11)$$

Since τ is assumed to be positive, we can write

$$\tau^{r-1} = e^{(r-1)\log \tau} = e^{(r-1)z},$$

where, as before, $z = \log \tau$. Substitution into (4.11) yields

$$\sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} a_{kl} \tau^{(r-1)l} \right) \tau^{\alpha+p} = \sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} a_{kl} e^{(r-1)lz} \right) \tau^{\alpha+p}. \quad (4.12)$$

The inner sum in (4.12) is finite and may be written as

$$\sum_{k+l=\alpha} a_{kl} e^{(r-1)lz} = \sum_{n=0}^{\alpha} a_n^{(\alpha)} e^{(r-1)nz},$$

where $a_n^{(\alpha)} = a_{\alpha-n,n}$. Defining

$$f_{\alpha}(z) = \sum_{n=0}^{\alpha} a_n^{(\alpha)} e^{(r-1)nz}$$

converts the double series in (4.10) into the single series (4.1), i.e.,

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+lr+p} = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha+p}. \quad (4.13)$$

Thus the resummation in this case is as in the logarithmic case, but with $\alpha = k + l$ and $f_{\alpha}(z)$ a polynomial in $e^{(r-1)z}$ of degree α .

In the third-order case, we considered three cases requiring psi-series. The first was the case where both resonances were positive integers but at least one was incompatible. The psi-series for this case is the logarithmic one, which has been dealt with. The others were the cases where the resonances were either complex or irrational conjugates. The psi-series for the latter cases consisted of the sum of two nonlogarithmic series, and this makes the resummation process more complicated.

Let us consider the psi-series for the case of complex conjugate resonances. Let $r, \bar{r}$ be the two complex conjugate resonances. We are assuming that the real part of the resonances is greater than 1. Let

$$\begin{aligned} r &= \lambda + i\mu \\ \bar{r} &= \lambda - i\mu. \end{aligned}$$

where $\lambda > 1$. The psi-series solution in this case takes the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+p} (\tau^{\bar{r}})^l. \quad (4.14)$$

where the a_{kl} and the $\bar{a}_{kl}$ are complex conjugates. Combining the series and using the definitions of r and $\bar{r}$, we obtain

$$\begin{aligned} u &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_{kl} \tau^{lr} + \bar{a}_{kl} \tau^{l\bar{r}}) \tau^{k+p} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_{kl} \tau^{i\mu l} + \bar{a}_{kl} \tau^{-i\mu l}) \tau^{\lambda l} \tau^{k+p}. \end{aligned} \quad (4.15)$$

Now write a_{kl} , $\bar{a}_{kl}$, $\tau^{i\mu l}$ and $\tau^{-i\mu l}$ as

$$\begin{aligned} a_{kl} &= a_{kl}^r + i a_{kl}^i, \quad \tau^{i\mu l} = \cos(\mu l z) + i \sin(\mu l z), \\ \bar{a}_{kl} &= a_{kl}^r - i a_{kl}^i, \quad \tau^{-i\mu l} = \cos(\mu l z) - i \sin(\mu l z), \end{aligned}$$

where a_{kl}^r and a_{kl}^i are the real and imaginary parts of a_{kl} , respectively, and $z = \log \tau$.

The quantity between parentheses in (4.15) then becomes

$$\begin{aligned} a_{kl} \tau^{i\mu l} + \bar{a}_{kl} \tau^{-i\mu l} &= (a_{kl} + \bar{a}_{kl}) \cos(\mu l z) + i(a_{kl} - \bar{a}_{kl}) \sin(\mu l z) \\ &= 2a_{kl}^r \cos(\mu l z) - 2a_{kl}^i \sin(\mu l z). \end{aligned}$$

Substituting the latter into (4.15) and writing $\tau^{k+p} \tau^{\lambda l} = \tau^{k+l+p} \tau^{(\lambda-1)l}$ yields

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} [A_{kl} \cos(\mu l z) + B_{kl} \sin(\mu l z)] \tau^{k+l+p} \tau^{(\lambda-1)l}, \quad (4.16)$$

where $A_{kl} = 2a_{kl}^r = 2\operatorname{Re}(a_{kl})$ and $B_{kl} = -2a_{kl}^i = -2\operatorname{Im}(a_{kl})$. Now letting $\alpha = k + l$, as before, we obtain

$$u = \sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} [A_{kl} \cos(\mu l z) + B_{kl} \sin(\mu l z)] \tau^{(\lambda-1)l} \right) \tau^{\alpha+p}.$$

Since for each α there are a finite number of pairs (k, l) such that $k + l = \alpha$ (see Table 4.1), the inner sum in the above is finite and can be written as

$$\sum_{k+l=\alpha} [A_{kl} \cos(\mu l z) + B_{kl} \sin(\mu l z)] \tau^{(\lambda-1)l} = \sum_{n=0}^{\alpha} [a_n^{(\alpha)} \cos(n\mu z) + b_n^{(\alpha)} \sin(n\mu z)] \epsilon^{n(\lambda-1)z},$$

where $a_n^{(\alpha)} = A_{\alpha-n, n} = 2\operatorname{Re}(a_{\alpha-n, n})$ and $b_n^{(\alpha)} = B_{\alpha-n, n} = -2\operatorname{Im}(a_{\alpha-n, n})$. Now define the function $f_{\alpha}(z)$ by

$$f_{\alpha}(z) = \sum_{n=0}^{\alpha} [a_n^{(\alpha)} \cos(n\mu z) + b_n^{(\alpha)} \sin(n\mu z)] \epsilon^{n(\lambda-1)z}.$$

so that the psi-series (4.14) converts into the single series (4.1), i.e.,

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+p} (\tau^{\bar{r}})^l = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha+p}.$$

Next, let us turn to the case of irrational conjugate resonances. The resummation procedure in this case is the same as in the complex conjugate case, except for the definition of $f_{\alpha}(z)$. The psi-series is of the form (4.14), but with the resonances r and $\bar{r}$ being irrational conjugates. The constants a_{kl} , $\bar{a}_{kl}$ are real constants and can always be defined to have a special form such that the psi-series can be converted into the single series. First, write the resonances in the form

$$r = \lambda + \mu$$

$$\bar{r} = \lambda - \mu,$$

where $\lambda > 0$, $\mu > 0$ and $r, \bar{r}$ are assumed to be greater than 1. When the series are combined, (4.14) becomes

$$\begin{aligned} u &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} (\tau^r)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+p} (\tau^{\bar{r}})^l \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_{kl} \tau^{\mu l} + \bar{a}_{kl} \tau^{-\mu l}) \tau^{(\lambda-1)l} \tau^{k+l-1}, \end{aligned}$$

and letting $\alpha = k + l$ gives

$$u = \sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} [a_{kl} \tau^{\mu l} + \bar{a}_{kl} \tau^{-\mu l}] \tau^{(\lambda-1)l} \right) \tau^{\alpha+p}. \quad (4.17)$$

Next, define the a_{kl} and the $\bar{a}_{kl}$ to have the forms

$$a_{kl} = a_{kl}^r + a_{kl}^i$$

$$\bar{a}_{kl} = a_{kl}^r - a_{kl}^i,$$

where a_{kl}^r and a_{kl}^i are constants. With these definitions of the a_{kl} and $\bar{a}_{kl}$, we have

$$\begin{aligned} a_{kl} \tau^{\mu l} + \bar{a}_{kl} \tau^{-\mu l} &= (a_{kl}^r + a_{kl}^i) \tau^{\mu l} + (a_{kl}^r - a_{kl}^i) \tau^{-\mu l} \\ &= a_{kl}^r (\tau^{\mu l} + \tau^{-\mu l}) + a_{kl}^i (\tau^{\mu l} - \tau^{-\mu l}) \\ &= 2a_{kl}^r \cosh(\mu l z) + 2a_{kl}^i \sinh(\mu l z). \end{aligned}$$

where $z = \log \tau$. Substitution into (4.17) gives

$$u = \sum_{\alpha=0}^{\infty} \left(\sum_{k+l=\alpha} \left[2a_{kl}^r \cosh(\mu l z) + 2a_{kl}^i \sinh(\mu l z) \right] \tau^{l(\lambda-1)} \right) \tau^{\alpha+p} \quad (4.18)$$

and, again, the inner sum is finite and can be written as

$$\sum_{n=0}^{\alpha} \left[a_n^{(\alpha)} \cosh(\mu n z) + b_n^{(\alpha)} \sinh(\mu n z) \right] e^{n(\lambda-1)z} \equiv f_{\alpha}(z), \quad (4.19)$$

converting the psi-series into the single series (4.1).

4.3 Preliminary Lemmas

In this section, we state and prove certain lemmas that are required in the convergence proofs.

Lemma 4.3.1 *Let*

$$S(\alpha) = \sum_{\beta=0}^{\alpha-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{\alpha-\beta}},$$

then

$$\lim_{\alpha \rightarrow \infty} S(\alpha) = \pi.$$

Proof:

Consider the function

$$f_{\alpha}(\beta) = \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{\alpha-\beta}}, \quad \text{for } -1 < \beta < \alpha.$$

The absolute minimum value of $f_{\alpha}(\beta)$ occurs at $\beta = \frac{\alpha-1}{2}$. Also, it is symmetric about the line $\beta = \frac{\alpha-1}{2}$. It is easy to see that

$$\int_0^{\alpha-1} f_{\alpha}(\beta) d\beta \leq S_{\alpha} \leq \int_0^{\alpha} f_{\alpha}(\beta) d\beta. \quad (4.20)$$

Let us first evaluate the indefinite integral

$$\int f_{\alpha}(\beta) d\beta = \int \frac{d\beta}{\sqrt{(\alpha-\beta)(\beta+1)}}.$$

Write

$$(\beta+1)(\alpha-\beta) = \alpha - [\beta^2 + (1-\alpha)\beta]$$

and complete the square to get

$$\begin{aligned} (\beta+1)(\alpha-\beta) &= \alpha + \left(\frac{1-\alpha}{2}\right)^2 - \left(\beta + \frac{1-\alpha}{2}\right)^2 \\ &= \frac{(1+\alpha)^2}{4} - \left(\beta + \frac{1-\alpha}{2}\right)^2 \end{aligned}$$

$\Rightarrow$

$$\int f_{\alpha}(\beta) d\beta = \int \frac{d\beta}{\sqrt{\frac{(1+\alpha)^2}{4} - \left(\beta + \frac{1-\alpha}{2}\right)^2}}.$$

Now let $\gamma = \beta + \frac{1-\alpha}{2} \Rightarrow d\gamma = d\beta$

$\Rightarrow$

$$\int f_{\alpha}(\beta) d\beta = \int \frac{d\gamma}{\sqrt{(\frac{1+\alpha}{2})^2 - \gamma^2}} .$$

Let $\gamma = (\frac{1+\alpha}{2}) \sin \theta \Rightarrow d\gamma = \frac{1+\alpha}{2} \cos \theta d\theta$

$\Rightarrow$

$$\begin{aligned} \int f_{\alpha}(\beta) d\beta &= \int \frac{\frac{1+\alpha}{2} \cos \theta}{\sqrt{(\frac{1+\alpha}{2})^2 - (\frac{1+\alpha}{2})^2 \sin^2 \theta}} d\theta \\ &= \int \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta}} d\theta \\ &= \int d\theta = \theta + C . \end{aligned}$$

Hence we have

$$\int f_{\alpha}(\beta) d\beta = \sin^{-1} \left(\frac{2\gamma}{1+\alpha} \right) + C = \sin^{-1} \left(\frac{2\beta - \alpha + 1}{1+\alpha} \right) + C$$

and the definite integrals in (4.20) become

$$\begin{aligned} \int_0^{\alpha} f_{\alpha}(\beta) d\beta &= \sin^{-1} \left(\frac{\alpha+1}{\alpha+1} \right) - \sin^{-1} \left(\frac{1-\alpha}{\alpha+1} \right) = \frac{\pi}{2} + \sin^{-1} \left(\frac{\alpha-1}{\alpha+1} \right) , \\ \int_0^{\alpha-1} f_{\alpha}(\beta) d\beta &= \sin^{-1} \left(\frac{\alpha-1}{\alpha+1} \right) + \sin^{-1} \left(\frac{\alpha-1}{\alpha+1} \right) = 2 \sin^{-1} \left(\frac{\alpha-1}{1+\alpha} \right) . \end{aligned}$$

We then have

$$2 \sin^{-1} \left(\frac{\alpha-1}{1+\alpha} \right) \leq S(\alpha) \leq \frac{\pi}{2} + \sin^{-1} \left(\frac{\alpha-1}{\alpha+1} \right) .$$

Since

$$\lim_{\alpha \rightarrow \infty} \sin^{-1} \left(\frac{\alpha-1}{\alpha+1} \right) = \frac{\pi}{2} ,$$

we have

$$\pi \leq \lim_{\alpha \rightarrow \infty} S(\alpha) \leq \frac{\pi}{2} + \frac{\pi}{2} = \pi ,$$

and the proof is complete. $\square$

Lemma 4.3.2 Let $z < 0$, q a positive integer greater than or equal to 2, $a > 0$, $b < 0$, $q > -\frac{1}{\alpha}$, $0 < \delta < 1$, $\alpha > \frac{q}{1-\delta}$, and consider the integral

$$I(z) = \int_{-\infty}^z e^{(\alpha-q)(x-z)} (a+bx)^{\alpha/q} dx.$$

Then

$$I(z) \leq \frac{(a+bz)^{\alpha/q}}{\delta\alpha} \left(1 + \frac{\varepsilon}{(q-\varepsilon)\delta}\right), \text{ where } \varepsilon = -\frac{1}{\alpha}.$$

Proof:

Integrate by parts once to get

$$\begin{aligned} I(z) &= e^{(\alpha-q)(x-z)} \frac{(a+bx)^{\alpha/q}}{\alpha-q} \Big|_{-\infty}^z - \frac{\frac{\alpha}{q}}{\alpha-q} b \int_{-\infty}^z e^{(\alpha-q)(x-z)} (a+bx)^{\frac{\alpha}{q}-1} dx \\ &= \frac{(a+bz)^{\alpha/q}}{\alpha-q} - \frac{\frac{\alpha}{q}}{\alpha-q} b \int_{-\infty}^z (a+bx)^{\frac{\alpha}{q}-1} e^{(\alpha-q)(x-z)} dx. \end{aligned}$$

A second integration by parts gives

$$\begin{aligned} I(z) &= \frac{(a+bz)^{\alpha/q}}{\alpha-q} - \frac{\frac{\alpha}{q}}{\alpha-q} b \left\{ \frac{(a+bz)^{\frac{\alpha}{q}-1}}{\alpha-q} - \frac{\frac{\alpha}{q}-1}{\alpha-q} b \int_{-\infty}^z (a+bx)^{\frac{\alpha}{q}-2} e^{(\alpha-q)(x-z)} dx \right\} \\ &= \frac{(a+bz)^{\frac{\alpha}{q}}}{\alpha-q} - (a+bz)^{\frac{\alpha}{q}-1} \frac{\frac{\alpha}{q}}{(\alpha-q)^2} b + \frac{\frac{\alpha}{q}(\frac{\alpha}{q}-1)}{(\alpha-q)^2} b^2 \int_{-\infty}^z (a+bx)^{\frac{\alpha}{q}-2} e^{(\alpha-q)(x-z)} dx, \end{aligned}$$

and after $n+1$ integrations by parts we obtain

$$\begin{aligned} I(z) &= \frac{(a+bz)^{\frac{\alpha}{q}}}{\alpha-q} - (a+bz)^{\frac{\alpha}{q}-1} \frac{\frac{\alpha}{q}}{(\alpha-q)^2} b + (a+bz)^{\frac{\alpha}{q}-2} \frac{\frac{\alpha}{q}(\frac{\alpha}{q}-1)}{(\alpha-q)^3} b^2 \\ &\quad + \dots + (-1)^n (a+bz)^{\frac{\alpha}{q}-n} \frac{\frac{\alpha}{q}(\frac{\alpha}{q}-1)(\frac{\alpha}{q}-2)\dots(\frac{\alpha}{q}-n+1)}{(\alpha-q)^{n+1}} b^n \\ &\quad + (-1)^{n+1} \frac{\frac{\alpha}{q}(\frac{\alpha}{q}-1)(\frac{\alpha}{q}-2)\dots(\frac{\alpha}{q}-n)}{(\alpha-q)^{n+1}} b^{n+1} \int_{-\infty}^z (a+bx)^{\frac{\alpha}{q}-n-1} e^{(\alpha-q)(x-z)} dx. \end{aligned}$$

Now let $n = [\frac{\alpha}{q}]$, the greatest integer less than or equal to $(\frac{\alpha}{q})$, $\gamma = \frac{\alpha}{q}$, $K = -b$, and $X = (a+bz)$. Then

$$I(z) = \frac{X^\gamma}{\alpha-q} + \frac{\gamma}{(\alpha-q)^2} K X^{\gamma-1} + \frac{\gamma(\gamma-1)}{(\alpha-q)^3} K^2 X^{\gamma-2} + \frac{\gamma(\gamma-1)(\gamma-2)}{(\alpha-q)^4} K^3 X^{\gamma-3}$$

$$\begin{aligned}
& + \dots + \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-n+1)}{(\alpha-q)^{n+1}} K^n X^{\gamma-n} \\
& + \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-n)}{(\alpha-q)^{n+1}} K^{n+1} \int_{-\infty}^z (a+bx)^{\gamma-n-1} e^{(\alpha-q)(x-z)} dx .
\end{aligned}$$

Now consider the integral

$$\int_{-\infty}^z (a+bx)^{\gamma-n-1} e^{(\alpha-q)(x-z)} dx .$$

If $n = [\gamma]$, then $0 \leq \gamma - n < 1$ and hence $-1 \leq \gamma - n - 1 < 0$. Also, since $x \leq z < 0$ and $b < 0$, we have

$$\begin{aligned}
a+bx & \geq a+bz \\
\Rightarrow \frac{a+bx}{a+bz} & \leq 1 \\
\Rightarrow \left(\frac{a+bx}{a+bz} \right)^{\gamma-n-1} & \geq 1 \\
\Rightarrow (a+bx)^{\gamma-n-1} & \leq (a+bz)^{\gamma-n-1} = X^{\gamma-n-1}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\int_{-\infty}^z (a+bx)^{\gamma-n-1} e^{(\alpha-q)(x-z)} dx & \leq \int_{-\infty}^z X^{\gamma-n-1} e^{(\alpha-q)(x-z)} dx \\
& = \frac{1}{\alpha-q} X^{\gamma-n-1} .
\end{aligned}$$

We then have

$$\begin{aligned}
I(z) & \leq \frac{X^\gamma}{\alpha-q} \left\{ 1 + \frac{\gamma}{\alpha-q} \left(\frac{K}{X} \right) + \frac{\gamma(\gamma-1)}{(\alpha-q)^2} \left(\frac{K}{X} \right)^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{(\alpha-q)^3} \left(\frac{K}{X} \right)^3 \right. \\
& \quad \left. + \dots + \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-n)}{(\alpha-q)^{n+1}} \left(\frac{K}{X} \right)^{n+1} \right\} \\
& = \frac{X^\gamma}{\alpha-q} \left\{ 1 + \frac{\gamma}{\alpha-q} \left(\frac{K}{X} \right) \left[1 + \sum_{j=1}^n \frac{\prod_{i=1}^j (\gamma-i)}{(\alpha-q)^j} \left(\frac{K}{X} \right)^j \right] \right\} . \tag{4.21}
\end{aligned}$$

Now since $b < 0$, $z < 0 \Rightarrow X = a + bz > a$, we have $\frac{K}{X} \leq \frac{K}{a} = -\frac{b}{a}$. Let $\varepsilon = -\frac{b}{a}$. Since $\left[\frac{\alpha}{q}\right] \leq \frac{\alpha}{q}$, for $0 < j \leq n = \left[\frac{\alpha}{q}\right]$, we have

$$0 < j \leq \frac{\alpha}{q} \Rightarrow 0 < jq \leq \alpha \Rightarrow \alpha - jq \geq 0,$$

and also

$$\alpha - jq \leq \alpha - q \quad (j \geq 1) \Rightarrow \frac{\alpha - j}{\alpha - q} \leq \frac{1}{q} \quad \text{or} \quad \frac{\gamma - j}{\alpha - q} \leq \frac{1}{q}.$$

Thus for $1 \leq j \leq n$, we have

$$\frac{\prod_{i=1}^j (\gamma - i)}{(\alpha - q)^j} \leq \left(\frac{1}{q}\right)^j.$$

Upon substitution of these estimates into (4.21), we obtain

$$\begin{aligned} I(z) &\leq \frac{X^\gamma}{\alpha - q} \left\{ 1 + \frac{\gamma}{\alpha - q} \varepsilon \left[1 + \sum_{j=1}^n \left(\frac{\varepsilon}{q} \right)^j \right] \right\} \\ &\leq \frac{X^\gamma}{\alpha - q} \left\{ 1 + \frac{\gamma}{\alpha - q} \varepsilon \left[1 + \sum_{j=1}^{\infty} \left(\frac{\varepsilon}{q} \right)^j \right] \right\} \\ &= \frac{X^\gamma}{\alpha - q} \left\{ 1 + \frac{\gamma}{\alpha - q} \varepsilon \left[\frac{1}{1 - \frac{\varepsilon}{q}} \right] \right\}, \quad \text{since } 0 < \varepsilon = -\frac{b}{a} < q. \end{aligned}$$

$$\text{Now } \alpha > \frac{q}{1-\delta} \Rightarrow \alpha - q > \delta \alpha$$

$\Rightarrow$

$$I(z) \leq \frac{X^{\frac{\alpha}{q}}}{\delta \alpha} \left(1 + \frac{\varepsilon}{\delta(q - \varepsilon)} \right) = \frac{(a + bz)^{\frac{\alpha}{q}}}{\delta \alpha} \left(1 + \frac{\varepsilon}{\delta(q - \varepsilon)} \right)$$

and the proof is complete. $\square$

Lemma 4.3.3 Let $\alpha > \lambda > 1$, $a > 0$, $b > 0$, $\frac{b}{a} < \frac{1}{2(\lambda-1)}$, $z < 0$, α an integer, and consider the integral

$$I(z) = \int_{-\infty}^z (a + be^{(\lambda-1)x})^\alpha e^{(\alpha-\lambda)(x-z)} dx.$$

Then

$$I(z) \leq \frac{(a + be^{(\lambda-1)z})^\alpha}{(\alpha - \lambda)(1 - 2(\lambda - 1)\varepsilon)}, \quad \text{where } \varepsilon = \frac{b}{a}.$$

Proof:

Let $y = e^{(\lambda-1)x} \Rightarrow \frac{dy}{dx} = (\lambda-1)y \Rightarrow dx = \frac{y^{-1}}{\lambda-1} dy$.

Write $e^{(\alpha-\lambda)(x-z)} = e^{-(\alpha-\lambda)z} e^{(\alpha-\lambda)x} = e^{-(\alpha-\lambda)z} (e^{(\lambda-1)x})^{\frac{\alpha-\lambda}{\lambda-1}} = e^{-(\alpha-\lambda)z} y^{\frac{\alpha-\lambda}{\lambda-1}}$, and let $\theta = \frac{\alpha-\lambda}{\lambda-1} > 0$ ($\alpha > \lambda > 1$), $H(z) = e^{(\lambda-1)z}$. Then $I(z)$ becomes

$$I(z) = \frac{e^{-(\alpha-\lambda)z}}{\lambda-1} \int_0^H y^{\theta-1} (a + by)^\alpha dy. \quad (4.22)$$

Now consider the integral

$$V(H) = \int_0^H y^{\theta-1} (a + by)^\alpha dy. \quad (4.23)$$

Integrate by parts once to get

$$\begin{aligned} V(H) &= \left. \frac{y^\theta}{\theta} (a + by)^\alpha \right|_0^H - \frac{\alpha b}{\theta} \int_0^H y^\theta (a + by)^{\alpha-1} dy \\ &= \frac{H^\theta}{\theta} (a + bH)^\alpha - \frac{\alpha b}{\theta} \int_0^H y^\theta (a + by)^{\alpha-1} dy, \end{aligned}$$

and a second integration by parts gives

$$V(H) = \frac{H^\theta}{\theta} (a + bH)^\alpha - \frac{\alpha b}{\theta(\theta+1)} H^{\theta+1} (a + bH)^{\alpha-1} + \frac{\alpha(\alpha-1)}{\theta(\theta+1)} b^2 \int_0^H y^{\theta+1} (a + by)^{\alpha-2} dy.$$

After α integrations by parts, we get

$$\begin{aligned} V(H) &= \frac{H^\theta}{\theta} (a + bH)^\alpha - \frac{\alpha b}{\theta(\theta+1)} H^{\theta+1} (a + bH)^{\alpha-1} + \frac{\alpha(\alpha-1)b^2}{\theta(\theta+1)(\theta+2)} H^{\theta+2} (a + bH)^{\alpha-2} \\ &\quad + \dots + (-1)^\alpha \frac{\alpha(\alpha-1)(\alpha-2) \dots 3.2.1}{\theta(\theta+1) \dots (\theta+\alpha-1)} b^\alpha \int_0^H y^{\theta+\alpha-1} dy. \end{aligned}$$

The integral

$$\int_0^H y^{\theta+\alpha-1} dy = \frac{H^{\theta+\alpha}}{\theta+\alpha}.$$

Let $X = a + bH$, then we have

$$\begin{aligned} V(H) &= \frac{H^\theta}{\theta} X^\alpha - \frac{\alpha b}{\theta(\theta+1)} H^{\theta+1} X^{\alpha-1} + \frac{\alpha(\alpha-1)b^2}{\theta(\theta+1)(\theta+2)} H^{\theta+2} X^{\alpha-2} \\ &\quad - \dots + (-1)^\alpha \frac{\alpha!}{\theta(\theta+1) \dots (\theta+\alpha)} b^\alpha H^{\theta+\alpha}. \end{aligned}$$

Factoring out $\frac{H^\theta}{\theta} X^\alpha$, we get

$$V(H) = \frac{H^\theta}{\theta} X^\alpha \left\{ 1 + \sum_{k=1}^{\alpha} (-1)^k \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{(\theta+1)(\theta+2)\cdots(\theta+k)} \left(\frac{bH}{X} \right)^k \right\}.$$

Now since $\theta = \frac{\alpha-\lambda}{\lambda-1} > 0$, we have, for $1 \leq k \leq \alpha$,

$$\frac{\alpha-k+1}{\theta+k} \leq \frac{\alpha-k+1}{\theta+1} = (\lambda-1) \frac{\alpha-k+1}{\alpha-1} \leq 2(\lambda-1).$$

Also, since for $z < 0$, $e^{-(\lambda-1)z} < 1$, we have

$$\frac{bH}{X} = \frac{bH}{a+bH} < \frac{bH}{a} < \frac{b}{a} = \varepsilon.$$

Using these estimates, we get

$$\begin{aligned} V(H) &\leq \frac{H^\theta}{\theta} X^\alpha \left\{ \sum_{k=0}^{\alpha} (2(\lambda-1)\varepsilon)^k \right\} \\ &\leq \frac{H^\theta}{\theta} X^\alpha \left\{ \sum_{k=0}^{\infty} (2(\lambda-1)\varepsilon)^k \right\}. \end{aligned}$$

Now the infinite series is a geometric series which converges to

$$\frac{1}{1-2(\lambda-1)\varepsilon} \quad \text{since } 0 < \varepsilon = \frac{b}{a} < \frac{1}{2(\lambda-1)}$$

$\Rightarrow$

$$V(H) \leq \frac{H^\theta}{\theta} X^\alpha \left(\frac{1}{1-2(\lambda-1)\varepsilon} \right).$$

Substituting back the expressions for H , θ , and X , and in view of (4.22), we have

$$\begin{aligned} I(z) &\leq \frac{e^{-(\alpha-\lambda)z}}{\lambda-1} \frac{e^{(\lambda-1)z} \frac{\alpha-\lambda}{\lambda-1} (a + b e^{(\lambda-1)z})^\alpha}{\frac{\alpha-\lambda}{\lambda-1}} \left(\frac{1}{1-2(\lambda-1)\varepsilon} \right) \\ &= \frac{(a + b e^{(\lambda-1)z})^\alpha}{(\alpha-1)(1-2(\lambda-1)\varepsilon)} \end{aligned}$$

which is the desired result, and the proof is complete. $\square$

Lemma 4.3.4 Let $a, b > 0$, $z < 0$, $\beta > \bar{\beta} > 1$, β and $\bar{\beta}$ irrational conjugates ($\beta = \lambda + \mu$, $\bar{\beta} = \lambda - \mu$), $\alpha > \beta$, $\frac{b}{a} < \frac{1}{\bar{\beta}}$, and consider the integral

$$I(z) = \int_{-\infty}^z e^{(\alpha-\beta)(x-z)} (a + be^{(\bar{\beta}-1)x})^\alpha dx .$$

Then

$$I(z) \leq \frac{(a + be^{(\bar{\beta}-1)z})^\alpha}{(\alpha - \beta)(1 - \beta\epsilon)} , \quad \text{where } \epsilon = \frac{b}{a}.$$

Proof:

Using the same notation and technique as in the proof of Lemma 4.3.3 with $y = e^{(\bar{\beta}-1)x}$, $\theta = \frac{\alpha-\beta}{\bar{\beta}-1}$, $H = e^{(\bar{\beta}-1)z}$, we obtain

$$I(z) \leq \frac{e^{-(\alpha-\beta)z}}{\bar{\beta}-1} \frac{H^\theta X^\alpha}{\theta} \left\{ 1 + \sum_{k=1}^{\alpha} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(\theta+1)(\theta+2)\cdots(\theta+k)} \left(\frac{bH}{X} \right)^k \right\} .$$

The difference from the proof of Lemma 4.3.3 is in the estimation of the quantity $\frac{\alpha-k+1}{\theta+k}$ for $1 \leq k \leq \alpha$. It is clear that $\frac{\alpha-k+1}{\theta+k} \leq \frac{\alpha}{\theta+1}$. With some manipulation, we obtain

$$\begin{aligned} \frac{\alpha}{\theta+1} &= \frac{\alpha}{\alpha-(1+2\mu)}(\bar{\beta}-1) \\ &= \left(1 + \frac{1+2\mu}{\alpha-(1+2\mu)} \right) (\bar{\beta}-1) . \end{aligned}$$

Now since $\alpha > \beta = \lambda + \mu \Rightarrow \alpha - (1 + 2\mu) \geq \lambda + \mu - 1 - 2\mu = \bar{\beta} - 1$, we have, for $1 \leq k \leq \alpha$,

$$\frac{\alpha-k+1}{\theta+1} \leq 1 + 2\mu + \bar{\beta} - 1 = \bar{\beta} + 2\mu = \beta .$$

Hence

$$I(z) \leq \frac{e^{-(\alpha-\beta)z}}{\bar{\beta}-1} \frac{H^\theta X^\alpha}{\theta} \sum_{k=0}^{\alpha} \left(\frac{\beta bH}{X} \right)^k .$$

Now for $z < 0$, $\frac{bH}{X} = \frac{bH}{a+bH} \leq \frac{bH}{a} = \frac{b}{a} e^{(\bar{\beta}-1)z} < \frac{b}{a} = \epsilon$. With these estimates we get the result

$$I(z) \leq \frac{(a + be^{(\bar{\beta}-1)z})^\alpha}{(\alpha - \beta)(1 - \beta\epsilon)}$$

and the proof is complete. $\square$

Lemma 4.3.5 Let z be either positive or negative and let $P_\alpha(z)$ be a sequence of polynomial functions of degree $n_\alpha = [\frac{\alpha}{q}]$, q a positive integer, i.e.,

$$P_\alpha(z) = \sum_{m=0}^{n_\alpha} c_m^{(\alpha)} z^m.$$

Given an integer $N > q$, $0 < M \leq 1$ and $p > 0$, there exists a real constant K , $\text{sign}(K) = \text{sign}(z)$, such that

$$|P_\alpha(z)| \leq M \frac{(p|K| + Kz)^{\frac{\alpha}{q}}}{\sqrt{\alpha+1}}, \quad \text{for } 1 \leq \alpha \leq N-1. \quad (4.24)$$

Proof:

For $1 \leq \alpha < q$, P_α are constants ($P_\alpha(z) = c_0^{(\alpha)}$). Then (4.24) is satisfied with $|K| = K_1$, K_1 is given by :

$$K_1 = \max_{1 \leq \alpha \leq q-1} \left\{ \frac{1}{p} \left(\frac{|c_0^{(\alpha)}| \sqrt{\alpha+1}}{M} \right)^{q/\alpha} \right\}.$$

For $\alpha \geq q$, we have

$$|P_\alpha(z)| \leq \sum_{m=0}^{n_\alpha} |c_m^{(\alpha)}| |z|^m.$$

Since for all $\alpha \geq q$, $(p|K_2| + K_2 z)^{\frac{\alpha}{q}} \geq (p|K_2| + K_2 z)^{n_\alpha}$ ($p|K_2| \geq 1$), it suffices to find K_2 such that

$$\sum_{m=0}^{n_\alpha} |c_m^{(\alpha)}| |z|^m \leq M \frac{(p|K_2| + K_2 z)^{n_\alpha}}{\sqrt{\alpha+1}}.$$

From the Binomial theorem, we have

$$M \frac{(p|K_2| + K_2 z)^{n_\alpha}}{\sqrt{\alpha+1}} = \frac{M}{\sqrt{\alpha+1}} \sum_{m=0}^{n_\alpha} \binom{n_\alpha}{m} p^{n_\alpha-m} |K_2|^{n_\alpha} |z|^m.$$

where $\binom{n_\alpha}{m} = \frac{n_\alpha!}{m!(n_\alpha-m)!}$. Hence it suffices to choose K_2 such that

$$p^{n_\alpha-m} |K_2|^{n_\alpha} \frac{M}{\sqrt{\alpha+1}} \binom{n_\alpha}{m} \geq |c_m^{(\alpha)}|, \quad \text{for } 0 \leq m \leq n_\alpha, \quad q \leq \alpha \leq N-1.$$

Thus choosing K_2 as

$$|K_2| = \max_{\substack{0 \leq m \leq n_\alpha \\ q \leq \alpha \leq N-1}} \left\{ \left(\frac{\sqrt{\alpha+1}}{M p^{n_\alpha-m} \binom{n_\alpha}{m}} |c_m^{(\alpha)}| \right)^{1/n_\alpha}, \frac{1}{p} \right\}, \quad (4.25)$$

(4.24) is satisfied for $q \leq \alpha \leq N-1$, and choosing K as

$$|K| = \max \{K_1, |K_2|\}$$

proves the lemma. $\square$

4.4 The Convergence Proofs

In Section 4.1, a general scheme was given for the proof of convergence of psi-series. The technicalities of the proof and, in particular, the boundedness of the f_α in the single series, depend upon the ODE under consideration. In this section, we shall use the examples from Chapter 3 to prove convergence for their psi-series solutions and to estimate their radii of convergence.

4.4.1 The Second-Order Case

CASE 1: Logarithmic psi-series

Let us consider the Korteweg-de Vries-Burgers (KdV-B) equation, i.e.,

$$u_t + uu_x + \gamma u_{xx} + u_{xxx} = 0, \quad (4.26)$$

where γ is a nonzero constant. We have seen, in Example 1.2.2, that seeking wave solutions transforms (4.26) into the second-order ODE

$$u'' + \gamma u' + \frac{1}{2}u^2 - cu = A, \quad (4.27)$$

where differentiation is with respect to the new variable $\xi = x - ct$. c is the wave speed and A is a constant of integration. The Laurent series analysis of (4.27) revealed that the leading order is $p = -2$ and the resonance is $r = 6$, which was found to be incompatible if the constant of integration A satisfies

$$A \neq [18(\frac{2}{5})^4 - \frac{1}{2}c^2].$$

For such A , (4.27) does not possess the Painlevé property and the logarithmic psi-series

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-2} (\tau^6 \log \tau)^l, \quad (4.28)$$

where $\tau = \xi - \xi_0$ and ξ_0 is the arbitrary singularity, was shown to contain $a_{6,0}$ as a free constant, in addition to ξ_0 , and hence to formally constitute a general solution.

Without loss of generality, set $A = 0$ and require the compatibility condition to fail.

We now proceed to prove the convergence of the logarithmic psi-series (4.28).

From Section 4.2.1 (resummation of psi-series), (4.28) converts into the single series

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha-2}, \quad (4.29)$$

where $z = \log \tau$ and the functions $f_{\alpha}(z)$ are given by

$$f_{\alpha}(z) = \sum_{n=0}^{[\frac{\alpha}{6}]} a_n^{(\alpha)} z^n, \quad (4.30)$$

polynomials in z of degree $[\frac{\alpha}{6}]$. The second step in the proof is to express the second-order ODE (4.27) as a system of first order ODEs.

Let $v = u' \Rightarrow v' = u'' = -\gamma u' + cu - \frac{1}{2}u^2 = -\gamma v + cu - \frac{1}{2}u^2$. Then the system is

$$(S1) = \begin{cases} u' = v \\ v' = cu - \frac{1}{2}u^2 - \gamma v. \end{cases}$$

From (4.29), we have

$$u' = v = \sum_{\alpha=0}^{\infty} \{f'_{\alpha} + (\alpha - 2)f_{\alpha}\} \tau^{\alpha-3} \equiv \sum_{\alpha=0}^{\infty} g_{\alpha} \tau^{\alpha-3}, \quad g_{\alpha} = f'_{\alpha} + (\alpha - 2)f_{\alpha},$$

$$u'' = v' = \sum_{\alpha=0}^{\infty} \{g'_{\alpha} + (\alpha - 3)g_{\alpha}\} \tau^{\alpha-4},$$

$$u^2 = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} f_{\beta} f_{\alpha-\beta} \tau^{\alpha-4},$$

which upon substitution into the system (S1), give

$$\begin{cases} \sum_{\alpha=0}^{\infty} \{f'_{\alpha} + (\alpha - 2)f_{\alpha}\} \tau^{\alpha-3} = \sum_{\alpha=0}^{\infty} g_{\alpha} \tau^{\alpha-3} \\ \sum_{\alpha=0}^{\infty} \{g'_{\alpha} + (\alpha - 3)g_{\alpha}\} \tau^{\alpha-4} = \sum_{\alpha=0}^{\infty} c f_{\alpha} \tau^{\alpha-2} - \frac{1}{2} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} f_{\beta} f_{\alpha-\beta} \tau^{\alpha-4} - \gamma \sum_{\alpha=0}^{\infty} g_{\alpha} \tau^{\alpha-3}. \end{cases}$$

Let $\alpha \rightarrow \alpha - 2$ in the first sum and $\alpha \rightarrow \alpha - 1$ in the last sum on the right-hand side of the second equation, and require the coefficients of $\tau^{\alpha-3}$ in the first equation

and $\tau^{\alpha-4}$ in the second to vanish to obtain, $\forall \alpha \geq 0$, the system

$$(S2) = \begin{cases} f'_\alpha + (\alpha - 2)f_\alpha = g_\alpha \\ g'_\alpha + (\alpha - 3)g_\alpha = cf_{\alpha-2} - \frac{1}{2} \sum_{\beta=0}^{\alpha} f_\beta f_{\alpha-\beta} - \gamma g_{\alpha-1} . \end{cases}$$

Since both f_α and $g_\alpha = f'_\alpha + (\alpha - 2)f_\alpha$ are polynomials of degree $[\frac{\alpha}{6}]$, $f'_\alpha = g'_\alpha = 0$ for $0 \leq \alpha \leq 5$. Solving the system (S2) for $0 \leq \alpha \leq 5$, we obtain :

$$\begin{aligned} \alpha = 0 & \quad f_0 = -12 \\ \alpha = 1 & \quad f_1 = 12\frac{7}{5} \\ \alpha = 2 & \quad f_2 = c + (\frac{7}{5})^2 \\ \alpha = 3 & \quad f_3 = (\frac{7}{5})^3 \\ \alpha = 4 & \quad f_4 = \frac{35}{20}(\frac{7}{5})^4 - \frac{1}{20}c^2 \\ \alpha = 5 & \quad f_5 = \frac{395}{60}(\frac{7}{5})^5 - \frac{11}{60}(\frac{22}{5})c^2 . \end{aligned}$$

Note that these are exactly the values of a_k for $0 \leq k \leq 5$ in the Laurent series and also the values of $a_{k,0}$ for $0 \leq k \leq 5$ in the psi-series. This is because the logarithmic terms first enter the series at $k = 6$ (the resonance).

The sum on the right-hand side of the second equation of (S2) may be expressed as

$$\sum_{\beta=0}^{\alpha} f_\beta f_{\alpha-\beta} = 2f_0 f_\alpha + \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} = -24f_\alpha + \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} ,$$

which upon substitution into (S2), gives, $\forall \alpha \geq 1$,

$$\begin{cases} f'_\alpha + (\alpha - 2)f_\alpha - g_\alpha = 0 \\ g'_\alpha - 12f_\alpha + (\alpha - 3)g_\alpha = cf_{\alpha-2} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} - \gamma g_{\alpha-1} \end{cases}$$

or, in matrix form,

$$\begin{pmatrix} f_\alpha \\ g_\alpha \end{pmatrix}' + \begin{pmatrix} \alpha - 2 & -1 \\ -12 & \alpha - 3 \end{pmatrix} \begin{pmatrix} f_\alpha \\ g_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ F_\alpha \end{pmatrix} .$$

where

$$F_\alpha = cf_{\alpha-2} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} - \gamma g_{\alpha-1} . \quad (4.31)$$

In vector notation, we have

$$\tilde{f}_\alpha' + A_\alpha \tilde{f}_\alpha = \tilde{F}_\alpha , \quad (4.32)$$

where

$$\tilde{f}_\alpha' = \begin{pmatrix} f_\alpha' \\ g_\alpha' \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} \alpha-2 & -1 \\ -12 & \alpha-3 \end{pmatrix}, \quad \tilde{F}_\alpha = \begin{pmatrix} 0 \\ F_\alpha \end{pmatrix} .$$

The eigenvalues λ_1, λ_2 of A_α are easily found by solving

$$\begin{vmatrix} \alpha-2-\lambda & -1 \\ -12 & \alpha-3-\lambda \end{vmatrix} = 0 .$$

They are $\lambda_1 = \alpha + 1$ and $\lambda_2 = \alpha - 6$. Note that the eigenvalues are precisely $\alpha - r$, where r is a resonance (-1 or 6). The corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} .$$

Hence A_α can be diagonalized as

$$A_\alpha = PD_\alpha P^{-1} \quad (4.33)$$

and $e^{A_\alpha(x-z)}$ is written as

$$e^{A_\alpha(x-z)} = P e^{D_\alpha(x-z)} P^{-1} , \quad (4.34)$$

where

$$P = \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} \alpha+1 & 0 \\ 0 & \alpha-6 \end{pmatrix}, \quad P^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} .$$

Since the eigenvalues of A_α are positive for $\alpha > 6$, the solution to the vector equation (4.32) is

$$\tilde{f}_\alpha(z) = \int_{-\infty}^z e^{A_\alpha(x-z)} \tilde{F}_\alpha(x) dx \quad (4.35)$$

⇒

$$\|\tilde{f}_\alpha(z)\| \leq \int_{-\infty}^z \|e^{A_\alpha(x-z)}\| \|\tilde{F}_\alpha(x)\| dx. \quad (4.36)$$

For $x \leq z$ and $\alpha > 6$, we have the following :

$$\begin{aligned} \|\tilde{F}_\alpha\| &= |F_\alpha| \\ \|e^{A_\alpha(x-z)}\| &\leq \|P\| \|e^{D_\alpha(x-z)}\| \|P^{-1}\| \\ &= 5 \left\| \begin{pmatrix} e^{(\alpha+1)(x-z)} & 0 \\ 0 & e^{(\alpha-6)(x-z)} \end{pmatrix} \right\| \\ &= 5e^{(\alpha-6)(x-z)}. \end{aligned}$$

Thus (4.36) becomes

$$\|\tilde{f}_\alpha(z)\| \leq 5 \int_{-\infty}^z e^{(\alpha-6)(x-z)} |F_\alpha(x)| dx. \quad (4.37)$$

Theorem 4.4.1 *The exists $K > 0$ such that $\forall z < 0$,*

$$\|\tilde{f}_\alpha(z)\| \leq \frac{(2K - Kz)^{\alpha/6}}{\sqrt{\alpha+1}} \quad \forall \alpha \geq 1. \quad (4.38)$$

Proof:

The idea is to estimate $|F_\alpha(x)|$ in (4.37) and then to make use of Lemma 4.3.2.

Since f_α and g_α are polynomials in z of degree $[\frac{\alpha}{6}]$, by Lemma 4.3.5 ($M = 1$, $q = 6$, $n_\alpha = [\frac{\alpha}{6}]$, $p=2$), given $N \geq 7$, there exist two real positive constants K_f , K_g , such that

$$|f_\alpha(z)| \leq \frac{(2K_f - K_f z)^{\alpha/6}}{\sqrt{\alpha+1}}, \quad |g_\alpha(z)| \leq \frac{(2K_g - K_g z)^{\alpha/6}}{\sqrt{\alpha+1}}, \quad \text{for } 1 \leq \alpha \leq N-1,$$

where K_f and K_g are given by equation (4.25) with the $c_m^{(\alpha)}$ being the coefficients of f_α and g_α , respectively.

Let $K = \max[K_f, K_g]$. Since $\|\tilde{f}_\alpha(z)\| = \max[|f_\alpha(z)|, |g_\alpha(z)|]$, (4.38) holds for $1 \leq \alpha \leq N-1$. Let this be our induction hypothesis and let us prove that the result holds for $\alpha = N$, and hence for any $\alpha \geq 1$.

Recall (by (4.31)) that

$$F_\alpha = cf_{\alpha-2} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} - \gamma g_{\alpha-1}$$

$\Rightarrow$

$$\begin{aligned} |F_N(z)| &\leq |c||f_{N-2}| + \frac{1}{2} \sum_{\beta=1}^{N-1} |f_\beta||f_{N-\beta}| + |\gamma||g_{N-1}| \\ &\leq |c| \frac{(2K - Kz)^{\frac{N-2}{6}}}{\sqrt{N-1}} + |\gamma| \frac{(2K - Kz)^{\frac{N-1}{6}}}{\sqrt{N}} \\ &\quad + \frac{1}{2} (2K - Kz)^{\frac{N}{6}} \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{N-\beta+1}} \\ &\leq (2K - Kz)^{N/6} \left\{ \frac{|c|}{\sqrt{N-1}} + \frac{|\gamma|}{\sqrt{N}} + \frac{1}{2} \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{N-\beta+1}} \right\}. \end{aligned}$$

By Lemma 4.3.1, we have

$$\sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{N-\beta+1}} \leq \sum_{\beta=0}^{N-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{N-\beta}} \leq \pi$$

$\Rightarrow$

$$|F_N(z)| \leq \left\{ \frac{|c|}{\sqrt{N-1}} + \frac{|\gamma|}{\sqrt{N}} + \frac{\pi}{2} \right\} (2K - Kz)^{\frac{N}{6}}.$$

Substituting this estimate into (4.37), we get

$$\|\tilde{f}_N(z)\| \leq 5 \left\{ \frac{|c|}{\sqrt{N-1}} + \frac{|\gamma|}{\sqrt{N}} + \frac{\pi}{2} \right\} \int_{-\infty}^z e^{(N-6)(x-z)} (2K - Kz)^{\frac{N}{6}} dx.$$

By Lemma 4.3.2 ($a = 2K$, $b = -K$, $q = 6$, $\varepsilon = \frac{1}{2}$), we have

$$\int_{-\infty}^z e^{(N-6)(x-z)} (2K - Kz)^{\frac{N}{6}} dx \leq \frac{1+11\delta}{11N\delta^2} (2K - Kz)^{N/6}$$

$\Rightarrow$

$$\|\tilde{f}_N(z)\| \leq 5 \left\{ \frac{|c|}{\sqrt{N-1}} + \frac{|\gamma|}{\sqrt{N}} + \frac{\pi}{2} \right\} \left(\frac{1+11\delta}{11N\delta^2} \right) (2K - Kz)^{N/6}.$$

We want (4.38) to hold for $\alpha = N$. This is so iff N satisfies

$$5 \frac{\sqrt{N+1}}{N} \left\{ \frac{|c|}{\sqrt{N-1}} + \frac{|\gamma|}{\sqrt{N}} + \frac{\pi}{2} \right\} \leq \frac{11\delta^2}{1+11\delta}, \quad (4.39)$$

where $0 < \delta < 1$. Also, we require that $N \geq \frac{6}{1-\delta}$. We refer to equation (4.39) as the *induction condition* (i.e., the condition on N so that the induction proof works). Therefore, if we choose N sufficiently large so that for any larger N the induction condition is satisfied, we have that (4.38) holds $\forall \alpha \geq 1$. Having found such an N , K_f and K_g can easily be found using (4.25), hence K can be found, and the theorem is proved.

Theorem 4.4.2 Consider the logarithmic psi-series solution (4.28). We claim that $\exists R > 0$ such that $\forall \tau$, $0 < \tau < R$, the series converges absolutely.

Proof:

By (4.29), we have

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha-2}.$$

Now since $\forall \alpha \geq 1$, $|f_{\alpha}(z)| \leq \|\tilde{f}_{\alpha}(z)\|$ and $\|f_{\alpha}(z)\|$ satisfies

$$\|f_{\alpha}(z)\| \leq \frac{(2K - Kz)^{\alpha/6}}{\sqrt{\alpha+1}},$$

we have

$$\sum_{\alpha=1}^{\infty} |f_{\alpha}(z) \tau^{\alpha-2}| \leq \sum_{\alpha=1}^{\infty} \frac{(2K - Kz)^{\alpha/6}}{\sqrt{\alpha+1}} \tau^{\alpha-2}.$$

Using the ratio test, the series on the right converges if

$$(2K - Kz)^{1/6} \tau < 1 \quad \text{or} \quad (2K - Kz) < \epsilon^{-6z}. \quad (4.40)$$

It is clear that there exists a unique $z_0 < 0$ such that $\forall z < z_0$, $(2K - Kz) < \epsilon^{-6z}$. This is shown in Fig. 4.2. Hence by the comparison test we have that the series (4.29) converges absolutely for $0 < \tau = \xi - \xi_0 < \epsilon^{\tau_0}$. This proves that the logarithmic

psi-series solution of the KdV-B equation has a positive radius of convergence R . From Fig. 4.2, R is at least e^{z_0} , where z_0 is the unique negative solution of the equation

$$(2K - Kz) = e^{-6z} ,$$

and the proof is complete. $\square$

CASE 2: Nonlogarithmic psi-series

We have seen in the case of a noninteger resonance, say r , that a nonlogarithmic psi-series contains the required number of free constants, and hence forms a general solution. It remains to prove that such a series has a positive radius of convergence. Let us consider the ODE

$$u'' + 4uu' + u^3 = 0 \quad (4.41)$$

(see Example 3.1.3). The leading order is $p = -1$ and the positive resonance is $\beta = 4\sqrt{2} - 4$, which is irrational, and the psi-series solution of (4.41) takes the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+\beta l-1} . \quad (4.42)$$

From Section 4.2.2, the nonlogarithmic psi-series may be expressed as

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha-1} , \quad (4.43)$$

where the $f_{\alpha}(z)$ are given by

$$f_{\alpha}(z) = \sum_{n=0}^{\alpha} a_n^{(\alpha)} e^{(\beta-1)nz} ,$$

and are polynomials in $e^{(\beta-1)z}$ of degree α , in contrast to the logarithmic psi-series (polynomials in z of degree $\lfloor \frac{\alpha}{r} \rfloor$). To prove convergence of the nonlogarithmic psi-series in (4.42), we proceed in the same way as for the logarithmic psi-series, up to (4.37). The vector solutions $\tilde{f}_{\alpha}(z)$ in the present case are dominated in norm as follows :

$$\|\tilde{f}_{\alpha}(z)\| \leq C \int_{-\infty}^z e^{(\alpha-\beta)(x-z)} |F_{\alpha}(x)| dx , \quad \forall \alpha > \beta , \quad (4.44)$$

where

$$C = \|P\| \|P^{-1}\| = 3 \frac{\beta + 3}{\beta + 1},$$

$$F_\alpha(z) = -4 \sum_{\gamma=1}^{\alpha-1} f_\gamma g_{\alpha-\gamma} - \sum_{\gamma=1}^{\alpha-1} f_0 f_\gamma f_{\alpha-\gamma} - \sum_{\gamma=1}^{\alpha-1} h_\gamma f_{\alpha-\gamma}, \quad (4.45)$$

$$h_\gamma = \sum_{\nu=0}^{\gamma} f_\nu f_{\alpha-\nu}, \quad (4.46)$$

$$g_\alpha = f'_\alpha + (\alpha - 1)f_\alpha,$$

$$f_0 = 2 - \sqrt{2}, \quad g_0 = -f_0 = \sqrt{2} - 2.$$

Theorem 4.4.3 *There exist $K > 0$ and M , $0 < M < 1$, such that $\forall z < 0$,*

$$\|\tilde{f}_\alpha(z)\| \leq M \frac{(3K + Ke^{(\beta-1)z})^\alpha}{\sqrt{\alpha+1}}, \quad \forall \alpha \geq 1. \quad (4.47)$$

Proof :

The proof is the same as the one for Theorem 4.4.1. We begin by noting that the function coefficients in (4.43) are polynomials in $e^{(\beta-1)z}$ of degree α , and since $g_\alpha(z) = f'_\alpha(z) + (\alpha - 1)f_\alpha(z)$, so are the $g_\alpha(z)$. Now given $N > \beta$, by Lemma 4.3.5 ($q = 1$, $n_\alpha = \alpha$, $q = 1$, $p = 3$) there exist 2 positive constants K_f and K_g such that

$$|f_\alpha(z)| \leq M \frac{(3K_f + K_f e^{(\beta-1)z})^\alpha}{\sqrt{\alpha+1}}, \quad |g_\alpha(z)| \leq M \frac{(3K_g + K_g e^{(\beta-1)z})^\alpha}{\sqrt{\alpha+1}}, \quad \text{for } 1 \leq \alpha \leq N-1.$$

Let $K = \max[K_f, K_g]$, then the theorem holds for $1 \leq \alpha \leq N-1$. Let this be our induction hypothesis and let us prove that, with the same K , the theorem holds for $\alpha = N$, and hence $\forall \alpha \geq 1$.

We begin by estimating $|F_N|$. By (4.46) we have

$$\begin{aligned} |h_\gamma| &\leq 2f_0|f_\gamma| + \sum_{\nu=1}^{\gamma-1} |f_\nu||f_{\gamma-\nu}| \\ &\leq 2f_0|f_\gamma| + M^2(3K + Ke^{(\beta-1)z})^\gamma \sum_{\nu=1}^{\gamma-1} \frac{1}{\sqrt{\nu+1}} \frac{1}{\sqrt{\gamma-\nu+1}} \end{aligned}$$

$$\begin{aligned}
&\leq 2Mf_0 \frac{(3K + Ke^{(\beta-1)z})^\gamma}{\sqrt{\gamma+1}} + \pi M^2(3K + Ke^{(\beta-1)z})^\gamma \\
&\leq M(f_0\sqrt{2} + M\pi)(3K + Ke^{(\beta-1)z})^\gamma.
\end{aligned}$$

which upon substitution into (4.45), gives the estimate for $|F_N|$,

$$|F_N| \leq M^2(3K + Ke^{(\beta-1)z})^N \left(4 + f_0 + f_0\sqrt{2} + M\pi + \sum_{\gamma=1}^{N-1} \frac{1}{\sqrt{N-\gamma+1}} \right).$$

The sum on the right-hand side of the above is estimated by

$$\sum_{\gamma=1}^{N-1} \frac{1}{\sqrt{N-\gamma+1}} \leq \int_1^N \frac{1}{\sqrt{\nu}} d\nu = 2(\sqrt{N} - 1)$$

to give

$$|F_N| \leq M^2(\sqrt{N} - 1)(8 + 2M\pi)(3K + Ke^{(\beta-1)z})^N.$$

With this estimate for $|F_N|$, (4.44) becomes

$$\|\tilde{f}_N(z)\| \leq M^2(\sqrt{N} - 1)(8 + 2M\pi)C \int_{-\infty}^z e^{(N-\beta)(x-z)} (3K + Ke^{(\beta-1)z})^N dx.$$

Now by Lemma 4.3.3 ($\alpha = N$, $\lambda = \beta \simeq 1.54$, $a = 3K$, $b = K$, $\varepsilon = \frac{1}{3} < \frac{1}{2(\beta-1)} \simeq 0.78$), we have

$$\int_{-\infty}^z e^{(N-\beta)(x-z)} (3K + Ke^{(\beta-1)x})^N dx \leq 3 \frac{(3K + Ke^{(\beta-1)z})^N}{(N-\beta)(5-2\beta)}.$$

This gives

$$\|\tilde{f}_N(z)\| \leq \frac{3M^2(\sqrt{N} - 1)(8 + 2M\pi)C}{(N-\beta)(5-2\beta)} (3K + Ke^{(\beta-1)z})^N.$$

Therefore, the theorem holds for $\alpha = N$ iff the induction condition

$$\frac{3M(8 + 2M\pi)C}{5-2\beta} \frac{(\sqrt{N} - 1)(\sqrt{N+1})}{N-\beta} \leq 1$$

is satisfied. As $N \rightarrow \infty$, the left-hand side of the above inequality approaches $\frac{3M(8+2M\pi)C}{5-2\beta}$. The induction condition is satisfied for some N by choosing the parameter M suitably. Hence the theorem holds for $\alpha = N$, and hence for all $\alpha \geq 1$, and the theorem is proved.

Theorem 4.4.4 Consider the nonlogarithmic psi-series solution (4.42). We claim that $\exists R > 0$ such that $\forall \tau$, $0 < \tau < R$, the series converges absolutely.

Proof

By (4.43), we have

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha-1},$$

and since $|f_{\alpha}(z)| \leq \|\tilde{f}_{\alpha}(z)\| \forall \alpha \geq 1$, in view of Theorem 4.4.3, we have

$$\sum_{\alpha=0}^{\infty} |f_{\alpha}(z) \tau^{\alpha-1}| \leq \sum_{\alpha=1}^{\infty} M \frac{(3K + Ke^{(\beta-1)z})^{\alpha}}{\sqrt{\alpha+1}} \tau^{\alpha-1}, \quad (4.48)$$

and by the ratio test, the series on the right converges if

$$(3K + Ke^{(\beta-1)z})|\tau| < 1, \text{ or } (3K + Ke^{(\beta-1)z}) < e^{-z}. \quad (4.49)$$

This is the condition for the existence of a positive radius of convergence. Equality holds in (4.49) for a unique z_0 , and $\forall z < z_0$ we have $(3K + Ke^{(\beta-1)z}) < e^{-z}$ (see Fig. 4.3). This shows that the series converges absolutely $\forall z < z_0$, i.e., $\forall t$ such that $0 < \tau = t - t_0 < e^{z_0}$. Therefore the nonlogarithmic psi-series has a positive radius of convergence $R \geq e^{z_0}$.

4.4.2 The Third-Order Case

For third-order equations, we have considered only those cases where the two positive resonances are either both integers (at least one incompatible) or conjugates (complex or irrational). The integer-resonance case required a logarithmic psi-series whose convergence proof is the same as the one for the second-order case. Actually, even in the complex or irrational conjugate resonance case, the proof is still the same. The only difference is in the forms of the f_α , the coefficients of the single series obtained by the resummation of the psi-series. Therefore, it suffices to prove that the f_α are dominated in absolute value by some polynomial functions.

Let us begin with the case of complex conjugate resonances. Let $r = \lambda + i\mu$, $\bar{r} = \lambda - i\mu$ be the two resonances. The psi-series solution is of the form

$$u = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+p} \tau^{rl} + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k+p} \tau^{\bar{r}l}, \quad (4.50)$$

where the $\bar{a}_{kl}$ are the complex conjugates of the a_{kl} , and p is the leading order. This series solution is just the sum of two nonlogarithmic psi-series. From Section 4.2.2, the series (4.50) resums to the single series

$$u = \sum_{\alpha=0}^{\infty} f_\alpha(z) \tau^{\alpha+p}, \quad (4.51)$$

where $f_\alpha(z)$ have the form

$$f_\alpha(z) = \sum_{n=0}^{\alpha} \left(a_n^{(\alpha)} \cos(n\mu z) + b_n^{(\alpha)} \sin(n\mu z) \right) e^{n(\lambda-1)z},$$

$a_n^{(\alpha)} = 2\operatorname{Re}(a_{\alpha-n,n})$ and $b_n^{(\alpha)} = -2\operatorname{Im}(a_{\alpha-n,n})$. Since $|\cos(n\mu z)| \leq 1$ and $|\sin(n\mu z)| \leq 1$, we have

$$|f_\alpha(z)| \leq \sum_{n=0}^{\alpha} C_n^{(\alpha)} e^{n(\lambda-1)z}, \quad (4.52)$$

where $C_n^{(\alpha)} = |a_n^{(\alpha)}| + |b_n^{(\alpha)}|$. The superscripts (α) simply indicate dependence upon the f_α . Therefore $|f_\alpha(z)|$ is dominated by some polynomial in $e^{(\lambda-1)z}$. The rest of

the proof follows along the lines of the proof for the second-order case, using Lemma 4.3.3.

In the case of irrational conjugate resonances, the psi-series solution has the form (4.50). Let the resonances be $r = \beta = \lambda + \mu$ and $\bar{r} = \bar{\beta} = \lambda - \mu$, $\mu > 0$. The coefficients a_{kl} , $\bar{a}_{kl}$ are defined by $a_{kl} = a_{kl}^r + a_{kl}^i$, $\bar{a}_{kl} = a_{kl}^r - a_{kl}^i$.

The psi-series solution in this case was shown in Section 4.2.2 to resum to the single series (4.51) with the function coefficients $f_\alpha(z)$ having the form

$$f_\alpha(z) = \sum_{n=0}^{\alpha} (a_n \cosh(n\mu z) + b_n \sinh(n\mu z)) e^{n(\lambda-1)z}, \quad (4.53)$$

where we have dropped the superscript (α). We shall now show that these functions are dominated in absolute value by certain polynomials in $e^{(\bar{\beta}-1)z}$. Since

$$|f_\alpha(z)| \leq \sum_{n=0}^{\alpha} (|a_n| |\cosh(n\mu z)| + |b_n| |\sinh(n\mu z)|) e^{n(\lambda-1)z},$$

let $C_n = \max[|a_n|, |b_n|]$ to obtain

$$|f_\alpha(z)| \leq \sum_{n=0}^{\alpha} C_n (|\cosh(n\mu z)| + |\sinh(n\mu z)|) e^{n(\lambda-1)z}.$$

Using

$$\cosh(n\mu z) = \frac{e^{n\mu z} + e^{-n\mu z}}{2}, \quad \sinh(n\mu z) = \frac{e^{n\mu z} - e^{-n\mu z}}{2},$$

we have, for $z < 0$,

$$|\cosh(n\mu z)| + |\sinh(n\mu z)| \leq e^{n\mu z} + e^{-n\mu z} \leq 1 + e^{-n\mu z}.$$

Therefore, for $z < 0$, we have

$$\begin{aligned} |f_\alpha(z)| &\leq \sum_{n=0}^{\alpha} C_n (e^{n(\lambda-1)z} + e^{n(\lambda-\mu-1)z}) \\ &= \sum_{n=0}^{\alpha} C_n (e^{n(\lambda-1)z} + e^{n(\bar{\beta}-1)z}). \end{aligned}$$

Now $\tilde{\beta} = \lambda - \mu$ and $\mu > 0 \Rightarrow \lambda - 1 > \tilde{\beta} - 1 \Rightarrow e^{n(\lambda-1)z} + e^{n(\tilde{\beta}-1)z} < 2e^{n(\tilde{\beta}-1)z}$
 $\Rightarrow$

$$|f_\alpha(z)| \leq \sum_{n=0}^{\alpha} D_n e^{n(\tilde{\beta}-1)z}.$$

Therefore, as claimed, $|f_\alpha(z)|$ are dominated in absolute value by polynomials in $e^{(\tilde{\beta}-1)z}$ of degree α , where $D_n = 2C_n = 2\max[|a_n|, |b_n|]$. The rest of the proof follows along the lines of the proof for the second-order case, using Lemma 4.3.4.

4.5 Conclusion and Discussion

In this chapter we have proved that psi-series solutions of second- and third-order ODEs of type (2.3) have positive radii of convergence. All of the proofs are basically the same, with minor differences. These differences, of course, arise from the differences in the nature of the resonances. We have seen that all of the psi-series considered can be transformed into single series with function coefficients, whose forms depend upon the type of resonance. Regardless of the type, however, the coefficient functions are dominated in absolute value by polynomials, and this is the basis of the proof.

We have made two assumptions: (1) $\tau = t - t_0 > 0$ and (2) the real parts of the resonances are greater than 1. The failure of the first does not rule out convergence. If $\tau = t - t_0 < 0$, then one may define $\log \tau = \log |\tau| + \pi i$ and replace τ by $|\tau|$. A real solution may still be obtained by redefining the free constants in the series. In the event that assumption 2 is invalid, we believe that the method of proof as outlined in this thesis is still applicable, but with certain minor modifications.

Even for equations for which (1) and (2) hold, certain steps in the convergence proof may fail. For example, it is not clear that the "induction condition" (4.39) can be satisfied for a second-order equation containing a fourth-degree nonlinearity, say u^4 . Such issues, as well as the significance of negative resonances, will be investigated in the future.

Chapter 5

THE KURAMOTO-SIVASHINSKY EQUATION

In this chapter we shall be concerned with the fourth-order nonlinear partial differential equation (NLPDE)

$$v_t + vv_x + \gamma v_{xx} + v_{xxxx} = 0 . \quad (5.1)$$

where γ is a nonzero constant. Equation (5.1) is known as the Kuramoto-Sivashinsky equation. It is one member of a family of wave equations that govern the nonlinear evolution of long waves on thin films, in various circumstances [8, 13]. It arises in certain chemical engineering problems as well, for example, the modeling of the nonlinear evolution of diffusion flame instabilities [14], certain chemical oscillations [15] and the Rayleigh-Taylor instability in a top-heavy two-layered film [16]. For derivations of the wave equations and the conditions under which (5.1) holds, see [8].

In Section 1 of this chapter, we investigate the Painlevé property for (5.1) and its integrability. In Section 2, we apply the ordinary series method to construct analytic solutions for various initial conditions. Plots depicting the behaviour of the solutions are also presented.

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5.1 The Painlevé Property

Recall that an ordinary differential equation is said to have the Painlevé property if its general solution has no movable singularities other than poles.

The Painlevé property for PDEs was defined by Weiss et al. [17]. Their definition is analogous to the one for ODEs, but singular points are now replaced by singular manifolds. The corresponding test states that an n -th order PDE for a function $u(x_1, x_2, \dots, x_m)$ of m variables passes the Painlevé test if an expansion of the form

$$u = \sum_{k=0}^{\infty} a_k(x_1, x_2, \dots, x_m) \phi^{k+p} \quad (5.2)$$

about a singular manifold $\phi(x_1, x_2, \dots, x_m) = 0$ is valid; that is, it has the following two properties:

1. p is a negative integer and
2. there exist $n - 1$ compatible “resonances,” i.e., values k , of the indices for which the coefficient recursion relations have the form

$$0 \cdot a_k = F(a_0, a_1, \dots, a_{k-1}, \phi) \equiv 0, \quad \text{for } j = 1, \dots, n - 1, \quad (5.3)$$

where F is a function of its arguments and possibly their derivatives.

The Painlevé analysis of (5.1), using the above definition, has been conducted [8, 11], and it has been found that (5.1) does not possess the Painlevé property.

In the following, we shall investigate (5.1) by applying the Painlevé test described in Chapter 2 (the ARS algorithm) to a reduced form of (5.1). In particular, we consider traveling-wave solutions. The fourth-order NLPDE (5.1) reduces to the third-order ODE

$$u''' + \gamma u' + \frac{1}{2}u^2 - cu = A, \quad (5.4)$$

where A is a constant. Equation (5.4) is obtained by setting $\xi = x - ct$, c is a constant, $u(\xi) = v(x, t)$ and integrating once with respect to the new independent variable ξ .

We begin by seeking solutions to (5.4) in the form

$$u(\xi) = \sum_{k=0}^{\infty} a_k (\xi - \xi_0)^{k+p}, \quad (5.5)$$

where p is the leading order, assumed to be negative (see Chapter 2), and ξ_0 is an arbitrary singularity.

Substitution of

$$u = a_0 \tau^p$$

into (5.4) ($\tau = \xi - \xi_0$) gives

$$p(p-1)(p-2)a_0\tau^{p-3} + \gamma p a_0 \tau^{p-1} + \frac{1}{2}a_0^2\tau^{2p} - ca_0\tau^p = 0,$$

and when the most singular terms (τ^{p-3} and τ^{2p}) are balanced, we obtain

$$\begin{aligned} p-3 &= 2p, \\ p(p-1)(p-2)a_0 + \frac{1}{2}a_0^2 &= 0, \end{aligned}$$

which give the leading order $p = -3$ and the leading coefficient $a_0 = 120$. The series expansion (5.5) then becomes

$$u = \sum_{k=0}^{\infty} a_k \tau^{k-3}. \quad (5.6)$$

Next, we determine the resonances and derive the recursion relation. Differentiation of (5.6) and substitution into (5.4) give

$$\sum_{k=0}^{\infty} \left\{ (k-3)(k-4)(k-5)a_k \tau^{k-6} + \gamma(k-3)a_k \tau^{k-4} + \frac{1}{2} \sum_{m=0}^k a_m a_{k-m} \tau^{k-6} - ca_k \tau^{k-3} \right\} = 0.$$

Changing indices to make τ^{k-6} a common factor, defining $a_n = 0$ if $n < 0$, and requiring the coefficients to balance, we obtain, $\forall k \geq 0$,

$$(k-3)(k-4)(k-5)a_k = -\gamma(k-5)a_{k-2} - \frac{1}{2} \sum_{m=0}^k a_m a_{k-m} + ca_{k-3} + A\delta_{k,6}, \quad (5.7)$$

where $\delta_{k,6}$ is the Kronecher delta,

$$\delta_{k,6} = \begin{cases} 1 & \text{if } k = 6, \\ 0 & \text{otherwise.} \end{cases}$$

The sum in (5.7) is expressed as

$$\sum_{m=0}^k a_m a_{k-m} = 2a_0 a_k + \sum_{m=1}^{k-1} a_m a_{k-m} = 240a_k + \sum_{m=1}^{k-1} a_m a_{k-m},$$

and when substituted back into (5.7), leads to the recursion relation for a_k , $k \geq 1$:

$$(k+1)(k^2 - 13k + 60)a_k = -\gamma(k-5)a_{k-2} - \frac{1}{2} \sum_{m=1}^{k-1} a_m a_{k-m} + ca_{k-3} + A\delta_{k,6}. \quad (5.8)$$

The resonances are those values of k for which the coefficients of a_k in the recursion relation (5.8) vanish, and are given by

$$r_1 = -1, \\ r_{2,3} = \frac{13}{2} \pm i\frac{\sqrt{71}}{2}.$$

The presence of two complex conjugate resonances implies that the Laurent series expansion (5.6) does not contain the required number of free constants (recall that free constants in such an expansion may arise only when the resonances are positive integers), and hence cannot be a representation of the general solution of (5.4) near its movable singularities. Therefore, the third-order ODE (5.4) obtained from equation (5.1) (and hence equation (5.1) itself) fails the Painlevé test, and is termed nonintegrable.

In Chapter 3 we dealt with nonintegrable ODEs and their appropriate series expansions. In the present case (complex conjugate resonances), the psi-series takes the form

$$\psi(\xi) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-3} \tau^{l\bar{r}} + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{a}_{kl} \tau^{k-3} \tau^{l\bar{r}}, \quad (5.9)$$

where

$$r = \frac{13}{2} + i\frac{\sqrt{71}}{2} \equiv \lambda + i\mu, \\ \bar{r} = \frac{13}{2} - i\frac{\sqrt{71}}{2} \equiv \lambda - i\mu,$$

$\tau = \xi - \xi_0$ and ξ_0 is the movable singularity. The constants a_{kl} and $\bar{a}_{kl}$ are complex conjugates $\forall (k, l)$. It was shown that series of the form (5.9) contain a_{01} and $\bar{a}_{01}$ as free constants. It remains to show that such a series admits a positive radius of convergence. This is the content of the next section

5.2 Convergence of Psi-series (Complex Resonances)

The first step is to write the ODE (5.4) as a system of first order ODEs. Letting $u_1 = u$, $u_2 = u'_1$ and $u_3 = u'_2$ transforms (5.4) to the system

$$(S1) = \begin{cases} u'_1 = u_2 \\ u'_2 = u_3 \\ u'_3 = -\gamma u_2 - \frac{1}{2}u_1^2 + cu_1 + A . \end{cases}$$

According to the discussion in Chapter 4, the series (5.9) is equivalent to the single series

$$u = \sum_{\alpha=0}^{\infty} f_{\alpha}(z) \tau^{\alpha-3} , \quad (5.10)$$

where $z = \log \tau$ and the functions $f_{\alpha}(z)$ are given by

$$f_{\alpha}(z) = \sum_{n=0}^{\alpha} \left[a_n^{(\alpha)} \cos(n\mu z) + b_n^{(\alpha)} \sin(n\mu z) \right] e^{n(\lambda-1)z} . \quad (5.11)$$

From (5.10) we have the following :

$$\begin{aligned} u'_1 = u' &= \sum_{\alpha=0}^{\infty} [(\alpha-3)f_{\alpha} + f'_{\alpha}] \tau^{\alpha-4} \equiv \sum_{\alpha=0}^{\infty} g_{\alpha} \tau^{\alpha-4} = u_2 , \\ u'_2 = u'' &= \sum_{\alpha=0}^{\infty} [(\alpha-4)g_{\alpha} + g'_{\alpha}] \tau^{\alpha-5} \equiv \sum_{\alpha=0}^{\infty} h_{\alpha} \tau^{\alpha-5} = u_3 , \\ u'_3 = u''' &= \sum_{\alpha=0}^{\infty} [(\alpha-5)h_{\alpha} + h'_{\alpha}] \tau^{\alpha-6} . \end{aligned}$$

Substituting the latter equations into (S1), changing indices and rearranging terms, we get

$$(S2) = \begin{cases} f'_\alpha + (\alpha - 3)f_\alpha - g_\alpha = 0 \\ g'_\alpha + (\alpha - 4)g_\alpha - h_\alpha = 0 \\ h'_\alpha + (\alpha - 5)h_\alpha = cf_{\alpha-3} - \gamma g_{\alpha-2} - \frac{1}{2} \sum_{\beta=0}^{\alpha} f_\beta f_{\alpha-\beta} + A\delta_{\alpha,6} . \end{cases}$$

Solving (S2) with $\alpha = 0$ (noting that f_0 , g_0 and h_0 are constants) gives $f_0 = 120$, $g_0 = -360$ and $h_0 = 14400$. Using $f_0 = 120$, the system (S2) becomes, $\forall \alpha \geq 1$,

$$(S3) = \begin{cases} f'_\alpha + (\alpha - 3)f_\alpha - g_\alpha = 0 \\ g'_\alpha + (\alpha - 4)g_\alpha - h_\alpha = 0 \\ h'_\alpha + 120f_\alpha + (\alpha - 5)h_\alpha = cf_{\alpha-3} - \gamma g_{\alpha-2} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} + A\delta_{\alpha,6} \end{cases}$$

which, in matrix notation, is equivalent to

$$\begin{pmatrix} f_\alpha \\ g_\alpha \\ h_\alpha \end{pmatrix}' + \begin{pmatrix} \alpha - 3 & -1 & 0 \\ 0 & \alpha - 4 & -1 \\ 120 & 0 & \alpha - 5 \end{pmatrix} \begin{pmatrix} f_\alpha \\ g_\alpha \\ h_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F_\alpha \end{pmatrix} ,$$

i.e.,

$$\vec{f}'_\alpha + A_\alpha \vec{f}_\alpha = \vec{F}_\alpha , \quad (5.12)$$

where

$$\vec{f}_\alpha = \begin{pmatrix} f_\alpha \\ g_\alpha \\ h_\alpha \end{pmatrix} , \quad A_\alpha = \begin{pmatrix} \alpha - 3 & -1 & 0 \\ 0 & \alpha - 4 & -1 \\ 120 & 0 & \alpha - 5 \end{pmatrix} , \quad \vec{F}_\alpha = \begin{pmatrix} 0 \\ 0 \\ F_\alpha \end{pmatrix}$$

and

$$F_\alpha = cf_{\alpha-3} - \gamma g_{\alpha-2} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} f_\beta f_{\alpha-\beta} + A\delta_{\alpha,6} . \quad (5.13)$$

Equation (5.12) is a first-order nonhomogenous vector ODE whose solution, for $\alpha \geq 7$, is

$$\tilde{f}_\alpha(z) = \int_{-\infty}^z e^{A_\alpha(z-x)} \tilde{F}_\alpha(x) dx . \quad (5.14)$$

The idea of the proof, as outlined in Chapter 4, is to dominate the vector solutions, $\tilde{f}_\alpha(z)$, in norm. To this end, we begin by diagonalizing the matrix A_α . The eigenvalues (ϵ_i , $i = 1, 2, 3$) of A_α and their corresponding eigenvectors are easily found to be

$$\begin{aligned} \epsilon_1 &= \alpha + 1 , & \epsilon_2 &= \alpha - \left(\frac{13}{2} + i \frac{\sqrt{71}}{2} \right) , & \epsilon_3 &= \alpha - \left(\frac{13}{2} - i \frac{\sqrt{71}}{2} \right) , \\ \vec{v}_1 &= \begin{pmatrix} 1 \\ -4 \\ 20 \end{pmatrix} , & \vec{v}_2 &= \begin{pmatrix} 1 \\ \frac{7}{2} + i \frac{\sqrt{71}}{2} \\ -9 + i3\sqrt{71} \end{pmatrix} , & \vec{v}_3 &= \begin{pmatrix} 1 \\ \frac{7}{2} - i \frac{\sqrt{71}}{2} \\ -9 - i3\sqrt{71} \end{pmatrix} . \end{aligned}$$

Note that the eigenvalues are precisely $\alpha - r_i$, where r_i are the resonances ($r_1 = -1$, $r_2 = \frac{13}{2} + i \frac{\sqrt{71}}{2} \equiv \lambda + i\mu$ and $r_3 = \frac{13}{2} - i \frac{\sqrt{71}}{2} \equiv \lambda - i\mu$). Hence A_α is diagonalizable as

$$A_\alpha = P D_\alpha P^{-1} , \quad (5.15)$$

where

$$P = [\vec{v}_1, \vec{v}_2, \vec{v}_3] ,$$

$$P^{-1} = \frac{1}{37} \begin{pmatrix} 15 & -3 & \frac{1}{2} \\ 11 + i \frac{17}{71} \sqrt{71} & \frac{3}{2} - i \frac{29}{142} \sqrt{71} & -\frac{1}{4} - i \frac{15}{284} \sqrt{71} \\ 11 - i \frac{17}{71} \sqrt{71} & \frac{3}{2} + i \frac{29}{142} \sqrt{71} & -\frac{1}{4} + i \frac{15}{284} \sqrt{71} \end{pmatrix} ,$$

$$D_\alpha = \begin{pmatrix} \alpha + 1 & 0 & 0 \\ 0 & \alpha - (\lambda + i\mu) & 0 \\ 0 & 0 & \alpha - (\lambda - i\mu) \end{pmatrix} .$$

It is interesting to note that P (and hence also P^{-1}) is independent of α .

By the above results, we have

$$e^{A_\alpha(x-z)} = P e^{D_\alpha(x-z)} P^{-1},$$

$$e^{D_\alpha(x-z)} = \begin{pmatrix} e^{(\alpha+1)(x-z)} & 0 & 0 \\ 0 & e^{(x-z)(\alpha-\lambda-i\mu)} & 0 \\ 0 & 0 & e^{(x-z)(\alpha-\lambda+i\mu)} \end{pmatrix}.$$

We also have

$$\|P\| = 20+24\sqrt{5}, \|P^{-1}\| = 1/2 \text{ and } \|e^{D_\alpha(x-z)}\| = e^{(\alpha-\lambda)(x-z)}, \text{ since } \alpha \geq 7 > \lambda = \frac{13}{2},$$

which give, $\forall \alpha \geq 7$, the following estimate on $\|\vec{f}_\alpha\|$:

$$\begin{aligned} \|\vec{f}_\alpha(z)\| &\leq C \int_{-\infty}^z e^{(\alpha-\lambda)(x-z)} \|\vec{F}_\alpha(x)\| dx \\ &= C \int_{-\infty}^z e^{(\alpha-\lambda)(x-z)} |F_\alpha(x)| dx, \end{aligned} \quad (5.16)$$

where the constant $C = \|P\| \|P^{-1}\| = 10 + 12\sqrt{5}$.

We now state and prove a theorem regarding the convergence of the psi-series (5.9).

Theorem 5.2.1 *There exists a positive number R such that the psi-series (5.9) converges $\forall \xi$, $0 < \xi - \xi_0 < R$.*

Proof :

We begin by the following :

Claim : *There exists a positive number K such that*

$$\|f_\alpha(z)\| \leq \frac{(12K + K e^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}}, \quad \forall \alpha \geq 1, z < 0. \quad (5.17)$$

Proof of the claim: Recall from Chapter 4 that

$$|f_\alpha(z)| \leq \sum_{n=0}^{\alpha} |c_n^{(\alpha)}| (e^{(\lambda-1)z})^n.$$

where the $c_n^{(\alpha)}$ are constants. Since $g_\alpha(z)$ and $h_\alpha(z)$ are linear combinations of $f_\alpha(z)$ and its derivatives, both $|g_\alpha(z)|$ and $|h_\alpha(z)|$ are dominated in the same manner as $|f_\alpha(z)|$, and by Lemma 4.3.5 ($n_\alpha = \alpha$, $M = 1$, $p = 12$), given $N \geq 7$ there exist three positive numbers K_f , K_g , K_h such that for $1 \leq \alpha \leq N-1$,

$$\begin{aligned} |f_\alpha(z)| &\leq \frac{(12K_f + K_f e^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}}, \\ |g_\alpha(z)| &\leq \frac{(12K_g + K_g e^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}}, \\ |h_\alpha(z)| &\leq \frac{(12K_h + K_h e^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}}. \end{aligned}$$

Therefore, by the definition of $\|\tilde{f}_\alpha(z)\|$, we have

$$\|\tilde{f}_\alpha(z)\| \leq \frac{(12K + K e^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}}, \quad 1 \leq \alpha \leq N-1, \quad (5.18)$$

where $K = \max[K_f, K_g, K_h]$. Let (5.18) be our induction hypothesis and let us prove that, with the same K , the result is true for $\alpha = N$. To this end, we estimate $|F_N(z)|$. By (5.13) we have, for $N \geq 7 = [\lambda] + 1$,

$$\begin{aligned} |F_N(z)| &\leq c|f_{N-3}(z)| + |\gamma||g_{N-2}(z)| + \frac{1}{2} \sum_{j=1}^{N-1} |f_\beta(z)| |f_{N-\beta}(z)| \\ &\leq \frac{c}{\sqrt{N-2}} (12K + K e^{(\lambda-1)z})^{N-3} + \frac{|\gamma|}{\sqrt{N-1}} (12K + K e^{(\lambda-1)z})^{N-2} \\ &\quad + \frac{1}{2} (12K + K e^{(\lambda-1)z})^N \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{N-\beta+1}} \\ &\leq \left[\frac{c}{\sqrt{N-2}} + \frac{|\gamma|}{\sqrt{N-1}} + \frac{\pi}{2} \right] (12K + K e^{(\lambda-1)z})^N \quad (\text{by Lemma 4.3.1}). \end{aligned}$$

Substitution of the latter estimate into (5.16) ($\alpha = N$) gives

$$\|\tilde{f}_N(z)\| \leq C \left(\frac{c}{\sqrt{N-2}} + \frac{|\gamma|}{\sqrt{N-1}} + \frac{\pi}{2} \right) \int_{-\infty}^z e^{(N-\lambda)(x-z)} (12K + K e^{(\lambda-1)x})^N dx. \quad (5.19)$$

Using Lemma 4.3.3 ($\alpha = N$, $\varepsilon = 1/12$, $a = 12K$, $b = K$, $\lambda = \frac{13}{2}$), the integral in (5.19) is estimated by

$$\int_{-\infty}^z e^{(N-\lambda)(x-z)} (12K + Ke^{(\lambda-1)x})^N dx \leq \frac{12}{N-\lambda} (12K + Ke^{(\lambda-1)z})^N$$

which, when substituted back into (5.19), gives

$$\|\tilde{f}_N(z)\| \leq \frac{12C}{N-\lambda} \left[\frac{c}{\sqrt{N-2}} + \frac{|\gamma|}{\sqrt{N-1}} + \frac{\pi}{2} \right] (12K + Ke^{(\lambda-1)z})^N.$$

Hence the claim holds for $\alpha = N$ iff N satisfies

$$12C \left[\frac{c}{\sqrt{N-2}} + \frac{|\gamma|}{\sqrt{N-1}} + \frac{\pi}{2} \right] \frac{\sqrt{N+1}}{N-\lambda} \leq 1. \quad (5.20)$$

This is what we referred to earlier as the induction condition, and is satisfied for sufficiently large N (for all c, γ), because the left-hand side of (5.20) goes to zero as $N \rightarrow \infty$. Therefore, by choosing N large enough (at least 7) such that for any larger value of N , equation (5.20) is satisfied, the claim is true for $\alpha = N$ and by induction true for $\alpha \geq 1$. This completes the proof of the claim. As for the calculation of K , once the appropriate value of N has been found, K_f, K_g, K_h are calculated using Lemma 4.3.5, namely $K = \max[K_f, K_g, K_h]$.

Returning now to the proof of the theorem, since $\forall \alpha \geq 1, |f_\alpha(z)| \leq \|\tilde{f}_\alpha(z)\|$, we have, by the claim and (5.10),

$$\sum_{\alpha=1}^{\infty} |f_\alpha(z)| \tau^{\alpha-3} \leq \sum_{\alpha=1}^{\infty} \frac{(12K + Ke^{(\lambda-1)z})^\alpha}{\sqrt{\alpha+1}} \tau^{\alpha-3}. \quad (5.21)$$

By the ratio test, the series on the right-hand side of (5.21), converges if

$$(12K + Ke^{(\lambda-1)z}) \tau < 1, \text{ or } 12K + Ke^{(\lambda-1)z} < \tau^{-1}. \quad (5.22)$$

Equality holds in (5.22) for a unique negative z_0 (see Fig. 4.3) such that $\forall z < z_0$ (5.22) is true, which implies that the series in (5.21) converges for $0 < \tau = \xi - \xi_0 < e^{z_0}$. Therefore the solution (5.10) converges absolutely for $0 < \xi - \xi_0 < R$, where the radius of convergence is at least e^{z_0} . This completes the proof of the theorem.

5.3 Analytic Solutions of the Kuramoto-Sivashinsky Equation

In Chapter 1, we described the series method for obtaining analytic solutions of non-linear ODEs. In this section, we apply the method to the ODE (equation (5.4)) obtained from (5.1) by symmetry reduction, i.e., to the third-order ODE

$$u''' + \gamma u' + \frac{1}{2}u^2 - cu = 0, \quad (5.23)$$

where, for convenience, the constant of integration A is taken to be zero. Let the initial conditions for (5.23) be

$$u(\xi_0) = I_0, \quad u'(\xi_0) = I_1, \quad u''(\xi_0) = I_2. \quad (5.24)$$

The series representation of the solution about any ordinary point ξ_0 is of the form

$$u(\xi) = \sum_{k=0}^{\infty} a_k \tau^k, \quad (5.25)$$

where $\tau = \xi - \xi_0$. The recursion relation resulting from the substitution of (5.25) into (5.23) is

$$k(k-1)(k-2)a_k = ca_{k-3} - \gamma(k-2)a_{k-2} + \frac{1}{2} \sum_{m=0}^{k-3} a_m a_{k-m-3}. \quad (5.26)$$

As can be seen from the above recursion relation, we have three free constants, namely, a_0 , a_1 and a_2 . To prove convergence for (5.25), and hence to find an estimate for the radius of convergence R , we follow the same proof as given for the KdV-B equation (Chapter 1) to find that $N \geq 3$, and K satisfies

$$6K^3 - |\gamma|K - \left(|c| + \frac{1}{2}M\right) \geq 0, \quad (5.27)$$

where $M = \max\{|a_0|, |a_1|, |a_2|\}$.

Therefore by choosing K to be at least the largest positive root of (5.27), say $K = K_{\min}$, we obtain a lower bound for the radius of convergence

$$R = \frac{1}{\max \left\{ \frac{|a_1|}{M}, \left(\frac{|a_2|}{M} \right)^{1/2}, K_{\min} \right\}}. \quad (5.28)$$

In all of the following, we take the initial point ξ_0 to be the origin ($\xi_0 = 0$), and proceed according to the following steps :

1. Let the free constants a_0, a_1, a_2 take the values

$$a_0 = I_0, \quad a_1 = I_1, \quad a_2 = \frac{I_2}{2},$$

so that the initial conditions at $\xi_0 = 0$ are satisfied. Find the value of $K_{\min}$ by solving (5.27) and hence find an estimate for R_0 using (5.28). The solution $u_0(\xi)$ valid for $0 \leq \xi < R_0$ is then given by

$$u_0(\xi) = \sum_{k=0}^{\infty} a_k \xi^k, \quad (5.29)$$

where a_0, a_1, a_2 are as above and $\forall k \geq 3$, the coefficients a_k are given by the recursion relation (5.26).

2. Take any point within the radius of convergence, say $\xi_1 = \alpha_0 R_0$, where $0 < \alpha_0 < 1$. Using the series solution (5.29), compute $u_0(\xi_1)$, $u'_0(\xi_1)$ and $u''_0(\xi_1)$ to any desired accuracy by taking a sufficient number of terms in the series and then expand about ξ_1 with

$$a_0 = u_0(\xi_1), \quad a_1 = u'_0(\xi_1), \quad a_2 = \frac{1}{2} u''_0(\xi_1),$$

to obtain the solution

$$u_1(\xi) = \sum_{k=0}^{\infty} a_k (\xi - \xi_1)^k, \quad (5.30)$$

valid for $\xi_1 \leq \xi < R_1$, where R_1 is the estimate of the radius of convergence of $u_1(\xi)$, computed using (5.28), with the coefficients $a_k, k \geq 3$, given by the same recursion relation (5.26).

3. Repeat step 2 until the desired end point ξ_{end} is reached.

After executing the above procedure, say m times, until the end point is reached, we obtain $m + 1$ local solutions $u_i(\xi)$, $i = 0, 1, 2, \dots, m$, each valid in the interval $\xi_i \leq \xi \leq \xi_{i+1}$. By collecting all $m + 1$ solutions, we obtain one solution $u(\xi)$ valid in the interval $0 \leq \xi \leq \xi_{\text{end}}$, where

$$u(\xi) = u_i(\xi) \text{ if } \xi_i \leq \xi \leq \xi_{i+1} . \quad (5.31)$$

Each one of the solutions $u_i(\xi)$ is a truncation of the series expansions about the points ξ_i .

For any desired accuracy ε , the minimum number of terms to be taken at each step i before truncation is

$$k_i = \frac{\log \left(\frac{\varepsilon(1-\alpha_i)}{M_i} \right)}{\log \alpha_i} - 1 ,$$

where

$$M_i = \max [|u_i(\xi_i)|, |u'_i(\xi_i)|, |u''_i(\xi_i)/2|]$$

and

$$\alpha_i = \frac{\xi_{i+1}}{\xi_i}, \quad \xi_i \neq 0 \quad (\text{see step 2}) .$$

Finally, a plot of the solution shows its behaviour over the interval $0 \leq \xi \leq \xi_{\text{end}}$. Before we present the results, we note that for certain initial conditions, it is possible that the desired end point ξ_{end} cannot be reached. This happens only if the radii of convergence R_i approach zero.

5.4 Results and Discussion

A computer program was written in Mathematica, running on the UNIX operating system, to perform the above procedures and to plot the solutions. The program prompts for input the following:

- The initial and end points (ξ_0, ξ_{end}) .
- The order of the ODE and the initial conditions at ξ_0 .
- The coefficient recursion relation [eq. (5.26)].
- The condition on K [eq. (5.27)].
- The accuracy ε .

The program was employed to compute various solutions of equation (5.23). The plot in Fig. 5.1 shows the behaviour of the solution satisfying the initial conditions $u(0) = 0.05948$, $u'(0) = 0.05928$ and $u''(0) = 0.05887$, with the parameters $\gamma = \frac{19}{11}$, $c = \frac{30}{11}$. These initial conditions are taken from an exact solution of (5.1) (see, e.g., [13]). The exact solution is

$$v(x, t) = \frac{60}{19} \gamma \sqrt{\frac{11}{19}} \gamma \frac{12e^\eta - 9e^{2\eta} + e^{3\eta}}{(1 + e^\eta)^3} . \quad (5.32)$$

where

$$\begin{aligned} \eta &= x - ct - \tau . \\ c &= 1 + \gamma . \quad \gamma > 0 . \end{aligned}$$

The solution in Fig. 5.1 is compared to the plot of the exact solution (5.32) at $t = 0$, and found to be identical.

Lin and Suryadevara [18] used the fourth-order Runge-Kutta method with step size $\Delta\xi = 0.005$ to obtain numerical solutions of (5.23) for various sets of initial conditions. Using the same sets of initial conditions, we applied the series method and obtained the solutions shown in Fig. 5.2-5.5. The initial conditions used and the parameter values γ and c are tabulated below.

	$u(0)$	$u'(0)$	$u''(0)$	γ	c
Fig. 5.2	0	0	0.001	1	1.21599
Fig. 5.3, Fig 5.4	0.00125	0.01	0.004	1	0.12161
Fig. 5.5	0.00125	0.01	0.004	1	0

For the first set of initial conditions, we noted that the solution (see Fig. 5.2) decreases indefinitely beyond the point $\xi = 43$, where the solution is plotted over the range $\xi = 0$ to $\xi = 45$. We also noted that the radii of convergence beyond $\xi = 43$ approach zero. These observations suggest that continuation beyond certain points may not be possible. The rapidly decreasing nature of the solution, as well as the rapid decrease of the radii of convergence, seem to indicate that as ξ approaches some value (greater than 43), the solution approaches $-\infty$, and this indicates the existence of a real-time singularity. The numerical solution obtained by [18] was only plotted up to ξ below 25, so comparison could not be made.

For the second set of initial conditions, a first application of the series method with the end point $\xi_{\text{end}} = 200$, produced the solution shown in Fig. 5.3; a second application with $\xi_{\text{end}} = 600$, revealed the behaviour of the solution beyond the point $\xi = 200$ (see Fig. 5.4). Fig. 5.4 shows an almost-periodic wave solution, which agrees with the numerical results obtained by [18]. For the same initial conditions but different parameter values ($\gamma = 1$, $c = 0$), the solution obtained is shown in Fig. 5.5, where we can see that the solution has the form of a small-amplitude periodic traveling-wave. This solution is compared to the one obtained numerically and is found to be identical.

FIGURES

Fig. 4.1. The single index $\alpha = k + rl$ versus the double index (k, l) for $r = 2$.

Fig. 4.2. A rough plot of the functions $(2K - Kz)$ and e^{-qz} , showing the radius of convergence of the logarithmic psi-series.

Fig. 4.3 A rough plot of the functions $(3K + Ke^{(\beta-1)z})$ and e^{-z} , showing the radius of convergence of the nonlogarithmic psi-series.

Fig. 5.1. Series solution of (5.4), $u(0) = 0.05948$, $u'(0) = 0.05928$, $u''(0) = 0.05887$.

$$A = 0, \quad \gamma = \frac{18}{11}, \quad c = \frac{30}{11}.$$

Fig. 5.2. Series solution of (5.4), $u(0) = u'(0) = 0$, $u''(0) = 0.001$.
 $A = 0$, $\gamma = 1$, $c = 1.21599$.

Fig. 5.3. Series solution of (5.4), $u(0) = 0.00125$, $u'(0) = 0.01$, $u''(0) = 0.004$.
 $A = 0$, $\gamma = 1$, $c = 0.12161$. Plotted in the range $[0, 200]$.

Fig. 5.4. Series solution of (5.4) with the same initial conditions as in Fig. 5.3.
Plotted in the range $[0, 600]$.

Fig. 5.5. Series solution of (5.4), $u(0) = 0.00125$, $u'(0) = 0.01$, $u''(0) = 0.004$.
 $A = 0$, $\gamma = 1$, $c = 0$.

CONCLUSION

In this thesis, we have dealt with series solutions of second- and third-order nonlinear ODEs about arbitrary ordinary points (ordinary series), movable poles (Laurent series) and movable branch points (psi-series). In each case, we have discussed the general features of the convergence proof, as well as having furnished detailed proofs for particular examples.

The ordinary series method turns out to be an invaluable tool for solving initial-value problems for nonlinear ODEs. Indeed, we can obtain solutions to any desired degree of accuracy, within the radius of convergence. In order to continue the solution beyond the (initial) radius R_0 , we can restart the problem at a new initial point and calculate the new radius R_1 . This process may be repeated indefinitely, unless $R_i \rightarrow 0$ and $\sum_{i=0}^{\infty} R_i < \infty$, in which case it is impossible to continue. Such an occurrence is highly indicative of the presence of a real singularity. If the restarting procedure can be repeated, say m times, then knowledge of a bound on the total cumulative error is of paramount importance. Such a bound can be found if we can bound (within a particular radius) the distance between two trajectories that are initially close. We have not dealt with this problem in this thesis. We note that the "strange attractor" for the Lorenz system [24] has been revealed by this method (see Zypchen [23]). Although the existence of ordinary (Taylor) series solutions about ordinary points is guaranteed by the "existence and uniqueness" theorem, we find that this method of determining trajectories, and especially the continuation beyond the radius of convergence, has not been amply exploited by other researchers.

Hille [12] has given a convergence proof for the (logarithmic) psi-series solution of a quadratic system. Although the appropriate forms of the psi-series for other nonintegrable equations had been known [10, 11], to our knowledge, convergence proofs had not been provided. In this thesis, we have shown that convergence can be proved by means of an appropriate resummation of psi-series.

It has been claimed, in the literature, that the ARS algorithm is incapable of detecting the presence of movable essential singularities. It is for this reason that an equation passing the Painlevé test is not automatically deemed to possess the Painlevé property. It follows that psi-series analyses must also suffer similar drawbacks. Our experience suggests, however, that if an equation possesses movable essential singularities, then any general solution exhibiting only pole or branch singularities (Laurent series or psi-series) ought to diverge. (Indeed, linear equations whose general solutions contain (fixed) essential singularities admit Frobenius series “solutions” (about irregular singular points); although the recursion relations for such series may be self-consistent, the series do not converge on any interval.)

We have seen that the presence of an irrational positive resonance β necessitates a psi-series involving powers of τ^β . If a negative resonance exists (other than the usual $r = -1$), then no psi-series can form a general solution. Is this a signal for the presence of movable essential singularities? We leave such tantalizing questions for future consideration.

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