Abstract

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Existing approximation techniques approach this problem in either a time/spacial domain or transform domain, but not both. This thesis, first of all, reviews various existing approximation techniques. Subsequently, we present a new strategy to obtain an approximation $\hat{f}(x)$ of $f(x)$ in such a way that it is *reasonably* close to the original function in the domain of the variable x , but it also *exactly* preserves some properties of the transformed domain. In this particular case, the properties of the transformed values that are preserved are geometric moments of the original function. The proposed technique has been applied to one-dimensional functions, two-dimensional planar curves, and two-dimensional images.

Keywords: signal processing, image processing, sampling, approximation, moment-preserving

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Moment-Preserving Piecewise Linear Approximations of Signals and Images

by
Thai B. Nguyen. B.C.S.

**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Computer Science**

**School of Computer Science
Carleton University
Ottawa, Ontario
November 1994**

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Approximations of
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submitted by Thai B. Nguyen, B.C.S.

in partial fulfillment of
the requirements for the degree of
Master of Computer Science

Thesis Supervisor

Chairman, School of Computer Science, Carleton University

Dec. 19, 1994
Date

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Chapter 1

1 Introduction

A signal, by definition from the 1990 Concise Oxford Dictionary, is “an electrical impulse or impulses, or radio waves ..., or a sequence of these.” The term signal processing is used to describe an operation or a sequence of operations carried out on a signal in order to simplify its analysis and storage. Approximation techniques are an important aspect of signal processing. For example, an analogue signal can be represented by a digitized signal which is an approximation of the original one. Electronic voice mail messages, photographic images, and X-ray images are often digitized and encoded to reduce their storage and transmission costs. Approximation and encoding is a small segment of the field of signal processing which also includes scaling, filtering, encoding, structural pattern recognition, feature extraction, scene analysis, motion detection, and image understanding [Pr 78]. The main focus of this thesis is on the approximation of digitized signals.

Approximation techniques are crucial in *practical* digital signal processing. For example, the typical bandwidth of digitized speech voice, varies from between 2.4 Kilobits per second (Kbps) to 64 Kbps, depending on the encoding algorithm used. Indeed, for compact disc quality audio this value may be up to 1.4 Megabits per second. Using these bandwidths, a 10-minute digitized speech sample could require from 180 Kilobytes to 100 Megabytes of storage.

The bandwidth requirements of digitized images and graphics are much greater than that of digitized speech. A still colour image with a resolution of 1024 by 1024 picture elements (pixels, or pels) at 24 bits per pixel requires 24 Megabits of storage

space in uncompressed form, and a sequence of still images used to represent a 10-minute moving-picture video requires close to 1.5 Gigabits. Clearly, approximation techniques for processing these types of signals can reduce the transmission time, storage requirements, and analysis time required in the pattern recognition or image understanding phase.

Approximation techniques are also used to remove excess noise or to simplify highly complex signals. A simple linear approximation technique is often applied to remove undesirable noise from the garbled version of the actual signal [Pa 73], [PH 74], [Sk 70], [Mo 70]. Indeed, unfiltered noisy signals can lead to gross misinterpretation of data. To clarify this, Ballard and Brown [BB 82] present a case where a boundary detection algorithm was applied to a radiograph of a chest in which an edge of a rib was obstructed by random noise. They show how the accurate detection of the rib was impossible due to the unfiltered noise.

1.1 Terminology and Concepts

Let $f(x)$ represent a one-dimensional function to be approximated, and let it be defined over a domain D . For a continuous function, D describes an interval $[a, b]$, and the function is defined for all values of x between a and b . For a discrete function, D is a set of N distinct values of $x_0, x_1, \dots, x_{N-1}$. In most cases, the function f is not given explicitly, but often is defined by a set of N distinct values $S = \{(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_{N-1}, f(x_{N-1}))\}$. The values in the domain D are strictly increasing in order and value, that is, for any two values x_i and x_j in D , if $i < j$ then $x_i < x_j$. Let the distance between any two adjacent values x_i and x_{i+1} be represented by Δx , then only one of the x_i 's is required, and the other x 's can be expressed in terms of the known values x_k and Δx . For example, suppose x_0 is given and the values x_i 's are equidistant apart, then $x_i = x_0 + i\Delta x$ for all $i = 0, 1, \dots, N-1$.

A two-dimensional function is often used to describe a planar curve or a digitized

image. A continuous planar curve, $f(x, y)$, describes the position of the curve on the xy -plane for all values of x defined in the interval $[A_x, B_x]$ and all values of y defined in the interval $[A_y, B_y]$. A discrete planar curve is usually defined by a set of points $S = \{p_i = (x_i, y_i) | i = 0, 1, 2, \dots, N - 1\}$. The main difference between a one-dimensional function and a planar curve is that for a one-dimensional function, there is a one-to-one mapping between the domain and range of f , whereas for a planar curve, there may be more than one value in the range for each value in the domain. If the function $f(x, y)$ is used to describe a digitized image whose width is W and whose height is H , then the value $f(x, y)$ represents the quantized intensity of the image (greyscale value) at position (x, y) where $0 \leq x \leq W$ and $0 \leq y \leq H$.

The purpose of an approximation technique is to find a set of n points, S , such that $\hat{S} = \{(\hat{x}_0, \hat{f}(\hat{x}_0)), (\hat{x}_1, \hat{f}(\hat{x}_1)), \dots, (\hat{x}_{n-1}, \hat{f}(\hat{x}_{n-1}))\}$, where $n < N$. If $x_i = \hat{x}_j$ for some $(x_i, f(x_i)) \in S$ and $(\hat{x}_j, \hat{f}(\hat{x}_j)) \in \hat{S}$, it is *not* necessary true that $f(x_i) = \hat{f}(\hat{x}_j)$. The approximated data points in $\hat{S}$ are joined by a polynomial of order at most $m - 1$ with an error norm less than a prespecified quantity ϵ . By restricting $m = 2$, approximation polynomials will be limited to linear equations.

When f describes a one-dimensional waveform such as an audio signal, it is the measured value of the signal at a specified time variable x . In the field of computer vision and pattern recognition, the function f often describes a curve or line segment which is dependent on the position variable x . Thus, f is referred to as a time or spacial domain function. A function can, therefore, be viewed as a small machine that converts numbers into other numbers; indeed, the input to the function is a number and the output of the function is another number. A *transform* can be viewed as a "bigger" machine in an analogous way; it converts functions into other functions. For example, the Fourier transform takes a continuous function of time (like a sound wave) or space (like an image), say $f(x)$, and converts it into a continuous function of frequency, $F(\omega)$. An approximation technique in the transform domain derives an approximation $\hat{F}(\omega)$ by quantizing or truncating

the transformation coefficients, and performs an inverse transformation on $F(\omega)$ to derive $\hat{f}$.

Certain techniques employ a set of orthogonal functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ to approximate $f(x)$. The term *orthogonal* derives from the Greek word *orthos* meaning *straight, upright*. Two intersecting lines are said to be orthogonal if they are at right angles to each other, and in general, two n-dimensional vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are orthogonal if their inner product defined by $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$ is equal to zero. The concept of orthogonal functions follows directly from that of orthogonal vectors.

There are several objective criteria which can be used to compare one approximation technique with another. Some of these criteria are:

1. Speed efficiency,
2. Memory efficiency, i.e. how much data is kept in order to retain a *good* approximation of the original function, and,
3. Minimal amount of error in the approximated function compared to the original function.

These evaluation standards will be applied to all of the techniques reviewed, and a special emphasis is made on how well each method performs in terms of error minimization. Since there are many different ways of looking at the error values, and since each method may favour one technique over another, the following three error calculating methods will be used:

1. maximum error (ME),
2. mean-squared error (MSE), and,
3. signal-to-noise ratio (SNR).

Figure 1: Illustration of parameters used in error calculation

Let ϵ_i be the error value of the i^{th} point approximation. For one-dimensional functions, $\epsilon_i = |f(x_i) - \hat{f}(x_i)|$, where $\hat{f}(x)$ is the approximating function. For two-dimensional functions, ϵ_i is the Euclidean distance between point $(x_i, f(x_i))$ and the approximating curve evaluated at point $(x_i, \hat{f}(x_i))$. If the approximating function is given as a linear function,

$$\sin \varphi \cdot x + \cos \varphi \cdot y = d$$

then,

$$\epsilon_i = |\sin \varphi \cdot x_i + \cos \varphi \cdot f(x_i) - d|$$

where d is the distance from the origin to the line and φ is the angle of line with the x-axis.

The maximum error and mean-squared error calculations can thus be defined in terms of ϵ_i 's as follows:

$$ME = \max_i \{\epsilon_i\}$$

$$MSE = \frac{1}{n} \sum_{i=1}^n \epsilon_i^2$$

The related measure of signal-to-noise ratio (SNR) is often used as a standard objective measure of coded waveform quality [RJ 91]. It is the ratio of the signal variance to the reconstruction error variance, and for historical reasons is measured in decibels (dB). Its computation is defined as:

$$SNR = 10 \log_{10} \left(\frac{\sum_{i=1}^n f(x_i)^2}{\sum_{i=1}^n \epsilon_i^2} \right).$$

Pavlidis and Horowitz [PH 74] showed that sensitivity to noise is an important difference between the ME and MSE error norms. If the curve $y = \cos(x)$ on the interval $[0, 2\pi k]$ is fitted by the line $y = 0$ by both methods the MSE value increases proportionately with increasing integral values of k whereas ME remains constant. Jayant and Noll [JN 84], and Wang [Wa 88] indicated that it is rather difficult to correlate between objective error measurements and the subjective quality of a human observer. Jayant and Noll even proceeded further by asserting that subjective measures of quality are more relevant than objective measures such as signal-to-noise ratio in the coding of signals such as speech and video. However, reliable values of subjective measurements are, in general, more difficult to obtain.

1.2 Motivation

Existing approximation techniques for one- and two-dimensional functions such as the ones presented by Stone[St 61], Gluss[Gl 61], Bellman[Be 61], Montanari[Mo 70], Ramer[Ram 72], Pavlidis and Horowitz [PH 74], Kurozumi and Davis [KD 82], Prusinkiewicz[PC 83], Dunham[Du 86], and many others derive a solution by working with constraints in the time/spacial domain. Some of them only work with continuous functions, while others work with both continuous and

discrete cases. These approximations are often a necessary part of the process in removing redundant data and noise before the pattern recognition phase. By reducing the amount of redundant data, the amount of *real* information that has to be processed is minimized. In many pattern recognition systems, for example, it is often better to perform educated hypothesis tests on the patterns made from the reduced amount of information rather than to be overwhelmed by the amount of original data which requires much more processing time.

From a practical standpoint, some of these techniques cannot be applied to current interactive applications due to their computational complexity and memory requirements. Many of the transform domain techniques such as the Fourier transform, discrete cosine transform, and Walsh-Hadamard transform, and others have made great strides in satisfying some of today's demanding interactive applications. With a few exceptions, most of the approximation techniques on one-dimensional functions can be extended and applied to planar curves and digitized images. One such notable technique is the discrete cosine transform; its use and acceptance as a standard by the Joint Photographic Experts Group (JPEG) and the Motion Picture Experts Group (MPEG) confirm that transform methods are one of the solutions to some of the problems currently faced by interactive applications.

As opposed to the methods referred to above, properties of geometric moments¹ have been explored by many researchers including, but not limited to, Hu [Hu 62], Dudani et al. [DBM 77], and [Pa 78], [TC 86], to perform pattern recognition and classification. Because of the close relationship between these geometric moments and the Fourier transform, they are considered to be in the transform domain. A finite number of these moments can, in fact, be used to reconstruct an approximation of the Fourier coefficients. Based on these properties, we introduce a new moment-preserving approximation method. This technique preserves a finite number of the geometric moments which are related to the transform domain. It is, howev-

¹The geometric moment of a function $f(x)$ is defined as $\int_{-\infty}^{\infty} x^k f(x) dx$ for $k = 0, 1, 2, \dots$

er, applied directly to the time/spacial domain approximation instead of the usual "*Transformation*→*Approximation*→*Inverse Transformation*" process. Our new approximation technique has been successfully applied to one-dimensional functions, planar curves, and digitized images.

1.3 Outline of thesis

The remainder of this introductory chapter consists of a summary of all the notations used in this thesis. In chapter 2, a review of some of the existing approximation techniques is done. The previous related works are grouped in either the time/spacial domain or the transform domain, and in each of these two categories, the work is presented in chronological order. Chapter 3 is a continuation of review of previous work from Chapter 2. All the methods reviewed in this section are divided into two-dimensional planar curves and digitized images, and the techniques are grouped according to their spacial or transform domain. Our new moment-preserving method is introduced in Chapter 4. Detailed derivations of this approximation technique and experimental results for one-dimensional functions are presented in this chapter. Chapter 5 contains an extension of this technique as it is used to approximate planar curves and digitized images. Results of planar curve and digitized image approximations are included at the end of this chapter. Chapter 6 concludes the thesis with a summary of the current work and some possible future research work.

The novel work done in chapters 4 to 6 are a result of a joint collaborative work with Professor Oommen. The work is currently available as a potential publication [NO 94]. A preliminary version of this paper will appear in the Proceedings of IAS-4, the Fourth International Conference on Intelligent Autonomous Systems, Karlsruhe, Germany, March 1995.

1.4 Summary of Notation

The notation found in the literature varies with the author, each being slightly distinct from the other. In this thesis, in the interest of uniformity, each author's notation will be modified as below. Unless it is explicitly stated in any particular section, the notation below applies uniformly to the entire thesis.

Δt or Δx = sampling interval

δ = decimation factor

D = domain of a function

ϵ = maximum allowable error limit

ϵ_i = pointwise error at (x_i, y_i)

$E[X]$ = expectation of random variable X

$f(x)$ = original function to be approximated

$f_i = f(x_i)$ = value of the function f at x_i

$f'_i = f'(x_i)$ = value of the first derivative of the function f at x_i

$f''_i = f''(x_i)$ = value of the second derivative of the function f at x_i

$f_i^{(k)} = f^{(k)}(x_i)$ = value of the k^{th} derivative of the function f at x_i

$\frac{d^n \phi(\omega)}{d\omega}$ = the n^{th} derivative of the function ϕ with respect to ω

$\hat{f}(x)$ = approximation of the original function $f(x)$

$\langle \mathbf{x}, \mathbf{y} \rangle$ = dot product of two vectors $\mathbf{x}$ and $\mathbf{y}$

$\phi_k(x)$ = polynomial approximation of $f(x)$ in segment k

$F(\omega)$ = transform of the function $f(x)$

$F^*(\omega)$ = complex-valued function

$\mathbf{F}$ = transformation matrix

$\mathbf{F}^T$ = transpose of matrix $\mathbf{F}$

$\mathbf{F}^{-1}$ = inverse of matrix $\mathbf{F}$

$\mu = E[X]$ = mean value of a function

m_k = k^{th} theoretical moment of a function

$\hat{m}_k$ = k^{th} moment of approximated function

MSE = mean squared error = $\frac{1}{N} \sum_{i=1}^N \epsilon_i$

ME = maximum error = $\max_{0 \leq i \leq N} \{\epsilon_i\}$

N = number of original sampled values

n = number of sampled values selected

Chapter 2

2 Related Work on One-dimensional (1-D) Functions

In this chapter, several previous approximation techniques will be reviewed including *hologram-like subsampling*, *piecewise linear*, *least squares polynomial*, *polynomial splines*, *Fourier transform*, *Walsh-Hadamard transform*, *Karhunen-Loeve transform*, *cosine transform*, and *fractals*. Each of these techniques falls into one of the following two categories:

1. *spacial or temporal* domain approximations, and,
2. *transformed* domain approximations.

The first group of techniques calculates its approximation, $\hat{f}(x)$, by directly using the given values of x and $f(x)$. They are considered to be approximation methods in the spacial or temporal domain. The second group determines the approximation of f by *transforming* it into another function, say F , approximating F , and exploiting an *inverse transform* to derive $\hat{f}(x)$. These approximation techniques operate on the *transformed* function F and in the domain of the transformed function. These two groups of approximation techniques will be described in detail in their respective sections below. A summary of these techniques will be presented at the end of this chapter, and a comparison of some of these techniques will also be made.

2.1 Spacial or Temporal Domain Function Approximation

The following four approximation techniques will be reviewed in this section: hologram-like subsampling, piecewise linear, least squares polynomial, and polynomial splines. All of these techniques derive their approximation, $\hat{f}(x)$, by directly manipulating the given values x and $f(x)$. Since the domain variable x represents either position or time, these techniques are thus considered to operate in the spacial or temporal domain.

2.1.1 Subsampling

In many signal processing applications, especially in digital audio, it is necessary to convert an analog signal into a discrete sequence of numbers. The conversion process is commonly done by sampling the continuous analogue signal, $f(x)$ at regular interval Δx to produce a sequence $S = \{f(x_0), f(x_1), f(x_2), \dots, f(x_{n-1})\}$, where $x_i = x_0 + i\Delta x$ and x_0 is some given starting time. There are special cases where the sampling interval is irregular, and in these circumstances, the set must specify both the time or position and the value of the signal at that particular time or position, such that $S = \{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), \dots, (x_{n-1}, f(x_{n-1}))\}$. For simplicity of expressing the dependent values $f(x_i)$'s without introducing any confusion or ambiguity, we shall use the notation $f_i = f(x_i)$.

The sampling theorem places a restriction on the magnitude of Δx as specified by the frequency content of the signal $f(x)$. The theorem, as described by Stearns and Hush [SH 90], is stated below:

Theorem 1 *To be able to recover $f(x)$ exactly, it is necessary to sample $f(x)$ at a rate greater than twice its highest frequency.*

For example, if the signal, $f(x)$, is a sine wave as shown in Figure 2, then more than two samples per cycle are required to reconstruct f . If two or less samples are used, then the samples are the same as those of sinusoid $g(x)$. *Aliasing* is a term

Figure 2: Sampling at less than twice the frequency of f

which is often used in signal processing to describe such frequency folding. Even with the sampling rate of exactly twice that of the highest frequency as shown in Figure 3, the samples still cannot be used to reconstruct function f correctly.

The sampling theorem, however, is not a good predictor of how easily objects can be recognized by computer programs. Often, objects can be more easily recognized in images that have a low sampling rate [BB 82]. Essentially, there are two fundamental reasons for this surprising result. First of all, the computations are fewer because of the reduction in dimensionality, and secondly, the confusing details contained in the high-resolution version of the images may not appear at the reduced resolution.

Subsampling, also known as decimation, consists of determining a sequence $\hat{S}$ which contains exactly n uniformly selected points from S , where $n < N$. Frequently, the value of n is not specified, but a decimation factor, δ , is provided, and in these cases, n can be calculated to as, $n = \lfloor N/\delta \rfloor$. If the given sequence S is

Figure 3: Sampling exactly twice the frequency of f

uniform, then the decimation by a factor of δ will generate an approximation sequence, $\hat{S} = \{f(x_j)\}$ where $j = 0, \delta, 2\delta, \dots, n\delta$. This technique requires practically no calculations on x_i or $f(x_i)$. For example, suppose that S contains 2000 uniform samples, where each sample is one millisecond apart. If S is decimated by a factor of five (i.e., $\delta = 5$), then $\hat{S}$ contains 400 samples where each sample is five milliseconds apart. Another way of looking at uniform decimation is that every sequence of five samples in S is used to form a single sample in $\hat{S}$.

If S contains a sequence of non-uniform samples, the subsampling process cannot be applied to S directly because subsampling can only be applied to a uniform sequence of values. For example, suppose S contains 2000 non-uniform samples such that

$$\begin{aligned}
 S = \{ & (0, 0), (20, 50), (21, 100), (22, 20), (23, -5), (24, 0), \dots, \\
 & (20i, 50), (20i + 1, 100), (20i + 2, 20), (20i + 3, -5), (20i + 4, 0), \\
 & \dots, \\
 & (7980, 50), (7981, 100), (7982, 20), (7983, -5), (7984, 0) \}
 \end{aligned}$$

for $1 \leq i < 400$. If one out of every five samples in S is used, starting at the first

sample (0,0), to derive $\hat{S}$, then $\hat{S}$ contains 400 samples whose values are zero. To overcome the non-uniformity problem, a new sequence $\bar{S}$ is calculated by performing interpolation of S such that all the values in $\bar{S}$ are equidistant apart by $\overline{\Delta x}$, where $\overline{\Delta x}$ is the smallest Δx in S . The decimation process is then performed on the sequence $\bar{S}$ to derive $\hat{S}$.

Several problems arise when n does not divide into N evenly, or in other words, the scale factor δ is non-integral. For example, suppose N is 640, and n is 100. A sample cannot be taken at every 6.4 samples to give exactly 100 values in $\hat{S}$. If the scale factor, δ is set to $\lfloor N/n \rfloor$, then the approximated sequence $\hat{S}$ will contain 106 samples; and if $\delta = \lceil N/n \rceil$ then $\hat{S}$ will contain 91 samples. In order to attain $\hat{S}$, an interpolation of S by a factor of k must be done to derive a new sequence $\bar{S}$, and a selection of one out of every δ samples from $\bar{S}$ is then performed. The main drawback with this method is that the n samples are relatively uniform with respect to $\bar{S}$ but not uniform with respect to S .

Prusinkiewicz [PC 85] proposed a progressive subsampling scheme which is intuitively based on the following two propositions:

1. A translated set of sampling points uniformly distributed in a line is a set of uniformly distributed points, and,
2. The union of appropriately translated sets of uniformly distributed points is also a set of uniformly distributed points, with a reduced distance between adjacent points.

If a given string of data S represents a signal of full resolution, it can be divided into substrings $S_0, S_1, \dots, S_n$ as shown in Figure 4(a) to Figure 4(h). Each substring S_i , representing a lower resolution of S , results from the traversal algorithm (subsampling technique) and the overlapping sampled areas.

Let $S = \langle f_0, f_1, f_2, \dots \rangle$ be a string of points of the one-dimensional function f , and $S(i, j)$ be a substring of S described by:

Figure 4: Prusinkiewicz's progressive uniform sampling technique

$$S(i, j) = \langle f_{j \cdot 2^i}, f_{j \cdot 2^i + 1}, \dots, f_{(j+1) \cdot 2^i - 1} \rangle$$

Furthermore, let $S_{\vec{x}}(i, j)$ denote the translation of the substring $S(i, j)$ by the vector $\vec{x}$.

Let N denote the size of the original sequence S which represents the sampled values of the function f . The string of sampled points is then defined as follows:

$$S(0, 0) = \langle f_N \rangle$$

$$S(k, 1) = S_{-N \cdot 2^{-k-1}}(k, 0)$$

$$S(k+1, 0) = S(k, 0) \dot{+} S(k, 1)$$

where $\dot{+}$ denotes the concatenation of strings, and $k = 0, 1, \dots$. Prusinkiewicz has proved the following result regarding the x-coordinate of the k^{th} point.

Theorem 1 *Let the binary word $r_{2^l-1}r_{2^l-2}\dots r_1r_0$ represent index k of a sample point (x_k, f_k) in the pure binary number system:*

$$k = \sum_{i=0}^{2^l-1} r_i 2^i$$

The x -coordinate of point f_k is:

$$x_k = T \left(1 - \sum_{i=0}^{l-1} r_{2^i} 2^{-(i+1)} \right)$$

This sampling method allows each substring, $\hat{S}$, to represent the whole string S at that particular resolution. These sampling techniques are easy to apply and are computationally inexpensive due to the absence of any complicated mathematical equations. This lack of computation is also the main downfall of subsampling techniques because it introduces aliasing into the approximated signals which, in turn, causes audio signals to lose their high frequency components and images to become blurry.

2.1.2 Least squares polynomial

One of the most well-known approximation techniques is the least squares polynomial approximation. It is based on the Legendre's principle of least squares which states that "when the available data in the domain D are either exact or of equal reliability, then the 'best approximation' over D is that one for which the aggregate (sum or integral) of the squared error in D is least." [Hi 87]. More generally, if $w(x_i)$ is the measure of dependability of the value of $f(x)$ when $x = x_i$, then the criterion also requires that the squared error at x_i be multiplied by the weight $w(x_i)$ before the aggregate sum is calculated. For a uniformly distributed set of discrete values, the weights are of equal significance, and for the sake of simplicity in the calculation, they are assumed to be unity.

This technique can be applied to both cases when the given function $f(x)$ is continuous or discrete; however, it is often applied to cases where $f(x)$ is a discrete function. The main concept of the least squares polynomial technique is to

employ a set of $n + 1$ orthogonal functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ to perform the approximation such that,

$$\hat{f}(x) \equiv \sum_{i=0}^n a_i \phi_i(x) \approx f(x).$$

The approximating polynomial $\hat{f}(x)$ is of degree n , and the values of $\hat{f}(x)$ on a set of $N + 1$ points agree "as well as possible" with known values of $f(x)$ at those points. If the residual function $r(x)$ is given as

$$\begin{aligned} r(x) &= f(x) - \hat{f}(x) \\ &= f(x) - \sum_{j=0}^n a_j \phi_j(x), \end{aligned}$$

then the best approximation is defined to be that for which the values a_j 's are determined so that the aggregate sum or integral of $w(x)r^2(x)$ over D is minimal. Thus, by differentiating the aggregate sum partially with respect to a_j for $j = 0, 1, \dots, n$ and setting it to zero, a system of $n + 1$ normal equations involving the $n + 1$ unknowns $a_0, a_1, \dots, a_n$ can be formulated as shown below:

$$\frac{\partial}{\partial a_k} \left\{ \sum_{i=0}^n w(x_i) \left[f(x_i) - \sum_{j=0}^n a_j \phi_j(x_i) \right]^2 \right\} = 0, \quad (k = 0, 1, \dots, n)$$

which yields:

$$\begin{aligned} a_0 \sum_{i=0}^N w(x_i) \phi_k(x_i) \phi_0(x_i) + a_1 \sum_{i=0}^N w(x_i) \phi_k(x_i) \phi_1(x_i) + \dots + a_k \sum_{i=0}^N w(x_i) \phi_k(x_i) \phi_k(x_i) \\ = \sum_{i=0}^N w(x_i) \phi_k(x_i) f(x_i), \quad (k = 0, 1, \dots, n). \end{aligned}$$

These normal equations can be obtained by first writing down the $N + 1$ equations where the $N + 1$ points $x_0, x_1, \dots, x_N$ exactly equal $f(x)$

$$\begin{array}{ccccccccc} a_0 \phi_0(x_0) & + & a_1 \phi_1(x_0) & + & \dots & + & a_n \phi_n(x_0) & = & f(x_0) \\ a_0 \phi_0(x_1) & + & a_1 \phi_1(x_1) & + & \dots & + & a_n \phi_n(x_1) & = & f(x_1) \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_0 \phi_0(x_N) & + & a_1 \phi_1(x_N) & + & \dots & + & a_n \phi_n(x_N) & = & f(x_N) \end{array}$$

The k^{th} normal equation can thus be obtained by multiplying each equation above by the coefficient of a_k in that equation and also by the weight associated with that equation, and summing the results.

A simple example below will show how these normal equations can be found using the steps described above. Let $f(x)$ be describe by the set of discrete values,

x	$f(x)$
0.0	1.4
0.2	1.8
0.4	2.7
0.6	3.4
0.8	4.4
1.0	4.8
1.2	6.1
1.4	8.0
1.6	9.4
1.8	12.1

$\hat{f}(x)$ be a polynomial approximation of degree two, and for the sake of simplicity, let the set of orthogonal functions be $\phi_0(x) = 1$, $\phi_1(x) = x$ and $\phi_2(x) = x^2$, then the following equations are formed,

$$\begin{aligned}
 a_0 + 0a_1 + 0a_2 &= 1.4 \\
 a_0 + 0.2a_1 + 0.04a_2 &= 1.8 \\
 a_0 + 0.4a_1 + 0.16a_2 &= 2.7 \\
 a_0 + 0.6a_1 + 0.36a_2 &= 3.4 \\
 a_0 + 0.8a_1 + 0.64a_2 &= 4.4 \\
 a_0 + 1.0a_1 + 1.0a_2 &= 4.8 \\
 a_0 + 1.2a_1 + 1.44a_2 &= 6.1 \\
 a_0 + 1.4a_1 + 1.96a_2 &= 8.0 \\
 a_0 + 1.6a_1 + 2.56a_2 &= 9.4 \\
 a_0 + 1.8a_1 + 3.24a_2 &= 12.1
 \end{aligned}$$

Under the assumption that each data value f_i is of equal significance, we derive the three normal equations,

$$\begin{aligned}
 10a_0 + 9.0a_1 + 11.4a_2 &= 52.7 \\
 9a_0 + 11.4a_1 + 34.2a_2 &= 57.14 \\
 11.4a_0 + 34.2a_1 + 24.5328a_2 &= 97.076
 \end{aligned}$$

By solving the system of three equations of three unknowns, the approximated function $\hat{f}(x)$ is determined to be,

$$\hat{f}(x) = 3.7334 + 1.4460x + 0.2063x^2$$

and the actual values of $f(x)$ and their corresponding approximated values are tabulated below.

x	$f(x)$	$\hat{f}(x)$
0.0	1.4	3.7334
0.2	1.8	4.0309
0.4	2.7	4.3448
0.6	3.4	4.6753
0.8	4.4	5.0222
1.0	4.8	5.3857
1.2	6.1	5.7657
1.4	8.0	6.1621
1.6	9.4	6.5751
1.8	12.1	7.0046

If the number of ordinates defining $f(x)$, n , is greater than the number of orthogonal functions, k , then an exact fit of n points is generally impossible. If $n = k$, however, it is possible to determine a function $\hat{f}(x)$ such that the error is reduced to, or close to, zero. This technique becomes impractical for polynomials of high degrees (ten or higher) due to the computational cost and the many local minima, maxima, and inflection points which inherently exist in polynomials.

2.1.3 Piecewise linear

In many cases, the original discrete function f is very complex and cannot be expressed by a simple closed-form equation. The function is thus divided up into smaller segments, and each segment is then approximated by a simple linear equation. The point where one segment joins the next is often referred to as a knot or a joining point. All of the techniques presented in this section are based on the same piecewise linear principle, but they do vary on their predefined conditions such as

using a fixed number of knots or variable number of knots, and the question of whether the spacing between the knots is of fixed size or of variable size.

Stone [St 61] presented the problem of trying to obtain a set of best-fit line segments to a curve and to give a closed form solution for the procedure. The problem formulated by Stone consists of the original given function $y = f(x)$ defined between the two endpoints u_0 and u_N , where $u_0 \leq x \leq u_N$. Formally, the problem reduces to determining an approximation, y , where

$$y = \begin{cases} a_1 + b_1x & u_0 \leq x \leq u_1 \\ a_2 + b_2x & u_1 \leq x \leq u_2 \\ \vdots & \\ a_N + b_Nx & u_{N-1} \leq x \leq u_N \end{cases}$$

such that

$$F(a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, u_1, u_2, \dots, u_{N-1}) = \sum_{j=1}^N \int_{u_{j-1}}^{u_j} [f(x) - a_j - b_jx]^2 dx$$

is minimized. If $u_1, u_2, \dots, u_N$ are specified in advance, then the problem reduces to the scenario where the best fit over each interval is obtained separately using the traditional least squares procedure (as presented in the previous section). If the specified function $f(x)$ is a quadratic, then the knots are equidistant apart and are defined by $u_p = u_0 + \frac{p(u_N - u_0)}{N}$, where $p = 1, 2, \dots, N-1$. The values of a 's and b 's can be calculated from:

$$a_p = r\alpha_p + t$$

$$b_p = r\beta_p + s$$

where r , s , and t are coefficients of the original quadratic equation $f(x) = rx^2 + sx + t$. The values of α and β are derived from the following formula.

$$\alpha_p = -\frac{1}{6}[(u_p + u_{p-1})^2 + 2u_p u_{p-1}]$$

$$\beta_p = u_p + u_{p-1}$$

Bellman [Be 61] made the following observation in Stone's derivation. For fixed $a = a_0$ and $b = a_{N+1}$, where $b \geq a$,

$$f_N(b) = \min_{[a,b,u_i]} F(a_1, \dots, a_N; b_1, \dots, b_N; u_1, \dots, u_N)$$

then

$$f_1(b) = \min \left[\int_a^{u_1} (g(x) - a_1 - b_1 x)^2 dx + \int_{u_1}^b (g(x) - a_2 - b_2 x)^2 dx \right]$$

where the minimum is computed over:

$$-\infty < a_1, a_2, b_1, b_2 < \infty, a \leq u_1 \leq b.$$

This function is readily determined since the minima can be computed over the a_i and b_i and then minimized over u_i by means of a discrete search. For $N \geq 2$,

$$f_N(b) = \min \left[\min_{[a_N, b_N]} \int_{u_N}^b (g(x) - a_N - b_N x)^2 dx + f_{N-1}(u_N) \right]$$

$$f_N(b) = \min_{a \leq u_N \leq b} [h(u_N, b) + f_{N-1}(u_N)].$$

Gluss [Gl 61] [Gl 62] reiterated the problem defined by Stone in simpler terms. If the number of points subdividing the given interval $[\alpha, \beta]$ is given, say n , then the main task is to find the n line segments which will give the best least squares approximation to a function $f(x)$ defined in that interval. Although Gluss' approach is similar to Bellman's, he also offered an equation for u_j which may be determined by an equality sign instead of a minimization. He also showed how dynamic programming produces computationally simple equations for the model in which the ends of the line segments are constrained to lie on the curve $f(x)$.

Phillips [Ph 68] used a different set of premises than Gluss, Bellman, and Stone. His approach was to determine an approximation to a given function $f(x)$ by k line segments with any pre-assigned accuracy. The approximation is obtained using the minimax norm². If $f''(x) > 0$ on the interval $[\alpha, \beta]$, then it is known that the

²Minimizing the maximum error norm.

straight line $px + q$ which minimizes the maximum error $|f(x) - (px + q)|$ on $[\alpha, \beta]$ satisfies the following equations,

$$f(\alpha) - (p\alpha + q) = \epsilon \quad (1)$$

$$f(\xi) - (p\xi + q) = -\epsilon \quad (2)$$

$$f(\beta) - (p\beta + q) = \epsilon \quad (3)$$

$$f'(\xi) - p = 0 \implies p = f'(\xi) \quad (4)$$

where ξ is an interior point of $[\alpha, \beta]$ and ϵ is the maximum error. Davis [DaP 63] showed that for a given α and β , the equations above have unique solutions for p , q , ξ , and ϵ , and thus, for a given α and ϵ , the equations have unique solutions for p , q , ξ , and β . The claim above can be informally proved by first subtracting equations (1) and (2), and then substituting (4) into the result to give,

$$G(\xi) = f'(\xi)(\xi - \alpha) + f(\alpha) - f(\xi) - 2\epsilon = 0 \quad (5)$$

$$G'(\xi) = f''(\xi)(\xi - \alpha) + 1 \cdot f'(\xi) - f'(\xi) \quad (6)$$

$$= f''(\xi)(\xi - \alpha) \quad (7)$$

so $G'(\xi) > 0$ for $\xi > \alpha$. Therefore, equation (5) has at most one solution $\xi > \alpha$. Since from (5), $G(\alpha) < 0$, a solution ξ of (5) will exist if and only if $G(b) \geq 0$. If (5) has a solution, then p can be found from (4), and q from (1). Equation (3) provides a way to determine β , say $H(\beta) = 0$ and $H'(\beta) = f'(\beta) - f'(\xi) > 0$ for $\beta > \xi$ implying that there is at most one solution for β . Since $H(\xi) < 0$, a solution will exist on $[\xi, \beta]$ if and only if $H(b) \geq 0$. Algorithms based on these ideas can be found in [Ph 68].

Pavlidis [Pa 73] also presented a waveform segmentation technique using linear approximation which is based on discrete optimization. Let $(x_0, f_0), (x_1, f_1), \dots, (x_N, f_N)$ be the pairs of points describing the waveform to be analysed in a quantized form. Also, let $u_1^k, u_2^k, \dots, u_n^k$ be the dividing points at the k^{th} iteration where $u_0 = x_0$ and $u_n = x_N$. Without loss of generality, let u_j^k be

last point of the j^{th} segment. At each iteration, a new set of dividing points is selected such that the error norm³, e_j , is minimized, where e_j is the error norm of the j^{th} segment. Since at each iteration the error norm is never getting larger and there is a finite number of discrete points describing $f(x)$, the dividing points will eventually converge on the local maxima or minima of the function. The number of iterations that must be carried out depends on the initial set of dividing points and the number of dividing points.

Stone's solution is derived from using classical mathematical techniques, and is applied only to continuous functions. Bellman's approach is based on Stone's work, but it also provided a dynamic programming technique to derive an approximated function. Gluss's work is similar to Bellman's, but he also introduced an additional constraint to the problem by requiring the endpoints of the approximated functions to be in a predefined and fixed position. Phillip's approach is based on a pre-assigned error accuracy of the approximation, and thus, given a maximum allowable error, his scheme yields the line segments that will best approximate a given function. Pavlidis provided an alternative to the dynamic programming technique introduced by Bellman by using a linear incremental search which is dependent on the initial dividing points and the size of the linear increment. All of these techniques, with the exception of Pavlidis's work, can only be applied to simple one-dimensional, continuous, well-defined functions where the approximated points do not necessarily lie on $f(x)$. Many other piecewise linear approximation methods including those introduced by Ramer[Ram 72], Pavlidis and Horowitz [PH 74], Kurozumi and Davis [KD 82], and Dunham[Du 86], apply to both one-dimensional functions and planar curves. These techniques will be examined in Chapter 3.

³For uniform approximation, error norm is defined as the minimum of the maximum absolute values of the pointwise errors, and for integral square error approximation, error norm is defined as the sum of squares of the pointwise errors.

2.1.4 Polynomial splines

Piecewise linear approximation techniques do not provide a visually aesthetically pleasing transition from one segment to the next, and thus, they lack the ability to smoothly interpolate the points that have been removed from the approximated functions. Let $f(x)$ be a discrete function defined in the interval $[a, b]$ where $D = \{a = x_0, x_1, \dots, x_{n-1} = b\}$. A smooth⁴ curve passing through these points is the Lagrange interpolating polynomial:

$$P(x) = \sum_{i=0}^{n-1} \frac{L(x)}{x - x_i} \frac{f_i}{L'(x_i)}$$

where

$$L(x) = \prod_{i=0}^{n-1} (x - x_i)$$

The main problem with the Lagrange polynomial approximation shown above is that as the number of points increases these polynomials tend to oscillate strongly with a direct dependence on the position of the points (x_i, f_i) . For example, if the set of points are given as $\{(0, 6), (6, 6), (8, 4), (10, 6), (14, 3), (16, 3), (20, 2)\}$, then P takes on functional values greater than 30 in the first interval $[0, 6]$.

As a compromise between a linear approximation and a polynomial of higher degree, a combination of low degree polynomials and a function that is often differentiable on the closed interval $[a, b]$ is used. If the set of polynomials of degree $2m + 1$ defined in the intervals $[x_i, x_{i+1}]$, where $m \leq n - 1$ and $i = 0, 1, \dots, n - 2$ are joined at nodes $x_1, \dots, x_{n-2}$ in such a way that the resulting function is always $2m$ times differentiable at these nodes, then the function defined in $[a, b]$ is called a spline function of degree $2m + 1$. If $s(x)$ represents the spline function and $s(x_i) = f_i$ for $i = 0, 1, \dots, n - 1$, then $s(x)$ is called an interpolating spline function. By this definition, a polygonal path through $\{(x_0, f_0), (x_1, f_1), \dots, (x_{n-1}, f_{n-1})\}$ is a first order interpolating spline. A third order spline or cubic spline consists of $n - 1$ third

⁴Smooth in the sense of being maximally often differentiable.

degree polynomials connected at the points $\{(x_1, f_1), (x_2, f_2), \dots, (x_{n-2}, f_{n-2})\}$ such that they are always twice differentiable at these nodes [Sp 74].

Let $s(x)$ be an interpolating spline function defined in the interval $[x_0, x_{n-1}]$ where $x_0 < x_2 < \dots < x_{n-1}$ and $s(x_i) = f_i$. The spline $s(x)$ consists of $n - 1$ cubic polynomials ϕ_k joined at $x_1, x_2, \dots, x_{n-2}$ such that it is twice differentiable at these joints. Furthermore, let the first and second derivatives of $s(x)$ evaluated at x_i be denoted by f'_i and f''_i , and consider a cubic polynomial $\phi_k(x)$ defined in the fixed interval $[x_k, x_{k+1}]$.

$$\phi_k(x) = A_k(x - x_k)^3 + B_k(x - x_k)^2 + C_k(x - x_k) + D_k$$

then there are $4(n - 1)$ coefficients A_k , B_k , C_k , and D_k to be determined. The function values and the first and second derivatives of the spline at the endpoints x_k and x_{k+1} are given as

$$\begin{aligned} f_k &= \phi_k(x_k) = D_k \\ f_{k+1} &= \phi_k(x_{k+1}) = A_k \Delta x_k^3 + B_k \Delta x_k^2 + C_k \Delta x_k + D_k \\ f'_k &= \phi'_k(x_k) = C_k \\ f'_{k+1} &= \phi'_k(x_{k+1}) = 3A_k \Delta x_k^2 + 2B_k \Delta x_k + C_k \\ f''_k &= \phi''_k(x_k) = 2B_k \\ f''_{k+1} &= \phi''_k(x_{k+1}) = 6A_k \Delta x_k + 2B_k \end{aligned}$$

The coefficients can now be described in terms of the function values and second derivatives,

$$\begin{aligned} A_k &= \frac{1}{6\Delta x_k}(f''_{k+1} - f''_k) \\ B_k &= \frac{1}{2}f''_k \\ C_k &= \frac{\Delta f_k}{\Delta x_k} - \frac{1}{6}\Delta x_k(f''_{k+1} + 2f''_k) \\ D_k &= f_k \end{aligned}$$

The first derivative values can thus be rewritten, with the substitution of the coefficients, as

$$\begin{aligned} f'_k &= \frac{\Delta f_k}{\Delta x_k} - \frac{1}{6}\Delta x_k(f''_{k+1} + 2f''_k) & (k = 0, \dots, n-2) \\ f'_{n-1} &= \frac{\Delta f_{n-1}}{\Delta x_{n-1}} - \frac{1}{6}\Delta x_{n-1}(2f''_{n-1} + f''_{n-2}) \end{aligned}$$

A set of $n - 2$ simultaneous equations for the n unknowns f_k'' can be formed by applying the requirement of continuity in the first derivative.

$$o'_{k-1}(x_k) = o'_k(x_k) \quad (k = 1, \dots, n - 2)$$

Since there are no requirements on the two endpoints x_0 and x_{n-1} , the second derivative values f_0'' and f_{n-1}'' are often assumed to be given, and the set of equations can be represented in a matrix form as follows.

$$\begin{bmatrix} 2(\Delta x_0 + \Delta x_1) & \Delta x_1 & & & \\ \Delta x_1 & 2(\Delta x_1 + \Delta x_2) & \Delta x_2 & & \\ & \Delta x_2 & 2(\Delta x_2 + \Delta x_3) & \Delta x_3 & \\ & & \ddots & \ddots & \\ & & & \Delta x_{n-3} & 2(\Delta x_{n-3} + \Delta x_{n-2}) \end{bmatrix} \cdot \begin{bmatrix} f_1'' \\ f_3'' \\ f_4'' \\ \vdots \\ f_{n-2}'' \end{bmatrix} = \begin{bmatrix} 6(\frac{\Delta f_1}{\Delta x_1} - \frac{\Delta f_0}{\Delta x_0}) - \Delta x_0 f_0'' \\ 6(\frac{\Delta f_2}{\Delta x_2} - \frac{\Delta f_1}{\Delta x_1}) \\ 6(\frac{\Delta f_3}{\Delta x_3} - \frac{\Delta f_2}{\Delta x_2}) \\ \vdots \\ 6(\frac{\Delta f_{n-2}}{\Delta x_{n-2}} - \frac{\Delta f_{n-1}}{\Delta x_{n-1}}) - \Delta x_{n-2} f_{n-1}'' \end{bmatrix}$$

Once the second derivative values are determined, the $n - 1$ polynomials can be found by substituting into the equations of the coefficients presented above. A more detailed analysis of the derivation of the cubic spline is given by Bellman [Be 61] and Späth [Sp 74].

B-splines are piecewise polynomial curves (usually cubic) which are related to a guiding polygon. B-spline approximations not only have good computational properties but also representational ones over linear and polynomial approximations [BB 82]. First, they are *variation diminishing*⁵; the curve is, therefore, assured to lie between the convex hull of groups of $k + 1$ consecutive points where k is the degree of the interpolating polynomial. Secondly, the interpolation of the curve is

⁵Variation diminishing means that the curve is guaranteed to “vary less” than its guiding polygons (for example, most polynomial approximation schemes have a tendency to oscillate between sample points).

Figure 5: B-spline of degree k must lie in the convex hull formed by consecutive groups of $k + 1$ points. (a) $k = 1$ (linear). (b) $k = 2$ (quadratic). (c) $k = 3$ (cubic).

local: if a point of the guiding polygon is shifted, then the effects are restricted to nearby points on the spline and not the entire approximation.

Let $\mathbf{x}(s)$ be a vector valued piecewise polynomial function interpolating through a discrete set of points $x_0, x_1, \dots, x_n$. Each piece of $\mathbf{x}$ is a cubic polynomial which implies that $\mathbf{x}(s)$ has three orders of continuity across the data points. Formally, $\mathbf{x}(s)$ is defined as

$$\mathbf{x}(s) = \sum_{i=0}^n v_i \mathbf{B}_i(s)$$

where v_i 's are not only the coefficients representing the curve $\mathbf{x}(s)$ but also the vertices of the guiding polygons, and $\mathbf{B}_i$'s are the basis functions.

The basis functions $\mathbf{B}_i$'s are non-negative, and each $\mathbf{B}_i$ is non-zero only for s between $i - 2$ and $i + 2$. Thus, on a given span $(i, i + 1)$ there are only four basis functions that are non-zero, namely: $\mathbf{B}_{i-1}$, $\mathbf{B}_i$, $\mathbf{B}_{i+1}$, and $\mathbf{B}_{i+2}$. For example, to determine $\mathbf{x}(s_0)$ for some s_0 , all that is required is to determine which span s_0 resides in, and a summation of the four terms is performed using the summation

equation presented above since only four basis functions are non-zero there.

The basis functions $B_i(s)$ are, themselves, piecewise cubic polynomials. If the spans under their support are of uniform size, then all the basis functions have the same form and are simply translates of each other. Moreover, each of the basis functions, on its non-zero support, is made of four pieces. For example, in the span $(i, i+1)$, the fourth piece of $B_{i-1}(s)$, the third piece of $B_i(s)$, the second piece of $B_{i+1}(s)$, and the first piece of $B_{i+2}(s)$ are used to determine $\mathbf{x}(s)$. Let these pieces be represented by $C_{i,0}$, $C_{i,1}$, $C_{i,2}$, and $C_{i,3}$ respectively. Then $\mathbf{x}(s)$ on the interval $(i, i+1)$ can be written as

$$\mathbf{x}(s) = C_{i-1,3}(s)v_{i-1} + C_{i,2}(s)v_i + C_{i+1,1}(s)v_{i+1} + C_{i+2,0}(s)v_{i+2}$$

Using the fact that the spans are uniform, $C_{i,j}$ will have the same shape no matter what the value of i happens to be. This property permits a simplification in calculations, and therefore, the four primitive basis functions can be defined as

$$C_{i,j}(s) = C_j(s-i) \quad i = 0, \dots, n+1; \quad j = 0, 1, 2, 3$$

and interpolation along the curve is done by parameter shifting. For example, to find $\mathbf{x}(s)$ where s is in the span $(i, i+1)$ then the following formula is used:

$$\mathbf{x}(s) = v_{i-1}C_3(s-i) + v_iC_2(s-i) + v_{i+1}C_1(s-i) + v_{i+2}C_0(s-i)$$

where the $C_i(t)$ are given by:

$$\begin{aligned} C_0(t) &= \frac{t^3}{6} \\ C_1(t) &= \frac{-3t^3 + 3t^2 + 3t + 1}{6} \\ C_2(t) &= \frac{3t^3 - 6t^2 + 4}{6} \\ C_3(t) &= \frac{-t^3 + 3t^2 - 3t + 1}{6} \end{aligned}$$

A more formal derivation of B-spline is given by [BR 74] and de Boor [deB 78].

2.2 Transformed Domain Function Approximation

Transform coding techniques determine the approximation of f by *transforming* it into another function, say, F , approximating F , and inverting the transformed approximation to derive $\hat{f}(x)$. These approximation techniques manipulate on the *transformed* function F and thus operate in the domain of the transformed function. Most of the lossy signal compression techniques currently used by industry employ a transform coding technique. The JPEG and MPEG compression schemes, picture phone transmission, and even some of the new high definition television standards (currently being considered by the United States and Canada), for example, employ a discrete cosine transform technique to reduce the transmission of data.

A transform technique often sub-divides the original signal into smaller blocks, and each block is then passed through a *unitary transform*⁶. The main purpose of the transform is to decorrelate the original signal, and the decorrelation generally results in the signal energy being redistributed among only a small set of transform coefficients. [RJ 91]. As indicated by Pavlidis [Pa 77], however, transform representations are economical if and only if the input signal can be represented by a set of orthonormal basis vectors whose coefficients are mostly zero or close to zero.

There are two views of looking at an n -point unitary transform. It can be interpreted as a decomposition of the original block of data into a set of n orthogonal basis functions. Another view is that the transform is a rotation of the n -dimensional coordinate axes defined by the signal values in a block. These views can be further explained by the examples below.

Suppose there is a sequence of two numbers, $x(0) = 2$ and $x(1) = 3$. This sequence can be approximately represented by $\{2.5, 2.5\}$, which can be written as

$$\{2.5, 2.5\} = \frac{5}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\} = \frac{5}{\sqrt{2}} \{b_{00}, b_{01}\}$$

The original sequence $x(n)$ can be exactly represented by adding to it a sequence

⁶A unitary transform is a reversible linear transform whose kernel describes a set of complete, orthonormal discrete basis functions.

$\{-0.5, +0.5\}$, which can be expressed as

$$\{-0.5, +0.5\} = \frac{-1}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\} = \frac{-1}{\sqrt{2}} \{b_{10}, b_{11}\}$$

Thus, the original sequence $x(0)$ and $x(1)$ can be represented as a linear combination of the basis vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ as

$$\begin{bmatrix} x(0) \\ x(1) \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

where $\mathbf{b}_0^T = [b_{00} \ b_{10}] = [\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}]$ and $\mathbf{b}_1^T = [b_{01} \ b_{11}] = [\frac{1}{\sqrt{2}} \ \frac{-1}{\sqrt{2}}]$

The example above shows that the two points are decorrelated, that is, they are separated out into high and low components. Thus, transform coding can be viewed as a decomposition of the original values into different weights of a set of orthogonal basis functions. The basis functions are different for each type of signal transform, and can be considered as elementary components that can be used to synthesize a signal given its corresponding transform coefficients.

An n -point transformation can also be viewed as an n -dimensional rotation. Suppose there is a set of n highly correlated points as displayed in Figure 6, where most of them lie along the 45° line defined by $y = x$. The variance in the x direction is defined as

$$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

and similarly, the variance in the y direction is defined as

$$\sigma_y^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2$$

A simple method of compression is to represent one of the components (either x or y) by its corresponding mean value $\bar{x}$ or $\bar{y}$, respectively, and the mean-squared error (MSE) introduced in the reconstructed signal is thus equal to σ_x^2 or σ_y^2 . Since the variance in both the x and y directions is large, an encoding technique using the mean value will result in large reconstruction errors. If each point, however, is

Figure 6: Rotation transform example

put through a unitary transformation which consists of a counter clockwise rotation about the origin by 45° .

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

then the variance in the y' direction is much less than that in the x' direction ($\sigma_{y'} \ll \sigma_{x'}$). An important feature of a unitary transform is that it is distance preserving; that is, the Euclidian distance between vectors is not changed by the transform. This fact can be used to show that the total variance is preserved by the rotation,

$$\sigma_x^2 + \sigma_y^2 = \sigma_{x'}^2 + \sigma_{y'}^2$$

Therefore, if the y' component is replaced by its mean value (which is zero) during encoding, the resultant MSE is only $\sigma_{y'}^2$, which is significantly less than either σ_x^2 or σ_y^2 .

In all transform coding schemes, the transform operation does not achieve any compression; the purpose of the transform is to decorrelate the original data and to compact a large portion of the signal energy into a few significant number of transform coefficients. Since most of these transform coefficients contain very little information, compression is mainly achieved by quantizing the coefficients. The level of compression strictly depends on the quantization factor; the coarser the quantization, the higher the compression ratio, but the degradation of the signal will also become more noticeable.

Another property of a unitary transform is that it is reversible; the original data can be recovered (assuming that the encoding process does not introduce any errors) by applying an *inverse transform* on the quantized transform coefficients. The inverse transform for the example above is a rotation about the origin by 45° in the clockwise direction.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Since the transform coefficients are quantized, the inverse transform on these coefficients will result in an approximation to the original data. As indicated by Rabbani and Jones [RJ 91], “because of the distance-preserving property of the rotation, the MSE between the original image and the reconstructed data is equal to the MSE introduced by the quantization process in the transform domain.”

For reasons of convenience and simplicity, all unitary transforms are expressed in a matrix notation. A sequence of n sample values $f_0, f_1, \dots, f_{n-1}$ is considered as a vector in n -dimensional space denoted by $\mathbf{f}$, and similarly, the sequence n transformed values $F_0, F_1, \dots, F_{n-1}$ is denoted by $\mathbf{F}$. Thus, $\mathbf{F}$ and $\mathbf{f}$ can be expressed as

$$\mathbf{F} = \mathbf{T} \cdot \mathbf{f}$$

$$\mathbf{f} = \mathbf{T}^{-1} \cdot \mathbf{F}$$

where $\mathbf{T}$ and $\mathbf{T}^{-1}$ are the forward and inverse transformation matrices, respectively.

The following four approximation techniques will be reviewed in this section: Karhunen-Loeve transform, Fourier transform, Walsh-Hadamard transform, and discrete cosine transform. Although most of these techniques can be applied to both continuous and discrete domains, only the discrete cases will be presented in details.

2.2.1 Karhunen-Loeve Transform

It is helpful to first consider the Karhunen-Loeve transform from an intuitive point of view. If a sinusoidal signal is transmitted over a medium in a sampled form, with each sampled data point being sent in a sequential manner, then a higher sampling rate will yield a superior reconstruction of the waveform. From a theoretical point of view, the information content of the sampled values is very low due to the fact that the sampled values are highly correlated. On the other hand, it is also well known that the reconstruction of the original signal can be accomplished by knowing its magnitude, phase, frequency, starting time, and the fact that it is sinusoid. Therefore, assuming that there are no errors in the transmission of these five uncorrelated values, the exact same signal can be reconstructed at the receiver end. Thus, the set of highly correlated sampled values and the five uncorrelated parameters represent the same signal. This fact naturally leads to the question of whether or not there is a transformation which can decorrelate the sampled values into an uncorrelated set of values that are equivalent to the five parameters listed above. The Karhunen-Loeve transform is considered to be in this category of transforms. A summary of the Karhunen-Loeve transform (KLT) will be presented below, and more detailed analyses of its derivation are done by Jain [Ja 85], Rao and Yip [RY 90], and Jayant and Noll [JN 84].

Let the N sampled values of a zero mean random vector $\mathbf{f}$ be given as $\mathbf{f} = \{f_0, f_1, \dots, f_{N-1}\}^T$. If there exists a set of linearly independent vectors, ϕ_k , spanning

the N -dimensional vector space, then $\mathbf{f}$ can be expressed in terms of ϕ_k as

$$\mathbf{f} = \sum_{k=0}^{N-1} F_k \phi_k$$

where F_k are the coefficients of expansion. Each coefficient itself is also stated in terms of the given input $\mathbf{f}$ and ϕ_k as

$$F_k = \frac{\langle \mathbf{f}, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}$$

where $k = 0, 1, \dots, N-1$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. Suppose that only the first n coefficients, where $n < N$, are significantly greater than zero, then the vector $\mathbf{f}$ can be well represented by only n coefficients in the ϕ_k space. The approximation of $\mathbf{f}$, $\hat{\mathbf{f}}$, can thus be expressed as

$$\hat{\mathbf{f}} = \sum_{k=0}^{n-1} F_k \phi_k$$

The Karhunen-Loeve transform's main objective is to determine the best approximation of $\mathbf{f}$ in the mean square error sense. If ϵ is the mean square error in the truncation, then it can be seen that,

$$\epsilon = E[(\mathbf{f} - \hat{\mathbf{f}})^2]$$

where $E[\cdot]$ is the expectation operator. By minimizing ϵ (i.e. differentiating with respect to ϕ_k partially, and setting the equations to zero), the Karhunen-Loeve transform of $\mathbf{f}$ is a matrix ϕ composed of the eigenvectors of $\mathbf{A}$ and is defined by the relation

$$\phi \mathbf{A} \phi^T = \lambda$$

where λ is a diagonal matrix of eigenvalues λ_k and $\mathbf{A}$ is the auto-covariance matrix (also referred to as the auto-correlation matrix), defined by $\mathbf{A} = E[\mathbf{x}\mathbf{x}^T]$. The set of basis vectors $\{\phi_0, \phi_1, \dots, \phi_{N-1}\}$ form the bases for the Karhunen-Loeve transform expansion.

As summarized by Rao and Yip [RY 90], the Karhunen-Loeve transform is considered to be an optimal transform because of the following properties:

1. It completely decorrelates the signal in the transform domain.
2. It minimizes the MSE in bandwidth reduction or data compression.
3. It contains the most variance (energy) in the fewest number of transform coefficients.
4. It minimizes the total representation entropy of the sequence

It is mainly used as a measuring tool against other suboptimal transforms such as the Fourier transform, the Walsh-Hadamard transform, and cosine transforms. In practice, the Karhunen-Loeve transform is not often applied because it is dependent on the input data; that is, it is difficult to determine the auto-covariance matrix and its diagonalization. Furthermore, the lack of pre-determined basis vectors in the transform domain has made it an ideal but impractical technique.

2.2.2 Fourier Transform

Fourier's theorem states that for any function $f(x)$ defined in the interval $-\infty < x < \infty$, it is possible to express it as a summation of a series of sine and cosine terms of increasing frequency. The Fourier transform of the function $f(x)$, $F(\omega)$, describes the amount of each frequency term that must be added together to make $f(x)$. It is often described as [RY 90] [JN 84],

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

where $j = \sqrt{-1}$. The exponential notation above relies on the mathematical identity (known as Euler's formula), $e^{-j\omega x} = \cos \omega x - j \sin \omega x$. In certain instances, it may be more convenient to express the Fourier transform in terms of sine and cosine as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \omega x dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j f(x) \sin \omega x dx$$

The original function $f(x)$ can be expressed in terms of $F(\omega)$ as

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega$$

The function $f(x)$ is generally a real-valued function such as time-varying voltage or spacially varying image brightness. The function $F(\omega)$, on the other hand, is generally complex, consisting of a real part and an imaginary part, and it is sometimes expressed as $F(\omega) = R(\omega) + jI(\omega)$. It can also be expressed in a convenient polar form as $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ where $|F|$ is called the magnitude and ϕ is called the phase.

An interesting characteristic of the Fourier transform is that if the symmetry of the input signal is known, its corresponding transform will be greatly simplified. A function $f(x)$ is defined to be *even* with respect to $x = 0$ if $f(-x) = f(x)$, and $f(x)$ is said to be *odd* if $f(-x) = -f(x)$. If $f(x)$ is even, then its corresponding Fourier transform will consist of only the cosine terms,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \omega x dx$$

On the other hand, if $f(x)$ is odd, then its corresponding Fourier transform will consist of only the sine terms,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j f(x) \sin \omega x dx$$

The integrals from $-\infty$ to $+\infty$ will in practice be converted to summations of discrete values of f_k and F_u . The discrete Fourier transform is written as

$$F_u = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_k e^{-j2\pi uk/n}$$

where $u = 0, 1, \dots, n-1$, and n is the number of uniformly spaced sampled points along the function $f(x)$. The inverse Fourier transform is similar to the forward transform except for the change in sign in the exponential term,

$$f_k = \frac{1}{\sqrt{n}} \sum_{u=0}^{n-1} F_u e^{j2\pi uk/n}$$

where $k = 0, 1, \dots, n-1$. The value u represents the number of discrete frequency components added together to construct the sampled value f_k .

As indicated earlier, the forward and inverse transformations can be expressed in a matrix notation as

$$\mathbf{F} = \mathbf{T} \cdot \mathbf{f}, \text{ and}$$

$$\mathbf{f} = \mathbf{T}^{-1} \cdot \mathbf{F}$$

where

$$\mathbf{T} = \left\{ \frac{1}{\sqrt{N}} e^{-j2\pi uk/n} \right\}_{k,u=0,1,\dots,n-1}.$$

Figure 7 shows the basis functions for the discrete Fourier transform where $n = 8$, and below are the forward and inverse Fourier transformation matrices of sizes $n = 2$ and $n = 4$,

$${}^2\mathbf{T} = {}^2\mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^0 & e^0 \\ e^0 & e^{-j\pi} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$${}^4\mathbf{T} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$${}^4\mathbf{T}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

The inverse transformation kernel is the conjugate transpose of the forward kernel. Although, in general, the matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$ have complex elements of the form $a + jb$, a *conjugate symmetry* property can be applied if the input values are real (i.e. $\mathbf{f} \in \mathbb{R}^n$). As a consequence, the discrete Fourier transform representation involves a total of only n and not $2n$ elements. The conjugate symmetry can be described by

$$F_u = F_{n-u}^*; u = 1, 2, \dots, (n/2) - 1$$

where F^* denotes the complex conjugate of F . The complex values of F_u^* can be expressed in separate real and imaginary values as

$$Re[F_u] = Re[F_{n-u}]; u = 1, 2, \dots, (n/2) - 1,$$

$$\text{Im}[F_u] = -\text{Im}[F_{n-u}]; u = 1, 2, \dots, (n/2) - 1, \text{ and,}$$

$$\text{Im}[F_0] = -\text{Im}[F_{n/2}] = 0.$$

In the case where the number of samples, n , is even, the number of nonredundant real-valued numbers in the Fourier transform representation is n , since F_0 and $F_{n/2}$ are real-valued, and the remaining $n - 2$ components have a total of $n - 2$ nonredundant numbers. Therefore, for n input samples, there are exactly n independent real values representing the transform coefficients that must be quantized and processed. For example, if $n = 4$ and the input values, expressed in a vector format, are $\mathbf{f}^T = (3, -1, 4, 2)$, then the transformed values are $\mathbf{F}^T = (4, 1/2 + 3j/2, 3, -1/2 - 3j/2)$. It can be clearly seen that the corresponding Fourier transform representation has exactly 4 nonredundant numbers, $-1/2$, $3/2$, 4, and 3.

2.2.3 Hadamard and Walsh-Hadamard Transforms

The Hadamard and Walsh-Hadamard transforms are not as efficient in an energy-packing sense as the other transforms described above. They are characterized by a symmetrical matrix whose elements are ± 1 , and their basis vectors are orthonormal. The term *sequency* is defined as one-half the number of changes in sign in one period of a sequence. For example, if the sequence is $\{2, -3, 1, -4\}$, then its sequency is $3/2$. The sequency of a rectangular basis waveform in the Walsh-Hadamard transform has the same connotation as the frequency of a trigonometric sine and cosine waveform as in the Fourier transform. The Hadamard transformation matrices are only defined for input sequence of sizes 2^p or $4p$ for any positive integral values of p . The Walsh transformation matrices are similar to the Hadamard matrices except that they can be of any size and that the ordering of the basis vectors is slightly different. The term Walsh-Hadamard transform is generally used to describe any transform that has basis vectors with elements ± 1 and that its size, n , is some

Figure 7: Basis functions for the discrete Fourier transform where $n = 8$.

power of 2.

Let $\mathbf{H}$ and $\mathbf{H}^{-1}$ be the Hadamard transformation matrix and its inverse, respectively. Then they can be defined as

$$\begin{aligned}
 {}^2\mathbf{H} &= {}^2\mathbf{H}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
 {}^4\mathbf{H} &= {}^4\mathbf{H}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\
 {}^{2n}\mathbf{H} &= {}^{2n}\mathbf{H}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} {}^n\mathbf{H} & {}^n\mathbf{H} \\ {}^n\mathbf{H} & -{}^n\mathbf{H} \end{bmatrix}
 \end{aligned}$$

for $n = 2, 3, \dots$. As indicated above, the recursive definition of the Hadamard transform requires that the number of sampled values are always 2^p for positive integral p . We note, in passing, that Hadamard matrices of other sizes such as

12, 16, ..., 0 modulo 4, ... may also exist.

Let $\mathbf{W}$ and $\mathbf{W}^{-1}$ be the Walsh transformation matrix and its inverse, respectively.

$$\begin{aligned} {}^2\mathbf{W} &= {}^2\mathbf{W}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ {}^3\mathbf{W} &= {}^3\mathbf{W}^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ {}^4\mathbf{W} &= {}^4\mathbf{W}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \end{aligned}$$

The Walsh transform, ${}^n\mathbf{W}$, and its inverse, ${}^n\mathbf{W}^{-1}$, are characterized by the basis vector sequences of $(0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{n-1}{2})$.

The Walsh-Hadamard transforms do not diagonalize covariance matrices which are Toeplitz, and thus, they are suboptimal in an energy-packing sense (not even asymptotically). They do possess some decorrelating characteristics, and they are quite popular due to their simplicity.

2.2.4 Discrete Cosine Transform

The discrete cosine transform is derived from the discrete Fourier transform. It is widely used in many lossy signal compression techniques, including the JPEG and MPEG standards adopted by industry in the image processing field. Although it is not as optimal as the Karhunen-Loeve transform, it is more efficient in an energy-packing sense than most of the other transform techniques such as the Fourier and the Walsh-Hadamard transforms.

The discrete cosine transform provides a way of performing frequency analysis, but yet it avoids the complex valued results generated from the Fourier transform by exploiting a unique feature of symmetric functions and their Fourier transforms. If the n input sample is given as $\{f_0, f_1, \dots, f_{n-1}\}$, then the discrete cosine transform

artificially generates a symmetric $2n$ -element signal by appending a reversed copy of the signal onto itself. The $2n$ input values, $\{f_0, f_1, \dots, f_{n-1}, f_{n-1}, f_{n-2}, \dots, f_1, f_0\}$ will be put through a variant of the discrete Fourier transform. As indicated earlier in Section 2.2.2, since the input values are known to be an even function, all of the coefficients for the sine waveforms will be zero, and thus, the transformed values, $\{F_0, F_1, \dots, F_{2n-1}\}$ will be real instead of complex. Moreover, only the first n of the $2n$ transformed coefficients need to be kept since the remaining n values are the conjugate of the previous n coefficients.

The discrete cosine transform and its inverse can be defined in a more formal setting than the description above as [ANR 85],

$$F_k = \sqrt{\frac{2}{n}} \alpha(k) \sum_{i=0}^{n-1} f_i \cos \frac{(2i+1)k\pi}{2n}; \quad k = 0, 1, \dots, n-1$$

and

$$f_i = \sqrt{\frac{2}{n}} \alpha(k) \sum_{k=0}^{n-1} F_k \cos \frac{(2k+1)i\pi}{2n}; \quad i = 0, 1, \dots, n-1$$

for $\alpha(0) = \frac{1}{\sqrt{2}}$ and $\alpha(k) = 1$ for $k \neq 0$. Similar to the Fourier transform, the discrete cosine transform and its inverse can be expressed in a matrix format as

$$\mathbf{F} = \mathbf{C}\mathbf{f}; \quad \mathbf{f} = \mathbf{C}^{-1}\mathbf{F}$$

where $\mathbf{C}$ is the forward transformation matrix whose basis vectors are

$$\mathbf{c}_k = \left\{ \sqrt{\frac{2}{n}} \alpha(k) \cos \frac{(2i+1)k\pi}{2n} \right\}_{i=0,1,\dots,n-1}$$

for $k = 0, 1, \dots, n-1$. The inverse transformation matrix is simply the transpose of the forward transformation matrix, that is,

$$\mathbf{C}^{-1} = \mathbf{C}^T$$

For $n = 2$, the forward and inverse discrete cosine transforms are identical to the Karhunen-Loeve and Fourier transforms,

$${}^2\mathbf{C} = {}^2\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos \frac{\pi}{4} & \cos \frac{3\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For $n = 4$, the discrete cosine transformation matrix and its inverse are,

$${}^4\mathbf{C} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & -b & -a \\ 1 & -1 & -1 & 1 \\ b & -a & a & -b \end{bmatrix}$$

$${}^4\mathbf{C}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & a & 1 & b \\ 1 & b & -1 & -a \\ 1 & -b & -1 & a \\ 1 & -a & 1 & -b \end{bmatrix}$$

where $a = \sqrt{2} \cos \frac{\pi}{8}$ and $b = \sqrt{2} \cos \frac{3\pi}{8}$. The basis functions for the discrete cosine transform for $n = 8$ is shown in Figure 8.

Figure 8: Basis functions for the discrete cosine transform where $n = 8$.

Let $\tilde{f}_j$ represent the $2n$ -element samples with even symmetry defined by

$$\tilde{f}_j = \begin{cases} f_j & j = 0, 1, \dots, n-1 \\ f_{2n-1-j} & j = n, n+1, \dots, 2n-1 \end{cases}$$

Then the $2n$ discrete Fourier transform of $\tilde{f}$ is,

$$\begin{aligned}\tilde{F}_k &= \frac{1}{\sqrt{2n}} \sum_{i=0}^{2n-1} \tilde{f}_i e^{-j(2\pi k i / 2n)} \\ &= \frac{1}{\sqrt{2n}} e^{jk\pi / 2n} \sum_{i=0}^{n-1} f_i \cos\left(\frac{(2i+1)k\pi}{2n}\right).\end{aligned}$$

Comparing the discrete Fourier transform, $\tilde{F}_k$, and the discrete cosine transform, F_k , it can be clearly seen that

$$F_k = 2\alpha(k) e^{-jk\pi / 2n} \tilde{F}_k.$$

There are several reasons why the discrete cosine transform is used in many signal processing applications. It is almost as optimal as the Karhunen-Loeve transform yet its basis vectors are independent of the input signals. Compared to the Fourier transform, it does not have to handle complex numbers, and it can be computed using a fast discrete Fourier transform technique.

2.3 Results of Previous Methods

In this chapter many different approximation techniques on one-dimensional functions have been reviewed. Of the methods that have been examined, the following algorithms are selected and used as a basis to compare the schemes against each other:

1. Linear search and split (Ramer)
2. Split-and-merge algorithm (Pavlidis and Horowitz)
3. Nearest-neighbour averaging algorithm (extension of Tanimoto)
4. Hologram-like approximation algorithm (Prusinkiewicz)
5. Minimax Polygon (Kurozumi and Davis)
6. Recursive binary search and split (Duda and Hart, Lowe)

7. Discrete Fourier transform (Rao, Yip, et al.)

8. Discrete cosine transform (Rao, Yip, et al.)

These methods provide a cross section of time/spacial domain and transform domain approximation techniques. The first six methods are categorized as the time/spacial domain techniques, and the last two (as their names imply) are transform domain techniques. Within the time/spacial domain category, whereas the hologram-like and nearest-neighbour techniques use uniform knot points, the other time/spacial domain techniques use non-uniform knots. In order to impartially compare the different techniques, they have been applied to the following six one-dimensional functions:

1. $f_1(x) = 5 \sin 4x + 4 \cos 3x$, defined in interval $[0, 10]$.

2. $f_2(x) = \frac{20}{\sqrt{2\pi}} e^{-2x^2}$ defined in interval $[-6, 6]$.

3. $f_3(x)$ = Sawtooth waves defined in interval $[-12, 12]$.

4. $f_4(x) = \frac{7 \sin x}{x}$ defined in interval $[-10.2, 12.1]$.

5. $f_5(x) = \frac{7 \sin x}{x} + \text{Sawtooth waves}$ defined in interval $[-10.2, 12.1]$.

6. $f_6(x) = 5 \sin 4x + 4 \cos 3x + 2 \cos 200x$ defined in interval $[0, 10]$.

One of the main reasons why these methods were chosen is because they can be applied to both one-dimensional functions and planar curves. The error estimation for the minimax and hologram-like techniques is aimed at planar curves and areas, and as such, the errors do not necessarily reflect the visual correctness of these approximations. The techniques provided by Ramer, Pavlidis and Horowitz, Kurozumi and Davis, and Lowe are not contained in this chapter; a review of these techniques is presented in Chapter 3 in which planar curve and digitized image approximation methods are discussed.

The results of these approximation techniques are displayed and tabulated at the end of this section. The column with the heading “# of Approx. Points” is not applicable to the discrete Fourier and discrete cosine transforms because their estimation is done in the transform domain. Both of these techniques divide the original sampled data into groups of 16 data values. The transformation is carried out using these techniques, and only three of the 16 quantized coefficients are retained. The inverse transformation is then performed on the quantized coefficients to derive the approximated values in the time/spacial domain.

From the results of the different approximation techniques, it is clear that the best time/spacial domain approximation techniques are the linear split algorithm proposed by Ramer [Ram 72] and the recursive split algorithm suggested by Duda and Hart [DH 73] and Lowe [Lo 87]. These techniques derive a piecewise linear approximation with the smallest error while simultaneously requiring a minimal amount of time. One major problem with this technique is that if the original signal is noisy, it will incorporate the noise as part of the signal as shown in the approximation of function f_6 (which is but a noisy version of function f_1). The nearest-neighbour algorithm, however, is computationally efficient, yet in terms of error minimization, it is the worst time/spacial domain technique. The hologram-like approximation proposed by Prusinkiewicz [Pr 84] only works if the original function is oversampled, and since all six sampled functions provided are oversampled, this algorithm performs quite well for these cases. The split and merge algorithm proposed by Pavlidis and Horowitz [PH 74] does not achieve its goal of minimizing the number of approximation points; in fact, it introduces more unnecessary computations and approximating line segments. The minimax method proposed by Kurozumi and Davis [KD 82], in most cases, achieves its goal of minimizing the number of line segments. However, in doing so, it is computationally inefficient compared to the linear and recursive split algorithms. An interesting point must be noted with regard to the minimax technique as applied to the sawtooth waves: even though the

Euclidean distance between the actual curve and the approximating line segments is within the allowable error limit, some pointwise errors $|f_i - \hat{f}_i|$ can be quite large causing it to be unacceptable in terms of error values. Otherwise, the minimax approximation of the sawtooth waves is visually acceptable. The discrete cosine transform provides a good approximation if the original function is a smooth curve obtained from an ensemble of, for example, sine and cosine waves. However, it tends to round out the sharp edges contained in functions such as the sawtooth waves.

Figure 9: (a) Original function f_1 . (b) to (e): Approximations of function f_1 using techniques (1) to (4) listed in Section 2.3.

Figure 10: (f) to (i): Approximations of function f_1 using techniques (5) to (8) listed in Section 2.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_m)</i>	<i>MSE (E_s/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	14	216	2.359	1.1337	-11.7755
Split/Merge (Pavlidis)	30	1004	4.9756	3.9396	-17.1851
Avg. neighbours (Tanimoto)	18	52	8.9249	19.2216	-24.0685
Hologram-like (Prusinkiewicz)	18	46	3.9049	2.4967	-15.2042
Minimax (Kurozumi)	14	1664	8.3923	6.7534	-19.5258
Recursive split (Love)	14	208	2.359	1.1337	-11.7755
DFT	N/A	236	9.5971	9.8354	-21.1585
DCT	N/A	193	2.1901	0.0662	-0.5873

Figure 11: Results of the various approximation techniques applied on function f_1 .

Figure 12: (a) Original function f_2 (normal bell curve). (b) to (e): Approximations of normal bell curve using techniques (1) to (4) listed in Section 2.3.

Figure 13: (f) to (i): Approximations of normal bell curve using techniques (5) to (8) listed in Section 2.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_L)</i>	<i>MSE (E_L/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	5	129	0.73505	0.1346	11.9091
Split/Merge (Pavlidis)	12	692	0.8403	0.133	11.9599
Avg. neighbours (Tanimoto)	34	62	0.8028	0.1633	11.0704
Hologram-like (Prusinkiewicz)	18	55	0.2442	0.0055	25.7988
Minimax (Kurozumi)	7	5901	1.4142	0.2721	8.8518
Recursive split (Lowe)	5	122	0.735	0.1346	11.9091
DFT	N/A	243	1.6673	0.2059	10.0619
DCT	N/A	198	0.1275	0.002	30.2411

Figure 14: Results of the various approximation techniques applied on normal bell curve.

Figure 15: (a) Original function f_3 (sawtooth waves). (b) to (e): Approximations of sawtooth waves using techniques (1) to (4) listed in Section 2.3.

Figure 16: (f) to (i): Approximations of sawtooth waves using techniques (5) to (8) listed in Section 2.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_{∞})</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	10	200	7.11E-15	3.86E-30	301.1069
Split/Merge (Pavlidis)	18	1122	9	2.1176	3.7139
Avg. neighbours (Tanimoto)	34	64	7.71	4.2873	0.6505
Hologram-like (Pusinkiewicz)	18	45	9.8182	6.0971	-0.8789
Minimax (Kurozumi)	10	754	3.24E+01	3.27E+01	-8.1678
Recursive split (Lowe)	10	192	7.11E-15	3.86E-30	301.1069
DFT	N/A	240	3.918	1.5705	5.0118
DCT	N/A	195	3.7447	0.7844	8.0268

Figure 17: Results of the various approximation techniques applied on sawtooth waves.

Figure 18: (a) Original function f_4 ($f(x) = \sin x/x$). (b) to (e): Approximations of function f_4 using techniques (1) to (4) listed in Section 2.3.

Figure 19: (f) to (i): Approximations of function f_4 ($f(x) = \sin x/x$) using techniques (5) to (8) listed in Section 2.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E₁)</i>	<i>MSE (E₂/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	8	155	0.998	0.1579	7.974
Split/Merge (Pavlidis)	24	887	1.0003	0.1377	8.5694
Avg. neighbours (Tanimoto)	34	62	1.1705	0.3105	5.0371
Hologram-like (Prusinkiewicz)	18	48	0.3833	0.021	16.7336
Minimax (Kurozumi)	12	2121	1.3175	0.0993	9.9887
Recursive split (Lowe)	8	148	0.998	0.1579	7.974
DFT	N/A	235	2.4872	0.3941	4.0023
DCT	N/A	197	0.2168	0.0037	24.2448

Figure 20: Results of the various approximation techniques applied on function f_4 ($f(x) = \sin x/x$).

Figure 21: (a) Original function f_5 (sum of sawtooth waves and $7 \sin x/x$). (b) to (e): Approximations of function f_5 using techniques (1) to (4) listed in Section 2.3.

Figure 22: (f) to (i): Approximations of function f_5 using techniques (5) to (8) listed in Section 2.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_m)</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	15	210	1.5017	0.2252	13.9691
Split/Merge (Pavlidis)	28	1011	8.3911	1.9748	4.5408
Avg. neighbours (Tanimoto)	34	62	8.3229	4.1081	1.3591
Hologram-like (Prusinkiewicz)	18	47	9.1788	5.3562	0.207
Minimax (Kurozumi)	16	1458	30.1181	26.2339	-6.6931
Recursive split (Lowe)	15	201	1.5017	0.2252	13.9691
DFT	N/A	238	4.4599	1.7246	5.1287
DCT	N/A	193	3.5019	0.8746	9.205

Figure 23: Results of the various approximation techniques applied on function f_5 .

Figure 24: (a) Original function f_6 (simulated noisy signal). (b) to (e): Approximations of function f_6 using techniques (1) to (4) listed in Section 2.3.

Figure 25: (f) to (i): Approximations of function f_6 using techniques (5) to (8) listed in Section 2.3.

Approximation Method	# of Approx. Points	Time Required	ME (E_{∞})	MSE (E_2/N)	SNR (dB)
Linear Split (Ramer)	14	252	4.6696	2.7531	-15.525
Split/Merge (Pavlidis)	28	873	6.3583	6.218	-19.0632
Avg. neighbours (Tanimoto)	34	62	6.9121	9.306	-20.8143
Hologram-like (Prusinkiewicz)	18	45	4.8693	3.9821	-17.1278
Minimax (Kurozumi)	16	1491	42.5632	115.6809	-31.7593
Recursive split (Lowe)	14	232	4.6696	2.7531	-15.525
DFT	N/A	237	9.5968	10.5518	-21.36
DCT	N/A	195	2.8431	1.0729	-11.4325

Figure 26: Results of the various approximation techniques applied on function f_0 .

Chapter 3

3 Related Work on Two-dimensional (2-D) Functions

Approximation of planar curves and digitized images is an important part of pattern recognition, image processing, and graphics applications. Not only does it provide a simple method of describing boundaries in a compact way, it also allows data features to be easily extracted by filtering out the undesirable noise that may be part of the original data. The problem of approximation for two-dimensional functions has received a lot of attention by many authors (see [Sk 70], [Mo 70], [Ram 72], [Gr 72], [ZR 72], [DH 73], [PH 74], [RW 72], [PF 77], [DBM 77], [DaL 79], [SG 80], [Wi 78], [Ta 79], [Pa 82], [FB 86], [KD 82], [Du 86], [Ab 89], [FWL 89], [DLTW 90], [AK 91], [IN 91], and [PD 94]). All of the results mentioned above are "philosophically" based on the one-dimensional techniques presented in the previous chapter, and they can be divided into two distinct categories:

1. Planar curve approximations, and
2. Digitized image approximation

Within each one of these categories, the approximation techniques will be grouped as either spacial or transformed domain approximation. These methods are *not* in any way a complete list of all approximation techniques. Nevertheless, they do provide a comprehensive sample of the different approaches to the approximation problem. A detailed analysis of most of these techniques will be reviewed in this chapter.

3.1 Planar Curve Approximation

Most of the planar curve approximation techniques are based on the assumption that a two-dimensional function is discrete, that is, the function $f(x, y)$ is described by a set of points $S = \{(x_i, y_i) | i = 0, 1, 2, \dots, N-1\}$. The main task of an approximation technique is to determine a minimum number, say n , such that S is divided into n subsets $S_0, S_1, \dots, S_n$ where on each of them the data points are approximated by a function, for example, a polynomial of order at most $m-1$ with an error norm less than a prespecified quantity ϵ . By restricting $m = 2$, the approximating polynomial will be limited to a linear form.

3.1.1 Time/Spatial Domain Approximation

Freeman's Chain Code Encoding

Freeman's [Fr 61] work provides a basis for encoding arbitrary geometric configurations in order to facilitate their analysis and manipulations. An arbitrary curve, C , is subdivided into segments of selected size, T . It can then be described by a sequence of angles and a standard distance as shown in Figure 27. The actual encoding can be simplified by limiting the number of permissible angles. For example, if the encoding is restricted to 16 standard angles in 22.5° increments, and these angles are numbered consecutively from 0 to 15, then the curve of Figure 27 can be represented by the following series of numbers: 1, 1, 0, 0, 2, 6, 7, 7, 6, 2, 1, 0, 0, 15, 14, 11. It can easily be seen that the desired precision can be obtained with a relatively coarse angle quantization by making the standard distance arbitrarily small. This method may be termed the unit-vector method for, in essence, the curve is represented by a sequence of small vectors of unit length and a limited set of possible directions.

For a rectangular grid, if a point on a continuous curve is known, then the next point can assume only one of eight possible adjacent positions. If these eight possible positions are each assigned a number from 0 to 7 starting with the one

Figure 27: Freeman's chain code encoding of an arbitrary curve

horizontally to the right and continue enumerating in a counter-clockwise direction. any arbitrary digitized curve on a rectangular grid can be described. Each point is described relative to the previous one, and so only the first point needs to be located in an absolute sense. There main advantage associated with this method is that the decoding process allows a given figure to be expanded, contracted, elongated, or rotated by an arbitrary amount without any significant additional cost.

Although Freeman's chain code is a natural way of representing curves generated on a uniform rectangular grid, it possesses two major drawbacks. First of all, its representation is not compact, and thus, an excess number of vertices and edges have to be stored. Consequently, the approximating polygon may not necessarily be of minimum length. Secondly, since the length of the edges is comparable to the noise introduced by quantization, concavities for polygons that represent locally convex curves can occur.

Work done by Montanari

Montanari [Mo 70] introduces the problem of approximating a digitized contour such that the polygonal approximation has a minimal length. This technique takes into consideration an initial point $P_{0_i} = (x_{0_i}, y_{0_i})$ and a shifted point $P_i = (x_i, y_i)$ such that C_i will be the domain centred in P_{0_i} . If the total number of points in the polygon under examination is N and assuming that $x_{N+1} = x_1$ and $y_{N+1} = y_1$, then the function

$$g(\mathbf{z}) = \sum_{k=1}^N \left[(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2 \right]^{\frac{1}{2}}$$

must be minimized with the constraints

$$P_i \in C_i, \quad i = 1, 2, \dots, N.$$

The function $g(\mathbf{z})$ is a "cost" function of $\mathbf{z}$, where $\mathbf{z}$ is assumed to be a vector in an Euclidean space with $2N$ dimensions with N constraints. For example, if the circle of radius r_0 , centred in P_{0_i} is assumed as C_i , then the constraints above become

$$(x_i - x_{0_i})^2 + (y_i - y_{0_i})^2 - r_0^2 \leq 0, \quad i = 1, 2, \dots, N$$

Thus, Montanari's method of computing the smoothed polygon of minimum length can be summarized as a nonlinear programming problem. The main drawback of this implementation is the excessive time required to determine a solution which satisfies the N constraints.

Work done by Jarvis

Jarvis [Jar 71] presents a method for fitting polygons with a small number of edges to boundary lines while still preserving the overall features of the boundaries. His technique uses the least-square fit criterion under the constraint that the vertices of the polygons must lie on a set of rays originating at the centre of mass. The initial set of rays is found from peaks and troughs of the polar coordinates of the boundary relative to the centre of mass. Additional rays are found iteratively at the

directions of maximum error until the fit criterion is satisfied. This method is not well-suited for complex boundaries since the computation time rises rapidly with increasing complexity of the boundary.

Work done by Ramer

Ramer's [Ram 72] objective in approximation of irregular planar curves is to provide an efficient representation of a digitized picture generated in sampling the gray-value function over the domain of an image on a regular square grid. The resulting curves can be interpreted as polygons whose vertices lie on the sampling points or whose edges coincide with the cell boundaries. Ramer argues that the works done by Stone[St 61], Bellman [Be 61], and Gluss[Gl 62] are only concerned with approximation of real, single-valued, continuous functions of one parameter polygons, and that they cannot be applied to image processing since they deal strictly with single-variable functions. He also argues that the methods presented by Freeman [Fr 70], Sklansky [Sk 70], Montanari [Mo 70], and Zahn [Za 66] generally result in polygons with too many edges, whereas in most cases, it is desirable to have approximating polygons which still retain the significant features with the least number of edges.

The parameter to the problem considered by Ramer consists of a given curve represented by a set $C = \{p_i = (x_i, y_i) | i = 1, 2, \dots, N + 1\}$, and $p_1 = p_{N+1}$ for a closed curve. The points p_i can also be interpreted as vertices of an open or closed polygon P with N edges. Let the approximated polygon be denoted by $\hat{P}$ with a reduced number of n edges, whose vertices $\hat{p}_k$ for $k = 1, 2, \dots, n$ coincide with vertices of P . The approximated polygon $\hat{P}$ then corresponds to an ordered set $\hat{C}$ of points such that $\hat{C} \subseteq C$. The set $\hat{C}$ of vertices $\hat{p}_k$ divides C into ordered subsets S_k of the following form:

$$S_k = \{p_i, p_{i+1}, \dots, p_j\}$$

where $p_i = \hat{p}_{k-1}$ and $p_j = \hat{p}_k$ and that p_i, p_{i+1} are consecutive vertices in polygon P , $i = 0, 1, \dots, N$; and $\hat{p}_{k-1}, \hat{p}_k$ are consecutive vertices in polygon $\hat{P}$, $k = 1, 2, \dots, n$

where $n \leq N$. Although a desirable property of $\hat{C}$ is that it be as small as possible, this is not the primary goal of this algorithm. $\hat{C}$ partitions the curve C into n curve segments, and each set S_k contains the points belonging to the k^{th} curve segment. The following properties always hold true for the subsets S_k :

$$\bigcup_{k=1}^n S_k = C$$

$$\left(\bigcup_{k=1}^{n-1} (S_k \cap S_{k+1}) \right) \cup \{p_0\} \cup \{p_N\} = \hat{C} \quad (\text{open polygon})$$

$$\left(\bigcup_{k=1}^{n-1} (S_k \cap S_{k+1}) \right) \cup (S_n \cap S_1) = \hat{C} \quad (\text{closed polygon}).$$

Figure 28: Ramer's curve approximation method.

The algorithm presented by Ramer is very similar to the recursive binary split algorithm. However, unlike the binary split algorithm, the open or closed state of a polygon must be explicitly specified. The algorithm starts with two empty sets; one of them is used as a temporary result, and the other is the final result, $\hat{C}$ (at the termination of the algorithm). If the polygon is open, then the first point is added to $\hat{C}$, and the last point is added to the temporary set. On the other hand, if the polygon is closed, then the rightmost point is added to both the temporary set and $\hat{C}$, and the leftmost point is added only to the temporary set. The line segment joining the last vertices entered in $\hat{C}$ and the temporary set is the linear

C	Temporary set	Considered segment	Generated vertex
A	B	(AB)	C
A	B.C	(AC)	D
A	B.C.D	(AD)	
A.D	B.C	(DC)	
A.D.C	B	(CB)	E
A.D.C	B.E	(CE)	
A.D.C.E	B	(EB)	F
A.D.C.E	B.F	(EF)	
A.D.C.E.F	B	(FB)	
A.D.C.E.F.B	-	-	-

Table 1: Vertex generation sequence for the curve AB .

approximation. If the maximum distance between the points of the curve is greater than the allowable limit, then the point with the largest distance is placed in the temporary set. If, however, the maximum distance is within the allowable limit, then the last point from the temporary set is moved to $\hat{C}$. This iterative process is continued until the temporary set becomes empty. Table 1 shows the steps taken by the approximation technique presented above for the arbitrary curve AB shown in Figure 28.

Ramer's technique is simple, yet it provides a good linear approximation of the given curve. There are a few problems with this algorithm. If the original curve contains undesirable noise or spikes, this technique has a tendency to include them as part of the approximation. Moreover, the number of approximating lines is not guaranteed to be minimal.

Work done by Pavlidis & Horowitz

Pavlidis and Horowitz [PH 74] present an algorithm which approximates a planar curve by straight line segments. The main goal of their algorithm is to describe boundaries of objects in a compact and meaningful way by filtering unwanted noise.

yet still retaining the major features of the objects. They consider the case of where the planar curve is described by a set of discrete points, $S = \{p_i = (x_i, y_i) | i = 1, 2, \dots, N\}$, and their main objective is to divide S into a sequence of smaller segments $S_1, S_2, \dots, S_n$. If the pointwise error, ϵ_i , is the Euclidean distance between p_i and the approximating function evaluated at the point (x_i, y_i) , then the maximum error norm for the segment S_k is defined as

$$E_k = \max_{S_k} \{\epsilon_i\}$$

and the integral square error for that same segment is given as

$$E_2 = \frac{1}{2} \{V_{xx} + V_{yy} - [(V_{xx} - V_{yy})^2 + 4V_{xy}^2]^{1/2}\}$$

where N_k is the number of points in S_k

$$\begin{aligned} V_x &= \frac{1}{N_k} \sum_{S_k} x_i & V_y &= \frac{1}{N_k} \sum_{S_k} y_i \\ V_{xx} &= \sum_{S_k} (x_i - V_x)^2 & V_{yy} &= \sum_{S_k} (y_i - V_y)^2 \\ V_{xy} &= \sum_{S_k} (x_i - V_x)(y_i - V_y). \end{aligned}$$

Their technique is based on the following two propositions:⁷

Proposition 1 *Let S_1 and S_2 be two adjacent segments such that the maximum error of approximation on each one of them does not exceed ϵ . Furthermore, let the approximation of each segment be represented by the polynomial of order $m - 1$,*

$$\sum_{i=1}^m a_i^1 \phi_i^1(x) \quad \text{and} \quad \sum_{i=1}^m a_i^2 \phi_i^2(x)$$

Then the maximum error approximation on $S_1 \cup S_2$ is bounded from above by

$$\epsilon + \delta L$$

⁷Formal proofs for these propositions can be found in [Pa 72] and [PH 74]

where

$$\delta = \left[\sum_{i=1}^m (a_i^1 - a_i^2)^2 \right]^{\frac{1}{2}}$$

and

$$L = \max \left[\sum_{i=1}^m \phi_i(t_j)^2 \right]^{\frac{1}{2}} \quad \text{for } t_j \in S_1 \cup S_2$$

For piecewise linear approximation, L is bounded from above by the arc length of $S_1 \cup S_2$.

Proposition 2 Let E_2^0 be the integral square error norm over S_1 and S_2 , and let δ and L be defined as in the proposition above. Then if E_2^u denotes the error norm over $S_1 \cup S_2$ the following holds,

$$E_2^u \leq 2E_2^0 + \delta^2 L^2 M$$

where M is the number of points in $S_1 \cup S_2$. For piecewise linear approximations,

$$E_2^u \leq 2E_2^0 + 2\delta^2 L^3.$$

Let $S_1, S_2, \dots, S_{n_k}$ be the segments at the k^{th} iteration and $E^1, E^2, \dots, E^{n_k}$ be the error norms,⁸ of the respective segments. The technique considers two cases. In the first, it attempts to maintain the error norm in each segment to be less than or equal to maximum allowable error, and so

$$E^i \leq E^{max} \quad i = 1, 2, \dots, n_k.$$

In the second case, it is necessary to have the overall error norm of the segments less than or equal to E^{max} . If the selected error norm is the maximum error, then these two cases coincide since the maximum error, E^i , of each segment is also the overall maximum error of the entire curve. If the selected error norm is the integral squared error, then the overall error norm is the sum of the error norms of all the segments, that is

$$\left(\sum_{i=1}^{n_k} E_i^1 \right) \leq E^{max}.$$

⁸An error norm can be either the mean squared error or the maximum error

The split-and-merge algorithm for the first case proceeds as follows. For each subsegment, if the error norm for the subsegment is greater than the allowable limit, then the subsegment is split into two pieces using the following rule: if two or more points are known where the pointwise error is maximum, use the midpoint between a pair of them as a dividing point; otherwise, divide it into two equal halves. Once the splitting phase is complete, error norms must be calculated for each new interval added. In order to minimize the number of segments, a merging phase is introduced, which performs the opposite operation of the splitting phase. It recombines two adjacent subsegments into one if the error norm of the new subsegment does not exceed that of the maximum limit. Subsequently, new error norms must be calculated for all newly formed segments. The entire process is repeated until no more splits or merges can occur.

The algorithm for the second case is based on the first algorithm. If the overall error norm is greater than the allowable limit, the segment with the largest error norm must be split using the same rule as in the first algorithm. Otherwise, a search is performed to find a pair of adjacent segments whose merge will cause the smallest increase in the overall error norm and still keep the error norm below the allowable limit. The algorithm is repeated until no further changes occur.

Work done by Kurozumi & Davis

Kurozumi and Davis [KD 82] approach the approximation of the planar curve problem using a different method of estimating the error between the given curve and the approximated line segments. They indicate that most of the techniques using the least squares estimation may miss some of the details of the data such as a "spike", or a sudden change of short duration. Their polygonal approximation algorithm uses the minimax method which obtains a set of line segments which minimize the maximum of the Euclidean distances between the lines and the given points.

A minimax approximated line for three points can be obtained as follows (see

Figure 29: Minimax approximated line for three points.

Figure 29). The line perpendicular to the longest side of the triangle passing through the opposite vertex is calculated. In this particular case, the longest side is $\overline{AB}$, and the perpendicular line segment is $\overline{CP}$. As shown in the diagram, it is not necessary that the point of intersection P lies on the line segment $\overline{AB}$. The line which bisects the line $\overline{CP}$ and is parallel to the longest side $\overline{AB}$ of the triangle is the minimax line of the triangle $\triangle ABC$.

Figure 30: Minimax approximated line for a convex polygon.

To find a minimax approximated line through a set of k points demands more work than the case of a simple triangle. From the set of k points, an m -sided convex

polygon containing the k points is ascertained, where $m \leq k$. For an m -sided convex polygon, each side must have at least $m-1$ distances⁹, and from the $m-1$ distances, a maximum distance is selected. Therefore, an m -sided convex polygon contains m maximum distances, and the minimum of the m maximum distances is referred to as the minimax distance. The minimax approximated line through the m -sided convex polygon bisects the minimax distance and is perpendicular to the minimax distance as shown in Figure 30.

The approximation algorithm presented by Kurozumi and Davis is based on the minimax line principle described above. The given curve is sequentially scanned, and the parameters of the line ℓ_k passing through the first two points of subsegment S_k are calculated. As each new point, p_j , is added to the subsegment S_k , it is determined if the point p_j is within the allowable error limit, ϵ , from the line ℓ_k . If it is, then the process is continued with the point p_{j+1} on the curve; otherwise, a minimax approximated line is calculated for the points in $S_k \cup p_j$. If the maximum distance from all the points in $S_k \cup p_j$ to the minimax line is within ϵ , then the minimax line becomes the line ℓ_k of subsegment S_k , and the point p_j is added to the subsegment S_k . On the other hand, if the distance from one or more points in $S_k \cup p_j$ to the minimax line is greater than ϵ , then a new subsegment S_{k+1} is formed with points p_j and p_{j+1} , and a new line ℓ_{k+1} is calculated. It can be safely assumed that if S_k contains i points where $i \geq 2$, all i points in S_k are within ϵ of the minimax approximated line segment.

Kurozumi and Davis also include several optimizations in their algorithm to speed up the minimax distance calculations. Using the assumption that the points are sequentially ordered, the sorting step required to determine a convex polygon can be entirely omitted. Furthermore, in order to minimize the number of line segments, an allowable gap is used. Although, two minimax lines are allowed to

⁹In this particular case, distance is defined to be the length of the perpendicular line segment from a vertex to an edge.

Figure 31: (a) Minimax approximated line for j points. (b) Minimax approximated line for $j + 1$ points.

intersect at some point not on the curve, the point of intersection must be within a gap that is contained in a circle centred by one of the points on the curve with radius ϵ (see Figure 31).

Work done by Dunham

Dunham's [Du 86] approach to the approximation of two-dimensional digital curves is based on Bellman's work. However, there several major differences between his work and that of Bellman's. First of all, Bellman uses the mean square error norm whereas Dunham uses the uniform error norm. Secondly, whereas Bellman considers a discontinuous solution, Dunham's approximations are continuous. Finally, Bellman fixes the number of line segments and minimizes the error while Dunham fixes the error and minimizes the number of segments. The knots of the piecewise linear approximation are constrained to lie on the original digital curve. Dunham categorizes his method as an optimal solution based on the criterion that his polygonal approximation contains the least number of edges and yet still satisfies the uniform error restriction.

A discrete planar curve, C_d , is defined as a set of points parameterized by $0 \leq$

Figure 32: Computing the approximation error using the translated and rotated values.

$i \leq N$.

$$C_d = \{p_i = (x_i, y_i) | 0 \leq i \leq N\}.$$

Its piecewise linear approximation, C_a , is a subset of the original points, and is defined as

$$C_a = \{p_{u_0}, p_{u_1}, \dots, p_{u_n} | 0 = u_0 < u_1 < u_2 < \dots < u_n = N\}.$$

Thus, the j^{th} interval of C_a consists of the straight line segment which connects the points $p_{u_{j-1}}$ and p_{u_j} , and is denoted as $\overline{p_{u_{j-1}}, p_{u_j}}$. The pointwise error between a point p_k on the curve C_d and the line segment $\overline{p_{u_{j-1}}, p_{u_j}}$ where $u_{j-1} \leq k < u_j$ is defined as the perpendicular distance between the point p_k and the line segment $\overline{p_{u_{j-1}}, p_{u_j}}$. Using elementary trigonometry, the pointwise error can be equated as follows.

$$\epsilon(u_{j-1}, u_j, k) = \begin{cases} \sqrt{(x_k - x_{u_{j-1}})^2 + (y_k - y_{u_{j-1}})^2} & \text{if } x'_k < 0 \\ \sqrt{(x_k - x_{u_j})^2 + (y_k - y_{u_j})^2} & \text{if } x'_k < x'_{u_j} \\ |y'_k| & \text{otherwise} \end{cases}$$

where

$$x'_k = (x_k - x_{u_{j-1}}) \cos \theta + (y_k - y_{u_{j-1}}) \sin \theta \quad (8)$$

$$y'_k = -(x_k - x_{u_{j-1}}) \sin \theta + (y_k - y_{u_{j-1}}) \cos \theta \quad (9)$$

and θ is the angle between the line segment $\overline{p_{u_{j-1}}, p_{u_j}}$ and the horizontal line parallel to the x-axis as shown in Figure 32. The error over a segment u_{j-1} and u_j is defined as

$$\epsilon(u_{j-1}, u_j) = \max_{u_{j-1} \leq k \leq u_j} \epsilon(u_{j-1}, u_j, k)$$

and similarly, the error between C_d and C_n is

$$\epsilon(C_d, C_n) = \max_{1 \leq j \leq n} \epsilon(u_{j-1}, u_j).$$

For a given index u , if $V(u)$ is defined as the set of all the points with $v \leq u$ where $\epsilon(v, u) \leq \epsilon$, and $F(u)$ is defined as the minimal number of segments needed to approximate the curve C_d from points p_0 to p_u within ϵ , then $F(u)$ satisfies

$$F(u) = \min_{v \in V(u)} 1 + F(v) \quad (10)$$

with the initial conditions

$$F(0) = 0 \quad \text{and} \quad F(1) = 1.$$

Since $u \geq 2$, there is one line segment $\overline{p_v, p_u}$ plus a minimal number of possible segments from 0 to v . $F(u)$ can, therefore, be recursively computed using dynamic programming with the given initial conditions and $F(N)$ will be the minimal number of segments needed for C_n to approximate C_d within ϵ . As the value of $F(u)$ is calculated, one of the $v \in V(u)$ which satisfies the minimum in (10) is stored. From $F(N) = F(u_n)$ the value of u_{n-1} can be derived, from $F(u_{n-1})$, u_{n-2} can be derived and so forth. Thus, an optimal continuous piecewise linear planar curve that approximates C_d with fixed endpoints u_0 and u_n within ϵ is generated.

Figure 33: Lines and angles from p_0 to p_k .

The evaluation of the set $V(u)$ is based on the Euclidean distance between the point p_0 and p_k and the angle subtended by the lines passing through the point p_0 tangent to the circle centred at point p_k with the radius ϵ as shown in Figure 33. If this angle is denoted by $2\psi_k$, and the angle between the line connecting $\overline{p_0, p_k}$ and the horizontal line parallel to the x-axis is denoted as ϕ_k , they can be evaluated by

$$\phi_k = \arctan\left(\frac{y_k - y_0}{x_k - x_0}\right), \text{ and,}$$

$$\psi_k = \arcsin\left(\frac{\epsilon}{\sqrt{(x_k - x_0)^2 + (y_k - y_0)^2}}\right).$$

The set S_1 , for example, represents all the lines through p_0 which passes within distance ϵ of p_1 . Similarly, $S_2, S_3, \dots, S_k$ represent all the lines through p_0 which passes within distance ϵ of $p_2, p_3, \dots, p_k$, respectively. The set S_k can be defined as

$$S_k = \{\theta | \phi_k - \psi_k \leq \theta \leq \phi_k + \psi_k\}.$$

The intersection of these sets, $T_k = S_1 \cap S_2 \cap \dots \cap S_k$ represents all the lines through p_0 which pass within distance ϵ of all the points from $p_1, \dots, p_k$. If T_k is empty, then there can be no line passing within ϵ of points $p_0, \dots, p_k$, and hence, no more points in $V(p_0)$. If T_k is not empty and $\phi_k \in T_k$, then all the points $p_0, p_1, \dots, p_k$ are

potentially within ϵ of the line $\overline{p_0 p_k}$. On the other hand, if T_k is not empty and $v_k \notin T_k$, then at least one point in $p_0, p_1, \dots, p_k$ is greater than ϵ away from the line $\overline{p_0 p_k}$ and $p_k \notin V(p_0)$. In Figure 34, for example, the points p_0, p_1 , and p_2 are within ϵ of the line $\overline{p_0 p_2}$, while the point p_3 is more than ϵ away from the line $\overline{p_0 p_3}$.

Figure 34: Scan-along procedure from p_0 to p_3 .

Apart from these, the Euclidean distance between p_0 and p_k must also be taken into consideration. It is defined as

$$d_k = \sqrt{(x_k - x_0)^2 + (y_k - y_0)^2}$$

and the maximum distance is

$$d_{\max, k} = \max_{1 \leq j \leq k} d_j.$$

If $d_{k+1} \geq d_{\max, k}$, then the x'_j coordinate of the translated and rotated point given by (8) and (9) will lie between 0 and x'_{k+1} for $0 \leq j \leq k$. Therefore, all the points $p_0, p_1, \dots, p_k$ are within ϵ of the line $\overline{p_0 p_{k+1}}$ and so $p_{k+1} \in V_{p_0}$. However, if $d_{k+1} < d_{\max, k}$, all the points whose distance is greater than d_{k+1} must be inspected to see if they are within ϵ of the line $\overline{p_0 p_{k+1}}$. If they are, then $p_{k+1} \in V(p_0)$, otherwise, $p_{k+1} \notin V(p_0)$.

Although Dunham offers a method of approximating a planar curve with the minimum number of line segments, there are several problems associated with this technique. If the planar curve is closed, the derived approximation is not necessarily optimum (compared to the open planar curves); it tackles the problem by invoking a costly exhaustive search using the open planar curve algorithm. Furthermore, in certain cases, the time required to derive an approximation increases inversely with the maximum error limit.

Work done by Lowe

Lowe [Lo 87] introduces a piecewise linear approximation technique which uses the ratio of the length of the line segment and the maximum deviation of any point from the line. The algorithm is based on the assumption that a digitized curve is derived from a bitmap, and as such, the maximum deviation is always at least two pixels in size to account for limitations on measurement accuracy. This error estimation, thus, provides a scale-independent measure of significance which places no prior expectations on the allowable deviations.

A curve segment is recursively subdivided at the point of maximum deviation from a line connecting its endpoints as shown in Figure 35. The process is repeated until each segment is no more than 4 pixels in length, producing a binary tree of possible divisions. On unwinding the recursion back up the tree, a decision is made at each junction to determine if the lower-level descriptions can be replaced by the single higher-level segment. The significance of every subsegment is calculated by its length-to-deviation ratio as mentioned above. If the maximum significance of any of the subsegments is greater than the significance of the complete segment, then the subsegments are returned; otherwise, the single segment is returned. By concatenating the returned subsegments together, a piecewise linear approximation is found.

This algorithm is similar to the binary split algorithm and to that of Ramer's.

Figure 35: Steps taken by algorithm for subdividing a curve into its most perceptually significant straight line segments.

and it does a reasonable job of detecting the perceptually most significant straight line groupings of the original data points. Whereas other traditional methods require setting of a prior threshold for the amount of “noise” to be removed from a curve, this estimation technique does not require such a threshold value. It tends to find the same structures regardless of the size at which an object appears in an image, and it avoids breaking up long lines if shorter constituents do not possess stronger perceptual elements. The major shortcoming of Lowe’s ratio of length-to-deviation is that if the deviation is zero, it causes a division-by-zero error. Rosin and West [RW 89] [RW 91] have generalized Lowe’s technique by inverting the ratio to avoid this problem.

3.1.2 Transformed Domain Approximation

Work done by Granlund

Granlund [Gr 72] uses the coefficients of Fourier transformations to represent features which are significant for a pattern. One of Granlund's requirements was that the processed patterns be described in an optimal way. Moreover, in the particular case of character recognition, his aim was to determine features which are invariant for different types of style, size, and slant of character.

Figure 36: Example of contour function.

A given curve γ , assumed to be closed, is defined by a sequence of points $Z(l) = (x(l), y(l))$ where l is the arc length along γ . A point moving along this curve generates the complex function $u(l) = x(l) + jy(l)$ which is periodic with a period L as shown in Figure 36. The Fourier descriptors of γ using the complex function $u(l)$ is defined as

$$a_k = \frac{1}{L} \int_0^L u(l) e^{-j(2\pi/L)kl} dl.$$

In the case where the curve is polygonal as outlined in Figure 37, the evaluation of

the descriptors a_k can be simplified [PF 77] to be:

$$a_k = \frac{1}{L \left(\frac{2\pi}{L} \right)^2} \sum_{i=1}^m (b_{i+1} - b_i) e^{-j(2\pi/L)k l_i}$$

where

$$l_i = \sum_{r=1}^i |V_r - V_{r-1}|, \quad \text{for } k \geq 0 \text{ and } l_0 = 0$$

and

$$b_i = \frac{V_{i+1} - V_i}{|V_{i+1} - V_i|}, \quad \text{such that } |b_i| = 1.$$

Figure 37: A polygonal boundary.

The original function, $u(l)$, can be reconstructed from the set of descriptors a_k using the following equation

$$u(l) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/L)kl}. \quad (11)$$

Some of the advantages of using this representation are that all sequences $\{a_k\}$ for which the series in (11) converges describe closed curves, and the inverse transform over a subset of the Fourier descriptors $\{a_k\}$ always produces closed curves. Among the disadvantages, this technique requires all N descriptors for a complete description because $u(l)$ is a function of complex values. Since the length is used as the

parameter to the function $u(l)$, the modulus of the derivative of $u(l)$ with respect to l equals 1, $|du/dl| \equiv 1$. This condition implies that the coefficients $\{a_k\}$ are not independent, and therefore, there is some redundancy in the description. Moreover, this technique (and in general, all transform techniques), has great difficulty describing local information [Pa 78].

Work done by Zahn & Roskies

The problem of extracting a finite set of numerical features from a closed planar curve was examined by Zahn and Roskies [ZR 72]. Their main focus is to employ a set of Fourier descriptors to characterize features in order to separate the shapes of different classes relative to the intraclass dispersion. The idea of using the Fourier descriptors was first suggested by Cosgriff [Co 60], and his work was soon followed by others including Borel [Bo 65], and Brill [Br 68]. A starting position on the curve is chosen, and a function $\theta(l)$ is defined. $\theta(l)$ measures the angular direction of the curve as a function of the arc length. After appropriate normalization, this periodic function is expanded in a Fourier series and the coefficients of a truncated expansion are used as shape features called Fourier descriptors. The idea presented by Zahn and Roskies is based on previous work done by Cosgriff and others, which differ in that they use a different normalizing function than the ones used previously.

Let γ represent a clockwise-oriented simple closed curve with parametric representation $Z(l) = (x(l), y(l))$ where l is the arc length and $0 \leq l \leq L$. Furthermore, let the function $\theta(l)$ denote the angular direction of γ at point l , and $\theta_0 = \theta(0)$ be the absolute angular direction at the starting point $Z(0)$. The cumulative angular function $\phi(l)$ is defined as the net amount of angular bend between the starting point and the point l as shown in Figure 38. By this definition, $\phi(0) = 0$ and $\phi(l) + \theta_0$ is identical to the $\theta(l)$ except for a possible multiple of 2π . It can be seen that for all smooth closed curves with clockwise orientation that $\phi(L) = -2\pi$ because all of these curves have a net angular bend of -2π , and thus, $\phi(l)$ does

Figure 38: Parametric representation of a planar curve with tangential direction $\theta(l)$ and cumulative angular bend function $\phi(l)$.

not convey any shape information. The domain of $\phi(l)$, $[0, L]$, contains absolute size information, and it is normalized to an interval of a typical periodic function of $[0, 2\pi]$. Hence, a normalized variant $\phi^*(t)$ whose domain is $[0, 2\pi]$ is defined as

$$\phi^*(t) = \phi\left(\frac{Lt}{2\pi}\right) + t$$

such that $\phi^*(0) = \phi^*(2\pi) = 0$. Applying the definition of $\phi^*(t)$ to a circle or any regular n -gon, it is noted $\phi^*(t) \equiv 0$, which indicates that the circle and the regular n -gons are the most shapeless closed curve. Another interesting feature of the function $\phi^*(t)$ is that it is invariant under translation, rotation, and changes in parameter L . Therefore, two planar curves have identical shape if they differ only by a combination of translation, rotation, and a change in size.

Figure 39: (a) Planar closed curve with starting point from boundary of character "H". (b) Cumulative angular bend function $\phi(l)$ for curve in (a). (c) Normalized cumulative angular bend function $\phi^*(t)$ for the curve in (a).

The function $\phi^*(t)$ can be expanded into a Fourier series as

$$\phi^*(t) = \mu_0 + \sum_{k=0}^{\infty} (a_k \cos kt + b_k \sin kt)$$

$$\phi^*(t) = \mu_0 + \sum_{k=0}^{\infty} A_k \cos(kt - \alpha_k)$$

where (A_k, α_k) are the polar coordinates of (a_k, b_k) . The numbers A_k and α_k are considered as the Fourier descriptors for the curve γ and are known as the k^{th} harmonic amplitude and phase angle, respectively.

One of the major advantages of using this representation is that there is no redundant information present in the set of Fourier descriptors $\{A_k, \alpha_k\}$ compared to the representation provided by Granlund [Gr 72]. Therefore, every sequence $\{A_k, \alpha_k, k = 1, 2, \dots\}$ describes one curve, and each curve has only one sequence

$\{A_k, \alpha_k\}$. There are several disadvantages of using this technique. Some sequences $\{A_k, \alpha_k\}$ describe non-closed curves. For polygonal curves, $\phi^*(t)$ contains discontinuities, and therefore, the descriptors A_k will decrease rather slowly as k increases. Moreover, the reconstruction of γ requires numerical integration.

3.2 Digitized Image Approximations

An image is often defined as a two-dimensional light intensity function $I(x, y)$ where x and y represent the spacial coordinates. A corresponding digitized image $f(i, k)$ can be obtained by sampling the analogue image $I(x, y)$ in the spacial coordinates and quantizing the brightness. Thus, a digitized image $f(x, y)$ can be defined as

$$f(i, k) = \hat{I}(x_0 + i\Delta x, y_0 + k\Delta y) \quad \text{for } i, k = 0, 1, 2, \dots, N$$

where $\hat{I}$ is the quantized brightness of the analogue image I , and Δx and Δy are the sampling intervals along the x - and y -axes, respectively. A digitized image is considered as an N by N matrix whose elements are the quantized brightness values at that point. Each element of an image is referred to as an *image element*, *picture element*, *pixel*, or *pel*.

3.2.1 Time/Spatial Domain Techniques

An image of size 512×512 with 8 bits per pixel (256 possible greyscale values) requires 256 Kilobytes of memory storage. If an image of such size is transmitted one scanline at a time, the amount of time required for the entire transmission makes it too slow for most interactive applications. The one-dimensional approximation technique presented in Chapter 2, which can be extended and applied to two-dimensional images, is the subsampling technique. Two variations of the subsampling schemes known as progressive transmission techniques will be presented in this section.

Work done by Tanimoto

Tanimoto [Ta 79] describes a hierarchical data structure for picture processing called the *pyramid data structure*. If an image is defined as $I(m, n)$, where $0 \leq m, n \leq N = 2^L$, then lower resolution approximations of the original $N \times N$ images can be defined as

$$I_{k-1}(i, j) = \frac{1}{4}[I_k(2i, 2j) + I_k(2i + 1, 2j) + I_k(2i + 1, 2j + 1) + I_k(2i, 2j + 1)]$$

where $0 \leq k \leq L$ and $0 \leq i, j \leq 2^{k-1}$ as shown in Figure 40.

Figure 40: Pyramid data structure for four levels 0, 1, 2, and 3.

A sequence of reduced resolution images from the original is formed by averaging 2×2 blocks of pixels. There are $L + 1$ “levels” of the pyramid, the most reduced of which is a 1×1 image (a single pixel) whose value is the overall average of the original image. The levels of the pyramid are transmitted top-down; thus, the average of the whole image is sent first, followed by the 2×2 image, 4×4 image, etc., until the $2^L \times 2^L$ original is sent. The reduced resolution images are merged

into one another to form an approximation of the original starting with the 1×1 root of the pyramid: the entire reconstructed image is filled with the root value of the pyramid. Each of the next four values received is used to fill one the quadrant of the reconstructed image. The squares become smaller each time a level is completed until in the last level the squares are the individual pixels. A decision as to what has to be done in any invocation is made by the user at the receiving end depending on how much detail he requires.

The main purpose of this technique is allow interactions between a user and the application handling a large amount of visual information. If images are stored in this manner, their approximations can be transmitted to allow interactions between the application and the users and to reduce communication cost. In most cases, only a small number of approximations is required to be transmitted before the user can see if the original image has to be sent or if the current transmission can be cancelled. There are several problems with this pyramid method. The construction of the pyramid approximations require a large amount of memory, and in the worst case scenario, the communication cost is doubled because the original image and all of its approximations need to be sent.

Work done by Prusinkiewicz

Prusinkiewicz's [Pr 84] one-dimensional subsampling technique can be easily extended to two-dimensional images. If S is the original signal at full resolution, then it can be divided into substrings $S_0, S_1, \dots, S_N$ where each substring, S_i , represents a reduced resolution of S . The main difference between this method and the binary tree, quadtree, or the pyramid structure [Ta 79] is that any substring, S_k , can be used to reconstruct the original signal S at the particular resolution at level k . As opposed to this, with the hierarchical structure representation only various initial substrings can be used to reconstruct the approximation of the entire string S - their non-initial substrings are meaningless. Moreover, the cost of the overhead of

node structures which are inherent in hierarchical representations is not incurred by Prusinkiewicz's algorithm.

Figure 41: String of sampling points. (a) to (d) represent the string of sampling points $S(0,0)$, $S(0,1)$, $S(0,2)$, and $S(0,3)$, respectively. (e) represents the substring $S(1,0)$ which is the concatenation of the substrings from (a) to (d).

Let $N > 0$ be the edge size of the square sampling region $A(N)$ whose vertices (corners) are given as $(0,0)$, $(0,N)$, (N,N) , and $(N,0)$. Let $S = \langle p_0, p_1, \dots, p_{N \times N} \rangle$ be a string of points in a plane, and $S(i, h)$ denote a substring of S containing exactly 4^h points

$$S(i, h) = \langle p_{h \cdot 4^h}, p_{h \cdot 4^h + 1}, p_{h \cdot 4^h + 2}, \dots, p_{(h+1)4^h - 1} \rangle$$

Also, let $S_A(i, 0)$ represent the translation of a substring $S(i, 0) = \langle p_0, p_1, p_2, \dots, p_{4^0 - 1} \rangle$ by vector $\vec{c}$ such that

$$S_A(i, 0) = \langle p_0 + \vec{c}, p_1 + \vec{c}, p_2 + \vec{c}, \dots, p_{4^0 - 1} + \vec{c} \rangle$$

where $\vec{c} = c_x \vec{1}_x + c_y \vec{1}_y$, c_x and c_y are the amount of translation and $\vec{1}_x$ and $\vec{1}_y$ are the unit vectors along the x - and y - axes, respectively. The string of sampling points is then defined as follows:

$$S(0,0) = \langle p_0 \rangle = \langle \text{point at } (N,N) \rangle$$

$$S(i,1) = S_{-\sqrt{N/2-i-1}\vec{1}_x}(i,0)$$

$$S(i,2) = S_{-\sqrt{N/2-i-1}\vec{1}_y}(i,0)$$

$$S(i,3) = S_{-\sqrt{N/2-i-1}(\vec{1}_x+\vec{1}_y)}(i,0)$$

$$S(i+1,0) = S(i,0) \cdots S(i,1) \cdots S(i,2) \cdots S(i,3)$$

where $\cdots$ denotes the concatenation of strings, and $i = 0, 1, \dots, \log_2 N$. Prusinkiewicz's traversal subsampling algorithm and the uniformity of the sampling technique are supported by the following two theorems:¹⁰

Theorem 2 *Let the index k of a sample point $p_k = (x_k, y_k)$ be represented as a binary word $r_{2q-1}r_{2q-2} \dots r_1r_0$ where $r_i = 0$ or 1 ,*

$$k = \sum_{i=0}^{2q-1} r_i 2^i.$$

Then the coordinates of point p_k are equal to

$$x_k = T \left(1 - \sum_{i=0}^{q-1} r_{2i} 2^{-(i+1)} \right)$$

$$y_k = T \left(1 - \sum_{i=0}^{q-1} r_{2i+1} 2^{-(i+1)} \right)$$

If a sampling region is given as $\mathcal{A}(N)$, the sampling lattice of $m \times m$ elements, with the origin in point (c_x, c_y) , is defined as the set

$$\mathcal{M}_{m \times m}(c_x, c_y) = \left\{ \left(c_x + (i-1) \frac{N}{m}, c_y + (j-1) \frac{N}{m} \right) \mid i, j = 1, 2, \dots, m \right\}$$

The theorem below is based on the definition of uniform distribution of $m \times m$ points in a square area $\mathcal{A}(N)$ formally introduced by Rosenfeld and Kak [RK 76].

¹⁰Their proofs can be found in [PC 85].

Figure 42: String of sampling points. (a) to (d) represent the string of sampling points $S(1,0)$, $S(1,1)$, $S(1,2)$, and $S(1,3)$, respectively. (e) represents the substring $S(2,0)$ which is the concatenation of the substrings from (a) to (d).

Theorem 3 Let $\tilde{S}$ denote the (unordered) set of elements of the substring $S(i, h)$.

$$\tilde{S}(i, h) = \{p_{h \cdot 4^i}, p_{h \cdot 4^i + 1}, p_{h \cdot 4^i + 2}, \dots, p_{(h+1)4^i - 1}\}.$$

Then for any $i, h = 0, 1, 2, \dots$, the set $\tilde{S}$ is a sampling lattice in the region $A(N)$:

$$(\forall i, h = 0, 1, 2, \dots) \left(\exists c_x, c_y \in \left[0, \frac{N}{2^i}\right) \right)$$

$$\tilde{S}(i, h) = M_{2^i \times 2^i}(c_x, c_y).$$

The sampling method proposed by Prusinkiewicz starts at the top right hand corner of an image and traverses to the bottom-left corner from one level to the next. One of the problems with this traversal algorithm is that if important information is contained in the lower left-hand corner of an image, then the first several substrings will be devoid of this information. Nevertheless, because the width and

height doubles at each successive stage, the sampling area is quadrupled, and therefore, converges very quickly. The reconstruction method proposed by Prusinkiewicz consists of linearly interpolating the unknown cells using the sampled values, and by doing so, the staircase effects caused by undersampling are minimized.

3.2.2 Transform Domain Techniques

Most, if not all, of the two-dimensional transform techniques are based on the one-dimensional transformed techniques presented in Chapter 2, and these techniques are extended to apply to two-dimensional images. Extensive amount of work has been done by many researchers in this field including (but not limited to) the following [ANR 85], [Ja 85], [RK 76], [Pr 78], [BB 82], [ER 82], [JN 84], [Ra 85], [Wa 88], [RY 90], [Ja 91], [RJ 91], [Ru 92]. Only two transform techniques will be presented in this section, namely the Fourier transform, and discrete cosine transform.

Fourier Transform of Two-dimensional Functions

The discrete Fourier transform of a two-dimensional function $f(i, k)$ is an extension of the one-dimensional transform, and is given as:

$$F(u, v) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f(i, k) e^{-\frac{j2\pi(u i + v k)}{N}} \quad (12)$$

where $0 \leq u, v < N$. Its inverse is defined as

$$f(i, k) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{\frac{j2\pi(u i + v k)}{N}}. \quad (13)$$

In both equations (12) and (13) above, the value of j is defined as $j = \sqrt{-1}$.

The main purpose of the discrete Fourier transform is to decompose an image block into its spectral components, and the indices u and v are referred to as the spacial frequencies of the transform. The two-dimensional transform kernel is separable, since

$$e^{-\frac{j2\pi(u i + v k)}{N}} = e^{-\frac{j2\pi u i}{N}} \cdot e^{-\frac{j2\pi v k}{N}}$$

which allows the two-dimensional transform to be implemented using two one-dimensional transforms.

Figure 43: Implicit periodicity of the discrete Fourier and cosine transforms.

The discrete Fourier transform is a Fourier series representation of a finite-duration sequence, and as such, there is an implicit periodicity of the sequence as shown in Figure 43. This periodicity is the result of sampling in the frequency domain. By replicating the original sequence in this manner the process often creates severe discontinuities between segments. Spurious high frequency components can result from these discontinuities, and therefore, the efficiency of the transform is greatly reduced. Although these spurious components are not really part of the original sequence, they are required to reconstruct the sharp boundaries in a periodic sequence. If some of these components are discarded in order to increase the transform efficiency, then the reconstructed signal would result in objectionable errors at the boundaries. In image coding, where an image is broken up into blocks of pixels to form two-dimensional sequences, these reconstruction errors result in

blocking artifacts (boundaries between adjacent blocks are highly visible).

Extensive study has been done on fast implementations of the DFT (known as FFT) which require $O(n \log n)$ for an n -point transform. The discrete Fourier transform coefficients are complex values consisting of real and imaginary components or magnitude and phase components. Their properties, as described in Chapter 2, still hold true for two-dimensional functions. Storage and manipulation of these complex values can be cumbersome. Another disadvantage of the discrete Fourier transform is that spurious spectral components are generated due to the implicit periodicity of the image blocks. When encoded at low bit rates, these spurious components may give rise to severe block artifacts leading to the so-called "staircase effect" [RJ 91].

Discrete Cosine Transform of Two-dimensional Functions

The discrete cosine transform for a two-dimensional function $f(i, k)$ can be described as

$$F(u, v) = \frac{4C(u)C(v)}{N^2} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f(i, k) \cos \left[\frac{(2i+1)u\pi}{2N} \right] \cos \left[\frac{(2k+1)v\pi}{2N} \right]$$

and the inverse two-dimensional discrete cosine transform is defined as

$$f(i, k) = \sum \sum C(u)C(v)F(u, v) \cos \left[\frac{(2i+1)u\pi}{2N} \right] \cos \left[\frac{(2k+1)v\pi}{2N} \right]$$

where

$$C(w) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } w = 0 \\ 1 & \text{for } w = 1, 2, \dots, N-1. \end{cases}$$

Similar to the discrete Fourier transform, the discrete cosine transform has a fast implementation with a computational complexity of $O(n \log n)$ for an n -point transform. However, it does not generate spurious spectral components, and therefore, it is very efficient in coding by compacting a large amount of information into a few coefficients. The discrete cosine transform has become the most widely used transform for audio and image compression, and has been adapted by the International Standards Organization (ISO) JPEG still-image coding scheme.

3.3 Results of Previous Planar Curve Approximation Methods

In this chapter many different approximation techniques on two-dimensional planar curves have been reviewed. Of all the methods that have been examined, the following algorithms are selected and used as a basis for comparison:

1. Linear search and split (Ramer)
2. Split-and-merge algorithm (Pavlidis and Horowitz)
3. Nearest-neighbour averaging algorithm (extension of Tanimoto)
4. Hologram-like approximation algorithm (Prusinkiewicz)
5. Minimax Polygon (Kurozumi and Davis)
6. Recursive binary search and split (Duda and Hart, Lowe)
7. Discrete Fourier transform (Rao, Yip, et al.)
8. Discrete cosine transform (Rao, Yip, et al.)

These methods provide a cross section of time/spacial domain and transform domain approximation techniques. As indicated in the previous chapter, these methods were chosen because they can be applied to both one-dimensional functions and two-dimensional planar curves. The first six methods are categorized as the time/spacial domain techniques, and the last two are transform domain techniques. Within the time/spacial domain category, the hologram-like and nearest-neighbour techniques use uniform knot points, whereas, the other time/spacial domain techniques use non-uniform knots. In order to accurately compare the different techniques, they were applied to the following five discrete planar curves:

1. Open ellipse:

$$\{(x_i, y_i) | x_i = 5 \cos(0.05i) + 3, y_i = 7 \sin(0.05i) + 4, i = 0, 1, \dots, 200\}$$

2. Rotated normal bell curve:

$$\left\{ (x_i, y_i) | x_i = x'_i \cdot \cos \frac{\pi}{3} - y'_i \cdot \sin \frac{\pi}{3}, y_i = x'_i \cdot \sin \frac{\pi}{3} + y'_i \cdot \cos \frac{\pi}{3} \right\}$$

where

$$x'_i = 0.05i - 6, y'_i = \frac{20}{\sqrt{2\pi}} e^{-x'^2/2}, i = 0, 1, \dots, 200$$

3. Complex closed curve:

$$\{(x_i, y_i) | x_i = 3 \cos(0.025i) - 4 \sin(0.025i) + 4, y_i = 5 \cos(0.025i) + \sin 2x + 6\}$$

for $i = 0, 1, \dots, 251$.

4. Simulated noisy complex closed curve:

$$\{(x_i, y_i) | x_i = 3 \cos(0.025i) - 4 \sin(0.025i) + 4, y_i = 5 \cos(0.025i) + \sin 2x + 6\}$$

for $i = 0, 1, \dots, 251$. The simulated noise is the value $\sin 200x_i$ added to y_i when $10 \leq i$ modulo 40 ≤ 18 .

5. Self-intersecting curve:

$$\{(x_i, y_i) | x_i = 3 \cos(0.02i) - 4 \sin(0.02i) + 4, y_i = 5 \cos(0.02i) + \sin 2x\}$$

for $i = 0, 1, \dots, 300$.

The results of some of these approximation techniques are displayed and tabulated at the end of this section. Similar to the output of one-dimensional functions, the column with the heading “**# of Approx. Points**” is not applicable to the discrete Fourier and discrete cosine transforms because their estimation is done in the transform domain. Both of these techniques divide the original sampled data into groups of 16 data values. The transformation is carried out using these techniques.

and only three of the 16 quantized coefficients are retained. The inverse transformation is then performed on the quantized coefficients to derive the approximated values in the time/spacial domain.

In reviewing the results of the planar curve approximation techniques, it is again clearly seen that the best time/spacial domain techniques are the linear split and the recursive split algorithms. Other time/spacial domain techniques, such as the hologram-like and nearest-neighbour approximations, do provide a better approximation in terms of errors for the cases of an open elliptical curve or rotated bell curve. However, the number of approximating points are almost two or three times that of the linear and recursive split approximations. The minimax method is the most computationally expensive algorithm: it requires 10 to 20 times as much time to derive a solution with the same error value as the hologram-like approximation.

3.4 Results of Digitized Image Approximation Methods

Several digitized image approximation methods have been reviewed including the subsampling techniques presented by Tanimoto [Ta 79] and Prusinkiewicz [PC 85] and the discrete Fourier and discrete cosine transforms. These methods provide a cross section of time/spacial domain and transform domain approximation techniques. The two subsampling methods presented by Tanimoto and Prusinkiewicz are categorized as the time/spacial domain techniques, and the last two are transform domain techniques. In order to equitably compare the different techniques, they are applied to the following four different digitized images:

1. A human face
2. A deer in a grey background
3. A squirrel in a dark background
4. A mountain hiking trail containing light background

Each digitized image is 512 pixels wide, 512 pixels high, and a pixel has 256 possible greyscale values. Since there is a lot of redundant data, the subsampling techniques perform extremely well. The nearest-neighbour technique as proposed by Tanimoto [Ta 79], for example, smoothenes the image by averaging the values of the neighbouring pixels. The discrete cosine transform performs extremely well; this result demonstrate why it is used in the JPEG compression algorithm.

Figure 44: (a) Original open ellipse. (b) to (e): Approximations of open ellipse using techniques (1) to (4) listed in Section 3.3.

Figure 45: (f) to (i): Approximations of open ellipse using techniques (5) to (8) listed in Section 3.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_{∞})</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	11	784	0.4035	0.0197	23.1234
SplitMerge (Pavlidis)	30	1004	4.9756	3.9396	-17.1851
Avg. neighbours (Tanimoto)	34	102	0.3231	0.0415	19.8924
Hologram-like (Prusinsiewicz)	18	72	0.0888	1.47E-03	34.4056
Minimax (Kurozumi)	16	2079	0.6281	0.1146	15.4858
Recursive split (Lowe)	13	169	0.1681	5.33E-03	28.8135
DFT	N/A	210	2.1044	0.5862	8.397
DCT	N/A	173	0.6462	0.0103	25.9476

Figure 46: Comparison table of eight previous techniques applied on open ellipse.

Figure 47: (a) Original rotated normal bell curve. (b) to (e): Approximations of rotated normal bell curve using techniques (1) to (4) listed in Section 3.3.

Figure 48: (f) to (i): Approximations of normal bell curve using techniques (5) to (8) listed in Section 3.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_1)</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	6	194	0.4044	0.0209	12.4223
Split/Merge (Pavlidis)	7	799	0.2637	0.011	15.2147
Avg. neighbours (Tanimoto)	34	133	0.3488	0.0234	11.9199
Hologram-like (Prusinkiewicz)	18	87	0.0825	4.24E-04	29.3477
Minimax (Kurozumi)	7	9227	1.03	0.1454	3.9925
Recursive split (Lowe)	6	183	0.4044	0.0209	12.4223
DFT	NA	237	2.1847	0.1348	4.3221
DCT	NA	228	1.4564	0.0152	13.8014

Figure 49: Comparison table of eight previous techniques applied on rotated normal bell curve.

Figure 50: (a) Original complex closed curve. (b) to (e): Approximations of complex closed curve using techniques (1) to (4) listed in Section 3.3.

Figure 51: (f) to (i): Approximations of complex closed curve using techniques (5) to (8) listed in Section 3.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	16	229	0.3013	0.0167	25.4022
Split/Merge (Pavlidis)	16	1017	0.5682	0.0355	22.1288
Avg. neighbours (Tanimoto)	34	107	0.6062	0.0567	20.0959
Hologram-like (Prusinkiewicz)	18	76	1.0082	0.0747	18.8942
Minimax (Kurozumi)	18	1511	1.1935	0.1519	15.8125
Recursive split (Lowe)	16	205	0.3013	0.0167	25.4022
DFT	N/A	231	7.2335	2.091	4.4256
DCT	N/A	187	5.2336	0.3676	11.9749

Figure 52: Comparison table of eight previous techniques applied on complex closed curve.

Figure 53: (a) Original complex closed curve with simulated noise. (b) to (e): Approximations of noisy complex closed curve using techniques (1) to (4) listed in Section 3.3.

Figure 54: (f) to (i) Approximations of noisy complex closed curve using techniques (5) to (8) listed in Section 3.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_{∞})</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	32	558	0.3718	0.0151	25.8644
Split/Merge (Pavlidis)	24	2252	1.0721	0.0578	20.0493
Avg. neighbours (Tanimoto)	34	160	1.1191	0.1124	17.1616
Hologram-like (Prusinkiewicz)	18	140	1.2565	0.1183	16.9391
Minimax (Kurozumi)	24	3479	0.9114	0.0676	19.3705
Recursive split (Lowe)	32	495	0.3718	0.0151	25.8644
DFT	N/A	471	7.1492	0.7526	8.9026
DCT	N/A	369	4.598	0.1729	15.2897

Figure 55: Comparison table of eight previous techniques applied on noisy complex closed curve.

Figure 56: (a) Original self-intersecting curve. (b) to (e): Approximations of self-intersecting curve using techniques (1) to (4) listed in Section 3.3.

Figure 57: (f) to (i): Approximations of self-intersecting curve using techniques (5) to (8) listed in Section 3.3.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_m)</i>	<i>MSE (E_s/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	23	342	0.3838	0.0182	5.1008
Split/Merge (Pavlidis)	22	1497	1.3491	0.153	-4.1346
Avg. neighbours (Tanimoto)	34	128	1.892	0.2253	-5.8149
Hologram-tiles (Prusinkiewicz)	18	87	3.262	0.6534	-10.4994
Minimax (Kurozumi)	24	1942	2.7917	0.6481	-10.4039
Recursive split (Lowe)	23	296	0.3838	0.0182	5.1008
DFT	N/A	296	3.4027	1.3516	-13.5966
DCT	N/A	224	1.3492	0.1778	-4.7833

Figure 58: Comparison table of eight previous techniques applied on self-intersecting curve.

Figure 59: (a) Original picture of a human face. (b) Nearest-neighbour approximation. (c) Hologram-like approximation. (d) Discrete Fourier transform approximation. (e) Discrete cosine transform approximation.

Figure 60: (a) Original picture of a deer. (b) Nearest-neighbour approximation. (c) Hologram-like approximation. (d) Discrete Fourier transform approximation. (e) Discrete cosine transform approximation.

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	198	545.0332	-6.5354
Nearest-neighbour	16:1	242 sec.	128	282.3885	-3.3803
DFT	8:1	710 sec.	255	21665.0011	-22.5288
DCT	16:1	680 sec.	69	220.1519	-2.5884

(a)

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	252	1839.1405	-13.8981
Nearest neighbour	16:1	244 sec.	211	816.3203	-10.3706
DFT	8:1	726 sec.	255	34371.1837	-26.6139
DCT	16:1	702 sec.	137	485.5841	-8.1146

(b)

Figure 61: (a) Results of approximation techniques applied to the human face image (image #1). (b) Results of approximation techniques applied to the deer image (image #2).

Figure 62: (a) Original picture of a squirrel. (b) Nearest-neighbour approximation. (c) Hologram-like approximation. (d) Discrete Fourier transform approximation. (e) Discrete cosine transform approximation.

Figure 63: (a) Original picture of a hiking trail. (b) Nearest-neighbour approximation. (c) Hologram-like approximation. (d) Discrete Fourier transform approximation. (e) Discrete cosine transform approximation.

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	119 sec.	244	806.4289	-12.7094
Nearest neighbour	16:1	228 sec.	180	366.7206	-8.2871
DFT	8:1	720 sec.	255	36126.7125	-29.222
DCT	16:1	689 sec.	132	179.9926	-6.1963

(a)

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	252	1828.7845	-11.9308
Nearest neighbour	16:1	228 sec.	180	869.7711	-8.7032
DFT	8:1	744 sec.	255	23828.5419	-23.0802
DCT	16:1	714 sec.	181	981.7904	-6.9566

(b)

Figure 64: (a) Results of approximation techniques applied to the squirrel image (image #3). (b) Results of approximation techniques applied to the hiking trail image (image #4).

Chapter 4

4 Moment-Preserving Method for 1-D Functions

As mentioned earlier, most if not all of the approximation techniques that have been proposed in the literature, tried to approximate $f(x)$ by some function $\hat{f}(x)$ in such a way that the two functions are *reasonably* close to each other in the domain of the variable x . This requirement will be extended so as to obtain an approximated function which is reasonably close in both the domain of variable x and its transform domain, where the transformation is the geometric moment function. Since the constraints to be satisfied are much more stringent, they will hopefully lead to a more interesting and powerful strategy.

The novel work done in chapters 4 to 6 are a result of a joint collaborative work with Professor Oommen. The work is currently available as a potential publication [NO 94]. A preliminary version of this paper will appear in the Proceedings of IAS-4, the Fourth International Conference on Intelligent Autonomous Systems, Karlsruhe, Germany, March 1995.

4.1 Introduction to Moments

In classical mechanics, the concept of moments is used extensively. For example, an object of mass m located at position x on the x -axis is said to have moment xm about the point $x = 0$, or more generally, moment $(x - x_c)m$ about the point x_0 . If several objects of masses $m_1, m_2, \dots, m_N$ are located at the points $x_1, x_2, \dots, x_N$, respectively, then the total moment of the system of masses about the point x_0 is

the sum of the individual moments.

$$M = (x_1 - x_0)m_1 + (x_2 - x_0)m_2 + \cdots + (x_N - x_0)m_N = \sum_{i=0}^N (x_i - x_0)m_i.$$

For a one-dimensional distribution of mass with continuously variable line density $\rho(x)$ lying along an interval $[0, a]$ of the x -axis, an element of length dx at position x contains mass $dm = \rho(x)dx$, has moment $dM = xdm = x\rho(x)dx$ about the origin. Therefore, the total moment about the origin is defined as

$$M = \int_0^a x\rho(x)dx.$$

In probability theory of random variables, moments play an important role. For example, if in a game of rolling a single die, a player can win X dollars where X is the number showing on top of the rolled die, then in each game the possible winnings are 1, 2, 3, 4, 5, or 6 dollars, each with probability $1/6$. In N number of games, a player can expect to win approximately

$$\frac{N}{6} + \frac{2N}{6} + \frac{3N}{6} + \frac{4N}{6} + \frac{5N}{6} + \frac{6N}{6} = \frac{21N}{6} = \frac{7N}{2} \text{ dollars.}$$

Thus, his expected average winnings per game are $7/2$ dollars, or \$3.50. By using this information, the gaming house can expect a profit if it charges the player more than \$3.50 for the privilege of rolling the die. The value $7/2$ is called the expectation or mean of the discrete random variable X . In general, if a random variable can take on values x_1 with probability p_1 , x_2 with probability p_2 , ..., x_N with probability p_N where $\sum_i^N p_i = 1$, then the mean μ or expectation $E[X]$ of that random variable is given by

$$\mu = E[X] = \sum_{i=1}^N x_i p_i.$$

For a continuous random variable X with probability density function $\rho(x)$ ¹¹ in the interval $[a, b]$, the analogous definition of the mean or the expectation of X is

$$\mu = E[X] = \int_a^b x\rho(x)dx.$$

¹¹If $\rho(x)$ were a mass density such as the case in the classical mechanics example, then μ would be the moment of the mass about 0, and since the total mass would be $\int_a^b \rho(x)dx = 1$, μ would in fact be the centre of mass.

More generally, the expectation of any function $g(X)$ of the random variable X is given as

$$E[g(X)] = \int_a^b g(x)\rho(x)dx$$

and the particular case of the mean of a random variable X , $g(x) = x$. The variance σ^2 of a random variable X with density $\rho(x)$ in the interval $[a, b]$ is the expectation of the square of the distance from X to the mean μ , and hence

$$\begin{aligned}\sigma^2 &= E[(X - \mu)^2] = \int_a^b (x - \mu)^2 \rho(x) dx \\ &= E[(X - \mu)^2] = \int_a^b x^2 \rho(x) dx - 2\mu \int_a^b x \rho(x) dx + \mu^2 \int_a^b \rho(x) dx.\end{aligned}$$

Since $\int_a^b \rho(x) dx = 1$, the expression above for σ^2 can be simplified as

$$\begin{aligned}\sigma^2 &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 \\ &= E[X^2] - E[X]^2.\end{aligned}$$

Typically, the first three moments of a function are very useful in describing some of the characteristics of a random variable X with density function $\rho(x)$. The zeroth moment is the area under the curve $\rho(x)$, which can be interpreted as the mass of an object or the total probability. The first moment describes the centre of mass of an object or equivalently the mean or average of the random variable X . The second moment in combination with the mean describes the moment of inertia for that object or the variance of the random variable X .

4.2 Relationship between Moments and the Characteristic Equations

The characteristic function of a random variable X is the Fourier transform of its density function $f(x)$ (with a reversal in sign) [Pap 65], and is defined as

$$\phi(\omega) = E[e^{j\omega X}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f(x) dx.$$

This function describes the expected value of the complex function $e^{j\omega X} = \cos \omega X + j \sin \omega X$ of X where $j = \sqrt{-1}$. If X is of discrete type, taking values x_k then

$$\phi(\omega) = \sum_k e^{j\omega x_k} P\{X = x_k\}$$

where $P\{X = x_k\}$ is the probability that the random variable X is equal to some value x_k .

If m_k represents the k^{th} geometrical moment of a function $f(x)$

$$m_k = E[X^k] = \int_{-\infty}^{\infty} x^k f(x) dx$$

where $f(x)$ is the density function of a random variable x , then the derivatives of the characteristic function of a random variable X are related to its moments by

$$\frac{d^k \phi(0)}{d\omega^k} = j^k m_k.$$

This correlation can be shown by expanding the characteristic function $\phi(\omega)$ and continuously differentiating it with respect to ω as follows:

$$\begin{aligned} \phi(\omega) &= E[e^{j\omega X}] \\ &= \int_{-\infty}^{\infty} e^{j\omega x} f(x) dx \\ &= \int_{-\infty}^{\infty} \left(1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \cdots + \frac{(j\omega x)^n}{n!} + \cdots \right) f(x) dx. \end{aligned}$$

Assuming that the termwise integration of the equation above is valid, the following is, thus, obtained,

$$\begin{aligned} \phi(\omega) &= \int_{-\infty}^{\infty} f(x) dx + \frac{j\omega}{1!} \int_{-\infty}^{\infty} x f(x) dx + \frac{(j\omega)^2}{2!} \int_{-\infty}^{\infty} x^2 f(x) dx \\ &\quad + \frac{(j\omega)^3}{3!} \int_{-\infty}^{\infty} x^3 f(x) dx + \cdots + \frac{(j\omega)^n}{n!} \int_{-\infty}^{\infty} x^n f(x) dx + \cdots \\ &= m_0 + \frac{j\omega}{1!} m_1 + \frac{(j\omega)^2}{2!} m_2 + \cdots + \frac{(j\omega)^n}{n!} m_n + \cdots. \end{aligned} \quad (14)$$

As indicated by Papoulis, the above operation holds if the moments m_n are finite and the series from equation (14) converges absolutely near $\omega = 0$. In this case $\phi(\omega)$ is given explicitly by (14); therefore, $f(x)$ is uniquely determined from its moments.

4.3 Moment-Preserving Approximation

If $f(x)$ and $g(x)$ are two functions defined between the interval $[a, b]$, then from the theory of moments all the k^{th} order moments $\int x^k f(x) dx$ and $\int x^k g(x) dx$ will be exactly equal for all values of k iff $f(x) \triangleq g(x)$. However, $\hat{f}(x)$ will be considered to be a *reasonably* good approximation of $f(x)$ if their respective moments match to some finite order, k . Another requirement in this approximation technique is that it must also be *reasonably* good in the transform domain. Although this provides much more stringent requirements to the strategy, these constraints will hopefully lead to a more interesting and powerful solution.

To initiate this strategy, it is assumed that the $f(x)$ is continuous¹², integrable and that all of its moments exist. The above assumption is identical to assuming that the appropriate transform of the function exists. In a practical situation, however, the explicit functional form of $f(x)$ will be unknown, and it is often specified by a set of S of N discrete sampling points as

$$S = \{p_i = (x_i, y_i) | i = 0, 1, \dots, N\}.$$

An approximation $\hat{f}(x)$ will be derived from S such that $\hat{f}(x)$ is specified a set $\hat{S}$ of n unique points where $n \ll N$.

$$\hat{S} = \{\hat{p}_k = (\hat{x}_k, \hat{y}_k) | k = 0, 1, \dots, n\}.$$

$\hat{f}(x)$ is a piecewise polynomial joining the points in the set $\hat{S}$, and in this particular case the polynomial is restricted to be linear functions. The notation $\hat{x}_k$ and $\hat{y}_k$ is used to emphasize the fact that the set $\hat{S}$ is not a subset of S . In other words, the approximation points do not necessarily lie on the original given curve, $f(x)$.

The main premise of this approximation technique is based on the idea that two functions will be approximately the same if a finite number of their moments

¹²For the sake of simplicity in the explanation, all of the derivations will be done on the assumption that $f(x)$ is continuous. In other words, the derivation of the approximation values is based on the assumption that a function $f(x)$ is given in its explicit form. The analogous case where $f(x)$ is a discrete function will also be shown.

are equal. Since the function of $f(x)$ is assumed to be known, the moments of the function, m_k , are also assumed to be known. The main task is to approximate $f(x)$ by piecewise linear functions. As indicated in the previous sections, there are many different ways of approaching this problem. However, our constraint here will be to determine an approximation function $\hat{f}(x)$ such that a finite number of its theoretical moments will be **exactly identical** to the moments of the original function $f(x)$.

Definition 1 Let $f(x)$ and $g(x)$ be two integrable functions between interval $[a, b]$. If $\int x^i f(x) dx = \int x^i g(x) dx$ for $a \leq x \leq b$ for all $0 \leq i \leq k$, then $f(x)$ and $g(x)$ are said to be k -order equivalent.

Consider a known function f and some function g which is used to approximate f . If we make f and g to be k -order equivalent, it implies that the moments, up to order k , of f and g are exactly identical which, in turn, implies that in the time/spacial domain and in the transform domain they are approximations.

Let $f(x)$ be the single variable function to be approximated. Then the k^{th} moment of the function $f(x)$ is defined as

$$m_k = \int x^k f(x) dx$$

where $k = 0, 1, 2, \dots$, are non-negative integers. It can be seen that by substituting in for $k = 0$ into the equation above, m_0 , the zeroth moment, is the area of under the curve $f(x)$. In the analogous case of a discrete function f defined by the set of points $\{(x_0, f_0), (x_1, f_1), \dots, (x_N, f_N)\}$, the k^{th} moment is

$$m_k = \sum_{i=0}^{N-1} x_i^k f_i \Delta x_{i+1}$$

where $\Delta x_{i+1} = x_{i+1} - x_i$.

Derivations for the moment-preserving approximation method will be done for three cases. The first case approximates a given function f by a single line segment;

the second case approximates it by two line segments; the third case is the general solution where the number of line segments is n . In order to show some insight into the moment-preserving technique, we will use a specific example to assist in the derivation of the approximation for one- and two-line approximation cases.

4.3.1 Single Straight Line (2-point) Approximation

Let $f(x)$ be a one-dimensional function to be approximated by two points (x_{i_0}, f_{i_0}) and $(x_{i_1}, \hat{f}_{i_1})$ where x_{i_0} and x_{i_1} are the knot values defined in the domain of $f(x)$ and are uniformly (or non-uniformly) selected. The values $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ must be calculated such that a finite number of moments of the line segment passing through the approximated points $(x_{i_0}, \hat{f}_{i_0})$ and $(x_{i_1}, \hat{f}_{i_1})$ are exactly identical to that of the given function f . Since there are two unknowns, a system of at least two analytic equations involving the moments and the two unknowns $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ is required.

Let $\hat{m}_0$ and $\hat{m}_1$ be the zeroth and first moments of the function $\hat{f}(x)$, respectively, where $\hat{f}(x)$ is an approximation of the function $f(x)$, and let m_0 and m_1 be the zeroth and first moments of the function $f(x)$. Then the first two moments of the approximated function $\hat{f}(x)$ are

$$\hat{m}_0 = \int_{x_{i_0}}^{x_{i_1}} \hat{f}(x) dx \quad (15)$$

$$\hat{m}_1 = \int_{x_{i_0}}^{x_{i_1}} x \hat{f}(x) dx \quad (16)$$

The equation of the straight line segment passing through the points $(x_{i_0}, \hat{f}_{i_0})$, and $(x_{i_1}, \hat{f}_{i_1})$ is given by:

$$\begin{aligned} \frac{\hat{f}(x) - \hat{f}_{i_1}}{x - x_{i_1}} &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \\ \hat{f}(x) &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}}(x - x_{i_1}) + \hat{f}_{i_1}. \end{aligned}$$

Substituting $\hat{f}(x)$ into the moment equations (15) and (16) above and proceeding

with the integration, the following equations are derived,

$$\begin{aligned}
 \widehat{m}_0 &= \int_{x_{i_0}}^{x_{i_1}} \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\
 &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{2(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^2 - x_{i_0}^2) - x_{i_1} \cdot \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x_{i_1} - x_{i_0}) \\
 &\quad + \hat{f}_{i_1} \cdot (x_{i_1} - x_{i_0}) \\
 &= \hat{f}_{i_0} \cdot \left[-\frac{x_{i_1}^2 - x_{i_0}^2}{2(x_{i_1} - x_{i_0})} + \frac{x_{i_1}(x_{i_1} - x_{i_0})}{x_{i_1} - x_{i_0}} \right] \\
 &\quad + \hat{f}_{i_1} \cdot \left[\frac{x_{i_1}^2 - x_{i_0}^2}{2(x_{i_1} - x_{i_0})} - \frac{x_{i_1}(x_{i_1} - x_{i_0})}{x_{i_1} - x_{i_0}} + (x_{i_1} - x_{i_0}) \right] \\
 &= \hat{f}_{i_0} \cdot \left[\frac{x_{i_1} - x_{i_0}}{2} \right] + \hat{f}_{i_1} \cdot \left[\frac{x_{i_1} - x_{i_0}}{2} \right] \tag{17}
 \end{aligned}$$

$$\begin{aligned}
 \widehat{m}_1 &= \int_{x_{i_0}}^{x_{i_1}} x \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\
 &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{3(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^3 - x_{i_0}^3) \\
 &\quad - x_{i_1} \cdot \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{2(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^2 - x_{i_0}^2) \\
 &\quad + \frac{\hat{f}_{i_1}}{2} \cdot (x_{i_1}^2 - x_{i_0}^2) \\
 &= \hat{f}_{i_0} \cdot \left[-\frac{x_{i_1}^3 - x_{i_0}^3}{3(x_{i_1} - x_{i_0})} + \frac{x_{i_1}(x_{i_1}^2 - x_{i_0}^2)}{2(x_{i_1} - x_{i_0})} \right] \\
 &\quad + \hat{f}_{i_1} \cdot \left[\frac{x_{i_1}^3 - x_{i_0}^3}{3(x_{i_1} - x_{i_0})} - \frac{x_{i_1}(x_{i_1}^2 - x_{i_0}^2)}{2(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^2 - x_{i_0}^2}{2} \right] \\
 &= \hat{f}_{i_0} \cdot \left[\frac{x_{i_1}^3 - 3x_{i_1}x_{i_0}^2 + 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} \right] \\
 &\quad + \hat{f}_{i_1} \cdot \left[\frac{-x_{i_1}^3 + 3x_{i_1}x_{i_0}^2 - 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^2 - x_{i_0}^2}{2} \right]. \tag{18}
 \end{aligned}$$

At this juncture we re-iterate our goal: the main purpose of this moment-preserving approximation technique is to find a good approximation which will also preserve the moments of the original function. In the particular case of a single straight line segment approximation, we therefore require that the first two moments of the original function be preserved, and we therefore equate, $\widehat{m}_0 = m_0$ and $\widehat{m}_1 = m_1$. By simplifying the equations (17) and (18) from above, the following

system of equations of two unknowns $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ are formed:

$$\begin{aligned} \left[\frac{x_{i_1} - x_{i_0}}{2} \right] \cdot \hat{f}_{i_0} + \left[\frac{x_{i_1} - x_{i_0}}{2} \right] \cdot \hat{f}_{i_1} &= m_0 \\ \left[\frac{x_{i_1}^3 - 3x_{i_1}x_{i_0}^2 + 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} \right] \cdot \hat{f}_{i_0} + \left[\frac{-x_{i_1}^3 + 3x_{i_1}x_{i_0}^2 - 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^2 - x_{i_0}^2}{2} \right] \cdot \hat{f}_{i_1} &= m_1. \end{aligned}$$

They can be expressed in a matrix notation as

$$\begin{bmatrix} \frac{x_{i_1} - x_{i_0}}{2} & \frac{x_{i_1} - x_{i_0}}{2} \\ \frac{x_{i_1}^3 - 3x_{i_1}x_{i_0}^2 + 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} & \frac{-x_{i_1}^3 + 3x_{i_1}x_{i_0}^2 - 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^2 - x_{i_0}^2}{2} \end{bmatrix} \cdot \begin{bmatrix} \hat{f}_{i_0} \\ \hat{f}_{i_1} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \quad (19)$$

Therefore, the values for $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ can be found by

$$\begin{bmatrix} \hat{f}_{i_0} \\ \hat{f}_{i_1} \end{bmatrix} = \begin{bmatrix} \frac{x_{i_1} - x_{i_0}}{2} & \frac{x_{i_1} - x_{i_0}}{2} \\ \frac{x_{i_1}^3 - 3x_{i_1}x_{i_0}^2 + 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} & \frac{-x_{i_1}^3 + 3x_{i_1}x_{i_0}^2 - 2x_{i_0}^3}{6(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^2 - x_{i_0}^2}{2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \quad (20)$$

The example below will further illustrate this technique. Suppose that the function $f(x)$ defined in the interval $[0, 2]$ is given as $f(x) = -(x - 2)^2 + 4$, and the two knot values are $x_{i_0} = 0$ and $x_{i_1} = 2$ as shown in Figure 65. Visual inspection of the function indicates that it is symmetrical about the line $x = 2$, and since one of knot values, x_{i_1} , lies directly on this line of symmetry while the other knot value is 2 units from this line, it can be expected that the line of approximation has a positive slope and that it will pass through the two end points of the curve. If the knot value x_{i_1} is moved away from the line $x = 2$ such that it is closer to the knot value x_{i_0} , the slope of the line of approximation should increase positively, becoming more vertical. On the other hand, if the knot value x_{i_0} is moved closer to the line $x = 2$, the opposite should occur, that is, the slope of the line of approximation will become more horizontal. If the knot x_{i_0} is moved closer to the line $x = 2$ and the other knot value x_{i_1} is moved away from the line $x = 2$ by the same amount *Delta*, then the values $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ will become larger because the zeroth moment or area under the function f and $\hat{f}$ are set to be equal so as to preserve the moments.

These visual inspections are easily verified by the actual calculations of the moments and parameters required by the matrix as shown in equation (19). The

Figure 65: Function $f(x) = -(x - 2)^2 + 4$ defined in the interval $[0, 2]$

zeroth and first moments of $f(x)$ are calculated to be $\frac{16}{3}$ and $\frac{20}{3}$, respectively. The matrix of equation (19) and its inverse are determined to be

$$\begin{bmatrix} 1 & 1 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -\frac{3}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

By substituting these values into equation (20),² the results $\hat{f}_{i_0}$ and $\hat{f}_{i_1}$ are found to be equal to $\frac{2}{3}$ and $\frac{14}{3}$, respectively. These values indicate that the line segment which passes through $(0, \frac{2}{3})$ and $(4, \frac{14}{3})$ is approximately the same as the function $f(x)$ in terms of the first two moments m_0 and m_1 as shown in Figure 66(a). The results of the actual calculations correspond with the visually observed properties indicated above, and these results are shown in Figure 66(b)-(d). It should be noted that as the values of x_{i_0} and x_{i_1} come closer together and the moments remain the same, then the magnitude of the slope of the approximating line segment becomes larger and larger (more vertical) causing the approximation $\hat{f}(x)$ to diverge from the function $f(x)$.

Figure 66: Result of single line approximation passing through the different knot values. (a) Knot values $x_{i_0} = 0$ and $x_{i_1} = 2$. (b) Knot values $x_{i_0} = 0$ and $x_{i_1} = 3/2$. (c) Knot values $x_{i_0} = 1$ and $x_{i_1} = 2$. (d) Knot values $x_{i_0} = 1/2$ and $x_{i_1} = 3/2$

4.3.2 Two-Piecewise-Linear (3-point) Approximation

The same approach can be applied when the number of approximated points is increased from two to three. The only difference between this case and the single line approximation is the location of the middle point which is required to join the two line segments together. In order to approximate $f(x)$ with two line segments passing through the points $(x_{i_0}, \hat{f}_{i_0})$, $(x_{i_1}, \hat{f}_{i_1})$, and $(x_{i_2}, \hat{f}_{i_2})$ where $\hat{f}_{i_k}$'s are the approximated values, there must be a third linearly independent equation in the new system of equations involving the three unknowns. By using the first three moments of the approximated and actual functions, a linear system of three equations involving the three unknowns $\hat{f}_{i_0}$, $\hat{f}_{i_1}$, and $\hat{f}_{i_2}$ can be set up. This motivates the following rule necessary for this moment-preserving technique:

Rule 1 Let $f(x)$ be the original function and $\hat{f}(x)$ be its approximation. If $\hat{f}(x)$ approximates $f(x)$ by $n - 1$ piecewise linear segments, it is necessary that the first n or more moments of $f(x)$ be preserved. This is equivalent to saying that at least the first n moments of $f(x)$ and $\hat{f}(x)$ must be equal.

To be more specific for our particular scenario, the first three moments of $\hat{f}(x)$ are defined as follows.

$$\begin{aligned}
 \hat{m}_0 &= \int_{x_{i_0}}^{x_{i_1}} \hat{f}(x) dx + \int_{x_{i_1}}^{x_{i_2}} \hat{f}(x) dx \\
 &= \int_{x_{i_0}}^{x_{i_1}} \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\
 &\quad + \int_{x_{i_1}}^{x_{i_2}} \left[\frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{x_{i_2} - x_{i_1}} \cdot (x - x_{i_2}) + \hat{f}_{i_2} \right] dx \\
 &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{2(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^2 - x_{i_0}^2) \\
 &\quad - x_{i_1} \cdot \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x_{i_1} - x_{i_0}) \\
 &\quad + \hat{f}_{i_1} \cdot (x_{i_1} - x_{i_0}) \\
 &\quad + \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{2(x_{i_2} - x_{i_1})} \cdot (x_{i_2}^2 - x_{i_1}^2) \\
 &\quad - x_{i_2} \cdot \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{x_{i_2} - x_{i_1}} \cdot (x_{i_2} - x_{i_1}) \\
 &\quad + \hat{f}_{i_2} \cdot (x_{i_2} - x_{i_1}).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \hat{m}_1 &= \int_{x_{i_0}}^{x_{i_1}} x \hat{f}(x) dx + \int_{x_{i_1}}^{x_{i_2}} x \hat{f}(x) dx \\
 &= \int_{x_{i_0}}^{x_{i_1}} x \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\
 &\quad + \int_{x_{i_1}}^{x_{i_2}} x \left[\frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{x_{i_2} - x_{i_1}} \cdot (x - x_{i_2}) + \hat{f}_{i_2} \right] dx \\
 &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{3(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^3 - x_{i_0}^3)
 \end{aligned}$$

$$\begin{aligned}
& -x_{i_1} \cdot \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{2(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^2 - x_{i_0}^2) \\
& + \frac{\hat{f}_{i_1}}{2} \cdot (x_{i_1}^2 - x_{i_0}^2) \\
& + \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{3(x_{i_2} - x_{i_1})} \cdot (x_{i_2}^3 - x_{i_1}^3) \\
& - x_{i_2} \cdot \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{2(x_{i_2} - x_{i_1})} \cdot (x_{i_2}^2 - x_{i_1}^2) \\
& + \frac{\hat{f}_{i_2}}{2} \cdot (x_{i_2}^2 - x_{i_1}^2).
\end{aligned}$$

Finally,

$$\begin{aligned}
\hat{m}_2 &= \int_{x_{i_0}}^{x_{i_1}} x^2 \hat{f}(x) dx + \int_{x_{i_1}}^{x_{i_2}} x^2 \hat{f}(x) dx \\
&= \int_{x_{i_0}}^{x_{i_1}} x^2 \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\
&\quad + \int_{x_{i_1}}^{x_{i_2}} x^2 \left[\frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{x_{i_2} - x_{i_1}} \cdot (x - x_{i_2}) + \hat{f}_{i_2} \right] dx \\
&= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{4(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^4 - x_{i_0}^4) \\
&\quad - x_{i_1} \cdot \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{3(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^3 - x_{i_0}^3) \\
&\quad + \frac{\hat{f}_{i_1}}{3} \cdot (x_{i_1}^3 - x_{i_0}^3) \\
&\quad + \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{4(x_{i_2} - x_{i_1})} \cdot (x_{i_2}^4 - x_{i_1}^4) \\
&\quad - x_{i_2} \cdot \frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{3(x_{i_2} - x_{i_1})} \cdot (x_{i_2}^3 - x_{i_1}^3) \\
&\quad + \frac{\hat{f}_{i_2}}{3} \cdot (x_{i_2}^3 - x_{i_1}^3).
\end{aligned}$$

The three moments of the approximation function $\hat{f}(x)$ must be equal to the three corresponding moments of the given function $f(x)$. By equating these respective quantities, a system of equations is formed and can be expressed using the

matrix notation as

$$\begin{bmatrix} \frac{x_{i_1}-x_{i_0}}{2} & \frac{x_{i_1}-x_{i_0}}{2} + \frac{x_{i_2}-x_{i_1}}{2} & \frac{x_{i_2}-x_{i_1}}{2} \\ \frac{(x_{i_1}^3-3x_{i_1}x_{i_0}^2+2x_{i_0}^3)}{6(x_{i_1}-x_{i_0})} & \frac{-x_{i_1}^3+3x_{i_1}x_{i_0}^2-2x_{i_0}^3}{6(x_{i_1}-x_{i_0})} + \frac{x_{i_1}^2-x_{i_0}^2}{2} + \frac{x_{i_2}^3-3x_{i_2}x_{i_1}^2+2x_{i_1}^3}{6(x_{i_2}-x_{i_1})} & \frac{-x_{i_2}^3+3x_{i_2}x_{i_1}^2-2x_{i_1}^3}{6(x_{i_2}-x_{i_1})} + \frac{x_{i_2}^2-x_{i_1}^2}{2} \\ \frac{(x_{i_1}^4-4x_{i_1}x_{i_0}^3+3x_{i_0}^4)}{12(x_{i_1}-x_{i_0})} & \frac{-x_{i_1}^4+4x_{i_1}x_{i_0}^3-3x_{i_0}^4}{12(x_{i_1}-x_{i_0})} + \frac{x_{i_1}^3-x_{i_0}^3}{3} + \frac{x_{i_2}^4-4x_{i_2}x_{i_1}^3+3x_{i_1}^4}{12(x_{i_2}-x_{i_1})} & \frac{-x_{i_2}^4+4x_{i_2}x_{i_1}^3-3x_{i_1}^4}{12(x_{i_2}-x_{i_1})} + \frac{x_{i_2}^3-x_{i_1}^3}{3} \end{bmatrix} \cdot \begin{bmatrix} \hat{f}_{i_0} \\ \hat{f}_{i_1} \\ \hat{f}_{i_2} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}$$

The three unknown values $\hat{f}_{i_0}$, $\hat{f}_{i_1}$, and $\hat{f}_{i_2}$ can easily be found by multiplying the inverse of the matrix by the known moments. Figure 67 shows the results of approximating the function $f(x) = -(x-2)^2 + 4$ with two, three, four and five piecewise linear segments where the x values of the knot points are uniformly distributed along the x -axis.

4.3.3 Approximation using n Piecewise Linear Segments

The general case in which the number of piecewise linear segments contained in the approximation function is n can be derived in the same manner as the cases for a single line and two-line approximations. Let the k^{th} moment of an n -piecewise linear approximation function $\hat{f}(x)$ be defined as the sum of the moments of the individual segments

$$\widehat{m}_k = S_{k,1} + S_{k,2} + \cdots + S_{k,n}.$$

Then the k^{th} moment of the individual segments can be obtained as follows,

$$\begin{aligned} S_{k,1} &= \int_{x_{i_0}}^{x_{i_1}} x^k \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot (x - x_{i_1}) + \hat{f}_{i_1} \right] dx \\ &= \int_{x_{i_0}}^{x_{i_1}} \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{x_{i_1} - x_{i_0}} \cdot x^{k+1} - \frac{x_{i_1}(\hat{f}_{i_1} - \hat{f}_{i_0})}{x_{i_1} - x_{i_0}} \cdot x^k + \hat{f}_{i_1} \cdot x^k \right] dx \\ &= \left[\frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{(k+2)(x_{i_1} - x_{i_0})} \cdot x^{k+2} - \frac{x_{i_1}(\hat{f}_{i_1} - \hat{f}_{i_0})}{(k+1)(x_{i_1} - x_{i_0})} \cdot x^{k+1} + \hat{f}_{i_1} \cdot \frac{x^{k+1}}{k+1} \right]_{x_{i_0}}^{x_{i_1}} \\ &= \frac{\hat{f}_{i_1} - \hat{f}_{i_0}}{(k+2)(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^{k+2} - x_{i_0}^{k+2}) \end{aligned}$$

Figure 67: Results of different number of approximating lines different knot values. (a) Three knot values $x_{i_0} = 0$, $x_{i_1} = 2$, and $x_{i_2} = 4$. (b) Four knot values $x_{i_0} = 0$, $x_{i_1} = 4/3$, $x_{i_2} = 8/3$, and $x_{i_3} = 4$. (c) Five knot values $x_{i_0} = 0$ and $x_{i_1} = 1$, $x_{i_2} = 2$, $x_{i_3} = 3$, and $x_{i_4} = 4$. (d) Six knot values $x_{i_0} = 0$, $x_{i_1} = 4/5$, $x_{i_2} = 8/5$, $x_{i_3} = 12/5$, $x_{i_4} = 16/5$, and $x_{i_5} = 4$.

$$\begin{aligned}
 & - \frac{x_{i_1}[\hat{f}_{i_1} - \hat{f}_{i_0}]}{(k+1)(x_{i_1} - x_{i_0})} \cdot (x_{i_1}^{k+1} - x_{i_0}^{k+1}) + \hat{f}_{i_1} \cdot \frac{x_{i_1}^{k+1} - x_{i_0}^{k+1}}{k+1} \\
 = & \hat{f}_{i_0} \cdot \left[-\frac{x_{i_1}^{k+2} - x_{i_0}^{k+2}}{(k+2)(x_{i_1} - x_{i_0})} + \frac{x_{i_1}(x_{i_1}^{k+1} - x_{i_0}^{k+1})}{(k+1)(x_{i_1} - x_{i_0})} \right] \\
 & + \hat{f}_{i_1} \cdot \left[\frac{x_{i_1}^{k+2} - x_{i_0}^{k+2}}{(k+2)(x_{i_1} - x_{i_0})} - \frac{x_{i_1}(x_{i_1}^{k+1} - x_{i_0}^{k+1})}{(k+1)(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^{k+1} - x_{i_0}^{k+1}}{k+1} \right] \\
 = & \hat{f}_{i_0} \cdot \left[\frac{x_{i_1}^{k+2} - (k+2)x_{i_1}x_{i_0}^{k+1} + (k+1)x_{i_0}^{k+2}}{(k+1)(k+2)(x_{i_1} - x_{i_0})} \right] \\
 & + \hat{f}_{i_1} \cdot \left[\frac{-x_{i_1}^{k+2} + (k+2)x_{i_1}x_{i_0}^{k+1} - (k+1)x_{i_0}^{k+2}}{(k+1)(k+2)(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^{k+1} - x_{i_0}^{k+1}}{k+1} \right] \\
 S_{k,2} = & \int_{x_{i_1}}^{x_{i_2}} x^k \left[\frac{\hat{f}_{i_2} - \hat{f}_{i_1}}{x_{i_2} - x_{i_1}} \cdot (x - x_{i_2}) + \hat{f}_{i_2} \right] dx
 \end{aligned}$$

$$\begin{aligned}
&= \hat{f}_{i_1} \cdot \left[\frac{x_{i_2}^{k+2} - (k+2)x_{i_2}x_{i_1}^{k+1} + (k+1)x_{i_1}^{k+2}}{(k+1)(k+2)(x_{i_2} - x_{i_1})} \right] \\
&\quad + \hat{f}_{i_2} \cdot \left[\frac{-x_{i_2}^{k+2} + (k+2)x_{i_2}x_{i_1}^{k+1} - (k+1)x_{i_1}^{k+2}}{(k+1)(k+2)(x_{i_2} - x_{i_1})} + \frac{x_{i_2}^{k+1} - x_{i_1}^{k+1}}{k+1} \right] \\
&\quad \vdots \\
S_{k,n-1} &= \int_{x_{i_{n-2}}}^{x_{i_{n-1}}} x^k \left[\frac{\hat{f}_{i_{n-1}} - \hat{f}_{i_{n-2}}}{x_{i_{n-1}} - x_{i_{n-2}}} \cdot (x - x_{i_{n-1}}) + \hat{f}_{i_{n-1}} \right] dx \\
&= \hat{f}_{i_{n-2}} \cdot \left[\frac{x_{i_{n-1}}^{k+2} - (k+2)x_{i_{n-1}}x_{i_{n-2}}^{k+1} + (k+1)x_{i_{n-2}}^{k+2}}{(k+1)(k+2)(x_{i_{n-1}} - x_{i_{n-2}})} \right] \\
&\quad + \hat{f}_{i_{n-1}} \cdot \left[\frac{-x_{i_{n-1}}^{k+2} + (k+2)x_{i_{n-1}}x_{i_{n-2}}^{k+1} - (k+1)x_{i_{n-2}}^{k+2}}{(k+1)(k+2)(x_{i_{n-1}} - x_{i_{n-2}})} + \frac{x_{i_{n-1}}^{k+1} - x_{i_{n-2}}^{k+1}}{k+1} \right] \\
S_{k,n} &= \int_{x_{i_{n-1}}}^{x_{i_n}} x^k \left[\frac{\hat{f}_{i_n} - \hat{f}_{i_{n-1}}}{x_{i_n} - x_{i_{n-1}}} \cdot (x - x_{i_n}) + \hat{f}_{i_n} \right] dx \\
&= \hat{f}_{i_{n-1}} \cdot \left[\frac{x_{i_n}^{k+2} - (k+2)x_{i_n}x_{i_{n-1}}^{k+1} + (k+1)x_{i_{n-1}}^{k+2}}{(k+1)(k+2)(x_{i_n} - x_{i_{n-1}})} \right] \\
&\quad + \hat{f}_{i_n} \cdot \left[\frac{-x_{i_n}^{k+2} + (k+2)x_{i_n}x_{i_{n-1}}^{k+1} - (k+1)x_{i_{n-1}}^{k+2}}{(k+1)(k+2)(x_{i_n} - x_{i_{n-1}})} + \frac{x_{i_n}^{k+1} - x_{i_{n-1}}^{k+1}}{k+1} \right]
\end{aligned}$$

Therefore, $\hat{m}_k$ can be rewritten as

$$\begin{aligned}
\hat{m}_k &= \hat{f}_{i_0} \cdot \left[\frac{x_{i_1}^{k+2} - (k+2)x_{i_1}x_{i_0}^{k+1} + (k+1)x_{i_0}^{k+2}}{(k+1)(k+2)(x_{i_1} - x_{i_0})} \right] \\
&\quad + \hat{f}_{i_1} \cdot \left[\frac{-x_{i_1}^{k+2} + (k+2)x_{i_1}x_{i_0}^{k+1} - (k+1)x_{i_0}^{k+2}}{(k+1)(k+2)(x_{i_1} - x_{i_0})} + \frac{x_{i_1}^{k+1} - x_{i_0}^{k+1}}{k+1} \right. \\
&\quad \quad \left. + \frac{x_{i_2}^{k+2} - (k+2)x_{i_2}x_{i_1}^{k+1} + (k+1)x_{i_1}^{k+2}}{(k+1)(k+2)(x_{i_2} - x_{i_1})} \right] \\
&\quad + \hat{f}_{i_2} \cdot \left[\frac{-x_{i_2}^{k+2} + (k+2)x_{i_2}x_{i_1}^{k+1} - (k+1)x_{i_1}^{k+2}}{(k+1)(k+2)(x_{i_2} - x_{i_1})} + \frac{x_{i_2}^{k+1} - x_{i_1}^{k+1}}{k+1} \right. \\
&\quad \quad \left. + \frac{x_{i_3}^{k+2} - (k+2)x_{i_3}x_{i_2}^{k+1} + (k+1)x_{i_2}^{k+2}}{(k+1)(k+2)(x_{i_3} - x_{i_2})} \right] \\
&\quad + \\
&\quad \vdots \\
&\quad + \hat{f}_{i_{n-1}} \cdot \left[\frac{-x_{i_{n-1}}^{k+2} + (k+2)x_{i_{n-1}}x_{i_{n-2}}^{k+1} - (k+1)x_{i_{n-2}}^{k+2}}{(k+1)(k+2)(x_{i_{n-1}} - x_{i_{n-2}})} + \frac{x_{i_{n-1}}^{k+1} - x_{i_{n-2}}^{k+1}}{k+1} \right. \\
&\quad \quad \left. + \frac{x_{i_n}^{k+2} - (k+2)x_{i_n}x_{i_{n-1}}^{k+1} + (k+1)x_{i_{n-1}}^{k+2}}{(k+1)(k+2)(x_{i_n} - x_{i_{n-1}})} \right]
\end{aligned}$$

$$+ \hat{f}_{i_n} \cdot \left[\frac{-x_{i_n}^{k+2} + (k+2)x_{i_n}x_{i_{n-1}}^{k+1} - (k+1)x_{i_{n-1}}^{k+2}}{(k+1)(k+2)(x_{i_n} - x_{i_{n-1}})} + \frac{x_{i_n}^{k+1} - x_{i_{n-1}}^{k+1}}{k+1} \right]$$

Let

$$C_{k,j} = \frac{x_{j+1}^{k+2} - (k+2)x_{j+1}x_j^{k+1} + (k+1)x_j^{k+2}}{(k+1)(k+2)(x_{j+1} - x_j)}$$

$$D_{k,j} = \frac{x_{j+1}^{k+1} - x_j^{k+1}}{k+1}$$

where $0 \leq j \leq (n-1)$ and $0 \leq k \leq n$, then

$$\begin{aligned} \hat{m}_k &= \hat{f}_{i_0} \cdot C_{k,0} \\ &+ \hat{f}_{i_1} \cdot (-C_{k,0} + D_{k,0} + C_{k,1}) \\ &+ \hat{f}_{i_2} \cdot (-C_{k,1} + D_{k,1} + C_{k,2}) \\ &+ \\ &\vdots \\ &+ \hat{f}_{i_{n-1}} \cdot (-C_{k,n-2} + D_{k,n-2} + C_{k,n-1}) \\ &+ \hat{f}_{i_n} \cdot (-C_{k,n-1} + D_{k,n-1}) \end{aligned}$$

If $\mathbf{E}_j$ is a column vector of n elements defined by

$$\mathbf{E}_j = \begin{cases} \begin{bmatrix} C_{0,j} \\ C_{1,j} \\ \vdots \\ C_{n,j} \end{bmatrix} & j = 0 \\ \begin{bmatrix} -C_{0,j-1} + D_{0,j-1} + C_{0,j} \\ -C_{1,j-1} + D_{1,j-1} + C_{1,j} \\ \vdots \\ -C_{n,j-1} + D_{n,j-1} + C_{n,j} \end{bmatrix} & 1 \leq j \leq n-1 \\ \begin{bmatrix} -C_{0,j-1} + D_{0,j-1} \\ -C_{1,j-1} + D_{1,j-1} \\ \vdots \\ -C_{n,j-1} + D_{n,j-1} \end{bmatrix} & j = n \end{cases}$$

then the system involving the n equations and n unknowns $f_{i_0}, f_{i_1}, \dots, f_{i_n}$ can be summarized in a matrix notation as

$$\begin{bmatrix} \mathbf{E}_0 & \mathbf{E}_1 & \mathbf{E}_2 & \cdots & \mathbf{E}_{n-1} & \mathbf{E}_n \end{bmatrix} \cdot \begin{bmatrix} \hat{f}_{i_0} \\ \hat{f}_{i_1} \\ \hat{f}_{i_2} \\ \vdots \\ \hat{f}_{i_{n-1}} \\ \hat{f}_{i_n} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ m_{n-1} \\ m_n \end{bmatrix}$$

4.3.4 Continuous versus Discrete Functions $f(x)$

All of the derivations described previously are based on the assumption that the given function $f(x)$ is continuous. However, in many practical applications, the function is never given in its closed form. In these situations, if $f(x)$ is described by a sequence of discrete points, then the same formula from above can be applied to the original function $f(x)$. Let $f(x)$ be given as a sequence of $N + 1$ points:

$$\{(x_i, f_i) | i = 0, 1, 2, \dots, N\}.$$

Then the k^{th} moment of the original function $f(x)$ is given by:

$$\begin{aligned} m_k &= f_0 \cdot C_{k,0} \\ &+ f_1 \cdot (-C_{k,0} + D_{k,0} + C_{k,1}) \\ &+ f_2 \cdot (-C_{k,1} + D_{k,1} + C_{k,2}) \\ &+ \\ &\vdots \\ &+ f_{N-1} \cdot (-C_{k,N-2} + D_{k,N-2} + C_{k,N-1}) \\ &+ f_N \cdot (-C_{k,N-1} + D_{k,N-1}) \end{aligned}$$

where

$$C_{k,j} = \frac{x_{j+1}^{k+2} - (k+2)x_{j+1}x_j^{k+1} + (k+1)x_j^{k+2}}{(k+1)(k+2)(x_{j+1} - x_j)}$$

$$D_{k,j} = \frac{x_{j+1}^{k+1} - x_j^{k+1}}{k+1}$$

for $0 \leq j \leq (N-1)$ and $0 \leq k \leq N$. Using this, the exact k^{th} moment of $f(x)$ and $\hat{f}(x)$ can be equated to yield the piecewise linear approximation.

4.4 Experimental Results on One-dimensional Functions

In this chapter, a new moment-preserving piecewise linear approximation technique has been presented. This approximation method is analysed in detail for specific cases of single-line-segment and two-piecewise-line-segment approximations. A general solution for N -piecewise-line-segments is also derived from the basic principle of moments.

In order to accurately compare this technique with the other techniques reviewed in Chapter 2, the same list of one-dimensional functions will be used:

1. $f_1(x) = 5 \sin 4x + 4 \cos 3x$, defined in interval $[0, 10]$.
2. $f_2(x) = \frac{20}{\sqrt{2\pi}} e^{-2x^2}$ defined in interval $[-6, 6]$.
3. $f_3(x) =$ Sawtooth waves defined in interval $[-12, 12]$.
4. $f_4(x) = \frac{7 \sin x}{x}$ defined in interval $[-10.2, 12.1]$.
5. $f_5(x) = \frac{7 \sin x}{x} +$ Sawtooth waves defined in interval $[-10.2, 12.1]$.
6. $f_6(x) = 5 \sin 4x + 4 \cos 3x + 2 \cos 200x$ defined in interval $[0, 10]$.

Results of this approximation technique as it is applied to the list of one-dimensional functions are shown and tabulated below. A comparison can be made between these moment-preserving approximations and the previous methods listed in Section 3.3. Using the SNR as the major criterion, results from the moment-preserving technique as applied to function f_1 are much better than the nearest-neighbour, split and merge, and minimax methods. It is not as good as the linear

and recursive split methods if the same number of line segments are used. However, using 25 points of approximation, this technique is comparable to that of the linear and recursive split methods. This technique outperforms all other time/spacial domain techniques for the approximation of the normal bell curve (function f_2); with exactly five points of approximation this technique achieves the SNR of over $14dB$, whereas the linear and recursive split methods has an SNR of under $12dB$. This technique has a lot of difficulty approximating the sawtooth waves; the SNR value of a 21-point moment-preserving approximation is less than the SNR value of an 18-point approximation of the split and merge method. This seems to indicate that the moment-preserving method cannot be applied to functions which have sharp edges (drastic changes). It is interesting to note that the approximation of the noisy function (function f_6) is identical to that of the approximation of the non-noisy function (function f_1). Our experimental experience tends to show that the moment-preserving approximation is capable of filtering undesirable high frequency noise in any signal.

Figure 68: (a) to (h) Original function f_1 and several moment-preserving approximations with the number of approximating points being 5, 8, 11, 14, 17, 20, and 25, respectively.

Approximation Method	# of Approx. Points	Time Required	ME (E_{∞})	MSE (E_2/N)	SNR (dB)
Linear Split (Ramer)	14	216	2.359	1.1337	-11.7755
Split/Merge (Pavlidis)	30	1004	4.9756	3.9396	-17.1851
Avg. neighbours (Tanimoto)	18	52	8.9249	19.2216	-24.0685
Hologram-like (Prusinkiewicz)	18	46	3.9049	2.4967	-15.2042
Minimax (Kurozumi)	14	1664	8.3923	6.7534	-19.5258
Recursive split (Lowe)	14	208	2.359	1.1337	-11.7755
DFT	N/A	236	9.5971	9.8354	-21.1585
DCT	N/A	193	2.1901	0.0862	-0.5873
Moment-Preserving	5	242	9.6563	24.2376	-25.0755
"	8	182	10.904	24.4884	-25.1198
"	11	206	9.5889	15.043	-23.0039
"	14	219	6.3154	5.8214	-18.8808
"	17	233	5.019	4.1455	-17.4064
"	20	260	4.7382	3.1301	-16.1881
"	25	308	1.7276	0.3549	-6.7314

Figure 69: Comparison table of various approximation techniques on function f_1 .

Figure 70: (a) to (h) Original function f_2 and several moment-preserving approximations with the number of approximating points being 3, 4, 5, 5, 6, 7, and 8, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E₁)</i>	<i>MSE (E₂/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	5	129	0.73505	0.1346	11.9081
Split/Merge (Pavlidis)	12	992	0.8403	0.133	11.9599
Avg. neighbours (Tanimoto)	34	62	0.8028	0.1633	11.0704
Hologram-like (Prusinkiewicz)	18	55	0.2442	0.0055	25.7988
Minimax (Kurozumi)	7	5901	1.4142	0.2721	8.8518
Recursive split (Lowe)	5	122	0.735	0.1346	11.9081
DFT	N/A	243	1.6673	0.2059	10.0619
DCT	N/A	198	0.1275	0.002	30.2411
Moment-Preserving	3	157	2.3925	1.695	0.9038
"	4	207	3.8433	3.3966	-2.1149
"	5	247	0.5522	0.0697	14.7607
"	5	181	0.8124	0.0828	14.0171
"	6	171	1.8403	0.9427	3.4518
"	7	212	0.7626	0.0866	14.9482
"	8	181	0.9755	0.0751	14.4392

Figure 71: Comparison table of various approximation techniques on function f_2 .

Figure 72: (a) to (h) Original function f_3 and several moment-preserving approximations with the number of approximating points being 2, 5, 6, 10, 14, 17, and 21, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_1)</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	10	200	7.11E-15	3.86E-30	301.1069
Split/Merge (Pavlidis)	18	1122	9	2.1178	3.7139
Avg. neighbours (Tanimoto)	34	64	7.71	4.2873	0.6505
Hologram-like (Prusinkiewicz)	18	46	9.8182	8.0971	-0.8789
Minimax (Kurozumi)	10	192	7.11E-15	3.86E-30	301.1069
Recursive split (Lowe)	5	122	0.735	0.1348	11.9091
DFT	N/A	240	3.918	1.5705	5.0118
DCT	N/A	196	3.7447	0.7844	8.0268
Moment-Preserving	2	123	5.7201	9.7143	-2.9018
"	5	177	6.4572	9.5026	-2.8061
"	6	173	6.6516	8.6691	-2.4074
"	10	201	7.1118	5.9342	-0.7813
"	14	211	6.4233	5.4649	-0.4036
"	17	239	6.7329	3.7983	1.1784
"	21	252	6.4968	3.0019	2.1983

Figure 73: Comparison table of various approximation techniques on function f_3 .

Figure 74: (a) to (h) Original function f_4 and several moment-preserving approximations with the number of approximating points being 2, 3, 5, 6, 8, 11, and 17, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E₁)</i>	<i>MSE (E₂/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	8	155	0.998	0.1579	7.974
SplitMerge (Pavlidis)	24	887	1.0603	0.1377	8.5694
Avg. neighbours (Tanimoto)	34	62	1.1705	0.3105	5.0371
Hologram-like (Prusinkiewicz)	18	48	0.3933	0.021	16.7336
Minimax (Kurozumi)	12	2121	1.3175	0.0993	9.9887
Recursive split (Lowe)	8	148	0.998	0.1579	7.974
DFT	N/A	235	2.4872	0.3941	4.0023
DCT	N/A	197	0.2168	0.0037	24.2448
Moment-Preserving	2	124	5.9463	5.7091	-7.5446
"	3	160	4.3754	3.7609	-5.7319
"	5	173	3.3175	2.322	-3.6376
"	6	165	1.2698	0.3886	4.1263
"	8	196	2.133	0.8652	0.6498
"	11	175	1.65	0.2788	5.5674
"	17	237	1.2462	0.2302	6.4007

Figure 75: Comparison table of various approximation techniques on function f_4 .

Figure 76: (a) to (h) Original function f_5 and several moment-preserving approximations with the number of approximating points being 5, 6, 8, 9, 11, 14, and 21, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_{∞})</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	15	210	1.5017	0.2252	13.9691
Split/Merge (Pavlidis)	28	1011	8.3811	1.9746	4.5408
Avg. neighbours (Tanimoto)	34	62	8.3229	4.1081	1.3591
Hologram-like (Prusinkiewicz)	18	47	9.1768	5.3562	0.207
Minimax (Kurozumi)	16	1458	30.1181	26.2039	-6.6931
Recursive split (Lowe)	15	201	1.5017	0.2252	13.9691
DFT	N/A	238	4.4599	1.7246	5.1287
DCT	N/A	193	3.5019	0.6746	9.205
Moment-Preserving	5	176	6.3611	11.1963	-3.0274
"	6	180	7.3	8.1608	-1.6541
"	8	191	5.4633	6.1083	-0.3959
"	9	201	6.9185	5.7785	-0.1549
"	11	216	5.6859	4.3837	1.0448
"	14	227	6.3185	4.8026	0.8486
"	21	273	6.0681	2.7291	3.1032

Figure 77: Comparison table of various approximation techniques on function f_5 .

Figure 78: (a) to (h) Original function f_6 and several moment-preserving approximations with the number of approximating points being 5, 8, 11, 14, 17, 20, and 25, respectively.

Approximation Method	# of Approx. Points	Time Required	ME (E_e)	MSE (E_s/N)	SNR (dB)
Linear Split (Ramer)	14	252	4.6696	2.7531	-15.525
Split/Merge (Pavlidis)	28	873	6.3583	6.218	-19.0632
Avg. neighbours (Tanimoto)	34	62	6.9121	9.306	-20.8143
Histogram-like (Prusinkiewicz)	18	45	4.8893	3.9821	-17.1278
Minimax (Kurozumi)	16	1491	42.5632	115.6809	-31.7593
Recursive split (Lowe)	14	232	4.6696	2.7531	-15.525
DFT	N/A	237	9.5968	10.5518	-21.36
DCT	N/A	196	2.8431	1.0729	-11.4325
Moment-Preserving	5	251	11.2025	25.2792	-25.1543
"	8	182	11.1581	25.5301	-25.1972
"	11	208	10.0301	16.0793	-23.1894
"	14	219	7.4988	6.9152	-19.5246
"	17	233	6.2464	5.0877	-18.1919
"	20	295	5.0844	4.0775	-17.2307
"	25	330	2.8278	1.3448	-12.4134

Fig. e 79: Comparison table of various approximation techniques on function f_6 .

Chapter 5

5 Moment-Preserving Methods for 2-D Functions

In this chapter, the moment-preserving approximation method will be used to approximate two-dimensional functions. The two categories of two-dimensional functions to be covered by this approximation method are planar curves and digitized images. Results of the moment-preserving technique as they are applied to planar curves and digitized images are summarized at the end of this chapter.

5.1 Approximating Two-dimensional Planar Curves Using Moments

The moment-preserving approximation technique can be extended and applied to planar curve approximations. However, in order to approximate two-dimensional planar curves using the moment-preserving method described above, the planar curve must be normalized to be a one-dimensional function. There are several ways to normalize a planar curve to a one-dimensional function. For example, the method described by Zahn and Roskies [ZR 72] uses the angular direction of the curve as a function of its arc length.

The normalizing function used by us in this thesis is different from that of Zahn and Roskies. For any arbitrary planar curve described by a discrete set of $N + 1$ points, $S = \{p_i = (x_i, y_i) | 0 \leq i \leq N\}$, the x - and y -coordinates of the centre of

the curve, denoted by $p_i = (x_i, y_i)$, are defined as

$$\begin{aligned}x_c &= \frac{1}{N} \sum_{i=0}^N x_i \\y_c &= \frac{1}{N} \sum_{i=0}^N y_i\end{aligned}$$

The normalizing function, $\varphi(\theta)$, of a planar curve is defined as the Euclidean distance of a point on the curve from its centre as a function of the normalized angle between 0 and 2π . Formally, it can be summarized as

$$\varphi(\theta_i) = | \langle p_i, p_c \rangle |$$

where

$$\theta_i = \frac{2\pi i}{N} \quad 0 \leq i \leq N.$$

There are two main reasons why this particular normalizing function is used instead of the one used by Zahn and Roskies. First of all, since the moments of the curve are of primary interest, the centre of mass (closely related to the zeroth moment) is the most logical position to be used as a reference point describing all of the other points on the curve. A circle, for example, when described by this normalizing function becomes a straight horizontal line. Secondly, by using the Euclidean distance, the normalized function is always positive. Finally, the normalizing function is invariant with respect to certain affine transformations.

5.2 Approximating Digitized Images Using Moments

A digitized image $f(x, y)$ is often considered as a two-dimensional matrix of size $N \times N$, whose elements are the quantized brightness values at that point. Thus, $f(x, y)$ is defined as

$$f(x_i, y_k) = \hat{I}(x_0 + i\Delta x, y_0 + k\Delta y) \quad \text{for } i, k = 0, 1, 2, \dots, N-1$$

where $\hat{I}$ is the quantized brightness of the analogue image I , and Δx and Δy are the sampling intervals along the x - and y -axis, respectively.

Figure 80: Subdividing an image into smaller components. (a) Dividing an image into rows. (b) Dividing an image into blocks. (c) Reorganizing pixel values into linear form.

Using this definition, it is possible to consider an image as N separate and independent one-dimensional functions. For example, if all the rows of the image are defined as

$$\begin{aligned}
 r_0(y_k) &= \hat{I}(x_0, y_0 + k\Delta y) \\
 r_1(y_k) &= \hat{I}(x_0 + \Delta x, y_0 + k\Delta y) \\
 &\vdots \\
 r_{N-1}(y_k) &= \hat{I}(x_0 + (N-1)\Delta x, y_0 + k\Delta y)
 \end{aligned}$$

for $k = 0, 1, 2, \dots, N-1$, then the moment-preserving method can be applied to the N independent rows. By separating the rows or columns in this manner, however,

the correlation between the rows is lost, introducing discontinuities between the rows in the image.

An often used strategy of subdividing an image is to break it up into 8×8 or (16×16) blocks as shown in Figure 80(b). By subdividing the image into such smaller blocks, the correlation between adjacent pixels can be maintained and can be used to assist in the approximation of other pixels. For example, in a row of a digitized image, each pixel (except for the two pixels at the end of the row) contains exactly two adjacent neighbours, whereas, in a 8×8 block the each corner pixel contains 3 neighbours, each edge pixel contains 5 neighbours, and each interior pixel contains 8 neighbours.

In order to apply the moment-preserving technique to the block of pixels, it is necessary to reorder these values so as to form a single-variable function. An important part of the reorganization is to preserve the *closeness* of the pixels from a square block into a linear form. If the rows in a block were concatenated together to form a line, then in most cases, there would be no correlation between the pixel at the end of one row and a pixel at the beginning of the next row even though they are placed adjacent to each other in a linear form. Results from the moment-preserving technique as it applied to one-dimensional functions indicate that noisy signals or "spikes" are removed by this technique. Therefore, it does not make any sense to place dissimilar pixels side-by-side and then attempt to preserve the moments of those pixel values. A well-known conversion technique called the zigzag scan is used in the moment-preserving approximation of images as shown in Figure 80(c). Once an approximation of the zigzag pixel values is found, the zigzag operation is reversed to restore the square block structure of the approximation image.

5.3 Experimental Results on Planar Curves and Digitized Images

In order to make an unbiased comparison between the moment-preserving technique and the previous techniques, the following five discrete planar curves were used:

1. Open ellipse:

$$\{(x_i, y_i) | x_i = 5 \cos(0.05i) + 3, y_i = 7 \sin(0.05i) + 4, i = 0, 1, \dots, 200\}$$

2. Rotated normal bell curve:

$$\left\{ (x_i, y_i) | x_i = x'_i \cdot \cos \frac{\pi}{3} - y'_i \cdot \sin \frac{\pi}{3}, y_i = x'_i \cdot \sin \frac{\pi}{3} + y'_i \cdot \cos \frac{\pi}{3} \right\}$$

where

$$x'_i = 0.05i - 6, y'_i = \frac{20}{\sqrt{2\pi}} e^{-x'^2/\pi}, i = 0, 1, \dots, 200$$

3. Complex closed curve:

$$\{(x_i, y_i) | x_i = 3 \cos(0.025i) - 4 \sin(0.025i) + 4, y_i = 5 \cos(0.025i) + \sin 2x + 6\}$$

for $i = 0, 1, \dots, 251$.

4. Simulated noisy complex closed curve:

$$\{(x_i, y_i) | x_i = 3 \cos(0.025i) - 4 \sin(0.025i) + 4, y_i = 5 \cos(0.025i) + \sin 2x + 6\}$$

for $i = 0, 1, \dots, 251$. The simulated noise is the value $\sin 200x_i$ added to y_i when $10 \leq i \text{ modulo } 40 \leq 18$.

5. Self-intersecting curve :

$$\{(x_i, y_i) | x_i = 3 \cos(0.02i) - 4 \sin(0.02i) + 4, y_i = 5 \cos(0.02i) + \sin 2x\}$$

for $i = 0, 1, \dots, 300$.

These planar curves are identical to the ones used in the review of previous planar curve methods from chapter 3. Using the SNR and the number of approximating points as the criteria, the moment-preserving method can approximate the open planar elliptical curve with the fewest number of points and the least amount of error; it outperforms all of the time/spacial domain techniques (except for the recursive split technique) and transform domain techniques. The moment-preserving method does not perform as well as the hologram-like technique in approximating the rotated bell curve, however, its results are comparable with the other techniques. The latter technique performs poorly on the approximation of a complex curve and the noisy complex curve compared to all the previous techniques, however, the moment-preserving approximations of these two planar curves are virtually identical. From our experiments we conclude that if the noise is relatively small compared to the actual signal, it will not adversely affect its moment-preserving approximation, and the latter will serve as a good low-pass filter. The moment preserving method outperforms all of the previous techniques in approximating self-intersecting curves.

The moment-preserving technique was also applied to the following four different digitized images presented in Chapter 3:

1. A human face
2. A deer in a grey background
3. A squirrel in a dark background
4. A mountain hiking trail containing light background

The results are fairly reasonable and typical of image approximation schemes. Since the moment-preserving method does not handle signals which contain drastic changes, it has a tendency to blur sharp edges contained in the images. In all four cases, the moment-preserving approximating images seem to appear more blurry than the hologram-like and discrete cosine transform approximations.

Figure 81: (a) to (g) Open elliptical curve and several moment-preserving approximations with the number of approximating points being 5, 7, 9, 11, 13, and 15, respectively.

Approximation Method	# of Approx. Points	Time Required	ME (E_1)	MSE (E_2/N)	SNR (dB)
Linear Split (Ramer)	11	784	0.4035	0.0197	23.1234
Split/Merge (Pavlidis)	30	1004	4.9758	3.9398	-17.1851
Avg. neighbours (Tanimoto)	34	102	0.3231	0.0415	19.8924
Hologram-like (Prusinkiewicz)	18	72	0.0088	1.47E-03	34.4066
Minimax (Kurozum)	16	2079	0.6281	0.1146	15.4858
Recursive split (Lowe)	13	169	0.1681	5.33E-03	28.8136
DFT	N/A	210	2.1044	0.5862	8.397
DCT	N/A	173	0.6462	0.0103	25.9476
Moment-Preserving	5	86	1.5727	0.3004	11.3023
"	7	106	0.8148	0.0803	17.0303
"	9	144	0.4322	0.0273	21.7232
"	11	177	0.303	0.0144	24.4982
"	13	216	0.2675	0.0084	26.8536
"	15	254	0.2191	0.0041	29.9987
xxx	xxx	xxx	xxx	xxx	xxx

Figure 82: Comparison table of various approximation techniques.

Figure 83: (a) to (h) Rotated bell curve and several moment-preserving approximations with the number of approximating points being 3, 5, 5, 8, 9, 11, and 14, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME ($\bar{\epsilon}_1$)</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	6	194	0.4044	0.0209	12.4223
Split/Merge (Pavlidis)	7	799	0.2837	0.011	15.2147
Avg. neighbours (Tanimoto)	34	133	0.3488	0.0234	11.9199
Hologram-like (Prusinsiewicz)	18	87	0.0825	4.24E-04	29.3477
Minimax (Kurozumi)	7	9227	1.03	0.1454	3.9925
Recursive split (Lowe)	6	183	0.4044	0.0209	12.4223
DFT	N/A	297	2.1847	0.1348	4.3221
DCT	N/A	228	1.4564	0.0152	13.8014
Moment-Preserving	3	108	5.3642	7.1352	-12.9149
"	5	118	0.5643	0.0314	10.651
"	5	110	1.7804	0.6584	-2.5658
"	8	121	0.8088	0.0831	7.5178
"	9	153	0.7592	0.0403	9.5673
"	11	161	0.4139	0.0201	12.5913
"	14	167	0.5642	0.035	10.1746

Figure S4: Comparison table of various approximation techniques.

Figure 85: (a) to (d) Complex closed curve and several moment-preserving approximations with the number of approximating points being 9, 12, and 14, respectively.

Figure 86: (e) to (h) Moment-preserving approximations of complex closed curve with the number of approximating points being 16, 19, 22, and 23, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_L)</i>	<i>MSE (E₂/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	16	229	0.3013	0.0167	25.4022
Split/Merge (Pavlidis)	16	1017	0.5882	0.0355	22.1288
Avg. neighbours (Tanimoto)	34	107	0.6062	0.0567	20.0959
Hologram-like (Prusinkiewicz)	18	76	1.0082	0.0747	18.8942
Minimax (Kurozumi)	18	1511	1.1935	0.1519	15.8125
Recursive split (Lowe)	16	205	0.3013	0.0167	25.4022
DFT	N/A	231	7.2335	2.091	4.4256
DCT	N/A	187	5.2336	0.3676	11.9749
Moment-Preserving	9	281	1.2716	0.3705	11.9409
"	12	306	2.9027	0.7139	9.099
"	14	305	2.9857	0.7234	9.0354
"	16	357	1.293	0.277	13.2043
"	19	340	1.7961	0.4305	11.2894
"	22	397	2.6872	0.374	11.9005
"	23	456	1.9107	0.2859	13.0674

Figure 87: Comparison table of various approximation techniques.

Figure 88: (a) Noisy complex closed curve and several moment-preserving approximations with the number of approximating points being 9, 12, and 14, respectively.

Figure 89: (e) to (h) Moment-preserving approximations of noisy complex closed curve with the number of approximating points being 16, 19, 22, and 23, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E₁)</i>	<i>MSE (E₂/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	32	558	0.3718	0.0151	25.8644
Split/Merge (Pavlidis)	24	2252	1.0721	0.0578	20.0493
Avg. neighbours (Tanimoto)	34	160	1.1191	0.1124	17.1616
Hologram-like (Prusinkiewicz)	18	140	1.2566	0.1183	16.9391
Minimax (Kurozumi)	24	3479	0.9114	0.0676	19.3705
Recursive split (Lowe)	32	495	0.3718	0.0151	25.8644
DFT	N/A	471	7.1492	0.7526	8.9026
DCT	N/A	369	4.599	0.1729	15.2897
Moment-Preserving	9	186	1.8844	0.4657	10.9317
"	12	201	2.8336	0.8079	8.5389
"	14	204	2.9241	0.7748	8.7202
"	16	252	1.6266	0.3597	12.0524
"	19	230	2.1161	0.4736	10.8582
"	22	277	2.5573	0.3903	11.0115
"	23	320	1.649	0.1833	14.981

Figure 90: Comparison table of various approximation techniques.

Figure 91: (a) to (h) Self-intersecting curve and several moment-preserving approximations with the number of approximating points being 4, 5, 7, 7, 9, 13, and 29, respectively.

<i>Approximation Method</i>	<i># of Approx. Points</i>	<i>Time Required</i>	<i>ME (E_1)</i>	<i>MSE (E_2/N)</i>	<i>SNR (dB)</i>
Linear Split (Ramer)	23	342	0.3838	0.0182	5.1008
Split/Merge (Pavlidis)	22	1497	1.3491	0.153	-4.1346
Avg. neighbours (Tanimoto)	34	128	1.892	0.2253	-5.8149
Hologram-like (Prusinkiewicz)	18	87	3.262	0.6624	-10.4994
Minimax (Kurozumi)	24	1942	2.7917	0.6481	-10.4039
Recursive split (Lowe)	23	296	0.3838	0.0182	5.1008
DFT	N/A	296	3.4027	1.3516	-13.5966
DCT	N/A	224	1.3492	0.1776	-4.7833
Moment-Preserving	4	200	4.4197	3.4505	-7.0962
"	5	186	3.4129	3.5516	-7.2216
"	7	203	5.2028	2.8076	-6.2007
"	7	235	4.9464	2.8408	-6.2516
"	9	251	4.1781	1.4715	-3.3948
"	13	268	1.8302	0.2788	3.8291
"	29	423	0.6756	0.0361	12.7084

Figure 92: Comparison table of various approximation techniques.

Figure 93: (a) Original image of human face. (b) Moment-preserving approximation with the number of approximating points being 32×32 . (c) Nearest-neighbour. (d) Hologram-like. (e) Discrete Fourier transform. (f) Discrete cosine transform.

Figure 94: (a) Original image of a deer. (b) Moment-preserving approximation with the number of approximating points being 32×32 . (c) Nearest-neighbour. (d) Hologram-like. (e) Discrete Fourier transform. (f) Discrete cosine transform.

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	198	545.0332	-6.5354
Nearest-neighbour	16:1	242 sec.	128	262.3665	-3.3803
DFT	8:1	710 sec.	255	21665.0011	-22.5288
DCT	16:1	660 sec.	69	220.1519	-2.5864
Moment-preserving	16:1	943 sec.	154	289.9788	-3.7949

(a)

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	252	1839.1405	-13.8981
Nearest neighbour	16:1	244 sec.	211	816.3203	-10.3706
DFT	8:1	726 sec.	255	34371.1837	-26.6139
DCT	16:1	702 sec.	137	485.5641	-5.1146
Moment-preserving	16:1	881 sec.	203	898.3696	-10.7865

(b)

Figure 95: (a) Comparison table of various approximation techniques on image #1.
 (b) Comparison table of various approximation techniques on image #2.

Figure 96: (a) Original image of a squirrel. (b) Moment-preserving approximation with the number of approximating points being 32×32 . (c) Nearest-neighbour. (d) Hologram-like. (e) Discrete Fourier transform. (f) Discrete cosine transform.

Figure 97: (a) Original image of a hiking trail. (b) Moment-preserving approximation with the number of approximating points being 32×32 . (c) Nearest neighbour. (d) Hologram-like. (e) Discrete Fourier transform. (f) Discrete cosine transform.

3 of/de 3

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<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	119 sec.	244	806.4289	-12.7094
Nearest neighbour	16:1	228 sec.	180	386.7206	-9.2871
DFT	8:1	720 sec.	255	36126.7125	-29.222
DCT	16:1	689 sec.	132	179.9826	-8.1983
Moment-preserving	16:1	886 sec.	197	463.7054	-10.3062

(a)

<i>Approximation Method</i>	<i>Compression Ratio</i>	<i>Time Required</i>	<i>ME (E_e)</i>	<i>MSE (E_e/N)</i>	<i>SNR (dB)</i>
Hologram-like	16:1	118 sec.	252	1828.7845	-11.9308
Nearest neighbour	16:1	228 sec.	180	889.7711	-8.7032
DFT	8:1	744 sec.	255	23628.5419	-23.0802
DCT	16:1	714 sec.	151	581.7504	-8.9588
Moment-preserving	16:1	989 sec.	197	981.8599	-9.2297

(b)

Figure 98: (a) Comparison table of various approximation techniques on image #3.
 (b) Comparison table of various approximation techniques on image #4.

Chapter 6

6 Conclusion and Open Problems

In this chapter, we present a summary of the moment-preserving approximation method, and compare it with previous techniques. We also present a short discussion of the implementation of these approximation techniques. We conclude this chapter and the thesis with a list of open problems which could be investigated by future research.

6.1 Review of Moment-preserving Technique

The geometric moments and the Fourier transform of a function are closely related in the sense that a finite number of these moments can be used to reconstruct an approximation of the Fourier coefficients. Because of the close relationship between these, the geometric moments can be considered to contain information typically found in the transform domain. In this thesis, we have introduced a new linear piecewise approximation technique which explores this interdependence.

The original function $f(x)$ and an approximation function $\hat{f}(x)$ are k^{th} order equivalent if their zeroth, first, second, ..., k^{th} moments are identical. In other words, by preserving the first $k + 1$ moments of the original function $f(x)$, the function $\hat{f}(x)$ is approximately equal to $f(x)$. By approximating $f(x)$ by a k^{th} order equivalent $\hat{f}(x)$, transform domain values are preserved while simultaneously providing an approximation in the time/spacial domain. Thus, we can avoid the usual “*Transformation*→*Approximation*→*Inverse Transformation*” process. Another interesting feature about the moment-preserving technique is that the knot values

Approximation Method	Type of orig. function	Type of Knot Values	Pre-specified ϵ Required
Linear split (Ramer)	continuous/discrete	non-uniform	yes
Split-and-Merge (Pavlidis)	continuous/discrete	non-uniform	yes
Nearest-neighbour (Tanimoto)	discrete	uniform	no
Hologram-like (Prusinkiewicz)	discrete	uniform	no
Minimax (Kurozumi)	discrete	non-uniform	yes
Recursive split (Duda & Hart, Lowe)	continuous/discrete	non-uniform	yes
DFT	discrete	N/A	yes
DCT	discrete	N/A	yes
Moment-preserving	continuous/discrete	uniform/non-uniform	no

Table 2: Comparison of different approximation techniques.

can be uniformly or non-uniformly selected.

6.2 Comparison with Other Techniques

The moment-preserving method is better than all existing spacial domain techniques which use uniform knot values such as the hologram-like and nearest-neighbour approximation techniques. Although it requires more computational time than these other techniques, the amount of error between the approximation and the actual function is significantly smaller. This property is clearly demonstrated in the approximation examples of one-dimensional functions and planar curves.

(Of the four approximation techniques which use non-uniform knots (linear split, split-and-merge, minimax, and recursive split), only the linear split and recursive

split methods consistently perform better than the moment-preserving technique. This conclusion is consistent with an observation made by Rice [Ri 69] that the key to the successful use of splines is to have the location of the knots as variables. However, the moment-preserving method consistently yields superior approximation results compared to the minimax and the split-and-merge methods. A comparative top-level survey of all these schemes is found in Table 2.

Only two transform domain approximation techniques are used in the comparison, namely the Fourier transform and discrete cosine transform. The Fourier coefficients contain too many spurious frequencies, and as such when a small number of truncated coefficients are used to approximate the original function, they introduce a lot of noise into the approximation. The moment-preserving method outperforms the Fourier transform on approximating one-dimensional and two-dimensional functions. The discrete cosine transform is very efficient in the energy-packing sense, and therefore, only a small number of discrete cosine coefficients are required to provide a good approximation. Results from the planar curve and digitized image approximations show that the moment-preserving method performs as well as or better than the discrete cosine transform. However, for one-dimensional functions, the discrete cosine transform is much more effective than the moment-preserving technique.

6.3 Discussion on Implementation

All the previous approximation techniques and the new moment-preserving method were fully implemented in Smalltalk. They have been applied to one-dimensional functions, two-dimensional planar curves, and digitized images. In the implementation of the moment-preserving technique, there is no maximum error limit requirement, ϵ , that is typically used by other time/spacial domain approximation techniques such as the linear split, recursive split, split-and-merge, and minimax methods. As opposed to them, for each moment-preserving approximation, the

number of knot values must be specified. Since there is no restriction on the location of the knot values, they can be uniformly or non-uniformly selected by the user. However, in order to make the current implementation easy to understand and maintain, the knot values are assumed to be uniformly distributed along the x -axis.

The major difficulty encountered by the current implementation is the cumulative errors from floating-point mathematical operations. For example, the function $f(x) = \frac{\sin x}{x}$ cannot be plotted correctly by the computer due to the fact that when $x = 0$, a division-by-zero error will occur. To overcome this problem, a very small positive value of 10^{-6} is added to each value of x input into the function. Initially, the error value may not appear to be very significant, but it is sometimes large enough to skew some of these approximation results.

6.4 Open Problems

The current moment-preserving technique is based on the assumption that the approximation points are connected by line segments. One common complaint about piecewise linear approximations is that interpolated values from linear functions are not as aesthetically pleasing as interpolated values from polynomials of higher degrees such as quadratic or cubic functions. It would be interesting to see if the current derivations can be extended from being valid for piecewise linear approximations to higher order polynomials. Also, it would be interesting to study whether the use of polynomials such as quadratic and cubic functions would provide a better approximation than linear functions in terms of the *amount of error* introduced between an approximation and the original function. Although the complexity of the problem would increase by using quadratic and cubic functions, would it be acceptable to double the time complexity in order to reduce the error by one-half?

As indicate by Rice [Ri 69], and Pavlidis [Pa 77], the selection of the position of the knots are very important in making a good approximation. Since our current

implementation uses a sequence of uniform knot positions, the question of whether non uniform knot positions would accomplish a better approximation remains unanswered. Certainly such a strategy combined with polynomials of higher order (such as quadratic and cubic functions) could be even more advantageous. The question of locating the knots in order to minimize the error between the approximation and the original function is also open. The possibility of combining our scheme with Prusinkiewicz's subsampling technique to provide progressively finer approximations is also an interesting avenue for future research. Finally, since the discrete cosine transform is related to the discrete Fourier transform, and the Fourier transform is, in turn, related to the moments, it would it be interesting to determine whether one could use moments to assist in the determination of the approximated discrete cosine transform, or vice versa.

6.5 The Final Word

There are currently many approximation techniques, and they can only be applied in either the time/spacial or the transform domain, but not both. Our new piecewise linear moment-preserving method bridges the gap between time/spacial and transform domain approximations. Experimental evidence obtained from the results of using this technique indicate that it is a viable option, and that it could well be a powerful approximation strategy for the future.

References

- [Ab 89] Abdelmalek, N. N., *Piecewise Linear Least-squares Approximation of Planar Curves Including Computer Programs*, National Research Council Canada, ERB-1020, NRCC No. 31069, May 1989.
- [ANR 85] Ahmed, H., Natarajan, T., and Rao, K. R., *Discrete Cosine Transform*, Discrete Transforms and Their Applications, Van Nostrand Reinhold Company Inc., pp 9-12, 1985.
- [AK 91] Aoyama, H., and Kawagoe, M., *A Piecewise Linear Approximation Method Preserving Visual Feature Points of Original Figure*, CVGIP:Graphical Models and Image Processing, Vol. 53, No. 5, pp. 435-446, Sept. 1991.
- [BB 82] Ballard, D. H., and Brown, C. M., *Computer Vision*, Prentice-Hall, Inc., 1982.
- [Be 61] Bellman, R., *On the approximation of curves by line segments using dynamic programming*, Comm. ACM, Vol. 4, No. 6, pg. 284, June 1961.
- [BR 74] Barnhill, R. E., and Riesenfeld, R. F., *Computer Aided Geometric Design*, Academic Press, New York, 1974.
- [BR 86] Bellman, R. E., and Roth, S. R., *Methods in Approximation - Techniques for Mathematical Modelling*, D. Reidel Publishing Company 1986.
- [Br 68] Brill, E. L., *Character Recognition via Fourier Descriptors*, in WESCON Technical Papers, Session 25, Qualitative Pattern Recognition Through Image Shaping, Los Angeles, California, 1968.
- [Co 60] Cosgriff, R. L., *Identification of Shape*, Ohio State University Research Foundation, Columbus, Ohio, Report 820-11, ASTIA AD 254 792, December 1960.
- [Bo 65] Borel, R. J., *A Mathematical Pattern Recognition Technique Based on Contour Shape Properties*, Ohio State University Research Foundation, Columbus, Ohio, Report 1801-11, ASTIA AD 176 113, December 1965.
- [BM 91] Buck, B., and Macaulay, V. A., *Maximum Entropy in Action*, Clarendon Press, 1991.
- [DaL 79] Davis, L. S., *Shape Matching Using Relaxation Techniques*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-1, pp. 60-72, 1979.
- [DaP 63] Davis, P. J., *Interpolation and Approximation*, Blaisdell, pg.151, 1963.

- [DLTW 90] Dobkin, D. P., Levy, S. V. F., Thurston, W. P., and Wilks, A. R., *Contour Tracing by Piecewise Linear Approximations*, A.C.M. Transactions on Graphics, Vol. 9, No. 4, pp. 389-421, Oct. 1990.
- [deB 78] de Boor, C., *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.
- [DH 73] Duda, R. O., and Hart, P. E., *Pattern Classification and Scene Analysis*, Wiley & Sons, Inc., pp. 328-339, 1973.
- [DBM 77] Dudani, S. A., Breeding, K. J., and McGhee, R. B., *Aircraft Identification by Moment Invariants*, I.E.E.E. Transactions on Computers, Vol. C-26, pp. 39-46, 1977.
- [Du 86] Dunham, J. G., *Optimum Uniform Piecewise Linear Approximation of Planar Curves*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-8, pp. 67-75, 1986.
- [Ek 84] Ekstrom, M. P., *Digital Image Processing Techniques*, Academic Press, 1984.
- [ER 82] Elliot, D. F., and Rao, K. R., *Fast Transforms - Algorithms, Analyses, Applications*, Academic Press, 1982.
- [FWL 89] Fahn, C. S., Wang, J. F., and Lee, J. Y., *An Adaptive Reduction Procedure for the Piecewise Linear Approximation*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-11, No.9, pp. 967-973, 1989.
- [FB 86] Fischler, M. A., and Bolles, R. C., *Perceptual Organization and Curve Partitioning*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-8, pp. 100-105, 1986.
- [Fr 61] Freeman, H., *On the Encoding of Arbitrary Geometric Configurations*, I.R.E. Transactions on Electronic Computers, Vol. EC-10, pp. 260-268, 1961.
- [Fr 70] Freeman, H., *Boundary Encoding and Processing*, Picture Processing and Psychopictorics, Lipkin and Rosenfeld, Eds., Academic Press, pp. 241-306, 1970.
- [Gl 61] Gluss, B., *Further remarks on line segment curve-fitting using dynamic programming*, Comm. ACM, Vol. 4, No. 8, pp 441-443, August 1961.
- [Gl 62] Gluss, B., *A line segment curve-fitting algorithm related to optimal encoding of information*, Information and Control, Vol. 5, pp261-267, 1962.

- [Gr 72] Granlund, G. H., *Fourier Preprocessing for Hand Print Character Recognition*, I.E.E.E. Transactions on Computers, Vol. C-21, pp. 195-201, 1972.
- [Gr 68] Greville, T. N. E., *Theory and Applications of Spline Functions*, Academic Press, 1969.
- [Hi 87] Hildebrand, F. B., *Introduction to Numerical Analysis*, 2nd Edition, Dover Publications, 1987.
- [Hu 62] Hu, M., Visual Pattern Recognition by Moment Invariants, I.E.E.E. Transactions on Information Theory, Vol. 2, pp. 179-187, 1962.
- [IN 91] Ihm, I., and Naylor, B., *Piecewise Linear Approximations of Digitized Space Curves with Applications*, Scientific Visualization of Physical Phenomena (Proceedings of Computer Graphics International 1991), Ed. Patrikalakis, N. M., Springer-Verlag, pp. 545-569, 1991.
- [Ja 91] Jahne, B., *Digital Image Processing - Concepts, Algorithms, and Scientific Applications*, Springer-Verlag, 1991.
- [Ja 85] Jain, A. K., *A Fast Karhunen-Loève Transform for a Class of Random Processes*, Discrete Transforms and Their Applications, Van Nostrand Reinhold Company Inc., pp. 180-186, 1985.
- [Jar 71] Jarvis, C. L., *A Method for Fitting Polygons to Figure Boundary Data*, Australian Computer Journal, Vol. 3, pp. 50-54, 1971.
- [JN 84] Jayant, N. S., and Noll, P., *Digital Coding of Waveforms - Principles and Applications to Speech and Video*, Prentice-Hall, 1984.
- [Ka 89] Karl, J. H., *An Introduction to Digital Signal Processing*, Academic Press, 1989.
- [Ke 87] Kemperman, J. H. B., *Geometry of the Moment Problem*, Moments in Mathematics, Proceedings of Symposia in Applied Mathematics, Volume 37, American Mathematical Society Short Course Lecture Notes, 1987.
- [KD 82] Kurozumi, Y., and Davis, W. A., *Polygonal Approximation by the Minimax Method*, Computer Graphics and Image Processing, Vol. 19, pp. 248-264, 1982.
- [La 87] Landau, H. J., *Classical Background of the Moment Problem*, Moments in Mathematics, Proceedings of Symposia in Applied Mathematics, Volume 37, American Mathematical Society Short Course Lecture Notes, 1987.
- [Le 90] Lewis, R., *Practical Digital Image Processing*, Ellis Horwood Ltd., 1990.

- [Lo 87] Lowe, D. G., *Three-dimensional Object Recognition from Single Two-Dimensional Images*, Artificial Intelligence, Vol. 31, pp. 355-395, 1987.
- [Mo 70] Montanari, U., *A Note on Minimal Length Polygonal Approximation to a Digitized Contour*, Communications of the A.C.M., Vol. 13, No. 1, pp. 41-47, January 1970.
- [NO 94] Nguyen, T. B., and Oommen, B. J., *Moment-Preserving Piecewise Linear Approximations of Signals and Images*, Technical Report TR#258, Carleton University, Ottawa, Ontario, 1994.
- [Or 85] Orfanidis, S. J., *Optimum Signal Processing: An Introduction*, Macmillan Publishing Company, 1985.
- [Pap 65] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill Book, Inc., 1965.
- [Pa 72] Pavlidis, T., *Piecewise Approximation of Functions of Two Variables through Regions with Variable Boundaries*, Proceedings of the A.C.M. Annual Conference, pp. 652-662, Aug. 1972.
- [Pa 73] Pavlidis, T., *Waveform Segmentation Through Functional Approximation*, I.E.E.E. Transactions on Computers, Vol. C-22, No. 7, pp. 689-697, July 1973.
- [PH 74] Pavlidis, T., and Horowitz, S. L., *Segmentation of Plane Curves*, I.E.E.E. Transactions on Computers, Vol. C-23, No. 8, pp. 860-870, August 1974.
- [Pa 77] Pavlidis, T., *Structural Pattern Recognition*, Springer-Verlag, 1977.
- [Pa 78] Pavlidis, T., *A Review of Algorithms for Shape Analysis*, Computer Graphics and Image Processing, Vol. 7, pp. 243-258, 1978.
- [Pa 82] Pavlidis, T., *Algorithms for Graphics and Image Processing*, Computer Science Press, pp. 275-297, 1982.
- [PF 77] Persoon, E., and Fu, K. S., *Shape Discrimination Using Fourier Descriptors*, I.E.E.E. Transactions on Systems, Man and Cybernetics, Vol. SMC-7, pp. 170-179, 1977.
- [Ph 68] Phillips, G. M., *Algorithm for piecewise straight line approximations*, Computer Journal, Vol. 11, pp. 211-212, 1968.
- [PD 94] Pikas, A., and Dinstein, I., *Using Simple Decomposition for Smoothing and Feature Point Detection of Noisy Digital Curves*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. 16, No. 8, pp. 808-813, 1994.

- [Pr 84] Prusinkiewicz, P., *A Hologram-like Digital Representation of Pictures*, Proceedings of the 22nd Allerton Conference on Communications, Control, and Computing, Monticello, pp. 796-797, 1984.
- [PC 85] Prusinkiewicz, P., and Christopher, M., *Hologram-like transmission of pictures*, The Visual Computer, Springer Verlag, pp. 185-193, 1985.
- [Pr 78] Pratt, W. K., *Digital Image Processing*, Wiley-Interscience, New York, 1978.
- [Ri 69] Rice, J. R., *The Approximation of Functions*, Vol. 1, 2, Reading, Mass., Addison-Wesley, 1964, 1969.
- [RJ 91] Rabbani, M., and Jones, P. W., *Digital Image Compression Techniques*, SPIE - The International Society for Optical Engineering, Washington, 1991.
- [RY 90] Rao, K. R., and Yip, P., *Discrete Cosine Transform - Algorithms, Advantages, Applications*, Academic Press, 1990.
- [Ram 72] Ramer, U., *An Iterative Procedure for the Polygonal Approximation of Plane Curves*, Computer Graphics and Image Processing, Vol. 1, pp. 244-256, 1972.
- [Ra 85] Rao, K. R., *Discrete Transforms and their Applications*, Van Nostrand Reinhold Co., 1985.
- [RW 72] Reuman, K., and Witkam, A. P. M., *Optimizing Curve Segmentation in Computer Graphics*, International Computer Symposium, pp. 467-472, 1972.
- [RK 76] Rosenfeld, A., and Kak, A. C., *Digital Picture Processing*, Academic Press, 1976.
- [RW 89] Rosin, P. L., and West, G. A. W., *Segmentation of Edges into Lines and Arcs*, Image and Vision Computing, Vol. 7, No. 2, pp. 109-111, 1989.
- [RW 91] Rosin, P. L., and West, G. A. W., *Techniques for Segmenting Image Curves into Meaningful Descriptions*, Pattern Recognition, Vol. 24, No. 7, pp. 643-652, 1991.
- [Ru 92] Russ, J. C., *The Image Processing Handbook*, CRC Press, Inc., 1992.
- [Sk 70] Sklansky, J., *Recognition of Convex Blobs*, Pattern Recognition, Pergamon Press, Vol.2, pp. 3-10, 1970.
- [SG 80] Sklansky, J., and Gonzalez, V., *Fast Polygonal Approximation of Digitized Curves*, Pattern Recognition, Vol. 12, pp. 327-331, 1980.

- [Sp 74] Späth, H., *Spline Algorithms for Curves and Surfaces*. Translated from German by Hoskins, W. D., and Sager, H. W., Utilitas Mathematica Publishing Incorporated, Winnipeg, 1974.
- [SH 90] Stearns, S. D., and Hush, D. R., *Digital Signal Analysis*, Prentice-Hall, 1990.
- [St 61] Stone, H., *Approximation of curves by line segments*, Mathematics of Computation, Vol. 15, pp 40-47, 1961.
- [Ta 79] Tanimoto, S. L., *Image Transmission with Cross Information First*, Computer Graphics and Image Processing, Vol. 9, pp. 72-76, 1979.
- [TC 86] Teh, C. H., and Chin, R. T., *On Digital Approximation of Moment Invariants*, Computer Vision, Graphics, and Image Processing, Vol. 33, pp. 318-326, 1986.
- [To 74] Tomek, I., *Piecewise Linear Approximations*, I.E.E.E. Transactions on Computers, Vol. C-23, pp. 445-448, 1974.
- [VK 82] Van Valkenburg, M. E., and Kinariwala, B. K., *Linear Circuits*, Prentice-Hall, Inc., 1982.
- [Wa 88] Wang, L., *Progressive Image Transmission - Ph.d Thesis Dissertation*, University of Ottawa, Ottawa, Ontario, 1988.
- [Wi 62] Wilkes, S. S., *Mathematical Statistics*, John Wiley & Sons, Inc., 1962.
- [Wi 78] Williams, C. M., *An Efficient Algorithm for the Piecewise Linear Approximation of Planar Curves*, Computer Graphics and Image Processing, Vol. 8, pp. 286-293, 1978.
- [Wu 84] Wu, L. D., *A Piecewise Linear Approximation Based on a Statistical Model*, I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-6, pp. 41-44, 1984.
- [YH 87] Yannakoudakis, E. J., and Hutton, P. J., *Speech Synthesis and Recognition Systems*, Ellis Horwood Ltd., 1987.
- [Za 66] Zahn, C. T., *Two-Dimensional Pattern Description and Recognition via Curvature Points*, Stanford Linear Accelerator Center, Stanford University, Stanford, California, SLAC Report No. 70, 1966.
- [ZR 72] Zahn, C. T., Roskies, R. Z., *A Fourier Description for Plane Closed Curves*, I.E.E.E. Transactions on Computers, Vol. C-21, pp. 269-281, 1972.

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