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ON THE LIMITING DISTRIBUTION OF AND CRITICAL VALUES FOR
THE HOEPPDING-BLUM-KIEFER-ROSENBLATT INDEPENDENCE CRITERION

by

DEREK S. COTTERILL, B.Sc. (Hons.), M.Sc.

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

September, 1980

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The undersigned recommend to the Faculty of Graduate Studies
and Research acceptance of the thesis

"On the Limiting Distribution of and Critical Values for the
Hoeffding-Blum-Kiefer-Rosenblatt Independence Criterion"

submitted by Derek S. Cotterill, B.Sc. (Hons.), M.Sc., in partial
fulfillment of the requirements for the degree of Doctor of Philosophy.

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September, 1980

ABSTRACT

Let F be the class of continuous distribution functions on d -dimensional Euclidean space R^d , $d \geq 2$, and let F_0 be the subclass consisting of every member of F which is a product of its associated one-dimensional marginal distribution functions. Let $X_1, \dots, X_n$ be independent random d -vectors with common unknown distribution $F \in F$, and suppose we want to test the null hypothesis $H_0: F \in F_0$ against the alternative $H_1: F \in F - F_0$. Let F_n be the empirical distribution function of $X_1, \dots, X_n$, and let F_{ni} be the marginal empirical distribution function of the i^{th} component of $X_j = (X_{j1}, \dots, X_{jd})$ ($j = 1, \dots, n$). Let $T_n(x) = F_n(x) - \prod_{i=1}^d F_{ni}(x_i)$, $d \geq 2$, and let $\{T(y); y \in I^d (d \geq 2)\} = \{B(y) - \prod_{i=1}^d B(\underline{1}, y_i, \underline{1}) \prod_{j \neq i} y_j; y = (y_1, \dots, y_d) \in I^d (d \geq 2)\}$, where $\{B(y); y \in I^d (d \geq 2)\}$ is a Brownian bridge over the d -dimensional unit interval I^d . Let $\Gamma_{n,d}$ be the distribution function of the rv $C_{n,d} = \int_R n T_n^2(x) \prod_{i=1}^d dF_i(x_i)$ and let Γ_d be that of the rv $C_d = \int_{I^d} T^2(y) dy$. Blum, Kiefer and Rosenblatt (Ann. Math. Statist. 1961) obtained the characteristic function of C_2 and tabulated critical values of Γ_2 . Here, with the help of our representation of the Gaussian process $T(y)$ above, we find the characteristic function of the rv C_d for $d \geq 2$ and give details as to how to

calculate critical values of Γ_d for all $d \geq 2$ and provide tables for the "usual" levels of significance for $d = 2$ to 20.

The latter are also approximate critical values for the rv

$$\hat{C}_{n,d} = \int_R n T_n^2(x) dF_n(x) \quad \text{and} \quad \check{C}_{n,d} = \int_R n T_n^2(x) \prod_{i=1}^d dF_{ni}(x_i) \quad \text{whose}$$

large values can be used as critical region for H_0 . Put

$$V_{n,d} = \sup_{0 \leq x \leq 1} |\hat{T}_{n,d}(x) - \Gamma_d(x)|, \quad d \geq 2. \quad \text{Using our characteristic}$$

function of C_d and earlier strong invariance results of Csörgő

(J. Multivariate Anal. 1979) for the process $n^{1/2}T_n(x)$ we prove

rates of convergence for the distance $V_{n,d}$. For example, for

$d = 2$ we have $V_{n,2} = O(n^{-1/2} \log^2 n)$.

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1. Introduction. Let F be the class of continuous distribution functions on d -dimensional Euclidean space R^d , $d \geq 2$, and let F_0 be the subclass consisting of every member of F which is a product of its associated one-dimensional marginal distribution functions. Let $X_1, \dots, X_n$ be independent random d -vectors with common unknown distribution $F \in F$, and suppose it is desired to test the null hypothesis

$$(1.1) \quad H_0 : F \in F_0 \quad \text{against the alternative} \quad H_1 : F \in F - F_0.$$

Let F_n be the empirical distribution function of $X_1, \dots, X_n$, i.e., for $x = (x_1, \dots, x_d) \in R^d$, $F_n(x)$ is n^{-1} times the number of $X_j = (X_{j1}, \dots, X_{jd})$, $j = 1, \dots, n$, all of whose components are less than or equal to the corresponding components of x , conveniently written as

$$(1.2) \quad F_n(x) = F_n(x_1, \dots, x_d) = n^{-1} \sum_{j=1}^n \prod_{i=1}^d I_{(-\infty, x_i]}(X_{ji}),$$

where

$$(1.3) \quad I_{(-\infty, x]}(u) = \begin{cases} 1 & \text{if } u \leq x \\ 0 & \text{if } u > x \end{cases}$$

Let F_{ni} be the marginal empirical distribution function of the i^{th} component of X_j , i.e.,

$$(1.4) \quad F_{ni}(x_i) = n^{-1} \sum_{j=1}^n I_{(-\infty, x_i]}(X_{ji}) \quad i = 1, \dots, d$$

and define

$$(1.5) \quad T_n(x) = T_n(x_1, \dots, x_d) = F_n(x) - \sum_{i=1}^d F_{ni}(x_i), \quad d \geq 2.$$

Hoeffding (1948), when $d = 2$, and Blum, Kiefer and Rosenblatt (1961), when $d \geq 2$, studied the problem of testing H_0 vs. H_1 via appropriate Cramér-von Mises type functionals of their multivariate empirical process $n^{1/2}T_n(x)$, obtaining the characteristic functions of the limiting distributions of these functionals and providing tables for the corresponding distribution functions in the bivariate case. One of the aims of this exposition is to continue their work for higher dimensions. Details are given in Sections 3 and 4. Strong invariance principles for the process $n^{1/2}T_n(x)$ were proved by Csörgő (1979). In order to summarize these asymptotic results and for the sake of proving further ones in Section 2, we need some further notation.

Let $y_i = F_i(x_i)$ be the i^{th} marginal distribution function of F and let $F_i^{-1}(\cdot)$ be its inverse. Define the mapping $L^{-1} : [0, 1]^d = I^d \rightarrow R^d$ by

$$(1.6) \quad L^{-1}(y_1, \dots, y_d) = (F_1^{-1}(y_1), \dots, F_d^{-1}(y_d)), \quad y = (y_1, \dots, y_d) \in I^d \quad (d \geq 2).$$

For our purposes it will be convenient sometimes to view T_n in terms of the latter, and we let

$$(1.7) \quad t_n(y) = T_n(L^{-1}(y)) = F_n(F_1^{-1}(y_1), \dots, F_d^{-1}(y_d)) = \prod_{i=1}^d F_{ni}(F_i^{-1}(y_i)).$$

Then $H_0 : F \in \mathcal{F}_0$ is equivalent to $H_0 : F(L^{-1}(y)) = \prod_{i=1}^d y_i = t(y)$,

i.e., under H_0 T_n is distribution free, so we may take F to be the uniform distribution on I^d ($d \geq 2$) and can then study T_n in terms of t_n .

Define the sequence of Gaussian processes $\{T^{(n)}(y); y \in I^d\}$

by

$$(1.8) \quad \{T^{(n)}(y); y \in I^d\} = \{B_n(y) - \sum_{i=1}^d B_n(\underline{1}, y_i, \underline{1}) \prod_{j \neq i} y_j; y = (y_1, \dots, y_d) \in I^d \quad (d \geq 2)\}$$

where $\{B_n(y); y \in I^d\}$ is a sequence of Brownian bridges, i.e., for each

n B_n is a separable Gaussian process with $EB_n(x) = 0$ and

$$EB_n(x)B_n(y) = \lambda(x, y) - \lambda(x)\lambda(y) \quad (x, y \in I^d).$$

Define the Gaussian process $\{T(y, t); y \in I^d, t \geq 0\}$ by

$$(1.9) \quad \{T(y, t); y \in I^d, t \geq 0\}$$

$$= \{K(y, t) - \sum_{i=1}^d K(\underline{1}, y_i, \underline{1}, t) \prod_{j \neq i} y_j; y \in I^d \quad (d \geq 2), t \geq 0\},$$

where $\{K(y,t); y \in I^d, t \geq 0\}$ is a Kiefer process, i.e., a separable Gaussian process with $EK(x,t) = 0$ and $EK(x,s)K(y,t) = (s \wedge t)(\lambda(x,y) - \lambda(x)\lambda(y))$ ($0 \leq s, t < \infty, x, y \in I^d$).

Obviously $ET^{(n)}(y) = ET(y,t) = 0$ and simple but somewhat tedious calculations yield the covariance functions

$$\begin{aligned} (1.10) \quad ET^{(n)}(x)T^{(n)}(y) &= \prod_{i=1}^d (x_i \wedge y_i) + (d-1) \prod_{i=1}^d x_i y_i - \sum_{i=1}^d (x_i \wedge y_i) \prod_{j \neq i} x_j y_j \\ &= \rho(x,y) \text{ for all } n, \end{aligned}$$

and

$$(1.11) \quad ET(x,s)T(y,t) = (s \wedge t)\rho(x,y),$$

where $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in I^d$ and $s, t \geq 0$.

As to $t_n(y) = T_n(L^{-1}(y))$, the following results are known (cf. Theorems 3 and 4 in Csörgő 1979).

THEOREM A (Csörgő 1979). Let $X_1, \dots, X_n$, $n = 1, 2, \dots$, be independent random d -vectors with a common distribution function $F \in F_Q$ and let $t_n(y)$ be as in (1.7). Then one can construct a probability space $(\Omega, \mathcal{A}, P)$ with $\{t_n(y); y \in I^d (d \geq 2), n=1, 2, \dots\}$, a sequence of Gaussian processes $\{T^{(n)}(y); y \in I^d (d \geq 2)\}$, defined as in (1.8), and a Gaussian process $\{T(y,t); y \in I^d (d \geq 2), t \geq 0\}$, defined as in (1.9), on it so that:

(i) if $d = 2$, then for any z and with C, D, μ positive absolute constants

$$(1.12) \quad P\left\{\sup_{y \in I^2} |n^{1/2} t_n(y) - \tau^{(n)}(y)| > n^{-1/2}(C \log n + z) \log n\right\} < D e^{-\mu z},$$

whence

$$(1.13) \quad \sup_{y \in I^2} |n^{1/2} t_n(y) - \tau^{(n)}(y)| = O(n^{-1/2} \log^2 n),$$

(ii) if $d \geq 3$, then for any $\mu > 0$, there exists a $C > 0$ such that

$$(1.14) \quad P\left\{\sup_{y \in I^d} |n^{1/2} t_n(y) - \tau^{(n)}(y)| > C(\log n)^{3/2} n^{-1/2(d+1)}\right\} \leq n^{-\mu},$$

whence

$$(1.15) \quad \sup_{y \in I^d} |n^{1/2} t_n(y) - \tau^{(n)}(y)| = O(n^{-1/2(d+1)} (\log n)^{3/2}),$$

and also

$$(1.16) \quad \sup_{1 \leq k \leq n} \sup_{y \in I^d} |k t_k(y) - \tau(y, k)| = O(n^{(d+1)/2(d+2)} \log^2 n), \quad d \geq 2.$$

It follows from (1.8) and (1.9) that for each n

$$(1.17) \quad \{\tau^{(n)}(y); y \in I^d\} \stackrel{D}{=} \{n^{-1/2} \tau(y, n); y \in I^d\} \\ \stackrel{D}{=} \{\tau(y, 1); y \in I^d\}.$$

Define the Gaussian process $\{T(y); y \in I^d (d \geq 2)\}$ via
(1.9) by

$$(1.18) \quad \{T(y); y \in I^d (d \geq 2)\} = \{T(y, t); y \in I^d (d \geq 2), t = 1\},$$

$$P \left(B(y) = \left\{ B(\underline{1}, y_1, \underline{1}) \prod_{j=1}^d y_j; y = (y_1, \dots, y_d) \in I^d (d \geq 2) \right\}, \right.$$

where $\{B(y); y \in I^d\}$ is a Brownian bridge. Thus $T(\cdot)$ has mean zero and the covariance function $\rho(x, y)$ of (1.10).

By (1.15) or (1.16) combined with (1.17) and (1.18) we get that under H_0

$$(1.19) \quad h(n^{1/2} t_n(\cdot)) \xrightarrow{D} h(T(\cdot))$$

for every continuous functional h on the space of real valued functions on I^d endowed with the supremum topology.

For example, given $f \in F_0$, we get

$$(1.20) \quad A_{n,d} = n^{1/2} \sup_{y \in I^d} |t_n(y)| \xrightarrow{D} A_d = \sup_{y \in I^d} |T(y)|, \quad d \geq 2,$$

and

$$(1.21) \quad C_{n,d} = n \int_{I^d} T_n^2(x) \prod_{i=1}^d dF_i(x_i) = n \int_{I^d} t_n^2(y) dy \xrightarrow{D} C_d = \int_{I^d} T^2(y) dy, \quad d \geq 2,$$

$$\text{with } dy = \prod_{i=1}^d dy_i.$$

As to how near the functionals $A_{n,d}$ and A_d , resp. $C_{n,d}$ and C_d , are to each other, we refer to (4.47), (4.48), resp. to

(4.49), (4.50), of Corollary 1 in Csörgő (1979), which, in turn, also prove (1.20), resp. (1.21), directly and they themselves are also a direct consequence of Theorem A.

The calculation of the distribution of the Kolmogorov-Smirnov type rv A_d seems very difficult. Indeed, one does not even know the distribution of $\sup_{y \in I^d} |B(y)|$; nor that of $\sup_{y \in I^d} W(y)$, where $\{B(y); y \in I^d\} = \{W(y) - \lambda(y)W(\underline{1}), y \in I^d\}$ is a Brownian bridge and $\{W(y); y \in I^d\}$ is a Wiener process. For sharp inequalities for the distribution of $\sup_{y \in I^d} W(y)$ we refer to Goodman (1976). It was falsely claimed by Zinchenko (1975) that, just like in the case of $d = 1$, we have $P(\sup_{y \in I^d} W(y) > a) = 2P(W(\underline{1}) > a)$ also for $d \geq 2$. This created a bit of confusion as to the relevancy of known results (cf. MR52#1912, MR54#11532, 11533).

The calculation of the distribution of the rv C_d is also not easy. For $d = 2$, Blum, Kiefer and Rosenblatt (1961) obtained the characteristic function of C_d and tabulated its distribution via numerical inversion of its characteristic function. In order to enable ourselves to tabulate approximate critical values of the rv $C_{n,d}$ for $d \geq 2$, we find the first five cumulants of C_d and then use the Cornish-Fisher asymptotic expansion. Details as to how to calculate approximate critical values for

all $d \geq 2$ and tables for the "usual" levels of significance for $d = 2$ to 20 are given in Section 5. Proofs for the statements of the latter are given in Section 6.

Let $\Gamma_{n,d}(\cdot)$ be the distribution function of the rv $C_{n,d}$ and let $\Gamma_d(\cdot)$ be that of the rv C_d . Then (1.21) reads

$$(1.22) \quad \lim_{n \rightarrow \infty} P(C_{n,d} \leq x) = \lim_{n \rightarrow \infty} \Gamma_{n,d}(x) = P(C_d \leq x) = \Gamma_d(x), \quad d \geq 2.$$

Since, for various values of n , nothing is known about the exact distribution function $\Gamma_{n,d}$, in addition to (1.22) it is also of interest to estimate the distance

$$(1.23) \quad \vartheta_{n,d} = \sup_{0 \leq x < \infty} |\Gamma_{n,d}(x) - \Gamma_d(x)|, \quad d \geq 2.$$

This will be done in Section 2.

The statistic $C_{n,d}$ of (1.21) cannot be used to test the hypothesis H_0 of (1.1) unless $F \in F_0$ is completely specified. In some other situations $C_{n,d}$ might come in handy of course (cf. e.g., Section 7). Hoeffding (1948), and Blum, Kiefer and Rosenblatt (1961) suggested, as critical region for H_0 , large values of

$$(1.24) \quad \hat{C}_{n,d} = n \int_{\mathbb{R}^d} T_n^2(x) dF_n(x), \quad d \geq 2,$$

or those of

$$(1.25) \quad \tilde{C}_{n,d} = n \int_{\mathbb{R}^d} T_n^2(x) \prod_{i=1}^d dF_{n_i}(x_i), \quad d \geq 2.$$

These two statistics are equivalent in that both converge in distribution to the rv C_d of (1.21). This was already remarked by Blum, Kiefer and Rosenblatt (1961). In Section 2 we are also going to outline a short proof of this statement.

Recently DeWet (1979) studied a version of (1.25) in the case of $d = 2$ with some nonnegative weight functions multiplying the integrand T_n^2 of $\tilde{C}_{n,d}$.

2. On Rates of Convergence for $\Gamma_{n,d}$ ($d \geq 2$).

THEOREM 1. For the distance $\Gamma_{n,d}$ of (1.23) we have

$$(2.1) \quad \Gamma_{n,d} = \begin{cases} O(n^{-\frac{1}{2}(d+1)} (\log n)^{3/2}) & \text{if } d \geq 3 \\ O(n^{-\frac{1}{2}} \log^2 n) & \text{if } d = 2. \end{cases}$$

As far as we know the rates of (2.1) for $\Gamma_{n,d}$ are the only available ones so far.

The proof of Theorem 1 is based on the respective statements of (1.12) and (1.14) and on the following two lemmas.

LEMMA 1. The distribution function Γ_d of the rv C_d ($d \geq 2$) is arbitrary many times differentiable and for an arbitrary integer p , $\Gamma_d^{(p)}(x) \rightarrow 0$ as $x \rightarrow \infty$.

LEMMA 2. For any real $p \geq 0$ and integers $q = 0, 1, 2, \dots$ the function $x^p \Gamma_d^{(q)}(x)$ ($d \geq 2$) is bounded on $(0, \infty)$.

Here and also in the sequel $\gamma_d^{(a)}$ stands for the a^{th} derivative of γ_d and $\gamma_d^{(*)}$ for that of the density function γ_d of C_d ($d \geq 2$).

PROOF OF THEOREM 1. Assuming for a moment that Lemmas 1 and 2 are already proved, and using the inequalities (1.12) and (1.14), the proof of Theorem 1 is, mutatis mutandis, exactly the same as that of Theorem 1 in Cotterill and Csörgő (1979) and will not, therefore, be repeated here.

Lemma 1 follows from the well known (cf., e.g., Theorem 11.6.2 in Kawata 1972)

LEMMA A. If the characteristic function f of an arbitrary distribution function F satisfies

$$(2.2) \quad \int_{-\infty}^{+\infty} |t|^p |f(t)| dt < \infty,$$

with some integer $p \geq 0$, then F has a bounded continuous $(p+1)^{\text{th}}$ derivative $F^{(p+1)}$ with $F^{(p+1)}(x) \rightarrow 0$ as $|x| \rightarrow \infty$, and

$$F^{(p+1)}(x) = ((-1)^{p/2} / 2\pi) \int_{-\infty}^{+\infty} e^{-itx} t^p f(t) dt.$$

Hence, in order to prove Lemma 1, we have to verify condition (2.2) for the characteristic function $\phi(t)$ of the rv C_d with an arbitrary nonnegative integer p .

As to the characteristic function of C_d of (1.21), first we must find an expression for it. Here we summarize our steps leading up to the latter. Detailed proofs will be given in Section 4. Recall (cf., e.g., D1 and D2 in Cotterill and Csörgő 1979) that a Brownian bridge $\{B(y); y \in I^d\}$ can be defined as

$$(2.3) \quad \{B(y); y \in I^d\} = \{W(y) - \sum_{i=1}^d y_i W(1, \dots, 1); y = (y_1, \dots, y_d) \in I^d\},$$

where $\{W(y); y \in I^d\}$ is a d-time parameter Wiener process. Thus, if we now define the Gaussian process $\{W^*(y); y \in I^d\}$ by

$$(2.4) \quad \{W^*(y); y \in I^d\} = \{W(y) - \sum_{i=1}^d W(1, y_i, 1) \prod_{j \neq i} y_j; y \in I^d\},$$

then $T(y)$ of (1.18) can be written as

$$(2.5) \quad \{T(y); y \in I^d (d \geq 2)\} = \{W^*(y) - \prod_{i=1}^d y_i W^*(1, \dots, 1); y \in I^d (d \geq 2)\},$$

and this representation is the key to getting the characteristic function of C_d . Just like we can say that a Brownian bridge is a tied down (at $\underline{0}$ and $\underline{1}$) Wiener process, so we can argue (compare (2.3) and (2.5)) that the Gaussian process $T(y)$ represented as in (2.5) is a tied down (at $\underline{0}$ and $\underline{1}$) W^* -process. Whence the distribution of the rv $C_d = \int_{I^d} T^2(y) dy$ is the same as that of

$$(2.6) \quad Q = \int_{I^d} (W^*(y))^2 dy$$

conditional on $W^*(1, \dots, 1) = 0$. Following Krivyakova, Martynov and Tyurin (1977) (cf. also (6) in Durbin (1970) and references given therein) we can express $\{W(y); y \in I^d \text{ (} d \geq 2 \text{)}\}$ as a weighted sum of independent $N(0,1)$ rv $Z_{n_1 \dots n_d}$ as follows:

$$(2.7) \quad W(y) = 2^{d/2} \prod_{n_1=0}^{\infty} \prod_{n_d=0}^{\infty} \prod_{i=1}^d \frac{\sin(n_i + \frac{1}{2})\pi y_i}{(n_i + \frac{1}{2})\pi} Z_{n_1 \dots n_d}$$

Now $W(\underline{1}, y_1, \underline{1})$ is simply the process $W(y)$ evaluated at $y = (1, \dots, 1, y_1, 1, \dots, 1)$, and $\sin(n_j + \frac{1}{2})\pi = (-1)^{n_j}$ for $j \neq 1$, i.e.,

$$(2.8) \quad W(\underline{1}, y_1, \underline{1}) = 2^{d/2} \prod_{n_1=0}^{\infty} \dots \prod_{n_d=0}^{\infty} \frac{\sin(n_1 + \frac{1}{2})\pi y_1}{\prod_{j=1}^d ((n_j + \frac{1}{2})\pi)} Z_{n_1 \dots n_d} \prod_{j \neq 1} (-1)^{n_j}$$

Since

$$2 \int_0^1 (\sin(n_1 + \frac{1}{2})\pi y_1) (\sin(m_1 + \frac{1}{2})\pi y_1) dy_1 = \begin{cases} 1 & \text{if } n_1 = m_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_0^1 y_1 \sin(n_1 + \frac{1}{2})\pi y_1 dy_1 = \frac{(-1)^{n_1}}{(n_1 + \frac{1}{2})^2 \pi^2},$$

we can easily integrate to obtain by (2.4), (2.6), (2.7) and (2.8)

$$(2.9) \quad Q = Q_1 + Q_2 + Q_{31} + Q_{32}$$

where

$$Q_1 = -2d \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \prod_{i=1}^d (n_i + \frac{1}{2})^{-2} Y_{n_1 \dots n_d}^2$$

$$Q_2 = -a \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \prod_{j=1}^d (n_j + \frac{1}{2})^{-3} (m_j + \frac{1}{2})^{-1}$$

$$\prod_{i=1}^d (n_i + \frac{1}{2})^{-2} S(n_i, m_i) Y_{n_1 \dots n_d} Y_{m_1 \dots m_d}$$

where

$$a = 2^{d-2-4d}, \quad S(n_i, m_i) = \begin{cases} 1 & \text{if } n_i = m_i \\ 0 & \text{otherwise} \end{cases}$$

$$Q_{31} = b \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \prod_{j=1}^d (n_j + \frac{1}{2})^{-1} (m_j + \frac{1}{2})^{-1} \sum_{i=1}^d S(n_i, m_i) Y_{n_1 \dots n_d} Y_{m_1 \dots m_d}$$

$$b = 3^{1-d} 2^{d-1} \pi^{-2d}$$

$$Q_{32} = c \sum_{i=1}^d \sum_{j=1}^d S_{i,j}, \quad c = 3^{2-d} 2^d \pi^{-2d-4}$$

$$S_i = \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} (n_i + \frac{1}{2})^{-2} \prod_{r=1}^d (n_r + \frac{1}{2})^{-1} Y_{n_1 \dots n_d}$$

and

$$Y_{n_1 \dots n_d} = (-1)^{n_1 + \dots + n_d} Z_{n_1 \dots n_d}$$

is a sequence of independent $N(0,1)$ rv.

Thus Q is simply a quadratic form in independent $N(0,1)$ rv.

To evade the difficulties of calculating the characteristic function of an infinite quadratic form, we define a finite quadratic form $Q^{(m)}$

$$Q^{(m)} = Q_1^{(m)} + Q_2^{(m)} + Q_{31}^{(m)} + Q_{32}^{(m)}$$

obtained by replacing the limit ∞ by m in each of the summations.

Then $\lim_{m \rightarrow \infty} Q^{(m)} \stackrel{a.s.}{=} Q$. Hence, if $\phi_m(t)$ is the characteristic function of $Q^{(m)}$, then

$$(2.10) \quad \phi_Q(t) = \lim_{m \rightarrow \infty} \phi_m(t).$$

Since $Q^{(m)}$ is a finite quadratic form in independent $N(0,1)$ random variables, it is a sum of chi-square random variables* and so must have the characteristic function

$$(2.11) \quad \phi_m(t) = \prod_{i=1}^k (1 - a_i t \sqrt{-1})^{-1/2}, \quad k \leq (m+1)^d,$$

and all we have to do is calculate the coefficients a_i . We have

$$\begin{aligned} (2.12) \quad \phi_m(t) &= E e^{tQ^{(m)}\sqrt{-1}} \\ &= (2\pi)^{-k/2} \int \dots \int e^{tQ^{(m)}\sqrt{-1}} e^{-\frac{1}{2} \sum_{i=1}^k Y_i^2} \prod_{i=1}^k dY_i, \\ &= (2\pi)^{-k/2} \int \dots \int e^{-\frac{1}{2} R} \prod_{i=1}^k dY_i, \quad k \leq (m+1)^d, \end{aligned}$$

* i.e. a weighted sum of independent chi-square random variables

where, for convenience, we kept the rv notation in the integral and

$$R = \int_{n_1=0}^{\infty} \dots \int_{n_d=0}^{\infty} Y^2_{n_1 \dots n_d} - 2tQ^{(m)} \sqrt{-1}$$

Suppose we express the $(m+1)^d$ random variables Y as a vector $\underline{Y}$. Then

$$Q^{(m)} = \underline{Y}^t \underline{A}^{(m)} \underline{Y}$$

where $\underline{A}^{(m)}$ is the coefficient matrix, and

$$R = \underline{Y}^t \underline{B}^{(m)} \underline{Y} = \underline{Y}^t (\underline{I} - 2t \sqrt{-1} \underline{A}^{(m)}) \underline{Y}$$

Now $\underline{A}^{(m)}$ is symmetrical, real, positive definite, thus $\underline{B}^{(m)}$ is also positive definite and can be diagonalised and expressed as

$$\underline{B}^{(m)} = \underline{J}^t \underline{J} \text{ for some matrix } \underline{J}$$

We define the vector $\underline{U} = \underline{J} \underline{Y}$. Then

$$R = \underline{U}^t \underline{U} = \sum_{i=1}^k U_i^2$$

and

$$\prod dU_i = ||\underline{J}|| \prod dY_i$$

Thus

$$(2.13) \quad \phi_{\underline{m}}(t) = ||\underline{J}||^{-1} = ||\underline{B}^{(m)}||^{-1/2}$$

$(|B^{(m)}|)$ is the absolute value of the determinant $B^{(m)}$.

Consider the eigenvalues (λ_i) of the matrix $\Delta^{(m)}$ (thus, of the quadratic form $Q^{(m)}$).

To calculate the λ_i we solve the determinant equation

$$|\Delta^{(m)} - \lambda I| = 0$$

and thus, as this is a polynomial in λ , we have

$$|\Delta^{(m)} - \lambda I| = C \prod_{i=1}^k (\lambda - \lambda_i), \quad k \leq (m+1)^d.$$

Thus

$$|B^{(m)}| = |I - 2t \sqrt{-1} \Delta^{(m)}| = C \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1}),$$

and hence by (2.13)

$$(2.14) \quad \phi_m(t) = \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1})^{-1}, \quad k \leq (m+1)^d,$$

since, by $\phi_m(0) = 1$ we must have $C = 1$.

Towards the characteristic function ϕ of the rv C_d of (1.21) we need

LEMMA B (Bateman 1949, Durbin 1970). Suppose that
 $x_1, x_2, \dots, x_k$ are independent $N(0,1)$ variables. The characteristic
function of

$$Q = \sum_{j=1}^k r_j x_j^2$$

conditional on

$$\sum_{j=1}^k x_j s_j = 0,$$

where

$$\sum_{j=1}^k s_j^2 = 1,$$

is

$$(2.15) \quad \phi_k(t) = \left\{ \prod_{j=1}^k (1 - 2r_j t \sqrt{-1}) \right\} \left\{ \sum_{j=1}^k \frac{s_j^2}{(1 - 2r_j t \sqrt{-1})} \right\}^{-1/2}.$$

We use this lemma to show that the form of ϕ is that of (2.15)

with $r_j = \lambda_j$ and evaluate the s_j . Our condition is $W^*(1,1,1) =$

$$W(1,1,1) - \sum_{i=1}^d W(1,1,1) = 0.$$

Hence $W(1,1,1) = 0$, i.e. (cf. (2.7) and the of (2.9)),

$$\tau^{-d} d^{d/2} \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \prod_{i=1}^d (n_i + \frac{1}{2})^{-1} Y_{n_1 \dots n_d} = 0$$

and, if the $Y_{n_1 \dots n_d}$ are ordered as in the vector $\underline{Y}$, we write the latter as

$$\underline{S}^t \underline{Y} = \sum_{j=1}^{\infty} s_j Y_j = 0$$

and show that $\sum_{j=1}^{\infty} s_j^2 = 1$.

We need to transform the vector $\underline{S}$ of constants s_j to a vector $\underline{V}$ in the coordinate system of the vector $\underline{U} = \underline{J} \underline{Y}$. We show that there exists a matrix $\underline{E}^{(m)}$ that diagonalises the matrix $\underline{\lambda}^{(m)}$ and hence the matrix $\underline{B}^{(m)}$, and that $\underline{V}^{(m)} = \underline{E}^{(m)t} \underline{S}$ is the required vector of constants v_j with the conditions

$$\sum_{j=1}^{\infty} v_j U_j = 0$$

and

$$\sum_{j=1}^{\infty} v_j^2 = 1.$$

Thus we can apply Durbin's lemma to obtain the required characteristic function

$$(2.16) \quad \phi(t) = \lim_{k \rightarrow \infty} \left\{ \prod_{j=1}^k (1 - \lambda_j u) \right\} \prod_{j=1}^k \frac{v_j^2}{1 - \lambda_j u} \Bigg|^{-1/2},$$

where the λ_j are the eigenvalues of $\underline{\lambda}^{(m)}$, $m \rightarrow \infty$, $u = 2t \sqrt{-1}$, $k = (m!)^d \rightarrow \infty$.

We note that

$$\prod_{j=1}^k (1 - \lambda_j u) \sum_{j=1}^k \frac{v_j^2}{1 - \lambda_j u}$$

is the Lagrange interpolation formula for a polynomial y which has the values y_j at the points $u = \lambda_j^{-1}$ with $y_j = v_j^2 \prod_{i \neq j} (1 - \frac{\lambda_i}{\lambda_j})$.

We summarize the above steps as

THEOREM 2. The asymptotic characteristic function of the rv $C_{n,d}$, $\hat{C}_{n,d}$ and $\tilde{C}_{n,d}$ (cf. (1.21), (1.24) and (1.25)) is the same as the characteristic function of C_d of (1.21), and it is given by (cf. (2.16))

$$(2.17) \quad \phi^{-2}(t) = \prod_{j=1}^{\infty} (1 - \lambda_j u) \sum_{j=1}^{\infty} \frac{v_j^2}{1 - \lambda_j u}, \quad u = 2it \sqrt{-1}, \quad \sum_{j=1}^{\infty} v_j^2 = 1,$$

where the eigenvalues λ_j of the quadratic form Q and the associated set of constants v_j can be calculated numerically as described above.

As mentioned earlier, a detailed proof of Theorem 2 will be given in Section 4.

With (2.17) at our disposal now, we can return to proving Lemmas 1 and 2. First we recall that the respective statements of the latter have already been verified for the asymptotic distribution of the multivariate Cramér-von Mises statistic (cf. Lemmas 1 and 2 in Cotterill and Csörgő (1979)). Since, just like the latter, the rv C_d is also a generalized chi-square type member of the exponential

family (indeed, not surprisingly at all, the very form of $\phi(\cdot)$ of (2.17) is the same as that of (2.9) in Cotterill and Csörgő (1979), only in the present case we did not succeed in finding the eigenvalues λ_j and the constants v_j in closed form), it is intuitively obvious that Lemmas 1 and 2 are true also in the present context. Unfortunately the details are somewhat tedious.

PROOF OF LEMMA 1. Given $\phi(\cdot)$ as in (2.17), by (2.2) we have to show that

$$(2.18) \quad \int_{-\infty}^{\infty} |t|^p |\phi(t)| dt < \infty \text{ for any integer } p \geq 0.$$

Since $\int_{-1}^1 |t|^p |\phi(t)| dt < \infty$, it suffices to show that

$$(2.19) \quad q_p = \int_1^{\infty} t^p |\phi(t)| dt < \infty.$$

Let

$$(2.20) \quad P(t) = \prod_{j=1}^{\infty} (1 - 2\lambda_j t i), \quad S(t) = \prod_{j=1}^{\infty} \frac{1 - 2\lambda_j^2 t^2}{(1 - 2\lambda_j t i)}, \quad 1 = \sqrt{-1}.$$

Then by (2.17)

$$(2.21) \quad |\phi(t)|^{-2} = |P(t)| |S(t)|,$$

where

$$(2.22) \quad |P(t)|^2 = P(t) P(-t) = \prod_{j=1}^{\infty} (1 + 4\lambda_j^2 t^2),$$

$$(2.23) \quad |S(t)|^2 = R^2(t) + I^2(t),$$

$$(2.24) \quad R(t) = \frac{1}{2} (S(t) + S(-t)) = \sum_{j=1}^{\infty} \frac{v_j^2}{(1 + 4\lambda_j^2 t^2)}$$

$$(2.25) \quad I(t) = \frac{1}{2i} (S(t) - S(-t)) = 2t \sum_{j=1}^{\infty} \frac{v_j^2 \lambda_j}{1 + 4\lambda_j^2 t^2}$$

Now for any integer $m > 0$

$$(2.26) \quad |P(t)|^2 > \prod_{j=1}^m (4\lambda_j^2 t^2) \prod_{j=m+1}^{\infty} (1 + 4\lambda_j^2 t^2) \\ > 4^m t^{2m} \prod_{j=1}^m \lambda_j^2 = K_m t^{2m}, \text{ for all real } t,$$

since $\prod_{j=m+1}^{\infty} (1 + 4\lambda_j^2 t^2) > 1$. By (2.24)

$$(2.27) \quad R(t) \geq \frac{v_1^2}{1 + 4\lambda_1^2 t^2} \geq t^{-2} \frac{v_1^2}{1 + 4\lambda_1^2} = K_0 t^{-2} \text{ for all } |t| \geq 1.$$

Similarly, by (2.25)

$$(2.28) \quad I(t) > K_1 t^{-1} \text{ for all } t \geq 1.$$

Thus by (2.23), (2.27) and (2.28) there exist a constant K such that

$$(2.29) \quad |S(t)|^2 > K t^{-4} \text{ for all } |t| \geq 1.$$

Whence by (2.21), (2.26) and (2.29) we have for all $|t| \geq 1$ and integer m that

$$(2.30) \quad |\phi(t)|^{-4} = |P(t)|^2 |S(t)|^2 > K_{\mu} K t^{2m-4}.$$

Therefore by (2.19)

$$(2.31) \quad q_p \leq \text{Const.} \int_1^x t^{p + \frac{2-m}{2}} dt < \infty,$$

provided for any given p we take m large enough. This also completes the proof of Lemma 1.

PROOF OF LEMMA 2. By Lemma 1 the statement of Lemma 2 holds with $p = 0$. Hence it suffices to verify the latter for an arbitrary integer $p \geq 1$. Again by Lemma 1 and the inversion formula of Lemma A

$$2\pi \gamma_d^{(q)}(x) = (-i)^q \int_{-\infty}^{+\infty} e^{-itx} t^q \phi(t) dt,$$

and integrating by parts p times

$$2\pi x^p \gamma_d^{(q)}(x) = (-i)^{q+p} \int_{-\infty}^{+\infty} e^{-itx} \frac{d^p}{dt^p} (t^q \phi(t)) dt.$$

Thus

$$(2.32) \quad |x^p \gamma_d^{(q)}(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{d^p}{dt^p} (t^q \phi(t)) \right| dt,$$

and hence Lemma 2 is true, provided that

$$(2.33) \quad \int_{-\infty}^{+\infty} \left| \frac{d^p}{dt^p} (t^q \phi(t)) \right| dt < \infty.$$

First we verify (2.33) for $p = 1$, i.e., we show that

$$(2.34) \quad Q_q = \int_{-\infty}^{\infty} \left| \frac{d}{dt} (t^q \phi(t)) \right| dt < \infty$$

We have

$$(2.35) \quad Q_q \leq q \int_{-\infty}^{\infty} t^{q-1} |\phi(t)| dt + \int_{-\infty}^{\infty} t^q \left| \frac{d}{dt} \phi(t) \right| dt$$

provided the indicated two integrals are finite. By (2.18) the first one is, and we are going to show now that the second one is also finite, i.e., that

$$(2.36) \quad \int_{-\infty}^{\infty} \left| t^q \frac{d}{dt} \phi(t) \right| dt < \infty \quad \text{for any integer } q \geq 0.$$

For the sake of the latter we consider

$$(2.37) \quad -2\phi^{-3}(t) \frac{d}{dt} \phi(t) = S(t) \frac{d}{dt} P(t) + P(t) \frac{d}{dt} S(t)$$

by (2.17) and (2.20). Whence,

$$(2.38) \quad \begin{aligned} -2\phi^{-3}(t) \frac{d}{dt} \phi(t) &= S(t)P(t) \frac{d}{dt} \log P(t) + P(t)S(t) \frac{d}{dt} \log S(t) \\ &= \phi^{-2}(t) \frac{d}{dt} \log P(t) + \phi^{-2}(t) \frac{d}{dt} \log S(t), \end{aligned}$$

where the last line is again by (2.17) and (2.20). Therefore

$$(2.39) \quad \frac{d}{dt} \phi(t) = -\frac{1}{2} \phi(t) \left(\frac{d}{dt} \log P(t) + \frac{d}{dt} \log S(t) \right)$$

and

$$(2.40) \quad \frac{d}{dt} \Phi(t) = \Phi(t) \left(\frac{d}{dt} \log P(t) + \frac{1}{S(t)} \frac{d}{dt} S(t) \right)$$

Now by (2.20) $\log P(t) = \sum_{j=1}^{\infty} \log(1 - 2^{-j} t)$ and so

$$(2.41) \quad \frac{d}{dt} \log P(t) = -2 \sum_{j=1}^{\infty} \frac{1}{1 - 2^{-j} t}$$

Whence,

$$(2.42) \quad \frac{d}{dt} \log P(t)^2 = 4 \sum_{j=1}^{\infty} \frac{1}{(1 + 4^{-j} t^2)} = 4 \sum_{j=1}^{\infty} \frac{1}{1 + 4^{-j} t^2} = 4X_2 = K_2 < \infty,$$

where X_2 is X_n of Corollary 1 of Section 3 with $n = 2$.

By $S(t)$ of (2.20) we have

$$(2.43) \quad \frac{d}{dt} S(t) = 2 \sum_{j=1}^{\infty} \frac{\lambda_j v_j^2}{(1 - 2^{-j} t)^2}$$

Whence, and remembering also that the $\lambda_j \geq 0$ since the quadratic form Q of (2.9) is non-negative definite, we have

$$(2.44) \quad \left| \frac{d S(t)}{dt} \right| \leq 2 \sum_{j=1}^{\infty} \frac{\lambda_j v_j^2}{|1 - 2^{-j} t|^2} = 2 \sum_{j=1}^{\infty} \frac{\lambda_j v_j^2}{(1 + 4^{-j} t^2)} \\ \leq 2 \sum_{j=1}^{\infty} \lambda_j v_j^2 = 2Z_1 = K_3 < \infty,$$

where Z_1 is Z_n of Corollary 1 of Section 3 with $n = 1$.

Combining (2.29), (2.30), (2.40), (2.42) and (2.44) we get

the original quadratic form Q evaluated at

$$y_j = s_j, \quad j = 1, 2, \dots$$

If we recall that $\underline{E}^t = \underline{E}^{-1}$, $\underline{E}^t \underline{E} = \underline{I} = \underline{E} \underline{E}^t$, then

$$\begin{aligned} z_n &= n! \underline{V}^t \underline{A}^n \underline{V} = n! \underline{V}^t (\underline{A} \underline{I})^n \underline{V} \\ &= n! \underline{S}^t \underline{E} (\underline{A} \underline{E}^t \underline{E})^n \underline{E}^t \underline{S} \\ &= n! \underline{S}^t (\underline{E} \underline{A} \underline{E}^t)^n (\underline{E} \underline{E}^t) \underline{S} \\ &= n! \underline{S}^t \underline{A}^n \underline{S}, \end{aligned}$$

a derived quadratic form in the s_j , $j = 1, 2, \dots$

This trivial manipulation would greatly facilitate computation, in that it is much less expensive to calculate the n^{th} power of a matrix $\underline{A}$ than it is to calculate its eigenvalues and eigenvectors. However, this possible method of calculation turned out to be less useful than hoped for, due to the fact that the results were to be the small differences of pairs of two large numbers (cf. K_n of Corollary 1). Hence we calculate the cumulants of Γ_d ($d \geq 2$) directly in Sections 5 and 6.

A look at (2.39) shows that for each r $|\frac{d^r}{dt^r} \phi(t)|$ is bounded above by a finite sum of terms like

$$(2.48) \quad \left| \frac{d^k}{dt^k} \phi(t) \right| \left(\left| \frac{d^{r-k}}{dt^{r-k}} \log P(t) \right| + \left| \frac{d^{r-k}}{dt^{r-k}} \log S(t) \right| \right), \quad 0 \leq k < r.$$

For example in case of $r = 2$, from (2.39) we get

$$(2.49) \quad \left| \frac{d^2}{dt^2} \phi(t) \right| \leq \left| \frac{d}{dt} \phi(t) \right| \left(\left| \frac{d}{dt} \log P(t) \right| + \left| \frac{d}{dt} \log S(t) \right| \right) + \left| \phi(t) \right| \left(\left| \frac{d^2}{dt^2} \log P(t) \right| + \left| \frac{d^2}{dt^2} \log S(t) \right| \right),$$

where on the right hand side we have already estimated everything (cf. (2.45) and (2.30)) but $\left(\left| \frac{d^2}{dt^2} \log P(t) \right| + \left| \frac{d^2}{dt^2} \log S(t) \right| \right)$. The latter, in turn, can be estimated by calculations similar to those of (2.42) and (2.43).

In case of (2.48) we proceed similarly and find that given an estimate from above for $\left| \frac{d^{r-1}}{dt^{r-1}} \phi(t) \right|$, the only terms we have to estimate when estimating $\left| \frac{d^r}{dt^r} \phi(t) \right|$ from above are the ones of

$$(2.50) \quad \left(\left| \frac{d^r}{dt^r} \log P(t) \right| + \left| \frac{d^r}{dt^r} \log S(t) \right| \right).$$

We find that

$$\frac{d^r}{dt^r} \log P(t) = -(r-1)! 2^r i^r \sum_{j=1}^{\infty} \frac{\lambda_j^r}{(1 - 2\lambda_j t)^r}.$$

and so

$$\left| \frac{d^r}{dt^r} \log P(t) \right| \leq (r-1)! 2^r \sum_{j=1}^{\infty} \frac{\lambda_j^r}{(1 + 4\lambda_j^2 t^2)^{r/2}}$$

$$\leq (r-1)! 2^r \sum_{j=1}^{\infty} \lambda_j^r = 2^r X_r < \infty,$$

where X_r is X_n of Corollary 1 of Section 3 with $n = r$.

As to estimating $\left| \frac{d^r}{dt^r} \log S(t) \right|$ of (2.50), in the light of (2.40) we have to estimate only $\left| \frac{d^r}{dt^r} S(t) \right|$. We find that

$$\frac{d^r}{dt^r} S(t) = r! 2^r i^r \sum_{j=1}^{\infty} \frac{\lambda_j^r v_j^2}{(1 - 2\lambda_j^2 t i)^{r+1}}$$

and so

$$\left| \frac{d^r}{dt^r} S(t) \right| \leq r! 2^r \sum_{j=1}^{\infty} \frac{\lambda_j^r v_j^2}{(1 + 4\lambda_j^2 t^2)^{(r+1)/2}}$$

$$\leq r! 2^r \sum_{j=1}^{\infty} \lambda_j^r v_j^2 = 2^r Z_r < \infty$$

where Z_r is Z_n of Corollary 1 of Section 3 with $n = r$. This also completes the proof of Lemma 2 and also the proof of Theorem 1 as indicated at the beginning of this section.

Now we return to the rv $\hat{C}_{n,d}$ of (1.24) and $\tilde{C}_{n,d}$ of (1.25) in order to study their asymptotic behaviour.

PROPOSITION 1. Given $F \in F_0$, we have

$$(2.51) \quad \lim_{n \rightarrow \infty} P\{\bar{C}_{n,d} \leq u\} = \lim_{n \rightarrow \infty} P\{\bar{C}_{n,d} \leq u\} = \Gamma_d(u), \quad u \geq 0 \quad (d \geq 2),$$

where Γ_d is the distribution function of the rv C_d (cf. (1.21) and (1.22)).

The proof of Proposition 1 is based on either of the next two corollaries to Theorem A, which might be of some interest on their own. Let $t_n(y)$ be as in (1.7) and $T^{(n)}(y)$ as in (1.8).

PROPOSITION 2. Given $F \in F_0$, we have for $\bar{C}_{n,d}$

$$(2.52) \quad \begin{aligned} &= \left| n \int_{I^d} t_n^2(y) dF_n(L^{-1}(y)) - \int_{I^d} (T^{(n)}(y))^2 dF_n(L^{-1}(y)) \right| \\ &\underset{\text{a.s.}}{=} \begin{cases} O(n^{-\frac{1}{2}} \log^2 n (\log \log n)^{\frac{1}{2}}) & \text{if } d = 2 \\ O(n^{-\frac{1}{2}(d+1)} (\log n)^{3/2} (\log \log n)^{\frac{1}{2}}) & \text{if } d \geq 3 \end{cases} \end{aligned}$$

and for $\bar{C}_{n,d}$

$$(2.53) \quad \begin{aligned} &= \left| n \int_{I^d} t_n^2(y) \prod_{i=1}^d dF_{ni}(F_i^{-1}(y_i)) - \int_{I^d} (T^{(n)}(y))^2 \prod_{i=1}^d dF_{ni}(F_i^{-1}(y_i)) \right| \\ &\underset{\text{a.s.}}{=} \begin{cases} O(n^{-\frac{1}{2}} \log^2 n (\log \log n)^{\frac{1}{2}}) & \text{if } d = 2 \\ O(n^{-\frac{1}{2}(d+1)} (\log n)^{3/2} (\log \log n)^{\frac{1}{2}}) & \text{if } d \geq 3. \end{cases} \end{aligned}$$

PROOF. Same as that of (4.49) in Csörgő (1979).

PROPOSITION 3. Given $F \in F_0$, we have

$$(2.54) \quad \sup_{0 < u < \infty} |P\{\tilde{C}_{n,d} \leq u\} - P\{\int_{I^d} (T^n(y))^2 dF_n(L^{-1}(y)) \leq u\}|$$

$$= \begin{cases} O(n^{-1/2} \log^2 n) & \text{if } d = 2 \\ O(n^{-1/2(d+1)} (\log n)^{3/2}) & \text{if } d \geq 3 \end{cases}$$

and

$$(2.55) \quad \sup_{0 < u < \infty} |P\{\tilde{C}_{n,d} \leq u\} - P\{\int_{I^d} (T^{(n)}(y))^2 \prod_{i=1}^d dF_{n_i}(F_i^{-1}(y_i)) \leq u\}|$$

$$= \begin{cases} O(n^{-1/2} \log^2 n) & \text{if } d = 2 \\ O(n^{-1/2(d+1)} (\log n)^{3/2}) & \text{if } d \geq 3 \end{cases}$$

PROOF. Same as that of Theorem 1.

PROOF OF PROPOSITION 1. By (2.52)

$$\lim_{n \rightarrow \infty} |\tilde{C}_{n,d} - \int_{I^d} (T^{(n)}(y))^2 dF_n(L^{-1}(y))| \xrightarrow{\text{a.s.}} 0, \quad d \geq 2.$$

Hence, or by (2.54), it suffices to show that for any given $\epsilon > 0$

and $0 < \delta < 1$ there exists an $n_0 = n_0(\epsilon, \delta)$ such that

$$(2.56) \quad P\left\{\left|\int_{I^d} (T^{(n)}(y))^2 dF_n(L^{-1}(y)) - \int_{I^d} (T^{(n)}(y))^2 dy\right| > \epsilon\right\} < \delta,$$

whenever $n \geq n_0$.

The rest of the proof for $\tilde{C}_{n,d}$ is the same, mutatis mutandis, as that of

Proposition 1 in Cotterill and Cdrgo (1979). A similar argument holds

for $\tilde{C}_{n,d}$ on the basis of (2.53) or that of (2.55). Hence we have Proposition 1.

3. Some Corollaries and a Possible Way for the Calculation of the Critical Values of the Hoeffding, Blum, Kiefer, Rosenblatt Independence Criterion. Here of course we are talking about the limiting distribution of the statistics $\hat{C}_{n,d}$ and $\tilde{C}_{n,d}$ which, under H_0 of (1.1), have the same asymptotic distribution function Γ_d (cf. Proposition 1). Our goal is to calculate the cumulants of this distribution. Our starting point is Theorem 2.

COROLLARY 1. The cumulants K_n of Γ_d are defined by

$$\log \phi(t) = \sum_{n=1}^{\infty} K_n \frac{(it)^n}{n!},$$

and are calculated from the equations:

$$K_n = 2^{n+1} (X_n - Y_n),$$

where

$$X_n = (n-1)! \sum_{j=1}^{\infty} \lambda_j^n,$$

$$Y_{n+1} = Z_{n+1} - \sum_{r=1}^n \binom{n}{r} Y_{n+1-r} Z_r,$$

and

$$Z_n = n! \sum_{j=1}^{\infty} v_j^2 \lambda_j^n.$$

COROLLARY 2. The numerical values of the moments M_n of Γ_d can be calculated from the equations:

$$M_n = 2^n p_n.$$

where

$$P_{n+1} = Q_{n+1} + \sum_{r=1}^n \binom{n}{r} Q_{n+1-r} P_r$$

and

$$Q_n = \frac{1}{2}(X_n - Y_n)$$

where X_n and Y_n are defined in Corollary 1.

COROLLARY 3. The mean μ and the variance σ^2 of Γ_d

are

$$\mu = K_1 = 4 \sum_{j=1}^{\infty} \lambda_j (1 - v_j^2)$$

$$\sigma^2 = K_2 = 8 \left(\sum_{j=1}^{\infty} \lambda_j^2 (1 - 2v_j^2) + \left(\sum_{j=1}^{\infty} v_j^2 \lambda_j \right)^2 \right)$$

A COMPUTATIONAL NOTE. As we examine the expressions for the cumulants in Corollary 1 we see that we do not really need the individual values of the eigenvalues λ_j nor of the weights v_j . What we need are

$$K_1 = \sum_{j=1}^{\infty} \lambda_j = \text{Tr}(\underline{\underline{\Lambda}})$$

$$K_n = (n-1)! \sum_{j=1}^{\infty} \lambda_j^n = (n-1)! \text{Tr}(\underline{\underline{\Lambda}}^n)$$

$$Z_1 = \sum_{j=1}^{\infty} v_j^2 \lambda_j = \underline{\underline{v}}^t \underline{\underline{\Lambda}} \underline{\underline{v}}$$

$$z_n = n! \sum_{j=1}^n v_j^2 \lambda_j^n = n! \underline{v}^t \underline{\Lambda}^n \underline{v}.$$

Now (cf. Proof of Theorem 2 in Section 4 and our steps leading up to the statement of Theorem 2 in the previous section) the λ_j are the eigenvalues of the symmetric matrix $\underline{\Lambda}$ defined by the quadratic form $Q = \underline{Y}^t \underline{\Lambda} \underline{Y}$, and there is a matrix $\underline{E}$ of eigenvectors such that $\underline{E}^{-1} = \underline{E}^t$ and

$$\underline{\Lambda} = \underline{E}^{-1} \underline{\Lambda} \underline{E}.$$

Hence

$$\text{Tr}(\underline{\Lambda}) = \text{Tr}(\underline{\Lambda}),$$

$$\text{Tr}(\underline{\Lambda}^n) = \text{Tr}(\underline{\Lambda}^n),$$

and

$$z_1 = \underline{v}^t \underline{\Lambda} \underline{v} = \underline{s}^t \underline{E} \underline{\Lambda} \underline{E}^t \underline{s} = \underline{s}^t \underline{\Lambda} \underline{s},$$

since

$$\underline{\Lambda} = \underline{E} \underline{\Lambda} \underline{E}^t \text{ and } \underline{v} = \underline{E}^t \underline{s} \text{ (cf. Proof of Theorem 2).}$$

Thus

$$z_1 = Q \Big|_{\underline{Y} = \underline{s}},$$

the original quadratic form Q evaluated at

$$Y_j = s_j, \quad j = 1, 2, \dots$$

If we recall that $E^t = E^{-1}$, $E^t E = I = E E^t$, then

$$\begin{aligned} z_n &= n! \underline{V}^t \underline{A}^n \underline{V} = n! \underline{V}^t (\underline{A} \underline{I})^n \underline{V} \\ &= n! \underline{S}^t \underline{E} (\underline{A} \underline{E}^t \underline{E})^n \underline{E}^t \underline{S} \\ &= n! \underline{S}^t (\underline{E} \underline{A} \underline{E}^t)^n (\underline{E} \underline{E}^t) \underline{S} \\ &= n! \underline{S}^t \underline{A}^n \underline{S}, \end{aligned}$$

a derived quadratic form in the s_j , $j = 1, 2, \dots$

This trivial manipulation would greatly facilitate computation, in that it is much less expensive to calculate the n^{th} power of a matrix $\underline{A}$ than it is to calculate its eigenvalues and eigenvectors. However, this possible method of calculation turned out to be less useful than hoped for, due to the fact that the results were to be the small differences of pairs of two large numbers (cf. K_n of Corollary 1). Hence we calculate the cumulants of Γ_d ($d \geq 2$) directly in Sections 5 and 6.

4. Proof of Theorem 2 and its Corollaries in Section 3.

In Section 2, beginning with the statement of (2.3), we have described a number of steps leading up to the statement of Theorem 2. The first one of these steps requiring a hint of proof is that of (2.7).

PROOF OF (2.7). The right hand side sum of (2.7) is a weighted sum of independent $N(0,1)$ rv. Hence, just like $\{W(y); y \in I^d\}$, it is a Gaussian process with mean zero and the two sides are identical in distribution, i.e., they are the same Gaussian processes, if and only if they have the same covariance function. Denoting the right hand side of (2.7) by $V(y)$ and using the fact that

$$E z_{n_1 \dots n_d} z_{m_1 \dots m_d} = \begin{cases} 1 & \text{if } n_i = m_i, i = 1, 2, \dots, d, \\ 0 & \text{otherwise,} \end{cases}$$

then standard calculations yield $E V(x) V(y) = \sum_{i=1}^d (x_i - y_i) = E W(x) W(y)$, as required.

PROOF OF (2.8). It is required that we prove (cf. (2.6) and

$$(4.1) \quad Q = \int_{I_d} (W^*(y))^2 dy = Q_1 + Q_2 + Q_{31} + Q_{32}.$$

where the latter summands are listed after (2.8). Write (cf. (2.4))

$$W^*(y) = W(y) - S(y).$$

where

$$s(y) = \prod_{i=1}^d w(\frac{1}{2}, y_i, \frac{1}{2}) \prod_{j \neq i} y_j, \quad y = (y_1, \dots, y_d) \in I^d.$$

Then

$$Q = Q_1 + Q_2 + Q_3.$$

where

$$Q_1 = \int_{I^d} w^2(y) dy,$$

$$Q_2 = -2 \int_{I^d} w(y) s(y) dy,$$

$$Q_3 = \int_{I^d} s^2(y) dy.$$

Hence, by (2.7) and using the orthogonal property of the sine functions of the latter,

$$\begin{aligned} Q_1 &= \int_{I^d} w^2(y) dy \\ &= 2^d \prod_{n_1=0}^{\infty} \dots \prod_{n_d=0}^{\infty} \int_{n_1 \dots n_d}^2 \prod_{i=1}^d \int_0^1 \frac{\sin^2(n_i + \frac{1}{2}) \pi y_i}{(n_i + \frac{1}{2})^2 \pi^2} dy_i \\ &= \pi^{-2d} \prod_{n_1=0}^{\infty} \dots \prod_{n_d=0}^{\infty} \prod_{i=1}^d (n_i + \frac{1}{2})^{-2} \pi^2 \int_{n_1 \dots n_d}^2. \end{aligned}$$

upon writing

$$\int_{n_1 \dots n_d}^2 = (-1)^{n_1 + \dots + n_d} \int_{n_1 \dots n_d}^2.$$

Thus Q_1 is as required. Next

$$Q_2 = -2 \int_{I^d} W(y) S(y) dy$$

$$= -2 \sum_{i=1}^d \int_{I^d} W(y) W(\underline{1}, y_i, \underline{1}) \prod_{j \neq i} y_j \prod_{j=1}^d dy_j$$

$$= -2 \sum_{i=1}^d I_i \text{ say.}$$

By (2.7) and (2.8), for the latter I_i we get that

$$I_i = \frac{2^d}{\pi^{2d}} \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} U_{i, n_1 \dots n_d, m_1 \dots m_d}$$

where

$$U_{i, n_1 \dots n_d, m_1 \dots m_d} = \prod_{j=1}^d \frac{(-1)^{m_j}}{(n_j + \frac{1}{2})(m_j + \frac{1}{2})} (-1)^{m_i} z_{n_1 \dots n_d} z_{m_1 \dots m_d} I_{i, n_1 \dots n_d, m_1 \dots m_d}$$

and

$$I_{i, n_1 \dots n_d, m_1 \dots m_d} = \int_{I^d} \prod_{j=1}^d \sin(n_j + \frac{1}{2}) \pi y_j \sin(m_j + \frac{1}{2}) \pi y_j \prod_{j \neq i} y_j \prod_{j=1}^d dy_j$$

$$= J_i \prod_{k \neq i} J_k$$

where

$$J_k = \int_0^1 y_k \sin(n_k + \frac{1}{2}) \pi y_k dy_k = \frac{(-1)^{n_k}}{(n_k + \frac{1}{2})^2 \pi^2}$$

and

$$J_1 = \int_0^1 \sin(n_1 + \frac{1}{2})\pi y_1 \sin(m_1 + \frac{1}{2})\pi y_1 dy_1$$

$$= \begin{cases} \frac{1}{2} & \text{if } n_1 = m_1 \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$S(n_1, m_1) = \begin{cases} 1 & \text{if } n_1 = m_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$J_1 = \frac{(-1)^{n_1}}{(n_1 + \frac{1}{2})^2 \pi^2} \cdot (-1)^{n_1} (n_1 + \frac{1}{2})^2 \frac{\pi^2}{2} S(n_1, m_1),$$

and hence

$$I_{1, n_1 \dots n_d, m_1 \dots m_d} = (-1)^{n_1} (n_1 + \frac{1}{2})^2 \frac{\pi^2}{2} S(n_1, m_1) \prod_{k=1}^d \frac{(-1)^{n_k}}{(n_k + \frac{1}{2})^2 \pi^2},$$

$$U_{1, n_1 \dots n_d, m_1 \dots m_d}$$

$$= \prod_{j=1}^d \frac{(-1)^{n_j + m_j}}{(n_j + \frac{1}{2})^3 (m_j + \frac{1}{2})} \frac{(-1)^{n_1 + m_1}}{\pi^{2d}} (n_1 + \frac{1}{2})^2 \frac{\pi^2}{2} S(n_1, m_1) z_{n_1 \dots n_d} z_{m_1 \dots m_d}.$$

Note that if, $S(n_1, m_1) \neq 0$ ($n_1 = m_1$), then $(-1)^{n_1 + m_1} = 1$. Write also

$$Y_{n_1 \dots n_d} = \prod_{j=1}^d (-1)^{n_j} z_{n_1 \dots n_d}.$$

Then $U_{1, n_1 \dots n_d, m_1 \dots m_d}$ can be written as

$$\frac{\pi^{2-2d}}{2} \prod_{j=1}^d (n_j + \frac{1}{2})^{-3} (m_j + \frac{1}{2})^{-1} (n_1 + \frac{1}{2})^2 S(n_1, m_1) Y_{n_1 \dots n_d} Y_{m_1 \dots m_d}.$$

and substituting the latter into the formula for I_1 above, with

$a = 2^d \pi^{2-4d}$ we get

$$Q_2 = -2 \int_d W(y) S(y) dy$$

$$= -2 \sum_{i=1}^d I_i$$

$$= -a \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \prod_{j=1}^d (n_j + \frac{1}{2})^{-3} (m_j + \frac{1}{2})^{-1} \sum_{i=1}^d (n_i + \frac{1}{2})^2.$$

$$S(n_1, m_1) Y_{n_1 \dots n_d} Y_{m_1 \dots m_d}$$

as required. Next consider

$$Q_3 = \int_d S^2(y) dy$$

$$= \sum_{i=1}^d \sum_{j=1}^d \int_d W(\underline{1}, y_i, \underline{1}) W(\underline{1}, y_j, \underline{1}) \prod_{r \neq i} y_r \prod_{s \neq j} y_s \prod_{k=1}^d dy_k$$

$$= Q_{31} + Q_{32}$$

where

$$Q_{31} = \sum_{i=1}^d \int_d W^2(\underline{1}, y_i, \underline{1}) \prod_{j \neq i} y_j^2 \prod_{j=1}^d dy_j$$

and by (2.8) $W(\underline{1}, y_i, \underline{1})$ can be written as

$$W(\underline{1}, y_i, \underline{1}) = \frac{2^{d/2}}{\pi^d} \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \prod_{j=1}^d \frac{(-1)^{n_j}}{(n_j + \frac{1}{2})} (-1)^{n_1} \sin(n_1 + \frac{1}{2}) \pi y_i z_{n_1 \dots n_d}$$

Thus

$$Q_{31} = \frac{2^d}{\pi^{2d}} \prod_{i=1}^d \sum_{n_i=0}^{\infty} \sum_{m_i=0}^{\infty} \prod_{j=1}^d \frac{(-1)^{n_j+m_j}}{(n_j+\frac{1}{2})(m_j+\frac{1}{2})} z_{n_1} \dots z_{n_d} z_{m_1} \dots z_{m_d} \prod_{j=1}^d J_j$$

with

$$J_1 = (-1)^{n_1+m_1} \int_0^1 \sin(n_1+\frac{1}{2})\pi y_1 \sin(m_1+\frac{1}{2})\pi y_1 dy_1$$

$$= \frac{(-1)^{n_1+m_1}}{2} S(n_1, m_1)$$

$$= \frac{1}{2} S(n_1, m_1)$$

$$= \begin{cases} \frac{1}{2} & \text{if } n_1 = m_1 \\ 0 & \text{if } n_1 \neq m_1 \end{cases}$$

$$\text{and for } j \neq 1 \quad J_j = \int_0^1 y_j^2 dy_j = \frac{1}{3}$$

Write $b = 3^{1-d} 2^{d-1} \pi^{-2d}$. Then

$$Q_{31} = b \prod_{n_1=0}^{\infty} \dots \prod_{n_d=0}^{\infty} \prod_{m_1=0}^{\infty} \dots \prod_{m_d=0}^{\infty} \prod_{j=1}^d (n_j+\frac{1}{2})^{-1} (m_j+\frac{1}{2})^{-1} \prod_{i=1}^d S(n_i, m_i) y_{n_1} \dots y_{n_d} y_{m_1} \dots y_{m_d}$$

are required. Now Q_{32} of Q_3 is

$$Q_{32} = \prod_{i=1}^d \prod_{j=1}^d \int_0^1 w(\underline{1}, y_i, \underline{1}) w(\underline{1}, y_j, \underline{1}) \prod_{r \neq i} y_r \prod_{s \neq j} y_s \prod_{k=1}^d dy_k$$

$$= \prod_{i=1}^d \prod_{j=1}^d H_{ij}$$

where

$$H_{ij} = 2^d \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \prod_{j=1}^d ((n_j+1)(m_j+1)\pi^2)^{-1} G_{(n,m)} z_{n_1} \dots z_{n_d} z_{m_1} \dots z_{m_d}$$

and

$$G_{(n,m)} = \int_I^d \sin(n_i+1)\pi y_i \sin(m_j+1)\pi y_j \prod_{r \neq i} (-1)^{n_r} y_r \prod_{s \neq j} (-1)^{m_s} y_s \prod_{k=1}^d dy_k$$

$$= F_i F_j \prod_{\substack{r \neq i \\ r \neq j}} E_r$$

with

$$F_i = \int_0^1 y_i \sin(n_i+1)\pi y_i dy_i = (-1)^{n_i} (n_i+1)^{-2} \pi^{-2}, \quad F_j = (-1)^{m_j} (m_j+1)^{-2} \pi^{-2}$$

and

$$E_r = \int_0^1 (-1)^{n_r+m_r} y_r^2 dy_r = \frac{1}{3} (-1)^{n_r+m_r}$$

Thus

$$G_{(n,m)} = \prod_{r=1}^d (-1)^{n_r+m_r} (n_r+1)^{-2} (m_r+1)^{-2} \pi^{-4} 3^{-(d-2)}$$

Write

$$c = 2^d 3^{2-d} \pi^{-2d-4}$$

Then

$$H_{ij} = c \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \prod_{r=1}^d (n_r+1)^{-1} (m_r+1)^{-1} (n_r+1)^{-2} (m_r+1)^{-2}$$

$$\prod_{r=1}^d (-1)^{n_r} z_{n_1} \dots z_{n_d} \prod_{r=1}^d (-1)^{m_r} z_{m_1} \dots z_{m_d}$$

We simplify by writing again

$$Y_{n_1 \dots n_d} = (-1)^{n_1 + \dots + n_d} Z_{n_1 \dots n_d}$$

$$Y_{m_1 \dots m_d} = (-1)^{m_1 + \dots + m_d} Z_{m_1 \dots m_d}$$

and letting

$$S_i = \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} (n_i + \frac{1}{2})^{-2} \prod_{r=1}^d (n_r + \frac{1}{2})^{-1} Y_{n_1 \dots n_d}$$

Then

$$H_{ij} = S_i S_j$$

and so

$$Q_{32} = c \sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d S_i S_j$$

as required.

For computational purposes we rearrange terms, in the respective summands of Q of (4.1), keeping the same lettering for the indices and noting that each off-diagonal term in this new arrangement results from two of those of Q of (4.1), and get

$$\begin{aligned} (4.2) \quad Q &= \int_{I^d} (W^*(y))^2 dy \\ &= \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} g_{n_1 \dots n_d} Y_{n_1 \dots n_d}^2 \\ &\quad + \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_d=0}^{\infty} k_{n_1 \dots n_d, m_1 \dots m_d} Y_{n_1 \dots n_d} Y_{m_1 \dots m_d} \end{aligned}$$

where the diagonal terms of the matrix of Q are

$$\begin{aligned}
 g_{n_1 \dots n_d} &= \pi^{-2d} \prod_{i=1}^d (n_i + \frac{1}{2})^{-2} && \text{from } Q_1 \\
 &- a \prod_{j=1}^d (n_j + \frac{1}{2})^{-4} \sum_{i=1}^d (n_i + \frac{1}{2})^2 && \text{from } Q_2 \\
 &+ bd \prod_{j=1}^d (n_j + \frac{1}{2})^{-2} && \text{from } Q_{31} \\
 &+ c \prod_{r=1}^d (n_r + \frac{1}{2})^{-2} \sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d (n_i + \frac{1}{2})^{-2} (n_j + \frac{1}{2})^{-2} && \text{from } Q_{32}
 \end{aligned}$$

and the off-diagonal terms k are

$$\begin{aligned}
 k_{n_1 \dots n_d, m_1 \dots m_d} &= -\frac{a}{2} \left\{ \prod_{j=1}^d (n_j + \frac{1}{2})^{-3} (m_j + \frac{1}{2})^{-1} \sum_{i=1}^d (n_i + \frac{1}{2})^2 S(n_i, m_i) + \prod_{j=1}^d (m_j + \frac{1}{2})^{-3} (n_j + \frac{1}{2})^{-1} \right. \\
 &\quad \left. \sum_{i=1}^d (m_i + \frac{1}{2})^2 S(n_i, m_i) \right\} && \text{from } Q_2 \\
 &+ b \prod_{j=1}^d (n_j + \frac{1}{2})^{-1} (m_j + \frac{1}{2})^{-1} \sum_{i=1}^d S(n_i, m_i) && \text{from } Q_{31} \\
 &+ c \prod_{j=1}^d (n_j + \frac{1}{2})^{-1} (m_j + \frac{1}{2})^{-1} \sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d (n_i + \frac{1}{2})^{-2} (m_j + \frac{1}{2})^{-2} && \text{from } Q_{32}
 \end{aligned}$$

where

$$a = 2^d \pi^{2-4d}, \quad b = 3^{1-d} 2^{d-1} \pi^{-2d}, \quad c = 3^{2-d} 2^d \pi^{-2d-4}.$$

In order to enable ourselves to calculate the characteristic function of the rv C_d of (1.21) from the quadratic form

$$Q = \int_D (W_n^*(y))^2 dy = Q_1 + Q_2 + Q_{31} + Q_{32}.$$

we consider the finite quadratic form

$$Q^{(m)} = Q_1^{(m)} + Q_2^{(m)} + Q_{31}^{(m)} + Q_{32}^{(m)}$$

obtained by replacing the limit of summation ∞ by the limit m in each of the above sums. Then by Kolmogorov's Three Series Theorem

$$(4.3) \quad \lim_{m \rightarrow \infty} Q^{(m)} \stackrel{a.s.}{=} Q.$$

Let $\phi_Q(t)$ be the characteristic function of the quadratic form Q and let $\phi_m(t)$ be the characteristic function of the finite quadratic form $Q^{(m)}$. Then by (4.3)

$$(4.4) \quad \lim_{m \rightarrow \infty} \phi_m(t) = \phi_Q(t).$$

Hence our first goal is to derive ϕ_m .

Since $Q^{(m)}$ is a finite quadratic form in independent $N(0,1)$ random variables, it is a sum of chi-square random variables and its characteristic function must be of the form as given in (2.11).

2

REMARK 4.1. In $Q^{(m)}$ the set $\{Y_{n_1 \dots n_d}; n_i = 0 \dots m; i = 1 \dots d\}$ has $(m+1)^d$ terms, i.e.

$$\{Y_{n_1 \dots n_d}\} = \{Y_j; j = 1 \dots k = (m+1)^d\}.$$

One can prove that the matrix corresponding to $Q^{(m)}$ is nonsingular.

Thus ϕ_m of (2.11) is actually of the form

$$(4.5) \quad \phi_m(t) = \prod_{i=1}^k (1 - a_i t \sqrt{-1})^{-1/2}, \quad k = (m+1)^d.$$

However, as Kendall and Stuart (1977) point out, it would not matter if the matrix in question were singular, since we do not require its inverse. The following derivation of $\phi_m(t)$ is based upon Kendall and Stuart (1977), Chapter 15.

The characteristic function $\phi_m(t)$ of the quadratic form $Q^{(m)} = \int_{I^d} (W^*(y))^2 dy$ is calculated as

$$(4.6) \quad \begin{aligned} \phi_m(t) &= E e^{tQ^{(m)}\sqrt{-1}} \\ &= (2\pi)^{-k/2} \int \dots \int e^{-1/2 R} \prod_{i=1}^k dY_i, \quad k = (m+1)^d, \end{aligned}$$

where

$$(4.7) \quad R = \sum_{n_1=0}^m \dots \sum_{n_d=0}^m Y_{n_1 \dots n_d}^2 - 2tQ^{(m)}\sqrt{-1},$$

and (4.6) and (4.7) follow from

$$\begin{aligned}\phi_m(t) &= E e^{tQ^{(m)}\sqrt{-1}} \\ &= (2\pi)^{-k/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{tQ^{(m)}\sqrt{-1}} \prod_{i=1}^k e^{-\frac{1}{2}y_i^2} dy_1, \end{aligned}$$

where, for convenience, we kept the rv notation in the integral, and

$$\{Y_i; i = 1 \dots k = (m+1)^d\} = \{Y_{n_1 \dots n_d}; n_j = 0 \dots m; j = 1 \dots d\}.$$

Next we prove (2.14), which we restate here as

PROPOSITION 4. The characteristic function $\phi_m(t)$ of the quadratic form $Q^{(m)} = \int_D (W^*(y))^2 dy$ is

$$(4.8) \quad \phi_m(t) = \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1})^{-\frac{1}{2}},$$

where $\{\lambda_i; i = 1 \dots k = (m+1)^d\}$ is the set of eigenvalues of the matrix $\Lambda^{(m)}$ defined by the quadratic form $Q^{(m)}$.

PROOF. Order the $N(0,1)$ random variables Y as a vector

$$(4.9) \quad \underline{Y} = (Y_1, Y_2, \dots, Y_k),$$

where

$$\{Y_i; i = 1 \dots k = (m+1)^d\} = \{Y_{n_1 \dots n_d}; n_j = 0 \dots m; j = 1 \dots d\}.$$

Then

$$Q^{(m)} = \underline{Y}^t \Lambda^{(m)} \underline{Y},$$

where $\Lambda^{(m)}$ is the matrix of coefficients of $Q^{(m)}$ and is chosen to be a symmetric matrix, hence Hermitian.

Then by (4.6) and (4.7)

$$\phi_{\mathbf{m}}(t) = (2\pi)^{-k/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-iR} \prod_{i=1}^k dy_i,$$

where

$$\begin{aligned} R &= \sum_{i=1}^k y_i^2 - 2tQ^{(m)} \sqrt{-1} \\ &= \underline{Y}^t (\underline{I} - 2t \sqrt{-1} \underline{A}^{(m)}) \underline{Y}. \end{aligned}$$

Then, as we shall prove, the matrix

$$\underline{B}^{(m)} = \underline{I} - 2t \sqrt{-1} \underline{A}^{(m)}$$

can be expressed as a product

$$\underline{B}^{(m)} = \underline{J}^t \underline{J} \text{ for some matrix } \underline{J}.$$

Then

$$R = \underline{Y}^t \underline{B}^{(m)} \underline{Y} = (\underline{Y}^t \underline{J}^t) (\underline{J} \underline{Y}) = \underline{U}^t \underline{U} = \sum_{j=1}^k u_j^2,$$

where the vector $\underline{U}$ is thus

$$\underline{U} = \underline{J} \underline{Y}.$$

For the sake of the discussion here we assume that $\underline{J}$ is non-singular, so that $\underline{J}^{-1}$ exists. Then

$$\underline{Y} = \underline{J}^{-1} \underline{U},$$

and

$$\prod_{i=1}^k dy_i = ||J^{-1}|| \prod_{j=1}^k du_j = ||J||^{-1} \prod_{j=1}^k du_j,$$

with $||J||^{-1}$ as the Jacobian of the transformation. Hence

$$(4.10) \quad \phi_{\underline{m}}(t) = ||J||^{-1} (2\pi)^{-k/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\frac{1}{2} \underline{u}^T \underline{A} \underline{u}} \prod_{j=1}^k du_j \\ = ||J||^{-1}.$$

On the other hand the matrix $\underline{A}^{(m)}$ is defined, as after

(4.9), by

$$(4.11) \quad Q^{(m)} = \underline{Y}^T \underline{A}^{(m)} \underline{Y},$$

being, by choice, symmetric and real, thus Hermitian. Hence it has a set of eigenvalues

$$\{\lambda_i; i = 1, 2, \dots, k; k = (m+1)^d\},$$

and corresponding eigenvectors

$$(4.12) \quad \{\underline{E}_i; i = 1, 2, \dots, k; k = (m+1)^d\}.$$

To calculate the eigenvalues λ_i we solve the determinant equation

$$|\underline{A}^{(m)} - \lambda \underline{I}| = 0 \dots$$

and, as this is a polynomial in λ , we have

$$|\underline{A}^{(m)} - \lambda \underline{I}| = C \prod_{i=1}^k (\lambda - \lambda_i), \quad k \leq (m+1)^d.$$

Thus

$$|z - \lambda^{-1} z^{(n)}| = c_1 \prod_{i=1}^k (1 - \lambda^{-1} \lambda_i)$$

for some constant c_1 . But

$$|z^{(n)}| = |z - 2t \sqrt{-1} z^{(n)}|$$

so

$$|z^{(n)}| = c_1 \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1})$$

Also

$$z^{(n)} = z^t z$$

so

$$|z^{(n)}| = |z|^2$$

Thus

$$||z|| = c_1^{-1} \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1})^{-1}$$

But by (4.10)

$$\phi_n(z) = ||z||^{-1}$$

Hence

$$\phi_n(z) = c_1^{-1} \prod_{i=1}^k (1 - \lambda_i 2t \sqrt{-1})^{-1}$$

But $\phi_n(0) = 1$ for a characteristic function, so $c_1^{-1} = 1$, and (4.8) is proved.

Notice that we have also proved that

$$|\underline{J}| \neq 0,$$

and hence that there exists a matrix $\underline{J}$ such that $\underline{B}^{(m)} = \underline{J}^t \underline{J}$.

PROOF OF THEOREM 2. According to (2.17) we are to show here that the characteristic function of the rv $C_d = \int_d T^2(y) dy$ is

$$(4.13) \quad \phi(t) = \left\{ \prod_{j=1}^m (1 - \lambda_j u) \sum_{j=1}^m \frac{v_j^2}{1 - \lambda_j u} \right\}^{-1/2}, \quad u = 2t \sqrt{-1},$$

where λ_j are the eigenvalues of the quadratic form Q and the v_j are the elements of the vector

$$(4.14) \quad \underline{v} = (v_1, v_2, \dots) = \lim_{m \rightarrow \infty} \underline{E}^{(m)} \underline{S}^{(m)},$$

where, with the $\underline{E}_1$ as in (4.12),

$$(4.15) \quad \underline{E}^{(m)} = (\underline{E}_1, \dots, \underline{E}_k),$$

and

$$(4.16) \quad \underline{S}^{(m)} = (s_1, s_2, \dots, s_k)^t$$

where

$$(4.17) \quad \{s_i, i = 1, 2, \dots, k\} = \{s_{n_1 \dots n_d}, n_j = 0, 1, 2, \dots, m (j = 1, 2, \dots, d)\}$$

are associated with the rv $Y_{n_1 \dots n_d}$ and are ordered in the same sequence as was used to form the vector $\underline{Y}$ of (4.9). As we proceed with the proof of (4.13) we will also gain expression for the s_i , and thus also for the v_i .

The form given for ϕ of (4.13) is that of Lemma B (cf. (2.15)). Hence, for the sake of deriving the former from the latter, we first must show that $Q^{(m)}$ of (4.11) can be written as

$$(4.18) \quad Q^{(m)} = \sum_{i=1}^k \lambda_i x_i^2, \quad k = (m+1)^d,$$

where the λ_i are still the eigenvalues of the matrix $A^{(m)}$ defined by the quadratic form $Q^{(m)} = \underline{Y}^t A^{(m)} \underline{Y}$, and where the x_i are independent $N(0,1)$ rv. i.e., in our case the x_i of Lemma B are equal to the λ_i . Secondly, we must evaluate the s_1 , given our condition that $W^*(1, \dots, 1) = 0$ for the distribution of Q of (2.6). The latter will result in (4.14) and

$$(4.19) \quad \sum v_i x_i = 0,$$

$$(4.20) \quad \sum v_i^2 = 1,$$

and hence, (4.18), (4.19), (4.20) combined will give (4.13).

As to the proof of (4.18), we start with the quadratic form $Q^{(m)}$ where (cf. (4.3))

$$\lim_{m \rightarrow \infty} Q^{(m)} \stackrel{a.s.}{=} Q = \int_{\mathbb{R}^d} (W^*(y))^2 dy$$

and we define again the real symmetric matrix $A^{(m)}$ by $Q^{(m)} = \underline{Y}^t A^{(m)} \underline{Y}$ (cf. (4.9)). Since Q is non-negative definite and $A^{(m)}$ is non-singular the set of eigenvalues $\{\lambda_i, i = 1, \dots, k\}$ of the matrix $A^{(m)}$ are positive.

Corresponding to the $\{\lambda_i\}$ we have a set of eigenvectors $\{\underline{E}_i ; i = 1, \dots, k\}$ which are orthonormal:

$$\underline{E}_i^t \underline{E}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

and for the matrix

$$\underline{E}^{(m)} = (\underline{E}_1, \underline{E}_2, \dots, \underline{E}_k)$$

we have

$$\underline{E}^{(m)t} \underline{E}^{(m)} = \underline{I}.$$

Because the $\underline{E}_i$ are eigenvectors of $\underline{A}^{(m)}$, we have

$$\underline{A}^{(m)} \underline{E}_1 = \lambda_1 \underline{E}_1 = \underline{E}_1 \lambda_1,$$

thus

$$\underline{A}^{(m)} \underline{E}^{(m)} = \underline{E}^{(m)} \underline{\Lambda},$$

where $\underline{\Lambda}$ is the diagonal matrix of eigenvalues

$$\underline{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_k \end{pmatrix}, \quad k = (m+1)^d - m.$$

Whence

$$\begin{aligned}
 Q^{(m)} &= \underline{Y}^t \underline{\Lambda}^{(m)} \underline{Y} \\
 &= \underline{Y}^t \underline{I} \underline{\Lambda}^{(m)} \underline{I} \underline{Y} \\
 &= \underline{Y}^t \underline{E}^{(m)} \underline{E}^{(m)t} \underline{\Lambda}^{(m)} \underline{E}^{(m)} \underline{E}^{(m)t} \underline{Y} \\
 &= \underline{Y}^t \underline{E}^{(m)} \underline{E}^{(m)t} \underline{E}^{(m)} \underline{\Lambda} \underline{E}^{(m)t} \underline{Y} \\
 &= (\underline{Y}^t \underline{E}^{(m)}) \underline{\Lambda} (\underline{E}^{(m)t} \underline{Y})
 \end{aligned}$$

We define the vector $\underline{X} = \underline{E}^{(m)t} \underline{Y}$ and consider the determinant $|\underline{E}^{(m)}|$ of $\underline{E}^{(m)}$. We have

$$\underline{E}^{(m)t} \underline{E}^{(m)} = \underline{I}$$

and

$$|\underline{E}^{(m)t}| \cdot |\underline{E}^{(m)}| = 1.$$

Thus

$$|\underline{E}^{(m)t}|^2 = 1 \quad \text{and} \quad ||\underline{E}^{(m)t}|| = 1,$$

i.e., the Jacobian of the transformation of variables $Y_1 \rightarrow X_1$ is 1,

and hence the X_1 of $\underline{X} = \{X_1, X_2, \dots, X_k\}$ are independent $N(0,1)$

rv and

$$\begin{aligned}
 Q^{(m)} &= \underline{X}^t \underline{\Lambda} \underline{X} \\
 &= \sum_{i=1}^k \lambda_i X_i^2.
 \end{aligned}$$

as required by (4.18).

Q

Towards a proof of (4.19) and (4.20), we first recall (cf. (2.6)) that the distribution of the rv $C_d = \int_I T^2(y) dy$ is the same as that of

$$Q = \int_I (W^*(y))^2 dy$$

conditional on $W^*(1,1,\dots,1) = 0$. By (2.4) the latter requirement means

$$W^*(1,1,\dots,1) = (1-d)W(1,1,\dots,1), \quad d \geq 2,$$

and hence our condition is

$$W(1,1,\dots,1) = 0.$$

By (2.8) and recalling again that

$$Y_{n_1 \dots n_d} = \prod_{i=1}^d (-1)^{n_i} Z_{n_1 \dots n_d},$$

our condition is

$$2^{d/2} \pi^{-d} \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \prod_{i=1}^d (n_i + \frac{1}{2})^{-1} Y_{n_1 \dots n_d} = 0.$$

Write this as

$$(4.21) \quad \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} s_{n_1 \dots n_d} Y_{n_1 \dots n_d} = 0,$$

with

$$s_{n_1 \dots n_d} = 2^{d/2} \pi^{-d} \prod_{i=1}^d (n_i + \frac{1}{2})^{-1}.$$

We must check that $\sum s_i^2 = 1$, i.e., that

$$S = \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} s_{n_1 \dots n_d}^2 = 1,$$

where

$$S = 2^d \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \prod_{i=1}^d (n_i + \frac{1}{2})^{-2} = 1,$$

and

$$\sum_{n=0}^{\infty} (n + \frac{1}{2})^{-2} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} (2n+1)^{-2} = \frac{8}{\pi^2} \cdot \frac{\pi^2}{8} = 1$$

(cf. e.g., Abramowitz and Stegun (1965) 23.2.28). Thus $\sum s_i^2 = 1$ as required.

Corresponding to each $Y_{n_1 \dots n_d}$ we have a coefficient $s_{n_1 \dots n_d}$. We have defined the vector $\underline{Y}$, ordering the former elements in some convenient sequence

$$\underline{Y} = (Y_1, Y_2, \dots, Y_k)^T, \quad k = (m+1)^d,$$

where

$$(Y_1, Y_2, \dots, Y_k) = (Y_{n_1 \dots n_d}; n_j = 0, 1, 2, \dots, m; j = 1, \dots, d).$$

Now we arrange the $s_{n_1 \dots n_d}$ in the same arbitrary sequence to form a vector $\underline{s}$,

$$\underline{s} = (s_1, s_2, \dots, s_k)^T.$$

and starting out from (4.21) we get with $k = (m+1)^d$

$$\begin{aligned} 0 &= \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} S_{n_1 \dots n_d} Y_{n_1 \dots n_d} = \lim_{m \rightarrow \infty} \sum_{i=1}^k s_i Y_i \\ &= \lim_{m \rightarrow \infty} \underline{S}^t \underline{Y} = \lim_{m \rightarrow \infty} \underline{S}^t \underline{I} \underline{Y} = \lim_{m \rightarrow \infty} \underline{S}^t \underline{E}^{(m)} \underline{E}^{(m)t} \underline{Y} \\ &= \lim_{m \rightarrow \infty} (\underline{S}^t \underline{E}^{(m)}) \underline{X} = \lim_{m \rightarrow \infty} \underline{V}^t \underline{X} = \lim_{m \rightarrow \infty} \sum_{i=1}^k v_i X_i, \end{aligned}$$

where we have defined the vector

$$\underline{V} = (v_1, v_2, \dots, v_k)^t$$

by the equation

$$\underline{V}^t = \underline{S}^t \underline{E}^{(m)},$$

or

$$\underline{V} = \underline{E}^{(m)t} \underline{S}.$$

(As with the λ_i , we are unable to obtain the elements of the matrix $\underline{E}^{(m)}$ in closed form, and hence the same holds also for the v_i).

Thus we have met the conditions of the Lemma 8 with

$$Q = \sum \lambda_i X_i^2$$

$$\sum v_i X_i = 0$$

and

$$\sum v_i^2 = 1.$$

since we have shown also that $\sum s_1^2 = 1$, i.e., that

$$\begin{aligned} 1 &= \underline{S}^t \underline{S} = \underline{S}^t \underline{I} \underline{S} = (\underline{S}^t \underline{E}^{(m)}) (\underline{E}^{(m)t} \underline{S}) \\ &= \underline{V}^t \underline{V} = \sum v_1^2 = 1 \text{ as required.} \end{aligned}$$

Hence Theorem 2 is proved.

REMARK 4.2. The result of Theorem 2 can be expressed as

$$\phi^{-2}(t) = \prod_{j=1}^k (1 - \lambda_j u) \sum_{j=1}^k \frac{v_j^2}{1 - \lambda_j u}, \quad k = (m+1)^d + \infty,$$

which is the Lagrange interpolation formula for a polynomial y which has the values y_j at the points $u = \lambda_j^{-1}$ with

$$y_j = v_j^2 \prod_{i \neq j} \left(1 - \frac{\lambda_i}{\lambda_j}\right).$$

This is helpful because it makes sense of what is otherwise a confusing formula. We had considered using this to evaluate the critical values of the distribution. We can infer that $\phi(t)$ has the values $y_j^{-1/4}$ at the points $u = \lambda_j^{-1}$ and that $\log \phi(t)$ has the values $\log y_j^{-1/4} = -1/4 \log y_j$ at these points, and then can use a similar interpolation formula, i.e.,

$$\log \phi(t) = \prod_{j=1}^k (1 - \lambda_j u) \sum_{j=1}^k (1 - \lambda_j u)^{-1} \log(v_j^2 \prod_{i \neq j} (1 - \frac{\lambda_i}{\lambda_j}))^{-1/4} \prod_{i \neq j} (1 - \frac{\lambda_i}{\lambda_j})^{-1}.$$

However the method of our Computational Note of Section 3 seems more convenient.

PROOF OF COROLLARY 1. We have

$$\kappa_n = 2^n \frac{d^n}{du^n} \log \phi(t) \Big|_{u=0}, \quad u = 2t \sqrt{-1}$$

and note that since $\phi(0) = 1$, $\log \phi(0) = 0$. Taking the logarithms of the formula for $\phi(t)$ of (4.13) we get

$$-2 \log \phi(t) = \sum_{j=1}^{\infty} \log(1 - \lambda_j u) + \log \sum_{j=1}^{\infty} \frac{v_j^2}{1 - \lambda_j u}.$$

We let

$$Q(t) = \log \phi(t), \quad X(t) = - \sum_{j=1}^{\infty} \log(1 - \lambda_j u),$$

$$Y(t) = \log \sum_{j=1}^{\infty} \frac{v_j^2}{1 - \lambda_j u}, \quad Z(t) = \sum_{j=1}^{\infty} \frac{v_j^2}{1 - \lambda_j u}.$$

Then

$$2Q(t) = X(t) - Y(t), \quad Y(t) = \log Z(t).$$

Let also

$$Q_n(t) = \frac{d^n}{du^n} Q(t), \quad Y_n(t) = \frac{d^n}{du^n} Y(t),$$

$$X_n(t) = \frac{d^n}{du^n} X(t), \quad Z_n(t) = \frac{d^n}{du^n} Z(t).$$

Then

$$2Q_n(t) = X_n(t) - Y_n(t)$$

and

$$X_1(t) = \sum_{j=1}^{\infty} \frac{\lambda_j}{1 - \lambda_j u}.$$

$$x_2(t) = \sum_{j=1}^{\infty} \frac{\lambda_j^2}{(1 - \lambda_j u)^2},$$

$$\vdots$$

$$x_n(t) = (n-1)! \sum_{j=1}^{\infty} \frac{\lambda_j^n}{(1 - \lambda_j u)^n}.$$

Similarly,

$$z_1(t) = \sum_{j=1}^{\infty} \frac{v_j^2 \lambda_j}{(1 - \lambda_j u)^2},$$

$$z_2(t) = 2 \sum_{j=1}^{\infty} \frac{v_j^2 \lambda_j^2}{(1 - \lambda_j u)^3},$$

$$\vdots$$

$$z_n(t) = n! \sum_{j=1}^{\infty} \frac{v_j^2 \lambda_j^n}{(1 - \lambda_j u)^{n+1}}.$$

where $|u|$ is taken to be so small that these series converge. Next consider

$$Y(t) = \log Z(t).$$

Then by differentiation

$$Y_1(t)Z(t) = z_1(t),$$

$$Y_2(t)Z(t) + Y_1(t)z_1(t) = z_2(t)$$

$\vdots$

$$\sum_{r=0}^n \binom{n}{r} Y_{n+1-r}(t) z_r(t) = z_{n+1}(t).$$

which, in turn, gives us

$$Y_{n+1}(t) = (Z_{n+1}(t) - \sum_{r=1}^n \binom{n}{r} Y_{n+1-r}(t) Z_r(t)) / Z(t).$$

We have

$$\begin{aligned} K_n &= 2^n \frac{d^n}{du^n} \log \phi(t) \Big|_{u=0} \\ &= 2^n Q_n(0) \\ &= 2^{n+1} (X_n(0) - Y_n(0)). \end{aligned}$$

Now write

$$Q_n(0) = Q_n, \quad X_n(0) = X_n, \quad Y_n(0) = Y_n, \quad Z_n(0) = Z_n.$$

Then

$$\begin{aligned} K_n &= 2^{n+1} (X_n - Y_n), \\ X_n &= (n-1)! \sum_{j=1}^n \lambda_j^n, \\ Y_{n+1} &= Z_{n+1} - \sum_{r=1}^n \binom{n}{r} Y_{n+1-r} Z_r. \end{aligned}$$

since

$$Z(0) = \sum_{j=1}^{\infty} v_j^2 = 1$$

as proved above and

$$Z_n = n! \sum_{j=1}^{\infty} \frac{v_j^2 \lambda_j^n}{j!}.$$

This also completes the proof of Corollary 1.

PROOF OF COROLLARY 2. In a formal way the moments M_n of the rv C_d may be expressed as

$$\phi(t) = \sum_{n=1}^{\infty} M_n \frac{(t \sqrt{-1})^n}{n!},$$

and

$$M_n = 2^n \frac{d^n}{du^n} \phi(t) \Big|_{u=0}, \quad u = 2t \sqrt{-1}.$$

We let

$$P(t) = \phi(t), \quad \text{with } P(0) = 1,$$

$$P_n(t) = \frac{d^n}{du^n} \phi(t)$$

and

$$P_n(0) = P_n.$$

Then

$$M_n = 2^n P_n.$$

But

$$\log \phi(t) = Q(t),$$

i.e.,

$$Q(t) = \log P(t).$$

Differentiating,

$$Q_1(t)P(t) = P_1(t)$$

$$Q_2(t)P(t) + Q_1(t)P_1(t) = P_2(t)$$

$$\vdots$$

$$P_{n+1}(t) = \sum_{r=0}^n \binom{n}{r} Q_{n+1-r}(t) P_r(t)$$

Thus

$$P_{n+1} = Q_{n+1} + \sum_{r=1}^n \binom{n}{r} Q_{n+1-r} P_r$$

as required, since

$$P(0) = \phi(t) \Big|_{t=0} = 1.$$

PROOF OF COROLLARY 3. By Corollary 1

$$K_1 = 4(X_1 - Y_1), \quad K_2 = 8(X_2 - Y_2),$$

$$Y_1 = X_1, \quad Y_2 = X_2 - X_1 X_1 = X_2 - X_1^2.$$

$$X_1 = \sum_{j=1}^{\infty} \lambda_j, \quad X_2 = \sum_{j=1}^{\infty} \lambda_j^2.$$

$$X_1 = \sum_{j=1}^{\infty} \sqrt{j} \lambda_j, \quad X_2 = 2 \sum_{j=1}^{\infty} \sqrt{j} \lambda_j^2.$$

Thus

$$r_1 = \sum_{j=1}^n v_j^2 \lambda_j.$$

$$r_2 = 2 \sum_{j=1}^n v_j^2 \lambda_j^2 - \left(\sum_{j=1}^n v_j^2 \lambda_j \right)^2.$$

and hence the result.

5. Calculation of the Critical Values of the Hoeffding,

Blum, Kiefer and Rosenblatt Independence Criterion.

In this section we calculate the cumulants of Γ_d (cf. (1.22)), the d -dimensional limiting distribution of the Hoeffding, Blum, Kiefer and Rosenblatt independence criterion, directly. To link these results together, for dimension $d = 2$ we have an interesting corollary to a theorem of Blum, Kiefer and Rosenblatt (1961).

THEOREM 3 (Blum, Kiefer and Rosenblatt 1961). The random variable $C_d = \int_d T^2(y) dy$ ($d = 2$) defined in (1.21) has the characteristic function

$$(5.1) \quad E e^{i t C_2} = \prod_{j,k=1}^{\infty} (1 - 2 i t / v_j^2 v_k^2)^{-1/2}.$$

COROLLARY 4. K_n , the n^{th} cumulant of the distribution Γ_2 of the random variable C_2 can be expressed as

$$(5.2) \quad K_n = 2^{2n-3} B_{2n}^2 (n-1)! ((2n)!)^{-2},$$

where B_{2n} is a Bernoulli number (cf. Abramowitz and Stegun 1964, 23. and Table 23.2). Also

$$(5.3) \quad K_n = 2^{2n-1} (n-1)! \sum_{k=1}^{\infty} \rho(2k) k^{-2n}.$$

where $\rho(2n) = \sum_{k=1}^{\infty} k^{-2n}$

(cf. Abramowitz and Stegun 1964, 23.2.18).

This second form of K_n in (5.3) is convenient for calculation since the $\rho(2n)$ converge rapidly.

PROOF. From Theorem B

$$\log \phi(z) = \log \sum_{j=1}^{\infty} \frac{1}{j^2} z^j$$

$$= -\frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \log(1 - 2iz/v^4 j^2 k^2)$$

Hence

$$\begin{aligned} k_n &= \frac{\partial^n}{\partial (iz)^n} \log \phi(z) \Big|_{z=0} \\ &= \frac{1}{2} 2^n (n-1)! \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (v^4 j^2 k^2)^{-n} \\ &= 2^{n-1} (n-1)! v^{-4n} \left(\sum_{k=1}^{\infty} k^{-2n} \right)^2, \end{aligned}$$

and the other form follows from Abramowitz and Stegun (1964), 23.2.16, which says

$$\rho(2n) = \frac{(2\pi)^{2n} |B_{2n}|}{2(2n)!}$$

COROLLARY 5. The first 6 cumulants of the limiting distribution
 Γ_d for dimension $d = 2$ are

$$K_1 = \text{mean } \mu = 1/36$$

$$K_2 = \text{Variance } \sigma^2 = 2(90)^{-2}$$

$$K_3, \text{ Skewness} = 8(945)^{-2}$$

$$K_4, \text{ Kurtosis} = 48(9450)^{-2}$$

$$K_5 = 32.12 (243.35.11)^{-2}$$

$$K_6 = 2^{27} 120 \left(\frac{691}{2730} \right)^2 (121)^{-2}$$

These values compare well with values one can calculate indirectly from the tabulated distribution function of Γ_2 , given in Blum, Kiefer and Rosenblatt (1961). Much of the discrepancy in the higher cumulants may be due to the inaccuracy inherent in calculating cumulants from such a table.

Having the values of the just indicated cumulants, we use the Cornish Fisher asymptotic expansion (cf. Abramowitz and Stegun 1964, 26.2.49 - 26.2.51) to calculate approximate critical values for Γ_2 . We tabulate here 3 such sets of values, using the first 4, first 5 and first 6 cumulants and compare them with the values given in Table 2 of Blum, Kiefer and Rosenblatt (1961), with each value divided by $\frac{\sqrt{4}}{2}$:

P	B.K.R. Values	Cornish-Fisher Critical Values		
		With First 4 Cumulants	With First 5 Cumulants	With First 6 Cumulants
.9	.04694	.04687	.04629	.04709
.95	.05840	.05957	.05717	.05927
.98	.07437	-	-	-
.99	.08685	.09187	.08315	.08882
.995	.09960	.10680	.09441	.10137
.998	.11662	-	-	-
.999	.12976	.14348	.12033	.12848
.9995	.14290	.16006	.13130	.13895
.9998	.16056	-	-	-
.9999	.17391	-	-	-

Compared with the figures of Blum, Kiefer and Rosenblatt, the Cornish-Fisher approximation seems to be quite good. So we may hope that it is equally good for higher dimensions, $d > 2$, where we have no such direct basis for comparison.

As mentioned already, Corollary 1 in Section 3 turned out to be less convenient for calculations than hoped for. While it is feasible (however tedious) to use the latter to calculate the cumulants K_n for any dimension $d \geq 2$, numerical calculations failed because the results were to be the small differences of pairs of two large numbers. Hence we prove here by direct calculations

THEOREM 3. The first five cumulants K_n of the limiting distribution of the statistic of Hoeffding, Blum, Kiefer and Rosenblatt, for dimension $d \geq 2$ can be expressed as.

$$(5.4) \quad K_n = (n-1)! 2^{n-1} M_{n,d} D_n^{-d},$$

where for $n = 1, 2, \dots, 5$ we have

$$D_1 = 6, \quad D_2 = 90, \quad D_3 = 945,$$

$$D_4 = 28350 = 2 \cdot 5 \cdot 5 \cdot 7 \cdot 9 \cdot 9, \quad D_5 = 467775 = 3 \cdot 25 \cdot 7 \cdot 9 \cdot 9 \cdot 11,$$

$$M_{1,d} = 3^d - (d+2)2^{d-1},$$

$$M_{2,d} = 15^d - 6(d+4)12^{d-1} + (4d^2 + 46d + 100) \cdot 10^{d-2}$$

$$M_{3,d} : \text{Table 1 and Proposition 22 below}^*$$

$$M_{4,d} : \text{Table 1 and Proposition 29 below}^*$$

$$M_{5,d} : \text{Table 1 and Proposition 38 below}^*.$$

These expressions become tedious. Using the methods of this section it is possible to calculate such cumulants K_n for any n and

TABLE 1. NUMERATORS OF CUMULANTS

Dimension d	$K_{1,d}$	$K_{2,d}$	$K_{3,d}$	$K_{4,d}$	$K_{5,d}$
2	1	1	1	9	25
3	7	67	559	124497	4748525
4	33	2481	135657	5.75632 (8)	2.30411 (11)
5	131	69631	2.20629 (7)	1.56635 (12)	5.76692 (15)
6	473	1.66071 (6)	2.82661 (9)	3.10254 (15)	9.70342 (19)
7	1611	3.55844 (7)	3.08952 (11)	4.97104 (18)	1.25217 (24)
8	5281	7.07000 (8)	3.01591 (13)	6.84697 (21)	1.33719 (28)
9	16867	1.32848 (10)	2.70650 (15)	8.42585 (24)	1.24012 (32)
10	52905	2.39229 (11)	2.27686 (17)	9.50859 (27)	1.03193 (36)
11	163835	4.16719 (12)	1.82059 (19)	1.00240 (31)	7.88520 (39)
12	502769	7.06975 (13)	1.39782 (21)	1.00069 (34)	5.62801 (43)
13	1.53288 (6)	1.17415 (15)	1.03847 (23)	9.55817 (36)	3.80086 (47)
14	4.65190 (6)	1.91657 (16)	7.50982 (24)	8.80501 (39)	2.45322 (51)
15	1.40704 (7)	3.08427 (17)	5.31131 (26)	7.87237 (42)	1.52526 (55)
16	4.24569 (7)	4.90557 (18)	3.68762 (28)	6.86585 (45)	9.19294 (58)
17	1.27895 (8)	7.72687 (19)	2.52110 (30)	5.86512 (48)	5.39873 (62)
18	3.84799 (8)	1.20728 (21)	1.70146 (32)	4.92391 (51)	3.10228 (66)
19	1.15676 (9)	1.87362 (22)	1.13589 (34)	4.07379 (54)	1.75036 (70)
20	3.47525 (9)	2.89141 (23)	7.51418 (35)	3.32922 (57)	9.72498 (73)

Note: By 3.47525 (9) we mean 3.47525×10^9 etc.

TABLE 2: Cumulants of Distribution Γ_d

Dimension d	$K_{1,d}$	$K_{2,d}$	$K_{3,d}$	$K_{4,d}$	$K_{5,d}$
2	2.77777 (-2)	2.46914 (-4)	8.95832 (-6)	5.37499 (-7)	4.38730 (-8)
3	3.24074 (-2)	1.63613 (-4)	5.29915 (-6)	2.62265 (-7)	1.78147 (-8)
4	2.54630 (-2)	7.56287 (-5)	1.36083 (-6)	4.27734 (-8)	1.84793 (-9)
5	1.68467 (-2)	2.35841 (-5)	2.34203 (-7)	4.10549 (-9)	9.88758 (-11)
6	1.01380 (-2)	6.24982 (-6)	3.17516 (-8)	2.86840 (-10)	3.55659 (-12)
7	5.75489 (-3)	1.48796 (-6)	3.67247 (-9)	1.62113 (-11)	9.81148 (-14)
8	3.14417 (-3)	3.28481 (-7)	3.79362 (-10)	7.87618 (-13)	2.23989 (-15)
9	1.67370 (-3)	6.85807 (-8)	3.60256 (-11)	3.41883 (-14)	4.44079 (-17)
10	8.74952 (-4)	1.37220 (-8)	3.20707 (-12)	1.36090 (-15)	7.89970 (-18)
11	4.51589 (-4)	2.65587 (-9)	2.71364 (-13)	5.06056 (-17)	1.29044 (-20)
12	2.30969 (-4)	5.00638 (-10)	2.20475 (-14)	1.78199 (-18)	1.96898 (-22)
13	1.17366 (-4)	9.23851 (-11)	1.73328 (-15)	6.00380 (-20)	2.84270 (-24)
14	5.93626 (-5)	1.67555 (-11)	1.32640 (-16)	1.95087 (-21)	3.92237 (-26)
15	2.99232 (-5)	2.99602 (-12)	9.92693 (-18)	6.15249 (-23)	5.21340 (-28)
16	1.50497 (-5)	5.29467 (-13)	7.29336 (-19)	1.89272 (-24)	6.71728 (-30)
17	7.55583 (-6)	9.26640 (-14)	5.27643 (-20)	5.70316 (-26)	8.43322 (-32)
18	3.78889 (-6)	1.60869 (-14)	3.76824 (-21)	1.68887 (-27)	1.03597 (-33)
19	1.89831 (-6)	2.77398 (-15)	2.66209 (-22)	4.92869 (-29)	1.24956 (-35)
20	9.50520 (-7)	4.75651 (-16)	1.86353 (-23)	1.42077 (-30)	1.48416 (-37)

Note: By 9.50520 (-7) we mean 9.50527×10^{-7} etc.

for any dimension d (the integrals are easy but tedious). We have found solutions for $n = 1, 2, \dots, 5$ and for any $d \geq 2$.

COROLLARY 6. From Theorem 3 and using the Cornish-Fisher asymptotic expansion (cf., e.g., Abramowitz and Stegun 1964, 26.2.49) with the first 5 cumulants, a table of critical values of T_d is computed and summarised in Table 3. (Refer also to Fisher and Cornish, 1960, and to Annex A.)

Table 4 compares critical values, using the first 4, 5 and 6 cumulants for $d = 2$, and for $d = 3, \dots, 20$, using the first 4 and 5 cumulants.

PROOF OF THEOREM 3. Here we present a sequence of propositions, amounting to a proof of Theorem 3. When proofs for these propositions themselves are required we relegate the details to Section 6 below.

PROPOSITION 5. For the random variable C_d of (1.21) we have

$$EC_d^n = \int_{I_d} \dots \int_{I_d} E(T^2(y^{(1)}) \dots T^2(y^{(n)})) dy^{(1)} \dots dy^{(n)},$$

where $dy^{(k)} = dy_1^{(k)} \dots dy_d^{(k)}$.

PROPOSITION 6. We have

$$E(T^2(y^{(1)}) \dots T^2(y^{(n)})) = 2^n a_n,$$

where a_n is the coefficient of $(-1)^n t_1^2 \dots t_n^2$ in the series expansion of $\phi_n(t)$, the characteristic function of the joint distribution of the Gaussian random variables $T(y^{(1)}), \dots, T(y^{(n)})$.

TABLE 1: RECOMMENDED LIMITING CRITICAL VALUES FOR 4-DIMENSIONAL STATISTIC OF HOEFFDING, BLUM, KIEFER AND ROSENBLATT

Dimension d	Probability of Exceeding Critical Value									
	.25	.10	.05	.025	.010	.005	.0025	.0010	.0005	
2	.03259	.04629	.05717	.06828	.08315	.09441	.10564	.12033	.13130	
3	.03697	.04833	.05737	.06668	.07929	.08828	.09877	.11160	.12169	
4	.02846	.03565	.04137	.04729	.05532	.06152	.06780	.07618	.08257	
5	.01853	.02252	.02570	.02899	.03346	.03691	.04041	.04508	.04864	
6	.01101	.01305	.01469	.01637	.01867	.02044	.02223	.02463	.02645	
7	.006180	.007176	.007971	.008793	.009910	.01077	.01164	.01281	.01369	
8	.003344	.003811	.004184	.004570	.005093	.005497	.005906	.006451	.006866	
9	.001765	.001978	.002148	.002324	.002563	.002747	.002934	.003182	.003371	
10	.0009161	.001011	.001087	.001165	.001272	.001354	.001438	.001549	.001633	
11	.0004698	.0005114	.0005447	.0005792	.0006261	.0006622	.0006988	.0007477	.0007849	
12	.0002389	.0002569	.0002713	.0002863	.0003066	.0003223	.0003382	.0003594	.0003756	
13	.0001208	.0001285	.0001347	.0001414	.0001498	.0001565	.0001634	.0001725	.0001795	
14	6.083(-5)	6.410(-5)	6.672(-5)	6.944(-5)	7.315(-5)	7.602(-5)	7.894(-5)	8.285(-5)	8.583(-5)	
15	3.055(-5)	3.193(-5)	3.303(-5)	3.418(-5)	3.574(-5)	3.696(-5)	3.820(-5)	3.986(-5)	4.113(-5)	
16	1.531(-5)	1.589(-5)	1.635(-5)	1.683(-5)	1.749(-5)	1.800(-5)	1.852(-5)	1.923(-5)	1.976(-5)	
17	7.668(-6)	7.907(-6)	8.100(-6)	8.300(-6)	8.575(-6)	8.790(-6)	9.009(-6)	9.304(-6)	9.531(-6)	
18	3.836(-6)	3.935(-6)	4.015(-6)	4.098(-6)	4.213(-6)	4.302(-6)	4.394(-6)	4.518(-6)	4.613(-6)	
19	1.918(-6)	1.959(-6)	1.992(-6)	2.026(-6)	2.074(-6)	2.111(-6)	2.149(-6)	2.201(-6)	2.241(-6)	
20	9.588(-7)	9.756(-7)	9.892(-7)	1.0034(-6)	1.0231(-6)	1.0385(-6)	1.0544(-6)	1.0760(-6)	1.0928(-6)	

Note: by 6.083(-5) we mean 6.083×10^{-5} , etc.

TABLE 4: LIMITING CRITICAL VALUES FOR d-DIMENSIONAL STATISTIC OF HOEFFDING, BLUM, KIEFER AND ROSENBLATT

LEVELS OF APPROXIMATION USING THE FIRST 4 AND THE FIRST 5 CUMULANTS

DIM d	Highest Cumulant	Probability of Exceeding Critical Value								
		.25	.10	.05	.025	.010	.005	.0025	.0010	.0005
2	4	.03159	.04687	.05957	.07304	.09187	.10680	.12227	.14348	.16006
	5	.03259	.04629	.05717	.06828	.08315	.09441	.10564	.12033	.13130
	6	.03238	.04709	.05927	.07193	.08882	.10137	.11347	.12848	.13895
3	4	.03610	.04898	.05963	.07091	.08668	.09919	.11215	.12994	.14386
	5	.03697	.04833	.05737	.06668	.07929	.08898	.09877	.11180	.12169
	6	.02790	.03609	.04287	.05006	.06012	.06810	.07639	.08776	.09667
4	5	.02846	.03565	.04137	.04729	.05312	.06152	.06780	.07618	.08257
	6	.01821	.02277	.02655	.03056	.03618	.04065	.04529	.05166	.05665
	5	.01853	.02252	.02570	.02899	.03346	.03691	.04041	.04508	.04864
5	4	.01084	.01319	.01513	.01720	.02010	.02240	.02480	.02809	.03067
	5	.01101	.01305	.01469	.01637	.01867	.02044	.02223	.02463	.02645
	6	.006097	.007240	.008189	.009199	.010616	.011744	.012917	.014530	.015794
6	5	.006180	.007176	.007971	.008793	.009910	.010772	.011645	.012808	.013695
	6	.003305	.003841	.004287	.004762	.005429	.005960	.006512	.007272	.007869
	7	.003344	.003811	.004184	.004570	.005093	.005497	.005906	.006451	.006866
7	4	.001747	.001992	.002196	.002413	.002718	.002961	.003214	.003562	.003835
	5	.001765	.001978	.002148	.002324	.002563	.002747	.002934	.003182	.003371
	6	.0009079	.0010172	.0011082	.0012053	.0013418	.0014506	.0015637	.0017196	.0018420
10	5	.0009161	.0010110	.0010869	.0011654	.0012721	.0013544	.0014177	.0015487	.0016332

TABLE 4: LIMITING CRITICAL VALUES FOR d-DIMENSIONAL STATISTICS OF
HOEFFDING, BLUM, KIEFER AND ROSENBLATT

LEVELS OF APPROXIMATION USING THE FIRST 4 AND THE FIRST 5 CUMULANTS

DIM d	Highest Cumulant	Probability of Exceeding Critical Value								
		.25	.10	.05	.025	.010	.005	.0025	.0010	.0005
11	4	.0004661	.0005141	.0005541	.0005968	.0006568	.0007047	.0007545	.0008231	.0008769
	5	.0004698	.0005114	.0005447	.0005792	.0006261	.0006622	.0006988	.0007477	.0007849
12	4	.0002373	.0002581	.0002754	.0002939	.0003200	.0003407	.0003623	.0003921	.0004155
	5	.0002389	.0002569	.0002713	.0002863	.0003066	.0003223	.0003382	.0003594	.0003756
13	4	.00012011	.00012903	.00013646	.00014439	.00015555	.00016445	.00017372	.00018650	.00019653
	5	.00012079	.00012850	.00013467	.00014107	.00014980	.00015654	.00016338	.00017252	.00017949
14	4	6.054(-5)	6.433(-5)	6.748(-5)	7.085(-5)	7.559(-5)	7.938(-5)	8.332(-5)	8.875(-5)	9.302(-5)
	5	6.083(-5)	6.410(-5)	6.672(-5)	6.944(-5)	7.315(-5)	7.602(-5)	7.894(-5)	8.285(-5)	8.583(-5)
15	4	3.043(-5)	3.203(-5)	3.336(-5)	3.478(-5)	3.678(-5)	3.837(-5)	4.003(-5)	4.232(-5)	4.412(-5)
	5	3.055(-5)	3.193(-5)	3.303(-5)	3.418(-5)	3.574(-5)	3.696(-5)	3.820(-5)	3.986(-5)	4.113(-5)
16	4	1.5264(-5)	1.5933(-5)	1.6490(-5)	1.7085(-5)	1.7923(-5)	1.8591(-5)	1.9388(-5)	2.0248(-5)	2.1003(-5)
	5	1.5315(-5)	1.5891(-5)	1.6352(-5)	1.6833(-5)	1.7491(-5)	1.8003(-5)	1.8525(-5)	1.9226(-5)	1.9763(-5)
17	4	7.646(-6)	7.925(-6)	8.158(-6)	8.406(-6)	8.755(-6)	9.033(-6)	9.324(-6)	9.724(-6)	1.0039(-5)
	5	7.668(-6)	7.907(-6)	8.100(-6)	8.300(-6)	8.575(-6)	8.790(-6)	9.009(-6)	9.304(-6)	9.531(-6)
18	4	3.827(-6)	3.943(-6)	4.039(-6)	4.142(-6)	4.287(-6)	4.403(-6)	4.523(-6)	4.689(-6)	4.820(-6)
	5	3.836(-6)	3.935(-6)	4.015(-6)	4.098(-6)	4.213(-6)	4.302(-6)	4.394(-6)	4.518(-6)	4.613(-6)
19	4	1.9144(-6)	1.9623(-6)	2.0021(-6)	2.0446(-6)	2.1045(-6)	2.1522(-6)	2.2020(-6)	2.2707(-6)	2.3247(-6)
	5	1.9180(-6)	1.9590(-6)	1.9920(-6)	2.0264(-6)	2.0740(-6)	2.1112(-6)	2.1494(-6)	2.2012(-6)	2.2412(-6)
20	4	9.573(-7)	9.770(-7)	9.934(-7)	1.0110(-6)	1.0356(-6)	1.0553(-6)	1.0758(-6)	1.1041(-6)	1.1263(-6)
	5	9.588(-7)	9.756(-7)	9.892(-7)	1.0034(-6)	1.0231(-6)	1.0385(-6)	1.0544(-6)	1.0760(-6)	1.0928(-6)

Notes: (1) by 6.054 (-5) we mean 6.054×10^{-5} , etc.

(2) note the asymptotic convergence for dimension 2. The values for the 5th cumulant termination seem best.

Notation

1. For compactness we will write (cf. (1.10))

$$\begin{aligned}\sigma_{ij} &= \sigma(y^{(i)}, y^{(j)}) = E T(y^{(i)}) T(y^{(j)}) \\ &= \prod_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) + (d-1) \prod_{r=1}^d y_r^{(i)} y_r^{(j)} - \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)}.\end{aligned}$$

For the special case of $d = 2$,

$$\sigma_{ij} = ((y_1^{(i)} \wedge y_1^{(j)}) - y_1^{(i)} y_1^{(j)}) (y_2^{(i)} \wedge y_2^{(j)}) - y_2^{(i)} y_2^{(j)})$$

2. Also for compactness, whenever we can, we will replace the dummy variables $y_r^{(i)}$ by such symbols as w, x, y, z .

PROPOSITION 7. For the quantity $E(T^2(Y^{(1)})) \dots T^2(Y^{(n)}) = 2^n a_n$, a_n is the coefficient of $s_1^2 \dots s_n^2$ in the product $\phi_n(s)$, defined by

$$\phi_n(s) = \prod_{i=1}^n (1 + \sum_{j=1}^n \sigma_{ij} s_j^2) \prod_{i=1}^{n-1} \prod_{j=i+1}^n (1 + \sigma_{ij} s_i s_j + \frac{1}{2} \sigma_{ij}^2 s_i^2 s_j^2)$$

PROOF. By writing $s_i = \sqrt{-1} t_i$ in the characteristic function $\phi_n(t)$, and by ignoring higher powers of t_i in the expansion of $\phi_n(t)$. For details we refer to Section 6 below.

PROPOSITION 8. The first 4 cumulants of the distribution of C_d are

$$K_1 = \text{mean } \mu = \int_{\mathcal{A}} \sigma_{11} \lambda_1 d\gamma$$

$$K_2 = \text{variance } \sigma^2 = 2 \int_{\mathcal{A}} \sigma_{12}^2 \lambda_2 d\gamma$$

$$K_3 = \mu_3 = 8 \int_I \sigma_{12} \sigma_{23} \sigma_{31} \lambda_3 dy$$

$$K_4 = 3.2^4 \int_I \sigma_{12} \sigma_{23} \sigma_{34} \sigma_{41} \lambda_4 dy$$

where for compactness we use the symbol $\lambda_n dy$ to denote the measure of integration over I^{nd} . Thus

$$\lambda_n dy = dy_1^{(1)} \dots dy_d^{(1)} \dots dy_1^{(n)} \dots dy_d^{(n)}$$

Details of proof in Section 6 below.

PROPOSITION 9. The n^{th} cumulant K_n of the distribution of C_d is given by

$$K_n = (n-1)! 2^{n-1} \int_I \sigma_{12} \sigma_{23} \dots \sigma_{n-1n} \lambda_n dy$$

where σ_{ij} and $\lambda_n dy$ are defined above.

Details of proof in Section 6 below.

The factor $(n-1)!$ in K_n of Proposition 9 above is the number of n cycles in the set of all products of n symbols

$$\sigma_{ij} \quad (i, j) \in \{1, 2, \dots, n\}$$

It is the number of ways in which the n integers $\{1, 2, \dots, n\}$ can be placed on the circumference of a circle, disregarding the distinction between clockwise and anti-clockwise. Thus there is $1 + \frac{1}{2} \cdot 2$ way of placing 3 integers, 3 locations for the fourth integer, and for each of these, 4 locations for the fifth integer, and so on.

PROPOSITION 10. The first cumulant K_1 or mean μ of the distribution Γ_d is

$$K_1 = \mu = 2^{-d} - \left(\frac{d}{2} + 1\right) 3^{-d} + 6^{-d} (3^d - (d+2) 2^{d-1}).$$

PROOF. By Proposition 8

$$K_1 = \int_1^d \sigma_{11} \lambda_1 dy.$$

and by (1.10)

$$\sigma_{11} = \pi^2(y) = \sum_{i=1}^d y_i + (d-1), \quad \sum_{i=1}^d y_i^2 = \sum_{i=1}^d y_i \sum_{j \neq i} y_j^2.$$

Thus

$$K_1 = \left(\frac{1}{2}\right)^d + (d-1) \left(\frac{1}{3}\right)^d + d \frac{1}{2} \left(\frac{1}{3}\right)^{d-1}.$$

For further details we refer to Section 6.

PROPOSITION 11.

$$\sigma_2^2 = \int_1^d \sum_{i=1}^d \left(\sum_{j=1}^{(1)} y_j^{(1)} y_j^{(2)} \right)^2 \lambda_2 dy = 3^{-2d}.$$

PROOF.

$$\sigma_2^2 = \left(\int_0^1 x^2 dx \right)^{2d} = 3^{-2d}.$$

PROPOSITION 12.

$$\sigma_2 = \int_0^1 \int_0^1 (x + y)^2 dx dy = \frac{1}{3}.$$

PROPOSITION 13.

$$\sigma_2 = \int_0^1 \int_0^1 (x + y)^2 dx dy = \frac{1}{3}.$$

PROPOSITION 14. The second moment K_2 of the variance σ^2 of the distribution Γ_4 is

$$K_2 = 2(100)^{-4} (15^4 - 6400112^{4-1} + (45^2 + 644 + 100)15^{4-2}).$$

PROOF. By Proposition 9

$$K_2 = 2 \int_{\Gamma} \sigma_{12}^2 \lambda_2 dV.$$

where $\sigma_{12} = \text{cov}(Y^{(1)}, Y^{(2)})$ is given by (1.10).

Then σ_{12}^2 is a sum of 9 terms, not all distinct, so that K_2 is the sum of 9 integrals each of which is a product of the elementary integrals a_1, a_2, a_3 . For details we refer to Section 6.

PROPOSITION 15.

$$\begin{aligned} J_1(u, v) &= \int_0^1 (x+u)(x+v) dx, \quad 0 \leq u \leq v \leq 1, \\ &= \frac{1}{6}(3uv - 3uv^2 - u^3). \end{aligned}$$

PROPOSITION 16.

$$J_2(v) = \int_0^1 x(x+v) dx = \frac{1}{6}(3v - v^3).$$

PROPOSITION 17.

$$J_3 = \int_0^1 \int_0^1 \int_0^1 (x+y)(y+z)(z+x) dx dy dz = \frac{1}{15}.$$

PROPOSITION 18.

$$S_3 = \int_0^1 \int_0^1 \int_0^1 (x+y)(y+z)xyz \, dx \, dy \, dz$$

$$= \int_0^1 J_2^2(y) dy = \frac{17}{115}$$

PROPOSITION 19.

$$S_4 = \int_0^1 \int_0^1 \int_0^1 xy^2z(x+z) \, dx \, dy \, dz = \frac{2}{45}$$

PROPOSITION 20.

$$S_2 = \int_0^1 \int_0^1 \int_0^1 x^2 y^2 z^2 \, dx \, dy \, dz = 3^{-3}$$

Next we have to integrate a number of terms that are products of the terms

$$C_{ij} = \sum_{r=1}^6 (y_r^{(i)} \wedge y_r^{(j)}) \wedge y_s^{(i)} \wedge y_s^{(j)}$$

Define the equivalence $f(y) \sim g(y)$ by

$$(3.5) \quad \int_{\Delta} f(y) h(y) \lambda_{\Delta} dy = \int_{\Delta} g(y) h(y) \lambda_{\Delta} dy$$

for the functions $h(y)$ encountered in these integrals, which themselves (i.e., the functions $h(\cdot)$) are of the form xy , and $x \wedge y$ and products of these.

PROPOSITION 21. In terms of the equivalence relation of (5.5)

$$-C_{ij} = \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)}$$

$$= d(y_1^{(i)} \wedge y_1^{(j)}) \prod_{s=2}^d y_s^{(i)} y_s^{(j)}$$

$$C_{ij} C_{km} = d(y_1^{(i)} \wedge y_1^{(j)}) (y_1^{(k)} \wedge y_1^{(m)}) \prod_{r=2}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)}$$

$$+ d(d-1) ((y_1^{(i)} \wedge y_1^{(j)}) y_1^{(k)} y_1^{(m)}) (y_2^{(i)} y_2^{(j)} y_2^{(k)} \wedge y_2^{(m)}) \prod_{r=3}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)}$$

$$-C_{ij} C_{km} C_{np} = d((y_1^{(i)} \wedge y_1^{(j)}) (y_1^{(k)} \wedge y_1^{(m)}) (y_1^{(n)} \wedge y_1^{(p)})) \prod_{r=2}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} y_r^{(n)} y_r^{(p)}$$

$$+ d(d-1) ((y_1^{(i)} \wedge y_1^{(j)}) (y_1^{(k)} \wedge y_1^{(m)}) y_1^{(n)} y_1^{(p)})$$

$$(y_2^{(i)} y_2^{(j)} y_2^{(k)} y_2^{(m)} y_2^{(n)} \wedge y_2^{(p)}) \prod_{r=3}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} y_r^{(n)} y_r^{(p)}$$

$$+ d(d-1) ((y_1^{(i)} \wedge y_1^{(j)}) y_1^{(k)} y_1^{(m)} (y_1^{(n)} \wedge y_1^{(p)}))$$

$$(y_2^{(i)} y_2^{(j)} (y_2^{(k)} \wedge y_2^{(m)}) y_2^{(n)} y_2^{(p)}) \prod_{r=3}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} y_r^{(n)} y_r^{(p)}$$

$$+ d(d-1) ((y_1^{(i)} \wedge y_1^{(j)}) y_1^{(k)} y_1^{(m)} y_1^{(n)} y_1^{(p)})$$

$$(y_2^{(i)} y_2^{(j)} (y_2^{(k)} \wedge y_2^{(m)}) (y_2^{(n)} \wedge y_2^{(p)})) \prod_{r=3}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} y_r^{(n)} y_r^{(p)}$$

$$+ d(d-1)(d-2) ((y_1^{(i)} \wedge y_1^{(j)}) y_1^{(k)} y_1^{(m)} y_1^{(n)} y_1^{(p)})$$

$$(y_2^{(i)} y_2^{(j)} (y_2^{(k)} \wedge y_2^{(m)}) y_2^{(n)} y_2^{(p)}) (y_3^{(i)} y_3^{(j)} y_3^{(k)} y_3^{(m)} (y_3^{(n)} \wedge y_3^{(p)}))$$

$$\prod_{r=4}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} y_r^{(n)} y_r^{(p)}$$

Similar equivalence relations hold for

$$C_{ij} C_{km} C_{np} C_{rs}$$

and

$$C_{ij} C_{jk} C_{km} C_{mn} C_{ni}$$

PROOF. Elementary rearrangement of integrals.

PROPOSITION 22. The third cumulant K_3 of the distribution

Γ_d is

$$K_3 = 3H_3,$$

where

$$\begin{aligned} H_3 = & \beta_1^d + (d-1)\beta_2^d - d\beta_1\beta_2^{d-1} - 3d(d-1)\beta_3\beta_4\beta_2^{d-2} - d(d-1)(d-2)\beta_4^3\beta_2^{d-3} \\ & + 3(d-1)\beta_3^d + 3(d-1)\beta_4^2\beta_2^d - 3d\beta_1\beta_3^{d-1} + 3d\beta_1\beta_4^{d-1} + 3d(d-1)\beta_3^2\beta_4^{d-2} \\ & - 3d(d-1)\beta_4\beta_2^{d-1} + 3d(d-1)\beta_3\beta_2^{d-1} + 3d(d-1)\beta_4^2\beta_2^{d-2} - 6d(d-1)\beta_3\beta_4^{d-1}, \end{aligned}$$

and

$$\beta_1 = \frac{1}{15}, \beta_2 = \frac{1}{27}, \beta_3 = \frac{17}{315}, \beta_4 = \frac{2}{45}.$$

Proof in Section 6 below.

In order to bring $\beta_1, \beta_2, \beta_3$ and β_4 of Proposition 22 in line with Theorem 3, with the common divisor $D_3 = 945$, they can be written as

$$B_1 = 63/D_3, \quad B_2 = 35/D_3, \quad B_3 = 51/D_3, \quad B_4 = 42/D_3.$$

PROPOSITION 23.

$$Y_1 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 (w \wedge x)(x \wedge y)(y \wedge z)(z \wedge w) dw dx dy dz = \frac{17}{630} = \frac{1}{45} + \frac{1}{210}.$$

PROPOSITION 24.

$$Y_2 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx(x \wedge y)(y \wedge z)(z \wedge w) dw dx dy dz = \frac{62}{45 \cdot 63} = \frac{1}{45} - \frac{1}{45 \cdot 63}.$$

PROPOSITION 25.

$$Y_3 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx^2 y(y \wedge z)(z \wedge w) dw dx dy dz = \frac{2}{3} \int_0^1 J_2^2(x) dx \\ = \frac{17}{21 \cdot 45} = \frac{1}{45} - \frac{4}{21 \cdot 45}.$$

where $J_2(\cdot)$ is as in Proposition 16.

PROPOSITION 26.

$$Y_4 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx(x \wedge y)yz(z \wedge w) dw dx dy dz = \frac{4}{225} = \frac{1}{45} - \frac{1}{225}.$$

PROPOSITION 27.

$$Y_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 (w \wedge x)wy^2 z^2 w dx dy dz = \frac{2}{135} = \frac{1}{45} - \frac{1}{135}.$$

PROPOSITION 28.

$$\gamma_6 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 w^2 x^2 y^2 z^2 dw dx dy dz = 3^{-4} = \frac{1}{81} = \frac{1}{45} - \frac{4}{405}$$

PROPOSITION 29. The fourth cumulant K_4 of the distribution

Γ_d is

$$K_4 = 48H_4$$

where

$$\begin{aligned} H_4 = & \gamma_1^d + (d-1)^4 \gamma_6^d + d\gamma_1 \gamma_6^{d-1} + 4d(d-1)\gamma_2 \gamma_5 \gamma_6^{d-2} + d(d-1)(2\gamma_3^2 + \gamma_4^2) \gamma_6^{d-2} \\ & + 2d(d-1)(d-2)(2\gamma_3 + \gamma_4) \gamma_5 \gamma_6^{d-3} + d(d-1)(d-2)(d-3) \gamma_5^2 \gamma_6^{d-4} \\ & + 4(d-1)\gamma_2^d + 4(d-1)^2 \gamma_5^d - 4d\gamma_1 \gamma_2^{d-1} - 4d\gamma_1 \gamma_5^{d-1} - 4d(d-1)(2\gamma_3 + \gamma_4) \gamma_2 \gamma_5^{d-2} \\ & - 4d(d-1)(d-2) \gamma_3^2 \gamma_5^{d-3} - 4d(d-1)^3 \gamma_5^{d-1} - 4d(d-1) \gamma_2 \gamma_6^{d-1} \\ & - 4d(d-1)^2 (2\gamma_3 + \gamma_4) \gamma_5 \gamma_6^{d-2} - 4d(d-1)^2 (d-2) \gamma_5^3 \gamma_6^{d-3} + 4(d-1)^2 \gamma_3^d \\ & + 2(d-1)^2 \gamma_4^d + 4d\gamma_1 \gamma_3^{d-1} + 4d(d-1) \gamma_2^2 \gamma_3^{d-2} + 2d\gamma_1 \gamma_4^{d-1} + 2d(d-1) \gamma_2^2 \gamma_4^{d-2} \\ & + 4d(d-1)^2 \gamma_3 \gamma_6^{d-1} + 4d(d-1)^3 \gamma_5^2 \gamma_6^{d-2} + 2d(d-1)^2 \gamma_4 \gamma_6^{d-1} + 2d(d-1)^3 \gamma_5^2 \gamma_6^{d-1} \\ & - 8d(d-1) \gamma_2 \gamma_3^{d-1} - 4d(d-1) \gamma_2 \gamma_4^{d-1} - 8d(d-1)^2 \gamma_3 \gamma_5^{d-1} - 4d(d-1)^2 \gamma_4 \gamma_5^{d-1} \\ & + 8d(d-1) \gamma_2 \gamma_5^{d-1} + 8d(d-1)^2 \gamma_3 \gamma_4 \gamma_5^{d-2} + 4d(d-1) \gamma_2 \gamma_5^{d-1} + 4d(d-1)^2 \gamma_3^2 \gamma_5^{d-2} \end{aligned}$$

$$\gamma_1 = \frac{17}{630} = \frac{765}{D_4} \quad \gamma_2 = \frac{62}{9.315} = \frac{620}{D_4}$$

$$\gamma_3 = \frac{17}{945} = \frac{510}{D_4} \quad \gamma_4 = \frac{4}{225} = \frac{504}{D_4}$$

$$\gamma_5 = \frac{2}{135} = \frac{420}{D_4} \quad \gamma_6 = \frac{1}{81} = \frac{350}{D_4}$$

where the common divisor $D_4 = 28350 = 2 \cdot 5 \cdot 5 \cdot 7 \cdot 9 \cdot 9$.

Proof in Section 6 below.

PROPOSITION 30.

$$\begin{aligned} \delta_1 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (v-w)(w-x)(x-y)(y-z)(z-v) dv dw dx dy dz \\ &= \frac{31}{2835} = \frac{1}{90} - \frac{1}{5670} = \frac{1}{135} + \frac{2}{567} \end{aligned}$$

PROPOSITION 31.

$$\begin{aligned} \delta_2 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw(w-x)(x-y)(y-z)(z-v) dv dw dx dy dz \\ &= \frac{1382}{25 \cdot 7 \cdot 9 \cdot 9 \cdot 11} = \frac{1382}{155925} = \frac{1}{135} + \frac{227}{155925} \end{aligned}$$

PROPOSITION 32.

$$\begin{aligned} \delta_3 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 w^2 x(x-y)(y-z)(z-v) dv dw dx dy dz \\ &= \frac{62}{8505} = \frac{62}{3 \cdot 5 \cdot 9 \cdot 9 \cdot 9} = \frac{1}{135} - \frac{1}{1 \cdot 4 \cdot 9 \cdot 9 \cdot 9} \end{aligned}$$

PROPOSITION 33.

$$\delta_4 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw(w+x)xy(y+z)(z+v)dvdxduydz$$

$$= \frac{34}{4725} = \frac{34}{3 \cdot 5 \cdot 5 \cdot 7 \cdot 9} = \frac{1}{135} - \frac{1}{4725}.$$

PROPOSITION 34.

$$\delta_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2x^2y(y+z)(z+v)dvdxduydz$$

$$= \frac{17}{2835} = \frac{17}{5 \cdot 7 \cdot 9 \cdot 9} = \frac{1}{135} - \frac{4}{5 \cdot 7 \cdot 9 \cdot 9}.$$

PROPOSITION 35.

$$\delta_6 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2x(x+y)yz(z+v)dvdxduydz$$

$$= \frac{4}{675} = \frac{4}{25 \cdot 27} = \frac{1}{135} - \frac{1}{25 \cdot 27}.$$

PROPOSITION 36.

$$\delta_7 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2x^2y^2z(z+v)dvdxduydz$$

$$= \frac{2}{405} = \frac{2}{5 \cdot 9 \cdot 9} = \frac{1}{135} - \frac{1}{405}.$$

PROPOSITION 37.

$$\delta_8 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2x^2y^2z^2dvdxduydz$$

$$= \frac{1}{1215} = \frac{1}{243} = \frac{1}{135} - \frac{4}{1215}.$$

PROPOSITION 38. The fifth cumulant K_5 of the distribution

Γ_d is

$$K_5 = 32.12H_5,$$

where

$$\begin{aligned} H_5 = & s_1^d + (d-1)s_8^d - ds_1s_8^{d-1} - 5d(d-1)s_2s_7s_8^{d-2} - 5d(d-1)s_3s_5s_8^{d-2} \\ & - 5d(d-1)s_4s_6s_8^{d-2} - 5d(d-1)(d-2)s_3s_7s_8^{d-3} - 5d(d-1)(d-2)s_4s_7s_8^{d-3} \\ & - 5d(d-1)(d-2)s_5s_7s_8^{d-3} - 5d(d-1)(d-2)s_5s_6s_7s_8^{d-3} - 5d(d-1)(d-2)s_6^2s_7s_8^{d-3} \\ & - 5d(d-1)(d-2)(d-3)s_6^3s_8^{d-4} - 5d(d-1)(d-2)(d-3)s_5^3s_8^{d-4} \\ & - d(d-1)(d-2)(d-3)(d-4)s_7^5s_8^{d-5} + 5(d-1)s_2^d + 5(d-1)s_7^d - 5ds_1s_2^{d-1} \\ & + 5ds_1s_7^{d-1} + 10d(d-1)s_2s_5s_7^{d-2} + 10d(d-1)s_2s_6s_7^{d-2} + 5d(d-1)s_3^2s_7^{d-2} \\ & + 5d(d-1)s_4^2s_7^{d-2} + 5d(d-1)s_3s_6s_7^{d-2} + 10d(d-1)(d-2)s_3s_5s_6s_7^{d-3} \\ & + 10d(d-1)(d-2)s_4s_5s_6s_7^{d-3} + 5d(d-1)(d-2)s_4s_5s_7^{d-3} \\ & + 5d(d-1)(d-2)s_3s_6s_7^{d-3} + 5d(d-1)(d-2)(d-3)s_5^2s_6s_7^{d-4} \\ & - 5d(d-1)s_7^4s_8^{d-1} + 5d(d-1)s_2^2s_8^{d-1} + 10d(d-1)s_3s_7s_8^{d-2} + 10d(d-1)s_4s_7s_8^{d-2} \\ & + 5d(d-1)s_5^2s_8^{d-2} + 5d(d-1)s_6^2s_8^{d-2} + 5d(d-1)s_5s_6s_8^{d-2} \\ & + 15d(d-1)^2(d-2)s_6^2s_8^{d-3} + 15d(d-1)^2(d-2)s_6^2s_7s_8^{d-3} \end{aligned}$$

$$\begin{aligned}
 & + 5d(d-1)^2(d-2)(d-3)s_7^4s_8^{d-4} + 5(d-1)^2s_3^d \\
 & + 5(d-1)^3s_5^d + 5ds_1s_3^{d-1} + 5d(d-1)s_2^2s_3^{d-2} \\
 & - 5ds_1s_5^{d-1} - 10d(d-1)s_2s_3s_5^{d-2} - 5d(d-1)s_2s_4s_5^{d-2} - 5d(d-1)(d-2)s_3s_4s_5^{d-3} \\
 & + 5d(d-1)^3s_5s_8^{d-1} + 5d(d-1)^4s_7^2s_8^{d-2} - 5d(d-1)^2s_3s_8^{d-1} - 10d(d-1)^3s_5s_7s_8^{d-2} \\
 & - 5d(d-1)^3s_6s_7s_8^{d-2} - 5d(d-1)^3(d-2)s_7^3s_8^{d-3} \\
 & + 5d(d-1)^2s_4^d + 5(d-1)^3s_6^d + 5ds_1s_4^{d-1} \\
 & + 5d(d-1)s_2^2s_4^{d-2} - 5ds_1s_6^{d-1} - 10d(d-1)s_2s_4s_6^{d-2} \\
 & - 5d(d-1)s_2s_3s_6^{d-2} - 5d(d-1)(d-2)s_3^2s_4s_6^{d-3} \\
 & + 5ds_6s_8^{d-1} + 5d(d-1)s_7^2s_8^{d-2} - 5d(d-1)^2s_4s_8^{d-1} - 10d(d-1)^3s_6s_7s_8^{d-2} \\
 & - 5d(d-1)^3s_5s_7s_8^{d-2} - 5d(d-1)^3(d-2)s_7^3s_8^{d-3} \\
 & - 10d(d-1)s_2s_3^{d-1} - 10d(d-1)s_2s_7^{d-1} \\
 & - 10d(d-1)^2s_4s_5s_7^{d-2} - 10d(d-1)^2s_4s_6s_7^{d-2} \\
 & - 10d(d-1)^2s_3s_6s_7^{d-2} - 10d(d-1)^2(d-2)s_5^2s_6s_7^{d-3} \\
 & - 10d(d-1)^3s_5s_7^{d-1} - 10d(d-1)s_2s_4^{d-1} - 10d(d-1)s_2s_7^{d-1} \\
 & - 10d(d-1)^2s_3s_6s_7^{d-2} - 10d(d-1)^2s_4s_6s_7^{d-2} - 10d(d-1)^2s_5s_6s_7^{d-2}
 \end{aligned}$$

$$\begin{aligned}
 & - 10d(d-1)^2(d-2)s_5^2s_6s_7^{d-3} - 10d(d-1)^3s_6s_7^{d-1} \\
 & + 10d(d-1)^2s_3s_7^{d-1} + 10d(d-1)^3s_5s_6s_7^{d-2} \\
 & + 10d(d-1)s_2s_5^{d-1} + 10d(d-1)^2s_3s_4s_5^{d-2} \\
 & - 10d(d-1)^2s_3s_5^{d-1} - 10d(d-1)^2s_4s_6^{d-1} \\
 & + 10d(d-1)^2s_4s_7^{d-1} + 10d(d-1)^3s_5s_6s_7^{d-2} \\
 & + 10d(d-1)s_2s_6^{d-1} + 10d(d-1)^2s_3s_4s_6^{d-2} \\
 & - 5d(d-1)^2s_4s_5^{d-1} + 5d(d-1)^2s_3s_7^{d-1} \\
 & + 5d(d-1)^3s_5^2s_7^{d-2} + 5d(d-1)s_2s_6^{d-1} \\
 & + 5d(d-1)^2s_4s_6^{d-2} - 5d(d-1)^2s_3s_6^{d-1} \\
 & + 5d(d-1)^2s_4s_7^{d-1} + 5d(d-1)^3s_6^2s_7^{d-2} \\
 & + 5d(d-1)s_2s_5^{d-1} + 5d(d-1)^2s_3^2s_5^{d-2} .
 \end{aligned}$$

where

$$s_1 = \frac{31}{2435} = 5115/d_3, \quad s_2 = \frac{1382}{159925} = 4146/d_3,$$

$$s_3 = \frac{62}{8995} = 3410/d_3, \quad s_4 = \frac{24}{4725} = 3366/d_3,$$

$$s_5 = \frac{17}{2435} = 2805/d_3, \quad s_6 = \frac{1}{4725} = 2772/d_3,$$

$$s_7 = \frac{2}{405} = 2310/D_5, \quad s_8 = \frac{1}{243} = 1925/D_5,$$

with the common divisor $D_5 = 467775 = 3 \cdot 25 \cdot 7 \cdot 9 \cdot 9 \cdot 11$.

Proof in Section 6 below.

REMARK. These calculations become even more tedious. However, because we employ these cumulants in the Cornish-Fisher asymptotic expansion, there is little to be gained by proceeding farther. If it were necessary to proceed further, one should attempt to find a less clumsy expansion for the terms $\sigma_{12}, \sigma_{23}, \dots, \sigma_{n-1, n}, \sigma_{n1}$ of Proposition 9.

PROPOSITION 39. For large dimensions, say $d > 50$, an adequate approximation is obtained by using only the largest term of each expansion, as

$$K_n \approx (n-1)! 2^{n-1} \int_{-\infty}^{\infty} \prod_{r=1}^d (y_r^{(1)} + y_r^{(2)}) \dots (y_r^{(n-1)} + y_r^{(n)}) (y_r^{(n)} + y_r^{(1)}) \lambda_n dy.$$

Thus, using the values calculated above, as approximations for $d > 50$ we have

$$K_1 \approx 2^{-d} \approx (.5)^d,$$

$$K_2 \approx 2.6^{-d} \approx 2 \cdot (.158657)^d,$$

$$K_3 \approx 8.15^{-d} \approx 8 \cdot (.086657)^d,$$

$$K_4 \approx 6.8 \cdot \left(\frac{17}{630}\right)^d \approx 48(2.07841)^d 10^{-24},$$

$$K_5 \approx 24.15 \cdot \left(\frac{21}{630}\right)^d \approx 104(1.08328)^d 10^{-44}.$$

Proof is by inspection.

REMARK. The evident comparison is with the χ^2 distribution with γ degrees of freedom for which

$$K_n = \Gamma_n^{-1} \gamma.$$

We would conclude that Γ_d distribution for large d lies between the χ^2 and normal distributions. Hence the Cornish-Fisher expansion should work even better for higher values of d than for $d = 2$.

For the latter it also gave very good critical values, as demonstrated in our table following Corollary 5 above.

6. RECALLED FORMS FOR SOME OF THE PROPOSITIONS OF SECTION 1.

PROOF OF PROPOSITION 7. Write $a_1 = \sqrt{1} a_1$ in the characteristic function $\phi_n(t)$, and ignore higher powers of a_1 . Then, from

$$\phi_n(t) = \exp - \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n a_{jk} t_j t_k$$

$$= \exp \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n a_{jk} t_j t_k$$

$$= \exp \frac{1}{2} \sum_{j=1}^n a_{jj} t_j^2 + \exp \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n a_{jk} t_j t_k$$

It follows that all the coefficients of $a_1^2 \dots a_n^2$ are contained in the product $\phi_n(\cdot)$.

PROOF OF PROPOSITION 8. The first 4 constants of F_n , the characteristic of ϕ_n , have the following values:

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Mean $K_1 = \mu : K_1 = \int_{\mathcal{D}} ET^2(Y^{(1)}) \lambda_1 dy$, $ET^2(Y^{(k)}) = 2a_1$, and a_1 is the coefficient of s_1^2 in $\psi_1(s)$. By inspection $a_1 = \frac{1}{2} \sigma_{11}$. Thus $K_1 = \mu = \int_{\mathcal{D}} \sigma_{11} \lambda_1 dy = EC_d$.

Variance $K_2 = \sigma^2$. Consider $EC_d^2 = \int_{\mathcal{D}} ET^2(Y^{(1)}) T^2(Y^{(2)}) \lambda_2 dy$, $ET^2(Y^{(1)}) T^2(Y^{(2)}) = 2^2 a_2$, and a_2 is the coefficient of $s_1^2 s_2^2$ in $\psi_2(t)$, i.e.,

$$\psi_2(t) = \prod_{i=1}^2 (1 + \frac{1}{2} \sigma_{ii} s_i^2) (1 + \sigma_{12} s_1 s_2 + \frac{1}{2} \sigma_{12}^2 s_1^2 s_2^2).$$

Thus

$$a_2 = \frac{1}{4} \sigma_{11} \sigma_{22} + \frac{1}{2} \sigma_{12}^2,$$

and so

$$EC_d^2 = \int_{\mathcal{D}} \sigma_{11} \sigma_{22} \lambda_2 dy + 2 \int_{\mathcal{D}} \sigma_{12}^2 \lambda_2 dy.$$

However, as we have seen, σ_{ii} is a function of $T(Y^{(i)})$ only, and these have identical distributions. Hence

$$\begin{aligned} EC_d^2 &= (\int_{\mathcal{D}} \sigma_{11} \lambda_1 dy)^2 + 2 \int_{\mathcal{D}} \sigma_{12}^2 \lambda_2 dy \\ &= (EC_d)^2 + 2 \int_{\mathcal{D}} \sigma_{12}^2 \lambda_2 dy. \end{aligned}$$

Since

$$K_2 = \sigma^2 = EC_d^2 - (EC_d)^2,$$

hence

$$K_2 = \sigma^2 = 2 \int_{\mathcal{D}} \sigma_{12}^2 \lambda_2 dy.$$

Third Cumulant K_3 . Consider

$$EC_d^3 = 2^3 \int_{I^3} a_3 \lambda_3 dy ,$$

with a_3 being the coefficient of $s_1^2 s_2^2 s_3^2$ in $\psi_3(s)$, i.e.,

$$\psi_3(s) = \prod_{i=1}^3 (1 + \frac{1}{2} \sigma_{ii} s_i^2) (1 + \sigma_{12} s_1 s_2 + \frac{1}{2} \sigma_{12}^2 s_1^2 s_2^2) .$$

$$(1 + \sigma_{13} s_1 s_3 + \frac{1}{2} \sigma_{13}^2 s_1^2 s_3^2) (1 + \sigma_{23} s_2 s_3 + \frac{1}{2} \sigma_{23}^2 s_2^2 s_3^2) .$$

Thus

$$a_3 = \frac{1}{8} \sigma_{11} \sigma_{22} \sigma_{33} + \frac{1}{4} \sigma_{11} \sigma_{23}^2 + \frac{1}{4} \sigma_{22} \sigma_{13}^2 + \frac{1}{4} \sigma_{33} \sigma_{12}^2 + \sigma_{12} \sigma_{13} \sigma_{23} ,$$

and so

$$\begin{aligned} EC_d^3 &= \int_{I^3} \sigma_{11} \sigma_{22} \sigma_{33} \lambda_3 dy \\ &+ 2 \int_{I^3} (\sigma_{11} \sigma_{23}^2 + \sigma_{22} \sigma_{13}^2 + \sigma_{33} \sigma_{12}^2) \lambda_3 dy , \\ &+ 8 \int_{I^3} \sigma_{12} \sigma_{23} \sigma_{31} \lambda_3 dy \\ &= K_1^3 + 3K_1 K_2 + 8 \int_{I^3} \sigma_{12} \sigma_{23} \sigma_{31} \lambda_3 dy , \end{aligned}$$

where K_1, K_2 are as above.

Now by the well-known formula $\mu_3' = EC_d^3 = K_3 + 3K_2 K_1 + K_1^3$, hence

$$K_3 = 8 \int_{I^3} \sigma_{12} \sigma_{23} \sigma_{31} \lambda_3 dy .$$

Fourth Cumulant K_4 . Consider

$$EC_d^4 = 2^4 \int_{I_4} a_4 \lambda_4 dy,$$

with a_4 as in Proposition 7. Next consider now the fourth central moment, $\mu_4 = E(C_d - EC_d)^4$. This is equal to $EC_d^4 = \mu_4'$ under the condition that the mean, $\mu = K_1 = EC_d$, is set to zero. This requires that $\sigma_{ii} = 0$, $i = 1, \dots, 4$. Thus

$$\mu_4' = E(C_d - EC_d)^4 = 2^4 \int_{I_4} a_4^* \lambda_4 dy,$$

where a_4^* is the coefficient of $S_1^2 S_2^2 S_3^2 S_4^2$ in

$$\psi_4^*(s) = \prod_{i=1}^3 \prod_{j=i}^4 (1 + \sigma_{ij} S_i S_j + \frac{1}{2} \sigma_{ij}^2 S_i^2 S_j^2)$$

$$= (1 + \sigma_{12} S_1 S_2 + \frac{1}{2} \sigma_{12}^2 S_1^2 S_2^2)$$

$$(1 + \sigma_{13} S_1 S_3 + \frac{1}{2} \sigma_{13}^2 S_1^2 S_3^2)$$

$$(1 + \sigma_{14} S_1 S_4 + \frac{1}{2} \sigma_{14}^2 S_1^2 S_4^2)$$

$$(1 + \sigma_{23} S_2 S_3 + \frac{1}{2} \sigma_{23}^2 S_2^2 S_3^2)$$

$$(1 + \sigma_{24} S_2 S_4 + \frac{1}{2} \sigma_{24}^2 S_2^2 S_4^2)$$

$$(1 + \sigma_{34} S_3 S_4 + \frac{1}{2} \sigma_{34}^2 S_3^2 S_4^2)$$

Thus

$$a_4^* = \sigma_{12} \sigma_{23} \sigma_{34} \sigma_{14} + \sigma_{13} \sigma_{23} \sigma_{24} \sigma_{14} + \sigma_{12} \sigma_{24} \sigma_{34} \sigma_{13}$$

$$+ \frac{1}{4} \sigma_{12}^2 \sigma_{34}^2 + \frac{1}{4} \sigma_{13}^2 \sigma_{24}^2 + \frac{1}{4} \sigma_{14}^2 \sigma_{23}^2$$

Thus, recalling that the $T(y^{(i)})$ have identical distributions,

$$\begin{aligned}\mu_4 &= 3.2^4 \int_I d\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}\lambda_4 dy + 3.2^2 \left(\int_I d\sigma_{12}\lambda_2 dy \right)^2 \\ &= 3.2^4 \int_I d\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}\lambda_4 dy + 3K_2^2.\end{aligned}$$

Since

$$\mu_4 = K_4 + 3K_2^2,$$

hence

$$K_4 = 3.2^4 \int_I d\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}\lambda_4 dy.$$

PROOF OF PROPOSITION 9. We prove by induction in steps 1 to 11 below that the n^{th} cumulant K_n of the distribution of C_d is given by

$$K_n = (n-1)! 2^{n-1} \int_I d\sigma_{12}\sigma_{23} \cdots \sigma_{n-1 n} \sigma_{n 1} \lambda_n dy.$$

1. We proved in Proposition 8 that this proposition is true for $n = 1, 2, 3, 4$.

Assuming now that the proposition is true for $k < n$, we prove it for n .

$$\begin{aligned}2. \text{ Let } C_{k,u} &= \prod_{m=1}^k \sigma_{i_m, j_m} \\ &= \prod_{m=1}^k \sigma(y^{(i_m)}, y^{(j_m)}), \quad u = 1, \dots, U_k,\end{aligned}$$

be the product of the σ_{i_m, j_m} , subject to the condition that each of the indices $i_m = j_m$ appears precisely twice in the product, or not at all. There is a finite number U_k of such products.

Then in Proposition 7 all the terms of a_n , thus all the coefficients of $s_1^2 \dots s_n^2$ in $\psi_n(s)$ must have the form $b_u C_{n,u}$ for some real numbers b_u .

3. Let
$$I_{k,u} = \int_{I^{kd}} C_{k,u} \lambda_k dy.$$

Then, by Proposition 5,

$$EC_d^n = 2^n \sum_{u=1}^{U_n} b_u I_{n,u},$$

for some real numbers b_u , which can be calculated from the expansion of $\psi_n(s)$ in picking out the coefficients of $s_1^2 \dots s_n^2$.

4. Let
$$r_{k,1} = \sigma_{12} \sigma_{23} \dots \sigma_{k-1,k} \sigma_{k,1}.$$

Let
$$J_k = \int_{I^{kd}} r_{k,1} \lambda_k dy.$$

Let $r_{k,v}$, $v = 2, \dots, v_k$, be the products $\prod_{m=1}^k \sigma_{i_m, j_m}$ obtained from $r_{k,1}$ by a cyclic permutation of the indices $\{1, \dots, k\}$.

Then

$$J_k = \int_{I^{kd}} r_{k,v} \lambda_k dy,$$

since such a permutation of the variables of integration does not change the value of an integral over the set $I^{kd} = [0,1]^{kd}$.

5. Hence we can renumber the integrals $I_{k,n}$ so that we can let

$$I_{k,1} = J_k.$$
 Hence in the expansion

$$EC_d^n = 2^n \sum_{u=1}^{U_n} b_u I_{n,u}.$$

the constant b_1 , the multiple of J_n , is equal to v_n , the number of such products $r_{n,v}$ in a_n , the coefficient of $s_1^2 \dots s_n^2$ in the symmetrical product $\psi_n(s)$. Since $\psi_n(s)$ is symmetrical, it must include every possible such distinct product. There are $\frac{1}{2}(n-1)!$ such products, namely the number of ways n integers can be placed on the circumference of a circle (the factor $\frac{1}{2}$ is there, because we equate clockwise and anti-clockwise placings). Thus, there is precisely $1 = \frac{1}{2} \cdot 2!$ way of placing 3 integers, leaving 3 locations for the fourth integer, and for each of these there are 4 locations for the fifth integer, and so on. Thus $b_1 = \frac{1}{2}(n-1)!$.

6. Considering the products $C_{n,u}$ defined above, either $C_{n,u} = r_{n,v}$ for some v , and so has already been counted, or else $C_{n,u}$ is a product of such "irreducible" products r_k .

Then

$$\begin{aligned} I_{nu} &= \int_{I_n} C_{n,u} \lambda_n dy \\ &= \int_{I_n} \prod_{w=1}^{W_u} r_{k_w, v_w} \lambda_n dy \quad \text{for some } W_u \\ &= \prod_{w=1}^{W_u} \int_{I_{k_w}} r_{k_w, v_w} \lambda_{k_w} dy \\ &= \prod_{w=1}^{W_u} J_{k_w} \end{aligned}$$

7. Note that in our case (i.e. for $d > 2$) $\sigma_{ij} = ET(y^{(i)})T(y^{(j)})$ cannot be factored as a product of two functions $g(y^{(i)}), h(y^{(j)})$, so that in this sense the J_k are "irreducible".

8. Consequently

$$EC_d^n = 2^{n-1} \frac{1}{2} (n-1)! J_n + \sum_{n=2}^Q b_u f_u \text{ for some } Q,$$

where the f_u are products of the J_k , $k < n$.

9. We have shown in Proposition 8 that

$$K_1 = J_1 = 2^1 \frac{1}{2} (0!) J_1$$

$$K_2 = 2J_2 = 2^2 \frac{1}{2} (1!) J_2$$

$$K_3 = 8J_3 = 2^3 \frac{1}{2} (2!) J_3$$

$$K_4 = 3 \cdot 2^4 J_4 = 2^4 \frac{1}{2} (3!) J_4$$

and by our induction assumption

$$K_k = 2^{k-1} (k-1)! J_k \text{ for } k < n.$$

10. For $u' = EC_d^n$ we have

$$u'_n = \sum_{m=1}^n \left[\frac{K_{p_1}}{p_1!} \right]^{\pi_1} \left[\frac{K_{p_2}}{p_2!} \right]^{\pi_2} \cdots \left[\frac{K_{p_m}}{p_m!} \right]^{\pi_m} \frac{n!}{\pi_1! \pi_2! \cdots \pi_m!}$$

where the second summation extends over all non-negative values

of the π_j such that $p_1 \pi_1 + p_2 \pi_2 + \cdots + p_m \pi_m = n$.

We can express this as

$$u'_n = EC_d^n = K_n + \sum_{v=2}^{Q'} e_v g_v$$

for some real numbers e_v and some integer Q' where the g_v

are products of the K_k , $k < n$.

11. Equating these two expressions for $EC_d^n = \mu_n'$ we have

$$(5.6) \quad K_n + \sum_{v=2}^{Q'} e_v g_v = 2^{n-1} (n-1)! J_n + \sum_{u=2}^Q b_u f_u,$$

where the e_v, b_u are real constants, Q, Q' are finite integers, the g_v are products of the K_k , $k < n$, and the f_u are products of the J_k , $k < n$, and

$$K_k = 2^{k-1} (k-1)! J_k, \quad k < n.$$

Referring back we see that none of Propositions 5 to 8 depends upon the functions $\sigma_{ij} = \sigma(y^{(i)}, y^{(j)})$. They are general propositions true for an infinite class of such functions. Thus equation (5.6) is to be true for an infinite set of values of the finite number of the products g_v, f_u of the finite number of functionals K_k, J_k . This can only be true if the corresponding terms are identical. Hence

$$K_n = 2^{n-1} (n-1)! J_n,$$

and Proposition 9 is proved.

PROOF OF PROPOSITION 10. From Proposition 8

$$K_1 = \mu = EC_d = \int_I \sigma_{11} \lambda_1 dy',$$

$$\sigma_{ij} = \prod_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) + (d-1) \prod_{r=1}^d y_r^{(i)} y_r^{(j)} - \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)},$$

so

$$\sigma_{11} = \prod_{r=1}^d y_r^{(1)} + (d-1) \prod_{r=1}^d (y_r^{(1)})^2 - \sum_{r=1}^d y_r^{(1)} \prod_{s \neq r} (y_s^{(1)})^2.$$

On integrating the latter term by term over I^d

$$\begin{aligned} K_1 &= \left(\frac{1}{2}\right)^d + (d-1) \left(\frac{1}{3}\right)^d - d \left(\frac{1}{2}\right) \left(\frac{1}{3}\right)^{d-1} \\ &= 2^{-d} + 3^{-d} (d-1 - \frac{3}{2}d) \\ &= 2^{-d} - \left(\frac{d}{2} + 1\right) 3^{-d} \\ &= 6^{-d} (3^d - (d+2) 2^{d-1}) \end{aligned}$$

as required.

PROOF OF PROPOSITION 11. Straight from the definition of α_3^k .

PROOF OF PROPOSITION 12. Define

$$\alpha_1 = \int_0^1 \int_0^1 (x \wedge y)^2 dx dy.$$

Then

$$\begin{aligned} \alpha_1 &= \int_0^1 \left[\int_0^y (x \wedge y)^2 dx + \int_y^1 (x \wedge y)^2 dx \right] dy \\ &= \int_0^1 \left[\int_0^y x^2 dx + \int_y^1 y^2 dx \right] dy \\ &= \int_0^1 \left[\frac{y^3}{3} + y^2 [1-y] \right] dy \\ &= \int_0^1 \left(y^2 - \frac{2}{3} y^3 \right) dy \\ &= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}. \end{aligned}$$

PROOF OF PROPOSITION 13. Define

$$a_2 = \int_0^1 \int_0^1 (x \wedge y) xy dx dy$$

Then

$$a_2 = \int_0^1 y \left[\int_0^y (x \wedge y) x dx + \int_y^1 (x \wedge y) x dx \right] dy$$

$$= \int_0^1 y \left[\int_0^y x^2 dx + \int_y^1 xy dx \right] dy$$

$$= \int_0^1 y \left(\frac{1}{3} y^3 + \frac{1}{2} y(1-y^2) \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} y^2 + \frac{1}{6} y^4 \right) dy$$

$$= \frac{1}{6} + \frac{1}{30} = \frac{4}{30}$$

PROOF OF PROPOSITION 14. From Proposition 8

$$K_2 = 2 \int_1^2 d \sigma_{12}^2 \lambda_2 dy$$

where

$$\sigma_{12} = ET(Y^{(1)})T(Y^{(2)})$$

$$= \prod_{i=1}^d (Y_i^{(1)} \wedge Y_i^{(2)}) + (d-1) \prod_{i=1}^d Y_i^{(1)} Y_i^{(2)}$$

$$- \sum_{i=1}^d (Y_i^{(1)} \wedge Y_i^{(2)}) \prod_{j \neq i} Y_j^{(1)} Y_j^{(2)}$$

$$= a + b + c$$

$$\text{So } \sigma_{12}^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac .$$

Write

$$K_2 = 2H_1 + 2H_2 + 2H_3 + 2H_4 + 2H_5 + 2H_6 ,$$

with H_i ($i = 1, 2, \dots, 6$) as defined in the sequel.

$$\begin{aligned} H_1 &= \int_I^{2d} a^2 \lambda_2 dy \\ &= \int_I^{2d} \prod_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)})^2 \lambda_2 dy \\ &= \left[\int_I^{2d} (y_i^{(1)} \wedge y_i^{(2)})^2 dy_i^{(1)} dy_i^{(2)} \right]^d \\ &= a_1^d = 6^{-d} = 15^d (90)^{-d} \end{aligned}$$

by Proposition 12.

$$\begin{aligned} H_2 &= \int_I^{2d} b^2 \lambda_2 dy \\ &= (d-1)^2 \int_I^{2d} \prod_{i=1}^d (y_i^{(1)} y_i^{(2)})^2 \lambda_2 dy \\ &= (d-1)^2 a_3^d = (d-1)^2 3^{-2d} = (d-1)^2 10^d 90^{-d} \end{aligned}$$

by Proposition 11.

$$H_3 = \int_1^{2d} C^2 \lambda_2 dy,$$

where

$$\begin{aligned} C^2 &= \left[\sum_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)}) \prod_{j \neq i} y_i^{(1)} y_j^{(2)} \right]^2 \\ &= \sum_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)})^2 \prod_{j \neq i} (y_i^{(1)} y_j^{(2)})^2 \\ &\quad + \sum_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)}) y_i^{(1)} y_i^{(2)} \sum_{j \neq i} y_j^{(1)} y_j^{(2)} (y_j^{(1)} \wedge y_j^{(2)}) \prod_{\substack{r \neq i \\ r \neq j}} (y_r^{(1)} y_r^{(2)})^2. \end{aligned}$$

Substituting the integrals $\alpha_1, \alpha_2, \alpha_3$ from Propositions 11, 12 and 13

$$\begin{aligned} H_3 &= d \alpha_1 \alpha_3^{d-1} + d(d-1) \alpha_2^2 \alpha_3^{d-2} \\ &= d \left(\frac{1}{9} \right)^{d-1} + d(d-1) \left(\frac{4}{30} \right)^2 \left(\frac{1}{9} \right)^{d-2} \\ &= \left(\frac{1}{90} \right)^d [15d 10^{d-1} + 144d(d-1) 10^{d-2}]. \end{aligned}$$

$$H_4 = \int_1^{2d} 2ab \lambda_2 dy,$$

where

$$2ab = 2(d-1) \prod_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)}) y_i^{(1)} y_i^{(2)}.$$

So

$$H_4 = 2(d-1) \alpha_2^d = 2(d-1) \left(\frac{4}{30} \right)^d = 2(d-1) 12^d (90)^{-d}.$$

$$H_5 = \int_I 2d \, 2bc \lambda_2 dy,$$

where

$$2bc = -2(d-1) \sum_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)}) y_i^{(1)} y_i^{(2)} \prod_{j \neq i} (y_j^{(1)} y_j^{(2)})^2.$$

So

$$H_5 = -d(d-1) a_2 a_3^{d-1}$$

$$= -2d(d-1) \left(\frac{4}{30}\right) \left(\frac{1}{9}\right)^{d-1} = -2d(d-1) 12 (10^{d-1}) (90)^{-d}$$

$$= -24d(d-1) 10^{d-1} (90)^{-d}$$

$$H_6 = \int_I 2d \, 2ac \lambda_2 dy,$$

where

$$2ac = -2 \sum_{i=1}^d (y_i^{(1)} \wedge y_i^{(2)})^2 \prod_{j \neq i} y_j^{(1)} y_j^{(2)} (y_j^{(1)} \wedge y_j^{(2)}).$$

So

$$H_6 = -2da_1 a_2^{d-1}$$

$$= -2d \left(\frac{1}{6}\right) \left(\frac{4}{30}\right)^{d-1} = -2d 15 (12^{d-1}) (90)^{-d} = -30d (12^{d-1}) (90)^{-d}.$$

Thus

$$\begin{aligned}
 \frac{K_2}{2} &= H_1 + H_2 + H_3 + H_4 + H_5 + H_6 \\
 &= 90^{-d} [15^d + (d-1)^2 10^d + 15d 10^{d-1} + 144d(d-1) 10^{d-2} \\
 &\quad + 2(d-1) 12^d - 24d(d-1) 10^{d-1} - 30d 12^{d-1}] \\
 &= 90^{-d} [15^d + 12^{d-1} [24(d-1) - 30d] \\
 &\quad + 10^{d-2} [100(d-1)^2 + 150d + 144d(d-1) - 240d(d-1)]] \\
 &= 90^{-d} [15^d + 12^{d-1} [-6d - 24] \\
 &\quad + 10^{d-2} [100d^2 - 200d + 100 + 150d + 144d^2 - 144d - 240d^2 + 240d]].
 \end{aligned}$$

Hence

$$K_2 = 2(90)^{-d} [15^d - 6(d+4) 12^{d-1} + (4d^2 + 46d + 100) 10^{d-2}],$$

as required.

PROOF OF PROPOSITION 15. Define

$$J_1(u, v) = \int_0^1 (x \wedge u) (x \wedge v) dx, \quad 0 \leq u \leq v \leq 1.$$

Then

$$\begin{aligned}
 J_1(u, v) &= \int_0^u (x \wedge u) (x \wedge v) dx + \int_u^v (x \wedge u) (x \wedge v) dx + \int_v^1 (x \wedge u) (x \wedge v) dx \\
 &= \int_0^u x^2 dx + \int_u^v u x dx + \int_v^1 u v dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{u^3}{3} + \frac{1}{2} u(v^2 - u^2) + uv(1-v) \\
 &= \frac{1}{3} u^3 - \frac{1}{2} u^3 + \frac{1}{2} uv^2 - uv^2 + uv \\
 &= \frac{1}{6} (6uv - 3uv^2 - u^3)
 \end{aligned}$$

as required.

PROOF OF PROPOSITION 16. Define

$$J_2(v) = \int_0^1 x(x \wedge v) dx, \quad 0 \leq v \leq 1.$$

hence

$$\begin{aligned}
 J_2(v) &= \int_0^v x(x \wedge v) dx + \int_v^1 x(x \wedge v) dx \\
 &= \int_0^v x^2 dx + \int_v^1 x v dx \\
 &= \frac{v^3}{3} + \frac{1}{2} v(1-v^2) \\
 &= \frac{1}{6} (3v - v^3)
 \end{aligned}$$

as claimed.

PROOF OF PROPOSITION 17. Define

$$\beta_1 = \int_0^1 \int_0^1 \int_0^1 (x \wedge y)(y \wedge z)(z \wedge x) dx dy dz.$$

Then

$$\beta_1 = \int_0^1 I(z) dz$$

with

$$\begin{aligned}
 I(z) &= \int_0^1 \int_0^1 (x \wedge y)(y \wedge z)(z \wedge x) dx dy \\
 &= \int_0^z (y \wedge z) \int_0^1 (x \wedge y)(x \wedge z) dx dy, \quad y \leq z \\
 &\quad + \int_z^1 (y \wedge z) \int_0^1 (x \wedge y)(x \wedge z) dx dy, \quad z \leq y \\
 &= \int_0^z (y \wedge z) J_1(y, z) dy + \int_z^1 (y \wedge z) J_1(z, y) dy \\
 &= \frac{1}{6} \int_0^z y(6yz - 3yz^2 - y^3) dy + \frac{z}{6} \int_z^1 (6zy - 3zy^2 - z^3) dy \\
 &= \frac{1}{30} [10z^4 - 5z^5 - z^5] + \frac{5z}{30} [2z - 4z^3 + 2z^4] \\
 &= \frac{1}{30} [10z^2 - 10z^4 + 4z^5].
 \end{aligned}$$

Thus

$$\begin{aligned}
 \beta_1 &= \frac{1}{30} \int_0^1 (10z^2 - 10z^4 + 4z^5) dz \\
 &= \frac{1}{30} \left(\frac{10}{3} - 2 + \frac{4}{6} \right) = \frac{1}{30} \left(\frac{12}{3} - 2 \right) = \frac{1}{15}
 \end{aligned}$$

as required.

PROOF OF PROPOSITION 19. Define

$$\beta_4 = \int_0^1 \int_0^1 \int_0^1 xy^2 z(z \wedge x) dx dy dz.$$

$$\begin{aligned}
 \beta_4 &= \left(\int_0^1 y^2 dy \right) \int_0^1 z \int_0^1 x(z \wedge x) dx dz \\
 &= \frac{1}{3} \int_0^1 z J_2(z) dz \\
 &= \frac{1}{18} \int_0^1 (3z^2 - z^4) dz \\
 &= \frac{1}{90} \int_0^1 (15z^2 - 5z^4) dz = \frac{1}{90}(5-1) = \frac{2}{45} .
 \end{aligned}$$

Notation and Addendum to Proposition 21. We can further simplify the lengthy expressions of the latter via introducing the non-commutative variables m, p , with

$$m = (y_r^{(i)} \wedge y_r^{(j)}) \quad \text{and} \quad p = y_r^{(i)} y_r^{(j)} ,$$

where the indices i, j, r are inferred from the positions of the m, p in the indicated products.

Thus we write:

$$- C_{ij} = d \prod_2^d p$$

$$C_{ij} C_{km} = d \prod_2^d pp + d(d-1)(mp)(pm) \prod_3^d pp$$

$$- C_{ij} C_{km} C_{np} = d \prod_2^d ppp + d(d-1)(mmp)(ppm) \prod_3^d ppp$$

$$+ d(d-1)(mpm)(pmp) \prod_3^d ppp + d(d-1)(mpp)(pmm) \prod_3^d ppp$$

$$+ d(d-1)(d-2)(mpp)(pmp)(ppm) \prod_4^d ppp ,$$

and similarly for larger products. We note also that for K_n , $n = 3$, of Proposition 22 the obvious further simplification holds (utilized in Q_3 of the proof of Proposition 23, coming next):

$$-C_{ij}C_{jk}C_{ki} = d \prod_{i=1}^d ppp + 3d(d-1)(mmp)(mmp) \prod_{i=1}^d ppp \\ + d(d-1)(d-2)(mpp)(mpp)(mpp) \prod_{i=1}^d ppp,$$

but not for larger products, as will be seen below.

PROOF OF PROPOSITION 22. The third cumulant K_3 of the distribution Γ_d is $K_3 = 8H_3$, where H_3 is calculated below. From Proposition 8

$$K_3 = 8 \int_1^d \sigma_{12} \sigma_{23} \sigma_{31} \lambda_3 dy,$$

where

$$\sigma_{ij} = ET_i(Y^{(i)})T(Y^{(j)}) \\ = \prod_{r=1}^d (Y_r^{(i)} \wedge Y_r^{(j)}) + (d-1) \prod_{r=1}^d Y_r^{(i)} Y_r^{(j)} \\ - \sum_{r=1}^d (Y_r^{(i)} \wedge Y_r^{(j)}) \prod_{s \neq r} Y_s^{(i)} Y_s^{(j)} \\ = a_{ij} + b_{ij} + c_{ij}.$$

Since the order of the three factors in each integral is not important, for the integrand of K_3 we may write

$$\sigma^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc,$$

so that K_3 itself can be written as

$$K_3 = 8(Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10})$$

Thus

$$\begin{aligned} Q_1 &= \int_{I^{3d}} a_{12} a_{23} a_{31} \lambda_3 dy \\ &= \int_{I^{3d}} \prod_{r=1}^d (mmm) \lambda_3 dy \\ &= \prod_{r=1}^d \int_{I^3} (y_1^{(r)} \wedge y_2^{(r)})(y_2^{(r)} \wedge y_3^{(r)})(y_3^{(r)} \wedge y_1^{(r)}) dy_1^{(r)} dy_2^{(r)} dy_3^{(r)} \\ &= \beta_1^d = 15^{-d} \end{aligned}$$

$$\begin{aligned} Q_2 &= \int_{I^{3d}} b_{12} b_{23} b_{31} \lambda_3 dy \\ &= (d-1)^3 \int_{I^{3d}} \prod_{r=1}^d (y_1^{(r)} y_2^{(r)} y_3^{(r)}) \lambda_3 dy \\ &= (d-1)^3 \beta_2^d = (d-1)^3 27^{-d} \end{aligned}$$

$$\begin{aligned} Q_3 &= \int_{I^{3d}} c_{12} c_{23} c_{31} \lambda_3 dy \\ &= -d \int_{I^{3d}} (mmm) \prod_2^d (ppp) \lambda_3 dy \\ &\quad - 3d(d-1) \int_{I^{3d}} (mmp) (mpp) \prod_3^d (ppp) \lambda_3 dy \\ &\quad - d(d-1)(d-2) \int_{I^{3d}} (mpp)^3 \prod_4^d (ppp) \lambda_3 dy \end{aligned}$$

$$= -d\beta_1\beta_2^{d-1} - 3d(d-1)\beta_3\beta_4\beta_2^{d-2} - d(d-1)(d-2)\beta_4^3\beta_2^{d-3}$$

$$\begin{aligned} Q_4 &= 3 \int_I 3d a_{12} a_{23} b_{31} \lambda_3 dy \\ &= 3(d-1) \int_I 3d \prod_1^d (mmp) \lambda_3 dy \\ &= 3(d-1)\beta_3^d \end{aligned}$$

$$\begin{aligned} Q_5 &= 3 \int_I 3d a_{12} a_{23} c_{31} \lambda_3 dy \\ &= -3d \int_I 3d (mmm) \prod_2^d (mmp) \lambda_3 dy \\ &= -3d\beta_1\beta_3^{d-1} \end{aligned}$$

$$\begin{aligned} Q_6 &= 3 \int_I 3d a_{12} b_{23} b_{31} \lambda_3 dy \\ &= 3(d-1)^2 \int_I 3d \prod_1^d (mpp) \lambda_3 dy \\ &= 3(d-1)^2\beta_4^d \end{aligned}$$

$$\begin{aligned} Q_7 &= 3 \int_I 3d b_{12} b_{23} c_{31} \lambda_3 dy = -3d(d-1)^2 \int_I 3d ppm \prod_2^d ppp \lambda_3 dy \\ &= -3d(d-1)^2\beta_4\beta_2^{d-1} \end{aligned}$$

$$Q_8 = 3 \int_I 3d a_{12} c_{23} c_{31} \lambda_3 dy$$

$$\begin{aligned}
 &= 3 \int_{I^{3d}} [d \text{mmmm} \cdot \prod_2 \text{mpp} + d(d-1) (\text{mmp}) (\text{mpm}) \prod_3 \text{mpp}] \lambda_3 dy \\
 &= 3d \beta_1 \beta_4^{d-1} + 3d(d-1) \beta_3^2 \beta_4^{d-2}
 \end{aligned}$$

$$Q_9 = 3 \int_{I^{3d}} b_{12} c_{23} c_{31} \lambda_3 dy$$

$$\begin{aligned}
 &= 3(d-1) \int_{I^{3d}} [d \text{pmm} \prod_2 (\text{ppp}) + d(d-1) (\text{pmp}) (\text{ppm}) \prod_3 (\text{ppp})] \lambda_3 dy \\
 &= 3d(d-1) \beta_3 \beta_2^{d-1} + 3d(d-1) \beta_4^2 \beta_2^{d-2}
 \end{aligned}$$

$$Q_{10} = 6 \int_{I^{3d}} a_{12} b_{23} c_{31} \lambda_3 dy$$

$$\begin{aligned}
 &= -6d(d-1) \int_{I^{3d}} (\text{mpm}) \prod_2 (\text{mpp}) \lambda_3 dy \\
 &= -6d(d-1) \beta_3 \beta_4^{d-1}
 \end{aligned}$$

Summing these results, Proposition 22 is proved.

PROOF OF PROPOSITION 23. Define

$$\begin{aligned}
 \gamma_1 &= \int_{I^4} (\text{mmmm}) dy \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 (w \wedge x) (x \wedge y) (y \wedge z) (z \wedge w) dw dx dy dz.
 \end{aligned}$$

Then

$$\gamma_1 = \int_0^1 I(z) dz,$$

where

$$\begin{aligned}
 I(z) &= \int_0^z \int_0^1 \int_0^1 (w \wedge x) (x \wedge y) (y \wedge z) (z \wedge w) dw dy dx \\
 &\quad + \int_z^1 \int_0^1 \int_0^1 (w \wedge x) (x \wedge y) (y \wedge z) (z \wedge w) dw dy dx \\
 &= \int_0^z \left(\int_0^1 (w \wedge x) (w \wedge z) dw \int_0^1 (y \wedge x) (y \wedge z) dy \right) dx \quad (x < z) \\
 &\quad + \int_z^1 (\dots)(\dots) dx \quad (z < x) \\
 &= \int_0^z J_1^2(x, z) dx + \int_z^1 J_1^2(z, x) dx \quad \text{by Proposition 15} \\
 &= I_1(z) + I_2(z), \quad \text{say.}
 \end{aligned}$$

Hence on utilizing $J_1(\cdot, \cdot)$ of Proposition 15 we get

$$I_1(z) = \frac{1}{36 \cdot 35} [152z^7 - 504z^6 + 420z^5],$$

$$Q_1 = \int_0^1 I_1(z) dz$$

$$= \frac{1}{36 \cdot 35} [19 - 72 + 70] = \frac{17}{36 \cdot 35}.$$

Since

$$Q_1 = \int_0^1 \int_0^z J_1^2(x, z) dx dz,$$

and

$$Q_2 = \int_0^1 \int_z^1 J_1^2(z, x) dx dz,$$

by symmetry $Q_2 = Q_1 = \frac{17}{36.35}$.

Hence

$$\gamma_1 = Q_1 + Q_2 = 2Q_1 = \frac{17}{18.35} = \frac{17}{630},$$

as required.

PROOF OF PROPOSITION 24. Define

$$\begin{aligned} \gamma_2 &= \int_I^4 (p_{\text{mmmm}}) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx(x \wedge y)(y \wedge z)(z \wedge w) dw dx dy dz. \end{aligned}$$

Then

$$\gamma_2 = \int_0^1 I(y) dy,$$

where

$$\begin{aligned} I(y) &= \int_0^1 x(x \wedge y) dx \int_0^1 w \int_0^1 (z \wedge y)(z \wedge w) dz dw \\ &= I_1(y) I_2(y). \end{aligned}$$

But

$$I_1(y) = J_2(y) = \frac{1}{6}(3y - y^3) \quad \text{by Proposition 16,}$$

and

$$I_2(y) = \int_0^y w \int_0^1 (z \wedge y)(z \wedge w) dz dw \quad (0 \leq w \leq y \leq 1)$$

$$\begin{aligned}
 & + \int_y^1 w(\dots)dw \\
 & = \int_0^y wJ_1(w,y)dw + \int_y^1 wJ_1(y,w)dw \\
 & = I_{21}(y) + I_{22}(y) \quad \text{from Proposition 15.}
 \end{aligned}$$

Hence

$$\begin{aligned}
 I_{21}(y) &= \frac{1}{6} \int_0^y w(6wy - 3wy^2 - w^3)dw \\
 &= \frac{1}{30} \int_0^y (30w^2y - 15w^2y^2 - 5w^4)dw \\
 &= \frac{1}{15} (5y^4 - 3y^5) ,
 \end{aligned}$$

and

$$\begin{aligned}
 I_{22}(y) &= \frac{1}{6} \int_y^1 w(6yw - 3yw^2 - y^3)dw \\
 &= \frac{1}{24} \int_y^1 (24yw^2 - 12yw^3 - 4y^3w)dw \\
 &= \frac{1}{24} (5y^4 - 2y^3 - 8y^4 + 5y^5) .
 \end{aligned}$$

So

$$\begin{aligned}
 I_2(y) &= I_{21}(y) + I_{22}(y) \\
 &= \frac{1}{120} (40y^4 - 24y^5 + 25y - 10y^3 - 40y^4 + 25y^5) \\
 &= \frac{1}{120} (25y - 10y^3 + y^5) .
 \end{aligned}$$

Hence,

$$\begin{aligned}
 I(y) &= I_1(y) I_2(y) \\
 &= \frac{1}{6}(3y - y^3) \frac{1}{120}(25y - 10y^3 + y^5) \\
 &= \frac{1}{720} (75y^2 - 30y^4 + 3y^6 - 25y^4 + 10y^6 - y^8) \\
 &= \frac{1}{63.720} (63.75y^2 - 63.55y^4 + 63.13y^6 - 63y^8).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \gamma_2 &= \int_0^1 I(y) dy \\
 &= \frac{1}{63.720} (21.75 - 63.11 + 9.13 - 7) \\
 &= \frac{1}{63.720} (1692 - 700) = \frac{992}{63.720} \\
 &= \frac{124}{63.90} = \frac{62}{63.45} = \frac{1}{45} - \frac{1}{45.63}.
 \end{aligned}$$

PROOF OF PROPOSITION 25. Define

$$\begin{aligned}
 \gamma_3 &= \int_I^4 (ppmm) dy \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx^2 Y(y \wedge z) (z \wedge w) dw dx dy dz.
 \end{aligned}$$

Then

$$\begin{aligned} \gamma_3 &= \int_0^1 x^2 dx \int_0^1 \left(\int_0^1 y(y \wedge z) dy \int_0^1 w(w \wedge z) dw \right) dz \\ &= \frac{1}{3} \int_0^1 J_2^2(z) dz \quad \text{by Proposition 16.} \end{aligned}$$

Hence

$$\begin{aligned} J_2^2(z) &= \frac{1}{36}(9z^2 - 6z^4 + z^6) \\ &= \frac{1}{36.35}(9.35z^2 - 6.35z^4 + .35z^6), \end{aligned}$$

so

$$\begin{aligned} \gamma_3 &= \frac{1}{3} \cdot \frac{1}{36.35} \cdot (3.35 - 42 + 5) \\ &= \frac{68}{3.36.35} = \frac{17}{27.35} = \frac{17}{21.45} \end{aligned}$$

PROOF OF PROPOSITION 26. Define

$$\begin{aligned} \gamma_4 &= \int_I (pmpm) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 wx(x \wedge y)yz(z \wedge w) dw dx dy dz. \end{aligned}$$

Then

$$\gamma_4 = \int_0^1 x \int_0^1 y(y \wedge x) dy dx \int_0^1 w \int_0^1 z(z \wedge w) dz dw$$

$$= I^2.$$

where

$$\begin{aligned}
 I &= \int_0^1 x \int_0^1 y(y \wedge x) dy dx \\
 &= \int_0^1 x J_2(x) dx \quad \text{by Proposition 16} \\
 &= \frac{1}{6} \int_0^1 x(3x - x^3) dx \\
 &= \frac{1}{30}(5-1) = \frac{2}{15},
 \end{aligned}$$

and so $\gamma_4 = \frac{4}{225}$.

PROOF OF PROPOSITION 27. Define

$$\begin{aligned}
 \gamma_5 &= \int_I (m p p p) dy \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 (w \wedge x) x y^2 z^2 w dw dx dy dz.
 \end{aligned}$$

Then,

$$\begin{aligned}
 \gamma_5 &= \int_0^1 y^2 dy \int_0^1 z^2 dz \int_0^1 w \int_0^1 x(w \wedge x) dx du \\
 &= \frac{1}{9} \int_0^1 w J_2(w) dw \quad \text{by Proposition 16} \\
 &= \frac{1}{54} \int_0^1 w(3w - w^3) dw \\
 &= \frac{1}{270}(5-1) = \frac{2}{135}.
 \end{aligned}$$

PROOF OF PROPOSITION 29. The next eleven steps.

1. From Proposition 8 we have

$$K_4 = 3.2^4 H_4 = 48H_4,$$

where

$$H_4 = \int_I 4d \sigma_{12} \sigma_{23} \sigma_{34} \sigma_{41} \lambda_4 dy$$

$$= \int_I 4d \theta(y) \lambda_4 dy, \text{ say.}$$

2. By definition

$$\sigma_{ij}^{\bullet} = ET(y^{(i)})T(y^{(j)})$$

$$= \prod_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) + (d-1) \prod_{r=1}^d y_r^{(i)} y_r^{(j)}$$

$$- \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)}$$

$$= a_{ij} + b_{ij} + c_{ij}.$$

3. If order were not important we could write $\theta(y) = (a + b + c)^4$.

Then we would have

$$\theta(y) = a^4 + b^4 + c^4 + 4a^3b + 4ab^3 + 4a^3c + 4ac^3 + 4b^3c + 4bc^3 +$$

$$+ 6a^2b^2 + 6b^2c^2 + 6a^2c^2 + 12a^2bc + 12ab^2c + 12abc^2,$$

which indicates that we have 81 terms to evaluate.

4. First consider terms in a, b only, where order is not important.

$$Q_1 = \int_I 4d a_{12} a_{23} a_{34} a_{41} \lambda_4 dy$$

$$= \prod_{r=1}^d \int_I 4 (\text{mmmm}) \lambda_1 dy$$

$$= \gamma_1^d = \left(\frac{17}{630}\right)^d, \quad \text{by Proposition 23.}$$

$$Q_2 = \int_I 4d b_{12} b_{23} b_{34} b_{41} \lambda_4 dy$$

$$= (d-1)^4 \prod_{r=1}^d \int_I 4 (\text{pppp}) dy$$

$$= (d-1)^4 \gamma_6^d = (d-1)^4 81^{-d} \quad \text{by Proposition 28.}$$

5. To integrate $\theta_3 = 4a^3b$ and $\theta_4 = 4ab^3$ we note again that the order of the factors does not affect the integral. So

$$Q_3 = \int_I 4d 4a^3 b \lambda_4 dy$$

$$= 4(d-1) \prod_{r=1}^d \int_I 4 (\text{mmp}) dy$$

$$= 4(d-1) \gamma_2^d \quad \text{by Proposition 24.}$$

$$Q_4 = \int_I 4d 4ab^3 \lambda_4 dy$$

$$= 4(d-1)^3 \prod_{r=1}^d \int_I 4 (\text{mppp}) dy$$

$$= 4(d-1)^3 \gamma_5^d \quad \text{by Proposition 27.}$$

6. The term $6a^2b^2$. Here order is important. It is of interest to list the 6 terms separately as:

$$a_{12}a_{23}b_{34}b_{41}, \quad b_{12}b_{23}a_{34}a_{41},$$

$$a_{12}b_{23}a_{34}b_{41}, \quad b_{12}a_{23}b_{34}a_{41},$$

$$a_{12}b_{23}b_{34}a_{41}, \quad b_{12}a_{23}a_{34}b_{41}.$$

Each is a product of d identical terms of two patterns (hence we will have two types of integrals):

$$\text{pattern } mmpm \approx pmmp \approx pmpm \approx mppm$$

as

$$a_{12}a_{23}b_{34}b_{41}, \quad b_{12}b_{23}a_{34}a_{41}$$

$$a_{12}b_{23}b_{34}a_{41}, \quad b_{12}a_{23}a_{34}b_{41},$$

$$\text{and pattern } mpmp \approx pmpm$$

as

$$a_{12}b_{23}a_{34}b_{41} \text{ and } b_{12}a_{23}b_{34}a_{41}$$

Hence we have the two integrals

$$Q_5 = 4 \int_I d a_{12} a_{23} b_{34} b_{41} \lambda_4 dy$$

$$= 4(d-1)^2 \prod_{r=1}^d \int_I (mmpm) dy$$

$$= 4(d-1) \gamma_3^d \quad \text{by Proposition 25,}$$

and

$$Q_6 = 2 \int_1^d a_{12} b_{23} a_{34} b_{41} \lambda_4 dy,$$

$$= 2(d-1)^2 \prod_{r=1}^d \int_1^d (mpmp) dy$$

$$= 2(d-1)^2 \gamma_4^d \quad \text{by Proposition 26.}$$

$$\text{Note: } \gamma_3 = \frac{17}{21.45} \neq \gamma_4 = \frac{4}{225}.$$

7. Consider now integrals in terms of C_{ij} to first power, where order is not important:

$$\theta_7 \approx 4a^3 c, \quad \theta_8 \approx 4b^3 c.$$

We have

$$C_{ij} = \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)},$$

which yields a sum of d identical integrals. Hence, using the equivalence relation defined in Proposition 21, we have

$$C_{ij} = -d(y_1^{(i)} \wedge y_1^{(j)}) \prod_{s=2}^d y_s^{(i)} y_s^{(j)}$$

$$= -dm \prod_2^d p.$$

Hence

$$\theta_7 = -4d(\text{mmmm}) \prod_2^d (\text{pmmmm}) ,$$

and so

$$\begin{aligned} Q_7 &= -4d \int_1^{4d} (\text{mmmm}) \prod_2^d (\text{pmmmm}) \lambda_4 dy \\ &= -4d \int_1^4 (\text{mmmm}) dy \prod_2^d \int_1^4 (\text{pmmmm}) dy \\ &= -4d \gamma_1 \gamma_2^{d-1} \quad \text{by Propositions 23 and 24.} \end{aligned}$$

Similarly

$$\begin{aligned} Q_8 &= -4d(d-1) \int_1^3 (\text{mppp}) dy \prod_2^d \int_1^4 (\text{pppp}) dy \\ &= -4d(d-1) \gamma_5 \gamma_6^{d-1} . \end{aligned}$$

8. Integrals with c_{ij} to first power where order is important.

Consider the terms

$$12a^2bc \quad \text{and} \quad 12ab^2c .$$

We can fix c as the fourth factor to obtain 3 cases, each occurring 4 times:

$$a_{12}a_{23}b_{34}c_{41} , \quad b_{12}b_{23}a_{34}c_{41}$$

$$a_{12}b_{23}a_{34}c_{41} , \quad b_{12}a_{23}b_{34}c_{41}$$

$$b_{12}a_{23}a_{34}c_{41} , \quad a_{12}b_{23}b_{34}c_{41}$$

and the first and last two of each give identical integrals.

Hence we have

$$\begin{aligned} Q_9 &= 8 \int_I 4d a_{12} a_{23} b_{34} c_{41} \lambda_4 dy \\ &= -8d(d-1) \int_I 4 (mmpm) dy \prod_2^d \int_I 4 (mmp) dy \end{aligned}$$

$$Q_9 = -8d(d-1) \gamma_2 \gamma_3^{d-1} \quad \text{by Propositions 24 and 25,}$$

and

$$\begin{aligned} Q_{10} &= 4 \int_I 4d a_{12} b_{23} a_{34} c_{41} \lambda_4 dy \\ &= -4d(d-1) \int_I 4 (mpmm) dy \prod_2^d \int_I 4 (mpm) dy \end{aligned}$$

$$Q_{10} = -4d(d-1) \gamma_2 \gamma_4^{d-1} .$$

Similarly,

$$\begin{aligned} Q_{11} &= 8 \int_I 4d b_{12} b_{23} a_{34} c_{41} \lambda_4 dy \\ &= -8d(d-1)^2 \int_I 4 (ppmm) dy \prod_2^d \int_I 4 (ppm) dy \\ &= -8d(d-1)^2 \gamma_3 \gamma_5^{d-1} \quad \text{by Proposition 25 and 27 ,} \end{aligned}$$

and

$$\begin{aligned}
 Q_{12} &= 4 \int_I 4d \, b_{12} a_{23} b_{34} c_{41}^{\lambda_4} dy \\
 &= -4d(d-1)^2 \int_I 4 \, (pmpm) dy \prod_{2} \int_I 4 \, (pmpp) dy \\
 &= -4d(d-1)^2 \gamma_4^{d-1} \gamma_5 \quad \text{by Propositions 26 and 27.}
 \end{aligned}$$

9. Integrals with c_{ij} to second power. Consider the terms

$$6a^2 c^2, \quad 6b^2 c^2, \quad 12abc^2.$$

From Proposition 21

$$\begin{aligned}
 c_{ij} c_{km} &= d(y_1^{(i)} \wedge y_1^{(j)})(y_1^{(k)} \wedge y_1^{(m)}) \prod_{r=2}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} \\
 &\quad + d(d-1)(y_1^{(i)} \wedge y_1^{(j)})(y_1^{(k)} \wedge y_1^{(m)}) y_2^{(i)} y_2^{(j)} y_2^{(k)} y_2^{(m)} \\
 &\quad + \prod_{r=3}^d y_r^{(i)} y_r^{(j)} y_r^{(k)} y_r^{(m)} \\
 &= dmm \prod_2 pp + d(d-1)(mp)(pm) \prod_3 pp.
 \end{aligned}$$

The form $u^2 c^2$ can arise as the following products

$$u_{12}^u u_{23}^c u_{34}^c u_{41}^c, \quad c_{12}^u u_{23}^u u_{34}^c u_{41}^c,$$

$$u_{12}^c u_{23}^c u_{34}^u u_{41}^u, \quad c_{12}^c u_{23}^u u_{34}^u u_{41}^u$$

and

$$u_{12}^c u_{23}^u u_{34}^c u_{41}^c, \quad c_{12}^u u_{23}^c u_{34}^u u_{41}^c$$

Hence we can define integrals $Q_{13}, Q_{14}, Q_{15}, Q_{16}$ below.

$$\begin{aligned} Q_{13} &= 4 \int_I 4d \, a_{12} a_{23} c_{34} c_{41} \lambda_4 dy \\ &= 4d \int_I 4 \, (nnmm) dy \prod_2^d \int_I 4 \, (mnp) dy \\ &\quad + 4d(d-1) \int_I 4 \, (mmpm) dy \int_I 4 \, (mmp) dy \prod_3^d \int_I 4 \, (mnp) dy . \end{aligned}$$

Thus

$$Q_{13} = 4d \gamma_1 \gamma_3^{d-1} + 4d(d-1) \gamma_2^2 \gamma_3^{d-2} .$$

$$\begin{aligned} Q_{14} &= 4d(d-1)^2 \int_I 4 \, (ppmm) dy \prod_2^d \int_I 4 \, (pppp) dy \\ &\quad + 4d(d-1)^3 \left(\int_I 4 \, (pppm) dy \right)^2 \prod_3^d \int_I 4 \, (pppp) dy . \end{aligned}$$

So

$$Q_{14} = 4d(d-1)^2 \gamma_3 \gamma_6^{d-1} + 4d(d-1)^3 \gamma_5^2 \gamma_6^{d-2} .$$

$$\begin{aligned} Q_{15} &= 2d \int_I 4 \, (nnmm) dy \prod_2^d \int_I 4 \, (mmp) dy \\ &\quad + 2d(d-1) \left(\int_I 4 \, (mmpm) dy \right)^2 \prod_3^d \int_I 4 \, (mmp) dy \\ &= 2d \gamma_1 \gamma_4^{d-1} + 2d(d-1) \gamma_2^2 \gamma_4^{d-2} . \end{aligned}$$

$$\begin{aligned}
 Q_{16} &= 2d(d-1)^2 \int_4 (mpmp) dy \prod_2^d \int_4 (pppp) dy \\
 &\quad + 2d(d-1)^3 \left(\int_4 (mppp) dy \right)^2 \prod_3^d \int_4 (pppp) dy \\
 &= 2d(d-1)^2 \gamma_4 \gamma_6^{d-1} + 2d(d-1)^3 \gamma_5 \gamma_6^{d-2}.
 \end{aligned}$$

As to integrating the terms represented as $12abc^2$, there are 12 of these instead of 6 because a and b are interchanged. This however does not affect the integral: there are still only two types of integrands, namely

$$\theta_{17} = 8a_{12}^b b_{23}^c c_{34}^c c_{41}$$

and

$$\theta_{18} = 4a_{12}^c c_{23}^b b_{34}^c c_{41}$$

giving

$$\begin{aligned}
 Q_{17} &= 8d(d-1) \int_4 (mpmm) dy \prod_2^d \int_4 (mppp) dy \\
 &\quad + 8d(d-1)^2 \int_4 (mpmp) dy \int_4 (mppm) dy \prod_3^d \int_4 (mppp) dy \\
 &= 8d(d-1) \gamma_2 \gamma_5^{d-1} + 8d(d-1)^2 \gamma_3 \gamma_4 \gamma_5^{d-2}.
 \end{aligned}$$

$$\begin{aligned}
 Q_{18} &= 4d(d-1) \int_4 (mmpm) dy \prod_2^d \int_4 (mppp) dy \\
 &\quad + 4d(d-1)^2 \int_4 (mmpm) dy \int_4 (mppm) dy \prod_3^d \int_4 (mppp) dy \\
 &= 4d(d-1) \gamma_2 \gamma_5^{d-1} + 4d(d-1)^2 \gamma_3^2 \gamma_5^{d-2}.
 \end{aligned}$$

10. Integrals with c_{ij} to third power in terms of

$$4ac^3, \quad 4bc^3.$$

Here again only one form is possible, but $c_{23}c_{34}c_{41}$ is complicated.

From Proposition 21

$$\begin{aligned} -c_{23}c_{34}c_{41} &= d(\text{mmm}) \prod_2^d (\text{ppp}) \\ &+ d(d-1)(\text{mmp})(\text{ppm}) \prod_3^d (\text{ppp}) + d(d-1)(\text{mpm})(\text{pmp}) \prod_3^d (\text{ppp}) \\ &+ d(d-1)(\text{mpp})(\text{pmm}) \prod_3^d (\text{ppp}) + d(d-1)(d-2)(\text{mpp})(\text{pmp})(\text{ppm}) \prod_4^d (\text{ppp}). \end{aligned}$$

Thus

$$Q_{19} = 4 \int_I^d a_{12} c_{23} c_{34} c_{41} \lambda^d dy.$$

The last term has then the integral

$$\prod_{k+1}^d \int_I^4 (\text{mppp}) dy = \gamma_5^{d-k}, \quad k = 1, 2, 3.$$

Thus

$$\begin{aligned} Q_{19} &= -4d \left(\int_I^4 (\text{mmm}) dy \right) \gamma_5^{d-1} \\ &- 4d(d-1) \left(\int_I^4 (\text{mmp}) dy \right) \int_I^4 (\text{mppm}) dy \gamma_5^{d-2} \\ &- 4d(d-1) \left(\int_I^4 (\text{mpm}) dy \right) \int_I^4 (\text{mpmp}) dy \gamma_5^{d-2} \\ &- 4d(d-1) \left(\int_I^4 (\text{mpp}) dy \right) \int_I^4 (\text{mpmm}) dy \gamma_5^{d-2} \end{aligned}$$

$$- 4d(d-1)(d-2) \left(\int_4 (mmpp) dy \int_4 (mpmp) dy \int_4 (mppm) dy \right) \gamma_5^{d-3}.$$

The second and fourth terms are identical as are the first and third integral factors of the last term. Thus

$$\begin{aligned} Q_{19} = & -4d\gamma_1\gamma_5^{d-1} - 8d(d-1)\gamma_2\gamma_3\gamma_5^{d-2} - 4d(d-1)\gamma_2\gamma_4\gamma_5^{d-2} \\ & - 4d(d-1)(d-2)\gamma_3\gamma_4\gamma_5^{d-3}. \end{aligned}$$

Similarly,

$$Q_{20} = 4 \int_4 b_{12} c_{23} c_{34} c_{41} \lambda_4 dy,$$

and the last term is then

$$\prod_{k=1}^d \int_4 (pppp) dy = \gamma_6^{d-k}, \quad k = 1, 2, 3.$$

Thus

$$\begin{aligned} Q_{20} = & -4d(d-1) \left(\int_4 (pmmm) dy \right) \gamma_6^{d-1} \\ & - 4d(d-1)^2 \left(\int_4 (pmmp) dy \int_4 (pppm) dy \right) \gamma_6^{d-2} \\ & - 4d(d-1)^2 \left(\int_4 (pmpm) dy \int_4 (ppmp) dy \right) \gamma_6^{d-2} \\ & - 4d(d-1)^2 \left(\int_4 (pmpp) dy \int_4 (ppmm) dy \right) \gamma_6^{d-2} \\ & - 4d(d-1)^2 (d-2) \left(\int_4 (mppp) dy \right) \gamma_6^{d-3}. \end{aligned}$$

So

$$Q_{20} = -4d(d-1)\gamma_2\gamma_6^{d-1} - 8d(d-1)^2\gamma_3\gamma_5\gamma_6^{d-2}$$

$$-4d(d-1)^2\gamma_4\gamma_5\gamma_6^{d-2} - 4d(d-1)^2(d-2)\gamma_5^3\gamma_6^{d-3}$$

11. We are left with the integral

$$Q_{21} = \int_I 4d (c_{12}c_{23}c_{34}c_{41})^{\lambda_4} dy$$

We evaluate this by expanding from the equivalent integral of $c_{23}c_{34}c_{41}$ in the previous paragraph. Below, in a table form, we write the integral in the second column for each term of the integrand. The last factor in each integral is

$$\prod_{k=1}^d \int_I (pppp) dy = \gamma_6^{d-k}, \quad k = 1, 2, 3, 4.$$

The integrand of Q_{21}	The integral
$\theta_{21} = c_{12}c_{23}c_{34}c_{41} = d(\text{mmmm}) \prod_2^d (pppp)$	$Q_{21} = d\gamma_1\gamma_6^{d-1}$
$+ d(d-1)(\text{mmpm})(pppm) \prod_3^d (pppp)$	$+ d(d-1)\gamma_2\gamma_5\gamma_6^{d-2}$
$+ d(d-1)(\text{mpmp})(ppmp) \prod_3^d (pppp)$	$+ d(d-1)\gamma_2\gamma_5\gamma_6^{d-2}$
$+ d(d-1)(\text{mmpm})(ppmm) \prod_3^d (pppp)$	$+ d(d-1)\gamma_3^2\gamma_6^{d-2}$
$+ d(d-1)(d-2)(\text{mmpm})(ppmp)(pppm) \prod_4^d p^4$	$+ d(d-1)(d-2)\gamma_3^2\gamma_5\gamma_6^{d-3}$
$+ d(d-1)(\text{mpmm})(pmpm) \prod_3^d p^4$	$+ d(d-1)\gamma_2\gamma_5\gamma_6^{d-2}$
$+ d(d-1)(\text{mpmp})(pmpm) \prod_3^d p^4$	$+ d(d-1)\gamma_4^2\gamma_6^{d-2}$

θ_{21} Cont'd.	Q_{21} Cont'd.
$+ d(d-1)(d-2)(mppm)(pmpp)(pppm) \prod_{4}^{d} p^4$	$+ d(d-1)(d-2)\gamma_4\gamma_5^2\gamma_6^{d-3}$
$+ d(d-1)(mppm)(pmpp) \prod_{3}^{d} p^4$	$+ d(d-1)\gamma_3\gamma_6^{d-2}$
$+ d(d-1)(mppp)(pmmm) \prod_{3}^{d} p^4$	$+ d(d-1)\gamma_2\gamma_5\gamma_6^{d-2}$
$+ d(d-1)(d-2)(mppp)(pmpp)(pppm) \prod_{4}^{d} p^4$	$+ d(d-1)(d-2)\gamma_3\gamma_5^2\gamma_6^{d-3}$
$+ d(d-1)(d-2)(mppm)(pmpp)(ppmp) \prod_{4}^{d} p^4$	$+ d(d-1)(d-2)\gamma_3\gamma_5^2\gamma_6^{d-3}$
$+ d(d-1)(d-2)(mppp)(pmpp)(ppmp) \prod_{4}^{d} p^4$	$+ d(d-1)(d-2)\gamma_4\gamma_5^2\gamma_6^{d-3}$
$+ d(d-1)(d-2)(mppp)(pmpp)(ppmm) \prod_{4}^{d} p^4$	$+ d(d-1)(d-2)\gamma_3\gamma_5^2\gamma_6^{d-3}$
$+ d(d-1)(d-2)(d-3)(mppp)(pmpp)$ $(ppmp)(pppm) \prod_{5}^{d} p^4$	$+ d(d-1)(d-2)(d-3)\gamma_5^4\gamma_6^{d-4}$

The above terms, when the like ones are collected, now amount to Proposition 29 of Section 5.

Intermediate Notes towards Calculating the fifth Cumulant K_5

- (1) It is tempting to collect together terms with common factors. This did not seem to introduce any great simplification, but it did introduce errors. So we eventually chose to evaluate the terms as they stand. We used integer arithmetic, reducing terms to a common denominator.

- (2) For calculating the fifth cumulant we chose to persist with a tedious method which is free of surprises, rather than spend time seeking a less laborious method. If we were to calculate the sixth and higher cumulants we would have to invest time in a search for a less laborious method. At present we expand the σ_{ij} in three terms as

$$\sigma_{ij} = a_{ij} + b_{ij} + c_{ij},$$

where c_{ij} is a finite series.

We can of course write

$$\begin{aligned} \sigma_{ij} &= \left(\prod_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \right) - \prod_{r=1}^d y_r^{(i)} y_r^{(j)} \\ &= \sum_{r=1}^d [(y_r^{(i)} \wedge y_r^{(j)}) - y_r^{(i)} y_r^{(j)}] \prod_{s \neq r} y_s^{(i)} y_s^{(j)}, \end{aligned}$$

so if we could express σ_{ij} in two convenient terms as, say,

$$\sigma_{ij} = g_{ij} + h_{ij},$$

then the n^{th} cumulant would only require the evaluation of 2^n terms instead of 3^n ; a great saving.

PROOF OF PROPOSITION 30. Define

$$\begin{aligned} \delta_Y &= \int_I (\text{xxxxxx}) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (v \wedge w) (w \wedge x) (x \wedge y) (y \wedge z) (z \wedge v) dv dw dx dy dz. \end{aligned}$$

and let

$$\delta_1 = \int_0^1 I(v) dv = \int_0^1 (I_1(v) + I_2(v)) dv$$

with

$$I_1(v) = \int_0^v I_{11}(v, x) dx,$$

$$I_2(v) = \int_v^1 I_{21}(v, x) dx,$$

$$I_{11}(v, x) = I_{111}(v, x) I_{112}(v, x),$$

$$I_{21}(v, x) = I_{211}(v, x) I_{212}(v, x),$$

where

$$\begin{aligned} I_{111}(v, x) &= \int_0^1 (v \wedge w) (w \wedge x) dw, \quad 0 \leq x \leq v \leq 1, \\ &= J_1(x, v) \end{aligned}$$

$$= \frac{1}{6}(6xv - 3xv^2 - x^3), \quad \text{by Proposition 15,}$$

$$\begin{aligned} I_{211}(v, x) &= \int_0^1 (v \wedge w) (w \wedge x) dw, \quad 0 \leq v \leq x \leq 1, \\ &= J_1(v, x) \end{aligned}$$

$$= \frac{1}{6}(6vx - 3vx^2 - v^3), \quad \text{By Proposition 15,}$$

$$I_{112}^{(v,x)} = \int_0^1 \int_0^1 (x \wedge y) (y \wedge z) (z \wedge v) dy dz, \quad 0 \leq x \leq v \leq 1,$$

$$= I_{1121}^{(v,x)} + I_{1122}^{(v,x)} + I_{1123}^{(v,x)},$$

$$I_{1121}^{(v,x)} = \int_0^x (z \wedge v) \int_0^1 (x \wedge y) (y \wedge z) dy dz, \quad 0 \leq z \leq x \leq v \leq 1,$$

$$= \int_0^x z J_1(z, x) dz$$

$$= \frac{1}{6} \int_0^x z (6zx - 3zx^2 - z^3) dz, \quad \text{by Proposition 15,}$$

$$= \frac{1}{120} \int_0^x (120z^2x - 60z^2x^2 - 20z^4) dz$$

$$= \frac{1}{120} (40x^4 - 24x^5),$$

$$I_{1122}^{(v,x)} = \int_x^v (z \wedge v) \int_0^1 (x \wedge y) (y \wedge z) dy dz, \quad 0 \leq x \leq z \leq v \leq 1,$$

$$= \int_x^v z J_1(x, z) dz$$

$$= \frac{1}{6} \int_x^v z (6xz - 3xz^2 - x^3) dz, \quad \text{by Proposition 15,}$$

$$= \frac{1}{120} \int_x^v (120xz^2 - 60xz^3 - 20x^3z) dz$$

$$= \frac{1}{120} (40xv^3 - 15xv^4 - 10x^3v^2 - 40x^4v + 25x^5),$$

$$\begin{aligned}
 I_{1123}(v, x) &= \int_v^1 (z \wedge v) \int_0^1 (x \wedge y) (y \wedge z) dy dz, \quad 0 \leq x \leq v \leq z \leq 1, \\
 &= \int_v^1 v J_1(x, z) dz \\
 &= \frac{v}{6} \int_v^1 (6xz - 3xz^2 - x^3) dz, \quad \text{by Proposition 15,} \\
 &= \frac{1}{6} v (2x - x^3 - 3xv^2 + xv^3 + x^3v) \\
 &= \frac{1}{120} (40xv - 20x^3v - 60xv^3 + 20xv^4 + 20x^3v^2).
 \end{aligned}$$

Also,

$$I_{1122}(v, x) = \frac{1}{120} (40xv^3 - 15xv^4 - 10x^2v^2 - 40x^4 + 25x^5)$$

$$I_{1121}(v, x) = \frac{1}{120} (40x^4 - 24x^5).$$

Hence

$$I_{112}(v, x) = \frac{1}{120} (40xv - 20x^3v - 20x^3 + 5xv^4 + 10x^3v^2 + x^5),$$

$$I_{111}(v, x) = \frac{1}{6} (6xv - 3xv^2 - x^3).$$

But

$$\begin{aligned}
 I_{11}(v, x) &= I_{111}(v, x) I_{112}(v, x) \\
 &= \frac{1}{720} (240x^2v^2 - 120x^4v^2 - 120x^2v^4 + 90x^2v^5 + 140x^4v^3 + 26x^6v \\
 &\quad - 120x^2v^3 - 15x^2v^6 - 35v^4v^4 - 13x^6v^2 - 40x^4v^5 - x^8).
 \end{aligned}$$

Hence

$$\begin{aligned} I_1(v) &= \int_0^v I_{11}(v, x) dx \\ &= \frac{1}{63.720} (80.63v^5 - 48.63v^6 - 64.63v^7 + 9.432v^8 - 880v^9). \end{aligned}$$

So

$$\begin{aligned} \int_0^1 I_1(v) dv &= \frac{1}{63.720} (40.21 - 48.9 - 8.63 + 432 - 88) \\ &= \frac{248}{63.720} = \frac{31}{5670}. \end{aligned}$$

It is evident that $\int_0^1 I_2(v) dv = \int_0^1 I_1(v) dv$, and hence

$$\delta_1 = \frac{31}{5670} + \frac{31}{5670} = \frac{31}{2835}.$$

PROOF OF PROPOSITION 31. Define

$$\begin{aligned} \delta_2 &= \int_0^1 (p_{11111}) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw(w \wedge x)(x \wedge y)(y \wedge z)(z \wedge v) dv dw dx dy dz. \end{aligned}$$

Then

$$\delta_2 = \int_0^1 I_1(y) I_2(y) dy = \int_0^1 I_1^2(y) dy,$$

since

$$\begin{aligned} I_1(y) &= \int_0^1 \int_0^1 w(w \wedge x)(x \wedge y) dx dw \\ &= I_2(y) = \int_0^1 \int_0^1 v(v \wedge z)(z \wedge y) dz dv. \end{aligned}$$

Let now

$$I_1(y) = I_{11}(y) + I_{12}(y) ,$$

where

$$I_{11}(y) = \int_0^y w \int_0^1 (w \wedge x) (x \wedge y) dx dw , \quad 0 \leq w \leq y \leq 1 ,$$

$$= \int_0^y w J_1(w, y) dw$$

$$= \frac{1}{6} \int_0^y w (6wy - 3wy^2 - w^3) dw , \quad \text{by Proposition 15,}$$

$$= \frac{1}{30} \int_0^y (30w^2y - 15w^2y^2 - 5w^4) dw$$

$$= \frac{1}{15} (5y^4 - 3y^5) ,$$

and

$$I_{12}(y) = \int_y^1 w \int_0^1 (w \wedge x) (x \wedge y) dx dw , \quad 0 \leq y \leq w \leq 1 ,$$

$$= \int_y^1 w J_1(y, w) dw , \quad \text{by Proposition 15,}$$

$$= \frac{1}{6} \int_y^1 w (6yw - 3yw^2 - y^3) dw$$

$$= \frac{1}{24} \int_y^1 (24yw^2 - 12yw^3 - 4y^3w) dw$$

$$= \frac{1}{120} (25y - 10y^3 - 40y^4 + 25y^5) .$$

So

$$\begin{aligned} I_1(y) &= I_{11}(y) + I_{12}(y) \\ &= \frac{1}{120}(25y - 10y^3 + y^5) \\ &= \frac{y}{120}(y^2 - 5)^2, \end{aligned}$$

and

$$I_1^2(y) = \left(\frac{y}{120}\right)^2 (y^2 - 5)^4.$$

So

$$\delta_2 = (120)^{-2} \int_0^1 y^2 (y^2 - 5)^4 dy,$$

and hence, as required,

$$\delta_2 = \frac{120^{-2}}{7.9.11} \cdot 88448 = \frac{1382}{7.9.11.15.15} = \frac{1382}{155925}.$$

PROOF OF PROPOSITION 32. Define

$$\begin{aligned} \delta_3 &= \int_5^1 (ppmm) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2 x(x \wedge y) (y \wedge z) (z \wedge v) dv dx dy dz. \end{aligned}$$

Since

$$\int_0^1 w^2 dw = \frac{1}{3},$$

so

$$\delta_3 = \frac{1}{3} \int_0^1 \int_0^1 \int_0^1 \int_0^1 vx(x \wedge y) (y \wedge z) (z \wedge v) dv dx dy dz$$

$$= \frac{1}{3} \gamma_2, \text{ by Proposition 24,}$$

$$= \frac{1}{3} \frac{62}{45.63} = \frac{62}{8505}$$

PROOF OF PROPOSITION 33. Define

$$\delta_4 = \int_5 (pmpmm) dy$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw(w \wedge x)xy(y \wedge z)(z \wedge v)dvdx dydz.$$

$$\delta_4 = I_1 I_2$$

$$I_1 = \int_0^1 \int_0^1 w(w \wedge x)xdxdw,$$

$$I_2 = \int_0^1 \int_0^1 \int_0^1 y(y \wedge z)(z \wedge v)vdvdydz$$

$$= \beta_3, \text{ by Proposition 18,}$$

$$= \frac{17}{315},$$

and

$$I_1 = \int_0^1 w \int_0^1 x(w \wedge x)xdxdw$$

$$= \int_0^1 wJ_2(w)dw$$

$$\begin{aligned}
 &= \frac{1}{6} \int_0^1 w(3w - w^3) dw, \text{ by Proposition 16,} \\
 &= \frac{2}{15}.
 \end{aligned}$$

So

$$\delta_4 = \frac{17}{315} \frac{2}{15} = \frac{34}{4725} \text{ as required.}$$

PROOF OF PROPOSITION 34. Define

$$\begin{aligned}
 \delta_5 &= \int_5 (pppmm) dy \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2 x^2 y(y \wedge z)(z \wedge v) dv dw dx dy dz.
 \end{aligned}$$

$$\delta_5 = \frac{1}{9} \int_0^1 \int_0^1 \int_0^1 vy(y \wedge z)(z \wedge v) dv dy dz$$

$$= \frac{1}{9} \beta_3, \text{ by Proposition 18,}$$

$$= \frac{1}{9} \frac{17}{315} = \frac{17}{2835}.$$

PROOF OF PROPOSITION 35. Define

$$\begin{aligned}
 \delta_6 &= \int_5 (ppmpm) dy \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2 x(x \wedge y)yz(z \wedge v) dv dw dx dy dz.
 \end{aligned}$$

Then

$$\begin{aligned}\delta_6 &= \frac{1}{3} \int_0^1 \int_0^1 x(x \wedge y) y dx dy \int_0^1 \int_0^1 z(z \wedge v) v dz dv \\ &= \frac{1}{3} I_1^2 = \frac{1}{3} \left(\frac{2}{15}\right)^2 = \frac{4}{675},\end{aligned}$$

as shown in proof of Proposition 33.

PROOF OF PROPOSITION 36. Define

$$\begin{aligned}\delta_7 &= \int_1^5 (ppppm) dy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 vw^2 x^2 y^2 z(z \wedge v) v dv dw dx dy dz.\end{aligned}$$

Since

$$\int_0^1 s^2 ds = \frac{1}{3},$$

so

$$\begin{aligned}\delta_7 &= \frac{1}{27} \int_0^1 \int_0^1 z(z \wedge v) v dv dz \\ &= \frac{1}{27} \frac{2}{15} = \frac{2}{405} \text{ as shown in proof of Proposition 33.}\end{aligned}$$

PROOF OF PROPOSITION 38. The fifth cumulant K_5 of the distribution Γ_d is

$$K_5 = 32.12 H_5, \text{ where } H_5 \text{ is listed in Section 5.}$$

1. From Proposition 9 we have

$$K_5 = 2^4 4! H_5 = 32.12 H_5,$$

where

$$H_5 = \int_I 5d \sigma_{12} \sigma_{23} \sigma_{34} \sigma_{45} \sigma_{51} \lambda_5 dy,$$

and here we have to evaluate H_5 .

2. We have

$$\begin{aligned} \sigma_{ij} &= \prod_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) + (d-1) \prod_{r=1}^d y_r^{(i)} y_r^{(j)} \\ &\quad - \sum_{r=1}^d (y_r^{(i)} \wedge y_r^{(j)}) \prod_{s \neq r} y_s^{(i)} y_s^{(j)} \end{aligned}$$

$$= a_{ij} + b_{ij} + c_{ij}, \text{ say,}$$

and we expand the product $\sigma_{12} \sigma_{23} \sigma_{34} \sigma_{45} \sigma_{51}$ as $243 = 3^5$ terms, group similar terms, and integrate term by term.

3. As a guide we have the expansion

$$\begin{aligned} (a + b + c)^5 &= a^5 + b^5 + c^5 \\ &\quad + 5a^4b + 5ab^4 + 5a^4c + 5ac^4 \\ &\quad + 5b^4c + 5bc^4 \\ &\quad + 10a^3b^2 + 10a^2b^3 + 10a^3c^2 + 10a^2c^3 \\ &\quad + 10b^3c^2 + 10b^2c^3 \\ &\quad + 20a^3bc + 20ab^3c + 20abc^3 \\ &\quad + 30a^2b^2c + 30a^2bc^2 + 30ab^2c^2, \end{aligned}$$

and note that the cross-product terms have to be subdivided to account for the different permutations of their factors, giving rise to different integrals as in the previous propositions.

4. The logic of the integration has been described in detail in Proposition 29, so here we will just write down the results. We represent the integrand as θ_r and its integral as Q_r .

5. To save repetition we use the functions $k_1 \dots k_8$ to represent the integrands corresponding to $\delta_1 \dots \delta_8$.

Thus

$$k_1 = (\text{mmmmmm}) = (v \wedge w)(w \wedge x)(x \wedge y)(y \wedge z)(z \wedge v),$$

$$\delta_1 = \frac{31}{2835} = 5115/D_5$$

where $D_5 = 3.25.7.9.9.11 = 467775$ is a common divisor.

$$k_2 = (\text{pmmmm}) = vw(w \wedge x)(x \wedge y)(y \wedge z)(z \wedge v),$$

$$\delta_2 = \frac{1382}{155925} = 4146/D_5$$

$$k_3 = (\text{ppmmmm}) = vw^2x(x \wedge y)(y \wedge z)(z \wedge v),$$

$$\delta_3 = \frac{62}{8505} = 3410/D_5$$

$$k_4 = (\text{pmpmm}) = vw(w \wedge x)xy(y \wedge z)(z \wedge v),$$

$$\delta_4 = \frac{34}{4725} = 3366/D_5$$

$$k_5 = (pppmm) = vw^2 x^2 y(y \wedge z)(z \wedge v) ,$$

$$\delta_5 = \frac{17}{2835} = 2805/D_5 .$$

$$k_6 = (ppmpm) = vw^2 x(x \wedge y)yz(z \wedge v) ,$$

$$\delta_6 = \frac{4}{675} = 2772/D_5 .$$

$$k_7 = (ppppm) = vw^2 x^2 y^2 z(z \wedge v) ,$$

$$\delta_7 = \frac{2}{405} = 2310/D_5 .$$

$$k_8 = (ppppp) = v^2 w^2 x^2 y^2 z^2 ,$$

$$\delta_8 = \frac{1}{243} = 1925/D_5 .$$

We also use the equivalence relation $\approx$ to represent as equivalent two integrands having the same integral. For the sake of compactness of presentation we also use the notation k_r^s to represent the product of s similar terms k_r , the integrand corresponding to the integral δ_r^s .

Then

$$\sigma_{12} \sigma_{23} \sigma_{34} \sigma_{45} \sigma_{51} = \theta_1 + \theta_2 + \dots + \theta_{39} , \text{ say,}$$

for

$$H_5 = Q_1 + Q_2 + \dots + Q_{39} ,$$

with

$$\theta_1 = a_{12} a_{23} a_{34} a_{45} a_{51} = \prod_1^d (\text{mmmm}) .$$

So

$$Q_1 = \delta_1^d .$$

Similarly,

$$\theta_2 = b_{12} b_{23} b_{34} b_{45} b_{51} = (d-1)^5 \prod_1^d (\text{pppp}) ,$$

$$Q_2 = (d-1)^5 \delta_8^d .$$

From here on we continue in similar vein and get:

$$\theta_3 = c_{12} c_{23} c_{34} c_{45} c_{51}$$

$$= - \text{dmmmm} k_8^{d-1}$$

$$Q_3 = - d \delta_1 \delta_8^{d-1}$$

$$-d(d-1) (\text{mmmp}) (\text{ppppm}) k_8^{d-2}$$

$$-d(d-1) \delta_2 \delta_7 \delta_8^{d-2}$$

$$-d(d-1) (\text{mmpm}) (\text{ppmp}) k_8^{d-2}$$

$$-d(d-1) \delta_2 \delta_7 \delta_8^{d-2}$$

$$-d(d-1) (\text{mmpm}) (\text{ppmm}) k_8^{d-2}$$

$$-d(d-1) \delta_3 \delta_5 \delta_8^{d-2}$$

$$-d(d-1) (d-2) (\text{mmpm}) k_7 k_8^{d-3}$$

$$-d(d-1) (d-2) \delta_3 \delta_7 \delta_8^{d-3}$$

θ_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)(mmpmm)(ppmp)k_8^{d-2} \\
 & -d(d-1)(mmpmp)(ppmp)k_8^{d-2} \\
 & -d(d-1)(d-2)(mmpmp)k_7k_8^{2,d-3} \\
 & -d(d-1)(mmpmm)(ppmp)k_8^{d-2} \\
 & -d(d-1)(mmpmp)(ppmp)k_8^{d-2} \\
 & -d(d-1)(d-2)(mmpmp)(ppmp)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)(mmpmp)k_7k_8^{2,d-3} \\
 & -d(d-1)(d-2)(mmpmp)(ppmp)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)(mmpmp)(ppmp)(ppmp)k_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)(mmpmp)k_7k_8^{3,d-4} \\
 & -d(d-1)(mmpmm)k_7k_8^{d-2} \\
 & -d(d-1)(mmpmp)(ppmp)k_8^{d-2} \\
 & -d(d-1)(d-2)(mmpmp)k_7k_8^{2,d-3} \\
 & -d(d-1)(mmpmp)(ppmp)k_8^{d-2} \\
 & -d(d-1)(mmpmp)(ppmp)k_8^{d-2} \\
 & -d(d-1)(d-2)(mmpmp)(ppmp)k_7k_8^{d-3}
 \end{aligned}$$

Q_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)\delta_2\delta_7\delta_8^{d-2} \\
 & -d(d-1)\delta_4\delta_6\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_4\delta_7\delta_8^{2,d-3} \\
 & -d(d-1)\delta_3\delta_5\delta_8^{d-2} \\
 & -d(d-1)\delta_3\delta_5\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_5\delta_7\delta_8^{2,d-3} \\
 & -d(d-1)(d-2)\delta_3\delta_7\delta_8^{2,d-3} \\
 & -d(d-1)(d-2)\delta_5\delta_6\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_5\delta_7\delta_8^{2,d-3} \\
 & -d(d-1)(d-2)(d-3)\delta_5\delta_7\delta_8^{3,d-4} \\
 & -d(d-1)\delta_2\delta_7\delta_8^{d-2} \\
 & -d(d-1)\delta_4\delta_6\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_4\delta_7\delta_8^{2,d-3} \\
 & -d(d-1)\delta_4\delta_6\delta_8^{d-2} \\
 & -d(d-1)\delta_4\delta_6\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_6\delta_7\delta_8^{2,d-3}
 \end{aligned}$$

θ_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)(d-2)(mpmp)k_7^2k_8^{d-3} \\
 & -d(d-1)(d-2)(mpmp)(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)(mpmp)k_7(ppmm)k_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)(mpmp)k_7^3k_8^{d-4} \\
 & -d(d-1)(mpmm)(pmpm)k_8^{d-2} \\
 & -d(d-1)(mpmp)(pmpm)k_8^{d-2} \\
 & -d(d-1)(d-2)(mpmp)(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(mpmm)(pmpm)k_8^{d-2} \\
 & -d(d-1)k_7(ppmm)k_8^{d-2} \\
 & -d(d-1)(d-2)k_7(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)(mpmm)(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)k_7(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)k_7(pmpm)(ppmm)k_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)k_7(pmpm)k_7k_8^{2,d-4} \\
 & -d(d-1)(d-2)(mpmm)k_7^2k_8^{d-3} \\
 & -d(d-1)(d-2)(mpmp)(pmpm)k_7k_8^{d-3} \\
 & -d(d-1)(d-2)(mpmp)k_7(ppmm)k_8^{d-3}
 \end{aligned}$$

Q_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)(d-2)\delta_4^2\delta_7^2\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_6^2\delta_7^2\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_5\delta_6\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)\delta_6^3\delta_7\delta_8^{d-4} \\
 & -d(d-1)\delta_3\delta_5\delta_8^{d-2} \\
 & -d(d-1)\delta_4\delta_6\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_5\delta_6\delta_7\delta_8^{d-3} \\
 & -d(d-1)\delta_3\delta_5\delta_8^{d-2} \\
 & -d(d-1)\delta_2\delta_7\delta_8^{d-2} \\
 & -d(d-1)(d-2)\delta_3^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_5^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_4^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_5^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)\delta_5\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)\delta_3^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_6^2\delta_7\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_6^2\delta_7\delta_8^{d-3}
 \end{aligned}$$

θ_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)(d-2)(d-3)(mppmp)k_7^3k_8^{d-4} \\
 & -d(d-1)(d-2)(mpppm)(pmpmp)k_7^d k_8^{d-3} \\
 & -d(d-1)(d-2)k_7(pmpmm)k_7^d k_8^{d-3} \\
 & -d(d-1)(d-2)k_7(pmpmp)(ppmpm)k_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)k_7(pmpmp)k_7^2k_8^{d-4} \\
 & -d(d-1)(d-2)(mpppm)(pmpmp)(ppmpm)k_8^{d-3} \\
 & -d(d-1)(d-2)k_7(pmppm)(ppmpm)k_8^{d-3} \\
 & -d(d-1)(d-2)k_7^2(ppmmm)k_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)k_7^2(ppmpm)k_7^d k_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)(mpppm)k_7^3k_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)k_7(pmppm)k_7^2k_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)k_7^2(ppmpm)k_7^d k_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)k_7^3(pppmm)k_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)(d-4)k_7^5k_8^{d-5}
 \end{aligned}$$

$$\theta_4 = 5b_{12}^a a_{23}^a a_{34}^a a_{45}^a a_{51}$$

$$= 5(d-1) \prod_{l=1}^d (p_{lmmmm}) , s$$

Q_3 (cont'd)

$$\begin{aligned}
 & -d(d-1)(d-2)(d-3)\delta_6\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)\delta_5\delta_6\delta_7^d\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_4\delta_7^2\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_6^2\delta_7^d\delta_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)\delta_6\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)\delta_5^2\delta_7^d\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_5\delta_6\delta_7^d\delta_8^{d-3} \\
 & -d(d-1)(d-2)\delta_3\delta_7^2\delta_8^{d-3} \\
 & -d(d-1)(d-2)(d-3)\delta_5\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)\delta_5\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)\delta_6\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)\delta_6\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)\delta_5\delta_7^3\delta_8^{d-4} \\
 & -d(d-1)(d-2)(d-3)(d-4)\delta_7^5\delta_8^{d-5}
 \end{aligned}$$

$$Q_4 = 5(d-1)\delta_2^d$$

$$\theta_5 = 5a_{12}b_{23}b_{34}b_{45}b_{51}$$

$$= 5(d-1)^4 \prod_1^d (mpppp)$$

$$\theta_6 = 5c_{12}a_{23}a_{34}a_{45}a_{51}$$

$$= -5d(\text{mmmmmm}) \prod_1^d (pmmmm)$$

$$\theta_7 = 5a_{12}c_{23}c_{34}c_{45}c_{51}$$

$$= 5c_{12}c_{23}c_{34}c_{45}a_{51}$$

$$= 5d(\text{mmmmmm})k_7^{d-1}$$

$$+ 5d(d-1)(\text{mmpm})(pppmm)k_7^{d-2}$$

$$+ 5d(d-1)(\text{mmpmm})(ppmpm)k_7^{d-2}$$

$$+ 5d(d-1)(\text{mpppm})(ppmmmm)k_7^{d-2}$$

$$+ 5d(d-1)(d-2)(\text{mmpmm})(ppmpm)(pppmm)k_7^{d-3}$$

$$+ 5d(d-1)(\text{mpmmmm})(pmpmm)k_7^{d-2}$$

$$+ 5d(d-1)(\text{mpmpm})(pmpmm)k_7^{d-2}$$

$$+ 5d(d-1)(d-2)(\text{mpmpm})(pmpmm)(pppmm)k_7^{d-3}$$

$$+ 5d(d-1)(\text{mppmm})(pmmmpm)k_7^{d-2}$$

$$+ 5d(d-1)(\text{mpppm})(pmmmp)k_7^{d-2}$$

$$Q_5 = 5(d-1)^4 \delta_7^d$$

$$Q_6 = -5d\delta_1\delta_2^{d-1}$$

$$Q_7 = 5d\delta_1\delta_7^{d-1}$$

$$+ 5d(d-1)\delta_2\delta_5\delta_7^{d-2}$$

$$+ 5d(d-1)\delta_2\delta_6\delta_7^{d-2}$$

$$+ 5d(d-1)\delta_3^2\delta_7^{d-2}$$

$$+ 5d(d-1)(d-2)\delta_3\delta_5\delta_6\delta_7^{d-3}$$

$$+ 5d(d-1)\delta_2\delta_6\delta_7^{d-2}$$

$$+ 5d(d-1)\delta_4^2\delta_7^{d-2}$$

$$+ 5d(d-1)(d-2)\delta_4\delta_5\delta_6\delta_7^{d-3}$$

$$+ 5d(d-1)\delta_3\delta_4\delta_7^{d-2}$$

$$+ 5d(d-1)\delta_2\delta_5\delta_7^{d-2}$$

θ_7 (cont'd)

$$+5d(d-1)(d-2)(mpppm)(pmppm)(pppmm)k_7^{d-3}$$

$$+5d(d-1)(d-2)(mppmm)(pmppm)(ppmpm)k_7^{d-3}$$

$$+5d(d-1)(d-2)(mpppm)(pmpmm)(ppmpm)k_7^{d-3}$$

$$+5d(d-1)(d-2)(mpppm)(pmppm)(ppmmmm)k_7^{d-3}$$

$$+5d(d-1)(d-2)(d-3)(mpppm)(pmppm) \\ (ppmpm)(pppmm)k_7^{d-4}$$

$$\theta_8 = 5c_{12}b_{23}b_{34}b_{45}b_{51}$$

$$= -5d(d-1)^4(mpppp) \prod_2^d (ppppp)$$

$$\theta_9 = 5b_{12}c_{23}c_{34}c_{45}c_{51}$$

$$= 5c_{12}c_{23}c_{34}c_{45}b_{51}$$

$$= 5d(d-1)(mmmp)k_8^{d-1}$$

$$+5d(d-1)^2(mmpp)k_7k_8^{d-2}$$

$$+5d(d-1)^2(mmpm)k_7k_8^{d-2}$$

$$+5d(d-1)^2(mmppp)(ppmp)k_8^{d-2}$$

$$+5d(d-1)^2(d-2)(mmppp)k_7k_8^{d-3}$$

Q_7 (cont'd)

$$+5d(d-1)(d-2)\delta_4\delta_5^2\delta_7^{d-3}$$

$$+5d(d-1)(d-2)\delta_3\delta_6^2\delta_7^{d-3}$$

$$+5d(d-1)(d-2)\delta_4\delta_5\delta_6\delta_7^{d-3}$$

$$+5d(d-1)(d-2)\delta_3\delta_5\delta_6\delta_7^{d-3}$$

$$+5d(d-1)(d-2)(d-3)\delta_5\delta_6^2\delta_7^{d-4}$$

$$Q_8 = -5d(d-1)^4\delta_7\delta_8^{d-1}$$

$$Q_9 = 5d(d-1)\delta_2\delta_8^{d-1}$$

$$+5d(d-1)^2\delta_3\delta_7\delta_8^{d-2}$$

$$+5d(d-1)^2\delta_4\delta_7\delta_8^{d-2}$$

$$+5d(d-1)^2\delta_5\delta_8^{d-2}$$

$$+5d(d-1)^2(d-2)\delta_5\delta_7\delta_8^{d-3}$$

θ_9 (cont'd)

$$\begin{aligned}
 &+5d(d-1)^2 (\text{mpmp}) k_7 k_8^{d-2} \\
 &+5d(d-1)^2 (\text{mpmp}) (\text{pmpmp}) k_8^{d-2} \\
 &+5d(d-1)^2 (d-2) (\text{mpmp}) k_7^2 k_8^{d-3} \\
 &+5d(d-1)^2 (\text{mpmp}) (\text{pmpmp}) k_8^{d-2} \\
 &+5d(d-1)^2 k_7 (\text{pmpmp}) k_8^{d-2} \\
 &+5d(d-1)^2 (d-2) k_7 (\text{pmpmp}) k_7 k_8^{d-3} \\
 &+5d(d-1)^2 (d-2) (\text{mpmp}) k_7^2 k_8^{d-3} \\
 &+5d(d-1)^2 (d-2) k_7 (\text{pmpmp}) k_7 k_8^{d-3} \\
 &+5d(d-1)^2 (d-2) k_7^2 (\text{pmpmp}) k_8^{d-3} \\
 &+5d(d-1)^2 (d-2) (d-3) k_7^4 k_8^{d-4}
 \end{aligned}$$

$$\begin{aligned}
 \theta_{10} &= 5b_{12}^b b_{23}^a a_{34}^a a_{45}^a a_{51} \\
 &= 5(d-1)^2 \prod_1^d (\text{ppmp})
 \end{aligned}$$

$$\begin{aligned}
 \theta_{11} &= 5a_{12}^a a_{23}^b b_{34}^b a_{45}^b a_{51} \\
 &= 5(d-1)^3 \prod_1^d (\text{mpmp})
 \end{aligned}$$

Q_9 (cont'd)

$$\begin{aligned}
 &+5d(d-1)^2 \delta_4 \delta_7 \delta_8^{d-2} \\
 &+5d(d-1)^2 \delta_6^2 \delta_8^{d-2} \\
 &+5d(d-1)^2 (d-2) \delta_6^2 \delta_7 \delta_8^{d-3} \\
 &+5d(d-1)^2 \delta_5 \delta_6 \delta_8^{d-2} \\
 &+5d(d-1)^2 \delta_3 \delta_7 \delta_8^{d-2} \\
 &+5d(d-1)^2 (d-2) \delta_5 \delta_7^2 \delta_8^{d-3} \\
 &+5d(d-1)^2 (d-2) \delta_6 \delta_7^2 \delta_8^{d-3} \\
 &+5d(d-1)^2 (d-2) \delta_6 \delta_7 \delta_8^{d-3} \\
 &+5d(d-1)^2 (d-2) \delta_5 \delta_7^2 \delta_8^{d-3} \\
 &+5d(d-1)^2 (d-2) (d-3) \delta_7^4 \delta_8^{d-4}
 \end{aligned}$$

$$Q_{10} = 5(d-1)^2 \delta_3^d$$

$$Q_{11} = 5(d-1)^3 \delta_5^d$$

$$\theta_{12} = 5c_{12}^c c_{23}^a a_{34}^a a_{45}^a a_{51}$$

$$\bullet = 6d(\text{mmmm}) \prod_2 (\text{ppmm})$$

$$+5d(d-1)(\text{mpmm})(\text{pmmmm}) \prod_3 (\text{ppmm})$$

$$\theta_{13} = 5a_{12}^a a_{23}^c c_{34}^c c_{45}^c c_{51}$$

$$= -5d(\text{mmmm})k_5^{d-1}$$

$$-5d(d-1)(\text{mmmp})(\text{mmpm})k_5^{d-2}$$

$$-5d(d-1)(\text{mmpm})(\text{mmpm})k_5^{d-2}$$

$$-5d(d-1)(\text{mmpm})(\text{mmpm})k_5^{d-2}$$

$$-5d(d-1)(d-2)(\text{mmpm})(\text{mmpm})(\text{mmpm})k_5^{d-3}$$

$$\theta_{14} = 5c_{12}^c c_{23}^b b_{34}^b b_{45}^b b_{51}$$

$$= 5d(d-1)^3(\text{mmpm})k_8^{d-1}$$

$$+5d(d-1)^4(\text{mmpm})(\text{mmpm})k_8^{d-2}$$

$$\theta_{15} = 5b_{12}^b b_{23}^c c_{34}^c c_{45}^c c_{51}$$

$$= -5d(d-1)^2(\text{ppmm})k_8^{d-1}$$

$$-5d(d-1)^3(\text{ppmp})k_7 k_8^{d-2}$$

$$Q_{12} = 5d\delta_1 \delta_d^{d-1} + 5d(d-1)\delta_2^2 \delta_3^{d-2}$$

$$Q_{13} = -5d\delta_1 \delta_5^{d-1}$$

$$-5d(d-1)\delta_2 \delta_3 \delta_5^{d-2}$$

$$-5d(d-1)\delta_2 \delta_4 \delta_5^{d-2}$$

$$-5d(d-1)\delta_2 \delta_3 \delta_5^{d-2}$$

$$-5d(d-1)(d-2)\delta_3^2 \delta_4 \delta_5^{d-3}$$

$$Q_{14} = 5d(d-1)^3 \delta_5 \delta_8^{d-1}$$

$$+ 5d(d-1)^4 \delta_7^2 \delta_8^{d-2}$$

$$Q_{15} = -5d(d-1)^2 \delta_3 \delta_8^{d-1}$$

$$-5d(d-1)^3 \delta_5 \delta_7 \delta_8^{d-2}$$

θ_{15} (cont'd)

$$-5d(d-1)^3 (ppmpm) k_7 k_8^{d-2}$$

$$-5d(d-1)^3 k_7 (pppmm) k_8^{d-2}$$

$$-5d(d-1)^3 (d-2) k_7 k_8^{d-3}$$

$$\theta_{16} = 5b_{12}^a b_{23}^b b_{34}^a b_{45}^a b_{51}$$

$$= 5(d-1)^2 \prod_1^d (pmpmm)$$

$$\theta_{17} = 5a_{12}^b b_{23}^a b_{34}^b b_{45}^b b_{51}$$

$$= 5(d-1)^3 \prod_1^d (mpmpp)$$

$$\theta_{18} = 5c_{12}^a c_{23}^c c_{34}^a c_{45}^a c_{51}$$

$$= 5d(\text{mmmm}) k_4^{d-1}$$

$$+5d(d-1) (\text{mmpmm}) (\text{pmmmm}) k_4^{d-2}$$

$$\theta_{19} = 5a_{12}^c c_{23}^a c_{34}^c c_{45}^c c_{51}$$

$$= -5d(\text{mmmm}) k_6^{d-1}$$

$$-5d(d-1) (\text{mmpmp}) (\text{mpmpm}) k_6^{d-2}$$

$$-5d(d-1) (\text{mmpmp}) (\text{mpmpm}) k_6^{d-2}$$

Q_{15} (cont'd)

$$-5d(d-1)^3 \delta_6 \delta_7 \delta_8^{d-2}$$

$$-5d(d-1)^3 \delta_5 \delta_7 \delta_8^{d-2}$$

$$-5d(d-1)^3 (d-2) \delta_7 \delta_8^{d-3}$$

$$Q_{16} = 5(d-1)^2 \delta_4^d$$

$$Q_{17} = 5(d-1)^3 \delta_6^d$$

$$Q_{18} = 5d\delta_1 \delta_4^{d-1} + 5d(d-1) \delta_2^2 \delta_4^{d-2}$$

$$Q_{19} = -5d\delta_1 \delta_6^{d-1}$$

$$-5d(d-1) \delta_2 \delta_4 \delta_6^{d-2}$$

$$-5d(d-1) \delta_2 \delta_4 \delta_6^{d-2}$$

θ_{19} (cont'd)

$$-5d(d-1)(mmpp)(mpmm)k_6^{d-2}$$

$$-5d(d-1)(d-2)(mmpp)(mpmp)(mpmp)k_6^{d-3}$$

$$\theta_{20} = 5c_{12}b_{23}c_{34}b_{45}b_{51}$$

$$= 5d(d-1)^3(mpmp)k_8^{d-1}$$

$$+ 5d(d-1)^4(mppp)(ppmp)k_8^{d-2}$$

$$\theta_{21} = 5b_{12}c_{23}b_{34}c_{45}c_{51}$$

$$= -5d(d-1)^2(pmpm)k_8^{d-1}$$

$$-5d(d-1)^3(pmpm)k_7k_8^{d-2}$$

$$-5d(d-1)^3(pppm)k_7k_8^{d-2}$$

$$-5d(d-1)^3k_7(pppm)k_8^{d-2}$$

$$-5d(d-1)^3(d-2)k_7k_8^{d-3}$$

$$\theta_{22} = 10b_{12}c_{23}a_{34}a_{45}a_{51}$$

$$= -10d(d-1)(pmmmm)k_3^{d-1}$$

Q_{19} (cont'd)

$$-5d(d-1)\delta_2\delta_3\delta_6^{d-2}$$

$$-5d(d-1)(d-2)\delta_3\delta_4\delta_6^{d-3}$$

$$Q_{20} = 5d(d-1)^3\delta_6\delta_8^{d-1}$$

$$+ 5d(d-1)^4\delta_7\delta_8^{d-2}$$

$$Q_{21} = -5d(d-1)^2\delta_4\delta_8^{d-1}$$

$$-5d(d-1)^3\delta_6\delta_7\delta_8^{d-2}$$

$$-5d(d-1)^3\delta_6\delta_7\delta_8^{d-2}$$

$$-5d(d-1)^3\delta_5\delta_7\delta_8^{d-2}$$

$$-5d(d-1)^3(d-2)\delta_7\delta_8^{d-3}$$

$$Q_{22} = -10d(d-1)\delta_2\delta_3^{d-1}$$

$$\theta_{23} = 10a_{12}b_{23}c_{34}c_{45}c_{51}$$

$$= -10d(d-1)(mpmm)k_7^{d-1}$$

$$-10d(d-1)^2(mppm)(mpppm)k_7^{d-2}$$

$$-10d(d-1)^2(mppm)(mppmp)k_7^{d-2}$$

$$-10d(d-1)^2(mppm)(mppmm)k_7^{d-2}$$

$$-10d(d-1)^2(d-2)(mppm)(mppmp)(mpppm)k_7^{d-3}$$

$$\theta_{24} = 10c_{12}a_{23}b_{34}b_{45}b_{51}$$

$$= -10d(d-1)^3(mppp)k_7^{d-1}$$

$$\theta_{25} = 10b_{23}a_{23}c_{34}a_{45}a_{51}$$

$$= -10d(d-1)(pmmmm)k_4^{d-1}$$

$$\theta_{26} = 10a_{12}c_{23}b_{34}c_{45}c_{51}$$

$$= -10d(d-1)(mpmm)k_7^{d-1}$$

$$-10d(d-1)^2(mppm)(mpppm)k_7^{d-2}$$

$$-10d(d-1)^2(mppm)(mppmp)k_7^{d-2}$$

$$-10d(d-1)^2(mppm)(mppmm)k_7^{d-2}$$

$$-10d(d-1)^2(d-2)(mppm)(mppmp)(mpppm)k_7^{d-3}$$

$$Q_{23} = -10d(d-1)\delta_2\delta_7^{d-1}$$

$$-10d(d-1)^2\delta_4\delta_5\delta_7^{d-2}$$

$$-10d(d-1)^2\delta_4\delta_6\delta_7^{d-2}$$

$$-10d(d-1)^2\delta_3\delta_6\delta_7^{d-2}$$

$$-10d(d-1)^2(d-2)\delta_5\delta_6^2\delta_7^{d-3}$$

$$Q_{24} = -10d(d-1)^3\delta_5\delta_7^{d-1}$$

$$Q_{25} = -10d(d-1)\delta_2\delta_4^{d-1}$$

$$Q_{26} = -10d(d-1)\delta_2\delta_7^{d-1}$$

$$-10d(d-1)^2\delta_4\delta_5\delta_7^{d-2}$$

$$-10d(d-1)^2\delta_3\delta_6\delta_7^{d-2}$$

$$-10d(d-1)^2\delta_3\delta_5\delta_7^{d-2}$$

$$-10d(d-1)^2(d-2)\delta_5\delta_6^2\delta_7^{d-3}$$

$$\begin{aligned}\theta_{27} &= 10c_{12}b_{23}a_{34}b_{45}b_{51} \\ &= -10d(d-1)^3(mppmp)k_7^{d-1}\end{aligned}$$

$$\begin{aligned}\theta_{28} &= 10a_{12}b_{23}b_{34}c_{45}c_{51} \\ &= 10d(d-1)^2(mppmm)k_7^{d-1} \\ &\quad + 10d(d-1)^3(mppmp)(ppppm)k_7^{d-2}\end{aligned}$$

$$\begin{aligned}\theta_{29} &= 10b_{12}c_{23}c_{34}a_{45}a_{51} \\ &= 10d(d-1)(ppmmmm)k_5^{d-1} \\ &\quad + 10d(d-1)^2(ppmpmm)(ppmmmm)k_5^{d-2}\end{aligned}$$

$$\begin{aligned}\theta_{30} &= 10c_{12}a_{23}a_{34}b_{45}b_{51} \\ &= -10d(d-1)^2(mmmpp)k_5^{d-1}\end{aligned}$$

$$\begin{aligned}\theta_{31} &= 10a_{12}b_{23}c_{34}a_{45}b_{51} \\ &= -10d(d-1)^2(mppmp)k_6^{d-1}\end{aligned}$$

$$\begin{aligned}\theta_{32} &= 10b_{12}c_{23}a_{34}b_{45}c_{51} \\ &= 10d(d-1)^2(ppmpm)k_7^{d-1} \\ &\quad + 10d(d-1)^3(ppmpmp)(ppmpm)k_7^{d-2}\end{aligned}$$

$$Q_{27} = -10d(d-1)^3\delta_6\delta_7^{d-1}$$

$$\begin{aligned}Q_{28} &= 10d(d-1)^2\delta_3\delta_7^{d-1} \\ &\quad + 10d(d-1)^3\delta_5\delta_6\delta_7^{d-2}\end{aligned}$$

$$\begin{aligned}Q_{29} &= 10d(d-1)\delta_2\delta_5^{d-1} \\ &\quad + 10d(d-1)^2\delta_3\delta_4\delta_5^{d-2}\end{aligned}$$

$$Q_{30} = -10d(d-1)^2\delta_3\delta_5^{d-1}$$

$$Q_{31} = -10d(d-1)^2\delta_4\delta_6^{d-1}$$

$$\begin{aligned}Q_{32} &= 10d(d-1)^2\delta_4\delta_7^{d-1} \\ &\quad + 10d(d-1)^3\delta_5\delta_6\delta_7^{d-2}\end{aligned}$$

$$\theta_{33} = 10c_{12}a_{23}b_{34}c_{45}a_{51}$$

$$= 10d(d-1)(mmpmm)k_6^{d-1}$$

$$+ 10d(d-1)^2(mpppm)(pmpmm)k_6^{d-2}$$

$$\theta_{34} = 5a_{12}b_{23}c_{34}b_{45}a_{51}$$

$$= -5d(d-1)^2(mpppm)k_5^{d-1}$$

$$\theta_{35} = 5b_{12}c_{23}a_{34}c_{45}b_{51}$$

$$= 5d(d-1)^2(pmmmp)k_7^{d-1}$$

$$+ 5d(d-1)^3(pmmppl)(ppmmp)k_7^{d-2}$$

$$\theta_{36} = 5c_{12}a_{23}b_{34}a_{45}c_{51}$$

$$= 5d(d-1)(mmpmm)k_6^{d-1}$$

$$+ 5d(d-1)^2(mppmp)(pmpmm)k_6^{d-2}$$

$$\theta_{37} = 5b_{12}a_{23}c_{34}a_{45}b_{51}$$

$$= -5d(d-1)^2(pmmpm)k_6^{d-1}$$

$$Q_{33} = 10d(d-1)\delta_2\delta_6^{d-1}$$

$$+ 10d(d-1)^2\delta_3\delta_4\delta_6^{d-2}$$

$$Q_{34} = -5d(d-1)^2\delta_4\delta_5^{d-1}$$

$$Q_{35} = 5d(d-1)^2\delta_3\delta_7^{d-1}$$

$$+ 5d(d-1)^3\delta_5\delta_7^{d-2}$$

$$Q_{36} = 5d(d-1)\delta_2\delta_6^{d-1}$$

$$+ 5d(d-1)^2\delta_4\delta_6^{d-2}$$

$$Q_{37} = -5d(d-1)^2\delta_3\delta_6^{d-1}$$

$$\theta_{38} = 5c_{12}b_{23}a_{34}b_{45}c_{51}$$

$$= 5d(d-1)^2 (mpmpm)k_7^{d-1}$$

$$+ 5d(d-1)^3 (mpmpp) (ppmpm)k_7^{d-2}$$

$$\theta_{39} = 5a_{12}c_{23}b_{34}c_{45}a_{51}$$

$$= 5d(d-1) (mmpmm)k_5^{d-1}$$

$$+ 5d(d-1)^2 (mmppm) (mppmm)k_5^{d-2}$$

$$Q_{38} = 5d(d-1)^2 \delta_4 \delta_7^{d-1}$$

$$+ 5d(d-1)^3 \delta_6^2 \delta_7^{d-2}$$

$$Q_{39} = 5d(d-1) \delta_2 \delta_5^{d-1}$$

$$+ 5d(d-1)^2 \delta_3^2 \delta_5^{d-2}$$

7. Concluding Remarks and Some Further Applications of the Empirical Process $T_n(x)$. As mentioned already at the end of Section 1, large values of the statistics $\hat{C}_{n,d}$ of (1.24) and $\tilde{C}_{n,d}$ of (1.25) are to be used to reject the null hypothesis H_0 of (1.1). Since the distribution of these statistics does not depend on F under H_0 , these critical regions are similar on F_0 . For the asymptotic distribution of these statistics under H_1 of (1.1) we refer to Section 4 of Blum, Kiefer and Rosenblatt (1961) (cf. also Remark 2 in Csörgö (1979)). Consistency and the power function of these tests are also discussed in Section 4 of the just mentioned paper of Blum, Kiefer and Rosenblatt. Basing their remarks on the general power properties of tests based on the sample distribution function as detailed in Kac, Kiefer and Wolfowitz (1955) and in Kiefer (1959), they find the behaviour of the power function of the $\hat{C}_{n,d}$ and $\tilde{C}_{n,d}$ tests against all alternatives to be far superior to that of the other nonparametric tests of independence. Hence the use of the statistics $\hat{C}_{n,d}$ and $\tilde{C}_{n,d}$, and that of their possible modifications via inserting a weight function in their definition (cf., e.g., DeWet (1979)), is strongly recommended for testing H_0 of (1.1). Also, just like in Kac, Kiefer and Wolfowitz (1955), Kiefer (1959), and Kiefer and Wolfowitz (1958), the results on sample size and minimum power are not affected if discontinuous distributions are admitted.

The statistic $C_{n,d}$ of (1.21) can only be used for H_0 when $F \in F_0$ is completely specified. There are also some other situations

when $C_{n,d}$ might come in handy. For example, Proposition 2 of Cotterill and Csörgö (1979) reduces the composite goodness-of-fit hypothesis

$$(7.1) \quad H'_0 : Y_j = (Y_{j1}, Y_{j2}, \dots, Y_{jm}) , \quad m \text{ fixed } \geq 6, \quad j = 1, 2, \dots, n$$

are n i.i.d. m -vectors of m i.i.d. $N(x; \mu, \sigma^2)$ rv

to the equivalent simple goodness-of-fit hypothesis

$$(7.2) \quad H''_0 : X_j = (X_{j1}, X_{j2}, \dots, X_{j,m-2}) \text{ of Proposition 2 in Cotterill and Csörgö (1979), } m \text{ fixed } \geq 6, \quad j = 1, 2, \dots, n$$

are n i.i.d. $(m-2)$ -vectors of independent Student rv with $1, 2, \dots, m-2$ degrees of freedom respectively.

Now define $T_n(x)$ of (1.5) in terms of the $(m-2)$ -vectors $X_j = (X_{j1}, X_{j2}, \dots, X_{j,m-2})$, $j = 1, \dots, n$, of Proposition 2 in Cotterill and Csörgö (1979) and consider the statistic

$$(7.3) \quad C_{n,m-2} = \int_{R^{m-2}} n T_n^2(x) \prod_{i=1}^{m-2} dF_i(x_i) , \quad m \geq d .$$

The latter, of course, is $C_{n,d}$ of (1.21) with $d = m-2$ and the F_i are Student distribution functions with respective degrees of freedom $i = 1, \dots, m-2$ and it converges in distribution to the rv C_{m-2} (cf. (1.21)). We can then base our critical region for H''_0 of (7.2) on large values of $C_{n,m-2}$. The latter would test H''_0 via testing for independence of the $(m-2)$ components of X_j in terms of $T_n(x)$. The independent Student nature of these $m-2$ components is also incorporated in this test via taking the Student product measure $\prod_{i=1}^{m-2} dF_i(x_i)$ as the measure of integration in the definition of $C_{n,m-2}$.

While at the topic of testing for normality in the presence of unknown parameters, we should also mention another possible way of using the empirical process $T_n(x)$ of (1.5). Let $Y_j = (Y_{j1}, \dots, Y_{jm})$, $j = 1, \dots, n$, be n independent random samples of size $m \geq 2$. Then it is well known that H'_0 of (7.1) with $m \geq 2$ is true if and only if

$$(7.4) \quad H'_0: \bar{Y}_{jm} = \frac{1}{m} \sum_{i=1}^m Y_{ji} \text{ and } S_{jm}^2 = \frac{1}{m} \sum_{i=1}^m (Y_{ji} - \bar{Y}_{jm})^2 \text{ are independent}$$

random variables for each $j = 1, \dots, n$.

Let $F_n(x) = F_n(x_1, x_2)$ be as in (1.2), but defined in terms of $(\bar{Y}_{jm}, S_{jm}^2)$, $j = 1, \dots, n$. Similarly, let $F_{n1}(x_1)$ be the empirical distribution function of the $\bar{Y}_{jm}$ ($j = 1, \dots, n$) and let $F_{n2}(x_2)$ be that of the S_{jm}^2 ($j = 1, \dots, n$). Define $T_n(x)$ of (1.5) in terms of the just defined $F_n(x) = F_n(x_1, x_2)$, $F_{n1}(x_1)$ and $F_{n2}(x_2)$ and consider the statistics

$$(7.5) \quad \hat{C}_{n,2} = \int_{R^2} n T_n^2(x) \prod_{i=1}^2 dF_{ni}(x_i)$$

$$(7.6) \quad \tilde{C}_{n,2} = \int_{R^2} n T_n^2(x) dF_n(x)$$

Both of them converge in distribution to the rv C_2 (cf. C_d of (1.21) with $d = 2$), and we can base our critical region for H'_0 of (7.4) on large values of any one of them.

It would be of some interest to study and compare the large sample power performance of the here proposed test statistics for

normality $C_{n,m-2}$, $\tilde{C}_{n,2}$, $\hat{C}_{n,2}$ and that of $W_{n,m-2}^2$ of (5.9) in Cotterill and Csörgö (1979). Our discussion of $W_{n,m-2}^2$ in the latter paper also applies to $C_{n,m-2}$ of this section.

The multivariate two-sample problem of H_0 of (5.10) in Cotterill and Csörgö (1979) can be also studied in terms of the statistic

$$(7.7) \quad C_{nm,d} = \frac{nm}{n+m} \int_{R^d} (T_n(x) - T_m(x))^2 dS_{n+m}(x),$$

with $S_{n+m}(x) = (nF_n(x) + mG_m(x))/(n+m)$, where F_n and G_m are the respective sample distribution functions of two independent random samples $X_j = (X_{j1}, \dots, X_{jd})$ ($j = 1, \dots, n$), $Y_j = (Y_{j1}, \dots, Y_{jd})$ ($j = 1, \dots, m$) with respective distribution functions F and G in F , and T_n and T_m are defined à la (1.5) in terms of these two random samples. Given $H_0: F = G$ and $F \in F_0$, $C_{nm,d}$ converges in distribution (as $n, m \rightarrow \infty$) to the rv C_d of (1.21), and we can base our critical region for the latter H_0 on large values of $C_{nm,d}$. Again, it would be of some interest to study and compare the large sample power performance of the here proposed $C_{nm,d}$ test, the $W_{nm,d}^2$ test of (2.48) and the $M_{nm,d}^2$ test of (5.18) (the latter two are defined in Cotterill and Csörgö (1979) and the two quoted numbers refer to there).

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Annex A. The Cornish-Fisher Asymptotic Expansion

As noted in Corollary 3 the Cornish-Fisher asymptotic expansion was employed to calculate a table of critical values of Γ_d , using the first 5 cumulants.

The primary reference is to Fisher and Cornish, (1960). For convenience ~~we~~ employed the coefficients listed in Abramowitz and Stegun (1964), 26.2.49. The following account is summarized from Kendall and Stuart (1977) 6.14 to 6.26.

Any probability density function $f(x)$ can be formally expressed in a series as

$$f(x) = \sum_{j=0}^{\infty} c_j H_j(x) e^{-x^2/2}$$

where $H_j(x)$ is the Chebyshev-Hermite polynomial of order j , defined by

$$(-1)^r \frac{d^r}{dx^r} e^{-x^2/2} = H_r(x) e^{-x^2/2}$$

and
$$c_r = \frac{1}{r!} \int_{-\infty}^{\infty} f(x) H_r(x) dx$$

$$= \frac{1}{r!} \left(\mu'_r - \frac{r(r-1)}{2 \cdot 1!} \mu'_{r-2} + \frac{r(r-1) \dots (r-4)}{2^2 \cdot 2!} \mu'_{r-4} - \dots \right)$$

where μ'_r denotes the r th moment of $f(x)$.

This is at once verified by multiplying each side of the equation by $H_r(x)$ and integrating over $(-\alpha, \alpha)$, because the $H_r(x)$ have an orthogonal property

$$\int_{-\alpha}^{\alpha} H_m(x) H_n(x) e^{-x^2/2} dx = \begin{cases} 0 & \text{if } m \neq n \\ n! & \text{if } m = n \end{cases}$$

and $\int_{-\alpha}^{\alpha} H_r(x) f(x) dx = c_r$

as is proved by expressing $H_r(x)$ as a polynomial. This expansion is known as a Gram-Charlier series of type A, or as an Edgeworth series.

Thus, for the distribution function $F(x)$ we have

$$F(x) = \int_{-\alpha}^x f(x) dx = \int_{-\alpha}^x e^{-x^2/2} dx - \sum_{j=0}^{\infty} c_j H_{j-1}(x) e^{-x^2/2}$$

By a theorem by Cramér this series is absolutely and uniformly convergent for all real x provided that $f(x)$ has a continuous derivative such that

$$\int_{-\alpha}^{\alpha} \left| \frac{df(x)}{dx} \right| e^{-x^2/2} dx \text{ converges}$$

and that $f(x)$ tends to zero as (x) tends to infinity.

In our problem we know the cumulants up to the fifth, and so we can easily calculate the value of x corresponding to a given $F(x)$. This could be done by standard numerical methods, however Cornish and Fisher inverted this expansion algebraically, to obtain a series expressing the critical value x_p corresponding to a probability p . We use the version given in Abramowitz and Stegun (1964) 26.2.49.

Unfortunately this is an asymptotic expansion. As higher cumulants k_r are used the approximation improves to a point, after which the approximation rapidly becomes less accurate. The rule is to select the value of r for which $x_p = x_p(k_1, \dots, k_r)$ is minimal.

The second disadvantage of this expansion is that there is no way of assessing the error of the approximation. As was noted in the table following corollary 5 the agreement with the values calculated by Blum, Kiefer and Rosenblatt (1961) is very good for probabilities $P = 0.9$ to 0.999 but it is less good for $P = 0.9995$. This is easily understood as it is in the tails of the distribution that the higher order cumulants have their greatest effect: if the Cornish-Fisher expansion precludes their use, it must be less accurate here.

It is intended to give further study to this problem in order to attempt to find a more accurate method of calculating these critical values.

END

1 6 1 2 8 1

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