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**Some generating functions and a projection formula
for the associated Askey-Wilson polynomials**

by

Qazi Mohammad Tariq

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

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submitted by
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Chairman

Thesis Supervisor

External Examiner

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Abstract

A generating function for the associated continuous q -Jacobi polynomials is obtained as an extension of a result of Gasper and Rahman, and, Bustoz and Ismail.

The bilinear generating function (the Poisson kernel) for the associated continuous q -ultraspherical polynomials, first studied by Bustoz and Ismail and later generalized by Ismail and Rahman, is obtained, essentially as a product of a ${}_2\phi_1$, a ${}_3\phi_2$ and a ${}_4\phi_3$ basic hypergeometric series. Some related generating functions are also given.

A projection formula for one of the two known families of associated Askey-Wilson polynomials is also obtained by summing an infinite series which exhibits the dual orthogonality of these polynomials in a certain limiting sense. The results also lead to a reproducing kernel for this system of polynomials as well as a weight function for orthogonality.

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Chapter 1

Introduction and preliminaries

1.1 Orthogonal polynomials

Let $\alpha(x)$ be a non decreasing, real valued bounded function with infinitely many points of increase, defined on $(-\infty, \infty)$ such that its moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x) \quad n = 0, 1, 2, \dots \quad (1.1.1)$$

are all finite. A sequence of polynomials $\{p_n(x)\}$, where $p_n(x)$ is of exact degree n in x , is said to be orthogonal with respect to a Lebesgue Stieltjes measure $d\alpha(x)$ if

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = 0 \quad m \neq n. \quad (1.1.2)$$

If $\alpha(x)$ also happens to be absolutely continuous with $d\alpha(x) = \omega(x)dx$, $\omega(x) > 0$, then (1.1.2) reduces to

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\omega(x)dx = 0 \quad m \neq n \quad (1.1.3)$$

and the sequence $\{p_n(x)\}$ is said to be orthogonal with respect to the weight function $\omega(x)$. If $\alpha(x)$ is a step function with jump w_j at $x = x_j$, $j = 0, 1, \dots$ then (1.1.2) takes the form of a sum:

$$\sum_{j=0}^{\infty} p_m(x_j)p_n(x_j)w_j = 0 \quad m \neq n. \quad (1.1.4)$$

In this case we refer to the sequence $\{p_n(x)\}$ as orthogonal polynomials of a discrete variable.

Every orthogonal system of polynomials $\{p_n(x)\}$ of real variable satisfies a three-term recurrence relation of the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad (1.1.5)$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, where A_n, B_n, C_n are real and $A_n C_{n+1} > 0$, Chihara [10]. Conversely if (1.1.5) holds for a sequence of polynomials $\{p_n(x)\}$ such that $p_{-1}(x) = 0$, $p_0(x) = 1$ and A_n, B_n, C_n are real with $A_n C_{n+1} > 0$, then there exists a positive measure $d\alpha(x)$ such that

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) d\alpha(x) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{C_n \cdots C_1}{A_{n-1} \cdots A_0} \int_{-\infty}^{\infty} d\alpha(x) & \text{if } m = n \geq 1, \end{cases} \quad (1.1.6)$$

and for $n = 0$ we have $\int_{-\infty}^{\infty} [p_n(x)]^2 d\alpha(x) = \int_{-\infty}^{\infty} d\alpha(x)$, Chihara [10]. This theorem usually goes under the name of Favard but it is much older. An equation of the form (1.1.5) is satisfied by the numerator and denominator polynomials of a continued fraction also, see [10], [22]. In general, if a sequence of orthogonal polynomials $\{p_n(x)\}$ satisfy (1.1.5), then the associated orthogonal polynomials $\{p_n^\alpha(x)\}$, $\alpha \geq 0$, are generated by the recurrence relation

$$xp_n^\alpha(x) = A_{n+\alpha} p_{n+1}^\alpha(x) + B_{n+\alpha} p_n^\alpha(x) + C_{n+\alpha} p_{n-1}^\alpha(x) \quad (1.1.7)$$

with $p_{-1}^\alpha(x) = 0$, $p_0^\alpha(x) = 1$. In general these polynomials are orthogonal with respect to some positive measure on the real line.

The associated polynomials are more than a laboratory curiosity. They are often vital for discussing the distribution function and spectrum of the original polynomials and analysing the nature of the Padé approximants based on the continued fraction defined by (1.1.5). The classical Laguerre polynomials $L_n^\beta(x)$ and the Hermite polynomials $H_n(x)$ have measure of orthogonality, $d\mu(x) = x^\beta e^{-x} dx$ on $(0, \infty)$

and $d\mu(x) = e^{-x^2}dx$ on $(-\infty, \infty)$, respectively. These polynomials satisfy the recurrence relation

$$-xL_n^\beta(x) = (n+1)L_{n+1}^\beta(x) - (2n+\beta+1)L_n^\beta(x) + (n+\beta)L_{n-1}^\beta(x), \quad (1.1.8)$$

$$L_{-1}^\beta(x) = 0, \quad L_0^\beta(x) = 1$$

and

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x), \quad (1.1.9)$$

$$H_{-1}(x) = 0, \quad H_0(x) = 1.$$

In view of (1.1.7) the associated Laguerre polynomials $L_n^{\beta,\alpha}(x)$ and the associated Hermite polynomials $H_n^\alpha(x)$ will satisfy the following recurrence relation

$$-xL_n^{\beta,\alpha}(x) \quad (1.1.10)$$

$$= (n+\alpha+1)L_{n+1}^{\beta,\alpha}(x) - (2n+2\alpha+\beta+1)L_n^{\beta,\alpha}(x) + (n+\alpha+\beta)L_{n-1}^{\beta,\alpha}(x).$$

$$L_{-1}^{\beta,\alpha}(x) = 0, \quad L_0^{\beta,\alpha}(x) = 1,$$

$$2xH_n^\alpha(x) = H_{n+1}^\alpha(x) + (2n+2\alpha)H_{n-1}^\alpha(x), \quad (1.1.11)$$

$$H_{-1}^\alpha(x) = 0, \quad H_0^\alpha(x) = 1.$$

Askey and Wimp [8] studied these polynomials. They gave the measures of orthogonality and the explicit representation for these polynomials. They proved that $L_n^{\beta,\alpha}(x)$ has the measure of orthogonality $d\mu^\alpha(x) = \frac{x^\beta e^{-x}}{|\Psi(\alpha, 1-\beta; x e^{-x})|^2} dx$ on $(0, \infty)$ and the explicit formula is

$$L_n^{\beta,\alpha}(x) = \frac{(\alpha+\beta+1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k x^k}{(\alpha+1)_k (\alpha+\beta+1)_k} \quad (1.1.12)$$

$$\times {}_3F_2 \left[\begin{matrix} k-n, \alpha, \alpha+\beta \\ \alpha+\beta+k+1, \alpha+k+1 \end{matrix} ; 1 \right]$$

where

$$\Psi(\alpha, 1 - \beta; x e^{-\pi i}) = \frac{1}{\Gamma(x)} \int_0^\infty t^{\alpha-1} (1+t)^{-\beta-\alpha} e^{tx} dt. \quad (1.1.13)$$

When $\alpha = 0$ it reduces to the Laguerre polynomials. The ${}_3F_2$ series in (1.1.12) is a special case of the generalized hypergeometric series

$$\begin{aligned} {}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; z) &\equiv {}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{n! (b_1)_n \cdots (b_s)_n} z^n \end{aligned} \quad (1.1.14)$$

where the shifted factorial $(a)_n$ is defined by

$$(a)_n = \begin{cases} a(1+a)(2+a) \dots (n-1+a) & n = 1, 2, \dots \\ 1 & n = 0. \end{cases} \quad (1.1.15)$$

They proved also that the measure of orthogonality for the associated Hermite polynomials, when $\alpha > 0$, is

$$\frac{[\Gamma(\alpha + 1)]^2 2^{-\alpha} e^{-x^2}}{|\alpha \int_0^\infty e^{-t^2 - 2ixt} t^{\alpha-1} dt|^2}. \quad (1.1.16)$$

They further obtained the following relationship between the associated Hermite and the associated Laguerre polynomials

$$H_n^\alpha(x) = \lim_{\beta \rightarrow 0} (-1)^n \left(\frac{2}{\beta} \right)^n (\alpha + 1)_n L_n^{\beta, \alpha} \left(x \sqrt{2\beta} + \beta \right). \quad (1.1.17)$$

Later J. Wimp [43] studied the associated Jacobi polynomials. He found the measure of orthogonality and an explicit representation for these polynomials which is

$$\begin{aligned} R_n^{(\beta, \gamma), \alpha}(x) &= \frac{(-1)^n (\delta + 2\alpha)_n (\gamma + \alpha + 1)_n}{(\delta + \alpha)_n n!} \\ &\times \sum_{k=0}^n \frac{(-n)_k (n + \delta + 2\alpha)_k}{(\alpha + 1)_k (\gamma + \alpha + 1)_k} x^k \end{aligned} \quad (1.1.18)$$

$$\times {}_4F_3 \left[\begin{matrix} k-n, n+\delta+k+2\alpha, \gamma+\alpha, \alpha \\ \gamma+\alpha+k+1, \alpha+k+1, \delta+2\alpha-1 \end{matrix} ; 1 \right]$$

where

$$\delta = \beta + \gamma + 1.$$

The polynomials $R_n^{(\beta, \gamma), \alpha}(x)$ in (1.1.18) are equal to $P_n^{(\beta, \gamma), \alpha}(2x-1)$ where $P_n^{(\beta, \gamma), \alpha}(x)$ are the associated Jacobi polynomials. To further generalize these polynomials to the q world we will have to define basic hypergeometric series.

1.2 Basic hypergeometric series

A basic hypergeometric series has the form $\sum_{n=1}^{\infty} a_n$ with a_{n+1}/a_n , a rational function of q^n for a fixed q . The standard notation for basic hypergeometric series, Gasper and Rahman [11], is

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z) &\equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n \end{aligned} \quad (1.2.1)$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where the q -shifted factorial $(a; q)_n$ is defined by

$$(a; q)_n = \begin{cases} (1-a)(1-aq) \cdots (1-aq^{n-1}) & n = 1, 2, \dots \\ 1 & n = 0. \end{cases} \quad (1.2.2)$$

The value $n = \infty$ is allowed in (1.2.2) provided $0 < |q| < 1$. Following Gasper and Rahman [11], we shall also use the notation $(a_1, \dots, a_r; q)_n$ to represent $(a_1; q)_n \cdots (a_r; q)_n$.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|z| <$

$|b_1 b_2 \cdots b_s|/|a_1 a_2 \cdots a_r|$. It diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s + 1$, and, if $|q| > 1$, and, $|z| > |b_1 b_2 \cdots b_s|/|a_1 a_2 \cdots a_r|$, unless it terminates. This latter case happens when one of the a 's is equal to q^{-k} for some positive integer k and none of the b 's are of this form. Throughout we will be dealing with the case when $r = s + 1$. Since we will be dealing with special types of basic hypergeometric series it is appropriate now to introduce some definitions and notations.

A basic hypergeometric series ${}_r\phi_r$ is called *well-poised* if

$$qa_1 = a_2 b_1 = \cdots = a_r b_{r-1}. \quad (1.2.3)$$

If the first or last equality fails in (1.2.3) then the series is called *nearly poised*. If in addition to (1.2.3) we have $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$, then the series is called *very well-poised*. The very well-poised series ${}_8\phi_7$ and ${}_{10}\phi_9$ will be used quite frequently in the chapters to come and it takes a lot of space to write such series. So following Bailey, see [11], we will use the contracted notation

$${}_{r+1}W_r(a; a_1, a_2, \dots, a_{r-2}; q, z)$$

to represent the series

$${}_{r+1}\phi_r \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, \dots, a_{r-2} \\ \sqrt{a}, -\sqrt{a}, qa/a_1, \dots, qa/a_{r-2} \end{matrix} ; q, z \right]. \quad (1.2.4)$$

There is another term which needs to be mentioned here. A basic hypergeometric series (1.2.1) is called *balanced* or *Saalschützian*, if $r = s + 1$ and

$$\prod b_i = q \prod a_j. \quad (1.2.5)$$

Most of the results regarding the basic hypergeometric series which we will be using are given in Gasper and Rahman [11]. We will refer to it as and when required.

1.3 Askey-Wilson and associated Askey-Wilson polynomials

The most general extension of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite was found by Askey and Wilson [7]. These are the ${}_4\phi_3$ polynomials with four free parameters and are defined

$$p_n(x) \equiv p_n(x; a, b, c, d) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, az, a/z \\ ab, ac, ad \end{matrix} ; q, q \right] \quad (1.3.1)$$

$$x = \frac{z + z^{-1}}{2}, \quad |z| \leq 1. \quad (1.3.2)$$

These polynomials satisfy the following orthogonality relation

$$\int_{-1}^1 w(x; a, b, c, d) p_n(x) p_m(x) dx = h_n(a, b, c, d) \delta_{m,n} \quad (1.3.3)$$

where

$$w(x; a, b, c, d) = \frac{h(x; 1)h(x; -1)h(x; \sqrt{q})h(x; \sqrt{q})}{h(x; a)h(x; b)h(x; c)h(x; d)} \frac{1}{\sqrt{1-x^2}}, \quad (1.3.4)$$

$$h(x; a) = \prod_{n=0}^{\infty} (1 - 2axq^n + q^{2n}) = (az, a/z; q)_{\infty}, \quad (1.3.5)$$

$$\begin{aligned} h_n(a, b, c, d) & \quad (1.3.6) \\ &= 2\pi \frac{(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}} \frac{(q, bc, bd, cd; q)_n}{(abcdq^{-1}, ab, ac, ad; q)_n} \frac{(1 - abcdq^{-1})}{(1 - abcdq^{2n-1})} a^{2n}, \end{aligned}$$

provided that $\max(|a|, |b|, |c|, |d|) < 1$. They satisfy the following three-term recurrence relation:

$$xp_n(x; a, b, c, d) = A_n p_{n+1}(x; a, b, c, d) + B_n p_n(x; a, b, c, d) + C_n p_{n-1}(x; a, b, c, d), \quad (1.3.7)$$

$$A_n = a^{-1} \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{(1 - abcdq^{2n-1})(1 - abcdq^{2n})}. \quad (1.3.8)$$

$$C_n = a \frac{(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})(1 - q^n)}{(1 - abcdq^{2n-1})(1 - abcdq^{2n-2})}. \quad (1.3.9)$$

$$B_n = a + a^{-1} - A_n - C_n. \quad (1.3.10)$$

If we specialize the parameters by taking $a = q^{\frac{3}{2} + \frac{1}{4}}$, $b = aq^{\frac{1}{2}}$, $c = -q^{\frac{7}{2} + \frac{1}{4}}$, $d = cq^{\frac{1}{2}}$ in (1.3.1), set

$$P_n^{(\beta, \gamma)}(x|q) = \frac{(q^{\beta+1}; q)_n}{(q; q)_n} p_n(x; q^{\frac{3}{2} + \frac{1}{4}}, q^{\frac{3}{2} + \frac{3}{4}}, -q^{\frac{7}{2} + \frac{1}{4}}, -q^{\frac{7}{2} + \frac{3}{4}})$$

and let $q \rightarrow 1$, we obtain the Jacobi polynomials

$$P_n^{(\beta, \gamma)}(x) = \frac{(\beta + 1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, n + \beta + \gamma + 1 \\ \beta + 1 \end{matrix}; \frac{(1 - x)}{2} \right]. \quad (1.3.11)$$

On the other hand, if we replace a, b, c, d by q^a, q^b, q^c, q^d , respectively, replace $e^{i\theta}$ by q^{it} and then take the limit $q \rightarrow 1$, we will get the Wilson polynomials

$$P_n(x; a, b, c, d) = {}_4F_3 \left[\begin{matrix} -n, a + b + c + d + n - 1, a + it, a - it \\ a + b, a + c, a + d \end{matrix}; 1 \right] \quad (1.3.12)$$

where

$$x = (a^2 + t^2). \quad (1.3.13)$$

At this point we must mention that Nasrallah and Rahman [27] used (2.10.18) and (2.10.19) of [11] along with (1.3.3) to derive the following integral representation of very-well poised ${}_8\phi_7$ series:

$$\begin{aligned}
& \int_{-1}^1 w(x; a, b, c, d) \frac{h(x; g)}{h(x; f)} dx \\
&= \frac{2\pi(ag, bg, cg, dg, fg, abcd f/g; q)_{\infty}}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, df, g^2; q)_{\infty}} \\
& \quad \times {}_8W_7(g^2/q; g/a, g/b, g/c, g/d, g/f; .q, abcd f/g),
\end{aligned} \tag{1.3.14}$$

which will be used extensively in our later work. If the series in (1.3.14) does not terminate then the condition $|abcd f/g| < 1$ must be imposed for convergence. In view of the Bailey transformation formula [11, (2.10.1)], many variants of (1.3.14) are possible.

In view of (1.1.7) the associated Askey-Wilson polynomials $p_n^{\alpha}(x; a, b, c, d)$ will then satisfy the recurrence relation:

$$x p_n^{\alpha}(x; a, b, c, d) = A_n^{\alpha} p_{n+1}^{\alpha}(x; a, b, c, d) + B_n^{\alpha} p_n^{\alpha}(x; a, b, c, d) + C_n^{\alpha} p_{n-1}^{\alpha}(x; a, b, c, d), \tag{1.3.15}$$

$$A_n^{\alpha} = a^{-1} \frac{(1 - abq^{n+\alpha})(1 - acq^{n+\alpha})(1 - adq^{n+\alpha})(1 - abcdq^{\alpha+n-1})}{(1 - abcdq^{2\alpha+2n-1})(1 - abcdq^{2\alpha+2n})}, \tag{1.3.16}$$

$$C_n^{\alpha} = a \frac{(1 - bcq^{\alpha+n-1})(1 - bdq^{\alpha+n-1})(1 - cdq^{\alpha+n-1})(1 - q^{\alpha+n})}{(1 - abcdq^{2\alpha+2n-1})(1 - abcdq^{2\alpha+2n-2})}, \tag{1.3.17}$$

$$B_n^{\alpha} = a + a^{-1} - A_n^{\alpha} - C_n^{\alpha}. \tag{1.3.18}$$

with $p_0^{\alpha}(x; a, b, c, d) = 1$ and $p_{-1}^{\alpha}(x; a, b, c, d) = 0$.

Using the theory of basic hypergeometric series and the integral representation (1.3.14) of ${}_8\phi_7$, Ismail and Rahman [17] first found a set of contiguous relations for ${}_8\phi_7$ series which they used to prove that the recurrence relation (1.3.15) has the following two independent solutions:

$$\begin{aligned}
r_{\alpha+n}(z; a, b, c, d) &= \frac{(abq^{\alpha+n}, acq^{\alpha+n}, adq^{\alpha+n}, bcdq^{\alpha+n}/z; q)_{\infty}}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, azq^{\alpha+n}; q)_{\infty}} \left(\frac{a}{z}\right)^{n+\alpha} \\
& \quad \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, qz/a)
\end{aligned} \tag{1.3.19}$$

and

$$\begin{aligned}
s_{\alpha+n}(z; a, b, c, d) & \quad (1.3.20) \\
&= \frac{(abcdq^{2\alpha+2n}, bzq^{\alpha+n+1}, czq^{\alpha+n+1}, dzq^{\alpha+n+1}, bcdzq^{\alpha+n}; q)_{\infty}}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, bcdzq^{2\alpha+2n+1}; q)_{\infty}} (az)^{n+\alpha} \\
&\quad \times {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, qz/a; q, az).
\end{aligned}$$

If $\alpha = 0$, then from Watson's formula [11, (2.5.1)] it follows that $r_n(z; a, b, c, d)$ is a multiple of the Askey-Wilson polynomials defined by (1.3.1).

By taking an appropriate linear combination of these solutions that yields the correct initial value, namely $p_0^\alpha(x; a, b, c, d) = 1$ and $p_{-1}^\alpha(x; a, b, c, d) = 0$. they proved that

$$p_n^\alpha(x) \equiv p_n^\alpha(x; a, b, c, d) = L_\alpha r_{\alpha+n}(z; a, b, c, d) + M_\alpha s_{\alpha+n}(z; a, b, c, d) \quad (1.3.21)$$

where

$$L_\alpha = W_\alpha^{-1} s_{\alpha-1}(z; a, b, c, d), \quad M_\alpha = -W_\alpha^{-1} r_{\alpha-1}(z; a, b, c, d) \quad (1.3.22)$$

and the Casorati determinant W_α is

$$r_\alpha(z; a, b, c, d) s_{\alpha-1}(z; a, b, c, d) - s_\alpha(z; a, b, c, d) r_{\alpha-1}(z; a, b, c, d). \quad (1.3.23)$$

By further evaluating the Casorati determinant they were able to give the following explicit representation for $p_n^\alpha(x; a, b, c, d)$:

$$\begin{aligned}
p_n^\alpha(x; a, b, c, d) &= \frac{a^n}{(1 - abcdq^{2\alpha-2})} \frac{(bc, bd, cd, az; q)_{\infty}}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}, bz, cz, dz, bcd/z; q)_{\infty}} \\
&\quad \left\{ \frac{(abcdq^{2\alpha-2}, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}; q)_{\infty}}{(bcdzq^{2\alpha-1}; q)_{\infty}} \right. \\
&\quad \times \frac{(abq^{\alpha+n}, acq^{\alpha+n}, adq^{\alpha+n}, bcdq^{\alpha+n}/z; q)_{\infty}}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, azq^{\alpha+n}; q)_{\infty}} z^{-n} \\
&\quad \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, qz/a) \\
&\quad \times {}_8W_7(bcdzq^{2\alpha-2}; bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha, zq/a; q, az)
\end{aligned}$$

$$\begin{aligned}
& - \frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bcdq^{\alpha-1}/z; q)_\infty}{(azq^{\alpha-1}; q)_\infty} \\
& \times \frac{(abcdq^{2\alpha+2n}, bzq^{\alpha+n+1}, czq^{\alpha+n+1}, dzq^{\alpha+n+1}, bcdzq^{\alpha+n}; q)_\infty}{(bcdzq^{2\alpha+2n+1}, q^{\alpha+n+1}, bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}; q)_\infty} z^{n+2} \\
& \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha-2}, q^{1-\alpha}; q, qz/a) \\
& \times {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, zq/a; q, az) \}.
\end{aligned} \tag{1.3.24}$$

They also gave the following orthogonality relation for the associated Askey-Wilson polynomials:

$$\int_{-1}^1 p_m^\alpha(x) p_n^\alpha(x) d\mu(x) = \xi_n \delta_{m,n} \tag{1.3.25}$$

where

$$\xi_n = \frac{1 - abcdq^{2\alpha-1}}{1 - abcdq^{2\alpha+2n-1}} \frac{(q^{\alpha+1}, bcq^\alpha, bdq^\alpha, cdq^\alpha; q)_n}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}; q)_n} a^{2n}. \tag{1.3.26}$$

By applying a well-known theorem of Markov [41, Th.(3.5.4)] and the Perron-Stieltjes inversion formula, Ismail and Rahman found the absolutely continuous component of measure of orthogonality

$$\begin{aligned}
\frac{d\mu}{dx} &= \frac{(abq^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}, q^{\alpha+1}; q)_\infty}{2\pi(abcdq^{2\alpha-1}, abcdq^{2\alpha}; q)_\infty} \\
&\times \left| \frac{(e^{2i\theta}, bcdq^{2\alpha-1}e^{i\theta}; q)_\infty}{(ae^{i\theta}, bq^\alpha e^{i\theta}, cq^\alpha e^{i\theta}, dq^\alpha e^{i\theta}, bcdq^{\alpha-1}e^{i\theta}; q)_\infty} \right|^2 \\
&\times |{}_8W_7(bcdq^{2\alpha-2}e^{i\theta}; bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha, qe^{i\theta}/a; q, ae^{i\theta})|^{-2},
\end{aligned} \tag{1.3.27}$$

where $x = \cos \theta$ and $\max \{|a|, |b|, |c|, |d|\} < 1$.

If we specialize the parameters by taking $a = q^{\frac{\theta}{2} + \frac{1}{4}}$, $b = aq^{\frac{1}{2}}$, $c = -q^{\frac{3}{2} + \frac{1}{4}}$, $d = cq^{\frac{1}{2}}$ in (1.3.24) and letting $q \rightarrow 1$ we obtain the associated Jacobi polynomials studied by Wimp [43]. On the other hand, if we replace a, b, c, d by q^a, q^b, q^c, q^d respectively, replace $e^{i\theta}$ by q^{it} and then take the limit as $q \rightarrow 1$, we will get the associated Wilson polynomials which is the associated extension of (1.3.12) and was studied by Ismail et al. [15].

A terminating polynomial form of $p_n^\alpha(x; a, b, c, d)$ which was also found by Ismail and Rahman [17] was used by Rahman [31] to obtain an integral representation for $p_n^\alpha(x; a, b, c, d)$ in terms of the Askey-Wilson polynomials $p_n(x; aq^{\frac{\alpha}{2}}, bq^{\frac{\alpha}{2}}, cq^{\frac{\alpha}{2}}, dq^{\frac{\alpha}{2}})$. More precisely Rahman proved:

$$p_n^\alpha(x; a, b, c, d) = C q^{-\alpha n} \frac{(abcdq^{2\alpha-1}, q^{\alpha+1}; q)_n}{(abcdq^{\alpha-1}, q; q)_n} \times \int_{-1}^1 dz K(x; z) p_n(z; aq^{\frac{\alpha}{2}}, bq^{\frac{\alpha}{2}}, cq^{\frac{\alpha}{2}}, dq^{\frac{\alpha}{2}}), \quad (1.3.28)$$

where

$$C = \frac{(q, q, q, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha; q)_\infty}{4\pi^2 (abcdq^{2\alpha-2}, q^{\alpha+1}; q)_\infty}, \quad (1.3.29)$$

and

$$K(x; z) = \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h(z; q^{\frac{\alpha}{2}} e^{i\theta}, q^{\frac{\alpha}{2}} e^{-i\theta})} \times \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, bq^{\frac{\alpha-1}{2}}, cq^{\frac{\alpha-1}{2}}, dq^{\frac{\alpha-1}{2}}) \frac{h(y; q^{\frac{\alpha+1}{2}} e^{i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta})}{h(y; \sqrt{q} e^{i\psi}, \sqrt{q} e^{-i\psi})}, \quad (1.3.30)$$

with

$$x = \cos \theta, \quad y = \cos \varphi, \quad z = \cos \psi. \quad (1.3.31)$$

In what follows we will see that (1.3.24) and (1.3.28) are the basis of most of our work. Though, Hahn [14] considered associated Laguerre polynomials as early as in the 1940's, the systematic studies of the associated classical polynomials and their q -analogues started in the early eighties with the work of Bustoz and Ismail [9]. The subject has developed since then and we have the measure of orthogonality for all associated classical polynomials up to the associated Askey-Wilson polynomials. Apart from these important results little else is known about these polynomials. For example we do not know the product formulas, addition formulas, Poisson kernels and other related kernels for these polynomials which may be useful for the harmonic analysis and further development of the theory of these polynomials. The

classical orthogonal polynomials of Jacobi, Hermite and Laguerre satisfy a second order differential equation of Sturm-Liouville type. Hahn [14] proved that the associated Laguerre polynomials are a sum of product of confluent hypergeometric functions. Since the confluent hypergeometric series satisfy a second-order differential equation, one would expect that the associated Laguerre polynomials will satisfy a fourth-order differential equation. Hahn found this differential equation explicitly. Later Wimp [43] found the fourth order-differential equation satisfied by the associated Jacobi polynomials. In view of the above, one would expect that the associated q -ultraspherical, associated continuous q -Jacobi and more generally associated Askey-Wilson polynomials, will satisfy fourth-order q -difference equations. Magnus [25] proved that this is indeed the case, but the exact form of the difference equation is yet to be found. This is a pretty difficult problem which I have tried to solve but without success, so far. However, some of the other problems mentioned here have been addressed in the subsequent chapters.

This thesis contains four chapters. In this chapter we have introduced some basic definitions, basic results and a description of the problems that will be considered in the following chapters. The second chapter deals with the generating functions of associated orthogonal polynomials. More precisely we will find a generating function of the associated continuous q -Jacobi polynomials. In chapter three, we will first observe that the associated Askey-Wilson polynomials (1.3.24) have yet another representation as a linear combination of the linearly independent solutions $s_{\alpha+n}(z; a, b, c, d)$ and $s_{\alpha+n}(z^{-1}; a, b, c, d)$ of (1.3.15), where $s_{\alpha+n}(z; a, b, c, d)$ is given by (1.3.20). Using this representation of the associated Askey-Wilson polynomials and a generating function of the associated q -ultraspherical polynomials derived by Bustoz and Ismail [9] we will find the Poisson kernel of the associated continuous q -ultraspherical polynomials. Besides, we will find several linear generating as well as bilinear generating functions for these polynomials.

Formulas of the form

$$q_n(x) = \int p_n(y) d\gamma_x(y), \quad d\gamma_x(y) \geq 0 \quad (1.3.32)$$

and

$$q_n(x) = \sum \alpha_{n,k} p_k(x), \quad \alpha_{n,k} \geq 0, \quad (1.3.33)$$

where $q_n(x)$ and $p_n(x)$ are polynomials of degree n and not necessarily the same, are called projection formulas and the connection coefficient formula respectively. We will find a projection formula and a reproducing kernel for the associated Askey-Wilson polynomials. The derivation of these formulas will be facilitated by the computation of the sum

$$\sum_{n=0}^{\infty} \frac{p_n^{\alpha}(x; a, b, c, d) p_n^{\alpha, \nu}(y; a, b, c, d)}{\xi_n},$$

where $p_n^{\alpha, \nu}(y; a, b, c, d) = p_n^{\alpha}(y; a\nu, b\nu, c\nu^{-1}, d\nu^{-1})$ and ξ_n is given by (1.3.26). This sum will enable us to prove the dual orthogonality relation also for the associated Askey-Wilson polynomials which in turns gives an alternative way of finding the weight function of the associated Askey-Wilson polynomials.

Chapter 2

A generating function for the associated continuous q -Jacobi polynomials

2.1 Introduction

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials. Define a function $F(x, t)$ by

$$F(x, t) = \sum_{n=0}^{\infty} p_n(x) t^n. \quad (2.1.1)$$

If $F(x, t)$ exists, it is called a generating function for the sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$. Two of the most well-known and useful generating functions for classical orthogonal polynomials are:

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x) t^n = (1 - 2xt + t^2)^{-\lambda}, \quad (2.1.2)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\beta + \gamma + 1)_n}{(1 + \beta)_n} P_n^{(\beta, \gamma)}(x) t^n \\ &= (1 - 2xt + t^2)^{-\frac{\beta + \gamma + 1}{2}} {}_2F_1 \left[\begin{matrix} \frac{\beta + \gamma + 1}{2}, & \frac{\beta - \gamma}{2} \\ \beta + 1 \end{matrix} ; \frac{2t(1 - x)}{1 - 2xt + t^2} \right] \end{aligned} \quad (2.1.3)$$

where $|t| < 1$ and

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, n+2\lambda \\ \lambda + \frac{1}{2} \end{matrix} ; \frac{(1-x)}{2} \right], \quad (2.1.4)$$

$$P_n^{(\beta, \gamma)}(x) = P_n^{(\beta, \gamma)}(1) {}_2F_1 \left[\begin{matrix} -n, n+\beta+\gamma+1 \\ \beta+1 \end{matrix} ; \frac{(1-x)}{2} \right] \quad (2.1.5)$$

are the ultraspherical and Jacobi polynomials respectively, [2], [35]. It is clear that (2.1.2) follows from (2.1.3) if we set $\beta = \gamma = \lambda - \frac{1}{2}$.

A q -analogue of (2.1.2) was given by Rogers [39] in 1895:

$$\sum_{n=0}^{\infty} C_n(x; a^2|q) t^n = \frac{(a^2 t e^{i\theta}, a^2 t e^{-i\theta}; q)_{\infty}}{(t e^{i\theta}, t e^{-i\theta}; q)_{\infty}}, \quad (2.1.6)$$

where $x = \cos \theta$, $0 \leq \theta \leq \pi$, $|t| < 1$ and

$$\begin{aligned} C_n(x; a^2|q) &= \frac{(a^4; q)_n}{(q; q)_n} a^{-n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, a^4 q^n, a e^{i\theta}, a e^{-i\theta} \\ a^2 \sqrt{q}, -a^2, -a^2 \sqrt{q} \end{matrix} ; q, q \right] \\ &= \sum_{m=0}^n \frac{(a^2; q)_m (a^2; q)_{n-m}}{(q; q)_m (q; q)_{n-m}} e^{i(n-2m)\theta} \end{aligned} \quad (2.1.7)$$

is Rogers' continuous q -ultraspherical polynomials. The properties of these polynomials were thoroughly studied by Askey and Ismail [4], [5]. It has the limiting property that

$$\lim_{q \rightarrow 1^-} C_n(x; q^\lambda|q) = C_n^\lambda(x).$$

The fact that (2.1.6) is a q -analogue of (2.1.2) follows from the fact that

$$\lim_{q \rightarrow 1^-} \frac{(t q^\lambda e^{i\theta}; q)_{\infty}}{(t e^{i\theta}; q)_{\infty}} = \lim_{q \rightarrow 1^-} (t e^{i\theta}; q)_{\lambda} = (1 - t e^{i\theta})^{-\lambda},$$

by the q -Binomial Theorem [11, (1.3.2)].

A q -analogue of (2.1.3) was found by Gasper and Rahman in 1986:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a^2 c^2; q)_n}{(q; q)_n} p_n(x; a, a\sqrt{q}, -c, -c\sqrt{q}) \left(\frac{t}{a}\right)^n \\ = \frac{(at, ac^2 t, acte^{i\theta}, acte^{-i\theta}; q)_{\infty}}{(ct, a^2 ct, te^{i\theta}, te^{-i\theta}; q)_{\infty}} \\ \times {}_8W_7(a^2 ct q^{-1}; ac, a/c, ct q^{-\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}; q, ct\sqrt{q}), \end{aligned} \quad (2.1.8)$$

where $x = \cos \theta$. Setting $a = q^{\frac{\alpha}{2} + \frac{1}{4}}$ and $c = q^{\frac{\gamma}{2} + \frac{1}{4}}$, we see that (2.1.8) approaches (2.1.3) in the limiting case $q \rightarrow 1^-$. The polynomials $p_n(x; a, a\sqrt{q}, -c, -c\sqrt{q})$ on the left-hand side of (2.1.8) are the continuous q -Jacobi polynomials of Askey and Wilson. It is a special case of the Askey-Wilson polynomials when we replace b, c, d in equation (1.3.1) by $a\sqrt{q}, -c, -c\sqrt{q}$, respectively. These polynomials were studied by Askey and Wilson [7], and Rahman [29].

Bustoz and Ismail [9], it seems, were the first who initiated a systematic study of the associated q -classical polynomials. They gave a thorough account of associated polynomials corresponding to ultraspherical, q -ultraspherical, q -Hermite and q -Laguerre polynomials. Amongst many interesting results, they gave the following generating function for the associated q -ultraspherical polynomials.

$$\sum_{n=0}^{\infty} C_n^{\alpha}(x; a^2 | q) t^n = \frac{1 - q^{\alpha}}{1 - 2xt + t^2} {}_3\phi_2 \left[\begin{matrix} a^2 te^{i\theta}, a^2 te^{-i\theta}, q \\ tqe^{i\theta}, tqe^{-i\theta} \end{matrix} ; q, q^{\alpha} \right], \quad (2.1.9)$$

where $x = \cos \theta$. To prove that this approaches (2.1.6) in the limiting case when $\alpha \rightarrow 0^+$, one needs to use the well-known formula of Abel, namely

$$\lim_{t \rightarrow 1^-} (1 - t) \sum_{n=0}^{\infty} a_n t^n = \lim_{n \rightarrow \infty} a_n. \quad (2.1.10)$$

Our primary interest in this chapter is to find the corresponding extension of (2.1.8) to the associated continuous q -Jacobi polynomials. We will show that

$$\begin{aligned}
P_t^\alpha(x; a, c) & \quad (2.1.11) \\
&:= \sum_{n=0}^{\infty} \frac{(a^2 c^2 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} p_n^\alpha(x; a, a\sqrt{q}, -c, -c\sqrt{q}) \left(\frac{t}{a}\right)^n \\
&= \frac{(c^2 q^{\alpha+\frac{1}{2}}; q)_\infty (1 - c^2 q^{\alpha-\frac{1}{2}})}{(a^2 q^{\alpha+\frac{1}{2}}; q)_\infty (1 - acq^{\alpha-\frac{1}{2}})} \frac{1 - q^\alpha}{1 - 2xt + t^2} \\
&\quad \times \sum_{k=0}^{\infty} \frac{|(acte^{i\theta}; q)_k|^2}{|(tqe^{i\theta}; q)_k|^2} \frac{(-ct, -ct\sqrt{q}; q)_k}{(-at, -at\sqrt{q}; q)_k} \frac{(a^2 t^2; q)_{2k}}{(act^2; q)_{2k}} q^{\alpha k} \mu_k \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} a^2 t^2 q^{2k}, -atq^{k+1}, a/c \\ -atq^k, act^2 q^{2k+1} \end{matrix} ; q, c^2 q^{\alpha-1/2} \right],
\end{aligned}$$

where

$$\mu_k = \sum_{n=0}^k \frac{1 - act^2 q^{2n-1}}{1 - act^2 q^{-1}} \frac{(act^2 q^{-1}, a/c, ctq^{-\frac{1}{2}}, ct, te^{i\theta}, te^{-i\theta}; q)_n}{(q, c^2 t^2, atq^{\frac{1}{2}}, at, acte^{-i\theta}, acte^{i\theta}; q)_n} (acq^{\frac{1}{2}})^n. \quad (2.1.12)$$

Let us see some special cases of this generating functions. When $a = c$, μ_k and

${}_3\phi_2$ series in (2.1.11) collapse and we get

$$P_t^\alpha(x; a, a) = \frac{1 - q^\alpha}{1 - 2xt + t^2} {}_3\phi_2 \left[\begin{matrix} a^2 te^{i\theta}, a^2 te^{-i\theta}, q \\ tqe^{i\theta}, tqe^{-i\theta} \end{matrix} ; q, q^\alpha \right],$$

which is the same as (2.1.9).

Note that

$$\lim_{k \rightarrow \infty} \mu_k = {}_8W_7(act^2 q^{-1}; a/c, ct, ctq^{-\frac{1}{2}}, te^{i\theta}, te^{-i\theta}; q, acq^{\frac{1}{2}}) \quad (2.1.13)$$

and

$$\lim_{k \rightarrow \infty} {}_3\phi_2 \left[\begin{matrix} a^2 t^2 q^{2k}, -atq^{k+1}, a/c \\ -atq^k, act^2 q^{2k+1} \end{matrix} ; q, c^2 q^{\alpha-1/2} \right] = \frac{(acq^{\alpha-1/2}; q)_\infty}{(c^2 q^{\alpha-1/2}; q)_\infty}. \quad (2.1.14)$$

Thus it is necessary for convergence of the series on the right hand side of (2.1.11) that $\alpha > 0$. When $\alpha \rightarrow 0^+$ we see, from Abel's Theorem (2.1.10), that (2.1.11) goes to (2.1.8). So (2.1.11) is an extension of (2.1.8).

2.2 Proof of (2.1.11)

As mentioned in the previous section, the continuous q -Jacobi polynomials are obtained from (1.3.1) if we replace b, c, d by $a\sqrt{q}, -c, -c\sqrt{q}$, respectively. Consider the generating function

$$P_t(x; a, c, \alpha) := \sum_{n=0}^{\infty} \frac{(a^2 c^2 q^{2\alpha}; q)_n}{(q; q)_n} p_n(x; a q^{\frac{\alpha}{2}}, a q^{\frac{\alpha+1}{2}}, -c q^{\frac{\alpha}{2}}, -c q^{\frac{\alpha+1}{2}}) \left(\frac{t q^{-\frac{\alpha}{2}}}{a} \right)^n, \quad (2.2.1)$$

$|t| < 1$. Then from (2.1.8)

$$\begin{aligned} P_t(x; a, c, \alpha) &= \frac{(a t q^{\frac{\alpha}{2}}, a c^2 t q^{\frac{3\alpha}{2}}, a c t q^{\alpha} e^{i\theta}, a c t q^{\alpha} e^{-i\theta}; q)_{\infty}}{(c t q^{\frac{\alpha}{2}}, a^2 c t q^{\frac{3\alpha}{2}}, t e^{i\theta}, t e^{-i\theta}; q)_{\infty}} \\ &\quad \times {}_8W_7(a^2 c t q^{\frac{3\alpha}{2}-1}; a c q^{\alpha}, a/c, c t q^{\frac{\alpha-1}{2}}, a q^{\frac{\alpha}{2}} e^{i\theta}, a q^{\frac{\alpha}{2}} e^{-i\theta}; c t q^{\frac{\alpha+1}{2}}). \end{aligned} \quad (2.2.2)$$

If we apply formula [11, (2.10.1)] on the above ${}_8W_7$ series, we get

$$\begin{aligned} P_t(x; a, c, \alpha) &= \frac{(a c q^{\alpha+\frac{1}{2}}, a t q^{\frac{\alpha}{2}}, a t q^{\frac{\alpha+1}{2}}, c^2 t^2 q^{\alpha}; q)_{\infty} |(a c t q^{\alpha} e^{i\theta}; q)_{\infty}|^2}{(a^2 q^{\alpha+\frac{1}{2}}, c t q^{\frac{\alpha}{2}}, c t q^{\frac{\alpha+1}{2}}, a c t^2 q^{\alpha}; q)_{\infty} |(t e^{i\theta}; q)_{\infty}|^2} \\ &\quad \times {}_8W_7(a c t^2 q^{\alpha-1}; a/c, c t q^{\frac{\alpha-1}{2}}, c t q^{\frac{\alpha}{2}}, t e^{i\theta}, t e^{-i\theta}; q, a c q^{\alpha+\frac{1}{2}}). \end{aligned} \quad (2.2.3)$$

From Rahman's integral representation of associated Askey-Wilson polynomials [31], we find that the associated continuous q -Jacobi polynomials are given by

$$\begin{aligned} &\frac{(a^2 c^2 q^{\alpha}; q)_n}{(q^{\alpha+1}; q)_n} p_n^{\alpha}(x; a, a\sqrt{q}, -c, -c\sqrt{q}) \\ &= C q^{-\alpha n} \frac{(a^2 c^2 q^{2\alpha}; q)_n}{(q; q)_n} \int_{-1}^1 dz K(x; z) p_n(z; a q^{\frac{\alpha}{2}}, a q^{\frac{\alpha+1}{2}}, -c q^{\frac{\alpha}{2}}, -c q^{\frac{\alpha+1}{2}}), \end{aligned} \quad (2.2.4)$$

where

$$C = \frac{(q, q, q, a^2 q^{\alpha-\frac{1}{2}}, -a c q^{\alpha-\frac{1}{2}}, -a c q^{\alpha-\frac{1}{2}}, -a c q^{\alpha-1}, -a c q^{\alpha}, c^2 q^{\alpha-\frac{1}{2}}, q^{\alpha}; q)_{\infty}}{4\pi^2 (a^2 c^2 q^{2\alpha-1}, q^{\alpha+1}; q)_{\infty}}, \quad (2.2.5)$$

$$K(x; z) = \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h(z; q^{\frac{\alpha}{2}} e^{i\theta}, q^{\frac{\alpha}{2}} e^{-i\theta})} \quad (2.2.6)$$

$$\times \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; q^{\frac{\alpha+1}{2}} e^{i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta})}{h(y; \sqrt{q} e^{i\psi}, \sqrt{q} e^{-i\psi})}.$$

$$x = \cos \theta, \quad y = \cos \varphi, \quad z = \cos \psi. \quad (2.2.7)$$

Multiplying both sides of (2.2.4) by $\left(\frac{tq^{\frac{\alpha}{2}}}{a}\right)^n$ and summing over n , we get

$$\begin{aligned} G_t^\alpha(x; a, c) &= C \int_{-1}^1 dz K(x; z) \sum_{n=0}^{\infty} \frac{(a^2 c^2 q^{2\alpha}; q)_n}{(q; q)_n} p_n(z; aq^{\frac{\alpha}{2}}, aq^{\frac{\alpha+1}{2}}, -cq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha+1}{2}}) \left(\frac{tq^{-\frac{\alpha}{2}}}{a}\right)^n. \end{aligned} \quad (2.2.8)$$

where

$$G_t^\alpha(x; a, c) = \sum_{n=0}^{\infty} \frac{(a^2 c^2 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} p_n^\alpha(x; a, a\sqrt{q}, -c, -c\sqrt{q}) \left(\frac{tq^{\frac{\alpha}{2}}}{a}\right)^n. \quad (2.2.9)$$

Thus we have

$$G_t^\alpha(x; a, c) = C \int_{-1}^1 dz K(x; z) P_t(z; a, c, \alpha). \quad (2.2.10)$$

From (2.2.3),

$$\begin{aligned} &\int_{-1}^1 dz K(x; z) P_t(z; a, c, \alpha) \\ &= \frac{(acq^{\alpha+\frac{1}{2}}, atq^{\frac{\alpha}{2}}, atq^{\frac{\alpha+1}{2}}, c^2 t^2 q^\alpha; q)_\infty}{(a^2 q^{\alpha+\frac{1}{2}}, ctq^{\frac{\alpha}{2}}, ctq^{\frac{\alpha+1}{2}}, act^2 q^\alpha; q)_\infty} \int_{-1}^1 dz K(x; z) \frac{|(actq^\alpha e^{i\psi}; q)_\infty|^2}{|(te^{i\psi}; q)_\infty|^2} \\ &\quad \times {}_8W_7(act^2 q^{\alpha-1}; a/c, ctq^{\frac{\alpha-1}{2}}, ctq^{\frac{\alpha}{2}}, te^{i\psi}, te^{-i\psi}; q, acq^{\alpha+\frac{1}{2}}) \\ &= \frac{(acq^{\alpha+\frac{1}{2}}, atq^{\frac{\alpha}{2}}, atq^{\frac{\alpha+1}{2}}, c^2 t^2 q^\alpha; q)_\infty}{(a^2 q^{\alpha+\frac{1}{2}}, ctq^{\frac{\alpha}{2}}, ctq^{\frac{\alpha+1}{2}}, act^2 q^\alpha; q)_\infty} \sum_{n=0}^{\infty} \frac{1 - act^2 q^{\alpha-1+2n}}{1 - act^2 q^{\alpha-1}} \frac{(act^2 q^{\alpha-1}, a/c; q)_n}{(q, c^2 t^2 q^\alpha; q)_n} \\ &\quad \times \frac{(ctq^{\frac{\alpha-1}{2}}, ctq^{\frac{\alpha}{2}}; q)_n}{(atq^{\frac{\alpha+1}{2}}, atq^{\frac{\alpha}{2}}; q)_n} (acq^{\alpha+\frac{1}{2}})^n \int_{-1}^1 dz K(x; z) \frac{|(actq^{\alpha+n} e^{i\psi}; q)_\infty|^2}{|(tq^n e^{i\psi}; q)_\infty|^2}. \end{aligned} \quad (2.2.11)$$

Since $|a| < 1$, $|c| < 1$ and $\alpha > 0$, the change of order of summation and integration can be justified. However,

$$\int_{-1}^1 dz K(x; z) \frac{|(actq^{\alpha+n} e^{i\psi}; q)_\infty|^2}{|(tq^n e^{i\psi}; q)_\infty|^2} \quad (2.2.12)$$

$$\begin{aligned}
&= \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) h(y; q^{\frac{\alpha+1}{2}} e^{i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta}) \\
&\quad \times \int_{-1}^1 dz w(z; q^{\frac{\alpha}{2}} e^{i\theta}, q^{\frac{\alpha}{2}} e^{-i\theta}, \sqrt{q} e^{i\varphi}, \sqrt{q} e^{-i\varphi}) \frac{|(actq^{\alpha+n} e^{i\psi}; q)_{\infty}|^2}{|(tq^n e^{i\psi}; q)_{\infty}|^2}.
\end{aligned}$$

by (2.2.6). The integral with respect to z in (2.2.12) is, by [11, (6.3.8)], equal to

$$\begin{aligned}
&\frac{2\pi |(actq^{\alpha+n+\frac{1}{2}} e^{i\varphi}; q)_{\infty}|^2 (actq^{n+\frac{3\alpha}{2}} e^{-i\theta}, tq^{n+1+\frac{\alpha}{2}} e^{-i\theta}, q^{\alpha+1}; q)_{\infty}}{|(tq^{n+\frac{1}{2}} e^{i\varphi}; q)_{\infty}|^2 (q, q, q^{\alpha}, tq^{n+\frac{\alpha}{2}} e^{-i\theta}, actq^{n+1+\frac{3\alpha}{2}} e^{-i\theta}; q)_{\infty} h(y; q^{\frac{\alpha+1}{2}} e^{i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta})} \\
&\times {}_8W_7(actq^{n+\frac{3\alpha}{2}} e^{-i\theta}; q, acq^{\alpha}, actq^{n+\frac{\alpha}{2}} e^{-i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta+i\varphi}, q^{\frac{\alpha+1}{2}} e^{-i\theta-i\varphi}; q, tq^{n+\frac{\alpha}{2}} e^{i\theta}).
\end{aligned} \tag{2.2.13}$$

We shall now transform this ${}_8W_7$ series by [11, (2.10.1)] so that the parameters q and $actq^{n+\frac{\alpha}{2}} e^{-i\theta}$ remain invariant in the numerator. Thus (2.2.13) reduces to

$$\begin{aligned}
&\frac{2\pi (act^2 q^{\alpha+2n}; q)_{\infty}}{(q, q, act^2 q^{\alpha+2n+1}; q)_{\infty}} \frac{|(tq^{\frac{\alpha}{2}+n+1} e^{i\theta}; q)_{\infty}|^2}{|(tq^{\frac{\alpha}{2}+n} e^{i\theta}; q)_{\infty}|^2} \frac{1}{h(y; q^{\frac{\alpha+1}{2}} e^{i\theta}, q^{\frac{\alpha+1}{2}} e^{-i\theta})} \\
&\times \sum_{m=0}^{\infty} \frac{1 - act^2 q^{\alpha+2n+2m}}{1 - act^2 q^{\alpha+2n}} \frac{|(actq^{\frac{\alpha}{2}+n} e^{i\theta}; q)_m|^2}{|(tq^{\frac{\alpha}{2}+n+1} e^{i\theta}; q)_m|^2} \frac{h(y; actq^{\alpha+n+m+\frac{1}{2}})}{h(y; tq^{n+m+\frac{1}{2}})} q^{\alpha m}
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{-1}^1 dz K(x; z) \frac{|(actq^{\alpha+n} e^{i\psi}; q)_{\infty}|^2}{|(tq^n e^{i\psi}; q)_{\infty}|^2} \\
&= \frac{2\pi |(tq^{1+\frac{\alpha}{2}} e^{i\theta}; q)_{\infty}|^2}{(q, q; q)_{\infty} |(tq^{\frac{\alpha}{2}} e^{i\theta}; q)_{\infty}|^2} \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \\
&\quad \times \frac{|(tq^{\frac{\alpha}{2}} e^{i\theta}; q)_n|^2}{|(tq^{1+\frac{\alpha}{2}} e^{i\theta}; q)_n|^2} \sum_{m=0}^{\infty} (1 - act^2 q^{\alpha+2n+2m}) \frac{|(actq^{\frac{\alpha}{2}+n} e^{i\theta}; q)_m|^2}{|(tq^{\frac{\alpha}{2}+n+1} e^{i\theta}; q)_m|^2} \frac{h(y; actq^{\alpha+n+m+\frac{1}{2}})}{h(y; tq^{n+m+\frac{1}{2}})} q^{\alpha m} \\
&= \frac{2\pi (1 - act^2 q^{\alpha}) |(tq^{\frac{\alpha}{2}+1} e^{i\theta}; q)_{\infty}|^2}{(q, q; q)_{\infty} |(tq^{\frac{\alpha}{2}} e^{i\theta}; q)_{\infty}|^2} \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \\
&\quad \times \frac{|(tq^{\frac{\alpha}{2}} e^{i\theta}; q)_n|^2}{|(actq^{\frac{\alpha}{2}} e^{i\theta}; q)_n|^2} \sum_{m=0}^{\infty} \frac{(1 - act^2 q^{\alpha+2n+2m})}{(1 - act^2 q^{\alpha})} \frac{|(actq^{\frac{\alpha}{2}} e^{i\theta}; q)_{m+n}|^2}{|(tq^{\frac{\alpha}{2}+1} e^{i\theta}; q)_{m+n}|^2} \frac{h(y; actq^{\alpha+n+m+\frac{1}{2}})}{h(y; tq^{n+m+\frac{1}{2}})} q^{\alpha m}.
\end{aligned} \tag{2.2.14}$$

Inserting (2.2.14) in (2.2.11) we find that

$$G_t^{\alpha}(x; a, c) \tag{2.2.15}$$

$$\begin{aligned}
&= C \frac{2\pi}{(q, q; q)_\infty} \frac{(acq^{\alpha+\frac{1}{2}}, atq^{\frac{\alpha}{2}}, atq^{\frac{\alpha+1}{2}}, c^2t^2q^\alpha; q)_\infty}{(a^2q^{\alpha+\frac{1}{2}}, ctq^{\frac{\alpha}{2}}, ctq^{\frac{\alpha+1}{2}}, act^2q^{\alpha+1}; q)_\infty} \frac{|(tq^{\frac{\alpha}{2}+1}e^{i\theta}; q)_\infty|^2}{|(tq^{\frac{\alpha}{2}}e^{i\theta}; q)_\infty|^2} \\
&\times \sum_{n=0}^{\infty} \frac{(1 - act^2q^{\alpha-1+2n})}{(1 - act^2q^{\alpha-1})} \frac{(act^2q^{\alpha-1}, a/c, ctq^{\frac{\alpha-1}{2}}, ctq^{\frac{\alpha}{2}}, tq^{\frac{\alpha}{2}}e^{i\theta}, tq^{\frac{\alpha}{2}}e^{-i\theta}; q)_n}{(q, c^2t^2q^\alpha, atq^{\frac{\alpha+1}{2}}, atq^{\frac{\alpha}{2}}, actq^{\frac{\alpha}{2}}e^{-i\theta}, actq^{\frac{\alpha}{2}}e^{i\theta}; q)_n} (acq^{\frac{1}{2}+\alpha})^n \\
&\times \sum_{m=0}^{\infty} \frac{(1 - act^2q^{\alpha+2n+2m})}{(1 - act^2q^\alpha)} \frac{|(actq^{\frac{\alpha}{2}}e^{i\theta}; q)_{n+m}|^2}{|(tq^{\frac{\alpha}{2}+1}e^{i\theta}; q)_{n+m}|^2} q^{\alpha m} \\
&\times \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; actq^{\alpha+n+m+\frac{1}{2}})}{h(y; tq^{n+m+\frac{1}{2}})}.
\end{aligned}$$

Now we replace t by $tq^{-\frac{\alpha}{2}}$ so that

$$P_t^\alpha(x; a, c) = G_{tq^{-\frac{\alpha}{2}}}^\alpha(x; a, c).$$

Then,

$$\begin{aligned}
P_t^\alpha(x; a, c) &:= \sum_{n=0}^{\infty} \frac{(a^2c^2q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} p_n^\alpha(x; a, a\sqrt{q}, -c, -c\sqrt{q}) \left(\frac{t}{a}\right)^n \quad (2.2.16) \\
&= C \frac{2\pi}{(q, q; q)_\infty} \frac{(acq^{\alpha+\frac{1}{2}}, at, atq^{\frac{1}{2}}, c^2t^2; q)_\infty}{(a^2q^{\alpha+\frac{1}{2}}, ct, ctq^{\frac{1}{2}}, act^2q; q)_\infty} \frac{|(tqe^{i\theta}; q)_\infty|^2}{|(te^{i\theta}; q)_\infty|^2} \\
&\times \sum_{n=0}^{\infty} \frac{(1 - act^2q^{-1+2n})}{(1 - act^2q^{-1})} \frac{(act^2q^{-1}, a/c, ctq^{-\frac{1}{2}}, ct, te^{i\theta}, te^{-i\theta}; q)_n}{(q, c^2t^2, atq^{\frac{1}{2}}, at, acte^{-i\theta}, acte^{i\theta}; q)_n} (acq^{\frac{1}{2}})^n \\
&\times \sum_{m=0}^{\infty} \frac{(1 - act^2q^{2n+2m})}{(1 - act^2)} \frac{|(acte^{i\theta}; q)_{n+m}|^2}{|(tqe^{i\theta}; q)_{n+m}|^2} q^{(n+m)\alpha} \\
&\times \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; actq^{n+m+\frac{\alpha+1}{2}})}{h(y; tq^{n+m+\frac{1-\alpha}{2}})}.
\end{aligned}$$

In the final stage of our computation we will do the integration with respect to y in (2.2.16). The Nasrallah and Rahman integral representation of the ${}_8\phi_7$ series, see [27], will again be found useful. The ${}_8\phi_7$ series, which we will get after performing the integration with respect to y , will then be transformed into a ${}_3\phi_2$ series by a tricky use of (2.10.10) and (3.4.10) of [11]. The details now follow.

From Nasrallah and Rahman formula (1.3.14), we have

$$\int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; actq^{k+\frac{\alpha+1}{2}})}{h(y; tq^{k+\frac{1-\alpha}{2}})} \quad (2.2.17)$$

$$\begin{aligned}
&= \frac{2\pi(a^2c^2q^{2\alpha-1}, a^2ctq^{\alpha+k}, a^2ctq^{\alpha+k+\frac{1}{2}}, -ac^2tq^{\alpha+k}; q)_\infty}{(q, a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, c^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, -acq^\alpha, atq^k, atq^{k+\frac{1}{2}}; q)_\infty} \\
&\times \frac{(-a^2ctq^{\alpha+k-\frac{1}{2}}; q)_\infty}{(-ctq^k, -a^3c^2tq^{k+2\alpha-\frac{1}{2}}; q)_\infty} \\
&\times {}_8W_7(-a^3c^2tq^{k+2\alpha-\frac{1}{2}}; a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, -acq^{\alpha-\frac{1}{2}}, -atq^{k+\frac{1}{2}}, acq^\alpha; q, -ctq^{k+\frac{1}{2}}),
\end{aligned}$$

where k is a non-negative integer. We assume, for the time being, that $|tq^{\frac{1-\alpha}{2}}| < 1$. Next, we apply [11, (2.10.1)] to the ${}_8W_7$ series in (2.2.17) with the invariant parameters $a^2q^{\alpha-\frac{1}{2}}$ and $-atq^{k+\frac{1}{2}}$. We obtain,

$$\begin{aligned}
&{}_8W_7(-a^3c^2tq^{k+2\alpha-\frac{1}{2}}; a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, -acq^{\alpha-\frac{1}{2}}, -atq^{k+\frac{1}{2}}, acq^\alpha; q, -ctq^{k+\frac{1}{2}}) \\
&= \frac{(-a^3c^2tq^{k+2\alpha-\frac{1}{2}}, c^2q^{\alpha-\frac{1}{2}}, -a^2ctq^{k+\alpha}, act^2q^{2k+1}; q)_\infty}{(-ac^2tq^{k+\alpha}, a^2c^2q^{2\alpha-1}, a^3ct^2q^{2k+\alpha+\frac{1}{2}} - ctq^{k+\frac{1}{2}}; q)_\infty} \\
&\times {}_8W_7(a^3ct^2q^{2k+\alpha-\frac{1}{2}}; a^2q^{\alpha-\frac{1}{2}}, -atq^{k+\frac{1}{2}}, -atq^{k+1}, atq^{k+\frac{1}{2}}, atq^k; q, c^2q^{\alpha-\frac{1}{2}}).
\end{aligned} \tag{2.2.18}$$

Submitting the ${}_8W_7$ series in (2.2.18) to the transformation formula [11, (2.10.10)], we get

$$\begin{aligned}
&{}_8W_7(a^3ct^2q^{2k+\alpha-\frac{1}{2}}; a^2q^{\alpha-\frac{1}{2}}, -atq^{k+\frac{1}{2}}, -atq^{k+1}, atq^{k+\frac{1}{2}}, atq^k; q, c^2q^{\alpha-\frac{1}{2}}) \tag{2.2.19} \\
&= \frac{(a^3ct^2q^{2k+\alpha+\frac{1}{2}}, acq^\alpha, -acq^\alpha, -acq^{\alpha-\frac{1}{2}}; q)_\infty}{(a^2ctq^{k+\alpha+\frac{1}{2}}, a^2ctq^{k+\alpha}, -a^2ctq^{k+\alpha}, -\frac{c}{t}q^{\alpha-k-\frac{1}{2}}; q)_\infty} \\
&\times {}_4\phi_3 \left[\begin{matrix} atq^k, atq^{k+\frac{1}{2}}, -atq^{k+\frac{1}{2}}, -ctq^k \\ -a^2ctq^{\alpha+k-\frac{1}{2}}, act^2q^{2k+1}, -\frac{t}{c}q^{\frac{3}{2}+k-\alpha} \end{matrix} ; q, q \right] \\
&+ \frac{(a^3ct^2q^{2k+\alpha+\frac{1}{2}}, -ctq^k, atq^k, atq^{k+\frac{1}{2}}, -atq^{k+\frac{1}{2}}, a^2c^2q^{2\alpha-1}; q)_\infty}{(-a^2ctq^{k+\alpha-\frac{1}{2}}, act^2q^{2k+1}, a^2ctq^{k+\alpha+\frac{1}{2}}, a^2ctq^{k+\alpha}, -a^2ctq^{k+\alpha}, c^2q^{\alpha-\frac{1}{2}})_\infty} \\
&\times \frac{(-ac^2tq^{k+\alpha+\frac{1}{2}}; q)_\infty}{(-\frac{t}{c}q^{\frac{1}{2}+k-\alpha}; q)_\infty} {}_4\phi_3 \left[\begin{matrix} acq^\alpha, -acq^\alpha, c^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}} \\ a^2c^2q^{2\alpha-1}, -ac^2tq^{\alpha+k+\frac{1}{2}}, -\frac{c}{t}q^{\alpha-k+\frac{1}{2}} \end{matrix} ; q, q \right].
\end{aligned}$$

Then, from (2.2.17), (2.2.18) and (2.2.19), we get

$$\begin{aligned}
& \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; actq^{k+\frac{\alpha+1}{2}})}{h(y; tq^{k+\frac{1-\alpha}{2}})} \quad (2.2.20) \\
&= 2\pi \left\{ \frac{(act^2q^{2k+1}, -a^2ctq^{\alpha+k-\frac{1}{2}}, acq^\alpha; q)_\infty}{(q, a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, atq^k, atq^{k+\frac{1}{2}}, -ctq^k, -ctq^{k+\frac{1}{2}}; q)_\infty} \right. \\
&\quad \times \frac{1}{(-\frac{c}{t}q^{\alpha-k-\frac{1}{2}}; q)_\infty} {}_4\phi_3 \left[\begin{matrix} atq^k, atq^{k+\frac{1}{2}}, -atq^{k+\frac{1}{2}}, -ctq^k \\ -a^2ctq^{\alpha+k-\frac{1}{2}}, act^2q^{2k+1}, -\frac{t}{c}q^{\frac{3}{2}+k-\alpha} \end{matrix} ; q, q \right] \\
&\quad + \frac{(a^2c^2q^{2\alpha-1}, -atq^{k+\frac{1}{2}}, -ac^2tq^{\alpha+k+\frac{1}{2}}; q)_\infty}{(q, a^2q^{\alpha-\frac{1}{2}}, c^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, -acq^\alpha, -ctq^{k+\frac{1}{2}}; q)_\infty} \\
&\quad \times \frac{1}{(-\frac{t}{c}q^{-\alpha+k+\frac{1}{2}}; q)_\infty} {}_4\phi_3 \left[\begin{matrix} acq^\alpha, -acq^\alpha, c^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}} \\ a^2c^2q^{2\alpha-1}, -ac^2tq^{\alpha+k+\frac{1}{2}}, -\frac{c}{t}q^{\alpha-k+\frac{1}{2}} \end{matrix} ; q, q \right] \Big\} \\
&= 2\pi \frac{(act^2q^{2k+1}, acq^\alpha; q)_\infty}{(q, a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, atq^k, atq^{k+\frac{1}{2}}, -ctq^k, -ctq^{k+\frac{1}{2}}; q)_\infty} \\
&\quad \left\{ \frac{(-a^2ctq^{\alpha+k-\frac{1}{2}}; q)_\infty}{(-\frac{c}{t}q^{\alpha-k-\frac{1}{2}}; q)_\infty} {}_4\phi_3 \left[\begin{matrix} atq^k, atq^{k+\frac{1}{2}}, -atq^{k+\frac{1}{2}}, -ctq^k \\ -a^2ctq^{\alpha+k-\frac{1}{2}}, act^2q^{2k+1}, -\frac{t}{c}q^{\frac{3}{2}+k-\alpha} \end{matrix} ; q, q \right] \right. \\
&\quad + \frac{(a^2t^2q^{2k}, acq^{\alpha-\frac{1}{2}} - ctq^k, -ac^2tq^{\alpha+k+\frac{1}{2}}; q)_\infty}{(act^2q^{2k+1}, c^2q^{\alpha-\frac{1}{2}}, -atq^k, -\frac{t}{c}q^{k-\alpha+\frac{1}{2}}; q)_\infty} \\
&\quad \times {}_4\phi_3 \left[\begin{matrix} acq^\alpha, -acq^\alpha, c^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}} \\ a^2c^2q^{2\alpha-1}, -ac^2tq^{\alpha+k+\frac{1}{2}}, -\frac{c}{t}q^{\alpha-k+\frac{1}{2}} \end{matrix} ; q, q \right] \Big\}.
\end{aligned}$$

The last equality in the above expression follows from the fact that

$$\begin{aligned}
(a^2c^2q^{2\alpha-1}; q)_\infty &= (a^2c^2q^{2\alpha-1}, a^2c^2q^{2\alpha}; q^2)_\infty \\
&= (acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, acq^\alpha, -acq^\alpha; q)_\infty
\end{aligned}$$

and

$$\begin{aligned}(a^2 t^2 q^{2k}; q)_\infty &= (a^2 t^2 q^{2k}, a^2 t^2 q^{2k+1}; q^2)_\infty \\ &= (atq^k, -atq^k, atq^{k+\frac{1}{2}}, -atq^{k+\frac{1}{2}}; q)_\infty.\end{aligned}$$

Now, if we take $A = a^2 t^2 q^{2k}$, $B = -q\sqrt{A}$, $C = a/c$ and $x = -\frac{c}{t}q^{\alpha-k-\frac{1}{2}}$, then the expresion inside the curly brackets in (2.2.20) reads

$$\begin{aligned}& \frac{(Ax; q)_\infty}{(x; q)_\infty} {}_5\phi_4 \left[\begin{matrix} \sqrt{A}, -\sqrt{A}, \sqrt{Aq}, -\sqrt{Aq}, Aq/BC \\ Aq/B, Aq/C, Ax, q/x \end{matrix} ; q, q \right] \\ & + \frac{(A, Aq/BC, Aqx/B, Aqx/C; q)_\infty}{(Aq/B, Aq/C, Aqx/BC, x^{-1}; q)_\infty} \\ & \times {}_5\phi_4 \left[\begin{matrix} x\sqrt{A}, -x\sqrt{A}, x\sqrt{Aq}, -x\sqrt{Aq}, Aqx/BC \\ Aqx/B, Aqx/C, xq, Ax^2 \end{matrix} ; q, q \right]\end{aligned}$$

which, by [11, (3.4.1)], is equal to

$${}_3\phi_2 \left[\begin{matrix} A, B, C \\ Aq/B, Aq/C \end{matrix} ; q, Aqx/BC \right]. \quad (2.2.21)$$

Substituting the values of A, B, C, x , we see, from (2.2.17), that

$$\begin{aligned}& \int_{-1}^1 dy w(y; aq^{\frac{\alpha-1}{2}}, aq^{\frac{\alpha}{2}}, -cq^{\frac{\alpha-1}{2}}, -cq^{\frac{\alpha}{2}}) \frac{h(y; actq^{k+\frac{\alpha+1}{2}})}{h(y; tq^{k+\frac{1-\alpha}{2}})} \\ & = \frac{2\pi(act^2 q^{2k+1}, acq^\alpha; q)_\infty}{(q, a^2 q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}, atq^k, atq^{k+\frac{1}{2}}, -ctq^k, -ctq^{k+\frac{1}{2}}; q)_\infty} \\ & \times {}_3\phi_2 \left[\begin{matrix} a^2 t^2 q^{2k}, -atq^{k+1}, a/c \\ -atq^k, act^2 q^{2k+1} \end{matrix} ; q, c^2 q^{\alpha-\frac{1}{2}} \right]\end{aligned} \quad (2.2.22)$$

and hence, from (2.2.16) and (2.2.22),

$$\begin{aligned}
P_t^\alpha(x; a, c) & \quad (2.2.23) \\
= & C \frac{2\pi}{(q, q; q)_\infty} \frac{(acq^{\alpha+\frac{1}{2}}, acq^\alpha; q)_\infty}{(a^2q^{\alpha+\frac{1}{2}}, q, a^2q^{\alpha-\frac{1}{2}}, -acq^{\alpha-\frac{1}{2}}, -acq^{\alpha-1}; q)_\infty} \frac{|(tqe^{i\theta}; q)_\infty|^2}{|(te^{i\theta}; q)_\infty|^2} \\
& \times \sum_{n=0}^{\infty} \frac{(1 - act^2q^{-1+2n})}{(1 - act^2q^{-1})} \frac{(act^2q^{-1}, a/c, ctq^{-\frac{1}{2}}, ct, te^{i\theta}, te^{-i\theta}; q)_n}{(q, c^2t^2, atq^{\frac{1}{2}}, at, acte^{-i\theta}, acte^{i\theta}; q)_n} (acq^{\frac{1}{2}})^n \\
& \times \sum_{m=0}^{\infty} \frac{(-ct, -ct\sqrt{q}; q)_{m+n}}{(-at, -at\sqrt{q}; q)_{m+n}} \frac{(a^2t^2; q)_{2m+2n}}{(act^2; q)_{2m+2n}} \frac{|(acte^{i\theta}; q)_{n+m}|^2}{|(tqe^{i\theta}; q)_{n+m}|^2} q^{(n+m)\alpha} \\
& \times {}_3\phi_2 \left[\begin{matrix} a^2t^2q^{2m+2n}, -atq^{m+n+1}, a/c \\ -atq^{m+n}, act^2q^{2m+2n+1} \end{matrix} ; q, c^2q^{\alpha-\frac{1}{2}} \right].
\end{aligned}$$

Now, taking $m + n = k$, $m = k - n$, substituting the value of C from (2.2.5) in (2.2.23) and changing the order of summation we get (2.1.11). We may now remove the temporary restriction $|tq^{\frac{1-\alpha}{2}}| < 1$. For the validity of (2.2.23) all we need is that $\max(|a|, |c|, |t|) < 1$. $\alpha > 0$ and $|c^2q^{\alpha-\frac{1}{2}}| < 1$. The last inequality is needed to ensure the convergence of the ${}_3\phi_2$ series in (2.2.23).

Chapter 3

Poisson kernel for the associated continuous q -ultraspherical polynomials

3.1 Introduction

The continuous q -ultraspherical polynomials, $C_n(x; \beta|q)$, introduced by L. J. Rogers [37-39], can be defined by the generating function (2.1.6)

$$\sum_{n=0}^{\infty} C_n(x; \beta|q) t^n = \frac{(\beta t e^{i\theta}, \beta t e^{-i\theta}; q)_{\infty}}{(t e^{i\theta}, t e^{-i\theta}; q)_{\infty}}. \quad (3.1.1)$$

where $0 < q < 1$, $x = \cos \theta$, $0 \leq \theta \leq \pi$, $|t| < 1$ [4], [5], [11]. Setting $\beta = 0$ in (3.1.1) one obtains the generating function for the continuous q -Hermite polynomials $H_n(x|q) = (q; q)_n C_n(x; 0|q)$:

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q; q)_n} t^n = \frac{1}{(t e^{i\theta}, t e^{-i\theta}; q)_{\infty}} \quad (3.1.2)$$

As a q -extension of Mehler's well known bilinear generating function for the classical Hermite polynomials:

$$\sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{n!} \left(\frac{t}{2}\right)^n = (1 - t^2)^{-\frac{1}{2}} \exp \left\{ \frac{2xyt - (x^2 + y^2)t^2}{1 - t^2} \right\}, \quad (3.1.3)$$

[40, Problem 23] and [41], Rogers [36] found the following extension of (3.1.3) for the q -Hermite polynomials:

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)H_n(y|q)}{(q; q)_n} t^n = \frac{(t^2; q)_{\infty}}{|(te^{i(\theta+\varphi)}, te^{i(-\theta+\varphi)}; q)_{\infty}|^2} \quad (3.1.4)$$

$x = \cos \theta$, $y = \cos \varphi$, $|t| < 1$. Its extension to q -ultraspherical polynomials was found by Gasper and Rahman [12]:

$$\begin{aligned} P_t(x, y; \beta|q) &:= \sum_{n=0}^{\infty} h_n C_n(x; \beta|q) C_n(y; \beta|q) t^n \\ &= \frac{(q, \beta q; q)_{\infty}}{2\pi(\beta, \beta t^2; q)_{\infty}} \left| \frac{(\beta t e^{i\theta+i\varphi}, \beta t e^{i\theta-i\varphi}; q)_{\infty}}{(t e^{i\theta+i\varphi}, t e^{i\theta-i\varphi}; q)_{\infty}} \right|^2 \\ &\quad \times {}_8W_7(\beta t^2/q; \beta/q, t e^{i\theta+i\varphi}, t e^{-i\theta-i\varphi}, t e^{i\theta-i\varphi}, t e^{-i\theta+i\varphi}; q, \beta q), \end{aligned} \quad (3.1.5)$$

β, t are assumed to be real and

$$h_n = \frac{(q, \beta^2; q)_{\infty}}{2\pi(\beta, \beta q; q)_{\infty}} \frac{(1 - \beta q^n)(q; q)_n}{(1 - \beta)(\beta^2; q)_n} \quad (3.1.6)$$

is the normalizing constant in the orthogonality relation

$$\int_0^{\pi} C_n(\cos \theta; \beta|q) C_m(\cos \theta; \beta|q) W_{\beta}(\cos \theta|q) d\theta = \frac{\delta_{m,n}}{h_n(\beta|q)}, \quad (3.1.7)$$

with

$$W_{\beta}(\cos \theta|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_{\infty}} \quad (3.1.8)$$

[5], [11]. One can see that the Poisson kernel in (3.1.5) reduces to (3.1.4) when $\beta = 0$, except for a constant factor. Formula (3.1.5) is admittedly more complicated than (3.1.4), but it does exhibit one of the basic properties of the Poisson kernel, namely, the positivity in some domain of β and t . The importance of positivity of these bilinear sums in Fourier analysis was pointed out by many authors, see for example, Askey [2]. But there is a renewed interest in Poisson kernels of orthogonal polynomials for a different reason. N. Wiener [42] showed in 1933 how the classical topic of Fourier integrals could be approached from the point of view of classical

orthogonal polynomials, in particular, by using Mehler's formula (3.1.3). One can show that the expression on the right-hand side of (3.1.3) approaches a multiple of $\exp(ixy)$, the usual kernel of Fourier transform, as $t \rightarrow i$ on the unit circle. This idea was suggested by Askey as a suitable starting point for a systematic q -extension of Fourier Analysis, which resulted in a number of recent publications [3], [6], [32]. It turns out that the most important piece of information that one needs from the Poisson kernel is the location of poles in the complex t -plane, as far as the Fourier transformation is concerned. One can see that the poles of the kernel in (3.1.5) are exactly the same as those of the simpler kernel in (3.1.4). Calculations with more general systems of polynomials, starting with the 2-parameter continuous q -Jacobi polynomials [13] up to the 4-parameter Askey-Wilson polynomials [34] reveal that this phenomenon persists through the entire Askey-Wilson family. The continuous q -ultraspherical polynomials are the special case of the Askey-Wilson polynomials that corresponds to setting $a = \sqrt{\beta}$, $b = \sqrt{\beta q}$, $c = -\sqrt{\beta}$, $d = -\sqrt{\beta q}$. [4], [5], [11] and are given by

$$C_n(x; \beta|q) = \frac{(\beta^2; q)_n}{(q; q)_n} \beta^{-\frac{n}{2}} {}_4\phi_3 \left[\begin{matrix} q^{-n}, \beta^2 q^n, \sqrt{\beta} e^{i\theta}, \sqrt{\beta} e^{-i\theta} \\ \beta\sqrt{q}, \beta, -\beta\sqrt{q} \end{matrix} ; q, q \right], \quad (3.1.9)$$

$x = \cos \theta$. In this chapter we wish to extend of (3.1.5) to the associated q -ultraspherical polynomials. The q -ultraspherical polynomials $C_n(x; \beta|q)$, [4], [5] satisfy the three-term recurrence relation

$$2xC_n(x; \beta|q) = \frac{1 - q^{n+1}}{1 - \beta q^n} C_{n+1}(x; \beta|q) + \frac{1 - \beta^2 q^{n-1}}{1 - \beta q^n} C_{n-1}(x; \beta|q) \quad (3.1.10)$$

with $C_{-1}(x; \beta|q) = 0$, $C_0(x; \beta|q) = 1$, $-1 \leq x \leq 1$. In view of the definition of associated polynomials, the associated q -ultraspherical polynomials $C_n^\alpha(x; \beta|q)$ will satisfy the recurrence relation

$$2xC_n^\alpha(x; \beta|q) = \frac{1 - q^{n+\alpha+1}}{1 - \beta q^{n+\alpha}} C_{n+1}^\alpha(x; \beta|q) + \frac{1 - \beta^2 q^{n+\alpha-1}}{1 - \beta q^{n+\alpha}} C_{n-1}^\alpha(x; \beta|q) \quad (3.1.11)$$

with $C_{-1}^\alpha(x; \beta|q) = 0$, $C_0^\alpha(x; \beta|q) = 1$. These polynomials were studied extensively by Bustoz and Ismail [9]. They found a measure $d\mu(x)$ with respect to which the orthogonality relation

$$\int_{-1}^1 C_n^\alpha(x; \beta|q) C_m^\alpha(x; \beta|q) d\mu(x) = \frac{1 - \beta q^\alpha}{1 - \beta q^{\alpha+n}} \frac{(\beta^2 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} q^{\alpha n} \delta_{m,n} \quad (3.1.12)$$

holds. By a straightforward manipulation of the recurrence relation (3.1.11) Bustoz and Ismail [9] found the generating function (2.1.9) which, for ready reference for this chapter is rewritten here.

$$\begin{aligned} G_t^\alpha(x; \beta|q) &:= \sum_{n=0}^{\infty} C_n^\alpha(x; \beta|q) t^n \\ &= \frac{(1 - q^\alpha)}{(1 - 2xt + t^2)^{3\phi_2}} \begin{bmatrix} \beta t e^{i\theta}, \beta t e^{-i\theta}, q \\ q t e^{i\theta}, q t e^{-i\theta}, \end{bmatrix} ; \quad q, q^\alpha \end{aligned} \quad (3.1.13)$$

In Section 3.4 we will find an alternative form of (3.1.13), as well as a few other generating functions. In Section 3.5 we will compute some bilinear generating functions. The main result of this chapter that follows from (3.5.1) and (3.5.5) is

$$\begin{aligned} K_t^\alpha(x, y; \beta|q) & \\ &:= \sum_{n=0}^{\infty} \frac{(1 - \beta q^{\alpha+n})}{(1 - \beta q^\alpha)} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} (t q^{-\alpha})^n C_n^\alpha(x; \beta|q) C_n^\alpha(y; \beta|q) \\ &= (1 - \beta q^\alpha)^{-1} \{ L_t^\alpha(x, y) - \beta q^\alpha L_{tq}^\alpha(x, y) \} \end{aligned} \quad (3.1.14)$$

where

$$\begin{aligned} L_t^\alpha(x, y) & \\ &= \frac{(1 - q^\alpha)^2}{(1 - e^{-2i\theta})(1 - 2ytq^{-\alpha}e^{i\theta} + t^2q^{-2\alpha}e^{2i\theta})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \\ &\quad \times {}_2\phi_1 \left[\begin{matrix} \beta e^{-2i\theta}, \beta \\ q e^{-2i\theta} \end{matrix} ; \quad q, q^\alpha \right] \end{aligned} \quad (3.1.15)$$

$$\begin{aligned}
& \times {}_3\phi_2 \left[\begin{matrix} \beta t q^{-\alpha} e^{i(\theta+\varphi)}, \beta t q^{-\alpha} e^{i(\theta-\varphi)}, q \\ t q^{1-\alpha} e^{i(\theta+\varphi)}, t q^{1-\alpha} e^{i(\theta-\varphi)} \end{matrix} ; q, q^\alpha \right] \\
& \times {}_4\phi_3 \left[\begin{matrix} \beta e^{2i\theta}, \beta, t q^{-\alpha} e^{i(\theta+\varphi)}, t q^{-\alpha} e^{i(\theta-\varphi)} \\ q e^{2i\theta}, \beta t q^{-\alpha} e^{i(\theta+\varphi)}, \beta t q^{-\alpha} e^{i(\theta-\varphi)} \end{matrix} ; q, q \right] \\
& + \theta \longleftrightarrow -\theta.
\end{aligned}$$

Note that K_t^α is a multiple of the Poisson kernel for $C_n^\alpha(x; \beta|q)$. This shows that the poles are only slightly shifted by a factor of q^α , but otherwise remain the same as in (3.1.5). Unfortunately, we are unable to simplify (3.1.15) to a form where the positivity would become obvious as it is in (3.1.5). It is only in the case $\alpha = 0$ that the two ${}_4\phi_3$ series in (3.1.15) combine to an ${}_8\phi_7$ which is what we have in (3.1.5). We will do this verification in Section 3.7, but otherwise the computation seems quite formidable.

In deriving the above mentioned formulas our main tool has been the generating function (3.1.13) as well as the following moment representation of $C_n^\alpha(x; \beta|q)$:

$$C_n^\alpha(x; \beta|q) = \int_{-\infty}^{\infty} \xi^n d\nu(\xi; \beta, x), \quad (3.1.16)$$

where ν is a step function whose jumps occur at the points $q^k e^{i\theta}$ and $q^k e^{-i\theta}$, $k = 0, 1, \dots$. The jumps are

$$\begin{aligned}
d\nu(q^k e^{i\theta}; \beta, x) &= \frac{(1 - q^\alpha)}{(1 - e^{-2i\theta})^2} {}_2\phi_1 \left[\begin{matrix} \beta e^{-2i\theta}, \beta \\ q e^{-2i\theta} \end{matrix} ; q, q^\alpha \right] \\
&\times \frac{(q e^{2i\theta}/\beta, q/\beta; q)_k}{(q, q e^{2i\theta}; q)_k} (\beta^2 q^\alpha)^k,
\end{aligned} \quad (3.1.17)$$

with $d\nu(q^k e^{-i\theta}; \beta, x)$ the same as (3.1.17) with θ replaced by $-\theta$. We prove this important result in Section 3.3. The idea of this moment representation of orthogonal

polynomials is not new. Karlin [23] used such a representation to study the sign regularity of determinants of classical orthogonal polynomials. Ismail and Stanton [19] exploited this idea and constructed several moment representations explicitly with special emphasis on the Al-Salam-Chihara and continuous q -ultraspherical polynomials. Implicit use of moment representations of the continuous q -ultraspherical polynomials was also present in formulas (7.4.4) and (7.4.7) of [11]. The derivation of (3.1.15) and (3.1.16) was facilitated by an alternate form of the polynomial solution of the three-term recurrence relation for the associated Askey-Wilson polynomials found in [17]. We carry out this alternate derivation in Section 3.2. In Section 3.6 we will give the $q \rightarrow 1$ limit of formulas (3.1.14) and (3.1.15) as well as the explicit form of the associated ultraspherical polynomials.

3.2 The associated Askey-Wilson polynomials

In [17] Ismail and Rahman found a representation of the associated Askey-Wilson polynomials, $p_n^\alpha(x; a, b, c, d)$, as a linear combination of the two linearly independent solutions of the three-term recurrence relation:

$$\begin{aligned} 2xp_n^\alpha(x; a, b, c, d) \\ = A_{n+\alpha}p_{n+1}^\alpha(x; a, b, c, d) + B_{n+\alpha}p_n^\alpha(x; a, b, c, d) + C_{n+\alpha}p_{n-1}^\alpha(x; a, b, c, d), \end{aligned} \quad (3.2.1)$$

$p_{-1}^\alpha(x; a, b, c, d) = 0$ and $p_0^\alpha(x; a, b, c, d) = 1$, namely,

$$\begin{aligned} r_{\alpha+n}(z; a, b, c, d) \\ = \frac{(abq^{\alpha+n}, acq^{\alpha+n}, adq^{\alpha+n}, bcdq^{\alpha+n}/z; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, azq^{\alpha+n}; q)_\infty} \left(\frac{a}{z}\right)^{n+\alpha} \\ \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, qz/a) \end{aligned} \quad (3.2.2)$$

and

$$s_{\alpha+n}(z; a, b, c, d) \quad (3.2.3)$$

$$\begin{aligned}
&= \frac{(abcdq^{2\alpha+2n}, bzq^{\alpha+n+1}, czq^{\alpha+n+1}, dzq^{\alpha+n+1}, bcdzq^{\alpha+n}; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, bcdzq^{2\alpha+2n+1}; q)_\infty} (az)^{n+\alpha} \\
&\quad \times {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n-1}, qz/a; q, az)
\end{aligned}$$

where, $z = e^{i\theta}$, $x = \cos \theta$, and

$$A_\alpha = a^{-1} \frac{(1 - abq^\alpha)(1 - acq^\alpha)(1 - adq^\alpha)(1 - abcdq^{\alpha-1})}{(1 - abcdq^{2\alpha-1})(1 - abcdq^{2\alpha})}, \quad (3.2.4)$$

$$C_\alpha = a \frac{(1 - bcq^{\alpha-1})(1 - bdq^{\alpha-1})(1 - cdq^{\alpha-1})(1 - q^\alpha)}{(1 - abcdq^{2\alpha-1})(1 - abcdq^{2\alpha-2})}, \quad (3.2.5)$$

$$B_\alpha = a + a^{-1} - A_\alpha - C_\alpha. \quad (3.2.6)$$

One can show by using Bailey's transformation formula [11, III.(37)] for very-well-poised ${}_8\phi_7$ series that $r_{\alpha+n}(z; a, b, c, d)$ is a linear combination of $s_{\alpha+n}(z; a, b, c, d)$ and $s_{\alpha+n}(z^{-1}; a, b, c, d)$ with coefficients independent of n . But, of course, $s_{\alpha+n}(z; a, b, c, d)$ and $s_{\alpha+n}(z^{-1}; a, b, c, d)$ are also independent solutions of (3.2.1) as long as $0 < \theta < \pi$, so one might seek a representation of the form

$$p_n^\alpha(x; a, b, c, d) = L_\alpha s_{\alpha+n}(z; a, b, c, d) + M_\alpha s_{\alpha+n}(z^{-1}; a, b, c, d). \quad (3.2.7)$$

The initial conditions then fix the coefficient L_α and M_α :

$$\begin{aligned}
&p_n^\alpha(x; a, b, c, d) \\
&= \frac{s_{\alpha+n}(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_{\alpha+n}(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d)}{s_\alpha(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_\alpha(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d)}.
\end{aligned} \quad (3.2.8)$$

Using (3.2.1) and an argument similar to the one used in [17] we can show that

$$\begin{aligned}
&\left\{ s_\alpha(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_\alpha(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d) \right\}^{-1} \\
&= \frac{(bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}; q)_\infty} \\
&\quad \times \frac{a^{1-2\alpha}}{(1 - abcdq^{2\alpha-2})} \frac{(az, a/z; q)_\infty}{(qz^2, q/z^2; q)_\infty} \frac{1}{(z - z^{-1})},
\end{aligned} \quad (3.2.9)$$

which is just the Casorati determinant of (3.2.1). Using the transformation formula [11, III.(23)] and substituting (3.2.9) in (3.2.8) we obtain

$$\begin{aligned}
p_n^\alpha(x; a, b, c, d) &= \frac{(1 - abcdq^{\alpha-2})}{(1 - abcdq^{2\alpha-2})} \\
&\times \frac{(aq^\alpha/z, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, azq^{\alpha+n+1}, bzq^{\alpha+n+1}; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}; q)_\infty} \\
&\times \frac{(czq^{\alpha+n+1}, dzq^{\alpha+n+1}, abcdq^{\alpha+n-1}; q)_\infty}{(q^{\alpha+n+1}, q^{\alpha+1}/z^2, z^2q^{\alpha+n+2}; q)_\infty} \frac{(az)^n}{(1 - z^{-2})} \\
&\times {}_8W_7(q^\alpha/z^2; q/az, q/bz, q/cz, q/dz, q^\alpha; q, abcdq^{\alpha-2}) \\
&\times {}_8W_7(z^2q^{\alpha+n+1}; zq/a, zq/b, zq/c, zq/d, q^{\alpha+n+1}; q, abcdq^{\alpha+n-1}) \\
&+ z \longleftrightarrow z^{-1}.
\end{aligned} \tag{3.2.10}$$

This is the form that will enable us to represent $C_n^\alpha(x; \beta|q)$ as the moment of a measure.

3.3 Associated q-ultraspherical polynomials as moments

Using the quadratic transformation formula [11, (3.4.7)] we have

$$\begin{aligned}
&{}_8W_7(t^2q^\lambda; tq/a, tq^{\frac{1}{2}}/a, -tq/a, -tq^{\frac{1}{2}}/a, q^\lambda; q, a^4q^{\lambda-1}) \\
&= \frac{(t^2q^{\lambda+1}, a^4q^{2\lambda-1}; q)_\infty}{(a^2q^\lambda, a^2t^2q^{2\lambda}; q)_\infty} {}_2\phi_1 \left[\begin{matrix} qt^2/a^2, q/a^2 \\ qt^2 \end{matrix} ; q, a^4q^{\lambda-1} \right].
\end{aligned} \tag{3.3.1}$$

Setting $b = aq^{\frac{1}{2}}$, $c = -a$, $d = -aq^{\frac{1}{2}}$ in (3.2.10), then using (3.3.1) and simplifying the coefficient we obtain

$$p_n^\alpha(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}) = \frac{1 - a^4q^{\alpha-1}}{1 - z^{-2}} \frac{(a^4q^{\alpha+n}; q)_\infty}{(q^{\alpha+n+1}; q)_\infty} (az)^n \tag{3.3.2}$$

$$\begin{aligned}
& \times {}_2\phi_1 \left[\begin{matrix} q/a^2 z^2, q/a^2 \\ q/z^2 \end{matrix} ; q, a^4 q^{\alpha-1} \right] \\
& \times {}_2\phi_1 \left[\begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix} ; q, a^4 q^{\alpha+n} \right] \\
& + z \longleftrightarrow z^{-1}.
\end{aligned}$$

Comparing (3.1.11) with (3.2.1) for these special values of the parameters b, c, d we find that

$$C_n^\alpha(x; a^2|q) = \frac{(a^4 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} a^{-n} p_n^\alpha(x; a, a q^{\frac{1}{2}}, -a, -a q^{\frac{1}{2}}). \quad (3.3.3)$$

So, from (3.3.2) and (3.3.3) we have

$$\begin{aligned}
C_n^\alpha(x; a^2|q) &= \frac{(a^4 q^{\alpha-1}; q)_\infty}{(q^{\alpha+1}; q)_\infty} \frac{z^n}{(1-z^{-2})} \\
& \times {}_2\phi_1 \left[\begin{matrix} q/a^2 z^2, q/a^2 \\ q/z^2 \end{matrix} ; q, a^4 q^{\alpha-1} \right] \\
& \times {}_2\phi_1 \left[\begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix} ; q, a^4 q^{\alpha+n} \right] \\
& + z \longleftrightarrow z^{-1}.
\end{aligned} \quad (3.3.4)$$

If we use [11, III.(3)] on the first ${}_2\phi_1$ series of the right hand-side we get

$$\begin{aligned}
C_n^\alpha(x; a^2|q) &= \frac{(1-q^\alpha)}{(1-z^{-2})} z^n \\
& \times {}_2\phi_1 \left[\begin{matrix} a^2/z^2, a^2 \\ q/z^2 \end{matrix} ; q, q^\alpha \right]
\end{aligned} \quad (3.3.5)$$

$$\begin{aligned} & \times {}_2\phi_1 \left[\begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix} ; q, a^4q^{\alpha+n} \right] \\ & + z \longleftrightarrow z^{-1}. \end{aligned}$$

Since

$$\lim_{\alpha \rightarrow 0^+} (1 - q^\alpha) {}_2\phi_1 \left[\begin{matrix} a^2/z^2, a^2 \\ q/z^2 \end{matrix} ; q, q^\alpha \right] = \frac{(a^2/z^2, a^2; q)_\infty}{(q, q/z^2; q)_\infty}, \quad (3.3.6)$$

one can see that (3.3.5) is a generalization of [11, (7.4.4)] as well as of the formula (5.11) of [19]. Replacing a by $\beta^{\frac{1}{2}}$ we find that (3.1.16) follows from (3.3.5).

3.4 Some generating functions for $C_n^\alpha(x; \beta|q)$

Formula (3.3.5) is obviously the most convenient form for deriving generating functions for $C_n^\alpha(x; \beta|q)$. For example, it is easy to see that for any λ

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} C_n^\alpha(x; \beta|q) t^n &= \frac{(1 - q^\alpha)}{(1 - z^{-2})} \frac{(\lambda z t; q)_\infty}{(z t; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^\alpha \right] \\ & \times {}_3\phi_2 \left[\begin{matrix} qz^2/\beta, q/\beta, zt \\ qz^2, \lambda zt \end{matrix} ; q, \beta^2 q^\alpha \right] \\ & + z \longleftrightarrow z^{-1}. \end{aligned} \quad (3.4.1)$$

Assume that $|t| < 1$. Setting $\lambda = q$ we get an alternative form of (3.1.13)

$$G_t^\alpha(x; \beta|q) = \frac{(1 - q^\alpha)}{(1 - z^{-2})(1 - zt)} {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^\alpha \right] \quad (3.4.2)$$

$$\begin{aligned} & \times {}_3\phi_2 \left[\begin{matrix} qz^2/\beta, q/\beta, zt \\ qz^2, qzt \end{matrix} ; q, \beta^2 q^\alpha \right] \\ & + z \longleftrightarrow z^{-1}. \end{aligned}$$

On the other hand, transforming the second ${}_2\phi_1$ series on the right-hand side of (3.3.5) by [11, III.(3)] we get

$$\begin{aligned} C_n^\alpha(x; a^2|q) &= \frac{1 - q^\alpha}{1 - z^{-2}} \frac{(q^{\alpha+n+1}; q)_\infty}{(\beta^2 q^{\alpha+n}; q)_\infty} z^n \\ & \times {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^\alpha \right] \\ & \times {}_2\phi_1 \left[\begin{matrix} \beta z^2, \beta \\ qz^2 \end{matrix} ; q, q^{\alpha+n+1} \right] \\ & + z \longleftrightarrow z^{-1}. \end{aligned} \tag{3.4.3}$$

Hence

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} C_n^\alpha(x; \beta|q) t^n \\ &= \frac{(1 - q^\alpha)}{(1 - z^{-2})(1 - zt)} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^\alpha \right] \\ & \times {}_3\phi_2 \left[\begin{matrix} \beta z^2, \beta, zt \\ qz^2, qzt \end{matrix} ; q, q^{\alpha+1} \right] \\ & + z \longleftrightarrow z^{-1}. \end{aligned} \tag{3.4.4}$$

More generally,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda, q^{\alpha+1}; q)_n}{(q, \beta^2 q^{\alpha}; q)_n} C_n^{\alpha}(x; \beta|q) t^n \\
&= \frac{(1 - q^{\alpha})}{(1 - z^{-2})} \frac{(q^{\alpha+1}; q)_{\infty}}{(\beta^2 q^{\alpha}; q)_{\infty}} \frac{(\lambda z t; q)_{\infty}}{(z t; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^{\alpha} \right] \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} \beta z^2, \beta, z t \\ q z^2, \lambda z t \end{matrix} ; q, q^{\alpha+1} \right] \\
&\quad + z \longleftrightarrow z^{-1}.
\end{aligned} \tag{3.4.5}$$

It also follows from (3.3.5) that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n+k}^{\alpha}(x; \beta|q) t^n &= \frac{(1 - q^{\alpha})}{(1 - z^{-2})(1 - t z)} z^k {}_2\phi_1 \left[\begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix} ; q, q^{\alpha} \right] \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} q z^2/\beta, q/\beta, z t \\ q z^2, q z t \end{matrix} ; q, \beta^2 q^{\alpha+k} \right] \\
&\quad + z \longleftrightarrow z^{-1}, \\
&\quad k = 0, 1, \dots
\end{aligned} \tag{3.4.6}$$

3.5 Some bilinear generating functions for associated q -ultraspherical polynomials

In this section we shall compute a few bilinear generating functions. First, let us consider the simplest one:

$$L_t^\alpha(x, y) := \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} (tq^{-\alpha})^n C_n^\alpha(x; \beta|q) C_n^\alpha(y; \beta|q), \quad (3.5.1)$$

which is symmetric in x and y , where $x = \cos \theta$, $y = \cos \varphi$, $0 < \theta, \varphi < \pi$. Let $e^{i\theta} = z_1$ and $e^{i\varphi} = z_2$. By assuming $|tq^{-\alpha}| < 1$ and using (3.4.3) in (3.5.1) we find that

$$\begin{aligned} L_t^\alpha(x, y) &= \frac{(1 - q^\alpha)}{(1 - z_1^{-2})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix} ; q, q^\alpha \right] \\ &\times \sum_{k=0}^{\infty} \frac{(\beta z_1^2, \beta; q)_k}{(q, q z_1^2; q)_k} q^{(\alpha+1)k} \sum_{n=0}^{\infty} (z_1 t q^{-\alpha+k})^n C_n^\alpha(y; \beta|q) \\ &+ z_1 \longleftrightarrow z_1^{-1}. \end{aligned} \quad (3.5.2)$$

However, by (3.1.13)

$$\begin{aligned} \sum_{n=0}^{\infty} (z_1 t q^{-\alpha+k})^n C_n^\alpha(y; \beta|q) &= \frac{(1 - q^\alpha)}{(1 - 2ytz_1 q^{k-\alpha} + z_1^2 t^2 q^{2k-2\alpha})} \\ &\times {}_3\phi_2 \left[\begin{matrix} q, \beta t z_1 q^{k-\alpha} e^{i\varphi}, \beta t z_1 q^{k-\alpha} e^{-i\varphi} \\ t z_1 q^{k-\alpha+1} e^{i\varphi}, t z_1 q^{k-\alpha+1} e^{-i\varphi} \end{matrix} ; q, q^\alpha \right], \end{aligned} \quad (3.5.3)$$

thus the sum over k in (3.5.2) is

$$\begin{aligned} (1 - q^\alpha) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\beta z_1^2, \beta; q)_k}{(q, q z_1^2; q)_k} q^{(\alpha+1)k+\alpha j} \\ \times \frac{(\beta t z_1 q^{-\alpha} e^{i\varphi}, \beta t z_1 q^{-\alpha} e^{-i\varphi}; q)_{j+k}}{(t z_1 q^{-\alpha} e^{i\varphi}, t z_1 q^{-\alpha} e^{-i\varphi}; q)_{j+k+1}} \frac{(t z_1 q^{-\alpha} e^{i\varphi}, t z_1 q^{-\alpha} e^{-i\varphi}; q)_k}{(\beta t z_1 q^{-\alpha} e^{i\varphi}, \beta t z_1 q^{-\alpha} e^{-i\varphi}; q)_k} \end{aligned} \quad (3.5.4)$$

$$\begin{aligned}
&= \frac{(1 - q^\alpha)}{(1 - 2ytz_1q^{-\alpha} + z_1^2t^2q^{-2\alpha})^3} \phi_2 \left[\begin{array}{c} q, \beta tz_1q^{-\alpha}e^{i\varphi}, \beta tz_1q^{-\alpha}e^{-i\varphi} \\ tz_1q^{-\alpha+1}e^{i\varphi}, tz_1q^{-\alpha+1}e^{-i\varphi} \end{array} ; q, q^\alpha \right] \\
&\quad \times \phi_3 \left[\begin{array}{c} \beta, \beta z_1^2, tz_1q^{-\alpha}e^{i\varphi}, tz_1q^{-\alpha}e^{-i\varphi} \\ qz_1^2, \beta tz_1q^{-\alpha}e^{i\varphi}, \beta tz_1q^{-\alpha}e^{-i\varphi} \end{array} ; q, q \right].
\end{aligned}$$

Hence

$$\begin{aligned}
L_t^\alpha(x, y) &= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2ytz_1q^{-\alpha} + z_1^2t^2q^{-2\alpha})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2q^\alpha; q)_\infty} \quad (3.5.5) \\
&\quad \times \phi_1 \left[\begin{array}{c} \beta/z_1^2, \beta \\ q/z_1^2 \end{array} ; q, q^\alpha \right] \\
&\quad \times \phi_2 \left[\begin{array}{c} q, \beta tz_1q^{-\alpha}e^{i\varphi}, \beta tz_1q^{-\alpha}e^{-i\varphi} \\ tz_1q^{-\alpha+1}e^{i\varphi}, tz_1q^{-\alpha+1}e^{-i\varphi} \end{array} ; q, q^\alpha \right] \\
&\quad \times \phi_3 \left[\begin{array}{c} \beta, \beta z_1^2, tz_1q^{-\alpha}e^{i\varphi}, tz_1q^{-\alpha}e^{-i\varphi} \\ qz_1^2, \beta tz_1q^{-\alpha}e^{i\varphi}, \beta tz_1q^{-\alpha}e^{-i\varphi} \end{array} ; q, q \right] \\
&\quad + z_1 \longleftrightarrow z_1^{-1}.
\end{aligned}$$

This leads to the expression (3.1.14) for the Poisson kernel for $C_n^\alpha(y; \beta|q)$. Note that the moment representation of $C_n^\alpha(y; \beta|q)$ also enables us to compute the non-symmetric kernel

$$\begin{aligned}
M_t^\alpha(x, y; \beta, \beta_1) &:= \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2q^\alpha; q)_n} (tq^{-\alpha})^n C_n^\alpha(x; \beta|q) C_n^\alpha(y; \beta_1|q) \quad (3.5.6) \\
&= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2ytz_1q^{-\alpha} + z_1^2t^2q^{-2\alpha})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2q^\alpha; q)_\infty}
\end{aligned}$$

$$\begin{aligned}
& \times {}_2\phi_1 \left[\begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix} ; q, q^\alpha \right] \\
& \times {}_3\phi_2 \left[\begin{matrix} q, \beta_1 t z_1 q^{-\alpha} e^{i\varphi}, \beta_1 t z_1 q^{-\alpha} e^{-i\varphi} \\ t z_1 q^{-\alpha+1} e^{i\varphi}, t z_1 q^{-\alpha+1} e^{-i\varphi} \end{matrix} ; q, q^\alpha \right] \\
& \times {}_4\phi_3 \left[\begin{matrix} \beta, \beta z_1^2, t z_1 e^{i\varphi} q^{-\alpha}, t z_1 e^{-i\varphi} q^{-\alpha} \\ q z_1^2, \beta_1 t z_1 e^{i\varphi} q^{-\alpha}, \beta_1 t z_1 e^{-i\varphi} q^{-\alpha} \end{matrix} ; q, q \right] \\
& + z_1 \longleftrightarrow z_1^{-1}.
\end{aligned}$$

Observe that the ${}_4\phi_3$ series in (3.5.5) is balanced, but the one in (3.5.6) is not, unless $\beta_1^2 = \beta^2$. The expression (3.5.3) may be seen as an extension of (5.5) of [19].

Finally, use of (3.3.5) and (3.1.13) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^\alpha(x; \beta|q) C_n^\alpha(y; \beta_1|q) t^n &= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2y t z_1 + z_1^2 t^2)} \quad (3.5.7) \\
& \times {}_2\phi_1 \left[\begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix} ; q, q^\alpha \right] \\
& \times {}_3\phi_2 \left[\begin{matrix} q, \beta_1 t z_1 e^{i\varphi}, \beta_1 t z_1 e^{-i\varphi} \\ q t z_1 e^{i\varphi}, q t z_1 e^{-i\varphi} \end{matrix} ; q, q^\alpha \right] \\
& \times {}_4\phi_3 \left[\begin{matrix} q z_1^2 / \beta, q / \beta, t z_1 e^{i\varphi}, t z_1 e^{-i\varphi} \\ q z_1^2, \beta_1 t z_1 e^{i\varphi}, \beta_1 t z_1 e^{-i\varphi} \end{matrix} ; q, q \right] \\
& + z_1 \longleftrightarrow z_1^{-1}.
\end{aligned}$$

3.6 Some limiting formulas

We will compute the limits of formulas (3.3.4) and (3.5.5) as $q \rightarrow 1$. We believe that these formulas are new.

It is clear from the expressions on the right-hand side of (3.3.4) and (3.5.5) that one cannot take the term-by-term limits. One has to do some transformations of the q -series involved in these formulas. Using (III. 1) and (III. 3) of [11], in that order, we find that

$$\begin{aligned} & {}_2\phi_1 \left[\begin{matrix} q/\beta z^2, q/\beta \\ q/z^2 \end{matrix} ; q, \beta^2 q^{\alpha-1} \right] \\ &= \frac{(\beta q^\alpha, \beta/z^2; q)_\infty}{(\beta^2 q^{\alpha-1}, q/z^2; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^\alpha, q/\beta \\ \beta q^\alpha \end{matrix} ; q, \beta/z^2 \right] \end{aligned} \quad (3.6.1)$$

and

$$\begin{aligned} & {}_2\phi_1 \left[\begin{matrix} qz^2/\beta, q/\beta \\ qz^2 \end{matrix} ; q, \beta^2 q^{\alpha+n} \right] \\ &= \frac{(\beta q^{\alpha+n+1}, \beta z^2; q)_\infty}{(\beta^2 q^{\alpha+n}, qz^2; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^{\alpha+n+1}, q/\beta \\ \beta q^{\alpha+n+1} \end{matrix} ; q, \beta z^2 \right]. \end{aligned} \quad (3.6.2)$$

These two formulas along with the definition of the q -gamma function [11, (1.10.1)] give

$$\begin{aligned} C_n^\alpha(x; q^\lambda | q) &= \frac{\Gamma_q(\alpha+1)\Gamma_q(2\lambda+\alpha+n)}{\Gamma_q(\alpha+\lambda)\Gamma_q(\alpha+n+\lambda+1)} \frac{(z^2 q^\lambda, q^\lambda/z^2; q)_\infty}{(qz^2, q/z^2; q)_\infty} \frac{z^n}{(1-z^{-2})} \\ & {}_2\phi_1 \left[\begin{matrix} q^\alpha, q^{1-\lambda} \\ q^{\alpha+\lambda} \end{matrix} ; q, q^\lambda/z^2 \right] {}_2\phi_1 \left[\begin{matrix} q^{\alpha+n+1}, q^{1-\lambda} \\ q^{\alpha+n+1+\lambda} \end{matrix} ; q, z^2 q^\lambda \right] \end{aligned} \quad (3.6.3)$$

$$+z \longleftrightarrow z^{-1},$$

where it is assumed that $\operatorname{Re} \lambda > 0$ and $\alpha \geq 0$.

So the associated ultraspherical polynomials have the following representation:

$$\begin{aligned} C_n^{\lambda, \alpha}(x) &= \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha+n)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+n+\lambda+1)} \frac{2^{2-2\lambda}}{2i} \frac{e^{i(n+1)\theta}}{(\sin \theta)^{2\lambda-1}} \\ &\quad \times {}_2F_1 \left[\begin{matrix} \alpha, 1-\lambda \\ \alpha+\lambda \end{matrix} ; e^{-2i\theta} \right] {}_2F_1 \left[\begin{matrix} \alpha+n+1, 1-\lambda \\ \alpha+n+1+\lambda \end{matrix} ; e^{2i\theta} \right] \\ &\quad +\theta \longleftrightarrow -\theta. \end{aligned} \quad (3.6.4)$$

For convergence on the unit circle of the two Gaussian series on the right hand side we need the stronger condition $\operatorname{Re} \lambda > \frac{1}{2}$. One can rewrite (3.6.4) in the form

$$\begin{aligned} C_n^{\lambda, \alpha}(x) &= \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha+n)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+n+\lambda+1)} \frac{2^{2-2\lambda}}{(\sin \theta)^{2\lambda-1}} \\ &\quad \times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_j (1-\lambda)_j (\alpha+n+1)_k (1-\lambda)_k}{j! k! (\alpha+\lambda)_j (\alpha+n+1+\lambda)_k} \sin(n+1-2j+2k)\theta, \end{aligned} \quad (3.6.5)$$

which is a generalization of [40, (4.9.22)] to which it reduces when we set $\alpha = 0$. In order to find the limit of $L_t^\alpha(x, y)$ as $q \rightarrow 1$ we do not use the final formula (3.5.5), rather we use the intermediate formula (3.5.2). First, by [11, (III.1)]

$${}_2\phi_1 \left[\begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix} ; q, q^\alpha \right] = \frac{(\beta q^\alpha, \beta/z_1^2; q)_\infty}{(q^\alpha, q/z_1^2; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^\alpha, q/\beta \\ \beta q^\alpha \end{matrix} ; q, \beta/z_1^2 \right].$$

Now, the series over k in (3.5.2) is

$$\begin{aligned} &\sum_{n=0}^{\infty} (z_1 t q^{-\alpha})^n C_n^\alpha(y; \beta|q) {}_2\phi_1 \left[\begin{matrix} \beta z_1^2, \beta \\ q z_1^2 \end{matrix} ; q, q^{\alpha+n+1} \right] \\ &= \frac{(\beta q^{\alpha+1}, \beta z_1^2; q)_\infty}{(q^{\alpha+1}, q z_1^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta q^{\alpha+1}; q)_n} (z_1 t q^{-\alpha})^n \end{aligned} \quad (3.6.6)$$

$$\times C_n^\alpha(y; \beta|q)_2 \phi_1 \left[\begin{matrix} q^{\alpha+n+1}, q/\beta \\ \beta q^{\alpha+n+1} \end{matrix} ; q, \beta z_1^2 \right].$$

We now substitute (3.6.3) and (3.6.6) into (3.5.2) and replace β by q^λ to get

$$\begin{aligned} L_t^\alpha(x, y) &= \frac{\Gamma_q(\alpha+1)\Gamma_q(2\lambda+\alpha)}{\Gamma_q(\alpha+\lambda)\Gamma_q(\alpha+\lambda+1)} \frac{(z_1^2 q^\lambda, q^\lambda/z_1^2; q)_\infty}{(q z_1^2, q/z_1^2; q)_\infty} (1 - z_1^{-2})^{-1} \\ &\quad \times {}_2\phi_1 \left[\begin{matrix} q^\alpha, q^{1-\lambda} \\ q^{\alpha+\lambda} \end{matrix} ; q, q^\lambda/z_1^2 \right] \\ &\quad \times \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\lambda+1}; q)_n} (z_1 t q^{-\alpha})^n C_n^\alpha(y; q^\lambda|q)_2 \phi_1 \left[\begin{matrix} q^{\alpha+n+1}, q^{1-\lambda} \\ q^{\alpha+\lambda+n+1} \end{matrix} ; q, z_1^2 q^\lambda \right] \\ &\quad + z_1 \longleftrightarrow z_1^{-1}, \end{aligned} \tag{3.6.7}$$

$Re \lambda > 0, Re \lambda \geq 0$. We now take the limit to obtain

$$\begin{aligned} \lim_{q \rightarrow 1^-} L_t^\alpha(x, y) &= \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+\lambda+1)} \frac{2^{2-2\lambda}}{(\sin \theta)^{2\lambda-1}} \frac{e^{i\theta}}{2i} {}_2F_1 \left[\begin{matrix} \alpha, 1-\lambda \\ \alpha+\lambda \end{matrix} ; e^{-2i\theta} \right] \\ &\quad \times \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(\alpha+\lambda+1)_n} (t e^{i\theta})^n C_n^{\lambda, \alpha}(y)_2 F_1 \left[\begin{matrix} \alpha+n+1, 1-\lambda \\ \alpha+n+1+\lambda \end{matrix} ; e^{2i\theta} \right] \\ &\quad + \theta \longleftrightarrow -\theta. \end{aligned} \tag{3.6.8}$$

To ensure convergence of the two ${}_2F_1$ series on the right hand side we assume, as before, that $Re \lambda > \frac{1}{2}$.

We shall now express the series over n on the right-hand side of (3.6.8) as a

double integral. First, by Euler's integral representation [11, (1.11.10)] we have

$$\begin{aligned} & \frac{\Gamma(\alpha + n + 1)\Gamma(\lambda)}{\Gamma(\alpha + 1 + n + \lambda)} {}_2F_1 \left[\begin{matrix} \alpha + n + 1, 1 - \lambda \\ \alpha + n + 1 + \lambda \end{matrix} ; e^{2i\theta} \right] \\ &= \int_0^1 u^{\alpha+n} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} du, \end{aligned} \quad (3.6.9)$$

$0 < \theta < \pi$, $n = 0, 1, \dots$. Since

$$\frac{\alpha}{\alpha + k} = \alpha \int_0^1 v^{\alpha+k-1} dv, \quad \operatorname{Re} \lambda > 0, \quad (3.6.10)$$

and, by [9, (3.13)]

$$C_n^{\lambda, \alpha}(y) = \sum_{k=0}^n \frac{\alpha}{\alpha + k} C_{n-k}^{\lambda}(y) C_k^{1-\lambda}(y), \quad (3.6.11)$$

we find that

$$\begin{aligned} & \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 + \lambda)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{(\alpha + \lambda + 1)_n} (te^{i\theta})^n C_n^{\lambda, \alpha}(y) {}_2F_1 \left[\begin{matrix} \alpha + n + 1, 1 - \lambda \\ \alpha + n + 1 + \lambda \end{matrix} ; e^{2i\theta} \right] \\ &= \frac{\alpha}{\Gamma(\lambda)} \int_0^1 \int_0^1 u^{\alpha} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} \\ & \quad \times \left\{ \sum_{n=0}^{\infty} (ute^{i\theta})^n \sum_{k=0}^n C_{n-k}^{\lambda}(y) C_k^{1-\lambda}(y) v^k \right\} dudv \\ &= \frac{\alpha}{\Gamma(\lambda)} \int_0^1 \int_0^1 u^{\alpha} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} (1-2utye^{i\theta} + u^2t^2e^{2i\theta})^{-\lambda} \\ & \quad \times (1-2uvtye^{i\theta} + u^2v^2t^2e^{2i\theta})^{\lambda-1} dudv, \end{aligned} \quad (3.6.12)$$

where the last line has been obtained by using the well-known generating function of the ultraspherical polynomials. Thus, we get

$$\lim_{q \rightarrow 1^-} L_t^{\alpha}(x, y) \quad (3.6.13)$$

$$= \frac{\alpha\Gamma(2\lambda + \alpha)}{\Gamma(\alpha + \lambda)\Gamma(\lambda)} \frac{2^{2-2\lambda}}{(\sin \theta)^{2\lambda-1}} \frac{e^{i\theta}}{2i} {}_2F_1 \left[\begin{matrix} \alpha, 1 - \lambda \\ \alpha + \lambda \end{matrix} ; e^{-2i\theta} \right]$$

$$\begin{aligned}
& \times \int_0^1 \int_0^1 u^\alpha (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} (1-2utye^{i\theta} + u^2t^2e^{2i\theta})^{-\lambda} \\
& \quad \times (1-2uvtye^{i\theta} + u^2v^2t^2e^{2i\theta})^{\lambda-1} dudv \\
& +\theta \longleftrightarrow -\theta,
\end{aligned}$$

with $|t| < 1$. Using (3.6.13) one can easily find the limit of $K_t^\alpha(x, y)$.

3.7 Limiting case when $\alpha \rightarrow 0$

In this section we will see that limit of $K_t^\alpha(x, y; \beta|q)$ as $\alpha \rightarrow 0$ indeed goes to the Poisson kernel of continuous q -ultraspherical polynomials. First, by Abel's formula (2.1.10) and equation (3.1.15)

$$\begin{aligned}
\lim_{\alpha \rightarrow 0} L_t^\alpha(x, y) &= \frac{(q^{\alpha+1}, \beta, \beta e^{-2i\theta}, \beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty}{(\beta^2 q^\alpha, q, e^{-2i\theta}, te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty} \\
&\quad \times {}_4\phi_3 \left[\begin{matrix} \beta e^{2i\theta}, \beta, te^{\theta+i\varphi}, te^{i\theta-i\varphi} \\ qe^{2i\theta}, \beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi} \end{matrix} ; q, q \right] \\
&+\theta \longleftrightarrow -\theta.
\end{aligned} \tag{3.7.1}$$

Combining the ${}_4\phi_3$ series by the formula [11, (2.10.10)] we get

$$\begin{aligned}
\lim_{\alpha \rightarrow 0} L_t^\alpha(x, y) &= \frac{(\beta, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
&\quad \times {}_8W_7(\beta t^2/q; \beta, te^{i\theta-i\varphi}, te^{i\theta+i\varphi}, te^{-i\theta-i\varphi}, te^{-i\theta+i\varphi}; q, \beta).
\end{aligned} \tag{3.7.2}$$

By (3.1.14),

$$\begin{aligned}
&\lim_{\alpha \rightarrow 0} K_t^\alpha(x, y; \beta|q) \\
&= \frac{(\beta q, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
&\quad \times {}_8W_7(\beta t^2/q; \beta, te^{i\theta-i\varphi}, te^{i\theta+i\varphi}, te^{-i\theta-i\varphi}, te^{-i\theta+i\varphi}; q, \beta) \\
&\quad - \beta \frac{(\beta q, t^2 q^2; q)_\infty}{(\beta^2, \beta t^2 q^2; q)_\infty} \frac{|(\beta tqe^{i\theta+i\varphi}, \beta tqe^{i\theta-i\varphi}; q)_\infty|^2}{|(tqe^{i\theta+i\varphi}, tqe^{i\theta-i\varphi}; q)_\infty|^2}
\end{aligned} \tag{3.7.3}$$

$$\begin{aligned}
& \times {}_8W_7(\beta t^2 q; \beta, tqe^{i\theta-i\varphi}, tqe^{i\theta+i\varphi}, tqe^{-i\theta-i\varphi}, tqe^{-i\theta+i\varphi}; q, \beta) \\
& = \frac{(\beta q, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
& + \frac{(\beta q, t^2 q; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \sum_{m=0}^{\infty} (1 - \beta t^2 q^{2m+1}) \frac{|(\beta tq^{m+1} e^{i\theta+i\varphi}, \beta tq^{m+1} e^{i\theta-i\varphi}; q)_\infty|^2}{|(tq^{m+1} e^{i\theta+i\varphi}, tq^{m+1} e^{i\theta-i\varphi}; q)_\infty|^2} \beta^{m+1} \\
& \times \left\{ \frac{(\beta t^2; q)_m (\beta; q)_{m+1}}{(qt^2; q)_m (q; q)_{m+1}} - \frac{(\beta t^2; q)_{m+1} (\beta; q)_m}{(qt^2; q)_{m+1} (q; q)_m} \right\}.
\end{aligned}$$

The term inside the curly brackets can be simplified to give

$$\frac{(\beta t^2, \beta; q)_{m+1}}{(qt^2, q; q)_{m+1}} \frac{(1 - t^2)(q - \beta)}{(1 - \beta t^2 q^m)(1 - \beta t^m)} q^m. \quad (3.7.4)$$

Substituting (3.7.4) in (3.7.3), we get

$$\begin{aligned}
& \lim_{\alpha \rightarrow 0} K_t^\alpha(x, y; \beta | q) \quad (3.7.5) \\
& = \frac{(\beta q, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
& + \frac{(\beta q, t^2 q; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \sum_{m=0}^{\infty} \frac{(1 - \beta t^2 q^{2m+1})}{(1 - \beta t^2 q^{-1})} \frac{|(\beta tq^{m+1} e^{i\theta+i\varphi}, \beta tq^{m+1} e^{i\theta-i\varphi}; q)_\infty|^2}{|(tq^{m+1} e^{i\theta+i\varphi}, tq^{m+1} e^{i\theta-i\varphi}; q)_\infty|^2} (\beta q)^{m+1} \\
& = \frac{(\beta q, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
& \times \sum_{m=0}^{\infty} \frac{(1 - \beta t^2 q^{2m-1})}{(1 - \beta t^2 q^{-1})} \frac{(\beta t^2/q, \beta/q)_m}{(q, qt^2)_m} \frac{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_m|^2}{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_m|^2} (\beta q)^m \\
& = \frac{(\beta q, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \frac{|(\beta te^{i\theta+i\varphi}, \beta te^{i\theta-i\varphi}; q)_\infty|^2}{|(te^{i\theta+i\varphi}, te^{i\theta-i\varphi}; q)_\infty|^2} \\
& \times {}_8W_7(\beta t^2/q; te^{i\theta+i\varphi}, te^{-i\theta+i\varphi}, te^{i\theta-i\varphi}, te^{-i\theta-i\varphi}, \beta/q; \beta q),
\end{aligned}$$

which is the Poisson kernel of continuous q -ultraspherical polynomials.

Chapter 4

A projection formula and a reproducing kernel for the associated Askey-Wilson polynomials

4.1 Introduction

Let us consider the infinite sum

$$K^\nu(x, y; a, b, c, d) = \sum_{n=0}^{\infty} \frac{p_n^\alpha(x; a, b, c, d) p_n^{\alpha, \nu}(y; a, b, c, d)}{\xi_n}, \quad (4.1.1)$$

where ν is a complex number with $|\nu| < 1$,

$$p_n^{\alpha, \nu}(y; a, b, c, d) \equiv p_n^\alpha(y; a\nu, b\nu, c\nu^{-1}, d\nu^{-1}) \quad (4.1.2)$$

is the associated Askey-Wilson polynomial of degree n with parameters $a\nu, b\nu, c\nu^{-1}, d\nu^{-1}$ and ξ_n is the squared norm (1.3.26) of the associated Askey-Wilson polynomials $p_n^\alpha(x; a, b, c, d)$. Clearly

$$\lim_{\nu \rightarrow 1^-} p_n^{\alpha, \nu}(y; a, b, c, d) = p_n^\alpha(y; a, b, c, d). \quad (4.1.3)$$

It was shown in [17] that $p_n^\alpha(x; a, b, c, d) \sim |A(\theta)|a^n \cos(n\theta + \epsilon(\theta))$ for large n where $A(\theta)$ and $\epsilon(\theta)$ are two bounded functions of θ . So the series in (4.1.1) is convergent

for all x, y in $[-1, 1]$ provided $|\nu| < 1$. For $\nu = 1$, of course, the series must diverge when $x = y$ since we have a complete system of orthogonal polynomials on a finite interval whose corresponding moment problem is determinate, Chihara [10]. From (1.3.26) and (4.1.1) it follows that

$$\int_{-1}^1 K^\nu(x, y; a, b, c, d) p_n^\alpha(x; a, b, c, d) d\mu(x) = p_n^{\alpha, \nu}(y; a, b, c, d), \quad (4.1.4)$$

which is an associated extension of the projection formula [27, (2.4)], [11, Ex.(6.9)].

Now, use of [17, (6.2)] gives

$$\begin{aligned} p_n^{\alpha, \nu}(y; a, b, c, d) &= p_n^\alpha(y; a\nu, b\nu, c\nu^{-1}, d\nu^{-1}) \\ &= \frac{(bcq^\alpha, cdq^\alpha\nu^{-2}; q)_n}{(adq^\alpha, abq^\alpha\nu^2; q)_n} \left(\frac{a\nu^2}{c} \right)^n p_n^\alpha(y; c\nu^{-1}, d\nu^{-1}, a\nu, b\nu). \end{aligned} \quad (4.1.5)$$

Interchanging $a \longleftrightarrow c$, $b \longleftrightarrow d$ in (4.1.4) we find that

$$\begin{aligned} \int_{-1}^1 K^\nu(x, y; c, d, a, b) p_n^\alpha(x; c, d, a, b) d\mu(x; c, d, a, b) \\ = \frac{(adq^\alpha, abq^\alpha\nu^{-2}; q)_n}{(bcq^\alpha, cdq^\alpha\nu^2; q)_n} \left(\frac{c\nu^2}{a} \right)^n p_n^\alpha(y; a\nu^{-1}, b\nu^{-1}, c\nu, d\nu). \end{aligned} \quad (4.1.6)$$

So,

$$\begin{aligned} \int_{-1}^1 K^\nu(y, z; a\nu^{-1}, b\nu^{-1}, c\nu, d\nu) d\mu(y; a\nu^{-1}, b\nu^{-1}, c\nu, d\nu) \\ \times \int_{-1}^1 K^\nu(x, y; c, d, a, b) p_n^\alpha(x; c, d, a, b) d\mu(x; c, d, a, b) \\ = \frac{(adq^\alpha, abq^\alpha\nu^{-2}; q)_n}{(bcq^\alpha, cdq^\alpha\nu^2; q)_n} \left(\frac{c\nu^2}{a} \right)^n p_n^\alpha(z; a, b, c, d). \end{aligned} \quad (4.1.7)$$

From here it follows that

$$\begin{aligned} \int_{-1}^1 G^\nu(x, z) p_n^\alpha(x; a, b, c, d) d\mu(x; a, b, c, d) \\ = \frac{(cdq^\alpha, abq^\alpha\nu^{-2}; q)_n}{(abq^\alpha, cdq^\alpha\nu^2; q)_n} \nu^{2n} p_n^\alpha(z; a, b, c, d), \end{aligned} \quad (4.1.8)$$

where

$$\begin{aligned} G^\nu(x, z) \\ = \int_{-1}^1 K^\nu(y, z; a\nu^{-1}, b\nu^{-1}, c\nu, d\nu) K^\nu(x, y; c, d, a, b) d\mu(y; a\nu^{-1}, b\nu^{-1}, c\nu, d\nu). \end{aligned} \quad (4.1.9)$$

It is obvious that $G^\nu(x, z)$ is the reproducing kernel for the associated Askey-Wilson polynomials. Any additional restrictions on the parameters that are needed for the integrals in (4.1.7)–(4.1.9) to converge are assumed to hold.

Our main objective in this chapter is to compute (4.1.1). We will prove that

$$\begin{aligned}
& K^\nu(x, y; a, b, c, d) \\
&= \frac{(abcdq^{\alpha-2}; q)_\infty}{(abcdq^{2\alpha-2}; q)_\infty} \frac{(abcdq^{2\alpha-1}, abcdq^{2\alpha}; q)_\infty}{(ab\nu^2q^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}, q^{\alpha+1}; q)_\infty} \\
&\quad \times \frac{(a/z, a\nu^2zq^\alpha, b\nu^2zq^\alpha, czq^\alpha, dzq^\alpha, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, bcdq^{\alpha-1}/z; q)_\infty}{(\nu^2z^2q^{\alpha+1}, bcdq^{2\alpha-1}/z, 1/z^2; q)_\infty} \\
&\quad \times \frac{|(\nu qze^{i\varphi}; q)_\infty|^2}{|(\nu ze^{i\varphi}; q)_\infty|^2} s'_{\alpha-1}(z^{-1}) S(z) \\
&\quad + z \longleftrightarrow z^{-1},
\end{aligned} \tag{4.1.10}$$

where $s'_{\alpha-1}(z)$ and $S(z)$ are given by (4.2.4) and (4.4.14) respectively. The polynomials in (1.3.24) form a closed set in the $L^2(-1, 1)$ space, [40, Th.(3.1.5). page 40]. Using that it is easy to see, in the distributional sense, that

$$\begin{aligned}
& \lim_{\nu \rightarrow 1^-} K^\nu(x, y; a, b, c, d) \\
&= \sum_{n=0}^{\infty} \frac{p_n^\alpha(x; a, b, c, d) p_n^\alpha(y; a, b, c, d)}{\xi_n} = \frac{\delta(x - y)}{\mu'(x; a, b, c, d)},
\end{aligned} \tag{4.1.11}$$

where $\delta(x)$ is the Dirac delta function. The equation (4.1.11) is an orthogonality relation dual to (1.3.25). We would like to remark that this computation of $K^\nu(x, y; a, b, c, d)$ provides us with an alternative way of computing the weight function for the associated Askey-Wilson polynomials. As a final remark we wish to mention that similar calculation can be done for the second set of associated Askey-Wilson polynomials considered in [17], which will lead to a second projection formula.

The proof of (4.1.10) is rather long which involves the evaluation of several sums and integrals. In order that the readers may not become overwhelmed, we have decided to break the proof in several sections. In Section 4.2 we will show

that $K^\nu(x, y; a, b, c, d)$ can be broken into a sum of three parts of similar type. The typical part will contain a triple sum and an integral. In Section 4.3 we will evaluate the triple sum and will show that it is an ${}_8W_7$ series. In Section 4.4 we will evaluate the integral which may be of independent interest and is very crucial for the computation of $K^\nu(x, y; a, b, c, d)$. In Section 4.5 we will again combine the three parts to get the formula (4.1.10) and finally in Section 4.6 we will evaluate the limit of $K^\nu(x, y; a, b, c, d)$ as $\nu \rightarrow 1^-$ to get formula (4.1.11).

4.2 First reduction of $K^\nu(x, y; a, b, c, d)$

In this section we will break $K^\nu(x, y; a, b, c, d)$ into three similar parts. The key to break it into three parts is that we use the representation (1.3.24) for $p_n^\alpha(x; a, b, c, d)$. and for $p_n^{\alpha, \nu}(x; a, b, c, d)$ we take the integral representation of Rahman (1.3.28). Then

$$\begin{aligned}
K^\nu(x, y; a, b, c, d) = & A_1 B_1(z) s'_{\alpha-1}(z) \int_{-1}^1 dt K(y; t) \\
& \times \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, abq^\alpha, acq^\alpha, adq^\alpha; q)_n}{(q, bcq^\alpha, bdq^\alpha, cdq^\alpha; q)_n} \\
& \times \frac{(abq^{\alpha+n}, acq^{\alpha+n}, adq^{\alpha+n}, bcdq^{\alpha+n}/z; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, azq^{\alpha+n}; q)_\infty} \left(\frac{q^{-\alpha}}{az} \right)^n \\
& \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, zq/a) \\
& \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right] \\
& + A_2 B_2(z) r'_{\alpha-1}(z) \int_{-1}^1 dt K(y; t) \\
& \times \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, abq^\alpha, acq^\alpha, adq^\alpha; q)_n}{(q, bcq^\alpha, bdq^\alpha, cdq^\alpha; q)_n} \left(\frac{zq^{-\alpha}}{a} \right)^n \\
& \times \frac{(abcdq^{2\alpha+2n}, bzq^{\alpha+n+1}, czq^{\alpha+n+1}, dzq^{\alpha+n+1}, bcdzq^{\alpha+n}; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, bcdzq^{2\alpha+2n+1}; q)_\infty}
\end{aligned} \tag{4.2.1}$$

$$\begin{aligned} & \times {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, zq/a; q, az) \\ & \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right] \end{aligned}$$

where

$$A_1 = \frac{1}{(1 - abcdq^{2\alpha-2})} \frac{(bc, bd, cd, abcdq^{2\alpha-2}; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}; q)_\infty}, \quad (4.2.2)$$

$$B_1(z) = \frac{(az, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}; q)_\infty}{(bz, cz, dz, bcd/z, bcdzq^{2\alpha-1}; q)_\infty}, \quad (4.2.3)$$

$$s'_{\alpha-1}(z) = {}_8W_7(bcdzq^{2\alpha-2}, bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha, zq/a; q, az), \quad (4.2.4)$$

$$A_2 = \frac{1}{(1 - abcdq^{2\alpha-2})} \frac{(bc, bd, cd, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, q^\alpha; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}; q)_\infty}, \quad (4.2.5)$$

$$B_2(z) = -z^2 \frac{(az, bcdq^{\alpha-1}/z; q)_\infty}{(bz, cz, dz, bcd/z, azq^{\alpha-1}; q)_\infty}, \quad (4.2.6)$$

$$r'_{\alpha-1}(z) = {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha-2}, q^{1-\alpha}; q, zq/a), \quad (4.2.7)$$

$$\begin{aligned} & K(y; t) \quad (4.2.8) \\ & = \frac{(q, q, q, ab\nu^2 q^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bcq^{\alpha-1}, bdq^{\alpha-1}, cd\nu^{-2} q^{\alpha-1}, q^\alpha; q)_\infty}{4\pi^2 (abcdq^{2\alpha-2}, q^{\alpha+1}; q)_\infty} \\ & \times \frac{(e^{2i\psi_1}, e^{-2i\psi_1}; q)_\infty}{h(t; q^{\frac{\alpha}{2}} e^{i\varphi}, q^{\frac{\alpha}{2}} e^{-i\varphi})} \int_{-1}^1 du w(u; a\nu q^{\frac{\alpha-1}{2}}, b\nu q^{\frac{\alpha-1}{2}}, c\nu^{-1} q^{\frac{\alpha-1}{2}}, d\nu^{-1} q^{\frac{\alpha-1}{2}}) \\ & \times \frac{h(u; q^{\frac{\alpha+1}{2}} e^{i\varphi}, q^{\frac{\alpha+1}{2}} e^{-i\varphi})}{h(u; \sqrt{q} e^{i\psi_1}, \sqrt{q} e^{-i\psi_1})} \end{aligned}$$

and

$$x = \cos \theta = \frac{z + z^{-1}}{2}, \quad y = \cos \varphi, \quad t = \cos \psi_1, \quad z = e^{i\theta}. \quad (4.2.9)$$

As we shall see later, the first part of (4.2.1) can be computed but in order to evaluate the second part we have to break the series

$${}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, zq/a; q, az).$$

into two ${}_8W_7$ series with the help of [11, (2.11.1)]. This can be done in many different ways but only one specific choice of the parameters seems to lead to the desired result. We found the software developed by C. Krattanthallur [24], to manipulate the basic hypergeometric series, very handy in locating that proper choice which leads to the desired simplification. The proper breaking is:

$$\begin{aligned} & {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, zq/a; q, az) \\ &= - \frac{(bcdzq^{2\alpha+2n+1}, q^{-\alpha-n}/bz, cd, dq/b, cq/b, acdq^{\alpha+n}/z, bcq^{\alpha+n}, bdq^{\alpha+n}; q)_\infty}{(bzq^{\alpha+n}, bcdzq^{\alpha+n}, dzq^{\alpha+n+1}, czq^{\alpha+n+1}, abcdq^{2\alpha+2n}, c/z, d/z, q^{1-\alpha-n}/ab; q)_\infty} \\ & \quad \times \frac{(qz/a, bz, q^{1-\alpha-n}/az, azq^{\alpha+n}; q)_\infty}{(bq/a, a/b, q^{-\alpha-n}, cdq/bz; q)_\infty} \\ & \quad \times {}_8W_7(cd/bz; c/z, d/z, q/bz, cdq^{\alpha+n}, q^{1-\alpha-n}/ab; q, az) \\ & \quad + \frac{(bcdzq^{2\alpha+2n+1}, qz/b, adq^{\alpha+n}, acq^{\alpha+n}, bd, zq^{1-\alpha-n}/a, q^{-\alpha-n}/bz, dq/a; q)_\infty}{(dzq^{\alpha+n+1}, czq^{\alpha+n+1}, abcdq^{2\alpha+2n}, a/b, q^{-\alpha-n}, bdqz/a, d/z, q^{1-\alpha-n}/ab; q)_\infty} \\ & \quad \times {}_8W_7(bdz/a; bz, dz, zq/a, bdq^{\alpha+n}, q^{1-\alpha-n}/ac; q, c/z). \end{aligned} \tag{4.2.10}$$

Then from (4.2.5), (4.2.6), (4.2.10), the second part of $K^\nu(x, y; a, b, c, d)$ is

$$\begin{aligned} &= -z^2 \frac{r'_{\alpha-1}(z)}{(1 - abcdq^{2\alpha-2})} \frac{(bc, bd, cd, q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bd, dq/a; q)_\infty}{(abq^\alpha, abcdq^{\alpha-1}, a/b, bcq^\alpha, bdq^\alpha, cdq^\alpha, q^{-\alpha}, q^{\alpha+1}, q^{1-\alpha}/ab; q)_\infty} \\ & \quad \times \frac{(az, bcdq^{\alpha-1}/z, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq^{1-\alpha}/a, bcdzq^\alpha; q)_\infty}{(bz, cz, dz, bcd/z, azq^{\alpha-1}, bdzq/a, d/z; q)_\infty} \\ & \quad \times \int_{-1}^1 dt K(y; t) \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, aq^\alpha/z; q)_n}{(q, bcdzq^\alpha; q)_n} \left(\frac{zq^{-\alpha}}{a} \right)^n \\ & \quad \times {}_8W_7(bdz/a; bz, dz, zq/a, bdq^{\alpha+n}, q^{1-\alpha-n}/ac; q, c/z) \\ & \quad \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right] \end{aligned}$$

$$\begin{aligned}
& + z^2 \frac{r'_{\alpha-1}(z)}{(1 - abcdq^{2\alpha-2})} \frac{(bc, bd, cd, q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, cd, dq/b; q)_\infty}{(abq^\alpha, abcdq^{\alpha-1}, a/b, acq^\alpha, adq^\alpha, cdq^\alpha, q^{-\alpha}, q^{\alpha+1}, q^{1-\alpha}/ab; q)_\infty} \\
& \times \frac{(cq/b; q)_\infty}{(bq/a; q)_\infty} \frac{(az, azq^\alpha, bcdq^{\alpha-1}/z, qz/a, bzq^{\alpha+1}, q^{-\alpha}/bz, bz, q^{1-\alpha}/az, acdq^\alpha/z; q)_\infty}{(bz, cz, dz, bcd/z, azq^{\alpha-1}, bzq^\alpha, c/z, d/z, cdq/bz; q)_\infty} \\
& \times \int_{-1}^1 dt K(y; t) \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, acq^\alpha, adq^\alpha, bzq^\alpha; q)_n}{(q, bcq^\alpha, bdq^\alpha, acdq^\alpha/z; q)_n} \left(\frac{q^{-\alpha}}{az} \right)^n \\
& \times {}_8W_7(cd/bz; c/z, d/z, q/bz, cdq^{\alpha+n}, q^{1-\alpha-n}/ab; q, az) \\
& \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right]. \tag{4.2.11}
\end{aligned}$$

The ${}_4\phi_3$ series

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right]$$

appearing in the above expression is balanced and terminating and hence transformable by Sears transformation formula [11, (2.10.4)]. Its use in this series, gives us

$$\begin{aligned}
& {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ \\ ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha \end{matrix} ; q, q \right] \\
& = \frac{(bdq^\alpha, bcq^\alpha; q)_n}{(acq^\alpha, adq^\alpha; q)_n} \left(\frac{a}{b} \right)^n {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, b\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, b\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ \\ ab\nu^2 q^\alpha, bdq^\alpha, bcq^\alpha \end{matrix} ; q, q \right]. \tag{4.2.12}
\end{aligned}$$

Then from (4.2.1), (4.2.11) and (4.2.12), we get

$$\begin{aligned}
& K^\nu(x, y; a, b, c, d) \\
& = K^{1,\nu}(x, y; a, b, c, d) + K^{2,\nu}(x, y; a, b, c, d) + K^{3,\nu}(x, y; a, b, c, d)
\end{aligned} \tag{4.2.13}$$

where

$$\begin{aligned}
& K^{1,\nu}(x, y; a, b, c, d) \\
&= \frac{s'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(bc, bd, cd, abcdq^{2\alpha-2}; q)_{\infty}}{(bcq^{\alpha}, bdq^{\alpha}, cdq^{\alpha}, abcdq^{\alpha-1}; q)_{\infty}} \\
&\quad \times \frac{(az, bzq^{\alpha}, czq^{\alpha}, dzq^{\alpha}, bcdzq^{\alpha-1}, bcdq^{\alpha}/z; q)_{\infty}}{(azq^{\alpha}, bz, cz, dz, bcd/z, bcdzq^{2\alpha-1}; q)_{\infty}} \\
&\quad \times \int_{-1}^1 dt K(y; t) \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, azq^{\alpha}; q)_n}{(q, bcdq^{\alpha}/z; q)_n} \left(\frac{q^{-\alpha}}{az} \right)^n \\
&\quad \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, zq/a) \\
&\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^{\alpha}, acq^{\alpha}, adq^{\alpha} \end{matrix} ; q, q \right],
\end{aligned} \tag{4.2.14}$$

$$\begin{aligned}
& K^{2,\nu}(x, y; a, b, c, d) \\
&= -z^2 \frac{r'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(bc, bd, cd, q^{\alpha}, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bd, dq/a; q)_{\infty}}{(abq^{\alpha}, bcq^{\alpha}, bdq^{\alpha}, cdq^{\alpha}, abcdq^{\alpha-1}, a/b, q^{-\alpha}, q^{\alpha+1}, q^{-\alpha+1}/ab; q)_{\infty}} \\
&\quad \times \frac{(az, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq^{1-\alpha}/a, bcdq^{\alpha-1}/z, bcdzq^{\alpha}; q)_{\infty}}{(bz, cz, dz, bcd/z, azq^{\alpha-1}, bdzq/a, d/z; q)_{\infty}} \\
&\quad \times \int_{-1}^1 dt K(y; t) \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, aq^{\alpha}/z; q)_n}{(q, bcdzq^{\alpha}; q)_n} \left(\frac{zq^{-\alpha}}{a} \right)^n \\
&\quad \times {}_8W_7(bdz/a; bz, dz, zq/a, bdq^{\alpha+n}, q^{1-\alpha-n}/ac; q, c/z) \\
&\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^{\alpha}, acq^{\alpha}, adq^{\alpha} \end{matrix} ; q, q \right]
\end{aligned} \tag{4.2.15}$$

and

$$\begin{aligned}
& K^{3,\nu}(x, y; a, b, c, d) \\
&= z^2 \frac{r'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(bc, bd, cd, q^{\alpha}, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, cd, dq/b; q)_{\infty}}{(abq^{\alpha}, acq^{\alpha}, adq^{\alpha}, cdq^{\alpha}, abcdq^{\alpha-1}, a/b, q^{-\alpha}, q^{\alpha+1}, q^{-\alpha+1}/ab; q)_{\infty}} \\
&\quad \times \frac{(cq/b; q)_{\infty}}{(bq/a; q)_{\infty}} \frac{(az, azq^{\alpha}, qz/a, bz, bzq^{\alpha+1}, q^{-\alpha}/bz, q^{1-\alpha}/az, bcdq^{\alpha-1}/z, acdq^{\alpha}/z; q)_{\infty}}{(bz, cz, dz, bcd/z, azq^{\alpha-1}, bzq^{\alpha}, cdq/bz, c/z, d/z; q)_{\infty}}
\end{aligned} \tag{4.2.16}$$

$$\begin{aligned}
& \times \int_{-1}^1 dt K(y; t) \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, bzq^\alpha; q)_n}{(q, acdq^\alpha/z; q)_n} \left(\frac{q^{-\alpha}}{bz} \right)^n \\
& \times {}_8W_7(cd/bz; c/z, d/z, q/bz, cdq^{\alpha+n}, q^{1-\alpha-n}/ab; q, az) \\
& \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, b\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, b\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^\alpha, bcq^\alpha, bdq^\alpha \end{matrix} ; q, q \right].
\end{aligned}$$

4.3 Evaluation of a typical sum

In this section we will evaluate the sum over n appearing in the expressions for $K^{1,\nu}(x, y; a, b, c, d)$, $K^{2,\nu}(x, y; a, b, c, d)$ and $K^{3,\nu}(x, y; a, b, c, d)$. Since the sum over n in $K^{2,\nu}(x, y; a, b, c, d)$ and $K^{3,\nu}(x, y; a, b, c, d)$ can be obtained by relabelling the parameters in the sum over n in $K^{1,\nu}(x, y; a, b, c, d)$, it is enough to compute the sum over n in $K^{1,\nu}(x, y; a, b, c, d)$. The key step in the evaluation of this sum is the use of formula [11, (2.10.10)] or equivalently of [11, (2.10.19)] for the ${}_8W_7$ series, and Watson formula [11, (2.5.1)] to write the ${}_4\phi_3$ series as an ${}_8W_7$ series. We have preferred formula [11, (2.10.19)] over [11, (2.10.10)] because the computation becomes somewhat simpler. By [11, (2.10.19)],

$$\begin{aligned}
& {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, zq/a) \\
& = \frac{(bz, b/z, cz, c/z, dz, d/z, q/az, bcd/z; q)_\infty}{(q, z^2, q/z^2, bc, bd, cd, bcdq^{\alpha+n}/z, q^{1-\alpha-n}/az; q)_\infty} \\
& \quad \times \frac{azH}{q(1-q)} \int_{\frac{zq}{aH}}^{\frac{q}{azH}} \frac{(aHu/z, aHzu, abcdHq^{\alpha+n-1}u, Hq^{-\alpha-n}u; q)_\infty}{(abHu/q, acHu/q, adHu/q, Hu; q)_\infty} d_q u
\end{aligned} \tag{4.3.1}$$

for an arbitrary parameter H . Taking $H = 1$, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, azq^\alpha; q)_n}{(q, bcdq^\alpha/z; q)_n} \left(\frac{q^{-\alpha}}{az} \right)^n \\
& \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, zq/a) \\
& = \frac{az}{q(1-q)} \frac{(bz, b/z, cz, c/z, dz, d/z, q/az, bcd/z; q)_\infty}{(q, z^2, q/z^2, bc, bd, cd, bcdq^\alpha/z, q^{1-\alpha}/az; q)_\infty} \\
& \quad \times \int_{\frac{zq}{a}}^{\frac{q}{az}} \frac{(au/z, azu, abcdq^{\alpha-1}u, q^{-\alpha}u; q)_\infty}{(abu/q, acu/q, adu/q, u; q)_\infty} d_q u
\end{aligned} \tag{4.3.2}$$

$$\times \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, q^{\alpha+1}/u; q)_n}{(q, abcdq^{\alpha-1}u; q)_n} (q^{-\alpha-1}u)^n.$$

The ${}_4\phi_3$ series

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^{\alpha}, acq^{\alpha}, adq^{\alpha} \end{matrix} ; q, q \right].$$

by Watson formula [11, (2.5.1)] is

$$\begin{aligned} &= \frac{(cdq^{\alpha}\nu^{-2}, q^{1-n}e^{-i\psi_1}/b\nu q^{\frac{\alpha}{2}}; q)_n}{(acd\nu^{-1}q^{\frac{3\alpha}{2}}e^{-i\psi_1}, q^{1-n}/ab\nu^2 q^{\alpha}; q)_n} \\ &\times {}_8W_7(acd\nu^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}; c\nu^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, d\nu^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, \\ &\quad a\nu q^{\frac{\alpha}{2}}e^{-i\psi_1}, abcdq^{2\alpha+n-1}, q^{-n}; q, \frac{q^{1-\frac{\alpha}{2}}}{b\nu}e^{i\psi_1}). \end{aligned} \quad (4.3.3)$$

With the help of (I.10), (I. 11) and (I. 17) of [11]

$$\frac{(q^{1-n}e^{-i\psi_1}/b\nu q^{\frac{\alpha}{2}}; q)_n}{(q^{1-n}/ab\nu^2 q^{\alpha}; q)_n} = \frac{(b\nu q^{\frac{\alpha}{2}}e^{i\psi_1}; q)_n}{(ab\nu^2 q^{\alpha}; q)_n} (a\nu q^{\frac{\alpha}{2}}e^{-i\psi_1})^n \quad (4.3.4)$$

and

$$\frac{(q^{-n}; q)_r (b\nu q^{\frac{\alpha}{2}}e^{i\psi_1}; q)_n}{(q^{1-n}e^{-i\psi_1}/b\nu q^{\frac{\alpha}{2}}; q)_r} = \frac{(q; q)_n (b\nu q^{\frac{\alpha}{2}}e^{i\psi_1}; q)_{n-r}}{(q; q)_{n-r}} (b\nu q^{\frac{\alpha}{2}-1}e^{i\psi_1})^r. \quad (4.3.5)$$

So from (4.3.3), (4.3.4) and (4.3.5), we get

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1} \\ ab\nu^2 q^{\alpha}, acq^{\alpha}, adq^{\alpha} \end{matrix} ; q, q \right] \\ &= \frac{(q, cdq^{\alpha}\nu^{-2}; q)_n}{(abq^{\alpha}\nu^2; q)_n} (a\nu q^{\frac{\alpha}{2}}e^{-i\psi_1})^n \\ &\times \sum_{r=0}^n \frac{(acd\nu^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}, q\sqrt{\cdot}, -q\sqrt{\cdot}, d\nu^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, c\nu^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, a\nu q^{\frac{\alpha}{2}}e^{-i\psi_1}; q)_r}{(q, \sqrt{\cdot}, -\sqrt{\cdot}, acq^{\alpha}, adq^{\alpha}, cd\nu^{-2}q^{\alpha}; q)_r} \\ &\quad \times \frac{(abcdq^{2\alpha-1+n}; q)_r (b\nu q^{\frac{\alpha}{2}}e^{i\psi_1}; q)_{n-r}}{(q; q)_{n-r} (acd\nu^{-1}q^{\frac{3\alpha}{2}}e^{-i\psi_1}; q)_{n+r}} (e^{2i\psi_1})^r. \end{aligned} \quad (4.3.6)$$

Then from (4.3.2) and (4.3.6) we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{(abcdq^{2\alpha-1}, azq^{\alpha}; q)_n}{(q, bcdq^{\alpha}/z; q)_n} \left(\frac{q^{-\alpha}}{az} \right)^n \\
& \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, zq/a) \\
& \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{2\alpha+n-1}, avq^{\frac{\alpha}{2}} e^{i\psi_1}, avq^{\frac{\alpha}{2}} e^{-i\psi_1} \\ abv^2q^{\alpha}, acq^{\alpha}, adq^{\alpha} \end{matrix} ; q, q \right] \\
& = \frac{az}{q(1-q)} \frac{(bz, b/z, cz, c/z, dz, d/z, q/az, bcd/z; q)_{\infty}}{(q, z^2, q/z^2, bc, bd, cd, bcdq^{\alpha}/z, q^{1-\alpha}/az; q)_{\infty}} \\
& \times \int_{\frac{za}{a}}^{\frac{a}{az}} \frac{(au/z, azu, abcdq^{\alpha-1}u, q^{-\alpha}u; q)_{\infty}}{(abu/q, acu/q, adu/q, u; q)_{\infty}} d_q u \\
& \times \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{1 - abcdq^{2\alpha+2n-1}}{1 - abcdq^{2\alpha-1}} \frac{1 - acdv^{-1}q^{\frac{3\alpha}{2}-1+2r}e^{-i\psi_1}}{1 - acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}} \frac{(q^{\alpha+1}/u, cdv^{-2}q^{\alpha}; q)_n}{(abcdq^{\alpha-1}u, abv^2q^{\alpha}; q)_n} \\
& \times \frac{(acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}, cv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, dv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}; q)_r}{(q, cdq^{\alpha}v^{-2}, adq^{\alpha}, acq^{\alpha}; q)_r} \\
& \times \frac{(abcdq^{2\alpha-1}; q)_{n+r}(bvq^{\frac{\alpha}{2}}e^{i\psi_1}; q)_{n-r}}{(acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}; q)_{n+r}(q; q)_{n-r}} (avuvq^{-\frac{\alpha}{2}-1}e^{-i\psi_1})^n (e^{2i\psi_1})^r.
\end{aligned} \tag{4.3.7}$$

Now changing the order of summation after taking $n \rightarrow n + r$, we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{r=0}^n (..) \\
& = \sum_{r=0}^{\infty} \frac{1 - acdv^{-1}q^{\frac{3\alpha}{2}-1+2r}e^{-i\psi_1}}{1 - acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}} \frac{(acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}, cv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}; q)_r}{(q, cdq^{\alpha}v^{-2}, adq^{\alpha}; q)_r} \\
& \times \frac{(dv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, q^{\alpha+1}/u, cdv^{-2}q^{\alpha}; q)_r (abcdq^{2\alpha}; q)_{2r}}{(acq^{\alpha}, abcdq^{\alpha-1}u, abv^2q^{\alpha}; q)_r (acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}; q)_{2r}} (avuvq^{-\frac{\alpha}{2}-1}e^{i\psi_1})^r \\
& \times {}_6W_5(abcdq^{2\alpha+2r-1}, q^{\alpha+1+r}/u, cdv^{-2}q^{\alpha+r}, bvq^{\frac{\alpha}{2}}e^{i\psi_1}; q, avuvq^{-\frac{\alpha}{2}-1}e^{-i\psi_1}).
\end{aligned} \tag{4.3.8}$$

For convergence of the above series we assume that $|\nu q^{-\frac{\alpha}{2}}| < 1$. The ${}_6W_5$ series in (4.3.8), is summable by [11, (2.7.1)]. After making some simplifications we find the sum over r in (4.3.8)

$$\begin{aligned}
& = \frac{(abcdq^{2\alpha}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}, acdv^{-1}q^{\frac{\alpha}{2}-1}e^{-i\psi_1}, abv^2u/q; q)_{\infty}}{(abcdq^{\alpha-1}u, abv^2q^{\alpha}, acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}, avq^{-\frac{\alpha}{2}-1}e^{-i\psi_1}u; q)_{\infty}} \\
& \times {}_6W_5(acdv^{-1}q^{\frac{3\alpha}{2}-1}e^{-i\psi_1}; cv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, dv^{-1}q^{\frac{\alpha}{2}}e^{-i\psi_1}, q^{\alpha+1}/u; q, avuvq^{-\frac{\alpha}{2}-1}e^{i\psi_1}).
\end{aligned}$$

Summing this series and making some obvious simplifications we find that the sum over n in $K^{1,\nu}(x, y; a, b, c, d)$ is

$$\begin{aligned}
&= \frac{az}{q(1-q)} \frac{(bz, b/z, cz, c/z, dz, d/z, q/az, bcd/z; q)_\infty}{(q, z^2, q/z^2, bc, bd, cd, bcdq^\alpha/z, q^{1-\alpha}/az; q)_\infty} \\
&\quad \times \frac{(abcdq^{2\alpha}; q)_\infty |(a\nu q^{\frac{\alpha}{2}} e^{-i\psi_1}; q)_\infty|^2}{(ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha; q)_\infty} \\
&\quad \times \int_{\frac{zq}{a}}^{\frac{a}{az}} \frac{(au/z, azu, ab\nu^2 q^{-1}u, q^{-\alpha}u; q)_\infty}{(abu/q, a\nu q^{-\frac{\alpha}{2}-1} e^{-i\psi_1}u, a\nu q^{-\frac{\alpha}{2}-1} e^{i\psi_1}u, u; q)_\infty} d_q u
\end{aligned} \tag{4.3.9}$$

and hence from [11, (2.10.19)], (4.2.14)

$$\begin{aligned}
&K^{1,\nu}(x, y; a, b, c, d) \\
&= \frac{s'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(abcdq^{2\alpha}, abcdq^{2\alpha-2}, \nu^2 q^{-\alpha}; q)_\infty}{(ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}; q)_\infty} \\
&\quad \times \frac{(az, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}, cz, c/z, dz, d/z, b\nu^2/z; q)_\infty}{(azq^\alpha, bz, cz, dz, bcdzq^{2\alpha-1}, b\nu^2 q^{-\alpha}/z; q)_\infty} \\
&\quad \times \int_{-1}^1 dt K(y; t) \frac{|(a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, b\nu q^{\frac{\alpha}{2}} e^{i\psi_1}; q)_\infty|^2}{|(\nu z q^{-\frac{\alpha}{2}} e^{i\psi_1}, \nu q^{-\frac{\alpha}{2}} e^{i\psi_1}/z; q)_\infty|^2} \\
&\quad \times {}_8W_7(b\nu^2 q^{-\alpha-1}/z; q^{-\alpha}, ab\nu^2/q, b/z, \nu q^{-\frac{\alpha}{2}} e^{-i\psi_1}/z, \nu q^{-\frac{\alpha}{2}} e^{i\psi_1}/z; q, zq/a).
\end{aligned} \tag{4.3.10}$$

Proceeding exactly the same way, we get

$$\begin{aligned}
&K^{2,\nu}(x, y; a, b, c, d) \\
&= -z^2 \frac{r'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(bc, bd, cd, q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, abcdq^{2\alpha}, \nu^2 q^{-\alpha}; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}, a/b, q^{-\alpha}; q)_\infty} \\
&\quad \times \frac{(az, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq^{1-\alpha}/a, bcdq^{\alpha-1}/z, q/az, b\nu^2 z; q)_\infty}{(q^{\alpha+1}, q^{-\alpha+1}/ab, bq/a, ab\nu^2 q^\alpha; q)_\infty (bz, bcd/z, azq^{\alpha-1}, b\nu^2 zq^{-\alpha}; q)_\infty} \\
&\quad \times \int_{-1}^1 dt K(y; t) \frac{|(a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, b\nu q^{\frac{\alpha}{2}} e^{i\psi_1}; q)_\infty|^2}{|(\nu z q^{-\frac{\alpha}{2}} e^{i\psi_1}, \nu q^{-\frac{\alpha}{2}} e^{i\psi_1}/z; q)_\infty|^2} \\
&\quad \times {}_8W_7(b\nu^2 zq^{-\alpha-1}; q^{-\alpha}, ab\nu^2/q, bz, \nu zq^{-\frac{\alpha}{2}} e^{-i\psi_1}, \nu zq^{-\frac{\alpha}{2}} e^{i\psi_1}; q, q/az)
\end{aligned} \tag{4.3.11}$$

and

$$\begin{aligned}
&K^{3,\nu}(x, y; a, b, c, d) \\
&= z^2 \frac{r'_{\alpha-1}(z)}{1 - abcdq^{2\alpha-2}} \frac{(bc, bd, cd, q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, abcdq^{2\alpha}, \nu^2 q^{-\alpha}; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}, a/b, bq/a, q^{-\alpha}; q)_\infty}
\end{aligned} \tag{4.3.12}$$

$$\begin{aligned}
& \times \frac{(azq^\alpha, qz/a, bzq^{\alpha+1}, q^{-\alpha}/bz, zq/b, av^2/z, q^{1-\alpha}/az, bcdq^{\alpha-1}/z; q)_\infty}{(q^{\alpha+1}, q^{-\alpha+1}/ab, abv^2q^\alpha; q)_\infty (bcd/z, azq^{\alpha-1}, bzq^\alpha, av^2q^{-\alpha}/z; q)_\infty} \\
& \times \int_{-1}^1 dt K(y; t) \frac{|(avq^{\frac{\alpha}{2}}e^{i\psi_1}, bvq^{\frac{\alpha}{2}}e^{i\psi_1}; q)_\infty|^2}{|(\nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}, \nu q^{-\frac{\alpha}{2}}e^{i\psi_1}/z; q)_\infty|^2} \\
& \times {}_8W_7(av^2q^{-\alpha-1}/z; q^{-\alpha}, abv^2/q, a/z, \nu q^{-\frac{\alpha}{2}}e^{-i\psi_1}/z, \nu q^{-\frac{\alpha}{2}}e^{i\psi_1}/z; q, q/bz).
\end{aligned}$$

4.4 Evaluation of an integral

In this section we will evaluate the integrals appearing in $K^{1,\nu}(x, y; a, b, c, d)$, $K^{2,\nu}(x, y; a, b, c, d)$ and $K^{3,\nu}(x, y; a, b, c, d)$. We notice that the integrals in $K^{2,\nu}(x, y; a, b, c, d)$ and $K^{3,\nu}(x, y; a, b, c, d)$ can be obtained from the integral in $K^{1,\nu}(x, y; a, b, c, d)$ by relabelling the parameters. So it is enough to evaluate

$$\begin{aligned}
& \int_{-1}^1 dt K(y; t) \frac{|(avq^{-\frac{\alpha}{2}}e^{i\psi_1}, bvq^{\frac{\alpha}{2}}e^{i\psi_1}; q)_\infty|^2}{|(\nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}, \nu/zq^{-\frac{\alpha}{2}}e^{i\psi_1}; q)_\infty|^2} \\
& \times {}_8W_7(bv^2q^{-\alpha-1}/z; abv^2/q, q^{-\alpha}, b/z, \nu/zq^{-\frac{\alpha}{2}}e^{i\psi_1}, \nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}; q, zq/a).
\end{aligned} \tag{4.4.1}$$

As we know the, kernel $K(y; t)$ itself is given in terms of integrals. This means that in order to compute (4.4.1) we will have to evaluate two integrals. The key step here is to break the ${}_8W_7$ series into a sum of two ${}_8W_7$ series by [11, (2.11.1)]. Indeed, we get

$$\begin{aligned}
& {}_8W_7(bv^2q^{-\alpha-1}/z; abv^2/q, q^{-\alpha}, b/z, \nu q^{-\frac{\alpha}{2}}e^{i\psi_1}/z, \nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}; q, zq/a) \\
& = \frac{(bv^2q^{-\alpha}/z, \nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}, \nu zq^{-\frac{\alpha}{2}}e^{-i\psi_1}, bz, \nu q^{1+\frac{\alpha}{2}}e^{i\psi_1}/z, \nu q^{1+\frac{\alpha}{2}}e^{-i\psi_1}/z, azq^\alpha, av^2/z; q)_\infty}{(\nu^2q^{-\alpha}, bvq^{-\frac{\alpha}{2}}e^{i\psi_1}, bvq^{-\frac{\alpha}{2}}e^{-i\psi_1}, z^2, q^{\alpha+1}, q\nu^2/z^2, avq^{\frac{\alpha}{2}}e^{i\psi_1}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}; q)_\infty} \\
& \times {}_8W_7(\nu^2/z^2; \nu^2, q/bz, q/az, \nu q^{-\frac{\alpha}{2}}e^{i\psi_1}/z, \nu q^{-\frac{\alpha}{2}}e^{-i\psi_1}/z; q, abq^\alpha) \\
& - \frac{(bv^2q^{-\alpha}/z, azq^\alpha, abv^2q^\alpha, av^2z, b/z, q/az, zq^{-\alpha}/a, aq^{\alpha+1}/z; q)_\infty}{(q^{-\alpha}/az, bv^2/z, \nu^2q^{-\alpha}, abq^\alpha, q/z^2, z^2, a^2bq^\alpha\nu^2z, q^{\alpha+1}; q)_\infty} \\
& \times \frac{(\nu q^{-\frac{\alpha}{2}}e^{i\psi_1}/z, \nu q^{-\frac{\alpha}{2}}e^{-i\psi_1}/z, ab\nu zq^{\frac{\alpha}{2}}e^{i\psi_1}, ab\nu zq^{\frac{\alpha}{2}}e^{-i\psi_1}; q)_\infty}{(avq^{\frac{\alpha}{2}}e^{i\psi_1}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}, bvq^{-\frac{\alpha}{2}}e^{i\psi_1}, bvq^{-\frac{\alpha}{2}}e^{-i\psi_1}; q)_\infty} \\
& \times {}_8W_7(a^2bv^2zq^{\alpha-1}; abv^2/q, az, abq^\alpha, avq^{\frac{\alpha}{2}}e^{i\psi_1}, avq^{\frac{\alpha}{2}}e^{-i\psi_1}; q, qz/a).
\end{aligned} \tag{4.4.2}$$

Now applying the Bailey transformation formula [11, (2.10.1)] twice on the second ${}_8W_7$ series on the right-hand side of (4.4.2), we get

$$\begin{aligned} & {}_8W_7(a^2b\nu^2zq^{\alpha-1}; ab\nu^2/q, az, abq^\alpha, a\nu q^{\frac{\alpha}{2}}e^{i\psi_1}, a\nu q^{\frac{\alpha}{2}}e^{-i\psi_1}; q, qz/a) \quad (4.4.3) \\ &= \frac{(a^2b\nu^2zq^\alpha, abq^\alpha, qz^2, b\nu^2z; q)_\infty |(\nu zq^{1+\frac{\alpha}{2}}e^{i\psi_1}; q)_\infty|^2}{(zq/a, azq^{\alpha+1}, ab\nu^2q^\alpha, q\nu^2z^2; q)_\infty |(ab\nu zq^{\frac{\alpha}{2}}e^{i\psi_1}; q)_\infty|^2} \\ & \times {}_8W_7(\nu^2z^2; \nu^2, zq/b, zq/a, \nu zq^{-\frac{\alpha}{2}}e^{i\psi_1}, \nu zq^{-\frac{\alpha}{2}}e^{-i\psi_1}; q, abq^\alpha). \end{aligned}$$

Then from (4.4.2) and (4.4.3), (4.4.1) is

$$\begin{aligned} &= \frac{(b\nu^2q^{-\alpha}/z, bz, azq^\alpha, a\nu^2/z; q)_\infty}{(\nu^2q^{-\alpha}, z^2, q^{\alpha+1}, q\nu^2/z^2; q)_\infty} \quad (4.4.4) \\ & \sum_{k=0}^{\infty} \frac{1 - \nu^2q^{2k}/z^2}{1 - \nu^2/z^2} \frac{(\nu^2/z^2, \nu^2, q/az, q/bz; q)_k}{(q, q/z^2, a\nu^2/z, b\nu^2/z; q)_k} (abq^\alpha)^k \int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z}q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z}q^{k-\frac{\alpha}{2}})} \\ & - \frac{(b\nu^2q^{-\alpha}/z, azq^\alpha, a\nu^2z, b/z, q/az, aq^{\alpha+1}/z, zq^{-\alpha}/a, b\nu^2z, qz^2; q)_\infty}{(azq^{\alpha+1}, q^{-\alpha}/az, q^{\alpha+1}, zq/a, z^2, q/z^2, q\nu^2z^2, \nu^2q^{-\alpha}, b\nu^2/z; q)_\infty} \\ & \times \sum_{k=0}^{\infty} \frac{1 - \nu^2z^2q^{2k}}{1 - \nu^2z^2} \frac{(\nu^2z^2, \nu^2, zq/a, zq/b; q)_k}{(q, qz^2, a\nu^2z, b\nu^2z; q)_k} (abq^\alpha)^k \int_{-1}^1 dt K(y; t) \frac{h(t; \nu zq^{k+1+\frac{\alpha}{2}})}{h(t; \nu zq^{k-\frac{\alpha}{2}})}. \end{aligned}$$

Observe that the two sums over k on the right hand side of (4.4.4) are complex conjugates of each other. First of all let us compute

$$\int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z}q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z}q^{k-\frac{\alpha}{2}})}.$$

From (4.2.8)

$$\begin{aligned} & \int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z}q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z}q^{k-\frac{\alpha}{2}})} \quad (4.4.5) \\ &= C \int_{-1}^1 ds w(s; a\nu q^{\frac{\alpha-1}{2}}, b\nu q^{\frac{\alpha-1}{2}}, c\nu^{-1}q^{\frac{\alpha-1}{2}}, d\nu^{-1}q^{\frac{\alpha-1}{2}}) h(s; q^{\frac{\alpha+1}{2}}e^{i\varphi}, q^{\frac{\alpha+1}{2}}e^{-i\varphi}) \\ & \times \int_{-1}^1 dt w(t; q^{\frac{\alpha}{2}}e^{i\varphi}, q^{\frac{\alpha}{2}}e^{-i\varphi}, q^{\frac{1}{2}}e^{i\psi_2}, q^{\frac{1}{2}}e^{-i\psi_2}) \frac{h(t; \frac{\nu}{z}q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z}q^{k-\frac{\alpha}{2}})}, \end{aligned}$$

where

$$C = \frac{(q, q, q, ab\nu^2q^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bcq^{\alpha-1}, bdq^{\alpha-1}, b\nu^{-2}q^{\alpha-1}, q^\alpha; q)_\infty}{4\pi^2(abcdq^{2\alpha-2}, q^{\alpha+1}; q)_\infty} \quad (4.4.6)$$

and

$$s = \cos \psi_2. \quad (4.4.7)$$

Notice that

$$\frac{q^{\frac{\alpha}{2}} e^{i\varphi} q^{\frac{\alpha}{2}} e^{-i\varphi} q^{\frac{1}{2}} e^{i\psi_2} q^{\frac{1}{2}} e^{-i\psi_2} \nu q^{-\frac{\alpha}{2}+k}/z}{\nu q^{\frac{\alpha}{2}+k+1}/z} = 1 = \lim_{\epsilon \rightarrow 0} (q^\epsilon). \quad (4.4.8)$$

By [11, (6.3.8)], we then have

$$\begin{aligned} & \int_{-1}^1 dt w(t; q^{\frac{\alpha}{2}} e^{i\varphi}, q^{\frac{\alpha}{2}} e^{-i\varphi}, q^{\frac{1}{2}} e^{i\psi_2}, q^{\frac{1}{2}} e^{-i\psi_2}) \frac{h(t; \frac{\nu}{z} q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z} q^{k-\frac{\alpha}{2}})} \\ &= \frac{2\pi |(\nu q^{\alpha+k+1} e^{i\varphi}/z, \nu q^{\frac{\alpha+3}{2}+k} e^{i\psi_2}/z; q)_\infty|^2}{(q, q, q^\alpha; q)_\infty h(s; q^{\frac{\alpha+1}{2}} e^{i\varphi}, q^{\frac{\alpha+1}{2}} e^{-i\varphi}) |(\nu q^k e^{i\varphi}/z, \nu q^{-\frac{\alpha-1}{2}+k} e^{i\psi_2}/z; q)_\infty|^2} \\ & \times \lim_{\epsilon \rightarrow 0} (q^\epsilon; q)_\infty \frac{(\nu^2 q^{2k+1}/z^2; q)_\infty}{(\nu^2 q^{2k+\alpha+2}/z^2; q)_\infty} \\ & \times {}_8W_7(\nu^2 q^{2k+\alpha+1}/z^2; \nu q^{k+1} e^{i\varphi}/z, \nu q^{k+1} e^{-i\varphi}/z, \\ & \quad \nu q^{k+\frac{\alpha+1}{2}} e^{i\psi_2}/z, \nu q^{k+\frac{\alpha+1}{2}} e^{-i\psi_2}/z, q^{\alpha+1}; q, q^\epsilon) \\ &= \frac{2\pi (q^{\alpha+1}; q)_\infty}{(q, q, q^\alpha; q)_\infty h(s; q^{\frac{\alpha+1}{2}} e^{i\varphi}, q^{\frac{\alpha+1}{2}} e^{-i\varphi})} \frac{|(\nu q^{k+1} e^{i\varphi}/z; q)_\infty|^2 h(s; \frac{\nu}{z} q^{k+\frac{\alpha+1}{2}})}{|(\nu q^k e^{i\varphi}/z; q)_\infty|^2 h(s; \frac{\nu}{z} q^{k-\frac{\alpha-1}{2}})}. \end{aligned} \quad (4.4.9)$$

by Abel's formula (2.1.10). Hence

$$\begin{aligned} & \int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z} q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z} q^{k-\frac{\alpha}{2}})} \\ &= C \frac{2\pi (q^{\alpha+1}; q)_\infty |(\nu q^{k+1} e^{i\varphi}/z; q)_\infty|^2}{(q, q, q^\alpha; q)_\infty |(\nu q^k e^{i\varphi}/z; q)_\infty|^2} \\ & \times \int_{-1}^1 ds w(s; a\nu q^{\frac{\alpha-1}{2}}, b\nu q^{\frac{\alpha-1}{2}}, c\nu^{-1} q^{\frac{\alpha-1}{2}}, d\nu^{-1} q^{\frac{\alpha-1}{2}}) \frac{h(s; \frac{\nu}{z} q^{k+\frac{\alpha+1}{2}})}{h(s; \frac{\nu}{z} q^{k-\frac{\alpha-1}{2}})}. \end{aligned} \quad (4.4.10)$$

Again by [11, (6.3.8)]

$$\begin{aligned} & \int_{-1}^1 ds w(s; a\nu q^{\frac{\alpha-1}{2}}, b\nu q^{\frac{\alpha-1}{2}}, c\nu^{-1} q^{\frac{\alpha-1}{2}}, d\nu^{-1} q^{\frac{\alpha-1}{2}}) \frac{h(s; \frac{\nu}{z} q^{k+\frac{\alpha+1}{2}})}{h(s; \frac{\nu}{z} q^{k-\frac{\alpha-1}{2}})} \\ &= \frac{2\pi (abcdq^{2\alpha-2}, b\nu^2 q^{k+\alpha}/z, cq^{k+\alpha}/z; q)_\infty}{(q, bcq^{\alpha-1}, bdq^{\alpha-1}, ab\nu^2 q^{\alpha-1}, cd\nu^{-2} q^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}; q)_\infty} \end{aligned} \quad (4.4.11)$$

$$\begin{aligned} & \times \frac{(dq^{k+\alpha}/z, bcdq^{k+\alpha-1}/z; q)_\infty}{(bcdq^{k+2\alpha-1}/z, b\nu^2q^k/z, cq^k/z, dq^k/z; q)_\infty} \\ & \times {}_8W_7(bcdq^{k+2\alpha-2}/z; bcq^{\alpha-1}, bdq^{\alpha-1}, cd\nu^{-2}q^{\alpha-1}, q^{k+1}/az, q^\alpha: q, a\nu^2q^k/z). \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1 - \nu^2 q^{2k}/z^2}{1 - \nu^2/z^2} \frac{(\nu^2/z^2, \nu^2, q/az, q/bz; q)_k}{(q, q/z^2, a\nu^2/z, b\nu^2/z; q)_k} (abq^\alpha)^k \int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z} q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z} q^{k-\frac{\alpha}{2}})} \\ & = C \frac{4\pi^2(q^{\alpha+1}, abcdq^{\alpha-2}, b\nu^2q^\alpha/z, cq^\alpha/z, dq^\alpha/z, a\nu^2q^\alpha/z, \nu^2q/z^2; q)_\infty}{(q, q, q, q^\alpha, bcq^{\alpha-1}, bdq^{\alpha-1}, ab\nu^2q^{\alpha-1}, cd\nu^{-2}q^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, \nu^2q^{\alpha+1}/z^2; q)_\infty} \\ & \times \frac{1}{(a\nu^2/z, b\nu^2/z, c/z, d/z; q)_\infty} \frac{|(\nu q e^{i\varphi}/z; q)_\infty|^2}{|(\nu e^{i\varphi}/z; q)_\infty|^2} \sum_{k=0}^{\infty} \frac{1 - \nu^2 q^{2k}/z^2}{1 - \nu^2/z^2} \frac{(\nu^2/z^2, \nu^2; q)_k}{(q, q/z^2; q)_k} \\ & \times \frac{(c/z, d/z; q)_k}{(\nu^2q/cz, \nu^2q/dz; q)_k} \frac{|(\nu e^{i\varphi}/z; q)_k|^2}{|(\nu q e^{i\varphi}/z; q)_k|^2} (abq^\alpha)^k \sum_{l=0}^{\infty} \frac{1 - \nu^2 q^{2k+2l+\alpha}/z^2}{1 - \nu^2 q^\alpha/z^2} \\ & \times \frac{(\nu^2 q^\alpha/z^2; q)_{2k+l} (q^\alpha; q)_l}{(\nu^2 q/z^2; q)_{2k+l} (q; q)_l} \frac{(q/az, q/bz, \nu^2q/cz, \nu^2q/dz; q)_{k+l}}{(a\nu^2q^\alpha/z, b\nu^2q^\alpha/z, cq^\alpha/z, dq^\alpha/z; q)_{k+l}} (abcdq^{\alpha-2})^l. \end{aligned} \quad (4.4.12)$$

Let $k+l = m$, substituting the value of C and changing the order of summation we finally obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1 - \nu^2 q^{2k}/z^2}{1 - \nu^2/z^2} \frac{(\nu^2/z^2, \nu^2, q/az, q/bz; q)_k}{(q, q/z^2, a\nu^2/z, b\nu^2/z; q)_k} (abq^\alpha)^k \int_{-1}^1 dt K(y; t) \frac{h(t; \frac{\nu}{z} q^{k+1+\frac{\alpha}{2}})}{h(t; \frac{\nu}{z} q^{k-\frac{\alpha}{2}})} \quad (4.4.13) \\ & = \frac{(abcdq^{\alpha-2}; q)_\infty}{(abcdq^{2\alpha-2}; q)_\infty} \frac{|(\nu q e^{i\varphi}/z; q)_\infty|^2}{|(\nu e^{i\varphi}/z; q)_\infty|^2} \frac{(a\nu^2q^\alpha/z, b\nu^2q^\alpha/z, cq^\alpha/z, dq^\alpha/z, \nu^2q/z^2; q)_\infty}{(a\nu^2/z, b\nu^2/z, c/z, d/z, \nu^2q^{\alpha+1}/z^2; q)_\infty} S(z), \end{aligned}$$

where

$$\begin{aligned} & S(z^{-1}) \quad (4.4.14) \\ & = \sum_{m=0}^{\infty} \frac{1 - \nu^2 q^{\alpha+2m}/z^2}{1 - \nu^2 q^\alpha/z^2} \frac{(\nu^2 q^\alpha/z^2, q/az, q/bz, \nu^2q/cz, \nu^2q/dz, q^\alpha; q)_m}{(q, a\nu^2q^\alpha/z, b\nu^2q^\alpha/z, cq^\alpha/z, dq^\alpha/z, \nu^2q/z^2; q)_m} (abcdq^{\alpha-2})^m \\ & \quad \times {}_{10}W_9(\nu^2/z^2; \nu^2, c/z, d/z, \nu^2q^{\alpha+m}/z^2, \nu e^{i\varphi}/z, \nu e^{-i\varphi}/z, q^{-m}; q, \frac{q^{3-\alpha}}{cd}). \end{aligned}$$

Then from (4.4.4) and (4.4.13), we get

$$\int_{-1}^1 dt K(y; t) \frac{|(a\nu q^{\frac{\alpha}{2}} e^{i\psi_1}, b\nu q^{-\frac{\alpha}{2}} e^{i\psi_1}; q)_\infty|^2}{|(\nu z q^{-\frac{\alpha}{2}} e^{i\psi_1}, \nu/z q^{-\frac{\alpha}{2}} e^{i\psi_1}; q)_\infty|^2} \quad (4.4.15)$$

$$\begin{aligned}
& \times {}_8W_7(b\nu^2 q^{-\alpha-1}/z; ab\nu^2/q, q^{-\alpha}, b/z, \nu q^{-\frac{\alpha}{2}} e^{i\psi_1}/z, \nu z q^{-\frac{\alpha}{2}} e^{i\psi_1}; q, zq/a) \\
& = \frac{(b\nu^2 q^{-\alpha}/z, bz, azq^\alpha, a\nu^2/z; q)_\infty}{(\nu^2 q^{-\alpha}, z^2, q^{\alpha+1}, q\nu^2/z^2; q)_\infty} S(z^{-1}) \\
& \quad - \frac{(b\nu^2 q^{-\alpha}/z, azq^\alpha, a\nu^2 z, b/z, q/az, aq^{\alpha+1}/z, zq^{-\alpha}/a, b\nu^2 z, qz^2; q)_\infty}{(azq^{\alpha+1}, q^{-\alpha}/az, q^{\alpha+1}, zq/a, z^2, q/z^2, q\nu^2 z^2, \nu^2 q^{-\alpha}, b\nu^2/z; q)_\infty} S(z).
\end{aligned}$$

4.5 Final step

In this section we will complete the deduction of formula (4.1.10). In view of (4.2.13), (4.3.10), (4.3.11), (4.3.12) and (4.4.15),

$$\begin{aligned}
& K^\nu(x, y; a, b, c, d) \tag{4.5.1} \\
& = \frac{1}{1 - abcdq^{2\alpha-1} (q^{\alpha+1}, ab\nu^2 q^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{2\alpha-2}, abcdq^{\alpha-1}; q)_\infty} \\
& \quad \left[\frac{|(\nu q e^{i\varphi}/z; q)_\infty|^2 (az, a\nu^2 q^\alpha/z, b\nu^2 q^\alpha/z, cq^\alpha/z, dq^\alpha/z; q)_\infty}{|(\nu e^{i\varphi}/z; q)_\infty|^2 (\nu^2 q^{\alpha+1}/z^2; q)_\infty} \left\{ (abcdq^{2\alpha-2}; q)_\infty s'_{\alpha-1}(z) \right. \right. \\
& \quad \times \frac{(bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}; q)_\infty}{(bcdzq^{2\alpha-1}, z^2; q)_\infty} + z^2 \frac{(bc, bd, cd, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, q^\alpha; q)_\infty}{(a/b, bq/a, abq^\alpha, q^{1-\alpha}/ab, q^{-\alpha}, q^{\alpha+1}; q)_\infty} \\
& \quad \times \frac{(bcdq^{\alpha-1}/z, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq/a; q)_\infty}{(bcd/z, azq^{\alpha-1}, c/z, d/z; q)_\infty} r'_{\alpha-1}(z) \\
& \quad \times \left(\frac{(zq^{1-\alpha}/a, aq^\alpha/z, azq^{\alpha+1}, q^{-\alpha}/az, q/z^2; q)_\infty}{(aq^{\alpha+1}/z, zq^{-\alpha}/a, 1/z^2, qz^2; q)_\infty} + \frac{(azq^\alpha, q^{1-\alpha}/az; q)_\infty}{(z^2; q)_\infty} \right) \Big\} S(z^{-1}) \\
& \quad - \frac{|(\nu q z e^{i\varphi}; q)_\infty|^2 (a\nu^2 z q^\alpha, b\nu^2 z q^\alpha, czq^\alpha, dzq^\alpha; q)_\infty}{|(\nu z e^{i\varphi}; q)_\infty|^2 (\nu^2 z^2 q^{\alpha+1}; q)_\infty} \left\{ (abcdq^{2\alpha-2}; q)_\infty s'_{\alpha-1}(z) \right. \\
& \quad \times \frac{(az, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}, b/z, c/z, d/z, q/az, aq^{\alpha+1}/z, zq^{-\alpha}/a, qz^2; q)_\infty}{(bz, cz, dz, bcdzq^{2\alpha-1}, azq^{\alpha+1}, q^{-\alpha}/az, zq/a, z^2, q/z^2; q)_\infty} \\
& \quad + z^2 \frac{(bc, bd, cd, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, q^\alpha, bcdq^{\alpha-1}/z; q)_\infty}{(a/b, bq/a, abq^\alpha, q^{1-\alpha}/ab, q^{-\alpha}, q^{\alpha+1}, bcd/z, azq^{\alpha-1}, cz, dz; q)_\infty} r'_{\alpha-1}(z) \\
& \quad \times \left(\frac{(az, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq^{1-\alpha}/a, aq^\alpha/z, q/az, b/z; q)_\infty}{(bz, 1/z^2; q)_\infty} \right. \\
& \quad \left. \left. + \frac{(zq/a, azq^\alpha, q^{1-\alpha}/az, a/z, q/bz, bq^{\alpha+1}/z, zq^{-\alpha}/a, qz^2; q)_\infty}{(z^2, q/z^2; q)_\infty} \right) \right\} S(z) \Big].
\end{aligned}$$

We now notice that

$$\begin{aligned}
& \frac{(zq^{1-\alpha}/a, aq^\alpha/z, azq^{\alpha+1}, q^{-\alpha}/az, q/z^2; q)_\infty}{(qz^2, 1/z^2, aq^{\alpha+1}/z, zq^{-\alpha}/a; q)_\infty} + \frac{(azq^\alpha, q^{1-\alpha}/az; q)_\infty}{(z^2; q)_\infty} \quad (4.5.2) \\
&= \frac{(azq^{\alpha+1}, q^{1-\alpha}/az; q)_\infty}{(z^2q; q)_\infty} \left[\frac{(1-azq^\alpha)}{1-z^2} + \frac{(1-aq^\alpha/z)(1-q^{-\alpha}/az)}{(1-zq^{-\alpha}/a)(1-1/z^2)} \right] \\
&= \frac{(azq^{\alpha+1}, q^{1-\alpha}/az; q)_\infty}{(z^2q; q)_\infty} \left[\frac{(1-azq^\alpha)}{1-z^2} - \frac{(1-azq^\alpha)}{1-z^2} \right] \\
&= 0.
\end{aligned}$$

By the use of the formula [11, (2.11.1)] on $s'_{\alpha-1}(z)$, we further see that

$$\begin{aligned}
& \frac{(az, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}, b/z, c/z, d/z, q/az, aq^{\alpha+1}/z, zq^{-\alpha}/a, qz^2; q)_\infty}{(bz, cz, dz, bcdzq^{2\alpha-1}, azq^{\alpha+1}, q^{-\alpha}/az, zq/a, z^2, q/z^2; q)_\infty} \\
& \times (abcdq^{2\alpha-2}; q)_\infty s'_{\alpha-1}(z) \\
&= -(abcdq^{2\alpha-2}; q)_\infty \frac{(a/z, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, bcdq^{\alpha-1}/z; q)_\infty}{(bcdq^{2\alpha-1}/z, 1/z^2; q)_\infty} s'_{\alpha-1}(z^{-1}) \\
& - \frac{(bc, bd, cd, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, q^\alpha, bcdq^{\alpha-1}/z, qz^2; q)_\infty}{(bcd/z, azq^{\alpha-1}, bz, cz, dz; q)_\infty} r'_{\alpha-1}(z). \quad (4.5.3)
\end{aligned}$$

Then (4.5.1), (4.5.2) and (4.5.3), we have

$$\begin{aligned}
& K^\nu(x, y; a, b, c, d) \quad (4.5.4) \\
&= \frac{(abcdq^{\alpha-2}; q)_\infty}{(abcdq^{2\alpha-2}; q)_\infty} \frac{(abcdq^{2\alpha-1}, abcdq^{2\alpha}; q)_\infty}{(q^{\alpha+1}, ab\nu^2q^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, acbdq^{\alpha-1}; q)_\infty} \\
& \left[\left\{ \frac{|\nu q e^{i\varphi}/z; q)_\infty|^2}{|\nu e^{i\varphi}/z; q)_\infty|^2} s'_{\alpha-1}(z) S(z^{-1}) \right. \right. \\
& \times \frac{(az, a\nu^2q^\alpha/z, b\nu^2q^\alpha/z, cq^\alpha/z, dq^\alpha/z, bzq^\alpha, czq^\alpha, dzq^\alpha, bcdzq^{\alpha-1}; q)_\infty}{(\nu^2q^{\alpha+1}/z^2, bcdzq^{2\alpha-1}, z^2; q)_\infty} \\
& + \frac{|\nu q z e^{i\varphi}; q)_\infty|^2}{|\nu z e^{i\varphi}; q)_\infty|^2} s'_{\alpha-1}(z^{-1}) S(z) \\
& \times \frac{(a/z, a\nu^2zq^\alpha, b\nu^2zq^\alpha, czq^\alpha, dzq^\alpha, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, bcdq^{\alpha-1}/z; q)_\infty}{(\nu^2z^2q^{\alpha+1}, bcdq^{2\alpha-1}/z, 1/z^2; q)_\infty} \Bigg\} \\
& - z^2 \frac{(bc, bd, cd, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, q^\alpha, bcdq^{\alpha-1}/z; q)_\infty}{(1/z^2, abcdq^{2\alpha-2}, a/b, bq/a, abq^\alpha, q^{1-\alpha}/ab, q^{-\alpha}, q^{\alpha+1}, bcd/z, azq^{\alpha-1}; q)_\infty} \\
& \times r'_{\alpha-1}(z) \Big\{ (az, qz/b, bzq^{\alpha+1}, q^{-\alpha}/bz, zq^{1-\alpha}/a, aq^\alpha/z, q/az, b/z; q)_\infty
\end{aligned}$$

$$+ \frac{1 - 1/z^2}{1 - z^2} (azq^\alpha, q^{1-\alpha}/az, a/z, q/bz, bq^{\alpha+1}/z, zq^{-\alpha}/b, zq/a, bz; q)_\infty \\ - (qz^2, abq^\alpha, q^{1-\alpha}/ab, a/b, bq/a, q^{-\alpha}, q^{\alpha+1}, 1/z^2; q)_\infty \Big\} .$$

The symmetry with respect to z and z^{-1} in the first part of (4.5.4) suggests that the second part is also either symmetric with respect to z and z^{-1} or else is zero. We shall see that it is the latter case which is true. In exercise (2.16) of [11], let

$$x = aq^{\frac{\alpha}{2}}, \quad \lambda = zq^{-\frac{\alpha}{2}}, \quad \mu = z^{-1}q^{-\frac{\alpha}{2}}, \quad \nu = bq^{\frac{\alpha}{2}}.$$

Then

$$\begin{aligned} & \left\{ (az, aq^\alpha/z, b/z, q^{-\alpha}/bz, q/az, qz/b, bzq^{\alpha+1}, zq^{1-\alpha}/a; q)_\infty \right. \\ & \quad \left. - (qz^2, abq^\alpha, q^{1-\alpha}/ab, a/b, bq/a, q^{-\alpha}, q^{\alpha+1}, 1/z^2; q)_\infty \right\} \\ & = z^{-2} (a/z, q/bz, bq^{\alpha+1}/z, zq^{-\alpha}/b, q^{1-\alpha}/az, azq^\alpha, zq/a, bz; q)_\infty \end{aligned} \quad (4.5.5)$$

and hence the second part of (4.5.4) is zero. Thus, finally, we have

$$\begin{aligned} & K^\nu(x, y; a, b, c, d) \\ & = \frac{(abcdq^{\alpha-2}; q)_\infty}{(abcdq^{2\alpha-2}; q)_\infty} \frac{(abcdq^{2\alpha-1}, abcdq^{2\alpha}; q)_\infty}{(ab\nu^2q^\alpha, acq^\alpha, adq^\alpha, bcq^\alpha, bdq^\alpha, cdq^\alpha, abcdq^{\alpha-1}, q^{\alpha+1}; q)_\infty} \\ & \quad \times \frac{(a/z, a\nu^2zq^\alpha, b\nu^2zq^\alpha, czq^\alpha, dzq^\alpha, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, bcdq^{\alpha-1}/z; q)_\infty}{(\nu^2z^2q^{\alpha+1}, bcdq^{2\alpha-1}/z, 1/z^2; q)_\infty} \\ & \quad \times \frac{|(\nu qze^{i\varphi}; q)_\infty|^2}{|(\nu ze^{i\varphi}; q)_\infty|^2} s_{\alpha-1}(z^{-1}) S(z) \\ & \quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (4.5.6)$$

4.6 Limit of $K^\nu(x, y; a, b, c, d)$ as $\nu \rightarrow 1^-$

In this section we will compute

$$\lim_{\nu \rightarrow 1^-} K^\nu(x, y; a, b, c, d).$$

First observe that from the Bailey transformation formula [11. (2.10.1)]

$$\lim_{\nu \rightarrow 1^-} S(z) = \frac{(az, bcdzq^{\alpha-1}, q^{\alpha+1}/z^2, abcdq^{2\alpha-2}, bcdq^{\alpha-1}/z; q)_{\infty}}{(a\nu^2 zq^{\alpha}, bcdzq^{2\alpha-1}, abcdq^{\alpha-2}, qz^2; q)_{\infty}} s'_{\alpha-1}(z) \quad (4.6.1)$$

and hence from (4.5.6) and (4.6.1)

$$\lim_{\nu \rightarrow 1^-} K^{\nu}(x, y; a, b, c, d) \quad (4.6.2)$$

$$\begin{aligned} &= \frac{(abcdq^{2\alpha-1}, abcdq^{2\alpha}; q)_{\infty}}{(q^{\alpha+1}, abq^{\alpha}, acq^{\alpha}, adq^{\alpha}, bcq^{\alpha}, bdq^{\alpha}, cdq^{\alpha}, abcdq^{\alpha-1}; q)_{\infty}} \\ &\quad \frac{(az, bzq^{\alpha}, czq^{\alpha}, dzq^{\alpha}, a/z, bq^{\alpha}/z, cq^{\alpha}/z, dq^{\alpha}/z, bcdzq^{\alpha-1}, bcdq^{\alpha-1}/z; q)_{\infty}}{(z^2, bcdzq^{2\alpha-1}, 1/z^2, bcdq^{2\alpha-1}/z; q)_{\infty}} \\ &\quad \times s'_{\alpha-1}(z) s'_{\alpha-1}(z^{-1}) \\ &\quad \times \lim_{\nu \rightarrow 1^-} \left\{ \frac{(q\nu e^{i\varphi}/z, q\nu e^{-i\varphi}/z, 1/z^2; q)_{\infty}}{(\nu e^{i\varphi}/z, \nu e^{-i\varphi}/z, q/z^2; q)_{\infty}} + \frac{(q\nu z e^{i\varphi}, q\nu z e^{-i\varphi}, z^2; q)_{\infty}}{(\nu z e^{i\varphi}, \nu z e^{-i\varphi}, qz^2; q)_{\infty}} \right\}. \end{aligned} \quad (4.6.3)$$

Case I $z \neq e^{i\varphi}$:

$$\begin{aligned} &\lim_{\nu \rightarrow 1^-} \left\{ \frac{(q\nu e^{i\varphi}/z, q\nu e^{-i\varphi}/z, 1/z^2; q)_{\infty}}{(\nu e^{i\varphi}/z, \nu e^{-i\varphi}/z, q/z^2; q)_{\infty}} + \frac{(q\nu z e^{i\varphi}, q\nu z e^{-i\varphi}, z^2; q)_{\infty}}{(\nu z e^{i\varphi}, \nu z e^{-i\varphi}, qz^2; q)_{\infty}} \right\} \\ &= \frac{(1 - z^{-2})}{(1 - e^{i\varphi}/z)(1 - e^{-i\varphi}/z)} + \frac{(1 - z^2)}{(1 - ze^{i\varphi})(1 - ze^{-i\varphi})} \\ &= 0. \end{aligned} \quad (4.6.4)$$

Case II $z = e^{i\varphi}$:

$$\begin{aligned} &\lim_{\nu \rightarrow 1^-} \left\{ \frac{(q\nu e^{i\varphi}/z, q\nu e^{-i\varphi}/z, 1/z^2; q)_{\infty}}{(\nu e^{i\varphi}/z, \nu e^{-i\varphi}/z, q/z^2; q)_{\infty}} + \frac{(q\nu z e^{i\varphi}, q\nu z e^{-i\varphi}, z^2; q)_{\infty}}{(\nu z e^{i\varphi}, \nu z e^{-i\varphi}, qz^2; q)_{\infty}} \right\} \\ &= \lim_{\nu \rightarrow 1^-} \left\{ \frac{1}{(1 - \nu e^{i(\theta-\varphi)})} + \frac{1}{(1 - \nu e^{i(-\theta+\varphi)})} \right\} \\ &= \lim_{\nu \rightarrow 1^-} \left\{ \frac{1}{u + it} + \frac{1}{u - it} \right\}. \end{aligned}$$

where $u = 1 - \nu \cos(\theta - \varphi)$ and $t = \nu \sin(\varphi - \theta)$.

Borrowing the following result from the theory of distribution, [44, p. 67],

$$\lim_{u \rightarrow 0^+} \frac{1}{u \pm it} = \pi \delta(t) \mp i P \frac{1}{t}, \quad (4.6.5)$$

where $\delta(t)$ is Dirac's delta function and Pv is the Cauchy principal value, we have

$$\lim_{u \rightarrow 0^+} \left\{ \frac{1}{u + it} + \frac{1}{u - it} \right\} = 2\pi\delta(t). \quad (4.6.6)$$

But $u \rightarrow 0^+$ when $\varphi = \theta$ and $\nu \rightarrow 1^-$. Then from (4.6.2) and (4.6.5)

$$\lim_{\nu \rightarrow 1^-} K^\nu(x, y; a, b, c, d) = \frac{\delta(\theta - \varphi)}{\mu'(x; a, b, c, d)}. \quad (4.6.7)$$

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