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Matroid 3-connectivity

by

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A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

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May, 1981

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Abstract

In this thesis, we study the 3-connectivity of matroids and derive related algorithms.

Two algorithms are presented: one for finding a non-trivial 3-connected minor of a matroid, the other for decomposing a matroid into 3-connected components.

We also obtain several theoretical results on 3-connectivity. Given a non-trivial 3-connected matroid N , not isomorphic to a wheel or a whirl, we have a characterization of those 3-connected matroids having a minor isomorphic to N . We also prove a similar characterization theorem under the weaker assumption that N is 2-connected, simple, and cosimple. We then show that the latter theorem is essentially equivalent to Seymour's "Splitter Theorem".

In addition, we have one theorem that strengthens Tutte's "Wheels and Whirls" Theorem, and another that generalizes it. The generalized Tutte's theorem is shown to be equivalent to Seymour's Splitter Theorem, which provides a connection between these two results.

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Chapter 1

Introduction and Conventions

1.1 Introduction

A matroid is an abstract structure over a finite set subject to certain axioms; it was originally introduced by H. Whitney [22]. In this introduction, a matroid can be temporarily thought of as a graph or a matrix over a field. Although not all matroids arise in these ways, these two examples give rise to two important classes of matroids. The connectivity properties of matroids form an important part of matroid theory; this concept was introduced by W.T. Tutte [17] based upon the corresponding idea for graphs. Let us explain some of the ideas and definitions by using the above two examples.

Let $G = (V(G), E(G))$ be a connected graph, where $V(G)$ is the set of vertices of G and $E(G)$ the set of edges of G . Let k be a positive integer. A k -separation of G is a pair $\{G_1, G_2\}$ of subgraphs of G such that $\{E(G_1), E(G_2)\}$ is a partition of $E(G)$, $|E(G_1)|, |E(G_2)| \geq k$ and $|V(G_1) \cap V(G_2)| \leq k$. The connectivity $\lambda(G)$ of G is the least integer k , if such exists, such that G has a k -separation. If there is no such integer k , we write

$\lambda(G) = \infty$. We say that G is n -connected, where n is a positive integer, if $n \leq \lambda(G)$. Thus a graph is 1-connected, if and only if it is connected. A vertex v of G is called a cut vertex, if removing v separates G into two or more parts. It is not difficult to show that a graph G has a unique decomposition into 2-connected "components" as illustrated by the following example, where figure (i) is the graph G and figure (ii) is the 2-connected components of G .

There is a very efficient algorithm for decomposing a graph into its 2-connected components. (See Aho, Hopcroft, Ullman [1].) Also the structure of 2-connected graphs is elementary and well-understood. Therefore, the next stage is to study the properties of 3-connected graphs or to decompose a 2-connected graph into 3-connected "components". These are precisely the goals of this thesis. (But we will consider matroids, not just graphs.)

As another important example of matroids, let us consider the case of matrices. Let A be a matrix over a field, and k be a positive integer. A k -separation of A is a pair $\{E_1, E_2\}$, such

that $\{E_1, E_2\}$ is a partition of the column-vectors of A , $|E_1|$, $|E_2| \geq k$, and $r(E_1) + r(E_2) \leq r(A) + k - 1$, where $r(E_1)$, $r(E_2)$, and $r(A)$ are the ranks of E_1 , E_2 , and A (as matrices) respectively.

A matrix A is n -connected, for n a positive integer, if A has no k -separation, for $1 \leq k < n$. A matrix A is in standard form, if it has a full-rank identity submatrix; that is, by permuting the columns, A has the form (I, A') . (Given any matrix A , we can perform elementary row operations, and then delete all the zero rows, such that the resulting matrix B is in standard form. From elementary linear algebra, we know that the corresponding columns of A and B possess the same linear dependence properties, and hence the connectivity properties of A are preserved in B . This is one of the reasons that we assume matrix A is in standard form in the following.) Let A be a matrix in standard form. A 1-separation $\{E_1, E_2\}$ of A has a very simple structure; by permuting the columns, A has the following "block diagonal" form:

$$\begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \left[\begin{array}{cc|cc} \hline & & & \\ A_1 & & 0 & \\ \hline 0 & & & A_2 \\ & & & \end{array} \right] \end{array}$$

A matrix in standard form has a unique such expression (unique up

to the permutation of these blocks), such that every block matrix has no 1-separation, and hence is 2-connected. These block matrices are called 2-connected components of A . This notion of 1-separation of a matrix is useful; for example, solving a system of linear equations $AX = b$, where A has 1-separations, the problem can be reduced to solving several smaller size problems. The existence of an efficient algorithm to express a matrix A (in standard form) as its (unique) block form is quite clear. This fact, just as in the case of graphs, leads us to the problem of decomposing a 2-connected matrix into 3-connected "components".

Now we explain the idea of decomposing a 2-connected matroid into its 3-connected "components", by considering the graph example. Let G be a 2-connected graph, and let $\{G_1, G_2\}$ be a 2-separation of G . Hence $V(G_1) \cap V(G_2) = \{u, v\}$ and $|E(G_i)| \geq 2$, $i = 1, 2$. Let e be an element not in $E(G)$. For $i = 1, 2$ we form a new graph G'_i by adding e to $E(G_i)$ as an edge joining u and v . Then $\{G'_1, G'_2\}$ is called the simple decomposition of G associated with the 2-separation $\{G_1, G_2\}$ and the marker e . The following diagram illustrates the above concept.

If we repeat the above decomposing process in each of the smaller graphs, then eventually we will obtain a collection of graphs, such that every one of them has no 2-separation, and hence is 3-connected. A decomposition of G consisting of 3-connected graphs is called a prime decomposition. (Although we have not defined rigorously what a decomposition of G is, it is intuitively obvious what we mean here.) Unfortunately, a 2-connected graph G may have several different prime decompositions, as can be seen from the following example.

However, for any 2-connected graph G , there is a unique decomposition in the following sense. Take a prime decomposition of G , and, whenever two graph pieces G_1, G_2 of the decomposition share a common marker edge, and they are both polygons (connected graphs in which

each vertex has degree 2) or bonds (connected graphs having 2 vertices and no loop), combine G_i and G_j to get a bigger polygon or bond. (That is, we reverse the operation of decomposing.) This process is repeated until we get a decomposition of G from the initial prime decomposition such that no two polygons or bonds share a marker edge. Then this last decomposition of G is uniquely determined, and is called the standard decomposition of G . The members (graphs) of the standard decomposition of G are called 3-connected components of G ; we note that they are 3-connected, polygons, or bonds. Cunningham and Edmonds [6] generalized the above decompositions idea to matroids (in fact, to an even more general system, called a set-family), and proved a similar uniqueness theorem of decomposition. One main result of this thesis is an algorithm for constructing this unique decomposition of a matroid. It should be pointed out that, for the special case of graphs, Hopcroft and Tarjan [8] have given an extremely efficient algorithm for dividing a graph into 3-connected components.

Why do we want to study 3-connectivity and to look for efficient algorithms to decompose a matroid into 3-connected components?

From a theoretical point of view, there are a number of important results about 3-connection; (See [2], [5], [6], [13], [17], and [21].),

three of which appear in this thesis under numbers (6.4.2), (6.5.4), and (6.6.2) due to Whitney, Tutte, and Seymour respectively. An algorithm for decomposing a graph into 3-connected components is useful for analyzing electrical circuits, for determining whether a graph is planar, and for determining whether two planar graphs are isomorphic. (Hopcroft and Tarjan [8] .) Such an algorithm in the matrix case may be used to decide whether a matrix is totally unimodular (See Seymour [13]), which is an important subject in integer linear programming. As for the general matroid case, to test a matroid for a certain property, it is often possible to test each of its 3-connected components for that property.

There are two kinds of results in this thesis; the first kind includes two recursive algorithms concerning 3-connectivity of matroids, and the second kind includes some theoretical results on 3-connectivity. Let us introduce them in turn as follows.

First, the algorithmic results. An algorithm for decomposing a matroid into 3-connected components is presented. For the special case of a matrix over the binary field (respectively real field), the computational work of the algorithm is bounded by the order of r^2c (respectively $r^2c + rc \log c$), where r, c are the numbers of the rows and columns of the matrix respectively.

Now, we explain one essential idea behind this algorithm.

Obviously, to get such an algorithm we have to be able to check whether a matroid is 3-connected and if not, to find a 2-separation of it. A recursive way to approach a given problem of testing a matroid for a certain property was introduced by Tutte [14, 15, 18]; in the following, we describe it by using the graph example. First, we need some definitions. Let $G = (V, E)$ be a graph; an edge e of G is said to be contracted from G , if it is deleted and its two end-vertices are identified. (By deleting an edge e , we simply mean to remove that edge without removing any vertex.) A collection A of edges is contracted (respectively deleted) from G , if each edge in A is contracted (respectively deleted) successively from G . If G is connected, an edge-cutset is a collection of edges of G whose deletion results in a disconnected graph. Let Y be a minimal (by set inclusion) edge-cutset of G ; we define the bridges of Y in G to be the 2-connected components, with at least one edge in it, of the resulting graph after deleting Y from G . If there is more than one such bridge, Y is said to be separating. A graph, obtained from G by contracting all the bridges of Y except one, (that is, contracting all the edges of those bridges) is called a Y -component of G . Now, if a certain property holds

in each of the Y-components and the Y-components are pasted together nicely, then the method suggests that the property holds in G . These ideas (contracting, deleting, Y, bridges, and Y-components) can be generalized to matroids and used to attack matroid problems. Using this recursive method, Tutte [14], [18], gave a characterization of graphic matroids (matroids arising from graphs), and an algorithm for recognizing them. By a similar approach, several other results [8], [9], [3], and [4] were obtained. (However, the first two were independent of Tutte's work.) In [3], Bixby and Cunningham obtained a recursive characterization of matroid 3-connectivity; it is this and Tutte's recursive approach which led us to our algorithm results.

We first apply the above ideas to get an algorithm for finding a 3-connected minor of a matroid. (A minor of a graph G is a graph obtained from G by successively deleting and/or contracting edges; a minor of a matroid is defined somewhat similarly as a generalization.) As applications, the algorithm is used to prove a theorem concerning K_4 minors (K_4 is the complete graph on 4 vertices), and to check if a matroid is a so-called "series-parallel" matroid. Then we use the same recursive approach to derive an algorithm for decomposing a matroid into 3-connected components.

Now, concerning the second kind of results, we first describe some previous results on characterization of 3-connectivity which

motivate our work. A wheel graph W_n of order n , where n is an integer ≥ 3 , is constructed from a polygon with n edges by adding one new vertex and n new edges joining the new vertex to the n vertices of the polygon. Tutte [16] proved that given any 3-connected graph G with at least 4 edges, if we successively delete and/or contract edges from G while at each stage the 3-connected property is preserved, until no more such operations can be performed, then the final resulting graph must be a wheel graph W_n , for some $n \geq 3$. We remark that a wheel graph is 3-connected, and the deletion or contraction of any edge of it destroys the 3-connected property. Tutte [17] also proved a matroid generalization of this result, which is known as the "wheels and whirls" theorem. Recently, Negami [11] proved an analogue of Tutte's theorem (in the graph case), that given two 3-connected graphs G and K , where K is not a wheel and has at least 4 edges, then G has a minor isomorphic to K if and only if we can successively delete and/or contract edges from G , while at each stage the 3-connected property is preserved, to get a minor isomorphic to K .

In this thesis, we successfully generalize Negami's result to matroids; our theorem is stated as follows: Given two 3-connected matroids M and K , where K is not a "wheel" or a "whirl" and has at least 4 elements, then M has a minor isomorphic to K if

and only if we can successively delete and/or contract elements from M , while at each stage the 3-connected property is preserved, to get a minor isomorphic to K . We further make a weaker assumption on K , that is, instead of being 3-connected, K is assumed to be 2-connected, "simple" and "cosimple"; we obtain a similar characterization theorem. We prove that the latter theorem is essentially equivalent to Seymour's theorem on "splitters" [13]; just to mention one importance of Seymour's theorem: it has been used to give a good characterization of totally unimodular matrices.

While proving the above results, we come across a proof of Tutte's wheels and whirls theorem; the interesting thing about our proof is that we get two new theorems, one strengthens Tutte's theorem and another one generalizes it. More interestingly, we show that the generalized Tutte's theorem is equivalent to Seymour's theorem on splitters; this provides a connection between these two results.

The contents of this thesis are as follows: In chapter 2, we develop the basic matroid theory which is needed. In chapter 3, we introduce the tools required in the recursive approach, and prove a theorem of Bixby and Cunningham which applies the recursive method and provides a characterization of matroid 3-connectivity. Chapter 4

presents an algorithm for finding a 3-connected minor of a matroid, and its applications. Chapter 5 derives an algorithm for decomposing a matroid into 3-connected components. In chapter 6, we give the theoretical results on 3-connection.

Chapter 6 essentially depends only on chapter 2. (Although a few results in chapter 4, results (4.1.6), (4.1.9), and (4.3.5), are used in chapter 6.)

1.2 Sets and general notations

It is assumed that the reader is familiar with basic set operations and conventional notations: $\cup$: union, $\cap$: intersection, $\setminus$: set difference, and $\in$: set membership. Some less standard notations: $\subseteq$: subset, $\subset$: proper subset, for a set A and an element x , $A+x$ means $A \cup \{x\}$, and $A-x$ means $A \setminus \{x\}$.

For two sets A and B , we say that A meets B if $A \cap B \neq \emptyset$. A partition of a set E is a family $\{A_i: i \in I\}$, such that $E = \bigcup (A_i: i \in I)$ and for $i \neq j$, $A_i \cap A_j = \emptyset$. Partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ of E are said to cross if A_i meets B_j , for $i, j = 1, 2$. Given a specified universal set E , for $A, B \subseteq E$, $\bar{A}$ denotes $E \setminus A$; A cuts B means that $A \cap B \neq \emptyset$ and $\bar{A} \cap B \neq \emptyset$.

The symbol Δ will be used to indicate that a proof has ended or, when inserted after the statement of a result, that no proof is required.

1.3 Principles of computation

The reader should have some familiarity with the principle of computation - including the concepts of an algorithm and the estimation of its complexity. A precise description of this may be found in Aho, Hopcroft and Ullman [1].

In this thesis, the complexity of an algorithm is expressed in terms of elementary computational steps. A step is elementary if the effort required to perform it is bounded by a constant, which does not depend on the particular problem being solved. For example, adding or comparing two numbers is an elementary step, and determining whether an edge of a graph meets a particular vertex of the graph is another. If an algorithm processes inputs of size n in no more than $k \cdot f(n)$ steps, for some constant k , then the algorithm is said to be $O(f(n))$, read "order of $f(n)$ ".

Chapter 2

Matroids

In this chapter, we develop the basic matroid theory which we need in this thesis. Some references for this subject are Whitney [22], Tutte [15], and Welsh [20].

2.1 Matroid axioms

(2.1.1) A Matroid $M = (E, C)$ is a finite set $E = E(M)$ together with a collection $C = C(M)$ of non-empty subsets of E called circuits such that

(C1) $X, Y \in C$ and $X \subseteq Y$ imply $X = Y$;

(C2) $X, Y \in C$, $X \neq Y$, and $a \in X \cap Y$ imply that there exists $Z \in C$ such that $Z \subseteq (X \cup Y) - \{a\}$.

Matroid M is said to be a matroid on E . The elements of E are called cells of M . A matroid on E is non-null if $E \neq \emptyset$. A subset of E is called independent if it does not contain any circuit; otherwise it is called dependent. Clearly a minimal dependent set is a circuit. For a set $A \subseteq E$, a basis of A is a maximal independent subset of A .

(2.1.2) Proposition. If I is independent in matroid M then for $e \in E(M)$, $I + e$ contains at most one circuit.

Proof. Suppose that I is an independent set such that there exists $x \in E(M)$ with two distinct circuits C_1, C_2 satisfying $C_1 \cup C_2 \subseteq I + x$. Then $x \in C_1 \cap C_2$ and hence by (C2) there exists a circuit C_3 of M , such that $C_3 \subseteq (C_1 \cup C_2) - x \subseteq I$, contradicting the independence of I . Δ

The following theorem is a characterization of the class of independent sets of a matroid.

(2.1.3) Theorem . . A collection $\mathcal{J}$ of subsets of E is the class of independent sets of a matroid on E if and only if conditions (I1), (I2), and (I3) are satisfied.

(I1) $\emptyset \in \mathcal{J}$

(I2) If $A \subseteq B \in \mathcal{J}$, then $A \in \mathcal{J}$

(I3) For every $A \subseteq E$, all bases of A have the same cardinality.

Proof. Suppose that $M = (E, C)$ is a matroid. It is clear that the set $\mathcal{J}$ of independent sets satisfies the properties (I1) and (I2).

Suppose that (I3) is not true. Then there exists $A \subseteq E$, and two

bases J_1, J_2 of A with $|J_1| > |J_2|$. Subject to this, choose

J_1, J_2 so that $|J_1 \cap J_2|$ is maximum. There exists $x \in J_2 \setminus J_1$.

Since $J_1 + x \notin \mathcal{J}$, there exists, by (2.1.2), a unique circuit C of

C such that $C \subseteq J_1 + x$. Moreover, since J_2 is independent,

$C \not\subseteq J_2$, and so there exists $y \in C \setminus J_2$. Then $J_1 + x - y$ is independent;

this contradicts the choice of J_2 . Thus (I3) is satisfied.

Now suppose that $(E, \mathcal{J})$ satisfies (I1), (I2), and (I3). Let C be the class of all non-empty subsets of E which are not members of $\mathcal{J}$ and are minimal with respect to this property. Clearly (E, C) satisfies property (C1). If (E, C) is not a matroid; there exist two members $C_1 \neq C_2$ of C and $a \in C_1 \cap C_2$ such that $(C_1 \cup C_2) - a$ does not contain any member of C . Then $(C_1 \cup C_2) - a \in \mathcal{J}$. Since $C_1 \neq C_2$, and by the definitions of C , there exists $b \in C_1 \setminus C_2$, and $C_1 - b \in \mathcal{J}$. By (I3), we can extend $C_1 - b$ to a basis B of $C_1 \cup C_2$. $C_1 - b \subseteq B$, so $b \notin B$. And there is at least one element of $C_2 \setminus C_1$ not in B , for otherwise $C_2 \subseteq B$, and by (I2), $C_2 \in \mathcal{J}$, a contradiction. Hence $|B| \leq |C_1 \cup C_2| - 2$. But $(C_1 \cup C_2) - a \in \mathcal{J}$, by (I3), a basis of $C_1 \cup C_2$ has cardinality at least $|C_1 \cup C_2| - 1$, this is a contradiction. Thus (C2) is satisfied, and (E, C) is a matroid. Δ

Hence the cardinality of a basis of A , where $A \subseteq E$, is a function r of A . We call $r(A)$ the rank of A .

Throughout this thesis, the letter M always stands for a matroid on E with rank function r . Furthermore, M_x or M^y stands for a matroid whose rank function is r_x or r^y , where x (respectively y) is a subscript (respectively a superscript). Where it is necessary for clarity, we may prefix matroid terms with the name of the relevant matroid; for example; "M-circuit". An M-basis of E

is called a basis of M. We call $r(E)$ the rank of M, written $r(M)$. Two matroids $M_1 = (E_1, C_1)$ and $M_2 = (E_2, C_2)$ are isomorphic, if there is a one-to-one function f from E_1 onto E_2 such that C is an M_1 -circuit if and only if $f(C)$ is an M_2 -circuit, and we write $M_1 \cong M_2$. It is clear that any of the rank function of M , the basis family of M , or the family of independent sets of M , together with E , determines M .

We give two examples of matroids here; one arises from graphs and the other from matrices.

A graph $G = (V, E)$ consists of a finite set $V \cong V(G)$ of vertices, a finite set $E = E(G)$ of edges, and a relation which associates with any edge two vertices called its ends. An edge is said to be incident to its ends. A path of G is a sequence $w = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$, $n \geq 1$, such that $V(w) = \{v_0, v_1, \dots, v_n\}$ is a subset of V , $E(w) = \{e_1, e_2, \dots, e_n\}$ is a subset of E , for $1 \leq i \leq n$, the ends of e_i are v_{i-1} and v_i , and $v_i \neq v_j$ if $i \neq j$, for $0 \leq i, j \leq n$. A polygon of G is a path of G except $v_0 = v_n$.

(2.1.4) Theorem. Let $G = (V, E)$ be a graph, and let C be the class of all subsets C of E such that C is the edge-set of a polygon. Then (E, C) is a matroid. (We call this matroid the

polygon matroid of G , and use $PM(G)$ to denote it).

Proof. It follows from the definition of polygons that the edge set of a polygon is non-empty and (C1) holds for $M(G)$. Let C_i be the edge set of a polygon, $i = 1, 2$ with $C_1 \neq C_2$ and $a \in C_1 \cap C_2$. We observe that the polygon with edge set C_1 contains a path L_1 which has only its end vertices v_0, v_n in common with the polygon of edge set C_2 . v_0 and v_n are joined by a path L_2 in C_2 such that a is not an edge of L_2 . Combining L_1 and L_2 we obtain a polygon whose edge set is contained in $(C_1 \cup C_2) - a$. Hence (C2) holds for $PM(G)$, and so it is a matroid. Δ

A matroid isomorphic to the polygon matroid of some graph is said to be graphic.

Our second example of matroid comes from matrices

(2.1.5) Let $A = (a_{ij} : i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ be a matrix over a field F . Let $E = \{1, 2, 3, \dots, n\}$. And let C be the class of all $J \subseteq E$ such that the columns of the submatrix $(a_{ij} : i = 1, 2, \dots, m; j \in J)$ form a minimal linearly dependent set (hence this submatrix has rank $|J| - 1$). Then it follows from elementary linear algebra that (E, C) is a matroid. We call this matroid the matric matroid of A . A binary matroid is a matric matroid of a matrix over the binary field.

(2.1.6) A well known result is that the polygon matroid of a graph G is isomorphic to the matroid of the incidence matrix of G . (The incidence matrix of a graph $G = (V, E)$ is a binary matrix $A = (a_{ij} : i \in V, j \in E)$ such that $a_{ij} = 1$ if edge j is incident to vertex i , and $a_{ij} = 0$, otherwise).

We now prove that a much stronger statement than (C2) can be made about the circuits of a matroid.

(2.1.7) Proposition (Strong circuit axiom) If C_1, C_2 are distinct circuits of a matroid M and $x \in C_1 \cap C_2$, then for any element y of $C_1 \setminus C_2$, there exists a circuit C such that $y \in C \subseteq (C_1 \cup C_2) \setminus x$.

Proof. Suppose that C_1, C_2, x, y are chosen such that the proposition is false, and that $|C_1 \cup C_2|$ is minimal with this property. By (C2) there exists a circuit C_3 with $C_3 \subseteq (C_1 \cup C_2) \setminus x$ but $y \notin C_3$. $C_3 \cap (C_2 \setminus C_1)$ cannot be null, or C_3 would be contained in C_1 . Let $z \in C_3 \cap (C_2 \setminus C_1)$. Consider C_2 and C_3 ; $z \in C_2 \cap C_3$, $x \in C_2 \setminus C_3$ and $C_2 \cup C_3$ in a proper subset of $C_1 \cup C_2$, since $y \notin C_2 \cup C_3$. By the minimality of $C_1 \cup C_2$, there exists a circuit C_4 such that $x \in C_4 \subseteq (C_2 \cup C_3) \setminus z$. Now consider C_1 and C_4 ; $x \in C_1 \cap C_4$, $y \notin C_2 \cup C_3$ and hence $y \in C_1 \setminus C_4$. Also $C_1 \cup C_4 \subseteq C_1 \cup C_2$. Thus by the minimality argument again, there exists a circuit C_5 such that

$y \in C_5 \subseteq (C_1 \cup C_4) \setminus x$. Since $C_1 \cup C_4 \subseteq C_1 \cup C_2$ we have found a circuit C_5 with $y \in C_5 \subseteq (C_1 \cup C_2) \setminus x$, a contradiction. Δ

We derive two basic properties of matroid rank functions.

(2.1.8) Proposition. The rank function of a matroid M is

submodular: $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ for $A, B \subseteq E$.

Proof. Let J_1 be an M -basis of $A \cap B$. Extend J_1 to an M -basis J_2 of A ; extend J_2 to an M -basis J_3 of $A \cup B$. Then $J_4 = J_3 \cap B$ is independent by (2.1.3). So $r(B) \geq |J_4|$. Now $r(A) + r(B) = r(A \cup B) - r(A \cap B) \geq |J_2| + |J_4| - |J_3| - |J_1| = 0$. Δ

(2.1.9) Proposition. The function f defined by $f(A, B) = r(A) + r(B) - r(A \cup B)$ for $A, B \subseteq E$ is non-negative and increasing in A, B .

Proof. $r(A) + r(B) - r(A \cup B) \geq$ (by 2.1.8) $r(A \cap B) \geq 0$, so f is non-negative.

Now $r(A \cup X) + r(B) - r(A \cup X \cup B) - (r(A) + r(B) - r(A \cup B)) = r(A \cup X) + r(A \cup B) - r(A) - r(A \cup X \cup B) \geq r(A \cup X) + r(A \cup B) - r((A \cup X) \cap (A \cup B)) - r(A \cup X \cup B) \geq 0$, by (2.1.8). Δ

2.2 Deletion and Contraction

In this section, we introduce two basic operations "deletion" and "contraction" on matroids which correspond in a natural way to the deletion and contraction of edges in a graph.

Let $G = (V, E)$ be a graph and let $A \subseteq E$ be a set of edges of G . We define $G \setminus A = (V, E \setminus A)$ to be a graph with vertex set V , edge set $E \setminus A$ and for each $e \in E \setminus A$, u, v are the ends of e in $G \setminus A$ if and only if they are the ends of e as in G . We say that $G \setminus A$ is obtained by deleting edges A from G . The following proposition is obvious.

(2.2.1) Proposition. The polygons of $G \setminus A$ are the polygons of G whose edge sets are contained in $E \setminus A$.

Let $G = (V, E)$ be a graph. An edge e of G is said to be contracted if it is deleted and its ends are identified. Let $A \subseteq E$ be a set of edges of G . Suppose each edge in A is contracted successively from G ; the resulting graph is denoted by G/A . We say that G/A is obtained by contracting edges A from G . It is not difficult to see that the order in which the edges are contracted does not affect the final result G/A .

(2.2.2) Proposition. Let $G = (V, E)$ be a graph, and let $A \subseteq E$. A non-empty subset S of $E \setminus A$ is the edge set of a polygon in G/A if and only if S is a minimal non-empty set of the form $C \cap (E \setminus A)$, where C is the edge set of a polygon in graph G .

Proof. We need two claims. Claim 1. If C is the edge set of a polygon in graph G and $C \cap (E \setminus A) \neq \emptyset$, then $C \cap (E \setminus A)$ contains a

polygon of G/A . This is obvious. Claim 2. If $\emptyset \neq S \subseteq E \setminus A$ and S is the edge set of a polygon in graph G/A , then S is of the form $C \cap (E \setminus A)$, where C is the edge set of a polygon in graph G . We prove Claim 2 by induction on the number $|A|$ of edges being contracted from G . For $|A| = 1$, Claim 2 is clearly true. Now suppose that the result is true whenever the number of edges being contracted from G is fewer than n , where $n \geq 2$ and suppose that n edges $e_1, e_2, \dots, e_n$ are contracted from G . Let $A = \{e_1, \dots, e_n\}$, $A' = \{e_1, \dots, e_{n-1}\}$, and S be the edge set of a polygon in $G/A = (G/A')/e_n$. Then by the induction hypothesis, there exists C' , where C' is the edge set of a polygon in G/A' , such that $S = C' \cap (E \setminus A' \setminus \{e_n\})$. Again by the induction hypothesis, there exists C , where C is the edge set of a polygon in G , such that $C' = C \cap (E \setminus A')$. Thus $S = C' \cap (E \setminus A) = C \cap (E \setminus A') \cap (E \setminus A) = C \cap (E \setminus A)$. This completes the induction proof. Now the Proposition follows from claims 1 and 2. Δ

We now define the operations "deletion" and "contraction" on matroids, which generalize the corresponding operations on graphs.

Let $M = (E, C)$ be a matroid, and let $A \subseteq E$. Let $C \setminus A = \{C : \bar{A} \supseteq C \in C\}$. It is easy to see that $M \setminus A = (\bar{A}, C \setminus A)$ is a matroid, called the restriction of M to $\bar{A}$. We say that $M \setminus A$ is

obtained by deleting A from M. If $e \in E$, we may denote $M \setminus \{e\}$ by $M \setminus e$. The next theorem shows that the deletion operation on matroids is indeed a generalization of the same operation on graphs.

(2.2.3) Theorem. Let $G = (V, E)$ be a graph, and let $A \subseteq E$. Then $PM(G) \setminus A = PM(G \setminus A)$.

Proof. This is obvious, by (2.2.1) and the definition of deletions. Δ

The following proposition follows immediately from the definition of deletion.

(2.2.4) Proposition. Let $M = (E, C)$ be a matroid with rank function r , and let $A \subseteq E$. Then I is an independent set of $M \setminus A$ if and only if I is an independent set of M and $I \subseteq \bar{A}$. Moreover, the rank function r' of $M \setminus A$ is given by $r'(X) = r(X)$ for $X \subseteq \bar{A}$. Δ

We now define another operation "contraction" on matroids.

Let $M = (E, C)$ be a matroid, and let $A \subseteq E$. Let $C/A = \{S: S \text{ is a minimal non-empty set of the form } C \cap \bar{A}, \text{ where } C \text{ is an M-circuit}\}$.

(2.2.5) Theorem. $(\bar{A}, C/A)$ is a matroid.

Proof. By definition of C/A , it is clear that $S \neq \emptyset$, for each

$S \in C/A$ and (C1) is satisfied. Let $C_1 \cap \bar{A} \in C/A$, $i = 1, 2$, and

let $x \in (C_1 \cap \bar{A}) \cap (C_2 \cap \bar{A})$. Then $(C_1 \cap \bar{A}) \cup (C_2 \cap \bar{A}) - x \subseteq ((C_1 \cup C_2) - x) \cap \bar{A}$.

Since $C_1, C_2 \in \mathcal{C}$ and M is a matroid by (C2), there is an M -circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) - x$. Thus $(C_1 \cap \bar{A}) \cup (C_2 \cap \bar{A}) - x \supseteq C_3 \cap \bar{A}$, and so there is a minimal non-empty set S of the form $C \cap \bar{A}$ such that $(C_1 \cap \bar{A}) \cup (C_2 \cap \bar{A}) - x \supseteq S$, where C is an M -circuit. Therefore (C2) holds and $(\bar{A}, C/A)$ is a matroid. Δ

We call the matroid $(\bar{A}, C/A)$ the contraction of M to $\bar{A}$, and use the notation M/A to denote it. We say M/A is obtained by contracting A from M . Where $e \in E$, we may denote $M/\{e\}$ by M/e .

(2.2.6) Theorem. Let $G = (V, E)$ be a graph and let $A \subseteq E$. Then $PM(G)/A = PM(G/A)$.

Proof. By (2.2.2) and the definition of contraction, this follows immediately. Δ

(2.2.7) Proposition. The rank function r' of M/A is given by $r'(X) = r(X \cup A) - r(A)$ for $X \subseteq \bar{A}$.

Proof. Let I be an (M/A) -basis of X . Then I is M -independent, for otherwise I contains an M -circuit C , and so $I \supseteq C \cap \bar{A}$, contradicting the independence of I in matroid M/A . Extend I to an M -basis $I \cup J$ of $I \cup A$. We will show that J is an M -basis of A , and $I \cup J$ is an M -basis of $X \cup A$; then it follows that $r'(X) = |I| = |I \cup J| - |J| = r(X \cup A) - r(A)$. Suppose that J

is not an M -basis of A . Then there is $x \in A \setminus J$ such that $J+x$ is M -independent but $(I \cup J)+x$ is not. Thus there is an M -circuit C contained in $(I \cup J)+x$ and $C \cap I \neq \emptyset$. So $I \supseteq C \cap \bar{A} \neq \emptyset$, contradicting the independence of I in matroid M/A . Hence J is an M -basis of A . Now suppose that $I \cup J$ is not an M -basis of $X \cup A$. Then there is $y \in X \setminus I$ such that $(I \cup J)+y$ is M -independent. Since I is an (M/A) -basis of X , $I+y$ is not (M/A) -independent, and so there is an M -circuit C with $I+y \supseteq C \cap \bar{A}$ and $y \in C$. We know that $(I \cup J)+y$ is M -independent and contained in $(I \cup J) \cup C$. Hence $r(I \cup J \cup C) \geq |I \cup J + y|$, and $C \neq \{y\}$. Thus, extending $C-y$ to an M -basis B of $I \cup J \cup C$, we have $|B| \geq |I \cup J + y|$. But $B \subseteq (I \cup J \cup C) - y \subseteq I \cup A$, contradicting the fact that $r(I \cup A) = |I \cup J|$. This completes the proof. Δ

A matroid M' obtained from M by deleting some cells of M and contracting others is called a minor of M .

2.3 Duality

The idea of dual matroid is due to Whitney. To define the dual of a matroid, it is probably easier to describe its independent sets.

(2.3.1) Theorem. Let M be a matroid on E , and let $\mathcal{J}^* = \{J \subseteq E: r(\bar{J}) = r(E)\}$. Then $\mathcal{J}^*$ is the class of independent sets of a matroid M^* on E . Moreover, its rank function r^* is

given by $r^*(A) = |A| + r(\bar{A}) - r(E)$, for every $A \subseteq E$.

We call M^* the dual of M .

Proof. Clearly $\emptyset \in \mathcal{B}^*$. If $I \subseteq J \in \mathcal{B}^*$, then $r(\bar{I}) = r(E)$. So the independent set axioms (I1) and (I2) are satisfied. Let $A \subseteq E$, and let J be an M^* -basis of A . Let I_1 be an M -basis of $\bar{A}$. Extend I_1 to an M -basis I_2 of $E \setminus J$. We will show that $I_2 \supseteq A \setminus J$. If not, there exists $x \in A \setminus J$ such that $x \notin I_2$. Then $r(E \setminus (J+x)) = r(E \setminus J) = r(E)$, so $J+x \in \mathcal{B}^*$ contradicting the maximality of J . Thus $I_2 = I_1 \cup \bar{J}$. So $|J| = |A| + |I_1| - |I_2| = |A| + r(\bar{A}) - r(E)$ independent of the particular M^* -basis of A which is chosen. Hence (I3) holds and so M^* is a matroid.

Furthermore, $r^*(A) = |J| = |A| + r(\bar{A}) - r(E)$. Δ

(2.3.2) The relation between M and M^* is symmetric, because the bases of M^* are the complements of the bases of M . Thus $(M^*)^* = M$.

A circuit of M^* is called a cocircuit of M .

The next theorem is a fundamental result relating contraction and deletion.

(2.3.3) Theorem. Let M be a matroid on E , and let $A \subseteq E$. Then (i) $(M \setminus A)^* = M^* / A$ and (ii) $(M / A)^* = M^* \setminus A$.

Proof. (i) Let r , r_1 , and r_2 be the rank functions of M , $M \setminus A$, and M^*/A respectively. Since the rank function determines the matroid, it is enough to show that $r_1^*(X) = r_2(X)$ for every $X \subseteq \bar{A}$.

$$\text{Now } r_1^*(X) = (\text{by (2.3.1)}) |X| + r_1(\bar{A} \setminus X) - r_1(\bar{A}).$$

$$\begin{aligned} \text{And } r_2(X) &= (\text{by (2.2.7)}) r^*(X \cup A) - r^*(A) \\ &= (\text{by (2.3.1)}) |X \cup A| + r(E \setminus (X \cup A)) - r(E) - |A| - r(E \setminus A) + r(E) \\ &= |X| + r(\bar{A} \setminus X) - r(\bar{A}). \end{aligned}$$

Since r_1 is r restricted to the subsets of $\bar{A}$, it is clear that $r_1^*(X) = r_2(X)$. Thus (i) is proved.

(ii) Putting M for M^* in (i) and taking duals gives (ii). Δ

(2.3.4) Corollary: The cocircuits of $M \setminus A$ are the minimal non-empty sets of the form $Y \cap \bar{A}$, where Y is an M -cocircuit. The cocircuits of M/A are the cocircuits of M which are contained in $\bar{A}$.

Proof. Since $(M \setminus A)^* = M^*/A$ and $(M \setminus A)^* = M^*/A$, this corollary follows trivially from the definition of contraction and deletion. Δ

(2.3.5) Proposition. Let C be an M -circuit, and let $A \subset C$. Then $C \setminus A$ is a circuit of M/A . Let Y be an M -cocircuit, and let $A \subset Y$. Then $Y \setminus A$ is a cocircuit of $M \setminus A$.

Proof. Suppose that $C \setminus A = C \cap \bar{A}$ is not a circuit of M/A . By definition of contraction, there is an M -circuit C_1 such that

$C_1 \cap \bar{A} \subset C \cap \bar{A}$. But then $C_1 \subset C$; this is a contradiction. Thus $C \setminus A$ is a circuit of M/A . And since $(M \setminus A)^* = M^*/A$, the second result follows. Δ

(2.3.6) Proposition. Deletions and contractions commute with each other and among themselves. More precisely, if A, B are disjoint subsets of E , then (i) $(M \setminus A) \setminus B = M \setminus (A \cup B)$; (ii) $(M/A)/B = M/(A \cup B)$; (iii) $(M \setminus A)/B = (M/B) \setminus A$.

Proof. The truth of (i) is obvious. By (2.3.3), we have

$$(M/A)/B = (M^* \setminus A)^*/B = (M^* \setminus A \setminus B)^* = (M^* \setminus (A \cup B))^* = M/(A \cup B),$$

proving (ii). It is easy to see that the rank function r' of either of the matroids in (iii) is given by, $r'(X) = r(X \cup A) - r(A)$, for $X \subseteq \overline{A \cup B}$. Δ

It follows from (2.3.6) that a minor of a minor of M is a minor of M .

In the remainder of this section, we derive some results on cocircuits which will be needed.

(2.3.7) Proposition. Let B be a basis of the matroid M . Then, for each $b \in B$, there exists a unique cocircuit $Y = Y(b, B)$ of M such that $b \in Y \subseteq (E \setminus B) + b$.

Proof. By (2.3.2), $E \setminus B$ is a basis of M^* , the dual of M . Hence for each $b \in B$, $E \setminus B + b$ contains a circuit Y of M^* , that is,

a cocircuit Y of M . Clearly $b \in Y \subseteq (E \setminus B) + b$. And the uniqueness of this cocircuit follows from (2.1.2). Δ

The unique cocircuit $Y(b, B)$ is called the fundamental cocircuit determined by b and basis B . Given a basis B of M , a B-fundamental cocircuit of M is a cocircuit having just one element from B .

(2.3.8) Proposition. Let M be a matroid on E . Then Y is a cocircuit of M if and only if Y is a minimal subset of E with $r(E \setminus Y) = r(E) - 1$.

Proof. Suppose that Y is a cocircuit of M . Then, by (2.3.1), where r^* is the rank function of M^* , $|Y| - 1 = r^*(Y) = |Y| + r(E \setminus Y) - r(E)$. Thus $r(E \setminus Y) = r(E) - 1$. And for any non-empty subset $A \subseteq Y$, $|Y \setminus A| = r^*(Y \setminus A) = |Y \setminus A| + r(E \setminus (Y \setminus A)) - r(E)$, so $r(E \setminus (Y \setminus A)) = r(E)$. Thus Y is a minimal subset of E with $r(E \setminus Y) = r(E) - 1$.

Conversely suppose that Y is a minimal subset of E with $r(E \setminus Y) = r(E) - 1$. Then $r^*(Y) = |Y| + r(E \setminus Y) - r(E) = |Y| - 1$, so Y is an M^* -dependent set. Let $Y' \subseteq Y$ be an M^* -circuit. Then by the first part of this proposition, we have $r(E \setminus Y') = r(E) - 1$. Therefore, by the minimality of Y , $Y = Y'$. Thus Y is a cocircuit of M . Δ

(2.3.9) Proposition. The intersection of an M -circuit C

and an M-cocircuit Y cannot be a single element. That is,

$$|C \cap Y| \neq 1.$$

Proof. Suppose that there exist an M-circuit C and an M-cocircuit Y such that $C \cap Y = \{e\}$. By (2.3.8), $r(E \setminus Y) = r(E) - 1$ but $r(E \setminus (Y - e)) = r(E)$. It is clear that $C - e$ is M-independent and contained in $E \setminus Y$. Extend $C - e$ to a basis B_1 of $E \setminus Y$, then extend B_1 to a basis B_2 of $E \setminus (Y - e)$. We see that $|B_1| = r(E \setminus Y) = r(E) - 1$, and $|B_2| = r(E \setminus (Y - e)) = r(E)$. Thus $e \in B_2$. But $B_2 \supseteq B_1 \supseteq C - e$ and so $B_2 \supseteq C$, contradicting the independence of B_2 . Δ

2.4 Separators

A separator of the matroid M is a subset A of E such that $r(A) + r(\bar{A}) = r(E)$.

(2.4.1) Proposition. For $A \subseteq E$, the following three conditions are equivalent:

- (1) A is a separator of M ;
- (2) A cuts no M-circuit;
- (3) $M/A = M \setminus A$.

Proof. "(1) implies (2)": Suppose that A is a separator of M but it cuts the circuit C of M . Then $0 = r(A) + r(\bar{A}) - r(E) \geq$
 (by 2.1.3) $r(C \cap A) + r(C \cap \bar{A}) - r(C) = |C \cap A| + |C \cap \bar{A}| - r(C) =$
 $|C| - r(C) > 0$, a contradiction.

"(2) implies (3)": Suppose A cuts no M -circuit, so neither does $\bar{A}$. Then a circuit of M has a non-null intersection with $\bar{A}$, if and only if it is itself a subset of $\bar{A}$. This implies $M/A = M \setminus A$.

"(3) implies (1)": Suppose $M/A = M \setminus A$; then $r'(\bar{A}) = r(\bar{A})$, where r' and r are the rank function of M/A and $M \setminus A$ respectively. Now $r'(\bar{A}) = r(\bar{A} \cup A) - r(A) = r(E) - r(A)$, so we have $r(A) + r(\bar{A}) = r(E)$. Thus A is a separator of M .

(2.4.2) Proposition. A is a separator of M if and only if A is a separator of M^* .

Proof. $r^*(A) + r^*(\bar{A}) - r^*(E) = (\text{by (2.3.1)}) |A| + r(\bar{A}) - r(E) + |\bar{A}| + r(A) - r(E) - (|E| - r(E)) = r(A) + r(\bar{A}) - r(E)$. The proposition follows. Δ

(2.4.3) Corollary. A is a separator of M if and only if A cuts no M -cocircuit. Δ

It follows from (2.4.1) that any union or intersection of separators is a separator; and the complement in E of a separator of M is also a separator. In particular, E and $\emptyset$ are separators of M . A minimal non-empty separator of M is called an elementary separator of M . A separator A of M is trivial if $A = \emptyset$ or E and non-trivial otherwise. A matroid is non-separable if it has

only trivial separators and is separable otherwise.

From the foregoing observations we have:

(2.4.5) The elementary separators of M partition E .

(2.4.6) Proposition. Let B be a separator of M . Then for each $A \subseteq E$, $B \cap \bar{A}$ is a separator of $M \setminus A$ and M/A .

Proof. Let C be a circuit of $M \setminus A$ or M/A . By definition of deletion and (2.3.4), there exists an M -circuit C' such that $C = C' \cap \bar{A}$. Since either $C' \subseteq B$ or $C' \subseteq E \setminus B$ we have $C \subseteq B \cap \bar{A}$ or $C \subseteq (E \setminus B) \cap \bar{A} = \bar{A} \setminus (B \cap \bar{A})$. Thus by (2.4.1), $B \cap \bar{A}$ is a separator of $M \setminus A$ and M/A . Δ

(2.4.7) Let B be an elementary separator of M . Let A be a subset of E disjoint from B . Then B is an elementary separator of $M \setminus A$ and M/A .

Proof. By (2.4.6), B is a separator of $M \setminus A$ and M/A . Suppose that B is not an elementary separator of $M \setminus A$ (respectively M/A). Let $B' \subseteq B$ be an elementary separator of $M \setminus A$ (respectively M/A), and let C be any M -circuit with $C \cap B' \neq \emptyset$. Since B is a separator of M and $A \cap B = \emptyset$, C is a circuit of $M \setminus A$ and M/A . Thus $C \subseteq B'$. This implies that B' is an elementary separator of M , a contradiction. Δ

(2.4.8) Proposition. Let B be an elementary separator of $(M \setminus A_1)/A_2$. Then B is contained in an elementary separator of M .

Proof. Let B_1 be an elementary separator of M with $B_1 \cap B \neq \emptyset$

By (2.4.6), $B_1 \setminus (A_1 \cup A_2)$ is a separator of $(M \setminus A_1)/A_2$. Now B is an elementary separator of $(M \setminus A_1)/A_2$ with $B \cap [B_1 \setminus (A_1 \cup A_2)] \neq \emptyset$; therefore $B \subseteq B_1 \setminus (A_1 \cup A_2)$, and so $B \subseteq B_1$. Thus $B = B_1$. Δ

Let M_i be a matroid on E_i , $i = 1$ and 2 . Suppose $E_1 \cap E_2 = \emptyset$.

We define a new matroid M on $E_1 \cup E_2$, by defining its circuit family $C(M) = C(M_1) \cup C(M_2)$, where $C(M_i)$ is the set of circuits of M_i , $i = 1$ and 2 . Clearly $(M, C(M))$ is a matroid; we call M the direct sum of M_1 and M_2 , and write $M = M_1 \oplus M_2$. It is easy to see that the direct sum operation is commutative and associative.

Let M be a matroid, and A be a non-trivial separator of M . By (2.4.1) and the definition of deletion, $M = (M \setminus A) \oplus (M \setminus \bar{A})$. By this (2.4.5) and (2.4.7) we have the following theorem.

(2.4.9) Theorem. Every non-null matroid has a unique expression as the direct sum of non-null non-separable matroids.

A loop of M is a cell e of M such that $\{e\}$ is an M -circuit; an isthmus is a cell e of M such that $\{e\}$ is an M -cocircuit. It is easy to see that $\{e\}$ is a separator of M if and only if e is a loop or an isthmus.

There is a simple and efficient algorithm for finding the elementary separators of a matroid. It is based on the following proposition.

(2.4.10) Proposition. Given any basis B of M , a set A is a separator of M if and only if every B -fundamental cocircuit is contained in A or in $\bar{A}$.

Proof. If A is a separator of M , then by (2.4.2) A is a separator of M^* . Hence by (2.4.1) A cuts no cocircuit of M . In particular, every B -fundamental cocircuit is contained in A or in $\bar{A}$.

Conversely, if every B -fundamental cocircuit is contained in A or in $\bar{A}$, then $B \cap A$ and $B \cap \bar{A}$ are maximal independent subsets of A and $\bar{A}$ respectively. So $r(A) + r(\bar{A}) = r(E)$, and A is a separator of M . Δ

(2.4.11). We now describe an algorithm for finding the elementary separators of a matroid M . Choose a basis B of M . Form a bipartite graph with one side of the vertices corresponding to the elements of B , and the other side of them corresponding to the elements of $E \setminus B$; two vertices b, x are joined by an edge, where $b \in B$ and $x \in E \setminus B$, if and only if the B -fundamental cocircuit determined by b contains x . By (2.4.10), the connected components of this bipartite graph correspond precisely to the elementary separators of M . Therefore, assuming the existence of efficient algorithm for constructing this bipartite graph, the amounts of

computation effort required to find the elementary separators of M is bounded by the order of $r(E)(|E| - r(E))$.

From now on, we shall deal with non-separable matroids in this thesis.

2.5 Connectivity.

In this section, we define n -connectivity for matroids; the ideas are due to Tutte. First we look at the special case of graphs. Where $G = (V, E)$ is a finite connected graph and $\{E_1, E_2\}$ is a partition of E , let G_i , $i = 1$ and 2 , denote the subgraph of G induced by E_i ; that is, $E(G_i) = E_i$ and G_i has no isolated vertex. Let k be a positive integer; $\{E_1, E_2\}$ is a k -separation of G if

- (1) $|V(G_1) \cap V(G_2)| \leq k$;
- (2) $|E_1| \geq k \leq |E_2|$.

where n is a positive integer, G is n -connected if for all k , $1 \leq k < n$, G has no k -separation.

Tutte introduced this notion of connectivity for graphs and then generalized the idea to n -connectivity for matroids. Let k be a positive integer; a partition $\{E_1, E_2\}$ of E is said to be a k -separation of the matroid M if

- (1) $r(E_1) + r(E_2) \leq r(E) + k - 1$;
- (2) $|E_1| \geq k \leq |E_2|$.

We can see that $\{A, \bar{A}\}$ is a 1-separation of M if and only if A is a non-trivial separator of M . Where n is a positive integer, matroid M is n -connected if for all k , $1 \leq k < n$, M has no k -separation. Tutte [17] proved a famous theorem which states that the polygon matroid of a connected graph G is n -connected if and only if G is n -connected. We observe that every matroid is 1-connected, and M is 2-connected if and only if it is non-separable. We now show some basic properties of n -connectivity.

(2.5.1) Proposition. Let M be a matroid on E , and let $\{E_1, E_2\}$ be a partition of E . Then $r(E_1) + r(E_2) - r(E) = r^*(E_1) + r^*(E_2) - r^*(E) = r(E_1) + r^*(E_1) - |E_1|$. (Hence a k -separation of M is a k -separation of M^* and conversely.)

Proof. $r^*(E_1) + r^*(E_2) - r^*(E) = (\text{by (2.3.1)}) |E_1| + r(E_2) - r(E) + |E_2| + r(E_1) - r(E) - |E| + r(E) = r(E_1) + r(E_2) - r(E)$. Moreover $r^*(E_1) + r^*(E_2) - r^*(E) = (\text{by (2.3.1)}) r^*(E_1) + |E_2| + r(E_1) - r(E) - |E| + r(E) = r(E_1) + r^*(E_1) - |E_1|$. Δ

(2.5.2) Corollary. Matroid M is n -connected if and only if M^* is n -connected. Δ

(2.5.3) Proposition. Let M be a matroid on E , $\{E_1, E_2\}$ be a partition of E , and S, T be disjoint subsets of E . If $r(E_1) + r(E_2) \leq r(E) + k - 1$, for some positive integer k , then

$r'(E_1 \setminus (S \cup T)) + r'(E_2 \setminus (S \cup T)) \leq r'(E \setminus (S \cup T)) + k - 1$, where r and r' are rank functions of M and $(M \setminus S)/T$ respectively.

Proof. It is sufficient to consider the case that one of S, T is the empty set.

If $T = \emptyset$, then $(M \setminus S)/T = M \setminus S$, and the rank function of $M \setminus S$ is r restricted to $E \setminus S$. $r(E_1 \setminus S) + r(E_2 \setminus S) - r(E \setminus S)$ (by (2.1.9)) $r(E_1) + r(E_2) - r(E) \leq k - 1$.

If $S = \emptyset$, then $(M \setminus S)/T = M/T$. Let r, r^*, r' , and r'^* be the rank function of $M, M^*, M/T$, and $(M/T)^*$ respectively. We note that $(M/T)^* = M^* \setminus T$; hence r'^* is just r^* restricted to $E \setminus T$. Now $r'(E_1 \setminus T) + r'(E_2 \setminus T) - r'(E \setminus T) =$ (by (2.5.1)) $r'^*(E_1 \setminus T) + r'^*(E_2 \setminus T) - r'^*(E \setminus T) = r^*(E_1 \setminus T) + r^*(E_2 \setminus T) - r^*(E \setminus T) \leq$ (by (2.1.9)). $r^*(E_1) + r^*(E_2) - r^*(E) =$ (2.5.1) $r(E_1) + r(E_2) - r(E) \leq k - 1. \Delta$

(2.5.4) Corollary. Let $\{E_1, E_2\}$ be a k -separation of M , and let S, T be disjoint subsets of E . If $|E_1 \setminus (S \cup T)| \geq k$, for $i = 1, 2$, then $\{E_1 \setminus (S \cup T), E_2 \setminus (S \cup T)\}$ is a k -separation of $(M \setminus S)/T. \Delta$

Chapter 3

Bridges and Y-Components

In this chapter, we introduce the "bridges" of a cocircuit Y and related properties; these ideas are due to Tutte [15]. We also prove a theorem of Bixby and Cunningham [3] which applies the concept of bridges and provides a characterization of matroid 3-connectivity. This theory forms the foundation of our results in chapters 4 and 5.

3.1 Definitions

Let M be a matroid on E . Two elements of E are in parallel (respectively in series), if they form a circuit (respectively cocircuit). Clearly the relation "in parallel" (respectively "in series") is transitive. A parallel set (respectively series set) of M is a subset of E , in which every pair of elements are in parallel (respectively in series). A parallel class (respectively series class) of M is a maximal parallel set (respectively maximal series set). A matroid is simple (respectively cosimple), if it has no one- or two- element circuit (respectively cocircuit).

(3.1.1) Proposition. Let Y be a cocircuit of M , P be a parallel class of M . Then either $P \subseteq Y$ or $P \cap Y = \emptyset$, and Y

contains no loop of M .

Proof. By proposition (2.3.9), the intersection of a cocircuit and a circuit cannot contain exactly one element. Δ

Let Y be a cocircuit of the matroid M on E . A bridge of Y in M is an elementary separator of $M \setminus Y$, the matroid obtained by deleting Y from M . A bridge is singleton, if it contains only one cell. If Y has more than one bridge, Y is a separating cocircuit; otherwise Y is non-separating. A Y -component of M is a matroid of the form $M/(E \setminus (B \cup Y))$, the contraction of M to $B \cup Y$, where B is a bridge of Y . Let $B_1, B_2, \dots, B_k$ be the bridges of Y , having corresponding Y -components $M_1, M_2, \dots, M_k$. A B_i -segment of Y is a parallel class of M_i contained in Y . By (3.1.1), a parallel class of M_i is either contained in Y or disjoint from it; therefore for a given bridge B_i of Y in M , the class of all B_i -segments is a partition of Y , we denote it by $\pi(M, B_i, Y)$, or simply by $\pi(B_i)$ if there is no ambiguity. Two bridges B_i and B_j are said to avoid each other, if there exists $S_i \in \pi(B_i)$ and $S_j \in \pi(B_j)$ such that $S_i \cup S_j = Y$, otherwise they are said to overlap.

We define a graph $G(Y)$, called the bridge graph of Y as follows: The vertices of $G(Y)$ are the bridges of Y with two

vertices joined by an edge if and only if the corresponding bridges overlap.

To illustrate the above concepts, we apply them to the polygon matroid and bond matroid of a non-separable graph G . (The bond matroid $BM(G)$ of a graph G is the dual of the polygon matroid $PM(G)$. The circuits of $BM(G)$ are the minimal edge-cutsets of G .)

(3.1.2) Let $PM(G)$ be the polygon matroid of a non-separable graph G . A cocircuit Y of $PM(G)$ is a minimal edge-cutset of G . By (2.2.3), $PM(G) \setminus Y = PM(G \setminus Y)$, the bridges of Y in $PM(G)$ are just the "blocks" of $G \setminus Y$, (A block of a graph G is a maximal 2-connected subgraph). A Y -component of $PM(G)$ is the polygon matroid of $G/(E \setminus (Y \cup B_1))$, where B_1 is a block of $G \setminus Y$. The elements are in parallel in $PM(G)$ if and only if they are parallel edges in graph G . Therefore the segment set $\pi(PM(G), B_1, Y)$ is a partition of Y into parallel edges in the graph $G/(E \setminus (Y \cup B_1))$.

(3.1.3) Let $BM(G)$ be the bond matroid of a non-separable graph G . A cocircuit Y of $BM(G)$ is the edge set of a polygon of G . By (2.3.3) and (2.2.6), $BM(G) \setminus Y = (PM(G))^* \setminus Y = (PM(G)/Y)^* = (PM(G/Y))^* = BM(G/Y)$. Thus the bridges of Y in $BM(G)$ are just the blocks of G/Y , the graph obtained by contracting Y from G . A Y -component of $BM(G)$ is the bond matroid of $G \setminus (E \setminus (Y \cup B_1))$;

where B_1 is a block of G/Y . The segment-set $\pi(BM(G), B_1, Y)$ is a partition of Y into paths, such that in the graph $G \setminus (E \setminus (Y \cup B_1))$ each end vertex of the paths has degree at least 3, and each inner vertex of the paths has degree exactly 2.

The reader may find it helpful to apply all the general theory in this thesis to these two particular cases.

3.2 Properties of bridges and segments

In this section, we prove some basic properties of segments and Y -components. The reference is Tutte [15].

(3.2.1) Proposition. Let A_1, A_2 be disjoint separators of M/Y and let $S, \emptyset \subset S \subset Y$, be such that the elements of S are in parallel in both $M/(E \setminus (A_1 \cup Y))$ and $M/(E \setminus (A_2 \cup Y))$. Then the elements of S are in parallel in $M/(E \setminus (A_1 \cup A_2 \cup Y))$.

Proof. Let a, b be distinct elements of S . Since a, b are in parallel in both $M/(E \setminus (A_1 \cup Y))$, $M/(E \setminus (A_2 \cup Y))$, by definition of contraction, these are circuits C_1, C_2 of M such that $C_1 \cap (A_1 \cup Y) = \{a, b\} = C_2 \cap (A_2 \cup Y)$. If $C_1 \cap A_2 = \emptyset$, then $C_1 \cap (A_1 \cup A_2 \cup Y) = \{a, b\}$. Hence, by (2.3.4) and (2.3.9), a, b are in parallel in $M/(E \setminus (A_1 \cup A_2 \cup Y))$, and we are done. If $C_1 \cap A_2 \neq \emptyset$, let $e \in C_1 \cap A_2$. By (2.1.7), there is a circuit C of M such that $e \in C \subseteq (C_1 \cup C_2) \setminus \{a\}$. But then $|C \cap Y| \leq 1$, so by (2.3.9)

$C \cap Y = \emptyset$. Since A_2 is a separator of $M \setminus Y$, by (2.4.1), this implies that $C \subseteq A_2$, whence $C \subseteq C_1$, a contradiction. The proof is complete. Δ

(3.2.2) Lemma. Let M_1 be a Y -component of M corresponding to the bridge B_1 . Let e, f be two distinct elements of B_1 . If e, f are in parallel in M_1 , then e, f are in parallel in M .

Proof. Since $M_1 \setminus Y = (M / (E \setminus (B_1 \cup Y))) \setminus Y = (M \setminus Y) \setminus (E \setminus (B_1 \cup Y)) = M \setminus (E \setminus B_1)$, it is easy to see, from the definition of deletion, that the lemma holds. Δ

It follows from (3.2.2), that if two elements are in parallel in M_1 but not in M , then both these two elements belong to Y .

(3.2.3) Theorem. Let Y be a cocircuit of the matroid M . Let P be a parallel class of M . Then either $P \subseteq Y$ or $P \cap Y = \emptyset$. Let $B_1, \dots, B_k$ be the bridges of Y , and $M_1, \dots, M_k$ be the corresponding Y -components of M . For $S \subseteq E(M)$, we have:

- (i) if $S \subseteq Y$, then S is a parallel class of M if and only if S is a parallel set in each M_i and $S = \bigcap_i (P_i(S) : P_i(S) \text{ is the parallel class of } M_i \text{ containing } S)$;
- (ii) if $S \cap Y = \emptyset$, then S is a parallel class of M if and only if $S \subseteq B_i$, for some i , and S is a parallel class of M_i .

Proof. Suppose P is a parallel class; by (3.1.1), $P \subseteq Y$ or $P \cap Y = \emptyset$.

To prove (i), it is enough to show that for two cells $a, b \in Y$, a and b are in parallel in M if and only if a and b are in parallel in every Y -component of M . But this is clearly true, by (3.2.1) and the fact that a Y -component is obtained from M by contracting all the bridges of Y except one.

To prove (ii), it is sufficient to show that for two cells $a, b \notin Y$, a and b are in parallel in M if and only if a and b belong to the same bridge B_i of Y with a, b in parallel in M_i . This follows from lemma (3.2.2) and the definition of Y -component. Δ

The next proposition is due to Bixby and Cunningham [3].

(3.2.4) Proposition. Let Y be a cocircuit of M . If C_1 and C_2 are complementary separators of $M \setminus Y$ and r' is the rank function of $M' = M/C_2$, then for any $A \subseteq C_1$

$$r'(A) + r'((C_1 \cup Y) \setminus A) - r'(C_1 \cup Y) = r(A) + r(E \setminus A) - r(E).$$

Proof. $r'(A) + r'((C_1 \cup Y) \setminus A) - r'(C_1 \cup Y)$
 $=$ (by (2.2.7)) $r(A \cup C_2) - r(C_2) + r(E \setminus A) - r(C_2) - (r(E) - r(C_2))$
 $= r(A \cup C_2) - r(C_2) + r(E \setminus A) - r(E)$
 $=$ (by (2.4.6)) $r(A) + r(C_2) - r(C_2) + r(E \setminus A) - r(E)$
 $= r(A) + r(E \setminus A) - r(E). \Delta$

It follows that, if $\{A_1, A_2\}$ is a k -separation of the Y -component M_1 with $A_1 \cap Y = \emptyset$, then $\{A_1, E(M) \setminus A_1\}$ is a k -separation of M . We shall use this important fact for the case of $k = 2$ later.

(3.2.5) Proposition. Let B be a bridge of Y in M , and A be a subset of E such that $(B \cup Y) \cap A = \emptyset$. Then Y is a cocircuit of M/A , B is a bridge of Y in M/A , and $\pi(M/A, B, Y) = \pi(M, B, Y)$.

Proof. By (2.3.4), Y is a cocircuit of M/A . Let $M' = M/A$. $M' \setminus Y = (M/A) \setminus Y =$ (by (2.3.6)) $(M \setminus Y)/A$. By (2.4.7) B is an elementary separator of the right hand side matroid, hence that of $M' \setminus Y$. So B is a bridge of Y in M/A . Now $M'/(E(M') \setminus (B \cup Y)) = (M/A)/((E \setminus A) \setminus (B \cup Y)) =$ (by (2.3.6)) $M/(E(M) \setminus (B \cup Y))$. By definition of segment, we have $\pi(M/A, B, Y) = \pi(M, B, Y)$. Δ

The next two results are consequences of (3.2.4) and (3.2.5).

(3.2.6) Corollary. Let β be a set of bridges of Y in M . Then the bridges of Y in $M/(\cup \{B : B \in \beta\})$ are precisely those bridges of Y in M that are not in β . In addition, $\pi(M/(\cup \{B : B \in \beta\}), B_1, Y) = \pi(M, B_1, Y)$ for all bridges $B_1 \notin \beta$. Finally, if M is non-separable then $M/(\cup \{B : B \in \beta\})$ is non-separable. Δ

(3.2.7) Corollary. Let Y be a cocircuit of M , then Y is a non-separating cocircuit of each Y -component. And if M is non-separable, then every Y -component is non-separable.

(3.2.8) Proposition. Let M be a non-separable matroid, and let Y be a cocircuit of M . Then $|\pi(M, B_i, Y)| \geq 2$, for each bridge B_i of Y . That is, every bridge has at least 2 segments.

Proof. Let B_i be an arbitrary bridge of Y , let M_i be the Y -component $M/(E \setminus (Y \cup B_i))$, and let r' be the rank function of M_i . By (3.2.6), $\pi(M, B_i, Y) = \pi(M_i, B_i, Y)$; hence it is enough to show that $r'(Y) \geq 2$. Since M is non-separable, by (3.2.7), every Y -component is also non-separable. So $r'(Y) + r'(B_i) \geq r'(B_i \cup Y) + 1$. And Y is a cocircuit of M_i we have $r'(B_i) = r'(B_i \cup Y) - 1$. Therefore, $r'(Y) \geq r'(B_i \cup Y) - r'(B_i) + 1 = 2$. Δ

(3.2.9) Proposition. Let B be a bridge of Y in M , and $A \subset Y$. Then $Y \setminus A$ is a cocircuit of $M \setminus A$, B is a bridge of $Y \setminus A$ in $M \setminus A$, and $\pi(M \setminus A, B, Y \setminus A)$ is the class of all non-empty intersections with $Y \setminus A$ of members of $\pi(M, B, Y)$.

Proof. By (2.3.5), $Y \setminus A$ is a cocircuit of $M \setminus A$. Let $M' = M \setminus A$ and $Y' = Y \setminus A$. $M' \setminus Y' = (M \setminus A) \setminus (Y \setminus A) =$ (by (2.3.6)) $M \setminus Y$. B is an elementary separator of $M \setminus Y$, hence that of $M' \setminus Y'$. So B is a bridge of $Y \setminus A$. Now the Y' -component of M' corresponding

to the bridge B is $M'/(E(M') \setminus (B \cup Y')) = (M \setminus A)/((E(M) \setminus A) \setminus (B \cup (Y \setminus A))) =$
 (by (2.3,6)) $(M/(E(M) \setminus (B \cup Y))) \setminus A$. The proposition follows from the
 definition of segment. Δ

The following result is due to Bixby and Cunningham [3].

(3.2.10) Proposition. Let B be a basis of the matroid M ,
 let Y be a B -fundamental cocircuit of M , and let M_1 be the Y -
 component of M corresponding to the bridge B_1 of Y . Then
 $B' = B \cap (B_1 \cup Y)$ is a basis of M_1 , and any B' -fundamental cocircuit
 Y' of M_1 is a B -fundamental cocircuit of M . Moreover if Y' is
 separating in M_1 , then Y' is separating in M .

Proof. Let Y be the B -fundamental cocircuit determined by b ,
 then $B \setminus Y = B \setminus \{b\}$ is independent in $M \setminus Y$. By (2.3.8), $r(E \setminus Y) =$
 $r(E) - 1$, so $B \setminus \{b\}$ is a basis of $E \setminus Y$, and thus $B \cap B_1$ is a
 basis of B_1 . Since Y is a cocircuit of M_1 , from (2.3.8),
 $B' = (B \cap B_1) \cup \{b\}$ is a basis of M_1 . Now by (2.3.4), Y' is a
 cocircuit of M , and it has just one element from B , so it is a
 B -fundamental cocircuit of M . Since Y, Y' are both B' -fundamental
 cocircuits of M_1 , from (2.3.4), it is easy to see that $Y \setminus Y'$ is
 a cocircuit of $M_1 \setminus Y'$. Therefore, if $\{A_1, A_2\}$ is a separator of
 $M_1 \setminus Y'$, we may assume that $A_2 \supseteq Y \setminus Y'$; moreover, by (2.4.6),
 $B_1 \setminus Y'$ is a separator of $M \setminus (Y \cup Y')$. Using (3.2.4) with $M \setminus Y'$

in place of M , we have where r_1 is the rank function of M_1 , $0 = r_1(A_1) + r_1(A_2) - r_1((B_1 \cup Y) \setminus Y') = r(A_1) + r((E \setminus Y') \setminus A_1) - r(E \setminus Y')$. Therefore, Y' is a separating cocircuit in M as desired. Δ

(3.2.11) Hence, in the context of (3.2.10), Y is a non-separating B' -fundamental cocircuit of M_1 for each Y -component M_1 . For any B -fundamental cocircuit Y' of M , $Y' \neq Y$, there exists exactly one Y -component M_1 of M such that Y' is a B' -fundamental cocircuit of M_1 , where M_1 is the Y -component of M corresponding to the bridge B_1 and $B' = B \cap (B_1 \cup Y)$.

(3.3) A characterization of 3-connectivity in terms of Y -components.

In this section, we prove a theorem of Bixby and Cunningham [3], which relates the 3-connectivity of a matroid to properties of its Y -components, for any cocircuit Y . From this theorem, they obtained an efficient algorithm for testing a matroid for 3-connectivity.

Let M be a matroid on E . Where $S \subseteq E$, we say that S is simple in M , if $M \setminus (E \setminus S)$ is simple. Where $S \subseteq E$, let D be a subset of S consisting of all the elements of S which are loops in M , and all the elements except only one of $P \cap S$, for each parallel class P of M ; consider the matroid $M[S]$ obtained by

deleting D from M , then $S \cap E(M[S])$ is simple in $M[S]$. Clearly, the matroid structure of $M[S]$ does not depend on the particular choice of such a D , that is, for two different such D_1, D_2 $M \setminus D_1$ is isomorphic to $M \setminus D_2$. We call $M[S]$ the S -simplification of M . We say that $M[S]$ has been obtained by simplifying S in M . If $S \not\subseteq E(M)$, to simplify S in M is just to simplify $S \cap E(M)$ in M .

(3.3.1) Proposition. Let $M[S]$ be the S -simplification of M . Let Y be a cocircuit of M . Then $Y \cap M[S]$ is a cocircuit of $M[S]$.

Proof. We may assume that either S contains only one element and S is a loop, or S contains more than one element and $S = P \cap S$ where P is a parallel class of M . In the former case the result is clear, since $a \notin Y$ by (2.4.1) and $M \setminus a = M/a$. In the latter case, suppose the result is not true; then there exists $A \subset Y \cap M[S]$ and $B \subset P \cap S$ such that $A \neq \emptyset$ and $A \cup B$ is a cocircuit of M by (2.3.4). Now suppose $B = \emptyset$; then cocircuit $A \cup B$ of M is strictly contained in Y , a contraction. Thus $B \neq \emptyset$. But again this is a contraction by (2.3.9), since P is a parallel class of M , $P \cap (A \cup B) \neq \emptyset$, and $P \not\subseteq (A \cup B)$. Δ

Let $M[S]$ be the S -simplification of M . For $T \subseteq E(M)$ we use the notation $\hat{T}$ to denote $T \cap E(M[S])$. For $T \subseteq E(M[S])$ we use the notation $(T)^{CE}$ to denote the maximal subset S of $E(M)$

with $\hat{S} = T$. (CE stands for the cross section expansion.) Let Y be a cocircuit of the matroid M . A simplified Y -component $M_1[Y]$ is the Y -simplification of a Y -component M_1 . Where M_1 is the Y -component corresponding to the bridge B_1 , we observe that the partition $\pi(B_1)$ is the set of all parallel classes of M_1 contained in Y , and the simplified Y -component $M_1[Y]$ is just the matroid obtained by deleting from M all the elements of S except one, for each $S \in \pi(B_1)$. Moreover, $\hat{Y}$ is a non-separating cocircuit of $M_1[Y]$.

The following proposition is obvious.

(3.3.2) Proposition. Let $M[S]$ be the S -simplification of M .

Let r and r_1 be the rank functions of M and $M[S]$ respectively.

Then $r_1(A) = r(A) = r((A)^{CE})$, for every $A \subseteq E(M[S])$ and

$r(B) = r_1(B)$ for every $B \subseteq E(M)$. Δ

(3.3.3) Proposition. If $\{E_1, E_2\}$ is a 2-separation of the

matroid M and the cocircuit Y of M meets both E_1 and E_2 ,

then $E_1 \setminus Y$ is a separator of both $M \setminus (E_1 \cap Y)$ and $M \setminus Y$. In ad-

dition, if Y is simple in M , then Y is separating.

Proof. First we prove that $r(E_1 \setminus Y) < r(E_1)$. Suppose this is not

true, that is, $r(E_1 \setminus Y) = r(E_1)$. Then in the matroid $M/(E_1 \setminus Y)$,

$Y \cap E_1$ is a set of loops, since $r'(Y \cap E_1) =$ (by (2.2.7)),

$r((Y \cap E_1) \cup (E_1 \setminus Y)) - r(E_1 \setminus Y) = r(E_1) - r(E_1 \setminus Y) = 0$, where r' is the rank function of $M/(E_1 \setminus Y)$. Also Y is a cocircuit of $M/(E_1 \setminus Y)$. But, by (2.3.9), this implies that $Y \cap E_1 = \emptyset$, a contradiction. Therefore $r(E_1 \setminus Y) < r(E_1)$. Then $r(E_1 \setminus Y) + r(E_2) \leq r(E_1) - 1 + r(E_2) = r(E) =$ (by (2.3.8)) $r(E \setminus (E_1 \cap Y))$; thus $E_1 \setminus Y$ is a separator of $M \setminus (E_1 \cap Y)$, and so, by (2.4.6), $E_1 \setminus Y$ is a separator of $M \setminus Y$. Now, if Y is non-separating, it must be that $Y \supseteq E_1$ (say). But then, since Y meets E_2 , by (2.3.8), we have $r(E_2) = r(E)$, so $r(E_1) = 1$. If Y is simple in M , this is not possible. Δ

(3.3.4) Proposition. Let M_1 be a Y -component of M . Let $M_1[Y]$ be the simplified Y -component corresponding to M_1 , and let $\{A_1, A_2\}$ be a 2-separation of $M_1[Y]$. Then either $Y \cap A_1 = \emptyset$ or $Y \cap A_2 = \emptyset$. Furthermore, if $Y \cap A_1 = \emptyset$, then $\{A_1, (A_2)^{CE}\} = \{A_1, E(M_1) \setminus A_1\}$ is a 2-separation of M_1 and $\{A_1, E(M) \setminus A_1\}$ is a 2-separation of M .

Proof. By (3.3.1), $\hat{Y}$ is a cocircuit of $M_1[Y]$. Obviously $\hat{Y}$ is non-separating and simple in $M_1[Y]$. Thus $\hat{Y}$ cannot meet both A_1 and A_2 , by (3.3.3). Hence $Y \cap A_1 = \emptyset$ or $Y \cap A_2 = \emptyset$, say $Y \cap A_1 = \emptyset$. Then clearly $(A_2)^{CE} = E(M_1)/A_1$, where r, r_1 and r_2 are the rank functions of $M, M_1, M_1[Y]$ respectively, we have:

$$\begin{aligned}
& r_2(A_1) + r_2(A_2) - r_2(E(M_1[Y])) \\
&= (\text{by (3.3.2)}) \quad r_1(A_1) + r_1(E(M_1) \setminus A_1) - r_1(E(M_1)) \\
&= (\text{by (3.2.4)}) \quad r(A_1) + r(E(M) \setminus A_1) - r(E(M)). \quad \Delta
\end{aligned}$$

The following theorem (3.3.5) is a slightly stronger form of the original theorem due to Bixby and Cunningham [3], with essentially the same proof. As we shall see, the proof of part (b) of (3.3.5) is based on lemma (3.3.6), which was proved by Bixby and Cunningham using induction principle. However, we are not giving its proof as we shall give a constructive proof of this lemma in chapter 5, which would lead us to an algorithm for finding all the 2-separations of a matroid.

Let Y be a cocircuit of M . We say that two singleton bridges B_1, B_2 of Y are in series, if b_1, b_2 are in series in M , where $B_i = \{b_i\}$, $i = 1, 2$.

(3.3.5) Theorem. Let M be a non-separable matroid, and Y be a cocircuit of M . The following two statements hold:

(a) Suppose that no two singleton bridges of Y are in series; then, M has a 2-separation $\{E_1, E_2\}$ such that $Y \subseteq E_2$ if and only if some simplified Y -component has a 2-separation.

(b) Suppose that Y is simple in M ; then, M has a 2-separation $\{E_1, E_2\}$ such that Y meets both E_1 and E_2 if and only if the bridge graph of Y is not connected.

Proof of (3.3.5(a)). Suppose that the simplified Y-component $M_1[Y]$ has a 2-separation $\{A_1, A_2\}$. By (3.3.4), we may assume that $Y \cap A_1 = \emptyset$; then $\{A_1, E(M) \setminus A_1\}$ is a 2-separation of M with $Y \subseteq E(M) \setminus A_1$.

Now suppose that M has a 2-separation $\{E_1, E_2\}$ such that $Y \subseteq E_2$. Let $A \subseteq E_1$ be such that, for each j , $A \cap B_j = E_1 \cap B_j$ or $A \cap B_j = \emptyset$. Then $r(A) + r(E_1 \setminus A) = r(E_1)$, so $r(A) + r(E \setminus A) \leq r(A) + r(E_2) + r(E_1 \setminus A) = r(E_1) + r(E_2) = r(E) + 1$. If $|B_j \cap E_1| = 1$ for two different choices of j , let $B_i \cap E_1 = \{b_i\}$, $i = 1, 2$, $B_1 \neq B_2$, and let $A = \{b_1, b_2\}$. So $r(A) + r(E \setminus A) \leq r(E) + 1$. Since b_1 and b_2 are in different bridges of Y , and M is non-separable, we have $r(A) = 2$; thus $r(E \setminus A) \leq r(E) - 1$. Then, by (2.3.8) and the fact that M is non-separable, A is a cocircuit of M . Then it is easy to see that $B_i = \{b_i\}$, $i = 1, 2$. This contradicts the assumption that no two singleton bridges of Y are in series. Therefore, since $|E_1| \geq 2$, there exists i such that $|E_1 \cap B_i| \geq 2$, where $A = E_1 \cap B_i$, we have $r(A) + r(E \setminus A) = r(E) + 1$, so by (3.2.4), where r_1 is the rank function of the Y-component M_1 corresponding to B_i , $r_1(A) + r_1((B_i \setminus A) \cup Y) = r_1(B_i \cup Y) + 1$. It is easy to check that $A \cap Y = \emptyset$ and $r_1((B_i \setminus A) \cup Y) \geq 2$, so the Y-simplification of M_1 is not 3-connected. This proves (3.3.5(a)).

To prove (3.3.5(b)), we need the following lemma.

(3.3.6) Lemma. If the bridge graph of Y is not connected, then there exists a connected component of the bridge graph having vertex-set C and a partition $\{L_1, L_2\}$ of Y such that:

- (i) L_1 contains all but one B -segment for each bridge $B \in C$;
- (ii) L_2 contains all but one B -segment for each bridge $B \notin C$.

For a proof, see Bixby and Cunningham [3].

Proof of Theorem (3.3.5(b)). Suppose that M has a 2-separation $\{E_1, E_2\}$ such that Y meets both E_1 and E_2 . Then by (3.3.3), $r(E_1 \setminus Y) + r(E_2 \setminus Y) = r(E \setminus Y)$, so each bridge B_i is contained in E_1 or E_2 . Let $L_i = E_i \cap Y$ for $i = 1, 2$. Let r', r'' be the rank functions of $M/(E_2 \setminus L_2)$, $M/(E_1 \setminus L_1)$ respectively. Then $r'(L_2) + r''(L_1) =$ (by (2.2.7))
 $r(E_2) - r(E_2 \setminus L_2) + r(E_1) - r(E_1 \setminus L_1) = r(E) + 1 - (r(E) - 1) = 2$.
 Since $L_1, L_2 \neq \emptyset$, it follows that the elements of L_2 are parallel in $M/(E_2 \setminus L_2)$ and the elements of L_1 are in parallel in $M/(E_1 \setminus L_1)$. Therefore, by (3.2.6), for every bridge $B_i \subseteq E_2$, L_1 is contained in a B_i -segment, and for every $B_i \subseteq E_1$, L_2 is contained in a B_i -segment. Thus every B_i contained in E_1 avoids every B_i contained in E_2 . Thus the bridge graph of Y is not connected, provided that each of E_1, E_2 contains at

least one bridge. Now suppose that E_1 (say) contains no B_1 , so that $E_1 = L_1$. Since $L_2 \neq \emptyset$, by (2.3.8) $r(E_2) = r(E)$, so $r(E_1) \leq 1$, and since M is Y -simple, $\{E_1, E_2\}$ is not a 2-separation of M .

Now, suppose that the bridge graph of Y is not connected.

Then we may choose C and $\{L_1, L_2\}$ as in (3.3.6). Let

$E_1 = L_1 \cup (\cup(B_1 : B_1 \in C))$ and let $E_2 = L_2 \cup (\cup(B_1 : B_1 \notin C))$.

Then the elements of L_1 are in parallel in $M/(E \setminus (B_1 \cup Y))$ for each $B_1 \notin C$, so by (3.2.1), the elements of L_1 are in parallel

in $M/(E \setminus (Y \cup (E_2 \setminus L_2))) = M/(E_1 \setminus L_1)$. Similarly the elements of

L_2 are in parallel in $M/(E_2 \setminus L_2)$. Letting r'', r' be the rank

function of these two minors, we have $2 = r''(L_1) + r'(L_2) =$

$r(E_1) - r(E_1 \setminus L_1) + r(E_2) - r(E_2 \setminus L_2) = r(E_1) + r(E_2) - r(E \setminus Y) =$

$r(E_1) + r(E_2) - r(E) + 1$. Therefore, $r(E_1) + r(E_2) = r(E) + 1$.

Since $L_1, L_2 \neq \emptyset$, and since each of E_1, E_2 contains at least one B_1 , we have $|E_1|, |E_2| \geq 2$, and the proof is complete. Δ

(3.3.7) Let Y be a cocircuit of the non-separable matroid

M , and let $\{E_1, E_2\}$ be a 2-separation of M such that Y meets

E_1 and E_2 . From the proof of theorem (3.3.5(b)). We have the

following conclusions:

- (1) Each bridge B_1 of Y is contained in either E_1 or E_2 ;

(ii) $Y \cap E_j$ is contained in one B_i -segment for each bridge

B_i with $B_i \subseteq E_{3-j}$, $j = 1$ and 2 ;

(iii) B_i avoids B_j , for every bridge $B_i \subseteq E_1$ and $B_j \subseteq E_2$.

Conversely, let $\{E_1, E_2\}$ be a partition of E such that Y meets E_1 and E_2 , and (i), (ii) are satisfied. Then $\{E_1, E_2\}$ is a 2-separation of M ; moreover, (iii) holds. (The reader may find it useful to visualize these statements in terms of the bond matroid of a non-separable graph G .)

Chapter 4

An algorithm for finding 3-connected minors

A non-trivial 3-connected matroid is a 3-connected matroid with at least 4 cells. In this chapter, we describe an algorithm which finds a non-trivial 3-connected minor of a matroid M , or shows that such a minor does not exist. Under appropriate assumptions, we can show that the computational effort is bounded by the order of $(r(M))^2 |E(M)|$. We also give two applications of this algorithm.

4.1 2-separations

In this section, we prove some properties of the matroid 2-separations which we need. The following theorem gives a characterization of the matroid 2-separation in terms of its circuit family.

(4.1.1) Theorem. Let M be a non-separable matroid on E , and $\{E_1, E_2\}$ be a partition of E with $|E_i| \geq 2$, $i = 1, 2$. Then $\{E_1, E_2\}$ is a 2-separation of M if and only if whenever circuit C_i meets both E_1 and E_2 , $i = 1, 2$, then $(C_i \cap E_1) \cup (C_i \cap E_2)$ is a circuit of M .

Proof. First we prove the "if" part. Let B_1 be a basis of E_1 . Extend B_1 to a basis B of E , and extend $B \cap E_2$ to a basis

B_2 of E . Hence $r(E_1) + r(E_2) - r(E) = |B_1| + |B_2| - |B| = |B_2| - |B \cap E_2| = |B_2 \setminus B|$. We have to show that $|B_2 \setminus B| \leq 1$.

If not, then there exist distinct elements x, y of $B_2 \setminus B$. By

(2.1.2), there exist circuits C_x, C_y such that $x \in C_x \subseteq B + x$,

$y \in C_y \subseteq B + y$, and both C_x and C_y must meet E_1 . Then

$C'_x = (C_x \cap E_1) \cup (C_x \cap E_2)$ is a circuit and there exists $z \in (C'_x \cap C_y) \cap E_1$.

By strong circuit axiom (2.1.7) there exists a circuit C such

that $x \in C \subseteq (C'_x \cup C_y) - z$. Now C must meet E_1 , for otherwise

$C \subseteq B_2$, but then $C' = (C \cap E_1) \cup (C'_x \cap E_2)$ is a circuit properly

contained in C'_x , a contradiction. Therefore, $\{E_1, E_2\}$ is a

2-separation of M .

To prove the "only if" part, we first prove Claim 1: If B and B' are two distinct bases of a set S , where $S \subseteq E$, then there exists a sequence of bases $B_0, B_1, \dots, B_n$ of S with $B_0 = B$ and $B_n = B'$ such that $|B_{i+1} - B_i| = 1$ and $|B_i - B_{i+1}| = 1$ for $i = 0, 1, \dots, n-1$. Proof. Let $b' \in B' \setminus B$, extend $\{b'\}$ to a basis B_1 of $B \cup \{b'\}$. Then $|B_1| = |B|$, since $B \cup \{b'\} \subseteq S$. Thus there is $b_1 \in B$ such that $B_1 = B - b_1 + b'$. Now it is not difficult to see we can prove claim 1 by induction (on $|B' \setminus B|$).

Next, we prove Claim 2. If B_i is a basis of E_i , $i = 1, 2$ where $\{E_1, E_2\}$ is a 2-separation of the non-separable matroid M

then $B_1 \cup B_2$ contains a unique M-circuit. Proof. For each $e \in E \setminus (B_1 \cup B_2)$, $B_1 \cup B_2 + e$ contains an M-circuit including e . Thus it is easy to see that $r(B_1 \cup B_2) = r(E)$. $\{E_1, E_2\}$ is a 2-separation of M , and by the non-separability of M , we have $r(E) = r(E_1) + r(E_2) - 1 = |B_1| + |B_2| - 1$. Thus $r(E) = r(B_1 \cup B_2) = |B_1 \cup B_2| - 1$, and so $B_1 \cup B_2 = B + x$ for some basis B of E and $x \notin B$. Hence $B_1 \cup B_2$ contains a unique M-circuit, by (2.1.2). This proves claim 2.

Now suppose that $\{E_1, E_2\}$ is a 2-separation of M , and suppose that C_1, C_2 are two M-circuits meeting E_1 and E_2 . Extend $C_1 \cap E_1$ to a basis B_1 of E_1 , and extend $C_2 \cap E_2$ to a basis B_2 of E_2 . By claim 2, $B_1 \cup B_2$ contains a unique M-circuit C . If we can prove that $C \cap E_1 = C_1 \cap E_1$ and $C \cap E_2 = C_2 \cap E_2$, then $C = (C_1 \cap E_1) \cup (C_2 \cap E_2)$ and we are done. We need only to prove $C \cap E_1 = C_1 \cap E_1$; the case $C \cap E_2 = C_2 \cap E_2$ is symmetric. Extend $C_1 \cap E_2$ to a basis B_3 of E_2 . Then C_1 is the unique circuit contained in $B_1 \cup B_3$. Suppose $B_3 = B_2$; then $C = C_1$ and so $C \cap E_1 = C_1 \cap E_1$. Suppose that $B_3 \neq B_2$; by claim 1, there exists a sequence of bases $B^0, B^1, B^2, \dots, B^n$ of E_2 with $B^0 = B_3$ and $B^n = B_2$ such that $|B^{i+1} - B^i| = 1$ and $|B^i - B^{i+1}| = 1$, for $i = 0, 1, \dots, n-1$. And by claim 2, $B_1 \cup B^i$ contains a unique

M-circuit C^i with $C^i \cap E_1 \neq \emptyset$ for $i = 0, 1, \dots, n$. Notice that

$C^1 = C_1$ and $C^n = C$. We claim $C^i \cap E_1 = C^{i+1} \cap E_1$, for $i =$

$0, 1, 2, \dots, n-1$. Choose any arbitrary i , $0 \leq i \leq n-1$ and suppose

that $B^{i+1} = B^i - a + b$. There is a circuit $F \subseteq B^i + a$ and $a \in F$.

For any $x \in (C^{i+1} \cap E_1) \setminus (C^i \cap E_1)$, there exists a circuit G such

that $x \in G \subseteq (C^{i+1} \cup F) - a$. But $G \subseteq B_1 \cup B^i$, so $G = C^i$; that is,

$C^i \supseteq C^{i+1} \cap E_1$. Similarly $C^{i+1} \supseteq C^i \cap E_1$, and the claim is proved.

Thus $C^1 \cap E_1 = C^2 \cap E_1 = \dots = C^n \cap E_1$. Since $C^1 = C_1$ and $C^n = C$,

we have $C_1 \cap E_1 = C \cap E_1$. This completes the proof of the theorem. Δ

(4.1.2) Corollary. Let M be a non-separable matroid on E ,

and let $\{E_1, E_2\}$ be a partition of E with $|E_i| \geq 2$, $i = 1, 2$.

Then $\{E_1, E_2\}$ is a 2-separation of M if and only if whenever co-

circuit Y_i meets both E_1 and E_2 , $i = 1, 2$, then $(Y_1 \cap E_1) \cup (Y_2 \cap E_2)$

is a cocircuit of M .

Proof. By (2.5.1), $\{E_1, E_2\}$ is a 2-separation of M if and only

if $\{E_1, E_2\}$ is a 2-separation of M^* . The result follows. Δ

(4.1.3) Corollary. Let $\{E_1, E_2\}$ be a 2-separation of the non-

separable matroid M . Let C_1, C_2 be two circuits (respectively

cocircuits) meeting E_1 and E_2 . If $C_1 \cap E_1 \subseteq C_2 \cap E_1$, then

$$C_1 \cap E_1 = C_2 \cap E_1.$$

Proof. If not, then $(C_1 \cap E_1) \cup (C_2 \cap E_2)$ is a circuit (respectively cocircuit) properly contained in C_2 , a contradiction. Thus

$$C_1 \cap E_1 = C_2 \cap E_1. \quad \Delta$$

(4.1.4) Proposition. Let $\{E_1, E_2\}$ be a 2-separation of the non-separable matroid M , and let C be an M -circuit meeting E_1 and E_2 . Then $C \cap E_2$ is a series set of the matroid $M \setminus (E_2 \setminus C)$.

Proof. Observe that a circuit of $M \setminus (E_2 \setminus C)$ is also a circuit of M . And by (4.1.3), it is easy to see that any circuit of $M \setminus (E_2 \setminus C)$ containing one element of $C \cap E_2$, contains all of $C \cap E_2$. Thus $C \cap E_2$ is a series set of $M \setminus (E_2 \setminus C)$. Δ

(4.1.5) Proposition. Let $\{E_1, E_2\}$ be a 2-separation of M . Let C_1, C_2 be two M -circuits meeting E_1 and E_2 , and let $x_1 \in C_1 \cap E_2$, $i = 1, 2$. Then the two matroids $M \setminus (E_2 \setminus C_1) / ((C_1 \cap E_2) - x_1)$ and $M \setminus (E_2 \setminus C_2) / ((C_2 \cap E_2) - x_2)$ are isomorphic (with x_1 corresponding to x_2).

Proof. For $i = 1, 2$ by (4.1.1) and (4.1.3), it is not difficult to check that the set of all circuits of $M \setminus (E_2 \setminus C_1)$ is $\{C \subseteq E_1 : C \text{ is an } M\text{-circuit}\} \cup \{(C \cap E_1) \cup (C_1 \cap E_2) : C \text{ is an } M\text{-circuit meeting } E_1 \text{ and } E_2\}$. By (4.1.4), $C_1 \cap E_2$ is a series set of $M \setminus (E_2 \setminus C_1)$. So the set of all circuits of $M \setminus (E_2 \setminus C_1) / ((C_1 \cap E_2) - x_1)$ is $\{C \subseteq E_1 : C \text{ is an } M\text{-circuit}\} \cup \{(C \cap E_1) + x_1 : C \text{ is an } M\text{-circuit meeting } E_1 \text{ and } E_2\}$.

meeting E_1 and E_2 . We see that, with x_1 replaced by x_2 , the two matroids $M \setminus (E_2 \setminus C_1) / ((C_1 \cap E_2) - x_1)$ and $M \setminus (E_1 \setminus C_2) / ((C_2 \cap E_1) - x_2)$ have the same circuit family, hence they are isomorphic. Δ

(4.1.6) Proposition. Let M be a non-separable matroid on E . And let X be a subset of E with $|X| = 2$. Then $\{X, E-X\}$ is a 2-separation of M , if and only if $|E| \geq 4$ and X is a circuit or cocircuit of M .

Proof. By (2.5.1) $\{X, E-X\}$ is a 2-separation of M , if and only if $1 \geq r(X) + r(E-X) - r(E) = r(X) + r^*(X) - |X|$ and $|X| \geq 2$, $|E-X| \geq 2$. Since $|X| = 2$, the last statement is true if and only if $r(X) + r^*(X) \leq 3$ and $|E| \geq 4$. Since M is non-separable, we have $0 < r(X) \leq 2$ and $0 < r^*(X) \leq 2$. Hence $r(X) + r^*(X) \leq 3$ if and only if at least one of $r(X), r^*(X)$ is 1. That is X is either a circuit or a cocircuit of M . Δ

For two matroids M, N we use the notation $M \geq N$ to denote that M has a minor isomorphic to N .

(4.1.7) Theorem. Let N be a non-trivial 3-connected minor of the non-separable matroid M . Suppose that $\{E_1, E_2\}$ is a 2-separation of M , and C is a circuit of M meeting E_1 and E_2 . Then either $M \setminus (E_2 \setminus C) / ((C \cap E_2) - x_2)$ or $M \setminus (E_1 \setminus C) / ((C \cap E_1) - x_1)$ has a minor isomorphic to N , where $x_i \in C \cap E_i$, $i = 1, 2$.

Proof. Since N is a 3-connected minor of M , by (2.5.4), to get a minor of M which is isomorphic to N , all the elements of E_2 (say) except possibly one, must be contracted or deleted from M .

Let A be a maximal subset of E_2 such that $M \setminus A \geq N$.

We claim that there is no circuit of $M \setminus A$ contained in $E_2 \setminus A$.

If not, say C_1 is a circuit of $M \setminus A$ with $C_1 \subseteq E_2 \setminus A$. Then, by the maximality of A , in order to get a minor of $M \setminus A$ which is isomorphic to N , all the elements except possibly one of C_1 must be contracted from $M \setminus A$. After contracting $|C_1| - 1$ elements of C_1 from $M \setminus A$, the last element e of C_1 is a loop (hence a separator). Thus $M \setminus A / (C_1 \setminus e) \setminus e \geq N$, and so $M \setminus (A + e) \geq N$.

This contradicts the maximality of A .

Now suppose that $E_2 \setminus A = \emptyset$. Since $C \cap E_2$ is a series set of $M \setminus (E_2 \setminus C)$, each element of $(C \cap E_2) - x_2$ is an isthmus of $(M \setminus (E_2 \setminus C)) \setminus x_2$, and so $M \setminus A = M \setminus E_2 = (M \setminus (E_2 \setminus C)) \setminus x_2 / ((C \cap E_2) - x_2) = (M \setminus (E_2 \setminus C)) \setminus x_2 / ((C \cap E_2) - x_2) = M \setminus (E_2 \setminus C) / ((C \cap E_2) - x_2)$. Thus $M \setminus (E_2 \setminus C) / ((C \cap E_2) - x_2) \geq N$.

Suppose that $E_2 \setminus A \neq \emptyset$. There must exist a circuit C_2 of $M \setminus A$ meeting E_1 and $E_2 \setminus A$, for otherwise $E_2 \setminus A$ is a separator of $M \setminus A$, and therefore $M \setminus A \setminus (E_2 \setminus A) \geq N$ contradicting the maximality of A . If $(E_2 \setminus A) \setminus C_2 \neq \emptyset$ let $f \in (E_2 \setminus A) \setminus C_2$. There

is a circuit C_3 of $M \setminus A$ containing f , for otherwise f is an isthmus of $M \setminus A$, and $M \setminus A \setminus f \geq N$, a contradiction. By the claim above, C_3 meets E_1 and $E_2 \setminus A$. By (4.1.3), we may assume that $C_3 \cap E_1 = C_2 \cap E_1$, and let $g \in C_3 \cap E_1$. By (2.1.5(C2')), $(C_3 \cup C_2) - g$ contains a circuit C_4 of $M \setminus A$. Now $C_4 \cap E_1 \subset C_3 \cap E_1$, and so $C_4 \cap E_1 = \emptyset$, by (4.1.3). Thus $C_4 \subseteq C_2 \setminus A$. This is contrary to the claim above. Therefore $(E_2 \setminus A) \setminus C_2 = \emptyset$ and so $E_2 \setminus C_2 = A$. Now C_2 is a circuit of $M \setminus A$, hence it is a circuit of M meeting E_1 and E_2 , and $M \setminus (E_2 \setminus C_2) = M \setminus A \geq N$. $E_2 \setminus C_2$ is a series set of $M \setminus (E_2 \setminus C_2)$, so $M \setminus (E_2 \setminus C_2) / ((C_2 \cap E_2) - y) \geq N$, where $y \in (C_2 \cap E_2)$. And by (4.1.5), we see that $M \setminus (E_2 \setminus C) / ((C \cap E_2) - x_2) \geq N$. This completes the proof. Δ

(4.1.8) Corollary. Let N be a non-trivial 3-connected minor of the non-separable matroid M . Suppose that $\{E_1, E_2\}$ is a 2-separation of M , and Y is a cocircuit of M meeting E_1 and E_2 , then either $M / (E_2 \setminus Y) \setminus ((Y \cap E_2) - x_2)$ or $M / (E_1 \setminus Y) \setminus ((Y \cap E_1) - x_1)$ has a minor isomorphic to N where $x_i \in Y \cap E_i$, $i = 1, 2$.

Proof. Since $M^* \geq N^*$, and by (4.1.7), either $M^* \setminus (E_2 \setminus Y) / ((Y \cap E_2) - x_2) = [M / (E_2 \setminus Y) \setminus ((Y \cap E_2) - x_2)]^* \geq N^*$ or $M^* \setminus (E_1 \setminus Y) / ((Y \cap E_1) - x_1) = [M / (E_1 \setminus Y) \setminus ((Y \cap E_1) - x_1)]^* \geq N^*$. The result follows. Δ

(4.1.9) Corollary. Let N be a non-trivial 3-connected minor of a non-separable matroid M . If there are two elements x, y of M in series (respectively in parallel); then $M/y \geq N$ (respectively $M \setminus y \geq N$).

Proof. By duality, it is enough to prove that if x, y are in series then $M/y \geq N$. Let $E_1 = \{x, y\}$ and $E_2 = E \setminus E_1$. By (4.1.6), $\{E_1, E_2\}$ is a 2-separation of M , and there is an M -circuit C meeting E_1 and E_2 . Since x, y are in series, any M -circuit containing one of them contains both of them. Thus by (4.1.7), $M/y \geq N$. Δ

(4.1.10) Corollary. Let M be a non-separable matroid on E , let N be a non-trivial 3-connected minor of M , and let $\{E_1, E_2\}$ be a 2-separation of M . Suppose that Y is a cocircuit meeting E_1 and E_2 . Then either $M/(E_1 \setminus Y)$ or $M/(E_2 \setminus Y)$ contains a minor isomorphic to N .

Proof. By (4.1.7). Δ

The following theorem is due to Bixby and Cunningham [3].

(4.1.11) Theorem. Let M be a non-separable matroid, and let B be a basis of M . If every B -fundamental cocircuit is simple in M and non-separating, then M is 3-connected.

Proof. Since M is non-separable, by (2.4.10), for every partition $\{E_1, E_2\}$ of E , there exists a B -fundamental cocircuit Y meeting E_1 and E_2 . If $\{E_1, E_2\}$ were a 2-separation, then by (3.3.3), Y would be separating, a contradiction. Δ

4.2. An algorithm for finding a non-trivial 3-connected minor

In this section, we describe an algorithm to find a non-trivial 3-connected minor in a matroid. First, we define some terminology. Let B be a basis of the matroid M . The display matrix M^0 of M with respect to B is the $(0,1)$ matrix whose rows are the incidence vectors of the B -fundamental cocircuits of M . It is convenient to consider that the rows of M^0 are indexed by B and then columns of M^0 are indexed by $E(M)$. The entry (b,e) of M^0 is 1 if and only if e is in the B -fundamental cocircuit determined by b . The following are some elementary properties of the display matrix.

(4.2.1) Proposition. Let M^0 be the display matrix of M with respect to the basis B . Then:

- (i) For each $e \in E \setminus B$, the column e of M^0 corresponds to a circuit of M . More precisely, $\{e\} \cup \{b_i \mid b_i \in B \text{ and the } B\text{-fundamental cocircuit determined by } b_i \text{ contains } e\}$ is a circuit of M .

- (ii) For any $S \subseteq E \setminus B$, the display matrix of $M \setminus S$ with respect to the basis B is just the matrix $M^0 \setminus S$ obtained from M^0 by deleting columns S .
- (iii) For any $T \subseteq B$, the display matrix of M/T with respect to the basis $B \setminus T$ is just the matrix M^0/T obtained from M^0 by deleting columns T and deleting rows corresponding to T .
- (iv) If two elements of M are in parallel, then the corresponding columns of M^0 are identical.
- (v) If every non-basic column of M^0 contains at least two 1's, and no two non-basic columns of M^0 are identical then M is simple.
- (vi) If every row of M^0 contains at least three 1's, and no two rows when restricted to $E \setminus B$, are identical then M is co-simple.

Proof. (i) Let e be an arbitrary non-basic element. Then $B+e$ contains a unique circuit $C(e,B)$, by (2.1.2). Let $b_1 \in B$, and $Y(b_1, B)$ be the B -fundamental cocircuit determined by b_1 . Clearly $C(e,B) \cap Y(b_1, B) \subseteq \{e, b_1\}$. Notice that the intersection of a circuit and a cocircuit cannot contain only one element by (2.3.9). Hence $b_1 \in C(e,B)$ if and only if $e \in Y(b_1, B)$. That is, $C(e,B) = \{e\} \cup \{b_1\}$ if $b_1 \in B$ and the B -fundamental cocircuit determined by b_1 contains e . This proves (i).

(ii) First we observe that B is a basis of $M \setminus S$, since $S \subseteq E \setminus B$. And each non-basic column of $M^0 \setminus S$ corresponds to a circuit of $M \setminus S$, since they corresponds a circuit of M . Now similar to the proof of (i), we can show that each row of $M^0 \setminus S$ is an incidence vector of a cocircuit (hence a B -fundamental cocircuit) of $M \setminus S$. And so $M^0 \setminus S$ is the display matrix of $M \setminus S$ with respect to the basis B .

(iii) Observe that $B \setminus T$ is a basis of M/T , since $T \subseteq B$. And for each $b \in B \setminus T$, the B -fundamental cocircuit $Y(b, B)$ of M determined by b is still a cocircuit of M/T . Thus (iii) is clearly true.

(iv) Suppose that e, f are in parallel. It is clear that at least one of e, f is not in B . If $e \notin B$ and $f \in B$, then by (i), $C(e, B) = \{e, f\}$, since $\{e, f\}$ is a circuit. So the columns of M^0 corresponding to e and f are identical. If neither of e, f is in basis B , by (i) and the fact that $\{e, f\}$ is a circuit, applying strong circuit axiom (2.1.7) twice, we see that (iv) holds.

(v) By (iv), it is easy to see that M has no loop and no parallel elements.

(vi) Similar to the proofs of (iv) and (v), we can prove that M has no isthmus and no series elements. Thus M is cosimple. Δ

M is a non-trivial 3 -connected matroid, if M is 3 -connected and $|E(M)| \geq 4$. Where n is a positive integer, a bridge B_1 of Y in M is a n -bridge, if $|\pi(M, B_1, Y)| = n$. Given a display matrix M^0 of the matroid M , a path in M^0 from row z_1 to row z_n , for $n \geq 0$, is a sequence $z_1, j_1, z_2, \dots, j_{n-1}, z_n$, where the z_k are row indices, the j_k are column indices, and for $1 \leq k \leq n-1$, M^0 has 1's in positions (z_k, j_k) and (z_{k+1}, j_k) . A path is minimal with respect to a certain property, if no proper subsequence of it is also a path with the same property. In the following, we prove 3 propositions which constitute the main body of our algorithm for finding a non-trivial 3 -connected minor in a matroid.

(4.2.2) Proposition. Let Y be a cocircuit of the non-separable matroid M , and B_1 be an n -bridge of Y with $n \geq 3$. Then the Y -component M_1 of M contains a non-trivial 3 -connected minor, where M_1 is the Y -component of M corresponding to bridge B_1 .

Proof. We prove this proposition by proposing an algorithm to actually find a non-trivial 3 -connected minor of M_1 .

(4.2.3) Algorithm:

Step 0. Choose a basis B of M_1 such that Y is a B -fundamental cocircuit, and let M_1^0 be the display matrix of M_1 with respect to B .

Let b, e, f be 3 distinct elements of Y chosen from

3 distinct B_1 -segments with $b \in \text{basis } B$.

(Comment: B_1 is an n -bridge of Y with $n \geq 3$.)

Step 1. If there is a row z of M^0 containing only one of e, f , go to step 2; otherwise, go to step 4.

Step 2. Find a path $z_1, j_1, z_2, j_2, \dots, j_{n-1}, z_n$ in M_1^0 minimal with respect to the property that $|z_1 \cap \{e, f\}| = 1$, each j_k is not in Y , and $x \in z_n$, where x is one of e, f which does not belong to z_1 .

(Comment: Such a row z_n must exist, for otherwise b and x are in the same B_1 -segment of Y .)

Let $S = (E \setminus B) \setminus \{e, f, j_1, j_2, \dots, j_{n-1}\}$ and $T = B \setminus \{b, b_1, b_2, \dots, b_n\}$ where b_1 is the basic element which determines the B -fundamental cocircuit $z_j, j = 1, 2, \dots, n$.

Step 3. Delete S and contract T from M_1 , and let $M'_1 = (M_1 \setminus S)/T$. Then stop; M'_1 is 3-connected, and hence is a non-trivial 3-connected minor of M . (We shall prove this later.)

Step 4. Let $\{z_0, z_1, z_2, \dots, z_n\}$ be the set of all rows of M_1^0 with $\{e, f\} \subseteq z_j, j = 0, 1, 2, \dots, n$, where $z_0 = Y$. And let $S = (E \setminus B) \setminus \{e, f\}, T = B \setminus \{b, b_1, b_2, \dots, b_n\}$, where b_1 is the basic element which determines the B -fundamental co-

circuit z_j , $j = 1, 2, \dots, n$.

Now delete S and contract T from M_1 , let $M'_1 = (M_1 \setminus S)/T$, and let $Y' = \{b, e, f\}$.

(Comment: We shall prove the following facts later: Y'

is a cocircuit of M'_1 , the bridges of Y' in M'_1 are precisely those singleton bridges $\{b_1\}, \{b_2\}, \dots, \{b_n\}$, and there is one bridge $B_k = \{b_k\}$ such that e, f are not in parallel in the Y' -component $M'_1 / \{b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n\}$.)

Now take a Y' -component $M'_1 / \{b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n\}$ such that e, f are not in parallel in it, stop; this Y' -component is 3-connected, and so it is a non-trivial 3-connected minor of M .

(~~Comment~~: This Y' -component is isomorphic to $U_{2,4}$, the matroid on 4 elements with every 2 elements subset forming a base.)

End of the algorithm.

To establish the validity of this algorithm, we need only to justify step 3 and step 4.

To justify step 3; we observe that the display matrix of M'_1 has one of the following two forms:

$$Wh_m = \begin{bmatrix} & b & & & e & f \\ 1 & & & & 1 & 1 \\ & 1 & & & 1 & 0 & 1 \\ & & 1 & & 0 & 0 & 1 & 1 \\ & & & \ddots & \vdots & \vdots & \ddots & 1 & 1 \\ & & & & 0 & 0 & & & 1 & 1 \\ & & & & 1 & 0 & 1 & & & 1 \end{bmatrix} \quad m \times 2m$$

$$H_m = \begin{bmatrix} & & & & 1 & 1 \\ & 1 & & & 1 & 0 & 1 \\ & & 1 & & 0 & 0 & 1 & 1 \\ & & & \ddots & \vdots & \vdots & \ddots & 1 & 1 \\ & & & & 0 & 0 & & & 1 & 1 \\ & & & & 1 & 1 & 1 & & & 1 \end{bmatrix} \quad m \times 2m$$

where Wh_m and H_m are $m \times 2m$ matrices, $m \geq 3$, the entries which do not show up in these matrices are zero.

Remark. As we shall see, by (6.2.1) in chapter 6, any matroid with display matrix of the form Wh_m is a "wheel" or a "whirl" matroid.

It is easy to check that any matroid whose display matrix has the form of Wh_m , is non-separable, simple, and has no separating B-fundamental cocircuit. By (4.1.11), M'_1 is 3-connected, provided its display matrix is Wh_m .

Suppose that the display matrix of M'_1 is H_m . Then M'_1 is non-separable, simple, and cosimple by (4.2.1 (v) and (vi)). It

is not difficult to check the following:

- (1) The last row z of H_m is the only separating B-fundamental cocircuit.
- (2) There are exactly two bridges of z in M'_1 .
- (3) The two bridges of z are overlapping; hence the bridge graph of z is connected.
- (4) If $m = 3$, then each of the two simplified z -components is simple and has rank 2, so they are 3-connected. (To see this we may either apply (4.1.11) or check that $r(E_1) + r(E_2) \geq r(E) + 2$ for every partition $\{E_1, E_2\}$ of E .)
- (5) If $m \geq 4$, then one of the simplified z -components has rank 2 and is simple, and the other has its display matrix of the form W_{m-1} ; hence both of them are 3-connected.

From the above 5 observations, we conclude that M'_1 is 3-connected by (3.3.5).

To justify step 4; we know that e, f are not in parallel in M_1 ; they are not in parallel in $M_1 \setminus S$. The display matrix of $M_1 \setminus S$, by (4.2.1 (ii)), has the following form:

$$\begin{array}{c}
 \begin{array}{c} b \\ \left[\begin{array}{ccccccc} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ & & & & & & & 1 \end{array} \right] \end{array}
 \end{array}
 \begin{array}{c}
 \begin{array}{cc} \overbrace{\quad}^T & \\ e & f \end{array} \\
 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}
 \end{array}$$

We note that T is the set of all basic elements whose corresponding row in M_1^0 is zero at the positions e and f ; thus T is a separator of $M_1 \setminus S$. Hence e and f are not in parallel in $M_1 \setminus S/T$, because $M_1 \setminus S/T \approx M_1 \setminus S \setminus T$. Now, from the display matrix of $M_1 \setminus S/T$, we see that $Y' = \{b, e, f\}$ is a cocircuit of $M'_1 = M_1 \setminus S/T$; the bridges of Y' in M'_1 are precisely those singleton bridges $\{b_1\}, \{b_2\}, \dots, \{b_n\}$. Suppose that e, f are in parallel in every Y' -component of M'_1 . By (3.2.3 (1)), e, f are in parallel in M'_1 , a contradiction. Thus there is one Y' -component $M'' = M'_1 / \{b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n\}$ such that e, f are not in parallel in it. The display matrix of M'' has the form:

$$\begin{bmatrix} & e & f \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

It is easy to check that M'' has 4 cells; the rank of M'' is 2, and any two-element set is independent. Thus M'' is isomorphic to $U_{2,4}$, and clearly $U_{2,4}$ is 3-connected. Δ

(4.2.4) Proposition. Let Y be a cocircuit of the non-separable matroid M , and let B_1, B_2 be overlapping 2-bridges of Y . Then the minor $M/(E \setminus (Y \cup B_1 \cup B_2))$ contains a non-trivial 3-connected minor.

Proof. Again the proof is algorithmic.

(4.2.5) Algorithm:

Step 0. Let $M_1 = M/(E \setminus (Y \cup B_1 \cup B_2))$. Choose a basis B of M_1 such that Y is a B -fundamental cocircuit, and let M_1^0 be the display matrix of M_1 with respect to B .

Step 1. Choose a row $Z_1 \neq Y$ such that the fundamental cocircuit Z_1 is determined by a basic element $b_i \in B \cap B_1$, and $Z_1 \cap Y \neq \emptyset$, $i = 1, 2$.

(Comment: Such Z_1 and Z_2 must exist; moreover

$(Z_1 \cap Y) \cap (Z_2 \cap Y) \neq \emptyset$, $(Z_1 \cap Y) \setminus (Z_2 \cap Y) \neq \emptyset$, and

$(Z_2 \cap Y) \setminus (Z_1 \cap Y) \neq \emptyset$, since B_1, B_2 are overlapping 2-bridges of Y .)

Step 2. Choose $e \in (Z_1 \cap Y) \cap (Z_2 \cap Y)$, $f \in (Z_1 \cap Y) \setminus (Z_2 \cap Y)$, and $g \in (Z_2 \cap Y) \setminus (Z_1 \cap Y)$. Let $S = (E \setminus B) \setminus \{e, f, g\}$ and $T = B \setminus \{b, b_1, b_2\}$, where b is the basic element which determines Y .

Now delete S and contract T from M_1 . Stop; $M_1 \setminus S/T$ is 3-connected, and hence is a non-trivial 3-connected minor of M_1 .

End of the algorithm.

We now justify the validity of this algorithm. The validity of step 1 is easily deduced from the comment. For step 2, we

observe that the display matrix of $M_1 \setminus S/T$ has the form:

$$\begin{array}{c} z_1 \\ z_2 \end{array} \begin{array}{c} e \quad f \quad g \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \end{array}$$

If we permute the first and last row of it, then we can see the resulting matrix is just H_3 (defined in Proposition (4.2.2)). We have proved that any matroid, whose display matrix is of the form H_m , $m \geq 3$, is 3-connected. So $M_1 \setminus S/T$ is 3-connected. Δ

(4.2.6) Proposition. Let Y be a cocircuit of the non-separable matroid M . Suppose that all the bridges of Y are mutually avoiding and every Y -component does not have a non-trivial 3-connected minor. Then M does not have a non-trivial 3-connected minor.

Proof. We prove this by induction on the number n of bridges of Y . For $n = 1$, it is true by assumption. Suppose that the result is true whenever the cocircuit has fewer than $k \geq 2$ bridges, and suppose that the cocircuit Y in question has k bridges. Let B_1 be a bridge of Y . Since all the bridges of Y are mutually avoiding, as shown in the proof of (3.3.5(b)), we can find a partition $\{L_1, L_2\}$ of Y (depending on the bridge B_1) such that $\{E_1, E_2\}$ is a 2-separation of M , where $E_1 = L_1 \cup B_1$.

$E_2 = L_2 \cup (\cup(B: B \text{ is a bridge of } Y \text{ and } B \neq B_1)), \text{ and } L_1 \neq \emptyset,$
 $i = 1, 2.$ Suppose that M has a non-trivial 3-connected minor N .
 Then by (4.1.10) either $M/(E_1 \setminus Y)$ or $M/(E_2 \setminus Y)$ contains a minor
 isomorphic to N . Since $M/(E_2 \setminus Y) = M/(E \setminus (B_1 \cup Y))$, this is the
 Y -component of M corresponding to B_1 , and we see that it does
 not have a non-trivial 3-connected minor by assumption. Now
 $M/(E_1 \setminus Y) = M/B_1$, by (3.2.6) Y is a cocircuit of M/B_1 , and
 the number of bridges of Y in M/B_1 is $k-1$. Hence by induction
 hypothesis $M/(E_1 \setminus Y)$ does not have a non-trivial 3-connected minor.
 This contradicts the assumption that one of $M/(E_1 \setminus Y)$, $M/(E_2 \setminus Y)$
 contains a minor isomorphic to N . Thus M does not have a non-
 trivial 3-connected minor. Δ

Now the main theorem of this section is the following:

(4.2.7) Theorem. Let M be a non-separable matroid, and let
 Y be a cocircuit of M . Then M has a non-trivial 3-connected
 minor if and only if at least one of the following three statements
 holds.

- (i) There is an n -bridge of Y with $n \geq 3$.
- (ii) There are two overlapping 2-bridges of Y .
- (iii) Some Y -component has a non-trivial 3-connected minor.

Proof. We prove the "if" part first. Suppose that (i) or (ii) holds. By proposition (4.2.2) and proposition (4.2.4), M has a non-trivial 3-connected minor. (In fact, the algorithms described in these propositions enable us to find such a minor). Suppose (iii) holds; then M has a non-trivial 3-connected minor, since a Y -component of M is a minor of M .

Conversely, suppose that M has a non-trivial 3-connected minor, and suppose that (iii) is not true. Then by (4.2.6) there are two bridges B_1, B_2 of Y which overlap. Since M is non-separable, by (3.2.8), bridge B_i has at least 2 segments of Y , for $i = 1, 2$. If one of B_1, B_2 has more than 2 segments, then (i) holds, otherwise (ii) holds. Δ

Now we consider the problem of finding algorithmically a non-trivial 3-connected minor of a matroid M . If M is separable, let $S_1, S_2, \dots, S_k$ be its elementary separators. By (2.5.4), we see that if ever M has a non-trivial 3-connected minor, it is contained in one of the $M \setminus (E \setminus S_i)$, $i = 1, 2, \dots, k$. Applying the algorithm described in (2.4.11), we can find all the elementary separators of M . Thus we just need an algorithm for finding a non-trivial 3-connected minor of a non-separable matroid M .

We now state such an algorithm:

(4.2.8) An algorithm to find a non-trivial 3-connected minor of a non-separable matroid. (Input is a non-separable matroid M .)

Step 1. If M has rank 1 or 0, or if M has 3 cells or less, stop; M does not have a non-trivial 3-connected minor.

Step 2. Choose a basis B of M , and choose a B -fundamental co-circuit Y (determined by the basic element b). Obtain all the bridges of Y in M .

If there is a bridge B_1 of Y which is a n -bridge with $n \geq 3$; go to step 3. If there are two bridges B_1, B_2 of Y which are overlapping 2-bridges; go to step 4. Otherwise; go to step 5.

Step 3. (B_1 is a n -bridge of Y in M with $n \geq 3$.)

Apply the algorithm described in (4.2.3) to the Y -component M_1 corresponding to bridge B_1 to get a non-trivial 3-connected minor of M ; stop.

Step 4. (B_1, B_2 are two overlapping 2-bridges of Y in M).

Apply the algorithm described in (4.2.5) to $M/(E \setminus Y \cup B_1 \cup B_2)$ to get a non-trivial 3-connected minor of M ; stop.

Step 5. (All the bridges of Y are mutually avoiding 2-bridges.)

From step 2, we have computed all the Y -components $M_1, M_2, \dots, M_k$, $k \geq 1$. Also we have the corresponding

B_i -segments S_i, T_i with $b \in S_i$, where b is the basic element which determines the B -fundamental cocircuit Y , $i = 1, 2, \dots, k$). For each Y -component M_i , choose an element $y_i \in T_i$. Apply the algorithm recursively to each of the matroids $M'_i = (M_i \setminus (Y - \{b, y_i\}))/b$, $i = 1, \dots, k$.

End of the algorithm.

The validity of each step of this algorithm is clear except step 5 which needs some explanation. In step 5, we know that all the bridges of Y are 2-bridges and mutually avoid. It follows from (4.2.7) that M has a non-trivial 3-connected minor if and only if some Y -component has such a minor. In the context of step 5, S_i, T_i are two parallel classes in M_i , b and y_i are in series in $M_i \setminus (Y - \{b, y_i\})$, so by (4.1.9), any non-trivial 3-connected minor of M_i is isomorphic to a minor of $(M_i \setminus (Y - \{b, y_i\}))/b$. $(M_i \setminus (Y - \{b, y_i\}))/b$ has rank at least one less than $r(M_i)$. This established the validity of the algorithm.

(4.2.9). We remark that, if the non-trivial 3-connected minor we found, by applying this algorithm, is not $U_{2,4}$ then its display matrix has the form of Wh_m or H_m , where $m \geq 3$.

(4.2.10) To analyse the efficiency of algorithm (4.2.8), we first make the following assumptions: there exist efficient algorithms for constructing the initial display matrix of M (with

respect to a basis B), for computing minors of a matroid, and for finding a parallel class in a matroid. We use the algorithm described in (2.4.11) as a subroutine to find the elementary separators of a matroid M ; the complexity of this algorithm is $O(r(M)|E(M)|)$.

Now we have the following 5 observations.

(1) In step 2, for each bridge B_1 of Y in M , we have only to find two members of $\pi(M, B_1, Y)$; for if $Y \setminus (S_1 \cup S_2) \neq \emptyset$ then B_1 is a n -bridge with $n \geq 3$, pick any $f \in Y \setminus (S_1 \cup S_2)$. S_1, S_2 and f are all the information we need from $\pi(M, B_1, Y)$ for algorithm (4.2.3) in step 3. To compute the B_1 -segments, we assume that there exists an efficient algorithm for finding a parallel class in a matroid; considering the overall computational effort $O((r(M))^2 |E(M)|)$ we obtain later, any algorithm for finding a parallel class in M within $O(r(M)|E(M)|)$ work is considered to be efficient. Let us check some particular (but useful) cases of matroids. For the case of real matric matroids or binary matric matroids, if the matroid is represented by a matrix, two elements are in parallel if and only if the corresponding columns of the matrix are proportional. Clearly, there is an $O(r(M)|E(M)|)$ algorithm to find a parallel class. For the case of graphic matroids; certainly there is no problem.

(2) In step 2, if all the bridges of Y in M are 2-bridges, we need to check whether there are two bridges overlapping. Suppose that there are k bridges $B_1, B_2, \dots, B_k$ of Y in M and $|\pi(B_i)| = 2$ for all $i = 1, 2, \dots, k$. We may examine every pair of $\pi(B_i), \pi(B_j)$ for overlapping, $i \neq j$, the computational work is bounded by $O(k^2|Y|)$. (Bixby and Cunningham [4, Theorem 6] give an $O(k|Y|)$ algorithm for testing mutual avoidance of the bridges or finding two overlapping bridges; in this algorithm $|\pi(B_i)|$ is not necessarily 2. But with the method we proposed here, the bound $O(k^2|Y|)$ given is sufficient to obtain the desired overall bound for the entire algorithm.)

(3) For $r \geq 1$, the total number of bridges encountered in step 2 is at most $r-1$, where r is the rank of the input matroid M . We prove this by induction on r . Denote the indicated number by $f(M)$. For $r = 1$, clearly $f(M) = 0 = r-1$. Suppose $r \geq 2$. Let $M_1, \dots, M_k$ be the Y -components of M obtained in step 2, with respective ranks $r_1, \dots, r_k$, $k \geq 1$. Notice that after step 2, we either carry out step 3 or 4 and then stop, or go to step 5 and apply the algorithm recursively on each of the M'_i defined as in step 5, $i = 1, 2, \dots, k$. Furthermore, the rank of M'_i is $r_i - 1$. Thus by induction, $f(M) \leq k + \sum_{i=1}^k f(M'_i) \leq k + \sum_{i=1}^k (r_i - 1 - 1) = k + (\sum_{i=1}^k r_i) - 2k = k + (r+k-1) - 2k = r-1$. This completes the induction proof.

(4) Let M be a matroid on E with a fixed basis B , M^0 be the display matrix of M with respect to basis B , Y be a B -fundamental cocircuit of M , and M_1 be a Y -component of M corresponding to the bridge B_1 . By (3.2.10), $B' = B \cap (B_1 \cup Y)$ is a basis for M_1 . By applying the operations we described in (4.2.1), it is easy to obtain the display matrix of M_1 with respect to the basis B' . (Just contract $B \setminus B'$ from M^0 , then delete zero columns if there are any.) Hence, once we have the initial display matrix of M , all the other display matrices of Y -components for recursive use can be obtained quite easily as submatrices.

(5) At most one of step 3 and step 4 need be applied, and is applied only once. Furthermore, suppose that step 3 (respectively step 4) is executed; the information needed in step 0 of algorithm (4.2.3) (respectively algorithm (4.2.5)) is provided by the main algorithm. This follows from observations 1 and 4 above.

By the foregoing observations, the total number of executions of step 2 need not exceed $r(M) - 1$, where M is the given matroid in which we want to find a non-trivial 3-connected minor. Therefore, under the assumptions stated in (4.2.10), we can assert that the total computational effort required by the elementary separators subroutine in step 2 is bounded by $O((r(M))^2 |E(M)|)$. We note that the bound

$(r(M))^2 |E(M)|$ is in fact achievable. Now we show that, under the same assumptions, the work required to execute the other parts of this algorithm is dominated by this bound, we remark that, in the case of matric matroids (if the input is a matrix), the total computational work for getting the segments π need not exceed $(r(M))^2 |E(M)|$; this can be seen from observations 1 and 3 above. Now the total computational work in step 2 for checking the avoidance of the bridges is bounded by $O((r(M))^2 |E(M)|)$, by observations 2 and 3 above. Step 3 (respectively step 4) is executed at most once; from observation 5, it is easy to see that the computational work for algorithm (4.2.3) (respectively algorithm (4.2.5)) is $O(r(M) |E(M)|)$. The rest of the work in this algorithm is obviously dominated by $O((r(M))^2 |E(M)|)$. Thus the overall computational effort of this algorithm is bounded by $O((r(M))^2 |E(M)|)$.

In the remainder of this chapter, we give two applications of algorithm (4.2.8).

4.3 $PM(K_4)$ minor

By applying algorithm (4.2.8), we can prove the following well-known result. We need this result in chapter 6.

(4.3.1) Theorem. Let M be a binary, non-trivial 3-connected matroid. Then M has a minor isomorphic to $PM(K_4)$ (the polygon matroid of the complete graph on 4 vertices.).

Before proving (4.3.1), we need some results on binary matroids. First we state one theorem of Tutte giving necessary and sufficient conditions for a matroid to be binary in terms of an excluded minor.

(4.3.2) Theorem. A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$, the matroid on 4 elements with every 2-element set forming a base.

For a proof, see [20]. The next result is an easy consequence of Tutte's theorem; it can also be proved independently

(4.3.3) Proposition. A minor of a binary matroid is binary.

(4.3.4) Proposition. Let M^0 be the display matrix of a matroid M with respect to the basis B . If M is binary, then the binary matrix matroid of M^0 is isomorphic to M (that is, M^0 is a binary matrix representation of M .)

Proof. Let M^1 be a binary matrix such that M is the matrix matroid of it. Since the columns of M^1 corresponding to basis B form a maximal linearly independent set, we can perform Gauss-Jordan elimination and delete all the zero rows such that the resulting binary matrix $M^2 = (a_{ij}; i = 1, 2, \dots, m, j \in E)$ has the property that $(a_{ij}; i = 1, 2, \dots, m, j \in B) = I$, the identity matrix. From elementary linear algebra, we know that the corresponding columns

of M^1 and M^2 possess the same linear dependence properties. Hence M^2 is a binary matrix representation of M . Now let $(a_j: j \in E)$ be an arbitrary row of M^2 , and let $Y = \{j \in E: a_j \neq 0\}$. Clearly, where r is the rank function of M , $r(E \setminus Y) = r(E) - 1$, and Y is minimal with this property. Thus by (2.3.8) Y is a cocircuit of M . Since Y contains only one element from the basis B , it is a B -fundamental cocircuit. Thus M^2 is a $(0,1)$ matrix whose rows are the incidence vectors of the B -fundamental cocircuits of M . We see that $M^0 = M^2$ and the binary matrix matroid of M^0 is isomorphic to M . Δ

A wheel graph of order n , where n is an integer ≥ 3 , is constructed from a polygon with n edges by adding one new vertex and n new edges joining the new vertex to the n vertices of the polygon.

We can now prove theorem (4.3.1).

Proof. Applying algorithm (4.2.8) to M , we find a non-trivial 3-connected minor N of M ; since M has such a minor. Since M is binary, by (4.3.3), so is N . Now by the remark (4.2.9), the display matrix N^0 of N has the form of Wh_m or H_m for some $m \geq 3$. Moreover the binary matrix matroid of N^0 is isomorphic to N , by (4.3.4).

Suppose that N^0 has the form of Wh_m , that is:

$$N^0 = \begin{bmatrix} 1 & & & & 1 & 1 & & & \\ & 1 & & & 1 & 0 & 1 & & \\ & & 1 & & 0 & 0 & 1 & 1 & \\ & & & \ddots & \vdots & \vdots & & 1 & \ddots \\ & & & & 0 & 0 & & & 1 & 1 \\ & & & & 1 & 0 & 1 & & & 1 \end{bmatrix} \quad m \times 2m$$

where $m \geq 3$.

By adding one new row, which is the sum of all the rows of N^0 , to the bottom of N^0 , we have a binary matrix A

$$A = \begin{bmatrix} 1 & & & & 1 & 1 & & & \\ & 1 & & & 1 & 0 & 1 & & \\ & & 1 & & 0 & 0 & 1 & 1 & \\ & & & \ddots & \vdots & \vdots & & \ddots & \\ & & & & 0 & 0 & & 1 & 1 \\ & & & & 1 & 0 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (m+1) \times 2m$$

We note that N^0 is a binary matrix, hence $1+1=0$. From elementary linear algebra, we know that the corresponding columns of A and N^0 possess the same linear dependence properties. Thus the binary matrix matroid of A is isomorphic to N . Since each column of A has exactly two 1's, it is not difficult to see that A is the incidence matrix of a wheel graph W_m of order m , where $m \geq 3$. By (2.1.6), the matrix matroid of A is isomorphic to the polygon matroid $PM(W_m)$ of W_m , so $N \cong PM(W_m)$. By (2.2.3) and (2.2.6), it is clear that $PM(W_m)$ contains a minor isomorphic to

2

where $m \geq 3$

A4

a

P

1

p

•

4.4 Series-Parallel matroids

As another application of algorithm (4.2.9), we use it to check if a matroid is a series-parallel matroid. First we state some definitions. Let M_1, M_2 be 2 matroids on E and $E+y$ respectively, where $y \notin E$, and let $x \in E$. We say that M_2 is a series extension (respectively parallel extension) of M_1 at x by y , if x, y are in series (respectively in parallel) in M_2 , or y is an isthmus or loop in M_2 , and furthermore $M_2/y = M_1$ (respectively $M_2 \setminus y = M_1$). We define a series-parallel extension of a matroid M_1 to be a matroid M_2 which can be obtained from M_1 by successive series and parallel extensions. A series-parallel matroid (defined by Minty [10]), is a matroid which can be expressed as a series-parallel extension of a matroid on one element. Let M be a non-null matroid. By (2.4.9), we can express M as the direct sum $M_1 \oplus M_2 \oplus \dots \oplus M_k$ of non-null non-separable matroids, where $E(M_i)$ is an elementary separator of M , $i = 1, \dots, k$. Then the following proposition is obvious.

(4.4.1) Proposition. M is a series-parallel matroid if and only if $M_1, M_2, \dots, M_k$ are series-parallel matroids.

We will prove the following theorem in section 2 of chapter 5.

2

(4.4.2) Theorem. A non-separable matroid is a series-parallel matroid if and only if it has no non-trivial 3-connected minor.

Incidentally, (4.4.2) is equivalent to the following generalization of Duffin's theorem which several people have observed.

(4.4.3) Theorem. A non-separable matroid is a series-parallel matroid if and only if it is binary and has no minor isomorphic to $PM(K_4)$.

Remark. The original Duffin's theorem is stated as the following: A non-separable graph is a series-parallel network if and only if it contains no subgraph homeomorphic to K_4 . See reference Welsh [20].

To prove that (4.4.2) and (4.4.3) are two equivalent theorems, we just have to see that M has no non-trivial 3-connected minor if and only if M is binary and has no minor isomorphic to $PM(K_4)$; the "only if" part is trivial, while "if" part is implied by (4.3.5).

We now explain how we can use algorithm (4.2.8) to check if a matroid M is a series-parallel matroid. By (4.4.1) and the algorithm described in (2.4.11), we check whether each $M \setminus (E \setminus S_1)$ is a series-parallel matroid, where S_1 is an elementary separator of M . Now we assume that M is non-separable. According to theorem (4.4.2), M is a series-parallel matroid if and only if it has a

non-trivial 3-connected minor. Therefore, we may apply algorithm (4.2.8) to check this. Notice that it is not necessary to actually find such a minor of M for our purpose here, as long as we know there is one. Hence, in step 2 of algorithm (4.2.8), once we find an n -bridge with $n \geq 3$, or two overlapping 2-bridges, we can stop there, and M is not a series-parallel matroid.

(4.4.4) There is another conceptually easy algorithm to check whether a non-separable matroid is a series-parallel matroid: First we delete all the parallel elements from M , then contract all the series elements of the remaining matroid; this process is repeated until no more elements can be deleted or contracted; if the remaining matroid has only one element left then M is a series-parallel matroid; otherwise, it is not. (By deleting a parallel element from a matroid, we mean finding two elements in parallel and deleting one of them; contracting a series element is similar.)

Apparently, a necessary condition for this algorithm to be efficient is that there be an efficient algorithm for finding parallel and series elements; otherwise, it is not competitive with our algorithm (4.2.8). For example, if A is a binary matrix, there is an $O(rc)$ algorithm, using the idea of lexicographic ordering, for finding all the parallel columns of A , where r and c are the numbers

of rows and columns of A , respectively. (See Aho, Hopcroft, Ullman [1], P.P.79, Theorem 3.1.) If A is a real matrix, by applying the same idea, we have an $O(rc \log c)$ algorithm for doing this. Now, if M is a binary matroid which is represented by a binary matrix having the form (I, A) , where I is an identity matrix, then it is not difficult to show that, testing whether M is a series-parallel matroid by applying the algorithm described in (4.4.4) requires $O((r(M))^2 |E(M)|)$ work in the worst case. Thus, in this case, the efficiency of algorithm (4.4.4) is about the same as that of our algorithm (4.2.8). On the other hand, if M is a real matrix matroid which is represented by a real matrix having the form (I, A) , then using algorithm (4.4.4) to check whether M is a series-parallel matroid, involves a worst case bound of $O((r(E))^2 |E(M)| \log |E(M)|)$.

Chapter 5

Decomposing a matroid into 3-connected components

This chapter describes a recursive algorithm for decomposing a matroid into 3-connected components, using the idea of cocircuits and their bridges. In the first two sections we define a decomposition of matroids, present the necessary theorems, and prove a unique decomposition theorem. Some references for these topics are Cunningham and Edmonds [6], and Cunningham [5]. In the rest of the chapter, we develop the decomposition algorithm.

5.1 Simple decomposition

In this section, we define the simple decomposition of the non-separable matroids and prove some of the related properties which we need.

(5.1.1) Let M be a non-separable matroid, and $\{E_1, E_2\}$ be a 2-separation of M . For $i = 1, 2$, we define a new matroid M_i on $E_i + e$, where e is not a cell of M , by defining its circuit family to be $C_i = \{C \subseteq E_i : C \text{ an } M\text{-circuit}\} \cup \{C \cap E_i + e : C \text{ an } M\text{-circuit meeting } E_1 \text{ and } E_2\}$. We shall show later that M_i defined in this way is indeed a matroid. We call $\{M_1, M_2\}$ the

simple decomposition of M , associated with the 2-separation

$\{E_1, E_2\}$ and the marker e ; we use $M(E_i; e)$ to denote the matroid M_i , $i = 1, 2$, and write $M \rightarrow \{M(E_1; e), M(E_2; e)\}$, or simply $M \rightarrow \{M_1, M_2\}$. To illustrate these ideas, we apply them to the polygon matroids of the non-separable graphs.

(5.1.2) Let $G = (V(G), E(G))$ be a non-separable graph.

Let $\{E_1, E_2\}$ be a 2-separation of G , $G(E_i)$ be the subgraph of G having edge set E_i and having no isolated vertices, $i = 1, 2$.

Hence $|V(G(E_1)) \cap V(G(E_2))| = 2$ and $|E_i| \geq 2$, $i = 1, 2$. It is

not difficult to see that a 2-separation $\{E_1, E_2\}$ of G is a

2-separation of $PM(G)$. (But not vice versa.) Let $V(G(E_1)) \cap V(G(E_2)) = \{u, v\}$

and e be an element not in $E(G)$ for $i = 1, 2$, we form graph G_1 by

adding e to $E(G_1)$ as an edge joining u and v . Then

$\{PM(G_1), PM(G_2)\}$ is the simple decomposition of $PM(G)$ associated

with the 2-separation $\{E_1, E_2\}$ and the marker e . We give an

example in the following.

We now prove that M_1 and M_2 defined in (5.1.1) are indeed matroids. Observe that, for any partition $\{E_1, E_2\}$ of a non-separable matroid M , there is an M -circuit meeting E_1 and E_2 , by (2.4.1).

(5.1.3) Theorem. Let $\{E_1, E_2\}$ be a 2-separation of the non-separable matroid M . Then M_1 defined in (5.1.1) is a matroid on $E_1 + e$, $i = 1, 2$. Furthermore, where C is an M -circuit meeting E_1 and E_2 , $D_1 = E_1 \cap C$, $x_1 \in D_1$, and $M'_1 = (M \setminus (E_{3-1} \setminus D_{3-1})) / (D_{3-1} - x_{3-1})$, then M_1 can be obtained from M'_1 by substituting e for x_{3-1} , $i = 1, 2$.

Remark. We write $M_1 \cong (M \setminus (E_{3-1} \setminus D_{3-1})) / (D_{3-1} - x_{3-1})$ with $e \rightarrow x_{3-1}$, $i = 1, 2$.

Proof. We prove this for $i = 1$; the case for $i = 2$ is similar.

From the proof of (4.1.5), we know that the circuit family C'_1 of

$M'_1 = (M \setminus (E_2 \setminus D_2)) / (D_2 - x_2)$ is $\{C \subseteq E_1 : C \text{ an } M\text{-circuit}\} \cup \{C \cap E_1 + x_2 : C, \text{ an } M\text{-circuit meeting } E_1 \text{ and } E_2\}$. Thus, replacing x_2 by e it is clear that (M_1, C_1) defined in (5.1.1) is a matroid, since

M'_1 is. Moreover $M_1 \cong M'_1$ with e corresponding to x_2 .

The next proposition shows that M can be recovered from M_1 and M_2 , if $M \rightarrow \{M_1, M_2\}$.

(5.1.4) Proposition. Let $M \rightarrow \{M_1, M_2\}$. Then the set of all circuits of M is $(C_1 \setminus e) \cup (C_2 \setminus e) \cup \{C_1 \cup C_2 - e : e \in C_1 \in C_1, i = 1, 2\}$, where C_i is the circuit family of M_i and $C_i \setminus e = \{C \in C_i : e \notin C\}$.

Proof. This follows from the definition of simple decomposition and (4.1.1). Δ

Where $M \rightarrow \{M_1, M_2\}$, the rank functions of M_1 and M_2 can be described in terms of the rank function of M ; to show this, we need a lemma.

(5.1.5) Lemma. Let $\{M_1, M_2\}$ be the simple decomposition of M associated with $\{E_1, E_2\}$ and the marker e . Then $M_1 \setminus e = M \setminus E_2$ and $M_1 / e = M / E_2$.

Proof. By the non-separability of M , there exists an M -circuit C meeting E_1 and E_2 . Let $D = E_2 \cap C$ and $x \in D$. By (5.1.3) $M_1 \cong (M \setminus (E_2 \setminus D)) / (D - x)$ with e corresponding to x . By (4.1.4) D is a series set in $M \setminus (E_2 \setminus D)$; thus $D - x$ is a set of isthmuses, hence a separator of $M \setminus (E_2 \setminus D) \setminus x$. So $M_1 \setminus e =$ (by (5.1.3)) $((M \setminus (E_2 \setminus D)) / (D - x)) \setminus x = (M \setminus (E_2 \setminus D) \setminus x) / (D - x) =$ (by (2.4.1)) $(M \setminus (E_2 \setminus D) \setminus x) \setminus (D - x) = M \setminus E_2$.

Now $M_1 / e =$ (by (5.1.3)) $(M \setminus (E_2 \setminus D)) / D = (M / D) \setminus (E_2 \setminus D) =$ (by (2.3.3)) $((M^* \setminus D) / (E_2 \setminus D))^*$. By (3.3.3), since C is a co-

circuit of M^* , we see that $E_2 \setminus D$ is a separator of $M^* \setminus D$.

Thus $((M^* \setminus D)/(E_2 \setminus D))^* = ((M^* \setminus D) \setminus (E_2 \setminus D))^* = (M^* \setminus E_2)^* = M/E_2$.

Hence $M_1/e = M/E_2$. Δ

(5.1.6) Theorem. Let $M \rightarrow \{M(E_1;e), M(E_2;e)\}$. Then the rank function r_1 of $M(E_1;e)$ can be derived from the rank function r of M by the formula: $r_1(A) = r(A)$, $r_1(A+e) = r(A \cup E_2) - r(E_2) + 1$ for $A \subseteq E_1$, and similarly for M_2 .

Proof. Suppose $A \subseteq E_1$. By lemma (5.1.5), $M_1 \setminus e = M \setminus E_2$, so clearly $r_1(A) = r(A)$. Let r, r_1, r' , and r'_1 be the rank functions of $M, M_1, M/E_2$, and M_1/e respectively. Then $r'_1 = r'$, since $M_1/e = M/E_2$. Now $r_1(A+e) =$ (by (2.2.7)) $r'_1(A) + r_1(e) = r'(A) + 1 =$ (by (2.2.7)) $r(A \cup E_2) - r(E_2) + 1$. Δ

(5.1.7) Proposition. Let $M \rightarrow \{M(E_1;e), M(E_2;e)\}$. Then $M(E_i;e)$ is non-separable, for $i = 1, 2$.

Proof. We prove this for $i = 1$. Let r_1 be the rank function of $M(E_1;e)$; and let $A \subseteq E_1$. Then $r_1(A) + r_1((E_1 \setminus A) + e) - r(E_1 + e) =$ (by (5.1.6)) $r(A) + r(E \setminus A) - r(E_2) + 1 - r(E) + r(E_2) - 1 = r(A) + r(E \setminus A) - r(E)$. Thus $M(E_1;e)$ is non-separable, since M is. Δ

Let $\{E_1, E_2\}$ be a 2-separation of M . By (2.5.1), $\{E_1, E_2\}$ is also a 2-separation of M^* , the dual of M . Hence corresponding

to a 2-separation $\{E_1, E_2\}$ of M , there are two simple decompositions associated with it; one is defined in M , the other one in M^* .

(5.1.8) Proposition. Let $\{M_1, M_2\}$ (respectively $\{N_1, N_2\}$) be the simple decomposition of M (respectively M^*) associated with $\{E_1, E_2\}$ and the marker e . Then $N_1 = M_1^*$ and $N_2 = M_2^*$.

Proof. Let r, r_1 , and r_0 be the rank functions of M, M_1 , and N_1 respectively. We have to show that $r_1^* = r_0$. Let $A \subseteq E_1$.

$$\begin{aligned} r_1^*(A) &= (\text{by (2.3.1)}) |A| + r_1(E_1 \setminus A + e) - r_1(E_1 + e) \\ &= (\text{by (5.1.6)}) |A| + r(E \setminus A) - r(E_2) + 1 - r(E) + r(E_2) - 1 \\ &= |A| + r(E \setminus A) - r(E) = (\text{by (2.3.1)}) r^*(A) = (\text{by (5.1.6)}) r_0(A) \\ r_1^*(A + e) &= |A + e| + r_1(E_1 \setminus A) - r_1(E_1 + e) = |A + e| + r(E_1 \setminus A) - r(E) + r(E_2) - 1 \\ &= |A| + r(E_1 \setminus A) - r(E) + r(E_2). \end{aligned}$$

On the other hand $r_0(A + e) = (\text{by (5.1.6)}) r^*(A \cup E_2) - r^*(E_2) + 1 =$

$(\text{by (2.3.1)}) |A \cup E_2| + r(E_1 \setminus A) - r(E) - |E_2| - r(E_1) + r(E) + 1 =$
 $|A| + r(E_1 \setminus A) - r(E_1) + 1$. We note that $r(E_1) + r(E_2) = r(E) + 1$,

since $\{E_1, E_2\}$ is a 2-separation of the non-separable matroid M .

Thus $r_1^*(A + e) = r_0(A + e)$. Δ

The following two results are consequences of (5.1.8).

(5.1.9) Corollary. Let $M \rightarrow \{M(E_1; e), M(E_2; e)\}$. Then the set of cocircuits of $M(E_1; e)$ is $\{Y \subseteq E_1; Y \text{ an } M\text{-cocircuit}\} \cup \{Y \cap E_1 + e; Y \text{ an } M\text{-cocircuit meeting } E_1 \text{ and } E_2\}$. Δ

(5.1.10) Corollary. Let $M \rightarrow \{M(E_1; e), M(E_2; e)\}$. Let Y be an M -cocircuit meeting E_1 and E_2 , $D = Y \cap E_1$, and $x_i \in Y \cap E_i$, $i = 1, 2$. Then $M(E_i; e) \cong (M/(E_{3-i} \setminus D_{3-i})) \setminus (D_{3-i} \setminus x_{3-i})$ with $e \rightarrow x_{3-i}$, $i = 1$ and 2 .

Proof. By (5.1.3) and (5.1.8), we have $(M(E_i; e))^* \cong (M^* \setminus (E_{3-i} \setminus D_{3-i})) / (D_{3-i} \setminus x_{3-i}) = (\text{by (2.3.3)}) ((M/(E_{3-i} \setminus D_{3-i})) \setminus (D_{3-i} \setminus x_{3-i}))^*$ with $e \rightarrow x_{3-i}$, $i = 1$ and 2 . Δ

Let $\{M_1, M_2\}$ be the simple decomposition of M associated with the 2-separation $\{E_1, E_2\}$ and the marker e . Let S, T be two disjoint subsets of E_2 such that $(M \setminus S)/T$ is non-separable and $|E_2 \setminus (S \cup T)| \geq 2$. By (2.5.4), $\{E_1, E_2 \setminus (S \cup T)\}$ is a 2-separation of $(M \setminus S)/T$. Now let $\{N_1, N_2\}$ be the simple decomposition of $(M \setminus S)/T$ associated with $\{E_1, E_2 \setminus (S \cup T)\}$ and the marker e . We have the following proposition:

(5.1.11) Proposition. Under the above assumptions, $N_1 = M_1$ and $N_2 = (M_2 \setminus S)/T$.

Proof. First we show $N_2 = (M_2 \setminus S)/T$. Since $(M \setminus S)/T$ is non-separable, there exists a circuit C of $(M \setminus S)/T$ meeting E_1 and $E_2 \setminus (S \cup T)$. By definition of contraction, there exists an M -circuit C' with $C' \cap (E \setminus S) = C$ and $C' \subseteq C \cup T$. Let $D_1 = E_1 \cap C$, $D'_1 = E_1 \cap C'$, and $x \in D_1$. Notice that $D_1 = D'_1$. By (5.1.3),

we have $N_2 \sim ((M \setminus S)/T) \setminus (E_1 \setminus D_1) / (D_1 - x_1) = ((M \setminus (E_1 \setminus D_1)) / (D_1 - x_1)) \setminus S/T$
and $M_2 \sim (M \setminus (E_1 \setminus D_1)) / (D_1 - x_1)$. Thus $N_2 = (M_2 \setminus S)/T$.

We use the rank functions to show that $N_1 = M_1$. Let r, r', r_1 , and r_0 be the rank functions of $M, M \setminus S/T, M_1$, and N_1 respectively. Let $A \subseteq E_1$. Then $r_0(A) =$ (by (5.1.6)) $r'(A) =$ (by (2.2.7)) $= r(A \cup T) - r(T)$. We want to show that $r(A \cup T) = r(A) + r(T)$. Since $M \setminus S/T$ is non-separable, there is a cocircuit Y of $M \setminus S/T$ meeting E_1 and $E_2 \setminus (S \cup T)$; hence there is a cocircuit Y' of M/T meeting E_1 and $E_2 \setminus T$. Now Y' is also a cocircuit of M meeting E_1 and E_2 with $Y' \cap T = \emptyset$. By (3.3.3), E_1 and $E_2 \setminus Y'$ are separators of $M \setminus (E_2 \cap Y')$. Then by (2.4.6), A and T are separators of $M \setminus (E \setminus (A \cup T))$, and thus $r(A \cup T) = r(A) + r(T)$. Now $r_0(A) = r(A \cup T) - r(T) = r(A) + r(T) - r(T) = r(A)$, and $r_1(A) =$ (by (5.1.6)) $= r(A)$. Thus $r_0(A) = r_1(A)$.

Let $A \subseteq E_1$, $r_0(A+e) =$ (by (5.1.6)) $r'(A \cup (E_2 \setminus (S \cup T))) - r'(E_2 \setminus (S \cup T)) + 1 =$ (by (2.2.7)) $r(A \cup (E_2 \setminus S)) - r(T) - r(E_2 \setminus S) + r(T) + 1 = r(A \cup (E_2 \setminus S)) - r(E_2 \setminus S) + 1$; and $r_1(A+e) = r(A \cup E_2) - r(E_2) + 1$. We observe that $r(A) + 1 \geq r_0(A+e) \geq r(A)$ and $r(A) + 1 \geq r_1(A+e) \geq r(A)$, since $r_0(A) = r_1(A) = r(A)$. Furthermore $r_0(A+e) \geq r_1(A+e)$, since $r(A \cup (E_2 \setminus S)) - r(E_2 \setminus S) - r(A) \geq$ (by (2.1.9)) $r(A \cup E_2) - r(E_2) - r(A)$. Suppose $r_0(A+e) \neq r_1(A+e)$,

by the above observations, we have $r_0(A+e) = r(A) + 1$ and $r_1(A+e) = r(A)$. Thus $r(A \cup (E_2 \setminus S)) = r(E_2 \setminus S) + 1 = r(A) + 1$ and $r(A \cup E_2) = r(E_2) + 1 = r(A)$, that is, $r(A) + r(E_2 \setminus S) = r(A \cup (E_2 \setminus S))$ and $r(A) + r(E_2) = r(A \cup E_2) + 1$. Therefore A is not a separator of $M \setminus (E \setminus (A \cup E_2))$, but A is a separator of $M \setminus (E \setminus (A \cup (E_2 \setminus S)))$. Then it is not difficult to see that there is a cocircuit Y of $M \setminus (E \setminus (A \cup E_2))$ with $Y \subseteq A \cup S$ and $Y \cap S \neq \emptyset$. By (2.3.4), there is a cocircuit Y' of M with $Y' \cap (A \cup E_2) = Y$. We note that Y' is a cocircuit of M meeting E_1 and E_2 with $Y' \cap E_2 \subseteq S$. Now by (3.3.3), E_1 and $E_2 \setminus Y'$ are separators of $M \setminus (E_2 \cap Y')$. Since $E_2 \cap Y' \subseteq S$, by (2.4.6), E_1 and $E_2 \setminus (S \cup T)$ are separators of $M \setminus S/T$, contradicting the non-separability of $M \setminus S/T$. Thus $r_0(A+e) = r_1(A+e)$. So $N_1 = M_1$. Δ

5.2 Uniqueness of matroid decomposition

In this section, we develop the uniqueness theory of the matroid decomposition. The references for this subject are Cunningham and Edmonds [6], Cunningham [5]. Most of the materials in this section are gathered from the above references.

Let M be a non-separable matroid. A decomposition D of M is defined inductively as follows:

- (1) either $D = \{M\}$;

(ii) or D is a set of matroids obtained from a decomposition

D' of M by replacing a member M_1 of D' by the members of a simple decomposition M_1 , where the marker of this simple decomposition is not a cell of any member of D' .

If a decomposition D is obtained from D' by a (non-empty) sequence of operations of the kind described in (ii), then D is called a (strict) refinement of D' . If the sequence consists of exactly one operation, the refinement is simple. A decomposition of M consisting of 3-connected matroids is called a prime decomposition. It is clear that every non-separable matroid has at least one prime decomposition. Two decompositions D, D' of M are equivalent, if D' can be obtained from D by replacing some of the markers of D by markers of D' .

We can associate a graph T with any decomposition D of a non-separable matroid M in the following way: The vertices of T are the members of D , two members of D are joined by an edge e if and only if they share a common marker e ; thus the edges of T correspond to markers of D . It is clear that T is a tree. This "decomposition tree" provides a useful way to visualize a decomposition.

A non-separable matroid may have two prime decompositions

which are not equivalent, as we can see from the following two important classes of matroids. A polygon is a matroid M such that E is a circuit (hence the only circuit) of M . A bond is a non-null matroid M such that every two-element subset of E is a circuit. Where M is a polygon or a bond, from (4.1.1), it is easy to see that every partition $\{E_1, E_2\}$ of $E(M)$ with $|E_1|, |E_2| \geq 2$ is a 2-separation of M . Thus, a polygon or a bond with 4 cells has three inequivalent prime decompositions. A polygon or a bond with 6 cells even has two non-equivalent prime decompositions with non-isomorphic decomposition tree structure. However, for any non-separable matroid M , there is a unique decomposition in the following sense. Take a prime decomposition D of M , and, whenever two members N_1, N_2 of D share a common marker, and they are both polygons or bonds, combine N_1 and N_2 to get a bigger polygon or bond N . (That is, $\{N_1, N_2\}$ is a simple decomposition of N , and we reverse the operation of decomposing.) This process is repeated until we get a decomposition D' of M from D such that no two polygons or bonds share a marker. Then this last decomposition D' of M is uniquely determined. We note that each member of D' is 3-connected, a polygon, or a bond.

(5.2.1) Theorem. Every non-separable matroid has a unique decomposition D (up to equivalence) consisting of 3-connected matroids, polygons, and bonds, such that no two polygons or bonds share a marker.

The proof of (5.2.1) appears at the end of this section. Before proving it, we derive some elementary properties of simple decompositions.

(5.2.2) Proposition. Let $M \rightarrow \{M(E_1;e), M(E_2;e)\}$ and $\{E_3, E_4\}$ be a partition of E such that, $E_3 \subset E_1$. Then $\{E_3, E_4\}$ is a 2-separation of M if and only if $\{E_3, E_1 \setminus E_3 + e\}$ is a 2-separation of $M(E_1;e)$.

Proof. It is clear that $|E_3| \geq 2 \leq |E_4|$ if and only if $|E_3| \geq 2 \leq |E_1 \setminus E_3 + e|$. Let r and r_1 be the rank functions of M and $M(E_1;e)$ respectively. Then $r_1(E_3) + r_1(E_1 \setminus E_3 + e) - r_1(E_1 + e) =$ (by (5.1.6)) $r(E_3) + r(E_4) - r(E_2) + 1 - r(E) + r(E_2) - 1 = r(E_3) + r(E_4) - r(E)$, the result follows. Δ

(5.2.3) Proposition. Let $\{E_1, E_2\}, \{E_3, E_4\}$ be 2-separations of the non-separable matroid M with $E_3 \subset E_1$. Then $M(E_1;e)(E_3;f) = M(E_3;f)$ and $M(E_1;e)(E_1 \setminus E_3 + e;f) = M(E_4;f)(E_1 \setminus E_3 + f;e)$.

Proof. Let r, r_1, r_5, r_6 , and r_7 be the rank functions of $M, M(E_1;g), M(E_1;e)(E_3;f), M(E_1;e)(E_1 \setminus E_3 + e;f)$, and $M(E_4;f)(E_1 \setminus E_3 + f;e)$

respectively, where $i = 1$ to 4 , $g = e$ or f . It is sufficient to show that $r_5 = r_3$ and $r_6 = r_7$. By using the formula in (5.1.6) and routine case-checking, the result follows. For example, let $A \subseteq E_3$, $r_5(A+f) = r_1(A \cup (E_1 \setminus E_3 + e)) - r_1(E_1 \setminus E_3 + e) + 1 = r(A \cup (E_1 \setminus E_3) \cup E_2) - r(E_2) + 1 - r((E_1 \setminus E_3) \cup E_2) + r(E_3) - 1 + 1 = r(A \cup E_4) - r(E_4) + 1 = r_3(A+f)$. Δ

A set of 2-separations of a matroid M is compatible if no two members of the set cross. Let D be a decomposition of M . A 2-separation $\{A, B\}$ of a member M' of D generates a unique simple refinement D' of D , and any other 2-separation of M' which does not cross $\{A, B\}$ induces, by (5.2.2), a unique 2-separation of a member of D' . Thus an ordered set $\{(A_i, B_i) : 1 \leq i \leq k\}$ of compatible 2-separations of M generates a unique decomposition of M . The next theorem says that this decomposition does not depend on the order. It is not difficult to prove it, but we shall just omit the proof.

(5.2.4) Theorem. For a non-separable matroid M , any set of compatible 2-separations of M generates a unique decomposition.

Conversely, by (5.2.2) and the definition of decomposition, every decomposition D of M gives rise to a set of compatible 2-separations of M , one for each simple refinement in a derivation of M .

A 2-separation of a non-separable matroid M is said to be good if it is crossed by no other 2-separation of M . Clearly a 3-connected matroid does not have any good 2-separation; but the converse is not true. For example, it is easy to see that polygons and bonds have no good 2-separation. We shall see later that if a matroid does not have a good 2-separation then it must be 3-connected, a polygon, or a bond. In the following, we derive some important properties of the good 2-separations.

(5.2.5) Proposition. Let $\{E_1, E_2\}$ and $\{E_3, E_4\}$ be two 2-separations of the non-separable matroid M on E . If $E_1 \cup E_3 \neq E$ then $r(E_1 \cap E_3) + r(E_2 \cup E_3) \leq r(E) + 1$. In particular, if $E_1 \cup E_3 \neq E$ and $|E_1 \cap E_3| \geq 2$, then $\{E_1 \cap E_3, E_2 \cup E_4\}$ is a 2-separation of M .

Proof. $2r(E) + 2 = r(E_1) + r(E_2) + r(E_3) + r(E_4) \geq$ (by (2.1.8)) $r(E_1 \cap E_3) + r(E_1 \cup E_3) + r(E_2 \cap E_4) + r(E_2 \cup E_4)$. We observe that $r(E_1 \cup E_3) + r(E_2 \cap E_4) > r(E)$, because M is non-separable. Thus $r(E_1 \cap E_3) + r(E_2 \cup E_4) \leq r(E) + 1$. Δ

(5.2.6) Proposition. Let $M \rightarrow \{M(E_1; e), M(E_2; e)\}$, and $\{E_3, E_4\}$ be a partition of E with $E_3 \supseteq E_1$. Then $\{E_3, E_4\}$ is a good 2-separation of M if and only if $\{E_3, E_1 \setminus E_3 + e\}$ is a good 2-separation of $M(E_1; e)$.

Proof. By (5.2.2), $\{E_3, E_4\}$ is a 2-separation of M if and only if $\{E_3, E_1 \setminus E_3 + e\}$ is a 2-separation of $M(E_1; e)$.

Suppose that $\{E_3, E_4\}$ is good, but $\{E_3, E_1 \setminus E_3 + e\}$ is not good. Then there exists a 2-separation $\{A, E_1 \setminus A + e\}$ of $M(E_1; e)$ which crosses $\{E_3, E_1 \setminus E_3 + e\}$. By (5.2.2), $\{A, E(M) \setminus A\}$ is a 2-separation of M , and it is easy to check that it crosses $\{E_3, E_4\}$, a contradiction. Thus $\{E_3, E_1 \setminus E_3 + e\}$ is good.

Now suppose that $\{E_3, E_1 \setminus E_3 + e\}$ is good, but $\{E_3, E_4\}$ is not.

Then there exists a 2-separation $\{E_5, E_6\}$ of M which crosses

$\{E_3, E_4\}$. Suppose that either $E_5 \subseteq E_1$ or $E_6 \subseteq E_1$, say the

former; then $\{E_5, E_1 \setminus E_5 + e\}$ is a 2-separation of $M(E_1; e)$ which

crosses $\{E_3, E_1 \setminus E_3 + e\}$, a contradiction. Thus $E_5 \cap E_2 \neq \emptyset$ and

$E_6 \cap E_2 \neq \emptyset$. Since $\{E_5, E_6\}$ is a partition of $E(M)$ and $E_1 \setminus E_3 \neq \emptyset$,

we may assume that $E_5 \cap (E_1 \setminus E_3) \neq \emptyset$. Considering the 2-separations

$\{E_5, E_6\}$ and $\{E_1, E_2\}$ of M , we see that $|E_5 \cap E_1| \geq 2$ and

$E_5 \cup E_1 \neq E(M)$, so $\{E_5 \cap E_1, E_2 \cup E_6\}$ is a 2-separation of M by

(5.2.5). Notice that $E_5 \cap E_1 \subseteq E_1$, by (5.2.2), $\{E_5 \cap E_1, E_1 \setminus (E_5 \cap E_1) + e\}$

is a 2-separation of $M(E_1; e)$. It is easy to see that $\{E_5 \cap E_1,$

$E_1 \setminus (E_5 \cap E_1) + e\}$ crosses $\{E_3, E_1 \setminus E_3 + e\}$, a contradiction. So

$\{E_3, E_4\}$ is good. Δ

The following theorem provides a useful characterization of the good 2-separations.

(5.2.7) Theorem. Let M be a non-separable matroid. A 2-separation $\{E_1, E_2\}$ of M is good if and only if at least one of $M \setminus E_1$, $M \setminus E_2$ and at least one of M/E_1 , M/E_2 are non-separable.

Proof. First we prove the "if" part. By interchanging E_1 with E_2 if necessary, we may assume that M/E_2 is non-separable. Suppose that $\{E_1, E_2\}$ is not good; then there exists a 2-separation $\{A_1, A_2\}$ of M which crosses it. Let r, r' be the rank functions of $M, M/E_2$ respectively, and let $x_{i,j} = E_i \cap A_j$, for $i = 1, 2$, and $j = 1, 2$. By the non-separability of M/E_2 , we have:

$$\begin{aligned} 0 &> r'(E_1) - r'(x_{1,1}) - r'(x_{1,2}) \\ &= (\text{by (2.2.7)}) \quad r(E) - r(E_2) - r(x_{1,1} \cup E_2) + r(E_2) - r(x_{1,2} \cup E_2) + r(E_2) \\ &= r(E) - r(x_{1,1} \cup E_2) - r(x_{1,2} \cup E_2) + r(E_2). \end{aligned}$$

So $r(E) + r(E_2) + 1 \leq r(x_{1,1} \cup E_2) + r(x_{1,2} \cup E_2)$. Now we prove that both of $M \setminus E_1$ and $M \setminus E_2$ are separable; if this is true, then, we have a contradiction, and hence $\{E_1, E_2\}$ must be good. Suppose

that $M \setminus E_1$ is non-separable, then $1 + r(E_1) \leq r(x_{1,1}) + r(x_{1,2})$:

Thus $1 + r(E_1) + r(E) + r(E_2) + 1 \leq r(x_{1,1}) + r(x_{1,2}) + r(x_{1,1} \cup E_2) + r(x_{1,2} \cup E_2) = (r(x_{1,1}) + r(x_{1,2} \cup E_2)) + (r(x_{1,2}) + r(x_{1,1} \cup E_2)) \leq (\text{by (5.2.5)}) \quad r(E_1) + 1 + r(E) + 1 = 2r(E) + 2$. But the left hand side of this inequality is $r(E_1) + r(E_2) + r(E) + 2 = r(E) + 1 + r(E) + 2 = 2r(E) + 3$, which is a contradiction. Suppose that $M \setminus E_2$ is non-separable; then $1 + r(E_2) \leq r(x_{2,1}) + r(x_{2,2})$.

Thus $1 + r(E_2) + r(E) + r(E_2) + 1 \leq r(x_{2,1}) + r(x_{2,2}) + r(x_{1,1} \cup E_2) + r(x_{1,2} \cup E_2) = (r(x_{2,1}) + r(x_{1,1} \cup E_2) + (r(x_{2,2}) + r(x_{1,2} \cup E_2))) \leq$ (by 2.1.8) $r(A_1) + r(E_2) + r(A_2) + r(E_2)$. So $r(E) + 2 \leq r(A_1) + r(A_2)$, contradicting the fact that $\{A_1, A_2\}$ is a 2-separation of M . This proves the "if" part.

Now we prove the "only if" part. Suppose that both $M \setminus E_1$ and $M \setminus E_2$ are separable, and let A and B (respectively C and D) be non-empty complementary separators of $M \setminus E_1$ (respectively $M \setminus E_2$). Then $r(A \cup C) + r(B \cup D) \leq$ (by (2.1.9)) $r(A) + r(C) + r(B) + r(D) = r(E_1) + r(E_2) = r(E) + 1$. Thus $\{A \cup C, B \cup D\}$ is a 2-separation, clearly it crosses $\{E_1, E_2\}$. This contradicts the assumption that $\{E_1, E_2\}$ is a good 2-separation of M . Thus at least one of $M \setminus E_1, M \setminus E_2$ is non-separable. By (2.5.1), the 2-separations and thus the good 2-separations, of M are the same as those of M^* . Hence at least one of $M^* \setminus E_1, M^* \setminus E_2$ is non-separable. By (2.3.3), $(M/E_i)^* = M^* \setminus E_i, i = 1, 2$. Thus by (2.4.2), at least one of $M/E_1, M/E_2$ is non-separable. Δ .

(5.2.8) Theorem. A non-separable matroid M has no good 2-separation if and only if M is 3-connected, a polygon, or a bond.

Proof. The "if" part is trivial. Suppose M is 3-connected; then clearly M has no good 2-separation. Suppose M is a polygon or a bond, but not 3-connected. Then $|E(M)| \geq 4$, and any partition $\{E_1, E_2\}$ of $E(M)$, with $|E_1|, |E_2| \geq 2$, is a 2-separation of M . Thus M has no good 2-separation.

Now suppose that M is non-separable and has no good 2-separation. We prove by induction on $|E(M)|$ that M is 3-connected, a polygon, or a bond, if $|E(M)| \leq 3$. Then clearly M is 3-connected. Suppose that the result is true for all M with $|E(M)| \leq n \geq 3$, and suppose that we are given M with $|E(M)| = n+1$, where M is non-separable and has no good 2-separation. Suppose that M is not 3-connected; then there exists a 2-separation $\{E_1, E_2\}$ of M . Let $\{M_1, M_2\}$ be the simple decomposition of M associated with $\{E_1, E_2\}$ and the marker e . Clearly $|E(M_i)| < |E(M)|$, for $i = 1, 2$. By (5.2.6), M_1 and M_2 have no good 2-separation. Thus, by induction hypothesis, M_i is 3-connected, a polygon, or a bond, for $i = 1, 2$.

We prove that M_i is a polygon, or a bond, for, $i = 1, 2$.

Notice that $|E_1| \geq 2$. If $|E_1| = 2$ then $|E(M_1)| = 3$. Since M_1 is non-separable, it is not difficult to see that a non-separable matroid with only 3 cells is a polygon, or a bond. Suppose that $|E_1| \geq 3$. Since M has no good 2-separation, there is a 2-separation

$\{E_3, E_4\}$ crossing $\{E_1, E_2\}$. One of $E_1 \cap E_3$, $E_1 \cap E_4$ has cardinality at least two, since $|E_1| \geq 3$, say the former. By (5.2.5) and (5.2.2), $\{E_1 \cap E_3, (E_1 \setminus (E_1 \cap E_3)) + e\}$ is a 2-separation of M_1 .

Thus M_1 is not 3-connected, and so it must be a polygon, or a bond, and similarly for M_2 .

Suppose that both M_1 and M_2 are polygons (respectively bonds), then by (5.1.4), M is a polygon (respectively bond). Suppose that one of M_1, M_2 is a polygon, and the other is a bond; say M_1 is a polygon and M_2 is a bond. By (5.1.5), we have $M/E_2 = M_1/e$ and $M \setminus E_1 = M_2 \setminus e$. It is clear that M_1/e is a polygon, since M_1 is, and $M_2 \setminus e$ is a bond, since M_2 is. Thus M/E_2 and $M \setminus E_1$ are non-separable. Now by (5.2.7), $\{E_1, E_2\}$ is a good 2-separation of M , contradicting the fact that M has no good 2-separation. This completes the induction proof. Δ

Now we prove theorem (5.2.1).

Proof. Let D be a decomposition of M consisting of 3-connected matroids, polygons, and bonds, such that no two polygons or bonds share a marker. Suppose that D is generated by the set S of 2-separations of M . Let D' be the decomposition of M generated by the set of all good 2-separations of M . We prove that D is equivalent to D' ; if this is true, then the uniqueness asserted

in theorem (5.2.1) is established. Suppose that there is a good 2-separation of M which is not in S ; then by (5.2.6), there is a member of D having a good 2-separation. By (5.2.8), this is a contradiction, since each member of M is 3-connected, a polygon, or a bond. Thus D is a refinement of D' . We observe that, by (5.2.6) and (5.2.8), each member of D' is 3-connected, a polygon, or a bond; a 3-connected matroid does not have any 2-separation, and a simple decomposition of a polygon (respectively bond) consists of two polygons (respectively two bonds) sharing a marker. Therefore, if D were not equivalent to D' , there would exist two polygons or bonds in D sharing a marker, a contradiction. Thus D is equivalent to D' . Δ

We call the decomposition of M whose uniqueness is asserted in (5.2.1), the standard decomposition of M . It follows from the proof of (5.2.1) that the standard decomposition of M is generated by the set of all the good 2-separations of M .

Cunningham and Edmonds (see [5]) have given an algorithm for finding a k -separation, if one exists, for any fixed k ; it requires the order of $|E(M)|^k$ applications of the major step of Edmonds' matroid partition algorithm. Considering the case that M is a matric matroid represented by a matrix, it is shown in [7]

that this algorithm for finding a k -separation of M has computation bound $O(rc^{k+1})$, where r and c are the numbers of rows and columns of M respectively. These methods lead to an $O(rc^3)$ algorithm for finding a prime decomposition of a matric matroid.

Having a prime decomposition of M , we then combine as much as possible polygons (respectively bonds) sharing a marker, and get the standard decomposition of M . It can be shown that the computational work for this combining procedure is dominated by $O(rc^3)$. Hence, there is an algorithm for finding the standard decomposition of a matric matroid; its computational effort is bounded by $O(rc^3)$. In the remainder of this chapter, we derive an algorithm for finding the standard decomposition of a matroid M , based on Bixby and Cunningham's theorem (3.3.5). Under appropriate assumptions, we can show that the computational effort of our algorithm is bounded by the order of $(r(M))^2 |E(M)|$; in particular, for a matric matroid, it is bounded by $O(r^2c)$. Another interesting point about our algorithm is that we always find good 2-separations instead of arbitrary 2-separations of M . Thus we avoid the last procedure (combining polygons and bonds).

To close this section, we prove theorem (4.4.2) which is stated without proof in chapter 4. The theorem is stated as follows:

A non-separable matroid is a series-parallel matroid if and only if it has no non-trivial 3-connected minor.

Proof. Suppose that M is a series-parallel matroid. We prove by induction on $|E|$ that M has no non-trivial 3-connected minor. This is clearly true for $|E| \leq 3$. Suppose that the result is true whenever the matroid has fewer than $k \geq 4$ elements, and suppose that the matroid M in question has k elements. Since M is a non-separable series-parallel matroid, there are two elements x, y of M in series (or in parallel), such that M is a series extension of M/y (or parallel extension of $M \setminus y$), and M/y (or $M \setminus y$) is a series-parallel matroid. If M had a non-trivial 3-connected minor N then by (4.1.9), $M/y \geq N$ (or $M \setminus y \geq N$), but by the induction hypothesis M/y (respectively $M \setminus y$) has no non-trivial 3-connected minor, a contradiction. Thus M has no non-trivial 3-connected minor.

Now suppose that M has no non-trivial 3-connected minor. Again we prove by induction on $|E|$ that M is a series-parallel matroid. If $|E| \leq 3$, since M is non-separable, it is easy to see that all the elements of M are either in parallel or in series, and so M is a series-parallel matroid. Suppose that the result is true for every matroid having fewer than $k \geq 4$ elements, and

suppose that the matroid in question has k elements. M is not 3-connected, for otherwise M itself is a non-trivial 3-connected minor of M . Let $\{E_1, E_2\}$ be a 2-separation of M with $|E_1|$ minimal. We claim that $|E_1| = 2$. Suppose not, then $|E_1| \geq 3$. Let $\{M_1, M_2\}$ be the simple decomposition of M associated with the 2-separation $\{E_1, E_2\}$ and the marker e . Clearly $|E(M_1)| \geq 4$. Suppose that M_1 is 3-connected. Then by (5.1.3), M_1 is isomorphic to a minor of M , contradicting the fact that M has no non-trivial 3-connected minor. Thus M_1 is not 3-connected. Let $\{A_1, A_2\}$ be a 2-separation of M_1 with $e \in A_2$ (the marker). Then by (5.2.2) $\{A_1, E \setminus A_1\}$ is a 2-separation of M . But $|A_1| < |E_1|$, and this contradicts the minimality of E_1 . Therefore E_1 has only two elements, say x, y . Now by (4.1.6), since $\{E_1, E_2\}$ is a 2-separation, $E_1 = \{x, y\}$ is either a circuit or a cocircuit of M . If $\{x, y\}$ is a circuit (respectively cocircuit), then M is a parallel extension of $M \setminus y$ (respectively a series extension of M/y). $M \setminus y$ (respectively M/y) has no non-trivial 3-connected minor, since M does not. By the induction hypothesis, $M \setminus y$ (respectively M/y) is a series-parallel matroid, and so is M . This completes the induction proof. Δ

5.3 Good 2-separations and simplified Y-components

Let M be a non-separable matroid. We seek an efficient algorithm for finding the standard decomposition of the matroid M . In order to do this, we need an efficient algorithm to find all the good 2-separations of M . Theorem (3.3.5) suggests a method, but there are two difficulties associated with it. First, if the matroid M is not 3-connected, the proof of (3.3.5) does not provide a particularly easy way actually to find the 2-separation. Second, there is no guarantee that the 2-separation we find is a good 2-separation. Therefore we shall study more carefully about the good 2-separations of a matroid, and their relationship with the Y-components and the bridge graph of Y , for any cocircuit Y .

(5.3.1) In this section and the remainder of this chapter, unless otherwise specified, we shall assume that M is a non-separable matroid, Y is a cocircuit of M , M_1 is the Y-component corresponding to the bridge B_1 of Y in M , and $M_1[Y]$ is the simplified Y-component corresponding to the bridge B_1 . As a first step, we have the following relationship between any 2-separation of M and the bridges of Y in M , for any cocircuit Y .

(5.3.2) Proposition. Under the assumptions of (5.3.1), let $\{E_1, E_2\}$ be a 2-separation of M . Then $\{E_1, E_2\}$ does not cross

$\{B_i, E(M) \setminus B_i\}$, for any bridge B_i of Y in M . Moreover, if $E_1 \subseteq B_i$, then $\{E_1, E(M_1[Y]) \setminus E_1\}$ is a 2-separation of $M_1[Y]$.

Proof. Suppose that $\{E_1, E_2\}$ crosses $\{B_i, E(M) \setminus B_i\}$, for some bridge B_i of Y . First we prove that Y cannot meet both E_1 and E_2 . For otherwise, by (2.5.3), $\{A_1, A_2\} = \{E_1 \cap (B_i \cup Y), E_2 \cap (B_i \cup Y)\}$ is a 2-separation of M_1 with Y meeting A_1 and A_2 , then $A_1 \setminus Y$ and $A_2 \setminus Y$ are non-null separators of $M_1 \setminus Y$ by (3.3.3), contradicting the fact that Y is a non-separating cocircuit of M_1 . So we may assume that $E_1 \cap Y = \emptyset$. Since $\{E_1, E_2\}$ crosses $\{B_i, E \setminus B_i\}$, there is a bridge B_j of Y in M , $B_j \neq B_i$, such that $B_j \cap E_1 \neq \emptyset$. By (2.5.3), we have $r'(E_1 \cap (B_i \cup B_j \cup Y)) + r'(E_2 \cap (B_i \cup B_j \cup Y)) \leq r'(B_i \cup B_j \cup Y) + 1$, where r' is the rank function of $M/(E \setminus (B_i \cup B_j \cup Y))$. Note that if $A \neq \emptyset$ is a proper subset of an elementary separator of M , then $r(A) + r(E \setminus A) \geq r(E) + 1$.

$$\begin{aligned}
 \text{Now } 1 &\geq r'(E_1 \cap (B_i \cup B_j \cup Y)) + r'(E_2 \cap (B_i \cup B_j \cup Y)) - r'(B_i \cup B_j \cup Y) \\
 &= r'(E_1 \cap (B_i \cup B_j)) + r'(E_2 \cap (B_i \cup B_j \cup Y)) - r'(B_i \cup B_j \cup Y) \\
 &= r'(E_1 \cap B_i) + r'(E_1 \cap B_j) + r'(E_2 \cap (B_i \cup B_j \cup Y)) - r'(B_i \cup B_j \cup Y) \\
 &\geq (\text{By the note above}) \quad r'(E_1 \cap B_j) + r'((E_1 \cap B_i) \cup (E_2 \cap (B_i \cup B_j \cup Y))) \\
 &\quad + 1 - r'(B_i \cup B_j \cup Y) \\
 &= r'(E_1 \cap B_j) + r'((E_2 \cap B_j) \cup B_i \cup Y) + 1 - r'(B_i \cup B_j \cup Y) \\
 &\geq (\text{By the note above}) \quad r'(B_i \cup B_j \cup Y) + 1 + 1 - r'(B_i \cup B_j \cup Y) \\
 &= 2.
 \end{aligned}$$

This is a contradiction. Hence $\{E_1, E_2\}$ does not cross $(B_1, E \setminus B_1)$.

Suppose $E_1 \subseteq B_1$, let r , r_1 , and r'_1 be the rank functions of M , M_1 , and $M_1[Y]$ respectively. Then $r'_1(E_1) + r'_1(E(M_1[Y]) \setminus E_1) - r'_1(E(M_1[Y])) = r_1(E_1) + r_1(E(M_1) \setminus E_1) - r_1(E(M_1)) =$ (by 3.2.4) $r(E_1) + r(E \setminus E_1) - r(E) = 1$. It is clear that $|E_1|, |E(M_1[Y]) \setminus E_1| \geq 2$, so $\{E_1, E(M_1[Y]) \setminus E_1\}$ is a 2-separation of $M_1[Y]$. Δ

(5.3.3) Proposition. Let $\{A_1, A_2\}$ be a 2-separation of $M_1[Y]$.

Then either $A_1 \subseteq B_1$ or $A_2 \subseteq B_1$. Furthermore, say $A_1 \subseteq B_1$, then

$\{A_1, E(M) \setminus A_1\}$ is a 2-separation of M .

Proof. Since $\hat{Y}$ is a non-separating cocircuit of $M_1[Y]$, and $\hat{Y}$ is simple in $M_1[Y]$, by (3.3.3), either $\hat{Y} \cap A_1 = \emptyset$ or $\hat{Y} \cap A_2 = \emptyset$, thus either $A_1 \subseteq B_1$ or $A_2 \subseteq B_1$.

Suppose $A_1 \subseteq B_1$; then clearly $\{A_1, E(M_1) \setminus A_1\}$ is a 2-separation of M_1 . Hence, it follows from (3.2.4) that $\{A_1, E(M) \setminus A_1\}$ is a 2-separation of M . Δ

Let $M_1[Y]$ be the simplified Y -component of M corresponding to the bridge B_1 . A 2-separation $\{E_1, E_2\}$ of M is said to be concealed in $M_1[Y]$, if $E_1 \subseteq B_1$ or $E_2 \subseteq B_1$. Since we are only interested in good 2-separations of M , the following result strengthens (5.3.2) and (5.3.3).

(5.3.4) Theorem. Let $A \subseteq B_i$; then $\{A, E(M_i[Y]) \setminus A\}$ is a good 2-separation of $M_i[Y]$ if and only if $\{A, E(M) \setminus A\}$ is a good 2-separation of M .

Proof. First we prove the "only if" part. By (5.3.3), $\{A, E(M) \setminus A\}$ is a 2-separation of M . Suppose that it is not a good 2-separation of M . Then there is a 2-separation $\{E_1, E_2\}$ of M that crosses $\{A, E(M) \setminus A\}$. Since $A \subseteq B_i$, and by (5.3.2), $\{E_1, E_2\}$ cannot cross $\{B_i, E(M) \setminus B_i\}$. Interchanging E_1 and E_2 if necessary, we may assume that $E_1 \subseteq B_i$. Then by (5.3.2), $\{E_1, E(M_i[Y]) \setminus E_1\}$ is a 2-separation of $M_i[Y]$. Now, A is not a proper subset of B_i , for otherwise $\{E_1, E(M_i[Y]) \setminus E_1\}$ would cross $\{A, E(M_i[Y]) \setminus A\}$, contradicting the assumption that $\{A, E(M_i[Y]) \setminus A\}$ is a good 2-separation of $M_i[Y]$. Thus $A = B_i$. But then, since $E_1 \subseteq B_i$, $\{E_1, E_2\}$ does not cross $\{A, E(M) \setminus A\}$, a contradiction. So $\{A, E(M) \setminus A\}$ is a good 2-separation of M .

Now we prove the "if" part. By (5.3.2), $\{A, E(M_i[Y]) \setminus A\}$ is a 2-separation of $M_i[Y]$. Suppose that it is not a good 2-separation. Then there is a 2-separation $\{E_1, E_2\}$ of $M_i[Y]$ that crosses $\{A, E(M_i[Y]) \setminus A\}$. Interchanging E_1 and E_2 if necessary, we may assume, by (5.3.3), that $E_1 \subseteq B_i$. Then $\{E_1, E(M) \setminus E_1\}$ is a 2-separation of M which crosses $\{A, E(M) \setminus A\}$. This is a contradiction,

since $\{A, E(M) \setminus A\}$ is a good 2-separation of M . This completes the proof. Δ

Theorem (5.3.4) provides us the first step of our recursive algorithm to find the standard decomposition of a matroid. The idea is the following. Choose a separating cocircuit Y of M . (We shall see later how we can handle the problem of finding a separating cocircuit.) Each simplified Y -component is a smaller matroid than M , so we may assume recursively that we have the standard decomposition, and thus all the good 2-separations, of each of the simplified Y -components. Now from (5.3.4), we know all the good 2-separations of M concealed in the simplified Y -components.

Let $M \rightarrow \{M(E_1; e), M(E_2; e)\}$. We say $M(E_2; e)$ has been obtained by splitting E_1 from M . Let $\{A_1, A_2\}$ be a 2-separation of $M_1[Y]$ with $A_1 \subseteq B_1$, where $M_1[Y]$ is the simplified Y -component of M corresponding to the bridge B_1 . Then $\{A_1, E(M) \setminus A_1\}$ is a 2-separation of M by (5.3.4). After splitting A_1 from M , let the resulting matroid be $M_2 = M(E(M) \setminus A_1; e)$, we shall see below that Y is still a cocircuit of M_2 . One natural question to ask is how the bridges and Y -components of Y in M_2 relate to those of Y in M ; the following theorem gives an answer to this.

(5.3.5) Theorem. Let $\{A_1, A_2\}$ be a 2-separation of $M_1[Y]$

with $A_1 \subseteq B_1$, where $M_1[Y]$ is a Y -component of M corresponding to the bridge B_1 . Let $\{M_1, M_2\}$ (respectively $\{N_1, N_2\}$) be the simple decomposition of M (respectively $M_1[Y]$) associated with $\{A_1, E(M) \setminus A_1\}$ (respectively $\{A_1, A_2\}$) and the marker e . Then

- (i) Y is a cocircuit of M_2 ;
- (ii) The bridges of Y in M_2 are precisely those bridges of Y in M with B_1 replaced by $(B_1 \setminus A_1) + e$;
- (iii) $\pi(M_2, B, Y) = \pi(M, B, Y)$, for each bridge $B \neq (B_1 \setminus A_1) + e$,
 $\pi(M_2, (B_1 \setminus A_1) + e, Y) = \pi(M, B_1, Y)$;
- (iv) The simplified Y -components of M_2 are precisely those simplified Y -components of M with $M_1[Y]$ replaced by N_2 ;
- (v) $M_1 = N_1$.

Proof. (i) Since Y is a cocircuit of M contained in $E \setminus A_1$, by (2.3.4), Y is a cocircuit of M_2 .

(ii) Let C be an M -circuit meeting both A_1 and $E(M) \setminus A_1$.

(The existence of such C is guaranteed by the non-separability

of M). Let $D = C \cap A_1$, and let $x \in D$. By (5.1.3), $M_2 \cong (M \setminus (A_1 \setminus D)) / (D - x)$

with $e \rightarrow x$. So $M_2 \setminus Y \cong ((M \setminus (A_1 \setminus D)) / (D - x)) \setminus Y = (M \setminus Y \setminus (A_1 \setminus D)) / (D - x)$.

Hence, by (2.4.7), every bridge B of Y in M is a bridge of Y

in M_2 , if $B \neq B_1$. It remains to show that $(B_1 \setminus A_1) + e$ is a

bridge of Y in M_2 , that is, it is an elementary separator of

$M_2 \setminus Y$. Let S_e be the elementary separator of $M_2 \setminus Y$ containing e .

We note that $S_e \subseteq (B_1 \setminus A_1) + e$, and we have $(M_2 \setminus Y) \setminus S_e = (M_2 \setminus Y)/S_e$,

by (2.4.1). Now $(M_2 \setminus Y) \setminus S_e = (M_2 \setminus S_e) \setminus Y = ((M_2 \setminus e) \setminus (S_e - e)) \setminus Y =$

(by (5.1.5)) $((M \setminus A_1) \setminus (S_e - e)) \setminus Y = (M \setminus Y) \setminus (A_1 \cup S_e - e)$. Similarly,

$(M_2 \setminus Y)/S_e = (M \setminus Y)/(A_1 \cup S_e - e)$. Therefore, by (2.4.1), $A_1 \cup S_e - e$

is a separator of $M \setminus Y$. Since $A_1 \cup S_e - e \subseteq B_1$, and B_1 is an

elementary separator of $M \setminus Y$, we have $A_1 \cup S_e - e = B_1$. And so

$S_e = (B_1 \setminus A_1) + e$, since $S_e \subseteq (B_1 \setminus A_1) + e$. This proves that

$(B_1 \setminus A_1) + e$ is a bridge of Y in M_2 .

(iii) $M_2 / ((B_1 \setminus A_1) + e) = (M_2 / e) / (B_1 \setminus A_1) =$ (by (5.1.5))

$(M/A_1) / (B_1 \setminus A_1) = M/B_1$. So, by (3.2.6), $\pi(M_2, B_1, Y) = \pi(M, B_1, Y)$,

for each bridge $B \neq (B_1 \setminus A_1) + e$. And $(M_2 / (E(M_2) \setminus ((B_1 \setminus A_1) + e) \cup Y)) \setminus (B_1 \setminus A_1) + e =$

$((M_2 \setminus e) \setminus (B_1 \setminus A_1)) / (E(M_2) \setminus ((B_1 \setminus A_1) + e) \cup Y) =$ (by (5.1.5)) $((M \setminus A_1) \setminus (B_1 \setminus A_1)) /$

$(E(M_2) \setminus ((B_1 \setminus A_1) + e) \cup Y) = (M \setminus B_1) / (E(M) \setminus (B_1 \cup Y)) = (M / (E(M) \setminus (B_1 \cup Y))) / B_1$

Thus $\pi(M_2, (B_1 \setminus A_1) + e, Y) = \pi(M, B_1, Y)$, by the definition of segment set.

(iv) From (iii), we see that $M_2 / ((B_1 \setminus A_1) + e) = M/B_1$, so we

have only to consider the simplified Y -component of M_2 corresponding

to the bridge $B_1 \setminus A_1 + e$. Since $\pi(M_2, (B_1 \setminus A_1) + e, Y) = \pi(M, B_1, Y)$,

N_2 is the simplified Y -component of M_2 corresponding to $B_1 \setminus A_1 + e$

by (5.1.11).

(v) This is a direct consequence of (5.1.11). Δ

G

G/Y

K₁

K₅

K₂

K₆

K₃

K₇

K₄

K₇

Figure 1

(5.3.6) As an illustration of the above concepts, consider the graphs in Figure 1, and their bond matroids. Let $M = BM(G)$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. (Recall that the bond matroid of a graph G is the dual of the polygon matroid of G , and hence a co-circuit of $BM(G)$ is the edge-set of a polygon of G . By (2.3.3), (2.2.3), and (2.2.6), $BM(G) \setminus A = BM(G/A)$ and $BM(G)/A = BM(G \setminus A)$; we shall use these in the example.) Then $M \setminus Y = BM(G/Y)$, and the bridges of Y are $B_1 = \{11, 12, 13\}$, $B_2 = \{14\}$, $B_3 = \{15\}$, $B_4 = \{16\}$, $B_5 = \{17\}$, $B_6 = \{18, 19, 20\}$, and $B_7 = \{21, 22, 23, 24, 25\}$. The Y -components are given by $M/(E(G) \setminus (B_i \cup Y)) = BM(G \setminus (E(G) \setminus (B_i \cup Y))) = BM(K_i)$, $i = 1, 2, \dots, 7$. $\pi(B_1) = \{\{3\}, \{4\}, \{1, 2, 5, 6, 7, 8, 9, 10\}\}$, $\pi(B_2) = \pi(B_3) = \{\{6, 7\}, \{1, 2, 3, 4, 5, 8, 9, 10\}\}$, $\pi(B_4) = \pi(B_5) = \{\{3, 4\}, \{1, 2, 5, 6, 7, 8, 9, 10\}\}$, $\pi(B_6) = \{\{8\}, \{9, 10\}, \{1, 2, 3, 4, 5, 6, 7\}\}$, and $\pi(B_7) = \{\{2\}, \{3, 4, 5, 6\}, \{1, 7, 8, 9, 10\}\}$. Considering the Y -component M_7 corresponding to the bridge B_7 , the simplified Y -component $M_7[Y]$ is the bond matroid of the graph $\bar{K}_7$. It is not difficult to see that $\{\{23, 24\}, E(M_7[Y]) \setminus \{23, 24\}\}$ is a good 2-separation of $M_7[Y]$ and $\{23, 24\} \subseteq B_7$. Thus $\{\{23, 24\}, E(G) \setminus \{23, 24\}\}$ is a good 2-separation of $BM(G)$ concealed in $M_7[Y]$. The follows graphs illustrate the simple decomposition of $BM(G)$ (respectively $M_7[Y]$) associated with the 2-separation $\{\{23, 24\}, E(G) \setminus \{23, 24\}\}$ (respectively

$\{23, 24\}$, $E(M_7[Y] \setminus \{23, 24\})$. The reader may find it useful to visualize each statement in Theorem (5.3.5) from this example.

We say that a matroid M_2 has been obtained by splitting a good 2-separation from M , if M_2 is obtained by splitting E_1 from M , for some good 2-separation $\{E_1, E_2\}$ of M . A matroid, obtained by splitting from M a set S of good 2-separations, is called a trunk matroid, (obtained by splitting S from M). Let Y be a cocircuit of M , and let $M_1[Y], M_2[Y], \dots, M_k[Y]$ be the simplified Y -components. Let D be the decomposition of M

generated by all the good 2-separations of M which are concealed in its simplified Y -components, and let M^r be the trunk matroid obtained by splitting from M all the good 2-separations of M concealed in the simplified Y -components. By (5.3.5) we know $M^r \in D$. By (5.3.3), (5.3.4) and (5.3.5), it is not difficult to see that, where D_i is the standard decomposition of $M_i[Y]$, there is a unique member $N_i^r \in D_i$ with $E(N_i^r) \cap Y \neq \emptyset$, for each $i = 1, 2, \dots, k$, and $D \setminus \{M^r\} = \bigcup_{i=1}^k (D_i \setminus \{N_i^r\})$. Moreover, the following hold:

- (i) Y is a cocircuit of M^r ;
- (ii) The bridges of Y in M^r are precisely $B_1^r, B_2^r, \dots, B_k^r$,
where $B_i^r = E(N_i^r) \setminus Y$, $i = 1, 2, \dots, k$;
- (iii) $\pi(M^r, B_i^r, Y) = \pi(M, B_i, Y)$;
- (iv) The simplified Y -components of M^r are precisely $N_1^r, N_2^r, \dots, N_k^r$.

Those matroids N_i^r , $i = 1, 2, \dots, k$, mentioned above are called knot matroids (with respect to the matroid M and the cocircuit Y).

We observe that, since $D \setminus \{M^r\} = \bigcup_{i=1}^k (D_i \setminus \{N_i^r\})$ and D_i is the standard decomposition of $M_i[Y]$, $i = 1, \dots, k$, each member of $D \setminus \{M^r\}$ is 3-connected, a polygon, or a bond. Now N_i^r has no good 2-separation, for every $i = 1, 2, \dots, k$, so M^r does not have any good 2-separation concealed in its simplified Y -components. In this sense, we know how to handle the good 2-separations of M concealed in its simplified Y -components, for any cocircuit Y . From now on, we

concentrate on finding the good 2-separations which are not concealed in its simplified Y-components.

Example. Consider the example in (5.3.6) with $M = BM(G)$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. It is easy to see that, for $i = 1$ to 6, the simplified Y-component $M_i[Y]$ is 3-connected, where $M_1 = BM(K_1)$. The standard decomposition of $M_7[Y]$ is illustrated in the next figure.

The decomposition of $BM(G)$, generated by all the good 2-separations of M which are concealed in its simplified Y-components, is illustrated in the following figure.

Before entering the next section, we prove one more proposition.

(5.3.7) Proposition. Let Y be a non-separating cocircuit of a non-separable matroid. Suppose Y is simple in M . Then M has no good 2-separation if and only if M has no 2-separation.

Proof. Suppose that M has no 2-separation; then clearly M has no good 2-separation.

Conversely, suppose that M has no good 2-separation, we present two ways to show that M has no 2-separation:

(i) By (5.2.8), a matroid has no good 2-separation if and only if it is 3-connected, a bond, or a polygon. It is easy to see that M cannot be a bond or a polygon with more than three cells. Thus M is 3-connected.

(ii) Suppose that M has a 2-separation $\{E_1, E_2\}$. Since Y is simple and non-separating, by (3.3.3) Y cannot meet both E_1 and E_2 , say $E_1 \cap Y = \emptyset$. Choose a 2-separation $\{E_1, E_2\}$ of M such that $E_1 \cap Y = \emptyset$, and $|E_1|$ is maximized. Since M has no good 2-separation, there exists another 2-separation $\{A_1, A_2\}$ of M which crosses $\{E_1, E_2\}$. By interchanging A_1 and A_2 if necessary, we may assume that $A_1 \cap Y = \emptyset$. Then, by (5.2.5), $\{E_1 \cup A_1, E_2 \cup A_2\}$ is a 2-separation of M with $(E_1 \cup A_1) \cap Y = \emptyset$ and $|E_1 \cup A_1| > |E_1|$, a contradiction. Thus M has no 2-separation. Δ

(5.3.8) Corollary. Let $M_1[Y]$ be a simplified Y -component of the non-separable matroid M . Then $M_1[Y]$ has no good 2-separation if and only if $M_1[Y]$ is 3-connected.

Proof. By (3.2.7), Y is a non-separating cocircuit of M_1 . Thus $\hat{Y}$ is a non-separating cocircuit of $M_1[Y]$, and is simple in $M_1[Y]$. Δ

5.4 Good 2-separations of type J

Let Y be a cocircuit of the non-separable matroid M . We call a good 2-separation of M which is not concealed in any of its simplified Y -components, a good 2-separation of type J (with respect to the cocircuit Y). (They are caused, roughly speaking, by the way in which Y and its bridges join together.) In this section, we show how to find a type J good 2-separation with respect to Y , for any cocircuit Y , by giving a constructive proof of (3.3.6) which is a key lemma for Bixby and Cunningham's theorem (3.3.5); these contain the main idea of our recursive algorithm for constructing the standard decomposition of a non-separable matroid.

We start with the following proposition.

(5.4.1) Proposition. Let Y be a cocircuit of the non-separable matroid M , and B_1 and B_2 be two bridges of Y with $\pi(B_1) = \pi(B_2)$ and $|\pi(B_1)| = 2$. Let $\{E_1, E_2\}$ be a type J good

2-separation of M with respect to Y . Then either $B_1 \cup B_2 \subseteq E_1$ or $B_1 \cup B_2 \subseteq E_2$.

Proof. Suppose not; then we may assume that $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$ by (5.3.2). Now $\{E_1, E_2\}$ is a good 2-separation of M , and hence at least one of $M \setminus E_1$ and $M \setminus E_2$ is non-separable by (5.2.7). Without loss of generality, we may assume that $M \setminus E_1$ is non-separable; then B_2 is not a separator of $M \setminus B_1$, for otherwise, by (2.4.6), B_2 is a separator of $M \setminus E_1 = M \setminus B_1 \setminus (E_1 \setminus B_1)$, and $E_2 \setminus B_2 \neq \emptyset$, contradicting the assumption that $M \setminus E_1$ is non-separable. Let r and r' be the rank functions of M and $M/(E \setminus (B_1 \cup B_2 \cup Y))$ respectively. It is easy to see that $r'(Y) = 2$ since $\pi(B_1) = \pi(B_2)$ and $|\pi(B_1)| = 2$. Now we have:

$$\begin{aligned} 1 &= r'(B_1 \cup B_2) + r'(Y) - r'(B_1 \cup B_2 \cup Y) \\ &= (\text{by (3.2.4)}) \quad r(B_1 \cup B_2) + r(E \setminus (B_1 \cup B_2)) - r(E) \\ &= r(B_1) + r(B_2) + r(E \setminus (B_1 \cup B_2)) - r(E) \\ &\geq r(B_1) + r(E \setminus B_1) + 1 - r(E) \\ &\geq 1 + 1 \quad (\text{by the non-separability of } M) \end{aligned}$$

This is a contradiction. Thus the result holds. Δ

Bixby and Cunningham's theorem (3.3.5) together with result (5.4.1) lead us to the following idea.

(5.4.2) Let Y be a cocircuit of the non-separable matroid M .

A collection C of the bridges of Y is called a connected component of the bridge graph of Y , if the bridge graph of Y restricted to C is connected and C is maximal with respect to this property. Let $\mathcal{C}$ be the class of all connected components of the bridge graph of Y in M . (From now on, we shall always restrict $\mathcal{C}$ and C to the meanings mentioned above. Notice that we use $\mathcal{C}$ and C to denote the circuit family and a circuit of a matroid respectively in chapter 2; but from this point onward we shall never need to consider the circuit family of a matroid, and therefore there will be no ambiguity.)

We partition $\mathcal{C}$ into two parts $\mathcal{C}_1$ and $\mathcal{C}_2$, such that $C \in \mathcal{C}_2$ if and only if C contains only one bridge B , and $|\pi(B)| = 2$. Therefore $C \in \mathcal{C}_1$ if and only if C contains more than one bridge or C contains only one bridge B and $|\pi(B)| \geq 3$. For each bridge B in $\bigcup\{C: C \in \mathcal{C}_2\}$, let $C(B) = \{B': B' \in \bigcup\{C: C \in \mathcal{C}_2\} \text{ and } \pi(B) = \pi(B')\}$, and let $\mathcal{C}_2^* = \{C(B): B \in \bigcup\{C: C \in \mathcal{C}_2\}\}$. That is, we group together those bridges in $\bigcup\{C: C \in \mathcal{C}_2\}$ having the same segment set π .

Let $\mathcal{C}^*(M, Y) = \mathcal{C}_1 \cup \mathcal{C}_2^*$, with $\mathcal{C}_1$ and $\mathcal{C}_2^*$ defined above, we call $\mathcal{C}^*(M, Y)$ the set of all good 2-components of the bridge graph of Y in M ; we write $\mathcal{C}^*$ or $\mathcal{C}^*(M)$ for $\mathcal{C}^*(M, Y)$, if there is no ambiguity.

We fix an element $f \in Y$. Let B_i and B_j be two bridges of Y , and define $\leq$ by $B_i \leq B_j$ if there exist segments S_i and T_j , $S_i \in \pi(B_i)$ and $T_j \in \pi(B_j)$, such that $f \in S_i$, $f \notin T_j$, and $S_i \cup T_j = Y$. Let C^* be the set of all good 2-components of the bridge graph of Y in M . We define a relation $\leq$ on C^* as follows. Given two sets C_1, C_2 in C^* , define that $C_1 \leq C_2$, if there exists a bridge $B_i \in C_1$, $i = 1, 2$, such that $B_i \leq B_2$. (This relation $\leq$ depends on the fixed element $f \in Y$. In case any ambiguity happens, we use $\stackrel{f}{\leq}$ instead of $\leq$.) We will prove that this relation $\leq$ on C^* is a partial order. (That is, it is transitive and anti-symmetric.)

Example Consider the example in (5.3.6) with $M = BM(G)$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The bridge graph of Y is given in the following.

The connected components of the bridge graph of Y are $\{B_1\}$, $\{B_4\}$, $\{B_5\}$, $\{B_6\}$, and $\{B_2, B_3, B_7\}$. It is easy to check that

$C_1 = \{\{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}\}$, $C_2 = \{\{B_4\}, \{B_5\}\}$, and $C_2^* = \{\{B_4, B_5\}\}$. Thus the set of all good 2-components of the bridge graph is $C^* = C_1 \cup C_2^* = \{\{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\}\}$.

Choose 1 as the fixed element of Y , and define " $\leq$ " on C^* as above. Then $\{B_1\} \leq \{B_4, B_5\} \leq \{B_2, B_3, B_7\}$, because $B_1 \leq B_4$ and $B_4 \leq B_7$. It follows from this relation that $\{B_6\}$ is not comparable with any other member of C^* .

(5.4.3) For ease of exposition, given any bridge B_i , we will denote by S_i the B_i -segment containing the fixed element f , by T_i a B_i -segment such that $f \notin T_i$, and by W_i a general B_i -segment (f may or may not be in W_i .)

We note that if bridge B_i avoids bridge B_j , by definition $W_i \cup W_j = Y$, for some $W_i \in \pi(B_i)$ and $W_j \in \pi(B_j)$, so $f \in W_i$ or $f \in W_j$. That is, $W_i = S_i$ or $W_j = S_j$ by our notations defined above.

(5.4.4) Lemma. Let $C_1 \in C^*$, let $B_1 \in C_1$, and let B_2 be a bridge not in C_1 such that there exists a B_2 -segment T_2 with $S_1 \cup T_2 = Y$. Then $S_i \cup T_2 = Y$ for all the bridges B_i in C_1 . Thus $B \leq B_2$ for some $B \in C_1$ implies $B \leq B_2$ for all $B \in C_1$. (Note our convention in (5.4.3)).

Proof. $C^* = C_1 \cup C_2^*$. The lemma is clearly true for $|C_1| = 1$ or $C_1 \in C_2^*$. Suppose that $C_1 \in C_1$ and $|C_1| \geq 2$. We first show

that for any bridge $B \in C_1$, there exists a B_2 -segment T'_2 (depending on B) such that $S \cup T'_2 = Y$, $f \in S \in \pi(B)$, and $f \notin T'_2 \in \pi(B_2)$. If this is not true, then since B_2 avoids all the bridges in C_1 , and C_1 is a connected component of the bridge graph, there exist adjacent bridges $B_j, B_k \in C_1$, and there exist $T''_2, S_2 \in \pi(B_2)$, $S_j \in \pi(B_j)$, and $W_k \in \pi(B_k)$ such that $S_j \cup T''_2 = Y = W_k \cup S_2$. But then $S_j \supseteq Y \setminus T''_2 \supseteq S_2 \supseteq Y \setminus W_k$, so $S_j \cup W_k = Y$, a contradiction.

Now we show that T'_2 (depending on B) is in fact T_2 . If not, then there exist adjacent bridges $B_j, B_k \in C_1$, and there exist $T'_2 \neq T_2$ in $\pi(B_2)$, $S_j \in \pi(B_j)$, and $S_k \in \pi(B_k)$ such that $S_j \cup T_2 = Y = S_k \cup T'_2$. But then $S_j \supseteq Y \setminus T_2 \supseteq T'_2 \supseteq Y \setminus S_k$, so $S_j \cup S_k = Y$, a contradiction. This completes the proof of the lemma. Δ

(5.4.5) Corollary. Let C_1, C_2 be two distinct elements in C^* . If $C_1 \leq C_2$, then there exists $B_2 \in C_2$ such that $B \leq B_2$ for all $B \in C_1$.

Proof. By definition of $C_1 \leq C_2$, we have $B_1 \leq B_2$ for some $B_1 \in C_1$ and $B_2 \in C_2$. Since $B_2 \notin C_1$, by (5.4.4), $B \leq B_2$ for all $B \in C_1$. Δ

(5.4.4) Theorem. For a fixed $f \in Y$, the relation $\leq$ on C^* is a partial order.

Proof. (i) Transitive property: Let $C_i \in C^*$, $i = 1, 2, 3$. If $C_1 \leq C_2$, $C_2 \leq C_3$, and $C_i \neq C_j$ for $i \neq j$, then by (5.4.5), there exist

$B_2 \in C_2$ and $B_3 \in C_3$ such that $B_1 \leq B_2$, and $B_2' \leq B_3$, for all $B_1 \in C_1$ and $B_2' \in C_2$. In particular, for any $B_1 \in C_1$, we have $B_1 \leq B_2$ and $B_2 \leq B_3$. Therefore $S_1 \cup T_2 = Y = S_2 \cup T_3$ for some T_1 , $f \notin T_1 \in \pi(B_1)$, $i = 2, 3$. Thus $S_1 \supseteq Y \setminus T_2 \supseteq S_2 \supseteq Y \setminus T_3$ and $S_1 \cup T_3 = Y$; that is, $B_1 \leq B_3$, and so $C_1 \leq C_3$.

(ii) Antisymmetric property: Let $C_i \in C^*$, $i = 1$, and 2 .

Suppose $C_1 \leq C_2$ and $C_2 \leq C_1$; then by (5.4.5) there exists $B_i \in C_i$, $i = 1$ and 2 , such that $B \leq B_i$, for all $B \in C_{3-i}$. In particular, $B_1 \leq B_2$ and $B_2 \leq B_1$. Therefore, $S_1 \cup T_2 = Y = S_2 \cup T_1$ for some T_1 , $f \notin T_1 \in \pi(B_1)$, $i = 1, 2$. Thus $S_1 \supseteq Y \setminus T_2 \supseteq S_2 \supseteq Y \setminus T_1$, so $\pi(B_1) = \{S_1, T_1\}$ and $S_1 = Y \setminus T_2 = S_2 = Y \setminus T_1$. Hence $\pi(B_1) = \pi(B_2)$, and $|\pi(B_1)| = 2$, $i = 1, 2$. If $C_1 \neq C_2$, it is easy to see that B_i avoids all the bridges of Y in M , $i = 1$ and 2 . So B_i is an isolated vertex in the bridge graph of Y , $i = 1$ and 2 . By our construction of the good 2-components C^* of the bridge graph of Y , B_1 and B_2 are in the same member of C^* . This is a contradiction. So, $C_1 = C_2$. Δ

(5.4.7) Proposition. Let C_1, C_2 be two distinct elements in C^* . If $C_1 \leq C_2$, then for any bridge $B_i \in C_i$, $i = 1, 2$, there exists $W_2 \in \pi(B_2)$ such that $S_1 \cup W_2 = Y$.

Proof. Suppose that this is not true; then there exists $B_i \in C_i$,

$i = 1, 2$ such that $S_1 \cup W_2 \neq Y$, for all $W_2 \in \pi(B_2)$. Since B_1 avoids B_2 , there exist $T_1, W_2, T_1 \in \pi(B_1), W_2 \in \pi(B_2)$, such that $T_1 \cup W_2 = Y$. Note that $f \notin T_1$, and hence $f \in W_2$, and by our notation $W_2 = S_2$. Therefore $B_2 \leq B_1$, and thus $C_2 \leq C_1$. By (5.4.6), $\leq$ is a partial order on C^* , so we have $C_1 = C_2$, a contradiction. Δ

Let Y be a cocircuit of the non-separable matroid M . We fix an element $f \in Y$, and define the partial order " $\leq$ " on C^* . Since C^* is a finite set, there exists a minimal element C of C^* .

(Remark. An element C of the partially ordered set C^* is called a minimal element of C^* , if there is no $C' \in C^*$ with $C' \leq C$ and $C' \neq C$).

(5.4.8) Lemma. Suppose $|C^*| \geq 2$. Given any minimal element C of $(C^*, \leq)$, let $L_2 = \cap (S_1 : f \in S_1 \in \pi(B_1), B_1 \in C)$ and $L_1 = Y \setminus L_2$. Then

- (1) L_1 contains all but one B -segment for each bridge $B \in C$;
- (2) L_2 contains all but one B -segment for each bridge $B \notin C$.

Proof. The truth of (1) is obvious, since $L_1 = Y \setminus L_2 = \cup (Y \setminus S_1 : f \in S_1 \in \pi(B_1), B_1 \in C)$, and $Y \setminus S_1$ contains all but one B_1 -segment for bridge B_1 .

Now given any bridge $B \notin C$, let $C' \in C^*$ be such that $B \in C'$.

(Thus $C' \neq C$). For any bridge $B_i \in C$, since B_i avoids B , there must exist $W \in \pi(B)$ such that $S_i \cup W = Y$, for otherwise $T_i \cup S = \overline{Y}$ for some $T_i \in \pi(B_i)$; this would imply that $B \leq B_i$, and thus $C' \leq C$, contradicting the assumption that C is a minimal element. Therefore S_i contains all but one B -segment. Since B is an arbitrary bridge not in C , and B_i is an arbitrary bridge in C , we conclude that L_2 contains all but one B -segment for each bridge $B \notin C$. Δ

Remark. Compare (5.4.8) with (3.3.6); it is not difficult to see that the former gives a constructive proof of the latter.

The following result follows from (3.3.7).

(5.4.9) Proposition. With all the assumptions in lemma (5.4.8), let $E_1 = L_1 \cup (\cup (B_i : B_i \in C))$ and $E_2 = L_2 \cup (\cup (B_i : B_i \notin C))$, where C, L_1, L_2 are defined as in (5.4.8). Then $\{E_1, E_2\}$ is a 2-separation of M . Δ

Example. Consider the example in (5.3.6) with $M = BM(G)$, $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Choose 1 as the fixed element of Y , and define the partial order " $\leq$ " on C^* . We have seen that $C^* = \{\{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\}\}$, $\{B_1\} \leq \{B_4, B_5\} \leq \{B_2, B_3, B_7\}$, and $\{B_6\}$ is not comparable with any other member of C^* . Thus $\{B_1\}$ and $\{B_6\}$ are minimal members of $(C^*, \leq)$. Hence,

letting $C = \{B_1\} = \{\{11, 12, 13\}\}$, $L_2 = \{1, 2, 5, 6, 7, 8, 9, 10\}$, $L_1 = Y \setminus L_2 = \{3, 4\}$, $E_1 = L_1 \cup (\cup \{B_i : B_i \in C\}) = \{3, 4, 11, 12, 13\}$, and $E_2 = E(G) \setminus E_1$, by (5.4.9), $\{E_1, E_2\}$ is a 2-separation of $BM(G)$. Similarly, by choosing $\{B_6\}$ as a minimal member of $(C^*; \leq)$, we see that, where $E_1 = \{8, 9, 10, 18, 19, 20\}$, $\{E_1, E(G) \setminus E_1\}$ is another 2-separation of $BM(G)$.

In fact $\{E_1, E_2\}$, in the context of (5.4.9), is a good 2-separation of M . To prove it, we need the following result.

(5.4.10) Proposition. Let M be a non-separable matroid on E , and Y be a cocircuit of M . Suppose that either the bridge graph of Y is connected or all the bridges of Y have the same segment set π with $|\pi| = 2$. Then M/P is non-separable, for every parallel class P contained in Y .

Proof. First we note that, by (3.1.1), any parallel class of M is either contained in Y or disjoint from it. Let P be a parallel class of M with $P \subseteq Y$. Suppose that M/P is separable, and let A_1 and A_2 be two non-null complementary separators of M . Where r and r' are the rank functions of M and M/P respectively, then we have:

$$\begin{aligned} 0 &= r'(A_1) + r'(A_2) - r'(E \setminus P) \\ &= (\text{by (2.2.7)}) \quad r(A_1 \cup P) - r(P) + r(A_2 \cup P) - r(P) - r(E) + r(P) \\ &= r(A_1 \cup P) + r(A_2 \cup P) - r(E) - 1. \end{aligned}$$

Suppose that one of A_1 and A_2 is contained in Y , say A_1 , then $0 = r(A_1 \cup P) + r(A_2 \cup P) - r(E) - 1 =$ (by (2.3.8)), $r(A_1 \cup P) + r(E) - r(E) - 1 = r(A_1 \cup P) - 1$. Thus $r(A_1 \cup P) = 1$, this contradicts the assumption that P is a parallel class. So we may assume that $A_i \not\subseteq Y$, for $i = 1, 2$. Since M is non-separable, $Y \neq P$, we may also assume that $A_1 \cap (Y \setminus P) \neq \emptyset$. Then $0 = r(A_1 \cup P) + r(A_2 \cup P) - r(E) - 1 \geq r(A_1) + r(A_2 \cup P) - r(E) - 1$. Hence $\{A_1, A_2 \cup P\}$ is a 2-separation of M with Y meeting A_1 and $A_2 \cup P$, moreover $A_1 \cup Y \neq \emptyset$ and $(A_2 \cup P) \setminus Y \neq \emptyset$. By (3.3.7), the bridge graph of Y is not connected and each bridge B_i of Y is either contained in A_1 or in $A_2 \cup P$. Thus all the bridges of Y have the same segment set π with $|\pi| = 2$. Let $F_1 = \cup \{B : B \text{ is a bridge of } Y \text{ with } B \subseteq A_1\}$. $M_2 = M/F_1$, and $M'_2 = M_2/P$. We note that M_2 is obtained from M by contracting some of the bridges of Y . Now, since $M'_2 = M_2/P = M/P/F_1$, by (2.4.6), $A_1 \setminus F_1$ and A_2 are separators of M'_2 . Where r_2 and r'_2 are the rank functions of M_2 and M'_2 respectively, we have

$$\begin{aligned}
 0 &= r'_2(A_1 \setminus F_1) + r'_2(A_2) - r'_2(E(M'_2)) \\
 &= \text{(by (2.2.7)) } r_2((A_1 \setminus F_1) \cup P) - r_2(P) + r_2(A_2 \cup P) - r_2(P) - r_2(E(M_2)) + r_2(P) \\
 &= r_2((A_1 \setminus F_1) \cup P) - 1 + r_2(A_2 \cup P) - 1 - r_2(E(M_2)) + 1 \\
 &= r_2((A_1 \setminus F_1) \cup P) + r_2(A_2 \cup P) - r_2(E(M_2)) - 1 \\
 &= \text{(by (2.3.8)) } r_2((A_1 \setminus F_1) \cup P) + r_2(E(M_2)) - r_2(E(M_2)) - 1 \\
 &= r_2((A_1 \setminus F_1) \cup P) - 1
 \end{aligned}$$

Thus $r_2((A_1 \setminus F_1) \cup P) = 1$. We note that P is a parallel class of M_2 with $P \subseteq Y$, since all the bridges of Y in M have the same segment set π with $|\pi| = 2$ and M_2 has been obtained from M by contracting some (but not all) of the bridges of Y in M . But $\emptyset \neq A_1 \setminus F_1 \subseteq Y$, and this is a contradiction. Hence M/P is non-separable. Δ

(5.4.11) Proposition. With the assumptions stated in (5.4.10), if P is a parallel class contained in Y with $|P| \geq 2$, then $\{P, E \setminus P\}$ is a good 2-separation of M .

Proof. Clearly any two cells of $M \setminus (E \setminus P)$ are in parallel, and hence it is non-separable. Now M/P and $M \setminus (E \setminus P)$ are non-separable, so by (5.2.7), $\{P, E \setminus P\}$ is a good 2-separation of M . Δ

(5.4.12) Let M be a non-separable matroid on E and Y be a cocircuit of M . Let f be a fixed element of Y . Let C^* be the set of all good 2-components of the bridge graph of Y . We define $(C^*, \overset{f}{\leq})$ as before. Let C be a minimal member of $(C^*, \overset{f}{\leq})$. Let $L_2 = \cap \{S_i : f \in S_i \in \pi(B_i), B_i \in C\}$, $L_1 = Y \setminus L_2$, $E_1 = L_1 \cup \{B_i : B_i \in C\}$, and $E_2 = L_2 \cup \{B_i : B_i \notin C\}$.

We have shown that, if $|C^*| \geq 2$, then $\{E_1, E_2\}$ is a 2-separation of M . Now we prove that it is in fact a good 2-separation of M .

(5.4.13) Theorem. With the assumption (5.4.12), if $|C^*| \geq 2$ then $\{E_1, E_2\}$ is a good 2-separation of M .

We prove a lemma first.

(5.4.14) Lemma. Let Y be a cocircuit of a matroid. Then M is separable if and only if there is a bridge B of Y in M having only one B -segment, that is, $\pi(B) = \{Y\}$.

Proof. Suppose M is separable. By (2.4.3), there is an elementary separator B of M disjoint from Y . Then, by (2.4.6), B is an elementary separator of $M \setminus Y$, and hence a bridge of Y . Now

$$(M/(E \setminus (B \cup Y))) \setminus B = (M \setminus B)/(E \setminus (B \cup Y)) = (\text{by (2.4.1)}) (M/B)/(E \setminus (B \cup Y)) =$$

$M/(E \setminus Y)$. Let r' be the rank function of $M/(E \setminus Y)$. Then

$$r'(Y) = (\text{by (2.2.7)}) r(E) - r(E \setminus Y) = (\text{by (2.3.8)}) r(E) - r(E) + 1 = 1.$$

Thus $\pi(B) = \{Y\}$.

Conversely suppose that B is a bridge of Y with $\pi(B) = \{Y\}$.

Let r' be the rank function of $M/(E \setminus (B \cup Y))$. Then $1 = r'(Y) =$

(by (2.2.7)) $r(E \setminus B) - r(E \setminus (B \cup Y))$, and thus $r(B) + r(E \setminus B) - r(E) =$

(by (2.3.8)) $r(B) + r(E \setminus B) - r(E \setminus Y) - 1 = r(B) + r(E \setminus B) - (r(B) + r(E \setminus (B \cup Y))) - 1$

$$= r(E \setminus B) - r(E \setminus (B \cup Y)) - 1$$

$$= 0.$$

So M is separable. Δ

Proof of Theorem (5.4.13)

The proof proceeds by checking the characterization of the good 2-separations stated in (5.2.7).

First we prove that M/E_2 is non-separable. Since $M/E_2 = (M/(E_2 \setminus L_2))/L_2 = (M/(\cup (B_i : B_i \notin C)))/L_2$. Let $M' = M/(\cup (B_i : B_i \notin C))$. By (3.2.6), Y is a cocircuit of M' , the bridges of Y in M' are precisely those bridges of Y in M that are in C , and $\pi(M', B_i, Y) = \pi(M, B_i, Y)$ for all bridges $B \in C$. Since $C \in C^*$, either the bridge graph of Y in M' is connected or all the bridges of Y in M' have exactly the same segment set π and $|\pi| = 2$. By (3.2.1) it is easy to see that L_2 is a parallel class of M' contained in Y . Thus, by (5.4.10), M'/L is non-separable, that is, M/E_2 is non-separable.

Now to prove $\{E_1, E_2\}$ is a good 2-separation of M , it is sufficient to prove the following claims:

- (i) If $C \in C_2^*$, then $M \setminus E_1$ is non-separable;
 - (ii) If $C \in C_1$, then $M \setminus E_2$ is non-separable.
- (Note that $C^* = C_1 \cup C_2^*$.)

Proof of (i). $M \setminus E_1 = (M \setminus L_1) \setminus (E_1 \setminus L_1) =$ (by (3.3.3)).

$$(M \setminus L_1) / (E_1 \setminus L_1) = (M / (E_1 \setminus L_1)) \setminus L_1 = (M / (\cup (B_i : B_i \in C))) \setminus L_1.$$

Let $M' = M / (\cup (B_i : B_i \in C))$. By (3.2.6) and (3.2.9), $Y \setminus L_1$ is a cocircuit of $M' \setminus L_1 (= M \setminus E_1)$; the bridges of $Y \setminus L_1$ in $M' \setminus L_1$ are precisely those bridges of Y in M that are not in C .

Suppose $M \setminus E_1 (= M' \setminus L_1)$ is separable. Then, by (5.4.14), there

exists a bridge B_0 of $Y \setminus L_1$ in $M' \setminus L_1$ such that $\pi(M' \setminus L_1, B_0, Y \setminus L_1) = \{Y \setminus L_1\} = \{L_2\}$. We note that B_0 is also a bridge of Y in M , and $B_0 \notin C$. By (3.2.9) and (3.2.6), $\pi(M', B_0, Y) = \pi(M, B_0, Y) = \{L_1, L_2\}$, since in matroid M , L_1 is contained in one B-segment for each bridge $B \notin C$. But since, by our assumption, $C \in C_2^*$, it is easy to see that $\pi(M, B, Y) = \{L_1, L_2\}$, for each bridge $B \in C$, and thus $B_0 \in C$, a contradiction. Therefore $M \setminus E_1$ is non-separable.

Proof of (ii). Similarly as in (i), we have the following.

$M \setminus E_2 = (M / (\cup (B_i : B_i \notin C))) \setminus L_2$, let $M' = M / (\cup (B_i : B_i \notin C))$.

$Y \setminus L_2$ is a cocircuit of $M' \setminus L_2$, and the bridges of $Y \setminus L_2$ in

$M' \setminus L_2$ are precisely those bridges of Y in M that are in C .

Suppose that $M \setminus E_2 (= M' \setminus L_2)$ is separable, then there exists a

bridge B_0 of $Y \setminus L_2$ in $M' \setminus L_2$ such that $\pi(M' \setminus L_2, B_0, Y \setminus L_2) =$

$\{Y \setminus L_2\} = \{L_1\}$. B_0 is also a bridge of Y in M , and $B_0 \in C$.

Again we have $\pi(M', B_0, Y) = \pi(M, B_0, Y) = \{L_1, L_2\}$, since in

matroid M , L_2 is contained in one B-segment for each bridge

$B \in C$. If $|C| \geq 2$, then we have found a bridge $B_0 \in C$ avoiding

any other bridge in C , this would imply that $C \in C_1 (= C^* \setminus C_2^*)$,

a contradiction. So $|C| = 1$, that is, $C = \{B_0\}$, but again $C \in C_1$,

since $|\pi(B_0)| = 2$, a contradiction. Therefore $M \setminus E_2$ is non-

separable. This completes the proof of Theorem (4.3.14). Δ

5.5 Development of the decomposition algorithm

In the context of (5.4.12), we have seen that $\{E_1, E_2\}$ is a good 2-separation of M provided $|C^*| \geq 2$. Two questions arise:

(1) Is there an efficient algorithm to find a minimal number of

$(C^*, \leq^f)$?

(2) After splitting E_1 from M , what happens to the resulting matroid?

The answer for question (1) is positive; we will propose such an algorithm later. For the present, we study the second question.

Where M_2 is the matroid resulting from splitting E_1 from M and e is the marker, the next theorem says that $L_2 + e$ is a cocircuit of M_2 , the good 2-components $C^*(M_2, L_2 + e)$ of $L_2 + e$ in M_2 are precisely those of Y in M with C deleted, and $C^*(M_2, L_2 + e)$ preserves the same partial ordering $\leq^f$. More precisely:

(5.5.1) Theorem with the assumption (5.4.12), and assuming

$|C^*(M)| \geq 2$, let $\{M_1, M_2\}$ be the simple decomposition of M associated with $\{E_1, E_2\}$ and the marker e , where $E_1 = L_1 \cup (\cup(B_i : B_i \in C))$ and $E_2 = L_2 \cup (\cup(B_i : B_i \notin C))$. Then the following properties hold.

(i) $L_i + e$ is a cocircuit of M_i , for $i = 1, 2$ and $f \in L_2 + e$.

(ii) The bridges of $L_1 + e$ in M_1 (respectively $L_2 + e$ in M_2) are precisely those bridges of Y in M that are in C (respectively that are not in C).

(iii) For each bridge B of $L_1 + e$ in M_1 , $\pi(M_1^*, B, L_1 + e)$ differs from $\pi(M, B, Y)$ only in the replacement of W by $W \cup L_{3-i} + e$,

where W is the B -segment of Y in M containing L_{3-i} ,

$i = 1, 2$.

(iv) The bridge graph of $L_1 + e$ in M_1 (respectively $L_2 + e$ in M_2) is precisely the bridge graph of Y in M restricted to those vertices corresponding to bridges in C (respectively corresponding to bridges not in C).

(v) The set of all good 2-components $C^*(M_1, L_1 + e)$ of $L_1 + e$ in M_1 (respectively $C^*(M_2, L_2 + e)$ of $L_2 + e$ in M_2) is precisely $\{C\}$ (respectively $C^*(M, Y) \setminus \{C\}$), where $C^*(M, Y)$ is the set of all good 2-components of Y in M .

(vi) With f as the fixed element in $L_2 + e$, we define $(C^*(M_2, L_2 + e), \overset{f}{\leq})$ in M_2 . Let C_1 and C_2 be two members of $C^*(M_2, L_2 + e)$, then $C_1 \overset{f}{\leq} C_2$ in $(C^*(M_2, L_2 + e), \overset{f}{\leq})$ if and only if $C_1 \overset{f}{\leq} C_2$ in $(C^*(M, Y), \overset{f}{\leq})$. (Hence, there is no ambiguity to use the same notation $\overset{f}{\leq}$ for the partial ordering in both $C^*(M_2, L_2 + e)$ and $C^*(M, Y)$).

Proof. Since Y is a cocircuit meeting E_1 and E_2 , and $L_1 = Y \cap E_1$,

$i = 1, 2$, we can choose $x_i \in L_i$, $i = 1, 2$. Then, by (5.1.10),

$M_i \cong (M / (E_{3-i} \setminus L_{3-i})) \setminus (L_{3-i} - x_{3-i})$ with $e \rightarrow x_{3-i}$, $i = 1, 2$.

Hence $M_1 \cong (M / (U(B_1 : B_1 \notin C))) \setminus (L_2 - x_2)$ with $e \rightarrow x_2$ and

$M_2 \cong (M / (U(B_1 : B_1 \in C))) \setminus (L_1 - x_1)$ with $e \rightarrow x_1$. Thus, by (3.2.6)

and (3.2.9), (i), (ii), (iii) follow.

To prove (iv) it is sufficient to show that, for any two bridges

B_1, B_2 of $L_1 + e$ in M_1 , there exists $W_j \in \pi(M_1, B_j, L_1 + e)$,

$j = 1, 2$, such that $W_1 \cup W_2 = L_1 + e$, if and only if there exists

$W'_j \in \pi(M, B_j, Y)$, $j = 1, 2$, such that $W'_1 \cup W'_2 = Y$, for $i = 1$ and 2 .

But this follows from (iii). Thus (iv) holds.

From (iii), it is easy to see that, for any two bridges B_1, B_2

of $L_1 + e$ in M_1 , $\pi(M_1, B_1, L_1 + e) = \pi(M_1, B_2, L_1 + e)$ and

$|\pi(M_1, B_1, L_1 + e)| = 2$ if and only if $\pi(M, B_1, Y) = \pi(M, B_2, Y)$

and $|\pi(M, B_1, Y)| = 2$. This establishes (v).

To prove (vi) it is enough to show that, for any two bridges

B_1, B_2 of $L_2 + e$ in M_2 , there exist S_1, T_2 with $f \in S_1 \in$

$\pi(M_2, B_1, L_2 + e)$ and $f \notin T_2 \in \pi(M_2, B_2, L_2 + e)$, such that

$S_1 \cup T_2 = L_2 + e$, if and only if there exist S'_1, T'_2 with $f \in S'_1 \in$

$\pi(M_2, B_1, L_2 + e)$ and $f \notin T'_2 \in \pi(M, B_2, Y)$. But this follows from

(iii). Thus (vi) holds. Δ

Example. Consider the example given in (5.3.6), $M = BM(G)$,

$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. With 1 as the fixed element of Y ,

we have seen that $C^*(M, Y) = \{ \{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\} \}$,

and that $\{B_1\}$ is a minimal member of $(C^*, \leq)$, where $B_1 = \{11, 12, 13\}$. Therefore, letting $C = \{B_1\}$, $L_1 = \{3, 4\}$, $L_2 = Y \setminus L_1$, $E_1 = L_1 \cup (\cup \{B_i : B_i \in C\}) = \{3, 4, 11, 12, 13\}$, and $E_2 = E(G) \setminus E_1$, $\{E_1, E_2\} = \{\{3, 4, 11, 12, 13\}, E(G) \setminus E_1\}$ is a good 2-separation of $BM(G)$. The simple decomposition of $BM(G)$ associated with this 2-separation and the marker e can be demonstrated in the following diagram. Notice that $L_2 + e = \{1, 2, e, 5, 6, 7, 8, 9, 10\}$ is a polygon of the graph G_2 , and hence is a cocircuit of $BM(G_2)$. Also, $C^*(BM(G_2), L_2 + e) = C^*(BM(G), Y) \setminus \{\{B_1\}\}$ and the partial ordering is preserved (with 1 as the fixed element in $L_2 + e$ and Y).

Because of result (5.5.1), we may now consider the case that $|C^*(M, Y)| = 1$, where Y is a cocircuit of the non-separable matroid M . So we assume $|C^*| = 1$, where C^* is the set of all good 2-components of the bridge graph of Y in M . We have shown in (5.4.11), that for each parallel class P contained in Y with $|P| \geq 2$, $\{P, E \setminus P\}$ is a good 2-separation of M ; after splitting these

parallel classes from M , we may consider the case that Y is simple in M .

(5.5.2) Theorem. Let Y be a cocircuit of a non-separable matroid M . Suppose that every simplified Y -component has no good 2-separation, $|C^*| = 1$, and Y is simple in M . Where $C^* = \{C\}$, we have the following:

- (i) If $C \in C_2^*$ (that is, all the bridges of Y have the same segment set π , and $|\pi| = 2$), then M is a polygon.
- (ii) If $C \in C_1$ (that is, the bridge graph of Y is connected) and no two singleton bridges of Y are in series, then M is 3-connected.

Proof. First we note that, by (5.3.8), every simplified Y -component is 3-connected.

Proof of (i). Since Y is simple in M , and all the bridges of Y have the same segment set π with $|\pi| = 2$, it follows that $|Y| = 2$. Now we prove that $|B_i| = 1$, for each bridge B_i of Y in M . Suppose not and let M_i be the Y -component of M corresponding to the bridge B_i with $|B_i| \geq 2$. The Y -simplification of M_i is clearly itself, since $|Y| = 2$ and $|\pi(B_i)| = 2$. It is easy to see that $r_i(B) + r_i(Y) = r_i(B \cup Y) + 1$, and $|B_i|, |Y| \geq 2$, where r_i is the rank function of M_i . Then M_i is not 3-connected,

contradicting the fact that every simplified Y -component is 3-connected. Thus $|B_i| = 1$ for every bridge B_i of Y in M . Now we prove, by induction on the number k of bridges of Y , that M is a polygon. If $k = 1$, then $|E| = 3$. It is easy to see that E is a circuit, and hence M is a polygon. Suppose this is true for $k = m \geq 1$, and suppose that we are given $k = m + 1$. Let B be a bridge of Y , so $B = \{b\}$. The cocircuit Y of M/b still satisfies all the desired conditions, and the number of bridges of Y in M/b is m . By induction hypothesis, M/b is a polygon, that is, $E - b$ is a circuit of M/b . Suppose that M is not a polygon. Since M is non-separable, there is an M -circuit F containing b such that F is not a loop nor the whole set E . But then $F - b$ is a circuit of M/b with $F - b \subset E - b$, a contradiction. So M is a polygon.

Proof of (ii). Since no two singleton bridges of Y are in series, every simplified Y -component is 3-connected, Y is simple in M , and the bridge graph of Y is connected, by (3.3.5), M is 3-connected. Δ

In the context of (5.5.2), if $C \in C_2^*$, then M is a polygon, and thus M does not have any good 2-separation. On the other hand, if $C \in C_1$, then M may not be 3-connected, for there may

be two singleton bridges in series. So we check this case in detail. First we make the following assumptions.

(5.5.3) Let Y be a cocircuit of the non-separable matroid M . Assume that every simplified Y -component has no good 2-separation, and assume that $|C^*| = 1$ with $C \in C_1$, where $C^* = \{C\}$.

We know from (5.4.11) that for every parallel class P contained in Y , $\{P, E \setminus P\}$ is a good 2-separation of M . Now suppose that Y is simple in M . We would like to find all the good 2-separations of M . Observe that if the bridge graph of Y is connected and Y is simple in M then, by (3.3.5(b)), Y cannot meet both E_1 and E_2 , for any 2-separation $\{E_1, E_2\}$ of M . We have the following theorem.

(5.5.4) Theorem. With the assumption (5.5.3), and assuming that Y is simple in M , $\{E_1, E_2\}$ is a 2-separation (respectively good 2-separation) with $E_1 \cap Y = \emptyset$ if and only if E_1 is a series set (respectively series class) with $|E_1| \geq 2$. Moreover, each element of E_1 is a singleton bridge of Y in M with the same segment set.

Proof. Suppose $\{E_1, E_2\}$ is a 2-separation of M with $E_1 \cap Y = \emptyset$.

We note that $|E_1 \cap B_i| \leq 1$ for each bridge B_i of Y . For if not, let $A = E_1 \cap B_i$, where B_i is a bridge of Y with $|E_1 \cap B_i| \geq 2$.

By (2.5.4), $r_1(A) + r_1((B_1 \setminus A) \cup Y) = r_1^*(B_1 \cup Y) + 1$, where r_1 is the rank function of the Y -component M_1 corresponding to B_1 .

Clearly, $r_1((B_1 \setminus A) \cup Y) \geq 2$; hence the Y -simplification of M_1 is not 3-connected. But by our assumptions, the Y -simplification of M_1 has no good 2-separation, so by (5.3.8), M_1 is 3-connected, a contradiction. Therefore $|E_1 \cap B_i| \leq 1$ for each bridge B_i of Y . We prove that $r(E_1) = |E_1|$. Suppose not, and let F be an M -circuit with $F \subseteq E_1$. Since $E_1 \cap Y = \emptyset$, F is contained in one bridge B_i of Y . But $|E_1 \cap B_i| \leq 1$, which implies that $|F| = 1$; that is, F is a loop of M , contradicting the fact that M is non-separable. Thus $r(E_1) = |E_1|$. Now $r^*(E_1) = |E_1| + r(E_2) - r(E) \leq |E_1| + 1 - r(E_1) = 1$, where r^* is the rank function of M^* , the dual matroid of M . So $r^*(E) = 1$. M does not have any coloop, since it is non-separable. Thus E_1 is a series class.

Suppose that E_1 is a series set with $|E_1| \geq 2$. Since $C^* = \{C\}$ and $C \in C_1$, either the number k of bridges of Y is strictly greater than one and the bridge graph of Y is connected, or $k = 1$ and $|\pi(B)| \geq 3$, where B is the only bridge of Y . Then, if two cells are in series, neither of them can be in Y . Thus $E_1 \cap Y = \emptyset$. So $|E_2| \geq 2$, since $E_2 \supseteq Y$. It is easy to check that $\{E_1, E_2\}$ is a 2-separation of M^* , the dual of M , so it is a 2-separation of M .

Suppose that $\{E_1, E_2\}$ is a good 2-separation of M . We have shown that E_1 is a series set. Let S be the series class containing E_1 . If $E_1 \subset S$, let $a \in S \setminus E_1$ and $b \in E_1$. Then $\{(E_1 - b) + a, (E_2 - a) + b\}$ is a 2-separation which crosses $\{E_1, E_2\}$, a contradiction. So $E_1 = S$, and thus E_1 is a series class.

Suppose that E_1 is a series class with $|E_1| \geq 2$. We know that $\{E_1, E_2\}$ is a 2-separation with $E_1 \cap Y = \emptyset$. If it is not a good 2-separation, then there exists a 2-separation $\{A_1, A_2\}$ that crosses $\{E_1, E_2\}$. By the observation we made before, Y cannot meet both A_1 and A_2 ; say $A_1 \cap Y = \emptyset$. Then A_1 is a series set. But $A_1 \cap E_1 \neq \emptyset$, $A_1 \cap E_2 \neq \emptyset$, contradicting the fact that E_1 is a series class. Thus $\{E_1, E_2\}$ is a good 2-separation.

Now we prove that each element of E_1 is a singleton bridge of Y in M . We have shown that $|E_1 \cap B_i| \leq 1$ for each bridge B_i of Y . Therefore, since $|E_1| \geq 2$, E_1 intersects at least two bridges of Y . By (5.3.2) a 2-separation cannot cross $\{B_i, E \setminus B_i\}$, where B_i is a bridge of Y . It follows, since $E_1 \cap Y = \emptyset$, that E_1 is a union of bridges of Y ; thus each element of E_1 is a bridge of Y .

Let $b_i \in E_1$, $i = 1, 2$; hence $B_i = \{b_i\}$ is a singleton

bridge of Y , $i = 1, 2$. Since b_1 and b_2 are in series, it is easy to see that $(M/b_2) \setminus b_1 = (M/b_2)/b_1 = (M/b_1) \setminus b_2$. Thus $(M/(E \setminus (Y + b_1))) \setminus b_1 = (M/(E \setminus (Y + b_1)))/b_1 = (M/(E \setminus (Y + b_2)))/b_2 = (M/(E \setminus (Y + b_2))) \setminus b_2$, and therefore $\pi(M, B_1, Y) = \pi(M, B_2, Y)$. Δ

We shall see later that this result still holds without the assumption of Y being simple.

Now we go back to the situation in (5.4.12) and assume that every simplified Y -component has no good 2-separation. Where C is a minimal member of $(C^*, \leq^f)$, $E_1 = L_1 \cup (U(B_i : B_i \in C))$, $E_2 = L_2 \cup (U(B_i : B_i \notin C))$, and $|E_2| \geq 2$, then we know that $\{E_1, E_2\}$ is a good 2-separation of M , and can split E_1 from M . But algorithmically, whenever a set A is split from M , (that is $\{A, E(M) \setminus A\}$ is a good 2-separation of M , and $M \rightarrow \{M_1(A; e), M_2(E(M) \setminus A; e)\}$), we want the "part" $M_1(A; e)$ being split to be 3-connected, a polygon, or a bond. So, before splitting E_1 from M , we first split the parallel classes of M contained in L_1 , and the series classes of M contained in $U(B_i : B_i \in C)$ if $C \in C_1$. (These are good 2-separations as will be seen below.) It is natural to check what happens after splitting them from M ; the answer is given in (5.5.5) and (5.5.7) below. Then, roughly speaking, we split the left part of E_1 from M ; we will see in (5.5.8), that

this "part" being split is 3-connected if $C \in C_1$, and is a polygon if $C \in C_2^*$. We can repeat the above process for the remaining trunk matroid, because of theorem (5.5.1).

(5.5.5) Theorem. With the assumption (5.4.12), let P be a parallel class contained in L_1 with $|P| \geq 2$. Then $\{P, E \setminus P\}$ is a good 2-separation of M . Furthermore, where $\{M_1, M_2\}$ is the simple decomposition of M associated with $\{P, E \setminus P\}$ and the marker e ; M_1 is a bond and the following properties hold.

- (i) $Y \setminus P + e$ is a cocircuit of M_2 and $f \in Y \setminus P + e$.
- (ii) The bridges of $Y \setminus P + e$ in M_2 are precisely those bridges of Y in M .
- (iii) For each bridge B of $Y \setminus P + e$ in M_2 , $\pi(M_2, B, Y \setminus P + e)$ differs from $\pi(M, B, Y)$ only in the replacement of W by $W \setminus P + e$, where W is the B -segment of Y in M containing P .
- (iv) The bridge graph of $Y \setminus P + e$ in M_2 is precisely the bridge graph of Y in M .
- (v) Let $C^*(M)$ (respectively, $C^*(M_2)$) be the set of all good 2-components of Y (respectively, $Y \setminus P + e$) in M (respectively, M_2); then $C^*(M_2) = C^*(M)$.
- (vi) With f as the fixed element in $Y \setminus P + e$, the partial ordering $(C^*(M_2), \overset{f}{\leq})$ (with respect to matroid M_2 and co-

circuit $Y \setminus P + e$) and the partial ordering $(C^*(M), \stackrel{f}{\leq})$ (with respect to matroid M and cocircuit Y) are the same.

(vii) Let $L'_1 = \{Y \setminus S_1 : f \in S_1 \in \pi(M_2, B_1, Y \setminus P + e), B_1 \in C\}$.

Then the set of all parallel classes of M_2 contained in L'_1 are precisely those parallel classes of M contained in L_1 with P replaced by $\{e\}$.

Remark: We note that any parallel class is either contained in L_1 or disjoint from it.

Proof. Suppose $|C^*(M)| = 1$. Then, by (5.4.11), $\{P, E \setminus P\}$ is a good 2-separation of M . Now suppose $|C^*(M)| \geq 2$, let $M \rightarrow \{M(E_1; g), M(E_2; g)\}$, where $E_1 = L_1 \cup (\cup(B_1 : B_1 \in C))$ and $E_2 = L_2 \cup (\cup(B_1 : B_1 \notin C))$, as defined in (5.4.12). By (5.5.1 (iii)), it is easy to check that P is a parallel class of $M(E_1; g)$ contained in $L_1 + e$. By (5.5.1 (v), (iii)) and (5.4.11), $\{P, E(M(E_1; g)) \setminus P\}$ is a good 2-separation of $M(E_1; g)$. Then, by (5.2.6), $\{P, E \setminus P\}$ is a good 2-separation of M .

Since Y is a cocircuit meeting P and $E \setminus P$ and $Y \cap P = P$ and $Y \cap (E \setminus P) = Y \setminus P$, then by (5.1.10), $M_1 \cong (M/(E \setminus Y)) \setminus (Y \setminus P - x_2)$ with $e \rightarrow x_2$, and $M_2 \cong M \setminus (P - x_1)$ with $e \rightarrow x_1$, where $x_1 \in P$ and $x_2 \in Y \setminus P$. Clearly M_1 is a bond. The rest of the properties follow easily from (3.2.9). Δ

Example. Consider the example given in (5.3.6), with $M = BM(G)$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Choose 1 as the fixed element in Y ; then $C^*(M, Y) = \{\{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\}\}$ and $\{B_6\}$ is a minimal member of $(C^*, \leq)$. Let $C = \{B_6\}$, where $L_1 = \{8, 9, 10\}$, $L_2 = Y \setminus L_1$, $E_1 = L_1 \cup (\cup \{B_i : B_i \in C\}) = \{8, 9, 10, 18, 19, 20\}$, and $E_2 = E(G) \setminus E_1$. Then $\{E_1, E_2\}$ is a good 2-separation of $BM(G)$. Now $\{9, 10\}$ is a parallel class contained in L_1 (Note that $\{9, 10\}$ is a minimal edge-cutset of G , and hence is a circuit of $BM(G)$.); thus $\{\{9, 10\}, E(G) \setminus \{9, 10\}\}$ is a good 2-separation of $BM(G)$. The simple decomposition of $BM(G)$ associated with this 2-separation and the marker e is illustrated in the following figure.

As we promised earlier, we prove the following theorem.

(5.5.6) Theorem. The result (5.5.4) is still true without the assumption that Y is simple in M .

Proof. Let k be the number of parallel classes contained in Y with cardinality at least 2. We prove, by induction on k , that $f(E_1, M, Y)$ is true, where $f(E_1, M, Y)$ stands for the statement "Under the assumptions of (5.5.3), $\{E_1, E_2\}$ is a 2-separation (good 2-separation, respectively) of M with $E_1 \cap Y = \emptyset$ if and only if E_1 is a series set (series class, respectively) of M . Furthermore, each element of E_1 is a singleton bridge of Y in M with the same segment set."

If $k = 0$, then $f(E_1, M, Y)$ is true by (5.5.4). Suppose that $f(E_1, M, Y)$ is true for $k = n \geq 0$, and suppose that we are given M and Y with $k = n+1$. Let P be a parallel class contained in Y with $|P| \geq 2$. Then $\{P, E \setminus P\}$ is a good 2-separation of M . Let $M \rightarrow (M(P;g), M(E \setminus P;g))$. It is easy to check, by (3.2.9), with matroid M replaced by $M(E \setminus P;g)$ and cocircuit Y replaced by $Y \setminus P + g$ that the conditions of (5.5.3) hold. Furthermore, the number of parallel classes of $M(E \setminus P;g)$ contained in $Y \setminus P + g$ with cardinality at least 2 is n . So, by induction hypothesis, $f(E_1, M(E \setminus P;g), Y \setminus P + g)$ holds. Now, by (5.2.2) and (5.2.6), $\{E_1, E_2\}$ is a 2-separation (respectively, good 2-separation) of M with $E_1 \cap Y = \emptyset$ if and only if $\{E_1, (E \setminus P) \setminus E_1 + g\}$ is a 2-separation (respectively, good 2-separation) of $M(E \setminus P;g)$. Also it is easy to see by the assumption $C \in C_1$, that E_1 is

a series set (respectively, series class) of M with $|E_1| \geq 2$ if and only if E_1 is a series set (respectively, series class) of $M(E \setminus P; g)$ with $|E_1| \geq 2$. Furthermore, if each element of E_1 is a singleton bridge of $Y \setminus P + g$ in $M(E \setminus P; g)$ with the same segment set, then by (3.2.9), each element of E_1 is a singleton bridge of Y in M . Therefore, $f(E_1, M, Y)$ holds. This completes the induction proof. Δ

(5.5.7) Theorem. In addition to the assumption (5.4.12), we assume that $C \in C_1$, and the simplified Y -component corresponding to bridge B_1 has no good 2-separation, for every $B_1 \in C$, where C is the minimal member of $(C^*, \leq)$ given in (5.4.12). Let Q be a series class contained in $U(B_1; B_1 \in C)$ with $|Q| \geq 2$. Then $(Q, E \setminus Q)$ is a good 2-separation of M , and each element of Q is a singleton bridge of Y with the same segment set π . Furthermore, where $[M_1, M_2]$ is the simple decomposition of M associated with $(Q, E \setminus Q)$ and the marker e , M_1 is a polygon and the following properties hold.

- (i) Y is a cocircuit of M_2 .
- (ii) The bridges of Y in M_2 are precisely those bridges of Y in M that are not singleton bridges contained in Q , plus a new singleton bridge $\{e\}$.

- (iii) $\pi(M_2, B_1, Y) = \pi(M, B_1, Y)$, for each bridge B_1 of Y in M_2 such that $B_1 \neq \{e\}$, and $\pi(M_2, \{e\}, Y) = \pi(M, \{q\}, Y)$ where $\{q\}$ is a singleton bridge of Y in M contained in Q .
- (iv) The bridge graph of Y in M_2 can be obtained from the bridge graph of Y in M by deleting all vertices except one corresponding to the singleton bridges contained in Q , and that single vertex left corresponds to the bridge $\{e\}$.
- (v) The set of all good 2-components $C^*(M_2)$ of Y in M_2 consists precisely of those good 2-components $C^*(M)$ of Y in M with C replaced by $C' = (C \setminus \{B_1 \mid B_1 \subset Q\}) \cup \{\{e\}\}$.
- (vi) With C' corresponding to C , the two partial orderings $(C^*(M_2), \stackrel{f}{\leq})$ and $(C^*(M), \stackrel{f}{\leq})$ are the same.
- (vii) The set of all series classes of M_2 contained in $U(B_1: B_1 \in C')$ consists precisely of those series classes of M contained in $U(B_1: B_1 \in C)$ with Q replaced by $\{e\}$.

Proof. Suppose that Q is a series class contained in $U(B_1: B_1 \in C)$.

Let $f(Q, M, Y)$ stand for the statement " $\{Q, E(M) \setminus Q\}$ is a good 2-separation of M , and each element of Q is a singleton bridge of Y with the same segment set." If $|C^*| = 1$ then, by (5.5.6),

$f(Q, M, Y)$ is true. Now suppose $|C^*| \geq 2$, and let $M \rightarrow \{M(E_1; g), M(E_2; g)\}$, where $E_1 = L_1 \cup (U(B_1: B_1 \in C))$ and $E_2 = L_2 \cup (U(B_1: B_1 \notin C))$,

as defined in (5.4.12). It is easy to check that Q is a series class of $M(E_1; g)$ contained in $\cup(B_i: B_i \in C)$, and by (5.5.1), the number of good 2-components of $L_1 + g$ in $M(E_1; g)$ is one. Thus, by (5.5.6), $f(Q, M(E_1; g), L_1 + g)$ holds. It follows from (3.2.6) and (5.5.6), that $f(Q, M, Y)$ also holds.

Now let $[M_1, M_2]$ be the simple decomposition of M associated with $(Q, E \setminus Q)$ and the marker e . Let Z be an M -circuit meeting Q and $E \setminus Q$; then $Q \subseteq Z$, since Q is a series class of M . Thus, by (5.1.3), $M_1 \cong (M \setminus (E \setminus Z)) / (Z \setminus Q - x_2)$ with $e \rightarrow x_2$, and $M_2 \cong M / (Q - x_1)$ with $e \rightarrow x_1$, where $x_1 \in Q$ and $x_2 \in Z \setminus Q$. Obviously M_1 is a polygon. The remaining parts of this theorem follow from (3.2.6). Δ

Example. Consider the following graph G and its bond matroid

$BM(G)$

$Y = \{1, 2, b, 5, 6, 7, c\}$ is a cocircuit of $M = BM(G)$. Choose 1 as the fixed element of Y ; it is easy to check that $B_2 = \{14\}$,

$B_3 = \{15\}$, and $B_7 = \{21, 22, a\}$ are the bridges of Y in $BM(G)$, that every simplified Y -component has no good 2-separation, and that $C^*(M, Y) = \{\{B_2, B_3, B_7\}\}$. Now B_2, B_3 are singleton bridges of Y , and $Q = \{14, 15\}$ is a series class contained in $\bigcup(B_i : B_i \in C)$, where $C = \{B_2, B_3, B_7\}$. Thus $(\{14, 15\}, E(G) \setminus \{14, 15\})$ is a good 2-separation of $BM(G)$. The simple decomposition of $BM(G)$ associated with this 2-separation and the marker e is shown in the following diagram.

Now, in addition to the assumption (5.4.12), we assume that, for each bridge $B_i \in C$, the simplified Y -component corresponding to B_i has no good 2-separation; that every parallel class of M contained in L_1 has cardinality one, and that $|C^*| \geq 2$, or $|C^*| = 1$ and $|E_2| \geq 2$. By (5.4.13), or (5.4.11) $\{E_1, E_2\}$ is a good 2-separation of M , where $E_1 = L_1 \cup (\bigcup(B_i : B_i \in C))$ and $E_2 = L_2 \cup (\bigcup(B_i : B_i \notin C))$. Let $M \rightarrow \{M(E_1; e), M(E_2; e)\}$; then we have the following theorem.

(5.5.8) Theorem. With the above assumptions,

- (i) if $C \in C_2^*$ then $M(E_1; e)$ is a polygon;
- (ii) if $C \in C_1$ and every series class of M contained in $U(B_1: B_1 \in C)$ has cardinality one, then $M(E_1; e)$ is 3-connected.

Proof. Applying (5.5.1), it is not difficult to check that, with M replacing by $M(E_1; e)$ and cocircuit Y , replaced by $L_1 + e$, the assumptions of theorem (5.5.2) are satisfied, and thus the theorem holds. Δ

Remark. In the context of (5.5.8), if $|C^*| = 1$ and $|E_2| \geq 2$ then clearly $M(E_2; e)$ is a bond.

(5.5.9) Finally, making all the assumptions of (5.5.8) except $|C^*| = 1$ and $|E_2| = 1$, the following two statements hold, by (5.5.2):

- (i) If $C \in C_2^*$ then M is a polygon.
- (ii) If $C \in C_1$ and every series class contained in $U(B_1: B_1 \in C)$ has cardinality one, then M is 3-connected.

Example. Consider the following graph G and its bond matroid $BM(G)$.

$Y = \{1, 2, b, 5, 6, 7, 8, c\}$ is a cocircuit of $M = BM(G)$. Choose 1 as the fixed element of Y . It is straightforward to check the following facts: The bridges of Y are $B_2 = \{14\}$, $B_3 = \{15\}$, $B_4 = \{16\}$, $B_5 = \{17\}$, $B_6 = \{18, 19, 20\}$, and $B_7 = \{21, 22, a\}$; every simplified Y -component has no good 2-separation; $C^*(M, Y) = \{\{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\}\}$ both $\{B_6\}$, $\{B_4, B_5\}$ are minimal members of $(C^*, \leq)$; $\{B_4, B_5\} \in C_2^*$; and $\{B_6\} \in C_1$. The reader may visualize theorem (5.5.8) from the following two simple decompositions of G .

5.6 Compatible ordering of the good 2-components

Let M be a non-separable matroid; and Y be a cocircuit of M with $f \in Y$. Let $C_1, C_2, \dots, C_m$ be the good 2-components of the bridge graph of Y . From the previous discussion, we need an efficient way of finding an ordering of the indices $1, 2, \dots, m$ of C_i 's so that if $C_i \stackrel{f}{\leq} C_j$, $i \neq j$, then $i < j$. Since $(C^*, \stackrel{f}{\leq})$ is a partially ordered set and C^* is finite, such an ordering does exist; we call an ordering (of the indices of C_i 's) of this kind an ordering compatible with $(C^*, \stackrel{f}{\leq})$ or simply a compatible ordering. In this section, we describe an algorithm to find such a compatible ordering. The following ideas (5.6.1) and (5.6.2), are due to Bixby and Cunningham [4]; by making a special ordering of the bridges of Y , they obtained an efficient algorithm to test the avoidance relation of the bridges in order to convert a linear program to a network flow problem.

(5.6.1) Let Y be a cocircuit of the non-separable matroid M . Fix $f \in Y$, for each bridge B_i of Y , let S_i be the B_i -segment containing f . Now order the indices $1, \dots, k$ of the bridges so that for $1 \leq i \leq j \leq k$ either $|S_i| > |S_j|$, or $|S_i| = |S_j|$ and $|\pi(B_i)| = 2$ implies $|\pi(B_j)| = 2$.

(5.6.2) Lemma. (Bixby and Cunningham [4]). Assume that the bridges $B_1, B_2, \dots, B_k$ are ordered as in (5.6.1). Let

$1 \leq i < j \leq k$. If $B_j \stackrel{f}{\leq} B_i$ then $\pi(B_i) = \pi(B_j)$ and $|\pi(B_i)| = 2$.
 (Thus $B_i \stackrel{f}{\leq} B_j$ and $\pi(B_i) = \pi(B_j) = \pi(B_k)$ for all k , $i \leq k \leq j$).

Proof. Since $B_j \stackrel{f}{\leq} B_i$, there exists $T_i \in \pi(B_i)$ such that $f \notin T_i$ and $S_j \cup T_i = Y$. Then $S_j \supseteq Y \setminus T_i \supseteq S_i$. But $i < j$ implies $|S_i| \geq |S_j|$; thus $S_j = S_i$ and $T_j = Y \setminus S_j = Y \setminus S_i = T_i$. So $\pi(B_i) = \pi(B_j)$ and $|\pi(B_i)| = 2$. Δ

It follows that, in the context of (5.6.2), if B_i avoids B_j , where $i < j$, then $S_i \cup W_j = Y$, where $f \in S_i \in \pi(B_i)$ and $W_j \in \pi(B_j)$.

In the following, we arrange a special ordering of the good 2-components of the bridge graph of Y , and prove that it is a compatible ordering.

(5.6.3) Suppose that the indices of the bridges of Y are ordered as in (5.6.1). Let $C_1, C_2, \dots, C_m$ be the good 2-components of the bridge graph, and the indices $1, 2, \dots, m$ be ordered so that for $1 \leq p < q \leq m$, $\max \{i: B_i \in C_p\} < \max \{i: B_i \in C_q\}$.

(5.6.4). Proposition. Assume that the good 2-components $C_1, C_2, \dots, C_m$ of the bridge graph are ordered as in (5.6.3). If $C_p \stackrel{f}{\leq} C_q$, and $p \neq q$, then $p < q$.

Proof. Since $C_p \stackrel{f}{\leq} C_q$, by (5.4.5), there exists $B_i \in C_q$ such that $B \stackrel{f}{\leq} B_i$, for all $B \in C_p$. If $p > q$, let $j = \max \{i: B_i \in C_p\}$.

Then we have $B_j \stackrel{f}{\leq} B_i$ and $j > i$. By (5.6.2), $\pi(B_i) = \pi(B_j)$ and $|\pi(B_j)| = 2$. Then B_i and B_j are in the same good 2-components and this is a contradiction. Thus $p < q$. Δ

Therefore, the ordering of the indices $1, 2, \dots, m$ of C_i 's described in (5.6.3) is a compatible ordering.

Let Y be a cocircuit of the non-separable matroid M with a fixed element $f \in Y$. We now describe an algorithm for finding an ordering of the good 2-components of the bridge graph of Y , which is compatible with $(C^*, \stackrel{f}{\leq})$.

(5.6.5) Algorithm. (Input is the segment sets $\pi(B_1), \pi(B_2), \dots, \pi(B_k)$, where $B_1, B_2, \dots, B_k$ are the bridges of Y in M , $k \geq 2$.)

Step 1. (Ordering the indices): For $i = 1, \dots, k$, let $S_i \in \pi(B_i)$ be such that $f \in S_i$. Now order the indices $1, \dots, k$ so that for $1 \leq i \leq j \leq k$ either $|S_i| > |S_j|$, or $|S_i| = |S_j|$ and $|\pi(B_i)| = 2$ implies $|\pi(B_j)| = 2$.

Step 2. Obtain the connected components of the bridge graph of Y .

Step 3. Obtain the good 2-components of the bridge graph of Y , and pick out the bridge with the largest index in each of the good 2-components.

Step 4. Sort the good 2-components $C_1, C_2, \dots, C_m$ in order, such that $\max\{i: B_i \in C_p\} < \max\{i: B_i \in C_q\}$, for $1 \leq p < q \leq m$.

End of the algorithm.

Example. Consider the example given in (5.3.6) with $M = BM(G)$, $Y = \{1,2,3,4,5,6,7,8,9,10\}$, and 1 as the fixed element in Y . The bridges of Y are $B_1 = \{11,12,13\}$, $B_2 = \{14\}$, $B_3 = \{15\}$, $B_4 = \{16\}$, $B_5 = \{17\}$, $B_6 = \{18,19,20\}$, and $B_7 = \{21,22,23,24,25\}$. Let $S_i \in \pi(B_i)$ be the B_i -segment with $1 \in S_i$, $i = 1,2,\dots,7$. Then $S_1 = Y \setminus \{3,4\}$, $S_2 = S_3 = Y \setminus \{6,7\}$, $S_4 = S_5 = Y \setminus \{3,4\}$, $S_6 = Y \setminus \{8,9,10\}$, and $S_7 = \{1,7,8,9,10\}$. Notice that $|S_1| = |S_2| = |S_3| = |S_4| = |S_5| > |S_6| > |S_7|$; thus we have already ordered the indices of the bridges as in (5.6.1). The set of all good 2-components of the bridge graph of Y is $C^* = \{\{B_1\}, \{B_6\}, \{B_2, B_3, B_7\}, \{B_4, B_5\}\}$. From (5.6.3) and (5.6.4), we see that the following is a compatible ordering of C^* : $C_1 = \{B_1\}$, $C_2 = \{B_4, B_5\}$, $C_3 = \{B_6\}$, $C_4 = \{B_2, B_3, B_7\}$.

(5.6.6) Now we analyse the complexity of algorithm (5.6.5). By applying the idea of (5.6.1), and (5.6.2), Bixby and Cunningham [4, page 332] has an efficient implementation of step 2; the computational work is bounded by $O(\max(k|Y|, k \log k))$, where k is the number of the bridges of Y . (Remark. For our purpose here, we need to make some minor modifications of their algorithm, which do not affect the complexity.) Clearly the amount of work required to execute step 1 and step 3 is bounded by $O(k|Y|)$. For step 4, we may apply radix sort to do it; the computational work is bounded by $O(k)$. (See Aho, Hopcroft, Ullman [1].) Therefore, algorithm (5.6.5) executes in time $O(\max(k|Y|, k \log k))$.

5.7 Outline of the decomposition algorithm

Based on the previous studies, we now outline a brief version of our algorithm for constructing the standard decomposition of a non-separable matroid. We will go into the details of the algorithm and revise it later.

(5.7.1) Outline of the main algorithm

Input is a non-separable matroid M .

Step 0. If M has rank 1 or less, stop; $\{M\}$ is the standard decomposition of M .

Step 1. Choose a basis B of M and find a B -fundamental cocircuit Y of M which is separating. If none exists, then apply the algorithm described in (5.7.2) below to find the standard decomposition of M ; stop.

Apply the algorithm recursively to the simplified Y -components to find their standard decompositions.

(Comment Let S be the set of all good 2-separations of M that are concealed in its Y -components. From this step, we obtain S .)

Step 3. Form the decomposition $D \cup \{M^T\}$ of M which is generated by S , where M^T is the trunk matroid obtained by splitting from M all the good 2-separations that are concealed in its simplified Y -components.

(Comment. Y is a cocircuit of M^r ; each simplified Y -component of M^r has no good 2-separation.)

Step 4. Construct the standard decomposition D^r of M^r which is generated by all the type J good 2-separations (with respect to the cocircuit Y) of M . Output $D \cup D^r$; this is the standard decomposition of M .

End of the algorithm.

(5.7.2) Let M be a non-separable matroid with rank at least 2, and let B be basis of M . Suppose every B -fundamental cocircuit of M is non-separating; then, by (5.4.11), for every B -fundamental cocircuit Y and every parallel class $P \subseteq Y$ with $|P| \geq 2$, $(P, E(M) \setminus P)$ is a good 2-separation of M . On the other hand, where $Y_1, Y_2, \dots, Y_k$ are the B -fundamental cocircuits of M , after simplifying $\bigcup_{i=1}^k Y_i$ in M , from (4.1.11), it is easy to see that the resulting matroid $M[\bigcup_{i=1}^k Y_i]$ is 3-connected. Thus, to construct the standard decompositions of M , we just have to find all the parallel classes $P_1, P_2, \dots, P_n$ of M with $|P_i| \geq 2$, $i = 1, \dots, n$, and from the decomposition D of M which is generated by $S = \{(P_i, E(M) \setminus P_i); i = 1, \dots, n\}$; then D is the standard decomposition of M .

In the following, we have 3 remarks about algorithm (5.7.1).

Remark 1. We use algorithm (2.4.11) as a subroutine for finding the bridges of a cocircuit. By refining the choice of the basis B

in step 1, we can show that the total number of applications of this subroutine is $r(M)$. The idea, motivated by Proposition (3.2.10), is due to Bixby and Cunningham; we shall see this later. Also, under appropriate assumptions, the total computational effort required by this subroutine is of the order of $(r(M))^2 |E(M)|$, and the work required to execute the other steps of the algorithm (5.7.1) is dominated by this bound.

Remark 2. We are going to work on the Y-components instead of simplified Y-components in the revised algorithm.

Remark 3. It is easy to see that the number of good 2-separations of M' is at most $|E(M')| - 3$, for any matroid M' . So, the number of simple decompositions needed, when step 3 and step 4 are applied to a matroid M' , is at most $|E(M')| - 3$. If M is the initial matroid, from remark 1, it might appear that the total number of simple decompositions performed by the straightforward algorithm (5.7.1) is of the order of $r(E)|E(M)|$. In the remainder of this chapter, we shall revise our algorithm so that the total number of simple decompositions required is exactly equal to the number of good 2-separations of M .

5.8 $\ast$ -($Y_1, Y_2, \dots, Y_k$)-subcomponents

Now we develop the details of decomposition algorithm (5.7.1).

Let M be a matroid on E , where n is a positive integer, a

$(Y_1, Y_2, \dots, Y_n)$ -subcomponent N of M is defined inductively as follows:

- (i) If $n = 1$, N is a Y_1 -component of M ;
- (ii) If $n \geq 2$, N is a Y_n -component of N' , where N' is a $(Y_1, Y_2, \dots, Y_{n-1})$ -subcomponent of M .

We remark that the definition of Y -component of M requires Y to be a cocircuit of M . A $(Y_1, Y_2, \dots, Y_n)$ -subcomponent of M is obtained from M by contracting elements; therefore, by induction, $Y_1, Y_2, \dots$, and Y_n are all cocircuits of M .

Similarly, we define a simplified $(Y_1, Y_2, \dots, Y_n)$ -subcomponent N of M inductively as follows:

- (i) If $n = 1$, N is a simplified Y_1 -component of M ;
- (ii) If $n \geq 2$, N is a simplified Y_n -component of N' , where N' is a simplified $(Y_1, Y_2, \dots, Y_{n-1})$ -subcomponent of M .

(5.8.1) Let M be a non-separable matroid on E . For $i = 1$ to n , let N_i be a $(Y_1, Y_2, \dots, Y_i)$ -subcomponent of M such that N_i is a Y_i -component of N_{i-1} , $1 \leq i \leq n$, and $N_0 = M$. In the following, we will show that, corresponding to each N_i in a natural way, there is a simplified $(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_i)$ -subcomponent $\hat{N}_i$ of M such that $\hat{N}_i$ is a simplified $\hat{Y}_i$ -component of $\hat{N}_{i-1}$, $\hat{Y}_i = Y_i \cap E(\hat{N}_{i-1})$, $1 \leq i \leq n$, and $\hat{N}_0 = M$. Furthermore, $\hat{N}_i$ is isomorphic to the matroid obtained by simplifying $\bigcup_{j=1}^i Y_j$ in N_i . We

call $\hat{N}_1$ the simplified $(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_k)$ -subcomponent associated with N_1 . To show this, we need the following lemma.

(5.8.2) Lemma. Let N_m be a $(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M . Then any parallel class P of N_m is either contained in $\bigcup_{i=1}^m Y_i$ or is disjoint from it.

Proof. We prove this by induction on m . Suppose $m = 1$; then Y_1 is a cocircuit of N_1 . By (3.1.1), the result holds. Now suppose the results holds for $m \geq 1$. For $i = m, m+1$ let N_{m+1} be a $(Y_1, Y_2, \dots, Y_{m+1})$ -subcomponent of M such that N_{m+1} is a Y_{m+1} -component of N_m . Let P be a parallel class of N_{m+1} with $P \cap \bigcup_{i=1}^{m+1} Y_i \neq \emptyset$. Suppose $P \cap Y_{m+1} \neq \emptyset$; then $P \subseteq Y_{m+1}$, since Y_{m+1} is a cocircuit of N_{m+1} . Suppose $P \cap Y_{m+1} = \emptyset$; then P is a parallel class of N_m with $P \subseteq E(N_m) \setminus Y_{m+1}$. By (3.2.3(ii)), P is a parallel class of N_m . Thus, by induction, $P \subseteq \bigcup_{i=1}^m Y_i$. This completes the proof. Δ

Now we construct $\hat{N}_1$ inductively, $i = 1, \dots, n$. Recall that

$M[S]$ is the matroid obtained from M by simplifying S in M .

First, let $\hat{N}_1 = N_1[Y_1]$ and $\hat{Y}_1 = Y_1 \cap E(\hat{N}_0) = Y_1 \cap E(M) = Y_1$; clearly

$\hat{N}_1$ is a simplified Y_1 -component of M . Let $\hat{Y}_2 = Y_2 \cap E(\hat{N}_1)$. Since

Y_2 is a cocircuit of N_1 and $\hat{N}_1$ has been obtained from N_1 by deleting some parallel elements, it is easy to see that $\hat{Y}_2$ is a

cocircuit of $\hat{N}_1$. Now suppose that we have defined simplified

$(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_i)$ -subcomponent $\hat{N}_i$ of M as required, for $i = 1, 2, \dots, k$,

where $1 \leq k < n$. We may assume $\hat{N}_k = N_k[\bigcup_{i=1}^k Y_i]$; note that, by (5.8.2), a parallel class P of N_k is either contained in $\bigcup_{i=1}^m Y_i$ or disjoint from it. Let $\hat{Y}_{k+1} = Y_{k+1} \cap E(\hat{N}_k)$; then $\hat{Y}_{k+1}$ is a cocircuit of $\hat{N}_k$, since Y_{k+1} is a cocircuit of N_k . Since N_{k+1} is a Y_{k+1} -component of N_k , it is easy to check that $E(N_{k+1}) \cap E(\hat{N}_k \setminus \hat{Y}_{k+1})$ is a bridge of $\hat{Y}_{k+1}$ in $\hat{N}_k$, let $\tilde{N}_{k+1}$ be the $\hat{Y}_{k+1}$ -component corresponding to it. Observe that if a_1 and a_2 are in parallel in N_k , then either both of them are in $E(N_{k+1})$ and a_1, a_2 are in parallel in N_{k+1} , or neither of them is in $E(N_{k+1})$; and if e_1 and e_2 are in parallel in N_{k+1} but not in N_k , then, by (3.1.1) and (3.2.3), $e_i \in Y_{k+1}$, $i = 1, 2$. From this observation, it is not difficult to see that the simplified $\hat{Y}_{k+1}$ -component $\tilde{N}_{k+1}[\hat{Y}_{k+1}]$ is isomorphic to $N_{k+1}[\bigcup_{i=1}^{k+1} Y_i]$. Letting $\hat{N}_{k+1} = \tilde{N}_{k+1}[\hat{Y}_{k+1}]$, this completes the inductive construction of $\hat{N}_i$, $i = 1, 2, \dots, n$.

In order to implement our decomposition algorithm (5.7.1) efficiently, we need to use a special kind of $(Y_1, Y_2, \dots, Y_n)$ -subcomponent; to introduce it, we first prove a proposition.

Let M be a non-separable matroid, and let $M_1[Y_1]$ be the simplified Y_1 -component of M corresponding to the bridge B_1 . To find all the good 2-separations of M concealed in $M_1[Y_1]$, we apply the recursive procedure to $M_1[Y_1]$, choose a cocircuit Y_2 of

$M_1[Y_1]$, and find all the simplified Y_2 -components. Let $M_2[Y_2]$ be the simplified Y_2 -component of $M_1[Y_1]$ corresponding to the bridge B_2 . Suppose that $\{A_1, A_2\}$ is a good 2-separation of $M_2[Y_2]$. By (5.3.3), either $A_1 \subseteq B_2$ or $A_2 \subseteq B_2$, say the former, then $\{A_1, E(M_1[Y_1]) \setminus A_1\}$ is a good 2-separation of $M_1[Y_1]$. Let $A_3 = E(M_1[Y_1]) \setminus A_1$. Now $\{A_1, A_3\}$ is a good 2-separation of $M_1[Y_1]$; at this point, again we have to check if $A_1 \subseteq B_1$ or $A_3 \subseteq B_1$, then apply (5.3.3) to get a good 2-separation of M . However, by choosing the cocircuit Y_2 of $M_1[Y_1]$ carefully, we can avoid these checking steps. More precisely, we have the following proposition.

(5.8.3) Proposition. In addition to the above assumptions, assume that $Y_2 \cap Y_1 \neq \emptyset$. Then $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M , for every good 2-separation $\{A_1, A_2\}$ of $M_2[Y_2]$ with $A_1 \subseteq B_2$.

Proof. By (5.3.3), $\{A_1, E(M_1[Y_1]) \setminus A_1\}$ is a good 2-separation of $M_1[Y_1]$. Again by (5.3.3), either $A_1 \subseteq B_1$ or $E(M_1[Y_1]) \setminus A_1 \subseteq B_1$; if the former holds then $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M , and we are done. So we just have to prove that $E(M_1[Y_1]) \setminus A_1 \not\subseteq B_1$. Since $A_1 \subseteq B_2$, we have $A_1 \cap Y_2 = \emptyset$, and so $Y_2 \subseteq E(M_1[Y_1]) \setminus A_1$. Now $Y_2 \cap Y_1 \neq \emptyset$; thus $E(M_1[Y_1]) \setminus A_1 \not\subseteq B_1$, as required. Δ

The above proposition and (3.2.10) motivate the following definition.

Let M be a non-separable matroid with a fixed basis B . Where $n \geq 1$, we define a $*(Y_1, Y_2, \dots, Y_n)$ -subcomponent N of M with inherited basis $B(N)$ and scanned set $Sc(N)$ as follows:

- (i) If $n = 1$, N is a Y_1 -component of M (say, corresponding to the bridge B_1), where Y_1 is a B -fundamental cocircuit of M (say, determined by $b_1 \in B$); inherited basis $B(N) = B \cap (B_1 \cup Y_1)$ and scanned set $Sc(N) = \{b_1\}$.
- (ii) If $n \geq 2$, N is a Y_n -component of N' (say, corresponding to the bridge B_n), where N' is a $*(Y_1, \dots, Y_{n-1})$ -subcomponent of M with inherited basis $B(N')$, scanned set $Sc(N')$, and $B(N') \setminus Sc(N') \neq \emptyset$, Y_n is a $B(N')$ -fundamental cocircuit (say, determined by $b_n \in B(N')$) satisfying (1) $b_n \in B(N') \setminus Sc(N')$; (2) $Y_n \cap (\cup (Y_i : Y_i \text{ is a } B(N')\text{-fundamental cocircuit determined by } b_i \in Sc(N')))) \neq \emptyset$. The inherited basis $B(N) = B(N') \cap (B_n \cup Y_n)$, and scanned set $Sc(N) = (Sc(N') \cap (B_n \cup Y_n)) + b_n$.

In the above definition, several points need to be justified; they will become clear after the next proposition. We remark that, in the context of the above definition, the existence of such a Y_n with $Y_n \cap (\cup (Y_i : Y_i \text{ is a } B(N')\text{-fundamental cocircuit determined by } b_i \in Sc(N')))) \neq \emptyset$ is guaranteed by the non-separability of N' . Recall that a B -fundamental cocircuit determined by b is denoted by $Y(b, B)$.

(5.8.4) Proposition. Let M be a non-separable matroid with a fixed basis B . Let N_m be a $*(Y_1, \dots, Y_m)$ -subcomponent of M with inherited basis $B(N_m)$ and scanned set $Sc(N_m)$. Then

- (1) $B(N_m)$ is a basis of N_m ;
- (2) $Sc(N_m) \subseteq B(N_m) \subseteq B$;
- (3) Every $B(N_m)$ -fundamental cocircuit of N_m is a B -fundamental cocircuit of M (hence, $Y(b, B(N_m)) = Y(b, B)$ for every $b \in B(N_m)$);
- (4) Every Y_i is a B -fundamental cocircuit of M ;
- (5) Where $b_i \in B$ is the basic element of M which determines Y_i , $1 \leq i < m$, $\{b_1, b_2, \dots, b_m\} \cap B(N_m) = Sc(N_m)$;
- (6) for every $b_i \in Sc(N_m)$, $Y(b_i, B(N_m)) = Y_i$ is a non-separating cocircuit of N_m ;
- (7) N_m is a non-separable matroid.

Proof. Applying (3.2.10), (3.2.11), and by induction on m , it is easy to see that all the results hold. Δ

(5.8.5) Proposition. Let M be a non-separable matroid with a fixed basis B . Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M with inherited basis $B(N_m)$ and scanned set $Sc(N_m)$. Let $b_i \in B$ be such that $Y_i = Y(b_i, B)$, $1 \leq i \leq m$. Then $(\bigcup_{i=1}^m Y_i) \cap E(N_m) = \bigcup(Y_i: Y_i = Y(b_i, B), b_i \in Sc(N_m))$.

Proof. We prove this by induction on m . For $m = 1$, it is obvious, since $Y_1 \subseteq E(N_1)$ and $b_1 \in Sc(N_1)$. Suppose that the result is

true for $m \geq 1$. For $i = m, m+1$, let N_i be a $*(Y_1, Y_2, \dots, Y_i)$ -subcomponent of M with inherited basis $B(N_i)$ and scanned set

$Sc(N_i)$, such that N_{m+1} is a Y_{m+1} -component of N_m . By the induction hypothesis, we have $(\bigcup_{i=1}^m Y_i) \cap E(N_m) = \bigcup (Y_i : Y_i = Y(b_i, B),$

$b_i \in Sc(N_m)$). Observe that $Y_i \subseteq E(N_{m+1})$, for every i with

$b_i \in Sc(N_{m+1})$. If $b_i \in Sc(N_m) \setminus Sc(N_{m+1})$, then Y_i is contained in

a Y_{m+1} -component of N_m other than N_{m+1} , and so $(Y_i \setminus Y_{m+1}) \cap E(N_{m+1}) = \emptyset$.

Now

$$\begin{aligned}
 & (\bigcup_{i=1}^m Y_i) \cap E(N_{m+1}) \\
 &= Y_{m+1} \cup ((\bigcup_{i=1}^m Y_i) \cap E(N_{m+1})) \\
 &= Y_{m+1} \cup ((\bigcup_{i=1}^m Y_i) \cap E(N_m) \cap E(N_{m+1})) \\
 &= (\text{by induction}) Y_{m+1} \cup (\bigcup (Y_i : Y_i = Y(b_i, B), b_i \in Sc(N_m)) \cap E(N_{m+1})) \\
 &= Y_{m+1} \cup (\bigcup (Y_i : Y_i = Y(b_i, B), b_i \in Sc(N_m) \cap Sc(N_{m+1}))) \cup (\bigcup (Y_i : Y_i = Y(b_i, B), \\
 &\quad b_i \in Sc(N_m) \setminus Sc(N_{m+1})) \cap E(N_{m+1})) \\
 &= \bigcup (Y_i : Y_i = Y(b_i, B), b_i \in Sc(N_{m+1})) \cup (\bigcup (Y_i \setminus Y_{m+1} : Y_i = Y(b_i, B), \\
 &\quad b_i \in Sc(N_m) \setminus Sc(N_{m+1})) \cap E(N_{m+1})) \\
 &= \bigcup (Y_i : Y_i = Y(b_i, B), b_i \in Sc(N_m)) \cup \emptyset.
 \end{aligned}$$

This completes the induction proof. Δ

(5.8.6) From now on, we assume that M is a non-separable matroid with a fixed basis B ; whenever a $*(Y_1, \dots, Y_m)$ -subcomponent of M is defined, it is defined with respect to this basis B . We shall use N_i to denote a $*(Y_1, \dots, Y_i)$ -subcomponent with N_i .

a Y_i -component of N_{i-1} , for $i \geq 1$, where $N_0 = M$, and use b_i to denote the basic element which determines the B-fundamental cocircuit Y_i . $\cup(Y_i: b_i \in \text{Sc}(N_m))$ is the short-hand notation for $\cup(Y_i: Y_i \text{ is a } B(N_m)\text{-fundamental cocircuit determined by } b_i \in \text{Sc}(N_m))$. In the following, we relate the good 2-separations of a $*(Y_1, \dots, Y_m)$ -subcomponent to those of M .

(5.8.7) Lemma. Let M be a non-separable matroid, let Y be a cocircuit of M , and let M_1 be a Y -component corresponding to the bridge B_1 . Let $A_1 \subseteq B_1$; then $\{A_1, E(M_1) \setminus A_1\}$ is a good 2-separation of M_1 if and only if $\{A_1, E(M_1[Y]) \setminus A_1\}$ is a good 2-separation of $M_1[Y]$.

Proof. Since Y is a non-separating cocircuit of M_1 , by (5.4.11), for each parallel class P of M_1 with $P \subseteq Y$ and $|P| \geq 2$, $\{P, E(M_1) \setminus P\}$ is a good 2-separation of M_1 . After splitting from M_1 all such parallel classes, the remaining matroid is just $M_1[Y]$. Now by (5.2.6), it is easy to see the result holds. Δ

(5.8.8) Theorem. Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M . Let $\{A_1, A_2\}$ be a good 2-separation of N_m with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$ (hence $A_2 \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$). Then $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M .

Proof. We prove this by induction on m . For $m = 1$, since N_1 is

a Y_1 -component of M and $A_1 \cap Y_1 = \emptyset$, by (5.8.7) and (5.3.3), $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M . Suppose the result is true for $m \geq 1$. For $i = m, m+1$, let N_i be a $*(Y_1, Y_2, \dots, Y_i)$ -subcomponent of M such that N_{m+1} is a Y_{m+1} -component of N_m . Let $\{A_1, A_2\}$ be a good 2-separation of N_{m+1} with $A_1 \cap (\bigcup_{i=1}^{m+1} Y_i) = \emptyset$. Since N_{m+1} is a Y_{m+1} -component of N_m and $A_1 \cap Y_{m+1} = \emptyset$, $\{A_1, E(N_m) \setminus A_1\}$ is a good 2-separation of N_m . Now $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$, because $A_1 \cap (\bigcup_{i=1}^{m+1} Y_i) = \emptyset$. Applying the induction hypothesis, we see that $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M . This completes the induction proof. Δ

In the context of (5.8.8), let $\{M^1, M^2\}$ (respectively $\{N_m^1, N_m^2\}$) be the simple decomposition of M (respectively N_m) associated with $\{A_1, E(M) \setminus A_1\}$ (respectively $\{A_1, A_2\}$) and the marker e ; with a similar proof as that of (5.3.5), and by induction on m , it is not difficult to see that the following theorem holds.

(5.8.9) Theorem. $M^1 = N_m^1$, N_m^2 is a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M^2 , the set of all parallel classes of Y_m in N_m^2 is the same as that of Y_m in N_m (that is, the segments of Y_m remain unchanged), and every $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M other than N_m is still a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M^2 . Moreover, the set of all $*(Y_1, Y_2, \dots, Y_m)$ -subcomponents of M^2

are precisely those mentioned above. Δ

Therefore, whenever we find a good 2-separation $\{A_1, A_2\}$ of N_m with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$, we can split A_1 directly from M .

(5.8.10) Theorem. Let N_m be a $*(Y_1, \dots, Y_m)$ -subcomponent of M . Then the simplified $(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_m)$ -subcomponent $\hat{N}_m$ of M associated with N_m (as defined in (5.8.1)) is 3-connected, if and only if for every good 2-separation $\{A_1, A_2\}$ of N_m , $\bigcup_{i=1}^m Y_i$ meets both A_1 and A_2 .

Before proving this theorem, we prove the following three lemmas.

(5.8.11) Lemma. Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M with scanned set $Sc(N_m)$. Let $\{A_1, A_2\}$ be a good 2-separation of N_m such that for each $b_k \in Sc(N_m)$; either $Y_k \subseteq A_1$ or $Y_k \subseteq A_2$, where Y_k is the B-fundamental cocircuit determined by b_k . Then there is a good 2-separation $\{E_1, E_2\}$ of M satisfying $A_j \subseteq E_j$, $j = 1, 2$, and for each i , $1 \leq i \leq m$, either $Y_i \subseteq E_1$ or $Y_i \subseteq E_2$.

Proof. We prove this by induction on m . Suppose $m = 1$ and

$Y_1 \subseteq A_2$. By (5.8.8), $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M , and the result clearly holds. Now assume the result holds for $m \geq 1$.

For $i = m, m+1$, let N_{m+1} be a $*(Y_1, Y_2, \dots, Y_{m+1})$ -subcomponent of M such that N_{m+1} is a Y_{m+1} -component of N_m . Let $\{A_1, A_2\}$ be a good 2-separation of N_{m+1} such that for each $b_k \in Sc(N_{m+1})$

either $Y_k \subseteq A_1$ or $Y_k \subseteq A_2$. We notice that $Y_{m+1} \subseteq E(N_{m+1})$ and $b_{m+1} \in \text{Sc}(N_{m+1})$. Without loss of generality we may assume that $Y_{m+1} \subseteq A_2$. By (5.8.8), $\{A_1, E(N_m) \setminus A_1\}$ is a good 2-separation of N_m . By the definition of scanned set, $\text{Sc}(N_m) \supseteq \text{Sc}(N_{m+1}) - b_{m+1}$. We claim that $Y_k \subseteq E(N_{m+1}) \setminus A_1$, for every $b_k \in \text{Sc}(N_m) \setminus \text{Sc}(N_{m+1})$. Suppose that the claim is true; then, by induction, there is a good 2-separation $\{E_1, E_2\}$ of M satisfying $A_1 \subseteq E_1$, $A_2 \subseteq E(N_{m+1}) \setminus A_1 \subseteq E_2$, and for each $1 \leq i \leq m$, either $Y_i \subseteq E_1$ or $Y_i \subseteq E_2$. As for Y_{m+1} , we know $Y_{m+1} \subseteq A_2 \subseteq E(N_{m+1}) \setminus A_1 \subseteq E_2$. This completes the induction proof.

Now we prove the claim. Suppose $b_k \in \text{Sc}(N_m) \setminus \text{Sc}(N_{m+1})$; then $Y_k \setminus Y_{m+1} \subseteq E(N_m) \setminus E(N_{m+1}) \subseteq E(N_m) \setminus A_1$. We know $Y_{m+1} \subseteq A_2 \subseteq E(N_m) \setminus A_1$, and thus $Y_k \subseteq E(N_m) \setminus A_1$. This proves the claim. Δ

(5.8.12) Lemma. Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M . Let $\{A_1, A_2\}$ be a good 2-separation of N_m with $A_j \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$, $j = 1$ and 2 . Then there is a $Y_k = Y(b_k, B)$, where $b_k \in \text{Sc}(N_m)$, $1 \leq k \leq m$, such that $A_1 \cap Y_k \neq \emptyset$ and $A_2 \cap Y_k \neq \emptyset$.

Proof. Suppose that this is not true; then for each $b_k \in \text{Sc}(N_m)$, either $Y_k \subseteq A_1$ or $Y_k \subseteq A_2$. By (5.8.11), there is a good 2-separation $\{E_1, E_2\}$ of M such that $A_j \subseteq E_j$, $j = 1$ and 2 , and for each i , $1 \leq i \leq m$, either $Y_i \subseteq E_1$ or $Y_i \subseteq E_2$. We observe that $\emptyset \neq A_j \cap (\bigcup_{i=1}^m Y_i) = A_j \cap ((\bigcup_{i=1}^m Y_i) \cap E(N_m)) =$ (by (5.8.5)) $A_j \cap (\bigcup_{i=1}^m (Y_i \cap E(N_m)))$ is

a B-fundamental cocircuit determined by $b_k \in \text{Sc}(N_m)$). Thus there is at least one $b_k \in \text{Sc}(N_m)$ with $Y_k \subseteq A_1$, and at least one $b_\ell \in \text{Sc}(N_m)$ with $Y_\ell \subseteq A_2$. Since $\text{Sc}(N_m) \subseteq \{b_1, b_2, \dots, b_m\}$, $A_1 \subseteq E_1$, and $A_2 \subseteq E_2$, we conclude that $m \geq 2$, and $\bigcup_{i=1}^m Y_i \not\subseteq E_j$, for $j = 1, 2$. Without loss of generality, we may assume that $Y_1 \subseteq E_1$. Let n be the smallest index such that $Y_n \subseteq E_2$, $n \leq m$. Then $Y_n \cap (\bigcup_{i=1}^{n-1} Y_i) = \emptyset$, because $\{E_1, E_2\}$ is a partition of $E(M)$. This contradicts the assumption that N_m is a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent. Thus the lemma holds. Δ

(5.8.13) Lemma. Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M . Let $\{A_1, A_2\}$ be a partition of N_m with $|A_j| \geq 2$, $j = 1, 2$. Then $\{A_1, A_2\}$ is a good separation of N_m with $A_j \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$, $j = 1$ and 2 , if and only if one of A_1, A_2 is a parallel class of N_m contained in Y_k , for some $b_k \in \text{Sc}(N_m)$, where Y_k is the B-fundamental cocircuit determined by b_k .

Proof. First we prove the "if" part. Suppose that A_1 is a parallel class contained in Y_k , for some $b_k \in \text{Sc}(N_m)$. By (5.8.4 (6)), Y_k is a non-separating cocircuit of N_m . Then by (5.4.11), $\{A_1, A_2\}$ is a good 2-separation of N_m , because A_1 is a parallel class of N_m with $|A_1| \geq 2$, and $A_1 \subseteq Y_k$. Now Y_k is a cocircuit of N_m , so $A_1 \not\subseteq Y_k$ unless $A_1 = Y_k = E(N_m)$ by the non-separability of N_m .

But the latter case is not possible, for $A_2 \neq \emptyset$. Thus $A_1 \not\subseteq Y_k$, and so $A_2 \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$. This proves the "if" part.

Now we prove the "only if" part. By (5.8.12), there is a $Y_k = Y(b_k, B)$, where $b_k \in \text{Sc}(N_m)$, such that Y_k meets A_1 and A_2 . By (3.3.3), $A_1 \setminus Y_k$ and $A_2 \setminus Y_k$ are separators of $N_m \setminus Y_k$. Now by (5.8.4(6)), Y_k is a non-separating cocircuit of N_m . Thus $A_1 \setminus Y_k = \emptyset$ or $A_2 \setminus Y_k = \emptyset$; say the former holds. Then, where r_m is the rank function of N_m , $r_m(A_2) = r_m(E(N_m))$, and $r_m(A_1) + r_m(A_2) = r_m(E(N_m)) + 1$. This implies that $r_m(A_1) = 1$; and thus A_1 is a parallel class of N_m . The "only if part" is established. Δ

Now we prove theorem (5.8.10).

Proof. By (5.8.5), $(\bigcup_{i=1}^m Y_i) \cap E(N_m) = \bigcup \{Y_i : Y_i = Y(b_i, B), b_i \in \text{Sc}(N_m)\}$, where $\text{Sc}(N_m)$ is the scanned set of N_m . By (5.8.4 (6)), for each $b_i \in \text{Sc}(N_m)$, $Y_i = Y(b_i, B)$ is a non-separating cocircuit of N_m ; and thus, by (5.4.11), for every parallel class P of N_m with $P \subseteq Y_i$ and $|P| \geq 2$, $\{P, E(N_m) \setminus P\}$ is a good 2-separation of N_m . Now, after splitting from N_m every parallel class P with $P \subseteq \bigcup_{i=1}^m Y_i$ and $|P| \geq 2$, by (5.8.1), the resulting matroid is isomorphic to $\hat{N}_m$. Moreover, $\hat{N}_m$ is a simplified $\hat{Y}_m$ -component of $\hat{N}_{m-1}$. Then by (5.3.8), $\hat{N}_m$ is 3-connected if and only if it does not have any good 2-separation. Therefore, $\hat{N}_m$ is 3-connected if and only if, for every good 2-separation $\{A_1, A_2\}$ of N_m , (say) A_1 is a

parallel class of N_m with $A_1 \subseteq Y_i$, for some $Y_i = Y(b_i, B)$ and $b_i \in \text{Sc}(N_m)$. By lemma (5.8.13), the latter statement holds if and only if for every good 2-separation $\{A_1, A_2\}$ of N_m , $\bigcup_{i=1}^m Y_i$ meets both A_1 and A_2 . Δ

(5.8.14) Theorem. Let N_m be a $*(Y_1, Y_2, \dots, Y_m)$ -subcomponent of M with inherited basis $B(N_m)$ and scanned set $\text{Sc}(N_m)$. Suppose $B(N_m) = \text{Sc}(N_m)$; then the simplified $(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_m)$ -subcomponent $\hat{N}_m$ associated with N_m is 3-connected.

Proof. Since $B(N_m) = \text{Sc}(N_m)$, by (5.8.4(6)), every $B(N_m)$ -fundamental cocircuit of N_m is non-separating. We note that, by (5.8.5),

$$\left(\bigcup_{i=1}^m Y_i\right) \cap E(N_m) = \bigcup \{Y_i : Y_i = Y(b_i, B(N_m)), b_i \in \text{Sc}(N_m)\}.$$

After simplifying $\bigcup_{i=1}^m Y_i$ in N_m , the resulting matroid is $\hat{N}_m$, and we

may assume that $B(N_m) \subseteq E(\hat{N}_m)$. Thus $B(N_m)$ is a basis of $\hat{N}_m$.

Clearly, the set of $B(N_m)$ -fundamental cocircuits of $\hat{N}_m$ is precisely $\{Y_i \cap E(\hat{N}_m) : Y_i \text{ is a } B(N_m)\text{-fundamental cocircuit of } N_m\}$.

Hence, it is not difficult to see that every $B(N_m)$ -fundamental cocircuit of $\hat{N}_m$ is simple and non-separating; by (4.1.11), $\hat{N}_m$ is 3-connected. Δ

5.9 Discussion of the details of the decomposition algorithm

Let M be a non-separable matroid. We consider the problem of constructing the standard decomposition of M , or equivalently finding

algorithmically all the good 2-separations of M . First of all, we choose a basis B of M . Where $m \geq 1$, let N_m be a $*(Y_1, \dots, Y_m)$ -subcomponent of M with inherited basis $B(N_m)$ and scanned set $Sc(N_m)$. In view of (5.8.8) and (5.8.10), our object is to find the good 2-separations $\{A_1, A_2\}$ of N_m with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$; whenever we find such a 2-separation, we can split A_1 directly from M . We will see that, if A_1 is chosen carefully, the "part" being split from M is always 3-connected, a polygon, or a bond.

(5.9.1) Suppose $B(N_m) = Sc(N_m)$; then by (5.8.14), the simplified $*(\hat{Y}_1, \dots, \hat{Y}_m)$ -subcomponent $\hat{N}_m$ is 3-connected, and nothing needs to be done. Suppose $B(N_m) \neq Sc(N_m)$; then we choose a $b_{m+1} \in B(N_m) \setminus Sc(N_m)$, where Y_{m+1} is the $B(N_m)$ -fundamental cocircuit of N_m determined by b_{m+1} , such that $Y_{m+1} \cap (\bigcup_{i=1}^m (Y_i : Y_i = Y(b_i, B(N_m)), b_i \in Sc(N_m))) \neq \emptyset$; existence of such a b_{m+1} is guaranteed by the non-separability of N_m . By the recursive procedure and (5.8.9), we may assume that all the Y_{m+1} -components of N_m do not have the kind of good 2-separation that we want, that is, for every good 2-separation $\{A_1, A_2\}$ of the $*(Y_1, \dots, Y_{m+1})$ -subcomponents, $A_j \cap (\bigcup_{i=1}^{m+1} Y_i) \neq \emptyset$, $j = 1$ and 2 . By (5.8.10), every corresponding $(\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_{m+1})$ -subcomponent $\hat{N}_{m+1}$ is 3-connected. We remark that, where N_{m+1} is a Y_{m+1} -component of N_m , the simplified Y_{m+1} -component $N_{m+1}[Y_{m+1}]$ is not necessarily 3-connected. However, if

$(\bigcup_{i=1}^{m+1} Y_i) \cap E(N_{m+1}) = Y_{m+1}$ then $N_{m+1}[Y_{m+1}]$ is isomorphic to $\hat{N}_{m+1}$, and thus $N_{m+1}[Y_{m+1}]$ is 3-connected. Now we choose an element $f \in Y_{m+1} \cap (\bigcup (Y_i : Y_i = Y(b_i, B(N_m)), b_i \in Sc(N_m)))$ as a fixed element of Y_{m+1} . Then we find all the good 2-components $C_1, C_2, \dots, C_k$ of the bridge graph of Y_{m+1} in N_m , $k \geq 1$, and order their indices in such a way that the ordering is compatible with $(C^*, \leq^f)$. Thus C_1 is a minimal member of C^* with respect to the partial ordering. Let $L_2(C_\ell) = \cap (S_i : f \in S_i \in \pi(B_j), B_j \in C_\ell)$ and $L_1(C_\ell) = Y_{m+1} \setminus L_2(C_\ell)$, $\ell = 1, \dots, k$.

(5.9.2) First we consider the case that $|C^*| \geq 2$. Where $E_1 = L_1(C_1) \cup (\bigcup (B_i : B_i \in C_1))$, and $E_2 = L_2(C_1) \cup (\bigcup (B_i : B_i \text{ is a bridge of } Y_{m+1}, B_i \notin C_1))$, it follows from (5.4.13), that $\{E_1, E_2\}$ is a good 2-separation of N_m . We note that $E_2 \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$. Since $|C^*| \geq 2$, it is not difficult to see that neither E_1 nor E_2 is a parallel class of N_m , and by (5.8.13), $E_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$. Thus by (5.8.8), $\{E_1, E(M) \setminus E_1\}$ is a good 2-separation of M . Furthermore, for every bridge $B_i \in C_1$, the simplified Y_{m+1} -component of N_m corresponding to B_i is 3-connected. We can split E_1 from M , but each time we split some "part" from M , we want that "part" to be 3-connected, a polygon, or a bond. So, as justified in (5.5.5) and (5.5.7), we first split from M the parallel classes P of N_m with $P \subseteq L_1(C_1)$ and $|P| \geq 2$; and if $C_1 \in C_1$, we split from M

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the series classes Q of N_m with $Q \subseteq U(B_i : B_i \in C_1)$ and $|Q| \geq 2$.

Hence we may assume that there are no such P 's and Q 's mentioned

above; then splitting E_1 from M , by (5.5.8), the "part" being

split is a polygon if $C_1 \in C_2^*$, and otherwise is 3-connected if

$C_1 \in C_1$. After splitting E_1 from M , C_2 becomes a minimal mem-

ber of $(C^* \setminus \{C_1\}, \leq^f)$. According to (5.5.1), we may do the pro-

cedure repeatedly until there is only one good 2-component of Y_{m+1}

in N_m left; this case has to be treated separately. (Remark:

$L_1(C_1)$ and Y_{m+1} have to be updated, because some elements of them

may have been split and replaced by markers; see (5.5.1)).

(5.9.3) Therefore, we assume $C^* = \{C\}$. Let $L_2(C) = \cap (S_i :$

$f \in S_i \in \pi(B_j), B_j \in C)$ and $L_1(C) = Y_{m+1} \setminus L_2(C)$. Recall that our

object is to find the good 2-separations $\{A_1, A_2\}$ of N_m with

$A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$. By (5.8.2), for every parallel class P with

$P \subseteq L_1(C)$, either $P \cap (\bigcup_{i=1}^m Y_i) = \emptyset$ or $P \subseteq \bigcup_{i=1}^m Y_i$; so we split

every parallel class P with $P \subseteq L_1(C)$, $|P| \geq 2$, and $P \cap (\bigcup_{i=1}^m Y_i) = \emptyset$.

From (5.2.6) and (5.8.13), it is not difficult to see that, if $C \in C_1$, for

every series class Q with $Q \subseteq U(B_i : B_i \in C)$ and $|Q| \geq 2$,

$\{Q, E(N_m) \setminus Q\}$ is a good 2-separation of N_m ; moreover, $Q \cap (\bigcup_{i=1}^m Y_i) = \emptyset$.

Thus we can split these Q 's from M . Now we may assume that N_m

has no such P 's and Q 's mentioned above. Observe that $f \in L_2(C)$

and $L_2(C)$ is a parallel class of N_m , and thus $L_2(C) \subseteq \bigcup_{i=1}^m Y_i$. Hence, it follows from (5.5.9), that the $(\bigcup_{i=1}^m Y_i)$ -simplification $N[\bigcup_{i=1}^m Y_i]$ is 3-connected or a polygon. (We remark that $N[\bigcup_{i=1}^m Y_i]$ cannot be a bond, for otherwise N_m is a bond, contradicting the fact that it is a $*(Y_1, \dots, Y_m)$ -subcomponent of M .) So, for every good 2-separation $\{A_1, A_2\}$ of N_m , $A_j \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$, $j = 1, 2$. In other words, the procedure for finding the good 2-separations $\{A_1, A_2\}$ of N_m with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$ has to stop at this stage, as we can find no more such good 2-separations.

(5.9.4) Now suppose that we are given a non-separable matroid M with a fixed basis B . Let Y_1 be a B -fundamental cocircuit of M . By applying the procedure described above, we may assume that, for every Y_1 -component N of M and every good 2-separation $\{A_1, A_2\}$ of N , $A_j \cap Y_1 \neq \emptyset$, $j = 1, 2$; then by (5.8.10), this implies that every simplified Y_1 -component of M is 3-connected. We can again apply the procedure described before to find all the type J good 2-separations of M (with respect to the cocircuit Y_1); however, the case is slightly different when $|C^*| = 1$, where C^* is the set of all good 2-components of Y_1 in M . Let $C^* = \{C\}$; we define $L_1(C)$, $L_2(C)$ as before. First, for every parallel class P of M with $P \subseteq L_1(C)$ and $|P| \geq 2$, $\{P, E(M) \setminus P\}$ is a good 2-separation of M . Second, if $|L_2(C)| \geq 2$, then $\{E(M) \setminus L_2(C),$

$L_2(C)$ is a good 2-separation of M by (5.4.11). The rest is the same as before.

(5.9.5) Now we discuss some technical problems. Let M be a non-separable matroid with a basis B . Let N_1 be a $*(Y_1, \dots, Y_1)$ -subcomponent of M with inherited basis $B(N_1)$ and scanned set $Sc(N_1)$, such that N_1 is a Y_1 -component of N_{1-1} , $1 \leq i \leq m$, and $N_0 = M$. Suppose that $\{A_1, A_2\}$ is a good 2-separation of N_m with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$; then $\{A_1, E(M) \setminus A_1\}$ is a good 2-separation of M . Instead of splitting A_1 from N_m , (this amounts to splitting A_1 from N_i , for every $i = 1, 2, \dots, m$, by recursive procedure,) we can avoid this trouble by splitting A_1 directly from M as follows. Suppose $B(N_m) \setminus Sc(N_m) = \emptyset$, then by (5.8.14), for every good 2-separation $\{A_1, A_2\}$ of N_m , $A_j \cap (\bigcup_{i=1}^m Y_i) \neq \emptyset$, $j = 1, 2$; and there is no problem in this case. Suppose $B(N_m) \setminus Sc(N_m) \neq \emptyset$; then choose a $B(N_m)$ -fundamental cocircuit Y_{m+1} of N_m which is determined by $b_{m+1} \in B(N_m) \setminus Sc(N_m)$ satisfying $Y_{m+1} \cap (U(Y_i : Y_i = Y(b_i, B(N_m)), b_i \in Sc(N_m))) \neq \emptyset$. Let $N_{m+1,1}, N_{m+1,2}, \dots, N_{m+1,k}$ be the Y_{m+1} -components of N_m . Let M^r (respectively N_m^r) be the resulting matroid after splitting from M (respectively N_m) every good 2-separation $\{A_1, A_2\}$ concealed in $N_{m+1,j}$ with $A_1 \cap (\bigcup_{i=1}^{m+1} Y_i) = \emptyset$, $j = 1, \dots, k$ (That is, we split all such A_1 's from M^r and N_m^r).

Let $N_{m+1,j}^r$ be the resulting matroid after splitting from $N_{m+1,j}$ every good 2-separation $\{A_1, A_2\}$ of $N_{m+1,j}$ with $A_1 \cap (\bigcup_{i=1}^{m+1} Y_i) = \emptyset$, $j = 1, \dots, k$. We call M^r the trunk matroid and $N_{m+1,j}^r$ the knot matroid of $N_{m+1,j}$ (with respect to $M, Y_1, \dots, Y_{m+1}$), $j = 1, \dots, k$. Notice that if we just split the above A_1 's from M , but not from $N_{m+1,j}$ and N_m , then we do not get the matroid structures of $N_{m+1,j}^r$ and N_m^r , $j = 1, \dots, k$. According to (5.8.9), N_m^r is a $*(Y_1, \dots, Y_m)$ -subcomponent of M^r , $N_{m+1,j}^r$ is a $*(Y_1, \dots, Y_{m+1})$ -subcomponent of M^r , $j = 1, \dots, k$, and $\{N_{m+1,1}^r, N_{m+1,2}^r, \dots, N_{m+1,k}^r\}$ is precisely the set of all Y_{m+1} -components of N_m^r . Moreover, the associated simplified $(\hat{Y}_1, \dots, \hat{Y}_{m+1})$ -subcomponent $\hat{N}_{m+1,j}^r$ is 3-connected, $j = 1, \dots, k$. Now we use the procedure described in (5.9.1)-(5.9.3), to split the good 2-separations $\{A_1, A_2\}$ of N_m^r with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$. Let us check how we can do this without knowing the matroid structures of N_m^r and $N_{m+1,j}^r$, $j = 1, 2, \dots, k$. We remark that all the Y -components together with the cocircuit Y do not determine the original matroid M uniquely, unless M is binary; therefore, knowing $\{N_{m+1,1}^r, N_{m+1,2}^r, \dots, N_{m+1,k}^r\}$ and Y_{m+1} does not necessarily give us N_m^r .

One way to solve this problem is to take the same cocircuit sequence $Y_1, \dots, Y_{m+1}$ in M^r , compute the $*(Y_1, \dots, Y_{m+1})$ -subcomponent of M^r , and actually get N_m^r and $N_{m+1,j}^r$, $j = 1, \dots, k$.

Another way to do it is to check carefully what information we need from N_m^r and $N_{m+1,j}^r$, $j = 1, \dots, k$. It turns out that we can obtain this information without knowing N_m^r and $N_{m+1,j}^r$, $j = 1, \dots, k$. This is more efficient and hence is the method we will use. We summarize the required information.

(1) We need the bridges of Y_{m+1} in N_m^r , that is, $E(N_{m+1,j}^r) \setminus Y_{m+1}$,

$\forall j = 1, \dots, k$;

(2) We need the segment set $\pi(N_m^r, B_{m+1,j}^r, Y_{m+1})$, $j = 1, \dots, k$,

where $B_{m+1,j}^r = E(N_{m+1,j}^r) \setminus Y_{m+1}$;

(3) Where C is a good 2-component of the bridge graph of Y_{m+1} in

N_m^r and a is a fixed element of Y_{m+1} , we need to know $L_1(C)$,

$L_2(C)$ and all the parallel classes P of $N_m^r / (E(N_m^r) \setminus (Y_{m+1} \cup (B_{i_1}$

B_{i_1} is a bridge in $C)))$ with $P \subseteq L_1(C)$, where $L_1(C) =$

$\cup \{S_i : \exists S_i \in \pi(B_j), B_j \in C\}$ and $L_2(C) = Y_{m+1} \setminus L_1(C)$;

(4) We have to be able to check whether two singleton bridges in C

are in series in N_m^r , if $C \in C_1$.

We now explain how we can obtain the above information, even if we do not have the matroid structures of N_m^r , $N_{m+1,j}^r$, $j = 1, 2, \dots, k$.

(1) We will obtain the bridges of Y_{m+1} in N_m^r in the recursive procedure. Suppose that $|B(N_m) \setminus Sc(N_m)| = 1$; then $B(N_{m+1,j}^r) =$
 $j = 1, 2, \dots, k$. By (5.8.14), $N_{m+1,j}^r = N_{m+1,j}$, $j = 1, \dots, k$.

$N_{m+1,j}$ has already been computed, and so has $E(N_{m+1,j}^r) \setminus Y_{m+1}$, $j = 1, \dots, k$. In general, suppose that we know $E(N_{m+1,j}^r)$, (but not the matroid $N_{m+1,j}^r$), $j = 1, \dots, k$. It is easy to compute the bridge $E(N_{m+1,j}^r) \setminus Y_{m+1}$ of Y_{m+1} in N_m^r , $j = 1, \dots, k$. It is also easy to see that, after finishing the procedure described in (5.9.1)-(5.9.3), the "remaining set" of $E(N_m^r) = Y_{m+1} \cup (\bigcup_{j=1}^k E(N_{m+1,j}^r))$ is just $E(N'_m)$, where N'_m is the knot matroid of N_m , that is, N'_m is the resulting matroid after splitting from N_m every good 2-separation $\{A_1, A_2\}$ with $A_1 \cap (\bigcup_{i=1}^m Y_i) = \emptyset$. Therefore, we can obtain the bridges of Y_{m+1} in N_m^r in the recursive procedure. $E(N_{m+1,j}^r)$ is called the knot set of $N_{m+1,j}$ with respect to M , $Y_1, \dots, Y_{m+1}$, $j = 1, \dots, k$.

(2) By (5.8.9), $\pi(N_m^r, B_{m+1,j}^r, Y_{m+1}) = \pi(N_m, B_{m+1,j}, Y_{m+1})$,

where $B_{m+1,j}^r = E(N_{m+1,j}^r) \setminus Y_{m+1}$ and $B_{m+1,j} = E(N_{m+1,j}) \setminus Y_{m+1}$,

$j = 1, \dots, k$. Hence we can obtain these segment sets from N_m ,

$B_{m+1,j}$ and Y_{m+1} , which have already been computed.

(3) Let C be a good 2-component of the bridge graph of Y_{m+1}

in N_m^r , and let f be a fixed element of Y_{m+1} . Considering the

matroid $N_m^r(C) = N_m^r / (E(N_m^r) \setminus (Y_{m+1} \cup (B_j : B_j \in C)))$, it is clear

that $\{P \mid P \text{ is a parallel class of } N_m^r(C) \text{ with } P \subseteq L_1(C)\} \cup \{L_2(C)\}$

is a partition of Y_{m+1} into parallel classes of $N_m^r(C)$. Knowing

the segment set $\pi(N_m^r, B_{m+1,j}^r, Y_{m+1})$, for every $B_{m+1,j}^r \in C$, it is not difficult to see; from (3.2.3), that we can partition Y_{m+1} into parallel classes of $N_m^r(C)$ in time $O(k_c |Y_{m+1}|)$, where k_c is the number of bridges in C .

(4) Let C be a good 2-component of the bridge graph of Y_{m+1} in N_m^r . First we know which bridges in C are singleton bridges, because we know the knot set $E(N_{m+1,j}^r)$, $j = 1, \dots, k$. Let $\{e_1\}$, $\{e_2\}$ be two singleton bridges in C . We notice that e_1, e_2 are in series in N_m^r if and only if e_1, e_2 are in series in the trunk matroid M^r , because N_m^r is a $*(Y_1, \dots, Y_m)$ -subcomponent of M^r , that is, N_m^r is obtained from M^r by contracting some elements. Therefore, we can check whether two singleton bridges in C are in series by using the matroid structures of M^r .

5.10 A decomposition algorithm

An algorithm to find the standard decomposition of a non-separable matroid

Main algorithm (Input is a non-separable matroid M with a basis $B = \{b_1, b_2, \dots, b_r\}$)

Step 0. If M has rank 1 or less, stop; $\{M\}$ is the standard decomposition of M .

(Comment: M is a bond or a loop).

Otherwise, let $N \leftarrow M$, inherited basis $B(N) \leftarrow \{b_1, b_2, \dots, b_r\}$, scanned set $Sc(N) \leftarrow \emptyset$, decomposition $D \leftarrow \emptyset$, and trunk matroid $M^T \leftarrow M$.

(Comment: $DU\{M^T\}$ is the current decomposition of M .)

Step 1. (Comment. We have a non-separable matroid N with inherited basis $B(N)$ and scanned set $Sc(N)$. Current decomposition is D , and current trunk matroid is M^T .)

If $Sc(N) = \emptyset$, choose a $B(N)$ -fundamental cocircuit Y , (say, determined by $b \in B(N)$), and arbitrarily choose $f \in Y$ as a fixed element of Y .

If $Sc(N) \neq \emptyset$ and $B(N) \setminus Sc(N) \neq \emptyset$, choose a $B(N)$ -fundamental cocircuit Y determined by the basic element $b \in B(N) \setminus Sc(N)$ such that $A = Y \cap (\cup \{Y_1 : Y_1 \text{ is a } B(N)\text{-fundamental cocircuit determined by } b_1 \in Sc(N)\}) \neq \emptyset$, and choose $f \in A$ as a fixed element of Y .

If $B(N) \setminus Sc(N) = \emptyset$, then declare that the current decomposition $DU\{M^T\}$ of M has used all the good 2-separations of M concealed in N . Set knot set of N to be $E(N)$. Decomposition D and trunk matroid M^T remain unchanged.

Step 2. (Comment. From step 1, we have a $B(N)$ -fundamental cocircuit Y of N determined by b with a fixed element $f \in Y$. We also have current decomposition D , current trunk matroid M^T .)

Obtain the bridges $B_1, B_2, \dots, B_k$ of Y in N ($k \geq 1$),
and the segment set $\pi(N, B_i, Y)$, for $i = 1$ to k .

Let N_i be the Y -component of N corresponding to bridge
 B_i , $i = 1$ to k .

For $i = 1$ to k , let inherited basis $B(N_i)$ of N_i be
 $B(N) \cap (B_i \cup Y)$, and let scanned set $Sc(N_i)$ of N_i be
 $(Sc(N) \cap (B_i \cup Y)) + b$.

Apply the algorithm (step 1,2,3) successively on each N_i
(with inherited basis $B(N_i)$ and scanned set $Sc(N_i)$, current
decomposition D , current trunk matroid M^T) to obtain an
updated decomposition D , updated trunk matroid M^T , and
knot set $E(N_i^T)$, such that the updated decomposition $D \cup \{M^T\}$
of M has used all the good 2-separations of M concealed
in N_i , for all $i = 1, 2, \dots, k$.

Step 3. Apply algorithm (II) to split from current trunk matroid M^T
type J good 2-separations (with respect to cocircuit Y) of
 N , obtain the updated decomposition D , trunk matroid M^T ,
and knot set $E(N^T)$.

(Comment: Current decomposition $D \cup \{M^T\}$ of M has used
all the good 2-separations concealed in N).

Step 4. Output $D \cup \{M^T\}$; this is the standard decomposition of M .

End of main algorithm.

Algorithm (II) (To split type J good 2-separations).

Input information: A matroid N , with inherited basis $B(N)$ and scanned set $Sc(N)$. A cocircuit Y of N with a fixed element $f \in Y$, the bridges $B_1, B_2, \dots, B_k$ of Y in N , segment set $\pi(N, B_1, Y)$, $i = 1, 2, \dots, k$, Y -components $N_1, N_2, \dots, N_k$, current decomposition D , trunk matroid M^r , and corresponding to each Y -component N_i there is a knot set $E(N_i^r)$, $i = 1, 2, \dots, k$.

Step 0. Use $E(N_i^r) \setminus Y$ as bridges of Y in N instead of B_i ,

that is, replace B_i by $E(N_i^r) \setminus Y$, $i = 1, 2, \dots, m$.

Apply algorithm (5.6.5) to obtain the good 2-components

$C_1, C_2, \dots, C_m$ of the bridge graph of Y in N with the

ordering of the indices $1, 2, \dots, m$ of C_i 's compatible

with $(C^*, \leq^f)$, determine $C_i \in C_1$ or $C_i \in C_2^*$ (as defined

in (5.4.2)), and get $L_1(C_i), L_2(C_i)$ for each $i = 1, 2, \dots, m$,

where $L_1(C_i) = \cup \{S_j : f \notin S_j \in \pi(B_j), B_j \in C_i\}$ and $L_2(C_i) =$

$Y \setminus L_1(C_i) = \cap \{S_j : f \in S_j \in \pi(B_j), B_j \in C_i\}$.

(Comment: Note that each bridge B_i of Y has been replaced by $E(N_i^r) \setminus Y$.)

Set $i \leftarrow 1$

Step 1. Whenever an element y in Y is identified as a marker e_x ,

for some marker e_x , and $y \in L_1(C_i)$, identify this y as

e_x in $L_1(C_i)$.

(Remark. For $i = 1$, this step is redundant.)

Step 2. While ($i < m$, there exists a parallel class P in trunk matroid M^r with $P \subseteq L_1(C_1)$ and $|P| \geq 2$) or ($i = m$, there exists a parallel class P in trunk matroid M^r with $P \subseteq L_1(C_1)$, $|P| \geq 2$, and $P \cap (U(Y: Y = Y(b, B(N), b \in S(N))) = \emptyset$)
Do split P from M^r , that is, $M^r \rightarrow (M^r(P; e_p), M^r(E(M^r) \setminus P; e_p))$, where e_p is a new marker; let $D \leftarrow D \cup (M^r(P; e_p))$, trunk matroid $M^r \leftarrow (M^r(E(M^r) \setminus P; e_p))$; in Y and $L_1(C_1)$, identify all the elements of P as e_p .
end While

Let $Q = U(B_j: B_j \in C_1)$.

If $C_1 \in C_1$ go to step 3, otherwise, if $C_1 \in C_2^*$, go to step 4.

Step 3. While there exists a series class Q in the trunk matroid

M^r with $Q \subseteq Q$ and $|Q| \geq 2$,

Do split Q from M^r , that is, $M^r \rightarrow (M^r(Q; e_Q), M^r(E(M^r) \setminus Q; e_Q))$,

where e_Q is a new marker;

let $D \leftarrow D \cup (M^r(Q; e_Q))$, trunk matroid $M^r \leftarrow M^r(E(M^r) \setminus Q; e_Q)$;

in Q identify all the elements of Q as e_Q .

end While.

Comment: By theorem (5.5.4), we know that two elements x, y

of $U(B_j: B_j \in C_1)$ are in series, only if $\{x\}, \{y\}$ are

two singleton bridges of Y ; we may make our computation

simpler by using this fact.)

Step 4. Let $E_1 = L_1(C_1) \cup Q$

If ($i < m$) or ($i = m$, $Sc(N) = \emptyset$ (this implies $N = M$), and

$|L_2(C_1)| \geq 2$), split E_1 from trunk matroid M^T , $M^T \rightarrow (M^T(E_1; e_{E_1}),$

$M^T(E(M^T) \setminus E_1; e_{E_1})$ where e_{E_1} is a new marker;

Let $D \leftarrow D \cup (M^T(E_1; e_{E_1}))$, trunk matroid $M^T \leftarrow (E(M^T) \setminus E_1; e_{E_1})$;

in Y , identify all the elements of $L_1(C_1)$ as e_{E_1} .

If $i < m$, let $i \leftarrow i+1$, and go to step 1.

Else. (Comment: $i = m$)

If $Sc(N) \neq \emptyset$, let knot set $E(N^T) = Q \cup Y$.

End of Algorithm (II).

The remainder of this section is devoted to deriving the computational complexity of the decomposition algorithm.

(5.10.1) For convenience of analysing the computational work, we make the following assumptions. Where M is the input matroid, there exist efficient algorithms:

- (i) for constructing the initial display matrix of M with respect to a basis B ;
- (ii) for finding the minors of a matroid;
- (iii) for computing the parallel class and series classes in M and its minor;
- (iv) for constructing a simple decomposition of a matroid. (That is

$$M \rightarrow (M_1, M_2).$$

Remark that algorithm (2.4.11) is used as a subroutine for finding the elementary separators of a matroid, which is needed in step 2 of the main algorithm. Display matrix of M is required for applying this subroutine; as pointed out in chapter 4, once we have the initial display matrix of M , all the other display matrices of Y -components for recursive use can be obtained quite easily as submatrices. We will discuss assumption (iii) by checking binary matroids and real matrix matroids. As for assumption (iv), notice that the number of simple decompositions needed in our algorithm is the same as the number of good 2-separations of M , and hence is the least we can expect.

Let M be the initial input matroid with a fixed basis B . Suppose that the rank of M is r , and hence $|B| = r$. Let N be a matroid encountered in step 1 of the main algorithm with inherited basis $B(N)$ and scanned set $Sc(N)$; N is called a leaf matroid, if $B(N) = Sc(N)$. We will need the following two results.

(5.10.2) Lemma. For $r \geq 2$, the total number of bridges encountered in Step 2 of the main algorithm is at most $2r - 2$.

(5.10.3) Lemma. For $r \geq 2$, the sum of all the leaf matroid ranks is at most $2r - 2$.

Proof of lemma (5.10.2)

Suppose that we apply the decomposition algorithm to M , suppose that recursively we are in the position of step 1 of the main algorithm, and suppose that N is the matroid encountered with inherited basis $B(N)$ and scanned set $Sc(N)$. We claim that the total number of bridges encountered in step 2, when the main algorithm is applied to N recursively, is at most $2r_N - 2 - |Sc(N)|$, where r_N is the rank of N . If the claim is true, since $|Sc(M)| = 0$, then lemma (5.10.2) is proved. Now we prove the claim by induction.

Denote the desired number by $f(N, Sc(N))$. Then it is easy to see that:

$$f(N, Sc(N)) = \begin{cases} 0 & \text{if } r_N = |Sc(N)|. \\ 1 & \text{if } r_N = 2 \text{ and } |Sc(N)| = 1. \end{cases}$$

Hence the claim holds for these simple cases. Now suppose that

$r_N - |Sc(N)| > 0$, let Y be the cocircuit of N chosen in step 1 of

the main algorithm, and let N_i be the Y -component of N with

scanned set $Sc(N_i)$ and rank r_i , $1 \leq i \leq k$. Note that

$$\sum_{i=1}^k |Sc(N_i)| = |Sc(N)| + k. \text{ Thus either } k > 1 \text{ and } r_i < r_N \text{ for}$$

$i = 1, \dots, k$, or $k = 1$ and $Sc(N_1) = Sc(N) + 1$. We have, by induction,

$$\begin{aligned}
f(N, Sc(N)) &= k + \sum_{i=1}^k f(N_i, Sc(N_i)) \\
&\leq k + \sum_{i=1}^k (2r_i - 2 - |Sc(N_i)|) \\
&= k + 2(r_N - k + 1) - 2k - (|Sc(N)| + k) \\
&= 2r_N - 4k + 2 - |Sc(N)| \\
&\leq 2r_N - 2 - |Sc(N)|. \quad \Delta
\end{aligned}$$

Proof of lemma (5.10.3)

The proof is similar to that of (5.10.2). Define $g(N, Sc(N))$ as expected; we want to prove that $g(N, Sc(N)) \leq 2r_N - 2$. First, we have:

$$g(N, Sc(N)) = \begin{cases} r_N & r_N = |Sc(N)| \\ 2 & r_N = 2 \end{cases}$$

Now suppose $r_N - |Sc(N)| > 0$, we have, by induction

$$\begin{aligned}
g(N, Sc(N)) &= \sum_{i=1}^k g(N_i, Sc(N_i)) \\
&\leq \sum_{i=1}^k (2r_i - 2) \\
&= 2(r_N + k - 1) - 2k \\
&= 2r_N - 2. \quad \Delta
\end{aligned}$$

To derive the computation bound for the decomposition algorithm, we discuss the steps of the algorithm in the following.

10.4) The total number of executions of Step 2 of the main algorithm is exactly $r(M)$ where $r(M)$ is the rank of the input

matroid M . Hence the total number of executions of algorithm (II) is also $r(M)$.

(5.10.5) In step 2 of the main algorithm, the B_i -segments set $\pi(N, B_i, Y)$ is computed, $i = 1, 2, \dots, k$. Let N_i be the Y -component of N with inherited basis $B(N_i)$ and scanned set $Sc(N_i)$, $i = 1, 2, \dots, k$. Note that computing all the parallel classes of N_i gives us $\pi(N, B_i, Y)$. If $B(N_i) \neq Sc(N_i)$, instead of working with the matroid N_i , we propose the following method to compute the B_i -segments $\pi(N, B_i, Y)$. First, if N_i is a leaf matroid, $B(N_i) = Sc(N_i)$, then we have no other choice but to work with the matroid N_i , and compute all the parallel classes of it. Now, by changing the notation, suppose that $B(N) \neq Sc(N)$, that we want to find all the parallel classes of N , and that for $i = 1, 2, \dots, k$ we have a partition of $E(N_i)$ which is the set of all parallel classes of N_i , where N_i is a Y -component of N corresponding to the bridge B_i . Then by (3.2.3), a parallel class P of N_i with $P \subseteq B_i$ is a parallel class of N , and the set of all parallel classes of N contained in Y is just the refinement of the partition $\pi(N, B_i, Y)$ $i = 1, 2, \dots, k$. Therefore, it is clear that knowing the set of parallel classes of N_i , $i = 1, \dots, k$, we can compute that of N in time $O(k|E(N)|)$.

(5.10.6) Step 0 of algorithm (II) is executed $r(M)$ times; by (5.6.6), each execution requires $O(\max(k|Y|, k \log k))$ computations, where k is the number of bridges of Y in N . Thus, by lemma (5.10.2), the overall bound for this step is $O(r(M)|E(M)|)$.

(5.10.7) In step 2 of Algorithm (II), the parallel classes of M^r contained in $L_1(C_1)$ are computed; although we write this statement there for the reason of clarity, they can be computed in step 0 in the following way. Let $C_1, C_2, \dots, C_m$ be the good 2-components of the bridge graph of Y in N . For each $i = 1, \dots, m$, consider the matroid $N(C_i) = N / (\cup B_j : B_j \text{ is a bridge of } Y \text{ and } B_j \notin C_i)$, and let $\mathcal{F}_i$ be the partition of Y into parallel classes of $N(C_i)$. Then, it is not difficult to see that $L_2(C_i)$ is a member of $\mathcal{F}_i$, and $\mathcal{F}_i - L_2(C_i)$ gives us the partition of $L_1(C_i)$ into parallel classes of M^r . Now, for each $B_j \in C_i$, we have already computed $\pi(N(C_i), B_j, Y)$; therefore, we may use the same idea described in (5.10.5) to find all the parallel classes of $N(C_i)$. This gives us the partition $\mathcal{F}_i$ defined above.

(5.10.8) By lemma (5.10.2), (5.10.5), and (5.10.7), the total work for computing the parallel classes in the decomposition algorithm is bounded by $O(r(M)|E(M)|)$ plus the work for computing the parallel classes for the leaf matroids. For the latter part of the work, let us consider some special classes of matroids. If A is

a binary matrix there is an $O(rc)$ algorithm, using the idea of lexicographic ordering, for finding all the parallel columns of A , where r and c are the numbers of rows and columns of A respectively. (See Aho, Hopcroft, Ullman [1], page 79, Theorem 3.1). If A is a real matrix, by applying the same idea, we have an $O(rc \log c)$ algorithm for doing this. Therefore, by Lemma (5.10.3) the total work for computing the parallel classes for the leaf matroids is bounded by $O(r(M)|E(M)|)$ (respectively $O(r(M)|E(M)|\log|E(M)|)$), if M is a binary matroid represented by a binary matrix (respectively, a matric matroid represented by a real matrix).

(5.10.9) In step 3 of Algorithm (II), some special series classes of the trunk matroid M^r are computed. Note that for each execution of algorithm (II), where k is the number of bridges of Y in N , at most k elements of M^r need to be checked for being in series. Also we note that, by lemma (5.10.2), the total number of bridges encountered in the whole algorithm is $2r(M) - 2$. Again, let us consider some useful classes of matroids; let M be the matric matroid represented by the matrix (I, A) , where A is a binary or a real matrix, and I is an identity matrix. Then it can be shown that the matric matroid of (A^t, I_0) is the dual of M , where A^t is the transpose of A , and I_0 is an identity matrix of

appropriate dimension. Therefore, in these cases, it is easy to see that the work for computing the series classes of M is bounded by $O((|E(M)| - r(M))r(M))$ (respectively $O((|E(M)| - r(M))r(M) \log r(M))$), if A is a binary matrix (respectively a real matrix). Thus, for each execution of algorithm (II), where k is the number of bridges of Y in N , the work for computing the series classes is bounded by $O(|E(M)|k)$ (respectively $O(|E(M)|k \log k)$). By lemma (5.10.2), the overall work for computing them is bounded by $O(|E(M)|r(M))$ (respectively $O(|E(M)|r(M) \log r(M))$).

Now, under the assumptions stated in (5.10.1), we summarize the above discussion as follows. Step 2 of the main algorithm is executed $r(M)$ times. Thus the overall work for the elementary separator subroutine (2.4.11) in this step is bounded by $O((r(M))^2 |E(M)|)$. It is not difficult to see that the work required to execute the decomposition algorithm is dominated by step 2 of the main algorithm and step 0, step 2, step 3 of Algorithm (II). Therefore, by (5.10.4)-(5.10.9), the computation bound for the whole algorithm is $O((r(M))^2 |E(M)|)$ plus the work for computing the parallel classes of the leaf matroids and for computing the series classes. (See (5.10.8) and (5.10.9).)

Let us consider the case that the input matroid is a matric matroid represented by a binary matrix (respectively a real matrix) of the

form (I, A) , then by (5.10.8) and (5.10.9), the decomposition algorithm executes in time $O((r(M))^2 |E(M)|)$ (respectively $O((r(M))^2 |E(M)| + r(M) |E(M)| \log |E(M)|)$).

Chapter 6

Chains of 3-connected minors

In this chapter, we study some theoretical properties of 3-connected matroids. The main results obtained are the following four:

(i) Given a non-separable, simple, cosimple, non-null matroid N , not isomorphic to a wheel or a whirl, we have a characterization of those 3-connected matroids having a minor isomorphic to N . We show that our first result together with one other result is equivalent to Seymour's "Splitter Theorem".

(ii) Given a non-trivial 3-connected matroid N , not isomorphic to a wheel or a whirl, we characterize those 3-connected matroids having a minor isomorphic to N .

(iii) We prove a strengthened form of Tutte's theorem on wheels and whirls.

(iv) We have a generalization of Tutte's theorem on wheels and whirls. We show that this result is equivalent to Seymour's "Splitter Theorem."

6.1 Introduction

We define two ~~classes~~ of matroids which are important in the theory of 3-connectivity.

A wheel graph of order n , where n is an integer ≥ 2 , is constructed from an n -gon called its "rim" by adding one new vertex, and n new edges or "spokes" joining the new vertex to the n vertices of the rim. For $n \geq 2$, the wheel matroid W_n is the polygon matroid of the wheel graph of order n .

For $n \geq 2$, the whirl matroid $W_{r,n}$ is a matroid on $E(W_n)$ having as its circuits all circuits of W_n other than the rim circuit, as well as all sets of edges formed by adding a single spoke to the set of edges of the rim. It is not difficult to see that this is indeed a matroid.

An element e of a 3-connected matroid M is essential if neither $M \setminus e$ nor M/e is 3-connected. There is a famous theorem

of Tutte (see [17]) which concerns the wheel and whirl matroids; we state it in the following:

Tutte's Theorem on wheels and whirls ((6.5.4) below): Let M be a non-null 3-connected matroid, not isomorphic to W_n or W_r_n , for $n \geq 3$. Then for some $e \in E(M)$, one of $M \setminus e$, M/e is 3-connected.

In other words, if every element of a non-null 3-connected matroid is essential, then M is isomorphic to a wheel W_n or a whirl W_r_n , with $n \geq 3$. We note that the converse holds obviously.

A non-trivial 3-connected matroid M is a 3-connected matroid with $|E(M)| \geq 4$. Let M be a matroid on E , and let $e \in E$. If e is either deleted or contracted from M , the resulting matroid is denoted by $M \circ e$. It is not difficult to see that the following theorem is an equivalent form of Tutte's theorem.

Theorem. Let M be a non-trivial 3-connected matroid. Then there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong W_n$ or W_r_n or is null, for some $n \geq 3$. Moreover, for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

It is obvious that, if M is a 3-connected matroid with $|E(M)| \leq 3$, the above result still holds; for convenience of comparing this result with ours, we do not include this case in the above statement.

In this chapter, we show some results which are analogies and/or generalizations of Tutte's theorem, and we relate our results to Seymour's theorem on splitters [13]. The purpose for doing so is, among other things, to provide a connection between Tutte's theorem and Seymour's theorem on splitters (see section 6 of this chapter). (Interestingly, this remark appears in [13]: "There is no connection between that theorem and the result we are concerned with here, as far as I can see.") The main results obtained are the following four theorems:

(for two matroids M, N , the notation $M \geq N$ means that M has a minor isomorphic to N).

(6.3.3) Theorem. Let N be a non-separable, simple, cosimple, non-null matroid not isomorphic to a wheel or a whirl, and let M be a 3-connected matroid. Then $M \geq N$, if and only if there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is non-separable, simple, and cosimple.

(6.4.4) Theorem. Let N be a non-trivial 3-connected matroid not isomorphic to a wheel or a whirl, and let M be a 3-connected matroid. Then $M \geq N$, if and only if there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$ such that,

$M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

The next result strengthens Tutte's theorem.

(6.5.6) Theorem. Let M be a non-trivial 3-connected matroid.

Then there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong W_m$ or W_{r_n} , for some $m \geq 3$, $n \geq 2$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

Moreover, if M is not binary (respectively binary), then such a sequence exists with $M \circ e_1 \circ \dots \circ e_k$ isomorphic to a whirl (respectively wheel).

The next result is a generalization of Tutte's theorem.

(6.5.7) Theorem. Let B be a 3-connected matroid, and let

M be a 3-connected matroid having a proper minor isomorphic to N . Suppose that M is not isomorphic to W_n or W_{r_n} , for $n \geq 3$. Then either there exists $e_1 \in E(M)$ such that $M \setminus e_1 \geq N$ and $M \setminus e_1$ is 3-connected, or there exists $e_2 \in E(M)$ such that $M/e_2 \geq N$ and M/e_2 is 3-connected.

We remark that, for the special case $N = \emptyset$, this theorem is just Tutte's theorem.

Now we discuss the relationship of our results to some other similar results. Negami [11] proved (6.4.4) for the graph case,

so our result (6.4.4) is a generalization of his theorem. For results (6.3.3) and (6.4.4), the special case where $|E(M)| - |E(N)| = 2$ is implicitly proved by Seymour [13, 6.2]. It was these results which led to our theorems. The proofs of our results depend very much upon two lemmas (6.3.1) and (6.3.2); these two lemmas are similar to Tutte's theorem on wheels and whirls (in fact (6.3.1) can be viewed as a generalization of Tutte's theorem), and their proofs are based on techniques similar to Tutte's. We show that our result (6.3.3) together with (6.5.3) is equivalent to Seymour's "Splitter Theorem"; and we explain that our generalized Tutte's theorem is also equivalent to it. We use theorems (6.3.3) and (6.4.4) to prove some known theorems on splitters, one of which has the consequence that K_5 is the only 3-connected non-planar graph which does not contain a subgraph contractible to $K_{3,3}$. (K_n is a complete graph on n vertices. $K_{m,n}$ is a complete bipartite graph, having vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $|V_1| = m$, $|V_2| = n$, and edge set $E = \{(u,v) \mid u \in V_1, v \in V_2\}$. A famous theorem of Kuratowski in graph theory states that a graph is non-planar if and only if it contains a subgraph contractible to K_5 or $K_{3,3}$.)

The contents of this chapter are as follows. The section following this one (section 2) contains preliminary results on wheels

and whirls. In section 3,4,5, we prove our results (6.3.3), (6.4.4), (6.5.6) and (6.5.7), respectively. In the last section, we relate our results to Seymour's results on splitters.

6.2 Preliminaries on wheels and whirls

This section includes some preliminary results which simplify the recognition of wheels and whirls. The following result is due to Tutte. (See the proof of Tutte's wheels and whirls Theorem in [17]).

(6.2.1) Proposition. Let M be a non-separable matroid on E and let $z_1, \dots, z_{2n}$ be distinct elements of E , where $n \geq 2$. Let $X_i = \{z_i, z_{i+1}, z_{i+2}\}$, ($1 \leq i \leq 2n$), reading suffices modulo $2n$. Suppose that for i odd, X_i is a circuit, and for i even, X_i is a cocircuit. Then $E = \{z_1, \dots, z_{2n}\}$ and M is isomorphic to W_n or Wr_n .

Proof. Let $Z = \{z_1, z_2, \dots, z_{2n}\}$. Then $r(Z) + r^*(Z) - |Z| \leq n + n - 2n = 0$. If $E \neq Z$, by (2.5.1) and the definition of separator, then Z is a non-empty proper separator of M , contradicting the fact that M is non-separable. Thus $E = Z$.

For $i \leq j$, $j \leq 2n$, $i \neq j$ and both i and j odd, we define $X_{ij} = \{z_i, z_{i+1}, z_{i+3}, \dots, z_{j-3}, z_{j-1}, z_j\}$ (again reading suffices

modulo $2n$). We claim that $X_{1,j}$ is a circuit of M . For $j = i+2$ this is true since $X_{1,i+2} = X_1$. In general suppose that $X_{1,j-2}$ is a circuit. Then there is a circuit $Z \subseteq (X_{1,j-2} \cup X_{j-2}) - \{z_{j-2}\}$. It is easy to see that $Z = X_{1,j}$ for otherwise $|Z \cap X_\ell| = 1$ for some $\ell \in \{i-1, i+1, \dots, j-1\}$ a contradiction.

We need three claims. Let $R = \{z_2, z_4, \dots, z_{2n}\}$.

Claim 1. For any odd i , $1 \leq i \leq 2n$, $R + z_i$ contains a circuit of M . This is obvious, because $X_{1,i+2} \cup X_{i+2,i} - \{z_{i+2}\} = R + z_i$ contains a circuit of M .

Claim 2. Let X be a circuit of M . Suppose that there is an odd i such that $z_i \in X$ and $X \not\supseteq R$. Then $X = X_{1,j}$ or $X_{j,i}$ for some odd j . Since $|X \cap X_{i-1}| \neq 1$, we may assume $z_{i+1} \in X$. Suppose we have known $z_{i+1}, z_{i+3}, \dots, z_{i+k} \in X$, k odd. Then $|X \cap X_{i+k}| \neq 1$ implies $z_{i+k+1} \in X$ or $z_{i+k+2} \in X$. But $X \not\supseteq R$ and so for some odd k we find that $X \supseteq X_{1,i+k+1}$. Thus $X = X_{1,i+k+1}$, since $X_{1,i+k+1}$ is a circuit. And claim 2 follows.

Claim 3. Either R is a circuit of M , or $R + z_i$ is a circuit of M for every odd i (but not both). By claim 1, for odd i , $R + z_i$ contains a circuit X . If $z_i \notin X$, then $X = R$, for otherwise

$|X \cap X_\ell| = 1$, for some even ℓ , and X_ℓ is a cocircuit, a contradiction. If $z_1 \in X$, then $X = R + z_1$, by claim 2. Thus claim 3 follows immediately.

Now let X be a circuit of M . Suppose $X = X_{i,j}$, for every pair of i, j , where i and j are odd. Then, by claim 2, either there is an odd i such that $z_1 \in X$ and $X \supseteq R$, or $R \supseteq X$. For the latter case, by claim 3, $X = R$, and so $M = W_n$. For the former case, again by claim 3, $X = R + z_1$, and $R + z_j$ is a circuit of M for every odd j , thus $M = W_n$. This completes the proof. Δ

(6.2.2) Corollary. The dual of a wheel or a whirl is a wheel or a whirl. Δ

In fact, the dual of a wheel is a wheel and the dual of a whirl is a whirl, but we will not need this fact.

The next proposition shows that the only non-separable, simple, cosimple, non-null minor of a wheel (respectively a whirl) is a wheel (respectively a whirl).

(6.2.3) Proposition. Let M be a wheel matroid (respectively a whirl matroid), and let N be a non-separable, simple, cosimple, non-null minor of M . Then N is isomorphic to a wheel (respectively a whirl). Furthermore $W_n \geq W_{n-1}$ and $W_r_n \geq W_r_{n-1}$ for $n \geq 3$.

Proof. Let $\{z_1, z_2, \dots, z_{2n}\}$ be an ordering of $E(M)$ such that $X_i = \{z_i, z_{i+1}, z_{i+2}\}$ is a circuit (respectively ~~cocircuit~~) for i even (respectively i odd), where $1 \leq i \leq n$, and reading suffices modulo $2n$. We claim that for any even i , $M \setminus z_i$ does not contain a minor isomorphic to N . If not, let $S \neq \emptyset$ be a maximal subset of E_2 where $E_2 = \{z_2, z_4, \dots, z_{2n}\}$, such that $M \setminus S \geq N$. Then $S \neq E_2$, for otherwise every element of $E \setminus S$ is a coloop of $M \setminus S$. Thus we may assume $z_{2i} \in S$ but $z_{2i+2} \notin S$ for some i . Then either $\{z_{2i+1}, z_{2i+2}\}$ is a cocircuit of M/S or a proper subset of $\{z_{2i+1}, z_{2i+2}\}$ is a cocircuit, since $X_{2i+1} = \{z_{2i+1}, z_{2i+2}, z_{2i+3}\}$ is a circuit of $M \setminus S$, the second alternative is not possible. Thus $\{z_{2i+1}, z_{2i+2}\}$ is a cocircuit of $M \setminus S$. And so $M \setminus S / z_{2i+1} \geq N$, because $M \setminus S \geq N$, N is cosimple, and $M \setminus S / z_{2i+1} \cong M \setminus S / z_{2i+2}$. Now $\{z_{2i+2}, z_{2i+3}\}$ is a circuit of $M \setminus S / z_{2i+1}$, and N is simple; therefore $M \setminus S / z_{2i+1} \setminus z_{2i+2} \geq N$. And we have $M \setminus (S + z_{2i+2}) \geq N$. This contradicts the maximality of S . Thus the claim holds. Dually we have that, for any odd i , $M/z_i \not\geq N$.

Suppose $M = W_k$ (respectively Wr_k), $k \geq 2$, we prove by induction on k that N is isomorphic to a wheel (respectively a whirl).

Suppose $k = 2$, then $|E(M)| = 4$. We note that $|E(N)| \geq 4$, since N is non-separable, simple, cosimple, and non-null. Thus $N = M$.

Suppose the result is true for $k = n-1$, $n \geq 3$, and suppose that we

are given $k=n$. We may assume that N is a proper minor of M , and let $x \in E(M) \setminus E(N)$. Without loss of generality, we may assume that $x = z_1$ or z_2 . If $x = z_1$, then $M \setminus z_1 \geq N$, by the claim. $\{z_{2n}, z_2\}$ is a cocircuit of $M \setminus z_1$, thus $M \setminus z_1 / z_2 \geq N$, since $M \setminus z_1 \geq N$, N is simple, and $M \setminus z_1 / z_2 \cong M \setminus z_1 / z_{2n}$. If $x = z_2$, then similarly we have $M / z_2 \setminus z_1 \geq N$. Now $\{z_3, \dots, z_{2n}\}$ is the set of all elements of $M / z_2 \setminus z_1$, for $1 \leq i \leq 2n-2$, we let $a_i = z_{i+2}$. By using the property that the intersection of a circuit and a cocircuit is not a single-element set, it is not difficult to see that $\{a_i, a_{i+1}, a_{i+2}\}$ is a circuit of $M \setminus z_1 / z_2$ if i is odd, and $\{a_i, a_{i+1}, a_{i+2}\}$ is a cocircuit of $M \setminus z_1 / z_2$ if i is even, where $1 \leq i \leq 2n-2$ and reading suffices modulo $2n-2$. By (6.2.1), $M \setminus z_1 / z_2 \cong W_{n-1}$ or Wr_{n-1} . It is not difficult to check that, if $M \cong W_n$ (respectively Wr_n), $n \geq 3$, then $M \setminus z_1 / z_2 \cong W_{n-1}$ (respectively Wr_{n-1}). Now $M \setminus z_1 / z_2 \geq N$. Thus by induction hypothesis, N is isomorphic to a wheel (respectively a whirl). Δ

Recall that we define a 3-connected matroid M to be non-trivial if $|E(M)| \geq 4$.

(6.2.4) Corollary. A non-trivial 3-connected minor of a wheel (respectively a whirl) is a wheel (respectively a whirl). Δ

The next result is implicit in Seymour [13] (result 6.2).

(6.2.5) Proposition. Let M be a non-separable matroid on E , and let $z_1, z_2, \dots$ be an infinite sequence of elements of E . Let $X_i = \{z_i, z_{i+1}, z_{i+2}\}$. Suppose that any 4 consecutive elements in this sequence are distinct, that for i odd, X_i is a circuit, and that for i even, X_i is a cocircuit. Then M is isomorphic to a wheel or a whirl.

Proof. Since E is a finite set, there is repetition in the sequence $z_1, z_2, \dots$. Choose $i < j$ with j minimum so that $z_i = z_j$. Then $j \geq i+4$, since any 4 consecutive elements are distinct.

Suppose that $j-i$ is odd. Then $j-i \geq 5$, one of X_i, X_{j-2} is a circuit and one is a cocircuit, but their intersection is $\{z_i\}$, a contradiction. Thus $j-i$ is even. Suppose that $i \geq 2$. Then one of X_{i-1}, X_{j-2} is a circuit and one is a cocircuit, and we get a contradiction as before.

Hence $i = 1$, and for some $n \geq 2$, $z_1 = z_{2n+1}$ and $z_1, z_2, \dots, z_{2n}$ are all distinct. By (6.2.1) to show $M \cong W_n$ or Wr_n , we need only to show that $\{z_{2n}, z_1, z_2\}$ is a cocircuit. Suppose that $n = 2$. $X_4 = \{z_4, z_1, z_6\}$ is a cocircuit, and $X_1 = \{z_1, z_2, z_3\}$ is a circuit. So $|X_4 \cap X_1| \neq 1$, and so $z_6 = z_2$ or z_3 . We note that z_6 cannot be z_3 , since any 4 consecutive elements in the sequence $z_1, z_2, \dots$ are distinct. Therefore $z_6 = z_2$, and $X_4 = \{z_4, z_1, z_2\}$

is a cocircuit, we are done. Suppose that $n \geq 3$. Then $X_{2n} = \{z_{2n}, z_1, z_{2n+2}\}$ is a cocircuit, $X_1 = \{z_1, z_2, z_3\}$ and $X_3 = \{z_3, z_4, z_5\}$ are circuits, and so $z_{2n+2} = z_2$. Hence $X_{2n} = \{z_{2n}, z_1, z_2\}$ is a cocircuit. Thus, by (6.2.1), M is isomorphic to W_n or Wr_n .

A triangle is a circuit of size 3, and a triad is a cocircuit of size 3.

(6.2.6) Lemma. Let M be a matroid on E . If X is both a triangle and a triad of M , and $|E| \geq 5$, then $(X, E-X)$ is a 2-separation of M .

Proof. By (2.5.1), $r(X) + r(E-X) - r(E) = r(X) + r^*(X) - |X| = 2 + 2 - 3 = 1$. And clearly $|X| \geq 2$ $|E-X| \geq 2$. Δ

(6.2.7) Lemma. Let M be a 3-connected matroid on E with $|E| \geq 5$, and let $z_1, z_2, \dots, z_n$ be elements of E (not necessarily distinct), $n \geq 5$. Let $X_i = \{z_i, z_{i+1}, z_{i+2}\}$, for $1 \leq i \leq n-2$. Suppose that for i odd, X_i is a circuit, and for i even X_i is a cocircuit. Then any 5 consecutive elements in this sequence are distinct.

Proof. Suppose there exist i, j such that $i < j$ and $x_i = x_j$. We choose such i, j with $j-i$ minimum. We note that any 3 consecutive elements are distinct, because they form either a circuit

or a cocircuit and M is 3-connected. Hence $j \geq i+3$. Suppose that $j = i+3$. Then $X_1 = \{z_i, z_{i+1}, z_{i+2}\}$ is both a triangle and a triad. By (6.2.6), M is not 3-connected, a contradiction. Now suppose that $j = i+4$. $X_1 = \{z_i, z_{i+1}, z_{i+2}\}$ and $X_{i+2} = \{z_{i+2}, z_{i+3}, z_i\}$ and both circuits (or both cocircuits). Then by (2.1.1 (C2)), $X_1 \cup X_{i+2} - \{z_i\} = \{z_{i+1}, z_{i+2}, z_{i+3}\}$ contains a circuit (or cocircuit) Y . Since M is 3-connected with $|E| \geq 5$, by (4.1.6), M cannot have a one or two elements circuit (or cocircuit), and so $Y = \{z_{i+1}, z_{i+2}, z_{i+3}\}$. But then $\{z_{i+1}, z_{i+2}, z_{i+3}\}$ is both a triangle and triad of M , and so by (6.2.6), M is not 3-connected, a contradiction. Thus $j \geq i+5$ and any 5 consecutive elements are distinct.

(6.2.8) Proposition. Let M be a non-trivial 3-connected matroid on E , let $z_1, z_2, \dots$ be an infinite sequence of elements of E , and let $X_i = \{z_i, z_{i+1}, z_{i+2}\}$, for positive integer i . Suppose that for i odd, X_i is a circuit, and for i even, X_i is a cocircuit. Then M is isomorphic to a wheel or a whirl.

Proof. By (6.2.7) and (6.2.5), this result holds for $|E| \geq 5$. Now suppose $|E| = 4$. Suppose that M is binary then by (4.3.5), $M \geq PM(K_4)$, a contradiction, since $PM(K_4)$ has 6 cells. Thus M is not binary, and so $M \geq U_{2,4}$ by (4.3.2). Now $U_{2,4}$ has 4 cells and $U_{2,4} \cong Wr_2$, so $M \cong Wr_2$. Δ

We will need the following two results of Tutte, which are the main tools he uses in proving his theorem on wheels and whirls. We omit the proof. For a proof see Tutte [17].

(6.2.9) Lemma. Let M be a 3-connected matroid, and let x be an essential cell of M . Then M is contained in a triangle or a triad of M .

(6.2.10) Lemma. (a) Let $\{x, y, z\}$ be a triangle of the 3-connected matroid M , and assume that neither $M \setminus x$ nor $M \setminus y$ is 3-connected. Then there is a triad including x and just one of $\{y, z\}$.

(b) Let $\{x, y, z\}$ be a triad of the 3-connected matroid M , and assume that neither M/x nor M/y is 3-connected. Then there is a triangle including x and just one of $\{y, z\}$.

To close this preliminary section, we prove the following two propositions:

(6.2.11) Proposition. Let M be a matroid which is not non-separable, simple, or cosimple. Then the following two hold:

- (i) If $M \setminus e$ is non-separable, simple, and cosimple, then e is a loop, coloop, or parallel element of M ;
- (ii) If M/e is non-separable, simple, and cosimple, then e is a loop, coloop, or series element of M .

Proof. By duality, it is enough to prove only (i). Suppose that $M \setminus e$ is non-separable, simple, and cosimple. And suppose that M is separable, let E_1, E_2 be non-null complementary separators of M with $e \in E_1$. By (2.4.6), $E_1 \setminus e$ and E_2 are complementary separators of $M \setminus e$, so $E_1 \setminus e = \emptyset$, and thus $E_1 = \{e\}$ is a separator of M . Therefore e is a loop or coloop of M . Now suppose that M is non-separable, then M has a 2-separation, since it is not simple or cosimple. Let $\{E_1, E_2\}$ be a 2-separation of M with $e \in E_1$. Then $|E_1| = 2$, for otherwise, by (2.5.4), $\{E_1 \setminus e, E_2\}$ is a 2-separation of $M \setminus e$, a contradiction. Now by (4.1.6), where $E_1 = \{e, x\}$, e and x are in parallel or in series in M . Since $M \setminus e$ is 3-connected; e and x are not in series. Thus e is a parallel element of M . Δ

(6.2.12) Proposition. Let M be a matroid which is not 3-connected. Then the following two hold:

- (i) If $M \setminus e$ is 3-connected, then e is a loop, coloop, or parallel element of M ;
- (ii) If M/e is 3-connected, then e is a loop, coloop, or series element of M .

Proof. Again by duality, it is sufficient to prove (i). Suppose $M \setminus e$ is 3-connected. By (6.2.11), we need only to show that M is

not non-separable, simple, or cosimple. Suppose that M is non-separable, then M has a 2-separation. Let $\{E_1, E_2\}$ be a 2-separation of M with $e \in E_1$. Then $|E_1| = 2$, for otherwise, by (2.5.4), $\{E_1 \setminus e, E_2\}$ is a 2-separation of $M \setminus e$, a contradiction. Now by (4.1.6), where $E_1 = \{e, x\}$, e and x are in parallel or in series in M . Thus M is not simple, or cosimple. Δ

6.3 A characterization of 3-connected matroids having a given non-separable minor

In this section, we prove Theorem (6.3.3). The following two lemmas are the keys to all of our results in this chapter. Their proofs use the same technique as that of Tutte's wheels and whirls Theorem.

(6.3.1) Lemma. Let N be a non-separable, simple, cosimple, non-null matroid, let M be a 3-connected matroid not isomorphic to a wheel or a whirl, and let $M \geq N$. If there is an $x \in E(M)$, such that $M \setminus x \geq N$ and $M/x \geq N$, then either there exists $e_1 \in E(M)$ such that $M \setminus e_1 \geq N$ and $M \setminus e_1$ is 3-connected, or there exists $e_2 \in E(M)$ such that $M/e_2 \geq N$ and M/e_2 is 3-connected.

Proof. Suppose not. Let $x \in E(M)$ be such that $M \setminus x \geq N$ and $M/x \geq N$. Then x must be essential, for otherwise one of $M \setminus x$,

M/x is 3-connected, a contradiction. By (6.2.9), x is contained in a triangle or triad of M . Replacing M by M^* , and N by N^* if necessary, we may assume that x is contained in a triangle

$C_1 = \{x_1, y_1, x_2\}$ with $x_2 \neq x$.

We define inductively a sequence $x_1, y_1, x_2, y_2, \dots$ of elements of M with the property:

(*) for $i \geq 1$, $\{x_i, y_i, x_{i+1}\}$ is a triangle, and $\{y_i, x_{i+1}, y_{i+2}\}$ is a triad.

To do so, we have x_1, y_1, x_2 already. If any $z \in C_1$ is not essential, then $M \setminus z$ is 3-connected, because M/z is not, and $z \neq x$. Since N is simple, $M/x \geq N$ and z is contained in a 2-element circuit of M/x ; therefore $(M/x) \setminus z \geq N$. Thus $M \setminus z \geq N$ and $M \setminus z$ is 3-connected, a contradiction. So x_1, y_1, x_2 are all essential and $M \setminus x_2 \geq N$, since $M/x \setminus x_2 \geq N$. By (6.2.10) there is a triad Y_1 including x_2 and just one of x_1, y_1 . Adjusting the notation, we may assume that $y_1 \in Y_1$ and let $Y_1 = \{y_1, x_2, y_2\}$. We see that $M \setminus x_2 / y_2 \geq N$, because $M \setminus x_2 \geq N$. N is cosimple, and y_1, y_2 are in series in $M \setminus x_2$. If y_2 is not essential, then M/y_2 is 3-connected and $M/y_2 \geq N$, a contradiction. Suppose that we have defined $x_1, y_1, \dots, x_n, y_n$, for some $n \geq 2$, with the property (*), that x_i, y_i are essential for all $i = 1$ to n , and

that $M \setminus x_n \geq N$, $M/y_n \geq N$. Since $Y_{n-1} = \{y_{n-1}, x_n, y_n\}$ is a triad, and y_{n-1}, x_n, y_n are all essential, by (6.2.10). There is a triangle C_n including y_n and just one of y_{n-1}, x_n . If $n = 2$, by interchanging y_1 and x_2 , we may assume that $x_2 \in C_n$. If $n \geq 3$ and $x_n \notin C_n$, then $y_{n-1} \in C_n$. By (6.2.8), $y_{n-2}, x_{n-1}, y_{n-1}, x_n$, and y_n are five distinct elements, $y_{n-1} \in C_n \cap Y_{n-2}$, and $|C_n \cap Y_{n-2}| \neq 1$, where $Y_{n-2} = \{y_{n-2}, x_{n-1}, y_{n-1}\}$ is a triad. Therefore either $x_{n-1} \in C_n$ or $y_{n-2} \in C_n$.

Suppose that $x_{n-1} \in C_n$, then $C_n = \{y_n, y_{n-1}, x_{n-1}\}$ and $C_{n-1} = \{x_{n-1}, y_{n-1}, x_n\}$ are two triangles, so $C_n \cup C_{n-1} - \{x_{n-1}\} = \{y_{n-1}, x_n, y_n\}$ contains a circuit of M . Now $|E(M)| \geq |E(N)| + 1 \geq 4 + 1 = 5$ because N is non-separable, simple, and cosimple. Thus $\{y_{n-1}, x_n, y_n\}$ is a circuit, by the 3-connectivity of M . Then $Y_{n-1} = \{y_{n-1}, x_n, y_n\}$ is both a triangle and a triad of M , which implies M is not 3-connected, a contradiction.

Hence $y_{n-2} \in C_n$, and $C_n = \{y_n, y_{n-1}, y_{n-2}\}$. We note that $M/y_{n-1} \geq N$, because y_{n-1} and y_n are in series in $M \setminus x_n$, $M \setminus x_n \geq N$, and N is cosimple. Since $C_{n-1} = \{x_{n-1}, y_{n-1}, x_n\}$ and $C_n = \{y_n, y_{n-1}, y_{n-2}\}$ are two triangles, we see that $\{x_{n-1}, x_n\}$ and $\{y_n, y_{n-2}\}$ are two parallel sets in M/y_{n-1} . Since $M/y_{n-1} \geq N$, and N is simple, $|E(M) \setminus E(N)| \geq 3$. So $|E(M)| \geq |E(N)| + 3 \geq 7$.

Let $S = \{y_{n-2}, x_{n-1}, y_{n-1}, x_n, y_n\}$. It is easy to see that $r(S) \leq 3$ and $r(E \setminus S) \leq r(E) - 2$, where r is the rank function of M , and $|S| \geq 2$, $|E \setminus S| \geq 2$. Thus $r(S) + r(E \setminus S) \leq 3 + r(E) - 2 = r(E) + 1$. This is contrary to the assumption that M is 3-connected.

So $x_n \in C_n$. In either case ($n = 2$ or $n \geq 3$), we have a circuit C_n containing x_n and y_n . Let $C_n = \{x_n, y_n, x_{n+1}\}$. As before $M/y_n \setminus x_{n+1} \geq N$, so $M \setminus x_{n+1} \geq N$. If x_{n+1} is not essential, then $M \setminus x_{n+1}$ is 3-connected, a contradiction. Hence x_{n+1} is essential.

Now by (6.2.10), there is a triad Y_n including x_{n+1} and just one of x_n, y_n , where $n \geq 2$. Suppose that $y_n \notin Y_n$; then $x_n \in Y_n$. Since $x_{n-1}, y_{n-1}, x_n, y_n, x_{n+1}$ are five distinct elements, we may argue similarly as before that M is not 3-connected, which is a contradiction. Therefore $y_n \in Y_n$. We let $Y_n = \{y_n, x_{n+1}, y_{n+1}\}$. Then $M \setminus x_{n+1}/y_{n+1} \geq N$, and so $M/y_{n+1} \geq N$. If y_{n+1} is not essential, then M/y_{n+1} is 3-connected. This is contrary to our assumptions that there does not exist any e such that M/e is 3-connected and contains a minor isomorphic to N . This completes the inductive definition of the sequence $x_1, y_1, \dots$. Now by (6.2.8), M is isomorphic to a wheel or a whirl, again a contradiction. Therefore the proof of the Lemma is complete. Δ

Remark. Lemma (6.3.1) still holds if $N = \emptyset$, because this case is just Tutte's theorem on wheels and whirls which we will prove later. (See (6.5.6)).

(6.3.2) Lemma. Let M, N be two non-separable, simple, co-simple matroids. Suppose that N is a proper minor of M , M is not a wheel or a whirl, and for each element x of $E(M)$ at least one of $M \setminus x$, M/x does not contain a minor isomorphic to N . (This implies that N is non-null). Then either there exists $e_1 \in E(M)$ such that $M \setminus e_1 \geq N$ and $M \setminus e_1$ is non-separable, simple, and cosimple, or there exists $e_2 \in E(M)$ such that $M/e_2 \geq N$ and M/e_2 is non-separable, simple, and cosimple.

Proof. Suppose not. First we claim that if $M \setminus x \geq N$, then $M \setminus x$ is non-separable and simple, and if $M/y \geq N$, then M/y is non-separable and cosimple. To prove this, suppose that $M \setminus x \geq N$ and $M \setminus x$ is separable. Let A, B be non-empty complementary separators of $M \setminus x$. Since N is non-separable, so N is isomorphic to a minor of $(M \setminus x) \setminus A$ or $(M \setminus x) \setminus B$, say the former, then $(M \setminus x)/A = (M \setminus x) \setminus A \geq N$. Since A is a separator of $M \setminus x$. Hence $(M \setminus x)/y \geq N$ and $(M \setminus x) \setminus y \geq N$, for any $y \in A$, and so $M/y \geq N$, $M \setminus y \geq N$. This contradicts the assumption that for each $y \in E(M)$, at least one of $M \setminus y$, M/y does not contain a minor isomorphic to N .

Therefore $M \setminus x$ is non-separable. It is clear that $M \setminus x$ is simple, since M is. Similarly we can prove that if $M/y \geq N$ then M/y is non-separable and cosimple.

Now suppose this lemma is not true. Since $M \geq N$ and $|E(M)| > |E(N)|$; by duality we may assume that there is a y such that $M/y \geq N$. We define inductively a sequence $x_1, y_1, x_2, y_2, \dots$ of elements of M with the following properties:

(i) for $i \geq 1$, $\{x_i, y_i, y_{i-1}\}$ is a triangle and $\{y_i, x_{i+1}, y_{i+1}\}$ is a triad;

(ii) for all $i \geq 1$, $M \setminus x_i \geq N$ and $M/y_i \geq N$.

To do so, since $M/y \geq N$, and by the claim, M/y is non-separable and cosimple. Hence M/y is not simple, and so there is a triangle

$C_1 = \{x_1, y_1, x_2\}$ of M containing y , let $y_1 = y$. We note that $M/y_1 \setminus x_1 \geq N$ for $i = 1, 2$, because $M/y_1 \geq N$; x_1, x_2 are

in series in M/y_1 ; and N is cosimple. So $M \setminus x_i \geq N$ for

$i = 1, 2$. So we have defined x_1, y_1, x_2 already. Inductively,

having defined $x_1, y_1, x_2, \dots, y_{n-1}, x_n$, for some $n \geq 2$, we have

that $M \setminus x_n \geq N$, and by the claim, $M \setminus x_n$ is non-separable and

simple. Hence $M \setminus x_n$ is not cosimple, and so there is a triad

Y_{n-1} containing x_n . Since $|Y_{n-1} \cap C_{n-1}| \neq 1$, where $C_{n-1} =$

$\{x_{n-1}, y_{n-1}, x_n\}$ is a triangle, one of x_{n-1}, y_{n-1} is in Y_{n-1} .

If $y_{n-1} \notin Y_{n-1}$, then $x_{n-1} \in Y_{n-1}$, and $M \setminus x_n / x_{n-1} \geq N$. Therefore $M/x_{n-1} \geq N$ and $M \setminus x_{n-1} \geq N$, a contradiction. Hence $y_{n-1} \in Y_{n-1}$. We let $Y_{n-1} = \{y_{n-1}, x_n, y_n\}$. Clearly $M \setminus x_n / y_n \geq N$, and so $M/y_n \geq N$. Similarly M/y_n is not simple, and so there is a triangle C_n containing y_n . We may argue similarly that $x_n \in C_n$. Let $C_n = \{x_n, y_n, x_{n+1}\}$. And $M/y \setminus x_{n+1} \geq N$, hence $M \setminus x_{n+1} \geq N$. This completes the inductive definition of the sequence $x_1, y_1, x_2, y_2, \dots$

Now if we can show that M is isomorphic to a wheel or a whirl, then we will have a contradiction. By (6.2.5) all we have to show is that any 4 consecutive elements of the sequence $x_1, y_1, x_2, y_2, \dots$ are distinct. First, by the definitions of triangle and triad, any 3 consecutive elements are distinct. Suppose that $x_1, y_1, x_{i+1}, y_{i+1}$ (or $y_1, x_{i+1}, y_{i+1}, x_{i+2}$) are 4 consecutive elements, but $x_1 = y_{i+1}$ or $(y_1 = x_{i+2})$. Then $M \setminus x_1 \geq N$, and $M/y_{i+1} \geq N$ (or $M/y_1 \geq N$, and $M \setminus x_{i+2} \geq N$). By property (ii) of this sequence, this is a contradiction. Hence any 4 consecutive elements of this sequence are distinct, and the proof is complete. Δ

Let N be a non-separable, simple, cosimple, non-null matroid not isomorphic to a wheel or a whirl. The following theorem is a characterization of those 3-connected matroids having a minor isomorphic to N .

(6.3.3) Theorem. Let N be a non-separable, simple, cosimple non-null matroid not isomorphic to a wheel or a whirl, and let M be a 3-connected matroid. Then $M \geq N$ if and only if there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is non-separable, simple, and cosimple.

Proof. The "if" part is trivial.

To prove the "only if" part, suppose that it is not true and take a counter example M with $|E(M)| - |E(N)|$ minimal. We note that $|E(M)| - |E(N)| \geq 1$, and M is not isomorphic to a wheel or a whirl, since a wheel or a whirl can have only non-separable, simple, cosimple (non-null) minors isomorphic to a wheel or a whirl.

If there is an $x \in E(M)$ such that $M \setminus x \geq N$ and $M/x \geq N$, then by (6.3-1), there exists $e_1 \in E(M)$ with $M \circ e_1$ 3-connected and $M \circ e_1 \geq N$. Now by the minimality of $|E(M)| - |E(N)|$, there is a sequence $e_2, e_3, \dots, e_k$ of distinct elements of $E(M) \setminus e_1$, $k \geq 0$ such that $(M \circ e_1) \circ e_2 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is non-separable, simple, cosimple. This contradicts the choice of M .

So we may assume that for each $x \in E(M)$, at least one of $M \setminus x$, M/x does not contain a minor isomorphic to N . Then, by (6.3.2),

there exists $e_1 \in E(M)$ with $M \circ e_1$ non-separable, simple, cosimple, and $M \circ e_1 \geq N$. Inductively, suppose we have a sequence $e_1, \dots, e_\ell$ of distinct elements of $E(M)$ such that $M \circ e_1 \circ \dots \circ e_\ell \geq N$, and for $1 \leq i \leq \ell$, $M \circ e_1 \circ \dots \circ e_i$ is non-separable, simple, and cosimple. Let $M_\ell = M \circ e_1 \circ \dots \circ e_\ell$. If $|E(M_\ell)| - |E(N)| = 0$ then we are done. If $|E(M_\ell)| - |E(N)| \geq 1$, we note that M_ℓ cannot have any element x with $M_\ell \setminus x \geq N$ and $M_\ell/x \geq N$, for otherwise $M \setminus x \geq N$ and $M/x \geq N$ for this same x . So by (6.3.2) again, there exists $e_{\ell+1} \in E(M_\ell) \subseteq E(M)$ with $M \circ e_{\ell+1}$ non-separable, simple, cosimple, and $M \circ e_{\ell+1} \geq N$. Since $|E(M)| - |E(N)|$ is finite, this process of finding $e_1, \dots, e_k$ has to stop with $M \circ e_1 \circ \dots \circ e_k \cong N$, for some $k \geq 0$. Again it is contrary to the assumption that (M, N) is taken as a counterexample. This completes the proof of the theorem. Δ

Remark: In (6.3.3), if M is just non-separable, simple, cosimple (not 3-connected), the result is not true. Consider the following example.

(ii)

Let M, N be the polygon matroids of graphs (i), (ii) respectively. It is easy to see that M and N are non-separable, simple, co-simple, not isomorphic to a wheel or a whirl, and $M \geq N$. Every cell of M is contained in a cocircuit of length 3. Hence $M \setminus e$ is not simple for every $e \in E(M)$. And every cell of M , except 1,2,3, 4,5 as shown in graph (i), is contained in a circuit of length 3. Thus M/e is not cosimple for every e except 1,2,3,4,5. But it is not difficult to see that $M/e \not\geq N$ for $e = 1,2,3,4,5$. Therefore, the conclusion of theorem (6.3.3) does not hold for this example.

Remark: In (6.3.3), if N is a wheel or a whirl, the result does not hold, because every pair $(M, N) = (W_m, W_n)$ or $(M, N) = (Wr_m, Wr_n)$, where $m > n$, is a counterexample.

6.4 A characterization of 3-connected matroids having a given 3-connected minor

In this section, applying theorem (6.3.3), we prove theorem (6.4.4). First we discuss the relationship between our result (6.4.4) and a result of Negami's [11]. We state without proofs two famous theorems, one due to Tutte, the other one due to Whitney.

(6.4.1) Theorem. (Tutte [17]). The polygon matroid of a connected graph G is n -connected if and only if G is n -connected.

We are particularly interested in 3-connectivity in graphs. Two non-isomorphic graphs can have the same polygon matroid. However, Whitney proved the following uniqueness theorem for graphic matroids.

(6.4.2) Theorem. (Whitney [21]) If the matroid M is the polygon matroid of the 3-connected graph G , then M uniquely determines G .

In other words, two 3-connected graphs are isomorphic if and only if their polygon matroids are isomorphic. We notice that an n -connected graph is defined to be connected.

A graph G is non-trivial 3-connected, if G is 3-connected and $|E(G)| \geq 4$. We say that G is contractible to a graph K , if there exists a finite sequence of contraction of edges which transforms G into K . Given a non-trivial 3-connected graph K not isomorphic to a wheel, Negami [11] gave a characterization of 3-connected graphs which contain a subgraph contractible to K . It follows from Whitney's theorem and elementary facts on deletion and contraction that Negami's theorem can be restated as follows:

(6.4.3) Theorem (Negami) Let N be a non-trivial 3-connected graphic matroid not isomorphic to a wheel, and let M be a 3-connected graphic matroid. Then M contains a minor isomorphic to N , if and only if there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements

of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

We now give a matroid generalization of Negami's theorem; the proof of it is based on our result (6.3.3).

(6.4.4) Theorem. Let N be a non-trivial 3-connected matroid not isomorphic to a wheel or a whirl, and let M be a 3-connected matroid. Then $M \geq N$, if and only if there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

Proof. The "if" part is obvious.

For the "only if" part, by (6.3.3), there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is non-separable, simple, and cosimple. Suppose that $M \circ e_1 \circ \dots \circ e_i$ is not 3-connected for some $1 \leq i \leq k$; let j be the largest index, $1 \leq j \leq k$, such that $M_j = M \circ e_1 \circ e_2 \circ \dots \circ e_j$ is not 3-connected. Then $j < k$, since N is 3-connected. Now $M_j \circ e_{j+1}$ is 3-connected, by (6.2.12), e_{j+1} is a loop, coloop, parallel element, or series element of M_j . This contradicts the fact that M_j is non-separable, simple, and cosimple. Thus $M \circ e_1 \circ e_2 \circ \dots \circ e_i$ is 3-connected, for every $1 \leq i \leq k$. Δ

Remark. The condition that $|E(N)| \geq 4$ is necessary. For example, let M be the polygon matroid of $K_{3,3}$, and N be the polygon matroid of K_3 . It is easy to see that M is 3-connected, $M \geq N$, and N is not a wheel or a whirl. Also it is not difficult to check that, for each $e \in E(M)$, $M \setminus e$ is not 3-connected, and $M/e \cong W_4$. Thus it is impossible to have a sequence $e_1, \dots, e_k$ of elements of $E(M)$ with $M \circ e_1 \circ \dots \circ e_k$ 3-connected, for $1 \leq i \leq k$, and $M \circ e_1 \circ \dots \circ e_k \cong N$.

Although (6.3.3) is used to prove (6.4.4), it does not directly contain (6.4.4). In (6.4.4) both the hypothesis and the conclusion are stronger.

In (6.4.4) or (6.3.3), if N is a wheel or a whirl then the result does not hold, because every pair $(M, N) = (W_m, W_n)$ or $(M, N) = (W_r_m, W_r_n)$ is a counterexample, where $m > n$. However, something interesting can still be said in this case. We explain this in the next section.

6.5 Generalization of Tutte's wheels and whirls theorem

In this section, we prove two theorems (6.5.6) and (6.5.7); one strengthens and the other generalizes Tutte's wheels and whirls Theorem. First, we need a lemma:

(6.5.1) Lemma. Let N be a non-trivial 3-connected matroid, and M be a non-separable, simple, cosimple matroid with $M \geq N$. Suppose that, for each $x \in E(M)$, at least one of $M \setminus x$, M/x does not contain a minor isomorphic to N . Then M is 3-connected.

Proof. We prove this by induction on $|E(M)| - |E(N)|$. If $|E(M)| - |E(N)| = 0$, the result is trivial. Now we assume that it is true whenever $|E(M) - E(N)| \leq k \geq 0$ and suppose that $|E(M)| - |E(N)| = k+1$. Suppose that M is not 3-connected; then M is not a wheel nor a whirl. By (6.3.2), there exists $e \in E(M)$ such that $M \circ e \geq N$ and $M \circ e$ is non-separable, simple, cosimple. We note that, for each $x \in E(M \circ e)$, at least one of $(M \circ e) \setminus x$, $(M \circ e)/x$ does not contain a minor isomorphic to N . By the induction hypothesis, $M \circ e$ is 3-connected. Since M is not 3-connected, by (6.2.12), e is a loop, coloop, parallel element, or series element of M , contradicting the fact that M is non-separable, simple, cosimple. Thus M is 3-connected. Δ

(6.5.2) Proposition. Let N be a non-trivial 3-connected matroid, and let M be a 3-connected matroid having a proper minor isomorphic to N . Suppose M is not a wheel or a whirl. Then either there exists $e_1 \in E(M)$ such that $M \setminus e_1 \geq N$ and $M \setminus e_1$ is 3-connected, or there exists $e_2 \in E(M)$ such that $M/e_2 \geq N$ and M/e_2 is 3-connected.

Proof. Suppose that there is an $x \in E(M)$ with $M \setminus x \geq N$ and $M/x \geq N$. Then, by (6.3.1), the result holds. Therefore, we may assume that, for each $x \in E(M)$, at least one of $M \setminus x$, M/x does not have a minor isomorphic to N . Then, by (6.3.2), there exists $e \in E(M)$ such that $M \circ e \geq N$ and $M \circ e$ is non-separable, simple, cosimple. We notice that, for each $x \in E(M \circ e)$, at least one of $(M \circ e) \setminus x$, $(M \circ e)/x$ does not have a minor isomorphic to N . By (6.5.1), $M \circ e$ is 3-connected. This completes the proof. Δ

We shall see that result (6.5.2) still holds, if N is a 3-connected matroid with $|E(N)| \leq 3$.

The next result is a consequence of (6.5.2).

(6.5.3) Proposition. Let N be a non-trivial 3-connected matroid, and M be a 3-connected matroid with $M \geq N$. Suppose that N is isomorphic to a wheel (respectively a whirl). Then there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k$ is isomorphic to a wheel (respectively a whirl), $M \circ e_1 \circ e_2 \circ \dots \circ e_k \geq N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

Proof. By (6.5.2) and the fact that $|E(M)| - |E(N)|$ is finite, it is clear that there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k$ is

isomorphic to a wheel or a whirl, $M \circ e_1 \circ \dots \circ e_k \geq N$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected. We note that N cannot be isomorphic to W_2 , since W_2 is not 3-connected. Since $M \circ e_1 \circ \dots \circ e_k \geq N$, by (6.2.4), if N is a wheel (respectively a whirl) then so is $M \circ e_1 \circ \dots \circ e_k$. Δ

Tutte [17] proved the following theorem on wheels and whirls.

(6.5.4) Theorem. (Tutte) Let M be a non-null 3-connected matroid, not isomorphic to W_n or Wr_n , for $n \geq 3$. Then for some $e \in E(M)$, one of $M \setminus e$, M/e is 3-connected.

It is not difficult to see that the following theorem is an equivalent form of Tutte's theorem.

(6.5.5) Theorem. Let M be a non-trivial 3-connected matroid. Then there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong W_n$ or Wr_n , or is null, for some $n \geq 3$. Moreover, for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected.

Remark 1. One simple observation is that such sequence exists with $M \circ e_1 \circ \dots \circ e_k \cong W_n$ or Wr_m for some $n \geq 3$, $m \geq 2$, because a non-trivial 3-connected matroid with 5 or 4 elements cannot be binary by (4.3.5), and a non-binary matroid with 4 elements is just Wr_2 .

Remark 2. If M is binary, it is easy to see that such a sequence exists with $M \circ e_1 \circ e_2 \circ \dots \circ e_k$ isomorphic to a wheel.

Using our result (6.5.3), we can prove theorem (6.5.5), and hence Tutte's theorem on wheels and whirls. The interesting thing about our proof is that we get a result which strengthens Tutte's result.

(6.5.6) Theorem. Let M be a non-trivial 3-connected matroid. Then there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ \dots \circ e_k \cong W_m$ or Wr_n , for some $m \geq 3$, $n \geq 2$, and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected. Moreover, if M is not binary (respectively binary), then such a sequence exists with $M \circ e_1 \circ \dots \circ e_k$ isomorphic to a whirl (respectively a wheel).

Proof. If M is not binary (respectively binary), then M has a minor isomorphic to $U_{2,4}$ by (4.3.2) (respectively $PM(K_4)$ by (4.3.5), where $PM(K_4)$ is the polygon matroid of the complete graph on 4 vertices). We note that $U_{2,4} \cong Wr_2$ and $PM(K_4) \cong W_3$. By (6.5.3), with $N = U_{2,4}$ (respectively $N = PM(K_4)$), there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ \dots \circ e_k \geq U_{2,4}$ (respectively $M \circ e_1 \circ \dots \circ e_k \geq PM(K_4)$), $M \circ e_1 \circ \dots \circ e_k \cong Wr_n$, for some $n \geq 2$ (respectively

W_m , for some $m \geq 3$), and for $0 \leq i \leq k$, $M \setminus e_1 \circ \dots \circ e_i$ is 3-connected. This completes the proof. Δ

Now we prove the following generalization of Tutte's wheels and whirls theorem.

(6.5.7) Theorem. Let N be a 3-connected matroid, and let M be a 3-connected matroid having a proper minor isomorphic to N . Suppose that M is not isomorphic to W_n or Wr_n , for $n \geq 3$. Then either there exists $e_1 \in E(M)$ such that $M \setminus e_1 \geq N$ and $M \setminus e_1$ is 3-connected, or there exists $e_2 \in E(M)$ such that $M/e_2 \geq N$ and M/e_2 is 3-connected.

Remark. For the special case $N = \emptyset$, this theorem is just Tutte's theorem.

Proof. Suppose that N is a non-trivial 3-connected matroid. Then $|E(M)| > |E(N)| \geq 4$, and thus M is not isomorphic to W_2 or Wr_2 . By (6.5.2), the result holds for this case. Now suppose that

$|E(N)| \leq 3$. If $|E(M)| \leq 4$, then the result is trivial. Therefore, we may assume that $|E(M)| \geq 5$. In particular, M is a non-trivial 3-connected matroid, $M \neq W_2$ and $M \neq Wr_2$. Thus M is not isomorphic to a wheel or a whirl. If M is binary, then $M \geq PM(K_4) \cong W_3$, and if M is non-binary then $M \geq U_{2,4} \cong Wr_2$. Hence M has a proper

minor N' isomorphic to either W_3 or Wr_2 . By (6.5.2), either there is $e_1 \in E(M)$ such that $M \setminus e_1 \geq N'$ and $M \setminus e_1$ is 3-connected, or there is $e_2 \in E(M)$ such that $M/e_2 \geq N'$ and M/e_2 is 3-connected. Moreover, it is not difficult to see that any trivial 3-connected matroid, (that is, with 3 cells or fewer) is isomorphic to a minor of W_5 and Wr_2 . Therefore $N' \geq N$, and the theorem follows. Δ

It is worthwhile noticing that in the generalized Tutte's theorem (6.5.7), the special case, in which N is a non-trivial 3-connected matroid not isomorphic to a wheel or a whirl, is equivalent to our result (6.4.4).

6.6 The Splitter Theorem

Finally, we relate our results to some results of Seymour's on splitters. The following definition is due to Seymour [13]. Let F be a class of matroids, closed under minors and under isomorphism. $N \in F$ is said to be a splitter for F if whenever $M \in F$ has a proper minor isomorphic to N , then M is not 3-connected. As an example of a splitter, we prove the following theorem using our result (6.4.4). (See Seymour [13], also Ore [12] Theorem 10.1.4.)

(6.6.1) Theorem. $PM(K_5)$ is a splitter for the class of graphic matroids without minors isomorphic to $PM(K_{3,3})$.

Before proving this theorem, let us explain a consequence of it.

Recall that it follows from Kuratowski's Theorem that a graph is non-planar if and only if G contains a subgraph contractible to either K_5 or $K_{3,3}$. Hence, by (6.6.1), K_5 is the only 3-connected non-planar graph which does not contain a subgraph contractible to $K_{3,3}$. In other words, given any 3-connected non-planar graph G , if $G \neq K_5$ then G contains a subgraph contractible to $K_{3,3}$.

Proof of (6.6.1). Suppose the result is not true, and let $PM(G)$ be a 3-connected graphic matroid without minors isomorphic to $PM(K_{3,3})$ but with a proper minor isomorphic to $PM(K_5)$. By (6.4.4), since $PM(K_5)$ is 3-connected and not isomorphic to a wheel or a whirl; there exists a sequence $e_1, e_2, \dots, e_k$ of edges of G , $k > 0$, such that $PM(G) \circ e_1 \circ e_2 \dots \circ e_k \cong PM(K_5)$, and for $1 \leq i \leq k$, $PM(G) \circ e_1 \circ \dots \circ e_i$ is 3-connected. We note that $PM(G) \setminus e = PM(G \setminus e)$ and $PM(G)/e = PM(G/e)$, by (2.2.3) and (2.2.6). Thus by (6.4.1) and (6.4.2), $G' \circ e_k \cong K_5$, where $G' = G \circ e_1 \circ \dots \circ e_{k-1}$ and is 3-connected. It is easy to see that $G' \setminus e_k \neq K_5$, because K_5 is a complete graph and G' is a 3-connected graph. So $G'/e_k \cong K_5$. It is also easy to check that there are only two 3-connected graphs (up to isomorphism), with the property that a graph isomorphic to K_5 is obtained by contracting a single edge; these are shown in the figure.

We notice that both graphs contain a subgraph isomorphic to $K_{3,3}$.

This is a contradiction, and the proof of the theorem is complete.

Seymour introduced the notion of splitter and proved a general result, called here the "Splitter Theorem", which can be used to recognize splitters. By using this result (together with others), Seymour gave a good characterization of regular matroids, and thus of totally unimodular matrices. (A regular matroid is a binary matroid without F_7 and F_7^* minors, where F_7 and F_7^* are the matric matroids of the following two binary matrices respectively:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A real matrix A is totally unimodular, if every square submatrix of A has determinant $-1, 0$, or 1 .)

We state Seymour's splitter theorem in the following:

(6.6.2) Splitter Theorem. (P.D. Seymour). Let $N \in F$ be non-null, non-separable, simple, cosimple. Suppose that the following statements are true:

- (i) for every $M \in F$, if $M \setminus x \cong N$ then x is a loop, coloop, or parallel element of M ;
- (ii) for every $M \in F$, if $M/x \cong N$ then x is a loop, coloop, or series element of M ;
- (iii) if $N \cong W_n$ or Wr_n for some $n \geq 2$, then $W_{n+1} \notin F$ or $Wr_{n+1} \notin F$ respectively.

Then N is a splitter for F .

We now show that our result (6.3.3) together with (6.5.3), is equivalent to Seymour's splitter theorem, that is, we can prove one from the other, and vice versa. In an early version of Seymour's paper [13], the condition (iii) of the Splitter Theorem was stated as follows: (iii) N is not isomorphic to a wheel or a whirl; it is worthwhile noticing that Seymour's splitter theorem in this form is equivalent to our result (6.3.3) alone, as can be seen in the following. (Seymour's own strengthening of the Splitter Theorem was done independently of this work.)

First we prove splitter theorem from (6.3.3), and (6.5.3):

Suppose that N is not isomorphic to a wheel or a whirl. Suppose

that $M \in F$ has a proper minor isomorphic to N , and suppose that M is 3-connected. By (6.3.3), there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k > 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, and $M \circ e_1 \circ \dots \circ e_{k-1}$ is non-separable, simple, cosimple. Let $M' = M \circ e_1 \circ \dots \circ e_{k-1}$; thus $M' \in F$ and $M' \circ e_k \cong N$. If the last element e_k is deleted from M' , that is, $M' \setminus e_k \cong N$, then, by (i), e_k is a loop, coloop, or parallel element of M' , contradicting the fact that M' is non-separable, simple, cosimple. If the last element e_k is contracted from M' , that is $M'/e_k \cong N$, then by (ii), e_k is a loop, coloop, or series element of M' , giving the same contradiction. Hence M is not 3-connected, and so N is a splitter for F . Now suppose that $N \cong W_n$ (respectively Wr_n) for some $n \geq 2$. We note that N cannot be isomorphic to W_2 , since W_2 is not simple or cosimple; thus N is a non-trivial 3-connected matroid. Suppose that $M \in F$ has a proper minor isomorphic to N , and M is 3-connected. By (6.5.3), there is a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$ such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$, $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong W_m$ (respectively Wr_m), and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected. Clearly $m \geq n$. Suppose $m > n$; then by (6.2.3) the fact that F is a class of matroids closed under minors and under isomorphism, $W_{n+1} \in F$ (respectively $Wr_{n+1} \in F$),

a contradiction. So $m = n$; that is $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong N$. But then, as before, we can prove that e_k is a loop, coloop, parallel element, or series element of $M \circ e_1 \circ e_2 \circ \dots \circ e_{k-1}$, again a contradiction. Thus N is a splitter for F .

Now we prove (6.3.3) from the splitter theorem. (We learned that this could be done from Professor R.E. Bixby). Let N be a non-separable, simple, cosimple, non-null matroid not isomorphic to a wheel or a whirl, and let M be a 3-connected matroid with $M \geq N$. Suppose that (6.3.3) does not hold, then, since $M \geq N$ and N is non-separable, simple, cosimple, there exists a positive integer j , $1 \leq j \leq k$, and a sequence of matroids $M_0, M_1, \dots, M_{j-1}, M_j, \dots, M_k$, where $k = |E(M)| - |E(N)|$, satisfying:

- (i) $M_0 = M$, $M_k \cong N$, and for $1 \leq i \leq k$, $M_i = M_{i-1} \setminus e_i$ or $M_i = M_{i-1} / e_i$;
- (ii) j is minimum with the property, that for $j \leq i \leq k$, M_i is non-separable, simple, cosimple, but M_{j-1} is not.

Let $F = \{M' \mid M \geq M'\}$; we claim that M_j is a splitter for F .

Suppose that the claim is true, then we have a contradiction, because M is 3-connected and has a proper minor isomorphic to M_j ; thus result (6.3.3) holds. Now we have only to prove the claim. We know that M_j is non-separable, simple, cosimple. By (6.2.3) and $M_j \geq N$, M_j is not isomorphic to a wheel or a whirl, since N is not. Now

by the minimum property of j , for any $M' \in F$ with $M' \setminus x \cong M_j$ (respectively $M'/x \cong M_j$), M' is not non-separable, simple, or cosimple; thus by (6.2.11), x is a loop, coloop, or parallel element (respectively series element) of M' . Thus by Seymour's splitter theorem (6.6.2), M_j is a splitter for F . This completes the proof.

We remark that, by using the similar method as above, we can prove our result (6.4.4) from Seymour's splitter theorem.

Now we prove (6.5.3) from the splitter theorem. Let N be a non-trivial 3-connected matroid and $N \cong W_n$ (respectively Wr_n) for some $n \geq 2$, and let M be a 3-connected matroid with $M \geq N$. Now let m be the largest positive integer such that $M \geq W_m$ (respectively Wr_m); clearly $m \geq n$. We claim that there exists a sequence $e_1, e_2, \dots, e_k$ of distinct elements of $E(M)$, $k \geq 0$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong W_m$ (respectively Wr_m), and for $1 \leq i \leq k$, $M \circ e_1 \circ \dots \circ e_i$ is 3-connected. Suppose the claim is true; then (6.5.3) holds, since $W_m \geq W_n$ (respectively $Wr_m \geq Wr_n$), and we are done. We now prove the claim. Suppose that the claim is not true; then there exists a positive integer j , $1 \leq j \leq k$, and a sequence of matroids $M_0, M_1, \dots, M_{j-1}, M_j, \dots, M_k$, where $k = |E(M)| - |E(W_m)|$ (respectively $k = |E(M)| - |E(Wr_m)|$), satisfying:

(i) $M_0 = M$, $M_k \cong W_m$ (respectively Wr_m), and for $1 \leq i \leq k$,

$$M_i = M_{i-1} \setminus e \text{ or } M_i = M_{i-1} / e_i;$$

(ii) j is minimum with the property that, for $j \leq i \leq k$, M_i is

3-connected, but M_{j-1} is not.

As before, by applying Seymour's splitter theorem, we can prove that

M_j is a splitter for the class $F = \{M' \mid M \geq M'\}$. (Note that M_j

could be W_m (respectively Wr_m), but $W_{m+1} \notin F$ (respectively,

$Wr_{m+1} \notin F$.) This gives the same contradiction as before, and so

the claim holds.

In fact, by using the similar techniques, we can prove that

Seymour's splitter theorem is equivalent to our result (6.5.2), and

hence to the generalized Tutte's theorem (6.5.7).

Seymour [13] showed the following three theorems:

(i) R_{10} is a splitter for the class of regular matroids, where R_{10}

is the matric matroid of the following binary matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(ii) $PM(K_5)$ is a splitter for the class of regular matroids with-

out minors isomorphic to $PM(K_{3,3})$.

(iii) F_7 is a splitter for the class of binary matroids without F_7^* minors.

Let us explain how we can prove these theorems based on our results (6.3.3) or (6.4.4). First we notice that R_{10} , $PM(K_5)$, and F_7 are all 3-connected, not isomorphic to a wheel or a whirl. Suppose R_{10} (respectively $PM(K_5)$, F_7) is not a splitter for the class F of matroids specified above. Then there is a 3-connected matroid $M \in F$ containing a proper minor isomorphic to R_{10} (respectively $PM(K_5)$, F_7). By (6.3.3) or (6.4.4) there exists a sequence $e_1, e_2, \dots, e_k$ of elements of $E(M)$, $k \geq 1$, such that $M \circ e_1 \circ e_2 \circ \dots \circ e_k \cong R_{10}$ (respectively $PM(K_5)$, F_7), and e_k is not a loop, coloop, parallel element, or series element of $M \circ e_1 \circ e_2 \circ \dots \circ e_{k-1}$. Let $M' = M \circ e_1 \circ e_2 \circ \dots \circ e_{k-1}$; we see that $M' \circ e_k \cong R_{10}$ (respectively $PM(K_5)$, F_7). Now we reduce the problem to case-checking, just as Seymour did in his proof of these three theorems; we take a binary matrix representation of R_{10} ($PM(K_5)$, $PM(K_5)^*$, F_7 , F_7^*), and examine the result of adding any other column vector to this representation. It will be found that either the vector added is the zero vector, or is one already in the representation, or the enlarged binary matroid produced will not be in class F . This contradicts the assumption that $M' \circ e_k \cong R_{10}$ (respectively $PM(K_5)$, F_7), $M' \in F$, and e_k is not a loop, coloop, parallel element, or series element of M' . (We remark that R_{10} is isomorphic to its dual.)

References

- [1] A.V. Aho, J.E. Hopcroft and J.D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass. (1975).
- [2] R.E. Bixby, Composition and Decomposition of Matroids and Related Topics, Ph.D. Thesis, Cornell University, 1972.
- [3] R.E. Bixby and W.H. Cunningham, "Matroids, graphs and 3-connectivity", in Graph Theory and Related Topics, J.A. Bondy and U.S.R. Murty, eds., Academic Press, New York, (1979), 91-103.
- [4] R.E. Bixby and W.H. Cunningham, "Converting linear programs to network problems", Math. of Operations Research 5 (1980), 321-357.
- [5] W.H. Cunningham, A Combinatorial Decomposition Theory, Ph.D. Thesis, University of Waterloo, 1973.
- [6] W.H. Cunningham and J. Edmonds, "A combinatorial decomposition theory", Canadian J. Math. 32 (1980), 734-765.
- [7] W.H. Cunningham and J. Edmonds, "Decomposition of linear systems", in preparation.
- [8] J. Hopcroft and R. Tarjan, "Dividing a graph into triconnected components", SIAM J. Computing 2 (1973), 135-158.

- [9] J. Hopcroft and R. Tarjan, "Efficient planarity testing", J. Association for Computing Machinery 21 (1974), 549-568.
- [10] G.J. Minty, "On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network programming", J. Math. Mech. 15 (1966), 485-520.
- [11] S. Negami, "A characterization of 3-connected graphs containing a given graph", preprint, 1980.
- [12] O. Ore, The Four-Color Problem, Academic Press, New York and London, 1967.
- [13] P.D. Seymour, "Decomposition of regular matroids", J. Combinatorial Theory B 28 (1980), 305-359.
- [14] W.T. Tutte, "An algorithm for determining whether a given binary matroid is graph", Proc. Amer. Math. Soc. 11 (1960), 905-917.
- [15] W.T. Tutte, "Lectures on matroids", J. Res. Nat. Bur. Standards B 69 (1965), 1-47.
- [16] W.T. Tutte, Connectivity in Graphs, University of Toronto Press, 1966.
- [17] W.T. Tutte, "Connectivity in matroids", Canad. J. Math. 18 (1966), 1301-1324.

- [18] W.T. Tutte, "On even matroids", J. Res. Nat. Bur. Stand. B 71 (1967), 213-214.
- [19] W.T. Tutte, Introduction to the Theory of Matroids, American Elsevier, New York, 1970.
- [20] D.J.A. Welsh, Matroid Theory, Academic Press, London, 1976.
- [21] H. Whitney, "Congruent graphs and the connectivity of graphs", Amer. J. Math. 54 (1932), 150-168.
- [22] H. Whitney, "On the abstract properties of linear dependence", Amer. J. Math. 57 (1935), 509-533.

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