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RADIATIVE CORRECTIONS TO DEEP INELASTIC ELECTRON-PROTON
SCATTERING

by

© K. C. WONG

A thesis submitted to the Faculty of Graduate Studies in
partial fulfillment of the requirement for the degree of
Master of Science.

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June 23, 1982

The undersigned recommend to the Faculty of Graduate Studies acceptance of the thesis "RADIATIVE CORRECTIONS TO DEEP INELASTIC ELECTRON-PROTON SCATTERING", submitted by K. C. Wong, B.Sc., in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

Radiative corrections of deep inelastic electron-proton scattering up to order α^3 have been calculated in a manifestly covariant manner. Numerical evaluation for radiative corrections is carried at $s=40,000 \text{ GeV}^2$. The expression for the radiance $R(\theta)$, which is essential to luminosity monitoring, is derived.

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CHAPTER 1
INTRODUCTION

(1.1) EP COLLIDER, CHEER AND RADIATIVE CORRECTIONS

One of the many alternatives among accelerators is the electron-proton (ep) collider. CHEER (Canadian High Energy Electron Ring) is a proposal for an ep collider and proposes to collide 10 GeV electrons with the 1000 GeV proton beam of the Fermilab's proton synchrotron, the Tevatron. Let us introduce a useful parameter to describe a reaction, namely, the centre-of-mass energy $\sqrt{s}$. If p_1 and p_2 are the four-momentum vector of the incoming particles, then s is defined as:

$$s = (p_1 + p_2)^2$$

The available centre-of-mass energy squared s of CHEER can attain values up to $40,000 \text{ GeV}^2$ if everything works perfectly. The luminosity is assumed to be $\sim 10^{31} \text{ cm}^{-2} \text{ s}^{-1}$.

There are several advantages of an ep collider over the fixed target machine, the e^+e^- collider and the $p\bar{p}$ collider. Firstly, s is quite large (although not as large as for the $p\bar{p}$ collider in CERN). The four momentum transfer Q^2 is bounded by s and with such a large s , we can test many theories in the large Q^2 region, e.g., validity of QED at high Q^2 , the Q^2 dependence of the fine structure constant $\alpha(Q^2)$, asymptotic freedom of the strong interaction predicted by the standard QCD model. The effects due to the weak intermediate bosons $W^\pm$, Z^0 predicted by the Weinberg-Salam's electro-weak model may also be seen at this large Q^2 value.

There is at least one major problem we will encounter in the analysing of the experimental cross-section. It is well known that both incoming and scattered electrons will bremsstrahlung, and this will alter the cross-section significantly. In the paper by Tsai^[2], he had calculated the radiative corrections of the scattering of 5 GeV electrons on stationary protons. It turns out that the bremsstrahlung effect can alter the cross-section by as much as 20%.

The s of the above ep collision is about 10 GeV². Thus, the radiative corrections for CHEER with an $s=40,000$ GeV² is expected to be larger. The main purpose of this thesis is to study the radiative corrections to electron-proton scattering under the conditions of the CHEER ep collider. In the next section, we will consider radiative corrections

(a)

(b)

(c)

Fig. 1. The diagram of the reaction $e+p \rightarrow e+X$

- (a) one-photon exchange;
- (b) charged weak boson $W^\pm$ exchange;
- (c) neutral boson Z^0 exchange.

INTRODUCTION

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and its necessity in greater detail with particular
reference to CHEER.

(1.2) NECESSITY OF RADIATIVE CORRECTIONS

Before any discussion we have to define the terminologies used in the literature on radiative corrections. By radiative correction we mean the correction to a basic process due to the emission and reabsorption of virtual photons and to the emission of any number of real photons. By inelastic scattering we refer to processes in which the proton is excited to a discrete level (or resonance)

$$e^- + p \rightarrow e^- + N^*, \quad (1)$$

or to a continuum of hadrons

$$e^- + p \rightarrow e^- + \text{hadrons} . \quad (2)$$

Inclusive processes involve final hadronic states X such that X can be anything including hadrons, photons and additional leptons,

$$e^- + p \rightarrow e^- + X . \quad (3)$$

In some earlier experiments on ep scattering only scattered electrons were detected. Thus the processes

$$e^- + p \rightarrow e^- + \gamma + p . \quad (4)$$

$$e^- + p \rightarrow e^- + \gamma + N^* . \quad (5)$$

$$e^- + p \rightarrow e^- + \gamma + \text{hadrons} . \quad (6)$$

give rise to corrections due to the radiative tails of the

elastic peak, resonances and continuous spectrum (or continuum) respectively, are indistinguishable from the 'useful' process, namely,

$$e + p \rightarrow e + \text{hadrons}.$$

Let $\vec{k}_1$, $\vec{k}_2$, $\vec{p}_1$, $\vec{p}_2$ be defined in Fig.2(a) with $\vec{k}_1$ and $\vec{p}_2$ on the x-z plane. The cross-section usually depends on the four-momentum transfer q^2 defined by

$$\begin{aligned} q^2 &= (\vec{k}_1 - \vec{k}_2)^2 \\ &= (\vec{p}_2 - \vec{p}_1)^2. \end{aligned}$$

Thus a determination of either $\vec{k}_2$ or $\vec{k}_1$ will give us the information necessary to define the cross-section. If the reaction has a radiated photon of four-momentum k emitted (assuming it is from the electron legs) then the four-momentum transfer is now given by

$$\begin{aligned} q^2 &= (\vec{k}_1 - \vec{k}_2 - \vec{k})^2 \\ &= (\vec{p}_2 - \vec{p}_1)^2. \end{aligned}$$

The new momentum configuration is shown in Fig.2(b). Now, $\vec{k}_1$ and $\vec{p}_2$ are no longer lie on the x-z plane. The energy and momentum distribution of the scattered electrons will depend on the energy and momentum distribution of the radiated photons.

Fig.2. Momentum configuration of particles of the non-radiative process (a), and the radiative process (b).

The complications are not only from the radiated photons but depend on the experimental set up as well. Consider the following apparatus typical to the study of structure function of proton . It consists of an incident beam of muons and a target flask of liquid hydrogen. The scattered muon is detected by several sets of hodoscope of scintillators. The charged tracks can be detected by the multi-wire proportional chambers (see Fig.3).

There are several possible photon emissions from the scattered muons:

(i) one hard photon radiated along the same direction of the scattered muon (same direction in the sense that the spatial resolution of the detector is larger than the angle between them);

(ii) one hard photon radiated at a large angle from the scattered muon;

(iii) many soft photons radiated along the direction of the scattered muon.

For the cases (i) and (iii), if we do not put the magnet M1 along the path of the scattered muon, the sum of the energy of the radiated photon and the muon will be recorded by the scintillators. For case (ii), only the energy of the scattered muon is recorded. If we apply radiative corrections to the measured data, it may be overly corrected. The reason is as follows. Although there are photons radiated from the scattered muons, for those photons in cases (i) and (iii), they are not 'recognized'. Since

Fig.3. Typical experimental set-up to detect scattered leptons.

the recorded energy and direction of the scattered muon is the same with or without these radiated photons. Thus no correction is required to those events of cases (i) and (iii). If we do not apply radiative correction at all, then events with radiated photons from the incoming muons and the radiated photons of case (ii) will alter the kinematics significantly.

The remedy to this problem is to apply a sufficiently strong magnetic field such that the radiated photon and the scattered muon are separated sufficiently so that the energy of the muon is recorded unambiguously. With this experimental set up, we can apply radiative corrections to the measured cross-section.

[1]
The detectors of the CHEER project are shown in Fig.4. consists of an outer hadron calorimeter and an inner calorimeter. Surrounding the interaction region is a Time Projection Chamber (TPC) which is responsible to measure the momenta of the charged particles. Superconducting solenoidal coil with a magnet field of 2 tesla is installed between the TPC and the electromagnetic calorimeter (EC). The EC is designed to measure the energy of photons and electrons with a spatial resolution better than 3 cm and an energy resolution of $8\%/\sqrt{E}$ (GeV). With these detectors, energy of the scattered electrons can be measured unambiguously. Hence, the radiative corrections calculated in this thesis are for application to the CHEER configuration.

Fig.4. CHEER detector.

Let us consider the following hypothetical case to illustrate the necessity of radiative corrections. Suppose the differential cross-section $d\sigma_0$ of the basic process $e+p \rightarrow e+\text{hadrons}$ has a form

$$\frac{d^2\sigma_0}{dx dy} = A(x) e^{-y}$$

$$x = \frac{-(k_1 - k_2)^2}{2 P_1 \cdot (k_1 - k_2)}, \quad y = \frac{P_1 \cdot (k_1 - k_2)}{P_1 \cdot k_2},$$

where x and y are the so called Bjorken scaling variables. Suppose an electron-proton scattering experiment has been performed with the following conditions:

- (i) only the outgoing electrons are measured;
- (ii) all events with a continuum final hadronic state will be measured.

We expect the experimental cross-section $d^2\sigma_{\text{expt}}/dx dy$ and the predicted cross-section $d^2\sigma_0/dx dy$ will be the same. But the experimental condition (ii) above implies that events with radiated photons which simulate the basic process will also be recorded.

Take $x=0.5$ as an example, it turns out that the predicted curve represented by the solid line of Fig.5(a) and the experimental curve represented by the dashed line do not agree with one another at y near 0 or 1 (assuming error bars are relatively small).

There are at least three possibilities to explain the discrepancy:

- (i) the theoretical prediction is incorrect;

- (ii) the experimental error is larger than expected;
- (iii) radiative corrections have not been taken into consideration.

Let us explore the third possibility. To be precise, we will divide the corrections into two parts:

(a) emission of photons from the hadron and electron currents (e.g., diagrams M_7 , M_8 and M_9 of Fig. 6). From now on, they are referred to as radiative corrections throughout the thesis.

(b) Vertex corrections to the hadron and electron currents (e.g., diagrams M_3 and M_5 of Fig. 6).

If we take radiative cross-section $d^2\sigma_R/dxdy$ and vertex cross-section $d^2\sigma_V/dxdy$ into account, then the measured $d^2\sigma_{\text{expt}}/dxdy$ is now given by

$$\begin{aligned}
 & \frac{d^2\sigma_{\text{expt}}}{dxdy} (e + p \rightarrow e + \text{hadrons}) \\
 & = \frac{d^2\sigma_0}{dxdy} (e + p \rightarrow e + \text{hadrons}) + \frac{d^2\sigma_V}{dxdy} (e + p \rightarrow e + \text{hadrons}) \\
 & + \frac{d^2\sigma_R}{dxdy} (e + p \rightarrow e + \gamma + \text{hadrons})
 \end{aligned} \tag{7}$$

In order to extract the basic cross-section from the experimental values, we have to calculate $d^2\sigma_R/dxdy$ and $d^2\sigma_V/dxdy$ theoretically. Define $\delta(x,y)$ to be

$$\delta(x,y) = \left(\frac{d^2\sigma_R}{dxdy} + \frac{d^2\sigma_V}{dxdy} \right) / \frac{d^2\sigma_0}{dxdy} \tag{8}$$

then

$$\frac{d^2\sigma_{\text{expt}}}{dx dy} = \frac{d^2\sigma_0}{dx dy} [1 + \delta(x, y)],$$

or

$$\frac{d^2\sigma_0}{dx dy} = C(x, y) \frac{d^2\sigma_{\text{expt}}}{dx dy},$$

where

$$C(x, y) = 1/[1 + \delta(x, y)].$$

Curves of $\delta(x, y)$ and $C(x, y)$ are shown in Fig. 5(b) and 5(c) respectively. It turns out that after the experimental curve (dashed curve) is properly corrected by $C(x, y)$, the resulting curve agree with the predicted one (solid curve). If there is still a discrepancy, other explanations have to be sought.

This thesis will concentrate on the derivations of the radiative and vertex corrections (i.e. $\delta(x, y)$) to the electron-proton scattering with the following assumptions:

- (i) the basic cross-section $d^2\sigma/dx dy$ is based on the first Born term (see Fig. 2(a));
- (ii) all weak effects like Fig. 2(b) will be ignored;
- (iii) diagrams contributing to the non-radiative cross-section up to order of α^3 will be included which implies only diagrams with one-photon emission will be

Fig.5. (a) Experimental cross-section $d^2\sigma_{\text{expt}}/dxdy$ (solid line) vs. theoretically predicted cross-section $d^2\sigma_0/dxdy$ (dash line) at $x=0.5$;
 (b) a plot of $\delta(x,y)$ at $x=0.5$;
 (c) a plot of $C(x,y)$ at $x=0.5$.

Fig. 5(b)

Fig. 5(c)

considered.

All the results are expressed in a manifestly covariant manner so that it is applicable to the CHEER configuration. Numerical results for radiative corrections for CHEER, which has not been previously done, are given in section (2.4).

In the numerical evaluation of the function $\delta(x,y)$, integrals of extremely peaked functions due to the smallness of the electron mass and the large value of Q^2 available in an ep collider, are involved. Methods to tackle this problem will be presented in section (2.3).

Making use of the radiated photons, we can apply it to the monitoring of the luminosity of the accelerator. A theoretical study of the feasibility of using this as a luminosity monitor is given in chapter 3. Relevant formulas are also given.

CHAPTER 2

RADIATIVE CORRECTION TO EP SCATTERING(2.1) DERIVATIONS OF THE BASIC CROSS-SECTION

There are quite a few papers on radiative corrections to the lepton-nucleon scattering, e.g., Tsai^[2], Mo and Tsai^[3], J. Kahane^[4], Akhundov et al.^{[5] [6] [7]}, Bardin and Shumeiko^[8], Shumeiko^[9] etc.. The earliest and the most extensive papers are by Mo and Tsai. There are several undesirable features in the early papers and these are indicated below:

(i) the useful formulas are not expressed in a manifestly covariant form and it only applies to the laboratory frame (stationary target) which is not applicable to the ep colliding frame,

(ii) an unphysical softness parameter Δ was introduced which is not a covariant quantity,

(iii) the integration was done in the so called peaking approximation which is not accurate for certain extreme kinematic regions.

The basic references on which this thesis is based are a series of papers by Akhundov et al. and will be referred to frequently. All their results have been derived in a manifestly covariant manner and no softness parameter Δ was introduced and no peaking approximation was used.

The Feynman diagrams contributing to the non-radiative part (up to order α^3) are shown in Fig.6. Notice that the final states for diagrams M_1 to M_6 are different from those of diagrams M_7 and M_8 , and there will be no interference between them. Therefore, the cross-section is simply the sum of the radiative and non-radiative parts.

The general formula for the differential cross-section can be written as

$$d\sigma = \frac{1}{4F} \left(\sum |\dot{M}|^2 \right) d\Gamma \quad (9)$$

$$M = M_1 + \dots + M_n.$$

The sum is over spin, polarization and all the final hadronic states. F is the Møller's invariant flux factor which is given by

$$F^2 = (p_1 \cdot k_1)^2 - m^2 M^2 \quad (10)$$

and $d\Gamma$ is the Lorentz invariant phase space. For a system of n particles in the final state:

$$d\Gamma = (2\pi)^4 \delta^{(4)}(p_1 + k_1 - \sum_{i=1}^n p_i) \frac{1}{(2\pi)^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{2E_{p_i}} \quad (11)$$

Fig.6. Feynman diagrams of the reaction $e+p \rightarrow e+X$ up to order α^3 .

Fig.6 (cont'd) Feynman diagrams of the reaction $e+p \rightarrow e+X$ up to order α^3 .

Fig.7 Definition of the angular variables.

The choice of Eq.(11) implies a particular normalization of any spinors that may occur in M_i , namely,

$$\bar{u}(p,s) u(p,s) = 2m \quad (12)$$

and hence

$$\sum_{s\bar{s}} u(p,s) \bar{u}(p,s) = \not{p} + m \quad (13)$$

The derivations of references [2] to [9], are based on two assumptions:

(i) smallness of the radiative corrections to the hadron current;

(ii) parton model was assumed so that the validity of the Callan-Gross relation of the structure functions $F_1(x)$ and $F_2(x)$

$$2x F_1(x) = F_2(x)$$

can be justified (we will comment more on this later).

The justification of the first assumption, suggested by Consoli and Greco^[10], is due to the following reason. The usual measurements of the hadronic structure functions are totally inclusive, regardless of whether the final hadronic states have or have not had any photons radiated. Therefore the only correction is due to bremsstrahlung from the initial hadronic states which can be absorbed into the definition of the structure function $F_1(x)$ and $F_2(x)$. With these assumptions, diagrams M_4 to M_7 can be ignored.

Consider the non-radiative terms, the invariant amplitude squared is simply

$$|M|^2 = |M_1 + M_2 + M_3|^2 \quad (14)$$

$$|M_1|^2 + 2 \operatorname{Re} M_1^* (M_2 + M_3) + O(\alpha^2).$$

Define the hadronic current to be

$$J_{had}^\mu = \langle X | J^\mu | p_i \rangle,$$

then the lowest order term $|M_1|^2$ is

$$|M_1|^2 = e^4 |\bar{u}(k_2) \gamma_\mu u(k_1)|^2 \frac{1}{q^2} |\langle X | J^\mu | p_i \rangle|^2, \quad (15)$$

where the momentum transfer q^2 is,

$$\begin{aligned} q^2 &= -Q^2 = (p_f - p_i)^2 \\ &= (k_1 - k_2)^2 \\ &= (2k_1 k_2 - 2m^2). \end{aligned} \quad (16)$$

The phase space element for the final state is

$$d\Gamma = \frac{d^3 k_2}{(2\pi)^3 2E_{k_2}} \prod_i \frac{d^3 S_i}{(2\pi)^3 2E_{S_i}} (2\pi)^4 \delta^{(4)}(k_1 + p_i - k_2 - \sum_i S_i), \quad (17)$$

and is in fact a many-particle final state. To simplify, we pretend that it is a 'two-particle' final state, then the phase space element becomes

$$d\Gamma = \frac{d^3 k_2}{(2\pi)^3 2E_{k_2}} \frac{d^3 p_f}{(2\pi)^3 2E_{p_f}} (2\pi)^4 \delta^{(4)}(k_1 + p_i - k_2 - p_f) d\Gamma_{had}, \quad (18)$$

where

$$p_f = \sum_i S_i, \quad (19)$$

and

$$d\Gamma_{had} = \pi \frac{dS_i}{(2\pi)^3 2E_{S_i}} \delta^{(4)}(P_f - \sum_i S_i). \quad (20)$$

Using the identities

$$\frac{d^3 p}{2E} = d^4 p \delta(p^2 - M^2) \theta(E),$$

$$\int d^4 \xi \delta^{(4)}(\xi) = 1,$$

we have

$$\frac{d^3 P_f}{2E_{P_f}} \delta^{(4)}(k_2 + P_f - k_1 - P_i) \rightarrow \delta[(P_i + k_1 - k_2)^2 - M_f^2]. \quad (21)$$

Combining Eqs. (15) and (21), we have

$$\frac{1}{2} \sum_{\text{spin}} \sum_X |M_i|^2 d\Gamma = \frac{4\alpha^2}{(Q^2)^2} \left(\frac{M}{2} L_{\mu\nu} H^{\mu\nu} \right) \frac{d^3 k_2}{2E_{k_2}}, \quad (22)$$

where $L_{\mu\nu}$, $H^{\mu\nu}$ are the leptonic and hadronic tensors respectively which are given by

$$\begin{aligned} L_{\mu\nu} &= \sum |\bar{u}(k_2) \gamma_\mu u(k_1)|^2 \\ &= 4 [k_{1\mu} k_{2\nu} + k_{2\mu} k_{1\nu} - g_{\mu\nu} (k_1 \cdot k_2 - m^2)], \end{aligned} \quad (23)$$

$$\begin{aligned} H_{\mu\nu} &= \frac{1}{M} \sum_X \langle P_i | J_\mu^\dagger | X \rangle \langle X | J_\nu | P_i \rangle \delta(P_f^2 - M_f^2) d\Gamma_{had} \\ &= W_1(Q^2, M_f^2) (-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) \\ &\quad + W_2(Q^2, M_f^2) \left[(P_{i\mu} - \frac{P_i \cdot q}{q^2} q_\mu) (P_{i\nu} - \frac{P_i \cdot q}{q^2} q_\nu) \right]. \end{aligned}$$

Observe that $L_{\mu\nu}$ and $H^{\mu\nu}$ are Lorentz covariant tensors. Their contraction will give an invariant:

$$L_{\mu\nu} H^{\mu\nu} = -4 (2k_1 \cdot k_2 - 4m^2) W_1(Q^2, M_f^2) + \frac{2}{M^2} W_2(Q^2, M_f^2) \{ 4(P_1 \cdot k_1)(P_1 \cdot k_2) - 2M^2(k_1 \cdot k_2 - m^2) \}. \quad (24)$$

Define the invariant variables S , X , λ_s as:

$$\begin{aligned} S &= 2P_1 \cdot k_1, \\ X &= 2P_1 \cdot k_2, \\ \lambda_s &= S^2 - 4m^2 M^2, \\ S_X &= S - X, \end{aligned} \quad (25)$$

then $L_{\mu\nu} H^{\mu\nu}$ is simplified to

$$\begin{aligned} \frac{M}{2} L_{\mu\nu} H^{\mu\nu} &= 2M(Q^2 - 2m^2) W_1(Q^2, M_f^2) \\ &+ \frac{1}{M} W_2(Q^2, M_f^2) (SX - M^2 Q^2). \end{aligned} \quad (26)$$

It is possible to express $d^3 k_2 / 2E_{k_2}$ in terms of the invariant variables X and Q^2 . The easiest way to do this is to evaluate everything in the proton rest frame (the laboratory frame) and then transform back to any frame we like through Lorentz transformations. In the laboratory frame, Q^2 and X are given by

$$X = 2ME_{k_2}, \quad (27)$$

$$Q^2 = 2(E_k E_{k_2} - |\vec{k}_1| |\vec{k}_2| \cos \theta_{k_2}).$$

where the direction of $\vec{k}_1$ is taken as the z-axis.

Evaluating the Jacobian we get

$$dX dQ^2 = 4M|\vec{k}_1| \frac{|\vec{k}_2|^2}{E_{k_2}} dk_2 d\cos\theta_{k_2} \quad (28)$$

Using the axial-symmetry of the non-radiative process, and integrating both sides over ϕ_{k_2} from $-\pi$ to $+\pi$ we get,

$$2\pi dX dQ^2 = 4M|\vec{k}_1| \int \frac{d^3k_2}{2E_{k_2}} \quad (29)$$

In this frame, $|\vec{k}_1|$ is simply $\sqrt{\lambda_S}/2M$, therefore

$$\frac{\pi}{2\sqrt{\lambda_S}} dX dQ^2 = \int \frac{d^3k_2}{2E_{k_2}} \quad (30)$$

hence

$$\begin{aligned} \frac{d^2\sigma}{dQ^2 dX} = & \frac{2\pi\alpha^2}{\lambda_S (Q^2)^2} \left\{ 2M(Q^2 - 2m^2) W_1(Q^2, M_f^2) \right. \\ & \left. + \frac{1}{M} W_2(Q^2, M_f^2) [SX - M^2 Q^2] \right\} \quad (31) \end{aligned}$$

The invariant mass squared has to satisfy the conservation equation, namely,

$$M_f^2 - (P_1 + k_1 - k_2)^2 = 0$$

For the non-radiative processes, $M_f^2 = W^2$ where W^2 is the invariant mass squared of the non-observed systems of particles (assuming that only the final scattered electrons are detected),

$$W^2 = (P_1 + k_1 - k_2)^2 = S - X - Q^2 + M^2 \quad (32)$$

hence

$$dW^2 dQ^2 = dx dQ^2$$

Thus, another useful representation of Eq.(31) can be written as:

$$\begin{aligned} \frac{d^2\sigma_0}{dQ^2 dW^2} &= \frac{2\pi\alpha^2}{\lambda_S(Q^2)^2} \left\{ 2M(Q^2 - 2m^2) W_1(Q^2, W^2) \right. \\ &\quad \left. + \frac{1}{M} W_2(Q^2, W^2) [SX - M^2 Q^2] \right\}. \end{aligned} \quad (34)$$

It was shown in 1969 by Bjorken that if Eq.(31) to (34) are expressed in terms of the so called scaling variables x and y , then the cross-section $d^2\sigma/dxdy$ scales, i.e., the form will not change with Q^2 or X but only on the scaling variables with x and y defined as

$$\begin{aligned} x &= \frac{-(k_1 - k_2)^2}{2P_1 \cdot (k_1 - k_2)} \\ &= \frac{Q^2}{S - X} \\ &= \frac{Q^2}{Q^2 + W^2 - M^2} \end{aligned} \quad (35)$$

$$\begin{aligned} y &= \frac{P_1 \cdot (k_1 - k_2)}{P_1 \cdot k_2} \\ &= \frac{S - X}{S} \\ &= \frac{Q^2 + W^2 - M^2}{S} \end{aligned} \quad (36)$$

From Eq.(35) and (36), we have

$$\begin{aligned} Q^2 &= Sxy, \\ X &= S(1-y), \\ W^2 &= Sy(1-x) + M^2, \\ dx dQ^2 &= dQ^2 dW^2 = S^2 y dx dy. \end{aligned} \quad (37)$$

Hence,

$$\frac{d^2\sigma_0}{dx dy} = \frac{2\pi\alpha^2}{\lambda_s(Q^2)^2} \left\{ 2M W_1(x, y) \left(1 - \frac{2m^2}{S_{xy}}\right) + \frac{1}{M} W_2(x, y) \left[\frac{S(1-y)}{xy} - M^2\right] \right\} \quad (38)$$

which does not have any explicit dependence on Q^2 , W^2 or x

which implies scaling.

(2.2) VERTEX CORRECTION

It will be shown later that the interference terms can be expressed in the form

$$\frac{\alpha}{\pi} \delta_{\text{interf}} \frac{d^2 \sigma_0}{dx dy} \quad (39)$$

So that

$$\frac{d^2 \sigma_v}{dx dy} = \frac{d^2 \sigma_0}{dx dy} \frac{\alpha}{\pi} \delta_{\text{interf}}. \quad (40)$$

It is known that the vacuum polarization diagram M_2 and the vertex diagram M_3 of the electron current contains infrared and ultra-violet divergences. The ultra-violet divergence can be removed by the mass and charge renormalization, while the infrared divergence of the vertex correction when combined with the infrared divergence of the radiative processes cancel each other. This will be shown in detail later. The derivation of the vertex correction has been given in many places, e.g., J. Kahane^[4], Bjorken and Drell^[15], Mo and Tsai^[3]. Using the notations of Mo and Tsai^[3], let us define

$$\begin{aligned} K(p_a, p_b) &= (p_a \cdot p_b) \int_0^1 \frac{dy}{y} \ln \left(\frac{p_y^2}{\lambda^2} \right) \\ L(p_a, p_b) &= (p_a \cdot p_b) \int_0^1 \frac{dy}{y} \ln \left(\frac{p_y^2}{m^2} \right) \\ \mu(p_a, p_b) &= \int_0^1 \frac{dy}{p_y^2} \end{aligned} \quad (41)$$

then

$$K(p_a, p_b) = L(p_a, p_b) + (p_a \cdot p_b) \ln \frac{m^2}{\lambda^2} \mu(p_a, p_b), \quad (42)$$

where

$$p_y = y p_a + (1-y) p_b,$$

and λ is the fictitious photon mass.

It was shown^[14] that the renormalized electron vacuum polarization contribution was found to be:

$$M_2 = \left(\frac{-\alpha}{2\pi} \right) \left[\frac{10}{9} - \frac{1}{3} (k_1 + k_2)^2 \mu(k_1, k_2) \right] M_1, \quad (43)$$

while the renormalized electron vertex diagram was found to be

$$M_3 = \left(\frac{-\alpha}{2\pi} \right) \left\{ K(k_1, k_2) - K(k_1, k_1) - \left[\frac{1}{4} q^2 + 2k_1 k_2 \right] \mu(k_1, k_2) + 2 \right\} M_1. \quad (44)$$

Collecting the terms which are infrared divergent (IR) and those which are finite (F), we have

$$\begin{aligned} \delta_V &= \delta_V^{IR} + \delta_V^F \\ \delta_V^{IR} &= K(k_1, k_2) - K(k_1, k_1) \\ \delta_V^F &= - \left\{ \frac{10}{9} - \frac{1}{3} (k_1 + k_2)^2 \mu(k_1, k_2) \right\} \\ &\quad - \left\{ \left[\frac{1}{4} q^2 + 2k_1 k_2 \right] \mu(k_1, k_2) + 2 \right\}. \end{aligned} \quad (45)$$

It is trivial to prove that

$$\begin{aligned}\mu(k_1, k_2) &= \frac{1}{m^2}, \\ L(k_1, k_2) &= 0.\end{aligned}\tag{46}$$

We want δ_v^{IR} , δ_v^F to be expressed in terms of S , X and Q^2 , thus

$$\begin{aligned}\mu(k_1, k_2) &= \int_0^1 \frac{d\alpha}{[\alpha(k_1 - k_2) + k_2]^2} \\ &= - \int_0^1 \frac{d\alpha}{[Q^2 \alpha^2 - Q^2 \alpha - m^2]^2} \\ &= - \frac{1}{\sqrt{\lambda_m}} \ln \left(\frac{2\alpha Q^2 - Q^2 - \sqrt{\lambda_m}}{2\alpha Q^2 - Q^2 + \sqrt{\lambda_m}} \right) \Big|_0^1 \\ &= - \frac{2}{\sqrt{\lambda_m}} \ln \left(\frac{Q^2 - \sqrt{\lambda_m}}{Q^2 + \sqrt{\lambda_m}} \right) \\ &= 2L_m,\end{aligned}\tag{47}$$

where

$$\lambda_m = Q^2(Q^2 + 4m^2),$$

and

$$L_m = \frac{1}{\sqrt{\lambda_m}} \ln \left(\frac{Q^2 + \sqrt{\lambda_m}}{Q^2 - \sqrt{\lambda_m}} \right).\tag{48}$$

Similarly,

$$\begin{aligned}L(k_1, k_2) &= (Q^2 + 2m^2) \left[L_m \ln \frac{\lambda_m}{m^2 Q^2} \right. \\ &\quad \left. + \frac{1}{\sqrt{\lambda_m}} \left[\Phi \left(\frac{Q^2 + \sqrt{\lambda_m}}{Q^2 - \sqrt{\lambda_m}} \right) - \Phi \left(\frac{Q^2 - \sqrt{\lambda_m}}{Q^2 + \sqrt{\lambda_m}} \right) \right] \right]\end{aligned}\tag{49}$$

where $\Phi(x)$ is the Spence function given by

$$\begin{aligned}\Phi(x) &= - \int_0^x \frac{dt}{t} \ln |1-t| \\ &= \int_1^x dt \frac{\ln t}{t-1}.\end{aligned}$$

Combining Eq.(46) to (49) and extracting the terms with a λ dependence, we have

$$\begin{aligned}\delta_V^{IR} &= K(k_1, k_1) - K(k_1, k_2) \\ &= L(k_1, k_1) - L(k_1, k_2) \\ &\quad + \{ (k_1, k_1) \mu(k_1, k_1) - (k_1, k_2) \mu(k_1, k_2) \} \ln \frac{m^2}{\lambda^2} \\ &\equiv \delta_V^\lambda + \delta_V^L,\end{aligned}\tag{50}$$

where

$$\begin{aligned}\delta_V^L &= -L(k_1, k_2), \\ \delta_V^\lambda &= [(Q^2 + 2m^2) L_m - 1] \ln \frac{\lambda^2}{m^2}.\end{aligned}\tag{51}$$

The finite part is simplified to

$$\begin{aligned}\delta_V^F &= [(\frac{3}{2} Q^2 + 4m^2) L_m - 2] \\ &\quad + \{ \frac{2}{3} (Q^2 + 4m^2) L_m - \frac{10}{9} \} + \{ m \rightarrow m_\mu \}.\end{aligned}\tag{52}$$

The terms in the curly brackets correspond to the vacuum polarization due to electron and muon respectively. The

reason that muon is introduced is that the virtual photon can decay either to electron-positron pair or muon-antimuon pair in the intermediate state (see Fig. 8). Now the interference term is written as

$$2 \operatorname{Re} (M_2 + M_3) M_1^\dagger = \frac{\alpha}{\pi} (\delta_V^F + \delta_V^\lambda + \delta_V^L) |M_1|^2$$

Since the final states of the vertex diagrams are the same as for the lowest order diagram, the kinematics are the same. Therefore the cross-section due to the vertex correction can be written as

$$\frac{d^2\sigma_V}{dx dy} = \frac{\alpha}{\pi} (\delta_V^F + \delta_V^\lambda + \delta_V^L) \frac{d^2\sigma_0}{dx dy}, \quad (53)$$

or equivalently, $d\sigma_V$ can be written as:

$$\begin{aligned} d\sigma_V &= d\sigma_V^F + d\sigma_V^{IR} \\ d\sigma_V^F &= \frac{\alpha}{\pi} (\delta_V^F + \delta_V^L) d\sigma_0, \\ d\sigma_V^{IR} &= \frac{\alpha}{\pi} \delta_V^\lambda d\sigma_0. \end{aligned} \quad (54)$$

Notice that the term δ_V^λ contains a factor $\ln(\lambda^2/m^2)$ which is divergent as $\lambda \rightarrow 0$. This term can be cancelled only if the radiative cross-section contains an identical term with the opposite sign.

Fig.8 Vacuum polarization diagrams involving electron and muon bubbles.

(2.3) RADIATIVE PROCESS

The Feynman diagrams for the radiative processes are shown in Fig.9. Let us define several invariant variables which are useful in simplifying the expression for the cross-section. These are,

$$\begin{aligned}
 t &= -(k_1 - k_2 - k)^2 = -q^2, \\
 z_1 &= 2k_1 \cdot k, \\
 z &= 2k_2 \cdot k, \\
 T &= 2P_1 \cdot q, \\
 M_f^2 &= (P_1 + k_1 - k_2 - k)^2, \\
 \lambda_x &= X^2 - 4m^2 M^2, \\
 \lambda_k &= (S_x - T)^2, \\
 \lambda_t &= T^2 + 4M^2 t, \\
 \lambda_y &= S_x^2 + 4M^2 Q^2.
 \end{aligned} \tag{55}$$

It is straight forward to write down the invariant amplitude for the processes. Observe that only corrections to the electron current are involved while the hadron current remains unchanged. This means that the hadronic tensor $H_{\mu\nu}$ of Eq.(23) remain unchanged while the leptonic tensor is corrected by the radiated photon. The other changes are the factor

$$e^4 \rightarrow e^6,$$

or

$$(4\pi\alpha)^2 \rightarrow (4\pi\alpha)^3.$$

Fig.9 Definition of kinematical variables of the radiative processes $e^+p \rightarrow e^+\gamma + \text{hadrons}$.

and the phase space factor which is changed into

$$d\Gamma = \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3p}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}(p + k_1 - k_2 - k - p_f) d\Gamma_{\text{had}}, \quad (56)$$

where ω is the energy of the radiated photon. From Fig.(9), we can write down the leptonic tensor to be

$$L_{\mu\nu}^R = \frac{1}{2} \sum_{\text{pol Spin}} \left| \bar{u}(k_2) \left[\gamma_\mu \frac{1}{\not{k}_1 - \not{k} - m} \not{\epsilon} + \not{\epsilon} \frac{1}{\not{k}_2 + \not{k} - m} \gamma_\mu \right] u(k_1) \right|^2 \quad (57)$$

Using the formula

$$\not{a} \not{b} = -\not{b} \not{a} + 2a \cdot b,$$

$$(\not{k}_1 - m) u(k_1) = 0,$$

$$\bar{u}(k_2) (\not{k}_2 - m) = 0.$$

then the square bracket sandwiched between $\bar{u}(k_2)$ and $u(k_1)$, after some manipulation, becomes

$$u(k_2) [\dots] u(k_1) = \bar{u}(k_2) e^\rho A_{\mu\rho} u(k_1),$$

where

$$A_{\mu\rho} = \gamma_\mu \left(\frac{k_{2\rho}}{k_2 \cdot k} - \frac{k_{1\rho}}{k_1 \cdot k} \right) + a \gamma_\mu \not{k} \gamma_\rho + b \gamma_\rho \not{k} \gamma_\mu,$$

$$a = \frac{1}{2} k_1 \cdot k = \frac{1}{2} E_1,$$

$$b = \frac{1}{2} k_2 \cdot k = \frac{1}{2} E_2.$$

If we sum over spin in Eq.(57), then

$$L_{\mu\nu}^R = \sum_{\text{pol}} \text{Tr} [A_{\mu\rho} e^\rho (\not{k}_1 + m) \tilde{A}_{\nu\sigma} e^\sigma (\not{k}_2 + m)], \quad (58)$$

$$\tilde{A}_{\nu\sigma} = \gamma_0 A_{\nu\sigma}^\dagger \gamma_0.$$

It is not hard to show that the 'current' $A_{\mu\rho}$ is conserved, i.e., $k^\mu A_{\mu\rho} = 0$. Hence,

$$\sum_{\text{pol}} e_\mu(k) e_\nu(k) = -g_{\mu\nu} + \frac{k^\mu k^\nu}{k^2},$$

and thus

$$\sum_{\text{pol}} A_{\mu\rho} A_{\nu\sigma} e^\rho e^\sigma = -A_{\mu\rho} A_{\nu}{}^\rho, \quad (59)$$

$$L_{\mu\nu}^R = \frac{1}{2} \text{Tr} [A_{\mu\rho}(K_1 + m) \tilde{A}_\nu{}^\rho(K_2 + m)].$$

After some lengthy and tedious manipulations, (or using the computer program ASHMEDAI) the trace was found to be

$$\begin{aligned} L_{\mu\nu}^R = & 4 \{ A_1^2 [k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} + (m^2 - K_1 \cdot K_2) g_{\mu\nu}] \\ & - (A_{1\mu} k_{2\nu} + A_{1\nu} k_{2\mu}) + \frac{2k_{1\mu} k_{2\nu} (A_1 \cdot k_1)}{(k_1 \cdot k)} \\ & - g_{\mu\nu} \frac{(A_1 \cdot k_1)(K_2 \cdot k)}{(K_1 \cdot k)} + g_{\mu\nu} (A_1 \cdot k_2) \\ & - (A_{1\mu} A_{1\nu} + A_{1\nu} K_{1\mu}) + \frac{2K_{1\nu} K_{1\mu} (A_1 \cdot K_2)}{K_2 \cdot k} \\ & - g_{\mu\nu} \frac{(A_1 \cdot K_2)(K_1 \cdot k)}{(K_2 \cdot k)} + g_{\mu\nu} (A_1 \cdot K_1) \\ & - \frac{K_{1\mu} k_{2\nu} + k_{2\nu} K_{1\mu}}{(K_1 \cdot k)} + g_{\mu\nu} \frac{(K_2 \cdot k)}{(K_1 \cdot k)} + \frac{8m^2 K_{1\mu} k_{2\nu}}{4(K_1 \cdot k)(K_2 \cdot k)} \\ & - \frac{K_{1\mu} K_{1\nu} + K_{1\nu} K_{1\mu}}{(K_2 \cdot k)} + g_{\mu\nu} \frac{(K_1 \cdot k)}{(K_2 \cdot k)}, \end{aligned} \quad (60)$$

where

$$A_{\mu} = \left(\frac{k_{2\mu}}{k_2 \cdot k} - \frac{k_{1\mu}}{k_1 \cdot k} \right).$$

After some lengthy algebra expressing everything in terms of S , X , Q^2 , t , z , z_1 , and T , $L_{\mu\nu}^R H^{\mu\nu}$ is found to be expressible as

$$\frac{1}{4} L_{\mu\nu}^R H^{\mu\nu} = 4 \mathcal{J}_1 W_1(t, M_f^2) + \frac{1}{M} \mathcal{J}_2 W_2(t, M_f^2), \quad (61)$$

where

$$\begin{aligned} \mathcal{J}_1 = & \{ m^2(2m^2 - t)/Z^2 + (4m^2 - t - Q^2)/2Z \\ & + (t - 4m^2)/(2Z_1 Z) + \frac{1}{2} \} + \{ Z_1 \leftrightarrow -Z \}, \end{aligned} \quad (62)$$

$$\begin{aligned} \mathcal{J}_2 = & \{ 2m^2 [M^2 t - S(S - T)] Z^{-2} \\ & + [M^2(t + Q^2) - XT] Z^{-1} \\ & + [t(S(S - T) + X(X + T) - 2M^2(t + 2m^2)) \\ & + 2m^2((S - T)(X + T) + SX)] (2Z Z_1)^{-1} - M^2 \} \\ & + \{ -Z_1 \leftrightarrow Z, S \leftrightarrow -X \}. \end{aligned}$$

The symbol $\{a \leftrightarrow b\}$ means that every a occurring in the first bracket is changed into b and vice versa.

The phase space $d^3k/2\omega$ can be expressed in terms of either $dt dz dM_f^2$ or $dt dz dT$. Again it is simple to evaluate t , z and M_f^2 in the laboratory frame. In this frame, (see Fig. 10)

$$\begin{aligned} z &= 2\omega (E_{k_2} - |R_2| \cos \theta_{\omega}), \\ t &= z_1 - z + Q^2 = 2\omega (E_{k_1} - |R_1| \cos \theta_{\omega}) - z + Q^2 \\ M_f^2 &= M^2 - t + S_X = 2P_1 k. \end{aligned} \quad (63)$$

Fig.10 Definition of the angular variables.

Therefore

$$\frac{\partial M_f^2}{\partial \omega} = 2M,$$

$$\frac{\partial M_f^2}{\partial \cos \theta_\omega} = \frac{\partial M_f^2}{\partial \phi_\omega} = \frac{\partial t}{\partial \phi_\omega} = 0,$$

$$\frac{\partial z}{\partial \phi_\omega} = -2\omega |\vec{k}_2| \frac{\partial \cos \theta_{\omega_2}}{\partial \omega}$$

$$\frac{\partial t}{\partial \cos \theta_\omega} = -2\omega |\vec{k}_1|,$$

and the Jacobian of transformation is simply

$$\frac{\partial(M_f^2, t, z)}{\partial(\omega, \cos \theta_\omega, \phi_\omega)} = \frac{\partial M_f^2}{\partial \omega} \frac{\partial t}{\partial \cos \theta_\omega} \frac{\partial z}{\partial \phi_\omega}, \quad (64)$$

or

$$dM_f^2 dt dz = 8M\omega |\vec{k}_1| |\vec{k}_2| \frac{\partial \cos \theta_{\omega_2}}{\partial \phi_\omega} (\omega d\omega d\cos \theta_\omega d\phi_\omega),$$

and $\cos \theta_{\omega_2}$ is related to $\cos \theta_\omega$ and $\cos \theta_{k_2}$ by

$$\begin{aligned} \cos \theta_{\omega_2} &= \cos \theta_\omega \cos \theta_{k_2} + \sin \theta_\omega \sin \theta_{k_2} \cos \phi, \\ \phi &= \phi_\omega - \phi_{k_2}. \end{aligned} \quad (65)$$

If we define the outgoing electron to be the x-z plane, that is $\phi_{k_2}=0$, then

$$\begin{aligned} R &= \frac{\partial \cos \theta_\omega}{\partial \phi_\omega} = \sin \theta_\omega \sin \theta_{k_2} \sin \phi_\omega \\ &= \sin \theta_\omega \sin \theta_{k_2} \left\{ 1 - \left(\frac{\cos \theta_{\omega_2} - \cos \theta_\omega \cos \theta_{k_2}}{\sin \theta_\omega \sin \phi_{k_2}} \right)^2 \right\}^{1/2} \\ &= \left\{ 1 - (\cos^2 \theta_{\omega_2} + \cos^2 \theta_{k_2} + \cos^2 \theta_\omega) + 2 \cos \theta_{\omega_2} \cos \theta_\omega \cos \theta_{k_2} \right\}^{1/2} \end{aligned} \quad (66)$$

But

$$\cos \theta_{\omega_2} = (2\omega E_{k_2} - Z)/2\omega |\vec{k}_2|,$$

$$\cos \theta_{\omega} = (2\omega E_{k_1} - Z_1)/2\omega |\vec{k}_1|,$$

$$\cos \theta_{k_2} = (2E_{k_1} E_{k_2} - Y')/2|\vec{k}_1||\vec{k}_2|,$$

$$Y' = Q^2 + 2m^2,$$

therefore

$$\begin{aligned} & 4\omega^2 |\vec{k}_1||\vec{k}_2|^2 R^2 \\ = & 4\omega^2 |\vec{k}_1|^2 |\vec{k}_2|^2 - |\vec{k}_1|^2 (2\omega E_{k_2} - Z)^2 - |\vec{k}_2|^2 (2\omega E_{k_1} - Z_1)^2 \\ & - \omega^2 (2E_{k_1} E_{k_2} - Y')^2 + (2\omega E_{k_1} - Z_1)(2\omega E_{k_2} - Z)(2E_{k_1} E_{k_2} - Y'). \end{aligned}$$

In the laboratory frame:

$$E_{k_1} = S/2M \quad ; \quad |\vec{k}_1| = \sqrt{\lambda_S}/2M$$

$$E_{k_2} = X/2M \quad ; \quad |\vec{k}_2| = \sqrt{\lambda_X}/2M.$$

To find ω , we use Eq.(55):

$$T = 2P_1 \cdot (k_1 - k_2 - k)$$

$$= S_X - 2P_1 k$$

$$= S_X - 2M\omega,$$

hence

$$\omega = (S_X - T)/2M = \sqrt{\lambda_X}/2M. \quad (67)$$

With these variables, R can be simplified to

$$R = \frac{\sqrt{R_z}}{4M\omega|\vec{k}_1||\vec{k}_2|}, \quad (68)$$

where

$$\begin{aligned} R_z &= -A_z z^2 + B_z z - C_z, \\ A_z &= \lambda_y, \\ B_z &= 2M^2 Q^2(Q^2 - t) + X(S_x t - Q^2 T) + S Q^2(S_x - T), \\ C_z &= [X t - Q^2(S - T)]^2 \\ &\quad + 4m^2[(S_x - T)(S_x t - Q^2 T) - M^2(t - Q^2)^2]. \end{aligned} \quad (69)$$

Combining Eq.(64) and (68), we have

$$\frac{dM_f^2 dt dz}{4\sqrt{R_z}} = \frac{d^3k}{2\omega}. \quad (70)$$

The phase space can now be written as:

$$\begin{aligned} d\Gamma &= \frac{1}{(2\pi)^5} \frac{d^3k_2}{2E_{k_2}} \frac{d^3k}{2\omega} \frac{d^3p_f}{2E_{p_f}} \int^{(4)} (p+k, -k, -k-p_f) d\Gamma_{had} \\ &= \frac{1}{(2\pi)^5} \frac{\pi}{8\sqrt{\lambda_s}} dX dQ^2 \frac{dM_f^2 dt dz}{\sqrt{R_z}} \\ &\quad \times \delta[(p+k, -k, -k)^2 - M_f^2] d\Gamma_{had}. \end{aligned} \quad (71)$$

The last term $\delta[\dots] d\Gamma_{had}$ will be absorbed into the definition of $H_{\mu\nu}$. The radiative cross-section is simplified to

$$\frac{d^2\sigma_R}{dX dQ^2} = \frac{\alpha^3}{\pi\lambda_s} \int dM_f^2 \frac{dt}{t^2} \frac{dz}{\sqrt{R_z}} \mathcal{S}, \quad (72)$$

where

$$\mathcal{J} = 2M W_1(t, M_f^2) \mathcal{J}_1 + \frac{1}{2M} W_1(t, M_f^2) \mathcal{J}_2. \quad (73)$$

There remain the limits of integrations to be found. Unlike the non-radiative case where there was axial symmetry, here there is no such symmetry but there is a symmetry such that the integral is symmetric under reflection about the plane defined by $\vec{k}_2$ and $\vec{k}_1$, i.e.,

$$\int_0^\pi d\phi_\omega = \int_\pi^{2\pi} d\phi_\omega.$$

An integration over z can be viewed as an integration over $d\phi_\omega$. Hence

$$\begin{aligned} \int_0^{2\pi} d\phi_\omega &= \int_0^\pi d\phi_\omega + \int_\pi^{2\pi} d\phi_\omega \\ &= 2 \int_0^\pi d\phi_\omega \\ &\rightarrow 2 \int_{z_{\min}}^{z_{\max}} dz. \end{aligned}$$

At $\phi_\omega = 0$ or π , $\sin \phi_\omega = 0$, hence $R_x = 0$. Therefore the roots of the trinomial $R_z = 0$ provide the limits of z -integration:

$$z_{\max, \min} = \frac{B_z \pm \sqrt{D_z}}{A_z}, \quad (74)$$

where

$$D_z = B_z^2 - A_z C_z. \quad (75)$$

In the laboratory frame, the limits of t are obtained when

the area enclosed by the triangle of momenta $\vec{k}$, $\vec{p}_f$ and $\vec{A} = \vec{k} - \vec{p}_f$ vanishes (see Fig.11). Using Eq.(55),

$$\begin{aligned} t &= -(\vec{p}_i - \vec{p}_f)^2 \\ &= -M^2 - M_f^2 + 2ME_{p_f}, \end{aligned}$$

therefore

$$\begin{aligned} |\vec{p}_f|^2 &= E_{p_f}^2 - M_f^2 \\ &= [t + M_f^2 - M^2]^2 + 4M^2t / 4M^2, \end{aligned}$$

or

$$|\vec{p}_f| = \sqrt{\lambda_t} / 2M, \quad (76)$$

and similarly

$$|\vec{A}| = \sqrt{\lambda_\lambda} / 2M. \quad (77)$$

It is trivial to show that the area of a triangle of sides a , b , and c is $\lambda(a^2, b^2, c^2)$ where

$$\lambda(x, y, z) = 2xy + 2xz + 2yz - (x^2 + y^2 + z^2).$$

Hence the vanishing of the area of the momentum triangle implies

$$\lambda(\lambda_\gamma, \lambda_t, \lambda_\kappa) = 0. \quad (78)$$

The above quantity is an Lorentz invariant, thus, it is true not only in the laboratory frame, but in any other frame.

$\lambda(\lambda_\gamma, \lambda_t, \lambda_\kappa)$ is a trinomial in t , so vanishing of $\lambda(\lambda_\gamma, \lambda_t, \lambda_\kappa)$ implies

$$\sqrt{\lambda_t} = \sqrt{\lambda_\gamma} \pm \sqrt{\lambda_\kappa}. \quad (79)$$

Fig.11 (a) Momentum triangle of $\vec{P}_f$, $\vec{k}$ and $\vec{\Lambda}$;
 (b) extreme configurations.

Solving for t , we have

$$t_{\max, \min} = \frac{1}{2W^2} [(W^2 - M_f^2)(S_x \pm \sqrt{\lambda_y}) + 2M_f^2 Q^2]. \quad (80)$$

The M_f^2 -integration is equivalent to the ω -integration. Referring to Eq.(55), note that when the photon has $\omega=0$, M_f^2 is maximum. The lowest lying hadron is the neutral pion. If the process leads to a continuum state, then the final hadronic state is at least composed of a proton and a pion. Therefore the limits for M_f^2 is

$$(M + m_\pi)^2 \leq M_f^2 \leq W^2. \quad (81)$$

The radiative cross-section thus has the form

$$\frac{d^2\sigma_R}{dQ^2 dW^2} = \frac{2\alpha^3}{\pi \lambda_s} \int_{(M+m_\pi)^2}^{W^2} dM_f^2 \int_{t_{\min}}^{t_{\max}} \frac{dt}{t^2} \int_{z_{\min}}^{z_{\max}} \frac{dz}{\sqrt{R_z}} \mathcal{S} \quad (82)$$

Just by counting powers of ω , the integral in Eq.(82) is seen to have a logarithmic divergence in it. Let $d\sigma_R$ be written as

$$\begin{aligned} d\sigma_R &= (d\sigma_R - d\sigma_R^{IR}) + d\sigma_R^{IR} \\ &= d\sigma_R^F + d\sigma_R^{IR}, \end{aligned}$$

where $d\sigma_R^F$ is free of infrared divergence. Combining with $d\sigma_V$, we want to prove that

$$d\sigma_R + d\sigma_V = d\sigma_R^F + d\sigma_V^F + (d\sigma_R^{IR} + d\sigma_V^{IR}),$$

is finite because $(d\sigma_R^{IR} + d\sigma_V^{IR})$ no longer contains $\log \lambda^2$.

The infrared part $d\sigma_R^{IR}$ is just the limit of $d\sigma_R$ when $k \rightarrow 0$. In this limit $t \rightarrow Q^2$ and $z_1, z \rightarrow 0$. Therefore

$$d\sigma_R^{IR} = \lim_{\substack{t \rightarrow Q^2 \\ z, z_1 \rightarrow 0}} d\sigma_R. \quad (83)$$

Thus by replacing all t by Q^2 , throwing away all z and z_1 in the numerator but not the denominator and identifying that terms containing $1/z^2$, $1/z_1^2$ and $1/zz_1$ are those which lead to infrared divergence, the infrared counterparts of $\mathcal{J}$, $\mathcal{J}_1$, $\mathcal{J}_2$ of Eq.(73) are reduced to

$$\mathcal{J} = \mathcal{J}^{IR} \rightarrow 2MW_1(Q^2, W^2)\mathcal{J}_1^{IR} + \frac{W_2(Q^2, W^2)}{2M}\mathcal{J}_2, \quad (84)$$

$$\mathcal{J}_1 = \mathcal{J}_1^{IR} \rightarrow (Q^2 - 2m^2)f^{IR},$$

$$\mathcal{J}_2 = \mathcal{J}_2^{IR} \rightarrow 2(SX - M^2 Q^2)f^{IR},$$

$$f^{IR} = \frac{Q^2 + 2m^2}{2z z_1} - m^2 \left(\frac{1}{z_1^2} - \frac{1}{z^2} \right). \quad (85)$$

Hence the Infrared free cross-section is

$$\frac{d^2\sigma_R^F}{dQ^2 dW^2} = \frac{2\alpha^3}{\pi \lambda_s} \int_{(M+m\pi)^2}^{W^2} dM_f^2 \int_{t_{min}}^{t_{max}} dt \int_{z_{min}}^{z_{max}} \frac{dz}{\sqrt{R_z}} \left(\frac{\mathcal{J}}{t^2} - \frac{\mathcal{J}^{IR}}{(Q^2)^2} \right) \quad (86)$$

The Infrared part can be expressed in the form:

$$\frac{d^2\sigma_R^{IR}}{dQ^2 dW^2} = \frac{d^2\sigma_0}{dQ^2 dW^2} \frac{\alpha}{\pi} \mathcal{J}_R^{IR} \quad (87)$$

If we put back the fictitious photon mass λ in m_f^2 in this limit, it becomes

$$\begin{aligned} M_f^2 &= (P + k_1 - k_2 - k)^2 \\ &= (P_1 + k_1 - k_2)^2 - k(P_1 + k_1 - k_2) + k^2 \\ \vec{\omega} &\rightarrow \vec{\lambda} \quad W^2 - \lambda \sqrt{W^2}. \end{aligned} \quad (88)$$

Thus

$$\delta_R^{IR} = \int_{(M+m_\pi)^2}^{W^2 - \lambda \sqrt{W^2}} \frac{dM_f^2}{(M+m_\pi)^2} J[f^{IR}] \quad (89)$$

where

$$J[A] = \frac{1}{\pi} \int_{t_{min}}^{t_{max}} dt \int_{z_{min}}^{z_{max}} \frac{dz}{\sqrt{R_2}} A \quad (90)$$

Since the integral in Eq. (90) does not involve $W_1(t)$ or $W_2(t)$, therefore it can be integrated exactly and was done in references [7] and [9]. We quote the result here:

$$\begin{aligned} \delta_R^\lambda &= [(Q^2 + 2m^2)L_m - 1] \ln \{ [W^2 - (M+m_f)^2 / \lambda^2 W^2] \}, \\ \delta_R^F &= \frac{1}{2} (S' L_{S'} + X' L_{X'}) + S_g(Q^2 + 2m^2, \lambda_m, S', X' - 2W^2, \lambda_{X'}) \\ \delta_R^{IR} &= \delta_R^\lambda + \delta_R^F, \end{aligned} \quad (91)$$

where

$$S' = X + Q^2, \quad X = S - Q^2,$$

$$L_{S'} = \frac{1}{\sqrt{\lambda_{S'}}} \ln \left(\frac{S' + \lambda_{S'}}{S' - \lambda_{S'}} \right), \quad (92)$$

$$L_{X'} = \frac{1}{\sqrt{\lambda_{X'}}} \ln \left(\frac{X' + \lambda_{X'}}{X' - \lambda_{X'}} \right), \quad (93)$$

$$\lambda_{S'} = S'^2 - 4m^2 W^2,$$

$$\lambda_{X'} = X'^2 - 4m^2 W^2,$$

and

$$\begin{aligned}
 & S_{\Sigma}(s_1, \lambda_1, s_2, s_3, \lambda_3) \\
 &= \frac{S_1}{2\sqrt{\lambda_1}} \left\{ \ln \frac{S_2 - \sqrt{\lambda_1}}{S_2 + \sqrt{\lambda_1}} \ln \frac{(t-t_1)(t-t_3)}{(t-t_2)(t-t_4)} \right. \\
 &+ \sum_{i,j=1}^4 (-1)^{ij} S_j \left(\delta_{ij} \frac{1}{2} \ln^2(t-t_i) + (1-\delta_{ij}) \right. \\
 &\left. \left. \times [\ln(t_i-t_j) \ln(t-t_j) - \Phi\left(\frac{t-t_j}{t_j-t_i}\right)] \right) \right\} \Big|_{t=t_1}^{t=t_4}
 \end{aligned} \tag{94}$$

$$t_{1,2} = 2M^2 \left(\frac{S_1 \pm \sqrt{\lambda_1}}{S_2 - \sqrt{\lambda_2}} \right) - S_3 - 2M^2 - \sqrt{\lambda_2},$$

$$t_{3,4} = 2M^2 \left(\frac{-S_1 \pm \sqrt{\lambda_1}}{S_2 + \sqrt{\lambda_2}} \right) + S_3 + 2M^2 - \sqrt{\lambda_2},$$

$$t_\ell = -\sqrt{\lambda_2} + \frac{1}{\sqrt{\lambda_2}} [S_2 S_3 + 2M^2 (S_2 - S_1)],$$

$$t_u = \sqrt{\lambda_2} - \sqrt{\lambda_2},$$

$$\lambda_{12} = S_2^2 - 4m^2 M^2,$$

$$S_j = (+1, +1, -1, -1).$$

Combining δ_R^{λ} and δ_V^{λ} of Eq.(92) and Eq.(51) respectively to form δ_k , we have

$$\delta_k = [(Q^2 + 2m^2)L_m - 1] \ln([W^2 - (M+m_\pi)^2]/W^2 m^2) \tag{95}$$

Now we can define the total correction $\delta(x,y)$ to the basic cross-section $d^2\sigma_0/dxdy$ to be

$$\delta(x,y) = \frac{\alpha}{\pi} (\delta_k + \delta_R^F + \delta_V^F + \delta_V^L) + \frac{d^2\sigma_k^F}{dxdy} / \frac{d^2\sigma_0}{dxdy} \tag{96}$$

where

$$\delta_V^L = (Q^2 + 2m^2) \left\{ L_m \ln \frac{\lambda_m}{m^2 Q^2} + \frac{1}{\sqrt{\lambda_m}} \left[\Phi \left(\frac{Q^2 + \sqrt{\lambda_m}}{Q^2 - \sqrt{\lambda_m}} \right) - \Phi \left(\frac{Q^2 + \sqrt{\lambda_m}}{Q^2 - \sqrt{\lambda_m}} \right) \right] \right\},$$

$$\delta_V^F = \left[\left(\frac{2}{3} Q^2 + 4m^2 \right) L_m - 2 \right] + \left\{ \frac{2}{3} (Q^2 + 2m^2) L_m - \frac{1}{3} \right\} - \{m \rightarrow m_\mu\},$$

$$\delta_R^F = \frac{1}{2} (S' L_{S'} + X' L_{X'}) + S_2(Q^2 + 2m^2, \lambda_m, S', X' - 2W^2, \lambda_{X'}).$$

Recall that Q^2 , W^2 are related to x , y through Eq.(37).

There arise two difficulties in the computation of $d^2\sigma_R^F/dQ^2 dw^2$:

(i) No experimental data are available for $W_1(Q^2, x)$ and $W_2(Q^2, x)$ up to $Q^2 \sim S = 40,000 \text{ GeV}^2$, thus it is impossible to compute $d\sigma_R^F$ numerically;

(ii) On closely examining the integrand, we see that it contains terms involving $1/z^2$, $1/z_1^2$, $1/zz_1$, $1/z$, and $1/z_1$ all of which will become very large when the incoming or outgoing electron is collinear with the photon. These are referred to as the k_1 and k_2 peak respectively. The widths of the peaks are estimated to be of the order of $m\sqrt{Q^2}$, which is extremely narrow for electron.

The first difficulty can be solved by using the Buras-Gaemers^[11] parametrizations of the quark and gluon (or partons) distributions with Q^2 dependence. These parametrizations are based on at least two models, namely, the parton model and the asymptotic freedom of gauge theories. We also assume that the Callan-Gross^o relation

$$2x F_1(x) = F_2(x),$$

is valid over the entire kinematic region. If $\nu = p \cdot q$ and Q^2 are sufficiently large, the structure function W_1 , and νW_2 will depend on the scaling variable

$$x = Q^2 / 2\nu = Q^2 / S_x.$$

That is,

$$2M W_1(x, Q^2) = F_1(x), \quad \nu W_2(x, Q^2) = F_2(x). \quad (97)$$

To get around the second difficulty, if we can locate the two peaks and estimate the widths accurately then we can subtract the peaks from the integrand and integrate the remainder with any method we like. Observe that the structure functions W_1, W_2 do not have a z -dependence, thus we can integrate over z analytically (see Appendix A for the z -integration). The result of this is:

$$\begin{aligned} \Sigma_1(t, M_f^2) &= \frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{dz}{R_z} \mathcal{S}_1 \\ &\quad \left\{ \frac{1}{\sqrt{C_z}} \left[\frac{t-Q^2}{2} + \frac{(t-2m^2)(Q^2+2m^2)}{t-Q^2} \right] \right. \\ &\quad \left. - m^2(t-2m^2) B_z C_z^{-3/2} \right\} - \{S \leftrightarrow -X\} + \frac{1}{\sqrt{\lambda_y}}. \end{aligned} \quad (98)$$

$$\begin{aligned} \Sigma_2(t, M_f^2) &= \frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{dz}{R_z} \mathcal{S}_2 \\ &= \left\{ \frac{1}{iC_z} [M^2(t+Q^2) - XT] + [t(S(S-T) + X(X+T)) \right. \\ &\quad \left. - 2M^2(t+2m^2) + 2m^2((S-T)(X+T))] / [t-Q^2] C_z^{1/2} \right. \\ &\quad \left. - 2m^2 B_z [S(S-T) - M^2 t] C_z^{-3/2} - \{S \leftrightarrow -X\} - \frac{2M^2}{\sqrt{\lambda_y}} \right\}. \end{aligned} \quad (99)$$

(See Appendix A for the explanation of the non-trivial minus sign between the braces.)

Similarly, $\mathcal{S}_1^{IR}$ and $\mathcal{S}_2^{IR}$ is found to be

$$\begin{aligned}\mathcal{S}_1^{IR} &= (Q^2 + 2m^2) F^{IR}(t, M_f^2), \\ \mathcal{S}_2^{IR} &= 2(SX - M^2 Q^2) F^{IR}(t, M_f^2).\end{aligned}\quad (100)$$

and

$$\begin{aligned}F^{IR}(t, M_f^2) &= \frac{1}{\pi} \int_{z_{min}}^{z_{max}} \frac{dz}{\sqrt{R_z}} f^{IR} \\ &= \left\{ \frac{Q^2 + 2m^2}{t - Q^2} \frac{1}{\sqrt{C_z}} - \frac{m^2 \beta_z}{C_z^{3/2}} \right\} - \{S \leftrightarrow -X\}.\end{aligned}\quad (101)$$

For the t -integration, the peaking structure is due to $C_z^{-1/2}$ and $C_z^{-3/2}$. The old methods to locate the peaks are through the peaking approximation (see Appendix B for the derivations) but it gives no information on the width.

For the k_1 peak, t is replaced by

$$t_1 = Q^2 \left(1 - \frac{v}{X'}\right), \quad (102)$$

for the k_2 peak, it is replaced by

$$t_2 = t_1 (S \leftrightarrow -X) = Q^2 \left(1 + \frac{v}{S'}\right), \quad (103)$$

where

$$\begin{aligned}v &= W^2 - M_f^2, \\ S' &= X + Q^2, \\ X' &= S - Q^2.\end{aligned}$$

Examining $C_z(S, X)$,

$$\begin{aligned}C_z(S, X) &= [Xt - Q^2(S - T)]^2 + 4mU, \\ U &= (S_x - T)(S_x t - Q^2 T) - M^2(t - Q^2)^2,\end{aligned}\quad (104)$$

the peak occurs when

$$[Xt - Q^2(S - T)] = 0,$$

that is,

$$t_{peak_2} = Q^2(S - M_f^2 + M^2)/(X + Q^2) = t_2,$$

Similarly for $C_2(-X, -S)$, the peak is where

$$t_{peak_1} = Q^2(X + M_f^2 - M^2)/(S - Q^2) = t_1.$$

At these two values,

$$C_2(S, X) = C_2(-X, -S) = 4m^2 U. \quad (105)$$

It is not hard to show that the full width half maximum (FWHM) of the function C_2^{-n} is found to be

$$\Gamma_1 \sim \frac{2m\sqrt{(2^n - 1)U}}{S - Q^2}, \quad (106)$$

and for $C_2^{-n}(-X, -S)$,

$$\Gamma_2 \sim \frac{2m\sqrt{(2^n - 1)U}}{X - Q^2}. \quad (107)$$

Equipped with the location t_i and width of the peaks, we can subtract from the integrand two Breit-Wigner resonance curves, that is,

$$B_i(t) = \frac{a_i}{(t - t_i)^2 + \Gamma_i^2/4} \quad i = 1, 2 \quad (108)$$

The remainder after subtraction can be easily handled by

straight forward numerical methods.

(2.4) NUMERICAL RESULTS

As mentioned in section (2.3), Buras-Gaemers parametrizations of the structure functions was used, but there is still a problem. Their parametrizations are theoretical extrapolations which are expected to work very well in the asymptotic region, i.e., the high Q^2 region. At low Q^2 region, say $Q^2 \leq 4 \text{ GeV}^2$, the parametrization does not fit the experimental data accurately. Therefore the computer program will use two different parametrizations: if $Q^2 \leq 10 \text{ GeV}^2$ then the parametrizations of Bodek et al.^[12] was used, otherwise that of Buras and Gaemers^[11]. Bodek's parametrizations are fits to the experimental data and has a very simple functional form:

$$\nu W_2(\nu, Q^2) = A(W^2, Q^2) f(\omega_\alpha) / \omega,$$

where

$$f(\omega_\alpha) = \omega_\alpha \sum_n C_n \left(1 - \frac{1}{\omega_\alpha}\right)^n,$$

and

$$\omega_\alpha = \frac{2M\nu + a^2}{Q^2 + b^2}$$

and $A(W^2, Q^2)$, a^2 , b^2 , C_n are the parameters extracted from the data. With this simple form, it is impossible to incorporate resonances of the structure function into the formula (see Fig.12).

This program was originated by R. J. McKee.

Fig.12 Structure function νW_2 at the resonance region.

In figure 13 we give the numerical results for the radiative corrections $\delta(x,y)$ to the process $\mu + p \rightarrow \mu + \text{hadrons}$ with muon energy $E_\mu = 250$ GeV when the proton is stationary. Comparing these results with those of reference [7], we see that they agree very well ($\sim 3\%$) in the high x region, and fairly well ($\sim 5-7\%$) in the low x region i.e., $x \leq 0.1$. In the region $x \leq 0.01$, $y \leq 0.05$, they agree only to within about 17%. The discrepancy at this low x and y region is not surprising. At $E_\mu = 250$ GeV, $x = 0.01$ and $y = 0.05$, $S = 469$ GeV² and $Q^2 = 0.23$ GeV². At this low Q^2 , resonance structure of the structure functions νW_2 and W_1 cannot be ignored. In reference [7], they have included the resonance contribution into consideration while we have not.

For the electron-proton scattering at high energy, the resonance structure of the structure functions do not contribute much. Using CHEER as an example, that is, $E_e = 10$ GeV and $E_p = 1000$ GeV, then at $x = 0.01$ and $y = 0.05$, $Q^2 = 20$ GeV² and $\sqrt{W^2} = 44.5$ GeV which is sufficiently far away from the resonance region. Even at $x = 0.01$ and $y = 0.01$, $Q^2 = 4$ GeV² and $\sqrt{W^2} = 20$ GeV, the resonance effect is expected to be small.

There is one more remark on the numerical computation of the integrals. With the supplied estimated widths and locations of the peaks, we can use an iterative searching procedure to approximate the 'exact' location successively. In the case of muon-proton scattering, since muon has mass of 0.1096 GeV which is 214 times larger than the electron

Fig.13 Radiative corrections to the process $\mu+p \rightarrow \mu+\text{hadrons}$
at $E=250$ GeV.

mass, the width of the peaks are considerably larger than for ep scattering. The average number of searches is 6 in the μ -p case while in the ep case, the average number of searches is only 1. It turns out that in the ep case, the estimated locations are almost exactly the predicted values and their widths agree very well.

Using the exact formulas and peaking approximation, figures 14(a), (b) and 15(a), (b) are respectively the numerical results of the radiative corrections $\delta(x,y)$ to the ep scattering with $E_t = 10$ GeV and $E_p = 1000$ GeV, i.e. $s = 40,000 \text{ GeV}^2$.

Fig.14 (a) & (b). Radiative corrections to the process $e+p \rightarrow e+\text{hadrons}$ at $s=40,000 \text{ GeV}^2$.

Fig.15(a) & (b). Radiative corrections, calculated by the peaking approximation, to the process $e+p \rightarrow e+\text{hadrons}$ at $s=40,000 \text{ GeV}^2$.

Fig.15(b)

CHAPTER 3

LUMINOSITY MONITOR(3.1) DIFFERENT TYPES OF LUMINOSITY MONITORS

In some experiments, only the relative magnitude of the cross-sections are of interest while in some experiments, the absolute value of the cross-sections σ has to be measured within an accuracy of 1 or 2 % or better. Thus the luminosity L of the beams have to be monitored accurately, where L is given by

$$\text{number of interactions/sec} = L \sigma. \quad (109)$$

There are several possible ways to monitor the luminosity of the ep collider suggested by the TRISTAN report [16]:

- (i) ep elastic scattering in the small Q^2 region,
- (ii) inclusive reactions in the small Q^2 region,
- (iii) radiative photon emission in the continuous spectrum.

The idea of monitoring the luminosity is as follows: if we can calculate the cross-section for a specific reaction 'exactly' then with the measured number of events for this specific reaction we can extract L by using Eq.(109).

In the first possibility (i) above, we will encounter several difficulties:

(a) the reconstruction of the elastic reaction will be limited by the efficiency of the detector because the proton cannot be easily identified and its energy measured if its energy exceeds several hundred GeV;

(b) the counting rates of the elastic reaction is known to be small. For a luminosity of $10^{31} \text{ cm}^{-2} \text{ s}^{-1}$, the counting rate can range from 0.1 event/sec to several events/sec. (Reaction rates at high Q^2 are orders of magnitude smaller than at low Q^2 , so that we have to count low Q^2 reactions);

(c) as mentioned in section (1.2), the radiative tail of the elastic peak can alter the cross-section considerably. Thus, if we cannot calculate the radiative cross-section of the elastic process within an accuracy of 1 to 2% or better, then we cannot monitor the luminosity to better than a few percent.

For the second possibility above, namely, the inclusive reactions in small Q^2 region, the counting rate is sufficiently high since we are not limited to elastic reaction. Because it is inclusive, radiative processes are

already included. Unfortunately, it is almost impossible to calculate the inclusive cross-section theoretically and reliably. Actually, it is not necessary to use a theoretically derived cross-section. We can use the known inclusive cross-section obtained from other experiments (such as those from FNAL or SLAC etc.) as long as the kinematic constraints are satisfied. Thus this method for luminosity monitoring relies on the accuracy with which the inclusive cross-sections can be measured.

There are several reasons which support the idea that it is feasible to use radiative photons to monitor the luminosity:

(I) The radiative correction can be calculated theoretically and exactly.

(II) From section (2.4), it was shown that the radiative cross-section $d\sigma_R$ is not small compared with the basic cross-sections $d\sigma_0$, therefore we expect the counting rate will be large.

(III) It is relatively easy to absorb and detect photons than to measure hadrons.

Using the fact that number of photons N_ω emitted between angles θ_1 and θ_2 is related to the differential cross-section $d\sigma/d\theta_\omega$ (or $d\sigma/d\cos\theta_\omega$) by

$$N_\omega = L \int_{\theta_1}^{\theta_2} \frac{d\sigma}{d\theta_\omega} d\theta_\omega. \quad (110)$$

can be used to determine L . Thus by placing a photon

detector in this angular range and counting the total number of photons detected, provided that $d\sigma/d\theta\omega$ is known, L can be found.

As proved in section (2.3) that the cross-section has a logarithmic divergence in ω . That is, if we measure all the photons with energy from 0 to ω_{max} , then N_ω is infinity. Therefore, we have to measure photons from some $\omega_{min} \neq 0$ such that

$$\int_{\theta_1}^{\theta_2} d\theta_\omega \int_{\omega_{min}}^{\omega_{max}} \frac{d^2\sigma}{d\omega d\theta_\omega} d\omega < \infty. \quad (111)$$

From the numerical results of section (2.4), it can be estimated that N_ω is of the order of 10^5 to 10^7 photons per second (depending on θ_1 , θ_2 and ω_{min}). In principle, the photons can be emitted at any angle over the 4π steradian. But in fact, most of the photons will be emitted in the electron forward direction. Since the integrand contains the terms like $1/z_1^2$, $1/z_1$ and $1/zz_1$, it will become enormously big when $\cos\theta_\omega = 1$, i.e., when the photons are almost parallel to the incoming electrons. The width of the peak, due to soft photon emission, is of the order

$$\Delta\theta \sim \frac{m}{E_{k_1}} = 0.05 \text{ mrad},$$

while for the collinear photon emission,

$$\Delta\theta \sim \sqrt{\frac{m}{E_{k_1}}} = 7 \text{ mrad}.$$

In order to capture a substantial portion of the radiated photons, the detector should be located sufficiently far away from the point of interaction. To ensure this, one might make a detector such that it subtends an angle of the order of 7 mrad and be outside the beam. A possible set up is shown in Fig.16. Suppose that we are able to construct a photon detector with these desired characteristics, still there are at least two technical problems we will encounter. The electronics has to be extremely fast to trigger and count 10^5 to 10^7 photons/sec. Secondly, an electromagnetic calorimeter has a better energy resolution than spatial resolution. With this huge burst of photons in a very narrow region, it is difficult for the calorimeter to resolve whether the signal is due to single photon absorption or multi-photon absorption. Therefore, in this method it is not possible to count the number of photons, and hence cannot be used to monitor L .

An alternative will be, instead of counting photons, to measure the total energy deposited in the detector. Let us define the Radiance $R(\theta)$ to be

$$R(\theta) = \int_0^{\omega_{\max}} \omega \frac{d^2\sigma}{d\cos\theta d\omega} d\omega \quad (112)$$

then the total energy deposited in the detector is given by

$$L \int_{\theta_1}^{\theta_2} R(\theta) d\theta \quad (113)$$

Fig.16 Photon detector for the monitoring of the luminosity.

where L is the luminosity. The advantages of this method are clearly:

(i) $R(\theta)$ is free of divergence. It is because $d^2\sigma/d\omega d\cos\theta_\omega$ leads to a logarithmic divergence, that is,

$$\frac{d^2\sigma}{d\omega d\cos\theta_\omega} d\omega \sim \frac{d\omega}{\omega}$$

Therefore, the integral

$$\int_0^{\omega_{\max}} \omega \frac{d^2\sigma}{d\omega d\cos\theta_\omega} d\omega \sim \int_0^{\omega_{\max}} \omega \frac{d\omega}{\omega}$$

is finite and so is $R(\theta)$.

(ii) With an electromagnetic calorimeter it is more efficient to measure total energy deposited than counting photons.

(iii) Extremely fast electronics is not required.

We now derive an expression for the radiance.

(3.2) DERIVATION OF THE RADIANCE

Making use the result of chapter two, the radiative differential cross-section can be expressed as:

$$d\sigma = \frac{2\alpha^3}{\pi\sqrt{\lambda_s}} \frac{d^3k}{\omega} \frac{d^3k_2}{E_{k_2}} \frac{d}{t^2}, \quad (114)$$

or

$$\omega \frac{d\sigma}{d\vec{k}} = \frac{2\alpha^3}{\pi^2\sqrt{\lambda_s}} \int \frac{d^3k_2}{E_{k_2}} \frac{d}{t^2}, \quad (115)$$

where

$$d\vec{k} = \omega^2 d\omega d\Omega_\omega,$$

$$\frac{d\vec{k}}{\omega} = \omega d\omega d\Omega_\omega,$$

(notice that both sides of Eq.(115) are Lorentz invariant quantities). Applying the same techniques as before, it is possible to express d^3k_2/E_{k_2} into either $dx dQ^2 dz$ or $dx dQ^2 dt$ or $dW^2 dt dz$ etc., but $dx dQ^2 dz$ has the simplest boundary conditions. It is not hard to show that in the laboratory frame,

$$\frac{\partial(X, Q^2, z)}{\partial(|\vec{k}_2|, \cos\theta_{k_2}, \phi_{k_2})} = 8M\omega |\vec{k}_1| \frac{|\vec{k}_2|^3}{E_{k_2}} \frac{\partial \cos\theta_{\omega_2}}{\partial \phi_{\omega_2}} \quad (116)$$

Again using Eq.(65) and defining the x-z plane to be the reference plane such that $\phi_\omega = 0$, then

$$H = \frac{\partial \cos\theta_{\omega_2}}{\partial \phi_{k_2}} = \sin\theta_\omega \sin\theta_{k_2} \sin\phi_{k_2}, \quad (117)$$

thus

$$dX dQ^2 dz = 8 M \omega |\vec{k}_1| |\vec{k}_2| H \frac{d^3 k_2}{E_{k_2}} \quad (118)$$

We define a new invariant variable G ,

$$\begin{aligned} G &= 2P_1 k \\ &= 2\omega (E_{p_1} + |\vec{p}_1| \cos \theta_\omega), \end{aligned} \quad (119)$$

which in the laboratory frame becomes

$$G = 2M\omega.$$

Following the same procedure as in section (2.3), it can be shown that

$$H = \frac{H_z}{4M\omega |\vec{k}_1| |\vec{k}_2|}, \quad (120)$$

where

$$\begin{aligned} H_z &= -\alpha_z z^2 + 2\beta_z z - \gamma_z, \\ \alpha_z &= \lambda_s, \\ \beta_z &= S(XZ_1 + GQ^2) - 2M^2 Z_1 Q^2 \\ &\quad + 2m^2 (GS_x - 2M^2 Z_1), \\ \gamma_z &= (Z_1 X - GQ^2)^2 \\ &\quad + 4m^2 [Z_1 (GS - M^2 Z_1) - G(Z_1 X - GQ^2)]. \end{aligned} \quad (121)$$

Hence

$$\frac{dX dQ^2 dz}{2\sqrt{H_z}} = \frac{d^3 k_2}{E_{k_2}} \quad (122)$$

which leads to

$$\omega \frac{d\sigma}{d\vec{k}} = \frac{2\alpha^3}{\pi^2 \sqrt{\lambda_s}} \int dX dQ^2 \frac{dz}{2\sqrt{H_z}} \frac{\mathcal{J}}{t^2} \quad (123)$$

where $\mathcal{J}$ is the same $\mathcal{J}$ as in Eq.(73) except that all T 's are replaced by $S_X - G$.

The limits for the z -integration are easily found to be

$$z_{\max, \min} = \frac{\beta_z \pm \sqrt{\beta_z^2 - \alpha_z \gamma_z}}{2\alpha_z} \quad (124)$$

while the boundary of X and Q^2 is such that the area of the momentum triangle of $\vec{k}_1$, $\vec{k}_2$ and $\vec{\lambda}$ vanishes, that is, (see Fig.17)

$$\lambda(\lambda_s, \lambda_y, \lambda_x) = 0 \quad (125)$$

Expanding $\lambda(\lambda_s, \lambda_y, \lambda_x)$ explicitly in terms of S , X and Q^2 , the resulting boundary is a hyperbola:

$$m^2 X^2 - SXQ^2 + M(Q^2)^2 - 2m^2 SX + 4m^2 M^2 Q^2 + m^2 S^2 = 0. \quad (126)$$

The other boundary conditions are obtained from the limiting values of W^2 for the unobserved system of particles:

$$M^2 \leq W^2 \leq (\sqrt{S} - m)^2 \quad (127)$$

Through Eq.(32), the above inequality can be expressed as:

$$S + M^2 - (\sqrt{S} - m)^2 \leq X + Q^2 \leq S. \quad (128)$$

Fig.17 (a) Momentum triangle of $\vec{k}_1$, $\vec{k}_2$ and $\vec{\lambda}$;
(b) extreme configurations.

The boundary and the physical region for X and Q^2 are shown in Fig.18, but it is still too complicated to do numerical calculations with. If we define a new variable η ,

$$\eta = X + Q^2 \quad (129)$$

then the hyperbola becomes

$$-\eta_a (Q^2)^2 + \eta_b Q^2 - \eta_c = 0, \quad (130)$$

where

$$\begin{aligned} \eta_a &= S + M^2 + m^2 = S, \\ \eta_b &= S\eta - 2m^2(S - \eta + 2M^2), \\ \eta_c &= m^2(S - \eta)^2, \end{aligned} \quad (131)$$

and thus

$$Q^2_{\max, \min} = \frac{\eta_b \pm \sqrt{\eta_b^2 - 4\eta_a \eta_c}}{2\eta_a} \quad (132)$$

The limits for η are simply

$$S + M^2 - (\sqrt{S} - m)^2 \leq \eta \leq S. \quad (133)$$

The boundary and the physical region for η and Q^2 are shown in Fig.19.

Using this set of variables Eq.(123) can be rewritten as:

$$\omega \frac{d\sigma}{dR} = \frac{2a^3}{\pi^2 \sqrt{\lambda_s}} \int_{S+M^2-(\sqrt{S}-m)^2}^S d\eta \int_{Q^2_{\min}}^{Q^2_{\max}} dQ^2 \int_{z_{\min}}^{z_{\max}} \frac{dz}{\sqrt{H_2}} \frac{d}{dt} \quad (134)$$

Fig.18 The boundary and the physical region for X and Q^2 .

Fig.19 The boundary and the physical region for η and Q^2 .

Although we have used the laboratory frame to derive Eq.(134), since both sides of Eq.(134) remain Lorentz invariant, the above expression is true in any frame, in particular, the CHEER's ep frame.

Again, using the axial-symmetry, integrating $\omega d\sigma/dk$ over $d\omega$, we have

$$\frac{d^2\sigma}{\omega d\omega d\cos\theta_\omega} = \frac{4\alpha^2}{\pi^2\sqrt{\lambda_s}} \int d\eta dQ^2 \frac{dz}{\sqrt{H_z}} \frac{d\ell}{t^2}, \quad (135)$$

thus $R(\theta)$ is

$$\begin{aligned} R(\theta) &= \int_0^{\omega_{\max}} \omega \frac{d\sigma}{d\omega d\cos\theta_\omega} d\omega \\ &= \frac{4\alpha^2}{\pi^2\sqrt{\lambda_s}} \int_0^{\omega_{\max}} \omega^2 d\omega \int d\eta dQ^2 \frac{dz}{\sqrt{H_z}} \frac{d\ell}{t^2}. \end{aligned} \quad (136)$$

The radiance $R(\theta)$ involves two non-Lorentz invariant variables ω , $\cos\theta_\omega$ in such a combination that $R(\theta)$ is a frame dependent quantity. But we have already proved that Eq.(135) is valid in the ep frame and so is Eq.(136).

The limit $\omega_{\max}$ can be expressed in terms of E_p and E_k . Let us define u to be

$$\begin{aligned} u &= (P_f + k) \\ &= (P_i + k_i - k_s), \end{aligned}$$

therefore

$$(u - k)^2 = P_f^2 = M_f^2,$$

or

$$\omega = \frac{(W^2 - M_f^2)}{2(u_0 - \vec{u}_1 \cos \theta_\omega)}$$

Thus, $\omega_{\max}$ implies W^2 is maximum and M_f^2 is minimum.

From Eq.(127) and Eq.(81), we have

$$\begin{aligned} \omega_{\max} &= \frac{(\sqrt{s} - m)^2 - (M + m_\pi)^2}{2[(E_{p_1} + E_{k_1} - m) - |\vec{p}_1 + \vec{k}_1| \cos \theta_\omega]} \\ &= \frac{(\sqrt{s} - m)^2 - (M + m_\pi)^2}{2[(E_{p_1} + E_{k_1} - m) + (|\vec{p}_1| - |\vec{k}_1|) \cos \theta_\omega]} \end{aligned} \quad (137)$$

With the formulas provided above, we can evaluate the integral $R(\theta)$ from θ_1 to θ_2 . The integral is a five dimensional one which means that a lot of computer time is required unless some of them can be integrated analytically. Unfortunately, with the set of variables chosen, none can be integrated analytically. For the previous problem in chapter 2, we could integrate over z analytically since the structure functions have no z dependence. But in Eq.136, $t = z_1 - z + Q^2$ is not a variable of integration and the structure functions have a t dependence which implies a z dependence.

The most difficult part of the numerical integrations is because of the peaked structure of the integrand. Noting that it is the Q^2 -integration which contains the peaks, we can use the same method described in chapter 2 to tackle the problem, i.e., to subtract the peaks by Breit-Wigner fitted

functions. For the z -integration, there is no peak but the integrand is singular at z_{max} and z_{min} . Fortunately, the singularities behave like $1/\sqrt{x}$:

$$\int_{z_{min}}^{z_{max}} \frac{dz}{\sqrt{H_z}} \sim \int_{z_{min}}^{z_{max}} \frac{dz}{\sqrt{(z_{min} - z)(z_{max} - z)}}$$

thus it can be easily handled by numerical means.

We observe that after the peaked function has been integrated, the rest of the integrations are smooth. (Smooth in the sense that the integrand is not peaked but it can be very steep or even singular at the limits of integration as long as the singularities behave like $1/\sqrt{x}$.)

Due to time limitation, no numerical results were obtained for the radiance and hence will not be given in this thesis.

CHAPTER 4

DISCUSSIONDISCUSSION OF RESULTS

This section will be devoted to the study of the accuracy of the results, the validity of the assumptions and the various factors which affected the calculations.

(I) Peaking Approximation Versus Exact Calculation

Comparing the radiative corrections derived from the exact result $\delta_{\text{exact}}(x,y)$ (Fig.14(a) and (b)) with the radiative corrections derived from the peaking approximation $\delta_{\text{peak}}(x,y)$ (Fig.15(a) and (b)), they agree in quite a large kinematic region. The shaded area of Fig.20 represents those regions such that the difference is less than 1%, that is,

$$|\delta_{\text{exact}}(x,y) - \delta_{\text{peak}}(x,y)| \leq 0.01 \quad (138)$$

Fig.20. The region of x and y in which the exact calculation differed from the peaking approximation by less than 1%.

Although we cannot use the peaking approximation in the entire kinematic region, at least for $\sim 70\%$ of the kinematic region the peaking approximation is applicable. Since the computer time required to evaluate the same number of points for the peaking approximation is ~ 100 times less than for the exact calculation, it is a significant saving of computer time.

(II) Two-Photon Corrections

Again from Fig. 14(a) and (b), $\delta(x,y)$ is not small at the region when y is near 0 or 1. If the radiative correction due to one-photon emission is large, then it is possible that two-photon emission is also important. Therefore, our original assumption in which we ignored diagrams of multi-photon emission is not valid in this region.

(III) Vacuum Polarization Effect

The vacuum polarization effect contribution to the radiative corrections is given by

$$\delta_{pol} = \left[\frac{2}{3} (Q^2 + 2m^2) L_m - \frac{10}{9} \right],$$

where

$$L_m = \frac{1}{\sqrt{\lambda_m}} \ln \left(\frac{\sqrt{\lambda_m} + Q^2}{\sqrt{\lambda_m} - Q^2} \right).$$

In our numerical work, we have included the contribution due to muon polarization, but there is no reason why we can ignore the vacuum polarization effect due to the heavy lepton, the tau, which has $m^2 = 3.18 \text{ GeV}^2$. Naively, vacuum polarization effects due to the u and d quarks are also possible. Assuming we take 0.33 GeV to be the 'masses' of the u and d quarks, then at $Q^2 = 10000 \text{ GeV}^2$, the vacuum polarization contributions due to the muon, tau lepton, u quark and d quark compared with the electron vacuum polarization have a ratio of 0.53 : 0.28 : 0.43 : 0.43 : 1, which is not insignificant.

With the high Q^2 accessible by the ep collider, we have to include the vacuum polarizations of other elementary particles unless we have valid theoretical reasons to ignore them.

(IV) Structure Functions

Although we have used the theoretical parametrizations of Buras and Gaemers, their formulas are not totally independent of experimental results. On closely examining the structure function $\nu W_2(x, Q^2)$, it is expressed as a linear combination of quark and gluon distributions:

$$\begin{aligned} \nu W_2(x, Q^2) = & x \left\{ \frac{5}{18} V_8(x, Q^2) + \frac{1}{6} V_3(x, Q^2) \right. \\ & \left. + \frac{2}{9} S(x, Q^2) + \frac{4}{9} C(x, Q^2) \right\}. \end{aligned}$$

Take $V_8(x, Q^2)$ as an example, it is given by

$$x V_8(x, Q^2) = 3 x^{\eta_1(3)} (1-x)^{\eta_2(3)} / B(\eta_1(3), 1 + \eta_2(3)).$$

where

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)},$$

$$\eta_i(\bar{s}) = \eta_i(0) + \eta'_i(0) G \bar{s} \quad i=1, 2,$$

and

$$\bar{s} = \ln \left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right).$$

(Because of these complicated expressions, most of the computer time is spent on the evaluation of the structure functions.)

The functions $\eta_1(\bar{s})$ and $\eta_2(\bar{s})$ depend on the parameters $\eta_1(0)$, $\eta'_1(0)$, $\eta_2(0)$, $\eta'_2(0)$, G , Λ^2 , Q_0^2 and these parameters are derived from 'low' energy experimental data. Hence $\nu W_2(x, Q^2)$ also depends on these 'low' energy data. In order to obtain accurate experimental results, we have to do radiative corrections. Therefore, an iterative scheme have to be devised so as to minimize the inter-dependence between radiative corrections and structure functions.

A possible iteration scheme is shown in Fig.21. Starting with the uncorrected experimental cross-section $d^2\sigma_{\text{exp}}/dx dy$, an approximate form for the structure function $\nu W_2(x, Q^2)$ can be obtained by fitting the data. With this crude structure function, $\delta(x, y)$ and the correction factor

$$C(x, y) = 1/[1 + \delta(x, y)],$$

Fig.21. An iterative scheme to do radiative corrections.

can be calculated.

Now, correct $d^2\sigma_{\text{expt}}/dxdy$ by $C(x,y)$ to give the first estimate of the basic cross-section $d^2\sigma_0/dxdy$:

$$\frac{d^2\sigma_0}{dxdy} = C(x,y) \frac{d^2\sigma_{\text{expt}}}{dxdy}.$$

Using this corrected cross-section, $\nu W_2(x,Q^2)$ can be fitted again and $C(x,y)$ recalculated. This iteration has to be continued until the newly corrected cross-section differed from the previous one by less than the tolerance requirement.

The Callan-Gross relation

$$2x F_1(x) = F_2(x),$$

has been used in the computation of $\delta(x,y)$. At present we do not have experimental data to check the validity of the relation in high Q^2 region. Therefore it may be a source of error to the calculation.

So far no strong interactions have been included into the calculation, except in the parametrization of the structure functions. It does not mean that strong interactions are not important. Since there will not be any gluon exchange between the proton and the electron, any QCD effect will be restricted to the hadron current. It is

possible that some QCD effects can be absorbed into the definition of the structure functions while other QCD effects have to be explored further.

APPENDIX A
Z-INTEGRATION

The z-integration involves five kinds of integral, namely,

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \left\{ 1, \frac{1}{z}, \frac{1}{z_1}, \frac{1}{z z_1}, \frac{1}{z^2}, \frac{1}{z^2} \right\} \frac{dz}{\sqrt{R_z}}, \quad (A1)$$

where

$$R_z = -A_z z^2 + 2 B_z z - C_z$$

$$z_{\max, \min} = \frac{B_z \pm \sqrt{B_z^2 - A_z C_z}}{A_z} \quad (A2)$$

The following formulas are found to be useful:

$$R = ax^2 + bx + c$$

$$\Delta = b^2 - 4ac$$

$$\int \frac{dx}{\sqrt{R}} = \frac{-1}{\sqrt{a}} \sin^{-1} \left(\frac{2ax + b}{\sqrt{\Delta}} \right) \quad [a < 0, \Delta > 0]$$

$$\int \frac{dx}{x\sqrt{R}} = \frac{1}{\sqrt{-c}} \tan^{-1} \left(\frac{2c + bx}{2\sqrt{-c}\sqrt{R}} \right) \quad [c < 0]$$

$$\int \frac{dx}{x^2\sqrt{R}} = \frac{-\sqrt{R}}{cx} - \frac{b}{2c} \int \frac{dx}{x\sqrt{R}}$$

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(I)

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{1}{z} \frac{dz}{\sqrt{R_z}} = \frac{1}{\pi C_z} \tan^{-1} \frac{(B_z z - C_z)}{\sqrt{C_z} \sqrt{R_z}} \Big|_{z_{\min}}^{z_{\max}}$$

$$= \pm \frac{1}{\sqrt{C_z}}$$

+ if $B_z z_{\max} - C_z \geq 0$ and $B_z z_{\min} - C_z \leq 0$,

- if $B_z z_{\max} - C_z \leq 0$ and $B_z z_{\min} - C_z \geq 0$.

From the positive-definiteness of z and R_z ,

$$D_z = B_z^2 - A_z C_z \geq 0.$$

Therefore

$$B_z z_{\max} - C_z = [B_z (B_z + \sqrt{D_z}) - A_z C_z] / A_z$$

$$= (D_z + B_z \sqrt{D_z}) / A_z$$

$$\geq 0.$$

Similarly,

$$B_z z_{\min} - C_z \leq 0.$$

Thus

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{dz}{z \sqrt{R_z}} = \frac{1}{\sqrt{C_z}}.$$
(A3)

(II)

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{dz}{z^2 \sqrt{R_z}} = \frac{\sqrt{R_z}}{z C_z} \Big|_{z_{\min}}^{z_{\max}} + \frac{B_z}{C_z} \int_{z_{\min}}^{z_{\max}} \frac{dz}{z \sqrt{R_z}}$$

$$= \frac{B_z}{C_z^{3/2}}$$
(A4)

(III)

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{1}{z} \frac{dz}{\sqrt{R_z}} = \int_{z_{\min}}^{z_{\max}} \frac{1}{z+d} \frac{dz}{\sqrt{R_z}},$$

where $d = t - \alpha^2$.

Change of variable, we have

$$\begin{aligned} & \frac{1}{\pi} \int_{z_{\min}+d}^{z_{\max}+d} \frac{d\xi}{\xi [-A_z(\xi-\alpha)^2 + 2B_z(\xi-\alpha) - C_z]^{1/2}} \\ &= \frac{1}{\pi} \int_{z_{\min}+d}^{z_{\max}+d} \frac{d\xi}{\xi [-A_z \xi^2 + B'_z \xi - C'_z]^{1/2}} \\ &= \frac{1}{\pi \sqrt{C'_z}} \tan^{-1} \frac{B'_z \xi - C'_z}{\sqrt{C'_z} \sqrt{R_z}} \Big|_{z_{\min}+d}^{z_{\max}+d} \\ &= + \frac{1}{\sqrt{C'_z}} \end{aligned}$$

where

$$\begin{aligned} B'_z &= B_z + A_z \alpha \\ C'_z &= A_z \alpha^2 + 2B_z \alpha + C_z \end{aligned} \quad (A5)$$

Similarly, the sign is proved to be positive, thus

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{1}{z} \frac{dz}{R_z} = + \frac{1}{\sqrt{C'_z}} \quad (A6)$$

hence

$$\frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{1}{z^2} \frac{dz}{\sqrt{R_z}} = + \frac{B_z}{C_z^{3/2}} \quad (A7)$$

(IV)

$$\begin{aligned} \frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \frac{1}{z} \frac{dz}{\sqrt{R_z}} &= \frac{1}{\pi \alpha} \int_{z_{\min}}^{z_{\max}} \left(\frac{1}{z} - \frac{1}{z_1} \right) \frac{dz}{\sqrt{R_z}} \\ &= \frac{1}{\alpha} \left[\frac{1}{\sqrt{C_z}} - \frac{1}{\sqrt{C'_z}} \right] \quad (A8) \end{aligned}$$

It can be shown that:

(a)

$$B'_z(S, X) = -B_z(-X, -S),$$

(b)

$$C'_z(S, X) = +C_z(-X, -S). \quad (A9)$$

Prove (a):

$$B_z(S, X) = SQ^2(S_x - T) + X(S_x t - Q^2 T) \\ + 2M^2 Q^2 (Q^2 - T).$$

Therefore

$$B'_z(S, X) = B_z(S, X) + (t - Q^2) A_z(S, X) \\ - 2M^2 Q^2 (Q^2 - t) + X Q^2 (S_x - T) \\ + S(S_x t - Q^2 T) \\ = -B_z(-X, -S).$$

Prove (b):

$$C_z = [Xt - Q^2(S - T)]^2 + 4m^2 U,$$

U is unchanged under $S \leftrightarrow -X$

$$C'_z = [(A_z \alpha + B_z) + B_z] \alpha + C_z \\ = [B'_z + B_z] \alpha + C_z \\ = (S + X) [Q^2 (S_x - T) + (S_x t - Q^2 T)] \\ + [Xt - Q^2(S - T)]^2 + 4m^2 U \\ = [St - Q^2(X + T)]^2 + 4m^2 U \\ = +C_z(-X, -S).$$

APPENDIX B

PEAKING APPROXIMATION

The peaking approximation method can be explained by the following example. Consider an integral of an extremely peaked function, the Dirac delta function $\delta(x)$, with an arbitrary function $f(x)$. Using the equality,

$$\int_a^b \delta(x - x_0) f(x) dx = f(x_0) \quad x_0 \in [a, b], \quad (B1)$$

and the normalization

$$\int_a^b \delta(x) dx = 1, \quad (B2)$$

we can see that even the function is replaced by a constant function $f(x_0)$, the result remain the same, i.e.,

$$\begin{aligned} \int_a^b f(x_0) \delta(x - x_0) dx &= f(x_0) \int_a^b \delta(x - x_0) dx \\ &= f(x_0). \end{aligned} \quad (B3)$$

Although $f(x)$ and $f(x_0)$ are completely different functions, as long as the kernel is sufficiently peaked, it makes very little difference. As mentioned in the text, the width of the k_1 and k_2 peaks are given by (see Eq.(106), (107))

$$\Gamma_1 \sim m\sqrt{U}/(S - Q^2)$$

$$\Gamma_2 \sim m\sqrt{U}/(X + Q^2)$$

which are quite narrow for a large part of the kinematic region.

At the k_1 peak, i.e., where $\vec{k}_1 // \vec{k}$, we want to prove that t has a value

$$t_1 = Q^2 (1 - v/X') = Q^2 X_1 \quad (B4)$$

$$X_1 = (1 - v/X')$$

where

$$\begin{aligned} v &= W^2 - M_f^2 = -2q \cdot k \\ &= 2\omega(E_f - |\vec{p}_f| \cos \theta_{fk}) \end{aligned}$$

$$\begin{aligned} X' &= S - Q^2 \approx 2k \cdot p_f \\ &\approx 2E_{k_1} (E_{p_f} - |\vec{p}_f| \cos \theta_{fk_1}) \end{aligned}$$

But $\vec{k}_1 // \vec{k}$, then $\cos \theta_{fk} = \cos \theta_{fk_1}$, hence

$$\frac{v}{X'} \approx \frac{\omega}{E_{k_1}}$$

Prove:

$$\begin{aligned}
 t_1 &= (k_1 - k_2 - k)^2 / z_1 \sim 0 \\
 &\approx (k_1 - k_2)^2 + 2kk_2 \\
 &\approx Q^2 \left(1 - \frac{k_2 k}{k_1 k_2}\right) \\
 &\approx Q^2 \left(1 - \omega / E_{k_1}\right) \\
 &\approx Q^2 \left(1 - v / \chi^2\right)
 \end{aligned}$$

Similarly, at the k_2 peak,

$$t = Q^2 \left(1 + \frac{v}{S}\right) = Q^2 \chi_2,$$

$$\chi_2 = \left(1 + \frac{v}{S}\right).$$

There are several assumptions made in the approximation^[9]:

(i) The terms $1/z_1$, $1/z_1^2$ contribute to the k_1 peak and terms $1/z$, $1/z^2$ contribute to the k_2 peak only.

(ii) The coefficients of $1/z_1$ and $1/z_1^2$ are evaluated at the k_1 peak, i.e., all the t 's appeared in the coefficients are replaced by t_2 while coefficients of $1/z$ and $1/z^2$ are evaluated at the k_2 peak.

(iii) The term $1/zz_1$ has equal contribution to the k_1 and k_2 peak, i.e., one-half will be evaluated at $t=t_1$ and the other half, $t=t_2$.

(iv) All non-logarithmic terms will be ignored and terms like $\log(X'^2/m^2 w^2)$ and $\log(S'^2/m^2 w^2)$ will be replaced by

$lm = \ln Q^2 / m^2$. The approximation that $|\vec{k}_1| \approx E_{k_1}$ and $|\vec{k}_2| \approx E_{k_2}$ will also be used. Under this assumptions, $\mathcal{J}_1$ and $\mathcal{J}_2$ of Eq.(73) when evaluated at the k_1 peak become

$$\mathcal{J}_1 = \frac{-t_1 m^2}{z_1^2} + \frac{t_1 + Q^2}{2z_1} + \frac{t_1^2}{2z_1 z}$$

$$\mathcal{J}_2 = \frac{2m^2}{z_1^2} [M^2 t_1 - X(X + T_1)] - \frac{[M^2(t_1 + Q^2) + ST_1]}{z} \quad (B5)$$

$$+ \frac{t_1}{2z_1 z} [S(S - T_1) + X(X + T_1) - 2M^2 t_1]$$

It was shown that :

(i)

$$J[z_1^{-2}] = J[z^{-2}] = \frac{1}{m^2 v} \quad (B6)$$

(ii)

$$J[z_1^{-1}] = L_X \approx lm/X' ; J[z^{-1}] = L_S \approx \frac{lm}{S'} \quad (B7)$$

(iii)

$$J\left[\frac{1}{z z_1}\right] = \frac{2Lm}{v} \approx \frac{2lm}{v Q^2} \quad (B8)$$

Hence

$$J[\mathcal{J}_1] \approx \frac{t_1}{X_1 v} \left[\frac{1}{2}(1 + X_1^2) lm - X_1 \right]$$

$$J[\mathcal{J}_2] \approx \frac{2}{v X_1} (S_1 X - M^2 t) \left[\frac{1}{2}(1 + X_1^2) lm - X_1 \right] \quad (B9)$$

$$S_1 = S X_1$$

Similarly the $J[\mathcal{S}_1]$ and $J[\mathcal{S}_2]$ evaluated at t_2 and $J[f^{IR}]$ are given by

$$J[\mathcal{S}_1] \cong \frac{t_2}{v} \left[\frac{1}{2} \left(\frac{1}{X_2} + X_2 \right) l_m - 1 \right],$$

$$J[\mathcal{S}_2] \cong \frac{2}{v} [SX - M^2 t_2] \left[\frac{1}{2} \left(\frac{1}{X_2} + X_2 \right) l_m - 1 \right], \quad (B10)$$

$$J[f^{IR}] \cong -2(1 - l_m).$$

Using the above results, $d^2 \sigma_R^F / dQ^2 dw^2$ can be factorized into

$$\frac{d^2 \sigma_R^F}{dQ^2 dw^2} = \int_{(M+M_H)^2}^{W^2} \frac{dM_f^2}{v} \left\{ X_1 t_1 \frac{d^2 \sigma_0}{dQ^2 dw^2}(S_1, t_1, M_f^2) \right. \quad (B11)$$

$$\left. + X_2 t_2 \frac{d^2 \sigma_0}{dQ^2 dw^2}(S, t_2, M_f^2) - 2t_r \frac{d^2 \sigma_0}{dQ^2 dw^2}(S, Q^2, W^2) \right\}$$

where

$$t_r = \frac{\alpha}{\pi} (l_m - 1),$$

(B12)

$$t_1 = \frac{\alpha}{\pi} \left[\frac{1}{2} (1 + X_1^2) l_m - X_1 \right],$$

$$t_2 = \frac{\alpha}{\pi} \left[\frac{1}{2} \left(1 + \frac{1}{X_2^2} \right) l_m - \frac{1}{X_2} \right].$$

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