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Generalized Sequence Spaces and Nuclearity

by

Wasfi Shatanawi

B.Sc. Yarmouk University 1994

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A thesis submitted to
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School of Mathematics and Statistics
Ottawa-Carleton Institute of
Mathematics and Statistics

Carleton University
Ottawa, Ontario
Canada

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acceptance of the thesis

Generalized Sequence Spaces and Nuclearity

submitted by

Wasfi Shatanawi, B.Sc., M.Sc.

in partial fulfilment of the requirements for
the degree of Doctor of Philosophy

Dr. C. Garner, Director
School of Mathematics and Statistics

Dr. C. K. Fong, Thesis Supervisor

Dr. A. Tallafha, External Examiner

Carleton University
Dec. 4, 2001

DEDUCTION

TO MY PARENTS, WIFE AND MY SON

Abstract

In this thesis we extend some work on nuclearity of smooth sequence spaces. Also, we give a generalization of the well-known result which says that the composition of two quasi-nuclear maps is nuclear by proving that the composition of a quasi-nuclear map with a 2-quasi-nuclear map is nuclear. Then we use sequence spaces to give result better than this result. Also, we give an improvement of Dubinsky and Ramanujan's result and Robinson's result by proving the following theorem: given a nuclear Köthe space $\lambda = \lambda(A)$, with the property that $\lambda(A)^\times$ is nuclear, a linear bounded map T from a Banach space E into a Banach space F is λ -nuclear if and only if it is pseudo- $\lambda\lambda^\times$ -nuclear. Finally, we improve Terzioğlu's decomposition theorem.

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Introduction

The study of nuclear spaces was largely motivated by the theory of distributions founded by Sobolev, L. Schwartz etc., which stimulated the development of general theory of topological vector spaces. It was through Grothendieck's pre found work on tensor products and absolute summing maps to establish the important role of nuclearity in Banach space theory. On the other hand, J. Schauder introduced the important notion of bases, causing such strong impact that many prominent Banach space theorists such as J. Lindenstrauss, A. Pełczyński, and H. Rosenthal took the study of Schauder bases as one of their main occupation. In some sense, a Banach space with a fixed base is equivalent to a sequence space. Thus the attraction of sequence spaces to a large number of Annalists is understandable, and among the pioneer workers in this area we should mention G. Köthe. An important class of sequence spaces is named after him : Köthe spaces. The interplay between bases theory for Banach spaces and the nuclearity naturally leads to the consideration of nuclear sequence spaces, which attracts many researchers to work in this area, e.g. T. Terzioğlu [35] introduced the notion of G_1 and G_∞ spaces and established their nuclearity criterion, E. Dubinsky and M. S. Ramanujan [13] studied λ -nuclearity, where λ is a sequence space. Earlier, Komoura-Komoura [22] showed that a locally convex space

E is nuclear if and only if there exists a set I such that E is isomorphic to a subspace of the product space S^I . Moreover, if E is Fréchet, then we may choose I to be $\mathbb{N}$. This clearly indicates the role of nuclear sequence spaces in the general theory of nuclear spaces.

In what follows we give a brief outline of the present thesis, describing the content of each chapter and indicating its relevance to the current research. In Chapter 1, we give the necessary definitions, facts, and material about locally convex spaces. For the standard theory of locally convex spaces we refer to [23], and for the theory of nuclear Fréchet spaces we refer to [11] and [27].

In Chapter 2, we extend some work on nuclearity of smooth sequence spaces. More precisely, we give a characterization of nuclearity of the sequence spaces of the form $\lambda(A) \vee \lambda(B)$, where $\lambda(A)$ and $\lambda(B)$ are smooth sequence spaces of the same type. Then we use our results to give a characterization of nuclearity of smooth sequence spaces.

In Chapter 3, we define p -quasi-nuclear maps between normed spaces E and F and we give a generalization of the well-known result which says that the composition of two quasi-nuclear maps is nuclear. More precisely we prove that the composition of a quasi-nuclear map with a 2-quasi-nuclear map is nuclear.

In Chapter 4, we use sequence spaces to give an improvement of our result in Chapter 3. We define a 2-quasi- λ -nuclear map, where λ is a sequence space which is assumed to be a subset of ℓ_1 , also we prove that every quasi- λ -nuclear map is 2-quasi- λ -nuclear. In this chapter, we prove that the composition of a quasi λ -nuclear map with a 2-quasi- λ -nuclear map is a

pseudo- λ -nuclear map. In case $\lambda = \ell_1$, we get our previous result in Chapter 3.

In [13], E. Dubinsky and M. S. Ramanujan defined $\lambda(P)$ -nuclear spaces, where the power set P is assumed to be countable, while W. Robinson in [32] studied the $\Lambda_1(\alpha)$ -nuclearity corresponding to a nuclear series space $\Lambda_1(\alpha)$ of finite type. In Chapter 5, we give an improvement of Robinson's result which says that a bounded linear map between Banach spaces is $\Lambda_1(\alpha)$ -nuclear if and only if it is a pseudo- $\Lambda_1(\alpha)^\times$ -nuclear. Also in [13], E. Dubinsky and M. S. Ramanujan proved that every λ -nuclear map from a Banach space into a reflexive Banach space is compact. We improve this result by proving that if T is a λ -nuclear map from a Banach space E into a Banach space F and if F contains no (isomorphic) copy of c_0 , then T is compact. We close this chapter by giving a characterization of a pseudo- $\lambda(P_0)$ -nuclearity for a sequence space of the form $\lambda_1 \vee \lambda_2$, where $\lambda(P_0)$ is assumed to be a stable G_∞ -space and λ_1, λ_2 are smooth sequence spaces of the same type.

In [3], A. Aytune, J. Krone and T. Terzioğlu defined what they called a local imbedding and they proved that if there is a local imbedding i from a nuclear stable G_∞ -space $\lambda(A)$ into $\lambda(A)^N$, then there is a complemented subspace G of $\lambda(A)^N$ which is isomorphic to $\lambda(A)$ and contained in $\overline{i(\lambda(A))}$. In [38], T. Terzioğlu generalized the result obtained by Aytune, Krone and Terzioğlu for spaces not necessarily metrizable. In Chapter 6, we define a stable local imbedding map i from a sequence space of the form $\lambda(A) \vee \lambda(B)$ into a locally convex space E , where $\lambda(A)$ and $\lambda(B)$ are nuclear G_∞ -spaces. Also, we prove that if there is a stable local imbedding map i from the sequence space of the form $\lambda(A) \vee \lambda(A)$ into E and if E is isomorphic to a closed subspace of $\lambda(A)$, then E has a complemented subspace isomorphic to $\lambda(A)$. We then have use

this result to give an improvement of Terzioğlu's result which states that, if i is a local imbedding map from a stable nuclear G_∞ -space $\lambda(A)$ into a locally convex space E , and if E is isomorphic to a closed subspace of $\lambda(A)$, then E has a complemented subspace isomorphic to $\lambda(A)$. Finally we close this chapter by proving that our result is an improvement of Tallafha's result.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we are going to give the necessary definitions and facts that will be used in later chapters. For the standard theory of a locally convex spaces we refer to [23]. For the theory of nuclear Frèchet spaces we refer to [11] and [27].

1.2 Semi-norms

Definition 1.2.1 A semi-norm on a linear space E over K , where ($K=\mathbf{R}$ or $\mathbf{C}$) is a real valued function p with the following properties :

1. $p(x) \geq 0 \ \forall x \in E$.
2. $p(x + y) \leq p(x) + p(y)$.
3. $p(\alpha x) = |\alpha|p(x)$ for $\alpha \in K$ and $x \in E$.

Definition 1.2.2 Let A be a subset of a linear space E . We say that A is **absolutely convex**, if $\alpha x + \beta y \in A$ for all $x, y \in A$ and $\alpha, \beta \in K$ with $|\alpha| + |\beta| \leq 1$. If A has the property that, for each element $x \in E$, there is a positive number λ with $x \in \lambda A = \{\lambda a : a \in A\}$, then A is said to be **absorbing**. Finally, we say that A is **central** if it is absorbing and absolutely convex subset of E . Notice that for a central set A , $\alpha A \subseteq A$ whenever $\alpha \in K$ with $|\alpha| \leq 1$.

1.3 Locally Convex Spaces

Definition 1.3.1 Let $\mathcal{U}_\tau(E)$ be a system of central subsets of a real or complex linear space E , which has the following properties :

1. For finitely many sets $U_1, \dots, U_n \in \mathcal{U}_\tau(E)$ there is a set $U \in \mathcal{U}_\tau(E)$ with $U \subseteq \bigcap_{i=1}^n U_i$.
2. For each set $U \in \mathcal{U}_\tau(E)$ there exists a set $V \in \mathcal{U}_\tau(E)$ with $V + V \subseteq U$.
3. For each element $x_0 \neq 0$ of E , there exists $U_0 \in \mathcal{U}_\tau(E)$ with $x_0 \notin U_0$.

The collection $\{x + U : U \in \mathcal{U}_\tau(E)\}$ form a fundamental neighbourhood system of an element $x \in E$. Each linear space E is made into a topological Hausdorff space in this way and becomes a **locally convex space**.

Definition 1.3.2 A semi-norm p defined on a locally convex space E is **fundamental** if it has the form $p_u(x) = \inf\{\lambda > 0 : x \in \lambda U\}$, where $U \in \mathcal{U}(E)$ is a central neighbourhood of zero.

It is easy to see that fundamental semi-norms are continuous.

The topology on E is thus determined by the collection of fundamental semi-norms P_u . A locally convex topology on a linear space E can also be constructed by means of a system $\beta_\tau(E)$ of fundamental semi-norms with the following properties :

1. For finitely many semi-norms $p_1, \dots, p_n \in \beta_\tau(E)$ there is a semi-norm $p \in \beta_\tau(E)$ with $p_r(x) \leq p(x)$ for $x \in E$ and $r = 1, 2, \dots, n$.
2. For each element $x_0 \neq 0$ of E there is a semi-norm $p_0 \in \beta_\tau(E)$ with $p_0(x_0) > 0$.

The sets $\varepsilon U_p = \{x \in E : p(x) < \varepsilon\}$ for $p \in \beta_\tau(E)$ and $\varepsilon > 0$ form a fundamental system $\mathcal{U}_\tau(E)$ of central zero neighbourhoods which has the properties described in definition 1.3.1.

Definition 1.3.3 *A linear subspace F of a locally convex space E is also a locally convex space. The topology on F can be constructed either by restricting the continuous semi-norm on E to F or from the zero neighbourhoods $U_0 = F \cap U$ with $U \in \mathcal{U}(E)$.*

Definition 1.3.4 *Let F be a closed linear subspace of a locally convex space E . A locally convex topology on the quotient space $Q = E/F$ is obtained from the semi-norms $p_U(x(F)) = \inf\{p_U(x + y) : y \in F\}$, $U \in \mathcal{U}(E)$.*

Definition 1.3.5 *A barrel in a locally convex spaces E is a subset which is central and closed. A linear space E is **barrelled** if each barrel in E is a neighbourhood of 0.*

1.4 Frèchet Spaces With Basis

A **Frèchet space** is a complete metrizable locally convex space. All subspaces of a topological vector space E will be assumed to be closed and

infinite dimensional unless otherwise stated. By a complemented subspace of E we mean a subspace F of E such that there exists another subspace G of E with $F \cap G = \{0\}$ and $F + G = E$.

A linear form a on a linear space E over K is a mapping which determines for each element $x \in E$ a number $\langle x, a \rangle \in K$ such that

$$\langle \alpha x + \beta y, a \rangle = \alpha \langle x, a \rangle + \beta \langle y, a \rangle.$$

The linear combination $\alpha a + \beta b$ of two linear forms a and b is defined by the equation

$$\langle x, \alpha a + \beta b \rangle = \alpha \langle x, a \rangle + \beta \langle x, b \rangle$$

for $x \in E$. The collection of all linear forms on E becomes a linear space E^* , and is called the algebraic dual of E .

The topological dual of a locally convex space E is the linear subspace E' of the algebraic dual E^* which consists of all continuous linear forms.

A subset B of a locally convex space E is called bounded if B is absorbed by every neighbourhood of zero. By $\beta(E)$ we denote the collection of all closed and absolutely convex bounded subsets of E .

The **strong topology** on E' is obtained from the system of semi-norms

$$p'_B(a) = \sup \{ |\langle x, a \rangle| : x \in B \}$$

with $B \in \beta(E)$. The locally convex space thus obtained is denoted by E'_b .

The **weak topology** on E' is obtained from the system of semi-norms

$$p_A(a) = \sup \{ |\langle x, a \rangle| : x \in A \}$$

where A ranges over all finite subsets of E . The locally convex space thus obtained is denoted by E'_s .

A **basis** in a locally convex space E is a sequence (x_n) in E such that every $x \in E$ determines a unique sequence (η_n) in K such that $x = \sum_n \eta_n x_n$ which is convergent with respect to the topology of E . By setting $\langle x, u_n \rangle = \eta_n$, we obtain a linear form on E . The linear form u_n are uniquely determined by the basis (x_n) . A basis (x_n) is called an **absolute basis** if for every continuous semi-norm p on E there is a continuous semi-norm q such that $\sum_n |\langle x, u_n \rangle| p(x_n) \leq q(x)$ for all $x \in E$.

1.5 Continuous Linear Maps

Definition 1.5.1 Let E and F be two arbitrary real or complex locally convex spaces. A map T from E into F is called a **linear map** if for all $x, y \in E$ and $\alpha, \beta \in K$ we have

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Definition 1.5.2 A linear map T from E into F is **continuous** if for each zero neighbourhood $V \in \mathcal{U}(F)$ there is a zero neighbourhood $U \in \mathcal{U}(E)$ with $T(U) \subseteq V$.

An **isomorphism** between two locally convex spaces E and F is a linear continuous bijection with a continuous inverse. By $E \cong F$ we mean E is isomorphic to F . A locally convex space E is said to be **stable** if $E \cong E \times E$. A linear map T from E into F is called an **imbedding** if T is injective and when considered as a map from E onto $T(E)$, T is an isomorphism. A linear map T from E into E is called a **projection** if $T^2 = T$.

A semi-norm p_U on a real or complex linear space E is said to be norm if $p_U(x) > 0$ for $x \neq 0$. The sets εU with $\varepsilon > 0$ and $U = \{x \in E : p_U(x) \leq 1\}$ form a fundamental system of zero neighbourhoods of E . A locally convex space E so constructed is called a normed space and U is called the closed unit ball of E . A **Banach space** is a complete normed space.

Definition 1.5.3 A linear map T from a normed space E into a normed space F is said to be **compact** if there is a compact subset K of F with $T(U) \subseteq K$, where U is the closed unit ball of E .

Proposition 1.5.1 Suppose that X , Y , and Z are Banach spaces, S is a bounded linear map from X into Y and T is a bounded linear map from Y into Z . If either S or T is compact, then TS is compact.

Definition 1.5.4 The dual map (or the adjoint map) T' of a linear map T from a normed space E into a normed space F is a linear map from F' into E' whose value at the linear form $b \in F'$ is the linear form $a \in E'$ defined by the equation $\langle x, a \rangle = \langle Tx, b \rangle$ for $x \in E$.

Theorem 1.5.1 (Schauder's Theorem) A bounded linear map between Banach spaces is compact if and only if its adjoint is compact.

Let U be an arbitrary closed, absolutely convex zero neighbourhood of the locally convex space E . Put $N(U) = \{x \in E : p_U(x) = 0\}$. Then we define the quotient space $E(U) = E/N(U) = \{x(U) : x \in E\}$, where $x(U)$ denotes the equivalence class corresponding to the element $x \in E$. The topology on $E(U)$ is defined by the norm $p(x(U)) = p_U(x)$ for $x \in E$.

Definition 1.5.5 Let U, V be two arbitrary zero neighbourhoods with V absorbed by U . Then we define the **canonical map** $E(V, U)$ from $E(V)$ into $E(U)$ by $E(V, U)(x(V)) = x(U)$.

1.6 The Diametral Dimension

For two normed spaces E and F and for integers $r \geq 0$, $\mathcal{A}_r(E, F)$ denotes the collection of all finite rank linear maps A from E into F whose range is at most r -dimensional.

Definition 1.6.1 [27; P. 120] *Let T be a linear map from a normed space E into a normed space F . The r -th approximation number $\alpha_r(T)$ of T is defined to be*

$$\inf\{\|T - A\| : A \in \mathcal{A}_r(E, F)\}.$$

Definition 1.6.2 [27; P. 144] *Let B be an arbitrary bounded subset in a normed space E with closed unit ball U . The infimum of all $\delta > 0$ for which there is a linear subspace F of E with dimension at most n such that $B \subset \delta U + F$ is called the n -th diameter of B and is denoted by $d_n(B)$.*

It is clear that $d_0(B) \geq d_1(B) \geq \dots \geq 0$.

Definition 1.6.3 [33] *Let $T : E \rightarrow F$ be a bounded linear map between normed spaces E and F with closed unit balls U and V respectively. The n -th diameter of T , denoted by $d_n(T)$, is defined to be $d_n(T(U))$.*

Definition 1.6.4 [27; P.149] *Let U, V be two arbitrary zero neighbourhoods in a locally convex space E with V absorbed by U . The infimum $d_n(V, U)$ of all positive numbers δ for which there is a linear subspace F of E of dimension at most n with $V \subset \delta U + F$ is called the n -th diameter of V with respect to U .*

Definition 1.6.5 [39] *Let E be a locally convex space and $\mathcal{U}(E)$ a basis of zero neighbourhoods consisting of absolutely convex sets. By the **diametral***

dimension of E we mean the set of all sequences (t_n) such that for each U in $\mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ with $\lim_{n \rightarrow \infty} t_n d_n(V, U) = 0$, where $d_n(V, U)$ is the n -th diameter of V with respect to U . The diametral dimension of E is denoted by $\Delta(E)$.

1.7 Nuclear and Quasinuclear Maps

Definition 1.7.1 [26] A bounded linear map T from a normed space E into a normed space F is said to be **absolutely p -summing**, where $p > 0$ if there is a positive constant c such that for an arbitrary sequence (x_n) in E we have

$$\left(\sum_n \|Tx_n\|^p \right)^{1/p} \leq c \sup \left\{ \left(\sum_n |\langle x_n, a \rangle|^p \right)^{1/p} : a \in E', \|a\| \leq 1 \right\}.$$

Definition 1.7.2 [26] A linear map T from a Hilbert space H_1 into a Hilbert space H_2 is called a **Hilbert Schmidt map** if $\sum_{i \in I} \|Te_i\|^2 < +\infty$ for some (equivalently, for each) orthonormal basis $(e_i)_{i \in I}$.

The class of all Hilbert-Schmidt operators from H_1 into H_2 (resp. of all absolutely p -summing operators) will be denoted by $\mathcal{C}_2(H_1, H_2)$ (resp. $\Pi_p(H_1, H_2)$).

Theorem 1.7.1 [26] (Pelczynski's Theorem) For $1 \leq p < +\infty$ we have

$$\mathcal{C}_2(H_1, H_2) = \prod_p(H_1, H_2).$$

Definition 1.7.3 [27; P. 49] Let E and F be two arbitrary normed spaces. A linear map T from E into F is called a **nuclear map** if there are sequences (a_n) , (y_n) in E' and F respectively with

$$\sum_n \|a_n\| \|y_n\| < +\infty$$

such that T has the form

$$T(x) = \sum_n \langle x, a_n \rangle y_n.$$

A locally convex space E is said to be **nuclear** if for every absolutely convex and closed neighbourhood U there is another such neighbourhood V absorbed by U such that the canonical mapping of the associated Banach space $\tilde{E}(V)$ into the associated Banach space $\tilde{E}(U)$ is nuclear where $\tilde{E}(U)$ is the completion of $E(U)$.

Definition 1.7.4 [27; P. 56] *Let E and F be two arbitrary normed spaces. A linear map T from E into F is called a **quasi-nuclear** map if there is a sequence (a_n) in E' with*

$$\sum_n \|a_n\| < +\infty$$

such that

$$\|T(x)\| \leq \sum_n |\langle x, a_n \rangle|.$$

Proposition 1.7.1 [27; P. 85] *Each linear subspace of a nuclear locally convex space is nuclear.*

Proposition 1.7.2 [27; P. 86] *Each quotient space $Q = E/F$ of a nuclear locally convex space E by a closed linear subspace F is nuclear.*

Proposition 1.7.3 [27; P. 90] *The topological product $\prod_{i \in E} E_i$ of an arbitrarily many nuclear locally convex spaces E_i is nuclear.*

Chapter 2

Köthe Spaces

2.1 Introduction

Smooth sequence spaces were introduced by Terzioğlu in [36] nearly forty years ago. In [36] and in [37], Terzioğlu gave a characterization of nuclearity of smooth sequence spaces. In the present chapter, we give the definitions and some important properties of Köthe spaces and nuclear Köthe spaces, and a characterization of the nuclearity of sequence spaces of the form $\lambda(A) \vee \lambda(B)$ as well, where $\lambda(A)$ and $\lambda(B)$ are smooth sequences spaces of the same type. Since each Köthe space $\lambda(A)$ can be written as a sequence space of the form $\lambda(A_1) \vee \lambda(A_2)$, where $A_1 = \{(a_{2n-1}) : a \in A\}$ and $A_2 = \{(a_{2n}) : a \in A\}$, our results are improvements of Terzioğlu's results.

2.2 Sequence Spaces

A sequence space is a vector space of sequences of scalars (real or complex) which includes every finitely non-zero sequence. If A is a set of sequences of

scalars, then the **normal hull** of A is the set A^N of sequences ζ such that there is $\eta \in A$ with $|\zeta_n| \leq |\eta_n|$ for all $n \in \mathbb{N}$. A sequence space is **normal** if it equals its **normal hull**. If λ is a sequence space then its **Köthe dual** is the sequence space

$$\lambda^\times = \{\zeta : \sum_n |\eta_n \zeta_n| < +\infty \quad \forall \eta \in \lambda\}.$$

If η, ζ are two sequences of scalars, we denote by $\eta\zeta$ the sequence whose n -th coordinate is $\eta_n \zeta_n$. We write $\zeta = O(\eta)$ if there exists $M > 0$ such that $|\zeta_n| \leq M |\eta_n|$ for all $n \in \mathbb{N}$.

The n -th coordinate sequence is the sequence e_n whose n -th term is 1 and all other terms are 0. We denote by ℓ_p , $0 < p < +\infty$, the space of sequences η such that $\sum_n |\eta_n|^p < +\infty$, by ℓ_∞ the space of all bounded sequences.

Definition 2.2.1 [23] *A set A of sequences of non-negative real numbers is called a **Köthe set**, if it satisfies the following conditions:*

1. *For each pair of elements $a, b \in A$ there is $c \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.*
2. *For every integer $r \in \mathbb{N}$ there exists $a \in A$ with $a_r > 0$.*

Given a Köthe set A , the space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n| a_n < +\infty$$

for all $a \in A$ is called the **Köthe space** generated by A , denoted by $\lambda(A)$. The semi-norms $\{p_a : a \in A\}$ define a locally convex Hausdorff topology on $\lambda(A)$, called the **normal topology** of $\lambda(A)$.

Definition 2.2.2 (Robinson [33; P. 294]) *Given sequence spaces λ_1 and λ_2 we define the sequence space $\lambda_1 \vee \lambda_2$ to be $\{(x_n): (x_{2n}) \in \lambda_1, (x_{2n-1}) \in \lambda_2\}$.*

Proposition 2.2.1 *If $\lambda(A)$ and $\lambda(B)$ are Köthe spaces, then $\lambda(B) \vee \lambda(A)$ is a Köthe space generated by the Köthe set*

$$(A, B) = \{(a_1, b_1, a_2, b_2, \dots): (b_n) \in B, (a_n) \in A\}.$$

Definition 2.2.3 [36] *A Köthe set P will be called a **power set of infinite type** if it satisfies the following conditions:*

1. *For each $a \in P$, $0 < a_n \leq a_{n+1}$ for all n .*
2. *For each $a \in P$, there exists $b \in P$ such that $a_n^2 = O(b_n)$.*

*A Köthe space of the form $\lambda(P)$ where P is a power set of infinite type is called a **G_∞ -space** or a **smooth sequence space of infinite type**.*

Remark. The word “smooth” here is suggested by considering Taylor coefficients of smooth functions in some function space. Condition 2 in Definition 2.2.3 can be replaced by the following condition: given $a, b \in P$ there is $c \in P$ such that $a_n b_n = O(c_n)$.

Definition 2.2.4 [36] *A Köthe set Q will be called a **power set of finite type** if it satisfies the following conditions:*

1. *Each $q \in Q$ is a positive non-increasing sequence.*
2. *For each $q \in Q$ there exists $p \in Q$ with $\sqrt{q_n} = O(p_n)$.*

*A Köthe space of the form $\lambda(Q)$ where Q is a power set of finite type is called a **G_1 -space** or a **smooth sequence space of finite type**.*

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then

$$P_\infty = \{(k^{\alpha_n}) : k \in \mathbf{N}\} \quad \text{and} \quad P_1 = \{((1 - 1/k)^{\alpha_n}) : k \in \mathbf{N}\}$$

are countable Köthe sets. The corresponding Köthe spaces $\Lambda_1(\alpha) = \lambda(P_1)$ and $\Lambda_\infty(\alpha) = \lambda(P_\infty)$ are called the **power series of finite** and **infinite type** respectively.

The space s of **rapidly decreasing sequence** is a G_∞ -space which is generated by

$$A = \{(n^k) : k = 1, 2, 3, \dots\}.$$

Theorem 2.2.1 (Grothendieck-Pietsch criterion for nuclearity) [37] *A Köthe space $\lambda(A)$ is nuclear if and only if for every $a \in A$, there is $b \in A$ such that $(a_n/b_n) \in \ell_1$.*

Theorem 2.2.2 (Komura-Komura)[22] *A locally convex space E is nuclear if and only if there exists a set I such that E is isomorphic to a subspace of the product s^I . Moreover, if E is Fréchet, then we may choose I to be $\mathbf{N}$.*

The topological dual $\lambda(A)'$ of a Köthe space $\lambda(A)$ is isomorphic to the space of all sequences $u = (u_n)$ for which $u_n = O(a_n)$ for some $a \in A$ [37].

The stability of a nuclear G_∞ -space $\lambda(P)$ is equivalent to the condition that, for each $a \in P$ there is $b \in P$ such that $a_{2n} = O(b_n)$, and the stability of a nuclear G_1 -space is equivalent to the condition that for each $q \in Q$ there is $p \in Q$ such that $q_n = O(p_{2n})$ [39].

The next proposition is elementary and hence its proof is omitted.

Proposition 2.2.2 *If A and B be Köthe sets, then we have the following:*

1. If $A \subseteq B$ then $\lambda(B) \subseteq \lambda(A)$.
2. If $A \subseteq B$ then $\lambda(A)' \subseteq \lambda(B)'$.
3. If $A \subseteq \lambda(B)$ then $\lambda(B)' \subseteq \lambda(A)$.
4. If $A \subseteq \lambda(B)'$ then $\lambda(B) \subseteq \lambda(A)$.
5. $A \subseteq \lambda(A)'$.

2.3 Properties of Köthe Spaces

The following two elementary lemmas will be needed to prove some theorems in the present thesis. We only present the proof of the first one.

Lemma 2.3.1 *Let $\lambda(Q)$ be a smooth sequence space of finite type and suppose that $x, y \in \lambda(Q)$. Then $xy \in \lambda(Q)$.*

Proof. Let $x, y \in \lambda(Q)$ be given, and let $b \in Q$. Then there exist $a \in Q$ and $\gamma > 0$ such that $b_n \leq \gamma a_n^2$ for all $n \in \mathbb{N}$. So

$$\sum_n |x_n| |y_n| b_n \leq \gamma \sum_n |x_n| |y_n| a_n^2.$$

Put $\beta = \sum_n |x_n| a_n$. Then

$$\sum_n |x_n| |y_n| b_n \leq \beta \gamma \sum_n |y_n| a_n < +\infty.$$

Therefore $xy \in \lambda(Q)$. ■

Lemma 2.3.2 *If x is an element of a G_∞ -space $\lambda(P)$ and if $a \in P$, then $xa \in \lambda(P)$.*

In general, a locally convex space with a basis can be regarded as a sequence space. More specifically we have

Proposition 2.3.1 [27] *Every Fréchet space E with an absolute basis is isomorphic to a Köthe space.*

2.4 Nuclearity of Köthe Spaces

We introduce the following definition in order to facilitate our subsequent arguments.

Definition 2.4.1 *If A is a set of sequences, then we define the set $B(A)$ by*

$$B(A) = \{x: x a \in \ell_\infty \ \forall \ a \in A\}.$$

Remark. Clearly if A is a Köthe set, then $\lambda(A) \subseteq B(A)$ but the converse in general is not true. However, we have:

Proposition 2.4.1 *If $\lambda(P)$ is a nuclear Köthe space, then $B(P) = \lambda(P)$.*

Proof. Let $x \in B(P)$ be given. For $a \in P$, we choose $b \in P$ such that $(a_n/b_n) \in \ell_1$. Since $x \in B(P)$ and $b \in P$, there is $\alpha > 0$ such that $|x_n| b_n \leq \alpha$ for each n . Let

$$N_a = \{n: a_n \neq 0\} \quad \text{and} \quad N_b = \{n: b_n \neq 0\}.$$

Then $N_a \subseteq N_b$. So

$$\begin{aligned} \sum_n |x_n| a_n &= \sum_{n \in N_a} |x_n| a_n \\ &\leq \sum_{n \in N_b} |x_n| a_n \\ &= \sum_{n \in N_b} |x_n| b_n \frac{a_n}{b_n} \\ &\leq \alpha \sum_{n \in N_b} \frac{a_n}{b_n} < +\infty. \end{aligned}$$

Therefore $x \in \lambda(P)$. ■

The following proposition will be needed for proving Theorem 5.3.4.

Proposition 2.4.2 *Suppose that $\lambda(P)$ is a nuclear Köthe space. If B is a bounded subset of $\lambda(P)$, then there exists a positive element $x \in \lambda(P)$ such that $\sup_{y \in B} |y_n| \leq x_n$ for all $n \in \mathbf{N}$.*

Proof. Let $x_n = \sup_{y \in B} |y_n|$. We claim that $x \in \lambda(P)$. For $b \in P$ there is $c > 0$ such that $B \subseteq cU_b$, here, $U_b = \{x \in \lambda(A) : p_b(x) \leq 1\}$, where p_b is the semi-norm given in Definition 2.2.1. For $y \in B$ we have

$$\sum_n |y_n| b_n \leq c.$$

So $|y_n| b_n \leq c$ for each $n \in \mathbf{N}$ and each $y \in B$. Hence $\sup_{y \in B} |y_n| b_n \leq c$ for each n which gives $b_n x_n \leq c$ for each n and for each $b \in P$. Therefore $x \in B(P) = \lambda(P)$. ■

2.5 Nuclearity of G_∞ -Spaces

In this section we give a characterization of nuclearity of Köthe spaces of the form $\lambda(B) \vee \lambda(A)$, where $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces. Then we use our result to obtain an improvement of a characterization of nuclear G_∞ -spaces.

Proposition 2.5.1 *A sequence space of the form $\lambda(B) \vee \lambda(A)$, where $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces, is nuclear if and only if there is $c = (a_1, b_1, \dots)$ in (A, B) such that $(1/c_n) \in \ell_1$.*

Proof ($\Rightarrow$) Assume that $\lambda(B) \vee \lambda(A) = \lambda(A, B)$ is nuclear. Then for $c = (a_1, b_1, a_2, b_2, \dots) \in (A, B)$, there is $c' = (a'_1, b'_1, a'_2, b'_2, \dots) \in (A, B)$ such that $(c_n/c'_n) \in \ell_1$. Let $t = \min\{a_1, b_1\}$. Then $t \leq c_n$ for all $n \in \mathbf{N}$. So

$$\frac{t}{c'_n} \leq \frac{c_n}{c'_n}$$

for all $n \in \mathbf{N}$. Therefore $(1/c'_n) \in \ell_1$.

($\Leftarrow$) Assume that there is $(c_n) = (a_1, b_1, a_2, b_2, \dots) \in (A, B)$ such that $(1/c_n) \in \ell_1$. Let $(c'_n) = (a'_1, b'_1, a'_2, b'_2, \dots) \in (A, B)$ be given. Then we choose $(c''_n) = (a''_1, b''_1, a''_2, b''_2, \dots) \in (A, B)$ such that $c'_n c_n = O(c''_n)$. Therefore there is $\alpha > 0$ such that $c_n c'_n \leq \alpha c''_n$ for all $n \in \mathbb{N}$. So

$$\frac{1}{c_n c'_n} \geq \frac{1}{\alpha c''_n}$$

for all $n \in \mathbb{N}$. Hence

$$\frac{c'_n}{\alpha c''_n} \leq \frac{1}{c_n}$$

for all $n \in \mathbb{N}$. Therefore $(c'_n/c''_n) \in \ell_1$. So $\lambda(B) \vee \lambda(A)$ is nuclear. ■

The following corollary is a consequence of our proposition.

Corollary 2.5.1 *A G_∞ -space $\lambda(P)$ is nuclear if and only if there is an $a \in P$ such that $(1/a_n) \in \ell_1$.*

Proof. Let $P_1 = \{(p_{2n-1}) : (p_n) \in P\}$ and $P_2 = \{(p_{2n}) : (p_n) \in P\}$. Then $\lambda(P) = \lambda(P_1, P_2) = \lambda(P_2) \vee \lambda(P_1)$. ■

Now, we give another characterization of the nuclearity of sequence spaces of the form $\lambda(B) \vee \lambda(A)$, where $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces.

Proposition 2.5.2 *A sequence space of the form $\lambda(B) \vee \lambda(A)$, where $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces, is nuclear if and only if for every $k \in \mathbb{N}$ there exist $b \in (A, B)$ and $\rho > 0$ such that $n^k \leq \rho b_n$ for all $n \in \mathbb{N}$.*

Proof ($\Rightarrow$) Assume that $\lambda(B) \vee \lambda(A)$ is nuclear. Then by Proposition 2.5.1, there is $c = (a_1, b_1, a_2, b_2, \dots) \in (A, B)$ such that $(1/c_n) \in \ell_1$. For each $n \in \mathbb{N}$, we have two cases. The first case, if n is even, then $n = 2t$ for some $t \in \mathbb{N}$. So

$$n \frac{1}{c_n} = 2t \frac{1}{c_{2t}} = 2t \frac{1}{b_t}.$$

Since $\lambda(B)$ is a G_∞ -space, we have

$$n \frac{1}{c_n} \leq 2 \sum_{i=1}^t \frac{1}{b_i} \leq 2 \sum_{i=1}^\infty \frac{1}{c_i}.$$

The second case, if n is odd, then $n = 2t - 1$ for some $t \in \mathbb{N}$. So, we have

$$n \frac{1}{c_n} = (2t - 1) \frac{1}{c_{2t-1}} \leq 2t \frac{1}{c_{2t-1}} = 2t \frac{1}{a_t}.$$

Since $\lambda(A)$ is a G_∞ -space, we have

$$n \frac{1}{c_n} \leq 2 \sum_{i=1}^t \frac{1}{a_i} \leq 2 \sum_{i=1}^\infty \frac{1}{a_i} \leq 2 \sum_{i=1}^\infty \frac{1}{c_i}.$$

in both cases, we put $\beta = \sum_{i=1}^\infty 1/c_i$. Then we have

$$n \frac{1}{c_n} \leq 2\beta$$

for all $n \in \mathbb{N}$. Hence $n \leq 2\beta c_n$ for all $n \in \mathbb{N}$. Therefore, for $k \in \mathbb{N}$, we have

$$n^k \leq n^{2^k} \leq (2\beta)^{2^k} (c_n)^{2^k}$$

for all $n \in \mathbb{N}$. Choose $d \in (A, B)$ such that $(c_n)^{2^k} = O(d_n)$. Then there is $\gamma > 0$ such that $n^k \leq \gamma d_n$ for all n .

($\Leftarrow$) Letting $k = 2$, we see that there are $b \in (A, B)$ and $\rho > 0$ such that $n^2 \leq \rho b_n$ for all $n \in \mathbb{N}$. Therefore $(1/b_n) \in \ell_1$. ■

Corollary 2.5.2 *A G_∞ -space $\lambda(P)$ is nuclear if and only if for every $k \in \mathbb{N}$ there exist $b \in P$ and $\rho > 0$ such that $n^k \leq \rho b_n$ for all $n \in \mathbb{N}$.*

Corollary 2.5.3 *If $\lambda(A)$ and $\lambda(B)$ are nuclear G_∞ -spaces, then for each $x \in \lambda(B) \vee \lambda(A)$ and $k \in \mathbb{N}$ we have $(n^k x_n) \in \lambda(B) \vee \lambda(A)$.*

Proof. Let $x \in \lambda(B) \vee \lambda(A)$ and $k \in \mathbf{N}$ be given. Since $\lambda(B) \vee \lambda(A)$ is nuclear, by Proposition 2.5.2, there are $b \in (A, B)$ and $\rho > 0$ such that

$$n^k \leq \rho b_n$$

for all $n \in \mathbf{N}$. For $c \in (A, B)$, we have

$$\sum_{n=1}^{\infty} |x_n| n^k c_n \leq \rho \sum_{n=1}^{\infty} b_n c_n |x_n|.$$

Since b and c are in (A, B) , there are $d \in P$ and $\alpha > 0$ such that $b_n c_n \leq \alpha d_n$ for all $n \in \mathbf{N}$. Therefore

$$\sum_{n=1}^{\infty} |x_n| n^k c_n \leq \rho \alpha \sum_{n=1}^{\infty} |x_n| d_n < +\infty.$$

Hence $(n^k x_n) \in \lambda(B) \vee \lambda(A)$. ■

Corollary 2.5.4 *If $\lambda(A)$ is a nuclear G_{∞} -space, then for each $x \in \lambda(A)$ and $k \in \mathbf{N}$ we have $(n^k x_n) \in \lambda(A)$.*

Proposition 2.5.3 *If $\lambda(B) \vee \lambda(A)$ is a nuclear space, where $\lambda(A)$ and $\lambda(B)$ are G_{∞} -spaces, then*

1. $\lambda(B) \vee \lambda(A) \subseteq s \subseteq \ell_1$, and
2. for each $a, b \in (A, B)$ and $k \in \mathbf{N}$ there is $c \in (A, B)$ such that $a_n b_n = O(c_n/n^k)$.

Proof. Let $k \in \mathbf{N}$ be given. Then by Proposition 2.5.2, there are $c \in (A, B)$ and $\rho > 0$ such that $n^k \leq \rho c_n$ for all $n \in \mathbf{N}$. For each $a, b \in (A, B)$, we have

$$a_n b_n n^k \leq \rho a_n b_n c_n$$

for all $n \in \mathbf{N}$. Choose $d \in A$ such that

$$a_n b_n c_n = O(d_n).$$

So we have $a_n b_n n^k = O(d_n)$. To prove (1), let $x \in \lambda(B) \vee \lambda(A)$ be given. We claim that $x \in s$. To prove our claim, let $k \in \mathbf{N}$ be given. Then by Proposition 2.5.2, there are $\rho > 0$ and $a \in (A, B)$ such that

$$n^k \leq \rho a_n$$

for all $n \in \mathbf{N}$. So we have

$$\sum_n |x_n| n^k \leq \rho \sum_n |x_n| a_n < +\infty.$$

Therefore $x \in s$. ■

Corollary 2.5.5 *If $\lambda(A)$ is a nuclear G_∞ -space, then*

1. $\lambda(A) \subseteq s \subseteq \ell_1$, and
2. for each $a, b \in A$ and $k \in \mathbf{N}$ there is $c \in A$ such that $a_n b_n = O(c_n/n^k)$.

Lemma 2.5.1 *If $\lambda(P)$ is a nuclear G_∞ -space then for every $y \in \lambda(P)$ with $y_n \neq 0$ for all n , there is $z \in \lambda(P)$ with $0 < |y_n| \leq z_n$ and $z_{n+1} \leq z_n$.*

Proof. Since $y \in \lambda(P)$, $y \in \ell_1$. Put $z_n = \sum_{j=n}^\infty |y_j|$. Then $0 < |y_n| \leq z_n$ and $z_{n+1} \leq z_n$. For $b \in P$, we have

$$\begin{aligned} \sum_{n=1}^\infty |z_n| b_n &= \sum_{n=1}^\infty \left(b_n \sum_{j=n}^\infty |y_j| \right) \\ &= \sum_{n=1}^\infty \left(|y_n| \sum_{j=1}^n b_j \right) \\ &\leq \sum_{n=1}^\infty n b_n |y_n| < +\infty. \end{aligned}$$

Therefore $z \in \lambda(P)$ with $0 < |y_n| \leq z_n$ and $z_{n+1} \leq z_n$. ■

2.6 Nuclearity of G_1 -spaces

We start the present section by giving the following characterization of sequence spaces of the form $\lambda(Q) \vee \lambda(P)$, where $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces. Then we use this to give a characterization of nuclear G_1 -spaces.

Proposition 2.6.1 *A sequence space of the form $\lambda(Q) \vee \lambda(P)$, where $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, is nuclear if and only if $(P, Q) \subseteq \ell_1$.*

Proof. ($\Rightarrow$) Assume that $\lambda(Q) \vee \lambda(P)$ is nuclear. Let $a = (p_1, q_1, p_2, q_2, \dots)$ be a sequence in (P, Q) . Then we choose $a' = (p'_1, q'_1, p'_2, q'_2, \dots) \in (P, Q)$ such that $(a_n/a'_n) \in \ell_1$. Let $t = \max\{p'_1, q'_1\}$. Then $t \geq a'_n$ for all $n \in \mathbb{N}$. Hence

$$\frac{a_n}{t} \leq \frac{a_n}{a'_n} \quad \text{for all } n \in \mathbb{N}.$$

So $a \in \ell_1$. Hence $(P, Q) \subseteq \ell_1$

($\Leftarrow$) Assume that $(P, Q) \subseteq \ell_1$. Let $a = (p_1, q_1, p_2, q_2, \dots)$ be a sequence in (P, Q) . Then we choose $c = (p'_1, q'_1, p'_2, q'_2, \dots) \in (P, Q)$ and $\rho > 0$ such that $a_n \leq \rho c_n^2$ for all $n \in \mathbb{N}$. So we have

$$\frac{a_n}{c_n} \leq \rho c_n \quad \text{for all } n \in \mathbb{N}.$$

Since $c \in (P, Q) \subseteq \ell_1$, $(a_n/c_n) \in \ell_1$. Hence $\lambda(Q) \vee \lambda(P)$ is nuclear. ■

As a corollary, we have the following characterization of nuclear G_1 -spaces.

Corollary 2.6.1 *A G_1 -space $\lambda(Q)$ is nuclear if and only if $Q \subseteq \ell_1$.*

Proof. Let $Q_1 = \{(p_{2n}) : p \in Q\}$ and $Q_2 = \{(p_{2n-1}) : p \in Q\}$. Then $\lambda(Q) = \lambda(Q_2, Q_1) = \lambda(Q_1) \vee \lambda(Q_2)$. Therefore by Proposition 2.6.1, $\lambda(Q)$ is nuclear if and only if $(Q_2, Q_1) \subseteq \ell_1$. Since $(Q_2, Q_1) \subseteq \ell_1$ if and only if $Q \subseteq \ell_1$, $\lambda(Q)$ is nuclear if and only if $Q \subseteq \ell_1$. ■

Now, we give another characterization for the sequence space of the form $\lambda(Q) \vee \lambda(P)$, where $\lambda(P)$ and $\lambda(Q)$ are sequence spaces of finite type.

Proposition 2.6.2 *A sequence space $\lambda(Q) \vee \lambda(P)$, where $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, is nuclear if and only if $(P, Q) \subseteq s$.*

Proof. The “if” part condition is clear. To prove the “only if” part, let $a = (p_1, q_1, p_2, q_2, \dots)$ be a sequence in (P, Q) . Let each $n \in \mathbb{N}$, let $n \in (\mathbb{N})$, if n is even, $n = 2t$ for some $t \in \mathbb{N}$. So we have

$$na_n = 2t a_{2t} = 2t q_t.$$

Since $\lambda(Q)$ is a G_1 -space, we have

$$na_n \leq 2 \sum_{i=1}^t q_i \leq 2 \sum_{i=1}^{\infty} q_i \leq 2 \sum_{i=1}^{\infty} a_i < +\infty.$$

If n is odd, then $n = 2t - 1$ for some $t \in \mathbb{N}$. So we have

$$na_n = (2t - 1)a_{2t-1} \leq 2ta_{2t-1} = 2tp_t.$$

Since $\lambda(P)$ is a G_1 -space, we have

$$na_n \leq 2 \sum_{i=1}^t p_i \leq 2 \sum_{i=1}^{\infty} p_i \leq 2 \sum_{i=1}^{\infty} a_i.$$

Therefore in both cases, we have $(na_n) \in \ell_{\infty}$. Since $(na_n) \in \ell_{\infty}$ for all $a \in (P, Q)$ and $\lambda(Q) \vee \lambda(P) = \lambda(P, Q)$ is nuclear, $(n) \in B(P, Q) = \lambda(P, Q)$. Since $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, $(n^k) \in \lambda(P, Q)$. So for each $a \in (P, Q)$, we have

$$\sum_n a_n n^k < +\infty$$

for all k . Hence $a \in s$. Therefore $(P, Q) \subseteq s$. ■

As a corollary we have the following characterization for nuclear G_1 -spaces.

Corollary 2.6.2 *A G_1 -space $\lambda(Q)$ is nuclear if and only if $Q \subseteq s$.*

Proof. Let $Q_1 = \{(q_{2n}) : q \in Q\}$ and $Q_2 = \{(Q_{2n-1}) : q \in Q\}$. Then $\lambda(Q) = \lambda(Q_1) \vee \lambda(Q_2) = \lambda(Q_2, Q_1)$. So by Proposition 2.6.2, $\lambda(Q)$ is nuclear if and only if $(Q_2, Q_1) \subseteq s$. Since $Q \subseteq s$ if and only if $(Q_2, Q_1) \subseteq s$, $\lambda(Q)$ is nuclear if and only if $Q \subseteq s$. ■

Proposition 2.6.3 *If the sequence space of the form $\lambda(Q) \vee \lambda(P)$, where $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, is nuclear, then*

1. $\ell_\infty \subseteq \lambda(Q) \vee \lambda(P) = \lambda(P, Q)$, and
2. for each $x \in \lambda(Q) \vee \lambda(P)$ and $k \in \mathbb{N}$, we have $(n^k x_n) \in \lambda(Q) \vee \lambda(P)$.

Proof. If $x \in \ell_\infty$, then there is $\alpha > 0$ such that $|x_n| \leq \alpha$ for each $n \in \mathbb{N}$. Since $(P, Q) \subseteq \ell_1$, $x \in \lambda(p, Q)$. To prove 2, let $B = \{(n^k) : k \in \mathbb{N}\}$. Then by Proposition 2.6.2, $(P, Q) \subseteq s = \lambda(B)$. So by Proposition 2.2.2,

$$B \subseteq \lambda(B)' \subseteq \lambda(P, Q).$$

Since $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, $(n^k x_n) \in \lambda(P, Q)$ for all $x \in \lambda(P, Q)$ and $k \in \mathbb{N}$. ■

Corollary 2.6.3 *If $\lambda(Q)$ is a nuclear G_1 -space, then*

1. for each $x \in \lambda(Q)$ and $k \in \mathbb{N}$ we have $(n^k x_n) \in \lambda(Q)$, and
2. $\ell_\infty \subseteq \lambda(Q)$.

Lemma 2.6.1 *Let $\lambda(Q)$ be a nuclear G_1 -space. Then for each $y \in \lambda(Q)$, there exists $x \in A_Q = \{z \in \lambda(Q) : 0 < z_1 \leq z_2 \leq \dots\}$ such that $|y_n| \leq x_n \forall n \in \mathbb{N}$.*

Proof. Let $y \in \lambda(Q)$ be given. Put $x_n = \sum_{j=1}^n |y_j|$ and $z_n = x_n + 1$. We claim : $z \in A_Q$. To prove this, let $b \in Q$, we choose $a \in Q$ such that $b_n = O(a_n^2)$. So there is $\rho > 0$ such that

$$\begin{aligned} \sum_{n=1}^{\infty} b_n x_n &= \sum_{n=1}^{\infty} \left(b_n \sum_{j=1}^n |y_j| \right) \\ &= \sum_{n=1}^{\infty} \left(|y_n| \sum_{j=n}^{\infty} b_j \right) \\ &\leq \rho \sum_{n=1}^{\infty} \left(|y_n| \sum_{j=n}^{\infty} (a_j)^2 \right) \\ &\leq \rho \sum_{n=1}^{\infty} \left(|y_n| a_n \sum_{j=n}^{\infty} a_j \right). \end{aligned}$$

Let $\gamma = \sum_{j=1}^{\infty} a_j$. Then

$$\sum_{n=1}^{\infty} b_n x_n \leq \rho \gamma \sum_{n=1}^{\infty} |y_n| a_n < +\infty.$$

Let $z_n = x_n + 1$. Then $0 < z_1 \leq z_2 \leq \dots$ and $|y_n| \leq x_n \leq z_n$. Moreover for $b \in Q$ we have

$$\sum_{n=1}^{\infty} z_n b_n = \sum_{n=1}^{\infty} x_n b_n + \sum_{n=1}^{\infty} b_n < +\infty.$$

Therefore z satisfies the required condition. ■

2.7 Some Relations Between G_{∞} -spaces and G_1 -spaces

In this section we are going to investigate relations between smooth sequence spaces of different types.

Proposition 2.7.1 *If the sequence space of the form $\lambda(B) \vee \lambda(A)$ is nuclear, where $\lambda(A)$ and $\lambda(B)$ are G_{∞} -spaces, and if the sequence space of the form $\lambda(Q) \vee \lambda(P)$ is nuclear, where $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, then*

$$\lambda(B) \vee \lambda(A) \subseteq \lambda(Q) \vee \lambda(P).$$

Proof. Since the sequence space $\lambda(B) \vee \lambda(A)$ is nuclear, $\lambda(B) \vee \lambda(A) \subseteq \ell_1$. Also, since $\lambda(Q) \vee \lambda(P)$ is nuclear, $\ell_\infty \subseteq \lambda(Q) \vee \lambda(P)$. Because $\ell_1 \subseteq \ell_\infty$, we have $\lambda(B) \vee \lambda(A) \subseteq \lambda(Q) \vee \lambda(P)$. ■

Now we prove the following relation between nuclear G_1 -spaces and nuclear G_∞ -spaces.

Theorem 2.7.1 *If $\lambda(Q)$ is a nuclear G_1 -space and if $\lambda(P)$ is a nuclear G_∞ -space, then $\lambda(P) \subseteq \lambda(Q)$.*

Proposition 2.7.2 ℓ_1 is not nuclear.

Proof. Let $P = \{(1, 1, 1, \dots), (2, 2, 2, \dots), \dots\}$. Then $\ell_1 = \lambda(P)$. If $\lambda(P)$ were nuclear, then $\lambda(P)$ would be both nuclear G_1 and G_∞ -spaces and consequently

$$\ell_1 = \lambda(P) \subseteq \ell_1 \subseteq \ell_\infty \subseteq \lambda(P) = \ell_1,$$

giving us $\ell_1 = \ell_\infty$, which is absurd. ■

Theorem 2.7.2 *If $\lambda(Q)$ is a nuclear G_1 -space then $\lambda(A_Q)$ is a nuclear G_∞ -space, where $A_Q = \{x \in \lambda(Q) : 1 \leq x_1 \leq x_2 \leq \dots\}$.*

Proof. Suppose that $\lambda(Q)$ is a nuclear G_1 -space. Then by Corollary 2.6.2, we have $Q \subseteq s$. Thus $(n) \in \lambda(Q)$. Therefore $A_Q \neq \emptyset$. It is clear that A_Q is a Köthe set and every sequence in A_Q is positive and non-decreasing. So the only remaining part is to show that for each $x \in A_Q$ there is $y \in A_Q$ such that $x_n^2 = O(y_n)$. Let $x \in A_Q$ be given. Then $(x_n^2) \in A_Q$ and hence we may take (y_n) to be (x_n^2) . So A_Q is a power set of infinite type. Since $(n^2) \in A_Q$ and $(1/n^2) \in \ell_1$, $\lambda(A_Q)$ is nuclear. ■

Theorem 2.7.3 *If $\lambda(A)$ is a nuclear G_∞ -space with the property that there is $y \in \lambda(A)$ with $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lambda(Q_A)$ is a nuclear G_1 -space where $Q_A = \{x \in \lambda(A) : 0 < x_{n+1} \leq x_n \forall n\}$.*

Proof. Let $x \in Q_A$ be given. For each $a \in A$, we choose $b \in A$ and $\mu > 0$ such that $a_n^2 \leq \mu b_n$ for all $n \in \mathbb{N}$. So

$$\sup_n x_n a_n^2 \leq \mu \sup_n x_n b_n \equiv \alpha < +\infty.$$

Therefore $x_n a_n^2 \leq \alpha$ for all $n \in \mathbb{N}$, which gives $\sqrt{x_n} a_n \leq \sqrt{\alpha}$ for all $n \in \mathbb{N}$ and consequently $(\sqrt{x_n} a_n) \in \ell_\infty$ for each $a \in A$. Thus

$$(\sqrt{x_n}) \in B(A) = \lambda(A).$$

Since (x_n) is positive and non-increasing, $(\sqrt{x_n}) \in Q_A$. Therefore Q_A is a power set of finite type. Finally, because $\lambda(A) \subseteq s$ and $Q_A \subseteq \lambda(A)$, we have the nuclearity of $\lambda(Q_A)$. ■

Proposition 2.7.3 *If $\lambda(A)$ is a stable nuclear G_∞ -space with the property that there is $y \in \lambda(A)$ with $y_n \neq 0$ for each positive integer n , then $\lambda(Q_A)$ is stable nuclear where $Q_A = \{x \in \lambda(A) : 0 < x_{n+1} \leq x_n \forall n\}$.*

Proof. By Theorem 2.7.3, we know that $\lambda(Q_A)$ is a nuclear G_1 -space. Let $x \in Q_A$ be given. Then $x_n = O(y_{2n})$, where $y = (x_1, x_1, x_2, x_2, \dots)$. We claim : $y \in \lambda(A)$. Given $a \in A$ we choose $b \in A$ and $\gamma > 0$ such that $a_{2n} \leq \gamma b_n$ for all $n \in \mathbb{N}$, which gives

$$\begin{aligned} \sum_{n=1}^{\infty} |y_n| a_n &\leq 2 \sum_{n=1}^{\infty} |x_n| a_{2n} \\ &\leq 2 \gamma \sum_{n=1}^{\infty} |x_n| b_n < +\infty. \end{aligned}$$

Therefore $y \in Q_A$. ■

Proposition 2.7.4 *If $\lambda(Q)$ is a stable nuclear G_1 -space, then $\lambda(A_Q)$ is stable nuclear where $A_Q = \{x \in \lambda(Q) : 1 \leq x_1 \leq x_2 \leq \dots\}$.*

Proof. By Theorem 2.7.2, we know that $\lambda(A_Q)$ is a nuclear G_∞ -space. Let $x \in A_Q$ be given. Then $x \in \lambda(Q)$ and $1 \leq x_1 \leq x_2 \leq \dots$. Let $y = (x_2, x_4, x_6, \dots)$. Then $1 \leq y_1 \leq y_2 \leq \dots$. We claim : $y \in \lambda(Q)$. For $p \in Q$ we choose $q \in Q$ and $\gamma > 0$ such that $p_n \leq \gamma q_{2n}$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} \sum_{n=1}^{\infty} |y_n| b_n &= \sum_{n=1}^{\infty} x_{2n} b_n \\ &\leq \gamma \sum_{n=1}^{\infty} x_{2n} a_{2n} \\ &\leq \gamma \sum_{n=1}^{\infty} x_n a_n < +\infty. \end{aligned}$$

Therefore $y \in \lambda(Q)$. ■

Chapter 3

2-Quasi-Nuclear Maps

3.1 Introduction

In [27], Pietsch proved that the composition of two quasi-nuclear maps is nuclear. In this chapter, we are going to define a 2-quasi-nuclear map between two normed spaces E and F . Also we are going to give all properties of this map and we prove that each quasi-nuclear map is a 2-quasi-nuclear map. In Section 3 of this chapter, we are going to give a characterization of 2-quasi-nuclear maps. Also, we are going to prove that a bounded linear map between Hilbert spaces is a 2-quasi-nuclear map if and only if it is a Hilbert-Schmidt map. Finally, in Section 4 of this chapter, we are going to prove that the composition of a quasi-nuclear map and a 2-quasi-nuclear map is nuclear, improving a result of Pietsch.

3.2 Definitions

Definition 3.2.1 A quasi-norm on a linear space E over a field K is a real-valued function p with the following properties:

1. $p(x) \geq 0$ for all $x \in E$.
2. $p(x) = 0$ if and only if $x = 0$.
3. $p(\alpha x) = |\alpha|p(x)$ for all $\alpha \in K$ and $x \in E$.
4. There is $t > 0$ such that $p(x + y) \leq t(p(x) + p(y))$ for all $x, y \in E$.

Definition 3.2.2 A linear map T from a normed space E into a normed space F is called a **p -quasi-nuclear map**, where p is a positive real number, if there is a sequence (a_n) in E' with

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

For each p -quasi-nuclear map we set

$$\nu_p(T) = \inf \left\{ \left(\sum_n \|a_n\|^p \right)^{1/p} \right\}$$

where the infimum is taken over all the sequences (a_n) in E' with

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p},$$

for all $x \in E$.

Lemma 3.2.1 For $\alpha, \beta \geq 0$, we have

$$\alpha^{1/p} + \beta^{1/p} \leq 2(\alpha + \beta)^{1/p} \leq 2^{(1+1/p)}(\alpha^{1/p} + \beta^{1/p}).$$

Proof. Since $\alpha, \beta \geq 0$, we have

$$\begin{aligned}\alpha^{1/p} + \beta^{1/p} &\leq 2\max\{\alpha^{1/p}, \beta^{1/p}\} \\ &\leq 2(\alpha + \beta)^{1/p}.\end{aligned}$$

Also, we have

$$\begin{aligned}(\alpha + \beta)^{1/p} &\leq (2\max\{\alpha, \beta\})^{1/p} \\ &= 2^{1/p}(\max\{\alpha, \beta\})^{1/p} \\ &= 2^{1/p}\max\{\alpha^{1/p}, \beta^{1/p}\} \\ &\leq 2^{1/p}(\alpha^{1/p} + \beta^{1/p}).\end{aligned}$$

So we have

$$\alpha^{1/p} + \beta^{1/p} \leq 2(\alpha + \beta)^{1/p} \leq 2^{(1+1/p)}(\alpha^{1/p} + \beta^{1/p}). \quad \blacksquare$$

Let $\mathcal{QN}_p(E, F)$ denote the collection of all p -quasi-nuclear maps from a normed space E into another F . Then we have the following.

Proposition 3.2.1 *The set $\mathcal{QN}_p(E, F)$ is a linear space and ν_p is a quasi-norm on $\mathcal{QN}_p(E, F)$.*

Proof. From the definition, it is easy to see that for each $T \in \mathcal{QN}_p(E, F)$ we have $\|T\| \leq \nu_p(T)$. Let T_1 and T_2 in $\mathcal{QN}_p(E, F)$. We claim that

$$T_1 + T_2 \in \mathcal{QN}_p(E, F) \quad \text{and} \quad \nu_p(T_1 + T_2) \leq 2^{(1+1/p)}(\nu_p(T_1) + \nu_p(T_2)).$$

Let $\varepsilon > 0$ be given. Then there are two sequences (a_n) and (b_n) in E' with

$$\left(\sum_n \|a_n\|^p\right)^{1/p} \leq \nu_p(T_1) + \varepsilon \quad \text{and} \quad \left(\sum_n \|b_n\|^p\right)^{1/p} \leq \nu_p(T_2) + \varepsilon$$

such that

$$||T_1 x|| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p} \quad \text{and} \quad ||T_2 x|| \leq \left(\sum_n |\langle x, b_n \rangle|^p \right)^{1/p}.$$

Now, for $x \in E$ we have

$$||(T_1 + T_2)x|| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p} + \left(\sum_n |\langle x, b_n \rangle|^p \right)^{1/p}.$$

So by Lemma 3.2.1, we have

$$||(T_1 + T_2)x|| \leq \left(\sum_n |\langle x, 2a_n \rangle|^p + |\langle x, 2b_n \rangle|^p \right)^{1/p}.$$

On the other hand

$$((\nu_p(T_1) + \varepsilon)^p + (\nu_p(T_2) + \varepsilon)^p)^{1/p} \leq 2^{1/p}(\nu_p(T_1) + \nu_p(T_2) + 2\varepsilon).$$

So we have

$$\left(\sum_n ||2a_n||^p + \sum_n ||2b_n||^p \right)^{1/p} \leq 2^{(1+1/p)}(\nu_p(T_1) + \nu_p(T_2) + 2\varepsilon).$$

Letting $\varepsilon \rightarrow 0$, we have

$$\nu_p(T_1 + T_2) \leq 2^{(1+1/p)}(\nu_p(T_1) + \nu_p(T_2)) \quad \text{and} \quad T_1 + T_2 \in \mathcal{QN}_p(E, F).$$

Let T be in $\mathcal{QN}_p(E, F)$ and $\alpha \in K$. We claim that $\alpha T \in \mathcal{QN}_p(E, F)$ and $|\alpha|\nu_p(T) = \nu_p(\alpha T)$. Let $\varepsilon > 0$ be given. Then there is a sequence (a_n) in E' with

$$\left(\sum_n ||a_n||^p \right)^{1/p} \leq \nu_p(T) + \varepsilon$$

such that

$$||Tx|| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

Now

$$||\alpha Tx|| = |\alpha| ||Tx|| \leq \left(\sum_n |\langle x, \alpha a_n \rangle|^p \right)^{1/p}.$$

on the other hand, we have

$$\left(\sum_n \|\alpha a_n\|^p\right)^{1/p} = |\alpha| \left(\sum_n \|a_n\|^p\right)^{1/p} \leq |\alpha|(\nu_p(T) + \varepsilon).$$

Letting $\varepsilon \rightarrow 0$ we get $\nu_p(\alpha T) \leq |\alpha|\nu_p(T)$ and $\alpha T \in \mathcal{QN}_p(E, F)$. Also we can show that $|\alpha|\nu_p(T) \leq \nu_p(\alpha T)$. So we get $\nu_p(\alpha T) = |\alpha|\nu_p(T)$. If $T = 0$ then it is clear that $\nu_p(T) = 0$. Conversely, if $\nu_p(T) = 0$, then from $\|T\| \leq \nu_p(T)$ we see that $\|T\| = 0$ and hence $T = 0$. ■

3.3 A Characterization of 2-quasi-nuclear maps

The following well-known results are essential for our subsequent arguments.

Lemma 3.3.1 (see Pietsch [27; P. 57]) *Each normed space F can be considered as a linear subspace of a Banach space $\ell_\infty(I)$.*

Lemma 3.3.2 [27; P. 58] *If G_0 is a linear subspace of the normed space G , then each continuous linear map S_0 of G_0 into a Banach space $\ell_\infty(I)$ can be extended to a continuous linear map S from G into $\ell_\infty(I)$ with $\|S\| = \|S_0\|$.*

Theorem 3.3.1 *If a p -quasi-nuclear map $T : E \rightarrow F$ between normed spaces E and F is regarded as a map from E into a Banach space $\ell_\infty(I)$ in which F is embedded as indicated in Lemma 3.3.1, then T can be written as*

$$x \longmapsto Tx = \sum_n \langle x, a_n \rangle y_n$$

for some sequences (a_n) in E' and (y_n) in $\ell_\infty(I)$ with $\sum_n \|a_n\|^p < +\infty$.

Proof. Since T is a p -quasi-nuclear map, there is a sequence (a_n) in E' with

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|T(x)\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

Let $G_0 = \{(\langle x, a_n \rangle) : x \in E\}$. Then G_0 is a linear subspace of ℓ_p . Define S_0 from G_0 into F by $S_0(\langle x, a_n \rangle) = Tx$. Then S_0 is continuous linear map with $\|S_0\| \leq 1$ because

$$\|S_0(\langle x, a_n \rangle)\| = \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p} = \|(\langle x, a_n \rangle)\|_p.$$

So by Lemma 3.3.2, the map S_0 , regarded as a map from G_0 into $\ell_\infty(I)$, can be extended to a continuous linear map S from ℓ_p into $\ell_\infty(I)$ with $\|S\| \leq 1$. Let $y_m = S(e_m)$, where (e_n) is the standard basis of ℓ_p . Then $\|y_m\|_\infty = \|Se_m\|_\infty \leq 1$ and the map S has the representation

$$S(\{\zeta_n\}) = \sum_n \zeta_n y_n$$

for $(\zeta_n) \in \ell_p$. Since $Tx = S(\langle x, a_n \rangle)$, the map T has the form

$$Tx = \sum_n \langle x, a_n \rangle y_n \quad \text{for } x \in E \text{ with } \sum_n \|a_n\|^p < +\infty. \quad \blacksquare$$

Theorem 3.3.2 *If $T : E \rightarrow F$ is a bounded linear map between normed spaces E and F and if S is a p -quasi-nuclear map from F into a normed space G , then ST is a p -quasi-nuclear map.*

Proof. Since S is a p -quasi-nuclear map, there is a sequence (b_n) in F' with

$$\sum_n \|b_n\|^p < +\infty \quad \text{such that} \quad \|Sy\| \leq \left(\sum_n |\langle y, b_n \rangle|^p \right)^{1/p}.$$

So we have

$$\begin{aligned} \|STx\| &\leq \left(\sum_n |\langle Tx, b_n \rangle|^p \right)^{1/p} \\ &= \left(\sum_n |\langle x, T'b_n \rangle|^p \right)^{1/p}. \end{aligned}$$

Let $a_n = T'(b_n)$. Then $(a_n) \in E'$ and

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|STx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

Therefore ST is p -quasi-nuclear map. ■

Theorem 3.3.3 *If $T : E \rightarrow F$ is a p -quasi-nuclear map between normed spaces E and F and if S is a bounded linear map from F into a normed space G , then ST is a p -quasi-nuclear map.*

Proof. Since T is a p -quasi-nuclear map, there is a sequence (a_n) in E' with

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

So we have

$$\|STx\| \leq \|S\| \|Tx\| \leq \|S\| \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

Let $b_n = \|S\| a_n$. Then $b_n \in F'$ with

$$\sum_n \|b_n\|^p < +\infty \quad \text{such that} \quad \|STx\| \leq \left(\sum_n |\langle x, b_n \rangle|^p \right)^{1/p}.$$

Therefore ST is a p -quasi-nuclear map. ■

Definition 3.3.1 *Suppose that $\mu = (\mu_1, \mu_2, \dots) \in \ell_p$. The diagonal map D_μ from ℓ_∞ into ℓ_p is given by $D_\mu(x) = (\mu_n x_n)$.*

Definition 3.3.2 *Let E be a normed space, a sequence (y_n) in E is called weakly 2-summing if*

$$\sum_n |\langle y_n, b \rangle|^2 < \infty$$

for all $b \in E'$.

Definition 3.3.3 Let E be a normed space, a sequence (a_n) in E' is called absolutely 2-summing if

$$\sum_n \|a_n\|^2 < \infty.$$

Theorem 3.3.4 If T is a bounded linear map from a normed space E into a Banach space F , then the following conditions are equivalent:

1. T is a 2-quasi-nuclear map.
2. T factors through $D_\mu : \ell_\infty \rightarrow \ell_2$ for some $\mu \in \ell_2$, that is, there are two bounded linear maps S_1 from E into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$.
3. There exists an absolutely 2-summing sequence (a_n) in E' and weakly 2-summing (y_n) in F such that

$$Tx = \sum_n \langle x, a_n \rangle y_n.$$

Proof. (1 $\Rightarrow$ 2) Assume that T is a 2-quasi-nuclear map. Let $\varepsilon > 0$ be given. Then there is a sequence (a_n) in E' with $a_n \neq 0$ for all n and

$$\left(\sum_n \|a_n\|^2 \right)^{1/2} \leq \nu_2(T) + \varepsilon \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Let $\mu_n = \|a_n\|$ and $b_n = \mu_n^{-1} a_n$. Then $\|b_n\| = 1$ and $(\mu_n) \in \ell_2$. Define $S_1 : E \rightarrow \ell_\infty$ by $S_1 x = (\langle x, b_n \rangle)$. Then $\|S_1\| \leq 1$ because

$$\|S_1 x\|_\infty = \|(\langle x, b_n \rangle)\|_\infty \leq \|x\|.$$

Also define $D_\mu : \ell_\infty \rightarrow \ell_2$ by $D_\mu(\zeta_n) = (\zeta_n \mu_n)$. Then $\|D_\mu\| \leq \nu_2(T) + \varepsilon$ because

$$\begin{aligned} \|D_\mu(\zeta_n)\|_2 &= \|(\zeta_n \mu_n)\|_2 \\ &= \left(\sum_n |\mu_n|^2 |\zeta_n|^2 \right)^{1/2} \\ &\leq \left(\sum_n |\mu_n|^2 \right)^{1/2} (\|(\zeta_n)\|_\infty). \end{aligned}$$

Let $H = \{(\langle x, a_n \rangle) : x \in E\}$. Then H is a linear subspace of ℓ_2 . Define $S : H \rightarrow F$ by $S(\langle x, a_n \rangle) = Tx$. Then $\|S\| \leq 1$ because

$$\|S(\langle x, a_n \rangle)\| = \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^2 \right)^{1/2} = \|(\langle x, a_n \rangle)\|_2.$$

Since ℓ_2 is a Hilbert space, there is a projection P on ℓ_2 with $\text{range}(P) = \overline{H}$. Put $S_2 = S_0 P$, where S_0 is the continuous extension of S to all the closed hull of H . Now it is clear that $T = S_2 D_\mu S_1$.

(2 $\Rightarrow$ 3) Let $a_n = \mu_n S'_1 \delta_n$ (where $S'_1 : \ell'_\infty \rightarrow E'$ and $\delta_k \in \ell'_\infty$ sends $(x_n) \in \ell_\infty$ to x_k) and $y_n = S_2 e_n$ (where (e_n) is the standard basis of ℓ_2). Then

$$\begin{aligned} Tx &= S_2 D_\mu S_1 x \\ &= S_2 D_\mu (\langle S_1 x, \delta_n \rangle) \\ &= S_2 (\langle S_1 x, \mu_n \delta_n \rangle) \\ &= S_2 \left(\sum_n \langle S_1 x, \mu_n \delta_n \rangle e_n \right). \end{aligned}$$

Therefore we have

$$Tx = \sum_n \langle x, a_n \rangle y_n.$$

(3 $\Rightarrow$ 1) For any $b \in F'$ we have

$$\begin{aligned} |\langle Tx, b \rangle|^2 &= \left| \sum_n \langle x, a_n \rangle \langle y_n, b \rangle \right|^2 \\ &\leq \sum_n |\langle x, a_n \rangle|^2 \sum_n |\langle y_n, b \rangle|^2. \end{aligned}$$

So we have

$$\|Tx\|^2 \leq M \sum_n |\langle x, a_n \rangle|^2 \quad \text{where} \quad M = \sup_{\|b_n\| \leq 1} \sum_n |\langle y_n, b \rangle|^2$$

which is finite because (y_n) is weakly 2-summable. ■

Theorem 3.3.5 *2-quasi-nuclear maps between Banach spaces are approximatable by finite rank operators and hence are compact.*

Proof. Let $T : E \rightarrow F$ be a 2-quasi-nuclear map between Banach spaces E and F . Then by Theorem 3.3.4, T factors through D_μ for some $\mu \in \ell_2$, that is, there are two bounded linear maps S_1 from E into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$. Define maps $T_k : \ell_2 \rightarrow \ell_2$ by putting

$$T_k(\eta_n) = (\eta_1, \eta_2, \dots, \eta_k, 0, 0, \dots).$$

Then the $T_k D_\mu$ have finite rank and $\|T_k D_\mu - D_\mu\| \rightarrow 0$ as $k \rightarrow +\infty$. ■

Next we determine 2-quasi-nuclear maps between Hilbert spaces.

Theorem 3.3.6 *A bounded linear map $T : H_1 \rightarrow H_2$ between Hilbert spaces H_1 and H_2 is 2-quasi-nuclear if and only if it is a Hilbert-Schmidt map.*

Proof. ($\Rightarrow$) Let $T : H_1 \rightarrow H_2$ be a 2-quasi-nuclear map between Hilbert spaces H_1 and H_2 . Then there is a sequence (a_n) in H_1 with

$$\sum_n \|a_n\|^2 < +\infty \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Therefore we have

$$\|Tx\|^2 \leq \sum_n |\langle x, a_n \rangle|^2.$$

For any orthonormal basis (e_n) in H_1 , we have

$$\sum_m \|Te_m\|^2 \leq \sum_m \sum_n |\langle e_m, a_n \rangle|^2 = \sum_n \|a_n\|^2 < +\infty,$$

and hence T is a Hilbert-Schmidt map.

($\Leftarrow$) Let $T : H_1 \rightarrow H_2$ be a Hilbert-Schmidt map. Then T is compact. So by the spectral decomposition theorem we have an orthonormal bases (f_n) and (g_n) in H_1 and H_2 respectively, as well as, a family of numbers (μ_n) in C_0 such that

$$Tx = \sum_n \mu_n \langle x, f_n \rangle g_n.$$

So we have $Tf_n = \mu_n g_n$. Since T is a Hilbert-Schmidt map, we have

$$\sum_n \|Tf_n\|^2 = \sum_n |\mu_n|^2 < +\infty.$$

On the other hand, we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \left\langle \sum_n \mu_n \langle x, f_n \rangle g_n, \sum_n \mu_n \langle x, f_n \rangle g_n \right\rangle \\ &= \sum_n |\mu_n|^2 |\langle x, f_n \rangle|^2 \\ &= \sum_n |\langle x, \mu_n f_n \rangle|^2. \end{aligned}$$

Let $a_n = \mu_n f_n$. Then (a_n) is a bounded sequence in H_1 with

$$\sum_n \|a_n\|^2 < +\infty$$

such that

$$\|Tx\| \leq \left(\sum_n |\langle x, \mu_n f_n \rangle|^2 \right)^{1/2}.$$

Therefore T is 2-quasi-nuclear . $\blacksquare$

Corollary 3.3.1 *Let T be a 2-quasi-nuclear map from a Hilbert space H_1 into another H_2 . Then there is an orthonormal base (e_i) in H_1 , an orthonormal set (f_i) in H_2 and a sequence (μ_i) in ℓ_2 such that $Te_i = \mu_i f_i$.*

Proof. Since T is compact, by the spectral decomposition theorem we have an orthonormal base (e_i) in H_1 and orthonormal set (f_i) in H_2 , as well as , a family of numbers (μ_i) in ℓ_2 such that

$$Tx = \sum_i \mu_i \langle x, e_i \rangle f_i.$$

So we have $Te_i = \mu_i f_i$. $\blacksquare$

Definition 3.3.4 A bounded linear map T from a Hilbert space H_1 into a Hilbert space H_2 is called of class $\mathcal{C}_p$ if there are two orthonormal bases (e_n) and (f_n) in H_1 and H_2 respectively, as well as, a sequence (μ_n) of real numbers with $\sum_n |\mu_n|^p < +\infty$ such that

$$Tx = \sum_n |\mu_n| \langle x, e_n \rangle f_n.$$

The set of all $\mathcal{C}_p$ class operators from a Hilbert space H_1 into a Hilbert space H_2 will be denoted by $\mathcal{C}_p(H_1, H_2)$.

Lemma 3.3.3 If $1 \leq p \leq 2$, and (t_n) is a sequence of nonnegative real numbers, then

$$\sum_n t_n^2 \leq \left(\sum_n t_n^p \right)^{2/p}.$$

Proof. It is clear for $p = 2$. If $p < 2$, then $2/p > 1$. So there is $\beta > 0$ such that $2/p = 1 + \beta$. Therefore we have

$$\begin{aligned} \left(\sum_n t_n^p \right)^{2/p} &= \left(\sum_n t_n^p \right)^{1+\beta} \\ &= (t_1^p + t_2^p + \dots) \left(\sum_n t_n^p \right)^\beta \\ &= t_1^p \left(\sum_n t_n^p \right)^\beta + t_2^p \left(\sum_n t_n^p \right)^\beta + \dots \end{aligned}$$

For $k \in N$ we have

$$t_k^p \left(\sum_n t_n^p \right)^\beta \geq t_k^p t_k^{p\beta} = t_k^{p(1+\beta)} = t_k^2.$$

Therefore

$$\begin{aligned} \sum_k t_k^2 &\leq \sum_k t_k^p \left(\sum_n t_n^p \right)^\beta \\ &= \left(\sum_k t_k^p \right)^{1+\beta} = \left(\sum_k t_k^p \right)^{2/p} \quad \blacksquare \end{aligned}$$

Remark: The above lemma can be generalized as follows, if $1 \leq p \leq r$ and $t_n \geq 0$, then

$$\left(\sum_n t_n^r \right)^{1/r} \leq \left(\sum_n t_n^p \right)^{1/p}.$$

Next, we give relations between r -quasi-nuclear maps and maps of class $\mathcal{C}_r$.

Theorem 3.3.7 *Suppose that $1 \leq r \leq 2$ and T is a bounded linear map from a Hilbert space H_1 into a Hilbert space H_2 . If T is of class $\mathcal{C}_r$, then T is r -quasi-nuclear.*

Proof. Assume that $1 \leq r \leq 2$ and T is of class $\mathcal{C}_r$. Then there are orthonormal bases (e_n) and (f_n) in H_1 and H_2 respectively, as well as, a sequence of positive real numbers (α_n) with $\sum_n \alpha_n^r < +\infty$ such that

$$Tx = \sum_n \alpha_n \langle x, e_n \rangle f_n.$$

Let $a_n = \alpha_n e_n$. Then

$$\sum_n \|a_n\|^r = \sum_n \alpha_n^r < +\infty.$$

On the other hand, we have

$$\|Tx\|^2 = \sum_n |\langle x, a_n \rangle|^2.$$

So by Lemma 3.3.3, we have

$$\|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^r \right)^{1/r}.$$

Since $\sum_n \|a_n\|^r < +\infty$, T is r -quasi-nuclear. ■

3.4 Relations Between 2-Quasi-Nuclear Maps and Quasi-Nuclear Maps

We start the present section by proving the following theorem.

Theorem 3.4.1 *If $0 < r < s < +\infty$, then r -quasi-nuclear maps are s -quasi-nuclear.*

Proof. Suppose $0 < r < s < +\infty$ and $T : E \rightarrow F$ is r -quasi-nuclear from a normed space E into a normed space F . Then there is a sequence (a_n) in E' with $\sum_n \|a_n\|^r < +\infty$ such that

$$\|Tx\|^r \leq \sum_n |\langle x, a_n \rangle|^r.$$

Let $\alpha = (s - r)/s$, $p = s/r$, $\lambda_n = \|a_n\|^\alpha$, and $a_n = \lambda_n b_n$. Then

$$\|Tx\|^r \leq \sum_n \lambda_n^r |\langle x, b_n \rangle|^r.$$

Let $q = p/(p - 1)$. Then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{p} = 1.$$

So, by the Holder inequality we have

$$\begin{aligned} \|Tx\|^r &\leq \left(\sum_n \lambda_n^{rq} \right)^{1/q} \left(\sum_n |\langle x, b_n \rangle|^{rp} \right)^{1/p} \\ &\leq \left(\sum_n \lambda_n^{rq} \right)^{1/q} \left(\sum_n |\langle x, b_n \rangle|^s \right)^{r/s}. \end{aligned}$$

Let $c = \sum_n \lambda_n^{rq}$. Then

$$\begin{aligned} c &= \sum_n \|a_n\|^{\alpha r q} \\ &= \sum_n \|a_n\|^r < +\infty. \end{aligned}$$

Therefore

$$\|Tx\| \leq c^{1/qr} \left(\sum_n |\langle x, b_n \rangle|^s \right)^{1/s}.$$

On the other hand we have

$$\begin{aligned} \sum_n \|b_n\|^s &= \sum_n \lambda_n^{-s} \|a_n\|^s \\ &= \sum_n \|a_n\|^{-\alpha s} \|a_n\|^s \\ &= \sum_n \|a_n\|^{s(1-\alpha)} \\ &= \sum_n \|a_n\|^r < +\infty. \end{aligned}$$

Therefore T is s -quasi-nuclear. ■

Next we give an example to show that the converse of Theorem 3.4.1 is not true.

Example 3.4.1 Let $1 < r < s < +\infty$. Define a map $D : \ell_{r'} \rightarrow \ell_s$ by putting $Dx = (x_n/n^{1/r})$, where $1/r + 1/r' = 1$. Then D is s -quasi-nuclear which is not r -quasi-nuclear.

Proof. To show that D is s -quasi-nuclear, let $a_n = e_n/n^{1/r}$. Then

$$\begin{aligned} \|Dx\|_s^s &= \sum_n \left| \frac{x_n}{n^{1/r}} \right|^s \\ &= \sum_n \left| \langle x, \frac{e_n}{n^{1/r}} \rangle \right|^s. \end{aligned}$$

Since (a_n) is sequence in $\ell_{r'} = \ell_r$ with

$$\sum_n \|a_n\|^s = \sum_n \frac{1}{n^{s/r}} < +\infty,$$

D is s -quasi-nuclear. To show that D is not r -quasi-nuclear, suppose the contrary that D is r -quasi-nuclear. So there is a sequence $(a_n = (a_{1n}, a_{2n}, \dots))$ in ℓ_r with $\sum_n \|a_n\|^r < +\infty$ such that

$$\|Dx\|^r \leq \sum_n |\langle x, a_n \rangle|^r.$$

Then for each $m \in N$, we have

$$\frac{1}{m} \leq \sum_n |a_{mn}|^r.$$

So we have

$$\sum_m \frac{1}{m} \leq \sum_m \sum_n |a_{mn}|^r = \sum_n \|a_n\|^r.$$

Therefore $\sum_n \|a_n\|^r = +\infty$, which is a contradiction. So D is not r -quasi-nuclear. ■

Corollary 3.4.1 *If $T : E \rightarrow F$ is a quasi-nuclear map between Banach spaces E and F , then T is compact.*

Corollary 3.4.2 *If $T : H_1 \rightarrow H_2$ is a quasi-nuclear map between Hilbert spaces H_1 and H_2 , then T is a Hilbert-Schmidt map.*

Corollary 3.4.3 *Each quasi-nuclear map T from a normed space E into a Banach space F factors through $D_\mu : \ell_\infty \rightarrow \ell_2$ for some $\mu \in \ell_2$, that is, there are bounded linear maps S_1 from E into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$.*

Theorem 3.4.2 *If T is a p -quasi-nuclear map from a normed space E into a normed space F , then T is absolutely p -summing.*

Proof. Since T is a p -quasi-nuclear map, there is a sequence (a_n) with $a_n \neq 0$ and

$$\sum_n \|a_n\|^p < +\infty \quad \text{such that} \quad \|Tx\| \leq \left(\sum_n |\langle x, a_n \rangle|^p \right)^{1/p}.$$

Let (x_m) be any arbitrary sequence in E . Then

$$\sum_m \|Tx_m\|^p \leq \sum_n \sum_m |\langle x_m, a_n \rangle|^p.$$

Let $b_n = \|a_n\|^{-1} a_n$. Then $\|b_n\| = 1$. So we have

$$\sum_m \|Tx_m\|^p \leq \sum_n \sum_m \|a_n\|^p |\langle x_m, b_n \rangle|^p.$$

Therefore we have

$$\sum_m \|Tx_m\|^p \leq \sum_n \|a_n\|^p \sup \left\{ \sum_m |\langle x_m, b \rangle|^p : \|b\| \leq 1, b \in E' \right\}.$$

Let $\alpha = \sum_n \|a_n\|^p$. Then

$$\left(\sum_m \|Tx_m\|^p \right)^{1/p} \leq \alpha^{1/p} \sup \left\{ \left(\sum_m |\langle x_m, b \rangle|^p \right)^{1/p} : \|b\| \leq 1, b \in E' \right\}.$$

So T is absolutely p -summing. ■

Now we present the last main result of this chapter: the composition of a quasi-nuclear map with a 2-quasi-nuclear map is nuclear. To prove this we need the following lemmas.

Lemma 3.4.1 (see Pietsch [27; P. 52]) *If $T : E \rightarrow F$ is a nuclear map between normed spaces and if S is a bounded linear map from F into a normed space G , then ST is nuclear.*

Lemma 3.4.2 (see Pietsch [27; P. 58]) *Each quasi-nuclear map $T : E \rightarrow F$ between normed spaces E and F is also nuclear if it is regarded as a map from E into a Banach space $\ell_\infty(I)$ containing F .*

Lemma 3.4.3 (see Pietsch [27; P. 59]) *If $T : E \rightarrow F$ is a quasi-nuclear map between normed spaces E and F , and if S is a bounded linear map from F into a normed space G . Then ST is a quasi-nuclear map.*

Lemma 3.4.4 (see Pietsch [27; P. 56]) *If $T : E \rightarrow F$ is a linear map from a normed space E into F , where F is a dense linear subspace of a normed space G and if T is nuclear as a map from E into G , then T is nuclear as a map from E into F .*

Theorem 3.4.3 *If $T : E \rightarrow F$ is a quasi-nuclear map between normed spaces E and F , and if S is a 2-quasi-nuclear map from F into a Banach space G , then ST is a nuclear map.*

Proof. Since T is a 2-quasi-nuclear map, by Theorem 3.3.4, T factors through D_μ for some $\mu \in \ell_2$. So $T = S_2 D_\mu S_1$ for some bounded linear maps $S_1 : F \rightarrow \ell_\infty$ and $S_2 : \ell_2 \rightarrow G$. Let $T_1 = S_1 T$ and $T_2 = S_2 D_\mu$. Then by Lemma 3.4.3, Lemma 3.4.2, and Lemma 3.4.1, ST is nuclear. ■

Corollary 3.4.4 *The composition of quasi-nuclear maps is nuclear.*

Proof. Let T be a quasi-nuclear map from a normed space E into a normed space F and S be a quasi-nuclear map from F into a normed space G . Considered S as a map from F into $\tilde{G}$ where $\tilde{G}$ is the completion of the G . Then by Theorem 3.4.3, and Lemma 3.4.4, ST is a nuclear map from E into G . ■

Now we will give an example of a 2-quasi-nuclear map which is not a quasi-nuclear map.

Example 3.4.2 *Define a map $D : c_0 \rightarrow \ell_2$ by $Dx = (x_n/n)$. Then D is a 2-quasi-nuclear map which is not quasi-nuclear.*

Proof. To Show that D is a 2-quasi-nuclear map, let $a_n = e_n/n$. Then we have

$$\begin{aligned} \|Dx\|_2^2 &= \|(x_n/n)\|_2^2 \\ &= \sum_n \frac{|x_n|^2}{n^2} \\ &= \sum_n |\langle x, e_n/n \rangle|^2 \\ &= \sum_n |\langle x, a_n \rangle|^2. \end{aligned}$$

Since (a_n) is sequence in $c'_0 = \ell_1$ with $\sum_n \|a_n\|^2 < +\infty$, D is a 2-quasi-nuclear map. To show that D is not a quasi-nuclear map, assume that the contrary that D is a quasi-nuclear map. Then there is a bounded sequence (b_n) in ℓ_1 with $\sum_n \|b_n\| < +\infty$ such that $\|Dx\|_2 \leq \sum_n |\langle x, b_n \rangle|$. One can show that $\sum_n \frac{1}{n} \leq \sum_n \|b_n\|$. So we get $\sum_n \|b_n\| = +\infty$, which is a contradiction. Therefore D is not a quasi-nuclear map. ■

Chapter 4

2-Quasi- λ -Nuclear Maps

4.1 Introduction

In this chapter, we are going to define 2-quasi- λ -nuclear maps between normed spaces. Also, we prove that a map T between two normed spaces is a 2-quasi-nuclear map if and only if T is a 2-quasi- ℓ_1 -nuclear map. In Section 3 of this chapter, we are going to prove that the composition of two quasi- λ -nuclear maps is a pseudo- λ -nuclear map, which gives us a better result than the known one which says that the composition of quasi-nuclear maps is a nuclear map. In Section 4 of this chapter, we are going to prove that each quasi- λ -nuclear map is a 2-quasi- λ -nuclear map and we will prove that the composition of a quasi- λ -nuclear map with a 2-quasi- λ -nuclear map is a pseudo- λ -nuclear map which gives a generalization of the result that the composition of a two quasi- λ -nuclear maps is a pseudo- λ -nuclear map. In case $\lambda = \ell_1$ we prove that a map T between two normed spaces is a nuclear map if and only if T is a pseudo- ℓ_1 -nuclear map, and we will prove that T is a quasi-nuclear map if and only if T is a quasi- ℓ_1 - nuclear map, which means

that Theorem 3.4.4 is a special case of the result that the composition of a quasi- λ -nuclear map with a 2-quasi- λ -nuclear map is a pseudo- λ -nuclear map.

4.2 Definitions

In the present chapter, letter λ stands for a fixed sequence space contained in ℓ_1 .

Definition 4.2.1 [13][28] *A bounded linear map T of a normed space E into a normed space F is called a **pseudo- λ -nuclear map** if there exist a sequence (α_n) in λ and a bounded sequences (a_n) and (y_n) in E' and F respectively such that*

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n,$$

for all x in E .

Definition 4.2.2 [13][28] *A bounded linear map T of a normed space E into a normed space F is called a **quasi- λ -nuclear map** if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that*

$$\|Tx\| \leq \sum_n |\alpha_n| |\langle x, a_n \rangle|,$$

for all x in E .

Definition 4.2.3 *A bounded linear map T of a normed space E into a normed space F is called a **2-quasi- λ -nuclear map** if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that*

$$\|Tx\| \leq \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2},$$

for all x in E .

Proposition 4.2.1 *Each pseudo- λ -nuclear map is a nuclear map.*

Proof. This is clear from definitions and our assumption that $\lambda \subseteq \ell_1$. ■

Next, we show

Proposition 4.2.2 *The dual map of a pseudo- λ -nuclear map is a pseudo- λ -nuclear map.*

Proof. Let $T : E \rightarrow F$ be a pseudo- λ -nuclear map between normed spaces. Then there exist a sequence (α_n) in λ and a bounded sequences $(a_n), (y_n)$ in E' and F respectively such that

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n,$$

for all x in E . Since the dual map T' of the map T from F' into E' is given by

$$\begin{aligned} (T'b)y &= b(Ty) \\ &= b\left(\sum_n \alpha_n \langle y, a_n \rangle y_n\right) \\ &= \sum_n \alpha_n \langle y, a_n \rangle \langle y_n, b \rangle \\ &= \sum_n \alpha_n \langle y_n, b \rangle a_n(y). \end{aligned}$$

So we have

$$T'b = \sum_n \alpha_n \langle y_n, b \rangle a_n.$$

Since (y_n) and (a_n) are bounded sequences in F'' and E' respectively, we get the pseudo- λ -nuclearity of T' . ■

Proposition 4.2.3 *Each quasi- λ -nuclear map is a quasi-nuclear map.*

Corollary 4.2.1 *Each quasi- λ -nuclear map T of a normed space E into a Banach space F is factored through D_μ for some $\mu \in \ell_2$, that is, there are bounded linear maps S_1 from E into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$, where D_μ is the diagonal map from ℓ_∞ into ℓ_2 .*

Proof. Since T is a quasi- λ -nuclear map from a normed space E into a Banach space F , by Proposition 4.2.3, T is a quasi-nuclear map. So by Corollary 3.4.3, we get our result. ■

Proposition 4.2.4 *Each 2-quasi- λ -nuclear map is a 2-quasi-nuclear map.*

4.3 Relations Between Pseudo- λ -Nuclear Maps and Quasi- λ -Nuclear Maps

We begin this section by proving the following proposition.

Proposition 4.3.1 *If $T : E \rightarrow F$ is a bounded linear map between normed spaces and if S is a pseudo- λ -nuclear map from F into a normed space G , then ST is a pseudo- λ -nuclear map.*

Proof. Since S is a pseudo- λ -nuclear map, there exist a sequence (α_n) in λ and bounded sequences (a_n) and (y_n) in F' and G respectively such that

$$Sy = \sum_n \alpha_n \langle y, a_n \rangle y_n,$$

for all y in F . Let $x \in E$ be given. Then we have

$$\begin{aligned} S(Tx) &= \sum_n \alpha_n \langle Tx, a_n \rangle y_n \\ &= \sum_n \alpha_n \langle x, T'a_n \rangle y_n. \end{aligned}$$

Let $b_n = T'a_n$. Then we have

$$S(Tx) = \sum_n \alpha_n \langle x, b_n \rangle y_n.$$

Since (b_n) is a bounded sequence in E' , ST is a pseudo- λ -nuclear map. ■

Proposition 4.3.2 *If $T : E \rightarrow F$ is a pseudo- λ -nuclear map between normed spaces and if S is a bounded linear map from F into a normed space G , then ST is a pseudo- λ -nuclear map.*

Proof. Similiar to Proposition 4.3.1.

Proposition 4.3.3 *If $T : E \rightarrow F$ is a quasi- λ -nuclear map between normed spaces and if S is a bounded linear map from F into a normed space G , then ST is a quasi- λ -nuclear map.*

Proof. Since T is a quasi- λ -nuclear map, there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \sum_n |\alpha_n| |\langle x, a_n \rangle|.$$

Let $x \in E$ be given. Then we have

$$\begin{aligned} ||STx|| &\leq ||S|| ||Tx|| \\ &\leq ||S|| \sum_n |\alpha_n| |\langle x, a_n \rangle|. \end{aligned}$$

Let $b_n = ||S|| a_n$. Then

$$||STx|| \leq \sum_n |\alpha_n| |\langle x, b_n \rangle|.$$

Since (b_n) is a bounded sequence in E' , ST is a quasi- λ -nuclear map. ■

Proposition 4.3.4 *If $T : E \rightarrow F$ is a bounded linear map between normed spaces and if S is a quasi- λ -nuclear map from F into a normed space G , then ST is a quasi- λ -nuclear map.*

Proof. Since S is a quasi- λ -nuclear map, there exist a sequence (α_n) in λ and a bounded sequence (a_n) in F' such that

$$||Sy|| \leq \sum_n |\alpha_n| |\langle y, a_n \rangle|,$$

for all y in F . Let $x \in E$ be given. Then we have

$$\begin{aligned} ||STx|| &\leq \sum_n |\alpha_n| |\langle Tx, a_n \rangle| \\ &= \sum_n |\alpha_n| |\langle x, T'a_n \rangle|. \end{aligned}$$

Let $b_n = T'a_n$. Then we have

$$||STx|| \leq \sum_n |\alpha_n| |\langle x, b_n \rangle|.$$

Since (b_n) is a bounded sequence in E' , ST is a quasi- λ -nuclear map. ■

Now, we give the relations between pseudo- λ -nuclear maps and quasi- λ -nuclear maps.

Proposition 4.3.5 *Each pseudo- λ -nuclear map is a quasi- λ -nuclear map.*

Before proving the following theorem we recall Lemma 3.3.1, which says that each normed space F can be considered as a linear subspace of the Banach space $\ell_\infty(I)$.

Theorem 4.3.1 *Each quasi- λ -nuclear map $T : E \rightarrow F$ between a normed spaces E and F is also pseudo- λ -nuclear if it is regarded as a map from E into a Banach space $\ell_\infty(I)$ containing F .*

Proof. Since T is a quasi- λ -nuclear map, there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \sum_n |\alpha_n| |\langle x, a_n \rangle|,$$

for all x in E . Let $G_0 = \{(\langle x, \alpha_n a_n \rangle) : x \in E\}$. Then G_0 forms a subspace of ℓ_1 . Define a map $S_0 : G_0 \rightarrow F$ by $S_0(\langle x, \alpha_n a_n \rangle) = Tx$. Then S_0 is a continuous linear map from G_0 into F with $\|S_0\| \leq 1$ because

$$\begin{aligned} \|S_0(\langle x, \alpha_n a_n \rangle)\| &= \|Tx\| \\ &\leq \sum_n |\alpha_n| |\langle x, a_n \rangle| \\ &= \sum_n |\langle x, \alpha_n a_n \rangle| \\ &= \|(\langle x, \alpha_n a_n \rangle)\|_1. \end{aligned}$$

So by Lemma 3.3.2, the map S_0 can be extended to a continuous linear map S from ℓ_1 into a Banach space $\ell_\infty(I)$ with $\|S\| \leq 1$. Let $y_n = Se_n$, where (e_n) is the standard basis of ℓ_1 . Then

$$\begin{aligned} \|y_n\|_\infty &= \|Se_n\|_\infty \\ &\leq \|S\| \|e_n\|_1 \leq 1. \end{aligned}$$

Also the map S has the representation $S(\zeta_n) = \sum_n \zeta_n y_n$, for $(\zeta_n) \in \ell_1$. Since $Tx = S(\langle x, \alpha_n a_n \rangle)$. Then the map T has the form

$$\begin{aligned} Tx &= \sum_n \langle x, \alpha_n a_n \rangle y_n \\ &= \sum_n \alpha_n \langle x, a_n \rangle y_n. \end{aligned}$$

Since $(\alpha_n) \in \lambda$ and (a_n) and (y_n) are bounded sequences in E' and $\ell_\infty(I)$ respectively, T is a pseudo- λ -nuclear map from E into $\ell_\infty(I)$. ■

Corollary 4.3.1 *Let $T : E \rightarrow F$ be a bounded linear map between normed spaces with the property that*

$$\|Tx\|^2 \leq \sum_n |\alpha_n|^2 |\langle x, a_n \rangle|^2,$$

for all x in E , where $(\alpha_n) \in \lambda$ and (a_n) is a bounded sequence in E' . Then the map T is a pseudo- λ -nuclear if it is regarded as a map from E into a Banach space $\ell_\infty(I)$.

Proof. Since T is a bounded linear map with the given property, T is a quasi- λ -nuclear map. So by Theorem 4.3.1, T is pseudo- λ -nuclear map from a normed space E into a Banach space $\ell_\infty(I)$. ■

Theorem 4.3.2 *If $T : E \rightarrow F$ is a quasi- λ -nuclear map between normed spaces and if S is a quasi- λ -nuclear map from F into a Banach space G , then ST is a pseudo- λ -nuclear map.*

Proof. Since S is a quasi- λ -nuclear map, by Corollary 4.2.1, S can be factored through $D_\mu : \ell_\infty \rightarrow \ell_2$ for some μ in ℓ_2 , that is, there are two bounded linear maps S_1 from F into ℓ_∞ and S_2 from ℓ_2 into G such that $S = S_2 D_\mu S_1$. So by Theorem 4.3.1, Proposition 4.3.3, and Proposition 4.3.2, we get the pseudo- λ -nuclearity of ST . ■

Theorem 4.3.3 *A bounded linear map T from a normed space E into a normed space F is a nuclear map if and only if it is a pseudo- ℓ_1 -nuclear map.*

Proof. ($\Rightarrow$) Assume that T is a nuclear map. Then there are two sequences (a_n) and (y_n) in E' and F respectively with $\sum_n \|a_n\| \|y_n\| < +\infty$ such that

$$Tx = \sum_n \langle x, a_n \rangle y_n,$$

for all x in E . Let $b_n = a_n / \|a_n\|$, $z_n = y_n / \|y_n\|$, and $\alpha_n = \|a_n\| \|y_n\|$. Then

$$Tx = \sum_n \alpha_n \langle x, b_n \rangle z_n,$$

for all x in E . Since $(\alpha_n) \in \ell_1$ and $(b_n), (z_n)$ are bounded sequences in E' and F respectively, T is a pseudo- ℓ_1 -nuclear map.

($\Leftarrow$) By using Proposition 4.2.1. ■

Theorem 4.3.4 *A bounded linear map $T : E \rightarrow F$ between normed spaces is a quasi-nuclear map if and only if it is a quasi- ℓ_1 -nuclear map.*

Proof. ($\Rightarrow$) Assume that T is a quasi-nuclear map. Then there exists a sequence (a_n) in E' with $\sum_n \|a_n\| < +\infty$ such that

$$\|Tx\| \leq \sum_n |\langle x, a_n \rangle|,$$

for all x in E . Let $b_n = a_n/\|a_n\|$ and $\alpha_n = \|a_n\|$. Then

$$\|Tx\| \leq \sum_n |\alpha_n| |\langle x, b_n \rangle|.$$

Since $(\alpha_n) \in \ell_1$ and (b_n) is bounded sequence in E' , T is a quasi- ℓ_1 -nuclear map.

($\Leftarrow$) By using Proposition 4.2.3. ■

The following Theorem in Pietsch [27] is consequence of Theorem 4.3.2.

Theorem 4.3.5 *If $T : E \rightarrow F$ is a quasi-nuclear map between a normed spaces and if S is a quasi-nuclear map from F into a Banach space G , then ST is a nuclear map.*

Proof. Since T and S are quasi-nuclear, they are quasi- ℓ_1 -nuclear. So by Theorem 4.3.2, we have the pseudo- ℓ_1 -nuclearity of ST . Therefore ST is a nuclear map. ■

4.4 Relations Between Quasi- λ -Nuclear Maps and 2-Quasi- λ -Nuclear Maps

We begin the present section by proving the following theorem

Theorem 4.4.1 *Each quasi- λ -nuclear map is a 2-quasi- λ -nuclear map.*

Proof. Let $T : E \rightarrow F$ be a quasi- λ -nuclear map between normed spaces E and F . Then there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \sum_n |\alpha_n| |\langle x, a_n \rangle|,$$

for all x in E . So by Hölder's inequality, we have

$$||Tx|| \leq \left(\sum_n |\alpha_n| \right)^{1/2} \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Let $\beta = \sum_n |\alpha_n|$. Then

$$||Tx|| \leq \left(\sum_n |\alpha_n| |\langle x, \sqrt{\beta} a_n \rangle|^2 \right)^{1/2}.$$

Since $(\alpha_n) \in \lambda$ and $(\sqrt{\beta} a_n)$ is a bounded sequence in E' , T is a 2-quasi- λ -nuclear map. ■

Theorem 4.4.2 *Each 2-quasi- λ -nuclear map T from a normed space E into a Banach space F can be written as*

$$Tx = \sum_n \sqrt{|\alpha_n|} \langle x, a_n \rangle y_n$$

for some $(\alpha_n) \in \lambda$ and a bounded sequences (a_n) and (y_n) in E' and F respectively.

Proof. Since T is a 2-quasi- λ -nuclear map, there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Let $G_0 = \{(\langle x, \sqrt{|\alpha_n|} a_n \rangle) : x \in E\}$. Then G_0 is a linear subspace of ℓ_2 . Define a map $S_0 : G_0 \rightarrow F$ by $S_0(\langle x, \sqrt{|\alpha_n|} a_n \rangle) = Tx$. Then S_0 is a bounded

linear map with $\|S_0\| \leq 1$ because

$$\begin{aligned} \|S_0(\langle x, \sqrt{|\alpha_n|} a_n \rangle)\| &= \|Tx\| \\ &\leq \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2} \\ &= \|(\langle x, \sqrt{|\alpha_n|} a_n \rangle)\|_2. \end{aligned}$$

Since ℓ_2 is a Hilbert space, S_0 can be extended to a continuous linear map S from ℓ_2 into F with $\|S\| \leq 1$. Let $y_n = Se_n$, where (e_n) is the standard basis of ℓ_2 . Then

$$\|y_n\| = \|Se_n\| \leq \|S\| \|e_n\|_2 \leq 1.$$

The map S has the representation $S(\zeta_n) = \sum_n \zeta_n y_n$ for $(\zeta_n) \in \ell_2$. Since $Tx = S(\langle x, \sqrt{|\alpha_n|} a_n \rangle)$, then the map T has the form

$$Tx = \sum_n \sqrt{|\alpha_n|} \langle x, a_n \rangle y_n,$$

where $(\alpha_n) \in \lambda$ and $(a_n), (y_n)$ are bounded sequences in E' and F respectively. ■

Theorem 4.4.3 *If $T : E \rightarrow F$ is a bounded linear map between normed spaces and if S is a 2-quasi- λ -nuclear map from F into a normed space G , then ST is a 2-quasi- λ -nuclear map*

Proof. Since S is a 2-quasi- λ -nuclear map, then there exist a sequence (α_n) in λ and a bounded sequence (a_n) in F' such that

$$\|Sy\| \leq \left(\sum_n |\alpha_n| |\langle y, a_n \rangle|^2 \right)^{1/2},$$

for all y in F . Let $x \in E$ be given. Then we have

$$\|STx\| \leq \left(\sum_n |\alpha_n| |\langle Tx, a_n \rangle|^2 \right)^{1/2}.$$

Let $b_n = T'a_n$. Then we have

$$||STx|| \leq \left(\sum_n |\alpha_n| |\langle x, b_n \rangle|^2 \right)^{1/2}.$$

Since (b_n) is a bounded sequence in E' , T is a 2-quasi- λ -nuclear map. ■

Theorem 4.4.4 *If T is a 2-quasi- λ -nuclear map from a normed space E into a normed space F and if S is a bounded linear map from F into a normed space G , then ST is a 2-quasi- λ -nuclear map.*

Proof. Since T is a 2-quasi- λ -nuclear map, there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Let $b_n = ||S||a_n$. Then

$$||STx|| \leq \left(\sum_n |\alpha_n| |\langle x, ||S||a_n \rangle|^2 \right)^{1/2},$$

for all x in E . Since (b_n) is a bounded sequence in E' , ST is a 2-quasi- λ -nuclear map. ■

Theorem 4.4.5 *A bounded linear map T from a normed space E into a normed space F is a 2-quasi- ℓ_1 -nuclear map if and only if it is a 2-quasi-nuclear map.*

Proof ($\Rightarrow$) By using Proposition 4.2.4.

($\Leftarrow$) Assume that T is a 2-quasi-nuclear map. Then there exists a sequence (a_n) in E' with $\sum_n ||a_n||^2 < +\infty$ such that

$$||Tx|| \leq \left(\sum_n |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

Let $b_n = a_n/||a_n||$ and $\alpha_n = ||a_n||^2$. Then

$$||Tx|| \leq \left(\sum_n |\alpha_n| |\langle x, b_n \rangle|^2 \right)^{1/2}.$$

Since $(\alpha_n) \in \ell_1$ and (b_n) is a bounded sequence in E' , T is a 2-quasi- ℓ_1 -nuclear map. ■

Corollary 4.4.1 *Let T be a 2-quasi- λ -nuclear map from a normed space E into a Banach space F . Then T can be factored through $D_\mu : \ell_\infty \rightarrow \ell_2$ for some $\mu \in \ell_2$, that is, there are two bounded maps S_1 from F into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$.*

Proof. This follows from Proposition 4.2.4 and Theorem 3.3.4. ■

Theorem 4.4.6 *Suppose $\lambda = \lambda(P_0)$ is a nuclear G_∞ -space, a bounded linear map between normed spaces is a quasi- λ -nuclear map if and only if it is a 2-quasi- λ -nuclear map.*

Proof. ($\Rightarrow$) By using Theorem 4.4.1.

($\Leftarrow$) Let $T : E \rightarrow F$ be a 2-quasi- λ -nuclear map between normed spaces E and F . Then there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum_n |\alpha_n| |\langle x, a_n \rangle|^2 \right)^{1/2}.$$

So we have

$$||Tx|| \leq \sum_n \sqrt{|\alpha_n|} |\langle x, a_n \rangle|.$$

It is enough to show that $(\sqrt{|\alpha_n|}) \in \lambda$. Let $a \in P_0$ be given. We choose $b \in P_0$ such that $(a_n)^2 = O(b_n)$. Therefore there is $\gamma > 0$ such that

$$|\alpha_n| (a_n)^2 \leq \gamma |\alpha_n| b_n$$

holds for all $n \in \mathbf{N}$. Since $(\alpha_n) \in \lambda$ there is a $\beta > 0$ such that $|\alpha_n|b_n \leq \beta$ for all $n \in \mathbf{N}$. Therefore we have $\sqrt{|\alpha_n|}a_n \leq \sqrt{\beta\gamma}$ for all $n \in \mathbf{N}$. So $(\sqrt{|\alpha_n|}) \in B(P_0) = \lambda(P_0)$. Therefore T is a quasi- λ -nuclear map. ■

Theorem 4.4.7 *If $T : E \rightarrow F$ is a quasi- λ -nuclear map between normed spaces and if S is a 2-quasi- λ -nuclear map from F into a Banach space G , then ST is a pseudo- λ -nuclear map.*

Proof. Since S is a 2-quasi- λ -nuclear map, S can be factored through a diagonal map $D_\mu : \ell_\infty \rightarrow \ell_2$ for some $\mu \in \ell_2$, that is, there are two bounded linear maps S_1 from F into ℓ_∞ and S_2 from ℓ_2 into G such that $S = S_2 D_\mu S_1$. Then by using Proposition 4.3.3, Theorem 4.3.1, and Proposition 4.3.2, we get the pseudo- λ -nuclearity of ST . ■

Remark. Theorem 3.4.3 from chapter 3 is a consequence of Theorem 4.4.7.

We use the following Proposition to give an example of a 2-quasi-nuclear map which is not a 2-quasi- λ -nuclear map.

Definition 4.4.1 *Let (d_n) be a non-negative bounded sequence. The diagonal map D on ℓ_1 into itself is defined by $D(x_n) = (x_n d_n)$.*

Proposition 4.4.1 [28] *A diagonal map $D = (d_n)$ with $d_n \geq 0$, is a pseudo- $\Lambda_\infty(\alpha)$ -nuclear map from ℓ_1 into ℓ_1 if and only if the d_n which are different from zero can be rearranged into a sequence in $\Lambda_\infty(\alpha)$.*

Now we give an example of a 2-quasi-nuclear map which is not a 2-quasi- λ -nuclear map.

Example 4.4.1 *Define a map $D : \ell_1 \rightarrow \ell_2$ by $Dx = (x_n/2^n)$. Then D is a 2-quasi-nuclear map which is not a 2-quasi- $\Lambda_\infty(n)$ -nuclear map.*

Proof. To show that D is a 2-quasi-nuclear map, let $a_n = (e_n/2^n)$. Then

$$\begin{aligned} \|Dx\|_2^2 &= \left\| \left(\frac{x_n}{2^n} \right) \right\|_2^2 \\ &= \sum_n \left| \frac{x_n}{2^n} \right|^2 \\ &= \sum_n |\langle x, a_n \rangle|^2. \end{aligned}$$

Since (a_n) is a sequence in ℓ_∞ with $\sum_n \|a_n\|_\infty^2 < +\infty$, D is a 2-quasi-nuclear map. To show that D is not a 2-quasi- $\Lambda(n)$ -nuclear map, define a map $A : \ell_2 \rightarrow \ell_1$ by putting $Ax = (x_n/2^n)$. Then A is quasi-nuclear. Therefore A is 2-quasi-nuclear. By Theorem 3.3.4, A can be factored through D_μ for some $\mu \in \ell_2$, that is, there are bounded maps $S_2 : \ell_2 \rightarrow \ell_\infty$, $D_\mu : \ell_\infty \rightarrow \ell_2$, and $S_1 : \ell_2 \rightarrow \ell_1$ such that $A = S_1 D_\mu S_2$. If we assume that D is a 2-quasi- $\Lambda_\infty(n)$ -nuclear, then by Theorem 4.4.6, D is quasi- $\Lambda_\infty(n)$ -nuclear. Therefore $S_2 D$ is quasi- $\Lambda_\infty(n)$ -nuclear. So by Theorem 4.3.1, $S_2 D$ is pseudo- $\Lambda_\infty(n)$ -nuclear. Thus AD is pseudo- $\Lambda_\infty(n)$ -nuclear. Since $AD : \ell_1 \rightarrow \ell_1$ is given by $ADx = (x_n/4^n)$ and AD is pseudo- $\Lambda_\infty(n)$ -nuclear, we have $(1/4^n) \in \Lambda_\infty(n)$, which is a contradiction. So A is not 2-quasi- $\Lambda_\infty(n)$ -nuclear. ■

Now, we give an example of a 2-quasi- λ -nuclear map which is not quasi- λ -nuclear. To prove our example we need the following result.

Lemma 4.4.1 [20; P.23] *Suppose that T is a linear map from a normed space E into a normed space F . Then $d_n(T) \leq \alpha_n(T) \leq \sqrt{n} d_n(T)$.*

Lemma 4.4.2 [20; P.23] *Suppose that T is a compact map from a Banach space X into a Banach space F . Then $\alpha_n(T) = \alpha_n(T')$.*

Lemma 4.4.3 [20; P.23] *Suppose that T is a compact map from a Hilbert space H_1 into a Hilbert space H_2 . Then $\alpha_n(T) = d_n(T)$.*

Now we will give an example of a 2-quasi- λ -nuclear map which is not a quasi- λ -nuclear.

Example 4.4.2 Let $P = \left\{ \left(n^{\ln(kn)} \right) : k = 1, 2, \dots \right\}$. Define the map D on ℓ_2 by $Dx = (\sqrt{\alpha_n} x_n)$, where

$$\alpha_n = \frac{1}{n^2 n^{\ln(n^2)}}.$$

Then we have the following assertions :

1. $\lambda(P)$ is a nuclear Köthe space which is subset of ℓ_1 .
2. There are no $\rho > 0$ and $m \in \mathbb{N}$ such that the inequality

$$n^{\ln(n^2)} \leq \rho n^{\ln(mn)}$$

holds for all $n \in \mathbb{N}$.

3. $(\alpha_n) \in \lambda(P)$ and $(\sqrt{\alpha_n}) \notin \lambda(P)$.
4. D is a 2-quasi- $\lambda(P)$ -nuclear map.
5. D is not a quasi- $\lambda(P)$ -nuclear map.

Proof. To prove 1, let $(n^{\ln(kn)}) \in \lambda(P)$ be given. Then choose $m = 9k$. So we have

$$\frac{n^{\ln(kn)}}{n^{\ln(9kn)}} = n^{-\ln 9} \leq \frac{1}{n^2}.$$

So we have

$$\left(\frac{n^{\ln(kn)}}{n^{\ln(mn)}} \right) \in \ell_1.$$

Therefore $\lambda(P)$ is a nuclear Köthe space. Also it is clear that $\lambda(P) \subseteq \ell_1$. To prove 2, assume that there are $\rho > 0$ and $m \in \mathbb{N}$ such that

$$n^{\ln(n^2)} \leq \rho n^{\ln(mn)},$$

for all $n \in \mathbb{N}$. Then we have

$$\frac{n^{\ln(n^2)}}{n^{\ln(mn)}} \leq \rho,$$

for all $n \in \mathbb{N}$. So we get

$$\lim_{n \rightarrow \infty} n^{\ln(\frac{n}{m})} = \infty \leq \rho,$$

which is a contradiction. To prove 3, let $k \in \mathbb{N}$ be given. Then we have

$$\lim_{n \rightarrow \infty} \frac{n^{\ln(k)}}{n^{\ln(n)}} = 0.$$

So we get

$$\frac{n^{\ln(kn)}}{n^2 n^{\ln(n^2)}} = O\left(\frac{1}{n^2}\right).$$

Therefore

$$\sum_n \frac{n^{\ln(kn)}}{n^2 n^{\ln(n^2)}} < +\infty.$$

So we get $(\alpha_n) \in \lambda(P)$. Since

$$\frac{n^{\ln(n)}}{n n^{\ln(n)}} = \frac{1}{n},$$

we get $(\sqrt{\alpha_n}) \notin \lambda(P)$. Since $(\alpha_n) \in \lambda(P)$ and (e_n) is a bounded sequence in ℓ_2 , D is a 2-quasi- $\lambda(P)$ -nuclear map. To prove 5, assume that D is a quasi- $\lambda(P)$ -nuclear map. Then there exist a sequence $(\beta_n) \in \lambda(P)$ and a bounded sequence (a_n) in ℓ_2 such that

$$\|Dx\| \leq \sum_n |\beta_n| |\langle x, a_n \rangle|.$$

Let $\gamma_n = \sum_{m=n}^{\infty} |\beta_m|$. We claim that, $\gamma = (\gamma_n) \in \lambda(P)$. For $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_n |\gamma_n| n^{\ln(kn)} &= \sum_{n=1}^{\infty} \left(\sum_{m=n}^{\infty} |\beta_m| \right) n^{\ln(kn)} \\ &= \sum_{n=1}^{\infty} \left(|\beta_n| \sum_{i=1}^n i^{\ln(ki)} \right) \\ &\leq \sum_n |\beta_n| n n^{\ln(kn)}. \end{aligned}$$

If we choose $m \in \mathbb{N}$ so that $n \leq n^{\ln m}$, then we have

$$\sum_n |\gamma_n| n^{\ln(kn)} \leq \sum_n |\beta_n| n^{\ln(mkn)} < +\infty.$$

Therefore $\gamma = (\gamma_n) \in \lambda(P)$. Let

$$M_n = \{x \in \ell_2 : \langle x, a_i \rangle = 0, i = 1, 2, \dots, n\}.$$

If $x \in M_n$, then

$$\|Dx\| \leq \sum_{m=n}^{\infty} |\beta_m| |\langle x, a_m \rangle| \leq \gamma_n \sup_n \|a_n\| \|x\|.$$

Hence $D(U \cap M_n) \subseteq \gamma_n U$, where U is the unit ball of ℓ_2 . Therefore

$$D'(U^\circ) \subseteq \gamma_n U^\circ + M_n^\perp,$$

which gives $\alpha_n(T) = d_n(D'(U^\circ), U^\circ) \leq \gamma_n$. Since $\alpha_n(T) = \sqrt{\alpha_n} \leq \gamma_n$ and $(\sqrt{\alpha_n}) \notin \lambda(P)$, we have $(\gamma_n) \notin \lambda(P)$, which is a contradiction. Therefore D is not a quasi- $\lambda(P)$ -nuclear map. ■

Chapter 5

λ -Nuclearity

5.1 Introduction

In [13], Dubinsky and Ramanujan proved that every λ -nuclear map from a Banach space E into a reflexive Banach space F is compact. In this chapter, we improve this result by proving that if T is a λ -nuclear map from a Banach space E into a Banach space F and if F contains no (isomorphic) copy of c_0 , then T is a compact map. Also, we give an example to show that the assumption that F does not contain a copy of c_0 is essential. Also in [32], Robinson proved that a map T from a Banach space E into a Banach space F is $\Lambda_1(\alpha)$ -nuclear if and only if it is pseudo- $\Lambda_1(\alpha)^\times$ -nuclear. In Section 3 of this chapter, we are going to prove that, given a nuclear G_1 -space $\lambda(Q)$, a map T from a Banach space E into a Banach space F is λ -nuclear if and only if T is pseudo- $\lambda(Q)^\times$ -nuclear, which gives us an improvement of Robinson's result. In the last section of this chapter, we are going to characterize pseudo- $\lambda(P_0)$ -nuclearity for a sequence space of the form $\lambda_1 \vee \lambda_2$, where $\lambda(P_0)$ is a stable nuclear G_∞ -space and λ_1, λ_2 are smooth sequence spaces of the same

type. Also, we prove that a G_1 -space $\lambda(Q)$ is nuclear if and only if it is s -nuclear. Also, we will prove that for given smooth sequence spaces λ_1 and λ_2 of the same type, the sequence space $\lambda_1 \vee \lambda_2$ is pseudo- $\lambda(P_0)$ -nuclear if and only if λ_1 and λ_2 are pseudo- $\lambda(P_0)$ -nuclear, where $\lambda(P_0)$ is a nuclear stable G_∞ -space.

5.2 Definitions

In present chapter, letter λ stands for a fixed sequence space.

Definition 5.2.1 [7] *We say that a Banach space E has the **Schur property** if every weakly convergent sequence in E is norm convergent.*

Definition 5.2.2 [13][28] *A linear map T from a normed space E into a normed space F is called a **λ -nuclear map** if there exist a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that (a_n) is bounded and (y_n) has the property that for each $b \in F'$, $(\langle y_n, b \rangle) \in \lambda^\times$ and such that*

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n,$$

for all $x \in E$.

Definition 5.2.3 [13][28] *A linear map T from a normed space E into a normed space F is called a **pseudo- λ -nuclear map** if there exist a sequence (α_n) in λ and a bounded sequences (a_n) and (y_n) in E' and F respectively such that*

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n,$$

for all $x \in E$.

Definition 5.2.4 [13][28][37] *A locally convex space E is called a **pseudo- λ -nuclear space** if for each barreled zero neighbourhood U in E there exists another one V absorbed by U such that the canonical map $E(V, U)$ is a pseudo- λ -nuclear map.*

Definition 5.2.5 [7] *A series $\sum_n x_n$ in a Banach space E is called **weakly unconditionally Cauchy** if*

$$\sum_n |\langle x_n, a \rangle| < +\infty,$$

for all $a \in E'$.

5.3 Relations Between λ -Nuclear Maps and Pseudo- λ -Nuclear Maps

The following well-known results in Banach space theory are crucial for our subsequent arguments.

Theorem 5.3.1 [7] *Every weakly unconditionally Cauchy series $\sum_n x_n$ in a Banach space E is unconditionally convergent if and only if E contains no (isomorphic) copy of c_0 .*

Lemma 5.3.1 [13] *Let F be a Banach space, λ sequence space and $(\alpha_n) \in \lambda$. Let (y_n) be a sequence in F such that $(\langle y_n, b \rangle) \in \lambda^\times$ for each $b \in F'$. Then*

$$\sup_{b \in F', \|b\| \leq 1} \sum_n |\alpha_n \langle y_n, b \rangle| < +\infty.$$

Remark. Let (α_n) be an element of a sequence space λ , (y_n) be a sequence in a Banach space F , and $(\langle y_n, b \rangle) \in \lambda^\times$ for all $b \in F$. Then by using Lemma 5.3.1, and Theorem 5.3.1, we know that series $\sum_n \alpha_n y_n$ is unconditionally convergent if F has no copy of c_0 .

Theorem 5.3.2 [7] (*Schur's Theorem*) *If a sequence (x_n) in ℓ_1 is weakly Cauchy (which means that $\lim_n \langle x_n, y \rangle$ exists for each $y \in \ell_\infty$), then (x_n) is norm convergent.*

Remark. According to Schur's Theorem, the space ℓ_1 has Schur's property.

Definition 5.3.1 [13] *We say that a Köthe set P of sequences is a **power set** if*

1. P is countable,
2. the locally convex space $\lambda(P)$ is nuclear,
3. there exists $a \in P$ such that $a_n \geq 1$ for all $n \in \mathbb{N}$,
4. for each $a, b \in P$ there exists $c \in P$ such that $a_n b_n = O(c_n)$, and
5. for each $a \in P$, $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 5.3.2 [13] *If P is a power set, then there exists $(\alpha_n) \in \lambda(P)$ with $\alpha_n \neq 0$ for all $n \in \mathbb{N}$.*

Lemma 5.3.3 *Let $\sum_n x_n$ be an unconditionally convergent series in a Banach space E . Define a map $S : \ell_\infty \rightarrow E$ by putting $S\eta = \sum_n \eta_n x_n$. Then S is a compact map. In fact, we have $\lim_n \|S - S_n\| = 0$, where S_n is the finite rank linear map from ℓ_∞ to E defined by $S_n\eta = \sum_{k=1}^n \eta_k x_k$.*

Proof. To prove this, let $R : c_0 \rightarrow E$ be the restriction of S to the subspace c_0 of ℓ_∞ . Then its adjoint $R' : E' \rightarrow \ell_1$ is given by $R'a = (\langle x_n, a \rangle)$. Indeed, writting $\eta_n = \langle x_n, a \rangle$, we have, for all $\zeta \in c_0$,

$$\begin{aligned} \langle \zeta, R'a \rangle &= \langle R\zeta, a \rangle \\ &= \sum_n \zeta_n \langle x_n, a \rangle \\ &= \sum_n \zeta_n \eta_n = \langle \zeta, \eta \rangle, \end{aligned}$$

and hence $R'a = \eta$. Clearly R' is continuous when both E' and ℓ_1 are equipped with weak*-topology. We claim that R' is also continuous when E' is equipped with weak*-topology while ℓ_1 is equipped with the weak topology. To establish this claim, it suffices to check that, for each $\zeta \in \ell_\infty$. The linear functional $a \mapsto \langle \zeta, R'a \rangle$ defined on E' is continuous with respect to the weak*-topology. This follows immediately from the identity $\langle \zeta, R'a \rangle = \langle S\zeta, a \rangle$ which is easily checked as follows:

$$\begin{aligned} \langle \zeta, R'a \rangle &= \sum_n \zeta_n \langle x_n, a \rangle \\ &= \left\langle \sum_n \zeta_n x_n, a \right\rangle = \langle S\zeta, a \rangle \end{aligned}$$

Now, the closed unit ball E'_1 is compact with respect to the weak*-topology. So, by the continuity of R' we have proved, its image $R'E'_1$ in ℓ_1 is weakly compact. By a version of Schur's Theorem, we see that $R'E'_1$ is compact in the norm topology. This shows that R' is compact. Hence its adjoint $R'' : \ell_\infty \rightarrow E''$ is also compact. Clearly $S : \ell_\infty \rightarrow E$ and $R'' : \ell_\infty \rightarrow E''$ coincide on ℓ_∞ . So S is also compact. Let $R_n : \ell_1 \rightarrow \ell_1$ be the projection defined by

$$R_n \zeta = (\zeta_1, \zeta_2, \dots, \zeta_n, 0, 0, \dots).$$

Using the compactness of $R'E'_1$ in the norm topology we can show that $\sup_{\zeta \in R'E'_1} \|\zeta - R_n \zeta\| \rightarrow 0$ as $n \rightarrow \infty$. Thus $\lim_n \|R' - R_n R'\| = 0$, which gives $\lim_n \|R'' - R'' R'_n\| = 0$. It is easy to check that $R'' R'_n$ coincides with S_n on ℓ_∞ . Therefore $\lim_n \|S - S_n\| = 0$. ■

Theorem 5.3.3 *If T is a λ -nuclear map from a Banach space E into a Banach space F and if F contains no copy of c_0 , then T is a compact map.*

Proof. Since T is a λ -nuclear map, there exists a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that (a_n) is bounded

in E' and $(\langle y_n, b \rangle) \in \lambda^\times$ for all $b \in F'$ and such that

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n.$$

Define a linear maps $R : \lambda \rightarrow F$, $L : E \rightarrow \ell_\infty$ and $D_\alpha : \ell_\infty \rightarrow \lambda$ by putting $R\zeta = \sum_n \zeta_n y_n$, $Lx = (\langle x, a_n \rangle)$, and $D_\alpha \eta = (\alpha_n \eta_n)$. Then clearly $T = RD_\alpha L$. Since $\sum_n \alpha_n y_n$ is unconditionally convergent series in F , by Lemma 5.3.3, the linear map from ℓ_∞ to F sending η to $\sum_n \eta_n \alpha_n y_n$ is compact. It is easy to check that $RD_\alpha \eta = \sum_n \alpha_n \eta_n y_n$ and hence RD_α is compact. Therefore T is compact as well. ■

Now we give an example to show that the assumption that F does not contain a copy of c_0 in Theorem 5.3.2 is essential.

Example 5.3.1 *Let $E = F = c_0$ and $\lambda = \ell_\infty$. Let T be the identity map on c_0 . Then T is λ -nuclear map which is noncompact.*

Proof. Let (e_n) denoted to the standard base of c_0 and (e'_n) denoted to the standard base of ℓ_1 . Then $Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n$, where $\alpha_n = 1$, $a_n = e'_n$, and $y_n = e_n$. Notice that $\alpha \in \ell_\infty = \lambda$, (a_n) is bounded sequence in $c'_0 = \ell_1$ and $(\langle y_n, b \rangle) \in \lambda^\times = \ell_1$ for all $b \in c'_0 = \ell_1$. Therefore T is λ -nuclear. Clearly T is noncompact. ■

Remark. Let $\lambda(P)$ be a Köthe space. Then $\lambda(P)^\times$ is a Köthe space generated by the Köthe set

$$A = \{(|x_n|) : x \in \lambda(P)\}.$$

Theorem 5.3.4 *Given a nuclear Köthe space $\lambda = \lambda(A)$, with the property that $y \in \lambda_\times$ is a nuclear Köthe space, a linear bounded map T from a Banach*

space E into a Banach space F is λ -nuclear if and only if it is a pseudo- $\lambda \lambda^\times$ -nuclear.

Proof. ($\Rightarrow$) Assume that $T : E \rightarrow F$ is λ -nuclear. Then there exist a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that the sequence (a_n) is bounded and $(\langle y_n, b \rangle) \in \lambda^\times$ for all $b \in F'$ and such that

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n.$$

Let $B = \{(\langle y_n, b \rangle) : b \in F', \|b\| \leq 1\}$. Then by Lemma 5.3.1, B is bounded subset of $\lambda^\times$. So by Proposition 2.4.2, there is a z in $\lambda^\times$ such that

$$|\langle y_n, b \rangle| \leq z_n$$

for all $b \in F'$ with $\|b\| \leq 1$. Therefore $\|y_n\| \leq z_n$ for all $n \in \mathbb{N}$. So $(\|y_n\|) \in \lambda^\times$. Let $\beta_n = \alpha_n \|y_n\|$ and $x_n = y_n / \|y_n\|$. Then

$$Tx = \sum_n \beta_n \langle x, a_n \rangle x_n.$$

Since the sequence (β_n) is in $\lambda \lambda^\times$, and since (a_n) and (x_n) are bounded sequences in E' and F respectively, T is a pseudo- $\lambda \lambda^\times$ -nuclear map.

($\Leftarrow$) Assume that T is a pseudo- $\lambda \lambda^\times$ -nuclear map. Then there exist sequences (α_n) in λ and (β_n) in $\lambda^\times$, and bounded sequences (a_n) and (y_n) in E' and F respectively such that

$$Tx = \sum_n \alpha_n \beta_n \langle x, a_n \rangle y_n.$$

Let $z_n = \beta_n y_n$. Then $Tx = \sum_n \alpha_n \langle x, a_n \rangle z_n$. Since $(\alpha_n) \in \lambda$, (a_n) is bounded sequence in E' and $(\langle z_n, b \rangle) \in \lambda^\times$ for all $b \in F'$, T is a λ -nuclear map. ■

Lemma 5.3.4 *If $\lambda(P)$ is a nuclear G_∞ -space, then $\lambda(P) = \lambda(P) \lambda(P)^\times$.*

Proof. Since $(1, 1, \dots) \in \lambda(P)^\times$, we have $\lambda(P) \subseteq \lambda(P) \lambda(P)^\times$. Also, since $xa \in \lambda(P)$ for all $x \in \lambda(P)$ and $a \in P$, we have $\lambda(P) \lambda(P)^\times \subseteq \lambda(P)$. ■

By using Lemma 5.3.2, Lemma 5.3.4, and Theorem 5.3.4, we obtain the following theorem given by Dubinsky and Ramanujan in [13].

Theorem 5.3.5 [13] *Given a power set P , a linear map T from a Banach space E into a Banach space F is a $\lambda(P)$ -nuclear map if and only if it is a pseudo- $\lambda(P)$ -nuclear map.*

Lemma 5.3.5 *If $\lambda(Q)$ is a nuclear G_1 -space, then $\lambda(Q)^\times = \lambda(Q) \lambda(Q)^\times$.*

Proof. Since $(1, 1, \dots) \in \lambda(Q)$, we have $\lambda(Q)^\times \subseteq \lambda(Q) \lambda(Q)^\times$. Also, since $xy \in \lambda(Q)$ for all $x, y \in \lambda(Q)$, we have $\lambda(Q) \lambda(Q)^\times \subseteq \lambda(Q)^\times$. ■

Theorem 5.3.6 *Given a nuclear G_1 -space $\lambda(Q)$, a linear map T from a Banach space E into a Banach space F is $\lambda(Q)$ -nuclear if and only if it is a pseudo- $\lambda(Q)^\times$ -nuclear.*

Proof. This follows from Theorem 5.3.4 and Lemma 5.3.5. ■

Remark. By using Theorem 5.3.6 and taking $\lambda(Q) = \Lambda_1(\alpha)$, we get the following theorem given by Robinson in [32].

Theorem 5.3.7 [32] *A linear map T from a Banach space E into a Banach space F is a $\Lambda_1(\alpha)$ -nuclear map if and only if it is a pseudo- $\Lambda_1(\alpha)^\times$ -nuclear map.*

5.4 Pseudo- λ -Nuclearity

In this section, the letter $\lambda(P_0)$ stands for a fixed nuclear G_∞ -space.

Lemma 5.4.1 [27] *Let P be a linear map from a normed space E into a normed space F with $\|P\| \leq 1$, $P^2 = P$ and $\dim(\text{Range}(P)) = r+1$. Let B be any bounded subset of E such that $U \cap \text{Range}(P) \subseteq \delta^{-1}B$. Then $\delta \leq d_r(B)$.*

Proposition 5.4.1 *If $\lambda(A)$ is a Köthe space, then for any $a, b \in A$ such that U_b absorbs U_a , we have $d_n(U_a(U_b)) \leq d_n(U_a, U_b)$ for all $n \in N$, where $U_a(U_b) = \{x(U_b) : x \in U_a\}$.*

Proof. Let $\delta > 0$ and n be given such that $d_n(U_a, U_b) \leq \delta$. Then there is a linear subspace F of $\lambda(A)$ with dimension at most n such that $U_a \subseteq \delta U_b + F$. Clearly

$$U_a(U_b) \subseteq \delta U_b(U_b) + F(U_b),$$

where $U_a(U_b) = \{x(U_b) : x \in U_a\}$ and $F(U_b) = \{x(U_b) : x \in F\}$. Since $U_b(U_b)$ is the closed unit ball of the normed space $\lambda(A)(U_b)$ and $F(U_b)$ is a linear subspace of $\lambda(A)(U_b)$ with dimension at most n , $d_n(U_a(U_b)) \leq \delta$. Therefore $d_n(U_a(U_b)) \leq d_n(U_a, U_b)$. ■

Proposition 5.4.2 *Suppose that $\lambda(P)$ is a Köthe space with the property that each element a in P is positive. If there is $\delta > 0$ such that*

$$p_a(x) \leq \delta p_b(x)$$

for all x in $L_n = \text{span}\{e_1, e_2, \dots, e_{n+1}\}$, then $1/\delta \leq d_n(U_a, U_b)$.

Proof. Define a map $T : \lambda(P)(U_b) \rightarrow \lambda(P)(U_b)$ by putting

$$Tx(U_b) = \sum_{i=1}^{n+1} x_i e_i(U_b).$$

Then clearly T is a linear map. To show that $\|T\| \leq 1$, let $x(U_b) \in \lambda(P)(U_b)$ be given such that $\|x(U_b)\| = 1$. Then

$$\|Tx(U_b)\| = \left\| \sum_{i=1}^{n+1} x_i e_i(U_b) \right\|$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n+1} |x_i| \|e_i(U_b)\| \\
&= \sum_{i=1}^{n+1} |x_i| p_b(e_i) \\
&= \sum_{i=1}^{n+1} |x_i| b_i.
\end{aligned}$$

Therefore $\|Tx(U_b)\| \leq p_b(x) = \|x(U_b)\|$. So we get $\|T\| \leq 1$. To show that $T^2 = T$, let $x \in \lambda(P)(U_b)$ be given. Then

$$\begin{aligned}
T^2x &= T \sum_{i=1}^{n+1} x_i e_i(U_b) \\
&= \sum_{i=1}^{n+1} x_i e_i(U_b) \\
&= Tx.
\end{aligned}$$

To establish $\dim(\text{Range}(T)) = n + 1$, we prove that

$$\{e_1(U_b), e_2(U_b), \dots, e_{n+1}(U_b)\}$$

is linearly independent. Let $\sum_{i=1}^{n+1} \alpha_i e_i(U_b) = 0$. Then $(\sum_{i=1}^{n+1} \alpha_i e_i)(U_b) = 0$. Therefore

$$p_b \left(\sum_{i=1}^{n+1} \alpha_i e_i \right) = 0 = \sum_{i=1}^{n+1} |\alpha_i| b_i.$$

Since $b_i > 0$ for all i , $|\alpha_i| = 0$. Therefore $\alpha_i = 0$ for all $i = 1, 2, \dots, n + 1$. So $\{e_1(U_b), e_2(U_b), \dots, e_{n+1}(U_b)\}$ is a basis for the range of T . Finally, we claim that

$$\frac{1}{\delta}(U_b(U_b) \cap \text{Range}(T)) \subseteq U_a(U_b).$$

To Show this, let $z \in (1/\delta)(U_b(U_b) \cap \text{Range}(T))$ be given. Then $z = (1/\delta)y$ for some $y = x(U_b) \in U_b(U_b)$. On the other hand

$$\begin{aligned}
y &= \sum_{i=1}^{n+1} \alpha_i e_i(U_b) \\
&= \left(\sum_{i=1}^{n+1} \alpha_i e_i \right) (U_b) \\
&= \alpha(U_b).
\end{aligned}$$

So $y = x(U_b) = \alpha(U_b)$. Therefore $p_b(x - \alpha) = 0$. So we have

$$p_b(\alpha) = p_b(x) \leq 1.$$

Since $p_a(\alpha) \leq \delta p_b(\alpha) \leq \delta$, we have $(1/\delta)\alpha \in U_a$. So $z \in U_a(U_b)$. Therefore

$$\frac{1}{\delta}(U_b(U_b) \cap \text{Range}(T)) \subseteq U_a(U_b).$$

So by Lemma 5.4.1 and Proposition 5.4.1, we have

$$1/\delta \leq d_n(U_a(U_b)) \leq d_n(U_a, U_b). \quad \blacksquare$$

Theorem 5.4.1 [37] *A locally convex space E is pseudo- $\lambda(P_0)$ -nuclear if and only if for every absolutely convex and closed neighbourhood U there is another one V such that $(d_n(V, U)) \in \lambda(P_0)$.*

Theorem 5.4.2 [37] *A locally convex space $\lambda(P)$ is pseudo- $\lambda(P_0)$ -nuclear if and only if for each $a \in P$ there exist a $b \in P$ with $a_n = O(b_n)$ and an injection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sigma(\mathbb{N}) = \{n : a_n \neq 0\}$ and $(a_{\sigma(n)}/b_{\sigma(n)})$ in $\lambda(P_0)$.*

Definition 5.4.1 [13][28][37] *A Köthe space $\lambda(K)$ is called **uniformly pseudo- $\lambda(P_0)$ -nuclear** if there is a 'universal' permutation σ such that for every $a \in K$ there are a $b \in K$ and $x \in \lambda(P_0)$ with $a_{\sigma(n)} \leq x_n b_{\sigma(n)}$ for all $n \in \mathbb{N}$.*

Recall the following definition.

Definition 5.4.2 *Given a two sequence spaces λ_1 and λ_2 we define the sequence space $\lambda_1 \vee \lambda_2$ to be $\{(x_n) : (x_{2n}) \in \lambda_1, (x_{2n-1}) \in \lambda_2\}$.*

Proposition 5.4.3 *If $\lambda(A)$ and $\lambda(B)$ are Köthe spaces, then $\lambda(A) \vee \lambda(B)$ is a Köthe space generated by a Köthe set*

$$(B, A) = \{(b_1, a_1, b_2, a_2, \dots) : (b_n) \in B, (a_n) \in A\}.$$

Proposition 5.4.4 *If $\lambda(A)$ and $\lambda(B)$ are Köthe spaces and if*

$$c = (b_1, a_1, b_2, a_2, \dots) \quad \text{and} \quad c' = (b'_1, a'_1, b'_2, a'_2, \dots)$$

are in (B, A) with $c_n = O(c'_n)$, then $d_n(U_{a'}, U_a) \leq 2d_n(U_{c'}, U_c)$.

Proof. Let $\delta > 0$ and $n \in N$ be given such that $d_n(U'_c, U_c) \leq \delta$. Then there is a linear subspace F of $\lambda(B, A)$ with dimension at most n such that $U_{c'} \subseteq \delta U_c + F$. Let $F_1 = \{(x_{2n}) : x \in F\}$. Then F_1 is a linear subspace of $\lambda(A)$ with dimension at most n . We claim that $U_{a'} \subseteq 2\delta U_a + F_1$. For $x \in U_{a'}$, we choose $y \in \lambda(B)$ such that $p_{\nu}(y) \leq 1$. Then $(y_1, x_1, y_2, x_2, \dots) \in 2U_{c'}$. Therefore

$$(y_1, x_1, y_2, x_2, \dots) = 2\delta(y'_1, x'_1, y'_2, x'_2, \dots) + (z_1, w_1, z_2, w_2, \dots)$$

for some $(y'_1, x'_1, y'_2, x'_2, \dots) \in U_c$ and $(z_1, w_1, z_2, w_2, \dots) \in F$. So

$$(x_1, x_2, \dots) = 2\delta(x'_1, x'_2, \dots) + (w_1, w_2, \dots) \in 2\delta U_a + F_1.$$

Therefore $U_{a'} \subseteq 2\delta U_a + F_1$. So $d_n(U_{a'}, U_a) \leq 2\delta$. Since δ is arbitrary, we have $d_n(U_{a'}, U_a) \leq 2d_n(U_{c'}, U_c)$. ■

In the rest of this section, we assume that $\lambda(P_0)$ is also stable.

Theorem 5.4.3 *If $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces, then the following conditions are equivalent:*

1. $\lambda(B, A)$ is pseudo- $\lambda(P_0)$ -nuclear.

2. For every $c \in (B, A)$ there is a $c' \in (B, A)$ with $c_n = O(c'_n)$ such that $(1/c'_n) \in \lambda(P_0)$.
3. There is a $c \in (B, A)$ with $(1/c_n) \in \lambda(P_0)$.
4. $\lambda(B, A)$ is uniformly pseudo- $\lambda(P_0)$ -nuclear.

Proof. $(1 \Rightarrow 2)$ Assume that $\lambda(B, A)$ is pseudo- $\lambda(P_0)$ -nuclear. Then for each $c = (b_1, a_1, b_2, a_2, \dots) \in (B, A)$ there is $c' = (b'_1, a'_1, b'_2, a'_2, \dots) \in (B, A)$ with $c_n = O(c'_n)$ such that $(d_n(U_{c'}, U_c)) \in \lambda(P_0)$. Therefore by Proposition 5.4.4, $(d_n(U_{a'}, U_a))$ and $(d_n(U_{b'}, U_b))$ are in $\lambda(P_0)$. Let

$$L_n = \text{span}\{e_1, e_2, \dots, e_{n+1}\}.$$

For $x \in L_n$ we have

$$\begin{aligned} p_{a'}(x) &= \sum_{i=1}^{n+1} |x_i| a'_i \frac{a_i}{a_i} \\ &\leq \frac{a'_{n+1}}{a_1} \sum_{i=1}^{n+1} |x_i| a_i \\ &= \frac{a'_{n+1}}{a_1} p_a(x). \end{aligned}$$

So by Proposition 5.4.2, we have $a_1/a'_{n+1} \leq d_n(U_{a'}, U_a)$, and hence $(1/a'_n)$ in $\lambda(P_0)$. By the same argument, one can show that $(1/b'_n) \in \lambda(P_0)$. To show that $(1/c'_n) \in \lambda(P_0)$, let $d \in P_0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{c'_n} d_n = \sum_{n=1}^{\infty} \frac{1}{b'_n} d_{2n-1} + \sum_{n=1}^{\infty} \frac{1}{a'_n} d_{2n}.$$

Since $\lambda(P_0)$ is a stable G_{∞} -space, there is an $e \in P_0$ such that $d_{2n} = O(e_n)$.

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{c'_n} d_n < +\infty.$$

So $(1/c'_n) \in \lambda(P_0)$.

(2 $\Rightarrow$ 3) Clear.

(3 $\Rightarrow$ 4) Assume that there is a $c \in (B, A)$ such that $(1/c_n) \in \lambda(P_0)$. Then for each $b \in (B, A)$, we choose $d \in (B, A)$ with $c_n = O(d_n)$ and $b_n = O(d_n)$. Therefore $(1/d_n) \in \lambda(P_0)$. We choose $a \in (B, A)$ with $(d_n)^2 = O(a_n)$. Since $b_n/a_n = O(1/d_n)$, we have $(b_n/a_n) \in \lambda(P_0)$. Define a map $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ by putting $\sigma(n) = n$. Then for each $b \in (B, A)$ there are an $a \in (B, A)$ and $x = (b_n/a_n) \in \lambda(P_0)$ such that $b_{\sigma(n)} \leq x_n a_{\sigma(n)}$. Therefore $\lambda(B, A)$ is uniformly pseudo- $\lambda(P_0)$ -nuclear.

(4 $\Rightarrow$ 1) Assume that $\lambda(B, A)$ is uniformly pseudo- $\lambda(P_0)$ -nuclear. Then there is a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $a \in (B, A)$ there are a $c \in (B, A)$ and an $x \in \lambda(P_0)$ such that $a_{\sigma(n)} = O(x_n c_{\sigma(n)})$. This implies that $a_{\sigma(n)}/c_{\sigma(n)} = O(x_n)$. Hence $(a_{\sigma(n)}/c_{\sigma(n)}) \in \lambda(P_0)$. Since σ is onto, and since $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces, $\sigma(\mathbb{N}) = \mathbb{N} = \{n : a_n \neq 0\}$. Therefore $\lambda(B, A)$ is pseudo- $\lambda(P_0)$ -nuclear. ■

Corollary 5.4.1 *If $\lambda(A)$ is a G_∞ -space, then the following conditions are equivalent:*

1. $\lambda(A)$ is pseudo- $\lambda(P_0)$ -nuclear.
2. For every $b \in A$, there is an $a \in A$ with $b_n = O(a_n)$ such that $(1/a_n)$ in $\lambda(P_0)$.
3. There is an $a \in A$ with $(1/a_n) \in \lambda(P_0)$.
4. $\lambda(A)$ is uniformly pseudo- $\lambda(P_0)$ -nuclear.

Proof. Let $A_1 = \{(a_{2n}) : a \in A\}$ and $A_2 = \{(a_{2n-1}) : a \in A\}$. Then clearly $\lambda(A) = \lambda(A_1) \vee \lambda(A_2)$. ■

Theorem 5.4.4 *If $\lambda(A)$ and $\lambda(B)$ are G_∞ -spaces, then the following conditions are equivalent:*

1. $\lambda(A, B)$ is pseudo- $\lambda(P_0)$ -nuclear.
2. $\lambda(A)$ and $\lambda(B)$ are pseudo- $\lambda(P_0)$ -nuclear.

Proof. $(1 \Rightarrow 2)$ Assume that $\lambda(B, A)$ is pseudo- $\lambda(P_0)$ -nuclear. Then there is a $c \in (B, A)$ such that $(1/c_n) \in \lambda(P_0)$. So there are an $a \in A$ and a $b \in B$ such that $c = (b_1, a_1, b_2, a_2, \dots)$. We claim that $(1/b_n)$ and $(1/a_n)$ are in $\lambda(P_0)$. To prove our claim, let $e \in P_0$ be given. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n} e_n &\leq \sum_{n=1}^{\infty} \frac{1}{b_n} e_{2n-1} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{c_n} e_n \\ &< +\infty. \end{aligned}$$

Also, one can use the same argument to prove that $(1/a_n) \in \lambda(P_0)$. Therefore by Corollary 5.4.1, $\lambda(A)$ and $\lambda(B)$ are pseudo- $\lambda(P_0)$ -nuclear.

$(2 \Rightarrow 1)$ Assume that $\lambda(A)$ and $\lambda(B)$ are pseudo- $\lambda(P_0)$ -nuclear. Then by Corollary 5.4.1, there are $a \in A$ and $b \in B$ such that $(1/a_n)$ and $(1/b_n)$ are in $\lambda(P_0)$. Let $c = (b_1, a_1, b_2, a_2, \dots)$. Then $c \in (B, A)$. We claim that $(1/c_n) \in \lambda(P_0)$. To prove this, let $e \in P_0$ be given. Then

$$\sum_{n=1}^{\infty} \frac{1}{c_n} e_n = \sum_{n=1}^{\infty} \frac{1}{c_n} e_{2n-1} + \sum_{n=1}^{\infty} \frac{1}{c_n} e_{2n}.$$

Since $\lambda(P_0)$ is stable, $(1/c_n) \in \lambda(P_0)$. Therefore by Theorem 5.4.3, $\lambda(B, A)$ is pseudo- $\lambda(P_0)$ -nuclear. ■

Theorem 5.4.5 *If $\lambda(P)$ and $\lambda(Q)$ are G_1 -spaces, then the following conditions are equivalent:*

1. $\lambda(Q, P)$ is $\text{pseudo-}\lambda(P_0)\text{-nuclear}$.
2. $(Q, P) \subseteq \lambda(P_0)$.
3. $\lambda(Q, P)$ is *uniformly* $\text{pseudo-}\lambda(P_0)\text{-nuclear}$.

Proof. $(1 \Rightarrow 2)$ Assume that $\lambda(Q, P)$ is $\text{pseudo-}\lambda(P_0)\text{-nuclear}$. Then for each $c = (q_1, p_1, q_2, p_2, \dots)$ in (Q, P) , there is a $c' = (q'_1, p'_1, q'_2, p'_2, \dots)$ in (Q, P) with $c_n = O(c'_n)$ such that $(d_n(U_{c'}, U_c)) \in \lambda(P_0)$. Let $p = (p_1, p_2, \dots)$, $p' = (p'_1, p'_2, \dots)$, $q = (q_1, q_2, \dots)$, and $q' = (q'_1, q'_2, \dots)$. Then by Proposition 5.4.5, $(d_n(U_{p'}, U_p))$ and $(d_n(U_{q'}, U_q))$ are in $\lambda(P_0)$. Let

$$L_n = \text{span}\{e_1, e_2, \dots, e_{n+1}\}.$$

For $x \in L_n$, we have

$$\begin{aligned} p_{p'}(x) &= \sum_{i=1}^{n+1} |x_n| p'_n \\ &= \sum_{i=1}^{n+1} |x_n| p'_n \frac{p_n}{p_n} \\ &\leq \frac{p'_1}{p_n} p_p(x) \end{aligned}$$

Therefore by Proposition 5.4.2, we have $(p_n) \leq d_n(U_{p'}, U_p)$. So $(p_n) \in \lambda(P_0)$. Also by the same argument, one can show that $(q_n) \in \lambda(P_0)$. Since $\lambda(P_0)$ is a stable, $(c_n) \in \lambda(P_0)$. Therefore $(Q, P) \subseteq \lambda(P_0)$.

$(2 \Rightarrow 3)$ Assume that $(Q, P) \subseteq \lambda(P_0)$. Define $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ by putting $\sigma(n) = n$. For $p \in (Q, P)$, we choose $q \in (Q, P)$ with $p_n = O(q_n^2)$. Let $x_n = b_n$. Then $a_{\sigma(n)} \leq x_n b_{\sigma(n)}$. Therefore $\lambda(Q, P)$ is uniformly $\text{pseudo-}\lambda(P_0)\text{-nuclear}$.

$(3 \Rightarrow 1)$ As in Theorem 5.4.3. ■

Corollary 5.4.2 *If $\lambda(Q)$ is a nuclear G_1 -space, then the following conditions are equivalent:*

1. $\lambda(Q)$ is pseudo- $\lambda(P_0)$ -nuclear.
2. $Q \subseteq \lambda(P_0)$.
3. $\lambda(Q)$ is uniformly pseudo- $\lambda(P_0)$ -nuclear.

Proof. Let $Q_1 = \{(q_{2n-1}) : (q_n) \in Q\}$ and $Q_2 = \{(q_{2n}) : (q_n) \in Q\}$. Then $\lambda(Q) = \lambda(Q_1, Q_2)$. Therefore by applying Theorem 5.4.5, we conclude that all conditions are equivalent. ■

Corollary 5.4.3 *If $\lambda(Q)$ is a G_1 -space, then the following conditions are equivalent:*

1. $\lambda(Q)$ is nuclear.
2. $\lambda(Q)$ is s -nuclear.
3. $\lambda(Q)$ is uniformly s -nuclear.
4. $Q \subseteq \ell_1$.
5. $Q \subseteq s$.

Proof. Conditions (2),(3),and (5) are equivalent by Theorem 5.4.4. Also conditions (1), (4), and (5) are equivalent by Corollary 2.6.1 and Corollary 2.6.2. ■

Theorem 5.4.6 *If $\lambda(Q)$ and $\lambda(P)$ are G_1 -spaces, then the following conditions are equivalent:*

1. $\lambda(Q, P)$ is pseudo- $\lambda(P_0)$ -nuclear.
2. $\lambda(P)$ and $\lambda(Q)$ are pseudo- $\lambda(P_0)$ -nuclear.

Proof. (1 $\Rightarrow$ 2) Assume that $\lambda(Q, P)$ is pseudo- $\lambda(P_0)$ -nuclear. Then by Theorem 5.4.5, $(Q, P) \subseteq \lambda(P_0)$. We claim that Q and P are subset of $\lambda(P_0)$. To prove this, let $p \in P$ be given. Then $c = (q_1, p_1, q_2, p_2, \dots) \in (Q, P)$ for some $q \in Q$. Therefore $c \in \lambda(P_0)$. For $e \in P_0$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} p_n e_n &\leq \sum_{n=1}^{\infty} p_n e_{2n} \\ &\leq \sum_{n=1}^{\infty} c_n e_n \\ &< +\infty. \end{aligned}$$

Therefore $P \subseteq \lambda(P_0)$. Also, by the same argument, one can show that $Q \subseteq \lambda(P_0)$. Therefore by Corollary 5.4.2, $\lambda(P)$ and $\lambda(Q)$ are pseudo- $\lambda(P_0)$ -nuclear.

(2 $\Rightarrow$ 1) Assume that $\lambda(P)$ and $\lambda(Q)$ are pseudo- $\lambda(P_0)$ -nuclear. Then by Corollary 5.4.2, Q and P are subset of $\lambda(P_0)$. We claim that $(Q, P) \subseteq \lambda(P_0)$. Let $c = (q_1, p_1, q_2, p_2, \dots) \in (Q, P)$. Then $q = (q_1, q_2, \dots)$ and $p = (p_1, p_2, \dots)$ are in $\lambda(P_0)$. Since $\lambda(P_0)$ is stable, $c \in \lambda(P_0)$. So $(Q, P) \subseteq \lambda(P_0)$. Therefore by Theorem 5.4.5, $\lambda(Q, P)$ is pseudo- $\lambda(P_0)$ -nuclear. ■

Chapter 6

Decomposition Theorems

6.1 Introduction

In [42], Vogt proved what is later called Vogt decomposition theorem and is one of the main tools to show that a certain nuclear Fréchet space is isomorphic to a stable power series space of infinite type. This theorem states that, if E is isomorphic to a complemented subspace of a stable nuclear power series space $\Lambda_\infty(\alpha)$, and $\Lambda_\infty(\alpha)$ in turn is isomorphic to a complemented subspace of E , then E is isomorphic to $\Lambda_\infty(\alpha)$. In [2], Aytune, Krone and Terzioğlu defined what they called a local imbedding and they used this definition to generalize the Vogt decomposition theorem. They proved that, if there is a local imbedding i from a nuclear stable G_∞ -space $\lambda(A)$ into $\lambda(A)^N$, then there is a complemented subspace G of $\lambda(A)^N$ which is isomorphic to $\lambda(A)$ and contained in $\overline{i(\lambda(A))}$. In [38], Terzioğlu generalized this result of Aytuna, Krone and Terzioğlu for spaces not necessarily metrizable. In this Chapter, we are going to define what we will call stable local imbedding and we give a further generalization.

6.2 Definitions

Definition 6.2.1 (see Terzioğlu [38; P. 308]) *A continuous linear map i from a stable nuclear G_∞ -space $\lambda(A)$ into a locally convex space E is called a **local imbedding** if there is a sequence $\sigma = (\sigma_n)$ and a continuous semi-norm $||\cdot||$ on E with*

$$|x|_\sigma := \sum_n |x_n| \sigma_n \leq ||ix||$$

where we assume σ satisfies the following condition:

$(L_1) : \sigma_n > 0$ and $\sigma_n = O(a_n)$, $1/\sigma_n = O(b_n)$ for some a and b in A .

Theorem 6.2.1 (see Terzioğlu [38; P. 309]) *Let $\lambda(A)$ be a nuclear stable G_∞ -space and suppose that there is a local imbedding of $\lambda(A)$ into E . Each one of the following assumptions implies that E has a complemented subspace isomorphic to $\lambda(A)$.*

1. *$\lambda(A)$ is a Fréchet space and E is isomorphic to a closed subspace of $\lambda(A)^N$.*
2. *E is barrelled and isomorphic to a closed subspace of $\lambda(A)$.*
3. *E is sequentially complete and isomorphic to a quotient space of $\lambda(A)$.*

Throughout the rest of this section we denoted by $\lambda(A)$ and $\lambda(B)$ two nuclear G_∞ -spaces.

Definition 6.2.2 *A continuous linear map j from $\lambda(B, A)$ into a locally convex space E is called a **stable local imbedding** if there is a sequence $\sigma = (\sigma_n)$ and a continuous semi-norm $||\cdot||$ on E with*

$$|x|_\sigma := \sum_n |x_n| \sigma_n \leq ||jx||$$

where σ satisfies the following condition:

$(L_2) : \sigma_n > 0$ and $\sigma_{2n} = O(a_n)$, $1/\sigma_{2n} = O(a'_n)$, $1/\sigma_{2n-1} = O(b_n)$ for some $a, a' \in A$ and $b \in B$

6.3 Decomposition Theorems

In this section we generalize the result obtained by Terzioğlu in [38], we prove the result for G_∞ -spaces not necessarily stable.

Theorem 6.3.1 *Suppose that $\lambda(A)$ is nuclear G_∞ -space. If there is a stable local imbedding j from $\lambda(A, A)$ into a locally convex space E and if E is isomorphic to a closed subspace of $\lambda(A)$, then E has a complemented subspace isomorphic to $\lambda(A)$.*

Proof. Since j is a stable local imbedding, there is a sequence $\sigma = (\sigma_n)$ which satisfies condition (L_2) and a continuous semi-norm p on E such that

$$|x|_\sigma = \sum_n |x_n| \sigma_n \leq p(jx).$$

Since E is nuclear, p is given by a semi-inner product $\langle \cdot, \cdot \rangle$ which is an inner product on the subspace $j(\lambda(A, A))$; (see Pietsch [27; P. 79]).

Let us to construct a basic sequence (g_n) in $j(\lambda(A, A))$, satisfying the following conditions:

1. $g_n \in \text{sp}\{je_1, je_2, \dots, je_{2n}\}$.
2. $\langle g_n, e_v \rangle = \langle g_n, g_i \rangle = 0$ for $v = 1, \dots, n$ and $i = 1, \dots, n-1$.
3. $p(g_n) = 1$.

where (e_n) is the canonical basis of $\lambda(A, A)$, which is assumed to be orthonormal with respect to $\langle, \rangle$. This construction is possible since $\text{sp}\{je_1, \dots, je_{2n}\}$ is a $2n$ -dimensional space and p is a norm on $j(\lambda(A, A))$. If $g_n = j(\sum_{i=1}^{2n} \mu_{n,i} e_i)$, then we have

$$\sum_{i=1}^{2n} |\mu_{n,i}| \sigma_i = \left| \sum_{i=1}^{2n} \mu_{n,i} e_i \right|_{\sigma} \leq p(g_n) = 1.$$

Let $\|\cdot\|$ be any continuous semi-norm on E . By the continuity of j we can find $c = (c_1, c_1, \dots) \in (A, A)$ which is of the form $c = (a_1, a_1, a_2, a_2, \dots)$ and $\rho > 0$ such that

$$\begin{aligned} \|g_n\| &= \left\| j\left(\sum_{i=1}^{2n} \mu_{n,i} e_i\right) \right\| \\ &\leq \rho \left| \sum_{i=1}^{2n} \mu_{n,i} e_i \right|_c \\ &= \rho \sum_{i=1}^{2n} |\mu_{n,i}| c_i. \end{aligned}$$

So we have

$$\begin{aligned} \|g_n\| &\leq \rho \sum_{i=1}^{2n} |\mu_{n,i}| c_i \frac{\sigma_i}{\sigma_i} \\ &\leq \rho \sup \left\{ \frac{c_i}{\sigma_i} : i = 1, \dots, 2n \right\} \end{aligned}$$

By condition (L_2) there is $b = (b_n) \in A$ such that

$$\|g_n\| = O(b_n a_n).$$

Since $\lambda(A)$ is a G_{∞} -space and $a, b \in A$, there are $d \in A$ and $\alpha > 0$ such that

$$b_n a_n \leq \alpha_n d_n \quad \text{for all } n \in \mathbb{N}. \quad (6.1)$$

For $x \in E$, write $x = \sum_v \zeta_v e_v$. We have

$$\langle g_n, x \rangle = \sum_v \zeta_v \langle g_n, e_v \rangle = \sum_{v > n} \langle g_n, e_v \rangle \zeta_v.$$

So for each $a \in A$,

$$\begin{aligned} |\langle g_n, x \rangle| a_n &\leq \sum_{v>n} |\langle g_n, e_v \rangle| |\zeta_v| a_v \\ &\leq \left(\sum_{v>n} |\zeta_v| \right) \left(\sum_{v>n} |\langle g_n, e_v \rangle| a_v \right) \\ &= \sum_{v>n} |\zeta_v| \|g_n\|_a. \end{aligned}$$

Since $\|\cdot\|_a$ is a continuous semi-norm on E , there are $d' \in A$ and $\beta > 0$ such that

$$\|g_n\|_a \leq \beta d'_n \quad \text{for all } n \in \mathbb{N}.$$

Therefore we have

$$|\langle g_n, x \rangle| a_n \leq \beta d'_n \sum_{v>n} |\zeta_v| \quad (6.2)$$

$$\leq \beta \sum_{v>n} |\zeta_v| d'_v \leq \beta \|x\|_{d'}. \quad (6.3)$$

Define a map $T : E \rightarrow E$ by putting $Tx = \sum_n \langle g_n, x \rangle g_n$. Then for any continuous seminorm $\|\cdot\|$ on E ,

$$\|Tx\| \leq \sum_n |\langle g_n, x \rangle| \|g_n\|.$$

Since $\|\cdot\|$ is a continuous seminorm on E , there are $a \in A$ and $\rho > 0$ such that

$$\|g_n\| \leq \rho a_n \quad \text{for all } n \in \mathbb{N}.$$

So we have

$$\|Tx\| \leq \rho \sum_n |\langle g_n, x \rangle| a_n.$$

By nuclearity of $\lambda(A)$ and inequality (6.3), there are $b \in A$, $d \in A$ and $\beta > 0$ such that $(a_n/b_n) \in \ell_1$ and $|\langle g_n, x \rangle| b_n \leq \beta \|x\|_d$. So we have

$$\begin{aligned} \|Tx\| &\leq \beta \sum_n |\langle g_n, x \rangle| \frac{a_n}{b_n} b_n \\ &\leq \beta \|x\|_d \sum_n \frac{a_n}{b_n}. \end{aligned}$$

Therefore T is continuous and. Moreover, since $Tg_n = g_n$ and (g_n) is a basic sequence, T is a projection of E onto $G = \overline{\text{sp}\{g_n : n \in \mathbb{N}\}}$. To conclude the proof it remains to show that G is isomorphic to $\lambda(A)$. Define a map $J : \lambda(A) \rightarrow G$ by putting $Je_n = g_n$. It is clear that J is one-to-one, continuous and well defined. So the only remaining part is to show that J is onto and open. To show J is onto, let $x \in G$ be given. Then $x = \sum_n \langle g_n, x \rangle g_n$. Let $c = (\langle g_n, x \rangle)$, which is in $\lambda(A)$ by nuclearity of $\lambda(A)$ and equation (6.3). Moreover $Jc = x$. Hence J is onto. For each $a \in A$

$$\|x\|_a = \sum_n |\langle g_n, x \rangle| a_n$$

defines a continuous seminorm on G . To show J is open, let U_a be any zero neighbourhood of $\lambda(A)$. We claim

$$J(U_a) = \{y \in G : \|y\|_a \leq 1\}.$$

To show our claim, let $z \in J(U_a)$. Then there is $y \in U_a$ such that $z = Jy$. Also there is $\zeta = (\langle g_n, z \rangle) \in \lambda(A)$ such that $z = J\zeta$. Since J is one-to-one, we have $\zeta = y$ and hence

$$p_a(\zeta) = p_a(y) = \sum_n |\langle g_n, z \rangle| a_n = \|z\|_a \leq 1.$$

Therefore $z \in \{y \in G : \|y\|_a \leq 1\}$. Conversely, let $y \in G$ with $\|y\|_a \leq 1$ be given. Then there is $\zeta = (\langle g_n, y \rangle) \in \lambda(A)$ such that $J\zeta = y$. Hence

$$p_a(\zeta) = \sum_n |\langle g_n, y \rangle| a_n = \|y\|_a \leq 1,$$

showing that $\zeta \in U_a$. Thus $y \in J(U_a)$. We have shown $J(U_a) = \{y \in G : \|y\|_a \leq 1\}$. Hence $\lambda(A) \cong G$. ■

Now we are ready to show that Terzioğlu theorem is a consequence of the above theorem.

Corollary 6.3.1 *Suppose that $\lambda(A)$ is a nuclear stable G_∞ -space. If there is a local imbedding of $\lambda(A)$ into a locally convex space E and if E is isomorphic to a closed subspace of $\lambda(A)$, then E has a complemented subspace isomorphic to $\lambda(A)$.*

Proof. Suppose that $\lambda(A)$ is a nuclear G_∞ -space and let $i : \lambda(A) \rightarrow E$ be a local imbedding. Since $\lambda(A)$ is stable, $\lambda(A) = \lambda(A, A)$. So it is enough to show that $i : \lambda(A, A) \rightarrow E$ is a stable local imbedding. Since i is local imbedding, there is a sequence $\sigma = (\sigma_n)$ and a continuous seminorm $||\cdot||$ on E with

$$|x|_\sigma = \sum_n |x_n| \sigma_n \leq ||ix||$$

where (σ_n) satisfies condition (L_1) , that is, $\sigma_n > 0$ and $\sigma_n = O(a_n)$, $1/\sigma_n = O(b_n)$ for some $a, b \in A$. Therefore $\sigma_{2n} = O(a_{2n})$ and

$$1/\sigma_{2n-1} = O(b_{2n-1}) = O(b_{2n}).$$

By the stability of $\lambda(A)$, we can choose $a', b' \in A$ such that $a_{2n} = O(a'_n)$ and $b_{2n} = O(b'_n)$. Therefore

$$\sigma_{2n} = O(a_{2n}) = O(a'_n), \quad 1/\sigma_{2n} = O(b_{2n}) = O(b'_n),$$

and $1/\sigma_{2n-1} = O(b_{2n-1}) = O(b'_n)$. So σ satisfies condition (L_2) . Hence i is a stable local imbedding. ■

Next, we give the following definition from [34].

Definition 6.3.1 [34] *A Köthe space $\lambda(A)$ is called a PN-space if the Köthe set A satisfies the additional assumption that every $a \in A$ is a positive, non-decreasing sequence.*

Remark. Every G_∞ -space is a PN-space.

Lemma 6.3.1 *If $\lambda(A)$ is a PN-space satisfying the following condition:*

(L₃) for all $a \in A$ there is $b \in B$ such that $a_{2n} = O(b_n)$, then

$$\lambda(A) \vee \lambda(A) = \lambda(A).$$

Proof. Let $x \in \lambda(A)$ be given. For each $a \in A$, we have

$$\begin{aligned} \sum_n |x_{2n}| a_n &\leq \sum_n |x_{2n}| a_{2n} \\ &\leq \sum_n |x_n| a_n < +\infty \end{aligned}$$

and hence $(x_{2n}) \in \lambda(A)$. By the same argument one can show that (x_{2n-1}) in $\lambda(A)$. Hence $(x_n) \in \lambda(A) \vee \lambda(A)$. Therefore $\lambda(A) \subseteq \lambda(A) \vee \lambda(A)$. To show that $\lambda(A) \vee \lambda(A) \subseteq \lambda(A)$, let $x \in \lambda(A) \vee \lambda(A)$ be given. Then $(x_{2n}) \in \lambda(A)$ and $(x_{2n-1}) \in \lambda(A)$. For $a \in A$, we have

$$\begin{aligned} \sum_n |x_n| a_n &= \sum_n |x_{2n}| a_{2n} + \sum_n |x_{2n-1}| a_{2n-1} \\ &\leq \sum_n |x_{2n}| a_{2n} + \sum_n |x_{2n-1}| a_{2n} \end{aligned}$$

Therefore, by condition (L₃), there are $b \in A$ and $\rho > 0$ such that

$$a_{2n} \leq \rho b_n \quad \text{for all } n \in \mathbb{N}.$$

So we have

$$\sum_n |x_n| a_n \leq \rho \sum_n |x_{2n}| b_n + \rho \sum_n |x_{2n-1}| b_n < +\infty.$$

Since (x_{2n}) and (x_{2n-1}) are in $\lambda(A)$, we have $(x_n) \in \lambda(A)$. Therefore $\lambda(A) = \lambda(A) \vee \lambda(A)$. ■

Our theorem [Theorem 6.3.1] is a modification of the following theorem.

Theorem 6.3.2 [34] *Suppose that $\lambda(A)$ is a nuclear, PN -space where the Köthe set A satisfies the condition (L_3) . If there is a local imbedding i from $\lambda(A)$ into a locally convex space E and if E is isomorphic to a closed subspace of $\lambda(A)$, then E has a complemented subspace isomorphic to $\lambda(A)$.*

Proof. This follows directly from Theorem 6.3.1, and Lemma 6.3.1.

Bibliography

- [1] M. Alpseyment, A generalization of Dragilev's Theorem, J. Reine Angew Math., **276**, 124-129 (1975).
- [2] A. Aytuna, J. Krone, and T. Terzioğlu, Complemented Infinite Type Power Series Subspaces of Nuclear Fréchet Spaces, Math. Ann., **283**, 193-202 (1989).
- [3] A. Aytuna, J. Krone, and T. Terzioğlu, Imbedding of Power Series and Spaces of Analytic Functions, Manuscripta Math., **67**, 125-142(1990).
- [4] A. Aytuna and T. Terzioğlu, On Certain Subspaces of a Nuclear Power Series Spaces of Finite Type, Studia Math., **69**, 79-86 (1980).
- [5] C. Bessaga, Some Remarks on Dragilev's Theorem, Studia Math., **31**, 307-318,(1968).
- [6] G. Crofts, Concerning Perfect Fréchet Spaces and Diagonal Transformations, Math. Ann., **182**, 67-76(1969).
- [7] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math., **92**, Springer Verlag, Heidelberg (1984).
- [8] M. Dragilev, On Regular Bases In Nuclear Spaces, Amer. Math. Soc. Transl., **93**, 61-82(1970).

- [9] E. Dubinsky, A Remark on a Paper by M. S. Ramanujan, *Math. Ann.*, **190**, 231-232 (1971).
- [10] E. Dubinsky, Basic Sequences in (s) , *Studia Math.*, **59**, 283-293 (1977).
- [11] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg, New York, **720**(1979).
- [12] E. Dubinsky and M. S. Ramanujan, Inclusion Theorems For Absolutely λ -Summing Maps, *Math. Ann.*, **192**, 177-190 (1971).
- [13] E. Dubinsky and M. S. Ramanujan, On λ -Nuclearity, *Mem. Amer. Math. Soc.*, **128** (1982)
- [14] E. Dubinsky and W. Robinson, Quotient Spaces of (s) With Basis, *Studia Math.*, **63**, 267-281 (1978).
- [15] L. Holmström, On Stable D_1 and D_2 Spaces, *Arch. Math.*, **36**, 546-553 (1981).
- [16] L. Holmström, Super Spaces of (s) With Basis, *Studia Math.*, **75**, 139-152 (1983).
- [17] L. Holmström, Super Spaces of (s) With Strong Finite Dimensional Decomposition, *Arch. Math.*, **42**, 58-66(1984).
- [18] H. Jarchow, *Locally Convex Spaces*, Teubner Verlag (1981).
- [19] P. K. Kamthan, M. Gupta, Manujul, M. A. Sofi, λ -Bases and thier Applications, *J. Math. Anal. Appl.*, **88**, 76-99 (1982).
- [20] P. K. Kamthan, M. Gupta, *Schauder Bases Behaviour and Stability*, Indian Institute of Technology, Kanpur, 1988.

- [21] P. K. Kamthan, M. A. Sofi, λ -Bases and λ -Nuclearity, J. Math. Anal. Appl., **99**, 164-188 (1984).
- [22] Y. Komura, T. Komura, Über die Einbettung der nuklearen Räume in (s^A) , Math. Ann., **162**, 284-288 (1966).
- [23] G. Köthe, *Topological Vector Spaces I and II*, Springer-Verlag, Berlin-Heidelberg, New York (1969) and (1979).
- [24] J. Lindenstruss, A. Pelczynski, Absolutely Summing Operators in ℓ_p -Spaces and their Applications, Studia Math., **29**, 275-326 (1968).
- [25] R. E. Megginson, *An Introduction to Banach Spaces Theory*, Graduate Texts in Math., **183**, Springer, New-York, Berlin-Heidelberg, 1998.
- [26] A. Pelczynski, A Characterization of Hilbert-Schmidt Operators, Studia Math., **28**, 355-360 (1967).
- [27] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer-Verlag, Berlin-Heidelberg, New York (1972).
- [28] M. S. Ramanujan, Power Series Spaces $\Lambda(\alpha)$ and Associated $\Lambda(\alpha)$ -Nuclearity, Math. Ann., **189**, 161-168 (1970).
- [29] M. S. Ramanujan, B. Rosenberger, On $\lambda(\phi, P)$ -Nuclearity, Compos. Math., **37**, 113-125 (1975).
- [30] M. S. Ramanujan, T. Terzioğlu, Diametral Dimensions of Cartesian Products, Stability of Smooth Sequence Spaces and Applications, J. Reine Angew Math., **280**, 167-171 (1976).
- [31] M. S. Ramanujan, T. Terzioğlu, Power Series Spaces $\Lambda_k(\alpha)$ of Finite Type and Related Nuclearities, Studia Math., **53**, 1-13 (1975).

- [32] W. Robinson, On $\Lambda_1(\alpha)$ -Nuclearity, *Duke Math. J.*, **40**, 541-546 (1973).
- [33] W. Robinson, Relationships Between λ -Nuclearity and Pseudo- μ -Nuclearity, *Trans. Amer. Math. Soc.*, **201**, 291-303 (1975).
- [34] A. Tallafha, Local Imbedding and PN-Spaces, *Tr. J. Of Mathematics*, **19**, 275-282 (1995).
- [35] T. Terzioğlu, Die Diametrale Dimension Von Lokal Konevexn Raumen, *Collect. Math.*, **20**, 49-99 (1969).
- [36] T. Terzioğlu, On the Diametreal Dimension of Some Classes Fréchet Spaces, *J. Kuradenis Uni. Ser. Math. Physics*, **8**, 1-19 (1985).
- [37] T. Terzioğlu, Smooth Sequence Spaces and Associated Nuclearity, *Proc. Amer. Math. Soc.*, **37**, 497-504 (1973).
- [38] T. Terzioğlu, Some Invariants of Fréchet Spaces and Imbedding of Smooth Sequence Spaces. *Advances in the Theorey Fréchet Spaces*, T. Terzioğlu ed. 305-324, Kluwer Academic Publishers (1989).
- [39] T. Terzioğlu, Stability of Smooth Sequences Spaces, *J. Reine Angew Math.*, **276**, 184-189 (1975).
- [40] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Publishing Company (1982).
- [41] D. Vogt, Ein Isomorphiesatz Für Polenzreihenraume, *Arch. Math.*, **38**, 540-548 (1982).
- [42] D. Vogt, M. J. Wangner, Charakterisierume Stbilen Potenzreihenraume Von Unendlichem TYP., *Studia Math.*, **70**, 63-80 (1981).