**Probabilistic Prediction of Coalescence Flutter Using Measurements:
Application to the Flutter Margin Method**

by

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“The most important maxim for data analysis to heed, and one which many statisticians seem to have shunned, is this: Far better an approximate answer to the right question, which is often vague, than an exact answer to the wrong question, which can always be made precise.”

Tukey, John W.

Abstract

The accurate prediction of flutter speed of an aeroelastic system is challenging. Flutter is a dynamic instability observed in flexible structures such as aircraft wings, tall chimneys, and long-span bridges. This dynamic instability originates from the interaction of aerodynamic, elastic and inertia forces. There exist various types of flutter mechanisms. The one treated in this thesis is the coalescence flutter, which is characterized by the coupling of two degrees of freedom. In turn, this coupling enables a net energy transfer from the flow to the structures. There are very few experimental approaches that closely estimate the flutter speed of complicated structures such as aircraft and long-span bridges. One particular approach is Zimmerman and Weissenburger flutter margin method. The crux of the method lies in the prediction of flutter onset using pre-flutter flight test data.

The models employed in this method are imperfect, and the measurements are corrupted by the noise that leads to uncertainty in the results. To quantify the uncertainty in results, a probabilistic approach is followed. Bayesian inference allows us to incorporate prior information of the system with current test data and increase the reliability of predictions. Previous research was conducted using a flat prior on a two degrees of freedom oscillating airfoil. The goal is to extend the methodology to informative priors for the quasi-steady aerodynamic model to improve the accuracy of the prediction of flutter onset. This study reveals that the use of prior knowledge with full statistical correlation significantly improves the prediction of flutter speed with few number of experiments at pre-flutter speed.

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Abbreviations

AC	A dvisory C ircular
AMCMC	A daptive M etropolis M arkov C hain M onte C arlo
CFD	C omputational F luid D ynamics
CSD	C omputational S tructural D ynamics
COV	C oefficient of V ariation
CPU	C entral P rocessing U nit
CG	C enter of G ravity
DOF	D egree O f F reedom
EA	E lastic A xis
FAA	F ederal A viation A dministration
FCE	F ourier C haos E xpansion
FM	F lutter M argin
FS	F lutter S peed
LCO	L imit C ycle O scillation
MCS	M onte C arlo S imulation
MH	M etropolis H asting
MCMC	M arkov C hain M onte C arlo
PCE	P olynomial C haos E xpansion
pdf	P robability D ensity F unction

Symbols

Symbols	Quantity	Unit
b	Semi-chord	m
c	Chord	m
m	Mass	kg
$\bar{m}$	Mass per unit span	kg/m
I_{EA}	Moment of Inertia at elastic axis	kg-m ²
U	True airspeed	m/s
L	Lift	N
M	Moment	N-m
α	Angle of attack	rad
$\omega_i^{(j)}$	Frequency of i^{th} mode and j^{th} test point	rad s ⁻¹
$\beta_i^{(j)}$	Decay rate of i^{th} mode and j^{th} test point	s ⁻¹
ρ	Free stream fluid density	kg/m ³
k_h	Bending stiffness	N-m ²
k_α	Torsional stiffness	N-m ²
x_α	Normalized static imbalance	
a_h	Normalized distance between EA and mid-chord	
ξ_1	Damping ratio (mode 1)	
ξ_2	Damping ratio (mode 2)	
μ	Mean	
q^0	True but unknown parameter	
q^*	Proposed Markov chain parameter	
q^{k-1}	Parameter at $k - 1$ step in Markov chain	
R	Cholesky decomposition	

ϕ	Parameter
d	Observation data
$\pi_0(\phi)$	Bayesian prior density
$\pi(\phi d)$	Bayesian posterior density
$\pi(d \phi)$	Bayesian likelihood function
$\pi_\gamma(d)$	Normalization factor
s_p	Design parameter

Dedicated to Chariji and my family

Chapter 1

Introduction

1.1 Aeroelasticity

Aeroelasticity is defined as the branch of physics studying the effect of coupling of inertial, elastic, aerodynamic as well as damping forces when (a structure is) exposed to fluid flow. Based on the interaction of forces, aeroelasticity is divided into **static aeroelasticity**, **dynamic aeroelasticity**, **dynamic stability** and **mechanical vibration**. The graphical representation of various interactions proposed by Collar [3] is shown in the following Figure 1.1. Static Aeroelasticity is a study of coupling between aerodynamic forces and elastic forces and the influence of this interaction on airplane or bridge design while dynamic aeroelasticity deals with the coupling of aerodynamic, elastic, inertia and damping forces.

FIGURE 1.1: Collar's aeroelastic triangle

1.2 Flutter

Flutter is the most notorious aeroelastic phenomenon occurring in flexible structures prone to aerodynamic forces, such as aircraft, tall chimneys, long-span bridges, etc. It is a dynamic aeroelastic phenomenon defined as an unstable self-excited vibration in which the structure extracts energy from the fluid flow and may result in catastrophic structural failure. Flutter study in bridges and aircraft are of prime engineering importance. Argentini [4], Mannini [5] presents an analysis of flutter using Monte Carlo simulations in long span bridge. The statistical approach to estimation of flutter speed of aircraft/bridges not only helps in simplifying the mathematical model but also gives the advantage of accommodating the experimental test data.

Theodorsen [6] first pioneered the comprehensive flutter analysis of 2-D wing with unsteady loads. Theodoresen and Garrick [7] presents an early comparison of the theoretical and experimental study. An in-depth study of historical development in aircraft flutter is submitted by Garrick [8].

1.2.1 Types of flutter

The main types of flutter are *classical flutter* occurring due to the critical coupling of two or more modes and *stall flutter* occurring under stall conditions with a single dominating degree of freedom. Classical wing flutter, also know as coalescence or coupled flutter, may be comprised of three main modes: flexure, torsion, and control surface rotation [9]. A flutter instability in which all three modes play a significant role is a trinary flutter, whereas binary flutter involves the contribution of two dominating modes and is a widely studied case because of its predominance in engineering applications.

More specifically classical flutter can be categorized as [10]

- Bending/torsional flutter → The aircraft wing or a long-span bridge undergoes coupled bending / torsional deformations.
- Bending/aileron flutter → This type of flutter occurs when aircraft wing undergoes bending in combination with aileron flapping mode.
- Bending/torsional/aileron flutter → A flutter combination includes the aircraft wing bending mode, wing torsional mode and aileron flapping mode.

Depending on the behavior of damping, flutter can also be categorized as hard flutter where a sudden decrease of damping is observed close to flutter point and soft flutter where damping decreases gradually [3, 10].

1.2.2 Engineering solution for flutter estimation

The methods discussed below are an overview of engineering practice followed in the aerospace industry to estimate flutter from CFD-CSD numerical models note that p method is not directly connected to flutter margin method adopted in this thesis. As an engineer a significant consideration in using these methods as describe by Hodges [10] are

- To estimate stability margin at flight conditions in the proximity of flutter boundary,
- To gain the understanding of the physical mechanism that plays a significant role in instability.

Based on the above considerations one can then propose design variations that may alleviate or even eliminate the instability, and prepare an appropriate plan for the flight test program. It is in the context of flight testing that this thesis is relevant.

1.2.2.1 p method

The flutter analysis based on the determination of stability using complex eigenvalues is known as the p method [10]. The name is derived from the dimensionless complex eigenvalue $p = bv/U$ also know as “*reduced eigenvalue*” where b is the semi-chord, v is the complex root and U is the freestream velocity. For accurate flutter prediction using the p method, one need an aerodynamic theory which incorporate loads induced by the transient motion of the lifting surface. The flutter can be described by a linear set of dynamic structural equations that includes a linear representation of the unsteady airloads in term of the elastic deformations, for detail derivation refer [10]. The equations of motion for a two-degree-of-freedom airfoil undergoing flutter are :

$$m(\ddot{h} + bx_\alpha\ddot{\alpha}) + k_h h = -L \quad (1.1)$$

$$I_{EA}\ddot{\alpha} + mbx_\alpha\ddot{h} + k_\alpha\alpha = M \quad (1.2)$$

where L is the lift and M is the pitching moment about the elastic axis. The motion is assumed to be simple harmonic with frequency ω . In p method the nontrivial general solution for number of modes for the above equation of motion is given by the flutter determinant as

$$\left| p^2[M] + \frac{b^2}{U^2}[M][\omega^2] - \rho_\infty[A(p)] \right| = 0 \quad (1.3)$$

where $[M]$ is the mass matrix and $[A(p)]$ is the unsteady aerodynamic matrix. For a given speed and altitude, this flutter determinant can be solved for p , resulting in

$$p = \gamma k \pm ik \quad (1.4)$$

where k is the reduced frequency

$$k = \frac{b\omega}{U} \quad (1.5)$$

and γ is the rate of decay

$$\gamma = \frac{1}{2\pi} \ln \left(\frac{a_{n+1}}{a_n} \right) \quad (1.6)$$

where a_n and a_{n+1} represent the amplitudes of successive cycles. In the p method, the flutter determinant Eq. 1.3 is solved for selected combinations of speed and altitude. The p method provides the most accurate solution.

1.2.2.2 k method

The underlying assumption in the k method is the consideration of the simple harmonic motion. Eq. 1.3 of flutter determinant is modified for the k method with the assumption $p = ik$ [10] given as

$$\left| -k^2[M] + \frac{b^2}{U^2}[M][\omega^2] - \rho_\infty[A(ik)] \right| = 0. \quad (1.7)$$

At preselected combinations of reduced frequency and altitude, Eq. 1.7 can be solved for the complex roots of $b^2/U^2 \lambda_r + i\lambda_i$. These roots may be interpreted as

$$\lambda_r + i\lambda_i = \left(\frac{b^2}{U^2} \right) (1 + ig) \quad (1.8)$$

where g is the structural damping required to sustain simple harmonic motion. This structural damping parameter can be related to the rate of decay parameter of the p method as $g \cong 2\gamma$. Although the k method has advantages over other approaches, it is not favored in most applications because of the following disadvantages [11] :

- The k method requires numerous iteration at constant density to ensure matching the Mach number with airspeed and altitude.
- The solution is valid only at $g = 0$. Other non-zero damping values are artificial and may not have physical meaning.
- The accuracy of the solution compared to p method and $p - k$ method is less.

1.2.2.3 $p - k$ method

The $p - k$ method is trade-off between the p and k methods. Irwin and Guyett first proposed this technique in 1965 [10, 12]. The notion behind the $p - k$ method is to conduct a p method analysis with the restriction that the unsteady aerodynamics matrix $[A(p)]$ is for simple harmonic motion. The flutter determinant Eq. 1.3 is modified for $p - k$ method as

$$\left| p^2[M] + \frac{b^2}{U^2}[M][\omega^2] - \rho_\infty[A(ik)] \right| = 0. \quad (1.9)$$

An iterative scheme is employed to find each eigenvalue at a given velocity and density [12, 13]. The accuracy of the $p - k$ method depends on the level of damping in any particular mode. Hassig [12] has given a detailed description of the superiority of the $p - k$ method over the k method. In this publication, Hassig presented the comparison of the p method and $p - k$ method for a twin-jet transport airplane and comparison between $p - k$ method and k method for a horizontal stabilizer with elevator. This comparison led to the conclusion that the $p - k$ method is an optimal choice for engineers that provides an edge over the k method concerning accuracy and reduce the computational cost compared to the p method.

1.3 Flutter Flight Test

Aircraft flight testing comprises multi-disciplinary field involving aeronautical, structural, mechanical and electrical in addition to man-machine interface [14]. The flight tests begin at the end of the aircraft design phase and undergo a series of controlled trials. The test phase is an essential component in the development of the aircraft since continuous optimizations are done to meet the standard airworthiness requirements for a plane. It is mandatory for all aircraft to undergo flutter flight tests under the *FAA regulations AC 25.629-1B, AC 23.629-1B, AC 25.672-1*. This advisory provides guidance in complying with the provisions of Title 14, Code of Federal Regulations part 25.629 associated with aeroelastic stability requirements, in order to ensure safety and stability of the aircraft. Aircraft flutter testing is considered to be the most hazardous testing in aircraft certification and requires utmost caution.

The flight flutter test consists of three main components, *excitation, measurement and data processing/analysis*. For measurement, it is essential to have good quality instrumentation that can withstand complex maneuvers and important vibrations of an aircraft. The instruments used in flutter tests are sensitive accelerometers and strain gauges. For data collection, telemetry system is used given its efficiency in reducing

schedule time and the number of flights, as it allows near real-time data analysis. Even with the best choice of instrumentation, flight testing is no exception to errors and uncertainty in data. The common sources of errors are instrument error (hysteresis and bias error), uncertainty due to atmospheric conditions and sensor location [14, 15].

The most common approach to flight flutter testing is to fly the aircraft at a number of airspeeds and excite it using any number of excitation systems, including control surface pulses and oscillations, thrusters, aerodynamic vanes or atmospheric turbulence [9]. The response of the aircraft measured at some stations on the structure and subsequently analyzed to yield the natural frequencies and damping of all the significant modes. Aircraft have a large number of modes. However, only two or three of them contribute directly to flutter and, hence, only these modes are tracked. The flutter speed is mathematically defined as the airspeed at which the damping of the weaker mode becomes zero. The calculated damping values can be plotted against airspeed and extrapolated to obtain the flutter condition. However, this method to estimate the flutter speed is not very robust because the damping measurement are very noisy, thus uncertain, and the variation of damping as a function of airspeed can be abrupt (hard flutter), which makes predictions/extrapolation unreliable [9, 14–16].

The first formal flight flutter test was conducted by Von Schlippe in Germany in 1935 [17]. With the advancement in computers and sensor, flight flutter test has become more reliable and accurate, though the fundamental problem of predicting the exact onset of flutter remain a significant challenge.

It became evident in the late 50's that the study of flutter testing was becoming more and more essential for aircraft, and an early example is the first *Flight Flutter Testing Symposium* held in 1958. The major advances and a comprehensive study of flutter testing were presented at the symposium held in 1975 [18]. In this symposium, several new techniques for testing and analysis came forth. In the mean time, Zimmerman and Weissenburger [19] developed the flutter margin method in 1964. This method is superior to the damping based approach because it relies more heavily on frequency measurements, which shows much less variability than damping. The projection in airspeed is also much more gradual. Further details of the method are given in Chapter 3.

In last two decades, there has been significant progress in techniques for flight flutter test. Cooper, in 1993, [20] introduced the *Envelope function* which relies on changes in the shapes of the envelope to predict any significant changes in the aircraft stability during the flight test at various speed using control jerk response data and impulse response. In 2000, Lind proposed the *Flutterometer* [21], an online tool to predict robust flutter margins during a flight test. Dimitriadis and Cooper [22] made the comparison

of five different approaches used in estimations of flutter speed namely *Damping fit*, *Flutter Margin method (FM)*, *Envelope Function*, *Nissim and Gilyard Method (NG)*, *Autoregressive moving average method (ARMA)*. In this publication, some interesting conclusions were drawn on the efficacy of various methods for a different aeroelastic model. The *flutter margin method* can predict acceptable results when applied to an aeroelastic model with some constraint on flight test data. The use of the *flutter margin method* and its extension to trinary flutter is well described and implemented by Lee and Price [23]. The flutter margin method holds potential and flexibility to extend and optimize. Poirel [24] extended the flutter margin method to account for uncertainties in modal parameters using a statistical approach. Additional details in flight flutter testing techniques in the last decade can be found in [25–31].

1.4 Flutter Estimation using Probabilistic Approach and Uncertainty Quantification

The utilization of uncertainty quantification on aeroelastic response prediction has received significant attention in the literature. Uncertainty quantification gives the leverage in the estimation of the relevant parameter in the absence of the well defined mathematical model. Beran and Stanford [32] present a comprehensive study of uncertainty quantification in aeroelasticity. Pettit [33–35] briefly described general sources of uncertainty that complicate airframe design and testing. Xiang [36] presented recent advancement in the nonlinear analysis of the aeroelastic system. In this publication, a review of nonlinear aeroelasticity of high-aspect-ratio wings and high altitude long endurance aircraft is presented to understand the behavior of nonlinear aeroelasticity coupled with nonlinear flight dynamics. A concise stochastic analysis of aeroelastic system dealing with buckling of joined wing is reported by Pettit [37]. A finite element model with the non-intrusive formulation is considered to include Gaussian randomness in Young’s modulus.

A probabilistic collocation method for modeling the long-term stochastic behavior of dynamical systems exhibiting a periodic response (LCO) is proposed by Witteveen [38]. Millman used *Fourier Chaos Expansion* (FCE) [39] to compute the subcritical and supercritical behavior of an aeroelastic system. Comparison between FCE and *Polynomial Chaos Expansion* (PCE) is done to illustrate the efficacy of FCE over PCE approach for pitch and plunge airfoil exhibiting LCO. Cooper used PCE in the design of composite wing including uncertainties into the probabilistic flutter model for different combinations of material [40]. In comparison with MCS, second and third order PCE seems to give good results with less computation. Apart from parametric uncertainties, work is

being conducted on modeling uncertainties. Riley [41] presented quantification of modeling uncertainties in the aeroelastic system. In this publication flutter in two degrees of freedom systems with unsteady aerodynamics is investigated to demonstrate modeling uncertainties. Sandhu [42] implemented Bayesian model selection method for nonlinear aeroelastic system taking advantage of wind tunnel data.

The Bayesian inference methodology built by Khalil [1, 43–47] on aeroelastic model and proof-of-concept developed by Poirel [24] estimates flutter speed pdf by estimating the joint pdf of coefficients of flutter prediction equation. Khalil first estimates posterior samples of modal parameters from corrupted measurement data at pre-flutter airspeeds. Once modal parameters are estimated, posterior samples of flutter margin pdf are estimated using flutter margin equation and finally flutter speed pdf is estimated from flutter margin using flutter prediction equation. In this work three assumptions were made:

- Flat prior, i.e., uniform prior with positive support is enforced for modal parameters.
- Independence among system parameter vector at different airspeeds.
- Flutter margin vector is independent at different airspeeds.

In this thesis, Bayesian inference with application of *Adaptive Metropolis Markov Chain Monte Carlo* (AMCMC) method developed by [1, 24, 47] is implemented to estimate flutter speed using the *flutter margin method*. It is extended to incorporate prior information of the system estimated using MCS and parametric uncertainty quantification [48]. For aeroelastic model and proof-of-concept used in this thesis refer to [2, 24]. The objective of this thesis is to extend Khalil's work by relaxing the assumptions stated above. Two different cases are considered to address the assumptions. 1: *Independent prior*, prior for modal parameters are considered independent at different airspeeds while considering correlation at a given airspeed, along with flutter margin vector independent at different airspeeds. This case relaxes the first assumption by using informative prior at specific airspeed. 2: *Joint prior*, the full statistical correlation among modal parameters at different airspeed as well as at a given airspeed is considered. Also, the flutter margin vector at various airspeeds is correlated. This case addresses all three assumptions. Further details are presented in Chapter 3.

The notions behind using statistical method are to bridge the gap between theoretical and experimental approach. In Bayesian framework, the test data acquired from flight flutter test is used to update current knowledge of the system. To understand the working principle of methodology, discussed in Chapter 2, a study of the mass-spring-damper

single degree of freedom system is conducted to estimate parameters using measurement, some interesting facts came to light. A strong correlation is observed between initial displacement and damping ratio.

1.5 Thesis Overview

The first chapter of the thesis is an introduction briefly elaborating the concepts of aeroelasticity, flutter and flutter flight test including a concise literature review. The second chapter is focused on the methodology and mathematics involved in the analysis of flutter using MCMC and AMMCMC algorithms, and examples are given to elaborate the utilization of the algorithm in practice. In the third chapter, we present our first case study of evaluation of flutter speed using flutter margin method for quasi-steady aerodynamics model. In the third chapter, two cases with two test points and three test points respectively, were studied. In each case a comparison between *flat prior*, *independent prior* and *joint prior* is conducted for the coefficient of variation values 5%, 7.5% and 10% in system parameters. The fourth chapter concludes and presents future work.

Chapter 2

Methodology

2.1 Introduction

This chapter lays the mathematical framework for the thesis. In this chapter, the statistical method used are explained in engineering perspective. The primary focus is on Bayesian Inference, Markov Chain Monte Carlo (MCMC) and Metropolis-Hasting Algorithm. Some examples are shown later in this chapter to demonstrate the implementation of the method and its efficacy.

2.2 Bayesian Inference

Bayesian inference is build on the presumption that probabilities of an observed phenomenon (current knowledge) can be updated on the availability of the new information. In the framework of parametric models, parameters are considered to be random variables [1, 49–54] .

Let A and B be two generic event whose joint probability is given as $P(AB)$. $P(AB)$ can be expressed in two ways namely

$$P(A|B)P(B) = P(B|A)P(A) \quad (2.1)$$

Depending on what we want to infer in our model one can rearrange Eq. 2.1. Supposing $P(B) > 0$ and dividing by $P(B)$ yields

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (2.2)$$

where Eq. 2.2 is the first form of Bayes theorem that relates the event A conditional on the occurrence of an event B .

Using Smith's notation [55] Bayes formula for joint distribution of parameter vector $\phi = [\phi_1, \dots, \phi_k]$ and observations data $d = [d_1, \dots, d_n]$ can be given by equation 2.3

$$\pi(\phi|d) = \frac{\pi(d|\phi)\pi_0(\phi)}{\pi_\gamma(d)} \quad (2.3)$$

where

1. $\pi(\phi|d)$ is known as inverse or posterior pdf, it contains updated information of the system parameters using the available observations d .
2. $\pi_0(\phi)$ is known as prior pdf which represents the available knowledge or information of the parameters before analyzing the observation vector d .
3. $\pi(d|\phi)$ is known as likelihood function which defines the conditional pdf of the observation vector d given the parameter vector ϕ .
4. $\pi_\gamma(d)$ is marginal density known as normalization factor, represented by $\pi_\gamma(d) = \int \pi(d|\phi)\pi_0(\phi)d\phi$.

The layout for Bayesian modeling [50, 56, 57] can be simplified as follows:

1. Identify the parameters that are observed d (e.g measurement of system response) and those parameters which have an impact on the process but not yet observed (latent parameters) ϕ (e.g. modal parameters).
2. Specify a joint density function $P(\phi, d)$ that describes how the variables interact $\pi(d|\phi)\pi_0(\phi)$ refer equation 2.1. In some cases, density function $\pi_0(\phi)$ is likely to make assumptions and may consist of the assumed analytical form of the distribution along with values for hyper-parameters (parameter of prior distribution). Since exact analytical form for $\pi_0(\phi)$ may not be available.
3. Perform inference based on observations. For a model with latent parameters, this involves forming a conditional density function $\pi(\phi|d)$, which states that latent parameters ϕ are updated upon observations d .
4. If required, update the value of the hyper-parameters of the model to improve the fit. A common approach is to aim to maximize the marginal likelihood of the observed values by the choice of the hyper-parameters.

2.3 Markov Chain Monte Carlo

MCMC method describes techniques to draw a representative sample from a complex distribution in a computationally efficient way [58, 59].

2.3.1 Markov chains

A sequence $\{x_1, x_2, \dots, x_n\}$ of random variables defined on a finite state space X is a *Markov chain* if it satisfies Markov property that x_{n+1} depends only on x_n [58]:

$$P(x_{n+1} \in X | x_0, x_1, \dots, x_n) = P(x_{n+1} \in X | x_n) \quad (2.4)$$

the value of x_{n+1} is dependent on its history only through its nearest past, x_n [55, 58, 60]. For a discrete state space, $P(x, y)$ is called transition probability when Markov chain is time-invariant (i.e. independent of n) and homogeneous (stationary transition probabilities) which satisfy following conditions [61]:

- $P(x, y) \geq 0, \forall x, y \in S$
- $\sum_{y \in S} P(x, y) = 1, \forall x \in S$

where S is the state space.

2.3.1.1 Stationarity

A distribution π is said to be stationary distribution of a chain with transition probability $P(x, y)$ [1, 61] if

$$\sum_{x \in S} \pi(x) P(x, y) = \pi(y), \quad \forall y \in S. \quad (2.5)$$

From above equation we can say that marginal distribution of x_n is independent of n . The distribution is also known as time-invariant or equilibrium distribution.

2.3.1.2 Reversibility, Irreducibility, Aperiodicity

A homogeneous, irreducible [55] and aperiodic Markov chain with transition probabilities $P(x, y)$ and stationary distribution π is said to be reversible with reversibility condition [61] given as

$$\pi(x) P(x, y) = \pi(y) P(y, x), \quad \forall x, y \in S \quad (2.6)$$

It can also be interpreted as detailed balance condition because it equates the rates of moves through states and it does it for every possible pair of states [1, 61].

2.3.2 Burn-in and Convergence diagnostics

Burn-in typically represents the initial transient period in the chain. The length of burn-in period depends on the initial parameter and the rate of convergence of distribution to the stationary distribution. To determine the length of burn-in period analytically, may not be computationally feasible in most situations [62]. Visual inspections of the output are the most commonly employed methods for determining burn-in period [58].

Several methods are proposed to determine the length of burn-in period and are categorized under Convergence diagnostics. The most popular and commonly used diagnostics are proposed by Gelman and Rubin [63], Raftery and Lewis [64, 65], Geweke [66]. Gelman and Rubin diagnostics [63] test relies on multiple simulated MCMC chains with the scattered initial values and compares the variances within each chain and the variances between the chains. Lack of convergence is observed when these two variances have large differences.

Geweke diagnostics tests whether the mean estimates have converged by comparing the means from the two time interval of the Markov chain [66]. In an another proposition by Geyer [67], evaluation of the length of burn-in might not be necessary if extreme starting values avoided. Geyer suggests burn-in length typically should be between 1% to 2% of the run length, where run length is sufficiently long enough to achieve adequate precision.

The convergence criteria adopted in this chapter for two algorithm discussed are:

- For simple *Markov Chain Monte Carlo* (MCMC) Geyer [67] approach is followed.
- For *Adaptive Metropolis Markov Chain Monte Carlo* (AMMCMC), Gelman and Rubin [63] diagnostic is implemented as convergence criteria during adaptation.

2.3.3 Metropolis Hasting Algorithm

The Metropolis algorithm was first proposed by Metropolis [68] for the case of symmetric proposal distributions (proposal distribution is the conditional probability that proposes next point from previous point). Later, Hasting extended Metropolis algorithm to the more general case [69]. The working principle behind Metropolis-Hasting algorithm is to generate a sequence of samples values such that when samples size increases, the distribution of samples closely approximates the desired distribution. The sample values are produced iteratively, at each iteration the algorithm accepts the candidate for next sample value based on current sample value and some probability based on acceptance ratio, in turn, making the sequence be the Markov chain. The algorithm is discussed in **Algorithm. 1**, and the flow chart is shown in Figure 2.1.

2.3.4 Adaptive Metropolis Hasting Algorithm

The adaptive Metropolis algorithm **Algorithm. 2** is developed by Haario [70]. It is based on random walk Metropolis algorithm [68, 69]. The parallel implementation of adaptive Metropolis algorithm is developed by Khalil [1]. In the adaptive Metropolis algorithm, the covariance of the proposal is calculated using all of the previous states, and this is achieved by using the recursive formula for covariances. An important advantage of using adaptive Metropolis algorithm as stated by [70] is that it start using the cumulative information right at the beginning of the simulation. The design parameter $s_p = 2.38^2/p$ is adopted from Gelman *et al.* [71]. This choice of design parameter optimizes the mixing properties of the chain in case of Gaussian targets and Gaussian proposals [70, 71]. The flow chart for adaptive Metropolis algorithm with parallel implementation is shown in Figure 2.2.

Algorithm 1 Metropolis-Hasting algorithm [55, 69]

1:

2: Initialization: Choose an initial parameter value q^0

3:

4: **for** $k \leftarrow 1, M$ **do**5: generate $z \sim N(0, 1)$

6:

7: Construct candidate

8:

$$q^* = q^{k-1} + Rz \quad (2.7)$$

9:

10: where R is the Cholesky decomposition of V (covariance matrix for parameter set q). It ensure that $q^* \sim N(q^{k-1}, V)$

11:

12: Because the construction of q^* takes into account q^{k-1} , this algorithm is know as random walk algorithm

13:

14: Compute the acceptance ratio

15:

$$r(q^*|q^{k-1}) = \frac{\pi(d|q^*)\pi_0(q^*)}{\pi(d|q^{k-1})\pi_0(q^{k-1})} \quad (2.8)$$

16:

17: The Metropolis-Hasting algorithm generalizes the Metropolis algorithm to include non symmetric proposal function $J(q^*|q^{k-1})$. In that case acceptance ratio becomes

18:

$$r(q^*|q^{k-1}) = \frac{\pi(d|q^*)\pi_0(q^*)J(q^{k-1}|q^*)}{\pi(d|q^{k-1})\pi_0(q^{k-1})J(q^*|q^{k-1})} \quad (2.9)$$

19:

20: Set

$$q^k = \begin{cases} q^*, & \text{with probability } \alpha = \min(1, r) \\ q^{k-1}, & \text{elsewhere.} \end{cases}$$

21:

That is we accept $q^k = q^*$ with probability α .

FIGURE 2.1: Flow chart of MH-MCMC algorithm [1]

Algorithm 2 Adaptive Metropolis-Hasting algorithm Part 1 [55, 70]

-
- 1: Initialization: Choose an initial parameter value q^0 , k_0 length of initial run and M number of chain iterates
 - 2: Set $eps = 0.1$
 - 3: Compute initial covariance estimate $V = s_0^2 \text{diag}[q^0]$ where $s_0 = 0.001$ and $R = \text{Chol}(V)$
 - 4: Compute initial posterior $PP_{q^0} \propto \pi(d|q^0)\pi_0(q^0)$
 - 5: Set $denominator = 1.0$
 - 6: Set current covariance $curcov = 0$
 - 7: Set old covariance $oldcov = 0$
 - 8: **procedure** *Adaptation*
 - 9: sample $z \sim N(0, I_p)$
 - 10: sample $u_\alpha \sim U(0, 1)$
 - 11: *loop*:
 - 12: **if** $i > k_0 + 1$ **then**
 - 13: Compute Frobenius norm $\|A\|_F$ of rms. error between old covariance and current covariance

$$\|A\|_F = \sqrt{\sum_{i=1}^p \sum_{j=1}^p |a_{ij}|^2} \quad (2.10)$$

- 14: **if** $\|A\|_F < eps$ **then** exit. This condition ensure that MCMC has reached stationarity. where a_{ij} represent covariance and p is dimension of parameter.
- 15: Update chain covariance matrix

$$V_k = s_p \text{curcov}(q^0, q^1, \dots, q^k) \quad (2.11)$$

- 16: where $s_p = (2.38^2/p)$, p is dimension of parameter and s_p is the design parameter.
 - 17: Update $R_k = \text{Chol}(V_k)$
 - 18: sample $z_k \sim N(0, I_p)$
 - 19: sample $u_\alpha \sim U(0, 1)$
 - 20: Set $oldcov = curcov$
 - 21: Set $i = 1$
 - 22: Construct candidate $q^* = q^{k-1} + R_k z_k$
 - 23: Compute posterior $PP_{q^*} \propto \pi(d|q^*)\pi_0(q^*)$
 - 24: Compute $numerator = e^{[PP_{q^*} - PP_{q^0}]}$
 - 25: Compute $\alpha = (numerator/denominator)$
 - 26: **if** $numerator < 1.0$ **then**
 - 27: Set $PP_q^0 = PP_q^*$
 - 28: Set $numerator = 1.0$
 - 29: **if** $u_\alpha < \alpha$ **then**
 - 30: Set $q^k = q^*$
 - 31: Set $denominator = numerator$
-

Algorithm 2 Part 2

32: Update mean and covariance matrix

33: Instead of using standard empirical covariance formula we will be using recursive relation which increases the computational efficiency of algorithm compare to standard relation [55, 72] and the ergodicity of this algorithm is established in [70].

$$V_{k+1} = \frac{k-1}{k}V_k + \frac{s_p}{k} \left[kq^{-k-1}(q^{-k-1})^T - (k+1)q^{-k}(q^{-k})^T + q^k(q^k)^T \right] \quad (2.12)$$

34: In a similar manner, the sample mean can be computed recursively as

$$q^{-k+1} = \frac{k}{k+1}q^{-k} + \frac{1}{k+1}q^k \quad (2.13)$$

35: Set $i = i + 1$

36: *Endloop:*

37: Compute posterior after adaptation is complete using latest proposal as initial value

$$PP_{q^k} \propto \pi(d|q^k)\pi_0(q^k)$$

38: Set *denominator* = 1.0

39: **procedure** *Sampling from stationary distribution*

40: **for** $i \leftarrow 1, M$ **do**

41: sample $z \sim N(0, I_p)$

42: sample $u_\alpha \sim U(0, 1)$

43: **for** $j \leftarrow 1, k_0$ **do**

44: $samples(p) = q^k(p)$

45: Construct candidate $q^* = q^k + R_k z$

46: Compute posterior $PP_{q^*} \propto \pi(d|q^*)\pi_0(q^*)$

47: Compute *numerator* = $e^{[PP_{q^*} - PP_q]}$

48: Compute $\alpha = (numerator/denominator)$

49: **if** $u_\alpha < \alpha$ **then**

50: Set $q^k = q^*$

51: Set *denominator* = *numerator*

FIGURE 2.2: Flow chart of AMCMC algorithm

2.4 Examples

Among the three examples presented in this section, in example 1 we would like to infer the value of the parameter x using samples of the observed variable y . In example two we want to investigate the effect of correlation between model and parameter and in example three we are doing the parametric estimation of the single-degree-of-freedom system. The purpose of doing single-degree-of-freedom system is to build our confidence in AMCMC methodology.

2.4.1 Example 1

TABLE 2.1: Statistical estimation problems

Case	F(x)	x_0	y_0	σ_x	σ_y	σ_q
1a	$y = x$	1	-1	1	0.3	1
1b	$y = x$	1	-1	1	0.3	0.1
2	$y = x^2$	1	-1	1	0.3	1
3	$y = x^2(x^2 - 2)$	1	-1	1	0.3	1
4a	$y = \cos(x)$	1	-1	1	0.3	1
4b	$y = \cos(x)$	1	-1	1	0.3	0.1
4c	$y = \cos(x)$	1	-1	4	0.3	1

The first example is taken from Evensen [73]. It is comprised of cases mentioned in the Table 2.1. The reason to select this problem is its simplicity and availability of analytical solution. The case 1 represents a simple scalar linear model, and weak non-linearity is introduced in case 2. Case 3 is the model with moderate nonlinearity and case 4 has strong nonlinearity. The results obtained from this simulation utilize two algorithm **Algorithm. 1** and **Algorithm. 2** and are in good agreement with results from Evensen [73].

The mathematical structure for the Example 1 is given below.

Parameters

$$x = x_0 + \epsilon_x \quad (2.14)$$

where x_0 is prior mean and ϵ_x is stochastic error with $\epsilon_x \sim N(0, \sigma_x^2)$.

Model

$$y = F(x) + \epsilon_q \quad (2.15)$$

where $F(x)$ is the nonlinear model operator ϵ_q is an additive stochastic error term with $\epsilon_q \sim N(0, \sigma_q^2)$.

Measurements

$$d = y_0 + \epsilon_y \quad (2.16)$$

where d is the data vector ϵ_y is the random measurement error with $\epsilon_y \sim N(0, \sigma_y^2)$

In this case x is parameter, y is the observed variable. For the sake of clarity and simplicity selected results are shown below.

The joint conditional pdf using Bayesian approach [73] is given by Eq. 2.17. The model error is assumed Gaussian.

$$p(x, y|d) = \frac{p(d|x, y)p(y|x)p(x)}{\int p(d|x, y)p(y|x)p(x)dx dy} \quad (2.17)$$

$$p(x, y|d) \propto \exp \left(-\frac{1}{2} \left(\frac{(x - x_0)^2}{\sigma_x^2} + \frac{(y - F(x))^2}{\sigma_q^2} + \frac{(d - y_0)^2}{\sigma_y^2} \right) \right) \quad (2.18)$$

The normalization constant $\int p(d|x, y)p(y|x)p(x)dx dy$ is estimated numerically.

The following results shown are a comparison of an analytical solution with results obtained using MCMC and AMMCMC algorithms. There are two sets of figures, one representing prior and second representing joint conditional pdf. The upper left figure is the joint conditional pdf of the analytical solution without measurement. The top right figure is analytical solution conditioned on measurement. The joint conditional pdf shown on lower left is obtained using MCMC algorithm **Algorithm. 1**. While, joint pdf shown in lower right is obtained using AMMCMC algorithm **Algorithm. 2**.

In case 1a and 1b shown in Figure 2.4 and Figure 2.6 the linear function is assumed $F(x) = x$. The result obtained using MCMC and AMMCMC are consistent with analytical results. In case 1a, one can notice model prediction has large uncertainty, and the effect of y measurement has less impact on the estimation of x . On the other hand in case 1b where the model has less uncertainty, pdf is narrow and aligned along the line $y = x$ and consistent with the pdf for the measurement y .

In case 2 shown in Figure 2.8 nonlinear function is assumed $F(x) = x^2$. In this case, joint conditional pdf appears to be non-Gaussian. The result obtained using MCMC and AMMCMC can capture the non-Gaussian effect and are consistent with analytical result. In case 3 shown in Figure. 2.10 nonlinear function is assumed $F(x) = x^2(x^2 - 2)$.

In this case, the joint conditional pdf appears to be unimodal and non-Gaussian. The result obtained using MCMC and AMMCMC are consistent with the analytical solution.

In cases 4a-4c shown in Figure. 2.12, 2.14 and 2.16 high nonlinearity is introduced using function $F(x) = \cos(x)$. The objective of this cases is to establish the working capability of MCMC and AMMCMC algorithm in an extremely non-linear scenario. In case 4a, the standard deviation of model error and the prior of x are set to one. Results from MCMC and AMMCMC contain the same characteristic as the analytical solution. In case 4b, the model is very accurate which leads to sharper estimate. Results from MCMC and AMMCMC are fairly consistent with the analytical solution. In case 4c, the standard deviation of the prior of x is set to four. The effect of low accuracy in the prior of x is seen in the bimodal structure of the joint conditional pdf. MCMC and AMMCMC algorithm have efficiently captured the bimodal structure of the joint conditional pdf.

The effectiveness of MCMC and AMMCMC algorithm in capturing the exact characteristics as compared to the analytical solution is established. Even though MCMC and AMMCMC algorithm were able to reproduce the results in comparison with the analytical solution, the AMMCMC has a significant advantage over MCMC in terms of computational efficiency and CPU time. The computational comparison was made on the supercomputer Guillimin servers from McGill University, managed by Calcul Qubec and Compute Canada. There are considerable time and cost saving while using AMMCMC algorithm shown in Table 2.2.

TABLE 2.2: Comparison of CPU time

Cases	MCMC (s)	AMMCMC (s)
1a	564.206 s	5.666 s
1b	563.616 s	5.811 s
2	519.79 s	5.811 s
3	542.046 s	5.66 s
4a	537.268 s	5.96 s
4b	499.835 s	5.96 s
4c	525.87 s	6.108 s

Case 1a

FIGURE 2.3: Marginal prior pdf for x and y FIGURE 2.4: Case 1a conditional and joint pdf for linear function $F(x) = x$

Case 1b

FIGURE 2.5: Marginal prior pdf for x and y FIGURE 2.6: Case 1b conditional and joint pdf for linear function $F(x) = x$ with minimum model noise

Case 2

FIGURE 2.7: Marginal prior pdf for x and y FIGURE 2.8: Case 2 conditional and joint pdf for quadratic function $F(x) = x^2$

Case 3

FIGURE 2.9: Marginal prior pdf for x and y

FIGURE 2.10: Case 3 conditional and joint pdf for nonlinear function $F(x) = x^2(x^2 - 2)$

Case 4a

FIGURE 2.11: Marginal prior pdf for x and y

FIGURE 2.12: Case 4a conditional and joint pdf for nonlinear function $F(x) = \cos(x)$

Case 4b

FIGURE 2.13: Marginal prior pdf for x and y

FIGURE 2.14: Case 4b conditional and joint pdf for nonlinear function $F(x) = \cos(x)$.

Case 4c

FIGURE 2.15: Marginal prior pdf for x and y

FIGURE 2.16: Case 4c conditional and joint pdf for nonlinear function $F(x) = \cos(x)$, for poor first guess resulting in bimodality.

2.4.2 Example 2

In previous example Case 1a, we assumed that model y and parameter x are independent. In this example, we want to investigate the effect of correlation between model and parameter. Also, to check whether the algorithm **Algorithm. 1** and **Algorithm. 2** can capture the exact characteristics as compared to the analytical solution in the presence of joint conditional prior, a simple example is considered $y = F(x)$ and $F(x) = x$. The joint conditional posterior pdf is given as:

$$p(x, y|d) \propto \exp\left(-\frac{1}{2} \left(\frac{(d - y_0)^2}{\sigma_y^2}\right)\right) p(x, y) \quad (2.19)$$

Joint conditional prior [74, 75] is given as:

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{1}{2(1-\rho^2)} \left[\frac{(x-x_0)^2}{\sigma_x^2} + \frac{(y-F(x))^2}{\sigma_q^2} - \frac{2\rho(x-x_0)(y-F(x))}{\sigma_x\sigma_q}\right]\right) \quad (2.20)$$

where ρ is correlation between x and y

$$\rho = \frac{\sigma_{xq}}{\sigma_x\sigma_q} \quad (2.21)$$

FIGURE 2.17: Case 1: Prior joint pdf estimated and analytical

FIGURE 2.18: Case 1: Posterior joint pdf for $\sigma_x = 1$, $\sigma_q = 1$ and $\sigma_y = 0.3$

FIGURE 2.19: Case 2: Prior joint pdf

FIGURE 2.20: Case 2: Posterior joint pdf for $\sigma_x = 0.7$, $\sigma_q = 1$ and $\sigma_y = 0.3$

Analytical solution

Analytical solution for Example 2 discussed previously, is obtained using *Kalman filter* update equation [1, 76] given below :

$$m_T = m_0^- + K[d - Hm_0^-] \quad (2.22)$$

$$P_T = [I - KH]P_0^- \quad (2.23)$$

where m_T is mean update and P_T is covariance update, d is measurement. m_0^- is predicted mean and P_0^- is predicted covariance of a Kalman filter at the time step 0 just before the measurement d .

$m_0^- = [x_0 \ y_0]$, Kalman gain K is given by Eq. 2.24

$$K = P_0^- H^T [\sigma^2 + H P_0^- H^T]^{-1} \quad (2.24)$$

$$P_0^- = \begin{bmatrix} \sigma_x^2 & \sigma_{xq} \\ \sigma_{xq} & \sigma_q^2 \end{bmatrix} \quad (2.25)$$

$$H = [0 \ 1] \quad (2.26)$$

Using equation 2.25 & 2.26 we get

$$K = \frac{1}{\sigma_q^2 + \sigma_y^2} \begin{Bmatrix} \sigma_{xq} \\ \sigma_q^2 \end{Bmatrix} \quad (2.27)$$

$$P_T = \frac{1}{\sigma_q^2 + \sigma_y^2} \begin{bmatrix} \sigma_x(1 - \rho^2)\sigma_q^2 + \sigma_y^2 & \sigma_{xq}\sigma_y^2 \\ \sigma_{xq}\sigma_y^2 & \sigma_q^2\sigma_y^2 \end{bmatrix} \quad (2.28)$$

where σ_y^2 is measurement variance and ρ correlation coefficient given by Eq. 2.21. The joint prior pdf for case 1 where $\sigma_x = \sigma_q = 1$ is shown in Figure 2.17 and the posterior joint pdf are shown in Figure 2.18 comparison of estimated and analytical solution shows reasonable accuracy. The joint prior pdf for case 2 where $\sigma_x = 0.7$ and $\sigma_q = 1$ are shown in Figure 2.19 and joint posterior pdf are shown in Figure 2.20. In comparison to analytical solution, estimated result obtained using AMMCMC are fairly accurate.

2.4.3 Example 3.1

A simple single-degree-of-freedom (SDOF) system is considered with uncertainties in measurement. In this case, we would like to infer system parameters having prior knowledge of system and measurement. we have consider our measurement to be function of frequency ω and damping ratio ξ .

Model

$$M\ddot{X} + C\dot{X} + KX = 0 \quad (2.29)$$

True response

$$Y = e^{-\omega\xi t_k} \sin(\omega\sqrt{1-\xi^2}t_k) \quad (2.30)$$

Measurement

$$Y_k = e^{-\omega\xi t_k} \sin(\omega\sqrt{1-\xi^2}t_k) + \eta_k \quad (2.31)$$

where η_k is Gaussian noise with zero mean and standard deviation $\sigma = 15\%$. The posterior of these unknown parameters is given using Bayes theorem as:

$$p(\omega, \xi, \sigma | Y_k) \propto p(Y_k | \omega, \xi, \sigma) p(\omega, \xi) p(\sigma) \quad (2.32)$$

where $p(\sigma) \propto 1$

Likelihood

$$p(Y_k | \omega, \xi, \sigma) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left[\sum_{K=1}^N -\frac{1}{2\sigma^2} (Y_k - \hat{Y}_k(\omega, \xi, \sigma))^2 \right] \quad (2.33)$$

Prior estimation $p(\omega, \xi)$

For prior estimation, we will use parametric uncertainty analysis of model.

We have considered uncertainty in mass, stiffness, and damping.

$$M = m + \epsilon_1. \quad (2.34)$$

$$K = k + \epsilon_2. \quad (2.35)$$

$$C = c + \epsilon_3. \quad (2.36)$$

To maintain positivity in mass, stiffness and damping, truncated Gaussian distribution is considered for errors ϵ_1 , ϵ_2 & ϵ_3 . Where $\epsilon_1 \sim N(0, 0.2^2)$, $\epsilon_2 \sim N(0, 10^2)$ and $\epsilon_3 \sim N(0, 0.5^2)$ where $m = 2 \text{ kg}$, $k = 200 \text{ N/m}$ and $c = 5 \text{ Ns/m}$

Using MCS and analytical expression for ω and ξ

$$\omega = \sqrt{\frac{K}{M}} \quad (2.37)$$

$$\xi = \frac{C}{2\sqrt{KM}}, \quad (2.38)$$

samples for ω and ξ are generated. Prior pdf $p(\omega, \xi)$ is obtained using multivariate kernel density estimation [74] [75].

FIGURE 2.21: System response

FIGURE 2.22: Prior joint pdf for ω and ξ

FIGURE 2.23: Single-degree-of-freedom system case 1: Marginal pdf of parameters ω , ξ , and σ

FIGURE 2.24: Single-degree-of-freedom system case 1: Joint pdf of parameters ω , ξ , and σ

In Figure 2.21 true response and noisy measurement are shown. The standard deviation in measurement noise is 15% of the root mean square (rms) value of the original signal and 100 data point are considered. Figure 2.22 shows the prior joint pdf for parameter ω and ξ . In Figure 2.23 marginal pdf of parameters ω, ξ and σ natural frequency, damping ratio and standard deviation of measurement respectively. In Figure 2.24 joint conditional pdf of parameters ω, ξ and σ .

2.4.4 Example 3.2

A single-degree-of-freedom system is considered with uncertainties in measurement. In previous example, true response Eq. 2.30 considered was a simple function of frequency ω and damping ratio ξ . While in this case, the true response is given by Eq. 2.40 is a function of initial displacement U_0 , frequency ω and damping ratio ξ . We would like to infer system parameter, initial displacement, and variance in the measurement having prior knowledge of system and measurement. For simplicity, initial velocity is set to zero.

Model

$$M\ddot{X} + C\dot{X} + KX = 0 \quad (2.39)$$

True response

$$Y = e^{-\omega\xi t_k} \left[U_0 \cos(\omega\sqrt{1-\xi^2}t_k) + \frac{V_0 + \omega\xi U_0}{\omega\sqrt{1-\xi^2}} \sin(\omega\sqrt{1-\xi^2}t_k) \right] \quad (2.40)$$

Measurement

$$Y_k = e^{-\omega\xi t_k} \left[U_0 \cos(\omega\sqrt{1-\xi^2}t_k) + \frac{V_0 + \omega\xi U_0}{\omega\sqrt{1-\xi^2}} \sin(\omega\sqrt{1-\xi^2}t_k) \right] + \eta_k \quad (2.41)$$

where initial velocity $V_0 = 0$, η_k is Gaussian noise with zero mean and standard deviation $\sigma = 15\%$. The posterior of these unknown parameters is given using Bayes theorem as:

$$p(\omega, \xi, U_0, \sigma | Y_k) \propto p(Y_k | \omega, \xi, U_0, \sigma) p(\omega, \xi) p(U_0) p(\sigma) \quad (2.42)$$

where $p(\sigma) \propto 1$

Likelihood

$$p(Y_k | \omega, \xi, U_0, \sigma) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left[\sum_{K=1}^N -\frac{1}{2\sigma^2} (Y_k - \hat{Y}_k(\omega, \xi, U_0, \sigma))^2 \right] \quad (2.43)$$

Prior estimation $p(\omega, \xi)$

For prior estimation, we will use parametric uncertainty analysis of model. we have considered uncertainty in mass, stiffness and damping.

$$M = m + \epsilon_1 \quad (2.44)$$

$$K = k + \epsilon_2 \quad (2.45)$$

$$C = c + \epsilon_3 \quad (2.46)$$

where $\epsilon_1 \sim N(0, 0.2^2)$, $\epsilon_2 \sim N(0, 10^2)$ and $\epsilon_3 \sim N(0, 0.5^2)$ where $m = 2 \text{ kg}$, $k = 200 \text{ N/m}$ and $c = 5 \text{ Ns/m}$

Using MCS and analytical expression for ω and ξ .

$$\omega = \sqrt{\frac{K}{M}} \quad (2.47)$$

$$\xi = \frac{C}{2\sqrt{KM}} \quad (2.48)$$

Samples for ω and ξ are generated and prior pdf $p(\omega, \xi)$ is obtained using multivariate kernel density estimation [74] [75]. Initial displacement U_0 is considered to be Gaussian with mean 1 and variance 0.04 ($U_0 \sim N(1, 0.04)$).

$$p(U_0) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{(U_0 - \mu)^2}{2\sigma^2} \right] \quad (2.49)$$

FIGURE 2.25: System response

FIGURE 2.26: Prior joint pdf for ω and ξ

FIGURE 2.27: Single-degree-of-freedom system: Posterior marginal pdf of parameters ω, ξ, U_0 and σ

FIGURE 2.28: Single-degree-of-freedom system: Joint conditional pdf between parameters ω, ξ, U_0 and σ

In Figure 2.25 true response and noisy measurement are shown. The standard deviation in measurement noise is 15% (signal-to-noise ratio 2.6 DB) and 100 data point are considered. Figure 2.26 is prior joint pdf for parameter ω and ξ . In Figure 2.27 posterior marginal pdf of parameters ω, ξ, U_0 and σ frequency, damping ratio, initial displacement and standard deviation respectively, are shown. The parameter estimation is in good agreement with true values. In Figure 2.28 joint conditional pdf between parameters ω, ξ, U_0 and σ are shown. The joint conditional pdf shows the correlation between system parameters, For example, the frequency is positively correlated with the damping ratio.

2.4.5 Example 3.3

In this example, we would like to investigate the effect of reduced measurement noise compared to the previous case. The standard deviation of measurement noise is reduced from 15% to 6% and 100 data point are considered.

FIGURE 2.29: System response

FIGURE 2.30: Prior joint pdf for ω and ξ

FIGURE 2.31: Single-degree-of-freedom system: Posterior marginal pdf of parameters ω, ξ, U_0 and σ

FIGURE 2.32: Single-degree-of-freedom system: Joint conditional pdf between parameters ω, ξ, U_0 and σ

In Figure 2.29 true response and noisy measurement are shown. Figure 2.30 is prior joint pdf for parameter ω and ξ . In Figure 2.31 posterior marginal pdf of parameters ω, ξ, U_0 and σ (frequency, damping ratio, initial displacement and standard deviation respectively) are shown. The parameter estimation is in good agreement with true values. In Figure 2.32 the joint conditional pdf between parameters ω, ξ, U_0 and σ are shown. The joint conditional pdf shows the correlation between system parameters. For example, frequency is positively correlated to damping ratio.

2.4.6 Example 3.4

In this example, we would like to investigate the effect of reduced measurement noise and dense measurement data on system considered in Example 3.2. The standard deviation of measurement noise is 6%, and the number of the data point is increased from 100 to 200.

FIGURE 2.33: System response

FIGURE 2.34: Prior joint pdf for ω and ξ

FIGURE 2.35: single-degree-of-freedom system: Posterior marginal pdf of parameters ω, ξ, U_0 and σ

FIGURE 2.36: Single-degree-of-freedom system: Joint conditional pdf between parameters ω, ξ, U_0 and σ

In Figure 2.33 true response and noisy measurement are shown. Figure 2.34 is prior joint pdf for parameter ω and ξ . In Figure 2.35 marginal pdf of parameters ω, ξ, U_0 and σ frequency, damping ratio, initial displacement and standard deviation respectively, are shown, the parameter estimation is in good agreement with true values. In Figure 2.36 joint conditional pdf between parameters ω, ξ, U_0 and σ are shown. The joint conditional pdfs show the correlation between system parameters. For example, frequency is positively correlated to damping ratio.

2.5 Conclusion

The purpose of this study was to check the working capabilities of AMMCMC algorithm with increasing complexity in the problem and to build confidence. In Example 1 we observed that MCMC and AMMCMC algorithm are capable of sampling from Gaussian as well as non-Gaussian distribution. Example 2 established that AMMCMC can capture exact characteristics when joint prior is considered. In Example 3 parameter estimation for the single-degree-of-freedom system was done, and effect of measurement noise and sparsity of measurement data were investigated. This study reveals that the accuracy of prediction improves as the measurement noise is reduced and further improvement is observed when dense measurement data is used. This is evident from following Table 2.3.

TABLE 2.3: Single-degree-of-freedom parameter estimation: Example 3

Parameters	True value	Example 3.1	Example 3.2	Example 3.3	Example 3.4
		mean/COV	mean/COV	mean/COV	mean/COV
ω	10	9.997/1.5015	9.8907/1.5566	10.0269/0.7204	10.0135/0.4891
ξ	0.1768	0.1510/6.7950	0.1437/8.5066	0.1713/4.1055	0.1720/3.2355
σ	0.15/0.06	0.1568/7.3020	0.1571/7.3251	0.0610/5.1109	0.0608/5.0847
U_0	1	-	0.8760/7.9274	0.9865/3.0456	0.9957/2.2427

Chapter 3

Application to Flutter Margin Method: Quasi Steady Aerodynamics 2D Airfoil

3.1 Introduction

The *flutter margin method* first proposed by Zimmerman and Weissenburger [19] has had significant impact on flutter flight test. In this publication, Zimmerman implemented Routh's stability criteria to derive a flutter margin equation for 2D airfoil with quasi-steady aerodynamics. Earlier use of the *flutter margin method* to wind tunnel data on aircraft wing is presented by Bennett [77]. In this chapter we will implement the *flutter margin method* to estimate flutter speed for the simple 2D aeroelastic model shown in Figure 3.1. The formulation for quasi-steady aerodynamics is adapted from [3, 9, 10] and lecture notes by Poirel [2]. The coefficient of variations (COV) considered in this chapter for structural parameters in prior estimation are 5%, 7.5%, and 10%. Extensive research has been done to study the effect of parametric uncertainty on the aeroelastic behavior of aircraft. Pettit [33, 34] reports the effect of parametric uncertainty on the aeroelastic behavior of airfoil undergoing limit cycle oscillations. Kurdi [78] performed uncertainty quantification of Goland's wing to quantify flutter boundaries using MCS to propagate variability in structural parameters. Castravete [79] reports the effect of uncertainty in stiffness ($\text{COV} \leq 10\%$) on the flutter of a cantilever wing. Ueda [80] has shown the effect of uncertainty (5% and 10 % COV) in the position of location of the center of gravity and elastic axis. Borello [81] studied the effect of uncertainty (COV: 7.5%, 10%, 12.5%, 15% and 20%) in modulus of elasticity, shear modulus and mass on

classical flutter prediction. Khodaparasi [82] used probabilistic and fuzzy methods to quantify uncertainty in flutter boundaries in the presence of structural uncertainties.

FIGURE 3.1: symmetric airfoil adapted from [2]

3.2 Assumptions

The following basic aeroelastic modeling assumption are used in this work:

1. The system considered is two-degree-of-freedom airfoil undergoing bending and torsion.
2. The physical quantities used are averaged for the unit span.
3. The quasi-steady thin-airfoil theory aerodynamic model is used.
4. Structural damping is assumed to be viscous and proportional Rayleigh damping.

3.2.1 Equation of Motion

The two equation of motion are given by Eq. 3.1 and Eq. 3.2 adapted from [2, 3, 9, 10, 48].

$$m \frac{d^2 h}{dt^2} + \frac{m c x_\alpha}{2} \frac{d^2 \alpha}{dt^2} + B_1 \frac{dh}{dt} + B_2 \frac{d\alpha}{dt} + k_h h = -L(t) \quad (3.1)$$

$$I_{EA} \frac{d^2 \alpha}{dt^2} + \frac{m c x_\alpha}{2} \frac{d^2 h}{dt^2} + B_3 \frac{dh}{dt} + B_4 \frac{d\alpha}{dt} + k_\alpha \alpha = M_{EA}(t) \quad (3.2)$$

$$L(t) = \frac{\rho U^2}{2} c 2\pi \left(\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha} c (\frac{1}{2} - a_h)}{2U} \right) \quad (3.3)$$

$$M_{EA} = \frac{\rho U^2}{2} c 2\pi \left(\alpha \frac{c(\frac{1}{2} + a_h)}{2} + \dot{h} \frac{c(\frac{1}{2} + a_h)}{2U} + \dot{\alpha} \left(\frac{c^2(\frac{1}{2} - a_h)(\frac{1}{2} + a_h)}{4U} - \frac{c^2}{16U} \right) \right) \quad (3.4)$$

$$\underbrace{\begin{bmatrix} m & \frac{mcx_\alpha}{2} \\ \frac{mcx_\alpha}{2} & I_{EA} \end{bmatrix}}_{\text{Structural inertia matrix}} \underbrace{\begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix}} + \underbrace{\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}}_{\text{Structural damping matrix}} \underbrace{\begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix}} + \underbrace{\begin{bmatrix} k_h & 0 \\ 0 & k_\alpha \end{bmatrix}}_{\text{Structural stiffness matrix}} \underbrace{\begin{Bmatrix} h \\ \alpha \end{Bmatrix}} = \underbrace{\begin{Bmatrix} -L(t) \\ M_{EA} \end{Bmatrix}}_{\text{Aerodynamic load vector}} \quad (3.5)$$

$$\begin{Bmatrix} -L(t) \\ M_{EA} \end{Bmatrix} = \frac{\rho U^2}{2} c 2\pi \left\{ \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & c/2(\frac{1}{2} + a_h) \end{bmatrix}}_{\text{Aerodynamic stiffness matrix}} \underbrace{\begin{Bmatrix} h \\ \alpha \end{Bmatrix}} + \frac{1}{U} \underbrace{\begin{bmatrix} -1 & -c/2(\frac{1}{2} - a_h) \\ c/2(\frac{1}{2} + a_h) & \frac{c^2(\frac{1}{2} - a_h)(\frac{1}{2} + a_h)}{4} - \frac{c^2}{16} \end{bmatrix}}_{\text{Aerodynamic damping matrix}} \underbrace{\begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix}} \right\} \quad (3.6)$$

The structural damping is assumed to be viscous and proportional Rayleigh damping [83]. The proportionality is set with respect to mass and stiffness matrices given by the following equation [83].

$$[B_{str}] = \alpha_0 [M_{str}] + \alpha_1 [K_{str}] \quad (3.7)$$

The coefficients α_0 and α_1 are evaluated using prescribed damping ratios ξ_i and ξ_j using the following relation [83]

$$\frac{1}{2} \begin{bmatrix} \frac{1}{\omega_i} & \omega_i \\ \omega_j & \omega_j \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \end{Bmatrix} = \begin{Bmatrix} \xi_i \\ \xi_j \end{Bmatrix} \quad (3.8)$$

where ω_i and ω_j are natural frequencies of the two modes taking part in coalescence flutter.

The state-space system matrix is given by Eq. 3.9 [2, 48]:

$$A = \begin{bmatrix} [0]_{2 \times 2} & [I]_{2 \times 2} \\ [M^{-1}K]_{2 \times 2} & [M^{-1}C]_{2 \times 2} \end{bmatrix} \quad (3.9)$$

where

$$[M] = \begin{bmatrix} m & \frac{m c x_\alpha}{2} \\ \frac{m c x_\alpha}{2} & I_{EA} \end{bmatrix} \quad (3.10)$$

$$[K] = \begin{bmatrix} k_h & \rho U^2 c \pi \\ 0 & k_\alpha - \frac{\rho U^2 c^2 \pi}{2} (\frac{1}{2} + a_h) \end{bmatrix} \quad (3.11)$$

$$[C] = \begin{bmatrix} B_1 + \frac{\rho U}{2} c 2\pi & B_2 + \frac{\rho U}{2} c^2 \pi (\frac{1}{2} - a_h) \\ B_3 - \frac{\rho U}{2} c^2 \pi (\frac{1}{2} + a_h) & B_4 - \frac{\rho U}{2} c 2\pi \left(\frac{c^2 (\frac{1}{2} - a_h)(\frac{1}{2} + a_h)}{4} - \frac{c^2}{16} \right) \end{bmatrix} \quad (3.12)$$

Solving the eigenvalue problem, we get two pairs of complex conjugate roots [2, 84]

$$s_i = \beta_i \pm i\omega_i \quad (3.13)$$

where β_i and ω_i are the negative decay rate and the frequency.

3.2.2 Flutter Margin Equation

Zimmerman and Weissenburger [19] derived the following flutter margin Eq. 3.14 for quasi-steady aerodynamics. The derivation of flutter margin equation is based on Routh's stability criteria and defined as the third sub-determinant of the fourth order characteristic equation of the 2D aeroelastic system [1, 2, 19, 85]. The flutter margin equation constitutes of frequencies ω_1 & ω_2 and the decay rates β_1 & β_2 of the aeroelastic modes participating in the flutter [19] at a given airspeed U . For the stable aeroelastic system, the flutter margin is a positive quantity given by [19].

$$\begin{aligned} FM(U) &= \left[\left(\frac{\omega_2^2 - \omega_1^2}{2} \right) + \left(\frac{\beta_2^2 - \beta_1^2}{2} \right) \right]^2 \\ &\quad + 4\beta_1\beta_2 \left[\left(\frac{\omega_2^2 + \omega_1^2}{2} \right) + 2 \left(\frac{\beta_2 + \beta_1}{2} \right)^2 \right] \\ &\quad - \left[\left(\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1} \right) \left(\frac{\omega_2^2 - \omega_1^2}{2} \right) + 2 \left(\frac{\beta_2 + \beta_1}{2} \right)^2 \right]^2 \end{aligned} \quad (3.14)$$

The flutter margin's dependence on airspeed is given by Eq. 3.15 [19].

$$FM(U) = B_1 U^4 + B_2 U^2 + B_3 \quad (3.15)$$

where coefficient $\{B_1, B_2, B_3\}$ are function of system parameters.

The airspeed at which $FM = 0$ is the flutter speed [19].

3.3 Application of Bayesian Inference in flutter speed estimation

The general response equation for the 2 DOF aeroelastic system with measurement error is given [1, 84]:

$$u_k = a_1 e^{-\beta_1 t_k} \cos(\omega_1 t_k + b_1) + a_2 e^{-\beta_2 t_k} \cos(\omega_2 t_k + b_2) + \eta_k \quad (3.16)$$

$$\hat{u}_k = a_1 e^{-\beta_1 t_k} \cos(\omega_1 t_k + b_1) + a_2 e^{-\beta_2 t_k} \cos(\omega_2 t_k + b_2) \quad (3.17)$$

where η_k is the Gaussian noise with zero mean and variance γ ; and it considers both measurement and model errors, for instance, due to the effect of higher aeroelastic modes and unsteady aerodynamics. While we have considered white noise to model the errors as the initial step, further studies are needed to investigate the benefits of colored, non-stationary and non-Gaussian noise [86]. Flutter prediction based on non-stationary measurement data is conducted by Torii [87]. The signal-to-noise ratio considered is 14 and 20 DB. Dimitriadis and Cooper [22] has studied the sensitivity of noise level to flutter margin method and other methods. The standard deviation in measurement noise considered are from 5% to 20% and it is shown that flutter margin gives good results below 15%. The standard deviation considered in this thesis is 12% for measurement noise and signal-to-noise ratio is 10.20 DB which is well within the acceptable range.

The likelihood function of these unknown parameters $\alpha_p = \{a_1, a_2, b_1, b_2, \beta_1, \beta_2, \omega_1, \omega_2, \gamma\}$ is :

$$p(u_{k=1:N}|\alpha_p) = \frac{1}{(2\pi\gamma)^{N/2}} \exp \left[\sum_{k=1}^N -\frac{1}{2\gamma} (u_k - \hat{u}_k(\alpha_p))^2 \right] \quad (3.18)$$

The posterior using Bayesian formulation is:

$$p(\alpha_p^{n_u} | u_{k=1:N}^{n_u}) \propto p(u_{k=1:N}^{n_u} | \alpha_p^{n_u}) p(\alpha_p^{n_u}) \quad (3.19)$$

Prior pdf $p(\alpha_p^{n_u})$ is estimated using parametric uncertainty quantification [48]. Samples of modal parameters are generated using Monte Carlo simulation by solving eigenvalue problem in Eq. 3.9. Prior pdf $p(\alpha_p^{n_u})$ is assumed to be Gaussian as given by

[1, 74] Eq. 3.20:

$$p(\alpha_p^{n_u}) = \frac{1}{\sqrt{(2\pi)^p |\Sigma|}} \exp \left(-\frac{1}{2} (\alpha_p^{n_u} - \mu)^T \Sigma^{-1} (\alpha_p^{n_u} - \mu) \right) \quad (3.20)$$

where Σ is the covariance matrix and μ is the mean vector of $\alpha_p^{n_u}$. Although Gaussian approximation for prior is adequate for this problem, this method is general and can handle any non-Gaussian prior. The estimation of joint posterior at given airspeeds using Bayesian formulation is given as:

$$p(\alpha_p^1, \alpha_p^2 | u_{k=1:N}^1, u_{k=1:N}^2) \propto p(\alpha_p^1, \alpha_p^2) p(u_{k=1:N}^2 | \alpha_p^2) p(u_{k=1:N}^1 | \alpha_p^1) \quad (3.21)$$

:

$$p(\alpha_p^1, \dots, \alpha_p^{n_u} | u_{k=1:N}^1, \dots, u_{k=1:N}^{n_u}) \propto p(\alpha_p^1, \dots, \alpha_p^{n_u}) \prod_{i=1}^{n_u} p(u_{k=1:N}^i | \alpha_p^i) \quad (3.22)$$

where, n_u is a index of airspeed at which experiments are conducted.

Prior pdf $p(\alpha_p^1, \dots, \alpha_p^{n_u})$ estimated using parametric uncertainty quantification [48] and multivariate normal distribution. Modal parameters $\beta_1, \beta_2, \omega_1$ and ω_2 are obtained from Eq. 3.22 using AMCMC sampling technique. Once we have parameters, samples of flutter margin is obtain using Eq. 3.14 and subsequently flutter margin pdf is generated. The flutter margin pdf $p(F|B)$ evaluated is used in estimation of unknown coefficient $B = \{B_1, B_2, B_3\}$ from Eq. 3.15 and Bayes theorem as follow [47, 85]:

$$p(B|F) \propto p(F|B)p(B) \quad (3.23)$$

where

$$p(F|B) = p(F^1, \dots, F^{n_u} | \alpha_p^1, \dots, \alpha_p^{n_u}) \quad (3.24)$$

In Eq. 3.15 flutter margin is quite sensitive to the accuracy of B_1 term [19, 77, 85]. To tackle the issue it is preferred to use second order polynomial as suggested by Zimmerman and Weissenburger [19] and also pointed out by Bennett [77] as

$$FM = B_2 U^2 + B_3 \quad (3.25)$$

The coefficient B_2 and B_3 can be estimated using least square fitting. The flutter speed is defined as the positive real root of Eq. 3.25

Moreover, the prior probability functions $p(B)$ is given by [47, 85]

$$p(B) \propto \begin{cases} 1, & \text{if } B_2, B_3 \text{ satisfy conditions(1) and (2).} \\ 0, & \text{otherwise.} \end{cases} \quad (3.26)$$

where

condition(1): $B_2 < 0$

condition(2): $B_3 > 0$

Condition 1 reflects that the flutter speed has real roots and condition 2 ensures the flutter margin is positive at zero airspeed [47, 85].

The flutter speed estimated using Bayesian inference methodology is earlier presented by Khalil [1, 43–47] on aeroelastic model and proof-of-concept developed by Poirel [24]. In this work three assumptions were made:

- Flat prior, i.e., uniform prior with positive support is enforced for modal parameters.

$$p(\alpha_p^{n_u}) \propto 1 \quad (3.27)$$

- Independence among system parameter vector is assumed at different airspeeds and the resulting modal parameter vector becomes

$$p(\alpha_p^{n_u}) = \prod_{i=1}^{n_u} p(\alpha_p^i)$$

- Consequently, the flutter margin vector is independent at different airspeeds.

$$p(F|B) = \prod_{i=1}^{n_u} p(F^i|\alpha_p^i) \quad (3.28)$$

In this thesis Bayesian inference with application of *Adaptive Metropolis Markov Chain Monte Carlo* (AMMCMC) method developed by [1, 24, 47] is implemented to estimate flutter speed using the *flutter margin method* and extended to incorporate prior information of the system. The objective of this thesis is to extend Khalil's work by relaxing the assumptions stated above.

To tackle the issue two cases are considered :

- *Independent prior*, prior for modal parameters are considered independent at different airspeeds while considering correlation at given airspeed. The flutter margin vector is independent at different airspeeds i.e.,

$$p(\alpha_p^{n_u}) = \prod_{i=1}^{n_u} p(\alpha_p^i) \quad (3.29)$$

where prior pdf $p(\alpha_p^i)$ is obtained using MCS at a given airspeed.

- *Joint prior*, full statistical correlation among modal parameters at different airspeed as well as at given airspeed is considered. Also, flutter margin vector at various airspeeds is correlated, leading to

$$p(\alpha_p^{n_u}) = p(\alpha_p^1, \dots, \alpha_p^{n_u}) \quad (3.30)$$

$$p(F|B) = p(F^1, \dots, F^{n_u} | \alpha_p^1, \dots, \alpha_p^{n_u}) \quad (3.31)$$

where joint prior pdfs $p(\alpha_p^{n_u})$ is obtained using MCS. This case relaxes all the three assumptions stated previously [47, 85].

In this thesis three different cases are considered for comparison :

1. Flat prior: prior for modal parameters is considered uniform with positive probability. This case is previously studied by Khalil [47].

$$p(\alpha_p^{n_u}) \propto 1 \quad (3.32)$$

2. Independent prior across airspeeds: Priors are independent from one airspeed to another airspeed with statistical correlation among modal parameters at given airspeed, i.e.,

$$p(\alpha_p^{n_u}) = \prod_{i=1}^{n_u} p(\alpha_p^i)$$

3. Joint prior across airspeeds: Full statistical correlation is consider among modal parameters at all airspeeds.

$$p(\alpha_p^{n_u}) = p(\alpha_p^1, \dots, \alpha_p^{n_u})$$

3.4 Deterministic result

The system parameters considered for 2D airfoil is given in Table 3.1. Flutter speed (FS) evaluated using the deterministic study for case 1 is 54.0137 m/s shown in Figure 3.4. The flutter observed, in this case, is coalescence type of flutter seen from the frequency response of Figure 3.2 and decay response of Figure 3.3. Figure 3.4 shows the comparison of flutter margin evaluated using the deterministic calculation shown in blue line using Eq. 3.15. This deterministic results provide true estimate of flutter speed which is used for comparison of results obtained using probabilistic approach. Flutter margin estimated using the least square fit (dashed line) with five flutter margin value (red dots) at a range of velocity from 50% to 70% of flutter speed using equation 3.25. The

least square estimate shows the bias in the estimate.

TABLE 3.1: Case 1: Parameters

parameter	Value
mass	50 kg
span	1 m
I_{EA}	0.25 kg-m ²
c	0.2
k_h	3000 N-m ²
k_α	150 N-m ²
x_α	0.25
a_h	-0.45
ρ	1.19 kg/m ³
ξ_1	0.02
ξ_2	0.02

FIGURE 3.2: Frequency versus airspeed

FIGURE 3.3: Decay versus airspeed

FIGURE 3.4: Flutter Margin

Following Figure 3.5 and Figure 3.6 show measurement and true response at airspeeds 50% and 70% of flutter speed. The standard deviation for measurement noise is 12% rms value of true signal.

FIGURE 3.5: Measurement at first airspeed

FIGURE 3.6: Measurement at second airspeed

3.5 Prior estimation for two airspeeds and 5% COV

In the first case, we consider two pre-flutter airspeeds, one at 50% and other at 70% of the flutter speed shown in Table 3.3. The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity for input COV

5%. The system parameters and input coefficient of variation are shown in Table 3.2. The COV in modal parameters estimated using MCS is shown in Table 3.4. Table 3.5, 3.6, and 3.7 show correlation among parameters at each airspeed and the joint correlation among two airspeeds respectively. The prior estimated using MCS incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. The prior marginal pdf of modal parameters is shown in Figure 3.7. The assumption of Gaussian distribution for prior pdf in Eq. 3.20 can be validated from Figure 3.7 which shows marginal pdf overlapped with corresponding Gaussian distribution and Figure 3.8 shows the quantile-quantile plot from which one can infer modal parameters closely follow the normal distribution.

TABLE 3.2: Case 1: Parameters with 5% COV

parameter	Value	σ for 5% COV
mass	50 kg	2.5
span	1 m	-
I_{EA}	0.25 kg-m ²	0.0125
c	0.2	
k_h	3000 N-m ²	150
k_α	150 N-m ²	7.5
x_α	0.25	0.0125
a_h	-0.45	0.0225
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.3: Case 1: Pre-flutter airspeeds

Flutter speed	54.0137 m/s
Airspeed 1 (50% Fs)	27.00686 m/s
Airspeed 2 (70% Fs)	37.8096 m/s

TABLE 3.4: COV in modal parameters for 5% input COV in system parameters

$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
3.5306	4.2608	3.8216	4.0104	3.6746	5.0851	4.2799	6.2295

TABLE 3.5: Correlation matrix for modal parameters airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8630	0.3985	0.3425
0.8630	1.0000	0.5087	0.4298
0.3985	0.5087	1.0000	0.9789
0.3425	0.4298	0.9789	1.0000

TABLE 3.6: Correlation matrix for modal parameters airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8224	0.2149	-0.0313
0.8224	1.0000	0.1725	-0.1724
0.2149	0.1725	1.0000	0.9140
-0.0313	-0.1724	0.9140	1.0000

TABLE 3.7: Full correlation matrix among modal parameters at airspeed 1 and 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .							
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8630	0.3985	0.3425	0.9866	0.7771	0.3319	0.1085
0.8630	1.0000	0.5087	0.4298	0.8620	0.9498	0.4136	0.1171
0.3985	0.5087	1.0000	0.9789	0.2991	0.3036	0.9841	0.8284
0.3425	0.4298	0.9789	1.0000	0.2270	0.1846	0.9979	0.9193
0.9866	0.8620	0.2991	0.2270	1.0000	0.8224	0.2149	-0.0313
0.7771	0.9498	0.3036	0.1846	0.8224	1.0000	0.1725	-0.1724
0.3319	0.4136	0.9841	0.9979	0.2149	0.1725	1.0000	0.9140
0.1085	0.1171	0.8284	0.9193	-0.0313	-0.1724	0.9140	1.0000

FIGURE 3.7: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.8: Normal probability plot

FIGURE 3.9: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.10: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

3.6 Flutter speed estimation using two airspeeds and 5% COV

FIGURE 3.11: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.12: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.13: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.14: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.15: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.16: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.17: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.18: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.19: Joint posterior pdf using flat prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.20: Joint posterior pdf using independent prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.21: Joint posterior pdf using joint prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.22: Joint posterior pdf flutter margin among two airspeeds: 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.23: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.24: Posterior of flutter speed for different priors

FIGURE 3.25: Posterior of flutter speed for different priors

Figures 3.11 – 3.18 are the posterior marginal pdfs of modal parameters estimated using AMCMC. The figures show comparison among posterior pdf obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters.

Observation on marginal pdfs are as follows:

- The posterior marginal pdfs obtained using *flat prior* with two airspeeds can capture frequency and decay to some extent at the expense of high bias and variance in estimation.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequency is comparatively better than the estimation of decay rates.
- To improve the performance of estimation using *flat prior* we need more number of airspeed and more data at a given airspeed.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variance in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequency using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.

Figure 3.19, 3.20 and 3.21 shows joint posterior pdfs estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show statistical correlation among parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior* is unable to consider underlying correlation among modal parameters across airspeeds. Figure 3.19 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian. Figure 3.22 shows joint posterior pdfs of flutter margin. The joint pdfs estimated using *joint prior* is able to predict the underlying correlation while, joint pdf estimated using *flat prior* is highly non-Gaussian. Figure 3.23 shows the joint posterior pdf among parameter B_2 and B_3 from the Eq. 3.25 used in estimation of flutter speed. Figure 3.24 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone* (uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit

using the true flutter margin value at given airspeeds. Figure 3.25 shows box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*. One can visualize the distribution and accuracy of flutter speed estimation from box plot.

3.7 Prior estimation for two airspeeds and 7.5% COV

The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity for input COV 7.5%. The system parameters and input coefficient of variation are shown in Table 3.8. The COV in modal parameters estimated using MCS is shown in Table 3.9. Table 3.10, 3.11, and 3.12 show correlation among parameters at each airspeed and the joint correlation among two airspeeds respectively. The prior estimated using MCS is incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. The prior marginal pdf of modal parameters is shown in Figure 3.26. The assumption of Gaussian distribution for prior pdf in Eq. 3.20 can be validated from Figure 3.26 which shows marginal pdf overlapped with corresponding Gaussian distribution. Figure 3.27 shows the quantile-quantile plot from which one can infer modal parameters closely follow the normal distribution.

TABLE 3.8: Case 1: Parameters with 7.5% COV

parameter	Value	σ for 7.5% COV
mass	50 kg	3.75
span	1 m	-
I_{EA}	0.25 kg-m ²	0.0188
c	0.2	
k_h	3000 N-m ²	225
k_α	150 N-m ²	11.25
x_α	0.25	0.0188
a_h	-0.45	0.0338
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.9: COV in modal parameters for 7.5% input COV in system parameters

$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
5.3219	6.4538	5.7568	6.0480	5.5482	7.7593	6.4555	9.5401

TABLE 3.10: Correlation matrix for modal parameters at airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8631	0.3998	0.3426
0.8631	1.0000	0.5085	0.4282
0.3998	0.5085	1.0000	0.9785
0.3426	0.4282	0.9785	1.0000

TABLE 3.11: Correlation matrix for modal parameters at airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8226	0.2113	-0.0423
0.8226	1.0000	0.1619	-0.1912
0.2113	0.1619	1.0000	0.9110
-0.0423	-0.1912	0.9110	1.0000

TABLE 3.12: Full correlation matrix for modal parameters at airspeed 1 and 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .							
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8631	0.3998	0.3426	0.9859	0.7745	0.3322	0.1022
0.8631	1.0000	0.5085	0.4282	0.8618	0.9466	0.4122	0.1078
0.3998	0.5085	1.0000	0.9785	0.2971	0.2947	0.9839	0.8241
0.3426	0.4282	0.9785	1.0000	0.2236	0.1742	0.9975	0.9161
0.9859	0.8618	0.2971	0.2236	1.0000	0.8226	0.2113	-0.0423
0.7745	0.9466	0.2947	0.1742	0.8226	1.0000	0.1619	-0.1912
0.3322	0.4122	0.9839	0.9975	0.2113	0.1619	1.0000	0.9110
0.1022	0.1078	0.8241	0.9161	-0.0423	-0.1912	0.9110	1.0000

FIGURE 3.26: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.27: Normal probability plot

FIGURE 3.28: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.29: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

3.8 Flutter speed estimation using two airspeeds and 7.5% COV

FIGURE 3.30: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.31: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.32: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.33: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.34: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.35: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.36: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.37: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.38: Joint posterior pdf using flat prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.39: Joint posterior pdf using independent prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.40: Joint posterior pdf using joint prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.41: Joint posterior pdf flutter margin among two airspeeds: 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.42: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.43: Flutter speed for different priors

FIGURE 3.44: Flutter speed for different priors

Figures 3.30 – 3.37 are the posterior marginal pdfs of modal parameters estimated using AMCMC. The figures show comparison among posterior pdf obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters. In this case, we have considered input coefficient of variation in system parameters to be 7.5%.

Observation on marginal pdfs are as follows:

- The posterior marginal pdfs obtained using *flat prior* with two airspeeds can capture frequencies and decay rates to some extent and shows high bias and variance in estimation.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequency is comparatively better than the estimation of decay rates.
- To improve the performance of *flat prior* we need more number of airspeed and more data at given airspeed.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variance in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequency using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.
- Although the uncertainty in prior information is increased from the previous case (5%COV), one can observe that the posterior estimated using *independent prior* and *joint prior* are still able to give fairly accurate results with the expected increment in variance.

Figure 3.38, 3.39 and 3.40 show the joint posterior pdfs estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show the statistical correlation among modal parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior* are unable to consider the underlying correlation among parameters across airspeeds. Figure 3.38 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian. Figure 3.41 show joint posterior

pdfs of flutter margin. The joint pdfs estimated using *joint prior* can predict the underlying correlation and joint pdf estimated using *flat prior* show non-Gaussian behavior. Figure 3.42 shows the joint posterior pdfs among parameter B_2 and B_3 from Eq. 3.25 used in estimation of flutter speed. Figure 3.43 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone* (uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit using the true flutter margin value at given airspeeds. Figure 3.44 shows the box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*. One can visualize the distribution and accuracy of flutter speed estimation from box plot.

3.9 Prior estimation for two airspeeds and 10% COV

The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity for input COV 10%. The system parameters and input coefficient of variation are shown in Table 3.13. The COV in modal parameters estimated using MCS is shown in Table 3.14. Table. 3.15, 3.16 and 3.17 show correlation among parameters at each airspeed and the joint correlation among two airspeeds respectively. The prior estimated using MCS incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. The prior marginal pdf of modal parameters is shown in Figure 3.45. The assumption of Gaussian distribution for prior pdf in Eq. 3.20 can be validated from Figure 3.45 which shows marginal pdf overlapped with corresponding Gaussian distribution. Figure 3.46 shows the quantile-quantile plot which shows modal parameters closely follow the normal distribution.

TABLE 3.13: Case 1: Parameters with 10% COV

parameter	Value	σ for 10% COV
mass	50 kg	5
span	1 m	-
I_{EA}	0.25 kg-m ²	0.025
c	0.2	
k_h	3000 N-m ²	300
k_α	150 N-m ²	15
x_α	0.25	0.025
a_h	-0.45	0.045
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.14: COV in modal parameters for 10% input COV in system parameters

$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
7.1456	8.7280	7.7245	8.1307	7.4720	10.6507	8.6805	13.1961

TABLE 3.15: Correlation matrix for modal parameter at airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8632	0.4015	0.3426
0.8632	1.0000	0.5081	0.4256
0.4015	0.5081	1.0000	0.9779
0.3426	0.4256	0.9779	1.0000

TABLE 3.16: Correlation matrix for modal parameter at airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8224	0.2049	-0.0617
0.8224	1.0000	0.1423	-0.2257
0.2049	0.1423	1.0000	0.9051
-0.0617	-0.2257	0.9051	1.0000

TABLE 3.17: Full correlation matrix for modal parameter at airspeed 1 and 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .							
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8632	0.4015	0.3426	0.9844	0.7686	0.3322	0.0912
0.8632	1.0000	0.5081	0.4256	0.8615	0.9394	0.4099	0.0923
0.4015	0.5081	1.0000	0.9779	0.2933	0.2780	0.9836	0.8160
0.3426	0.4256	0.9779	1.0000	0.2174	0.1552	0.9970	0.9097
0.9844	0.8615	0.2933	0.2174	1.0000	0.8224	0.2049	-0.0617
0.7686	0.9394	0.2780	0.1552	0.8224	1.0000	0.1423	-0.2257
0.3322	0.4099	0.9836	0.9970	0.2049	0.1423	1.0000	0.9051
0.0912	0.0923	0.8160	0.9097	-0.0617	-0.2257	0.9051	1.0000

FIGURE 3.45: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.46: Normal probability plot

FIGURE 3.47: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at two airspeeds

FIGURE 3.48: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

3.10 Flutter speed estimation using two airspeeds and 10% COV

FIGURE 3.49: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.50: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.51: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.52: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.53: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.54: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.55: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.56: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.57: Joint posterior pdf using flat prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.58: Joint posterior pdf using independent prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.59: Joint posterior pdf using joint prior of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among two airspeeds

FIGURE 3.60: Joint posterior pdf flutter margin among two airspeeds: 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.61: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.62: Flutter speed for different priors

FIGURE 3.63: Flutter speed for different priors

Figures 3.49 – 3.56 are the posterior marginal pdfs of modal parameters estimated using AMCMC. The figures show comparison among posterior pdf obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters. In this case, we have considered input coefficient of variation in system parameters to be 10%.

Observations from statistical perspective are as follows :

- The posterior marginal pdfs obtained using *flat prior* with two airspeeds can capture frequencies and decay rates shows high bias and variance in estimation.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequencies is comparatively better than the estimation of decay rates.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates better than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variance in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequency using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.
- Although the uncertainty in prior information is increased from the previous case (5% and 7.5% COV), one can observe that the posterior estimated using *independent prior* and *joint prior* are still able to give fairly accurate results.

Figures 3.57, 3.58 and 3.59 shows joint posterior pdfs estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show the statistical correlation among modal parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior* are unable to consider the underlying correlation among modal parameters across airspeeds. Figure 3.57 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian. Figure 3.60 shows joint posterior pdfs of flutter margin. The joint pdf estimated using *joint prior* can predict the underlying correlation and joint pdf estimated using *flat prior* is non-Gaussian. Figure 3.61 shows the joint posterior pdf among parameter B_2 and B_3 from the Eq. 3.25 used in estimation of flutter speed. Figure 3.62 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone*

(uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit using the true flutter margin value at given airspeeds. Figure 3.63 shows box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*. One can visualize the distribution and accuracy of flutter speed estimation from box plot.

TABLE 3.18: Coefficient of variation in posterior of modal parameters

Input COV	$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$	$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
posterior with flat prior	3.1036	56.1170	0.3966	17.5200	5.3250	51.6323	0.3272	14.4966
5% CV in prior	3.5306	4.2608	3.8216	4.0104	3.6746	5.0851	4.2799	6.2295
posterior with independent prior	2.1684	2.8641	0.3742	0.8960	2.9050	4.4595	0.3087	2.3643
posterior with joint prior	1.9252	2.4718	0.3234	0.3831	2.0161	3.0469	0.2905	1.5892
7.5%CV in prior	5.3219	6.4538	5.7568	6.0480	5.5482	7.7593	6.4555	9.5401
posterior with independent prior	2.5170	3.8675	0.3771	1.2778	3.6534	6.2489	0.3111	3.4224
posterior with joint prior	2.1851	3.2768	0.3455	0.5043	2.2793	4.0064	0.2988	1.8574
10% CV in prior	7.1456	8.7280	7.7245	8.1307	7.4720	10.6507	8.6805	13.1961
posterior with independent prior	2.6902	4.8599	0.3783	1.6905	4.1011	7.9948	0.3124	4.5690
posterior with joint prior	0.7500	0.4947	0.1930	0.4419	0.8765	0.6731	0.1568	1.0840

TABLE 3.19: COV in flutter speed estimation with three different input coefficients of variation in priors

Coefficient of variation in flutter speed			
Input COV for prior	Flat prior $p(\alpha) \propto 1$	Independent prior $p(\alpha) = p(\alpha^1)p(\alpha^2)$	Joint prior $p(\alpha) = p(\alpha^1, \alpha^2)$
5%	4.6591	2.8993	1.8993
7.5%	4.6591	3.3691	2.1471
10%	4.6591	3.6735	1.1862

3.11 Discussion

The Table 3.18 shows variability in estimation of posterior of modal parameters using AMCMC algorithm with (a) *flat prior* where $p(\alpha_p) \propto 1$, (b) *independent prior* where

statistical correlation among modal parameters for given airspeed is considered while Independence across airspeeds is assumed $p(\alpha_p) = p(\alpha_p^1)p(\alpha_p^2)$ and lastly (c) *joint prior* where full statistical correlation among modal parameters for given airspeed and among airspeeds is considered. The *flat prior* case shows the high coefficient of variation in estimation of decay rates compare to frequency estimation. Modal parameters show less variability when prior information is available. Compared to the independent prior case, joint prior shows a significant reduction in variability. The coefficient of variation in the posterior of flutter speed for three cases is shown in Table 3.19

3.12 Prior estimation using parametric uncertainty and flutter estimations for three airspeeds

In this second case, we consider three pre-flutter airspeeds, one at 50%, second at 60% and third at 70% of the flutter speed shown in Table 3.20. Compared to previous case we have added pre-flutter speed at 60% because after 70% of flutter speed flight testing poses higher risk and we need to consider additional assumption arising from near flutter dynamics. The lower limit of 50% is suggested by Price [23]. Also, we want to study the effect of additional information in the same range of airspeed (50% - 70%) as compared to previous case. Measurement and true response are shown in Figure 3.64 , 3.65 and 3.66. The standard deviation in measurement is 12% of rms value of the true signal.

TABLE 3.20: Case 2: Pre-flutter airspeeds

Flutter speed	54.0137 m/s
Airspeed 1 (50% Fs)	27.00686 m/s
Airspeed 2 (60% Fs)	32.4082 m/s
Airspeed 3 (70% Fs)	37.8096 m/s

FIGURE 3.64: Measurement at first airspeed

FIGURE 3.65: Measurement at second airspeed

FIGURE 3.66: Measurement at third airspeed

3.13 Prior estimation for three airspeeds and 5% COV

The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity shown in Table 3.21. Table 3.22, 3.23, and 3.24 show correlation among parameters at each airspeed respectively. The prior estimated using MCS incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. Figure 3.67 shows marginal prior pdf compared with normal distribution of modal parameters. Figure 3.68 shows normal probability plot. Figure 3.69 and Figure 3.70 are plots of joint prior pdfs of modal parameters.

TABLE 3.21: Case 2: Parameters with 5% COV

parameter	Value	σ for 5% COV
mass	50 kg	2.5
span	1 m	-
I_{EA}	0.25 kg-m ²	0.0125
c	0.2	
k_h	3000 N-m ²	150
k_α	150 N-m ²	7.5
x_α	0.25	0.0125
a_h	-0.45	0.0225
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.22: Correlation matrix for modal parameter at airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8630	0.3985	0.3425
0.8630	1.0000	0.5087	0.4298
0.3985	0.5087	1.0000	0.9789
0.3425	0.4298	0.9789	1.0000

TABLE 3.23: Correlation matrix for modal parameter at airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8414	0.3283	0.1961
0.8414	1.0000	0.3821	0.1909
0.3283	0.3821	1.0000	0.9563
0.1961	0.1909	0.9563	1.0000

TABLE 3.24: Correlation matrix for modal parameter at airspeed 3

$\omega_1^{(3)}$	$\beta_1^{(3)}$	$\omega_2^{(3)}$	$\beta_2^{(3)}$
1.0000	0.8224	0.2149	-0.0313
0.8224	1.0000	0.1725	-0.1724
0.2149	0.1725	1.0000	0.9140
-0.0313	-0.1724	0.9140	1.0000

FIGURE 3.67: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds for 5% input COV

FIGURE 3.68: Normal probability plot

FIGURE 3.69: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds

FIGURE 3.70: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among three airspeeds

3.14 Flutter speed estimation using three airspeeds

FIGURE 3.71: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.72: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.73: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.74: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.75: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.76: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.77: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.78: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.79: Comparison of posterior of $\omega_1^{(3)}$ for different priors

FIGURE 3.80: Comparison of posterior of $\omega_2^{(3)}$ for different priors

FIGURE 3.81: Comparison of posterior of $\beta_1^{(3)}$ for different priors

FIGURE 3.82: Comparison of posterior of $\beta_2^{(3)}$ for different priors

FIGURE 3.83: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using flat prior among three airspeeds

FIGURE 3.84: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using independent prior among three airspeeds

FIGURE 3.85: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using joint prior among three airspeeds

FIGURE 3.86: Joint posterior pdf of flutter margin using flat prior : column 1, Independent prior : column 2 and Joint prior : column 3 among three airspeeds

FIGURE 3.87: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.88: Flutter speed for different priors

FIGURE 3.89: Flutter speed for different priors

Figures 3.71 – 3.82 are the posterior marginal pdfs of modal parameters estimated using AMCMC at three airspeeds. The figures show comparison among posterior pdfs obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters. In this case, we have considered input coefficient of variation in system parameters to be 5%. Observation on marginal pdfs are as follows:

- The posterior marginal pdfs obtained using *flat prior* with three airspeeds can capture frequencies and decay rates.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequencies are comparatively better than the estimation of decay rates.
- With the increase in number of airspeeds from two to three, the posterior estimated using flat prior shows improvement in the estimation of decay rates.
- The nonlinear behavior previously observed in posterior of decay rates with two airspeeds is reduced.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variance in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequencies using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.
- The overall performance for posterior estimated using flat prior, independent prior and joint prior show significant improvement.

Figures 3.83, 3.84 and 3.85 shows joint posterior pdfs of modal parameters estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show the statistical correlation among modal parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior* is unable to consider the underlying correlation among modal parameters across the airspeeds. Figure 3.83 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian across the airspeeds. Figure 3.86 shows joint posterior pdfs of flutter margin. The joint pdfs estimated using *flat prior* are highly non-Gaussian. The joint pdfs estimated

using *joint prior* can predict the underlying correlation. Figure 3.87 shows the joint posterior pdf among parameter B_2 and B_3 from Eq. 3.25 used in estimation of flutter speed. Figure 3.88 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone* (uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit using the true flutter margin value at given airspeeds. Figure 3.89 shows box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*.

3.15 Prior estimation for three airspeeds and 7.5% COV

The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity shown in Table. 3.25. Table. 3.26, 3.27, 3.28 show correlation among parameters at each airspeed respectively. The prior estimated using MCS incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. Figure 3.90 shows marginal prior pdf compared with normal distribution of modal parameters. Figure 3.91 shows normal probability plot. Figure 3.92 and Figure 3.93 are plots of joint prior pdfs of modal parameters.

TABLE 3.25: Case 2: Parameters with 7.5% COV

parameter	Value	σ for 7.5% COV
mass	50 kg	3.75
span	1 m	-
I_{EA}	0.25 kg-m ²	0.0188
c	0.2	
k_h	3000 N-m ²	225
k_α	150 N-m ²	11.25
x_α	0.25	0.0188
a_h	-0.45	0.0338
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.26: Correlation matrix for modal parameter at airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8631	0.3998	0.3426
0.8631	1.0000	0.5085	0.4282
0.3998	0.5085	1.0000	0.9785
0.3426	0.4282	0.9785	1.0000

TABLE 3.27: Correlation matrix for modal parameter at airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8416	0.3282	0.1925
0.8416	1.0000	0.3789	0.1836
0.3282	0.3789	1.0000	0.9552
0.1925	0.1836	0.9552	1.0000

TABLE 3.28: Correlation matrix for modal parameter at airspeed 3

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8226	0.2113	-0.0423
0.8226	1.0000	0.1619	-0.1912
0.2113	0.1619	1.0000	0.9110
-0.0423	-0.1912	0.9110	1.0000

FIGURE 3.90: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds for 7.5% input COV

FIGURE 3.91: Normal probability plot

FIGURE 3.92: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds

FIGURE 3.93: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among three airspeeds

3.16 Flutter speed estimation using three airspeeds and 7.5% COV

FIGURE 3.94: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.95: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.96: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.97: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.98: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.99: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.100: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.101: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.102: Comparison of posterior of $\omega_1^{(3)}$ for different priors

FIGURE 3.103: Comparison of posterior of $\omega_2^{(3)}$ for different priors

FIGURE 3.104: Comparison of posterior of $\beta_1^{(3)}$ for different priors

FIGURE 3.105: Comparison of posterior of $\beta_2^{(3)}$ for different priors

FIGURE 3.106: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using flat prior among three airspeeds

FIGURE 3.107: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using independent prior among three airspeeds

FIGURE 3.108: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using jointprior among three airspeeds

FIGURE 3.109: Joint posterior pdf of flutter margin using flat prior : column 1, independent prior : column 2 and joint prior : column 3 among three airspeeds

FIGURE 3.110: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.111: Flutter speed for different priors

FIGURE 3.112: Flutter speed for different priors

Figures 3.71 – 3.82 are the posterior marginal pdfs of modal parameters estimated using AMCMC at three airspeeds. The figures show comparison among posterior pdf obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters. In this case, we have considered input coefficient of variation in system parameters to be 7.5%.

Observation on marginal pdfs are as follows:

- The posterior marginal pdfs obtained using *flat prior* with three airspeeds can capture frequencies and decay rates to the large extent with some bias and variance in estimation.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequencies is comparatively better than the estimation of decay rates.
- With the increase in number of airspeeds from two to three, the posterior estimated using *flat prior* shows improvement in the estimation of decay rates.
- The nonlinear behavior previously observed in posterior of decay rates with two airspeeds is reduced.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variability in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequencies using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.
- Although the uncertainty in prior information is increased from the previous case (5% COV), one can observe that the posterior estimated using *independent prior* and *joint prior* are still able to give fairly accurate results with the expected increment in variance.

Figures 3.106, 3.107 and 3.108 shows joint posterior pdfs of modal parameters estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show the statistical correlation among modal parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior*

are unable to consider the underlying correlation among modal parameters across airspeeds. Figure 3.106 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian across the airspeeds. Figure 3.109 shows joint posterior pdfs of flutter margin. The joint pdfs estimated using *flat prior* are non-Gaussian across the airspeeds. The joint pdfs estimated using *joint prior* can predict the underlying correlation. Figure 3.110 shows the joint posterior pdfs among parameter B_2 and B_3 from the Eq. 3.25 used in estimation of flutter speed. Figure 3.111 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone* (uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit using the true flutter margin value at given airspeeds. Figure 3.112 shows box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*.

3.17 Prior estimation for three airspeeds 10% COV

The uncertainty is considered in mass, stiffness, location of the elastic axis from mid-chord and from the center of gravity shown in Table 3.29. Table 3.30, 3.31, 3.32 show correlation among parameters at each airspeed respectively. The prior estimated using MCS incorporated into AMCMC algorithm is shown in Eq. 3.19 and Eq. 3.21. Figure 3.113 shows marginal prior pdf compared with normal distribution of modal parameters. Figure 3.114 shows normal probability plot. Figure 3.115 and Figure 3.116 are plots of joint prior pdfs of modal parameters.

TABLE 3.29: Case 2: Parameters with 10% COV

parameter	Value	σ for 10% COV
mass	50 kg	5
span	1 m	-
I_{EA}	0.25 kg-m ²	0.025
c	0.2	
k_h	3000 N-m ²	300
k_α	150 N-m ²	15
x_α	0.25	0.025
a_h	-0.45	0.045
ρ	1.19 kg/m ³	
ξ_1	0.02	
ξ_2	0.02	

TABLE 3.30: Correlation matrix for modal parameter at airspeed 1

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(1)}$	$\beta_1^{(1)}$	$\omega_2^{(1)}$	$\beta_2^{(1)}$
1.0000	0.8632	0.4015	0.3426
0.8632	1.0000	0.5081	0.4256
0.4015	0.5081	1.0000	0.9779
0.3426	0.4256	0.9779	1.0000

TABLE 3.31: Correlation matrix for modal parameter at airspeed 2

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8418	0.3276	0.1866
0.8418	1.0000	0.3737	0.1716
0.3276	0.3737	1.0000	0.9534
0.1866	0.1716	0.9534	1.0000

TABLE 3.32: Correlation matrix for modal parameter at airspeed 3

Correlation matrix for uncertainty in mass, k_h , k_α , x_α and a_h .			
$\omega_1^{(2)}$	$\beta_1^{(2)}$	$\omega_2^{(2)}$	$\beta_2^{(2)}$
1.0000	0.8224	0.2049	-0.0617
0.8224	1.0000	0.1423	-0.2257
0.2049	0.1423	1.0000	0.9051
-0.0617	-0.2257	0.9051	1.0000

FIGURE 3.113: Marginal prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds for 10% input COV

FIGURE 3.114: Normal probability plot

FIGURE 3.115: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 at three airspeeds

FIGURE 3.116: Joint prior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 among three airspeeds

3.18 Flutter speed estimation using three airspeeds and 10% COV

FIGURE 3.117: Comparison of posterior of $\omega_1^{(1)}$ for different priors

FIGURE 3.118: Comparison of posterior of $\omega_2^{(1)}$ for different priors

FIGURE 3.119: Comparison of posterior of $\beta_1^{(1)}$ for different priors

FIGURE 3.120: Comparison of posterior of $\beta_2^{(1)}$ for different priors

FIGURE 3.121: Comparison of posterior of $\omega_1^{(2)}$ for different priors

FIGURE 3.122: Comparison of posterior of $\omega_2^{(2)}$ for different priors

FIGURE 3.123: Comparison of posterior of $\beta_1^{(2)}$ for different priors

FIGURE 3.124: Comparison of posterior of $\beta_2^{(2)}$ for different priors

FIGURE 3.125: Comparison of posterior of $\omega_1^{(3)}$ for different priors

FIGURE 3.126: Comparison of posterior of $\omega_2^{(3)}$ for different priors

FIGURE 3.127: Comparison of posterior of $\beta_1^{(3)}$ for different priors

FIGURE 3.128: Comparison of posterior of $\beta_2^{(3)}$ for different priors

FIGURE 3.129: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using flat prior among three airspeeds

FIGURE 3.130: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using independent prior among three airspeeds

FIGURE 3.131: Joint posterior pdf of parameters $\omega_1, \beta_1, \omega_2$ and β_2 using joint prior among three airspeeds

FIGURE 3.132: Joint posterior pdf of flutter margin using flat prior : column 1, independent prior : column 2 and joint prior : column 3, among three airspeeds

FIGURE 3.133: Joint posterior pdf B_2 and B_3 parameters of equation 3.25 : 1) Flat prior 2) Independent prior 3) Joint prior

FIGURE 3.134: Flutter speed for different priors

FIGURE 3.135: Flutter speed for different priors

Figures 3.117 – 3.128 are the posterior marginal pdfs of modal parameters estimated using AMCMC at three airspeeds. The figures show comparison among posterior pdfs obtained using *flat prior*, *independent prior* and *joint prior* of modal parameters. In this case, we have considered input coefficient of variation in system parameters to be 10%.

Observation on marginal pdfs are as follows:

- The posterior marginal pdfs obtained using *flat prior* with three airspeeds can capture frequencies and decay rates.
- The posterior of modal parameters for *flat prior* are governed by measurement alone, it shows that the estimation of modal frequencies is comparatively better than the estimation of decay rates.
- With the increase in number of airspeeds from two to three, the posterior estimated using flat prior shows improvement in the estimation of decay rates.
- The nonlinear behavior previously observed in posterior of decay rates with two airspeeds is reduced.
- The posterior marginal pdfs obtained using *independent prior* can better capture frequencies and decay rates than *flat prior*.
- The estimation of decay rates improved with *independent prior* compared to *flat prior*. The variance in estimation is significantly reduced.
- The posterior marginal pdfs obtained using *joint prior* show that the accuracy of estimation is significantly improved when prior information is considered with full statistical correlation.
- The bias in the estimation of modal frequencies using *independent prior* and *joint prior* is perhaps due to insufficient measurement data.
- Although the uncertainty in prior information is increased from the previous case (5% and 7.5%COV), one can observe that the posterior estimated using *independent prior* and *joint prior* are still able to give fairly accurate results with the expected increment in variance.

Figure 3.129, 3.130 and 3.131 show joint posterior pdfs of modal parameters estimated using *flat prior*, *independent prior* and *joint prior* respectively. Joint posterior pdfs estimated using *joint prior* show the statistical correlation among modal parameters across the airspeeds, while joint pdfs estimated using *flat prior* and *independent prior*

is unable to consider the underlying correlation among modal parameters across airspeeds. Figure 3.129 shows joint pdf of β_1 estimated using *flat prior* to be non-Gaussian across the airspeeds. Figure 3.132 shows joint posterior pdfs of flutter margin. The joint pdfs estimated using *joint prior* can predict the underlying correlation. The joint pdf estimated using *flat prior* show highly non-Gaussian behavior. Figure 3.133 shows the joint posterior pdfs among parameter B_2 and B_3 from the Eq. 3.25 used in estimation of flutter speed. Figure 3.134 shows the comparison of posterior of flutter speed estimated using *flat prior*, *independent prior*, *joint prior* and *prior alone* (uncertainty quantified using propagation of prior information only without measurement data) with the true flutter speed estimated using the deterministic calculation and least square fit using the true flutter margin value at given airspeeds. Figure 3.135 shows box plot of flutter speed estimated using *flat prior*, *independent prior* and *joint prior*.

TABLE 3.33: COV in flutter speed estimated using three airspeeds with three different input coefficients of variation in priors

Input COV for prior	Coefficient of variation in flutter speed		
	Flat prior $p(\alpha) \propto 1$	Independent prior $p(\alpha) = p(\alpha^1)p(\alpha^2)p(\alpha^3)$	Joint prior $p(\alpha) = p(\alpha^1, \alpha^2, \alpha^3)$
5%	4.5903	2.995	2.0213
7.5%	4.5903	3.4324	2.3638
10%	4.5903	3.7115	2.5625

The coefficient of variation in the posterior of flutter speed for three cases is shown in Table 3.33. The variability observed as from posterior flutter speed estimated using joint prior is comparatively less than independent prior and flat prior.

3.19 Discussion

The flutter speed estimation using measurement at two and three pre-flutter airspeed shows that estimation using joint prior and Independent prior give better results. The predicted flutter speed in both cases shows bias as compared to true flutter speed Figure 3.24, 3.43, 3.62, 3.88, 3.111 and 3.134. The main reason for this bias is the simplified assumption of reduce order model of flutter margin equation Eq. 3.15 i.e., the quadratic flutter margin equation Eq. 3.25.

3.20 Robustness of flutter margin method with AMM-CMC

To establish the robustness of methodology, five different data realizations were generated using different random seeds for both measurement and prior for 10% COV in system parameters. The flutter speed is estimated for two cases (a) *joint prior*. (b) *prior alone*. The posterior pdfs of flutter speed estimated for all five experiments using joint prior and prior alone are shown in Figure 3.136. The flutter speed estimates are well within the range of acceptance. The flutter speed pdf is showing variability in the case of *joint prior* is perhaps due to different data realization of measurements. In the absence of measurement data flutter speed pdf estimated using *prior alone* shows consistent results.

FIGURE 3.136: 1) posterior of flutter speed estimated using joint prior and measurement 2) posterior of flutter speed estimated using prior alone

Chapter 4

Conclusion

In this thesis, Bayesian inference method is applied using Adaptive Metropolis Markov Chain Monte Carlo (AMMCMC) methods to estimate flutter speed with quasi-steady aerodynamics. The parallel implementation of AMMCMC significantly reduces simulation time. The AMMCMC method is used for parameter estimation of a simple mass-spring-damper single-degree-of-freedom system as a test case in Chapter 2. In this case, we observed that AMMCMC is capable of estimating parameters with accuracy and also capable of capturing the underlying statistical correlation between parameters given prior knowledge. In Chapter 3 application of Bayesian inference method in estimation of flutter speed of two-degree-of-freedom airfoil system, using the flutter margin method is studied. In this chapter, two cases using two and three pre-flutter airspeeds, respectively, were examined. In each case comparison is made between *flat prior*, *independent prior* and *joint prior* for different input coefficient of variation values (5%, 7.5%, and 10%) in system parameters which in turn propagated to modal parameters. The priors are estimated using Monte Carlo simulation and parametric uncertainty. The following observations emerged from this investigation:

- Flutter speed estimation using *flat prior*
 - Although *flat prior* is capable of predicting flutter speed, the estimate shows high bias and variance.
 - *Flat prior* case shows a better estimation of frequencies in comparison to decay rates.
 - Joint posterior pdf of parameter shows *flat prior* is unable to consider the underlying statistical correlation between modal parameters among airspeeds.
 - Joint posterior pdf of flutter margin shows independence from one airspeed to another.

- Performance of flutter speed estimation using *flat prior* improves when the number of airspeeds is increased.
- Flutter speed estimation using *independent prior*
 - In this case, prior information considered is independent from one airspeed to another, while the correlation among parameters is considered at given airspeed.
 - The posterior estimation of flutter speed and modal parameters using the *independent prior* results in better prediction compared to the estimation using *flat prior*. Also, reduction in variance of estimates is observed.
 - Joint posterior pdf of modal parameters using *independent prior* is unable to capture the underlying statistical correlation between modal parameters among airspeeds. However it can capture correlation between the modal parameters at a given airspeed.
 - The estimation of flutter margin is improved, but the joint posterior pdf of flutter margin shows independence from one airspeed to another.
- Flutter speed estimation using *joint prior*
 - In this case, full statistical correlation is considered among modal parameters at a given airspeed as well as among the airspeeds.
 - The *joint prior* case tries to represent the physical behavior of the system accurately.
 - The posterior estimation of flutter speed and modal parameters using *joint prior* shows significant improvement in prediction, when compared with *flat prior* and *independent prior*.
 - Joint posterior pdf of modal parameters using *joint prior* is now able to capture underlying correlation which was not observed previously with *flat prior* and *independent prior*.
 - The estimates of the joint posterior pdf of flutter margin is now able to capture the correlation among flutter margin from one airspeed to another.

This investigation led to the conclusion that in presence of prior information the estimate of flutter speed and modal parameters is significantly improved and the need for number of flight test can be reduced. It is also found that the assumption of reduce order model of flutter margin equation lead to over-prediction of flutter onset.

4.1 Future work

- The methodology adopted in this thesis can be used in the estimation of flutter speed for unsteady aerodynamics model.
- The scope of this methodology is not limited to 2D airfoil model but can be further extended to 3D wing model.
- Optimization and quantification of testing requirements can be done based on the amount of prior information.
- Further, advance machine learning algorithm based on this methodology can be directly implemented in system onboard to monitor continuously and predict flutter speed.

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