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BRANCHING DIFFUSION PROCESSES

by

Barbara Gail Ivanoff, B.Sc., M.Sc.

A thesis submitted to the Faculty of
Graduate Studies in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Carleton University

Ottawa, Ontario

November, 1976

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The undersigned hereby recommend to the
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ABSTRACT

At some random time, a mutant gene is introduced into a population of "normals" which migrate, give birth, and die in some domain $D \subseteq \mathbb{R}^d$. For genetical computations, it is important to study the evolution of the mutant over time and space. This dissertation deals with the mathematical model which describes this problem. The model is known as a branching diffusion process. The random variable under consideration is $N_A(t)$ = (the number of the mutants found in a set $A \subseteq D$ at time t). The set of random variables $\{N_A(t): A \subseteq D, 0 \leq t \leq \infty\}$ is known as a point process.

Chapter 1 discusses the genetical problems which can be modelled mathematically by means of branching diffusion processes. The branching diffusion is defined heuristically and mathematically, and relevant results of Stanley Sawyer are stated.

In Chapter 2, it is proven that under the assumptions made in Chapter 1, the point process $N_A(t)$ can have no multiple points, and a characterization of $N_A(t)$ is given based on this property.

Chapter 3 discusses the single-type branching diffusion (one type of particle), first with one particle initially in D , and second with an initial Poisson random field of particles in D . Chapter 4 deals with the multitype branching diffusion (more than one type of particle) with one particle initially. Chapter 5 considers a single-type branching diffusion with immigration, D initially empty.

For each of the above cases, recursion formulae are developed for the n^{th} order factorial moment density functions of $N_A(t)$. For the special case in which the domain D is R^d and the diffusion process is d -dimensional Brownian motion, the behaviour of the first two factorial moment densities as $t \rightarrow \infty$ is considered. Again in the special case of Brownian motion, for the single-type process beginning with one particle initially and in the immigration case, approximations of

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$$

are calculated for large compact symmetric sets $A \subseteq D$.

Using these approximations in the immigration case, a spatial Central Limit Theorem is proven in Chapter 6. In particular, it is shown that

$$\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var } N_A(t)}} \rightarrow N(0,1)$$

as $A \uparrow R^d$, A compact and symmetric.

In Chapter 5, the existence of a steady-state distribution as $t \rightarrow \infty$ is proven for the immigration process with subcritical branching. The Central Limit Theorem for the steady-state distribution follows as an immediate corollary to the fixed-time Central Limit Theorem in Chapter 6.

A second corollary is that for any fixed bounded symmetric set $A \subseteq D$, if the immigration is speeded up and the migration is slowed down in a suitable way, then the distribution of $N_A(t)$ becomes approximately normal.

TABLE OF CONTENTS

<u>CHAPTER</u>		<u>PAGE</u>
1	INTRODUCTION.	1
	1.1 The Model	3
	1.2 Statement of Results	6
	1.3 Theorems	8
2	PROPERTIES OF $N_A(t)$	11
	2.1 Multiple Points	11
	2.2 A Characterization of $N_A(t)$	13
3	THE SINGLE-TYPE BRANCHING DIFFUSION	19
	3.1 Initially One Particle in D	19
	3.2 Behaviour of $N_A(t)$ for Large A	31
	3.3 Poisson Random Field Initially in D	44
4	THE MULTITYPE BRANCHING DIFFUSION	56
	4.1 Initially One Particle in D	56
	4.2 Behaviour of The First and Second Factorial Moment Density Functions	73
5	THE BRANCHING DIFFUSION WITH IMMIGRATION	83
	5.1 The Single-Type Branching Diffusion with Immigration	83
	5.2 Behaviour of $N_A(t)$ for Large A	91
	5.3 The Subcritical Case -- The Existence of a Steady-State Distribution	111

<u>CHAPTER</u>		<u>PAGE</u>
5	5.4 Renormalization of the Immigration Process	116
	5.5 The Multitype Branching Diffusion with Immigration	123
6	CENTRAL LIMIT THEOREMS	138
	6.1 The Central Limit Theorem for the Single-Type Process with Immigration	138
	6.2 Effects of Rescaling the Process	159
7	CONCLUSION	161
	APPENDIX A	
	APPENDIX B	

1. INTRODUCTION

A subject of some concern in genetics is the appearance of a rare mutant gene in a population of "normals", and the numbers and locations of the mutant present at future times. The intrinsic importance of mutants and their effect on genetic computations make a knowledge of their distribution a prime problem.

Unfortunately, the mathematical expressions of such phenomena are often complicated, and do not readily yield quantities of interest. However, two important cases may be described mathematically by means of a branching diffusion process in the sense of Sawyer [22], or Ikeda-Nagasawa-Watanabe [13].

These cases are as follows:

1. We are dealing with a haploid population which is infinite but countable. The haploid individuals undergo dispersion, birth, and death independently of each other in some domain D . Time is continuous and generations may overlap. A mutant individual appears spontaneously at some location in D . At some later time, the mutant disappears and is replaced spontaneously by $0, 1, 2, \dots$ identical mutant offspring with probabilities $p_0, p_1, p_2, \dots$ respectively.

Each of the mutant offspring migrates and produces further identical progeny according to the same laws as the original mutant, but independently of each other and the rest of the population. The reproduction is described by a branching process, and the movement by a diffusion process.

2. We consider now a countably infinite population of diploid dioecious creatures. (Dioecious means that there are two sexes, and that mating is intersexual. Diploid means that the genetic information of any individual is carried on pairs of chromosomes, and that during mating, each parent contributes one chromosome from each pair to the offspring.) The individuals independently undergo dispersion, random mating, and death in a domain-D. Time is continuous and generations may overlap. As before, a mutant appears in the population. As long as the frequency of the mutant gene remains locally finite, then the distribution of the number of genes present in the population converges to a branching diffusion process. (Sawyer [21]).

The branching is described as in the haploid case in the following manner: any mutant will produce $0, 1, 2, \dots$ identical mutants with probabilities $p_0, p_1, p_2, \dots$. The dispersion of the mutants is described by a diffusion process.

In both the haploid and the diploid cases, the selective advantage of the mutant is defined to be the expected number of offspring of any one mutant gene. This expected number is designated by s , where $s = \sum_{n=0}^{\infty} np_n$. We will see later that the behaviour of the number distribution of the mutants is affected greatly by the value of s . The gene has a selective advantage if $s > 1$, selective neutrality if $s = 1$, and a selective disadvantage if $s < 1$.

In both examples, if more than one mutant type is under consideration, and spontaneous mutation from one genotype to another is possible, then the birth-death-mutation-migration process is described by means of a multitype branching diffusion process.

Immigration of the mutant gene into the population, or a constant mutational flux from a constant-density population of "normals" will be handled by means of a branching diffusion process with immigration.

A branching diffusion process is heuristically described in the following way: at time $t = 0$ we are given a fixed particle or a distribution of particles in a domain $D \subseteq \mathbb{R}^d$. Each particle moves about independently according to the same migration law, until a random time t , when it branches. All particles, whether present at $t = 0$ or produced by branching, diffuse and reproduce independently, and have the same migration and birth-death behaviour. Frequently, the migration law is Brownian motion. The problem is to study the properties of the point process $N_A(t)$, where $N_A(t) =$ (number of particles in $A \subseteq D$ at time t).

1.1 THE MODEL

A branching diffusion is a Markov point process on $X = \bigcup_{n=0}^{\infty} D^n = \emptyset \cup D \cup (D \times D) \cup (D \times D \times D) \cup \dots$. See Appendix B for a general description of a point process. A branching diffusion is completely defined by an initial distribution of particles, a migration process, a branching rate, and branching offspring distributions.

The initial distribution of particles will generally consist of a single particle at a known location or a discrete but random initial distribution. When the initial distribution is random in this study it will usually be assumed to be a Poisson random field, where

$$P(N_A(0) = n) = \frac{e^{-\lambda|A|} (\lambda|A|)^n}{n!}$$

λ is the intensity of the Poisson random field and $|A|$ is the Lebesgue measure of $A \subseteq D$.

The migration process will be assumed to have a density with respect to Lebesgue measure on R^d , thus giving rise to the following transition probabilities:

$$P[X_t \in A | X_0 = a] = \int_A p(t, a, b) db$$

where $\{X_t\}$ is the stochastic process describing the movement of the particle until the time that it branches, and " $X_0 = a$ " indicates that we are conditioning on exactly one particle initially present in the population, located at $a \in D$. We require that $\int_D p(t, a, b) db = 1$. By the Markovian nature of the transition probabilities, the operators

$$T_t f(a) = \int_D p(t, a, b) f(b) db$$

form a semigroup. If A is the infinitesimal generator of the semigroup $\{T_t\}$, then in the case of Brownian motion, A is the Laplacian.

It is assumed that as each particle migrates in D , it branches in the interval $(t, t + dt)$ with probability $V dt$. In other words, the probability that a particle present initially does not branch before time t is e^{-Vt} .

Given that there is a branch, there are n offspring produced with probability p_n , $n = 0, 1, 2, \dots$, and we also assume that $\sum_{n=0}^{\infty} n^2 p_n < \infty$.

Branching is local; i.e., all offspring are born at the same position in D at which the parent branched. For most of the results in this study, we will assume that on branching, a particle either splits or dies; i.e., $p_0 = 1 - \alpha$, $p_2 = \alpha$, $p_k = 0$, $k \neq 0, 2$.

We note here that the branching rate and the branching offspring distribution do not depend on location or time.

This model can be generalized. First, we can consider a multitype branching process, in which there are not one, but several types of particles. At the time of a branch a particle may produce offspring identical to itself, or offspring of different types of particles. In this study, we will assume that on branching, a particle dies, splits into two particles of the same type as the original, or mutates spontaneously into a particle of another type. The multitype process can be treated in the same way as the usual branching

diffusion process by considering the domain of migration to be $D_1 = D \times \{1, \dots, k\}$, if there are k types of particles. A particle is specified by an ordered pair (a, i) where $a \in D$ gives its location in D , and i its type. Branching is no longer local, since a mutation at $a \in D$ from type i to type j can be described as a death at (a, i) and a simultaneous birth at (a, j) .

NOTE: All subsets of R^d in this paper will be assumed to be measurable with respect to Lebesgue measure on R^d .

1.2 STATEMENT OF RESULTS

In much of this study, we will be making use of the n^{th} order factorial moments of the random variable $N_A(t)$. We will show that there exists a density function of order n associated with the probability of having particles located at $b_1, \dots, b_n$ in D at time t , regardless of what happens elsewhere. By Fisher ([10], page 474, Lemma 2.1), if this n^{th} order density is designated by $\hat{P}_n(t, b_1, \dots, b_n)$, then

$$\begin{aligned} & E(N_A(t)(N_A(t) - 1) \dots (N_A(t) - (n-1))) \\ &= \int_A \dots \int_A \hat{P}_n(t, b_1, \dots, b_n) db_1 \dots db_n. \end{aligned}$$

Recursion formulae for these densities are developed for the following cases:

- (a) The single-type branching diffusion, with initially one particle in D (§ 3.1);

- (b) the single-type branching diffusion, with a Poisson random field of particles initially in D (§ 3.3) ;
- (c) the multitype branching diffusion, with initially one particle in D (§ 4.1);
- (d) the single-type branching diffusion with immigration, D initially empty (§ 5.1).

One further case is considered: the multitype branching diffusion with immigration (§ 5.5). Due to the complicated combinatorics involved in the higher moments, only the first two moment density functions are given.

The random variable $N_A(t)$ in the special case of a branching diffusion with Brownian motion as the transition function is studied extensively. The behaviour as $t \rightarrow \infty$ of the first two product moment densities of $N_A(t)$ is discussed, and physical interpretations are given. In the subcritical case with immigration, as $t \rightarrow \infty$ the existence of a steady-state distribution is proven in § 5.3.

Again considering the Brownian motion case, approximations of

$$E(N_A(t)(N_A(t)-1) \dots (N_A(t)-(n-1)))$$

are given for large symmetric sets $A \subseteq D$ for the single-type process beginning initially with one particle (§ 3.2) and for the single-type process with immigration (§ 5.2). Using these approximations, a central limit theorem for the single-type branching diffusion with immigration is proven in § 6.1. In particular, it is shown that as $A \uparrow D$, $|A| < \infty$,

$$\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var} N_A(t)}} \xrightarrow{d} N(0,1)$$

The theorem is also shown to be true for the steady-state distribution in the subcritical process. It is shown in § 6.2 that this theorem is mathematically equivalent to a Central Limit Theorem for $N_A(t)$ as the immigration is speeded up and the migration is slowed down in a suitable way.

Knowledge of the moments of the point process associated with the branching diffusion model described in § 1.1 gives valuable insight into the related biological problems. The Central Limit Theorem is especially useful in the problem of prediction and estimation of the growth behaviour of a population over time and space.

1.3 THEOREMS

The following results from Sawyer [22] will be required:

Theorem 1 (Sawyer [22], Theorem 2.1).

$E(N_A(t) | X_0 = a)$ has a density function

$$E(N_A(t) | X_0 = a) = \int_A m(t, a, b) db$$

where $m(t, a, b) = e^{-mt} p(t, a, b)$, and $m = V \cdot (1 - \sum_{n=0}^{\infty} np_n)$. Notice that

" $X_0 = a$ " indicates that we are conditioning on the fact that there was initially exactly one particle in D , and that particle was located at $a \in D$.

Theorem 2 (Sawyer [22], Theorem 2.2)

$$E(N_A(t)N_B(t)|X_0=a) - E(N_{A \cap B}(t)|X_0=a) = \iint_{AB} w(t, a, x, y) dx dy$$

where $w(t, a, x, y) dx dy$

$$= (N_{dx}(t)N_{dy}(t)|X_0=a) - E(N_{dx}(t)|X_0=a)\delta_x(y)$$

$$= dx dy [V(\sum_{n=0}^{\infty} n(n-1)p_n)e^{-mt} \int_0^t e^{-ms} \int_D p(t-s, a, b) p(s, b, x) p(s, b, y) db ds]$$

Again, $m = V(1 - \sum_0^{\infty} np_n)$.

Theorem 3 (Sawyer [22], Theorem 2.4)

For fixed $a \in D$ and $t > 0$,

$$\lim_{A \downarrow x} \frac{P(N_A(t) \geq 1 | X_0=a)}{|A|} = \lim_{A \downarrow x} \frac{P(N_A(t)=1 | X_0=a)}{|A|} = m(t, a, x),$$

and

$$\lim_{A \downarrow x, B \downarrow y} \frac{P(N_A(t) \geq 1, N_B(t) \geq 1 | X_0=a)}{|A||B|}$$

$$= \lim_{A \downarrow x, B \downarrow y} \frac{P(N_A(t) = N_B(t) = 1 | X_0=a)}{|A||B|} = w(t, a, x, y) \quad (x \neq y)$$

where $|A| = \int_A db$.

Theorem 4 (Sawyer [22], Theorem 4.1)

Assume $\{N_A(0)\}$ is a Poisson random field with $E(N_A(0)) = \lambda|A|$ for every $A \subseteq D$, and that given the initial state, the particles move and branch independently. Then $E(N_{dx}(t)N_{dy}(t)) - E(N_{dx}(t))\delta_x(y) = w(t,x,y)dxdy$ where $w(t,x,y) = m(t,x)m(t,y) + \int_D w(t,a,x,y)da$, and $m(t,x) = \int_D m(t,a,x)da$.

2. PROPERTIES OF $N_A(t)$

2.1 MULTIPLE POINTS

A point process $\{N\}$ on D is said to have no multiple points if $\Pr(\exists a \in D; N(a) \geq 2) = 0$.

This simply says that the probability of finding more than one particle at any one location in D at a given time is 0.

Considering the single-type branching diffusion process, suppose we initially have one particle at $a \in D$. Then by § 1, Theorem 2,

$$\begin{aligned} & E(N_A(t)N_A(t) - N_A(t) | X_0 = a) \\ &= \iint_{AA} w(t, a, x, y) dx dy \\ &= o(|A|). \end{aligned}$$

But $E(N_A(t)N_A(t) - N_A(t) | X_0 = a)$

$$= \sum_{k=2}^{\infty} k(k-1) P(N_A(t) = k | X_0 = a)$$

$$\geq 2 \sum_{k=2}^{\infty} P(N_A(t) = k | X_0 = a)$$

$$= 2P(N_A(t) \geq 2 | X_0 = a)$$

$$\therefore P(N_A(t) \geq 2 | X_0 = a) = o(|A|).$$

With this property, by Jagers ([14], Proposition 2.4), we have that

$\{(N_A(t) | X_0 = a)\}$ can have no multiple points.

If at time $t = 0$, there is initially a Poisson random field of particles, then by using § 1, Theorem 4, and an argument identical to that used in the single particle case, we can show that

$$E(N_A(t)N_A(t) - N_A(t)) = o(|A|).$$

Therefore $\{(N_A(t) | N_A(0) \text{ is a Poisson random field})\}$ also has no multiple points.

In summary, we have the following:

LEMMA 1:

For the single-type process if there is initially one particle at a known location in D , or a Poisson random field of particles, then $\{N_A(t)\}$ has no multiple points.

COROLLARY:

The same conclusion holds for the multitype branching diffusion for one particle initially, or for an initial Poisson random field of one type of particle.

PROOF OF COROLLARY:

Simply consider the multitype process as a single-type process on $D \times \{1, \dots, k\}$. Theorems 2 and 4 of § 1 still hold, and so the corollary follows immediately.

2.2 A CHARACTERIZATION OF $N_A(t)$

We will consider a single-type process without immigration. According to Jagers ([14], Proposition 2.3), the distribution of $N_A(t)$ is uniquely determined by all $\Pr[N_A(t) = 0]$ for bounded $A \in B(D)$, ($B(D) = \{\text{Borel sets in } D\}$), if $N_A(t)$ has no multiple points. But if initially there is one particle with a known location or an initial Poisson random field, then by § 2, Theorem 1, $N_A(t)$ has no multiple points.

We can therefore characterize $N_A(t)$ by $\Pr[N_A(t) = 0]$ for bounded $A \in B(D)$. We will consider first $P(N_A(t) = 0 | X_0 = a)$. For notational convenience we shall write $P(a, t) = P(N_A(t) = 0 | X_0 = a)$. If we begin with one particle at $a \in D$, and there is no branching, then

$$P(a, t) = \int_{D-A} p(t, a, b) db.$$

If $V > 0$, indicating that there can be a branch between 0 and t , then the time of the first branch must be considered. Either there is no branch up to time t , which occurs with probability e^{-Vt} , or the first branch occurs at time s , $0 < s < t$, at some point $y \in D$. There are n particles produced by this branch with probability p_n .

$$\begin{aligned} \therefore P(N_A(t) = 0 | X_0 = a) \\ = P(a, t). \end{aligned}$$

$$\begin{aligned}
 &= e^{-Vt} \int_{D-A} p(t, a, b) db + \int_0^t \int_D \sum_{n=0}^{\infty} V e^{-Vs} p(s, a, y) p_n [P(N_A(t-s) = 0 | X_0 = y)]^n dy ds \\
 &= e^{-Vt} \int_{D-A} p(t, a, b) db + \int_0^t \int_D \sum_{n=0}^{\infty} V e^{-Vs} p(s, a, y) p_n [P(y, t-s)]^n dy ds.
 \end{aligned}$$

We now have an equation for $P(a, t)$ which we will show to have a substochastic solution, which can be found by iteration.

$$\text{Let } P_0(a, t) \equiv 0.$$

$$\text{Let } P_k(a, t) = e^{-Vt} \int_{D-A} p(t, a, b) db + \int_0^t \int_D \sum_{n=0}^{\infty} V e^{-Vs} p(s, a, y) p_n [P_{k-1}(y, t-s)]^n dy ds.$$

$$\therefore P_1(a, t) = e^{-Vt} \int_{D-A} p(t, a, b) db \leq e^{-Vt} \times 1 \leq 1.$$

By induction, if $P_k(a, t) \leq 1$, then

$$\begin{aligned}
 P_{k+1}(a, t) &= e^{-Vt} \int_{D-A} p(t, a, b) db + \sum_{n=0}^{\infty} \int_0^t \int_D V e^{-Vs} p(s, a, y) p_n [P_k(y, t-s)]^n dy ds \\
 &\leq e^{-Vt} + \sum_{n=0}^{\infty} \int_0^t \int_D V e^{-Vs} p(s, a, y) p_n dy ds \\
 &= e^{-Vt} + \sum_{n=0}^{\infty} p_n \int_0^t V e^{-Vs} \int_D p(s, a, y) dy ds \\
 &= e^{-Vt} + \sum_{n=0}^{\infty} p_n \int_0^t V e^{-Vs} ds \\
 &= e^{-Vt} + (1 \times (1 - e^{-Vt})) \\
 &= 1.
 \end{aligned}$$

Hence, $0 \leq P_k(a, t) \leq 1$ for $\forall k, t, a \in D$.

Also, $P_1 \geq 0 = P_0$, and by induction if $P_k \geq \dots \geq P_1 \geq P_0$, then

$$\begin{aligned} P_{k+1}(a, t) &= e^{-Vt} \int_{D-A} p(t, a, b) db + \sum_{n=0}^{\infty} \int_0^t \int_D V e^{-Vs} p(s, a, y) p_n[P_k(y, t-s)]^n dy ds \\ &\geq e^{-Vt} \int_{D-A} p(t, a, b) db + \sum_{n=0}^{\infty} \int_0^t \int_D V e^{-Vs} p(s, a, y) p_n[P_{k-1}(y, t-s)]^n dy ds \\ &= P_k(a, t). \end{aligned}$$

We have shown that $0 \leq P_0 \leq P_1 \leq \dots \leq P_k \leq \dots \leq 1$. So a substochastic solution $P(a, t) = \lim_{k \rightarrow \infty} P_k(a, t)$ does exist. These probabilities

$P(N_A(t) = 0 | X_0 = a)$ characterize the point process.

We now assume an initial Poisson random field.

As mentioned in § 1.1, a Poisson random field with intensity λ is the random field defined by the following probabilities:

$$P(N_A = n) = \frac{(\lambda |A|)^n e^{-\lambda |A|}}{n!}, \text{ where } |A| \text{ is the Lebesgue measure of } A.$$

If $A \cap B = \emptyset$, then N_A and N_B are independent Poisson variates with means $\lambda |A|$ and $\lambda |B|$, respectively. Clearly, if $|D| < \infty$, then with probability 1, there are at most finitely many points in D .

Let g be a real-valued function on the state space $X = \bigcup_{n=0}^{\infty} D^n$ of the Poisson random field. If there are n particles in D , then the distribution of their locations is uniform over D^n . Denote by $E_{\pi}(g(x))$ the expected value of $g: X \rightarrow \mathbb{R}$ with respect to the Poisson random field. Assuming $|D| < \infty$, we have $E(g(x) | n \text{ particles in } D)$

$$= \frac{1}{|D|^n} \int \dots \int_{D^n} g(x_1, \dots, x_n) dx_1 \dots dx_n.$$

From this, we see that

$$E_{\pi}(g(x)) = E(E(g(x) | n \text{ particles in } D))$$

$$= E\left(\frac{1}{|D|^n} \int \dots \int_{D^n} g(x_1, \dots, x_n) dx_1 \dots dx_n\right)$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda|D|} (\lambda|D|)^n}{n!} \frac{1}{|D|^n} \int \dots \int_{D^n} g(x_1, \dots, x_n) dx_1 \dots dx_n.$$

For an initial Poisson random field with intensity λ , we assume that

$|D| < \infty$, and so with probability 1, there are finitely many particles in D at $t = 0$.

$$P(N_A(t) = 0 | \text{Poisson random field at } t = 0)$$

$$= P(N_A(t) = 0)$$

$$= E_{\pi}(P(N_A(t) = 0 | (a_1)))$$

where E_{π} is the expectation relative to the Poisson random field, taken over all possible n -tuples $(a_1, \dots, a_n)$, $n = 0, 1, \dots$.

$$\begin{aligned}
 \therefore P(N_A(t) = 0) &= E_{\pi} [P(N_A(t) = 0 | (a_i))] \\
 &= \sum_{n=0}^{\infty} e^{-\lambda |D|} \frac{(\lambda |D|)^n}{n!} \frac{1}{|D|^n} \int \dots \int_D^n [P(N_A(t) = 0 | (a_i))] da_1 \dots da_n \\
 &= \sum_{n=0}^{\infty} e^{-\lambda |D|} \frac{(\lambda |D|)^n}{n!} \frac{1}{|D|^n} \left[\int_D P(N_A(t) = 0 | X_0 = a) da \right]^n \\
 &= \sum_{n=0}^{\infty} e^{-\lambda |D|} \frac{\lambda^n}{n!} \left[\int_D P(a, t) da \right]^n \\
 &= \exp[-\lambda (|D| - \int_D P(a, t) da)] \\
 &= \exp[-\lambda (\int_D (1 - P(a, t)) da)] .
 \end{aligned}$$

We now have a characterization of $N_A(t)$ depending on the initial conditions.

THEOREM 1:

The distribution of $N_A(t)$ is characterized by $P(N_A(t) = 0)$ for all bounded $A \in B(D)$. For the single-type process

i) if there is one particle initially at $a \in D$, then

$$P(N_A(t) = 0) = P(a, t), \text{ and}$$

ii) if there is an initial Poisson random field with intensity λ ,

$$\text{and } |D| < \infty, \text{ then } P(N_A(t) = 0) = \exp[-\lambda (\int_D (1 - P(a, t)) da)] ,$$

where $P(a, t)$ is the minimal substochastic solution of the equation

$$P(a, t) = e^{-Vt} \int_{D-A} p(t, a, b) db + \int_0^t \int_D \sum_{n=0}^{\infty} v e^{-Vs} p(s, a, y) p_n[P(y, t-s)]^n dy ds .$$

3. THE SINGLE-TYPE BRANCHING DIFFUSION

In this section, we will consider the factorial moments of $N_A(t)$ for the single-type process. We use the fact that there can be no multiple points, and so the probability that there is more than one particle in a small increment of space $dy \subseteq D$ is negligible. We will be examining the special case where the particle either dies or splits into two on branching: i.e., $p_0 = 1 - \alpha$; $p_2 = \alpha$; $p_i = 0$, $i \neq 0, 2$. By definition, the process is supercritical if $\alpha > \frac{1}{2}$, critical if $\alpha = \frac{1}{2}$, and subcritical if $\alpha < \frac{1}{2}$. Here we recall that for a genetic model, the supercritical case corresponds to a gene with a selective advantage, the critical to a gene with selective neutrality, and the subcritical to a selective disadvantage.

3.1 INITIALLY ONE PARTICLE IN D

In this section, we consider the probability that particles exist at $db_1, \dots, db_n$ in D at time t , given one particle initially at $a \in D$, ignoring what happens elsewhere. We designate

$$P[\text{there exist particles in } db_1, \dots, db_n \text{ at } t | X_0 = a] = \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$$

Recall from § 1.2 that

$$E[N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1))] = \int_A \dots \int_A \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$$

so in effect, we will be finding the factorial moment density functions of $N_A(t)$.

THEOREM 1:

For the model described previously, if $n = 1$, $\hat{P}_1(t, a, b) = e^{-mt} p(t, a, b)$,
where $m = V(1-2\alpha)$. If $n \geq 2$,

$$\hat{P}_n(t, a, b_1, \dots, b_n) = 2V\alpha \int_0^t \int_D e^{-ms} p(s, a, y) f_n(t-s, y, b_1, \dots, b_n) dy ds,$$

where $m = V(1-2\alpha)$, and

$$\begin{aligned} f_n(t, a, b_1, \dots, b_n) &= \frac{1}{2} [\hat{P}_1(t, a, b_1) \hat{P}_{n-1}(t, a, b_2, \dots, b_n) + \dots + \hat{P}_1(t, a, b_n) \hat{P}_{n-1}(t, a, b_1, \dots, b_{n-1}) \\ &+ \hat{P}_2(t, a, b_1, b_2) \hat{P}_{n-2}(t, a, b_3, \dots, b_n) + \dots + \hat{P}_2(t, a, b_{n-1}, b_n) \hat{P}_{n-2}(t, a, b_1, \dots, b_{n-2}) \\ &+ \dots + \hat{P}_k(t, a, b_1, \dots, b_k) \hat{P}_{n-k}(t, a, b_{k+1}, \dots, b_n) + \dots + \hat{P}_k(t, a, b_{n-k+1}, \dots, b_n) \\ &\quad \times \hat{P}_{n-k}(t, a, b_1, \dots, b_{n-k}) \\ &+ \dots + \hat{P}_{n-1}(t, a, b_1, \dots, b_{n-1}) \hat{P}_1(t, a, b_n) + \dots + \hat{P}_{n-1}(t, a, b_2, \dots, b_n) \hat{P}_1(t, a, b_1)] \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \left(\sum_{\rho \in S_k} \hat{P}_k(t, a, b_{\rho(1)}, \dots, b_{\rho(k)}) \hat{P}_{n-k}(t, a, b_{\rho(k+1)}, \dots, b_{\rho(n)}) \right) \end{aligned}$$

where $\sum_{\rho \in S_k}$ indicates summation over the $\binom{n}{k}$ possible combinations of $b_1, \dots, b_n$ into two subsets of size k and $n-k$, respectively.

(For example,

$$\begin{aligned} f_2(t, a, b_1, b_2) &= \frac{1}{2} [\hat{P}_1(t, a, b_1) \hat{P}_1(t, a, b_2) + \hat{P}_1(t, a, b_2) \hat{P}_1(t, a, b_1)] \\ &= \hat{P}_1(t, a, b_1) \hat{P}_1(t, a, b_2). \end{aligned}$$

$$f_3(t, a, b_1, b_2, b_3) = \hat{P}_1(t, a, b_1) \hat{P}_2(t, a, b_2, b_3) \\ + \hat{P}_1(t, a, b_2) \hat{P}_2(t, a, b_1, b_3) \\ + \hat{P}_1(t, a, b_3) \hat{P}_2(t, a, b_1, b_2) .$$

PROOF:

i) $n = 1$: It is necessary to consider this case separately, as there is the possibility of no branching, whereas for $n \geq 2$, at least one branch must occur between 0 and t .

$\hat{P}_1(t, a, db) = \text{Pr}(\text{there is no branching, and the particle migrates from } a \text{ to } db \text{ in } (0, t)) \\ + \text{Pr}(\text{at some time } s < t, \text{ the first branch occurs at } y \in D, \text{ does not result in the death of the particle, and one of the two particles produced by the branch is the ancestor of a particle in } db \text{ at time } t).$

Notice that since there are no multiple points, we can neglect the possibility that both particles produced by the first branch are responsible for particles in db at time t ; i.e., that there is more than one particle in db at time t .

$$\therefore \hat{P}_1(t, a, db) = e^{-Vt} p(t, a, b) db + \int_0^t \int_D 2V \alpha e^{-Vs} p(s, a, y) \hat{P}_1(t-s, y, db) dy ds .$$

Direct substitution shows that $e^{-V(1-2\alpha)t} p(t, a, b) db$ is a solution of this equation:

$$\text{LHS} = e^{-V(1-2\alpha)t} p(t, a, b) db$$

$$\text{RHS} = e^{-Vt} p(t, a, b) db + \int_0^t \int_D 2V \alpha e^{-Vs} p(s, a, y) e^{-V(1-2\alpha)(t-s)} p(t-s, y, db) dy ds$$

$$= e^{-Vt} p(t, a, b) db + \int_0^t 2V\alpha e^{-V(1-2\alpha)t} e^{-2V\alpha s} \int_D p(s, a, y) p(t-s, y, b) dy ds.$$

But by the Markovian nature of the migration process,

$$\int_D p(s, a, y) p(t-s, y, db) dy = p(t, a, db) = p(t, a, b) db.$$

$$\begin{aligned} \therefore \text{RHS} &= e^{-Vt} p(t, a, b) db + p(t, a, b) db \times 2V\alpha e^{-V(1-2\alpha)t} \int_0^t e^{-2V\alpha s} ds \\ &= e^{-Vt} p(t, a, b) db + p(t, a, b) db e^{-Vt} 2V\alpha t (1 - e^{-2V\alpha t}) \\ &= e^{-Vt} p(t, a, b) db (1 - e^{2V\alpha t} - 1) \\ &= e^{-V(1-2\alpha)t} p(t, a, b) db \\ &= \text{LHS}. \end{aligned}$$

Thus, a factorial moment density function of order '1' does exist,

and $\hat{p}_1(t, a, b) = e^{-mt} p(t, a, b)$, where $m = V(1-2\alpha)$.

ii) $n \geq 2$: For notational convenience, it will be noted here that if factorial moment densities of order $n-1$ exist, then the factorial moment densities of order n exist, by the form of equation 1 which follows. So, since we can write $\hat{p}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$, we will consider the density function alone.

Now, since we begin at $t = 0$ with exactly one particle in D , and we want to consider probabilities of more than one particle at $t > 0$, then the initial particle must branch sometime between 0 and t , and this first branch cannot result in a death. If the first branch occurs at $s < t$, at $y \in D$, then with probability α , two offspring are produced. There are two possibilities. One of the particles can be the common ancestor of all n particles at $db_1, \dots, db_n$ at time t . This occurs with probability $2 \times \hat{P}_n(t-s, y, b_1, \dots, b_n) db_1, \dots, db_n$. On the other hand, both particles can be ancestors. This occurs with probability $2 \times f_n(t-s, y, b_1, \dots, b_n) db_1, \dots, db_n$, since one offspring is the common ancestor of k of the n particles ($1 \leq k < n$), and the other offspring is the common ancestor of the other $n - k$ particles with probability

$$2 \times \sum_{p \in S_k} \hat{P}_k(t-s, y, b_p(1), \dots, b_p(k)) \hat{P}_{n-k}(t-s, y, b_p(k+1), \dots, b_p(n)) .$$

" $\sum_{p \in S_k}$ " is as explained in the statement of the theorem. From this the

following renewal-type equation for $\hat{P}_n$ is obtained:

$$\begin{aligned} & \hat{P}_n(t, a, b_1, \dots, b_n) \\ &= \int_0^t \int_D v e^{-vs} p(s, a, y) \alpha 2[\hat{P}_n(t-s, y, b_1, \dots, b_n) + f_n(t-s, y, b_1, \dots, b_n)] dy ds \quad (1) \end{aligned}$$

In order to solve this equation, take Laplace transforms of both sides:

$$L_{\lambda} \hat{P}_n = \int_0^{\infty} e^{-\lambda t} P_n(t, a, b_1, \dots, b_n) dt.$$

$$= 2V \alpha \int_0^{\infty} \int_0^t \int_D e^{-\lambda t} e^{-Vs} p(s, a, y) [\hat{P}_n(t-s, y, b_1, \dots, b_n) + f_n(t-s, y, b_1, \dots, b_n)] dy ds dt.$$

Let $u = t-s$

$s=s$.

$$\therefore L_{\lambda} \hat{P}_n = 2V \alpha \int_0^{\infty} \int_D e^{-\lambda s} e^{-Vs} p(s, a, y) \int_0^{\infty} e^{-\lambda u} [\hat{P}_n + f_n](u, y, b_1, \dots, b_n) du dy ds$$

$$= 2V \alpha \int_0^{\infty} \int_D e^{-\lambda s} e^{-Vs} p(s, a, y) (L_{\lambda} \hat{P}_n + L_{\lambda} f_n) dy ds = 2V \alpha R_{\lambda}^{\circ} L_{\lambda} \hat{P}_n + 2V \alpha R_{\lambda}^{\circ} L_{\lambda} f_n \quad (2)$$

where R_{λ}° is the Laplace transform of the semigroup

$$T_t^{\circ} f(a) = \int_D e^{-Vt} p(t, a, y) f(y) dy.$$

But if A is the infinitesimal generator of the semigroup.

$$T_t f(a) = \int_D p(t, a, y) f(y) dy,$$

then $A^{\circ} = A - VI$ is the infinitesimal generator of T_t° , and so

$$(A^{\circ} - \lambda) R_{\lambda}^{\circ} = R_{\lambda}^{\circ} (A^{\circ} - \lambda) = -I. \text{ Multiplying both sides of (2) by } (A - V - \lambda) = A^{\circ} - \lambda,$$

$$\begin{aligned}(A-V-\lambda)L_{\lambda}\hat{P}_n &= 2V\alpha[(A-V-\lambda)R_{\lambda}L_{\lambda}P_n + (A-V-\lambda)R_{\lambda}L_{\lambda}f_n] \\ &= -2V\alpha L_{\lambda}\hat{P}_n - 2V\alpha L_{\lambda}f_n\end{aligned}$$

$$\Rightarrow (A-V(1-2\alpha) - \lambda)L_{\lambda}\hat{P}_n = -2V\alpha L_{\lambda}f_n. \quad (3)$$

If Q_{λ}^{-1} is the Laplace transform of the semigroup T_t^{-1} where

$$T_t^{-1}f(a) = \int_0^{\infty} \int_D e^{-V(1-2\alpha)t} p(t,a,b)f(b)dbdt$$

then $Q_{\lambda}^{-1}(A-V(1-2\alpha)-\lambda) = -I$, since $A-V(1-2\alpha)I$ is the infinitesimal generator of T_t^{-1} .

Multiplying (3) by Q_{λ}^{-1} ,

$$Q_{\lambda}^{-1}(A-V(1-2\alpha)-\lambda)L_{\lambda}\hat{P}_n = -2V\alpha Q_{\lambda}^{-1}L_{\lambda}f_n$$

$$\Rightarrow L_{\lambda}\hat{P}_n = 2V\alpha Q_{\lambda}^{-1}L_{\lambda}f_n$$

$$\Rightarrow \int_0^{\infty} e^{-\lambda t} P_n(t,a,b_1,\dots,b_n)dt$$

$$= 2V\alpha \int_0^{\infty} e^{-\lambda s} e^{-V(1-2\alpha)s} \int_D p(s,a,y) \int_0^{\infty} e^{-\lambda u} f_n(u,y,b_1,\dots,b_n)dudyds.$$

Let $s = s$

$t = u+s$.

Then

$$\begin{aligned} & \int_0^{\infty} e^{-\lambda t} P_n(t, a, b_1, \dots, b_n) dt \\ &= 2V \alpha \int_0^{\infty} \int_0^t \int_D e^{-\lambda s} e^{-V(1-2\alpha)s} p(s, a, y) e^{-\lambda(t-s)} f_n(t-s, y, b_1, \dots, b_n) dy ds dt \\ &= 2V \alpha \int_0^{\infty} e^{-\lambda t} \left[\int_0^t \int_D e^{-V(1-2\alpha)s} p(s, a, y) f_n(t-s, y, b_1, \dots, b_n) dy ds \right] dt . \end{aligned}$$

Therefore, by inverting Laplace transforms we have proven for $n \geq 2$

$$\hat{P}_n(t, a, b_1, \dots, b_n) = 2V \alpha \int_0^t \int_D e^{-ms} p(s, a, y) f_n(t-s, y, b_1, \dots, b_n) dy ds$$

for $m = V(1-2\alpha)$.

q.e.d.

For example, for $n = 2$

$$\hat{P}_2(t, a, b_1, b_2) = 2V \alpha \int_0^t \int_D e^{-ms} p(s, a, y) \hat{P}_1(t-s, y, b_1) \hat{P}_1(t-s, y, b_2) dy ds .$$

For $n = 3$,

$$\begin{aligned} \hat{P}_3(t, a, b_1, b_2, b_3) &= 2V \alpha \int_0^t \int_D e^{-ms} p(s, a, y) [\hat{P}_1(t-s, y, b_1) \hat{P}_2(t-s, y, b_2, b_3) \\ &+ \hat{P}_1(t-s, y, b_2) \hat{P}_2(t-s, y, b_1, b_3) + \hat{P}_1(t-s, y, b_3) \hat{P}_2(t-s, y, b_1, b_2)] dy ds . \end{aligned}$$

Still considering the case where there is one particle initially, we assume that $D = R^d$ and that the migration is d -dimensional Brownian motion. We will consider the behaviour of the first two factorial moment densities as $t \rightarrow \infty$.

$$a) \hat{P}_1(t, a, b) = e^{-mt} p(t, a, b) = e^{-mt} \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t).$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \hat{P}_1(t, a, b) &= \frac{1}{(2\pi)^{d/2}} \times \lim_{t \rightarrow \infty} \exp(-|a-b|^2/2t) \times \lim_{t \rightarrow \infty} \frac{e^{-mt}}{(2\pi t)^{d/2}} \\ &\sim \frac{1}{(2\pi)^{d/2}} \lim_{t \rightarrow \infty} \frac{e^{-mt}}{(2\pi t)^{d/2}}. \end{aligned}$$

Subcritical branching: ($m > 0$)

If $m > 0$, then $\hat{P}_1(t, a, b)$ dies off exponentially fast.

Critical branching: ($m = 0$).

If $m = 0$, then $\hat{P}_1(t, a, b)$ decays like $t^{-d/2}$.

Supercritical branching: ($m < 0$)

If $m < 0$, then $\hat{P}_1(t, a, b) \rightarrow \infty$ as $t \rightarrow \infty$.

$$\begin{aligned}
 b) \quad \hat{P}_2(t, a, b_1, b_2) &= 2V \alpha \int_0^t \int_D e^{-ms} p(s, a, y) \hat{P}_1(t-s, y, b_1) \hat{P}_1(t-s, y, b_2) dy ds \\
 &= 2V \alpha \int_0^t e^{-m(t-s)} e^{-2ms} \int_D p(t-s, a, y) p(s, y, b_1) p(s, y, b_2) dy ds \\
 &= 2V \alpha e^{-mt} \int_0^t e^{-ms} \times \frac{\exp(a - \frac{b_1 + b_2}{2})^2 / (2t-s))}{(\pi(2t-s))^{d/2}} \times \frac{\exp(-(b_1 - b_2)^2 / 4s)}{(4\pi s)^{d/2}} ds.
 \end{aligned}$$

Letting $k_1 = \left(\frac{a - (b_1 + b_2)}{2} \right)^2$

$k_2 = (b_1 - b_2)^2,$

$$\hat{P}_2(t, a, b_1, b_2) = 2V \alpha e^{-mt} \int_0^t e^{-ms} \frac{e^{-k_1 / (2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-k_2 / 4s}}{(4\pi s)^{d/2}} ds.$$

Subcritical branching: ($m > 0$)

We notice that $\frac{e^{-k_1 / (2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-k_2 / 4s}}{(4\pi s)^{d/2}}$ is bounded

above by some positive constant K for every $t > 0, s < t$.

$$\hat{P}_2(t, a, b_1, b_2) \leq 2V \alpha K e^{-mt} \frac{(1 - e^{-mt})}{m} \quad \text{for } \forall t.$$

Since $m > 0$, as $t \rightarrow \infty$, $\hat{P}_2(t, a, b_1, b_2) \rightarrow 0$ at least exponentially fast.

Critical branching: ($m = 0$)

Notice that $\frac{e^{-k_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \leq \frac{1}{(\pi t)^{d/2}}$ for $s < t$.

$$\therefore \hat{P}_2(t, a, b_1, b_2) \leq 2V \alpha \int_0^t \frac{1}{(\pi t)^{d/2}} \cdot \frac{e^{-k_2/4s}}{(4\pi s)^{d/2}} ds$$

$$= \frac{2V \alpha}{(\pi t)^{d/2}} \int_0^t \frac{e^{-k_2/4s}}{(4\pi s)^{d/2}} ds$$

By considering the behaviour of $\int_0^t \frac{e^{-k_2/4s}}{(4\pi s)^{d/2}} ds$, we obtain

$$\hat{P}(t, a, b_1, b_2) \leq \begin{cases} \frac{2V \alpha}{(\pi t)^{1/2}} \sqrt{\frac{t}{\pi}} + O(1) \times \frac{1}{(\pi t)^{1/2}}, & d = 1 \text{ (Bateman [3], page 146)} \\ \frac{2V \alpha}{(\pi t)} \frac{\log t}{4\pi} + O(1) \times \frac{1}{\pi t}, & d = 2 \text{ (obtained by noticing} \\ & \text{that the error is of the same order} \\ & \text{as in the case } d = 1) \\ \frac{2V \alpha}{(\pi t)^{3/2}} \frac{1}{4\pi} \frac{1}{\sqrt{k_2}} + O\left(\frac{1}{\sqrt{t}}\right) \times \frac{1}{(\pi t)^{3/2}}, & d = 3 \end{cases}$$

(Bateman [3], page 146).

Thus, for $D = R^1$, $\hat{P}_2(t, a, b_1, b_2)$ remains bounded, and for $D = R^2$ or R^3 , $\hat{P}_2(t, a, b_1, b_2)$ dies off as $t \rightarrow \infty$.

Supercritical branching: ($m < 0$)

Since $s \leq t \leq 2t - s \leq 2t$, we have that

$$\frac{e^{-k_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-k_2/4s}}{(4\pi s)^{d/2}} \geq \frac{e^{-k/s}}{(2\pi)^d (2t)^d}$$

where $k = k_1 + k_2/4$.

$$\begin{aligned} \therefore \hat{P}_2(t, a, b_1, b_2) &\geq 2V \alpha \frac{e^{-mt}}{(2\pi)^d (2t)^d} \int_0^t e^{-ms} e^{-k/s} ds \\ &\geq 2V \alpha \frac{e^{-mt} e^{-k}}{(2\pi)^d (2t)^d} \int_1^t e^{-ms} ds \end{aligned}$$

$\therefore$ as $t \rightarrow \infty$, $\hat{P}_2(t, a, b_1, b_2)$ blows up exponentially fast, since $m < 0$.

Thus, dimension affects the moments of $N_A(t)$ as $t \rightarrow \infty$, for bounded sets $A \subseteq R^d$, $E(N_A(t))$ dies off as $t \rightarrow \infty$ in the subcritical and critical cases, and explodes in the supercritical case. If $A \cap B = \emptyset$, then as $t \rightarrow \infty$, $E(N_A(t)N_B(t))$ dies off in the subcritical case, as well as in the critical case for $d = 2, 3$, remains bounded in the critical case for $d = 1$, and explodes in the supercritical case.

At this point it is interesting to introduce two results of S. Watanabe for the supercritical branching process. Still assuming the same model, with $D = R^d$ and Brownian motion for the migration process, if we begin with a single particle at $a \in D$, then:

1. For bounded sets $A \subseteq D$,

$$P(\lim_{t \rightarrow \infty} (2\pi t)^{d/2} \frac{N_A(t)}{N_D(t)} = |A|) = 1.$$

2.
$$P(\lim_{t \rightarrow \infty} \frac{N_{\sqrt{t}A}(t)}{N_D(t)} = \int_A \frac{\exp(-|x|^2/2)}{(2\pi)^{d/2}} dx) = 1.$$

$\sqrt{t}A$ is simply the dilation of A by a factor of $\sqrt{t}$. (Athreya and Ney, [2], page 243-5.)

3.2. BEHAVIOUR OF $N_A(t)$ FOR LARGE A , FOR ONE PARTICLE INITIALLY IN D .

For this section, we assume again that $D = R^d$, and that migration is d -dimensional Brownian motion. We will consider the behaviour of the factorial moments of $N_A(t)$ for large compact symmetric sets $A \subseteq D$.

Theorem 2.

Assume that $D = R^d$, and $p(t, a, b) = \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t)$.

Then for noncritical branching ($m = V(1-2\alpha) \neq 0$), if there is one particle initially at $a \in D$,

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1))) = n! \times \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} \left(\frac{1}{|A|^{n_1}} \right).$$

A compact and symmetric about the origin, $n_1 > 0$, n_1 arbitrary. The error term is uniform in t for subcritical branching.

Proof:

We notice first that

$$E(N_A(t)) = \int_A \hat{P}_1(t, a, b) db = \int_A e^{-mt} p(t, a, b) db \leq e^{-mt}.$$

$$\begin{aligned} E(N_A(t)(N_A(t)-1)) &= \iint_{AA} \hat{P}_2(t, a, b_1, b_2) db_1 db_2 \\ &= 2V \alpha \int_0^t \int_{QAAD} e^{-ms} p(s, a, y) f_2(t-s, y, b_1, b_2) dy db_1 db_2 ds \\ &= 2V \alpha \int_0^t \int_{QD} e^{-ms} p(s, a, y) \int_A e^{-m(t-s)} p(t-s, y, b_1) db_1 \\ &\quad \times \int_A e^{-m(t-s)} p(t-s, y, b_2) db_2 dy ds \end{aligned}$$

$$\begin{aligned}
 &\leq 2V \alpha \int_0^t \int_{OD} e^{-ms} p(s, a, y) e^{-2m(t-s)} dy ds \\
 &= 2V \alpha e^{-mt} \int_0^t e^{-m(t-s)} ds \\
 &= 2V \alpha \frac{(1-e^{-mt})}{m} e^{-mt}
 \end{aligned}$$

Assume that for $k \leq n-1$

$$\begin{aligned}
 E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(k-1))) &= \int_{A^k} \dots \int \hat{P}_k(t, a, b_1, \dots, b_k) db_1 \dots db_k \\
 &\leq k! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{k-1} e^{-mt}
 \end{aligned}$$

Then

$$\begin{aligned}
 &E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1))) \\
 &= \int_{A^n} \dots \int \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n \\
 &= 2V \alpha \int_0^t \int_{OD} e^{-ms} p(s, a, y) \int_{A^n} \dots \int f_n(t-s, y, b_1, \dots, b_n) db_1 \dots db_n dy ds
 \end{aligned}$$

$$= 2V \alpha \int_{OD}^t e^{-ms} p(s, a, y)^{\frac{1}{2}} \times \sum_{k=1}^{n-1} \sum_{\rho \in S_k} \int_{A^n} \dots \int \hat{P}_k(t-s, y, b_{\rho(1)}, \dots, b_{\rho(k)}) \\ \times \hat{P}_{n-k}(t-s, y, b_{\rho(k+1)}, \dots, b_{\rho(n)}) db_{\rho(1)} \dots db_{\rho(n)} dy ds$$

$$= V \alpha \int_{OD}^t e^{-ms} p(s, a, y) \sum_{k=1}^{n-1} \binom{n}{k} \int_{A^k} \hat{P}_k(t-s, y, b_1, \dots, b_k) db_1 \dots db_k$$

$$\times \int_{A^{n-k}} \hat{P}_{n-k}(t-s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n dy ds$$

$$\leq V \alpha \int_{OD}^t e^{-ms} p(s, a, y) \sum_{k=1}^{n-1} \binom{n}{k} k! \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{k-1} e^{-m(t-s)}$$

$$\times (n-k)! \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{n-k-1} e^{-m(t-s)} dy ds$$

$$= V \alpha \sum_{k=1}^{n-1} \int_0^t e^{-m(t-s)} e^{-mt} \frac{n!}{k! (n-k)!} k! (n-k)! \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{n-2} ds$$

$$= n! V \alpha e^{-mt} \left(\frac{1-e^{-mt}}{m} \right)^{n-1} (V \alpha)^{n-2} \frac{n-1}{n-1}$$

$$= n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt}$$

∴ by induction we have shown that

$$E(N_A(t) (N_A(t)-1) \dots (N_A(t)-(n-1))) \leq n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt}$$

We will now show that the n^{th} order factorial moment actually converges to this quantity, and we will obtain a rough estimate of the rate of convergence.

Without loss of generality, we will consider only sets A that are symmetrical cubes with sides of length $2k$, so that $|A| = (2k)^d$, and let $k \uparrow \infty$. To handle symmetrical sets A that are not cubes, simply consider the largest such cube $K \subseteq A$.

If $D = R^d$, then $x = (x_1, \dots, x_d) \forall x \in R^d$. Choose k large enough that $k > \max_{1 \leq i \leq d} |a_i|$, where $a = (a_1, \dots, a_d)$ is the location of the initial particle in D .

We have

$$\begin{aligned} e^{-mt} &\geq E(N_A(t)) = \int_A \hat{P}_1(t, a, b) db \\ &= \int_A e^{-mt} p(t, a, b) db \\ &= \left(\frac{e^{-mt}}{(2\pi t)^{d/2}} \right) \int_{-k}^k \dots \int_{-k}^k \prod_{i=1}^d e^{-|a_i - b_i|^2 / 2t} db_1 \dots db_d \\ &= \frac{e^{-mt}}{(2\pi)^{d/2}} \int_{\frac{a_1-k}{\sqrt{t}}}^{\frac{a_1+k}{\sqrt{t}}} \dots \int_{\frac{a_d-k}{\sqrt{t}}}^{\frac{a_d+k}{\sqrt{t}}} \prod_{i=1}^d e^{-x_i^2 / 2} dx_1 \dots dx_d \end{aligned}$$

$$\leq e^{-mt} \prod_{i=1}^d \left[1 - \frac{1}{\frac{a_i+k}{\sqrt{t}}} \phi\left(\frac{a_i+k}{\sqrt{t}}\right) - \frac{1}{\frac{|a_i-k|}{\sqrt{t}}} \phi\left(\frac{|a_i-k|}{\sqrt{t}}\right) \right]$$

for $k \geq |a_i|$, $i = 1, \dots, d$, and ϕ the normal density function. (Feller I [9], page 175).

$$\therefore E(N_A(t)) = e^{-mt} - \text{error } A \quad (1)$$

$$\text{where error } A \leq e^{-mt} \left[1 - \prod_{i=1}^d \left[1 - \frac{\sqrt{t}}{k+a_i} \phi\left(\frac{a_i+k}{\sqrt{t}}\right) - \frac{\sqrt{t}}{k-a_i} \phi\left(\frac{|a_i-k|}{\sqrt{t}}\right) \right] \right]$$

$$\text{But } \phi\left(\frac{(k+a_i)}{\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k+a_i)^2}{2t}\right).$$

$$\therefore \text{for fixed } a \in D, \text{ error } A = O\left(e^{-mt} \frac{\sqrt{t}}{k} e^{-k^2/2t}\right) \quad (2)$$

Recall that $|A| = (2k)^d$.

$$\frac{\sqrt{t}}{k} e^{-k^2/2t} = \frac{\sqrt{t}}{k e^{k^2/2t}} = \frac{\sqrt{t}}{\sum_{j=0}^{\infty} \frac{(k^2)^j}{(2t)^j j!}} \times k < \frac{t^{\frac{1}{2}} (2t)^j j!}{k^{2j+1}} \text{ for every } j > 0. \quad (3)$$

∴ for fixed $a \in D$ and $t > 0$, from (2)

$$\text{error } A = o\left(\frac{1}{n^d}\right) = o\left(\frac{1}{|A|^n}\right) \text{ for every } n > 0.$$

For the subcritical case ($m > 0$), since

$$e^{-mt} = \frac{1}{\sum_{j=0}^{\infty} \frac{(mt)^j}{j!}} \leq \frac{j!}{(mt)^j} \text{ for every } j > 0,$$

$$\text{we have from (2) and (3), error } A = o\left(\frac{1}{t^{j+1}} \times \frac{t^{j+\frac{1}{2}}}{k^{2j}}\right) = o\left(\frac{1}{k^{2j}}\right)$$

uniformly in t for every $j > 0$.

$$\therefore E(N_A(t)) = e^{-mt} = o\left(\frac{1}{|A|^{n_1}}\right) \text{ for any } n_1 > 0, \text{ uniformly in } t \text{ if}$$

$m > 0$, for fixed t if $m < 0$, and for $a = (a_1, \dots, a_d)$ fixed.

Assuming that for $k \leq n-1$,

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(k-1))) = \int_A \hat{P}_k(t, a, b_1, \dots, b_k) db_1 \dots db_k$$

$$= k! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{k-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right)$$

for any $n_1 > 0$, fixed $a \in D$, and uniformly in t if $m > 0$, then

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$$

$$= 2V \alpha \int_0^t \int_D e^{-ms} p(s, a, y) \times \frac{1}{2} \times \left[\sum_{k=1}^{n-1} \binom{n}{k} \int_{A^k} \hat{P}_k(t-s, y, b_1, \dots, b_k) db_1 \dots db_k \right. \\ \left. \times \int_{A^{n-k}} \hat{P}_{n-k}(t-s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n \right] dy ds.$$

Choose $|D^1| < \infty$, $D^1 \subset D$, D^1 symmetric, so that for fixed $\epsilon > 0$, and $s \leq t$

$\int_{D^1} p(s, a, b) db \geq 1-\epsilon$. Therefore, uniformly for $y \in D^1$, $s \leq t$,

$$\int_{A^k} \hat{P}_k(t-s, y, b_1, \dots, b_k) db_1 \dots db_k$$

$$= k! \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{k-1} e^{-m(t-s)} - o\left(\frac{1}{|A|^{n_1}}\right), \quad 1 \leq k \leq n-1.$$

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$$

$$= 2V \alpha \int_0^t \int_{OD^1} e^{-ms} p(s, a, y) \times \frac{1}{2} \times \left[\sum_{k=1}^{n-1} \binom{n}{k} k! (n-k)! \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{n-2} e^{-2m(t-s)} \right. \\ \left. - o\left(\frac{1}{|A|^{n_1}}\right) \right] dy ds \\ + \text{error } B$$

$$\geq (1-\epsilon)n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} - \left(\frac{1-e^{-mt}}{m} \right) (1-\epsilon) o\left(\frac{1}{|A|^{n_1}}\right) \quad (4)$$

+ error B

where error B is bounded above by $n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} \times \epsilon$.

But (4) is true for every $\epsilon > 0$, so by induction we have proven

$$E(N_A(t) \dots (N_A(t) - (n-1))) = n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} + o\left(\frac{1}{|A|^{n_1}}\right)$$

for $n_1 > 0$, n , arbitrary.

From (4), we see that if $m > 0$ (the branching is subcritical) then the error term is uniform in t .

$$E(N_A(t) \dots (N_A(t) - (n-1))) = n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right), \text{ any } n_1 > 0$$

for $m \neq 0$, fixed $a \in D$, and fixed $t > 0$. If $m > 0$ (subcritical branching) t need not be fixed, as the error term is uniform in t .

So as $A \uparrow D$, A symmetric,

$$E(N_A(t) \dots (N_A(t) - (n-1))) \rightarrow n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} \quad \text{q.e.d.}$$

Theorem 3.

Assume that $D = \mathbb{R}^d$, and $p(t, a, b) = \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t)$. Then

for critical branching ($m = V(1-2\alpha) = 0$), if there is one particle initially

$$\text{at } a \in D, E(N_A(t)(N_A(t)-1) \dots (N_A(t)-(n-1))) = \frac{n!}{2^{n-1}} (Vt)^{n-1} - o\left(\frac{1}{|A|^{n-1}}\right),$$

A compact and symmetric about the origin, $n_1 > 0$, n_1 arbitrary.

Proof.

Here, since $\alpha = \frac{1}{2}$, we have $m = V(1-2\alpha) = 0$.

$$E(N_A(t)) = \int_A \hat{P}_1(t, a, b) db = \int_A p(t, a, b) db \leq 1.$$

But by exactly the same arguments as in the proof of theorem 2,

$$E(N_A(t)) = \int_A p(t, a, b) db \geq 1 - o\left(\frac{1}{|A|^{n_1}}\right) \text{ for any } n_1 > 0.$$

$$\Rightarrow E(N_A(t)) = 1 - o\left(\frac{1}{|A|^{n_1}}\right) \text{ for any } n_1 > 0.$$

Assume for $k \leq n-1$,

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(k-1)))$$

$$= \int \dots \int_{A^k} \hat{P}_k(t, a, b_1, \dots, b_k) db_1 \dots db_k$$

$$= \frac{k!}{2^{k-1}} (vt)^{k-1} - o\left(\frac{1}{|A|^{n_1}}\right). \text{ For any } n_1 > 0, \text{ fixed } t > 0$$

and fixed $A \subseteq D$.

Then $E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$

$$= \int \dots \int_{A^n} \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$$

$$= v \int_0^t \int_{OD} p(s, a, y) \int \dots \int_{A^n} f_n(t-s, y, b_1, \dots, b_n) db_1 \dots db_n dy ds$$

$$= v \int_0^t \int_{OD} p(s, a, y) \times \frac{1}{2} \left[\sum_{k=1}^{n-1} \binom{n}{k} \int_{A^k} \hat{P}_k(t-s, y, b_1, \dots, b_k) db_1 \dots db_k \right.$$

(1)

$$\left. \times \int_{A^{n-k}} \hat{P}_{n-k}(t-s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n \right] dy ds.$$

$$\therefore \int_{A^n} \dots \int \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$$

$$\leq V \int_{OD}^t p(s, a, y) \times \frac{1}{2} \left[\sum_{k=1}^{n-1} \binom{n}{k} \int_{D^k} \dots \int \hat{P}_k(t-s, y, b_1, \dots, b_k) db_1 \dots db_k \right.$$

$$\left. \times \int_{D^{n-k}} \dots \int \hat{P}_{n-k}(t-s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n \right] dy ds$$

$$= V \int_{OD}^t p(s, a, y) \times \frac{1}{2} \left[\sum_{k=1}^n \frac{n!}{k! (n-k)!} k! \left(\frac{V(t-s)}{2} \right)^{k-1} (n-k)! \left(\frac{V(t-s)}{2} \right)^{(n-k-1)!} \right] dy ds$$

$$= \left(\frac{V}{2} \right)^{n-1} \sum_{k=1}^n n! \int_0^t (t-s)^{n-2} ds$$

$$= n! \left(\frac{Vt}{2} \right)^{n-1} \frac{n-1}{n-1}$$

$$= n! \left(\frac{Vt}{2} \right)^{n-1}$$

(2)

We must now show that the n^{th} factorial moment actually converges to this quantity as $A \uparrow D$. Assume for $k \leq n-1$, and any $n_1 > 0$ that

$$\int_{A^k} \dots \int \hat{P}_k(t, a, b_1, \dots, b_k) db_1 \dots db_k = k! \left(\frac{Vt}{2} \right)^{k-1} - o\left(\frac{1}{|A|^{n_1}} \right). \text{ As in the proof}$$

of Theorem 2, choose $D^1 \subseteq D$, D^1 symmetric, such that $\int_{D^1} p(s, a, b) db \geq 1 - \epsilon$

for some $\epsilon > 0$, for every $s \leq t$ so the error term

$O\left(\frac{1}{|A|^{\frac{n-1}{2}}}\right)$ in $\int \dots \int_{A^k} \hat{P}_k(s, a, b_1, \dots, b_k) db_1 \dots db_k$ is uniform for $s \leq t, y \in D^1$,

$$1 \leq k \leq n-1.$$

From (1), $E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$

$$= V \int_0^t \int_{OD^1} p(s, a, y) \times \frac{1}{2} \left[\sum_{k=1}^{n-1} \binom{n}{k} \frac{k! (n-k)!}{2^{k-1} 2^{n-k-1}} (V(t-s))^{k-1} (V(t-s))^{(n-k-1)} O\left(\frac{1}{|A|^{\frac{n-1}{2}}}\right) \right] dy ds$$

+ error C

$$= \frac{V^{n-1}}{2^{n-1}} n! (n-1) \frac{t^{n-1}}{n-1} \times (1-\epsilon) + Vt(1-\epsilon) O\left(\frac{1}{|A|^{\frac{n-1}{2}}}\right) \quad (3)$$

+ error C

where |error C| is bounded above by

$$\frac{n!}{2^{n-1}} (Vt)^{n-1} \epsilon.$$

But (3) is true for every $\epsilon > 0$

$$\therefore \text{from (2) and (3), } E(N_A(t) \dots (N_A(t)-(n-1))) = \frac{(Vt)^{n-1}}{2^{n-1}} n! + O\left(\frac{1}{|A|^{\frac{n-1}{2}}}\right),$$

proving the theorem.

q.e.d.

3.3. POISSON RANDOM FIELD INITIALLY IN D.

Suppose that instead of one particle with a known location in D at $t = 0$, there is an initial distribution of particles which is discrete, but random. In particular, we will consider the case of an initial Poisson random field with intensity λ . Define

$$\begin{aligned} \hat{P}_n(t, b_1, \dots, b_n) db_1 \dots db_n \\ = P [\text{there exist particles in } db_1, \dots, db_n \text{ at } t, \\ \text{given an initial Poisson random field of particles}]. \end{aligned}$$

We will need the following lemma:

Lemma 1.

Assume that $D \subseteq R^d$ is either bounded or that $D = R^d$. If $f: D^k \rightarrow R$, then for a Poisson random field of particles with intensity λ ,

$$E \left(\sum_{i_1 \neq \dots \neq i_k} f(a_{i_1}, \dots, a_{i_k}) \right) = \lambda^k \int \dots \int_{D^k} f(a_1, \dots, a_k) da_1 \dots da_k,$$

where E_π is the expectation relative to the Poisson random field over all possible n -tuples $(a_1, \dots, a_n)$, $n = 0, 1, 2, \dots$.

Proof.

We note first that if the locations of the particles are given by the vector $(a_1, a_2, \dots)$, then $\sum_{i_1 \neq \dots \neq i_k} f(a_{i_1}, \dots, a_{i_k}) = g((a_i))$, where

$g: X \rightarrow \mathbb{R}$, and X is the state space of the population. (See Appendix B).

Assume first that $|D| < \infty$, and so there are finitely many particles in D with probability 1. Referring to the discussion of Poisson random fields in § 2.2, we obtain

$$\begin{aligned} & E_{\pi} \left(\sum_{i_1 \neq \dots \neq i_k} f(a_{i_1}, \dots, a_{i_k}) \right) \\ &= \sum_{n=k}^{\infty} \frac{(\lambda |D|)^n}{n!} \frac{e^{-\lambda |D|}}{|D|^n} \int \dots \int_{D^n} \left[\sum_{i_1 \neq \dots \neq i_k} f(a_{i_1}, \dots, a_{i_k}) \right] da_1 \dots da_n \\ &= \lambda^k \sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{n!} e^{-\lambda |D|} n(n-1) \dots (n-k+1) \times |D|^{n-k} \int \dots \int_{D^k} f(a_1, \dots, a_k) da_1 \dots da_k \\ &= \left(\lambda^k \int \dots \int_{D^k} f(a_1, \dots, a_k) da_1 \dots da_k \right) \times \sum_{n=k}^{\infty} \frac{\lambda^{n-k} |D|^{n-k}}{(n-k)!} e^{-\lambda |D|} \\ &= \lambda^k \int \dots \int_{D^k} f(a_1, \dots, a_k) da_1 \dots da_k. \end{aligned}$$

We have proven the theorem for $|D| < \infty$.

Now assume that $D = R^d$. D can be expressed as the disjoint union $D = \bigcup_{j=1}^{\infty} D_j$ of bounded sets such that $|D_j| = 1$, $j = 1, 2, \dots$. D_j then contains n particles with probability $e^{-\lambda} \frac{\lambda^n}{n!}$.

Notice first that

$$D^k = \bigcup_{j_1 \neq \dots \neq j_k} (p_1 + \dots + p_k = k) D_{j_1}^{p_1} \times \dots \times D_{j_k}^{p_k}, \quad j_k \geq 1, p_k \geq 0. \quad (1)$$

When calculating $E_{\pi_{a_{i_1} \neq \dots \neq a_{i_k}}}(f(a_{i_1}, \dots, a_{i_k}))$, we must integrate over

a countable sum of Cartesian products, considering the number of a_i 's which are in D_j simultaneously, (say, p_j of them), and the total number of particles in each D_j (say n_j , and we must have $n_j \geq p_j$).

$$\begin{aligned} & E_{\pi_{a_{i_1} \neq \dots \neq a_{i_k}}}(f(a_{i_1}, \dots, a_{i_k})) \\ &= \sum_{j_1 \neq \dots \neq j_k} \sum_{p_1 + \dots + p_k = k} \sum_{n_1 \geq p_1} \dots \sum_{n_k \geq p_k} \frac{e^{-\lambda} \lambda^{n_1}}{n_1!} \dots \frac{e^{-\lambda} \lambda^{n_k}}{n_k!} \\ & \quad \times \int_{D_{j_1}}^{n_1} \dots \int_{D_{j_k}}^{n_k} \sum_{i_1 \neq \dots \neq i_k} f(a_{i_1}, \dots, a_{i_k}) da_1 \dots da_N \end{aligned}$$

where $N = \sum_{i=1}^k n_i$.

$$\begin{aligned}
 & \therefore E_{\pi} \left(\sum_{j_1, \dots, j_k} f(a_{j_1}, \dots, a_{j_k}) \right) \\
 &= \sum_{j_1, \dots, j_k} \sum_{p_1 + \dots + p_k = k} \sum_{n_1 \geq p_1} \dots \sum_{n_k \geq p_k} [n_1 \dots (n_1 - p_1 + 1)] \dots [n_k \dots (n_k - p_k + 1)] \\
 &\quad \times \left(\frac{e^{-\lambda} \lambda^{n_1}}{n_1!} \right) \times \dots \times \left(\frac{e^{-\lambda} \lambda^{n_k}}{n_k!} \right) \times \int_{D_{j_1}}^{p_1} \dots \int_{D_{j_k}}^{p_k} f(a_1, \dots, a_k) da_1 \dots da_k \\
 &= \sum_{j_1, \dots, j_k} \sum_{p_1 + \dots + p_k = k} \lambda^{p_1} \dots \lambda^{p_k} \sum_{n_1 \geq p_1} e^{-\lambda} \frac{\lambda^{n_1 - p_1}}{(n_1 - p_1)!} \dots \sum_{n_k \geq p_k} e^{-\lambda} \frac{\lambda^{n_k - p_k}}{(n_k - p_k)!} \\
 &\quad \times \int_{D_{j_1}}^{p_1} \dots \int_{D_{j_k}}^{p_k} f(a_1, \dots, a_k) da_1 \dots da_k \\
 &= \sum_{j_1, \dots, j_k} \sum_{p_1 + \dots + p_k = k} \lambda^{k-1} \dots \lambda^1 \int_{D_{j_1}}^{p_1} \dots \int_{D_{j_k}}^{p_k} f(a_1, \dots, a_k) da_1 \dots da_k \\
 &= \lambda^k \int_{D^k} f(a_1, \dots, a_k) da_1 \dots da_k, \text{ from (1), above.}
 \end{aligned}$$

q.e.d.

Theorem 4.

For an initial Poisson random field with intensity λ ,

$$\hat{P}_n(t, b_1, \dots, b_n) = \sum_{j=1}^n \lambda^j \int_{D^j} \dots \int \hat{P}_{n_j}(t, b_1, \dots, b_n, (a_1, \dots, a_j)) da_1 \dots da_j$$

where

$$\begin{aligned} & \hat{P}_{n_j}(t, b_1, \dots, b_n, (a_1, \dots, a_j)) \\ &= \sum_{\rho(k_1, \dots, k_j)} \hat{P}_{k_1}(t, a_1, b_{\rho(1)}, \dots, b_{\rho(k_1)}) \hat{P}_{k_2}(t, a_2, b_{\rho(k_1+1)}, \dots, b_{\rho(k_1+k_2)}) \\ & \quad \times \dots \times \hat{P}_{k_j}(t, a_j, b_{\rho(k_1+\dots+k_{j-1}+1)}, \dots, b_{\rho(n)}) , \end{aligned}$$

where $\hat{P}_k(t, a, b_1, \dots, b_k)$ is as in Theorem 1 of this chapter, and

$\rho(k_1, \dots, k_j)$ indicates summation over all the S_n^j possible different groupings of n distinguishable objects into j nonempty subsets.

(S_n^j is a Stirling number of the second kind. See Appendix A.)

Proof.

If there is more than one particle present at $t = 0$, then the n particles in $db_1, \dots, db_n$ at time t can have $1, 2, \dots$, or n distinct ancestors. This gives us that

$$\hat{P}_n(t, db_1, \dots, db_n) = \sum_{j=1}^n \hat{P}_{n_j}(t, db_1, \dots, db_n), \text{ where } \hat{P}_{n_j}(t, db_1, \dots, db_n)$$

= Pr [particles exist in $db_1, \dots, db_n$ at t | an initial Poisson random field, and the n particles come from j distinct ancestors at $t = 0$].

If the initial locations of the particles are known, say $(a_1, a_2, \dots)$, then we must consider the various ways in which the n particles in $db_1, \dots, db_n$ can have exactly j distinct ancestors.

$$\hat{P}_{n_j}(t, db_1, \dots, db_n | (a_1, a_2, \dots))$$

$$= \hat{P}_{n_j}(t, b_1, \dots, b_n | (a_1, a_2, \dots)) db_1 \dots db_n$$

$$= \sum_{i_1 \neq \dots \neq i_j} \left(\sum_{\rho(k_1, \dots, k_j)} \hat{P}_{k_1}(t, a_{i_1}, db_{\rho(1)} \dots db_{\rho(k_1)}) \right.$$

$$\times \hat{P}_{k_2}(t, a_{i_2}, db_{\rho(k_1+1)} \dots db_{\rho(k_1+k_2)})$$

$$\times \dots \times \hat{P}_{k_j}(t, a_{i_j}, db_{\rho(k_1+\dots+k_{j-1}+1)} \dots db_{\rho(n)}) db_1 \dots db_n$$

where $\sum_{\rho(k_1, \dots, k_j)}$ is as described in the statement of the theorem.

(Note that $\sum_{i=1}^j k_i = n$, and $k_i > 0, i = 1, \dots, j$). Summation over

$i_1, \dots, i_j$ indicates exactly which j of the particles $(a_1, a_2, \dots)$

present at $t = 0$ are the ancestors. $\hat{P}_k(t, a, b_1, \dots, b_k)$ is as in

§ 3.1 Theorem 1.)

Thus, $\hat{P}_{n_j}(t, b_1, \dots, b_n | (a_1, \dots, a_k, \dots))$

$$= \sum_{i_1, \dots, i_j} \hat{P}_{n_j}(t, b_1, \dots, b_n, (a_{i_1}, \dots, a_{i_j}))$$

where $\hat{P}_{n_j} : R^j \rightarrow R$. So by Lemma 1, we have

$$E_{\pi} P_{n_j}(t, b_1, \dots, b_n | (a_1, a_2, \dots))$$

$$= \lambda^j \int \dots \int_{D^j} \hat{P}_{n_j}(t, b_1, \dots, b_n, (a_1, \dots, a_j)) da_1 \dots da_j$$

$$\therefore \hat{P}_{n_j}(t, b_1, \dots, b_n) = \lambda^j \int \dots \int_{D^j} \hat{P}_{n_j}(t, b_1, \dots, b_n, (a_1, \dots, a_j)) da_1 \dots da_j$$

q.e.d.

For example,

$$\hat{P}_1(t, b) = \int_D \hat{P}_1(t, a, b) da$$

$$= \lambda \int_D e^{-\lambda t} p(t, a, b) da$$

$$= \lambda e^{-\lambda t}$$

$$\begin{aligned}
 \hat{P}_2(t, b_1, b_2) &= \lambda \int_D \hat{P}_2(t, a, b_1, b_2) da + \lambda^2 \iint_{DD} \hat{P}_1(t, a_1, b_1) \hat{P}_1(t, a_2, b_2) da_1 da_2 \\
 &= \lambda 2V \alpha \int_0^t \iiint_{DOD} e^{-ms} p(s, a, y) e^{-2m(t-s)} p(t-s, y, b_1) p(t-s, y, b_2) dy ds da \\
 &\quad + \lambda^2 \int_D e^{-mt} p(t, a_1, b_1) da_1 \int_D e^{-mt} p(t, a_2, b_2) da_2 \\
 &= 2\lambda V \alpha \int_0^t \iint_{OD} e^{-mt} e^{-m(t-s)} p(t-s, y, b_1) p(t-s, y, b_2) dy ds + \lambda^2 e^{-2mt} \\
 &= 2\lambda V \alpha e^{-mt} \int_0^t e^{-m(t-s)} \int_D p(t-s, y, b_1) p(t-s, y, b_2) dy ds \\
 &\quad + \lambda^2 e^{-2mt} .
 \end{aligned}$$

$$\begin{aligned}
 \hat{P}_3(t, b_1, b_2, b_3) &= \int_D \hat{P}_3(t, a, b_1, b_2, b_3) da \\
 &\quad + \lambda^2 \iint_{DD} [\hat{P}_2(t, a_1, b_1, b_2) \hat{P}_1(t, a_2, b_3) \\
 &\quad + \hat{P}_2(t, a_1, b_1, b_3) \hat{P}_1(t, a_2, b_2) \\
 &\quad + \hat{P}_2(t, a_1, b_2, b_3) \hat{P}_1(t, a_2, b_1)] da_1 da_2 \\
 &\quad + \lambda^3 \iiint_{DDD} \hat{P}_1(t, a_1, b_1) \hat{P}_1(t, a_2, b_2) \hat{P}_1(t, a_3, b_3) da_1 da_2 da_3 .
 \end{aligned}$$

Now, as in the initial single particle case, we assume that $D = R^d$, and the migration process is Brownian motion. We consider the behaviour of the first two factorial moment densities as $t \rightarrow \infty$.

$$a) \hat{P}_1(t, a, b) = \lambda e^{-mt}$$

Subcritical branching: ($m > 0$)

As $t \rightarrow \infty$, $\hat{P}_1(t, a, b) \rightarrow 0$ exponentially fast.

Critical branching: ($m = 0$)

$$\hat{P}_1(t, a, b) = \lambda \text{ for every } t.$$

Supercritical branching: ($m < 0$)

As $t \rightarrow \infty$, $\hat{P}_1(t, a, b)$ grows exponentially fast.

$$b) \hat{P}_2(t, b_1, b_2) = 2\lambda V \alpha \times e^{-mt} \int_0^t e^{-m(t-s)} \int_D p(t-s, y, b_1) p(t-s, y, b_2) dy ds + \lambda^2 e^{-2mt}$$

$$= 2\lambda V \alpha e^{-mt} \int_0^t e^{-m(t-s)} p(2(t-s), b_1, b_2) ds + \lambda^2 e^{-2mt} \quad (1)$$

$$= 2\lambda V \alpha e^{-mt} \int_0^t \frac{e^{-mu}}{(4\pi u)^{d/2}} \exp(-|b_1 - b_2|^2/4u) du + \lambda^2 e^{-2mt}$$

Subcritical branching: ($m > 0$)

As $t \rightarrow \infty$, by the Bateman Manuscript Project ([3], page 146)

$$\int_0^t \frac{e^{-mu}}{(4\pi u)^{d/2}} \exp(-|b_1 - b_2|^2/4u) du$$

$$\rightarrow \begin{cases} \frac{1}{4m^{\frac{1}{2}}} \exp(\sqrt{m}(|b_1 - b_2|)) & d = 1 \\ \frac{1}{4\pi} K_0(\sqrt{m}(|b_1 - b_2|)) & d = 2 \text{ (} K_0 \text{ is a Bessel} \\ & \text{function of the third kind.)} \\ \frac{1}{4\pi(|b_1 - b_2|)} \exp(\sqrt{m}(|b_1 - b_2|)) & d = 3 \end{cases}$$

$\therefore$ since $\int_0^t e^{-mu} p(2u, b_1, b_2) du$ remains bounded for $d = 1, 2, 3$ as $t \rightarrow \infty$,

then $\hat{p}_2(t, b_1, b_2) \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

Critical branching: ($m = 0$)

$$\hat{p}_2(t, b_1, b_2) = 2\lambda V \alpha \int_0^t \frac{\exp(-|b_1 - b_2|^2/4u) du}{(4\pi u)^{d/2}}$$

Again by Bateman [3], page 146.

$$\int_0^t p(2u, b_1, b_2) du = \begin{cases} \sqrt{\frac{t}{\pi}} + o(1) & d = 1 \\ \frac{\log t}{4\pi} + o(1) & d = 2 \text{ (obtained by noticing that} \\ & \text{the error is of the same order} \\ & \text{as in the case } d = 1). \\ \frac{1}{4\pi |b_1 - b_2|} + o\left(\frac{1}{\sqrt{t}}\right) & d = 3. \end{cases}$$

So $\hat{P}_2(t, b_1, b_2)$ grows as $\sqrt{t}$ for $d = 1$, $\hat{P}_2(t, b_1, b_2)$ grows as $\log t$ for $d = 2$, and remains positive and bounded for $d = 3$, as long as $b_1 \neq b_2$.

Supercritical branching: ($m < 0$)

From (1), $\hat{P}_2(t, b_1, b_2) > \lambda^2 e^{-2mt}$.

∴ for supercritical branching by considering (1), we see that $\hat{P}_2(t, b_1, b_2)$ always grows exponentially quickly.

Again, dimension affects the behaviour of the moments. For bounded sets $A \subseteq \mathbb{R}^d$, as $t \rightarrow \infty$, $E(N_A(t))$ dies off for supercritical branching, remains bounded for critical branching, and explodes for supercritical branching.

For $A \cap B = \emptyset$, $E(N_A(t)N_B(t))$ dies off for subcritical branching in

R^1, R^2 , and R^3 . For critical branching, $E(N_A(t)N_B(t))$ explodes in R^1 and R^2 , and remains bounded in R^3 . If $A \cap B \neq \emptyset$, $E(N_A(t)N_B(t))$ explodes in R^3 for critical branching due to the singularity in

$$\frac{1}{4\pi|b_1 - b_2|} \text{ at } b_1 = b_2.$$

For supercritical branching, $E(N_A(t)N_B(t))$ grows exponentially as $t \rightarrow \infty$.

4. THE MULTI-TYPE BRANCHING DIFFUSION

We now extend the model to include a multi-type branching process with diffusion. We assume that there are k distinguishable types of particles. All types obey the same migration law in the domain $D \subseteq \mathbb{R}^d$, with density function $p(t, a, b)$, as before. A particle of type i branches in $(t, t+dt)$ with probability $v_i dt$, regardless of its location in D . Given that a particle of type i branches, it mutates to type j with probability $p_{ij} (j \neq i)$, it dies with probability w_i , or it produces two offspring of type i with probability α_i . Clearly, $\sum_{j \neq i} p_{ij} + w_i + \alpha_i = 1$.

We would like to study the factorial moment density functions as before, but now it is necessary to designate a particle by both its type and its location. Thus, the ordered pair (a, i) indicates a particle of type i at $a \in D$.

4.1 INITIALLY ONE PARTICLE IN D .

In this section we consider the following density functions for the multi-type process:

$$\hat{P}_k(t, (a, i), (b_1, j_1), \dots, (b_k, j_k)) db_1 \dots db_k$$

= P [at t there are particles of type $j_1, \dots, j_k$ in $db_1, \dots, db_k$, respectively, given that initially there was exactly one particle of type i in D , located at point a].

Theorem 1.

$$\hat{p}_1(t, (a, i), (b, j)) = c_{ij}(t) p(t, a, b),$$

where $c_{ij}(t)$ is the ij^{th} component of the matrix

$$c(t) = \begin{bmatrix} c_{11}(t) & \dots & c_{1k}(t) \\ \vdots & & \vdots \\ c_{k1}(t) & \dots & c_{kk}(t) \end{bmatrix}$$

and $c(t)$ satisfies the following system of differential equations:

$$\frac{\partial}{\partial t} c(t) = M c(t), \quad c(0) = I, \quad \text{with}$$

$$M = \begin{bmatrix} -m_1 & v_{1p12} & \dots & v_{1p1k} \\ v_{2p21} & -m_2 & \dots & v_{2p2k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{kpkl} & v_{kp k2} & \dots & -m_k \end{bmatrix}$$

$$\text{and } m_i = v_i(1 - 2\alpha_i),$$

$$i = 1, \dots, k.$$

Proof.

We consider the time and place of the first branch of the type i particle. If $i \neq j$, it must branch (or mutate) at some time $s < t$. If the result of the branch is two particles of type i , then one of the two will be the ancestor of the particle in db at t . If the result of the branch is a mutation, say to type h , then the type h particle is the ancestor of the particle in db . However, if $i = j$, then there may never be a branch before time t , with the original type i particle simply migrating from a to db in time t . With these considerations, we arrive at the following equation:

$$\begin{aligned}
 & \hat{P}_1(t, (a, i), (db, j)) \\
 &= \hat{P}_1(t, (a, i), (b, j)) db \\
 &= \delta_{ij} e^{-V_i t} p(t, a, b) db + \left[\int_0^t \int_{OD} p(s, a, y) e^{-V_i s} V_i \right. \\
 & \quad \left. \times (2 \alpha_{i1} \hat{P}_1(t-s, (y, i), (b, j)) db + \sum_{h \neq i} p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) db) dy ds \right]. \quad (1)
 \end{aligned}$$

By direct substitution, it will be shown that

$$\hat{P}_1(t, (a, i), (b, j)) = e^{-m_i t} p(t, a, b) \delta_{ij} \quad (2)$$

$$+ \sum_{h \neq i} \int_0^t \int_{OD} V_i e^{-m_i(t-s)} p(t-s, a, y) \hat{P}_1(s, (y, h), (b, j)) dy ds,$$

where $m_i = V_i(1-2\alpha_i)$.

Substituting (2) in the right-hand side of (1), we obtain

$$\delta_{ij} p(t, a, b) e^{-V_i t} + \int_0^t \int_{OD} p(s, a, y) e^{-V_i s} V_i [2\alpha_i \hat{P}_1(t-s, (y, i), (b, j))$$

$$+ \sum_{h \neq i}^k P_{ih} \hat{P}_1(t-s, (y, h), (b, j))] dy ds$$

$$= \delta_{ij} p(t, a, b) e^{-V_i t}$$

$$+ \int_0^t \int_{OD} p(s, a, y) e^{-V_i s} V_i 2\alpha_i e^{-V_i(1-2\alpha_i)(t-s)} p(t-s, y, b) \delta_{ij} dy ds$$

$$+ \int_0^t \int_{OD} p(s, a, y) e^{-V_i s} V_i 2\alpha_i \left(\sum_{h \neq i}^k \int_0^{t-s} \int_{OD} V_i e^{-m_i(t-s-u)} p(t-s-u, y, z) \right.$$

$$\times P_{ih} \hat{P}_1(u, (z, h), (b, j)) dz du dy ds$$

$$+ \int_0^t \int_{OD} p(s, a, y) e^{-V_1 s} \left(\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right) dy ds$$

$$= \delta_{ij} p(t, a, b) e^{-V_1 t}$$

$$+ \delta_{ij} \int_0^t 2 \alpha_{11} V_1 e^{-V_1 (s+t-s-2 \alpha_1 t + 2 \alpha_1 s)} \int_D p(s, a, y) p(t-s, y, b) dy ds$$

$$+ \int_0^{t-t-s} 2 \alpha_{11} V_1^2 e^{-V_1 (s+t-s-u-2 \alpha_1 t + 2 \alpha_1 s + 2 \alpha_1 u)} \left(\sum_{h \neq i}^k p_{ih} \int_D \hat{P}_1(u, (z, h), (b, j)) \right.$$

$$\left. \times \int_D p(s, a, y) p(t-s-u, y, z) dy dz \right) du ds$$

$$+ \int_0^t \int_{OD} p(s, a, y) e^{-V_1 s} \left(\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right) dy ds$$

$$= \delta_{ij} p(t, a, b) e^{-V_1 t}$$

$$+ \delta_{ij} \int_0^t 2 \alpha_{11} V_1 e^{-m t - 2 \alpha_{11} V_1 s} p(t, a, b) ds$$

$$+ \int_0^t 2 \alpha_{11} V_1^2 \int_0^{t-s} e^{-m t - 2 \alpha_{11} V_1 s - m u} \left(\sum_{h \neq i}^k p_{ih} \int_D \hat{P}_1(u, (z, h), (b, j)) \right.$$

$$\left. \times p(t-u, a, z) dz \right) du ds$$

$$+ \int\limits_{OD}^t p(s, a, y) e^{-V_1 s} V_1 \left(\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right) dy ds$$

(where $m_i = V_1(1-2\alpha_i)$)

$$= \delta_{ij} p(t, a, b) \left(e^{-V_1 t} + 2\alpha_i V_1 e^{-m_i t} \int_0^t e^{-2\alpha_i V_1 s} ds \right)$$

$$+ \int\limits_{OD}^t p(t-u, a, z) 2\alpha_i V_1^2 e^{-m_i(t-u)} \int_0^{t-u} e^{-2\alpha_i V_1 s} ds$$

$$\times \left[\sum_{h \neq i}^k p_{ih} \hat{P}_1(u, (z, h), (b, j)) \right] dz du$$

$$+ \int\limits_{OD}^t V_1 e^{-V_1 s} p(s, a, y) \left[\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right] dy ds$$

$$= \delta_{ij} p(t, a, b) \left[e^{-V_1 t} + e^{-V_1(1-2\alpha_i)t} (1 - e^{-2\alpha_i V_1 t}) \right]$$

$$+ \int\limits_{OD}^t p(t-u, a, z) e^{-V_1(1-2\alpha_i)(t-u)} V_1 (1 - e^{-2\alpha_i V_1(t-u)})$$

$$\times \left[\sum_{h \neq i}^k p_{ih} \hat{P}_1(u, (z, h), (b, j)) \right] dz du$$

$$+ \int\limits_{OD}^t V_1 e^{-V_1 s} p(s, a, y) \left[\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right] dy ds$$

$$\begin{aligned}
 &= \delta_{ij} p(t, a, b) e^{-V_1 (1-2\alpha_1)t} \\
 &+ \int_0^t p(s, a, y) \left[\sum_{h \neq i}^k p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) \right] V_1 \times [e^{-V_1 s} + e^{-V_1 (1-2\alpha_1)s} - e^{-V_1 s}] dy ds \\
 &= \delta_{ij} p(t, a, b) e^{-m_1 t} + \sum_{h \neq i}^k \int_0^t p(s, a, y) V_1 e^{-m_1 s} p_{ih} \hat{P}_1(t-s, (y, h), (b, j)) dy ds.
 \end{aligned}$$

This shows that (2) does indeed satisfy (1).

By differentiating (2), we will obtain a system of partial differential equations for $\hat{P}_1(t, (a, i), (b, j))$. For notational convenience, let

$$\hat{P}_1(t, (a, i), (b, j)) = \hat{P}_{ij}(t, a, b).$$

∴ from (2),

$$\begin{aligned}
 &\frac{\partial}{\partial t} \hat{P}_{ij}(t, a, b) \\
 &= \frac{\partial}{\partial t} [e^{-m_1 t} p(t, a, b) \delta_{ij} + \sum_{h \neq i}^k \int_0^t V_1 e^{-m_1 (t-s)} p(t-s, a, y) p_{ih} \hat{P}_{hj}(s, y, b) dy ds] \\
 &= -m_1 e^{-m_1 t} p(t, a, b) + e^{-m_1 t} \frac{\partial}{\partial t} p(t, a, b) \\
 &\quad - \sum_{h \neq i}^k \int_0^t V_1 m_1 e^{-m_1 (t-s)} p(t-s, a, y) p_{ih} \hat{P}_{hj}(s, y, b) dy ds
 \end{aligned}$$

$$+ \sum_{h \neq i}^k \int_0^t \int_D v_i e^{-m_i(t-s)} \frac{\partial}{\partial t} p(t-s, a, y) p_{ih} \hat{p}_{hj}(s, y, b) dy ds$$

$$+ \sum_{h \neq i}^k \int_D v_i p(0, a, y) p_{ih} \hat{p}_{hj}(t, y, b) dy .$$

Note that $p(0, a, y) = \begin{cases} 1 & \text{if } y = a \\ 0 & \text{if } y \neq a . \end{cases}$

Recall that if $T_t f(a) = \int_D p(t, a, b) f(b) db$ forms a semigroup, and

if A is the infinitesimal generator of T_t , $\frac{\partial}{\partial t} p(t, a, b) = A_a p(t, a, b)$.

Then $\frac{\partial}{\partial t} \hat{p}_{ij}(t, a, b)$

$$= -m_i (e^{-m_i t} p(t, a, b) \delta_{ij}$$

$$+ \sum_{h \neq i}^k \int_0^t \int_D v_i e^{-m_i(t-s)} p(t-s, a, y) p_{ih} \hat{p}_{hj}(s, y, b) dy ds]$$

$$+ A_a [e^{-m_i t} p(t, a, b) \delta_{ij} + \sum_{h \neq i}^k \int_0^t \int_D v_i e^{-m_i(t-s)} p(t-s, a, y) p_{ih} \hat{p}_{hj}(s, y, b) dy ds]$$

$$+ \sum_{h \neq i}^k v_i p_{ih} \hat{p}_{hj}(t, a, b)$$

$$= -m_i \hat{p}_{ij}(t, a, b) + A_a \hat{p}_{ij}(t, a, b) + \sum_{h \neq i}^k v_i p_{ih} \hat{p}_{hj}(t, a, b) . \quad (3)$$

$\hat{p}_{1j}$ has the initial condition $\hat{p}_{1j}(0, a, b) = \delta_a(b) \delta_1(j)$.

In order to obtain a solution for (3), we return to the integral equation (2).

Let $\hat{p}_j(t, a, b) = \begin{bmatrix} \hat{p}_{1j}(t, a, b) \\ \vdots \\ \hat{p}_{kj}(t, a, b) \end{bmatrix}$, and $e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ jth position.

Let $K(t) = \begin{bmatrix} 0 & e^{-m_1 t} v_{1p_{12}} & \dots & e^{-m_1 t} v_{1p_{1k}} \\ e^{-m_2 t} v_{2p_{21}} & 0 & \dots & e^{-m_2 t} v_{2p_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-m_k t} v_{kp_{k1}} & e^{-m_k t} v_{kp_{k2}} & \dots & 0 \end{bmatrix}$

Using this notation, we write (2) in matrix form:

$$\hat{p}_j(t, a, b) = e_j p(t, a, b) e^{-m_j t} + \int_0^t p(t-s, a, y) K(t-s) \hat{p}_j(s, y, b) dy ds$$

Setting $\hat{\tilde{p}}_j^{(0)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and recursively defining

$$\hat{\tilde{p}}_j^{(n+1)}(t, a, b) = e_{\tilde{j}} p(t, a, b) e^{-m_j t} + \int_0^t \int_{OD} p(t-s, a, y) K(t-s) \hat{\tilde{p}}_j^{(n)}(s, y, b) dy ds,$$

we see that the minimal solution $\hat{\tilde{p}}_j = \lim_{n \rightarrow \infty} \hat{\tilde{p}}_j^{(n)}$ will have the form

$$p(t, a, b) \tilde{c}_j(t), \text{ where } \tilde{c}_j(t) = \begin{bmatrix} c_{1j}(t) \\ \vdots \\ c_{kj}(t) \end{bmatrix}. \text{ This follows because for}$$

each n , if $\hat{\tilde{p}}_j^{(n)} = p(t, a, b) \tilde{c}_j^{(n)}(t)$, then

$$\begin{aligned} \hat{\tilde{p}}_j^{(n+1)} &= e_{\tilde{j}} p(t, a, b) e^{-m_j t} + \int_0^t \int_{OD} p(t-s, a, y) K(t-s) p(s, y, b) \tilde{c}_j^{(n)}(s) dy ds \\ &= e_{\tilde{j}} p(t, a, b) e^{-m_j t} + p(t, a, b) \int_0^t K(t-s) \tilde{c}_j^{(n)}(s) ds \\ &= p(t, a, b) \tilde{c}_j^{(n+1)}(t), \text{ where } \tilde{c}_j^{(n+1)}(t) = e^{-m_j t} e_{\tilde{j}} + \int_0^t K(t-s) \tilde{c}_j^{(n)}(s) ds. \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \hat{\tilde{p}}_j^{(n)}(t, a, b) = p(t, a, b) \lim_{n \rightarrow \infty} \tilde{c}_j^{(n+1)}(t)$$

$$= p(t, a, b) \tilde{c}_j(t)$$

$$= \hat{\tilde{p}}_j(t, a, b).$$

Returning to the differential equation (3), substitute

$$\hat{p}_{ij}(t, a, b) = p(t, a, b) c_{ij}(t).$$

$$\text{Thus, } \frac{\partial}{\partial t} c_{ij}(t) p(t, a, b) + c_{ij}(t) \frac{\partial}{\partial t} p(t, a, b)$$

$$= -m_i c_{ij}(t) p(t, a, b) + c_{ij}(t) A_a p(t, a, b)$$

$$+ \sum_{h \neq i}^k v_{ih} p_{ih} c_{hj}(t) p(t, a, b).$$

But since $\frac{\partial}{\partial t} p(t, a, b) = A_a p(t, a, b)$, and $p(0, a, b) = \delta_a(b)$, $c_{ij}(t)$ satisfies

$$\frac{\partial}{\partial t} c_{ij}(t) = -m_i c_{ij}(t) + \sum_{h \neq i}^k v_{ih} p_{ih} c_{hj}(t) \quad (4)$$

with initial condition $c_{ij}(0) = \delta_i(j)$.

In matrix notation, if

$$c(t) = [c_1(t), \dots, c_k(t)] = \begin{bmatrix} c_{11}(t) & \dots & c_{1k}(t) \\ \vdots & & \vdots \\ c_{k1}(t) & \dots & c_{kk}(t) \end{bmatrix},$$

then from (4),

$$\frac{\partial}{\partial t} C(t) = MC(t), \quad C(0) = I,$$

where $M =$

$$\begin{bmatrix} -m_1 & v_{1p_{12}} & \dots & v_{1p_{1k}} \\ v_{2p_{21}} & -m_2 & \dots & v_{2p_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ v_{kp_{k1}} & v_{kp_{k2}} & \dots & -m_k \end{bmatrix} \quad (5)$$

By Bellman and Cooke ([4], theorem 2.1), (5) has a unique solution, namely $C(t) = \exp(Mt) = \sum_{r=0}^{\infty} \frac{t^r M^r}{r!}$.
q.e.d.

It should be pointed out here that this solution $\hat{P}_1(t, (a, i), (b, j)) = C_{ij}(t)p(t, a, b)$ seems intuitively reasonable, as it can easily be seen that the matrix M is the infinitesimal generator of the mean semigroup of the multitype branching process alone, ignoring diffusion. (See, for example Athreya-Ney [2], page 202.) Therefore, $C(t)$ is the mean matrix of the branching process alone.

We now consider higher order factorial moment density functions:

$$\begin{aligned} & \hat{P}_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n)) db_1 \dots db_n \\ &= P \{ \text{at } t \text{ there are particles of type } j_1, \dots, j_n \} \end{aligned}$$

in $db_1, \dots, db_n$ respectively, given that at $t = 0$ there was one particle of type i at $a \in D$.

As in § 3, Theorem 1, we make use of the function

$$\begin{aligned} f_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n)) \\ = \frac{1}{2} \sum_{k=1}^n \sum_{\rho \in S_k} \hat{P}_k(t, (a, i), (b_{\rho(1)}, j_{\rho(1)}), \dots, (b_{\rho(k)}, j_{\rho(k)})) \\ \times \hat{P}_{n-k}(t, (a, i), (b_{\rho(k+1)}, j_{\rho(k+1)}), \dots, (b_{\rho(n)}, j_{\rho(n)})) \end{aligned}$$

where $\rho \in S_k$ indicates summation over the $\binom{n}{k}$ possible combinations of $(b_1, j_1), \dots, (b_n, j_n)$ into two subsets of size k and $n-k$.

Theorem 2. For $n \geq 2$,

$$\begin{aligned} \hat{P}_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n)) \\ = \sum_{j=1}^k 2 \alpha_j v_j \int_0^t \int_D \hat{P}_1(t-s, (a, i), (c, j)) f_n(s, (c, j), (b_1, j_1), \dots, (b_n, j_n)) dc ds. \end{aligned} \quad (1)$$

Proof.

It is necessary to consider the time and location of the first branch, and the type of particle produced by that branch. If the particle mutates

to a different type j at time s , at $y \in D$, then the mutant must be the common ancestor of the particles at $b_1, \dots, b_n$ at time t . This occurs with probability

$$\sum_{h \neq i}^k \hat{P}_n(t-s, (y, h), (b_1, j_1), \dots, (b_n, j_n)) .$$

If the original particle splits at time s at $y \in D$, either we have that one of the offspring is the common ancestor of all n particles at $b_1, \dots, b_n$, or that both offspring are ancestors. The former occurs with probability $2 \times \hat{P}_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n))$, and the latter with probability $2 \times f_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n))$. From these considerations we obtain the following:

$$\begin{aligned} & \hat{P}_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n)) \\ &= \iint_{QD} v_i e^{-V_i s} p(s, a, y) \left\{ \sum_{h \neq i}^k p_{ih} \hat{P}_n(t-s, (y, h), (b_1, j_1), \dots, (b_n, j_n)) \right. \\ & \quad \left. + 2 \alpha_i (\hat{P}_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n)) \right. \\ & \quad \left. + f_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n))) \right\} dy ds . \end{aligned} \tag{2}$$

We will show by direct substitution that (1) is the solution of (2)

$$\hat{P}_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n))$$

$$= \int\limits_{OD}^t v_i e^{-v_i s} p(s, a, y) \left[\sum_{h \neq i}^k p_{ih} \hat{P}_n(t-s, (y, h), (b_1, j_1), \dots, (b_n, j_n)) \right.$$

$$+ 2\alpha_i (\hat{P}_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n))$$

$$+ f_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n))) dy ds$$

$$= \int\limits_{OD}^t v_i e^{-v_i s} p(s, a, y) \left[\sum_{h \neq i}^k p_{ih} \left(\sum_{j=1}^k 2\alpha_j v_j \int\limits_{OD}^{t-s} \hat{P}_1(t-s-u, (y, h), (c, j)) \right. \right.$$

$$\times f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) dc du dy ds$$

$$+ \int\limits_{OD}^t v_i e^{-v_i s} p(s, a, y) 2\alpha_i \times \left[\sum_{j=1}^k 2\alpha_j v_j \int\limits_{OD}^{t-s} \hat{P}_1(t-s-u, (y, i), (c, j)) \right.$$

$$\times f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) dc du dy ds$$

$$+ \int\limits_{OD}^t v_i e^{-v_i s} p(s, a, y) 2\alpha_i f_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n)) dy ds$$

$$\begin{aligned}
 &= \sum_{h \neq i}^k p_{ih} \sum_{j=1}^k 2 \alpha_j V_j \iiint_{ODO}^{t \ t-s} V_i e^{-V_i s} C_{hj}(t-s-u) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) \\
 &\quad \times \left(\int_D p(s, a, y) p(t-s-u, y, c) dy \right) du dc ds \\
 &+ 2 \alpha_i V_i \iiint_{ODO}^{t \ t-s} e^{-V_i s} \sum_{j=1}^k 2 \alpha_j V_j C_{ij}(t-s-u) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) \\
 &\quad \times \left(\int_D p(s, a, y) p(t-s-u, y, c) dy \right) du dc ds \\
 &+ 2 \alpha_i V_i \iint_{OD}^t e^{-V_i s} p(s, a, y) f_n(t-s, (y, i), (b_1, j_1), \dots, (b_n, j_n)) dy ds.
 \end{aligned}$$

The expression above was obtained by making use of Theorem 1 of this section where it was proven that $\hat{P}_1(t, (a, i), (b, j)) = C_{ij}(t) p(t, a, b)$.

$$\therefore \hat{P}_n(t, (a, i), (b_1, j_1), \dots, (b_n, j_n))$$

$$\begin{aligned}
 &= \sum_{j=1}^k 2 \alpha_j V_j \iint_{OD}^t p(t-u, a, c) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) \\
 &\quad \times \int_0^{t-u} \sum_{h \neq i}^k p_{ih} V_i e^{-V_i s} C_{hj}(t-s-u) ds dc du \\
 &+ \sum_{j=1}^k 2 \alpha_j V_j \iint_{OD}^t p(t-u, a, c) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) \\
 &\quad \times \int_0^{t-u} 2 \alpha_i V_i e^{-V_i s} C_{ij}(t-s-u) ds dc du
 \end{aligned}$$

$$+ \iint_{OD}^t 2\alpha_1 v_1 e^{-v_1(t-u)} p(t-u, a, y) f_n(u, (y, i), (b_1, j_1), \dots, (b_n, j_n)) dy ds$$

$$= \iint_{OD}^t 2\alpha_1 v_1 e^{-v_1(t-u)} p(t-u, a, y) f_n(u, (y, i), (b_1, j_1), \dots, (b_n, j_n)) dy ds$$

$$+ \sum_{j=1}^k 2\alpha_j v_j \iint_{OD}^t p(t-u, a, c) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n))$$

(3)

$$\times \left(\int_0^{t-u} e^{-v_1 s} (2\alpha_1 v_1 C_{1j}(t-s-u) + \sum_{h \neq i}^k p_{ih} v_i C_{hj}(t-s-u)) ds \right) dc du.$$

But $m_1 = v_1(1-2\alpha_1)$, so $2\alpha_1 v_1 C_{1j}(t-s-u) = -m_1 C_{1j}(t-s-u) + v_1 C_{1j}(t-s-u)$.

$$\therefore \int_0^{t-u} e^{-v_1 s} (2\alpha_1 v_1 C_{1j}(t-s-u) + \sum_{h \neq i}^k p_{ih} v_i C_{hj}(t-s-u)) ds$$

$$= \int_0^{t-u} e^{-v_1 s} (-m_1 C_{1j}(t-s-u) + v_1 C_{1j}(t-s-u) + \sum_{h \neq i}^k p_{ih} v_i C_{hj}(t-s-u)) ds$$

$$= \int_0^{t-u} e^{-v_1 s} \left(-\frac{\partial}{\partial s} C_{1j}(t-s-u) + v_1 C_{1j}(t-s-u) \right) ds$$

(from (4) in the proof of Theorem 1 of this section)

$$= C_{1j}(t-u) e^{-v_1(t-u)} \delta_1(j) \quad (\text{since } C_{1j}(0) = \delta_1(j)).$$

$$\text{Hence, (3) } = \int_0^t \int_0^t 2 \alpha_i v_i e^{-V_i(t-s)} p(t-u, a, y) f_n(u, (y, i), (b_1, j_1), \dots, (b_n, j_n)) dy ds$$

$$+ \sum_{j=1}^k 2 \alpha_j v_j \int_0^t \int_0^t p(t-u, a, c) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) C_{ij}(t-u) dc du$$

$$- 2 \alpha_i v_i \int_0^t \int_0^t p(t-u, a, c) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) e^{-V_i(t-u)} dc du$$

$$= \sum_{j=1}^k 2 \alpha_j v_j \int_0^t \int_0^t \hat{P}_1(t-u, (a, i), (c, j)) f_n(u, (c, j), (b_1, j_1), \dots, (b_n, j_n)) dc du$$

So by substitution, we have proven the theorem.

q.e.d.

4.2 BEHAVIOUR OF THE FIRST AND SECOND FACTORIAL MOMENT DENSITY FUNCTIONS.

In the previous section, we have proven that

$$\hat{P}_1(t, (a, i), (b, j)) = C_{ij}(t) p(t, a, b), \text{ and}$$

$$\hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2))$$

$$= \sum_{j=1}^k 2 \alpha_j v_j \int_0^t \int_0^t \hat{P}_1(t-s, (a, i), (c, j)) \hat{P}_1(s, (c, j), (b_1, j_1)) \hat{P}_1(s, (c, j), (b_2, j_2)) dc ds$$

$$= \sum_{j=1}^k 2 \alpha_j v_j \int_0^t C_{ij}(t-s) C_{j11}(s) C_{j22}(s) \int_D p(t-s, a, c) p(s, c, b_1) p(s, c, b_2) dc ds$$

We would like to consider the behaviour of $\hat{P}_1$ and $\hat{P}_2$ as $t \rightarrow \infty$.

As mentioned previously, $C_{ij}(t)$ is the ij th component of the matrix $C(t)$, which is the mean matrix of the multitype branching process without diffusion.

It is assumed that there exists t_0 , $0 < t_0 < \infty$, such that $C_{ij}(t_0) > 0$ for all i, j . (In other words, the multitype process is positive regular.)

We can then deduce the existence of a real eigenvalue λ of M (where M is the infinitesimal generator of $C(t)$ - see Theorem 1 of § 4.1) such that, if $\lambda_1, \dots, \lambda_p$ are the other eigenvalues of M , then

$$\lambda > \operatorname{Re} \lambda_1 \geq \dots \geq \operatorname{Re} \lambda_p.$$

(Athreya-Ney [2], pg. 202).

The multitype process is said to be supercritical if $\lambda > 0$, critical if $\lambda = 0$, and subcritical if $\lambda < 0$. The multitype process with $k = 1$ is

the single-type process, and it is clear that the only eigenvalue is

$-m$, ($m = V(1-2\alpha)$). Thus the supercritical case occurs if $-m > 0$ ($m < 0$),

the critical occurs if $m = 0$, and the subcritical occurs if $-m < 0$ ($m > 0$).

As a result of the Frobenius Theorem (Athreya-Ney [2], Theorem 1, pg. 185),

we can find left and right eigenvectors $\underline{v}$ (with j th component v_j),

and $\underline{u}$ (with i th component u_i) of λ with all coordinates strictly

positive, and with $\underline{u} \cdot \underline{v} = 1$, $\underline{u} \cdot \underline{1} = 1$. We also have from the Frobenius

Theorem that

$$\lim_{t \rightarrow \infty} C(t)e^{-\lambda t} = P = ((u_i v_j))$$

We assume that $D = \mathbb{R}^d$ and that the migration process is Brownian motion.

✓ We have that

$$\begin{aligned} \lim_{t \rightarrow \infty} \hat{P}_1(t, (a, i), (b, j)) \\ &= \lim_{t \rightarrow \infty} C_{ij}(t) p(t, a, b) \\ &= \lim_{t \rightarrow \infty} u_i v_j e^{\lambda t} \frac{1}{(2\pi t)^{d/2}} e^{-(|a-b|^2)/2t} \quad (\text{using the Frobenius Theorem}) \\ &= \frac{u_i v_j}{(2\pi)^{d/2}} \lim_{t \rightarrow \infty} \frac{e^{\lambda t}}{t^{d/2}} \end{aligned}$$

Subcritical branching: $(\lambda < 0)$

If $\lambda < 0$, then $\hat{P}_1(t, (a, i), (b, j)) \rightarrow 0$ as $t \rightarrow \infty$.

Critical branching: $(\lambda = 0)$

If $\lambda = 0$, then $\hat{P}_1(t, (a, i), (b, j)) \rightarrow 0$ as $t \rightarrow \infty$.

Supercritical branching: $(\lambda > 0)$

If $\lambda > 0$, then $\hat{P}_1(t, (a, i), (b, j)) \rightarrow \infty$ as $t \rightarrow \infty$.

$$b) \hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2))$$

$$= \sum_{j=1}^k 2 \alpha_j V_j \int_0^t C_{ij}(t-s) C_{jj_1}(s) C_{jj_2}(s) \int_D \frac{1}{(2\pi(t-s))^{d/2}} \frac{1}{(2\pi s)^d}$$

$$e^{-|a-c|^2/2(t-s)} e^{-|c-b_1|^2/2s} e^{-|c-b_2|^2/2s} dc ds$$

$$= \sum_{j=1}^k 2 \alpha_j V_j \int_0^t C_{ij}(t-s) C_{jj_1}(s) C_{jj_2}(s) \exp\left(-\left|a - \frac{b_1+b_2}{2}\right|^2/(2t-s)\right) \times \frac{1}{(\pi(2t-s))^{d/2}} \\ \times \frac{\exp(-|b_1-b_2|^2/4s)}{(4\pi s)^{d/2}} ds$$

$$= \sum_{j=1}^k 2 \alpha_j V_j \int_0^t C_{ij}(t-s) C_{jj_1}(s) C_{jj_2}(s) \frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}} ds,$$

$$\text{where } K_1 = \left|a - \frac{b_1+b_2}{2}\right|^2,$$

$$\text{and } K_2 = |b_1-b_2|^2.$$

$$\text{Let } F_j(s) = C_{ij}(t-s) C_{jj_1}(s) C_{jj_2}(s) \frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}}.$$

$$\text{Thus, } \hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2))$$

$$= \sum_{j=1}^k 2 \alpha_j V_j \int_0^t F_j(s) ds,$$

(1)

and so we must consider the behaviour of

$$\int_0^t F_j(s) ds \text{ as } t \rightarrow \infty.$$

Recall that $\lim_{t \rightarrow \infty} C_{ij}(t)e^{-\lambda t} = u_i v_j$.

Subcritical branching: ($\lambda < 0$)

Since $\lim_{t \rightarrow \infty} C_{ij}(t)e^{-\lambda t} = u_i v_j$, $C_{ij}(t)e^{-\lambda t}$ is bounded above by some constant C for every value of t . Similarly, $C_{jj_1}(t)e^{-\lambda t} < C_1$ and

$$C_{jj_2}(t)e^{-\lambda t} < C_2. \text{ Also, } \frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \times \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}}$$

is bounded above by some constant K .

$$\text{Thus, } e^{-\lambda t} \int_0^t F_j(s) ds$$

$$= \int_0^t e^{-\lambda(t-s)} C_{ij}(t-s)e^{-\lambda s} C_{jj_1}(s)e^{-\lambda s} C_{jj_2}(s)e^{\lambda s}$$

$$\times \frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \times \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}} ds$$

$$\leq \int_0^t C C_1 C_2 K e^{\lambda s} ds.$$

$$\therefore \text{ as } t \rightarrow \infty, e^{-\lambda t} \int_0^t F_j(s) ds \leq \int_0^\infty CC_1 C_2 K e^{\lambda s} ds$$

$$= \frac{CC_1 C_2 K}{-\lambda}$$

So by (1), we see that

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2))$$

$$\leq \sum_{j=1}^k 2 \alpha_{jj} v_j \frac{CC_1 C_2 K}{-\lambda}$$

Since $\lambda < 0$, this shows that $\hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2))$

decays exponentially quickly as $t \rightarrow \infty$.

Critical branching: ($\lambda = 0$)

Since $\lim_{t \rightarrow \infty} C_{ij}(t) = u_i v_j$, $C_{ij}(t)$ and similarly $C_{jj_1}(t)$ and $C_{jj_2}(t)$

are bounded above. There exists some constant C such that

$$C_{ij}(t-s) C_{jj_1}(s) C_{jj_2}(s) < C, \text{ for every } t, \text{ and } s < t.$$

Also, since $(2t-s) > t$ for $s < t$, then

$$\frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \leq \frac{1}{(\pi t)^{d/2}}.$$

$$\text{Therefore, } \int_0^t F_j(s) ds \leq \int_0^t \frac{C}{(\pi t)^{d/2}} \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}} ds.$$

By Bateman [3], page 146,

$$\int_0^t F_j(s) ds \leq \begin{cases} \frac{C}{(\pi t)^{1/2}} \left(\frac{t}{\pi}\right)^{1/2} + O\left(\frac{1}{t^{3/2}}\right) & d=1 \\ \frac{C}{\pi t} \frac{\log t}{4\pi} + O\left(\frac{1}{t}\right) & d=2 \\ \frac{C}{(\pi t)^{3/2}} \frac{1}{4\pi(K_2)^{1/2}} + O\left(\frac{1}{t^{5/2}}\right) & d=3 \end{cases}$$

From (1), we see that $\hat{P}_2(t, (a, 1), (b_1, j_1), (b_2, j_2))$ remains bounded for $d=1$, and converges to 0 for $d=2$ and $d=3$.

Supercritical branching: ($\lambda > 0$)

$$e^{-\lambda t} \int_0^t F_j(s) ds = \int_0^t e^{-\lambda(t-s)} C_{jj_1}(t-s) e^{-\lambda s} C_{jj_1}(s) e^{-\lambda s} C_{jj_2}(s) e^{\lambda s} \frac{e^{-K_1/(2t-s)}}{(\pi(2t-s))^{d/2}} \frac{e^{-K_2/4s}}{(4\pi s)^{d/2}} ds.$$

Since we have that $s < t < 2t-s < 2t$, for $0 < s < t$,

then

$$e^{-\lambda t} \int_0^t F_j(s) ds > \frac{1}{(2\pi t)^{d/2}} \frac{1}{(4\pi t)^{d/2}} \int_0^t e^{-\lambda s} C_{jj_1}(s) e^{-\lambda s} C_{jj_2}(s) e^{-\lambda(t-s)} C_{ij}(t-s) e^{\lambda s} e^{-K/s} ds$$

where $K = K_1 + K_2$

$$\text{Let } G(s) = e^{-\lambda s} C_{jj_1}(s) e^{-\lambda s} C_{jj_2}(s) e^{-\lambda(t-s)} C_{ij}(t-s) e^{\lambda s} e^{-K/s}$$

$$\therefore \lim_{t \rightarrow \infty} e^{-\lambda t} \int_0^t F_j(s) ds \geq \lim_{t \rightarrow \infty} \frac{1}{(2\pi)^{d/2}} \frac{1}{(2t)^d} \int_0^t G(s) ds$$

Making the change of variable $s = t-T$ and defining

$$G_n(T) = \begin{cases} \frac{1}{(2n)^d} e^{-\lambda(n-T)} C_{jj_1}(n-T) C_{ij}(T) e^{-\lambda T} C_{jj_2}(n-T) e^{-\lambda(n-T)} \\ \quad \times e^{-\lambda(n-T)} e^{-K/(n-T)}; & \text{for } T \leq n \\ 0 & \text{for } T > n \end{cases}$$

we have that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} G_n(T) dT = \lim_{t \rightarrow \infty} \frac{1}{(2t)^d} \int_0^t G(T) dT .$$

By Fatou's Lemma,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_0^{\infty} G_n(T) dT &\geq \int_0^{\infty} \liminf_{n \rightarrow \infty} G_n(T) dT \\ &= \int_0^{\infty} u_{j_1 j_1}^v c_{ij}(T) e^{-\lambda T} u_{j_2 j_2}^v \lim_{n \rightarrow \infty} \frac{e^{\lambda(n-T)}}{(2n)^d} dT . \end{aligned}$$

= ∞ since $\lambda > 0$.

$$\begin{aligned} &\lim_{t \rightarrow \infty} e^{-\lambda t} \int_0^t F_j(s) ds \\ &\geq \frac{1}{(2\pi)^{d/2}} \lim_{t \rightarrow \infty} \frac{1}{(2t)^d} \int_0^t G(s) ds \\ &\geq \liminf_{n \rightarrow \infty} \int_0^{\infty} G_n(T) dT \times \frac{1}{(2\pi)^{d/2}} \end{aligned}$$

From (1), we have that $\hat{P}_2(t, (a, i), (b_1, j_1), (b_2, j_2)) \rightarrow \infty$ as $t \rightarrow \infty$

for supercritical branching.

These results are the same as those in the single-type case.

We designate by $N_A(t, j)$ the number of type j particles in $A \subseteq D$ at time t , given that there was initially one type i particle.

For bounded sets $A \subseteq D$, $E(N_A(t, j))$ dies off as $t \rightarrow \infty$ in the subcritical and critical cases, and explodes in the supercritical case.

If $A \cap B = \emptyset$, then $E(N_A(t, j_1), N_B(t, j_2))$ dies off in the subcritical case, and in the critical case for $d = 2$ and $d = 3$, and remains bounded in the critical case for $d = 1$. $E(N_A(t, j_1) N_B(t, j_2))$ explodes as $t \rightarrow \infty$ in the supercritical case.

5. THE BRANCHING DIFFUSION WITH IMMIGRATION

5.1 THE SINGLE-TYPE BRANCHING DIFFUSION WITH IMMIGRATION

This model is a simple extension of the single-type branching diffusion. There is only one type of particle, and we assume that initially the domain D is empty. Particles are introduced into D with immigration rate r . (This means that a particle appears in an element of volume dx in the time interval $(t, t+dt)$ with probability $rdxdt$.) Once the first immigrant has appeared in D , further particles are introduced through either immigration or birth. Once a particle is in D , it obeys the original branching diffusion process: its migration process is defined by the migration density $p(t, a, b)$, its branching rate is λ , and branching is local. If a particle branches, it produces two offspring with probability α , or it dies with probability $1-\alpha$. As before, $N_A(t)$ will be the random variable specifying the number of particles in a set $A \subseteq D$ at time t .

We will consider the n^{th} order factorial moment density function of $N_A(t)$ for the immigration process:

$$\hat{P}_n^I(t, b_1, \dots, b_n) db_1 \dots db_n$$

= Pr [at time t there are particles at $db_1, \dots, db_n$ respectively, regardless of the total number of particles in the system and their locations] at $t = 0$, there are no particles in the system].

Theorem 1.

$$\hat{P}_n^I(t, b_1, \dots, b_n) = \int_0^t \int_{OD} g_n(s, y, b_1, \dots, b_n) dy ds$$

where $g_n(s, y, b_1, \dots, b_n)$

$$= \sum_{k=1}^{n-1} \sum_{\rho \in S_k} \hat{P}_k(s, y, b_{\rho(1)}, \dots, b_{\rho(k)}) \hat{P}_{n-k}^I(s, b_{\rho(k+1)}, \dots, b_{\rho(n)}) + \hat{P}_n(s, y, b_1, \dots, b_n).$$

As in § 3.1, Theorem 1, $\sum_{\rho \in S_k}$ indicates summation over the $\binom{n}{k}$ possible

combinations of $b_1, \dots, b_n$ into two subsets of size k and $n-k$, respectively.

$\hat{P}_k(s, y, b_1, \dots, b_k)$ is the k^{th} order factorial moment density for the single-type process without immigration, as developed in § 3.1 Theorem 1.

Proof.

We begin by assuming that $|D| < \infty$.

We consider the time s and location of y of the first immigration.

The probability that a particle does not appear before time s is $e^{-r|D|s}$.

Given that a particle does immigrate, it appears in the set element dx

with probability $\frac{dx}{|D|}$.

We now have that the probability that the first immigrant appears in $dx \subseteq D$ in $(s, s+ds)$ is $e^{-r|D|s} \frac{r|D|}{|D|} ds dx$. This particle may be the common ancestor of $0, 1, \dots, n-1$, or n of the particles at $db_1, \dots, db_n$ at time t . Recalling that the probability of multiple points is 0, the first immigrant is the common ancestor of

0 with probability $\hat{P}_n^I(t-s, b_1, \dots, b_n) db_1 \dots db_n$

1 with probability $[\hat{P}_1(t-s, y, b_1) \hat{P}_{n-1}^I(t-s, b_2, \dots, b_n)$

$+ \hat{P}_1(t-s, y, b_2) \hat{P}_{n-1}^I(t-s, b_1, b_3, \dots, b_n)$

$+ \dots + \hat{P}_1(t-s, y, b_n) \hat{P}_{n-1}^I(t-s, b_1, \dots, b_{n-1})] db_1 \dots db_n$

$= [\sum_{\rho \in S_1} \hat{P}_1(t-s, y, b_{\rho(1)}) \hat{P}_{n-1}^I(t-s, b_{\rho(2)}, \dots, b_{\rho(n)})] db_1 \dots db_n$

$\vdots$
 $n-1$ with probability $\sum_{\rho \in S_{n-1}} \hat{P}_{n-1}(t-s, y, b_{\rho(1)}, \dots, b_{\rho(n-1)}) \hat{P}_1^I(t-s, b_{\rho(n)}) db_1 \dots db_n$

n with probability $\hat{P}_n(t-s, y, b_1, \dots, b_n) db_1 \dots db_n$. Define $g_n(s, y, b_1, \dots, b_n)$

as in the statement of the theorem, and we see that if $|D| < \infty$,

$$\hat{P}_n^I(t, b_1, \dots, b_n) = \int_{OD}^t e^{-r|D|s} r|D| [\hat{P}_n^I(t-s, y, b_1, \dots, b_n) + g_n(t-s, y, b_1, \dots, b_n)] \frac{dy ds}{|D|}$$

Applying Laplace transforms to both sides of this equation, we obtain

$$\begin{aligned} L_{\lambda} \hat{P}_n^I(\cdot, b_1, \dots, b_n) &= \int_{OD}^{\infty} \int_{OD}^{\infty} e^{-\lambda t} e^{-r|D|s} r|D| \\ &\quad \times [\hat{P}_n^I(t-s, b_1, \dots, b_n) + g_n(t-s, y, b_1, \dots, b_n)] \frac{dy ds dt}{|D|} \\ &= \int_{OD}^{\infty} \int_{OD}^{\infty} e^{-s(r|D|+\lambda)} r|D| e^{-\lambda(t-s)} \\ &\quad \times [\hat{P}_n^I(t-s, b_1, \dots, b_n) + g_n(t-s, y, b_1, \dots, b_n)] \frac{dy ds dt}{|D|} \end{aligned}$$

Let $u = t-s$

$s = s$

$$\begin{aligned} \therefore L_{\lambda} \hat{P}_n^I &= \int_{OD}^{\infty} \int_{OD}^{\infty} e^{-s(r|D|+\lambda)} r|D| \int_0^{\infty} e^{-\lambda u} [\hat{P}_n^I(u, b_1, \dots, b_n) + g_n(u, y, b_1, \dots, b_n)] \frac{du dy ds}{|D|} \\ &= \int_{OD}^{\infty} \int_{OD}^{\infty} e^{-s(r|D|+\lambda)} r|D| [L_{\lambda} \hat{P}_n^I(\cdot, b_1, \dots, b_n) + L_{\lambda} g_n(\cdot, y, b_1, \dots, b_n)] \frac{dy ds}{|D|} \end{aligned}$$

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$$= L_{\lambda} \hat{P}_n^I \frac{|D|}{|D|} \times \frac{r|D|}{r|D|+\lambda} + \int_D L_{\lambda} g_n(\cdot, y, b_1, \dots, b_n) \frac{dy}{|D|} \frac{r|D|}{r|D|+\lambda} \dots$$

$$\therefore (r|D|+\lambda) L_{\lambda} \hat{P}_n^I = r|D| L_{\lambda} \hat{P}_n^I + \int_D r L_{\lambda} g_n(\cdot, y, b_1, \dots, b_n) dy$$

$$\Rightarrow \lambda L_{\lambda} \hat{P}_n^I = \int_D r L_{\lambda} g_n(\cdot, y, b_1, \dots, b_n) dy.$$

Inverting Laplace transforms, we obtain $\hat{P}_n^I(t, b_1, \dots, b_n)$

$$= \int_0^t \int_D r g_n(s, y, b_1, \dots, b_n) dy ds.$$

This proves the theorem for $|D| < \infty$. If $|D| = \infty$, since D is σ -compact, we can choose a sequence of subsets $B_1, B_2, \dots$ of D such that $|B_k| < \infty \forall k$, and $B_k \uparrow D$. Define

$$p_k(t, a, b) = \begin{cases} p(t, a, b) & \text{if } a \in B_k \text{ and } b \in B_k \\ 0 & \text{otherwise.} \end{cases}$$

We now have that the theorem is true for all B_k and $p(t, a, b)$ restricted to B_k , and by letting $B_k \uparrow D$, we have the theorem for $D = R^d$ in general.

q.e.d.

For example, if $\int_D p(t, a, b) da = \int_D p(t, a, b) db = 1$,

$$\begin{aligned}\hat{P}_1^I(t, b) &= r \int_{OD}^t \int \hat{P}_1(s, y, b) dy ds \\ &= r \int_0^t e^{-ms} \int_D p(s, y, b) dy ds\end{aligned}$$

$$= \begin{cases} r \frac{(1-e^{-mt})}{m} & m \neq 0 \\ rt & m = 0 \end{cases}$$

$$\begin{aligned}\hat{P}_2^I(t, b_1, b_2) &= r \int_{OD}^t \int \hat{P}_1(s, y, b_1) \hat{P}_1^I(s, b_2) + \hat{P}_1(s, y, b_2) \hat{P}_1^I(s, b_1) dy ds \\ &\quad + r \int_{OD}^t \int \hat{P}_2(s, y, b_1, b_2) dy ds.\end{aligned}$$

If $m \neq 0$, $\hat{P}_2^I(t, b_1, b_2)$

$$= r \int_0^t e^{-ms} r \frac{(1-e^{-ms})}{m} \int_D (p(s, y, b_1) + p(s, y, b_2)) dy ds$$

$$+ r \int_{OD}^t \int_{OD}^s 2V \alpha \int e^{-mu} p(u, y, z) f_2(s-u, z, b_1, b_2) dz du dy ds$$

$$= r^2 \left(\frac{1-e^{-mt}}{m} \right)^2 + r \int_{OD}^t \int_{OD}^s 2V \alpha e^{-mu} e^{-2m(s-u)}$$

$$\times \int_D p(s-u, z, b_1) p(s-u, z, b_2) dz du ds.$$

$$\text{If } m = 0, \hat{P}_2^I(t, b_1, b_2) = r \int_0^t rs \int_D (p(s, y, b_1) + p(s, y, b_2)) dy ds$$

$$+ r \int_0^t \int_{OD} 2V \alpha \int_{OD} p(u, y, z) f_2(s-u, z, b_1, b_2) dz du dy ds$$

$$= r^2 t^2 + r \int_0^t \int_{OD} 2V \alpha \int_D p(s-u, z, b_1) p(s-u, z, b_2) dz du ds$$

If $D = R^d$ and the migration is Brownian motion, then considering the first two factorial moment density functions as $t \rightarrow \infty$, we have:

$$a) \hat{P}_1^I(t, b) = r M_t,$$

$$\text{where } M_t = \begin{cases} \frac{1-e^{-mt}}{m} & m \neq 0 \\ t & m = 0 \end{cases}$$

Subcritical branching: ($m > 0$)

$$\hat{P}_1^I(t, b) \rightarrow \frac{r}{m} \text{ as } t \rightarrow \infty.$$

Critical branching: ($m = 0$)

$$\hat{P}_1^I(t, b) \text{ grows as } t \rightarrow \infty.$$

Supercritical branching: ($m < 0$)

$\hat{P}_1^I(t, b)$ grows exponentially as $t \rightarrow \infty$.

$$b) \quad \hat{P}_2^I(t, b_1, b_2) = r M_t^2 + 2rV \alpha \int_0^{ts} \int_0^s e^{-ms} e^{-mu} p(2u, b_1, b_2) du ds.$$

Subcritical branching: ($m > 0$)

Since $p(2u, b_1, b_2)$ is bounded above some constant, say C , we have

$$\begin{aligned} \hat{P}_2^I(t, b, b_2) &\leq r^2 \left(\frac{1-e^{-mt}}{m} \right)^2 + 2rV \alpha C \int_0^{ts} \int_0^s e^{-ms} e^{-mu} du ds \\ &= r^2 \left(\frac{1-e^{-mt}}{m} \right)^2 + rV \alpha C \left(\frac{1-e^{-mt}}{m} \right)^2 \\ &\rightarrow \frac{r}{m^2} (r+V \alpha C). \end{aligned}$$

$\therefore \hat{P}_2^I(t, b_1, b_2)$ remains bounded if $m > 0$.

Critical branching: ($m = 0$)

Since $M_t^2 = t^2 \rightarrow \infty$, $\hat{P}_2^I(t, b_1, b_2) \rightarrow \infty$ if $m = 0$.

Supercritical branching: ($m < 0$)

$$M_t^2 = \left(\frac{1 - e^{-mt}}{m} \right)^2 \rightarrow \infty \text{ if } m < 0, \text{ so } \hat{P}_2^I(t, b_1, b_2) \rightarrow \infty.$$

These results indicate that for bounded sets A , as $t \rightarrow \infty$,

$E(N_A(t)) \rightarrow \frac{r|A|}{m}$ for subcritical branching, and explodes for critical and supercritical branching. For bounded sets $A, B \subseteq D$, $A \cap B = \emptyset$, $E(N_A(t)N_B(t))$ remains bounded in the subcritical case, and becomes unbounded in the critical and supercritical cases.

5.2 BEHAVIOUR OF $N_A(t)$ FOR LARGE A

Here we assume that $D = \mathbb{R}^d$ and that migration is d -dimensional Brownian motion. As in the non-immigration case, we will study the behaviour of the factorial moments of $N_A(t)$ for large compact symmetric sets $A \subseteq D$.

At this point, we recall from § 3.2, Theorems 2 and 3, that as $A \uparrow D$, A symmetric and compact,

$$\int_{A^n} \dots \int \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n$$

$$\begin{cases} n! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{n-1} e^{-mt} - o\left(\frac{1}{|A|^{n-1}}\right) & \text{- in the noncritical case} \\ & (m = V(1-2\alpha) \neq 0) \\ n! \left(\frac{Vt}{2} \right)^{n-1} - o\left(\frac{1}{|A|^{n-1}}\right) & \text{- in the critical case} \\ & (m = V(1-2\alpha) = 0) . \end{cases}$$

$n_1 > 0$ is arbitrary.

Define

$$\begin{aligned} \mu'_{[n]}(A) &= E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1))) \\ &= \int \dots \int_{A^n} \hat{P}_n^I(t, b_1, \dots, b_n) db_1 \dots db_n . \end{aligned}$$

Notice that for notational convenience, dependence of $\mu'_{[n]}(A)$ on t has been suppressed.

Theorem 2.

Assume that $D = R^d$ and that $p(t, a, b) = \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t)$.

Then for supercritical and subcritical branching ($m = V(1-2\alpha) \neq 0$),

as $A \uparrow D$, A symmetric and bounded,

$$\mu'_{[n]}(A) = \sum_{k=1}^n (-1)^{n-k} S_n^k (M_t r|A|)^k (M_t V \alpha)^{n-k} - o\left(\frac{1}{|A|^{n-1}}\right) \text{ where } n_1 > 0 \text{ is}$$

arbitrary, $M_t = \frac{1-e^{-mt}}{m}$, and the S_n^k 's are Stirling numbers of the first kind.

The convergence is uniform in t for subcritical branching.

Proof.

Recall from § 3.2 Theorem 2 that

$$\int_{A^n} \dots \int \hat{P}_n(t, a, b_1, \dots, b_n) db_1 \dots db_n \quad (1)$$

$$= n! (V \alpha M_t)^{n-1} e^{-mt} - o\left(\frac{1}{|A|^{n-1}}\right) \text{ for any } n_1 > 0, \text{ fixed } a \in D, \text{ and uniformly}$$

in t if $m > 0$, where $M_t = \frac{1-e^{-mt}}{m}$.

By § 5.1 Theorem 1

$$\begin{aligned} \mu_{[n]}(A) &= \int_{A^n} \dots \int_{OD}^t r g_n(s, y, b_1, \dots, b_n) db_1 \dots db_n \\ &= \int_{A^n} \dots \int_{OD}^t \sum_{k=1}^{n-1} \sum_{\rho \in S_k} r \int_{OD}^t \hat{P}_k(s, y, b_{\rho(1)}, \dots, b_{\rho(k)}) \hat{P}_{n-k}^I(s, b_{\rho(k+1)}, \dots, b_{\rho(n)}) dy ds db_1 \dots db_n \\ &\quad + \int_{A^n} \dots \int_{OD}^t \hat{P}_n(s, y, b_1, \dots, b_n) dy ds db_1 \dots db_n \\ &= r \sum_{k=1}^n \binom{n}{k} \int_{OD}^t \dots \int_{A^n} \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k \quad (2) \\ &\quad \times \int_{A^{n-k}} \dots \int \hat{P}_{n-k}^I(s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n ds \end{aligned}$$

From (2), it is clear that we need to know the behaviour as $A \uparrow D$ of

$$\begin{aligned}
 & \int_{A^k} \dots \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k \\
 &= \int_{A^k} \dots \int_{\text{ODD}} 2V \alpha \int_{\text{ODD}} e^{-\mu u} p(u, y, z) f_k(s-u, z, b_1, \dots, b_k) dz dy du db_1 \dots db_k \\
 &= 2V \alpha \int_0^s e^{-\mu u} \int_{A^k} \dots \int_D f_k(s-u, z, b_1, \dots, b_k) dz db_1 \dots db_k du \quad (3)
 \end{aligned}$$

((3) was obtained from § 3.1, Theorem 1).

So we now must consider $\int_{A^k} \dots \int_D f_k(s-u, z, b_1, \dots, b_k) dz db_1 \dots db_k$

for large symmetric sets A .

$$\begin{aligned}
 & \int_{A^k} \dots \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k \\
 &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \sum_{\rho \in S_j} \int_{A^k} \dots \int_D \hat{P}_j(t, z, b_{\rho(1)}, \dots, b_{\rho(j)}) \hat{P}_{k-j}(t, z, b_{\rho(j+1)}, \dots, b_{\rho(k)}) \right. \\
 & \quad \left. dz db_1 \dots db_k \right] \\
 &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \binom{k}{j} \int_D \dots \int_{A^j} \hat{P}_j(t, z, b_1, \dots, b_j) db_1 \dots db_j \right. \\
 & \quad \left. \times \int_{A^{k-j}} \dots \int \hat{P}_{k-j}(t, z, b_{j+1}, \dots, b_k) db_{j+1} \dots db_k dz \right].
 \end{aligned}$$

For any $\epsilon > 0$, choose a symmetric bounded set $D^1 \subseteq D$ such that for $s \leq t$, $\int_{D^1} p(s, z, b) dz \geq 1 - \epsilon$, for all $b \in A$. For $z \in D^1$, the error terms in (1) are uniform.

$$\begin{aligned}
 & \text{For } k \geq 2, \int_{A^k} \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k \\
 &= \frac{1}{2} \sum_{j=2}^{k-2} \binom{k}{j} \int_{D^1} \left[\int_{A^j} \int_{OD}^t 2V \alpha e^{-ms} p(s, z, b) f_j(t-s, b, b_1, \dots, b_j) db ds db_1 \dots db_j \right] \\
 & \times \left[\int_{A^{k-j}} \hat{P}_{k-j}(t, z, b_{j+1}, \dots, b_k) db_{j+1} \dots db_k \right] dz \\
 &+ 2k \int_{D^1} \int_{A^{k-1}} \hat{P}_{k-1}(t, z, b_1, \dots, b_{k-1}) db_1 \dots db_{k-1} \int_A \hat{P}_1(t, z, b_k) db_k dz + \text{error } A \\
 &= \frac{1}{2} \sum_{j=2}^{k-2} \binom{k}{j} \int_{D^1} \left[\int_{A^j} \int_{OD}^t 2V \alpha e^{-ms} p(s, z, b) f_j(t-s, b, b_1, \dots, b_j) db ds db_1 \dots db_j \right] \\
 & \times \left[(k-j)! (V \alpha M_t)^{k-j-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right) \right] dz \\
 &+ (2k(k-1)! (V \alpha M_t)^{k-2} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right)) e^{-mt} \int_{AD^1} p(t, z, b_k) dz db_k + \text{error } A
 \end{aligned}$$

$$= \frac{1}{2} \left[\sum_{j=2}^{k-2} \binom{k}{j} \left[(k-j)! (V \alpha M_t)^{k-j-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1-1}}\right) \right] \right]$$

$$\times \int_0^t \int_{A^j} \dots \int_D 2V \alpha e^{-ms} f_j(t-s, b, b_1, \dots, b_j) db db_1 \dots db_j ds$$

$$+ 2k(k-1)! (V \alpha M_t)^{k-2} e^{-2mt} |A| - o\left(\frac{1}{|A|^{n_1-1}}\right) e^{-mt} \Big] + \text{error A} + \text{error B},$$

where both |error A| and |error B| are less than

$$\epsilon \frac{k-1}{2} \sum_{j=1}^{k-1} \binom{k}{j} (k-j)! (V \alpha M_t)^{k-j-1} e^{-mt} 2V \alpha \int_{A^j} \dots \int_{OD}^t e^{-ms} f_j(t-s, b, b_1, \dots, b_j) db ds db_1 \dots db_j$$

Since ϵ is arbitrary,

$$\int_{A^k} \dots \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k$$

$$= \frac{1}{2} \left[\sum_{j=2}^{k-2} \binom{k}{j} \left[(k-j)! (V \alpha M_t)^{k-j-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1-1}}\right) \right] \right] \quad (4)$$

$$\times \left[\int_0^t 2V \alpha e^{-ms} \int_{A^j} \dots \int_D f_j(t-s, b, b_1, \dots, b_j) db db_1 \dots db_j ds \right]$$

$$+ 2k! (V \alpha M_t)^{k-2} e^{-2mt} |A| - o\left(\frac{1}{|A|^{n_1-1}}\right) e^{-mt} \Big]$$

Note that for the subcritical case, because of (1) the error terms will be uniform in t . (Refer to the proof of § 3.2 Theorem 2).

From first principles, we have that

$$\begin{aligned}
 & \iiint_{AAD} r_2(t, z, b_1, b_2) dz db_1 db_2 \\
 &= \iint_{AD} \hat{P}_1(t, z, b_1) \int_A \hat{P}_1(t, z, b_2) db_2 dz db_1 \\
 &= \iint_{AD} e^{-mt} p(t, z, b_1) \int_A e^{-mt} p(t, z, b_2) db_2 dz db_1 \\
 &\leq |A| e^{-2mt}
 \end{aligned}$$

Choose $D^1 \subseteq D$, D^1 bounded and symmetric so that for fixed $\epsilon > 0$,

$$\int_{D^1} p(t, z, b_2) dz \geq 1 - \epsilon, \text{ for } b_2 \in A.$$

$$\text{Uniformly for } z \in D^1, e^{-mt} \int_A p(t, z, b_2) db_2 = e^{-mt} - o\left(\frac{1}{|A|^{\frac{1}{n-1}}}\right).$$

As in Theorem 2 of § 3.2, if $m > Q$ the error term is uniform in t .

$$\therefore \iiint_{AAD} f_2(t, z, b_1, b_2) dz db_1 db_2$$

$$= e^{-mt} \iint_{AD} p(t, a, b_1) (e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right)) db_1 + \text{error } C$$

$$\geq e^{-2mt} |A| (1 - \epsilon) - e^{-mt} o\left(\frac{1}{|A|^{n_1}}\right) (1 - \epsilon) + \text{error } C$$

where $|\text{error } C| \leq \epsilon e^{-2mt} \times |A|$.

But ϵ is arbitrary, so

$$\iiint_{AAD} f_2(t, z, b_1, b_2) dz db_1 db_2 = e^{-2mt} \times |A| - o\left(\frac{1}{|A|^{n_1-1}}\right) e^{-mt}.$$

for arbitrary $n_1 > 0$.

In the subcritical case, the error term is of course uniform in t .

Now, if we assume that for $j \leq k-1$,

$$\begin{aligned} & \int \dots \int_{A^j} f_j(t, b, b_1, \dots, b_j) db db_1 \dots db_j \\ &= j! \frac{(j-1)}{2} e^{-2mt} \left(\vee \alpha \frac{(1-e^{-mt})}{m} \right)^{j-2} |A| - t^{j-1} e^{-mt} o\left(\frac{1}{|A|^{n_1}}\right) \text{ for any } n_1 > 0, \end{aligned}$$

then by induction we have from (4)

$$\begin{aligned}
 & \int \dots \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k \\
 & = \frac{1}{2} \left[\sum_{j=2}^{k-2} \binom{k}{j} \left[(k-j)! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{k-j-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right) \right] \right. \\
 & \quad \times \left[\int_0^t 2V \alpha e^{-ms} j! \frac{(j-1)}{2} e^{-2m(t-s)} \left(V \alpha \frac{(1-e^{-m(t-s)})}{m} \right)^{j-2} |A| \right. \\
 & \quad \left. \left. - 2V \alpha e^{-ms} (t-s)^{j-1} e^{-m(t-s)} o\left(\frac{1}{|A|^{n_1}}\right) ds \right] \right. \\
 & \quad \left. + 2k! \left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{k-2} e^{-2mt} |A| e^{-mt} o\left(\frac{1}{|A|^{n_1-1}}\right) \right] \\
 & = \frac{1}{2} \left[\sum_{j=1}^{k-1} \frac{k!}{(k-j)!(j)!} (k-j)! (j)! \left[\left(V \alpha \frac{(1-e^{-mt})}{m} \right)^{k-j-1} e^{-mt} - o\left(\frac{1}{|A|^{n_1}}\right) \right] \right. \\
 & \quad \times \left[\left(V \alpha \right)^{j-1} e^{-mt} \frac{j-1}{j-1} \left(\frac{1-e^{-mt}}{m} \right)^{j-1} |A|^{-t^j} e^{-mt} o\left(\frac{1}{|A|^{n_1-1}}\right) ds \right] \\
 & = \frac{1}{2} \sum_{j=1}^{k-1} \left[k! \left(V \alpha M_t \right)^{k-2} e^{-2mt} |A|^{-t^j} e^{-mt} o\left(\frac{1}{|A|^{n_1-1}}\right) \right] \\
 & = k! \left(V \alpha M_t \right)^{k-2} e^{-2mt} |A| \sum_{j=1}^{k-1} \frac{1}{2} - t^{k-1} e^{-mt} o\left(\frac{1}{|A|^{n_1-1}}\right) \\
 & = k! \frac{(k-1)}{2} \left(V \alpha M_t \right)^{k-2} e^{-2mt} |A| - t^{k-1} e^{-mt} o\left(\frac{1}{|A|^{n_1-1}}\right) \text{ for any } n_1 > 0.
 \end{aligned}$$

In the subcritical case, ($m > 0$) the error term will clearly be uniform in t . By induction we have shown that

$$= \int_{A^k} \int_D r_k(s-u, z, b_1, \dots, b_k) dz db_1 \dots db_k \quad (5)$$

$$= k! \frac{(k-1)}{2} |A| (V \alpha M_{s-u})^{k-2} e^{-2m(s-u)} (s-u)^{k-1} e^{-m(s-u)} o\left(\frac{1}{|A|^{n_1-1}}\right)$$

for any $n_1 > 0$.

Substituting (5) in (3),

$$\int_{A^k} \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1, \dots, db_k$$

$$= 2V \alpha |A| \int_0^s e^{-mu} (k! \frac{(k-1)}{2} (V \alpha \frac{(1-e^{-m(s-u)})}{m})^{k-2} e^{-2m(s-u)} + e^{-mu} (s-u)^{k-1} e^{-m(s-u)})$$

$$\times o\left(\frac{1}{|A|^{n_1-1}}\right) ds$$

$$= |A| k! \left(V \alpha \frac{(1-e^{-ms})}{m} \right)^{k-1} e^{-ms} + s^k e^{-ms} o\left(\frac{1}{|A|^{n_1-1}}\right) \quad (6)$$

Again, we note that for the subcritical case the error term will be uniform in s .

Substituting (6) in (2), for $m \neq 0$

$$\begin{aligned} \mu_{[n]}(A) = & r \sum_{k=1}^n \binom{n}{k} \int_0^t \left[|A|^k \left(V_\alpha \frac{(1-e^{-ms})}{m} \right)^{k-1} e^{-ms} + s e^{-ms} \left(\frac{1}{|A|^{n-1}} \right) \right] \\ & \times \int_{A^{n-k}} \hat{P}_{n-k}^I(s, y, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n ds. \end{aligned} \quad (7)$$

By direct computation, we have that

$$\begin{aligned} \int_A \hat{P}_1^I(t, b) db &= \int_{AOD} \int_0^t r \hat{P}_1(s, y, b) dy ds db \\ &= r \int_{AO} \int_0^t e^{-ms} ds db \\ &= r|A| \left(\frac{1-e^{-mt}}{m} \right). \end{aligned}$$

$$\begin{aligned} \iint_{AA} \hat{P}_2^I(t, b_1, b_2) db_1 db_2 &= 2r \int_{OA} \int_0^t \hat{P}_1^I(s, b_1) \int_{AD} \hat{P}_1(s, y, b_2) dy db_2 db_1 ds \\ &+ r \int_{OAAD} \int_0^t \hat{P}_2(s, y, b_1, b_2) dy db_1 db_2 ds \\ &= 2r \int_0^t r|A| \left(\frac{1-e^{-ms}}{m} \right) e^{-ms} |A| ds \end{aligned}$$

$$\begin{aligned}
 & + r \int_0^t |A|^2 \left(V \alpha \frac{(1-e^{-ms})}{m} \right) e^{-ms} + e^{-ms} o \left(\frac{1}{|A|^{n_1}} \right) ds \\
 & = \frac{2(r|A|)^2}{2} \left(\frac{1-e^{-mt}}{m} \right)^2 + 2r|A| \frac{V \alpha}{2} \left(\frac{1-e^{-mt}}{m} \right)^2 + \left(\frac{1-e^{-mt}}{m} \right) \times o \left(\frac{1}{|A|^{n_1}} \right) \\
 & = M_t^2 ((r|A|)^2 + r|A|V \alpha) + \left(\frac{1-e^{-mt}}{m} \right) o \left(\frac{1}{|A|^{n_1}} \right).
 \end{aligned}$$

Suppose for $k \leq n-1$

$$\begin{aligned}
 & \int_{A^k} \dots \int \hat{P}_k^I(s, b_1, \dots, b_k) db_1 \dots db_k \\
 & = \left(\frac{(1-e^{-ms})}{m} \right)^k \sum_{j=0}^k (-1)^{k-j} S_k^j (r|A|)^j (V \alpha)^{k-j} + o \left(\frac{1}{|A|^{n_1}} \right),
 \end{aligned}$$

for any $n_1 > 0$ and uniformly in t for $m > 0$, where the S_k^j 's are Stirling numbers of the first kind. Then, by induction, if $m \neq 0$, from (7)

$$\begin{aligned}
 & \int_{A^n} \dots \int \hat{P}_n(t, b_1, \dots, b_n) db_1 \dots db_n \\
 & = r \int_0^t \sum_{k=1}^n \sum_{j=0}^{n-k} \binom{n}{k} \left[|A|^k e^{-ms} \left(V \alpha \frac{(1-e^{-ms})}{m} \right)^{k-1} + e^{-ms} o \left(\frac{1}{|A|^{n_1}} \right) \right] \\
 & \quad \times \left[\left(\frac{(1-e^{-ms})}{m} \right)^{n-k} (-1)^{n-k-j} S_{n-k}^j (r|A|)^j (V \alpha)^{n-k-j} + o \left(\frac{1}{|A|^{n_1}} \right) \right] ds
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^n \sum_{j=0}^{n-k} \binom{n}{k} k! (r|A|)^{j+1} (v\alpha)^{n-j-1} (-1)^{n-k-j} s_{n-k}^j \\
 &\quad \times \int_0^t \left(\frac{1-e^{-ms}}{m} \right)^{n-1} e^{-ms} + s^k e^{-ms} o\left(\frac{1}{|A|^{n-1-j}}\right) ds \\
 &= \frac{1}{n} \left(\frac{1-e^{-mt}}{m} \right)^n \times \sum_{k=1}^n \left(\sum_{j=0}^{n-k} \binom{n}{k} k! (r|A|)^{j+1} (v\alpha)^{n-j-1} (-1)^{n-k-j} s_{n-k}^j \right. \\
 &\quad \left. + o\left(\frac{1}{|A|^{n-1}}\right) \left[\frac{k!}{m^{k+1}} - e^{-mt} \left(\sum_{h=1}^k \frac{t^{k-h}}{m^{h+1}} \frac{k!}{(k-h)!} \right) \right] \right) .
 \end{aligned}$$

For $m > 0$, this error term is uniform in t .

We have shown for the noncritical process,

$$\mu_{[n]}^1(A) \approx \frac{1}{n} M_t^n \sum_{k=1}^n \sum_{j=0}^{n-k} \binom{n}{k} k! (r|A|)^{j+1} (v\alpha)^{n-j-1} (-1)^{n-k-j} s_{n-k}^j \quad (8)$$

with an error term of $o\left(\frac{1}{|A|^{n_1}}\right)$, $n_1 > 0$, n_1 arbitrary, and the error term

is uniform in t in the subcritical case ($m > 0$).

We now consider the coefficient of $M_t^n (r|A|)^{j+1} (v\alpha)^{n-j-1}$ in (8) for fixed j , as j goes from 0 to $n-1$. From (8) this coefficient is

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n-j} \binom{n}{k} k! (-1)^{n-k-j} S_{n-k}^j \\ = \sum_{k=1}^{n-j} \frac{(n-1)!}{(n-k)!} (-1)^{n-k-j} S_{n-k}^j \end{aligned} \quad (9)$$

But note that (see Appendix A, equation (1))

$$\begin{aligned} S_n^{j+1} &= S_{n-1}^j - (n-1) S_{n-1}^{j+1} \\ &= S_{n-1}^j - (n-1) S_{n-2}^j + (n-1)(n-2) S_{n-2}^{j+1} \\ &= \dots = \sum_{k=1}^{n-j} \frac{(n-1)!}{(n-k)!} S_{n-k}^j (-1)^{k+1} \end{aligned}$$

$$\therefore (9) = (-1)^{n-j-1} S_n^{j+1}$$

So from (8), we have for arbitrary $n_1 > 0$

$$\begin{aligned} \mu_{[n]}'(A) &= M_t^n \sum_{j=0}^{n-1} (-1)^{n-j-1} S_n^{j+1} (r|A|)^{j+1} (v\alpha)^{n-j-1} + o\left(\frac{1}{|A|^{n_1}}\right) \\ &= \sum_{k=1}^n (-1)^{n-k} S_n^k (M_t r|A|)^k (M_t v\alpha)^{n-k} + o\left(\frac{1}{|A|^{n_1}}\right) \end{aligned}$$

q.e.d.

Theorem 3: Assume that $D = \mathbb{R}^d$ and that $p(t, a, b) = \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t)$.

Then for critical branching, ($m = V(1-2\alpha) = 0$), as $A \uparrow D$, A symmetric and bounded,

$$\mu'_{[n]}(A) = \sum_{k=1}^n (-1)^{n-k} S_n^k (\text{tr}|A|)^k \left(t \frac{V}{2}\right)^{n-k} + o\left(\frac{1}{|A|^{n-1}}\right).$$

$n_1 > 0$ is arbitrary.

Proof.

$$\begin{aligned} \mu'_{[n]}(A) = & r \sum_{k=1}^n \binom{n}{k} \int_0^t \left[\int_{A^k} \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k \right. \\ & \left. \times \int_{A^{n-k}} \hat{P}_{n-k}^I(s, b_{k+1}, \dots, b_n) db_{k+1} \dots db_n \right] ds. \end{aligned} \quad (1)$$

From (1), we must consider the behaviour of

$$\int_{A^k} \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k \quad \text{as } A \uparrow D.$$

$$\int_{A^k} \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k$$

$$= V \int_{DOL} \int_{\dots} \int_{\dots} p(u, y, z) \int_{A^k} f_k(s-u, z, b_1, \dots, b_k) db_1 \dots db_k dz du dy$$

$$= V \int_0^s \int_{A^k} \int_D f_k(s-u, z, b_1, \dots, b_k) dz db_1 \dots db_k du, \quad (2)$$

For arbitrary $\epsilon > 0$ choose $D^1, |D^1| < \infty$ such that D^1 is symmetric and $\int_{D^1} p(s, a, b) da \geq 1 - \epsilon$ for $s < t$ and $b \in A$. Then by § 3.2 Theorem 3,

$$\begin{aligned} & \int_{A^k} \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k \\ &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \binom{k}{j} \int_D \int_{A^j} \hat{P}_j(t, z, b_1, \dots, b_j) db_1 \dots db_j \right. \\ & \quad \times \left. \int_{A^{k-j}} \hat{P}_{k-j}(t, z, b_{j+1}, \dots, b_k) db_{j+1} \dots db_k dz \right] \\ &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \binom{k}{j} \int_{A^j} \int_D \int_{OD^1} p(s, z, y) f_j(t-s, y, b_1, \dots, b_j) dz ds dy db_1 \dots db_j \right. \\ & \quad \times \left. (k-j)! \left(\frac{Vt}{2} \right)^{k-j-1} - o\left(\frac{1}{|A|^{n_1}}\right) \right] + \text{error } A \end{aligned}$$

$$= \frac{1}{2} \left[\sum_{j=1}^{k-1} \left((k-j)! \left(\frac{Vt}{2} \right)^{k-j-1} - o \left(\frac{1}{|A|^{\frac{1}{n-1}}} \right) \right) \binom{k}{j} V \int_0^t \int_{A^j} \int_D f_j(t-s, y, b_1, \dots, b_j) dy db_1 \dots db_j ds \right] + \text{error A} + \text{error B}, \quad (3)$$

where both |error A| and |error B| are less than

$$\in \sum_{j=1}^{k-1} \binom{k}{j} (k-j)! \left(\frac{Vt}{2} \right)^{k-j-1} V \int_0^t \int_{A^j} \int_D f_j(t-s, y, b_1, \dots, b_j) dy db_1 \dots db_j ds.$$

$$\int_0^t \int_{A^2} f_2(t, y, b_1, b_2) dy db_1 db_2$$

$$= \int_0^t \int_{A^2} p(t, y, b_1) p(t, y, b_2) dy db_1 db_2$$

$$= \int_0^t \int_{AA} p(2t, b_1, b_2) db_1 db_2$$

$$= |A| - o \left(\frac{1}{|A|^{\frac{1}{n-1}}} \right).$$

If for $j \leq k-1$

$$\int_0^t \int_{A^j} \int_D f_j(t, y, b_1, \dots, b_j) dy db_1 \dots db_j = j! \frac{(j-1)}{2} \left(\frac{Vt}{2} \right)^{j-2} |A| + o \left(\frac{1}{|A|^{\frac{1}{n-1}}} \right)$$

then from (3), we obtain by induction

$$\begin{aligned}
 & \int_{A^k} \dots \int_D f_k(t, z, b_1, \dots, b_k) dz db_1 \dots db_k \\
 &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \left((k-j)! \cdot \left(\frac{Vt}{2} \right)^{k-j-1} - o \left(\frac{1}{|A|^{n_1}} \right) \right) \binom{k}{j} \left(V \int_0^t j! \cdot \frac{(j-1)}{2} \left(\frac{V(t-s)}{2} \right)^{j-2} |A| \right. \right. \\
 & \quad \left. \left. + o \left(\frac{1}{|A|^{n_1}} \right) ds \right) \right] \\
 &= \frac{1}{2} \left[\sum_{j=1}^{k-1} \frac{k!}{j! (k-j)!} (k-j)! j! \left(\frac{Vt}{2} \right)^{k-j-1} \left(\frac{Vt}{2} \right)^{j-1} |A| \right] + o \left(\frac{1}{|A|^{n_1-1}} \right) \\
 &= \frac{k!}{2} \left(\frac{Vt}{2} \right)^{k-2} |A| \sum_{j=1}^{k-1} 1 + o \left(\frac{1}{|A|^{n_1-1}} \right) \\
 &= \frac{k!}{2} \frac{(k-1)}{2} \left(\frac{Vt}{2} \right)^{k-2} |A| + o \left(\frac{1}{|A|^{n_1-1}} \right), \text{ for arbitrary } n_1 > 0. \quad (4)
 \end{aligned}$$

Substituting (4) in (2),

$$\begin{aligned}
 & \int_{A^k} \dots \int_D \hat{P}_k(s, y, b_1, \dots, b_k) dy db_1 \dots db_k \\
 &= V \int_0^s k! \frac{(k-1)}{2} \left(\frac{V(s-u)}{2} \right)^{k-2} |A| + o \left(\frac{1}{|A|^{n_1}} \right) du \\
 &= \left(\frac{Vs}{2} \right)^{k-1} k! |A| + o \left(\frac{1}{|A|^{n_1}} \right). \quad (5)
 \end{aligned}$$

Substituting (5) in (1),

$$\begin{aligned} \mu_{[n]}^I(A) &= r \sum_{k=1}^n \binom{n}{k} \int_0^t \left[\int_{A^{n-k}} \dots \int_{A^{n-k}} \hat{P}_{n-k}^I(s, b_1, \dots, b_{n-k}) db_1 \dots db_{n-k} \right] \\ &\quad \times \left(\left(\frac{Vs}{2} \right)^{k-1} k! |A| + o\left(\frac{1}{|A|^{n_1}}\right) \right) ds. \end{aligned} \quad (6)$$

By direct computation, we have that

$$\begin{aligned} \int_A \hat{P}_1^I(t, b) db &= \iiint_{AOD} r p(s, y, b) dy ds db \\ &= rt |A|. \end{aligned}$$

$$\begin{aligned} \iint_{AA} \hat{P}_2^I(t, b_1, b_2) db_1 db_2 &= 2r \iint_{OA} \hat{P}_1^I(s, b_1) \iint_{AD} \hat{P}_1(s, y, b_2) dy db_2 db_1 ds \\ &\quad + r \iiint_{OAAD} \hat{P}_2(s, y, b_1, b_2) dy db_1 db_2 ds \\ &= 2r \int_0^t r |A| s |A| ds + r \int_0^t Vs |A| ds - o\left(\frac{1}{|A|^{n_1}}\right) \\ &= t^2 ((r|A|)^2 + r|A|\frac{V}{2}) - o\left(\frac{1}{|A|^{n_1}}\right). \end{aligned}$$

Suppose for $k \leq n-1$,

$$\int \dots \int_{A^k} \hat{P}_k^I(s, b_1, \dots, b_k) db_1 \dots db_k = s^k \sum_{j=0}^k (-1)^{k-j} S_k^j (r|A|)^j \left(\frac{V}{2}\right)^{k-j} + o\left(\frac{1}{|A|^{n_1}}\right)$$

where the S_k^j 's are Stirling numbers of the first kind. Then, by substituting in (6), we obtain by induction

$$\begin{aligned} \mu'_{[n]}(A) &= r \sum_{k=1}^n \sum_{j=0}^{n-k} \binom{n}{k} \int_0^t \left(s^{n-k} (-1)^{n-k-j} S_{n-k}^j (r|A|)^j \left(\frac{V}{2}\right)^{n-k-j} + o\left(\frac{1}{|A|^{n_1}}\right) \right) \\ &\quad \times \left(\left(\frac{Vs}{2}\right)^{k-1} k! |A| + o\left(\frac{1}{|A|^{n_1}}\right) \right) ds \\ &= \sum_{k=1}^n \sum_{j=0}^k \frac{n!}{(n-k)!} (-1)^{n-k-j} S_{n-k}^j (r|A|)^{j+1} \left(\frac{V}{2}\right)^{n-j-1} \int_0^t s^{n-1} ds + o\left(\frac{1}{|A|^{n_1-j}}\right) \\ &\sim t^n \sum_{k=1}^n \sum_{j=0}^k \frac{(n-1)!}{(n-k)!} (-1)^{n-k-j} S_{n-k}^j (r|A|)^{j+1} \left(\frac{V}{2}\right)^{n-j-1} \end{aligned} \quad (7)$$

with an error term of $o\left(\frac{1}{|A|^{\frac{n}{2}}}\right)$ for arbitrary $n_2 > 0$, $n_2 = n_1 - j$.

By exactly the same argument as used in Theorem 2 of this section, we have that the coefficient of $(r|A|)^{j+1} \left(\frac{V}{2}\right)^{n-j-1}$ in (7) is

$$(-1)^{n-j-1} g_n^{j+1}$$

Therefore, by induction, we have as $A \uparrow D$, A bounded and symmetric, for arbitrary $n_1 > 0$,

$$\begin{aligned} & \int \dots \int_{A_n} \hat{P}_n^I(t, b_1, \dots, b_n) db_1 \dots db_n \\ &= t^n \sum_{j=0}^{n-1} (-1)^{n-j-1} S_n^{j+1} (r|A|)^{j+1} \left(\frac{V}{2}\right)^{n-j-1} + o\left(\frac{1}{|A|^{n_1}}\right) \\ &= \sum_{k=1}^n (-1)^{n-k} S_n^k (\text{tr}|A|)^k \left(\frac{tV}{2}\right)^{n-k} + o\left(\frac{1}{|A|^{n_1}}\right). \end{aligned}$$

q.e.d.

5.3 THE SUBCRITICAL CASE - THE EXISTENCE OF A STEADY-STATE DISTRIBUTION

Holding $A \subseteq D$ fixed, we will now study the behaviour of the random variable $N_A(t)$ as $t \rightarrow \infty$ for the subcritical case.

We recall that $N_A(t)$ = the number of particles in $A \subseteq D$ at time t , given that the process began at time 0. We could say that "the immigration was turned on at time 0." All particles present in A at t are descendants of particles which immigrated into D after $t = 0$.

Suppose that the immigration was "turned on" at some time before 0, say at time $-s$. Then the particles in A at time $t > 0$ are descendants not only of particles which immigrated after time 0, but also during the

interval $-s$ to 0 . Designate this new random variable $N_A(t, s)$. It is clear that $N_A(t, s)$ is equal in distribution to $N_A(t+s)$.

By the Markovian nature of the process, the immigration, branching, and diffusion of particles after $t = 0$ is unaffected by any particles present in D before $t = 0$. Thus, for every $w \in \Omega$ (where Ω is the underlying probability space), for $s_1 \leq s_2$

$$N_A(t, s_1)(w) \leq N_A(t, s_2)(w).$$

We have that for every $w \in \Omega$, $N_A(t, s)$ is a nondecreasing function in s .

Let

$$N_A(w) = \lim_{s \rightarrow \infty} N_A(t, s)(w).$$

By the Monotone Convergence Theorem,

$$E(N_A) = \lim_{s \rightarrow \infty} E(N_A(t, s)(w))$$

$$= \lim_{s \rightarrow \infty} E(N_A(t+s)(w))$$

$$= \lim_{s \rightarrow \infty} \left(\frac{1 - e^{-m(t+s)}}{m} \right) r|A|$$

$$= \frac{r|A|}{m}, \text{ when } m > 0, \text{ as it is in the subcritical}$$

case.

Since $N_A(t)$ is a non-negative random variable, N_A is also, and since its expectation is finite, $N_A < \infty$ almost everywhere. Therefore, we have that there exists a finite-valued random variable N_A , such that

$$N_A(t) \rightarrow N_A \text{ in distribution as } t \rightarrow \infty$$

where N_A has a steady-state distribution. It will be seen that the stationarity of N_A is an immediate consequence of the Markovian nature of the branching diffusion process.

To prove the stationarity of the distribution of N_A , we will use the population probability space approach to the definition of a point process. (See Appendix B.) We will refer to a state of the population by a possibly infinite set $\{x_i\}$ of points in D , which designates the number and locations of the particles present in the population. By Appendix B, the point process is the class X_ω of all countable sets of points $\{n_i\}$ in D such that $\{x_i, \dots\} \cap A$ is finite for all compact $A \subseteq D$. The measurable sets in X_ω are as defined in Appendix B.

If $Q \subseteq X_\omega$ is measurable, then the process is characterized by the transition probability

$$P_s^*(Q | \{x_i\}).$$

$$= P \left(\text{at } t+s, \text{ the population is in some state in } Q | \{x_i\} \in X_\omega \text{ at time } t \right).$$

If $P_t(Q)$ is the unconditional probability of some state in Q at t , then

$$P_{t+s}(Q) = \int_{X_\omega} P_s(Q | \{x_i\}) P_t(\{dx_i\}) .$$

We know that $\lim_{t \rightarrow \infty} P_t(Q)$ exists, and if it is designated by $P_\infty(Q)$,

to show its stationarity, we must show that

$$P_\infty(Q) = \int_{X_\omega} P_s(Q | \{x_i\}) P_\infty(\{dx_i\})$$

for every $s > 0$.

We will again use the technique of "turning the immigration on" at $-t$. Then at time $s \geq 0$, $P_s^t(Q) = P_{t+s}(Q)$. The superscript t indicates the time at which the immigration was started. We have that

$$\begin{aligned} \lim_{t \rightarrow \infty} P_s^t(Q) &= \lim_{t \rightarrow \infty} P_{t+s}(Q) \\ &= P_\infty(Q) . \end{aligned}$$

Note that this relation is true for any $s \geq 0$.

Let $\lim_{t \rightarrow \infty} P_s^t(Q) = P_s^\infty(Q) = P_s(\text{state } Q \text{ at } s | \text{the immigration started at } -\infty)$.

By the Markovian nature of the process,

$$P_s^\infty(Q) = \int_{X_\omega} P_s(Q | \{x_i\}) P_0^\infty(\{dx_i\}) .$$

$$\text{But } P_s^\infty(Q) = P_\infty(Q) = P_0^\infty(Q) .$$

Therefore,

$$P_\infty(Q) = \int_{X_\omega} P_s(Q | \{x_i\}) P_\infty(\{dx_i\})$$

for every $s > 0$, proving the stationarity of the limiting distribution.

We have proven the following:

Theorem 4.

The random variable $N_A(t)$ in the subcritical process converges in distribution to a nonnegative, finite-valued random variable N_A whose distribution is a steady state.

From this theorem, we know that the moments of N_A are the limits of the moments of $N_A(t)$. Recalling from § 3.2 that, for the subcritical process

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$$

$$= \sum_{k=1}^n (-1)^{n-k} S_n^k (M_t r|A|)^k (M_t v \alpha)^{n-k} + o\left(\frac{1}{|A|^{n-1}}\right)$$

where $n_1 > 0$ is arbitrary, $M_t = \frac{1-e^{-mt}}{m}$, and the error term is uniform in t , we have that

$$\begin{aligned} E(N_A(N_A-1)\dots(N_A-(n-1))) &= \lim_{t \rightarrow \infty} E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1))) \\ &= \sum_{k=1}^n (-1)^{n-k} S_n^k \left(\frac{r|A|}{m}\right)^k \left(\frac{v\alpha}{m}\right)^{n-k} + o\left(\frac{1}{|A|^{n_1}}\right). \end{aligned} \quad (1)$$

We now have an approximation for the factorial moments of the steady-state distribution.

5.4 RENORMALIZATION OF THE IMMIGRATION PROCESS

It is interesting to conjecture about the behaviour of a process such as the branching diffusion with immigration when more and more particles are found in increasingly smaller sets. In particular, we will see that when the migration process is d -dimensional Brownian motion, $\text{Cov}(N_A(t), N_B(t))$ tends to 0 for nonintersecting closed sets A and B .

Let us increase the rate of immigration by a factor of k^d , and decrease the variance t of the Brownian motion by a factor of $\frac{1}{k^2}$.

This means that for a set A , the probability that a particle enters A by immigration in $(t, t+dt)$ is $k^d r|A|dt = r|kA|dt$, where kA is the set produced by a dilation of A by the factor k . For Brownian motion,

$$p_k(t, a, b) = \frac{1}{\left(\frac{2\pi t}{k^2}\right)^{d/2}} \exp(-|a-b|^2/(2t/k^2))$$

$$= \frac{k^d}{(2\pi t)^{d/2}} \times \exp(-|ka-kb|^2/2t)$$

where $p_k(t, a, b)$ is the rescaled Brownian motion. For both the immigration and migration processes, the rescaling effectively inflates distances in $D = R^d$ by a factor of k . Define $N_A^k(t)$ to be the number of particles in A at time t under the rescaled process, where the rescaling factor is k .

Let A and B be two nonintersecting closed sets. We will show that under the rescaling, with Brownian motion as the migration process

$$\lim_{k \rightarrow \infty} \text{Cov}(N_A^k(t) N_B^k(t)) = 0.$$

Proof.

$$\text{Cov}(N_A^k(t) N_B^k(t)) = E(N_A^k(t) N_B^k(t)) - E(N_A^k(t)) E(N_B^k(t)). \quad (1)$$

From § 5.1, Theorem 1, recalling that the new immigration rate is $k^d r$, we have

$$E(N_A^k(t)) E(N_B^k(t)) = k^{2d} M_t^2 r^2 |A| |B|,$$

where $M_t = \begin{cases} \frac{1-e^{-mt}}{m} & \text{if } m \neq 0 \\ t & \text{if } m = 0. \end{cases} \quad (2)$

Since. $A \cap B = \emptyset$,

$$\begin{aligned}
 E(N_A^k(t)N_B^k(t)) &= \iint_{AB} \hat{P}_2^I(t, b_1, b_2) db_1 db_2 \\
 &= k^d r \int_0^t \iiint_{ABOD} \hat{P}_1^I(s, b_1) \hat{P}_1(s, y, b_2) dy ds db_1 db_2 \\
 &\quad + k^d r \int_0^t \iiint_{ABOD} \hat{P}_1^I(s, b_2) \hat{P}_1(s, y, b_1) dy ds db_1 db_2 \\
 &\quad + k^d r \int_0^t \iiint_{ABOD} \hat{P}_2(s, y, b_1, b_2) dy ds db_1 db_2 \\
 &= 2k^d r \int_0^t \iiint_{ABO} k^d r M_s e^{-ms} ds db_1 db_2 \quad (3) \\
 &\quad + k^d r \int_0^t \iiint_{ABOD} 2V \alpha \int_0^s e^{-mu} p_k(u, y, z) e^{-m(s-u)} p_k(s-u, z, b_1) \\
 &\quad \times e^{-m(s-u)} p_k(s-u, z, b_2) dz du dy ds db_1 db_2 .
 \end{aligned}$$

We now consider the noncritical case: $m \neq 0$.

From (3), $E(N_A^k(t)N_B^k(t))$

$$\begin{aligned}
 &= 2k \frac{2d}{2} \frac{r^2}{2} M_t^2 |A| |B| \\
 &+ k^d r \int_0^t \int_0^s \int_0^u \int_0^v 2V \alpha \int_0^s e^{-mu} e^{-2m(s-u)} p_k(s-u, z, b_1) p_k(s-u, z, b_2) du dz ds db_1 db_2 \\
 &= k \frac{2d}{2} \frac{r^2}{2} M_t^2 |A| |B| \quad (4)
 \end{aligned}$$

$$+ k^d r \int_0^{ts} \int_0^s \int_0^u \int_0^v 2V \alpha e^{-ms} e^{-m(s-u)} p_k(2(s-u), b_1, b_2) du ds db_1 db_2.$$

Substituting (4) and (2) in (1),

$$\begin{aligned}
 \text{Cov}(N_A^k(t) N_B^k(t)) &= 2k^d r V \alpha \int_0^{ts} \int_0^s \int_0^u \int_0^v e^{-ms} e^{-m(s-u)} p_k(2(s-u), b_1, b_2) du ds db_1 db_2 \\
 &= 2k^d r V \alpha \int_0^{ts} \int_0^s \int_0^u \int_0^v e^{-ms} e^{-mu} p_k(2u, b_1, b_2) du ds db_1 db_2 \quad (5)
 \end{aligned}$$

$$p_k(2u, b_1, b_2) = \frac{k^d}{(4\pi u)^{d/2}} \exp(-k^2 |b_1 - b_2|^2 / 4u).$$

Since A and B are both closed, then

$$\begin{aligned}
 d(A, B) &= \min(|x-y| \mid x \in A, y \in B) \\
 &= \epsilon > 0.
 \end{aligned}$$

Then $|b_1 - b_2|^2 \geq \epsilon^2$ for any $b_1 \in A$, $b_2 \in B$.

We would like to consider the behaviour of

$$p_k(2u, b_1, b_2) \text{ for } 0 < u \leq t.$$

$$\begin{aligned} \frac{\partial}{\partial u} p_k(2u, b_1, b_2) &= \frac{-d}{2} \frac{k^d}{(4\pi)^{d/2}} \frac{1}{u^{(d+2)/2}} \exp(-k^2 |b_1 - b_2|^2 / 4u) \\ &\quad + \frac{k^2 |b_1 - b_2|^2}{4u^2} \frac{k^d}{(4\pi u)^{d/2}} \exp(-k^2 |b_1 - b_2|^2 / 4u) \\ &= \frac{k^d}{(4\pi)^{d/2}} \frac{1}{u^{(d+2)/2}} \exp(-k^2 |b_1 - b_2|^2 / 4u) \left[-\frac{d}{2} + \frac{k^2 |b_1 - b_2|^2}{4u} \right]. \end{aligned}$$

Therefore, $p_k(2u, b_1, b_2)$ is increasing for $0 < u < \frac{k^2 |b_1 - b_2|^2}{2d}$

and decreasing for $u > \frac{k^2 |b_1 - b_2|^2}{2d}$.

But $|b_1 - b_2|^2 \geq \epsilon^2$ for $b_1 \in A$, $b_2 \in B$. If k is chosen large enough

that $\frac{k^2 \epsilon^2}{2d} > t$, then since $u \leq t < \frac{k^2 \epsilon^2}{2d} < \frac{k^2 |b_1 - b_2|^2}{2d}$, we have that

$p_k(2u, b_1, b_2)$ is increasing for $0 < u \leq t$, and $p_k(2u, b_1, b_2)$

$$\leq \frac{k^d}{(4\pi t)^{d/2}} \exp(-k^2 |b_1 - b_2|^2 / 4t)$$

$$\leq \frac{k^d}{(4\pi t)^{d/2}} \exp(-k^2 \epsilon^2 / 4t).$$

Returning to (5),

$$\begin{aligned}
 \text{Cov}(N_A^k(t), N_B^k(t)) &\leq 2k^d rV \alpha \int_{ABOO}^{ts} e^{-ms} e^{-mu} \frac{k^d}{(4\pi t)^{d/2}} \exp(-k^2 \epsilon^2 / 4t) du ds db_1 db_2 \\
 &= \frac{2k^{2d}}{2} \frac{rV \alpha}{(4\pi t)^{d/2}} \exp(-k^2 \epsilon^2 / 4t) \left(\frac{1 - e^{-mt}}{m} \right)^2 \\
 &= rV \alpha \frac{M_t^2}{(4\pi t)^{d/2}} k^{2d} \exp(-k^2 \epsilon^2 / 4t).
 \end{aligned}$$

But as $k \rightarrow \infty$, $k^{2d} \exp(-k^2 \epsilon^2 / 4t) \rightarrow 0$, and so in the noncritical case,

$$\text{Cov}(N_A^k(t), N_B^k(t)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

The result remains to be proven for the critical case: $m = 0$.

From (3),

$$\begin{aligned}
 &= E(N_A^k(t) N_B^k(t)) \\
 &= 2k^{2d} \frac{rt^2}{2} |A| |B| \\
 &\quad + 2k^d rV \alpha \int_{ABOO}^{ts} p_k(s-u, z, b_1) p_k(s-u, z, b_2) dz du ds db_1 db_2 \\
 &= k^{2d} rt^2 |A| |B|
 \end{aligned}$$

$$\begin{aligned}
 & + 2k^d rV \alpha \int_{ABOO} \int_0^{ts} p_k(2(s-u), b_1, b_2) du ds db_1 db_2 \\
 & = k^{2d} r t^2 |A||B| + 2k^d rV \alpha \int_{ABOO} \int_0^{ts} p_k(2u, b_1, b_2) du ds db_1 db_2. \quad (6)
 \end{aligned}$$

Substituting (6) and (2) in (1),

$$\text{Cov}(N_A^k(t), N_B^k(t)) = 2k^d rV \alpha \int_{ABOO} \int_0^{ts} p_k(2u, b_1, b_2) du ds db_1 db_2. \quad (7)$$

As proven in the noncritical case, we have that for k sufficiently large, for $0 < u \leq t$,

$$p_k(2u, b_1, b_2) \leq \frac{k^d}{(4\pi t)^{d/2}} \exp(-k^2 \epsilon^2 / 4t),$$

where $\epsilon = d(A, B)$.

Returning to (7),

$$\begin{aligned}
 \text{Cov}(N_A^k(t), N_B^k(t)) & \leq 2k^d rV \alpha \int_{ABOO} \int_0^{ts} \frac{k^d}{(4\pi t)^{d/2}} \exp(-k^2 \epsilon^2 / 4t) du ds db_1 db_2 \\
 & = \frac{2k^{2d}}{2} \cdot \frac{rV \alpha}{(4\pi t)^{d/2}} t^2 \exp(-k^2 \epsilon^2 / 4t).
 \end{aligned}$$

But as $k \rightarrow \infty$, $k^{2d} \exp(-k^2 \epsilon^2 / 4t) \rightarrow 0$, and so $\text{Cov}(N_A^k(t), N_B^k(t)) \rightarrow 0$ as $k \rightarrow \infty$ in the critical case.

q.e.d.

5.5 THE MULTITYPE BRANCHING DIFFUSION WITH IMMIGRATION

The multitype immigration model is a simple extension of the single-type process. There are k types of distinguishable particles and initially the domain D is empty. Particles of type $1, \dots, k$ are introduced into D with immigration rates $r_1, \dots, r_k$ respectively.

Once a particle has been introduced into D through immigration or birth, it obeys the laws governing the multitype branching diffusion described in § 4.

In this section we shall study $\hat{P}_1^I(t, (b, j))db$

$= P$ [there exists a particle of type j in db at time t], and

$$\hat{P}_2^I(t, (b_1, j_1), (b_2, j_2))db_1db_2$$

$= P$ [there exist particles of type j_1 and j_2 in db_1 and db_2 , respectively, at time t].

These density functions are the first and second order factorial moment densities. As the higher order density functions involve very complicated combinatorial expressions, they will not be considered.

Theorem 5.

$$\text{For } D \subseteq R^d, \hat{P}_1^I(t, (b, j)) = \sum_{i=1}^k r_i \int_0^t \int_{OD} \hat{P}_1(s, (y, i), (b, j)) dy ds$$

where $\hat{P}_1(s, (y, i), (b, j))$ is as defined in § 4.1, Theorem 1.

Proof.

We begin by assuming that $|D| < \infty$ and that only particles of type i can enter D by immigration. Their immigration rate is r_i . Thus, the probability of a type i particle immigrating into D in $(t, t+dt)$ is $r_i |D| dt$. Given that a particle immigrates, it appears in the set element dx with probability $\frac{dx}{|D|}$.

We consider the time $s < t$ that the first particle appears in D . The particle in db at time t is either descended directly from this particle, or from another particle which immigrates into D at some time between s and t .

The former case occurs with probability

$$\int_0^t \int_D e^{-r_i |D| s} r_i |D| \hat{P}_1(t-s, (y, i), (db, j)) \frac{dy ds}{|D|}.$$

The latter case occurs with probability

$$\begin{aligned} & \int_0^t \int_D e^{-r_i |D| s} r_i |D| \hat{P}_1^I(t-s, (db, j)) \frac{dy ds}{|D|} \\ &= \int_0^t e^{-r_i |D| s} r_i |D| \hat{P}_1^I(t-s, (db, j)) ds. \end{aligned}$$

Thus, $\hat{P}_1^I(t, (db, j))$

$$= \int_0^t e^{-r_i |D|s} r_i |D| \hat{P}_1^I(t-s, (db, j)) ds + \iint_{OD} e^{-r_i |D|s} r_i \hat{P}_1(t-s, (y, i), (db, j)) dy ds.$$

So we obtain the following renewal-type equation:

$$\hat{P}_1^I(t, (b, j)) = \int_0^t e^{-r_i |D|s} r_i (|D| \hat{P}_1^I(t-s, (b, j)) + \int_D \hat{P}_1(t-s, (y, i), (b, j)) dy) ds.$$

Taking Laplace transforms of both sides of this equation, we obtain

$$L_\lambda \hat{P}_1^I(., (b, j)) = \iint_{00}^{\infty} e^{-\lambda t} e^{-r_i |D|s} r_i (|D| \hat{P}_1^I(t-s, (b, j)) + \int_D \hat{P}_1(t-s, (y, i), (b, j)) dy) ds dt.$$

Let $s = s$

$u = t-s$

$$L_\lambda \hat{P}_1^I(., (b, j)) = \int_0^\infty e^{-\lambda s} e^{-r_i |D|s} r_i \left(\int_0^\infty e^{-\lambda u} (|D| \hat{P}_1^I(u, (b, j))) + \int_D \hat{P}_1(u, (y, i), (b, j)) dy \right) ds$$

$$= \int_0^\infty e^{-s(\lambda + r_i |D|)} r_i (|D| L_\lambda \hat{P}_1^I(., (b, j)) + L_\lambda \int_D \hat{P}_1(., (y, i), (b, j)) dy) ds$$

$$= \frac{r_i}{\lambda + r_i |D|} (|D| L_\lambda \hat{P}_1^I(., (b, j))) + L_\lambda \int_D \hat{P}_1(., (y, i), (b, j)) dy$$

$$\Rightarrow \lambda L_\lambda \hat{P}_1^I(., (b, j)) = r_i L_\lambda \int_D \hat{P}_1(., (y, i), (b, j)) dy.$$

Inverting Laplace transforms,

$$\begin{aligned} \frac{\partial}{\partial t} \hat{P}_1^I(t, (b, j)) &= r_i \int_D \hat{P}_1(t, (y, i), (b, j)) dy \\ \Rightarrow \hat{P}_1^I(t, (b, j)) &= r_i \int_0^t \int_{OD} \hat{P}_1(s, (y, i), (b, j)) dy ds \\ &= r_i \int_0^t C_{ij}(s) \int_0^s p(t, y, b) dy ds. \end{aligned} \quad (1)$$

Up to here, we have assumed that $|D| < \infty$. If $|D| = \infty$, since D is σ -compact, we can choose a sequence of subsets $B_1, B_2, \dots$ of D such that $|B_k| < \infty$ for all k , and $B_k \uparrow D$. For each set B_k , the immigration rate is $r_i |B_k|$, and define

$$p_k(t, a, b) = \begin{cases} p(t, a, b) & \text{if } a \in B_k \text{ and } b \in B_k \\ 0 & \text{otherwise} \end{cases}$$

Therefore, (1) is true for all B_k , with immigration and migration restricted to B_k . Thus, by letting $B_k \uparrow D$, we obtain (1) for D in general.

We have considered immigration of only one type of particle. If each of the k types can enter D by immigration with immigration rates $r_1, \dots, r_k$, then by independence of the immigration, migration, and branching of the particles, the probabilities are additive, and so

$$\hat{P}_1^I(t, (b, j)) = \sum_{i=1}^k r_i \int_0^t \int_{OD} \hat{P}_1(s, (y, i), (b, j)) dy ds.$$

q.e.d.

Theorem 6.

For $D \subseteq R^d$,

$$\begin{aligned} & \hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) \\ &= \hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2)) \\ &+ \sum_{i=1}^k 2 \alpha_i v_i \int_0^t \int_{OD} \hat{P}_1^I(t-s, (c, i)) \hat{P}_1(s, (c, i), (b_1, j_1)) \hat{P}_1(s, (c, i), (b_2, j_2)) dc ds. \end{aligned}$$

Proof.

Again, we begin by assuming that $|D| < \infty$. Consider the time and place of the first immigration. There are various possibilities. This first particle can be the common ancestor of both the particles in db_1 and db_2 at time t , it may be the ancestor of only one, or it may be the ancestor of neither. It is the ancestor of:

a) both particles with probability density

$$\sum_{i=1}^k \int_0^t \int_{OD} e^{-r|D|s} r_i |D| \hat{P}_2(t-s, (y, i), (b_1, j_1), (b_2, j_2)) \frac{dy ds}{|D|}$$

where $r = \sum_{j=1}^k r_j$,

b) one particle with probability density

$$\sum_{i=1}^k \int_{OD}^t e^{-r|D|s} r_i |D| \left(\hat{P}_1^I(t-s, (b_1, j_1)) \hat{P}_1(t-s, (y, i), (b_2, j_2)) \right. \\ \left. + \hat{P}_1^I(t-s, (b_2, j_2)) \hat{P}_1(t-s, (y, i), (b_1, j_1)) \right) \frac{dy}{|D|} ds,$$

c) neither particle with probability density

$$\sum_{i=1}^k \int_{OD}^t e^{-r|D|s} r_i |D| \hat{P}_2^I(t-s, (b_1, j_1), (b_2, j_2)) \frac{dy ds}{|D|}.$$

Thus,

$$\hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) \\ = \sum_{i=1}^k \int_{OD}^t e^{-r|D|s} r_i |D| \left[\hat{P}_2^I(t-s, (b_1, j_1), (b_2, j_2)) \right. \\ + \hat{P}_1^I(t-s, (b_1, j_1)) \hat{P}_1(t-s, (y, i), (b_2, j_2)) \\ + \hat{P}_1^I(t-s, (b_2, j_2)) \hat{P}_1(t-s, (y, i), (b_1, j_1)) \\ \left. + \hat{P}_2(t-s, (y, i), (b_1, j_1), (b_2, j_2)) \right] \frac{dy}{|D|} ds.$$

Taking Laplace transforms of both sides, we obtain

$$\begin{aligned}
 & L_{\lambda} \hat{P}_2^I(\cdot, (b_1, j_1), (b_2, j_2)) \\
 &= \sum_{i=1}^k \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-r|D|s} r_i |D| [\hat{P}_2^I(t-s, (b_1, j_1), (b_2, j_2)) \\
 &\quad + \frac{1}{|D|} \hat{P}_1^I(t-s, (b_1, j_1)) \int_D \hat{P}_1(t-s, (y, i), (b_2, j_2)) dy \\
 &\quad + \frac{1}{|D|} \hat{P}_1^I(t-s, (b_2, j_2)) \int_D \hat{P}_1(t-s, (y, i), (b_1, j_1)) dy \\
 &\quad + \frac{1}{|D|} \hat{P}_2(t-s, (y, i), (b_1, j_1), (b_2, j_2)) dy] ds dt.
 \end{aligned}$$

Letting $u = t-s$

$s = s,$

$$\begin{aligned}
 & L_{\lambda} \hat{P}_2^I(\cdot, (b_1, j_1), (b_2, j_2)) \\
 &= \sum_{i=1}^k \int_0^{\infty} e^{-s(\lambda+r|D|)} r_i |D| \int_0^{\infty} e^{-\lambda u} [\hat{P}_2^I(u, (b_1, j_1), (b_2, j_2)) \\
 &\quad + \frac{1}{|D|} \hat{P}_1^I(u, (b_1, j_1)) \int_D \hat{P}_1(u, (y, i), (b_2, j_2)) dy \\
 &\quad + \frac{1}{|D|} \hat{P}_1^I(u, (b_2, j_2)) \int_D \hat{P}_1(u, (y, i), (b_1, j_1)) dy \\
 &\quad + \frac{1}{|D|} \hat{P}_2(u, (y, i), (b_1, j_1), (b_2, j_2)) dy] ds du.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{|D|} \hat{P}_1^I(u, (b_2, j_2)) \int_D \hat{P}_1(u, (y, i), (b_1, j_1)) dy \\
 & + \frac{1}{|D|} \int_D \hat{P}_2(u, (y, i), (b_1, j_1), (b_2, j_2)) dy \, du \, ds \\
 = & \sum_{i=1}^k \frac{r_i |D|}{\lambda + r_i |D|} [L_\lambda \hat{P}_2^I(\cdot, (b_1, j_1), (b_2, j_2)) \\
 & + \frac{1}{|D|} L_\lambda \hat{P}_1^I(\cdot, (b_1, j_1)) \int_D \hat{P}_1(\cdot, (y, i), (b_2, j_2)) dy \\
 & + \frac{1}{|D|} L_\lambda \hat{P}_1^I(\cdot, (b_2, j_2)) \int_D \hat{P}_1(\cdot, (y, i), (b_1, j_1)) dy \\
 & + \frac{1}{|D|} L_\lambda \int_D \hat{P}_2(\cdot, (y, i), (b_1, j_1), (b_2, j_2)) dy] , \text{ where } r = \sum_{j=1}^k r_j .
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow & \lambda L_\lambda \hat{P}_2^I(\cdot, (b_1, j_1), (b_2, j_2)) \\
 = & \sum_{i=1}^k r_i L_\lambda [\hat{P}_1^I(\cdot, (b_1, j_1)) \int_D \hat{P}_1(\cdot, (y, i), (b_2, j_2)) dy + \hat{P}_1^I(\cdot, (b_2, j_2)) \\
 & + \hat{P}_1^I(\cdot, (b_2, j_2)) \int_D \hat{P}_1(\cdot, (y, i), (b_1, j_1)) dy \\
 & + \int_D \hat{P}_2(\cdot, (y, i), (b_1, j_1), (b_2, j_2)) dy]
 \end{aligned}$$

Inverting Laplace transforms, we obtain

$$\begin{aligned} & \hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) \\ &= \sum_{i=1}^k r_i \int_0^t \int_{OD} (\hat{P}_1^I(s, (b_1, j_1)) \hat{P}_1(s, (y, i), (b_2, j_2)) \\ & \quad + \hat{P}_1^I(s, (b_2, j_2)) \hat{P}_1(s, (y, i), (b_1, j_1))) dy ds \\ & \quad + \sum_{i=1}^k r_i \int_0^t \hat{P}_2(s, (y, i), (b_1, j_1), (b_2, j_2)) dy ds. \end{aligned} \quad (1)$$

Notice that the first expression on the right hand side of (1) is simply the integral of the derivative of $\hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2))$.

From § 4.1, Theorem 2,

$$\begin{aligned} & \hat{P}_2(s, (y, i), (b_1, j_1), (b_2, j_2)) \\ &= \sum_{h=1}^k 2 \alpha_h \int_0^s \int_{OD} \hat{P}_1(s-u, (y, i), (c, h)) \hat{P}_1(u, (c, h), (b_1, j_1)) \\ & \quad \times \hat{P}_1(u, (c, h), (b_2, j_2)) dc du. \end{aligned}$$

From this and making the transformation $T = s-u$, $u = u$, the second expression on the right hand side of (1) is

$$\sum_{h=1}^k 2 \alpha_h V_h \iint_{OD}^t \hat{P}_1(u, (c, h), (b_1, j_1)) \hat{P}_1(u, (c, h), (b_2, j_2)) \\ \times \left(\sum_{i=1}^k r_i \iint_{OD}^{t-u} \hat{P}_1(T, (y, i), (c, h)) dy dT \right) dc du .$$

$$\text{But } \sum_{i=1}^k r_i \iint_{OD}^{t-u} \hat{P}_1(T, (y, i), (c, h)) dy dt \\ = \hat{P}_1^I(t-u, (c, h)) .$$

Therefore, from (1)

$$\hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) \\ = \hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2)) \\ + \sum_{h=1}^k 2 \alpha_h V_h \iint_{OD}^t \hat{P}_1^I(t-u, (c, h)) \hat{P}_1(u, (c, h), (b_1, j_1)) \hat{P}_1(u, (c, h), (b_2, j_2)) dc du .$$

As in the proof of Theorem 5 of this chapter, the result for $|D| = \infty$ follows readily by approximating D by compact sets.

q.e.d.

We now assume that $D = R^d$, and that the migration is d -dimensional Brownian motion. We consider the behaviour of the first two factorial moment densities as $t \rightarrow \infty$.

As in the case of the multitype process without immigration (see § 4.2), we let λ be the maximal real eigenvalue of the matrix $C(t)$, and we recall that

$$\lim_{t \rightarrow \infty} C_{ij}(t)e^{-\lambda t} = u_i v_j,$$

where $\tilde{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ and $\tilde{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ are the eigenvectors of $G(t)$

described in § 4.2.

$$\begin{aligned} \text{a) } \hat{P}_1^I(t, (b, j)) &= \sum_{i=1}^k r_i \int_0^t \int_{OD} \hat{P}_1(s, (y, i), (b, j)) dy ds \\ &= \sum_{i=1}^k r_i \int_0^t C_{ij}(s) \int_D p(s, y, b) dy ds \\ &= \sum_{i=1}^k r_i \int_0^t C_{ij}(s) ds. \end{aligned}$$

Clearly, we need to know $\lim_{t \rightarrow \infty} \int_0^t C_{ij}(s) ds$.

Subcritical branching: ($\lambda < 0$)

For arbitrary $\epsilon > 0$, choose T large enough that for $s > T$,

$$C_{ij}(s)e^{-\lambda s} < u_i v_j + \epsilon.$$

Then

$$\int_0^{\infty} C_{ij}(s) ds = \int_0^T C_{ij}(s) ds + \int_T^{\infty} C_{ij}(s) ds$$

$$\begin{aligned}
 &= \int_0^T C_{ij}(s) ds + \int_T^\infty C_{ij}(s) e^{-\lambda s} e^{\lambda s} ds \\
 &< \int_0^T C_{ij}(s) ds + \int_T^\infty (u_i v_j + \epsilon) e^{\lambda s} ds \\
 &= \int_0^T C_{ij}(s) ds + (u_i v_j + \epsilon) \frac{e^{\lambda T}}{-\lambda}, \text{ since } \lambda < 0.
 \end{aligned}$$

Therefore, since $\int_0^t C_{ij}(s) ds$ is a nondecreasing function of t which is bounded above, $\lim_{t \rightarrow \infty} \int_0^t C_{ij}(s) ds$ exists. So we have that as $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} \hat{P}^I(t, (b, j)) = \sum_{i=1}^k r_i \lim_{t \rightarrow \infty} \int_0^t C_{ij}(s) ds.$$

Critical branching: ($\lambda = 0$)

For arbitrary $\epsilon > 0$, choose T large enough that for $s > T$, $|C_{ij}(s) - u_i v_j| < \epsilon$. Then for $t > T$, $\frac{1}{t} \int_0^t C_{ij}(s) ds = \frac{1}{t} \int_0^T C_{ij}(s) ds + \frac{1}{t} \int_T^t C_{ij}(s) ds$, and $\frac{1}{t} (t-T)(u_i v_j - \epsilon) < \frac{1}{t} \int_T^t C_{ij}(s) ds < \frac{1}{t} (t-T)(u_i v_j + \epsilon)$.

Since ϵ is arbitrary, we have

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t C_{ij}(s) ds &= \lim_{t \rightarrow \infty} \frac{1}{t} (t-T) u_i v_j \\
 &= u_i v_j.
 \end{aligned}$$

$\therefore \int_0^t c_{ij}(s)ds$, and hence $\hat{P}_1^I(t, (b, j))$ grow as t .

Supercritical branching: ($\lambda > 0$)

For arbitrary $\epsilon > 0$, choose T large enough that for $s > T$,

$$c_{ij}(s)e^{-\lambda s} > u_i v_j - \epsilon.$$

$$\begin{aligned} \therefore \int_0^\infty c_{ij}(s)ds &= \int_0^T c_{ij}(s)ds + \int_T^\infty c_{ij}(s)ds \\ &= \int_0^T c_{ij}(s)ds + \int_T^\infty c_{ij}(s)e^{-\lambda s}e^{\lambda s}ds \\ &> \int_0^T c_{ij}(s)ds + \int_T^\infty (u_i v_j - \epsilon)e^{\lambda s}ds \\ &= \infty. \end{aligned}$$

Thus, $P_2(t, (b, j)) \rightarrow \infty$ as $t \rightarrow \infty$.

$$\begin{aligned} \text{b) } \hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) &= \hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2)) \\ &+ \sum_{h=1}^k 2\alpha_h v_h \int_0^t \int_{OD} \hat{P}_1^I(t-u, (c, h)) \hat{P}_1(u, (c, h), (b_1, j_1)) \hat{P}_1(u, (c, h), (b_2, j_2)) dc du. \end{aligned}$$

Subcritical branching: ($\lambda < 0$)

Since $\hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2))$ remains bounded as $t \rightarrow \infty$, we need only consider the behaviour of:

$$\begin{aligned} & \int_0^t \int_{OD} \hat{P}_1^I(t-u, (c, h)) \hat{P}_1(u, (c, h), (b_1, j_1)) \hat{P}_1(u, (c, h), (b_2, j_2)) dc du \\ &= \sum_{i=1}^k r_i \int_0^{t-u} \int_{ODO} C_{ih}(s) C_{hj_1}(u) p(u, c, b_1) C_{hj_2}(u) p(u, c, b_2) ds dc du \\ &= \sum_{i=1}^k r_i \int_0^t \left[\int_0^{t-u} C_{ij}(s) ds \right] C_{hj_1}(u) C_{hj_2}(u) p(2u, b_1, b_2) du. \end{aligned} \quad (1)$$

But from the discussion of the behaviour of $\hat{P}_1(t, (b, j))$, we have that $\int_0^{t-u} C_{ij}(s) ds$ is bounded above, say by K_1 . $p(2u, b_1, b_2)$ is also bounded,

say by K_2 .

$$\begin{aligned} (1) &\leq K_1 K_2 \sum_{i=1}^k r_i \int_0^t C_{hj_1}(u) C_{hj_2}(u) du \\ &= K_1 K_2 \sum_{i=1}^k r_i \int_0^t C_{hj_1}(u) e^{-\lambda u} C_{hj_2}(u) e^{-\lambda u} e^{2\lambda u} du \\ &\leq K K_1 K_2 \sum_{i=1}^k \int_0^t e^{2\lambda u} du, \end{aligned}$$

where K is some constant greater than $C_{hj_1}(u)e^{-\lambda u}C_{hj_2}e^{-\lambda u}$. Since $\lambda < 0$,

we have that
$$\int_0^t \int_{OD} \hat{P}_1^I(t-u, (c, h)) \hat{P}_1^I(u, (c, h), (b_1, j_1)) \hat{P}_1^I(u, (c, h), (b_2, j_2)) dc du$$

is bounded. This gives us that $\hat{P}_2^I(t, (b_1, j_1), (b_2, j_2))$ remains bounded as $t \rightarrow \infty$.

Critical branching: ($\lambda = 0$)

Since $\hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2)) \rightarrow \infty$ as $t \rightarrow \infty$, $\hat{P}_2^I(t, (b_1, j_1), (b_2, j_2)) \rightarrow \infty$ also.

Supercritical branching: ($\lambda > 0$)

Since $\hat{P}_1^I(t, (b_1, j_1)) \hat{P}_1^I(t, (b_2, j_2)) \rightarrow \infty$ as $t \rightarrow \infty$, $\hat{P}_2(t, (b_1, j_1), (b_2, j_2)) \rightarrow \infty$ also.

As before, these results show that as $t \rightarrow \infty$, for bounded sets A ,

$$E(N_A(t), j) \rightarrow |A| \sum_{i=1}^k r_i \lim_{t \rightarrow \infty} \int_0^t C_{ij}(s) ds, \text{ where } (N_A(t), j) \text{ is the number}$$

of type j particles in A . For subcritical branching, $E(N_A(t), j)$

remains bounded. For critical and supercritical branching, $E(N_A(t), j)$

becomes unbounded as $t \rightarrow \infty$. For $A \cap B = \emptyset$, $E(N_A(t), j_1) (N_B(t), j_2)$

remains bounded for subcritical branching, and blows up for critical and supercritical branching.

6. CENTRAL LIMIT THEOREMS

6.1 THE CENTRAL LIMIT THEOREM FOR THE SINGLE-TYPE PROCESS WITH IMMIGRATION

In this chapter, we return to the single-type branching diffusion model with immigration. We assume that $D = R^d$ and that the migration process is Brownian motion. We want to consider $N_A(t)$ for large but bounded sets $A \subseteq R^d$.

Theorem 1 (Central Limit Theorem)

Under the assumptions of the model described in § 5.1, if $D = R^d$ and $p(t, a, b) = \frac{1}{(2\pi t)^{d/2}} \exp(-|a-b|^2/2t)$, then as $A \uparrow D$, $|A| < \infty$, and A symmetric about the origin,

$$\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var} N_A(t)}} \xrightarrow{f} N(0, 1).$$

Proof.

From § 5.2, Theorems 2 and 3, we have that

$$E(N_A(t)(N_A(t)-1)\dots(N_A(t)-(n-1)))$$

$$= \int \dots \int_A \hat{p}_n^I(t, b_1, \dots, b_n) db_1 \dots db_n$$

$$= \sum_{k=1}^n (-1)^{n-k} S_n^k (M_t r|A|)^k (M_t v \alpha)^{n-k} + o\left(\frac{1}{|A|^{n_1}}\right) \quad (1)$$

for arbitrary $n_1 > 0$, where the S_n^k 's are Stirling numbers of the first kind, and

$$M_t = \begin{cases} \frac{(1-e^{-mt})}{m}, & \alpha \neq \frac{1}{2}, m = v(1-2\alpha) \\ t, & \alpha = \frac{1}{2}. \end{cases}$$

We will use these factorial moments to calculate the usual moments about the origin, and then after normalization we will see that

$$E \left[\left(\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var} N_A(t)}} \right)^n \right] \rightarrow \begin{cases} (n-1)(n-3)\dots 5 \cdot 3 \cdot 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Since these are the moments of the standard normal distributions, which is uniquely determined by its moments, we have that as $A \uparrow D$,

$$\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var} N_A(t)}} \xrightarrow{L} N(0,1).$$

Using (1), we can calculate the usual moments about the origin:

$$E[N_A(t)^n] = \sum_{j=1}^n S_n^j \mu_{[j]}'(A)$$

$$= \mu_n'(A)$$

where the S_n^j 's are Stirling numbers of the second kind. Notice that the dependence on t is suppressed in this notation. From (1),

$$\mu_n'(A) = \sum_{j=1}^n \sum_{k=1}^j S_n^j S_j^k (-1)^{j-k} (M_t r|A|)^k (M_t v \alpha)^{j-k} + o\left(\frac{1}{|A|^{n_1}}\right)$$

where $n_1 > 0$ is arbitrary.

From § 5.2 we have that $E(N_A(t)) = M_t r|A|$, so converting to moments about the mean, we obtain

$$E[(N_A(t) - E(N_A(t)))^n]$$

$$= \mu_n(A).$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n-h} \mu_h'(A) (M_t r|A|)^{n-h} + o\left(\frac{1}{|A|^{n_1}}\right)$$

$$= \sum_{h=0}^n \sum_{j=1}^h \sum_{k=1}^j \binom{n}{h} (-1)^{n-h} (M_t r|A|)^{n-h} S_h^j S_j^k (-1)^{j-k} (M_t r|A|)^k (M_t v \alpha)^{j-k} + o\left(\frac{1}{|A|^{n_1}}\right)$$

$$= \sum_{h=0}^n \sum_{j=1}^h \sum_{k=1}^j \binom{n}{h} (-1)^{n-h+j-k} \tilde{S}_h^j S_j^k P^{n-h+k} Q^{j-k} + o\left(\frac{1}{|A|^{n_1}}\right),$$

where $P = M_t r|A|$ and $Q = M_t V \alpha$. Note that Q does not depend on $|A|$.

Also, define $\tilde{S}_n^k = S_n^k = 0$ for $k > n$ or $k \leq 0$.

$$E(N_A(t)) = M_t r|A| = P.$$

$$\text{Var}(N_A(t)) = E(N_A(t)(N_A(t)-1)) + E(N_A(t)) - (E(N_A(t)))^2$$

$$= M_t^2 r|A| V \alpha + (M_t r|A|)^2 + M_t r|A| - (M_t r|A|)^2 + o\left(\frac{1}{|A|^{n_1}}\right)$$

$$= P(1 + Q) + o\left(\frac{1}{|A|^{n_1}}\right) \text{ for arbitrary } n_1 > 0.$$

So we want to show that as $A \uparrow D$,

$$E\left[\left(\frac{N_A(t)-P}{\sqrt{P(1+Q)}}\right)^n\right] \rightarrow \begin{cases} (n-1)(n-3)\dots 5\cdot 3\cdot 1 & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

or equivalently, since Q does not depend on $|A|$,

$$\frac{\mu_n(A)}{P^{n/2}} \rightarrow \begin{cases} (1+Q)^{n/2} (n-1)\dots 5\cdot 3\cdot 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

$$\text{Let } \mu_n = \sum_{h=0}^n \sum_{j=1}^h \sum_{k=1}^j \binom{n}{h} (-1)^{n-h+j-k} s_h^j s_p^{j-k} Q^{j-k} \quad (2)$$

$\mu_n(A) = \mu_n + o\left(\frac{1}{|A|^{n_1}}\right)$, where $n_1 > 0$ is arbitrary, and so

$$\lim_{A \uparrow D_P} \frac{\mu_n(A)}{p^{n/2}} = \lim_{A \uparrow D_P} \frac{\mu_n}{p^{n/2}}.$$

μ_n is a polynomial in P of degree at most n . If we can show that the coefficient of P^s (which in turn is a polynomial in Q) is 0 for $s > \left[\frac{n}{2}\right]$ ($\left[\frac{n}{2}\right]$ is the largest integer less than or equal to $\frac{n}{2}$), then for

n odd, since $\left[\frac{n}{2}\right] < \frac{n}{2}$, μ_n will be a polynomial of degree less than $\frac{n}{2}$.

Thus, $\frac{\mu_n}{p^{n/2}} \rightarrow 0$, as $|A| \rightarrow \infty$, and also $P \rightarrow \infty$. For n even, we will have

that μ_n is a polynomial of degree $\frac{n}{2}$, and that $\frac{\mu_n}{p^{n/2}} \rightarrow R(n)$, where

$R(n)$ is the coefficient of $P^{n/2}$ in μ_n . As indicated previously, $R(n)$ is a polynomial in Q , and we would like to show that

$$R(n) = (n-1)(n-3)\dots 5 \cdot 3 \cdot 1 (1+Q)^{n/2}. \quad (3)$$

Going back to (2), we will consider the coefficient of P^{n-x} , for

$x = h-k$, x fixed. This coefficient is

$$\sum_{h=0}^n \sum_{j=1}^h \binom{n}{h} (-1)^{n-x-j} s_h^j s_p^{j-h+x} Q^{x+j-h}. \quad (4)$$

We are now dealing with a polynomial in Q of degree at most x .
Consider the coefficient in (4) of Q^{x-y} , where $y = h-j, y$ fixed.

This coefficient is $K_n(x, y)$, where

$$K_n(x, y)$$

$$= \sum_{h=0}^n \binom{n}{h} (-1)^{n-x+y-h} \tilde{S}_h^{h-y} S_{h-y}^{h-x}$$

$$= (-1)^{x+y} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} \tilde{S}_h^{h-y} S_{h-y}^{h-x} \quad (5)$$

According to Jordan ([15], pg. 171), $\tilde{S}_h^{h-y}$ is a polynomial of variable h of degree $2y$ (with coefficients depending on y). Again by Jordan, pg. 150, S_{h-y}^{h-x} is a polynomial of variable $(h-y)$ of degree $2(x-y)$ (with coefficients depending on $(x-y)$). So $\tilde{S}_h^{h-y} S_{h-y}^{h-x}$ is a polynomial of variable h of degree $2(x-y)$ with coefficients depending on x and y .

As a result, we have that $\tilde{S}_h^{h-y} S_{h-y}^{h-x}$ is a polynomial of variable h of degree $2y+2x-2y = 2x$, and so can be expressed uniquely in the following way:

$$\tilde{S}_h^{h-y} S_{h-y}^{h-x} = \sum_{k=0}^{2x} c_k(x, y) \binom{h}{2x-k} \quad (6)$$

Note here that $\tilde{S}_h^{h-y} S_{h-y}^{h-x} = 0$ for $h \leq x$.

For any sequence of numbers $a_0, a_1, \dots$ we have the n^{th} order difference

$$\Delta^n a_0 = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} a_h.$$

Using this notation, (5) gives us

$$K_n(x, y) = (-1)^{x+y} \Delta_{\sim 0}^{n, 0-y} S_{0-y}^{0-x}. \quad (7)$$

But $\Delta^n \binom{h}{2x-k} = \binom{h}{2x-k-n}$, and so from (6),

$$\begin{aligned} \Delta_{\sim 0}^{n, 0-y} S_{0-y}^{0-x} &= \sum_{k=0}^{2x} c_k(x, y) \Delta^n \binom{0}{2x-k} \\ &= \sum_{k=0}^{2x} c_k(x, y) \binom{0}{2x-k-n} \\ &= \begin{cases} c_{2x-n}(x, y) & \text{if } 2x \geq n \\ 0 & \text{if } 2x < n, \end{cases} \end{aligned}$$

since $\binom{0}{2x-k-n} = 0$ unless $2x-k-n = 0$, or equivalently, $k = 2x-n$.

From (7)

$$\Delta^n S_0^{0-y} S_{0-y}^{0-x} = (-1)^{x+y} K_n(x, y)$$

$$= \begin{cases} c_{2x-n}(x, y) & \text{if } 2x \geq n \\ 0 & \text{if } 2x < n. \end{cases} \quad (8)$$

Thus, if $2x < n$, the coefficient $K_n(x, y)$ of q^{x-y} is 0 for every value of y . From (4), this in turn implies that the coefficient of p^{n-x} is 0 if $2x < n$, or for $x < \frac{n}{2}$.

We now have that the coefficient of p^s in μ_n is 0 for $s > \lfloor \frac{n}{2} \rfloor$.

This gives us that the odd order moments $\frac{\mu_n(A)}{p^{n/2}} \rightarrow 0$ as $|A| \rightarrow \infty$.

The even order moments $\frac{\mu_n(A)}{p^{n/2}} \rightarrow R(n)$, where $R(n)$ is the coefficient of $p^{n/2}$ in μ_n . Letting $x = \frac{n}{2}$ in (4), we see that $R(n)$ will be a polynomial in q of degree $\frac{n}{2}$. The coefficient of $q^{n/2-y}$ in $R(n)$

is $K_n(\frac{n}{2}, y)$. From (8), $K_n(\frac{n}{2}, y) = c_0(\frac{n}{2}, y) (-1)^{n/2+y}$.

Recalling (3), we would like to show that

$$c_0\left(\frac{n}{2}, y\right) = (n-1)(n-3)\dots 5.3.1 \binom{n/2}{y} (-1)^{n/2+y}$$

$$= (2x-1)(2x-3)\dots 5.3.1 \binom{x}{y} (-1)^{x+y}, \text{ for } x = \frac{n}{2}.$$

Consider (6): from this we have that

$$\begin{aligned}\Delta \tilde{S}_h^{h-y} S_{h-y}^{h-x} &= \sum_{k=0}^{2x} C_k(x,y) \Delta \binom{h}{2x-k} \\ &= \sum_{k=0}^{2x} C_k(x,y) \binom{h}{2x-k-1} \quad (9) \\ &= C_0(x,y) \binom{h}{2x-1} + C_1(x,y) \binom{h}{2x-2} + \dots + C_{2x-1}(x,y) \binom{h}{0}.\end{aligned}$$

(Notice that the differencing is with respect to the variable h .)

On the other hand,

$$\Delta \tilde{S}_h^{h-y} S_{h-y}^{h-x} = \tilde{S}_{h+1}^{h+1-y} S_{h+1-y}^{h+1-x} - \tilde{S}_h^{h-y} S_{h-y}^{h-x}.$$

By Jordan (see Appendix A)

$$\tilde{S}_{h+1}^{h+1-y} = \tilde{S}_h^{h-y} + (h+1-y) \tilde{S}_h^{h+1-y}$$

and also

$$S_{h+1-y}^{h+1-x} = S_{h-y}^{h-x} - (h-y) S_{h-y}^{h+1-x}$$

$$\begin{aligned}
 \therefore \Delta \tilde{S}_h^{h-y} S_{h-y}^{h-x} &= \tilde{S}_h^{h-y} S_{h+1-y}^{h+1-x} + (h+1-y) \tilde{S}_h^{h+1-y} S_{h+1-y}^{h+1-x} - \tilde{S}_h^{h-y} S_{h-y}^{h-x} \\
 &= \tilde{S}_h^{h-y} (S_{h+1-y}^{h+1-x} - S_{h-y}^{h-x}) + (h+1-y) \tilde{S}_h^{h+1-y} S_{h+1-y}^{h+1-x} \\
 &= -(h-y) \tilde{S}_h^{h-y} S_{h-y}^{h+1-x} + (h+1-y) \tilde{S}_h^{h+1-y} S_{h+1-y}^{h+1-x}
 \end{aligned}$$

Using (6) in the right hand side of this equation,

$$\begin{aligned}
 \Delta \tilde{S}_h^{h-y} S_{h-y}^{h-x} &= - \sum_{k=0}^{2(x-1)} C_k(x-1, y) (h-y) \binom{h}{2(x-1)-k} \\
 &\quad + \sum_{k=0}^{2(x-1)} C_k(x-1, y-1) (h+1-y) \binom{h}{2(x-1)-k} \\
 &= \sum_{k=0}^{2x-2} (C_k(x-1, y-1) - C_k(x-1, y)) \binom{h}{2(x-1)-k} \\
 &\quad + \sum_{k=0}^{2x-2} (y C_k(x-1, y) - (y-1) C_k(x-1, y-1)) \binom{h}{2(x-1)-k}
 \end{aligned}$$

Notice that $\binom{h}{2(x-1)-k} = \binom{2x-1-k}{2x-1-k} \binom{h}{2x-2-k}$

$$\therefore \Delta \tilde{S}_h^{h-y} S_{h-y}^{h-x}$$

$$\begin{aligned}
 &= \sum_{k=0}^{2x-2} (C_k(x-1, y-1) - C_k(x-1, y)) \left((2x-1-k) \binom{h}{2x-1-k} + (2x-2-k) \binom{h}{2x-2-k} \right) \\
 &+ \sum_{k=0}^{2x-2} (yC_k(x-1, y) - (y-1)C_k(x-1, y-1)) \binom{h}{2x-2-k} \\
 &= \sum_{k=0}^{2x-2} (2x-1-k) (C_k(x-1, y-1) - C_k(x-1, y)) \binom{h}{2x-1-k} \\
 &+ \sum_{k=0}^{2x-2} ((2x-2-k) (C_k(x-1, y-1) - C_k(x-1, y)) + yC_k(x-1, y) - (y-1)C_k(x-1, y-1)) \binom{h}{2x-2-k}.
 \end{aligned} \tag{10}$$

(9) and (10) gives us two equivalent expressions for $\Delta S_h^{h-y} S_h^{h-x}$.

Equating the coefficients of $\binom{h}{2x-1}$ in (9) and (10) we obtain

$$C_0(x, y) = (2x-1) (C_0(x-1, y-1) - C_0(x-1, y)). \tag{11}$$

We have already shown that $\mu_2 = P(1+Q)$, so $R(2) = (1+Q)$. Thus, $K_2(1, 0) = K_2(1, 1) = 1$. From (8), $K_2(1, y) = C_0(1, y) (-1)^{1+y}$, so $-C_0(1, 0) = C_0(1, 1) = 1$. In other words, $C_0(1, 0) = (-1)^{(1+0)} \binom{1}{0}$, and $C_0(1, 1) = (-1)^{(1+1)} \binom{1}{1}$.

Assume that $C_0(k, y) = \begin{cases} (-1)^{k+y} (2k-1) \dots 5 \cdot 3 \cdot 1 \binom{k}{y} & \forall y \leq k \\ 0 & \forall y > k \end{cases}$
for every $k \leq x$.

Using (11), by induction, for $y \leq x$ we have

$$\begin{aligned}
 c_0(x+1, y) &= (2x+1)(c_0(x, y-1) - c_0(x, y)) \\
 &= (2x+1)(2x-1)\dots 5 \cdot 3 \cdot 1 \cdot ((-1)^{x+y-1} \binom{x}{y-1} - (-1)^{x+y} \binom{x}{y}) \\
 &= (2x+1)\dots 5 \cdot 3 \cdot 1 \cdot (-1)^{x+y+1} \left(\binom{x}{y-1} + \binom{x}{y} \right) \\
 &= (-1)^{x+y+1} (2x+1)(2x-1)\dots 5 \cdot 3 \cdot 1 \cdot \binom{x+1}{y} .
 \end{aligned}$$

For $y = x+1$,

$$\begin{aligned}
 c_0(x+1, x+1) &= (2x+1)(c_0(x, x) - c_0(x, x+1)) \\
 &= (2x+1)(2x-1)\dots 5 \cdot 3 \cdot 1 \cdot ((-1)^{x+x} \binom{x}{x} - 0) \\
 &= (2x+1)(2x-1)\dots 5 \cdot 3 \cdot 1 \cdot (-1)^{2(x+1)} \binom{x+1}{x+1} .
 \end{aligned}$$

For $y > x+1$,

$$\begin{aligned}
 c_0(x+1, y) &= (2x+1)(c_0(x, y-1) - c_0(x, y)) \\
 &= (2x+1)(0-0) \\
 &= 0 .
 \end{aligned}$$

Therefore, by induction we have shown that

$$c_0(x,y) = (2x-1)(2x-3)\dots 5\cdot 3\cdot 1 \cdot \binom{x}{y} (-1)^{x+y}.$$

But recall that for even n , $R(n) = \sum_{y=0}^{n/2} q^{(n/2-y)} K_n\left(\frac{n}{2}, y\right),$

from (8).
$$= \sum_{y=0}^{n/2} q^{(n/2-y)} (-1)^{(n/2+y)} c_0\left(\frac{n}{2}, y\right),$$

So
$$R(n) = \sum_{y=0}^{n/2} (n-1)(n-3)\dots 5\cdot 3\cdot 1 \binom{n/2}{y} q^{(n/2-y)}$$

$$= (n-1)(n-3)\dots 5\cdot 3\cdot 1 (1+q)^{n/2}.$$

Thus, we have shown that

$$\frac{\mu_n(A)}{p^{n/2}} = \frac{\mu_n}{p^{n/2}} + o\left(\frac{1}{|A|^{n/2}}\right)$$

$$\begin{cases} (n-1)(n-3)\dots 5\cdot 3\cdot 1 (1+q)^{n/2} & n \text{ even order} \\ 0 & n \text{ odd} \end{cases}$$

This proves that as $|A| \rightarrow \infty$

$$E\left[\left(\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var}N_A(t)}}\right)^n\right] \rightarrow \begin{cases} (n-1)(n-3)\dots 5\cdot 3\cdot 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

so
$$\frac{N_A(t) - E(N_A(t))}{\sqrt{\text{Var}N_A(t)}} \xrightarrow{d} N(0,1).$$

q.e.d.

Corollary 1.

For the subcritical process with immigration, the Central Limit Theorem is valid for the stationary distribution of N_A .

Proof.

In § 5.3, the existence of a stationary distribution for N_A as $t \rightarrow \infty$ was proven, and approximations were found for the factorial moments of this distribution for large compact sets A . From § 5.3, equation (1), these moments are:

$$E(N_A(N_A-1)\dots(N_A-(n-1))) = \sum_{k=1}^n (-1)^{n-k} S_n^k \left(\frac{r|A|}{m}\right)^k \left(\frac{v\alpha}{m}\right)^{n-k}$$

But this is the same form as equation (1) in the Central Limit Theorem replacing M_t with $\frac{1}{m}$, so by exactly the same proof, the theorem holds for the steady-state distribution.

q.e.d.

For the second corollary, we consider a sequence of concentric increasing compact spheres A_k in R^d , with the radius of A_k equal to k . We will be using the population probability space notation for a point process.

(See Appendix B).

Let Q be the event that some compact set A is empty. Then if

$$P_s(Q|x_1) = P(\text{at } t+s \text{ the population is in some state in}$$

$$Q|x_1 \in X_u \text{ at time } t),$$

we have

Corollary 2.

$$P_s(Q|x_1) = \lim_{k \rightarrow \infty} P_s(Q|\Gamma_k(x_1)) \text{ almost surely with respect to } P_s,$$

where $\Gamma_k(x_1)$ is that subset of points in x_1 belonging to A_k .

Note: This corollary simply states that the effect on A of points outside large compact sets containing A tends to be negligible.

Before proving this corollary, we need the following lemma:

Lemma.

If $C_s = ((M_s r \omega_d (1 + M_s V \alpha))^{\frac{1}{2}} + M_s r \omega_d)$ where $D = R^d$, and $\omega_d k^d = |A_k|$, then $P(N_{A_k}(s) > C_s k^d \text{ infinitely often}) = 0$.

Proof.

Without loss of generality, assume that the centre of the spheres is the origin.

Let $z(k, s) = \frac{N_{A_k}(s) - P}{(P(1+Q))^{\frac{1}{2}}}$, where $P = M_s r |A_k|$, and $Q = MsV \alpha$. Recall that

$E(N_{A_k}(s)) = P$ and $\text{Var}(N_{A_k}(s)) = P(1+Q)$. Dependence of P and Q on s and k is suppressed for notational convenience.

From the Central Limit Theorem, we know that $E(z(k, s)) = 0$ and $\text{Var}(z(k, s)) = 1 + o\left(\frac{1}{|A_k|^{\frac{1}{n_1}}}\right)$ for an arbitrary $n_1 > 0$. Therefore, for any

constant $\rho > 1$, we can choose K_1 so that for every $k > K_1$,

$$\text{Var}(z(k, s)) = \int_{-\infty}^{\infty} z^2(k, s) P(dz(k, s)) < \rho. \quad (1)$$

Now consider $P(z(k, s) > k)$.

$$P(z(k, s) > k) = P\left(\frac{N_{A_k}(s) - P}{(P(1+Q))^{\frac{1}{2}}} > k\right) = \epsilon_k \quad (2)$$

We need an upper bound on ϵ_k .

Note that we have suppressed the dependence of ϵ_k on s . From (1), if

$k > K_1$,

$$\begin{aligned} \rho &> \int_{-\infty}^{\infty} z^2(k, s) P(dz(k, s)) \\ &\geq \int_k^{\infty} z^2(k, s) P(dz(k, s)) \\ &\geq k^2 \int_k^{\infty} P(dz(k, s)) \\ &= k^2 \epsilon_k, \text{ from (2).} \end{aligned}$$

Therefore, $\epsilon_k < \frac{\rho}{k^2}$.

Using the upper bounds just calculated, we can now consider $\sum_{k=K_1}^{\infty} P(z(k,s) > k)$.

$$\begin{aligned} & \sum_{k=K_1}^{\infty} P(z(k,s) > k) \\ &= \sum_{k=K_1}^{\infty} \epsilon_k \\ &\leq \sum_{k=K_1}^{\infty} \frac{\rho}{k^2} \\ &< \infty. \end{aligned}$$

(3)

$$\delta \quad \text{Now, } z(k,s) > k \Leftrightarrow \frac{N_{A_k}(s) - P}{(P(1+Q))^{\frac{1}{2}}} > k$$

$$\Leftrightarrow N_{A_k}(s) > k(M_s r |A_k| (1+M_s V \alpha))^{\frac{1}{2}} + M_s r |A_k|$$

$$= k^{(d+2)/2} (M_s r \omega_d (1+M_s V \alpha))^{\frac{1}{2}} + M_s r \omega_d k^d,$$

where $|A_k| = \omega_d k^d$.

But $k^{(d+2)/2} (M_s r \omega_d (1+M_s V \alpha))^{\frac{1}{2}} + M_s r \omega_d k^d$

$$< k^d (M_s r \omega_d (1+M_s V \alpha))^{\frac{1}{2}} + M_s r \omega_d$$

for $k > 1, d \geq 2$

$$= C_s k^d.$$

Therefore, $N_{A_k}(s) > C_s k^d \Rightarrow z(k, s) > k$, and so $P(N_{A_k}(s) > C_s k^d) \leq P(z(k, s) > k)$.

By (3) above, $\sum_{k=K_1}^{\infty} P(N_{A_k}(s) > C_s k^d) < \infty$. By the ~~Borel~~ Borel-Cantelli Lemma this implies that $P(N_{A_k}(s) > C_s k^d \text{ i.o.}) = 0$. q.e.d.

Proof of Corollary 2.

Since A is compact, $A \subset A_k$ for all $k > K$, for some finite K .

If $j > K$, then $d(A, \bar{A}_j) \geq j - K$.

Without loss of generality, suppose the particles $\{x_i\}$ are ordered by their distance from the origin.

Let $P_s(Q|x_i) = P(\text{at time } t+s, \text{ none of the progeny of the particle at } x_i \text{ at time } t \text{ are in } A)$,

$P_s(Q) = P(\text{at time } t+s, \text{ none of the progeny of particles which immigrated in } (t, t+s) \text{ are in } A)$.

By independence of the particles,

$$P_s(Q|\{x_i\}) = P_s(Q) \prod_{i=1}^{\infty} P_s(Q|x_i).$$

Let $k > K$.

Suppose $\Gamma_k\{x_i\} = \{x_1, \dots, x_{n_k}\}$.

Then $\prod_{i=1}^{\infty} P_s(Q|x_i)$

= $P(\text{none of the particles in } \bar{A}_k \text{ at time } t \text{ have progeny in } A \text{ at time } t+s)$

= $1-P$ (at least one of the particles in $\bar{A}_k$ at time t has an offspring in A at time $t+s$)

$$= 1-Q_s(k) \quad (1)$$

Now suppose that we have a particle at x_1 at time t , where $x_1 \in \bar{A}_j \cap A_{j+1}$. Then if x_1 has x progeny alive at $t+s$, then since $d(A, \bar{A}_j) \geq j-K$, the probability that any of these progeny are in A is

$$\begin{aligned} & x \int_A p(s, x_1, b) db \\ & \leq x \int_A \frac{1}{(2\pi s)^{d/2}} \exp(-(j-K)^2/2s) db \\ & = x \frac{|A|}{(2\pi s)^{d/2}} \exp(-(j-K)^2/2s) \end{aligned}$$

Let the probability of any particle alive at t having exactly x progeny alive at $t+s$ be $p_s(x)$.

$$\begin{aligned} (1-P_s(Q|x_1)) & < \sum_{x=0}^{\infty} x p_s(x) \frac{|A|}{(2\pi s)^{d/2}} \exp(-(j-K)^2/2s) \\ & = e^{-ms} \frac{|A|}{(2\pi s)^{d/2}} \exp(-(j-K)^2/2s) \end{aligned}$$

$(\sum_{x=0}^{\infty} x p_s(x) = E [\# \text{ progeny alive in } D \text{ at } t+s \text{ of a single particle alive at } t, \text{ at } a \in D])$

$$= e^{-ms} \int_D p(s, a, b) db$$

$$= e^{-ms}, \text{ by } \S 3.1, \text{ Theorem 1).}$$

Let $l_j = 0, 1, 2, \dots$ be the total number of the x_i 's in $\bar{A}_j \cap A_{j+1}$.
If n_j is the total number of particles in A_j , then $\sum_{i=0}^{j-1} l_i = n_j$.

Trivially, this gives us that $l_j \leq n_{j+1}$, for every j . Therefore,
 $P(\text{at least one of the particles in } \bar{A}_j \cap A_{j+1} \text{ at time } t \text{ has an offspring}$

$$\text{in } A \text{ at } t+s) < l_j e^{-ms} \frac{|A|}{(2\pi s)^{d/2}} \exp(-(j-k)^2/2s).$$

This gives us that

$Q_s(k) = P(\text{at least one of the particles in } \bar{A}_k \text{ at time } t \text{ has an offspring in } A \text{ at } t+s)$

$$< |A| \sum_{j=k}^{\infty} l_j \frac{e^{-ms}}{(2\pi s)^{d/2}} \exp(-(j-k)^2/2s)$$

$$\leq |A| \frac{e^{-ms}}{(2\pi s)^{d/2}} \sum_{j=k}^{\infty} n_{j+1} \exp(-(j-k)^2/2s). \quad (2)$$

We want to show that with probability 1, (2) converges. But by the Lemma, $n_{j+1} > C_s (j+1)^d$ infinitely often with probability 0. This implies that with probability 1, the state $\{x_1\}$ will be such that we can choose K_1 so that for $\forall j > K_1$, $n_{j+1} \leq C_s (j+1)^d$.

For $k > \max(K, K_1)$

$$\begin{aligned} Q_s(k) &< |A| \frac{e^{-ms}}{(2\pi s)^{d/2}} \sum_{j=k}^{\infty} n_{j+1} \exp(-(j-K)^2/2s) \\ &\leq |A| \frac{e^{-ms} C_s}{(2\pi s)^{d/2}} \sum_{j=k}^{\infty} (j+1)^d \exp(-(j-K)^2/2s) \end{aligned}$$

But since this sum converges, for any $\epsilon > 0$, we can choose K_2 large enough that if $k > K_2$, $Q_s(k) < \epsilon$.

Therefore, from (1), if k is sufficiently large

$$\begin{aligned} \prod_{n_k+1}^{\infty} P_s(Q|x_1) &= 1 - Q_s(k) \\ &> 1 - \epsilon, \end{aligned}$$

$$\text{and so } P_s(Q|\Gamma_k(x_1)) = P_s(Q) \prod_{1}^{n_k} P_s(Q|x_1)$$

$$> P_s(Q|x_1)$$

$$= P_s(Q) \prod_{1}^{\infty} P_s(Q|x_1)$$

$$> P_s(Q) \prod_{1}^{n_k} P_s(Q|x_1) (1-\epsilon)$$

$$= P_s(Q|\Gamma_k(x_1)) (1-\epsilon).$$

Since ϵ is arbitrary, $P_s(Q|(x_1)) = \lim_{k \rightarrow \infty} P_s(Q|\Gamma_k(x_1))$.

q.e.d.

6.2 EFFECTS OF RESCALING THE PROCESS

In § 5.4, renormalization of the single-type process was discussed. In particular, for $D = R^d$ and Brownian motion as the migration process the branching rate is increased by a factor of k^d and the variance of the diffusion process is decreased by a factor of $\frac{1}{k^2}$. It was shown that the effect is mathematically equivalent to inflating distances in R^d by a factor of k .

In § 5.4, $N_A^k(t)$ was defined to be the number of particles in A at time t under the rescaled process. Because of the mathematical equivalence, we have $P(N_A^k(t) = n) = P(N_{kA}(t) = n)$ where $N_A(t)$ is the original process before rescaling, and kA is the dilation of the set A by a factor of k . So $N_A^k(t) \stackrel{f}{=} N_{kA}(t)$. But by the Central Limit Theorem, for a symmetric compact set $A \subseteq D$,

$$\frac{N_{kA}(t) - E(N_{kA}(t))}{\sqrt{\text{Var} N_{kA}(t)}} \xrightarrow{f} N(0,1)$$

as $k \rightarrow \infty$. We now have the following corollary to § 6.1, Theorem 1:

Corollary 3.

For the branching diffusion process with immigration, under the rescaling described in this section, if $N_A^k(t)$ is the point process for the scaling factor k , then

$$\frac{N_A^k(t) - E(N_A^k(t))}{(\text{Var} N_A^k(t))^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$$

as $k \rightarrow \infty$, for any compact symmetric set $A \subseteq D$.

Intuitively, the corollary simply says that by increasing the rate of immigration and slowing down the migration sufficiently the distribution of the number of particles in any compact symmetric set can be approximated by the normal distribution.

7. CONCLUSION

In this study, the factorial moment density functions of the random variable $N_A(t)$ were calculated for a few special cases of branching diffusion processes. Since $N_A(t)$ is integral-valued, its distribution function has been completely characterized.

However, many restrictions were placed on the model in order to obtain the factorial moment densities.

1. The branching process was restricted to death, splitting and mutation.
2. Immigration rates, branching rates, and branching offspring distributions were assumed to be space and time-homogeneous.
3. All particles were assumed to move and branch independently.
4. All processes were assumed to be Markovian.

Such extensive assumptions limit the applicability of the model to actual biological phenomena, but it does appear that it will be possible to generalize the model somewhat. Weakening the first two restrictions seems to be the most promising step for further research. It would also be interesting to study a model in which interactions between the particles was allowed.

It is clear that more study should be done on multitype processes, and on the effects of generalizing the initial conditions on the population. Central Limit Theorems for subcritical processes with initial random fields seem probable. More work would be useful in characterizing the behaviour of the joint distribution functions of the random variables $N_{A_i}(t)$ for an arbitrary number of sets $A_i \subseteq D$.

It is unfortunate that the form of the recursion relations developed for the factorial moment densities make it difficult to obtain exact expressions for the higher order moments of $N_A(t)$. For this reason, the Central Limit Theorems proven in the immigration case are the most useful results in this study. Numerical approximations for the underlying distribution of $N_A(t)$ are needed for any kind of prediction or estimation problems.

APPENDIX A

STIRLING NUMBERS

Unless otherwise stated, all of the material in this section can be found in Jordan [15], Chapter 4.

1. STIRLING NUMBERS OF THE FIRST KIND

Stirling numbers of the first kind will be designated by S_n^m .
 $(-1)^{n-m} S_n^m$ is the number of permutations of n symbols which have exactly m cycles. (Abramowitz and Stegun [1], page 824). S_n^m is defined by:

$$\sum_{m=0}^n S_n^m x^m = x(x-1)\dots(x-(n-1))$$

Special cases:

$$\begin{aligned} S_n^n &= 1 \\ S_n^{n-1} &= -\binom{n}{2} \\ S_n^1 &= (-1)^{n-1} (n-1)! \\ S_n^0 &= \delta_{0n} \end{aligned}$$

Properties:

$$(a) \quad S_{n+1}^m = S_n^{m-1} + n S_n^m \quad (1)$$

(b) S_n^{n-m} is a polynomial of the variable n of degree $2m$, and so can be written uniquely in the form

$$S_n^{n-m} = C_{m,0} \binom{n}{2m} + C_{m,1} \binom{n}{2m-1} + \dots + C_{m,2m} \quad (2)$$

2. STIRLING NUMBERS OF THE SECOND KIND

Stirling numbers of the second kind will be designated by S_n^m . S_n^m is the number of ways of partitioning a set of n elements into m nonempty subsets.

(Abramowitz and Stegun [1], page 824). S_n^m is defined by

$$\sum_{m=0}^n S_n^m x(x-1)\dots(x-(m-1)) = x^n.$$

In closed form,

$$S_n^m = \frac{1}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^n.$$

Special cases.

$$S_n^0 = \delta_{0n}.$$

$$S_n^1 = S_n^n = 1$$

$$S_n^{n-1} = \binom{n}{2}.$$

Properties.

$$(a) \quad S_{n+1}^m = S_n^{m-1} + m S_n^m. \quad (3)$$

(b) S_n^{n-m} is a polynomial of the variable n of degree $2m$, and so can be written uniquely in the form

$$S_n^{n-m} = C_{m,0} \binom{n}{2m} + C_{m,1} \binom{n}{2m-1} + \dots + C_{m,2m}. \quad (4)$$

FORMULAE INVOLVING STIRLING NUMBERS OF BOTH KINDS

$$(a) \quad S_n^m = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} S_{n-m+k}^k. \quad (\text{Abramowitz and Stegun [1], page 824.})$$

$$(b) \quad x(x-1)\dots(x-(n-1)) = \sum_{i=1}^n \sum_{m=1}^i S_n^i S_{i-1}^m x(x-1)\dots(x-(m-1)).$$

$$(c) \quad x^n = \sum_{i=1}^n \sum_{m=1}^i S_n^i S_i^m x_m.$$

$$(d) \quad \sum_{k=m}^n S_k^m S_n^k = \sum_{k=m}^n S_n^k S_k^m = \delta_{mn}.$$

STIRLING NUMBERS AND MOMENTS

The usual n^{th} moment about the origin of a random variable X is $E(X^n) = \mu_n'$. The n^{th} order factorial moment of X about the origin is defined to be $E(X(X-1)\dots(X-(n-1))) = \mu_{[n]}'$. Stirling numbers enable us to

convert the usual moments into factorial moments and vice versa.

Since $x^n = \sum_{m=0}^n S_n^m x(x-1)\dots(x-(m-1))$, we have

$$\mu_n' = \sum_{m=0}^n S_n^m \mu_m' \quad (5)$$

Conversely, since $x(x-1)\dots(x-(n-1)) = \sum_{m=0}^n S_n^m x^m$, we have

$$\mu_{[n]}' = \sum_{m=0}^n S_n^m \mu_m' \quad (6)$$

APPENDIX B

POINT PROCESSES

All of the material in this section can be found in Moyal [20].

Assume that we have a domain D and a σ -field of sets $\mathcal{B}$ defined on D . Point processes on D may be defined in two different ways. In the first, we consider a "counting process" N_A where N_A is the number of particles in a subset A of D , where $A \in \mathcal{B}$. In the second, we consider the actual numbers and locations of particles present in D . This will be called a "population probability space".

1. Counting Processes

For each set $A \subseteq D$, associate the number N_A of members of the population in A . We assume first that $N_D < \infty$ with probability one.

Let $\mathcal{N}$ be the set of all non-negative, finite, integral-valued and completely additive measures on D . $\mathcal{N}$ is called the class of counting measures on the class of all subsets of D .

Let $\mathcal{B}_N$ be the smallest σ -field containing all sets

$$\{N|N_{A_i} = k_i, i = 1, \dots, n\}$$

in $\mathcal{N}$ where $A_i \in \mathcal{B}$, and k_i is a non-negative integer, $i = 1, \dots, n$.

A triplet $N = (\mathcal{N}, \mathcal{B}_N, P)$ where P is a probability measure on $\mathcal{B}_N$ specifies a point process.

We now allow for a countably infinite population with a non-zero probability. Let η_w be the set of all integral-valued completely additive measures N_w on $\mathcal{B}$ such that $N_w(D_k) < \infty$ for each D_k of a given fixed nondecreasing sequence of sets $D_k \in \mathcal{B}$, where $D_k \uparrow D$. If C_w is a suitably defined σ -field on η_w , and P_w a probability distribution on C_w , then the triplet $N_w = (\eta_w, C_w, P_w)$ is a σ -finite counting process on D .

In particular, C_w is the σ -field generated by all sets of the form

$$\{N_A \leq k\} \quad (k \in \mathbb{Z}_+, A \in \mathcal{B}).$$

(Daley and Vere-Jones, [7].)

Now, let η_k be the restriction of η_w to D_k ; i.e., η_k is the space of all finite counting measures on the σ -field $\mathcal{B}_k$ of measurable subsets of D_k . If C_k is the restriction of C_w to D_k , then let $Q_k(S_k) = P_w(S_k^w)$, where $S_k \in C_k$ and S_k^w is the continuation of S_k to D . $N_k = (\eta_k, C_k, Q_k)$ is a finite counting process on D_k . The sequence $\{N_k\}$ uniquely defines N_w and vice versa (Moyal [20], Theorem 6.1).

2. Population Probability Spaces.

To begin with, we will assume again that the population is finite with probability one. The population state space X is defined to be the union of all unordered sets $\{x_1, \dots, x_n\} \subseteq D^n$, $n = 0, 1, 2, \dots$, of individual states;

i.e., a state in which the population has n members, with one each in location $x_1, x_2, \dots$.

Let $\mathcal{B}$ be a σ -field of sets on D , and $\mathcal{B}^n$ the minimal σ -field of sets in D^n containing all rectangles $\prod_{i=1}^n A_i, A_i \in \mathcal{B}, i=1, \dots, n$.

Let $\mathcal{B}^*$ be the minimal σ -field of sets in X containing all sets

$A_n \in \mathcal{B}^n, n = 0, 1, 2, \dots$. Since we are dealing with unordered sets in X ,

let $\mathcal{B}_s$ be the σ -subfield of $\mathcal{B}^*$ of all symmetric sets. If $Q \in \mathcal{B}_s$,

then $Q = \sum_{n=0}^{\infty} Q_n$, where Q_n is a symmetric set in $\mathcal{B}^n$.

A triplet $(X, \mathcal{B}_s, P)$ where X is the state space, $\mathcal{B}_s$ is the σ -field on X described above, and P a probability distribution, is a point process. If P^n is the restriction of P to D^n , then

$$P(Q) = \sum_{n=0}^{\infty} P^n(Q_n),$$

where $Q = \sum_{n=0}^{\infty} Q_n$.

Moyal proves that the relation

$$(N_A | x_1, \dots, x_n) = \sum_{i=1}^n \delta(A | x_i) \quad (\delta \text{ is the characteristic function of } A)$$

defines a one-to-one correspondence between X and $\mathcal{N}$. This is intuitively clear, since for each fixed $A \subseteq \mathcal{B}$, $(N_A | \cdot)$ is a function on X , and for each fixed

$\{x_1, \dots, x_n\} \in X$, $(N_\omega | x_1, \dots, x_n)$ is a function on $\mathcal{B}$.

For σ -finite population processes, the D_k -restriction N_k of the σ -finite counting process N_ω will be a finite symmetric point process. The sample space of the σ -finite point process is the class X_ω of all countable sets of points $\{x_1, \dots\}$ in D such that $\{x_1, \dots\} \cap D_k$ is finite for $k = 1, 2, \dots$. The relation between X_ω and N_ω is defined by

$$(N_{\omega_A} | (x_1)) = \sum_{j=1}^{\infty} \delta(A|x_j).$$

By Moyal, this relation is one-to-one from N_ω to X_ω , and will be designated by T . The measurable sets in X_ω will simply be

$$(Q|Q = \tau^{-1}m, m \in C_\omega).$$

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