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Interacting Branching Particle Systems and Superprocesses

by

Hao Wang, B.SC., M.SC.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for
the degree of Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute for Graduate Studies and Research
in Mathematics and Statistics

Carleton University
Ottawa, Ontario
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Sept. 1995

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submitted by

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in partial fulfilment of the requirements for
the degree of Doctor of Philosophy

for Dr. V. Dlab, Chairman,
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Abstract

The central theme of this thesis is the construction and study of a new class of interacting measure-valued branching diffusions (IMBD) which arise as limits in distribution of a sequence of interacting branching particle systems. The pregenerators of these IMBD are given by

$$\begin{aligned} \mathcal{L}F(\mu) := & \frac{1}{2}\gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx) + \frac{1}{2} \int_{\mathbb{R}} \rho_{\epsilon} \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy), \end{aligned}$$

where $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(E)$, $\rho(z) = \int_{\mathbb{R}} g(z-x)g(x)dx$, $\rho_{\epsilon} = \rho(0) + \epsilon^2$, $\gamma \geq 0$, $\epsilon \in \mathbb{R}$ and $g(\cdot)$ is a symmetric square-integrable function.

Our model clearly includes super-Brownian motion (super-BM) as a special case, but essentially differs from super-BM because of the interacting term. This interacting mechanism causes the loss of multiplicative property and in consequence we can not apply log-Laplace functional technique to our model. Under the assumption that $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0, the IMBD are constructed and characterized by an operator perturbation technique and a generalized duality method. On the other hand, the interacting mechanism gives our IMBD a rich and interesting spatial structure. The effective state space of our IMBD is contained in the set of purely atomic measures or the set of absolutely continuous measures according as $\epsilon = 0$ or $\epsilon \neq 0$ if $g(\cdot)$ is a smooth function. As a result of the existence of density processes, we derive a new class of stochastic partial differential equations based on two orthogonal white noises for our IMBD.

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Chapter 1

Introduction

1.1 Prelude and Related Works

Since the initial work of Galton and Watson, the theory of branching processes has been developing now for over a century (cf. Harris [31], Athreya-Ney [1], Dynkin [20], Ethier-Kurtz [23] and Dawson [12]). Beginning in the 1950's, both practical needs and theoretical interest led to the development of spatially distributed models, diffusion approximations and their extension to the space-time context (eg Feller [25], Jirina [39] [40], Sevastyanov [52], Lamperti [43], Moyal [47], Ikeda-Nagasawa-Watanabe [35], Skorohod [56], Dawson [8] [9] [10], Watanabe [62], Silverstein [55], and Fleming-Viot [27]. There is a huge amount of literature on this topic. This list is by no means exhaustive). Similar models were used to describe free (boson) fields in relativistic quantum mechanics, a hydromagnetic dynamoprocess in cosmology, diffraction in random-heterogeneous media in statistical physics, the dynamics of populations with geographical distribution in population genetics, etc. In particular spatially interacting models are closer to many realistic situations than purely branching models and therefore are of considerable interest (eg. Dawson-Vaillancourt [15], Perkins [49], Gärtner [28], Borkar [5], Holley-Stroock

[33], Oelschläger [48], Méléard-Roelly [45], Vaillancourt [59], to name only a few references). In the present dissertation, a spatial interacting model is formulated; the existence, uniqueness, spatial structure, and associated stochastic partial differential equations of the limiting measure-valued diffusions are investigated. These measure-valued processes arise naturally as the limits in distribution of a sequence of interacting branching particle systems. As an introduction to our model, at first we would like to briefly review the historical development of two important spatially distributed models and their limiting characteristics: the Dawson-Watanabe processes and the Fleming-Viot processes. Secondly we would like to briefly present two related works of Kotelenetz [42] and Dawson-Vaillancourt [15] on stochastic McKean-Vlasov equations.

1.1.1 Dawson-Watanabe processes

Usually Dawson-Watanabe processes refer to the measure-valued processes which arise as limits in distribution of a sequence of independent branching particle systems. These processes were initially constructed in the Feller process setting but now have been generalized by Fitzsimmons [26] to a right process setting and Dynkin [19] to a time inhomogeneous motion, in which the branching rate is given by a continuous additive functional.

To illustrate, here we consider (α, d, β) -superprocesses. In this special case the motion process W is a symmetric α -stable process in $\mathbb{R}^d$ with semigroup $\{S_t^\alpha\}$, generator $A_\alpha = -(-\Delta)^\alpha$, $0 < \alpha \leq 2$, and with paths in $D_{\mathbb{R}^d} := D([0, \infty), \mathbb{R}^d)$. The branching mechanism in this case is given by $\Phi(x, \lambda) = -\gamma\lambda^{1+\beta}$, $0 < \beta \leq 1$. For fixed (α, d, β) , by Ito's formula, we can obtain the formal generator of (α, d, β) -

superprocess as follows: If $\beta = 1$

$$(1.1) \quad LF(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} \mu(dx) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} + \int_{\mathbb{R}^d} \mu(dx) A_\alpha \frac{\delta F(\mu)}{\delta \mu(x)},$$

and if $0 < \beta < 1$

$$(1.2) \quad LF(\mu) = \int_{\mathbb{R}^d} \mu(dx) A_\alpha \frac{\delta F(\mu)}{\delta \mu(x)} + \frac{\beta(1+\beta)}{\Gamma(1-\beta)} \int_{\mathbb{R}^d} \mu(dx) \int_0^\infty \frac{1}{u^{\beta+2}} [F(\mu + u\delta_x) - F(\mu) - u \frac{\delta F(\mu)}{\delta \mu(x)}] du.$$

Then the (α, d, β) -superprocess μ_t , which is a $M_F(\mathbb{R}^d)$ -valued process, can be constructed by exploiting the martingale problem method (refer to Roelly-Coppoletta [51]). Another way to construct this process is to use Trotter's product formula. This idea is given by Dawson [9] and in that paper the author proposed a sequence of open problems and measure-valued processes related to the study of stochastic partial differential equations, which actually lead to the flourishing of superprocesses.

In the following, we want to outline the log-Laplace technique to construct (α, d, β) -superprocesses (Dawson [9] [10], Watanabe [62], Silverstein [55]). This method consists of two facts: The first one is Jirina's log-Laplace functional formula which provides a one-to-one correspondence between (α, d, β) -superprocesses $\{\mu_t\}$ and log-Laplace semigroups V_t ; the correspondence is given by

$$(1.3) \quad \mathbb{E}_\mu e^{<\mu_t, \phi>} = \exp\{-<V_t \phi, \mu>\},$$

where $\phi \in C_0(\mathbb{R}^d)_+$, the space of continuous, strictly positive functions vanishing at infinity on $\mathbb{R}^d$, $<\mu_t, \phi> = \int_{\mathbb{R}^d} \phi(x) \mu_t(dx)$. This formula is based on the multiplicative property, which is the fact that if two branching Markov processes start at m_1 and m_2 and evolve independently, then their sum is equal in law to the same

process starting at $m_1 + m_2$. We will see this formula fails to hold if processes are dependent. The second fact is that the log-Laplace semigroup $V_t\phi$ is the unique solution of

$$(1.4) \quad V_t\phi = S_t^\alpha\phi(x) - \gamma \int_0^t S_u^\alpha[(V_{t-u}\phi)^{1+\beta}]du.$$

(1.4) comes from the consideration of a single particle's first branching. Essentially log-Laplace functional technique transforms a measure-valued problem into a function-valued problem and usually it is easy to prove that (1.4) has a unique solution which in turn gives the construction and characterization of $\{\mu_t\}$.

There are a host of beautiful results on the processes' spatial structure properties, long-time behaviors, family structures, sample path properties, relations with Fleming-Viot processes, associated stochastic partial differential equations, generalizations and so on. Most of these results are obtained by making use of (1.4). For example, Konno-Shiga [41] proved that for any $t > 0$, if $d = 1, 1 < \alpha \leq 2$, μ_t is absolute continuous with respect to dx almost surely, and then they derive stochastic partial differential equations for these processes as suggested by Dawson [9]. We omit the details. Interested readers are referred to Dawson [12].

1.1.2 Fleming-Viot processes

Fleming-Viot processes are probability-measure-valued diffusion processes that come from the generalization of Moran diffusion processes, in which the finite type space is replaced by a compact metric space initially given by Fleming-Viot [27]. These processes also arise as diffusion approximations for Wright-Fisher models or diploid models in population genetics (see the survey article Ethier-Kurtz [23]).

These processes have been generalized to time inhomogeneous setting, with recombination and selection, interaction and so on (see Ethier-Kurtz [22] [23], Dawson-

March [14], Vaillancourt [59] [60], Handa [30] and so on). But here we only consider the simplest case and briefly describe Moran model and outline the original construction and characterization of the processes given by Fleming-Viot [27].

Consider a population with J possible types $x_1, \dots, x_J$ and let s denote time measured on some scale and $n_j(s)$ be the number of type x_j individuals at time s . The total number N is assumed fixed:

$$N = n_1(s) + \dots + n_J(s).$$

In the model, the vector

$$n(s) = (n_1(s), \dots, n_J(s))$$

is a continuous time Markov chain, with the transition rule

$$\begin{aligned} \mathbb{P}(n_i(s) = k - 1, n_j(s) = m + 1 | n_i(s) = k, n_j(s) = m) \\ = Nb_{ij}\Delta s + o(\Delta s), \quad (k + m \leq N), \end{aligned}$$

which means each individual jumps at rate $Nb_{ij} + O(\Delta s)$ to the type of one of the remaining $N - 1$ individuals and all other transitions have probability $o(\Delta s)$. Let $p_j = N^{-1}n_j$, then $p = (p_1, \dots, p_J)$ takes values in the simplex

$$\Delta_J = \{p = (p_1, \dots, p_J) : p_1 \geq 0, \dots, p_J \geq 0, p_1 + \dots + p_J = 1\}.$$

The generator of Markov chain $p(s)$ is

$$\mathcal{K}f(p) = N \sum_{i \neq j} b_{ij} [f(p^{ij}) - f(p)],$$

where p^{ij} is the vector with components

$$p_k^{ij} = \begin{cases} p_k - 1/N & \text{if } k = i \\ p_k + 1/N & \text{if } k = j \\ p_k & \text{if } k \neq i, j \end{cases}$$

and

$$\begin{aligned} b_{ij} &= b_{ij}(p) = a(p_i p_j + N^{-1} p_i \theta_{ij}) \quad a > 0 \\ \theta_{ii} &= - \sum_{i \neq j} \theta_{ij} \quad \theta_{ij} \geq 0 \quad \text{for } i \neq j. \end{aligned}$$

Let $\mathcal{H}_N^J = Na^{-1}\mathcal{K}$ denote the generator on the new time scale $t = aN^{-1}s$, then using Taylor expansion of $f(p^{ij})$, we have

$$\mathcal{H}^J := \lim_{N \rightarrow \infty} \mathcal{H}_N^J = \sum_{j=1}^J L_j(p) f_{p_j} + \sum_{i,j=1}^J (p_i \delta_{ij} - p_i p_j) f_{p_i p_j},$$

where $L_j(p) = \sum_{i=1}^J p_i \theta_{ij}$. $\mathcal{H}^J$ is the generator of J -type Moran (or Wright-Fisher) diffusion process in population genetics. Replacing $\{1, \dots, J\}$ by a compact metric space S , Δ_J by $\mathcal{P}(S)$, the set of Borel probability measures on S with the topology of weak convergence, and $\mathcal{H}^J$ by

$$\begin{aligned} (1.5) \quad \mathcal{H}F(\mu) &= \frac{1}{2} \int_S \int_S \mu(dx) (\delta_x(dy) - \mu(dy)) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \\ &\quad + \int_S A \frac{\delta F(\mu)}{\delta \mu(y)} \mu(dy), \\ &\quad F(\mu) \in \mathcal{D}(\mathcal{H}) \subset C(E), \end{aligned}$$

where A , which describes the mutation mechanism in the genetics model, is a generator of Feller semigroup on $C(S)$. Fleming and Viot got this uncountably infinite type diffusion process by exploiting martingale problems. The existence is established by discretizing generator A and verifying a Prokhorov-type compactness condition. The uniqueness is proved by a moment measure equation method (refer to Fleming-Viot [27] for more details).

There are extensive studies and plenty of results related to the processes' qualitative behavior, ergodicity, transition function, atomic measure state, genealogical

structure, infinite particle representation, comparison with Dawson-Watanabe processes, associated stochastic partial differential equation, generalizations, and so on. Interested readers are referred to Dawson [12] and Ethier-Kurtz [23] for more details and references listed there.

1.1.3 Two Models on McKean-Vlasov Limits

Kotelenez [42] and Dawson-Vaillancourt [15] have introduced a class of exchangeable particle systems and have discussed the corresponding stochastic McKean-Vlasov equations and McKean-Vlasov limits as their weak solutions. The latter arises as strong or weak limits in probabilistic sense of a sequence of conservative empirical processes consisting of mean-field interacting branching-free particle systems.

In Dawson-Vaillancourt [15], the particle systems considered have the following form:

$$(1.6) \quad dX_i^m(t) = \sigma(X_i^m(t), \frac{1}{m} \sum_{j=1}^m \delta_{X_j^m(t)}) dW_i(t) + b(X_i^m(t), \frac{1}{m} \sum_{j=1}^m \delta_{X_j^m(t)}) dt$$

for $i = 1, \dots, m$, where $\{W_i\}_{1 \leq i \leq m}$ are Brownian motions and their quadratic variational processes satisfy

$$(1.7) \quad \langle W_i, W_j \rangle_t = C_{i,j} t,$$

where $C_{i,j} = C_{\Pi i, \Pi j} \neq 0$ for $i, j = 1, \dots, m$, and Π is a permutation of $\{1, \dots, m\}$.

Since the particle motions in this model depend on the feature of all present particles, the function-valued dual processes are not independent of measure-valued processes. This dependence leads to the failure of the duality method in proving the uniqueness of the corresponding martingale problem. In Dawson-Vaillancourt [15], the uniqueness of McKean-Vlasov limits is established by using embedding and coupling techniques under assumption that the coefficients of generator are continuous, strictly positive, and satisfy growth, coercivity and monotonicity conditions.

(1.7) implies that these particle systems are exchangeable. We will see that the particle systems in our model are also exchangeable, but only have the "particle-particle" interaction, so the duality method still works for our model.

In Kotelenetz [42], the particle systems are assumed to satisfy the following stochastic ordinary differential equations:

$$(1.8) \quad \begin{aligned} dr^i(t) &= F(\Xi_N(t), r^i(t))dt + \int \mathcal{J}(\Xi_N(t), r^i(t) - p)W(dp, dt), \\ r^i(0) &= r_0^i, i = 1, \dots, N, \Xi_N(t) = \sum_{i=1}^N a_i \delta_{r^i(t)}, \quad a_i \in \mathbb{R}, \quad N \in \mathbb{N}, \end{aligned}$$

where $W(\cdot, \cdot)$ is a cylindrical Brownian motion.

Under the assumption of some kind of Lipschitz conditions on coefficients F and $\mathcal{J}$, the existence and uniqueness of strong solutions in the probabilistic sense of (1.8) are established, and the McKean-Vlasov limit is constructed by using a limiting extension procedure and characterized as a unique solution of the following stochastic partial differential equation:

$$(1.9) \quad \begin{aligned} d\Xi &= \left\{ \frac{1}{2} \sum_{k,l=1}^d \partial_{kl}^2 (D_{kl}(\Xi)\Xi) - \nabla \cdot (\Xi F(\Xi)) \right\} dt \\ &\quad - \nabla \cdot \left(\Xi \int \mathcal{J}(\Xi, \cdot - p)W(dp, dt) \right), \end{aligned}$$

where $D_{kl}(\mu) := D_{kl}(\mu, \cdot) \in C_b^2(\mathbb{R}^d, \mathbb{R})$. This equation does not satisfy the type of coercivity assumption used in Dawson-Vaillancourt [15]. The L^2 -continuity of the McKean-Vlasov limit is proved by very subtle functional analytical techniques.

In the next section we will introduce a new class of exchangeable particle systems in which the particles can undergo branching. In the branching-free limit they include special cases of the exchangeable systems of Dawson-Vaillancourt [15] and Kotelenetz [42]. As in the case of Kotelenetz's model, if our model becomes branching-free, then the particle systems can be defined on a same probability space and a same

cylindrical Brownian motion. It is by making use of this property that the state classification (Chapter 6) and the derivation of stochastic partial differential equations (Chapter 7) for our model are obtained. Our model does not include "particle-medium-particle" interaction (mean field interaction) as in Dawson-Vaillancourt [15] and Kotelenetz [42], while their models are branching-free. This means that we must follow another route to derive the stochastic partial differential equations for our limiting measure-valued processes.

1.2 The Basic Model and Questions

1.2.1 An Interacting Branching Particle Model

In this subsection we begin to describe our proposed interacting branching particle model which consists of a sequence of branching particle systems in a Brownian medium. More precisely, the model studied is as follows: for any integer n , we consider a system of particles initially with m_0^n particles which move, die and produce offspring in a Brownian medium on $\bar{\mathbb{R}} := \mathbb{R} \cup \{\Delta\}$, one point compactification of $\mathbb{R}$. Such a system has the form

$$(2.10) \quad dx_i(t) = \int_{\mathbb{R}} g(y - x_i(t)) W(dy, dt) + \epsilon dB_t^i,$$

where $W(\cdot, \cdot)$ is a cylindrical Brownian motion (refer to Chapter 3 or Walsh [61] and Da Prato-Zabczyk [7]), $\{B_t^i\}$ are independent one-dimensional Brownian motions which are independent of W , ϵ is a real constant, and g is a symmetric, square-integrable function on $\bar{\mathbb{R}}$ such that

$$(2.11) \quad \lim_{x \rightarrow \pm\infty} g(x) = g(\Delta) = 0, \quad \int_{\mathbb{R}} g^2(x) dx < \infty.$$

The quadratic variational process

$$(2.12) \quad \langle x_i, x_j \rangle_t = \int_0^t \rho(x_i(s) - x_j(s)) ds + \epsilon^2 t \delta_{ij},$$

where $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$ and

$$(2.13) \quad \rho(z) = \int_{\mathbb{R}} g(z - y)g(y)dy.$$

Here $x_i(t)$ is the location of i^{th} particle. We assume that each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. In the case of critical branching, when a particle dies, it produces k particles with probability p_k ; $k = 0, 1, 2, \dots$. We assume that offspring distribution satisfies

$$(2.14) \quad p_1 = 0, \quad \sum_{k=0}^{\infty} k p_k = 1, \quad \text{and} \quad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

After branching, the resulting set of particles evolve in the same way and they start off from the parent particle's branching site. Let m_t^n denote the total number of particles at time t . Define

$$(2.15) \quad \mu_t^n(\cdot) := \frac{1}{\theta^n} \sum_{i=1}^{m_t^n} \delta_{x_i^n(t)}(\cdot).$$

We assume that $m_0^n/\theta^n \leq \xi$ for $n \geq 0$, where $\xi > 0$ is a fixed constant, $\mu_0^n \Rightarrow \bar{\mu}$ (weak convergence), and $\bar{\mu}$ is a finite measure with compact support, and there are no interactions in the branching mechanism. Let $E := \mathbf{M}_F(\bar{\mathbb{R}})$ be the space of all bounded Radon measures on $\bar{\mathbb{R}}$ with weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad \langle \mu^n, f \rangle \rightarrow \langle \mu, f \rangle \text{ for } \forall f \in C(\bar{\mathbb{R}}).$$

Then E is a Polish space (see Prokhorov [50]).

By Ito's formula and the independence of motion and branching, we can obtain the following formal generators (usually called pregenerator) of the limiting measure-valued processes:

$$(2.16) \quad \mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu),$$

$$(2.17) \quad \mathcal{B}F(\mu) := \frac{1}{2}\gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx),$$

and

$$(2.18) \quad \begin{aligned} \mathcal{A}F(\mu) := & \frac{1}{2} \int_{\mathbb{R}} \rho_{\epsilon} \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{\epsilon}(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) \end{aligned}$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(E)$, where the variational derivatives are defined by

$$(2.19) \quad \frac{\delta F(\mu)}{\delta \mu(x)} := \lim_{\epsilon \downarrow 0} \frac{F(\mu + \epsilon \delta_x) - F(\mu)}{\epsilon},$$

$$(2.20) \quad \begin{aligned} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} := \\ \lim_{\epsilon_2 \downarrow 0} \lim_{\epsilon_1 \downarrow 0} \frac{F(\mu + \epsilon_1 \delta_x + \epsilon_2 \delta_y) - F(\mu + \epsilon_2 \delta_y) - F(\mu + \epsilon_1 \delta_x) + F(\mu)}{\epsilon_1 \epsilon_2}, \end{aligned}$$

ρ_{ϵ} is a constant satisfying

$$(2.21) \quad \rho_{\epsilon} = \rho(0) + \epsilon^2,$$

and $\mathcal{D}(\mathcal{L})$ is the domain of pregenerator $\mathcal{L}$.

Clearly our model includes super-Brownian motion as a special case ($\rho(\cdot) \equiv 0$ and $\epsilon \neq 0$). Note, in the following we often use terms "in the degenerate case" which means $\epsilon = 0$ and "in the uniformly elliptic case" which means $\epsilon \neq 0$. These terms coincide with those in the finite dimensional cases.

1.2.2 Difficulties Encountered

In the usual model (for example, (α, d, β) -superprocesses) in which the motions of particles are independent and the motions are independent of branchings, the particle systems have the multiplicative property. From Section 1.1.1 it is well

known that log-Laplace functional (or evolution equation) technique can be used to this model to construct the limiting measure-valued process . But as for our model and pregenerators, it is obvious that the motions of particles are not independent and this destroys the multiplicative property. In fact, if $\rho(\cdot)$ is a smooth function and

$$(2.22) \quad \rho_\epsilon = \rho(0) = \int_{\mathbb{R}} y^2(x) dx \quad (\text{i.e. } \epsilon = 0).$$

Then we can show the finite dimensional diffusion process $(x_1(t), \dots, x_N(t))$ governed by generator

$$(2.23) \quad G^N f(x_1, \dots, x_N) := \frac{1}{2} \sum_{i,j=1}^N \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_N)$$

has a property that for any $i \neq j$ and $t \geq 0$, if $x_i(0) \neq x_j(0)$, then $x_i(t)$ never collides with $x_j(t)$ (cf. Chapter 2). Here we simply take $\gamma = 0$ which means branching-free. So if the multiplicative property were to hold, then for any strictly positive function $f \in C(\mathbb{R})$, we have

$$(2.24) \quad \mathbb{E}_{\delta_{x_i(0)} + \delta_{x_j(0)}} e^{-\langle \delta_{x_i(t)} + \delta_{x_j(t)}, f \rangle} = \mathbb{E}_{\delta_{x_i(0)}} e^{-\langle \delta_{x_i(t)}, f \rangle} \mathbb{E}_{\delta_{x_j(0)}} e^{-\langle \delta_{x_j(t)}, f \rangle};$$

However, from right hand side of (2.24) it clearly follows that the particles can be simultaneously located at same position. The contradiction implies the multiplicative property is lost. In consequence, we cannot apply Jirina's formula and the evolution equation method. We can check that the discretization method used in Fleming-Viot [27] does not work for our case either since (2.18) has an interacting term. Therefore, questions such as existence, uniqueness, spatial structure properties and so on arise, and new techniques are needed to handle this interacting model. In this thesis, we will give solutions to some of these questions.

1.3 Main Original Results

This thesis includes the following main original results:

1.3.1 Construction of Super-Brownian Motion in a Random Medium

Let $\rho(\cdot)$ as defined by (2.13) in Chapter 1 be continuous and locally Lipschitz continuous away from 0 which means that $\rho(\cdot)$ is Lipschitz outside each interval $(-c, c)$ for $c > 0$. Then for any finite measure μ_0 with compact support, the $(\mathcal{L}, \delta_{\mu_0})$ -martingale problem has a unique solution which is a measure-valued diffusion process.

1.3.2 Spatial Structure and State Classification

As for the spatial structure and state classification, we obtained the following criterion.

(a) Let $\epsilon = 0$ and $g \not\equiv 0$ be a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$. Then for any finite measure ν on $\mathbb{R}$ with compact support, the unique solution $\{\nu_t\}$ to $(\mathcal{A} + \mathcal{B}, \delta_\nu)$ -martingale problem has the form $\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)}$ where $a_i(t) \geq 0$, $\{x_i(t)\}_{i \geq 0}$ are continuous in t , $x_i(t) \neq x_j(t)$ for $t \geq 0$ and $i \neq j$, and $I(t)$ is a nonincreasing random subset of $\mathbb{N}$.

(b) Suppose $\epsilon \neq 0$ and $g(\cdot) \not\equiv 0$ is a square-integrable, smooth, and symmetric function in the definition of $\mathcal{A} + \mathcal{B}$. Let μ be a finite measure on $\mathbb{R}$ with compact support and $\mathbb{P}_\mu$ denote the unique solution to the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem with canonical process μ_t on the space $D([0, \infty), M_F(\bar{\mathbb{R}}))$. Then μ_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ for almost all $t > 0$, $\mathbb{P}_\mu$ -almost surely.

1.3.3 Associated Stochastic Partial Differential Equations

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of C^∞ -functions on $\mathbb{R}$ which are rapidly decreasing at infinity, $\mathcal{S}'(\mathbb{R})$ be the topological dual i.e. the space of Schwartz distributions. As a result of the existence of the density process, a new class of stochastic partial differential equations are derived for the density process.

Assume $\rho(\cdot) \not\equiv 0$ is a smooth function and $\epsilon \neq 0$. Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu_t)$ be the interacting measure-valued branching diffusions governed by $\mathcal{A} + \mathcal{B}$ and $\ell_t(x)$ be their density processes constructed in Chapter 6. Then there exist two orthogonal $\mathcal{S}'(\mathbb{R})$ -valued cylindrical Brownian motions V_t and W_t defined on an extended probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}_\mu)$ of the original space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ such that for every $\phi \in \mathcal{S}(\mathbb{R})$,

$$(3.25) \quad \begin{aligned} \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle &= \frac{1}{2} \rho_\epsilon \int_0^t \langle \mu_s, \Delta \phi \rangle ds \\ &+ \int_0^t \int_{\mathbb{R}} \sqrt{\gamma(m_2 - 1)} \ell_s(x) \phi(x) V(dx, ds) \\ &+ \int_0^t \int_{\mathbb{R}} \langle g(y - x) \nabla \phi(x), \mu_s(dx) \rangle W(dy, ds) \end{aligned}$$

holds for every $t \geq 0$, $\bar{\mathbb{P}}_\mu$ -almost surely.

1.4 Methodology Developed

In order to prove the main results, we have developed the following new methods and ideas:

1.4.1 Bounded Generator Approximation and Perturbation

As we mentioned before, we cannot apply log-Laplace functional technique to construct the limiting measure-valued processes in our model. For above defined two

unbounded operators $\mathcal{A}$ and $\mathcal{B}$, in order to establish the existence of the martingale problem for $\mathcal{A} + \mathcal{B}$, in Chapter 4 we have developed a new idea to construct a sequence of interacting branching particle systems as follows:

- (a) Define a sequence of simple bounded generators $\{\mathcal{B}_n\}$ which converges to $\mathcal{B}$ such that $\mathcal{B}_n$ is the generator of a jump Markov process and simultaneously we obtain that $\mathcal{B}$ is a dissipative operator.
- (b) Use bounded perturbation technique and tightness argument to establish the existence of the martingale problem for $\mathcal{A} + \mathcal{B}$.

In this way, we have avoided the hard work to check the dense range condition and domain condition in the Trotter's perturbation theorem which is the usual way to approach problems of this kind.

1.4.2 Limiting Duality Method

As for the uniqueness of the martingale problem for $\mathcal{A} + \mathcal{B}$, If $\rho(\cdot)$ is a smooth function, we can directly obtain the following dual identity

$$\mathbb{E}_\mu[F_f(\mu_t)] = \mathbb{E}_f[F(\mu, Y(t)) \exp(\frac{1}{2}\gamma(m_2 - 1) \int_0^t N(Y(u))(N(Y(u)) - 1)du)]$$

which implies the uniqueness. However in Chapter 5 we consider the case in which $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0. In this case we cannot directly construct the dual process such that the dual identity holds. In order to prove the uniqueness of the martingale problem for $\mathcal{A} + \mathcal{B}$, we constructed a generalized dual process and used a limiting argument to lead to the dual identity. In this way we have successfully developed a new method; namely, the limiting duality method.

1.4.3 Commutativity of Semigroups

In the Chapter 6 at first we proved that the action of $\mathcal{B}$ operator only changes the masses of atoms and the action of $\mathcal{A}$ operator only changes the positions of atoms if $\epsilon = 0$. Secondly we established the commutativity of the spatial motion semigroup and the branching semigroup if $\epsilon = 0$. Combining these results, we reach the conclusion that if $\epsilon = 0$ and $\rho(\cdot)$ is a smooth function, then the constructed limiting measure-valued process takes values in the set of purely atomic measures.

1.4.4 Application of Malliavin Calculus

Konno-Shiga successfully used evolution equation technique to estimate the moments of the super-Brownian motion and to prove the existence of the density process of super-Brownian motion. However, in our case we don't have evolution equation. In order to estimate the moments of the limiting measure-valued process, we make use of the dual identity to change the question into estimating the moments of the dual process. Then the solution comes from the following main result of the application of Malliavin calculus. Given a finite dimensional, second order differential operator G^d with smooth coefficients and uniformly elliptic diffusion coefficient, then the transition probability of the G^d -diffusion admits a bounded, smooth density.

1.4.5 Quasi-SPDE and L^p -convergence

In Chapter 7, we have made use of the stochastic partial differential equation for the empirical measure processes(Quasi-SPDE or formal SPDE) and L^p -convergence to yield the tightness of several process sequences, the orthogonality of two martingale measures, and the derivation of SPDE for the density process. This method seems useful for similar problems.

1.5 Organization of the Thesis

The thesis is organized as follows. In Chapters 2 and 3, we prepare some basic knowledge and results such as weak convergence, C-diffusions, duality method, moment problems, continuous state branching processes, orthogonal martingale measures, space-time stochastic integrals and their generalizations, for later use. Chapters 4 and 5 are devoted to the establishment of existence and uniqueness of the $\mathcal{L}$ -martingale problem under assumption that $\rho(\cdot)$ is a smooth function and a function which is continuous and locally Lipschitz continuous away from 0, respectively. Then we turn to discuss the spatial structure properties of the processes in Chapter 6. In the case in which $\epsilon = 0$ and $\rho(\cdot)$ is a smooth function, then for any $t > 0$, the fact that the limiting measure-valued processes at t take values in the set of purely atomic measures is proved by using Feller's accessibility criterion. On the other hand, if $\epsilon \neq 0$ and $\rho(\cdot)$ is a smooth function, it is proved that the effective state space of the limiting measure-valued processes at any time $t > 0$ is contained in the set of absolutely continuous measures. As a result of the existence of the density processes, in Chapter 7 a new class of stochastic partial differential equations based on two orthogonal space-time white noises are derived for the density processes.

Chapter 2

Definitions and Preliminaries

The purpose of this chapter is to give basic definitions and gather some preliminary results for later use.

2.1 Martingale Problem and Infinitesimal Generators

Let (E, r) be a metric space, $B(E)$ be the Banach space of real-valued, bounded, and Borel measurable functions on E with $\|f\| = \sup_{x \in E} |f(x)|$ and $\bar{C}(E) (\subset B(E))$ be the space of bounded, continuous function on the metric space E . If E is locally compact, $\hat{C}(E)$ denotes the Banach space of all continuous functions vanishing at infinity with norm $\|f\| = \sup_{x \in E} |f(x)|$.

Definition 2.1.1 *Given a subset $A \subset B(E) \times B(E)$ and $\mu \in \mathcal{P}(E)$, the space of all probability measures on E , by a solution to (A, μ) -martingale problem, we mean a measurable stochastic process X with values in E defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $(f, g) \in A$,*

$$f(X(t)) - \int_0^t g(X(s)) ds$$

is a martingale with respect to the filtration

$$\hat{\mathcal{F}}_t^X := \mathcal{F}_t^X \vee \sigma\left(\int_0^s h(X(u))du, \quad s \leq t, h \in B(E)\right)$$

and $\mathbb{P}X(0)^{-1} = \mu$.

Consider the canonical processes on the space $D([0, \infty), E)$ or $C([0, \infty), E)$, then we have following definition.

Definition 2.1.2 A family of probability measures $\{\mathbb{P}_x : x \in E\}$ on $D([0, \infty), E)$ (or $C([0, \infty), E)$) is said to be a solution to martingale problem for A if for every $(f, g) \in A$,

- (1) and for each $x \in E$, $\mathbb{P}_x(X(0) = x) = 1$;
- (2) for each probability measure $\mathbb{P}_x$, $\{X(t) : t \geq 0\}$ is a solution to (A, δ_x) -martingale problem, where $\{X(t) : t \geq 0\}$ is the canonical process on $D([0, \infty), E)$ (or $C([0, \infty), E)$).

Theorem 2.1.1 Let (E, r) be complete and separable, and let $\{X_\alpha\}$ be a family of processes with sample paths in $D_E[0, \infty)$. Suppose that the compact containment condition holds; that is, for every $\eta > 0$ and $T > 0$, there exists a compact set $\Gamma_{\eta, T} \subset E$ for which

$$(1.1) \quad \inf_{\alpha} P\{X_\alpha(t) \in \Gamma_{\eta, T} \text{ for } 0 \leq t \leq T\} \geq 1 - \eta.$$

Let H be a dense subset of $\bar{C}(E)$ in the topology of uniform convergence on compact sets. Then X_α is relatively compact iff $\{f \circ X_\alpha\}$ is relatively compact (as a family of processes with sample paths in $D_{\mathbb{R}}[0, \infty)$) for each $f \in H$.

Proof: See Ethier-Kurtz [21] P142 Theorem 9.1. ■

For each α , let $X_\alpha(t)$ be a process with sample paths in $D_{\mathbb{R}}[0, \infty)$ defined on a probability space $(\Omega_\alpha, \mathcal{F}^\alpha, \mathbb{P}_\alpha)$ and adapted to a filtration $\{\mathcal{F}_t^\alpha\}$. Let $\mathcal{L}_\alpha$ be the Banach space of real-valued, $\{\mathcal{F}_t^\alpha\}$ -progressive processes with norm $\|Y\| = \sup_{t \geq 0} \mathbb{E}[|Y(t)|] < \infty$. Let

$$(1.2) \quad \hat{\mathcal{A}}_\alpha = \{(Y, Z) \in \mathcal{L}_\alpha \times \mathcal{L}_\alpha : Y(t) - \int_0^t Z(s)ds \text{ is an } \{\mathcal{F}_t^\alpha\}\text{-martingale}\}.$$

Theorem 2.1.2 *Let (E, r) be arbitrary, $\{X_\alpha\}$ be a family of processes as above, C_α be a subalgebra of $\tilde{C}(E)$ (e.g., the space of bounded uniformly continuous functions with bounded support), and D be the collection of $f \in \tilde{C}(E)$ such that for every $\epsilon > 0$ and $T > 0$, there exists $(Y_\alpha, Z_\alpha) \in \hat{\mathcal{A}}_\alpha$ with*

$$(1.3) \quad \sup_\alpha \mathbb{E} \left[\sup_{t \in [0, T] \cap Q} |Y_\alpha(t) - f(X_\alpha(t))| \right] < \epsilon$$

and

$$(1.4) \quad \sup_\alpha \mathbb{E}[\|Z_\alpha\|_{p, T}] < \infty \quad \text{for some } p \in (1, \infty],$$

where Q is the set of all rational numbers in $\mathbb{R}$, $\|h\|_{p, T} := [\int_0^T |h(t)|^p dt]^{1/p}$ if $p < \infty$, and $\|h\|_{\infty, T} := \text{ess sup}_{0 \leq t \leq T} |h(t)|$ if $p = \infty$. If C_α is contained in the closure of D (in the sup norm), then $\{f \circ X_\alpha\}$ is relatively compact for each $f \in C_\alpha$; more generally, $\{(f_1, \dots, f_k) \circ X_\alpha\}$ is relatively compact in $D_{\mathbb{R}^k}[0, \infty)$ for all $f_1, \dots, f_k \in C_\alpha, 1 \leq k \leq \infty$.

Proof: see Ethier-Kurtz [21] p.145 Theorem 9.4. ■

Theorem 2.1.3 *Let $A \subset \tilde{C}(E) \times \tilde{C}(E)$ and $A_n \subset B(E) \times B(E), n = 1, 2, \dots$. Suppose that for each $(f, g) \in A$, there exists $(f_n, g_n) \in A_n$ such that*

$$(1.5) \quad \lim_{n \rightarrow \infty} \|f_n - f\| = 0, \quad \lim_{n \rightarrow \infty} \|g_n - g\| = 0.$$

If for each n, X_n is a solution of the martingale problem for A_n with sample paths in $D_E[0, \infty)$ and if $X_n \Rightarrow X$, then X is a solution of the martingale problem for A .

Proof: see Ethier-Kurtz [21] p.196 Lemma 5.1. ■

Corollary 2.1.4 *Suppose that (E, r) is complete and separable and that $\overline{D(A)}$ contains an algebra that separates points and vanishes nowhere (and hence is dense in $\bar{C}(E)$ in the topology of uniform convergence on compact sets). Then $\{X_n\}$ is relative compact if (1.5) holds for each $(f, g) \in A$ and if for every $\epsilon, T > 0$, there exists a compact $K_{\epsilon, T} \subset E$ such that*

$$(1.6) \quad \inf_n \mathbb{P}\{X_n(t) \in K_{\epsilon, T} \text{ for } 0 \leq t \leq T\} \geq 1 - \epsilon.$$

Proof: The conclusion follows from Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.1.3. ■

Consider operator

$$(1.7) \quad \hat{G}^d = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.$$

Let S_d denote the space of $d \times d$ nonnegative-definite matrices. About infinitesimal generators, we need the following results.

Theorem 2.1.5 *Let $a : \mathbb{R}^d \rightarrow S_d$ satisfy $a_{i,j} \in C^2(\mathbb{R}^d)$ with $\frac{\partial^2}{\partial x_k \partial x_l} a_{i,j}$ bounded for $i, j, k, l = 1, \dots, d$, and let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Lipschitz continuous (i.e., $\sup_{x,y \in \mathbb{R}^d, x \neq y} |b(x) - b(y)|/|x - y| < \infty$). Then, with $\hat{G}^d$ defined by (1.7), the closure of $\{(f, \hat{G}^d f) : f \in C_c^\infty(\mathbb{R}^d)\}$ is single-valued and generates a Feller semigroup on $\hat{C}(\mathbb{R}^d)$.*

Proof: see Ethier-Kurtz [21] p.373 Theorem 2.5 . ■

Theorem 2.1.6 *Let A be a linear operator on a Banach space L such that $\bar{A}$, the closure of A , is single-valued and generates a strong continuous contraction*

semigroup on L . Let B be a dissipative linear operator on L such that $\mathcal{D}(A) \subset \mathcal{D}(B)$.

If

$$(1.8) \quad \|Bf\| \leq \alpha \|Af\| + \beta \|f\|, \quad f \in \mathcal{D}(A),$$

where $0 \leq \alpha < 1$ and $\beta \geq 0$, then $\overline{A+B}$ is single-valued and generates a strongly continuous semigroup on L . Moreover, $\overline{A+B} = \bar{A} + \bar{B}$.

Proof: See Ethier-Kurtz [21]p.37 Theorem 7.1. ■

Theorem 2.1.7 Let $\{T(t)\}$, $\{S(t)\}$ and $\{V(t)\}$ be strongly continuous contraction semigroups on Banach space L with generators A, B , and C respectively. Let C be a core for A , and assume that $C \subset \mathcal{D}(B) \cap \mathcal{D}(C)$ and that $A = B + C$ on C . Then for each $f \in L$

$$(1.9) \quad \lim_{n \rightarrow \infty} [S(\frac{t}{n})V(\frac{t}{n})]^n f = T(t)f$$

for all $t \geq 0$, uniformly on bounded intervals.

Proof: See Ethier-Kurtz [21]p.33 Corollary 6.7 ■

2.2 Continuous State Branching Processes

In order to construct our interacting branching particle systems, prove their tightness, and prove the uniqueness of their limiting measure-valued processes, we must estimate the moments of a class of continuous state branching processes and discuss the uniqueness of the moment problem. So for this purpose, let us begin to discuss a special class of branching processes and their diffusion approximation. Given $n \in \mathbb{N}$ and $0 \leq \gamma < \infty$, $2 \leq \theta < \infty$. let $\{Q_n(t); t \geq 0\}$ be a one dimensional continuous time Markov branching process with state space $\Theta_n := \{\frac{k}{\theta^n}; k \in \mathbb{N} \cup \{0\}\}$. Its time homogeneous transition functions

$$P_{ij}^n(t) = P_{ij}^n(s, s+t) = P\{Q_n(s+t) = \frac{j}{\theta^n} | Q_n(t) = \frac{i}{\theta^n}\}$$

are determined by the following infinitesimal probabilities:

$$\sum_{k=0}^{\infty} p_k = 1, \quad \sum_{k=0}^{\infty} k p_k = 1, \\ m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty,$$

as solution of the Kolmogorov forward equation

$$(2.10) \quad \frac{d}{dt} P_{ij}^n(t) = -j\gamma\theta^n P_{ij}^n(t) + \gamma\theta^n \sum_{k=1}^{j+1} k P_{ik}^n(t) p_{j-k+1}$$

or backward equation

$$(2.11) \quad \frac{d}{dt} P_{ij}^n(t) = -i\gamma\theta^n P_{ij}^n(t) + \gamma\theta^n \sum_{k=i-1}^{\infty} i p_{k-i+1} P_{kj}^n(t)$$

with boundary condition

$$P_{ij}^n(0+) = \delta_{ij}.$$

Then we have the following results.

Lemma 2.2.1 *Let $k \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ and $t > 0$, $\mathbb{E}Q_n(t)^k < \infty$ if and only if $\sum_{j=0}^{\infty} j^k p_j < \infty$.*

Proof: refer to Athreya-Ney([1] p.111, Corollary 1). ■

Following the notations and assumptions given in Section 1.2 of Chapter 1. Let $Q_n(0) := \frac{m_n^n}{\theta^n}$. Then $Q_n(0) \leq \xi$ for $\forall n \in \mathbb{N}$. According to the theory of continuous state branching processes (see [1], [25], [43]), the limiting process $\{Q(t); t \geq 0\}$ of $\{Q_n(t); t \geq 0\}$ as $n \rightarrow \infty$ exists and is a diffusion process.

Lemma 2.2.2 *If $\sum_{k=0}^{\infty} k p_k = 1, m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty$, then $\mathbb{E}(Q(t)^k) < \infty$, for any $k \in \mathbb{N}$ and $t \geq 0$.*

Proof: By direct calculation, we know that the density function of transition function $P_t(x, A)$ of $Q(t)$ is the solution of the backward Kolmogorov equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}(m_2 - 1)\gamma x \frac{\partial^2 \varphi}{\partial x^2}.$$

According to the branching property, the Laplace transforms of $P_t(x, A)$ can be written in the form

$$\int_0^\infty \exp(-\lambda y) P_t(x, dy) = \exp(-x\psi_t(\lambda)), \quad \text{for } \lambda \geq 0$$

and

$$\psi_t(\lambda) = \lambda \left[1 + \frac{1}{2}(m_2 - 1)\gamma t \lambda \right]^{-1}.$$

Our conclusion follows directly from

$$\mathbb{E}[Q(t)^k | Q(0) = x] = (-1)^k \frac{\partial^k}{\partial \lambda^k} \{ \exp(-x\psi_t(\lambda)) \} |_{\lambda=0}$$

and $\psi_t(\lambda)$ is infinitely differentiable with respect to λ . ■

2.3 C-Diffusions

In the singular case (i.e. g is a singular function and $\rho(\cdot)$ is locally Lipschitz continuous away from 0. refer to Chapter 5), we need basic results of C-diffusions (coalescing diffusions) which were first introduced by T.E.Harris in [32] for constructing Brownian flow. Let us begin to talk about them. Let $\Omega = \Omega_d = C([0, \infty), \mathbb{R}^d)$ be the space of continuous mapping $\omega : [0, \infty) \rightarrow \mathbb{R}^d$ with the topology of uniform convergence on compact sets. Putting $\xi_t = \omega(t) = \omega_t$, let $\mathcal{F}_t = \mathcal{F}_{d,t} = \sigma(\xi_s, 0 \leq s \leq t)$, $\mathcal{F}_d = \vee_t \mathcal{F}_t$

and

$$(3.12) \quad \tilde{G}^d f(x) = \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \beta_i(x) \frac{\partial f(x)}{\partial x_i},$$

$$x = (x_1, \dots, x_d), \text{ for } f \in C_d^\infty(\mathbb{R}^d),$$

where $B = (b_{ij})$ is nonnegative definite and $B, \beta = (\beta_1, \dots, \beta_d)$ are bounded and continuous.

Definition 2.3.1 A probability measure $\mathbb{P}_x$ on $\mathcal{F}_d$, governing a process $\{\xi_t : t \geq 0\}$ in $\mathbb{R}^d$, is a solution to the martingale problem (MP) for $\tilde{G}^d$ from $x \in \mathbb{R}^d$ if $\mathbb{P}_x(\xi_0 = x) = 1$ and

$$f(\xi_t) - \int_0^t \tilde{G}^d f(\xi_s) ds$$

is a $(\mathbb{P}_x, \mathcal{F}_t)$ -martingale for every $f \in C_b^\infty(\mathbb{R}^d)$.

Definition 2.3.2 Define a shift operator $S_t : \Omega_d \rightarrow \Omega_d$ by

$$(S_t \omega)(s) = \omega(t + s), \text{ for } s, t \geq 0.$$

By a C-diffusion, we mean a probability measure $\mathbb{P}$ on $\mathcal{F}_d$ which solves the martingale problem for $\tilde{G}^d$ and there is a $C \in \mathcal{F}_d$ such that $S_t C \subset C$ and $\mathbb{P}(C) = 1$. Henceforth C is specialized to be the set of ω in Ω_d such that if $\omega_i(t) = \omega_j(t)$ for some $i \neq j$ and $t \geq 0$, then $\omega_i(t + s) = \omega_j(t + s)$ for all $s \geq 0$.

Let

$$D_k := \{(x_1, \dots, x_k) : (x_1, \dots, x_k) \in \mathbb{R}^k, x_i \neq x_j, \text{ for } i \neq j\} \text{ and } H_k := \mathbb{R}^k \setminus D_k.$$

Consider the following conditions on a real-valued function $b(\cdot)$:

(c.1) $b(0) = a > 0$, a is a constant,

(c.2) $b(x)$ is continuous on $\mathbb{R}$ and locally Lipschitz continuous away from 0 (i.e., Lipschitz continuous on the complement of every open interval in $\mathbb{R}$ centred at the origin),

(c.3) For every $k \geq 2$ and $(x_1, \dots, x_k) \in D_k$, matrix $(b(x_i - x_j))$ is strictly positive definite. Define

$$(3.13) \quad G^d f(x) = \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad \text{for } f \in C_b^\infty(\mathbb{R}^d),$$

where $x = (x_1, \dots, x_d)$ and $b_{ij}(x) := b(x_i - x_j)$.

Theorem 2.3.1 Suppose $b(\cdot)$ satisfies conditions (c.1) to (c.3), then for each $x \in \mathbb{R}^d$, there is a unique solution, C -diffusion $\mathbb{P}_x$, to the MP for G^d from x . If $A \in \mathcal{F}_d$, $\mathbb{P}_x(A)$ is measurable in x . The family $\{\mathbb{P}_x : x \in \mathbb{R}^d\}$ is strong Markov and Feller.

Proof: see T.E.Harris [32]p.191, Lemma(3.2). ■

Remark Suppose $\{(x_1(s), \dots, x_n(s)) : s \geq 0\}$ is a diffusion process with generator G^n . Let $\eta_t = x_i(t) - x_j(t)$, $i \neq j$, then $\{\eta_t\}$ is a diffusion process with state space $\mathbb{R}$, absorbing state 0 and generator

$$(3.14) \quad \mathcal{G}f(y) = (b(0) - b(y))f''(y), \quad f \in C_b^\infty(\mathbb{R}).$$

From Feller's criterion for accessibility, the probability that η reaches 0 is 0 or 1 according as

$$\int_0^1 \frac{y}{(b(0) - b(y))} dy$$

is ∞ or $< \infty$. But

$$0 \leq \sup \frac{b(0) - b(y)}{y^2} \leq \frac{1}{2}|b''(0)|.$$

So if $b''(0)$ is finite, state 0 is inaccessible.

2.4 Duality Method and Moment Problem

The duality method is a quite powerful tool to prove the uniqueness for the solutions to a martingale problem. This method is developed from an original idea due to Stroock and Varadhan. The idea is stated as follows.

Theorem 2.4.1 *Let Φ be a subset of $B(E)$ which generates $B(E)$ under bounded pointwise convergence (e.g., $C_b(E)$). Assume that for each $t > 0$ and $F \in \Phi$ there is a function $\Lambda_{F,t} \in B(E)$ such that for each $x \in E$,*

$$(4.15) \quad \mathbb{E}_x F(X(t)) = \Lambda_{F,t}(x)$$

holds for any solution $\{\mathbb{P}_x : x \in E\}$ to the martingale problem for A . Then there is at most one solution to the martingale problem for A and if such a solution exists, then it is measurable and satisfies the strong Markov property.

Proof: refer to Stroock-Varadhan [58]p.147 or Ethier-Kurtz [21]p.184. ■

In Chapter 4, we will take E as a subset of $M_F(\bar{\mathbb{R}})$ and use the above theorem to prove the uniqueness of the martingale problem for $A + B$ in which a certain function $\Lambda_{F,t}(\cdot)$ is obtained by using monomial function (defined in Chapter 4). In the realization of this procedure, an important step is to use the Stone-Weierstrass theorem to get the algebra generated by polynomials being dense in $C(E)$ if E is compact. But $M_F(\bar{\mathbb{R}})$ itself is not compact, so we need a certain generalized Theorem 2.4.1 which is given by Dawson-Kurtz [13]. In order to use this theorem, we need the following concept and a result of moment problem in one-dimensional case.

Definition 2.4.1 *Given a sequence of real constants $\{C_n\}$, a solution to the moment problem for $\{C_n\}$ is a distribution function $F(x)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that*

$$C_n = \int_{-\infty}^{\infty} x^n dF(x).$$

The problem of moments is of fairly old origin and the subject has an extensive literature. Interested readers are referred to Shohat-Tamarkin [54]. Here we only cite some sufficient conditions to guarantee a moment problem having uniqueness.

Theorem 2.4.2 (Carleman) *A sufficient condition that the moment problem for $\{C_n\}$ having uniqueness is that*

$$\sum_{n=1}^{\infty} (C_{2n})^{-1/2n} = \infty.$$

Proof: refer to Carleman [6] or Zessin [63]. ■

Theorem 2.4.3 *Let μ be a probability measure on the line having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k \mu(dx)$ of all orders. If the power series*

$$\sum_k \frac{\alpha_k r^k}{k!}$$

has a positive radius of convergence, then μ is the only probability measure with the moments $\{\alpha_k; k \geq 0\}$.

Proof: refer to Billingsley ([4] p.342, Theorem 30.1). ■

Chapter 3

Introduction to Space-time Stochastic Calculus

Before starting to discuss our model, we need to prepare some further basic concepts and preliminaries on orthogonal martingale measures, measure-valued and distribution-valued stochastic calculus. Let us introduce them now.

3.1 Orthogonal Martingale Measures

Let $(E, \mathcal{E})$ be a Lusin space, i.e. a measurable space homeomorphic to a Borel subset of the line. This includes all Euclidean space and, more generally, all Polish space. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space with a right continuous filtration. Suppose $U(A, \omega)$ is a function defined on $\mathcal{A} \times \Omega$, where $\mathcal{A} \subset \mathcal{E}$ is a ring, and such that $\mathbb{E}\{U(A)^2\} < \infty$, $A \in \mathcal{A}$. Suppose that U is finitely additive: if $A \cap B = \emptyset$, $A, B \in \mathcal{A}$, then $U(A \cup B) = U(A) + U(B)$ a.s. Let $\|U(A)\|_2 = \mathbb{E}\{U^2(A)\}^{1/2}$ be the L^2 -norm of $U(A)$. We say U is σ -finite if there exists an increasing sequence $\{E_n\} \subset \mathcal{E}$ whose union is E , such that for all n ,

$$(c-1.1) \quad \mathcal{E}_n \subset \mathcal{A}, \text{ where } \mathcal{E}_n = \mathcal{E}|_{E_n},$$

$$(c-1.2) \quad \sup\{\|U(A)\|_2 : A \in \mathcal{E}_n\} < \infty.$$

Define $\mu(A) = \|U(A)\|_2^2$. The set function U is said to be countably additive if for every sequence $\{A_j\}$ decreasing to $\emptyset$, $\mu(A_j)$ tends to zero. This means that U as an L^2 -valued set function is countably additive; but note that this does not mean that for every ω , $U(\cdot, \omega)$ is countably additive as a real-valued set function. Let

$$\hat{\mathcal{E}} := \{A \in \mathcal{E} \mid \lim_{n \rightarrow \infty} U(A \cap E_n) \text{ exists in } L^2\},$$

then $\hat{\mathcal{E}}$ is a ring, and U is countably additive on $\hat{\mathcal{E}}$. Indeed, if $A_i \in \hat{\mathcal{E}}$ and $\cup_{i=1}^{\infty} A_i \in \hat{\mathcal{E}}$, without loss of generality, we can suppose $A_i \cap A_j = \emptyset$, $i \neq j$; then we have

$$\begin{aligned} \mu(\cup_{i=1}^{\infty} A_i) &= \|U(\cup_{i=1}^{\infty} A_i)\|_2^2 \\ &= \lim_{n \rightarrow \infty} \|U(\cup_{i=1}^{\infty} A_i \cap E_n)\|_2^2 \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|U(A_i \cap E_n)\|_2^2 \\ &= \sum_{i=1}^{\infty} \|U(A_i)\|_2^2, \end{aligned}$$

where the limit can pass the sum sign because

$$\sup_n \sum_{i=1}^{\infty} \|U(A_i \cap E_n)\|_2^2 < \infty.$$

By general measure theory, μ can be extended on $\tilde{\mathcal{E}} := \sigma(\hat{\mathcal{E}})$. Equivalently, $U(\cdot)$ can be extended on $\tilde{\mathcal{E}}$. In the following, we will assume that all our countably additive set functions can be extended in this way. A set function which satisfies all these properties is called a σ -finite L^2 -valued measure.

Definition 3.1.1 A process $\{M_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{A}\}$ is a martingale measure if

$$(c-1.3) \quad M_0 = 0;$$

$$(c-1.4) \quad \text{if } t \geq 0, M_t \text{ is a } \sigma\text{-finite } L^2\text{-valued measure};$$

$$(c-1.5) \quad \text{for every } A \in \mathcal{A}, \{M_t(A), \mathcal{F}_t, t \geq 0\} \text{ is a martingale.}$$

The covariance functional of M is defined by

$$Q_M(t, A, B) := \langle M(A), M(B) \rangle_t \quad \text{for } A, B \in \mathcal{A}.$$

$\langle M(A), M(B) \rangle_t$ is the compensator of the process $M_t(A)M_t(B)$ under the sense of Doob-Meyer's decomposition theorem. It defines a covariance measure Q_M on $E \times E \times (\mathbb{R}^+)$.

The martingale measure M is worthy if there exists a random σ -finite measure $K(\Lambda; \omega)$, $\Lambda \in \mathcal{E} \times \mathcal{E} \times \mathcal{B}(\mathbb{R}^+)$, $\omega \in \Omega$, such that

(c-1.6) K is symmetric and positive definite; that is, for any $f \in b\mathcal{E} \times \mathcal{B}(\mathbb{R}^+)$.

$$\int_0^t \int_E \int_E f(x, s) f(y, s) K(dx, dy, ds) \geq 0,$$

(c-1.7) $\exists E_n \uparrow E$ such that

$$\mathbb{E}\{K(E_n \times E_n \times [0, T])\} < \infty, \quad \forall n,$$

(c-1.8) for fixed $A, B \in \mathcal{A}$, $\{K(A \times B \times (0, t]); t \geq 0\}$ is $\mathcal{F}_t$ -predictable,

(c-1.9) $|Q_M(t, A, B)| \leq K(A \times B \times [0, t])$ for $A, B \in \mathcal{A}$.

K is called the dominating measure for M .

Definition 3.1.2 M is an orthogonal martingale measure if it is a martingale measure and if $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $M_t(A)$ and $M_t(B)$ are orthogonal martingales.

If M is a martingale measure and if for all $A \in \mathcal{A}$, the map $t \rightarrow M_t(A)$ is continuous, we say that M is continuous.

Theorem 3.1.1 Let $\{M_t(A), \mathcal{F}_t, 0 \leq t \leq T, A \in \mathcal{A}\}$ be an orthogonal martingale measure. Then there exists a family $\{\nu_t(\cdot), 0 \leq t \leq T\}$ of random σ -finite measures

on $(E, \mathcal{E})$ such that

(c-1.10) $\{\nu_t(A), 0 \leq t \leq T\}$ is predictable for $A \in \mathcal{A}$.

(c-1.11) for all $A \in \mathcal{A}$, $t \rightarrow \nu_t(A)$ is right continuous and increasing;

(c-1.12) $\mathbb{P}(\nu_t(A) = \langle M(A) \rangle_t) = 1$ for all $t \geq 0$ and $A \in \mathcal{A}$.

The measure ν is called the intensity of the orthogonal martingale measure M .

(c-1.13) An orthogonal martingale measure M is worthy.

(c-1.14) If the orthogonal martingale measure M is continuous, then $\nu(\{t\} \times E_n) = 0$, $\forall t \geq 0, \forall n \in \mathbb{N}$.

(c-1.15) A worthy martingale measure N is orthogonal iff Q_N is supported by diagonal $\{(x, y); x = y \in E\}$ for all $t \geq 0$.

Proof: . see Walsh ([61] p.299-302). ■

Definition 3.1.3 A martingale measure M has nuclear covariance if there exists a finite measure η on $(E, \mathcal{E})$ and a complete ortho-normal system $\{\phi_k\}$ in $L^2(E, \mathcal{E}, \eta)$ such that if $\eta(A) = 0$, then $\mu(A) = 0$ for all $A \in \mathcal{E}$ and

$$\sum_k \mathbb{E}[M_t(\phi_k)^2] < \infty,$$

where $M_t(\phi_k) = \int \phi_k(x) M_t(dx)$ is a Bochner integral (see Ethier-Kurtz [21] p.473).

3.2 Space-time Stochastic Integral and Preliminaries

Now we turn to define the stochastic integral with respect to a martingale measure. Let M be a worthy martingale measure on the Lusin space $(E, \mathcal{E})$, and let Q_M and K_M be its covariance and dominating measures, respectively. Let T_n be a positive real number and $T_n \uparrow \infty$. We restrict ourselves to $E_n \times [0, T_n]$, so that M is finite on $E_n \times [0, T_n]$. Let $M^n = M|_{E_n \times [0, T_n]}$.

Definition 3.2.1 A function $f(x, s, \omega)$ is elementary on $E_n \times [0, T_n]$ if it is of the form

$$f(x, s, \omega) = X(\omega)1_{(a,b]}(s)1_A(x),$$

where $0 \leq a < b \leq T_n$, X is bounded and $\mathcal{F}_a$ -measurable, and $A \in \mathcal{E}_n$. f is simple if it is a finite sum of elementary functions.

The class of simple functions is denoted by

$$\mathcal{V}^n := \{h(x, s, \omega) = \sum_{i=1}^k h_i(\omega)1_{(u_i, v_i]}(s)1_{B_i}(x), B_i \in \mathcal{E}_n, h_i \in b\mathcal{F}_{u_i}, v_i \leq T_n\}.$$

Then $\mathcal{V}^n$ is dense in

$$L_{K^n}^2 := \{f | f \text{ is } \mathcal{P} \otimes \mathcal{E}\text{-measurable, and } \|f\|_{M^n} < \infty\},$$

where $\mathcal{P}$ is the predictable σ -field on $\Omega \times E_n \times [0, T_n]$ generated by $\mathcal{V}^n$ and

$$(f, g)_{K^n} := \int_{E_n \times E_n \times [0, T_n]} f(x, s)g(y, s)K(dx, dy, ds),$$

$\|f\|_{M^n} := \mathbb{E}\{(|f|, |f|)_{K^n}\}^{1/2}$. If $h \in \mathcal{V}^n$, we can define a new martingale measure

$$(2.1) \quad h \cdot M_t^n(A) := \sum_{i=1}^k h_i(M_{u_i \wedge t}^n(A \cap B_i) - M_{u_i \wedge t}^n(A \cap B_i)) \text{ for any } A \in \mathcal{E}_n.$$

Lemma 3.2.1 For any $f \in \mathcal{V}^n$, $f \cdot M^n$ is a worthy martingale measure. Its covariance $Q_{f \cdot M^n}$ and dominating measure $K_{f \cdot M^n}$ are given by

$$(2.2) \quad Q_{f \cdot M^n}(dx, dy, ds) = f(x, s)f(y, s)Q_{M^n}(dx, dy, ds),$$

$$(2.3) \quad K_{f \cdot M^n}(dx, dy, ds) = |f(x, s)f(y, s)|K_{M^n}(dx, dy, ds).$$

Moreover,

$$(2.4) \quad \mathbb{E}[(f \cdot M_t^n(B))^2] \leq \|f\|_{M^n}^2 \text{ for all } B \in \mathcal{E}_n, t \leq T_n.$$

Proof: see Walsh ([61] p.294, Lemma 2.4). ■

Suppose now that $f \in L^2_{K^n}$, since $\mathcal{V}^n$ is dense in $L^2_{K^n}$, there exist $\{f_n\} \subset \mathcal{V}^n$ such that $\|f - f_n\|_{M^n} \rightarrow 0$. By (2.4), if $A \in \mathcal{E}_n$, and $t < T_n$, then

$$(2.5) \quad \mathbb{E}\{(f_m \cdot M_t^n(A) - f_l \cdot M_t^n(A))^2\} \leq \|f_m - f_l\|_{M^n}^2 \rightarrow 0 \text{ as } l, m \rightarrow \infty.$$

It follows that $\{f_l \cdot M_t^n(A)\}_{l \geq 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, so it converges in L^2 to a martingale which is denoted by $f \cdot M_t^n(A)$. The limit is independent of the choice of sequence $\{f_l\}$. It is easy to see $f \cdot M_t^m(A) = f \cdot M_t^n(A)$ for $n < m, t \leq T_n$ and $A \in \mathcal{E}_n$. Therefore, there exists a unique process $f \cdot M_t(A)$ corresponding to any given $f \in \bigcap_n L^2_{K^n}$ such that $f \cdot M_t(A) = f \cdot M_t^n(A)$ for any $n, t \leq T_n, A \in \mathcal{E}_n$. $f \cdot M$ is called the stochastic integral martingale measure of f with respect to M .

Theorem 3.2.2 *If $f \in \bigcap_n L^2_{K^n}$, then $f \cdot M$ is a worthy martingale measure. It is orthogonal if M is. If $g \in \bigcap_n L^2_{K^n}$, and $A, B \in \mathcal{E}_n$ then*

$$(2.6) \quad \langle f \cdot M(A), g \cdot M(A) \rangle_t = \int_{A \times B \times [0, t]} f(x, s) g(y, s) Q_M(dx, dy, ds),$$

$$(2.7) \quad \mathbb{E}[(f \cdot M_t(A))^2] \leq \|f\|_{M^n}^2 \text{ for all } t \leq T_n.$$

Proof: see Walsh ([61] p.295, Theorem 2.5). ■

Definition 3.2.2 *Let ν be a σ -finite measure on $(E, \mathcal{E})$. A Gaussian (orthogonal) martingale measure W based on $\nu(dx)dt$ is a random set function on $\mathcal{E} \times \mathcal{B}(\mathbb{R}^+)$ such that*

- (a) *For any $n, B \in \mathcal{E}_n, a, b \leq T_n, W(B \times [a, b])$ is a Gaussian random variable with mean zero and variance $\nu(B) \cdot |b - a|$.*
- (b) *W is an orthogonal martingale measure.*

Theorem 3.2.3 *A continuous orthogonal martingale measure is a Gaussian orthogonal martingale measure iff its covariance is deterministic.*

Proof: see Walsh ([61] p.302 Proposition 2.10). ■

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of C^∞ -functions on $\mathbb{R}$ which are rapidly decreasing at infinity, $\mathcal{S}'(\mathbb{R})$ be the topological dual i.e. the space of Schwartz distributions.

Definition 3.2.3 *On a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, a $\mathcal{S}'(\mathbb{R})$ -valued process $\{W_t : t \geq 0\}$ is called a $(\mathcal{F}_t)$ -cylindrical Brownian motion if $W_t(\phi)$ is a real-valued Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with increasing process $\langle W_t(\phi) \rangle_t = t \int \phi^2(x) dx$ for every $\phi \in \mathcal{S}(\mathbb{R})$. Since $W_t(\phi)$ can be extended by L^2 -continuity to $L^2(\mathbb{R})$, there corresponds a Gaussian orthogonal martingale measure $W(dx, ds)$ on $\mathcal{B}(\mathbb{R} \times [0, \infty))$ satisfying that*

$$W_t(\phi) = \int_0^t \int_{\mathbb{R}} \phi(x) W(dx, ds) \equiv \int_0^t \int_{\mathbb{R}} \phi(x) \dot{W}_s(x) dx ds,$$

where $\dot{W}_s(x)$ is called space-time white noise.

The following two theorems are needed later on, so we cite them here:

Theorem 3.2.4 (Burkholder-Davis-Gundy) *Let $p \geq 1$. If Φ is a $(\mathcal{F}_t)$ -predictable function satisfying*

$$\mathbb{E} \left(\int_0^t \int_{\mathbb{R}} \Phi(x, s) dx ds \right)^p < \infty \quad \text{for any } t \geq 0,$$

then for any $t \geq 0$,

$$(2.8) \quad \mathbb{E} \left(\int_0^t \int_{\mathbb{R}} \Phi(x, s) \dot{W}_s(x) dx ds \right)^{2p} \leq C_{2p} \mathbb{E} \left(\int_0^t \int_{\mathbb{R}} \Phi(x, s)^2 dx ds \right)^p,$$

where C_{2p} is an absolute constant.

Proof: see Walsh ([61] p.377, Theorem 7.11). ■

Theorem 3.2.5 (Mitoma) *Let $\{X_t^n; 0 \leq t \leq T\}$, $n = 1, 2, \dots$ be a sequence of processes whose sample paths are in $D([0, T], S'(\mathbb{R}))$ a.s. Then the sequence (X^n) is tight iff for each $\phi \in S(\mathbb{R})$, the sequence of real-valued processes $\{X_t^n(\phi); t \geq 0\}$, $n = 1, 2, \dots$, is tight in $D([0, T], \mathbb{R})$.*

Proof: see Mitoma [46]. ■

Chapter 4

Superprocesses in a Brownian Medium (I)

4.1 Introduction

Measure-valued processes have been an active area of research and there are considerable number of papers on this topic in the current literature. In this field we recommend Dawson [12] as an excellent general reference.

This chapter is devoted to the construction and characterization of the limiting measure-valued processes of a sequence of interacting branching particle systems which are formulated in Section 1.2 of Chapter 1. So the notations here coincide with those given in Section 1.2 of Chapter 1 if we don't specify.

As already mentioned in Section 1.2 of Chapter 1, we can not apply Jirina's formula and log-Laplace functional (or evolution equation) technique to construct the limiting measure-valued processes in our model since the motions of particles are not independent in this case. Therefore, we encounter a natural question : Given any finite measure μ with compact support on $\mathbb{R}$, does $(\mathcal{L}, \delta_\mu)$ -martingale problem have a unique solution ? This chapter is designed to answer this question. First let's give the main ideas in our solution to the question and this chapter's organization.

In Section 2, we will establish the fact that for any finite measure ν_0 with $\nu_0(\bar{\mathbb{R}}) \leq \xi$, the $(\mathcal{A}, \delta_{\nu_0})$ -martingale problem has a unique solution and a subspace $\mathcal{D}(\mathcal{A}) \subset C(E)$ can be chosen such that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is a generator of a Feller semigroup. In Section 3, a sequence of finite particle systems will be constructed and for any given finite measure $\tilde{\mu}$ with compact support, we obtain a solution to $(\mathcal{L}, \delta_{\tilde{\mu}})$ -martingale problem by tightness argument. The uniqueness of $(\mathcal{L}, \delta_{\tilde{\mu}})$ -martingale problem is proved by duality method in Section 4.

4.2 The Generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$

In this section, we prove that for any finite measure μ on $\bar{\mathbb{R}}$ with compact support, the $(\mathcal{A}, \delta_{\mu})$ -martingale problem is uniquely solvable, and a subspace $\mathcal{D}(\mathcal{A}) \subset C(E)$ can be chosen such that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is an infinitesimal generator of a Feller semigroup $\{V_t : t \geq 0\}$. Now we start introducing a general setting for measure-valued processes and their dual processes. For any positive ℓ , let $E_\ell := \{\mu \in E : \langle \mu, 1 \rangle \leq \ell\}$,

$$(2.1) \quad \Pi_1^D(E_\ell) := \{F_f(\mu) : F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}), \\ f(x_1, \dots, x_N) \in C^\infty(\bar{\mathbb{R}}^N), \\ \mu^{\otimes N}(d\tilde{x}) = \prod_{i=1}^N \mu(dx_i), \mu \in E_\ell, N \geq 0\},$$

and $\Pi^D(E_\ell)$ be the smallest algebra of functions on E_ℓ which contains $\Pi_1^D(E_\ell)$, then it follows from Stone-Weierstrass theorem that $\Pi^D(E_\ell)$ is dense in $C(E_\ell)$ (see Dawson-Kurtz [13] p.93, Lemma 2.2). Let

$$(2.2) \quad D(\bar{\mathbb{R}}^N) := C^\infty(\bar{\mathbb{R}}^N),$$

where $\bar{\mathbb{R}}^N$ is one point compactification of $\mathbb{R}^N$ and $C^\infty(\bar{\mathbb{R}}^N)$ is the space of all infinitely differentiable functions on $\bar{\mathbb{R}}^N$.

It is obvious that $D(\bar{\mathbb{R}}^N)$ is dense in $C(\bar{\mathbb{R}}^N)$.

Let $\tilde{C} := \cup_{N=0}^{\infty} C(\bar{\mathbb{R}}^N)$ (disjoint union), where $C(\bar{\mathbb{R}}^0) := \mathbb{R}^1$. We define a metric d on $\tilde{C}$ by

$$d(f, g) := \begin{cases} \|f - g\|_n & \text{if } N(f) = N(g) = n \\ |m - n| & \text{if } N(f) = m, \quad N(g) = n, \text{ and } m \neq n, \end{cases}$$

where $N(f) := n$ if $f \in C(\bar{\mathbb{R}}^n)$ and $f \notin C(\bar{\mathbb{R}}^{n-1})$, and $\|f - g\|_n$ is the uniform norm on $C(\bar{\mathbb{R}}^n)$. Then $(\tilde{C}, d)$ is a Polish space. Let $\tilde{D} := \cup_{N=0}^{\infty} D(\bar{\mathbb{R}}^N)$ (disjoint union). Then $\tilde{D}$ is a subspace of $\tilde{C}$. We will see $\tilde{C}$ is the state space of the dual processes.

$$(2.3) \quad Q_0(\tilde{D}) := \left\{ F_{\mu}(f) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}), \right. \\ \left. f \in D(\bar{\mathbb{R}}^N), \mu \in E, N \geq 0 \right\}.$$

Let $Q(\tilde{D})$ denote the linear space generated by $Q_0(\tilde{D})$.

The following facts are needed for proving the existence of $(\mathcal{A}, \delta_{\mu})$ -MP.

Lemma 4.2.1 *Let $X = (x_t, \mathcal{F}_t, P_x)$ be a Markov process on the state space $(H, \mathcal{H})$ with transition function $P(t, x, A)$. Let γ be a measurable transformation of $(H, \mathcal{H})$ into the state space $(\tilde{H}, \tilde{\mathcal{H}})$ such that $\gamma H = \tilde{H}$, $\tilde{\mathcal{H}} \supseteq \gamma(\mathcal{H})$ and such that the following condition is satisfied: For all $A \in \tilde{\mathcal{H}}$ and $x, x' \in H$ such that $\gamma x = \gamma x'$,*

$$P(t, x, \gamma^{-1}A) = P(t, x', \gamma^{-1}A).$$

Set $\tilde{x}_t = \gamma x_t$ and denote by $\mathcal{N}$ the σ -algebra generated by the sets $\{\tilde{x}_t \in A\}$ ($t \geq 0, A \in \tilde{\mathcal{H}}$). Further, let $\tilde{\mathcal{F}}_t = \mathcal{F}_t \cap \mathcal{N}$ and

$$\tilde{P}_{\gamma x}(B) = P_x(B) \quad \text{for} \quad B \in \mathcal{N}.$$

The system $\tilde{X} = (\tilde{x}_t, \tilde{\mathcal{F}}_t, \tilde{P}_x)$ defines a Markov process on the state space $(\tilde{H}, \tilde{\mathcal{H}})$ with transition function

$$\tilde{P}(t, \gamma x, A) = P(t, x, \gamma^{-1}A).$$

If X is a strong Markov process, then so is the process $\tilde{X}$.

Proof: see Dynkin([18] v1, p.325, Theorem 10.13) ■

Let $C_b^2(\mathbb{R}^N)$ be the space of all bounded twice continuously differentiable functions on $\mathbb{R}^N$, $C_c^\infty(\mathbb{R}^N)$ be the space of all infinitely differentiable functions on $\mathbb{R}^N$ with compact support and $C_0(\mathbb{R}^N)$ be the space of all continuous functions on $\mathbb{R}^N$ vanishing at infinity.

Lemma 4.2.2 For any $N \in \mathbb{N}$, let

$$(2.4) \quad G^N f(x_1, \dots, x_N) := \frac{1}{2} \sum_{i=1}^N \epsilon^2 \frac{\partial^2}{\partial x_i^2} f(x_1, \dots, x_N) \\ + \frac{1}{2} \sum_{i,j=1}^N \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_N),$$

where $\rho(\cdot)$ defined by (2.13) of Chapter 1. Suppose $\rho(\cdot)$ is a smooth function, then

(i) There exist uniquely a system of probabilities $\{P_x, x \in \mathbb{R}^N\}$ on $(\bar{W}^N, \mathcal{B}(\bar{W}^N))$ which is strongly Markovian and satisfies

$$P_x\{w; w(0) = x\} = 1 \text{ for every } x \in \mathbb{R}^N,$$

$$f(w(t)) - f(w(0)) - \int_0^t (G^N f)(w(s)) ds$$

is a $(P_x, \mathcal{B}_t(\bar{W}^N))$ -martingale for every $f \in C^2(\bar{\mathbb{R}}^N)$, where

$$\bar{W}^N = \{w : w \in C([0, \infty), \bar{\mathbb{R}}^N) \text{ such that if } w(t) = \Delta,$$

$$\text{then } w(u) = \Delta, \text{ for all } u \geq t\},$$

$\mathcal{B}_t(\bar{W}^N)$ is the σ -field generated by Borel cylinder sets before t and $\mathcal{B}(\bar{W}^N) := \sigma(\cup_t \mathcal{B}_t(\bar{W}^N))$.

(ii) Define

$$\tilde{G}^N \tilde{f}(x) := \begin{cases} G^N f(x) & \text{if } x \neq \Delta \\ 0 & \text{if } x = \Delta, \end{cases}$$

where $\tilde{f} \in C^\infty(\bar{\mathbb{R}}^N)$ and $f := \tilde{f} - \tilde{f}(\Delta) \in C_0^\infty(\mathbb{R}^N)$. Then the closure of $(\hat{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ is single-valued and generates a Feller semigroup on $C(\bar{\mathbb{R}}^N)$.

Remark Given any initial value $x \in \mathbb{R}^N$, by Ikeda-Watanabe([34] p.163, Theorem 2.4), then the explosion time of the G^N -diffusion $e(\omega) = \infty$ a.s. This means that the G^N -diffusion always lives in $\mathbb{R}^N$.

Proof: (i) If $\epsilon \neq 0$, then this is the conclusion of Ikeda-Watanabe([34] p.155, Theorem 2.2 and p.177, Theorem 3.3). So we only need to consider the case in which $\epsilon = 0$. Let

$$a^{ij}(x) := \rho(x_i - x_j) \quad \text{for } x = (x_1, \dots, x_N) \in \mathbb{R}^N$$

and

$$a(x) := (a^{ij}(x)).$$

Obviously $a(x)$ is symmetric, also it is a non-negative definite matrix.

Indeed, for $\forall b = (b_1, \dots, b_N) \in \mathbb{R}^N$,

$$\begin{aligned} ba(x)b^T &= \sum_{i,j=1}^N b_i b_j \int_{\mathbb{R}} g(x_i - x_j - z)g(z)dz \\ &= \sum_{i,j=1}^N b_i b_j \int_{\mathbb{R}} g(x_i - t)g(x_j - t)dt \\ &= \int_{\mathbb{R}} \left(\sum_{i=1}^N b_i g(x_i - t) \right)^2 dt \geq 0. \end{aligned}$$

Given a matrix $\sigma(x)$ such that

$$(2.5) \quad \begin{cases} x \rightarrow \sigma(x) \text{ is continuous} \\ a^{ij}(x) = \sum_{k=1}^N \sigma^{ik}(x)\sigma^{jk}(x), \end{cases}$$

let's consider the following stochastic differential equation

$$(2.6) \quad dX^i(t) = \sum_{k=1}^N \sigma^{ik}(X(t))dB^k(t),$$

where $(B^1(t), \dots, B^N(t))$ are independent Brownian motions.

It is well known that if (2.5) holds, then G^N -diffusion $\{P_x, x \in \bar{\mathbb{R}}^N\}$ exists uniquely if and only if (2.6) is uniquely solvable.

Since $\rho(\cdot)$ is a smooth function, the result follows from the corollary of Ikeda-Watanabe([34] p.203).

(ii) Since $a^{ij}(x) \in C^2(\mathbb{R}^N)$ with $\partial_k \partial_l a^{ij}$ bounded for $i, j, k, l = 1, 2, \dots, N$, by Theorem 2.1.5, the closure of $(G^N, C_0^\infty(\mathbb{R}^N))$ is single-valued and generates a Feller semigroup $\{\hat{S}_t^N : t \geq 0\}$ on $C_0(\mathbb{R}^N)$. Let $\Delta \notin \mathbb{R}^N$ be the point at infinity and $\bar{\mathbb{R}}^N = \mathbb{R}^N \cup \{\Delta\}$. Let $\tilde{f} \in C(\bar{\mathbb{R}}^N)$, We define

$$S_t^N \tilde{f} = \tilde{f}(\Delta) + \hat{S}_t^N(\tilde{f} - \tilde{f}(\Delta)).$$

Then by Ethier-Kurtz ([21] p.166, Lemma 2.3), $\{S_t^N : t \geq 0\}$ is a Feller semigroup on $C(\bar{\mathbb{R}}^N)$ and for $\tilde{f} \in C^\infty(\bar{\mathbb{R}}^N)$, let $f := \tilde{f} - \tilde{f}(\Delta) \in C_0^\infty(\mathbb{R}^N)$ and define $(\tilde{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ by

$$(2.7) \quad \tilde{G}^N \tilde{f}(x) := \begin{cases} G^N f(x) & \text{if } x \neq \Delta \\ 0 & \text{if } x = \Delta, \end{cases}$$

then the closure of $(\tilde{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ is the generator of $\{S_t^N : t \geq 0\}$. ■

In the following we will not distinguish notationally between $\tilde{G}^N$ and G^N .

Theorem 4.2.3 Suppose $\rho(\cdot)$ defined by (2.13) of Chapter 1 is a smooth function, then we have: (I) For any finite measure μ on $\mathbb{R}$ with compact support, the $(\mathcal{A}, \delta_\mu)$ -martingale problem is uniquely solvable.

(II) A subspace $\mathcal{D}(\mathcal{A}) \subset C(E)$ exists such that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is a generator of a Feller semigroup.

Remark We only need to consider the measure on $\mathbb{R}$ in (I), because $x(u) = \Delta$ then $x(t) = \Delta$ for any $t \geq u$.

Proof: (I) As for the existence, since μ is a finite measure, there exists a positive constant ξ such that $\langle \mu, 1 \rangle \leq \xi$.

(a) first we suppose measure $\mu = \frac{1}{\theta^n} \sum_{i=1}^m \delta_{x_i}$, and for any

$$F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}) \in \Pi_1^D(E_\ell),$$

define

$$(2.8) \quad \rho_\epsilon(x) := \begin{cases} \rho_\epsilon = \rho(0) + \epsilon^2 & \text{if } x = 0 \\ \rho(x) & \text{if } x \neq 0. \end{cases}$$

Let $f'_\alpha(x_1, \dots, x_n)$ denote the derivative of $f(x_1, \dots, x_n)$ with respect to α^{th} variable.

Now we change the form of $\mathcal{A}F_f(\mu)$ as follows.

$$\begin{aligned} (2.9) \quad \mathcal{A}F_f(\mu) &= \frac{1}{2} \rho_\epsilon \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^N f''_{ii}(y_1, \dots, y_N) \mu^{\otimes N}(d\tilde{y}) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1; i \neq j}^N \rho(y_i - y_j) f''_{ij}(y_1, \dots, y_N) \mu^{\otimes N}(d\tilde{y}) \\ &= \frac{1}{2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1}^N \rho_\epsilon(y_i - y_j) f''_{ij}(y_1, \dots, y_N) \mu^{\otimes N}(d\tilde{y}) \\ &= \frac{1}{2} \left(\frac{1}{\theta^n} \right)^N \sum_{\alpha=1}^N \rho_\epsilon \sum_{l_1, \dots, l_N=1}^m f''_{\alpha\alpha}(x_{l_1}, \dots, x_{l_N}) \\ &\quad + \frac{1}{2} \left(\frac{1}{\theta^n} \right)^N \sum_{\alpha, \beta=1; \alpha \neq \beta}^N \rho_\epsilon \sum_{l_1, \dots, l_N=1; l_\alpha=l_\beta}^m f''_{\alpha\beta}(x_{l_1}, \dots, x_{l_N}) \\ &\quad + \frac{1}{2} \left(\frac{1}{\theta^n} \right)^N \sum_{i,j=1; i \neq j}^m \rho_\epsilon(x_i - x_j) \sum_{\alpha, \beta=1; \alpha \neq \beta}^N \sum_{l_1, \dots, [l_\alpha], \dots, [l_\beta], \dots, l_N=1}^m f''_{\alpha\beta}(x_{l_1}, \\ &\quad \dots, x_i, \dots, x_j, \dots, x_{l_N}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=1}^m \rho_i \frac{\partial^2}{\partial x_i \partial x_i} \left\{ \left(\frac{1}{\theta^n} \right)^N \sum_{l_1, \dots, l_N=1}^m f(x_{l_1}, \dots, x_{l_N}) \right\} \\
&+ \frac{1}{2} \sum_{i,j=1; i \neq j}^m \rho_i(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} \left\{ \left(\frac{1}{\theta^n} \right)^N \sum_{l_1, \dots, l_N=1}^m f(x_{l_1}, \dots, x_{l_N}) \right\} \\
&= \frac{1}{2} \sum_{i=1}^m \rho_i \frac{\partial^2}{\partial x_i \partial x_i} g(x_1, \dots, x_m) \\
&+ \frac{1}{2} \sum_{i,j=1; i \neq j}^m \rho_i(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} g(x_1, \dots, x_m) \\
&= G^m g(x_1, \dots, x_m).
\end{aligned}$$

Here $[\alpha]$ means excluding variable α ,

$$(2.10) \quad g(x_1, \dots, x_m) = \left(\frac{1}{\theta^n} \right)^N \sum_{l_1, \dots, l_N=1}^m f(x_{l_1}, \dots, x_{l_N}),$$

and G^m is defined by (2.4).

For fixed integers n, m , let

$$(2.11) \quad E^{n,m} := \{ \mu \in E \mid \mu = \frac{1}{\theta^n} \sum_{i=1}^m \delta_{x_i}, x_i \in \bar{\mathbb{R}}, i = 1, \dots, m \},$$

and

$$(2.12) \quad E^n := \bigcup_{m=0}^{\infty} E^{n,m}.$$

Both $E^{n,m}$ and E^n are endowed with subspace topologies of E .

Definition 4.2.1 Let $\varphi(m)$ be the set of all permutations of $\{1, \dots, m\}$. A family of probability measures $\{P_y^m : y \in \mathbb{R}^m\}$ on $C([0, \infty), \mathbb{R}^m)$ (or on $D([0, \infty), \mathbb{R}^m)$) is called exchangeable if it satisfies $P_{\Pi y}^m = P_y^m \circ \tilde{\Pi}^{-1}$ for every $y \in \mathbb{R}^m$ and every $\Pi \in \varphi(m)$, with $\Pi y = \Pi(y_1, y_2, \dots, y_m) := (y_{\Pi 1}, y_{\Pi 2}, \dots, y_{\Pi m})$ for each $y \in \mathbb{R}^m$ and the corresponding $\tilde{\Pi}(X)(t) := \Pi(X(t))$ for each $X \in C([0, \infty), \mathbb{R}^m)$ (or $D([0, \infty), \mathbb{R}^m)$).

Given a second order differential operator

$$A = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b^i(x) \frac{\partial}{\partial x_i},$$

we define

$$\Pi A := \frac{1}{2} \sum_{i,j=1}^d a^{\Pi_i \Pi_j}(x) \frac{\partial^2}{\partial x_{\Pi_i} \partial x_{\Pi_j}} + \sum_i b^{\Pi_i}(x) \frac{\partial}{\partial x_{\Pi_i}}.$$

A is said symmetric if $\Pi A = A$.

We define a transformation from $\mathbb{R}^m$ to $E^{n,m}$ by

$$(2.13) \quad \Phi_n : (x_1, \dots, x_m) \rightarrow \frac{1}{\theta^n} \sum_{i=1}^m \delta_{x_i}.$$

Since G^m is symmetric, the law of the corresponding diffusion process is exchangeable and the conditions of Lemma 4.2.1 are satisfied. By Lemma 4.2.1 and Lemma 4.2.2, we obtain a solution to the $(\mathcal{A}, \delta_\mu)$ -martingale problem.

(b) Now suppose μ is any finite measure with compact support, we define

$$(2.14) \quad B_n^k := B\left(\frac{k}{2^n}, \frac{1}{2^{n+1}}\right) := \left\{x \in \mathbb{R} : \frac{k}{2^n} - \frac{1}{2^{n+1}} \leq x < \frac{k}{2^n} + \frac{1}{2^{n+1}}\right\},$$

$$(2.15) \quad N(\mu(B_n^k)) := \begin{cases} i & \text{if existing } i \in \mathbb{N} \text{ such that} \\ & \frac{i}{\theta^n} \leq \mu(B_n^k) < \frac{i+1}{\theta^n} \text{ and } |k| \leq n2^n \\ 0 & \text{otherwise,} \end{cases}$$

$$(2.16) \quad N_n := \sum_{|k| \in \mathbb{N}} N(\mu(B_n^k)),$$

$$x_i^n = \begin{cases} \frac{-n2^n}{2^n} & \text{if } 1 \leq i \leq N(\mu(B_n^{-n2^n})) \\ \frac{-n2^n+1}{2^n} & \text{if } N(\mu(B_n^{-n2^n})) + 1 \leq i \leq N(\mu(B_n^{-n2^n})) + N(\mu(B_n^{-n2^n+1})) \\ \vdots & \vdots \\ \frac{n2^n}{2^n} & \text{if } \sum_{k \leq (n2^n-1)} N(\mu(B_n^k)) + 1 \leq i \leq \sum_{k \leq n2^n} N(\mu(B_n^k)), \end{cases}$$

$$(2.17) \quad \mu_n := \frac{1}{\theta^n} \sum_{i=1}^{N_n} \delta_{x_i^n}(\cdot),$$

then it is obvious $\mu_n \Rightarrow \mu$. Without loss of generality, we can assume $\langle \mu_n, 1 \rangle \leq \xi$ for $n \in \mathbb{N}$. From the above argument, suppose Y_n is the solution to the $(\mathcal{A}, \delta_{\mu_n})$ -martingale problem. Since $\mathcal{A}$ does not change particles' masses by Feller's criterion for accessibility, for any $n \in \mathbb{N}$ and $t \geq 0$, $\langle Y_n(t), 1 \rangle \leq \xi$. It follows from Ethier-Kurtz ([21] p.196, Lemma 5.1) that $(\mathcal{A}, \delta_{\mu})$ -martingale problem has a solution which is the weak limit of Y_n .

As for the uniqueness, let

$$(2.18) \quad \mathcal{A}^* F_{\mu'}(f) := F_{\mu'}(G^{N(f)} f) \quad \text{for } F_{\mu'}(f) \in Q_0(\tilde{D}),$$

where $N(\cdot)$ is a function of f , i.e.,

$$N(f) = m \quad \text{if } f \in D(\bar{\mathbb{R}}^m) \text{ and } f \notin D(\bar{\mathbb{R}}^{m-1}).$$

Then we have

$$(2.19) \quad \begin{aligned} \mathcal{A}F_f(\mu') &= \frac{1}{2} \rho_\epsilon \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^N f''_{ii}(x_1, \dots, x_N) \mu'^{\otimes N}(d\tilde{x}) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1; i \neq j}^N \rho(x_i - x_j) f''_{ij}(x_1, \dots, x_N) \mu'^{\otimes N}(d\tilde{x}) \\ &= \mathcal{A}^* F_{\mu'}(f). \end{aligned}$$

From Lemma 4.2.2, we know that for any $N \in \mathbb{N}$, the closure of $(\tilde{G}^N, D(\bar{\mathbb{R}}^N))$ is a generator of a Feller semigroup $\{S_t^N : t \geq 0\}$ and

$$S_t^N : D(\bar{\mathbb{R}}^N) \rightarrow D(\bar{\mathbb{R}}^N).$$

So $(\mathcal{A}^*, Q(\tilde{D}))$ has the structure of an infinitesimal generator of a conservative $\tilde{D}$ -valued Markov process restricted to functions in $Q(\tilde{D})$. If the initial state of this

process is $f \in D(\bar{\mathcal{R}}^N)$, then it keeps evolving in a continuous and deterministic way on $D(\bar{\mathcal{R}}^N)$; namely

$$(2.20) \quad f \rightarrow S_t^N f.$$

So by Dawson-Kurtz([13] p.101, Theorem 4.4), $(\mathcal{A}, \delta_\mu)$ -martingale problem is well posed.

(II) First since $(\mathcal{A}, \delta_\mu)$ -martingale problem has a unique solution, by Ethier-Kurtz([21] p.178, Prop. 3.5), $\mathcal{A}$ is a dissipative operator. Secondly, given any real number $\lambda > 0$ and any monomial function

$$\int \cdots \int g(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}),$$

we consider the range condition of $(\lambda - \mathcal{A})$. Since G^N is a generator of a Feller semigroup, for above $g \in C(\mathbb{R}^N)$ and $\lambda > 0$, by Hille-Yosida's theorem, there exists $f \in \mathcal{D}(G^N)$ such that $(\lambda - G^N)f = g$. Therefore,

$$\begin{aligned} & (\lambda - \mathcal{A}) \int \cdots \int f(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}) \\ &= \int \cdots \int (\lambda - G^N)f(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}) \\ &= \int \cdots \int g(x_1, \dots, x_N) \mu^{\otimes N}(d\tilde{x}). \end{aligned}$$

Since $\Pi^D(E_l)$ is dense in $C(E_l)$, $(\lambda - \mathcal{A})$ satisfies the range condition. Note that the l is an arbitrary positive number, thus the closure of $\mathcal{A}$ generates a strongly continuous contraction semigroup V_t on $C(M_F(\bar{\mathbb{R}}))$ by Hille-Yosida's theorem. It is obvious that V_t is conservative. Since for any nonnegative $F(\mu) \in C(M_F(\bar{\mathbb{R}}))$, we have

$$\|F(\mu)\| - V_t F(\mu) = V_t(\|F(\mu)\| - F(\mu)) \leq \| \|F(\mu)\| - F(\mu) \| \leq \|F(\mu)\|$$

for all $\mu \in M_{F \setminus \bar{\mathbb{R}}}$ and $t \geq 0$, the positivity of V_t follows. So the proof of (II) is complete. ■

4.3 Finite Particle Systems and Tightness

In this section, first we construct a sequence of finite particle systems and prove that they satisfy tightness conditions, then we will show that their limiting measure-valued processes are solutions to the $(\mathcal{L}, \delta_\mu)$ -martingale problem. Now we begin to construct a sequence of finite particle systems. Define $\mathcal{B}_n : B(E^n) \rightarrow B(E^n)$ by

$$(3.21) \quad \mathcal{B}_n F(\nu_n) := \gamma \theta^n (\theta^{2n} \wedge m) \int_{E^n} (F(\nu') - F(\nu_n)) \Gamma(\nu_n, d\nu'),$$

where E^n is defined by (2.12), $B(E^n)$ is the space of all bounded measurable functions on E^n , $x \wedge y := \min(x, y)$ and define

$$\Gamma(\nu_n, d\nu') := \frac{1}{m} \sum_{i=1}^m \frac{1}{2} \delta_{(\nu_n + \frac{1}{2\theta^n} \delta_{x_i})}(d\nu') + \frac{1}{m} \sum_{i=1}^m \frac{1}{2} \delta_{(\nu_n + \frac{1}{2\theta^n} \delta_{x_i})}(d\nu')$$

$$\text{for } \nu_n = \frac{1}{\theta^n} \sum_{i=1}^m \delta_{x_i} \in E^n.$$

So by definition, $\mathcal{B}_n$ is a bounded operator. Define $\mathcal{A}_n := \mathcal{A}|_{E^n}$.

Lemma 4.3.1 *For μ_n defined by (2.17), the $(\mathcal{A}_n + \mathcal{B}_n, \delta_{\mu_n})$ -martingale problem has a unique solution with state space E^n .*

Proof: Let $\mathcal{B}_n F_f(\mu_n)$ be defined by (3.21), then $(\mathcal{B}_n, B(E^n))$ is the generator of a pure jump process. It follows from Theorem 2.1.6, Theorem 2.1.7, and Ethier-Kurtz ([21] p.166, Lemma 2.3, p.169. Theorem 2.7 and p.182, Theorem 4.1) that the $(\mathcal{A}_n + \mathcal{B}_n, \delta_{\mu_n})$ -martingale problem has a unique solution with state space E^n which is denoted by X_n . ■

Suppose $X_n(t) = \frac{1}{\theta^n} \sum_{i=1}^{m_t^n} \delta_{x_i}$. This means that m_t^n is the total number of particles at time t with initial $m_0^n = N_n$.

Lemma 4.3.2 *Under above notations, we have the following results:*

(1) *For any positive η , we have*

$$(3.22) \quad \tilde{\mathbb{P}}(\sup_t \langle X_n(t), 1 \rangle > \eta) \leq \frac{\langle \mu_n, 1 \rangle}{\eta}$$

and $\{X_n(t) : t \geq 0\}$ is tight.

(2) Suppose $\tilde{\mathbb{P}}^{X_n}$, the law of X_n , weakly converges to a probability $\tilde{\mathbb{P}}$ on $D([0, \infty), E)$, then there are a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $D([0, \infty), E)$ -valued random variables $Z_n, n = 1, 2, \dots$, and Z defined on it such that $\tilde{\mathbb{P}}^{X_n} = \mathbb{P}^{Z_n}$, $\tilde{\mathbb{P}} = \mathbb{P}^Z$ and $\{Z_n : n > 0\}$ converges to Z on $D([0, \infty), E)$ almost surely with respect to probability $\mathbb{P}$.

(3)

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t} \frac{\langle Z_n(t), 1 \rangle}{\theta^n} = 0 \quad \mathbb{P} - a.s.$$

Proof: (1). Let us first prove inequality (3.22). For $\lambda > 0$, let $F_\lambda(\mu) = e^{-\langle \lambda, \mu \rangle}$ and $\vartheta(z) = \frac{1}{2} + \frac{1}{2}z^2$, then B_n is the generator of the pure jump process $|X_n(t)| := \langle X_n(t), 1 \rangle$ and

$$(3.23) \quad \begin{aligned} M_\lambda(t) &:= e^{-\lambda|X_n(t)|} - \int_0^t B_n F_\lambda(X_n(s)) ds \\ &= e^{-\lambda|X_n(t)|} + \int_0^t \gamma \theta^{2n} (\theta^n \wedge |X_n(s)|) e^{-\lambda|X_n(s)|} [1 - e^{\lambda/\theta^n} \vartheta(e^{-\lambda/\theta^n})] ds \end{aligned}$$

is a martingale. So

$$\tilde{\mathbb{E}} e^{-\lambda|X_n(t)|} = e^{-\lambda|\mu_n|} - \int_0^t \tilde{\mathbb{E}} \gamma \theta^{2n} (\theta^n \wedge |X_n(s)|) e^{-\lambda|X_n(s)|} [1 - e^{\lambda/\theta^n} \vartheta(e^{-\lambda/\theta^n})] ds$$

and

$$\begin{aligned}
& \tilde{\mathbb{E}}(e^{-\lambda|X_n(t)|} \lambda |X_n(t)|) \\
& \leq \tilde{\mathbb{E}}(1 - e^{-\lambda|X_n(t)|}) \\
& = 1 - e^{-\lambda|\mu_n|} + \int_0^t \tilde{\mathbb{E}} \gamma \theta^{2n} (\theta^n \wedge |X_n(s)|) e^{-\lambda|X_n(s)|} [1 - e^{\lambda/\theta^n} \vartheta(e^{-\lambda/\theta^n})] ds \\
& \leq 1 - e^{-\lambda|\mu_n|} + \int_0^t \tilde{\mathbb{E}} \gamma \theta^{2n} (\theta^n \wedge |X_n(s)|) e^{-\lambda|X_n(s)|} [1 - \vartheta(e^{-\lambda/\theta^n})] \frac{\lambda}{\theta^n} ds \\
& \leq 1 - e^{-\lambda|\mu_n|} + \int_0^t \gamma \theta^n [1 - \vartheta(e^{-\lambda/\theta^n})] \tilde{\mathbb{E}} \lambda |X_n(s)| e^{-\lambda|X_n(s)|} ds.
\end{aligned}$$

By Gronwall's inequality,

$$\tilde{\mathbb{E}} e^{-\lambda|X_n(t)|} \lambda |X_n(t)| \leq (1 - e^{-\lambda|\mu_n|}) \exp\{t \gamma \theta^n [1 - \vartheta(e^{-\lambda/\theta^n})]\}.$$

Let $\lambda \rightarrow 0$, we have

$$(3.24) \quad \tilde{\mathbb{E}} |X_n(t)| \leq |\mu_n|$$

and $|X_n(t)| = \lim_{\lambda \rightarrow 0} \frac{1 - M_\lambda(t)}{\lambda}$ is a martingale. Hence (3.22) follows from Doob's supermartingale inequality. Now we use Theorem 2.1.1 and Theorem 2.1.2 to prove the tightness. First let us check the compact containment condition (1.1) of Theorem 2.1.1. Since $\bar{\mathbb{R}}$ is compact, by Prokhorov ([50] Theorem 1.12), a set $\ell \subset M_F(\bar{\mathbb{R}})$ is compact if and only if $\sup_{\mu \in \ell} \mu(\bar{\mathbb{R}}) < \infty$. Recall the construction of μ_n given by (2.17), $\mu_n \Rightarrow \mu$, and μ is a finite measure, there exists a constant $\xi > 0$ such that $\langle \mu_n, 1 \rangle \leq \xi$ for any $n \in \mathbb{N}$. By (3.22), the compact containment condition (1.1) of Theorem 2.1.1 holds. Now let us check the conditions of Theorem 2.1.2. By (3.24),

the condition (1.2) of Theorem 2.1.2 is satisfied. Since

$$\begin{aligned}
 (3.25) \quad & \mathcal{B}_n F_f(X_n(t)) \\
 &= \gamma \theta^n (\theta^{2n} \wedge m_t^n) \frac{1}{m_t^n} \sum_{i=1}^{m_t^n} \frac{1}{2} (F_f(X_n(t) + \frac{-1}{\theta^n} \delta_{x_i}) - F_f(X_n(t))) \\
 &+ \gamma \theta^n (\theta^{2n} \wedge m_t^n) \frac{1}{m_t^n} \sum_{i=1}^{m_t^n} \frac{1}{2} (F_f(X_n(t) + \frac{1}{\theta^n} \delta_{x_i}) - F_f(X_n(t))) \\
 &= \frac{1}{2} \gamma (m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F_f(X_n(t))}{\delta X_n(t)(x)^2} X_n(t)(dx) + o(\frac{1}{\theta^{2n}})
 \end{aligned}$$

and

$$(3.26) \quad (\mathcal{A} + \mathcal{B})F_\mu(f) := F_\mu(G^{N(f)}f) + \frac{1}{2} \gamma (m_2 - 1) \sum_{j,k=1; j \neq k}^{N(f)} F_\mu(B^{(jk)}f),$$

where $G^{N(f)}$ is defined by (2.4) and its closure is the generator of a Feller semigroup $\{S_t^{N(f)} : t \geq 0\}$.

$$B^{(ji)}f(x_1, \dots, x_N) := \begin{cases} f(x_1, \dots, x_j, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_N) & \text{if } j < i \\ f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_j, \dots, x_N) & \text{if } i < j. \end{cases}$$

Lemma 2.2.2 implies the condition (1.4) of Theorem 2.1.2 and the tightness is proved.

(2). These results follow from Ikeda-Watanabe([34] p.9, Theorem 2.7).

(3). (3.22) implies $\sup_{0 \leq t} Z(t) < \infty$ almost surely with respect to $\mathbb{P}$. So

$$\lim_{n \rightarrow \infty} \sup_{t \geq 0} \frac{\langle Z_n(t), 1 \rangle}{\theta^n} = 0 \quad \mathbb{P} - a.s. \quad \blacksquare$$

Theorem 4.3.3 Suppose $\rho(\cdot)$ defined by (2.13) of Chapter 1 is a smooth function, then for any finite measure μ with $\mu(\bar{\mathbb{R}}) \leq \xi$, the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem has at least one solution.

Proof: First from Lemma 4.3.2, for $\forall t \in D(Z) := \{t \geq 0 : P\{Z(t) = Z(t-)\} = 1\}$ and any

$$F_f(\mu) = f(\langle \mu, \varphi \rangle), \quad \text{for } f, \varphi \in D(\tilde{\mathbb{R}}),$$

by Ethier-Kurtz ([21] p.118, Proposition 5.2), we have

$$(3.27) \quad \lim_{n \rightarrow \infty} F_f(Z_n(t)) = F_f(Z(t)) \quad \mathbb{P} - a.s.$$

Using Taylor's expansion and (3) of the Lemma 4.3.2, we have

$$\begin{aligned} (3.28) \quad & \lim_{n \rightarrow \infty} B_n F_f(Z_n(t)) \\ &= \lim_{n \rightarrow \infty} \gamma \theta^n (\theta^{2n} \wedge m_t^n) \sum_{i=1}^{m_t^n} \frac{1}{2} \left[\{f(\langle Z_n(t) + \frac{-1}{\theta^n} \delta_{x_i}, \varphi \rangle) \right. \\ & \quad \left. - f(\langle Z_n(t), \varphi \rangle)\} + \{f(\langle Z_n(t) + \frac{1}{\theta^n} \delta_{x_i}, \varphi \rangle) - f(\langle Z_n(t), \varphi \rangle)\} \right] \\ &= \frac{1}{2} \gamma (m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F_f(Z(t))}{\delta Z(t)(x)^2} Z(t)(dx) \quad \mathbb{P} - a.s. \\ &= B F_f(Z(t)). \end{aligned}$$

From Ethier-Kurtz ([21] p.131, Lemma 7.7), we know that the complement in $[0, \infty)$ of $D(Z)$ is at most countable. Let Λ be Lebesgue measure on $[0, \infty)$, then from (3.28) we have

$$(3.29) \quad \lim_{n \rightarrow \infty} (\mathcal{A}_n + \mathcal{B}_n) F_f(Z_n(u)) = (\mathcal{A} + \mathcal{B}) F_f(Z(u)) \quad \mathbb{P} \otimes \Lambda - a.s..$$

For any $0 \leq t_i \leq s < t$, $t_i, s, t \in D(X)$ and any $h_i \in C(E)$, $i = 1, \dots, k$, by Lemma 2.2.2,

$$\{[F_f(Z(t)) - F_f(Z(s)) - \int_s^t (\mathcal{A} + \mathcal{B}) F_f(Z(u)) du] \prod_{i=1}^k h_i(Z(t_i))\}$$

is integrable. So by dominated convergence theorem, we have

$$\mathbb{E}\{[F_f(Z(t)) - F_f(Z(s)) - \int_s^t (\mathcal{A} + \mathcal{B})F_f(Z(u))du] \prod_{i=1}^k h_i(Z(t_i))\} =$$

$$\lim_{n \rightarrow \infty} \mathbb{E}\{[F_f(Z_n(t)) - F_f(Z_n(s)) - \int_s^t (\mathcal{A}_n + \mathcal{B}_n)F_f(Z_n(u))du] \prod_{i=1}^k h_i(Z_n(t_i))\} = 0.$$

The right continuity of Z implies the above equality holds for all $0 \leq t_i \leq s < t$, and hence Z is a solution to the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem. ■

4.4 Uniqueness

In this section, we want to prove that the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem is well posed by duality method.

Theorem 4.4.1 *Suppose $\rho(\cdot)$ defined by (2.13) of Chapter 1 is a smooth function, then the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem is well posed, so the solution of the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem is a strong Markov process.*

For proving the above theorem, we need the following lemma which is discussed in Dawson-Kurtz([13]). We omit its proof here.

Lemma 4.4.2 *Given $A \subset B(E) \times B(E)$, suppose (A, δ_μ) -martingale problem has at least one solution. If any solution to the (A, δ_μ) -martingale problem satisfies*

(i) *for $t_0 > 0$ and $0 \leq t \leq t_0$, $f \in \tilde{D}$ ($\tilde{D}$ is defined in Section 4.2) and*

$$F_f(\nu) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \cdots, x_{N(f)}) \nu^{\otimes N(f)}(d\bar{x}),$$

there is a function $\aleph_{f,t}(\cdot)$ which only depends on f and t and is a measurable function on $E := M_F(\bar{\mathbb{R}})$, bounded on $E_\xi := \{\mu \in E \mid \langle \mu, 1 \rangle \leq \xi\}$ for $0 < \xi < \infty$, and

$$\mathbb{E}^{\mathbf{P}^\mu}(F_f(\mu_t)) = \aleph_{f,t}(\mu)$$

holds for any solution $\mathbb{P}_\mu$ to the (A, δ_μ) -MP.

(ii) for $t_0 > 0$ and $0 \leq t \leq t_0$, the moment problem for $\langle \mu(t), 1 \rangle$ with distribution determined by $\{\mathbb{P}_\mu : \mu \in E\}$ has a unique solution.

Then the (A, δ_μ) -martingale problem is well posed. The solution is measurable and satisfies strong Markov property.

Proof of the Theorem 4.4.1 :

Let $\tau_\ell := \inf\{t : \mu_t \notin E_\ell\}$. For $F_f(\nu) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_{N(f)}) \nu^{\otimes N(f)}(d\tilde{x})$, let

$$(4.30) \quad \mathcal{L}^* F_\mu(f) := F_\mu(G^{N(f)} f) + \frac{1}{2} \gamma(m_2 - 1) \sum_{j,k=1; j \neq k}^{N(f)} (F_\mu(B^{(jk)} f) - F_\mu(f)),$$

where $G^{N(f)}$ is defined by (2.4),

$$B^{(ji)} f(x_1, \dots, x_N) := \begin{cases} f(x_1, \dots, x_j, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_N) & \text{if } j < i \\ f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_j, \dots, x_N) & \text{if } i < j, \end{cases}$$

and

$$(4.31) \quad \mathcal{L}^* F_\mu(f) := \mathcal{L}^* F_\mu(f) + \frac{1}{2} \gamma(m_2 - 1) N(f)(N(f) - 1) F_\mu(f).$$

Then $\mathcal{L}^*$ has the structure of an infinitesimal generator of a $\tilde{D}$ -valued Markov process restricted to function in $Q(\tilde{D})$. The dynamics of the $\tilde{D}$ -valued Markov process $\{Y(t) : t \geq 0\}$ with generator $\mathcal{L}^*$ involves two basic mechanisms:

(m1) (Jump mechanism)

$$f \rightarrow B^{(jk)} f \quad \text{with jump rate} \quad \frac{1}{2} \gamma(m_2 - 1),$$

where $j, k = 1, \dots, N(f); j \neq k$.

(m2) (Spatial diffusion semigroup) between jumps, $Y(t)$ evolves in a continuous and deterministic way; namely

$$f \rightarrow S_t^{N(f)} f.$$

By direct calculation, we can get

$$\mathcal{L}F_f(\mu) = \mathcal{L}^*F_\mu(f).$$

Let

$$\aleph_{f,t}(\mu) := \mathbb{E}_f[F(\mu, Y(t)) \exp(\frac{1}{2}\gamma(m_2 - 1) \int_0^t N(Y(u))(N(Y(u)) - 1)du)].$$

So by Feynman-Kac formula (see ([21] p.189, Proposition 4.7 and p.184, Theorem 4.2), we have

$$(4.32) \quad \mathbb{E}_\mu[F_f(\mu_{t \wedge \tau_\ell})] = \aleph_{f,t}(\mu).$$

Since $\mathbb{E}_\mu(\langle \mu_t, 1 \rangle) < \infty$, $\lim_{\xi \rightarrow \infty} \mathbb{P}_\mu(\tau_\xi \leq t) = 0$. Note that G^N is conservative and $\mathcal{L}F_{1_{(N)}}(\mu) \geq 0$ for $F_{1_{(N)}}(\mu) \in \Pi_1^D$ and $1_{(N)}$ is the function of N variables which assumes the constant value 1. By Lemma 2.2.2, $(\langle \mu_t, 1 \rangle)^N$ is a submartingale for each $N \geq 1$. Therefore,

$$\begin{aligned} & \lim_{\xi \rightarrow \infty} \mathbb{E}_\mu(\langle \mu_{t \wedge \tau_\xi}, 1 \rangle^N 1_{\{\tau_\xi \leq t\}}) \\ & \leq \lim_{\xi \rightarrow \infty} \mathbb{E}_\mu(\langle \mu_t, 1 \rangle^N 1_{\{\tau_\xi \leq t\}}) = 0. \end{aligned}$$

Then let $\xi \rightarrow \infty$ in (4.32), we obtain

$$(4.33) \quad \mathbb{E}_\mu[F_f(\mu_t)] = \aleph_{f,t}(\mu).$$

Since $\aleph_{f,t}(\cdot)$ is a measurable function which is bounded on the sets E_ℓ , it remains to verify that the moment problem for $\langle \mu_t, 1 \rangle$ has a unique solution. Let $Q(t)$ be the continuous state branching process constructed in Section 2 of Chapter 2 and

$$\psi_t(\lambda) = \lambda[1 + \frac{1}{2}(m_2 - 1)\gamma t \lambda]^{-1}.$$

From the proof of Lemma 2.2.2 and $Q(t) = \langle \mu_t, 1 \rangle$, it is easy to conclude that the moment power series

$$\sum_k \alpha_k \frac{\lambda^k}{k!} = \exp(x\psi_t(\lambda)).$$

It follows from Theorem 2.4.3 that the moment problem for $\langle \mu_t, 1 \rangle$ has uniqueness. This completes the proof of uniqueness of $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -martingale problem. ■

Chapter 5

Superprocesses in a Brownian Medium (II)

5.1 Introduction

In the preceding chapter, we have constructed and characterized the limiting measure-valued processes of a sequence of interacting branching particle systems with smooth function $g(\cdot)$. In this chapter, we will consider the same model and questions as before; however, $g(\cdot)$ becomes singular function. By a singular function, we mean that there exists at least one point at which it assumes infinite value.

Here we assume that $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0 which means that $\rho(\cdot)$ is Lipschitz outside each interval $(-c, c)$, $c > 0$. From Chapter 4, we know $\rho(\cdot)$ is nonnegative definite.

In this chapter, our operators $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{B}$ are defined same as (2.16), (2.18) and (2.17) in Chapter 1 respectively except the assumption that $\rho(\cdot)$ is a smooth function is replaced by assuming that $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0. If $F(\mu) \in \Pi_1^D(E)$, then

$$(1.1) \quad \mathcal{A}F(\mu) = F_\mu(G^N f),$$

where

$$(1.2) \quad G^N f(x_1, \dots, x_N) = \frac{1}{2} \sum_{i,j=1}^N \rho_\epsilon(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_N),$$

and

$$(1.3) \quad \rho_\epsilon(x) := \begin{cases} \rho_\epsilon = \rho(0) + \epsilon^2 & \text{if } x = 0 \\ \rho(x) & \text{if } x \neq 0. \end{cases}$$

From Chapter 4 we see that the spatial diffusion semigroup is one of the two structural elements of the dual processes. Since in the present case in which $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0, we cannot show that the closure of $(G^N, C_0^\infty(\mathbb{R}^N))$ is the generator of a Feller semigroup. In consequence, we cannot apply the duality method used in Chapter 4 to directly construct the dual process such that the dual identity (cf. (4.32) of Chapter 4) holds. A limiting duality method (or a generalized duality method) is developed in this chapter for proving the uniqueness of solution to $(\mathcal{L}, \delta_{\mu_0})$ -martingale problem. The main idea of the limiting duality method is to establish the dual identity by making use of the limiting argument and the existence of generalized dual processes.

5.2 Existence

First let us introduce the following result.

Definition 5.2.1 *Let*

$$(2.4) \quad \alpha(x) := \begin{cases} \frac{1}{c} \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where c is a constant such that $\int_{\mathbb{R}^n} \alpha(x) dx = 1$ and

$$(2.5) \quad \alpha_\epsilon(x) := \epsilon^{-n} \alpha\left(\frac{x}{\epsilon}\right).$$

For any locally integrable function u on $\mathbb{R}^n$, define

$$J_\epsilon u(x) := \int_{\mathbb{R}^n} u(x-y) \alpha_\epsilon(y) dy.$$

$J_\epsilon u$ is called mollifier of u .

Lemma 5.2.1 Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0. Define $g_k(x) := J_{1/k} \rho(x)$ on $\mathbb{R}$ and $\rho_k(z) := \int_{\mathbb{R}} g_k(z-y) g_k(y) dy$, then $\{\rho_k(x)\}$ are smooth functions and $\{g_k(x)\}$ are symmetric functions. Also

$$\rho_k \rightarrow \rho \quad \text{uniformly as} \quad k \rightarrow \infty.$$

Proof: The fact that $\rho_k \in C^\infty$ comes from the results of Jose Barros-Neto ([3] p.6). Since

$$(2.6) \quad g_k(-x) = \int_{|y|<1} g(-x-y) \alpha_{1/k}(y) dy = g_k(x),$$

g_k is symmetric. The uniform convergence follows from the following inequalities:

$$\begin{aligned} (2.7) \quad |\rho_k(x) - \rho(x)| &= \left| \int_{\mathbb{R}} g_k(x-y) g_k(y) dy - \int_{\mathbb{R}} g(x-y) g(y) dy \right| \\ &\leq \left| \int_{\mathbb{R}} g_k(x-y) g_k(y) dy - \int_{\mathbb{R}} g_k(x-y) g(y) dy \right| \\ &\quad + \left| \int_{\mathbb{R}} g_k(x-y) g(y) dy - \int_{\mathbb{R}} g(x-y) g(y) dy \right| \\ &\leq \left| \int_{\mathbb{R}} g_k(x-y)^2 dy \right| \left| \int_{\mathbb{R}} |g_k(y) - g(y)|^2 dy \right| \\ &\quad + \int_{\mathbb{R}} g^2(y) dy \int_{\mathbb{R}} |g_k(x-y) - g(x-y)|^2 dy. \end{aligned}$$

■

In (2.16) of Chapter 1, we substitute ρ_n for ρ , the resulting operator is denoted by $\mathcal{L}_n F(\mu)$.

Theorem 5.2.2 Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0, then for any finite measure μ_0 with compact support, the $(\mathcal{L}, \delta_{\mu_0})$ -MP has at least one solution.

Proof: By Theorem 4.3.3 and Theorem 4.4.1, the $(\mathcal{L}_n, \delta_{\mu_0})$ -MP has a unique solution. Since the present branching mechanism is the same as in Chapter 4, a similar argument to the proofs of Lemma 4.3.2 and Theorem 4.3.3 of Chapter 4 can reach the conclusion. ■

5.3 Uniqueness

In this section, we want to prove that the $(\mathcal{L}, \delta_{\mu_0})$ -MP has at most one solution by a generalized duality method.

Let

$$(3.8) \quad D(\bar{\mathbb{R}}^N) := C^\infty(\bar{\mathbb{R}}^N),$$

where $\bar{\mathbb{R}}^N$ is the one point compactification of $\mathbb{R}^N$ and $C^\infty(\bar{\mathbb{R}}^N)$ is the space of all infinitely differentiable functions on $\bar{\mathbb{R}}^N$. Let $C(\bar{\mathbb{R}}^N)$ be the space of all continuous functions on $\bar{\mathbb{R}}^N$. It is obvious that $D(\bar{\mathbb{R}}^N)$ is dense in $C(\bar{\mathbb{R}}^N)$.

Let $\tilde{C} := \cup_{N=0}^\infty C(\bar{\mathbb{R}}^N)$ (disjoint union), where $C(\bar{\mathbb{R}}^0) := \mathbb{R}^1$. We define a metric d on $\tilde{C}$ by

$$d(f, g) := \begin{cases} \|f - g\|_n & \text{if } N(f) = N(g) = n \\ |m - n| & \text{if } N(f) = m, N(g) = n, \text{ and } m \neq n, \end{cases}$$

where $N(f) := n$ if $f \in C(\bar{\mathbb{R}}^n)$ and $f \notin C(\bar{\mathbb{R}}^{n-1})$, and $\|f - g\|_n$ is the uniform norm on $C(\bar{\mathbb{R}}^n)$. Then $(\tilde{C}, d)$ is a Polish space. Let $\tilde{D} := \cup_{N=0}^\infty D(\bar{\mathbb{R}}^N)$ (disjoint union). Then $\tilde{D}$ is a subspace of $\tilde{C}$. We will see $\tilde{C}$ is the state space of the dual processes.

We define $F : E \times \hat{D} \rightarrow \mathbb{R}$ by $F(\mu, f) = \langle f, \mu^m \rangle$ for $f \in D(\bar{\mathbb{R}}^m)$, then

$$\begin{aligned}
 (3.9) \quad \mathcal{L}F(\mu, f) &= \langle G^m f, \mu^m \rangle + \frac{1}{2}\gamma(m_2 - 1) \sum_{j,k=1, j \neq k}^m \langle \Phi_{j,k} f, \mu^{m-1} \rangle \\
 &= \langle G^m f, \mu^m \rangle + \frac{1}{2}\gamma(m_2 - 1) \sum_{j,k=1, j \neq k}^m \{ \langle \Phi_{j,k} f, \mu^{m-1} \rangle \\
 &\quad - \langle f, \mu^m \rangle \} + \frac{1}{2}\gamma(m_2 - 1)m(m-1) \langle f, \mu^m \rangle,
 \end{aligned}$$

where G^m is defined by (1.2),

$$(3.10) \quad \Phi_{j,k} f(x_1, \dots, x_m) = \begin{cases} f(x_1, \dots, x_{k-1}, x_j, x_{k+1}, \dots, x_m) & \text{if } j < k \\ f(x_1, \dots, x_{j-1}, x_k, x_{j+1}, \dots, x_m) & \text{if } j > k. \end{cases}$$

In the following, we want to use Lemma 4.4.2 of Chapter 4 to prove uniqueness. Since there is no difference in branching mechanism between the present model and the model discussed in Chapter 4, if we can prove condition (i) of Lemma 4.4.2 of Chapter 4 holds, then the condition (ii) of the Lemma 4.4.2 of Chapter 4 will hold automatically. In proving the condition (i) of Lemma 4.4.2 of Chapter 4, the key point is to establish the duality identity and then the question changes to whether or not we can construct the dual processes of the measure-valued processes, the solutions of $(\mathcal{L}, \delta_{\mu_0})$ -MP. In order to construct the dual processes, we need the following preparations.

Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0, then $\rho(\cdot)$ satisfies all the conditions of Theorem 2.3.1. By Theorem 2.3.1, suppose $\{\mathbb{P}_z\}$ is the C-diffusion governed by G^n and suppose $\xi(t) = w(t)$ is the canonical process in the space $C([0, \infty), \mathbb{R}^n)$. Define

$$(3.11) \quad T_n(t)f(z) = \mathbb{E}^{\mathbb{P}_z} f(\xi(t)) \quad \text{for } f \in C(\bar{\mathbb{R}}^n),$$

we have the following lemma.

Lemma 5.3.1 Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0, then $\{T_n(t)\}$ defined by (3.11) is a strongly continuous non-negative contraction semigroup on $C(\bar{\mathbb{R}}^n)$. Let A^n be its infinitesimal generator, then $A^n|_{C^2(\mathbb{R}^n)} = G^n$.

Proof: (i) Since for $f \in C^2(\bar{\mathbb{R}}^n)$,

$$\begin{aligned}\|T_n(t)f - f\| &= \left\| \int_0^t \mathbb{E}^{\mathbf{P}} G^n f(\xi(s)) ds \right\| \\ &= \left\| \int_0^t \int_{\mathbb{R}^n} G^n f(y) p(s, \cdot, dy) ds \right\| \\ &\leq \|G^n f\| t,\end{aligned}$$

where $p(s, x, dy)$ is the transition probability corresponding to $\mathbb{P}_x$, and $C^2(\bar{\mathbb{R}}^n)$ is dense in $C(\bar{\mathbb{R}}^n)$, $T_n(t)$ is strongly continuous on $C(\bar{\mathbb{R}}^n)$.

(ii) The semigroup property follows from

$$\begin{aligned}T_n(t)(T_n(s)f(z)) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(y) p(s, x, dy) \right] p(t, z, dx) \\ &= \int_{\mathbb{R}^n} f(y) p(s+t, z, dy) \\ &= T_n(s+t)f(z).\end{aligned}$$

(iii) Non-negativeness and contraction are obvious.

(iv) For any $f \in C^2(\bar{\mathbb{R}}^n)$, we have $G^n f \in C(\bar{\mathbb{R}}^n)$. Since

$$\|T_n(s)G^n f - G^n f\| \rightarrow 0 \quad \text{as } s \rightarrow 0,$$

we have

$$\left\| \frac{1}{t}(T_n(t)f - f) \right\| = \frac{1}{t} \left\| \int_0^t \mathbb{E}^{\mathbf{P}} G^n f(\xi(s)) ds \right\|$$

(solution of the (G^n, δ) -MP) and

$$\begin{aligned} & \frac{1}{t} \left\| \int_0^t \mathbb{E}^{\mathbb{P}} G^n f(\xi(s)) ds - G^n f \right\| \\ & \leq \frac{1}{t} \int_0^t \left\| \mathbb{E}^{\mathbb{P}} G^n f(\xi(s)) - G^n f \right\| ds \\ & = \frac{1}{t} \int_0^t \|T_n(s) G^n f - G^n f\| ds \rightarrow 0, \end{aligned}$$

we obtain $A^n|_{C^2(\mathbb{R}^n)} = G^n$. ■

Remark $T_n(t) : C(\bar{\mathbb{R}}^n) \rightarrow C(\bar{\mathbb{R}}^n)$ because $\mathbb{P}_x$ is a Feller process.

Lemma 5.3.2 Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0. If $\{\mu_t\}$ is a solution to the $(\mathcal{L}, \delta_{\mu_0})$ -martingale problem on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, define filtration

$$\mathcal{F}_t^\mu := \sigma(\mu_s, s \leq t) \vee \sigma\left(\int_0^s h(\mu_u) du, s \leq t, h \in B(E)\right),$$

where $B(E)$ is the space of all bounded measurable functions on E . Then

$$\langle f, \mu_t^n \rangle - \int_0^t [\langle A^n f, \mu_u^n \rangle + \sigma \sum_{i,j=1, i \neq j}^n \langle \Phi_{ij} f, \mu_u^{n-1} \rangle] du$$

$$\cdot \quad \text{for any } f \in C(\bar{\mathbb{R}}^n) \cap \mathcal{D}(A^n)$$

is a $\mathcal{F}_t^\mu$ -martingale, where $\sigma := \frac{1}{2}\gamma(m_2 - 1)$.

Proof: Since $A^n|_{C^2(\mathbb{R}^n)} = G^n$ and $\{\mu_t\}$ is a solution to the $(\mathcal{L}, \delta_{\mu_0})$ -MP, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}_2} \left\{ \langle f, \mu_t^n \rangle - \int_0^t [\langle A^n f, \mu_u^n \rangle + \sigma \sum_{i,j=1, i \neq j}^n \langle \Phi_{ij} f, \mu_u^{n-1} \rangle] du \right. \\ (3.12) \quad & \left. - \langle f, \mu_s^n \rangle + \int_0^s [\langle A^n f, \mu_u^n \rangle + \sigma \sum_{i,j=1, i \neq j}^n \langle \Phi_{ij} f, \mu_u^{n-1} \rangle] du \middle| \mathcal{F}_s^\mu \right\} = 0 \end{aligned}$$

for $t \geq s \geq 0$ and any $f \in D(\bar{\mathbb{R}}^n)$.

For any $g \in C(\bar{\mathbb{R}}^n) \cap \mathcal{D}(A^n)$, there exist a sequence of smooth functions $g_n \in D(\bar{\mathbb{R}}^n)$ such that g_n uniformly converges to g on $\bar{\mathbb{R}}^n$. Since A^n is the generator of the semigroup $\{T_n(t)\}$, by Hille-Yosida Theorem, A^n is a closed linear operator (refer to Ethier-Kurtz [21] p.8 for definition). Therefore $A^n g_k \rightarrow A^n g$ in $C(\bar{\mathbb{R}}^n)$ as $k \rightarrow \infty$. In (3.12), replace f by g_k and let $k \rightarrow \infty$, we reach the conclusion. ■

Now we turn to construct the dual process. Let $M = \{M(t) : t \geq 0\}$ be a jump Markov process on a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ with state space $\mathbb{N}$ and transition intensities $q_{m,m-1} = \sigma m(m-1)$, $q_{i,j} = 0$ for all other $|i-j| \neq 1$. Let $\{\tau_k\}$ be the sequence of jump times of M (take $\tau_0 = 0$) and $\{\Gamma_k\}$ be a sequence of random operators which are conditionally independent given M and satisfy

$$(3.13) \quad \mathbb{P}_1(\Gamma_k = \Phi_{i,j} | M) = \frac{1}{m(m-1)} \mathbf{1}_{\{M(\tau_k-) = m, M(\tau_k) = m-1\}}$$

for $1 \leq i, j \leq m$ and $j \neq i$, then the dual process is given by

$$(3.14) \quad Y(t) = T_{M(\tau_k)}(t - \tau_k) \Gamma_k T_{M(\tau_{k-1})}(\tau_k - \tau_{k-1}) \Gamma_{k-1} \cdots \Gamma_1 T_{M(0)}(\tau_1) Y(0)$$

$$\text{for } \tau_k \leq t < \tau_{k+1} \quad k = 0, 1, 2, \dots,$$

where $Y(0) \in D(\bar{\mathbb{R}}^{M(0)})$.

For any $f \in D(\bar{\mathbb{R}}^m)$, take $Y(0) = f$, $M(0) = m$, let $\{\mu_t\}$ be a solution to $(\mathcal{L}, \delta_{\mu_0})$ -MP defined on $(\Omega_2, \mathcal{F}_2, \mathcal{F}_t^\mu, \mathbb{P}_2)$ and let $\mathbb{E} = \mathbb{E}^{\mathbb{P}_1 \times \mathbb{P}_2}$ be the expectation on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$. If the following duality identity

$$(3.15) \quad \mathbb{E}[\langle f, \mu_t^m \rangle] = \mathbb{E}[\langle Y(t), \mu_0^{M(t)} \rangle \exp\{\sigma \int_0^t M(u)(M(u) - 1) du\}]$$

holds for any solution $\{\mu_t\}$ to $(\mathcal{L}, \delta_{\mu_0})$ -MP, then $(\mathcal{L}, \delta_{\mu_0})$ -MP has uniqueness by Lemma 4.4.2 of Chapter 4.

Theorem 5.3.3 Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0, then for any finite measure μ_0 with compact support, the $(\mathcal{L}, \delta_{\mu_0})$ -MP has a unique solution.

Proof: We only need to establish the identity (3.15). Writing the difference of the two sides of (3.15) as a telescoping sum, we only need to show that each term of the telescoping sum is $o(h)$ and this is equivalent to showing

$$(3.16) \quad \begin{aligned} & \mathbb{E}\{[\langle Y(s+h), \mu_{t-s-h}^{M(s+h)} \rangle \exp\{\sigma \int_0^{s+h} M(u)(M(u)-1)du\}] \\ & - [\langle Y(s), \mu_{t-s}^{M(s)} \rangle \exp\{\sigma \int_0^s M(u)(M(u)-1)du\}]\} = o(h). \end{aligned}$$

Let

$$\mathcal{F}_t^Y := \sigma(Y(s), s \leq t) \vee \sigma\left(\int_0^s h(Y(u))du, s \leq t, h \in B(\tilde{D})\right),$$

where $B(\tilde{D})$ is the space of all bounded measurable functions on $\tilde{D}$. Using the definition of Y and some basic results of Q -process $\mathbf{M}$, we can decompose the first term in (3.16) in terms of the values of $M(s+h) - M(s)$ to obtain

$$\begin{aligned} & \mathbb{E}\left\{\mathbb{E}[\langle Y(s+h), \mu_{t-s-h}^{M(s+h)} \rangle \exp\{\sigma \int_0^{s+h} M(u)(M(u)-1)du\} | \mathcal{F}_s^Y]\right\} \\ &= \mathbb{E}\left[\langle T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} \rangle \exp\{\sigma \int_0^s M(u)(M(u)-1)du\}\right] \\ &+ \mathbb{E}\left\{\int_0^h \sum_{i \neq j} \langle T_{M(s)-1}(h-r)\Phi_{ij} T_{M(s)}(r)Y(s), \mu_{t-s-h}^{M(s)-1} \rangle \cdot \right. \\ & \quad \exp\{\sigma \int_0^s M(u)(M(u)-1)du + \sigma M(s)(M(s)-1)r \\ & \quad \left. + \sigma(M(s)-1)(M(s)-2)(h-r) - \sigma M(s)(M(s)-1)r\} \sigma dr\right\} + o(h) \\ &= \mathbb{E}[\langle T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} \rangle \exp\{\sigma \int_0^s M(u)(M(u)-1)du\}] \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left[\int_0^h \sigma \sum_{i \neq j}^{M(s)} \langle T_{M(s)-1}(h-r) \Phi_{ij}(r) T_{M(s)}(r) Y(s), \mu_{i-s-h}^{M(s)-1} \rangle \right. \\
(3.17) \quad & \left. \exp \left\{ \sigma \int_0^s M(u)(M(u)-1) du + \sigma(M(s)-1)(M(s)-2)(h-r) \right\} dr \right] + o(h).
\end{aligned}$$

Using Lemma 5.3.2, we have

Lemma 5.3.4 For above $\{T_m(u)\}$, and any positive t, s provided $t \geq s$,

$$\begin{aligned}
(3.18) \quad & \langle T_{M(s)}(t-s-v) Y(s), \mu_v^{M(s)} \rangle \\
& - \int_0^v \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-l) Y(s), \mu_l^{M(s)-1} \rangle dl
\end{aligned}$$

is a $\{\mathcal{F}_v^\mu\}$ -martingale for every $\omega_1 \in \Omega_1$ and $0 \leq v \leq t-s$ on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, where $\{\mathcal{F}_v^\mu\}$ is defined in Lemma 5.3.2.

Proof of Lemma 5.3.4: Our conclusion is equivalent to showing that for $t-s \geq u \geq v \geq 0$,

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}_2} \left[\langle T_{M(s)}(t-s-u) Y(s), \mu_u^{M(s)} \rangle \right. \\
& - \int_0^u \sigma \sum_{i,j=1, i \neq j}^{M(s)} [\langle \Phi_{ij} T_{M(s)}(t-s-l) Y(s), \mu_l^{M(s)-1} \rangle] dl \\
& - \langle T_{M(s)}(t-s-v) Y(s), \mu_v^{M(s)} \rangle \\
& \left. + \int_0^v \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-l) Y(s), \mu_l^{M(s)-1} \rangle dl \middle| \mathcal{F}_v^\mu \right] = 0.
\end{aligned}$$

Above equality holds if for any $h > 0$ satisfying $t-s \geq v+h$, we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}_2} \left[\langle T_{M(s)}(t-s-v-h) Y(s), \mu_{v+h}^{M(s)} \rangle \right. \\
& - \int_0^{v+h} \sigma \sum_{i,j=1, i \neq j}^{M(s)} [\langle \Phi_{ij} T_{M(s)}(t-s-r) Y(s), \mu_r^{M(s)-1} \rangle] dr
\end{aligned}$$

$$\begin{aligned}
& - \langle T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} \rangle \\
& + \int_0^v \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-r)Y(s), \mu_r^{M(s)-1} \rangle dr | \hat{\mathcal{F}}_v^\mu \Big] = o(h)
\end{aligned}$$

or

$$\begin{aligned}
(3.19) \quad & \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \mathbb{E}^{\mathbb{P}^2} \left[\langle T_{M(s)}(t-s-v-h)Y(s), \mu_{v+h}^{M(s)} \rangle \right. \right. \\
& \quad - \langle T_{M(s)}(t-s-v)Y(s), \mu_{v+h}^{M(s)} \rangle \\
& \quad + \langle T_{M(s)}(t-s-v)Y(s), \mu_{v+h}^{M(s)} \rangle - \\
& \quad \left. \left. \langle T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} \rangle - \right. \right. \\
& \quad \left. \left. \int_v^{v+h} \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-r)Y(s), \mu_r^{M(s)-1} \rangle dr | \mathcal{F}_v^\mu \right] \right\} = 0.
\end{aligned}$$

By bounded convergence theorem and Lemma 5.3.2, (3.19) is equivalent to

$$\begin{aligned}
(3.20) \quad & \mathbb{E}^{\mathbb{P}^2} \left\{ - \langle A^{M(s)} T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} \rangle \right. \\
& + \langle A^{M(s)} T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} \rangle \\
& + \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)-1} \rangle \\
& \left. - \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)-1} \rangle \right\} = 0.
\end{aligned}$$

(3.20) is obvious, So Lemma 5.3.4 is proved.

From Lemma 5.3.4, the second term on left hand side of (3.16) can be decomposed as

$$\begin{aligned}
(3.21) \quad & \mathbb{E} \left[\langle Y(s), \mu_{t-s}^{M(s)} \rangle \exp \left\{ \sigma \int_0^s M(u)(M(u)-1)du \right\} \right] \\
& = \mathbb{E} \left\{ \left[\langle T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} \rangle \right. \right.
\end{aligned}$$

$$+ \int_0^h \sigma \sum_{i,j=1, i \neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-r}^{M(s)-1} > dr \Big] \\ \cdot \exp \left[\sigma \int_0^s M(u) (M(u) - 1) du \right] \Big\}.$$

Indeed, by Lemma 5.3.4, we have

$$(3.22) \quad \mathbb{E} \left\{ \left[< T_{M(s)}(t-s-v-h) Y(s), \mu_{v+h}^{M(s)} > \right. \right. \\ \left. \left. - < T_{M(s)}(t-s-v) Y(s), \mu_v^{M(s)} > \right. \right. \\ \left. \left. - \int_v^{v+h} \sigma \sum_{i,j=1, i \neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(t-s-r) Y(s), \mu_r^{M(s)-1} > dr \right] \middle| \mathcal{F}_v^\mu \right\} = 0.$$

Let $h = t - s - v$, then above equality is equivalent to

$$(3.23) \quad \mathbb{E} \left\{ \left[< T_{M(s)}(0) Y(s), \mu_{t-s}^{M(s)} > - < T_{M(s)}(h) Y(s), \mu_{t-s-h}^{M(s)} > \right. \right. \\ \left. \left. - \int_{t-s-h}^{t-s} \sigma \sum_{i,j=1, i \neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(t-s-r) Y(s), \mu_r^{M(s)-1} > dr \right] \middle| \mathcal{F}_{t-s-h}^\mu \right\} = 0$$

or

$$(3.24) \quad \mathbb{E} \left\{ \left[< Y(s), \mu_{t-s}^{M(s)} > - < T_{M(s)}(h) Y(s), \mu_{t-s-h}^{M(s)} > \right. \right. \\ \left. \left. - \int_0^h \sigma \sum_{i,j=1, i \neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(u) Y(s), \mu_{t-s-u}^{M(s)-1} > du \right] \middle| \mathcal{F}_{t-s-h}^\mu \right\} = 0,$$

which implies (3.21).

Comparing (3.16), (3.17) and (3.21), we see that (3.16) is equivalent to

$$(3.25) \quad \frac{1}{h} \mathbb{E} \left\{ \int_0^h \left[\sigma \sum_{i,j=1, i \neq j}^{M(s)} < T_{M(s)-1}(h-r) \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-h}^{M(s)-1} > \right. \right.$$

$$\exp\left\{\sigma \int_0^s M(u)(M(u)-1)du + \sigma(M(s)-1)(M(s)-2)(h-r)\right\} dr$$

$$- \int_0^h \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-r}^{M(s)-1} \rangle$$

$$\cdot \exp\left[\sigma \int_0^s M(u)(M(u)-1)du\right] dr \Big\} \rightarrow 0$$

or

$$(3.26) \quad \frac{1}{h} \mathbb{E} \left\{ \int_0^h \left[\sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle T_{M(s)-1}(h-r) \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-h}^{M(s)-1} \rangle \right. \right.$$

$$\left. \left. - \sigma \sum_{i,j=1, i \neq j}^{M(s)} \langle \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-r}^{M(s)-1} \rangle \right] \right.$$

$$\left. \cdot \exp\left[\sigma \int_0^s M(u)(M(u)-1)du\right] dr \right\} \rightarrow 0.$$

From Lemma 5.3.4, similar to the proof of (3.21), we can get

$$\mathbb{E} \left\{ \left[\langle T_{M(s)-1}(v) \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-v-r}^{M(s)-1} \rangle \right. \right.$$

$$\left. \left. - \langle \Phi_{ij} T_{M(s)}(r) Y(s), \mu_{t-s-r}^{M(s)-1} \rangle \right] | \mathcal{F}_{t-s-v-r}^\mu \right\}$$

$$(3.27) \quad = \mathbb{E} \left\{ - \left[\int_0^v \sigma \sum_{k,l, k \neq l}^{M(s)-1} \langle \Phi_{kl} T_{M(s)-1}(u) \Phi_{ij} T_{M(s)}(r) Y(s), \right. \right.$$

$$\left. \left. \mu_{t-s-r-u}^{M(s)-2} \rangle du \right] | \mathcal{F}_{t-s-v-r}^\mu \right\}.$$

From (3.27), we see that (3.26) holds. So does (3.16) and we reach the conclusion of Theorem 5.3.3. ■

5.4 Continuity

In this Section, we want to prove that if $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0, then the sample paths

of $\{\mu_t; t \geq 0\}$, the solution to $(\mathcal{L}, \delta_{\mu_0})$ -MP, are continuous. Therefore, $\{\mu_t; t \geq 0\}$ is a diffusion process. Let Δ and ∇ denote Laplacian and gradient operators on $\mathbb{R}$, respectively. Let μ_0 be a finite measure on $\mathbb{R}$ with compact support. $\mathbb{P}_{\mu_0}$ denotes the solution to the $(\mathcal{L}, \delta_{\mu_0})$ -MP with canonical process μ_t on the space $D([0, \infty), M_F(\bar{\mathbb{R}}))$. First, let us cite the following Lemma. Let $D(Q)$ denote the class of functions on $M_F(\bar{\mathbb{R}})$ of the form

$$F(\mu) = \psi(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle),$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, $f_i \in C^2(\mathbb{R})$, and $n \geq 1$. Suppose Q is an operator on space $C(M_F(\bar{\mathbb{R}}))$ and for given $\mu \in M_F(\bar{\mathbb{R}})$

$$M_t^F := F(X_t) - F(X_0) - \int_0^t QF(X_s) ds \quad \text{for all } F \in D(Q), \text{ and } t \geq 0$$

is a locally bounded $\mathbb{P}_\mu$ -martingale and is càdlàg a.s. $\mathbb{P}_\mu$, where X_t is the canonical process on space $D([0, \infty), M_F(\bar{\mathbb{R}}))$. Define

$$\mathcal{R}_Q(F, G) := Q(FG) - FQ(G) - GQ(F) \quad \text{for any } F, G \in D(Q).$$

Lemma 5.4.1 *Suppose that $\mathcal{R}_Q$ has the derivation property:*

$$(4.28) \quad \mathcal{R}_Q(FG, H) := F\mathcal{R}_Q(G, H) + G\mathcal{R}_Q(F, H) \quad \text{for any } F, G, H \in D(Q).$$

Then the process M^F and $F(X_\cdot)$ have continuous paths a.s. $\mathbb{P}_\mu$, for all $F \in D(Q)$.

Proof: see Bakry-Emery ([2] Prop.2) ■

Theorem 5.4.2 *Suppose $\rho(\cdot)$ defined by (2.13) in Chapter 1 is continuous and locally Lipschitz continuous away from 0. Let $\mathcal{L}$ be the generator defined by (2.16) in Chapter 1 and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_{\mu_0}, \mu_t)$ be the branching measure-valued Markov process which is unique solution to $(\mathcal{L}, \delta_{\mu_0})$ -MP. Then $\{\mu_t\}$ is a measure-valued diffusion process.*

Proof: Since

$$\begin{aligned} \mathcal{R}_{\mathcal{L}}(F, G)(\mu) &= \gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta F(\mu)}{\delta \mu(x)} \frac{\delta G(\mu)}{\delta \mu(x)} \mu(dx) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x - y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \frac{\delta G(\mu)}{\delta \mu(y)} \mu(dx) \mu(dy), \end{aligned}$$

$\mathcal{L}$ has the derivation property. Therefore, the continuity of $\{\mu_t; t \geq 0\}$ follows from the Lemma 5.4.1 and the fact that $C^\infty(\bar{\mathbb{R}})$ is convergence determining. ■

Since μ_t is a measure-valued diffusion process, we have following corollary.

Corollary 5.4.3 *The following two conditions are equivalent:*

(1) *For any $\phi \in S(\bar{\mathbb{R}})$*

$$(4.29) \quad M_t^\phi := \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle - \frac{1}{2} \int_0^t \rho_s \langle \mu_s, \Delta \phi \rangle ds$$

is a $\mathbb{P}_{\mu_0}$ -martingale with increasing process

$$(4.30) \quad \gamma(m_2 - 1) \int_0^t \langle \mu_s, \phi^2 \rangle ds + \int_0^t \int_{\mathbb{R}} \langle g(y - x) \nabla \phi, \mu_s \rangle^2 dy ds.$$

(2) *For any $\phi \in S(\bar{\mathbb{R}})$ and $G \in C^2(\bar{\mathbb{R}})$*

$$(4.31) \quad G(\langle \mu_t, \phi \rangle) - G(\langle \mu_0, \phi \rangle) - \int_0^t \mathcal{L}G(\langle \mu_s, \phi \rangle) ds$$

is a $\mathbb{P}_{\mu_0}$ -martingale.

Proof: (1) $\Rightarrow$ (2). This follows from Ito's formula.

(2) $\Rightarrow$ (1). Let $G(x) = x$, by Ito's formula, we have

$$M_t^\phi = \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle - \frac{1}{2} \int_0^t \rho_s \langle \mu_s, \Delta \phi \rangle ds$$

is a $\mathbb{P}_{\mu_0}$ -martingale. Then

$$\langle \mu_t, \phi \rangle = M_t^\phi + \langle \mu_0, \phi \rangle + \frac{1}{2} \int_0^t \rho_s \langle \mu_s, \Delta \phi \rangle ds$$

is a semimartingale. By Ito's formula, we have

$$\begin{aligned}
 (4.32) \quad G(<\mu_t, \phi>) - G(<\mu_0, \phi>) &= \int_0^t G'(<\mu_s, \phi>) dM_s^\phi \\
 &+ \frac{\rho_t}{2} \int_0^t G'(<\mu_s, \phi>) <\mu_s, \Delta\phi> ds \\
 &+ \frac{1}{2} \int_0^t G''(<\mu_s, \phi>) d<M^\phi>_s,
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 (4.33) \quad \int_0^t G'(<\mu_s, \phi>) dM_s^\phi &= G(<\mu_t, \phi>) - G(<\mu_0, \phi>) \\
 &- \frac{\rho_t}{2} \int_0^t G'(<\mu_s, \phi>) <\mu_s, \Delta\phi> ds \\
 &- \frac{1}{2} \int_0^t G''(<\mu_s, \phi>) d<M^\phi>_s
 \end{aligned}$$

is a $\mathbb{P}_{\mu_0}$ -martingale.

By uniqueness, comparing (4.31) with (4.33), we get

$$<M^\phi>_t = \gamma(m_2 - 1) \int_0^t <\mu_s, \phi^2> ds + \int_0^t \int_{\mathbb{R}} <g(y-x) \nabla \phi, \mu_s>^2 dy ds.$$

So the proof is complete. ■

Chapter 6

Spatial Structures and State Classification

6.1 Atomic Measure States

6.1.1 Introduction

In Chapters 4 and 5, a class of measure-valued branching diffusions in a Brownian medium have been constructed and characterized. Now we begin to investigate their spatial structure properties. Recall the definitions of generators.

$$(1.1) \quad \mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu),$$

$$(1.2) \quad \mathcal{B}F(\mu) := \frac{1}{2}\gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx),$$

and

$$(1.3) \quad \begin{aligned} \mathcal{A}F(\mu) := & \frac{1}{2} \int_{\mathbb{R}} \rho_{\epsilon} \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy), \end{aligned}$$

where $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(E)$ and

$$\rho(z) := \int_{\mathbb{R}} g(z-y)g(y)dy, \quad \rho_{\epsilon} := \rho(0) + \epsilon^2, \quad \epsilon \in \mathbb{R}.$$

For any finite measure ξ with compact support on $\bar{\mathbb{R}} = \mathbb{R} \cup \Delta$, the one point compactification of $\mathbb{R}$, suppose that $\{Z_t : t \geq 0\}$ is the unique solution to $(\mathcal{L}, \delta_\xi)$ -martingale problem (MP). Noting, the notations here coincide with those given in Section 1.2 of Chapter 1 if we don't specify. Our main results are as follows: first, under the assumptions that g is a square-integrable, smooth, and symmetric function on $\mathbb{R}$ satisfying

$$g(\Delta) := \lim_{|x| \rightarrow \infty} g(x) = 0,$$

and $\epsilon = 0$ (These conditions will be called smooth degenerate case), then for any $t > 0$, Z_t is a purely atomic measure. Secondly, on the other hand, let g be a square-integrable, smooth, and symmetric function on $\mathbb{R}$ satisfying $g(\Delta) = 0$ and $\epsilon \neq 0$ (These conditions will be called smooth uniformly elliptic case), then for same initial state ξ , Z_t , the unique solution to $(\mathcal{L}, \delta_\xi)$ -martingale problem (MP), is an absolutely continuous measure-valued process.

Before entering rigorously detailed proofs, we want to present the rough idea of how we obtained these conclusions and to outline the organization of this chapter.

Let $\{U_t : t \geq 0\}$, $\{T_t : t \geq 0\}$ and $\{S_t : t \geq 0\}$ be the semigroups corresponding to measure-valued Markov processes which are solutions to $\mathcal{L}$ -MP, $\mathcal{A}$ -MP and $\mathcal{B}$ -MP, respectively.

In Subsection two of Section one, we prove that for any finite measure ν with compact support, the unique solution $\{\nu_t\}$ to $(\mathcal{B}, \delta_\nu)$ -MP has the form

$$\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)}, \quad t > 0,$$

where $I(t)$ is a random subset of the integers $\mathbb{N}$, $x_i(t)$ represents the position of the i^{th} particle at time t and $a_i(t) \geq 0$ represents particle's mass. The $\{a_i(t)\}_i$ are independent and given by the unique solution to $(\frac{1}{2}\gamma(m_2 - 1)x \frac{d^2}{dx^2}, \delta_{a_i(0)})$ -MP. This

explicitly shows that ν_t is a purely atomic measure for $t > 0$, and the action of $\mathcal{B}$ operator only changes the masses of atoms.

The Subsection three of Section one is devoted to prove that if $\epsilon = 0$, then for any finite atomic measure

$$\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)},$$

where $J \subset \mathbb{N}$, $a_i(0) \geq 0$, $x_i(0) \in \mathbb{R}$ and $x_i(0) \neq x_j(0)$, for any $i \neq j$, $i, j \in J$, the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP has the form

$$\mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)} \quad \text{for } t > 0,$$

where $x_i(t) \neq x_j(t)$, for any $i \neq j$, $i, j \in J$, $t \geq 0$. This reflects the fact that if $\epsilon = 0$, then the action of $\mathcal{A}$ operator only changes the positions of atoms.

In Subsection four of Section one, we establish the commutativity of $\{T_t\}$ and $\{S_t\}$ which implies the conclusion that for any $t > 0$, Z_t is a purely atomic measure for the smooth degenerate case. Generally, the semigroup $\{T_t\}$ does not commute with the semigroup $\{S_t\}$, but in the extreme case in which g is a smooth function and $\epsilon = 0$ it does indeed. Once this step is completed, then for above measure ν , by the uniqueness of the solutions to $(\mathcal{L}, \delta_\nu)$ -MP, we have

$$Z_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)}, \quad I(t) \subset \mathbb{N} \quad \text{for } t > 0$$

and the conclusion for the smooth degenerate case is obvious.

In Section two, we turn to discuss the smooth uniformly elliptic case. First we give the necessary preparation, and then we use the duality method and a result of application of Malliavin calculus to directly construct the density process. Therefore, the conclusion that Z_t is an absolutely continuous measure-valued process is an easy corollary.

6.1.2 Behavior of the Generator $\mathcal{B}$

In this subsection, we will generalize the idea given by Ethier-Kurtz ([21] p.441) to achieve our first target. Let E be a Polish space, and $B(E)$ be the space of all bounded measurable functions on E . Define

$$\mathcal{F}_t^Y := \sigma(Y(s), s \leq t) \vee \sigma\left(\int_0^s h(Y(u))du, s \leq t, h \in B(E)\right).$$

Lemma 6.1.1 *For any finite measure $\nu = \nu_0$ on $\mathbb{R}$ with compact support, the $(\mathcal{B}, \delta_\nu)$ -MP has a unique solution $\{\nu_t : t \geq 0\}$ with distribution $\mathbb{P}_\nu^1$ on probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \{\nu_t : t \geq 0\})$ is a measure-valued diffusion process.*

Proof: The existence and uniqueness of the $(\mathcal{B}, \delta_\nu)$ -MP follow from the conclusion of Theorem 4.3.3 and Theorem 4.4.1 of Chapter 4 by taking $g \equiv 0$.

Let $D(\mathcal{B})$ denote the class of functions on $M_F(\bar{\mathbb{R}})$ of the form

$$F(\mu) = \psi(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle),$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, $f_i \in C^2(\mathbb{R})$, and $n \geq 1$. Since

$$\mathcal{R}_{\mathcal{B}}(F, G)(\mu) = \gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta F(\mu)}{\delta \mu(x)} \frac{\delta G(\mu)}{\delta \mu(x)} \mu(dx) \quad \text{for any } F, G \in D(\mathcal{B})$$

and $\mathcal{R}_{\mathcal{B}}$ has the derivative property, the continuity of $\{\nu_t : t \geq 0\}$ follows from Lemma 5.4.1 of Chapter 5. ■

In the following, in order to prove that $\{\nu_t : t \geq 0\}$ takes values in the set of purely atomic measures, we use a random time change, which was introduced by Konno-Shiga [41] and Shiga [53], to transform $\{\nu_t : t \geq 0\}$ into a probability measure-valued process. For reader's convenience, we follow the argument and notation of Section 8.1 of Dawson([12]).

Let $\tau_\infty := \inf\{t : \langle \nu_t, 1 \rangle = 0\}$ and define $C_t := \int_0^t \langle \nu_s, 1 \rangle^{-1} ds$, then C_t is a homeomorphism between $[0, \tau_\infty)$ and $[0, \infty)$. Let $D_t : [0, \infty) \rightarrow [0, \tau_\infty)$ be the continuous strictly increasing inverse to C_t .

Now define $\tilde{\nu}_t := \nu_{D_t}$, $\hat{\nu}_t := \tilde{\nu}_{\text{nor}}(t) := \tilde{\nu}_t / \langle \tilde{\nu}_t, 1 \rangle$ and $\mathcal{G}_t := \mathcal{F}_{D_t}^\nu$, then $\{\hat{\nu}_t\}$ is a probability measure-valued process.

Since in the present case the particle's motion generator is equal to zero, by (1.2), we see that

$$(1.4) \quad M_t(\phi) := \langle \nu_t, \phi \rangle - \langle \nu_0, \phi \rangle$$

is a $\mathbb{P}_\nu^1, \{\mathcal{F}_t^\nu\}$ -martingale for $\phi \in C(\mathbb{R})$. By Dellacherie-Meyer ([17] XV22. XV26) or Fitzsimmons ([26]p355 corollary(4.3)) and noting that

$$(1.5) \quad \mathcal{B}(\langle \nu_t, \phi \rangle \langle \nu_t, \psi \rangle) - \langle \nu_t, \phi \rangle \mathcal{B} \langle \nu_t, \psi \rangle - \langle \nu_t, \psi \rangle \mathcal{B} \langle \nu_t, \phi \rangle \\ = \langle \nu_t, c\phi\psi \rangle,$$

where $c = \frac{1}{2}\gamma(m_2 - 1)$ and $\phi, \psi \in C(\mathbb{R})$, we obtain

$$(1.6) \quad \langle M(\phi) \rangle_t = c \int_0^t \langle \nu_s, \phi^2 \rangle ds$$

and

$$(1.7) \quad \langle M(\phi), M(\psi) \rangle_t = c \int_0^t \langle \nu_s, \phi\psi \rangle ds.$$

Using random time change, by optional sampling theorem (see Ikeda-Watanabe [34]p34), it follows that

$$(1.8) \quad \tilde{M}_t(\phi) := \langle \tilde{\nu}_t, \phi \rangle - \langle \tilde{\nu}_0, \phi \rangle$$

is a $\mathcal{G}_t$ -martingale,

$$(1.9) \quad \langle \tilde{M}(\phi) \rangle_t = \langle M(\phi) \rangle_{D_t} = c \int_0^{D_t} \langle \nu_s, \phi^2 \rangle ds$$

$$\begin{aligned}
&= c \int_0^{D_t} \langle \nu_s, \phi^2 \rangle \langle \nu_s, 1 \rangle \frac{1}{\langle \nu_s, 1 \rangle} ds \\
&= c \int_0^{D_t} \langle \nu_{D_{C_s}}, \phi^2 \rangle \langle \nu_{D_{C_s}}, 1 \rangle dC_s \\
&= c \int_0^t \langle \nu_{D_u}, \phi^2 \rangle \langle \nu_{D_u}, 1 \rangle du = c \int_0^t \langle \tilde{\nu}_u, \phi^2 \rangle \langle \tilde{\nu}_u, 1 \rangle du,
\end{aligned}$$

and

$$(1.10) \quad \langle \tilde{M}(\phi), \tilde{M}(\psi) \rangle_t = c \int_0^t \langle \tilde{\nu}_u, \phi \psi \rangle \langle \tilde{\nu}_u, 1 \rangle du.$$

Note that $\langle \tilde{\nu}_t, 1 \rangle \neq 0$ for any $t \geq 0$. Let $f(x, y) = x/y$. Then by Ito's formula,

$$\begin{aligned}
(1.11) \quad &f(\langle \tilde{\nu}_t, \phi \rangle, \langle \tilde{\nu}_t, 1 \rangle) - f(\langle \tilde{\nu}_0, \phi \rangle, \langle \tilde{\nu}_0, 1 \rangle) \\
&= \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle} d\tilde{M}_s(\phi) - \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle}{\langle \tilde{\nu}_s, 1 \rangle^2} d\tilde{M}_s(1) \\
&\quad + \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle}{\langle \tilde{\nu}_s, 1 \rangle^3} d\langle \tilde{M}(1) \rangle_s - \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle^2} d\langle \tilde{M}(\phi), \tilde{M}(1) \rangle_s \\
&= \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle} d\tilde{M}_s(\phi) - \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle}{\langle \tilde{\nu}_s, 1 \rangle^2} d\tilde{M}_s(1) \\
&\quad + \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle}{\langle \tilde{\nu}_s, 1 \rangle^3} c \langle \tilde{\nu}_s, 1 \rangle^2 ds \\
&\quad - \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle^2} c \langle \tilde{\nu}_s, 1 \rangle \langle \tilde{\nu}_s, \phi \rangle ds \\
&= \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle} d\tilde{M}_s(\phi) - \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle}{\langle \tilde{\nu}_s, 1 \rangle^2} d\tilde{M}_s(1) =: N_t.
\end{aligned}$$

Therefore, $N_t = \langle \tilde{\nu}_t, \phi \rangle - \langle \tilde{\nu}_0, \phi \rangle$ is a $\mathcal{G}_t$ -martingale and

$$\begin{aligned}
(1.12) \quad \langle N \rangle_t &= \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle^2} c \langle \tilde{\nu}_s, 1 \rangle \langle \tilde{\nu}_s, \phi^2 \rangle ds \\
&\quad + \int_0^t \frac{\langle \tilde{\nu}_s, \phi \rangle^2}{\langle \tilde{\nu}_s, 1 \rangle^4} c \langle \tilde{\nu}_s, 1 \rangle^2 ds
\end{aligned}$$

$$\begin{aligned}
& -2 \int_0^t \frac{1}{\langle \tilde{\nu}_s, 1 \rangle^3} c \langle \tilde{\nu}_s, \phi \rangle^2 \langle \tilde{\nu}_s, 1 \rangle ds \\
& = \int_0^t c [\langle \tilde{\nu}_s, \phi^2 \rangle + \langle \tilde{\nu}_s, \phi \rangle^2 - 2 \langle \tilde{\nu}_s, \phi \rangle^2] ds \\
& = c \int_0^t [\langle \tilde{\nu}_s, \phi^2 \rangle - \langle \tilde{\nu}_s, \phi \rangle^2] ds.
\end{aligned}$$

For $\beta \geq 2$, let $g(x, y) = (x/y)^\beta$, by Ito's formula, similarly we obtain

$$\begin{aligned}
(1.13) \quad & g(\langle \tilde{\nu}_t, \phi \rangle, \langle \tilde{\nu}_t, 1 \rangle) - g(\langle \tilde{\nu}_0, \phi \rangle, \langle \tilde{\nu}_0, 1 \rangle) \\
& = \int_0^t g'_x(\langle \tilde{\nu}_s, \phi \rangle, \langle \tilde{\nu}_s, 1 \rangle) d\tilde{M}_s(\phi) \\
& \quad + \int_0^t g'_y(\langle \tilde{\nu}_s, \phi \rangle, \langle \tilde{\nu}_s, 1 \rangle) d\tilde{M}_s(1) \\
& \quad + \frac{1}{2} \int_0^t g''_{xx}(\langle \tilde{\nu}_s, \phi \rangle, \langle \tilde{\nu}_s, 1 \rangle) d\langle \tilde{M}(\phi) \rangle_s \\
& \quad + \frac{1}{2} \int_0^t g''_{yy}(\langle \tilde{\nu}_s, \phi \rangle, \langle \tilde{\nu}_s, 1 \rangle) d\langle \tilde{M}(1) \rangle_s \\
& \quad + \int_0^t g''_{xy}(\langle \tilde{\nu}_s, \phi \rangle, \langle \tilde{\nu}_s, 1 \rangle) d\langle \tilde{M}(\phi), \tilde{M}(1) \rangle_s \\
& = \int_0^t \beta \frac{\langle \tilde{\nu}_s, \phi \rangle^{\beta-1}}{\langle \tilde{\nu}_s, 1 \rangle^\beta} d\tilde{M}_s(\phi) - \int_0^t \beta \frac{\langle \tilde{\nu}_s, \phi \rangle^\beta}{\langle \tilde{\nu}_s, 1 \rangle^{\beta+1}} d\tilde{M}_s(1) \\
& \quad + \frac{1}{2} \int_0^t \beta(\beta-1) \frac{\langle \tilde{\nu}_s, \phi \rangle^{\beta-2}}{\langle \tilde{\nu}_s, 1 \rangle^\beta} c \langle \tilde{\nu}_s, \phi^2 \rangle \langle \tilde{\nu}_s, 1 \rangle ds \\
& \quad + \frac{1}{2} \int_0^t \beta(\beta+1) \frac{\langle \tilde{\nu}_s, \phi \rangle^\beta}{\langle \tilde{\nu}_s, 1 \rangle^{\beta+2}} c \langle \tilde{\nu}_s, 1 \rangle^2 ds \\
& \quad - \int_0^t \beta^2 \frac{\langle \tilde{\nu}_s, \phi \rangle^{\beta-1}}{\langle \tilde{\nu}_s, 1 \rangle^{\beta+1}} c \langle \tilde{\nu}_s, \phi \rangle \langle \tilde{\nu}_s, 1 \rangle ds \\
& = \tilde{M}_t^\beta(\phi) + \frac{1}{2} \int_0^t [\beta(\beta-1) c \langle \tilde{\nu}_s, \phi \rangle^{\beta-2} \langle \tilde{\nu}_s, \phi^2 \rangle
\end{aligned}$$

$$\begin{aligned}
& +\beta(\beta+1)c \langle \hat{\nu}_s, \phi \rangle^\beta - 2\beta^2 \langle \hat{\nu}_s, \phi \rangle^\beta] ds \\
& = \bar{M}_t^\beta(\phi) + \frac{1}{2} \int_0^t \beta(\beta-1)c [\langle \hat{\nu}_s, \phi^2 \rangle - \langle \hat{\nu}_s, \phi \rangle^2] \langle \hat{\nu}_s, \phi \rangle^{\beta-2} ds,
\end{aligned}$$

where

$$(1.14) \quad \bar{M}_t^\beta(\phi) := \int_0^t \beta \frac{\langle \hat{\nu}_s, \phi \rangle^{\beta-1}}{\langle \hat{\nu}_s, 1 \rangle^\beta} d\bar{M}_s(\phi) - \int_0^t \beta \frac{\langle \hat{\nu}_s, \phi \rangle^\beta}{\langle \hat{\nu}_s, 1 \rangle^{\beta+1}} d\bar{M}_s(1).$$

Consequently,

$$\begin{aligned}
(1.15) \quad & \langle \bar{M}^\beta(\phi), \bar{M}^\beta(\psi) \rangle, \\
& = \beta^2 \int_0^t \frac{\langle \hat{\nu}_s, \phi \rangle^{\beta-1}}{\langle \hat{\nu}_s, 1 \rangle^\beta} \frac{\langle \hat{\nu}_s, \psi \rangle^{\beta-1}}{\langle \hat{\nu}_s, 1 \rangle^\beta} c \langle \hat{\nu}_s, \phi \psi \rangle \langle \hat{\nu}_s, 1 \rangle ds \\
& - \beta^2 \int_0^t \frac{\langle \hat{\nu}_s, \phi \rangle^\beta}{\langle \hat{\nu}_s, 1 \rangle^{\beta+1}} \frac{\langle \hat{\nu}_s, \psi \rangle^{\beta-1}}{\langle \hat{\nu}_s, 1 \rangle^\beta} c \langle \hat{\nu}_s, \psi \rangle \langle \hat{\nu}_s, 1 \rangle ds \\
& - \beta^2 \int_0^t \frac{\langle \hat{\nu}_s, \psi \rangle^\beta}{\langle \hat{\nu}_s, 1 \rangle^{\beta+1}} \frac{\langle \hat{\nu}_s, \phi \rangle^{\beta-1}}{\langle \hat{\nu}_s, 1 \rangle^\beta} c \langle \hat{\nu}_s, \phi \rangle \langle \hat{\nu}_s, 1 \rangle ds \\
& + \beta^2 \int_0^t \frac{\langle \hat{\nu}_s, \phi \rangle^\beta}{\langle \hat{\nu}_s, 1 \rangle^{\beta+1}} \frac{\langle \hat{\nu}_s, \psi \rangle^\beta}{\langle \hat{\nu}_s, 1 \rangle^{\beta+1}} c \langle \hat{\nu}_s, 1 \rangle^2 ds \\
& = \beta^2 \int_0^t [\langle \hat{\nu}_s, \phi \rangle^{\beta-1} \langle \hat{\nu}_s, \psi \rangle^{\beta-1} c \langle \hat{\nu}_s, \phi \psi \rangle \\
& - \langle \hat{\nu}_s, \phi \rangle^\beta \langle \hat{\nu}_s, \psi \rangle^{\beta-1} c \langle \hat{\nu}_s, \psi \rangle \\
& - \langle \hat{\nu}_s, \psi \rangle^\beta \langle \hat{\nu}_s, \phi \rangle^{\beta-1} c \langle \hat{\nu}_s, \phi \rangle + \langle \hat{\nu}_s, \phi \rangle^\beta \langle \hat{\nu}_s, \psi \rangle^\beta c] ds \\
& = \beta^2 c \int_0^t \langle \hat{\nu}_s, \phi \rangle^{\beta-1} \langle \hat{\nu}_s, \psi \rangle^{\beta-1} [\langle \hat{\nu}_s, \phi \psi \rangle \\
& - \langle \hat{\nu}_s, \phi \rangle \langle \hat{\nu}_s, \psi \rangle] ds.
\end{aligned}$$

(1.13) implies that

$$(1.16) \quad \bar{M}_t^\beta(\phi) = \langle \hat{\nu}_t, \phi \rangle^\beta - \langle \hat{\nu}_0, \phi \rangle^\beta$$

$$-\frac{1}{2} \int_0^t \beta(\beta-1) c[\langle \hat{\nu}_s, \phi^2 \rangle - \langle \hat{\nu}_s, \phi \rangle^2] \langle \hat{\nu}_s, \phi \rangle^{\beta-2} ds$$

is a continuous, square-integrable $\mathcal{G}_t$ -martingale.

Let $\mathcal{P}[0, 1]$ be the set of all probability measures on $[0, 1]$ and $\mathcal{P}_a[0, 1]$ be the set of all purely atomic probability measures on $[0, 1]$.

Lemma 6.1.2 *For any $t > 0$, ν_t is almost surely a purely atomic measure.*

Proof: Without loss of generality, we can suppose the support of ν is $[0, 1]$ and $\nu([0, 1]) = 1$. Define $\phi_{n,\beta} : \mathcal{P}[0, 1] \rightarrow [0, \infty)$ for each $\beta > 0$ and $n = 1, 2, \dots$ by

$$(1.17) \quad \phi_{n,\beta}(\mu) = \mu([0, \frac{1}{2^n}])^\beta + \mu((\frac{1}{2^n}, \frac{2}{2^n}])^\beta + \dots + \mu((1 - \frac{1}{2^n}, 1])^\beta.$$

It follows from (1.16) for $\beta \geq 2$ that

$$(1.18) \quad Z_{n,\beta}(t) \equiv \phi_{n,\beta}(\hat{\nu}_t) - \phi_{n,\beta}(\hat{\nu}_0) - \frac{1}{2} \int_0^t c\beta(\beta-1)[\phi_{n,\beta-1} - \phi_{n,\beta}](\hat{\nu}_s) ds$$

is a continuous, square-integrable $\mathcal{G}_t$ -martingale with increasing process

$$(1.19) \quad I_{n,\beta}(t) = c\beta^2 \int_0^t (\phi_{n,2\beta-1} - \phi_{n,\beta}^2)(\hat{\nu}_s) ds.$$

For $\beta > 1$, let $g_\epsilon(x, y) := \frac{(x+\epsilon)^\beta}{y^\beta}$. In (1.13) and (1.15), we substitute $g_\epsilon(x, y)$ for $g(x, y)$. Since $x \geq 0$, (1.16) holds for $g_\epsilon(x, y)$ with $\beta > 1$. The fact that $g_\epsilon(x, y)$ converges to $g(x, y)$ on $[0, \infty) \times [0, \infty)$ uniformly as $\epsilon \downarrow 0$ implies (1.18), (1.19) hold for each $\beta > 1$.

Define $\phi_\beta : \mathcal{P}[0, 1] \rightarrow [0, \infty]$ for each $\beta > 0$ by

$$(1.20) \quad \phi_\beta(\mu) := \begin{cases} \sum_{0 \leq x \leq 1} \mu(\{x\})^\beta & \text{if } \beta \neq 1 \\ 1 & \text{if } \beta = 1. \end{cases}$$

We have $\text{bp-lim}_{n \rightarrow \infty} \phi_{n,\beta} = \phi_\beta$ (bounded pointwise limit; see [21]p495) for each $\beta \geq 1$, while $\phi_{n,\beta} \uparrow \phi_\beta$ as $n \rightarrow \infty$ for $0 < \beta < 1$. We conclude that for each $\beta > 1$

$$(1.21) \quad Z_\beta(t) \equiv \phi_\beta(\hat{\nu}_t) - \phi_\beta(\hat{\nu}_0) - \frac{1}{2} \int_0^t c\beta(\beta-1)[\phi_{\beta-1} - \phi_\beta](\hat{\nu}_s) ds$$

is a continuous, square-integrable $\mathcal{G}_t$ -martingale with increasing process

$$(1.22) \quad I_\beta(t) = c\beta^2 \int_0^t (\phi_{2\beta-1} - \phi_\beta^2)(\dot{\nu}_s) ds;$$

here we are using

$$(1.23) \quad \lim_{n \rightarrow \infty} \mathbb{E}_\nu^1[Z_{n,\beta}(t)^2] = \lim_{n \rightarrow \infty} \mathbb{E}_\nu^1[I_{n,\beta}(t)] = \mathbb{E}_\nu^1[I_\beta(t)]$$

and monotone convergence theorem to show that, when $1 < \beta < 2$,

$$(1.24) \quad \mathbb{E}_\nu^1\left[\int_0^t \phi_{\beta-1}(\dot{\nu}_s) ds\right]^2 < \infty, \quad t \geq 0.$$

Letting $\phi_{1+}(\mu) = \text{bp-lim}_{\beta \rightarrow 1+} \phi_\beta(\mu) = \sum_{0 \leq x \leq 1} \mu(\{x\})$, we have

$$(1.25) \quad \begin{aligned} 0 &= \lim_{\beta \rightarrow 2+} \mathbb{E}_\nu^1[Z_\beta(0) - Z_2(0)] \\ &= \lim_{\beta \rightarrow 2+} \mathbb{E}_\nu^1[Z_\beta(t) - Z_2(t)] = \mathbb{E}_\nu^1\left[\int_0^t c(1 - \phi_{1+})(\dot{\nu}_s) ds\right] \end{aligned}$$

for all $t \geq 0$, so

$$\mathbb{P}_\nu^1\{\dot{\nu}_t \in \mathcal{P}_a[0, 1] \text{ for almost every } t > 0\} = 1.$$

Since

$$(1.26) \quad \begin{aligned} \lim_{\beta \rightarrow 1+} \mathbb{E}_\nu^1[Z_\beta(t)^2] &= \lim_{\beta \rightarrow 1+} \mathbb{E}_\nu^1[I_\beta(t)] \\ &= \mathbb{E}_\nu^1\left[c \int_0^t \phi_{1+}(\dot{\nu}_s)(1 - \phi_{1+})(\dot{\nu}_s) ds\right] = 0 \end{aligned}$$

for each $t \geq 0$ by (1.25). Fix $t_0 > 0$. By Doob's martingale inequality,

$$\sup_{0 \leq t \leq t_0} |Z_\beta(t)| \rightarrow 0$$

in probability as $\beta \rightarrow 1+$, so there exists a sequence $\beta_n \rightarrow 1+$ such that

$$\sup_{0 \leq t \leq t_0} |Z_{\beta_n}(t)| \rightarrow 0 \quad \text{a.s. } (\mathbb{P}_\nu^1).$$

Letting

$$(1.27) \quad \eta(t) = \overline{\lim}_{n \rightarrow \infty} \frac{1}{2} c \beta_n (\beta_n - 1) \int_0^t \phi_{\beta_n - 1}(\hat{\nu}_s) ds,$$

we obtain from (1.21) and (1.25) that

$$(1.28) \quad \phi_{1+}(\hat{\nu}_t) - \phi_{1+}(\hat{\nu}_0) - \eta(t) = 0 \quad \text{a.s. } (\mathbb{P}_\nu^1)$$

for every $t \in [0, t_0]$. Since $\eta(t)$ is nondecreasing in t , we conclude that

$$\mathbb{P}_\nu^1 \{ \hat{\nu}_t \in \mathcal{P}_a[0, 1] \text{ for all } t > 0 \} = 1$$

as required and this implies ν_t is a purely atomic measure for all $t > 0$. ■

Theorem 6.1.3 *For any finite measure ν on $\mathbb{R}$ with compact support, the unique solution $\{\nu_t\}$ to $(\mathcal{B}, \delta_\nu)$ -MP has the form*

$$(1.29) \quad \nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(0)}, \quad x_i(0) \in \mathbb{R}, \quad x_i(0) \neq x_j(0) \text{ for } i \neq j,$$

where $I(t)$ is a random subset of $\mathbb{N}$, given ω , $I(t, \omega)$ is decreasing in t in terms of set inclusion order, and $\{a_i(t)\}$ is a one dimensional diffusion process with state space $\mathbb{R}^+ := \{x : x \geq 0\}$.

Proof: By Lemma 6.1.2, we know that for $\forall t > 0$, ν_t is a purely atomic measure. Since this is a motion-free superprocess, the decreasing property of $I(t)$ is obvious.

From Chapter 4, we know that $\mathcal{B}_n \rightarrow \mathcal{B}$ and from the construction of $\mathcal{B}_n$ we know at each position $x_i(0)$, the particle's branching is independent of that of the other positions. So looking at atom $a_i(t) \delta_{x_i(0)}$ and recalling the basic results of continuous state branching processes, we see that $a_i(t)$ is just the solution to the martingale problem for generator $(\frac{1}{2} \gamma(m_2 - 1) x \frac{d^2}{dx^2})$. ■

6.1.3 Behavior of the Generator $\mathcal{A}$

In this subsection, we will show that suppose $\epsilon = 0$, $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function, and the initial value is an atomic measure

$$(1.30) \quad \mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)},$$

where $a_i(0) \geq 0$, $x_i(0) \in \mathbb{R}$, and $x_i(0) \neq x_j(0)$ for $i \neq j$ and $J \subset \mathbb{N}$, then the unique solution to $(\mathcal{A}, \delta_\mu)$ -MP has the form

$$(1.31) \quad \mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)},$$

where $x_i(t)$ is continuous in t and $x_i(t) \neq x_j(t)$ for $\forall t \geq 0$ and $i \neq j$. Let us begin with considering the case in which J has finite elements.

Lemma 6.1.4 *Suppose in (1.3), $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$ and $\epsilon = 0$. If $J = \{1, 2, \dots, n\}$, then for any $(x_1(0), x_2(0), \dots, x_n(0)) \in \mathbb{R}^n$ and*

$$\mu^n = \sum_{i=1}^n a_i(0) \delta_{x_i(0)}, \quad a_i(0) \geq 0, \quad x_i(0) \neq x_j(0) \text{ for } i \neq j,$$

there exists a unique $\mathbb{R}^n$ -valued diffusion process $\{(x_1(t), x_2(t), \dots, x_n(t)) : t \geq 0\}$ such that

$$(1.32) \quad \mu_t^n = \sum_{i=1}^n a_i(0) \delta_{x_i(t)}, \quad x_i(t) \neq x_j(t), \quad \text{for } \forall t \geq 0 \text{ and } i \neq j,$$

is the unique solution to $(\mathcal{A}, \delta_{\mu^n})$ -MP.

Proof: For $\mu_s^n = \sum_{i=1}^n a_i(0) \delta_{x_i(s)}$, same as (2.9) of Chapter 4, we can change the form of $\mathcal{A}F_f(\mu_s^n)$ as following:

$$(1.33) \quad \mathcal{A}F_f(\mu_s^n) = G^n g(x_1(s), x_2(s), \dots, x_n(s))$$

for

$$(1.34) \quad F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_N) \mu^{\otimes N}(d\bar{x}),$$

where

$$(1.35) \quad G^n f = \frac{1}{2} \sum_{i,j=1}^n \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f$$

$$\mu^{\otimes N}(d\bar{x}) := \mu(dx_1) \cdots \mu(dx_N)$$

and

$$(1.36) \quad g(x_1(s), \dots, x_n(s)) = \sum_{l_1, \dots, l_N}^n a_{l_1}(0) \cdots a_{l_N}(0) f(x_{l_1}(s), \dots, x_{l_N}(s)).$$

So $\{(x_1(s), \dots, x_n(s)) : s \geq 0\}$ is a diffusion process with generator G^n . Let $\eta_i = x_i(t) - x_j(t)$ for $i \neq j$, then $\{\eta_i\}$ is a diffusion process with state space $\mathbb{R}$, absorbing state 0, and generator

$$(1.37) \quad \mathcal{G}f(y) = (\rho(0) - \rho(y))f''(y) \quad \text{for} \quad f \in C_b^\infty(\mathbb{R}).$$

From Feller's criterion of accessibility, the probability that η reaches 0 is 0 or 1 according as

$$\int_0^1 \frac{y}{(\rho(0) - \rho(y))} dy$$

is ∞ or $< \infty$. But

$$0 \leq \sup_y \frac{(\rho(0) - \rho(y))}{y^2} \leq \frac{1}{2} |\rho''(0)|.$$

Since g is smooth and $\rho''(0)$ is finite, state 0 is inaccessible. Define

$$(1.38) \quad \Phi_n : (x_1, \dots, x_n) \longrightarrow \sum_{i=1}^n a_i(0) \delta_{x_i}.$$

Since G^n is symmetric, the law of the corresponding diffusion process is exchangeable (see Chapter 4 Definition (4.2.1)) and the conditions of Lemma 4.2.1 of Chapter 4 are satisfied. Our conclusion follows from the Lemma 4.2.1 of Chapter 4 and the

above facts. ■

Remark Let $e(\omega)$ be the explosion time of G^n -diffusion. By Ikeda-Watanabe ([34] p.163 Theorem 2.4), With any initial point $x \in \mathbb{R}^n$, the explosion time of G^n -diffusion $e(\omega) = \infty$ almost surely. This means the G^n -diffusion always lives in $\mathbb{R}^n$ and this is why we often consider questions related to $\mathbb{R}^n$ instead of $\bar{\mathbb{R}}^n$.

In order to achieve our objective of this subsection, we need some results related to weak atomic topology which was introduced by Ethier-Kurtz [24]. Let us introduce the following concepts first.

Let (E, r) be a complete, separable metric space. For $\mu \in M_F(E)$, the space of finite, positive, Borel measures on E with weak topology, define χ_μ to be the purely atomic measure given by $\chi_\mu = \sum \mu(\{x\})^2 \delta_x$. Let λ denote Prohorov metric on $M_F(E)$ and $\Phi : [0, \infty) \rightarrow [0, 1]$ be a continuous and nonincreasing function with $\Phi(0) = 1$ and $\Phi(1) = 0$. Define

$$(1.39) \quad \lambda_a(\mu, \nu) = \lambda(\mu, \nu) + \sup_{0 < \epsilon \leq 1} \left| \int_E \int_E \Phi\left(\frac{r(x, y)}{\epsilon}\right) \mu(dx) \mu(dy) - \int_E \int_E \Phi\left(\frac{r(x, y)}{\epsilon}\right) \nu(dx) \nu(dy) \right|.$$

By Lemma 2.3 of Ethier-Kurtz [24], $(M_F(E), \lambda_a)$ is a Polish space.

In order to use the results of Ethier-Kurtz [24], we must prove our branching-free particle systems are defined on a common probability space. This result is given in the following lemma.

Definition 6.1.1 Let W be a cylindrical Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose g is a square-integrable, smooth, and symmetric function on $\mathbb{R}$. For any $n \in \mathbb{N}$, consider the following system of stochastic integral equations

$$(1.40) \quad x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} g(y - x_i(s)) W(dy, ds), \quad i = 1, \dots, n.$$

A continuous process $(x_1(t), \dots, x_n(t))$ which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a strong solution of (1.40) if (1.40) holds almost surely with respect to probability $\mathbb{P}$.

Lemma 6.1.5 Assume $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$. Let W be a cylindrical Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following system of stochastic integral equations

$$x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} g(y - x_i(s)) W(dy, ds) \quad i = 1, \dots, n$$

has a unique strong solution.

Proof: We prove the existence and uniqueness by using the method of successive approximation as follows.

We define a sequence $x^k(t) = \{(x_1^k(t), \dots, x_n^k(t))\}$ of n -dimensional continuous processes by

$$(1.41) \quad \begin{cases} x^0(t) = x_0 \\ x^k(t) = x_0 + \left(\int_0^t \int_{\mathbb{R}} g(y - x_1^{k-1}(s)) W(dy, ds), \right. \\ \quad \left. \dots, \int_0^t \int_{\mathbb{R}} g(y - x_n^{k-1}(s)) W(dy, ds) \right), \end{cases}$$

where $x_0 \in \mathbb{R}^n$ is fixed. Since

$$\begin{aligned} (1.42) \quad & \mathbb{E}\{\sup_{0 \leq t \leq T} |x^{k+1}(t) - x^k(t)|^2\} \\ &= \mathbb{E}\left\{\sup_{0 \leq t \leq T} \sum_{i=1}^n \left[\int_0^t \int_{\mathbb{R}} g(y - x_i^k(s)) W(dy, ds) \right. \right. \\ & \quad \left. \left. - \int_0^t \int_{\mathbb{R}} g(y - x_i^{k-1}(s)) W(dy, ds) \right]^2 \right\} \\ &\leq \mathbb{E} \sum_{i=1}^n \left[\int_0^T \int_{\mathbb{R}} (g(y - x_i^k(s)) - g(y - x_i^{k-1}(s))) W(dy, ds) \right]^2 \end{aligned}$$

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$$\begin{aligned}
&= \mathbb{E} \sum_{i=1}^n \int_0^T \int_{\mathbb{R}} (g(y - x_i^k(s)) - g(y - x_i^{k-1}(s)))^2 dy ds \\
&= \sum_{i=1}^n \mathbb{E} \int_0^T 2[\rho(0) - \rho(x_i^k(s) - x_i^{k-1}(s))] ds \\
&\leq \sum_{i=1}^n \rho''(0) \mathbb{E} \int_0^T [x_i^k(s) - x_i^{k-1}(s)]^2 ds, \\
&= \rho''(0) \mathbb{E} \int_0^T |x^k(s) - x^{k-1}(s)|^2 ds,
\end{aligned}$$

we have

$$\begin{aligned}
(1.43) \quad &\mathbb{E}[\sup_{0 \leq s \leq T} |x^{k+1}(s) - x^k(s)|^2] \\
&\leq (\rho''(0))^k \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \mathbb{E}(|x^1(t_k) - x^0(t_k)|^2) \\
&\leq (\rho''(0))^k \frac{T^k}{k!} n \rho(0) T.
\end{aligned}$$

Consequently,

$$\mathbb{P}\left\{\sup_{0 \leq s \leq T} |x^{k+1}(s) - x^k(s)| > \frac{1}{2^k}\right\} \leq \text{const} (\rho''(0))^k \frac{T^k}{k!}.$$

By the Borel-Cantelli lemma, we see that $x^k(s)$ converges uniformly on $[0, T]$ almost surely with respect to $\mathbb{P}$, since T is arbitrary, $\lim_{k \rightarrow \infty} x^k(s) = x(s)$ determines a n -dimensional continuous process which is clearly a strong solution of (1.40). The uniqueness is obvious from the above argument. ■

Theorem 6.1.6 *Suppose $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$ and $\epsilon = 0$. For any finite, purely atomic measure*

$$(1.44) \quad \mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)}, \quad a_i(0) \geq 0, \quad x_i(0) \neq x_j(0) \text{ for } J \subset \mathbb{N} \text{ and } i \neq j$$

with compact support, the unique solution to $(\mathcal{A}, \delta_\mu)$ -MP has the form

$$(1.45) \quad \mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)} \quad x_i(t) \neq x_j(t) \text{ for } i \neq j \text{ and } \forall t \geq 0,$$

and $x_i(t)$ is continuous in t for $i \geq 1$.

Remark Compare $\mathcal{A}$ with the motion generator of super-Brownian motion

$$(1.46) \quad \tilde{\mathcal{A}}F(\mu) := \int_{\mathbb{R}} \Delta \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx).$$

Even if initial state is an atomic measure with finite number of atoms, say,

$$\kappa^n = \sum_{i=1}^n a_i \delta_{x_i},$$

the solution to $(\tilde{\mathcal{A}}, \delta_{\kappa^n})$ -MP is an absolutely continuous measure-valued process for $t > 0$ by Konno-Shiga ([41]p210.Lemma 2.3).

Proof: We only need to consider that J is an infinite set. Without loss of generality, we can suppose that J is the set of all natural integers. Let $\mu^n := \sum_{i=1}^n a_i(0) \delta_{x_i(0)}$. By Lemma 6.1.5, let $(x_1(t), \dots, x_n(t))$ be the unique solution of (1.40) with initial state $(x_1(0), \dots, x_n(0))$. Define $\mu_t^n := \sum_{i=1}^n a_i(0) \delta_{x_i(t)}$. Then by Lemma 4.2.1, μ_t^n is the unique solution to $(\mathcal{A}, \delta_{\mu^n})$ -MP and all $\{\mu_t^n\}$ are defined on a common probability space. Since for any $n \geq 1$,

$$\mu_t^n \in C_{(M_F(\mathbb{R}), \lambda_a)}[0, \infty).$$

If we can prove that given $\epsilon > 0$, there exists a N_0 such that, almost surely,

$$(1.47) \quad \sup_{0 \leq t < \infty} \lambda_a(\mu_t^n, \mu_t^m) \leq \epsilon \quad \text{for any } n, m \geq N_0,$$

then

$$\mu_t := \lim_{n \rightarrow \infty} \mu_t^n \in C_{(M_F(\mathbb{R}), \lambda_a)}[0, \infty)$$

and this implies our conclusion.

Indeed, since μ is a finite measure, there exists N_0 such that

$$(1.48) \quad \sum_{i \geq N_0} a_i(0) < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{i \geq N_0} a_i^2(0) < \frac{\epsilon}{2}.$$

From this, we get

$$\sup_{0 \leq t < \infty} |\chi_{\mu_t^n}(\mathbb{R}) - \chi_{\mu_t^m}(\mathbb{R})| \leq \epsilon \quad \text{for } n, m \geq N_0 \text{ a.s.}$$

By lemma 2.1 and lemma 2.2 of Ethier-Kurtz [24], (1.47) follows and we are done.

■

6.1.4 Commutativity of Semigroups

For any finite measures ν, μ on $\mathbb{R}$ with compact supports, let $\{Z_t^1 : t \geq 0\}$, $(\{Z_t^2 : t \geq 0\})$ be the unique solutions to $(\mathcal{B}, \delta_\nu)$ -MP, $((\mathcal{A}, \delta_\mu)$ -MP) with distribution $\mathbb{P}_\nu^1$, $(\mathbb{P}_\mu^2)$ on probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, $((\Omega^2, \mathcal{F}^2, \mathbb{P}^2))$, respectively. Define

$$(1.49) \quad S_t F(\nu) := \mathbb{E}_\nu^1 F(Z_t^1) := \int F(\omega^1(t)) \mathbb{P}_\nu^1(d\omega^1)$$

$$T_t F(\mu) := \mathbb{E}_\mu^2 F(Z_t^2) := \int F(\omega^2(t)) \mathbb{P}_\mu^2(d\omega^2)$$

for $F \in C(M_F(\bar{\mathbb{R}}))$.

Lemma 6.1.7 $\{S_t\}$ and $\{T_t\}$ are Feller semigroups on $C(M_F(\bar{\mathbb{R}}))$. Let $\epsilon = 0$ and $g \not\equiv 0$ be a square-integrable, smooth, and symmetric function, then $\{T_t\}$ commutes with $\{S_t\}$. Furthermore, for any finite measure ξ on $\mathbb{R}$ with compact support, suppose $\{Z_t : t \geq 0\}$ is the unique solution to $(\mathcal{A} + \mathcal{B}, \delta_\xi)$ -MP with distribution $\mathbb{P}_\xi$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$; if we define

$$U_t F := T_t S_t F \quad \text{for } F \in C(M_F(\bar{\mathbb{R}}))$$

then

$$(1.50) \quad U_s F(Z_t) = \mathbb{E}_t[F(Z_{t+s}) | \mathcal{F}_t^Z]$$

for all $s, t \geq 0$ and $F \in C(M_F(\bar{\mathbb{R}}))$.

Proof: The assertion that $\{S_t\}$ and $\{T_t\}$ are strongly continuous, positive, contraction semigroups can be proved similarly as the proofs of Theorem 4.2.3 of Chapter 4 or Lemma 5.3.1 of Chapter 5. The assertion of Feller semigroups follows from Ethier-Kurtz ([21] p166, Lemma 2.3). Let us turn to the proof of commutativity.

Given $\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)}$. By Theorem 6.1.3, let $\nu_t = \sum_{i \in J} a_i(t) \delta_{x_i(t)}$ be the unique solution to $(\mathcal{B}, \delta_\mu)$ -MP. Define $\mu^n := \sum_{i=1}^n a_i(0) \delta_{x_i(0)}$ and $\nu_t^n := \sum_{i=1}^n a_i(t) \delta_{x_i(t)}$, then $\{\nu_t^n\}_{n \geq 0}$ is defined on a common probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$. If we take $a_i(0) = 0$ for $i \notin \{1, \dots, n\}$, then by Theorem 6.1.3, ν_t^n is the unique solution to $(\mathcal{B}, \delta_{\mu^n})$ -MP. Let $\mu_t^n = \sum_{i=1}^n a_i(t) \delta_{x_i(t)}$ be the unique solution to $(\mathcal{A}, \delta_{\mu^n})$ -MP constructed by Lemma 6.1.5, then all the $\{\mu_t^n\}_{n \geq 1}$ are defined on a common probability space $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$. Define

$$\tilde{\Omega} := \Omega^1 \times \Omega^2, \quad \tilde{\mathcal{F}} := \mathcal{F}^1 \times \mathcal{F}^2, \quad \tilde{\mathbb{P}} := \mathbb{P}^1 \times \mathbb{P}^2, \quad \mathcal{F}_t^{\mu^n, \nu^n} := \mathcal{F}_t^\mu \vee \mathcal{F}_t^\nu.$$

We claim that $Z_t^n = \sum_{i=1}^n a_i(t) \delta_{x_i(t)}$ is the unique solution to $(\mathcal{A} + \mathcal{B}, \delta_{\mu^n})$ -MP on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

To prove the claim, we only need to show that for any $f \in C^\infty(\bar{\mathbb{R}}^l)$ and $\phi_i \in C^\infty(\bar{\mathbb{R}})$, $i = 1, 2, \dots, l$,

$$(1.51) \quad f(\langle Z_t^n, \phi_1 \rangle, \dots, \langle Z_t^n, \phi_l \rangle) - f(\langle Z_0^n, \phi_1 \rangle, \dots, \langle Z_0^n, \phi_l \rangle)$$

$$- \int_0^t (\mathcal{A} + \mathcal{B})f(\langle Z_s^n, \phi_1 \rangle, \dots, \langle Z_s^n, \phi_l \rangle) ds$$

is a $\tilde{\mathbb{P}}$, $\mathcal{F}_t^{\mu^n, \nu^n}$ -martingale. Let

$$(1.52) \quad g(t, s) = f(\langle \sum_{i=1}^n a_i(t) \delta_{x_i(s)}, \phi_1 \rangle, \dots, \langle \sum_{i=1}^n a_i(t) \delta_{x_i(s)}, \phi_l \rangle),$$

then (1.51) is equivalent to, for any $t > s \geq 0$,

$$(1.53) \quad \tilde{\mathbb{E}}[g(t, t) - g(s, s) - \int_s^t (\mathcal{A} + \mathcal{B})g(u, u)du | \mathcal{F}_s^{\mu^n, \nu^n}] = 0.$$

Given a partition $s = t_0 < t_1 < \dots < t_m = t$ of the interval $[s, t]$, we can rewrite the left hand side of (1.53) as follows:

$$\begin{aligned} (1.54) \quad & \tilde{\mathbb{E}}[g(t, t) - g(s, s) - \int_s^t (\mathcal{A} + \mathcal{B})g(u, u)du | \mathcal{F}_s^{\mu^n, \nu^n}] \\ &= \tilde{\mathbb{E}}[\sum_{i=1}^m \{g(t_i, t_i) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} (\mathcal{A} + \mathcal{B})g(u, u)du\} | \mathcal{F}_s^{\mu^n, \nu^n}] \\ &= \tilde{\mathbb{E}}[\sum_{i=1}^m \{g(t_i, t_i) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} (\mathcal{A} + \mathcal{B})g(u, u)du - g(t_i, t_{i-1}) \\ &\quad + \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u)du + g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u)du - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1})du \\ &\quad + \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1})du\} | \mathcal{F}_s^{\mu^n, \nu^n}] \\ &= \tilde{\mathbb{E}}[\sum_{i=1}^m \{(g(t_i, t_i) - g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u)du) \\ &\quad + (g(t_i, t_{i-1}) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1})du) \\ &\quad + \int_{t_{i-1}}^{t_i} (\mathcal{A}g(t_i, u) - \mathcal{A}g(u, u))du + \int_{t_{i-1}}^{t_i} (\mathcal{B}g(u, t_{i-1}) - \mathcal{B}g(u, u))du | \mathcal{F}_s^{\mu^n, \nu^n}]. \end{aligned}$$

Note that for $i = 1, \dots, m$,

$$(1.55) \quad \mathbb{E}_{\mu^n}^2[g(t_i, t_i) - g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u)du | \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n}] = 0$$

and

$$(1.56) \quad \mathbb{E}_{\mu^n}^1[g(t_i, t_{i-1}) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1})du | \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n}] = 0.$$

Therefore, to prove (1.53), it suffices to prove

$$(1.57) \quad \tilde{\mathbb{E}} \left[\int_{t_{i-1}}^{t_i} (\mathcal{A}g(t_i, u) - \mathcal{A}g(u, u)) du \right. \\ \left. + \int_{t_{i-1}}^{t_i} (\mathcal{B}g(u, t_{i-1}) - \mathcal{B}g(u, u)) du | \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n} \right] = o(|t_i - t_{i-1}|).$$

But (1.57) follows from the continuity of $\{a_i(t)\}$, $\{x_i(t)\}$, $\mathcal{A}g(s, t)$, and $\mathcal{B}g(s, t)$. So the claim is proved. From this result combined with Theorem 6.1.3 and Theorem 6.1.6, we have

$$(1.58) \quad T_t S_t F(\mu^n) = S_t T_t F(\mu^n) \quad \text{for } F \in C(M_F(\bar{\mathbb{R}})).$$

Since $\nu_t \in M_F(\bar{\mathbb{R}})$, $\{Z_t^n(\phi)\}$ is a Cauchy sequence for any $\phi \in C_c^\infty(\mathbb{R})$. By a limiting argument, we conclude that $Z_t := \sum_{i=1}^\infty a_i(t) \delta_{x_i(t)}$ is the unique solution to $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -MP. Since $\{S_t : t \geq 0\}$ and $\{T_t : t \geq 0\}$ are Feller semigroups, we obtain

$$(1.59) \quad T_t S_t F(\mu) = S_t T_t F(\mu) \quad \text{for } F \in C(M_F(\bar{\mathbb{R}})).$$

In fact, (1.59) holds for any $\mu \in M_F(\bar{\mathbb{R}})$ because purely atomic measures are dense in $M_F(\bar{\mathbb{R}})$ and (1.59) holds for any purely atomic measures.

(1.50) follows from the uniqueness of solution to $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -MP for any finite measure μ with compact support. ■

Theorem 6.1.8 *Let $\epsilon = 0$ and $g \not\equiv 0$ be a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$. Then for any finite measure ν on $\mathbb{R}$ with compact support, the unique solution $\{\nu_t\}$ to $(\mathcal{A} + \mathcal{B}, \delta_\nu)$ -MP has the form $\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)}$, where $a_i(t) \geq 0$, $\{x_i(t)\}_{i \geq 0}$ are continuous in t , $x_i(t) \neq x_j(t)$ for $t \geq 0$ and $i \neq j$, and $I(t)$ is a nonincreasing random subset of $\mathbb{N}$.*

Proof: The conclusion follows from above Theorem 6.1.3, Theorem 6.1.6, and Lemma 6.1.7. ■

6.2 Absolutely Continuous Measure States

6.2.1 Preliminaries

The objective of this section is to prove that suppose $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$ with $g(\Delta) = 0$ and $\epsilon \neq 0$ in (1.1), then the effective state space of the limiting measure-valued process constructed in Chapter 4 at any $t > 0$ is contained in the set of absolutely continuous measures.

First let us recall some of the basic definitions and results of measure theory. Let $(X, \mathcal{S}, \mu)$ be a finite measure space and for $1 \leq p < \infty$, let $L^p(X, \mathcal{S}, \mu)$ be the Banach space of all real-valued $\mathcal{S}$ -measurable functions $f(x)$ defined μ -a.e. on X such that $|f|^p$ is μ -integrable over X with norm

$$\|f\| = \left\{ \int_X |f|^p \mu(dx) \right\}^{1/p}.$$

Definition 6.2.1 Let $f_n \in L^1, n = 1, 2, \dots$. $\{f_n\}$ is L^1 -bounded if $\sup_n \int_X |f_n| d\mu < \infty$. $\{f_n\}$ is called uniformly absolutely continuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_A |f_n| d\mu < \epsilon$$

holds for every $n \geq 0$ and $A \in \mathcal{S}$ satisfying $\mu(A) < \delta$. $\{f_n\}$ is uniformly integrable if

$$\sup_n \int_X |f_n| d\mu < \infty$$

and for every $\epsilon > 0$, there exists a $k > 0$ such that

$$\sup_n \int_{\{x: |f_n| > k\}} |f_n| d\mu < \epsilon.$$

The following two theorems and their proofs can be found in any standard probability or measure theory textbook; for example, Loève [44] or Halmos [29]. So we omit their proofs.

Theorem 6.2.1 *Let $f_n \in L^1, n = 1, 2, \dots$, then $\{f_n\}$ is uniformly integrable if and only if $\{f_n\}$ is L^1 -bounded and uniformly absolutely continuous.*

Theorem 6.2.2 *Let $p \geq 1$ and $f_n \in L^p, n = 1, 2, \dots$, then $\{f_n\}$ is a Cauchy sequence in L^p -norm if and only if $\{|f_n|^p\}$ is uniformly integrable and $\{f_n\}$ is a Cauchy sequence in measure.*

Moreover, we need a result obtained by the application of Malliavin calculus due to Stroock [57] to attain our objective of this section. Since the proof of this result needs the whole theory of Malliavin calculus, we only cite the theorem in the following. The interested reader is referred to Stroock [57] for more details. Let

$$\Theta_0 := \{\theta \in C([0, \infty), \mathbb{R}^d) : \theta(0) = 0\},$$

and

$$(2.60) \quad p_t(x, y) \equiv p(t, x, y) := (2\pi t)^{-d/2} \exp[-|x - y|^2/2t] \\ t > 0, \quad x \in \mathbb{R}^d.$$

Let $X = (X_t)_{t \geq 0}$ be a d -dimensional continuous process such that for every $0 < t_1 < \dots < t_m$ and $E_i \in \mathcal{B}(\mathbb{R}^d), i = 1, 2, \dots, m$,

$$(2.61) \quad P\{X_{t_1} \in E_1, X_{t_2} \in E_2, \dots, X_{t_m} \in E_m\} \\ = \int_{\mathbb{R}^d} \mu(dx) \int_{E_1} p(t_1, x_1 - x, 0) dx_1 \int_{E_2} p(t_2 - t_1, x_2 - x_1, 0) dx_2 \dots \\ \int_{E_m} p(t_m - t_{m-1}, x_m - x_{m-1}, 0) dx_m.$$

Here, μ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Let $\mu = \delta_0$, a process $X = (X_t)$ with the above finite dimensional distribution is a d -dimensional (standard) Brownian motion (or Wiener process) with the initial distribution δ_0 . The probability law P^X on $(\Theta_0, \mathcal{B}(\Theta_0))$ is called the d -dimensional Wiener measure.

Theorem 6.2.3 Let $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be C^∞ -functions such that $\frac{\partial \sigma}{\partial x_j}$ and $\frac{\partial b}{\partial x_j}$ are C_b^∞ -functions for all $1 \leq j \leq d$. Let

$$\hat{G}^d = 1/2 \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i},$$

where $a = \sigma \sigma^*$. Then there is a unique transition probability function $P(t, x, \cdot)$ on $\mathbb{R}^d$ such that

$$\int f(y) P(t, x, dy) - f(x) = \int_0^t ds \int \hat{G}^d f(y) P(s, x, dy)$$

for all $f \in C_b^2(\mathbb{R}^d)$. Furthermore, if $x \in \mathbb{R}^d$ and $y(\cdot, x)$ is given by :

$$y(t, x) = x + \int_0^t \sigma(y(s, x)) d\theta(s) + \int_0^t b(y(s, x)) ds, \quad t \geq 0,$$

where $\theta(\cdot)$ is the canonical process on Θ_0 , then $P(t, x, \cdot)$ is the distribution of $y(t, x)$ under Wiener measure P^X .

Assume that $\sigma \sigma^* \geq \varrho I$ for some $\varrho > 0$. Then for each $t > 0$, $P(t, x, \cdot)$ admits a $C_b^\infty(\mathbb{R}^d)$ density $p(t, x, y)$.

6.2.2 Existence of Density Process

In this subsection, we will use the duality method to establish the existence of the density process. To this end, we need to recall the construction of the dual process given in Chapter 5. Let $\mathbf{M} = \{M(t) : t \geq 0\}$ be a jump Markov process on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with state space $\mathbb{N}$ and transition intensities $q_{m, m-1} = \sigma m(m-1)$, $q_{i,j} = 0$ for all other $|i-j| \neq 1$. Let $\{\tau_k\}$ be the sequence of jump times of $\mathbf{M}$ (take $\tau_0 = 0$) and $\{\Gamma_k\}$ be a sequence of random operators which are conditionally independent given $\mathbf{M}$ and satisfy

$$(2.62) \quad \hat{\mathbb{P}}(\Gamma_k = \Phi_{ij} | M) = \frac{1}{m(m-1)} \mathbf{1}_{\{M(\tau_k-) = m, M(\tau_k) = m-1\}}$$

for $1 \leq i, j \leq m$ and $j \neq i$. Then for the generator $\mathcal{L}$, the dual process is given by

$$(2.63) \quad Y(t) = T_{M(\tau_k)}(t - \tau_k) \Gamma_k T_{M(\tau_{k-1})}(\tau_k - \tau_{k-1}) \Gamma_{k-1} \cdots \Gamma_1 T_{M(0)}(\tau_1) Y(0)$$

$$\tau_k \leq t < \tau_{k+1} \quad k = 0, 1, 2, \dots,$$

where $Y(0) \in D(\mathbb{R}^{M(0)}) = C^\infty(\mathbb{R}^{M(0)})$.

For any $f \in D(\mathbb{R}^m)$, take $Y(0) = f$, $M(0) = m$, let $\{\mu_t\}$ be a solution to $(\mathcal{L}, \delta_\mu)$ -MP defined on $(\Omega, \mathcal{F}, \mathcal{F}_t^{\mu}, \mathcal{P})$. In Chapter 5, we have proved that the following duality identity

$$(2.64) \quad \mathbb{E}_\mu[\langle f, \mu_t^m \rangle] = \mathbb{E}[\langle Y(t), \mu^{M(t)} \rangle \exp\{\sigma \int_0^t M(u)(M(u) - 1) du\}]$$

holds for any solution $\{\mu_t\}$ to $(\mathcal{L}, \delta_\mu)$ -MP. Now we turn to the central topic of this subsection.

Theorem 6.2.4 *Suppose $c \neq 0$ and $g(\cdot) \not\equiv 0$ is a square-integrable, smooth, and symmetric function in the definition of $\mathcal{L}$. Let μ be a finite measure on $\mathbb{R}$ with compact support and $\mathbb{P}_\mu$ denote the solution to the $(\mathcal{L}, \delta_\mu)$ -MP with canonical process μ_t on the space $D([0, \infty), M_F(\mathbb{R}))$. Then μ_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ for almost all $t > 0$, $\mathbb{P}_\mu$ -almost surely.*

Proof: Let

$$p_h(x, y) \equiv p(h, x, y) := (2\pi h)^{-1/2} \exp[-(x - y)^2/2h] \\ h > 0, \quad x, y \in \mathbb{R}^1.$$

First we claim that for every $T > 0$,

$$(2.65) \quad \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu[\langle \mu_t, p_h(x, \cdot) \rangle^2] dx dt < \infty$$

and

$$(2.66) \quad \lim_{h \downarrow 0} \lim_{h \uparrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu[\langle \mu_t, p_h(x, \cdot) \rangle - \langle \mu_t, p_h(x, \cdot) \rangle^2] dx dt = 0.$$

Once the claim is established, by the Theorem 6.2.2, we can find a subsequence of h if necessary such that there exists a jointly measurable function

$$\ell_t(x, \omega) : (0, \infty) \times \mathbb{R} \times \Omega \rightarrow [0, \infty)$$

satisfying

$$(2.67) \quad \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(\ell_t(x))^2 dx dt < \infty,$$

for every $T > 0$, and

$$(2.68) \quad \lim_{h \downarrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(\langle \mu_t, p_h(x, \cdot) \rangle - \ell_t(x))^2 dx dt = 0$$

for every $T > 0$. Moreover, for every $\phi \in C_c^\infty(\mathbb{R})$

$$\begin{aligned} (2.69) \quad & \mathbb{E}_\mu |\langle \mu_t, \phi \rangle - \int \ell_t(x) \phi(x) dx|^2 \\ & \leq \lim_{h \downarrow 0} \mathbb{E}_\mu |\langle \mu_t, \phi \rangle - \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx|^2 \\ & + \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx - \int \ell_t(x) \phi(x) dx \right|^2 \\ & = \lim_{h \downarrow 0} \mathbb{E}_\mu |\langle \mu_t, \phi \rangle - \int_{\mathbb{R}} [\int_{\mathbb{R}} p_h(x, y) \phi(x) dx] \mu_t(dy)|^2 \\ & + \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx - \int \ell_t(x) \phi(x) dx \right|^2 \\ & = \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx - \int \ell_t(x) \phi(x) dx \right|^2 \\ & \leq \lim_{h \downarrow 0} \int_{\mathbb{R}} \mathbb{E}_\mu |\langle \mu_t, p_h(x, \cdot) \rangle - \ell_t(x)|^2 dx \int_{\mathbb{R}} \phi^2(x) dx. \end{aligned}$$

By (2.68), we conclude that $\mathbb{P}_\mu(\mu_t(dx))$ is absolutely continuous with respect to dx , whose density coincides with $\ell_t(x)$ for almost all $t > 0$ (c.f. [1]). Now let us turn to

prove our claim. First let us consider (2.65). By dual identity, we have

$$\begin{aligned}
(2.70) \quad & \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu \langle \mu_t, p_h(x, \cdot) \rangle^2 dx dt \\
&= \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu \left[\int_{\mathbb{R}} \int_{\mathbb{R}} p_h(x, y_1) p_h(x, y_2) \mu_t(dy_1) \mu_t(dy_2) \right] dx dt \\
&= \int_0^T \int_{\mathbb{R}} \mathbb{E}[\langle Y(t), \mu^{M(t)} \rangle \exp\{\sigma \int_0^t M(u)(M(u) - 1) du\}] dx dt \\
&= \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t < \tau_1)} \left[\langle \mathbb{E}^{(x_1, x_2)} \frac{1}{2\pi h} \exp\left\{-\frac{(x - X_t^1)^2}{2h} \right. \right. \\
&\quad \left. \left. - \frac{(x - X_t^2)^2}{2h}\right\}, \mu^{M(t)}(dx_1, dx_2) \rangle \exp\{\sigma \int_0^t M(u)(M(u) \right. \\
&\quad \left. - 1) du\} \right] dx dt + \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t \geq \tau_1)} \left[\langle \mathbb{E}^{(x_1)} \mathbb{E}^{(X_{t-\tau_1}^1, X_{t-\tau_1}^2)} \frac{1}{2\pi h} \right. \\
&\quad \left. \exp\left\{-\frac{(x - X_{\tau_1}^1)^2}{2h} - \frac{(x - X_{\tau_1}^2)^2}{2h}\right\}, \mu(dx_1) \rangle \right] dx dt \\
&= \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t < \tau_1)} \left[\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi h} \exp\left\{-\frac{(x - y_1)^2}{2h} - \frac{(x - y_2)^2}{2h}\right\} \right. \\
&\quad \left. f_t^{(x_1, x_2)}(y_1, y_2) dy_1 dy_2, \mu^{M(t)}(dx_1, dx_2) \rangle \exp\{\sigma 2t\} \right] dx dt \\
&+ \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t \geq \tau_1)} \left[\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi h} \exp\left\{-\frac{(x - y_1)^2}{2h} - \frac{(x - y_2)^2}{2h}\right\} \right. \\
&\quad \left. f_{\tau_1}^{(z, z)}(y_1, y_2) dy_1 dy_2 f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \rangle \right] dx dt \\
&\rightarrow \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t < \tau_1)} \left[\langle f_t^{(x_1, x_2)}(x, x), \mu(dx_1) \mu(dx_2) \rangle \exp\{\sigma 2t\} \right] dx dt \\
&+ \int_0^T \int_{\mathbb{R}} \mathbb{E} 1_{(t \geq \tau_1)} \left[\langle f_{\tau_1}^{(z, z)}(x, x) f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \rangle \right] dx dt \\
&\quad \text{as } h \rightarrow 0
\end{aligned}$$

$$\begin{aligned}
&= < \int_0^T \int_{\mathbb{R}^1} f_t^{(x_1, x_2)}(x, x) dx dt, \mu(dx_1) \mu(dx_2) > \\
&+ \int_0^T [< \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} 2\sigma \exp\{-2\sigma s\} f_s^{(z, z)}(x, x) dx f_t^{(x_1)}(z) ds dz, \mu(dx_1) >] dt,
\end{aligned}$$

where $\tilde{\mathbb{P}}^{(x_1, x_2)}$ ($\tilde{\mathbb{P}}^{(x_1)}$) is the solution to $(G^2, \delta_{(x_1, x_2)})$ (MP) ($(G^1, \delta_{(x_1)})$ (MP)) with canonical processes $\{X_t^1, X_t^2\}$ ($\{X_t^1\}$) on $C([0, \infty), \mathbb{R}^2)$ ($C([0, \infty), \mathbb{R}^1)$), $f_s^{(x_1, x_2)}(\cdot, \cdot)$ ($f_s^{(x_1)}(\cdot)$) is the transition density corresponding to $\tilde{\mathbb{P}}^{(x_1, x_2)}$ ($\tilde{\mathbb{P}}^{(x_1)}$), and

$$G^N := \frac{1}{2} \sum_{i,j=1}^N \rho_i(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Generally, for a finitely dimensional, uniformly elliptic operator of second order, by Theorem 6.2.3 we only know that its transition density function is a bounded smooth function for each $t > 0$. But for above special operator G^N we can find that

$$\begin{aligned}
(2.71) \quad &f_t^{(x_1, x_2)}(y_1, y_2) \\
&= \frac{1}{2\Pi t \sqrt{\det A(x_1, x_2)}} \exp\left\{-\frac{1}{2t} \sum_{i,j=1}^2 A^{(i,j)}(x_1, x_2)(y_i - x_i)(y_j - x_j)\right\}, \\
&f_t^{(x_1)}(y_1) = \frac{1}{\sqrt{2\Pi t \rho_i}} \exp\left\{-\frac{1}{2t\rho_i}(y_1 - x_1)^2\right\},
\end{aligned}$$

where $\rho_\epsilon = \rho(0) + \epsilon^2$.

$$A(x_1, x_2) = \begin{pmatrix} \rho_i & \rho(x_1 - x_2) \\ \rho(x_1 - x_2) & \rho_i \end{pmatrix},$$

and $A^{(i,j)}(x_1, x_2)$, the element of the inverse matrix of $A(x_1, x_2)$, is given by

$$A^{(i,j)}(x_1, x_2) = \begin{cases} \frac{\rho_i}{\rho_i^2 - \rho^2(x_1 - x_2)} & \text{if } i = j \\ \frac{-\rho(x_1 - x_2)}{\rho_i^2 - \rho^2(x_1 - x_2)} & \text{if } i \neq j. \end{cases}$$

Therefore, (2.71) yields that

$$\begin{aligned} & \int_{\mathbb{R}} f_t^{(x_1, x_2)}(y, y) dy \\ &= \int_{\mathbb{R}} \frac{1}{2\sqrt{11t} \sqrt{\rho_t^2 - \rho^2(x_1 - x_2)}} \exp\left\{-\frac{1}{2t} \left[\frac{\rho_t}{\rho_t^2 - \rho^2(x_1 - x_2)} (y - x_1)^2 \right. \right. \\ & \quad \left. \left. - 2 \frac{\rho(x_1 - x_2)}{\rho_t^2 - \rho^2(x_1 - x_2)} (y - x_1)(y - x_2) + \frac{\rho_t}{\rho_t^2 - \rho^2(x_1 - x_2)} (y - x_2)^2 \right] \right\} dy \\ &= \frac{1}{2\sqrt{11t(\rho(0) + \epsilon^2 - \rho(x_1 - x_2))}} \exp\left\{-\frac{\rho(x_1 - x_2)}{4t[\rho_t^2 - \rho^2(x_1 - x_2)]} (x_1 - x_2)^2\right\}. \end{aligned}$$

This proves that the supremum of the left hand side of (2.70) is finite.

To prove (2.66), it suffices to show that

$$(2.72) \quad \lim_{h \downarrow 0} \lim_{h \downarrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(< \mu_t, p_h(x, \cdot) > < \mu_t, p_h(x, \cdot) >) dx dt = N_T$$

and N_T is independent of the ways of h and h approaching to zero. Recall τ_1 is the first jump time of $\mathbf{M}$. Similarly to the (2.70), we can get

$$\begin{aligned} (2.73) & \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(< \mu_t, p_h(x, \cdot) > < \mu_t, p_h(x, \cdot) >) dx dt \\ &= \int_0^T \int_{\mathbb{R}} \mathbb{E}[1_{(t < \tau_1)}] < \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi(hh)^{1/2}} \exp\left\{-\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2h}\right\} \\ & \quad f_t^{(x-x_2)}(y_1, y_2) dy_1 dy_2, \mu^{M(t)}(dx_1, dx_2) > \exp\{\sigma 2t\} \bigg] dx dt \\ &+ \int_0^T \int_{\mathbb{R}} \mathbb{E}[1_{(t \geq \tau_1)}] < \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi(hh)^{1/2}} \exp\left\{-\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2h}\right\} \\ & \quad f_{\tau_1}^{(z-x_2)}(y_1, y_2) dy_1 dy_2 f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) > \bigg] dx dt \end{aligned}$$

$$\begin{aligned} & \rightarrow \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} 1_{(t < \tau_1)} [< f_t^{(x_1, x_2)}(x, x), \mu(dx_1) \mu(dx_2) > \\ & \exp\{\sigma 2t\}] dx dt \\ & + \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} 1_{(t \geq \tau_1)} [< \int_{\mathbb{R}} f_{\tau_1}^{(z, z)}(x, x) f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) >] dx dt \end{aligned}$$

and by (2.71), clearly the above limit is finite and independent of the ways of h and $\hbar$ converging to zero. Therefore, (2.66) holds, as desired. ■

Remark . In the degenerate case ($\epsilon = 0$), if two particles have already met, then they never separate. So

$$\begin{aligned} (2.74) \quad & \hat{\mathbb{E}}^{(x_1, x_1)} \frac{1}{2\pi h} \exp\left\{-\frac{(x - X_t^1)^2}{2h} - \frac{(x - X_t^2)^2}{2h}\right\} \\ & = \hat{\mathbb{E}}^{(x_1, x_1)} \frac{1}{2\pi h} \exp\left\{-2\frac{(x - X_t^1)^2}{2h}\right\} \\ & = \int_{\mathbb{R}} \frac{1}{2\pi h} \exp\left\{-\frac{(x - y)^2}{h}\right\} f_t^{(x_1)}(y) dy, \quad x \in \mathbb{R}. \end{aligned}$$

Suppose $f_t^{(x_1)}(y_0) > 0$ (Note $\sup_{y \in \mathbb{R}} f_t^{(x_1)}(\cdot) \neq 0$, so such y_0 exists). Since

$$(2.75) \quad \lim_{h \downarrow 0} \int_{\mathbb{R}} \frac{1}{(\pi h)^{1/2}} \exp\left\{-\frac{(y_0 - y)^2}{h}\right\} f_t^{(x_1)}(y) dy = f_t^{(x_1)}(y_0) > 0,$$

we have

$$\begin{aligned} (2.76) \quad & \lim_{h \downarrow 0} \int_{\mathbb{R}} \frac{1}{2\pi h} \exp\left\{-\frac{(x - y)^2}{h}\right\} f_t^{(x_1)}(y) dy \\ & = \lim_{h \downarrow 0} \frac{1}{2(\pi h)^{1/2}} \int_{\mathbb{R}} \frac{1}{(\pi h)^{1/2}} \exp\left\{-\frac{(x - y)^2}{h}\right\} f_t^{(x_1)}(y) dy = \infty. \end{aligned}$$

This suggests that in degenerate case, it is impossible to prove that $\mu_t(dx)$ is absolutely continuous with respect to dx — the result which we proved above.

Chapter 7

Associated Stochastic Partial Differential Equations

7.1 Introduction and Main Results

With the above preparation, in this chapter we intend to derive a class of stochastic partial differential equations (SPDE) for the interacting measure-valued branching diffusions governed by $\mathcal{L}$. As an introduction to our work, we would like to briefly review a beautiful result given by Konno and Shiga [41]. In [41], they proved that the $(\alpha, 1, 1)$ -superprocess has a density which, in an appropriate sense, satisfies the stochastic partial differential equation

$$(1.1) \quad dX_t(x) = \Delta_\alpha X_t(x) dt + \sqrt{X_t(x)} W(dx, dt),$$

where $W(\cdot, \cdot)$ is a cylindrical Brownian motion, Δ_α is the fractional power of the Laplace operator.

Their derivation of the stochastic partial differential equation (1.1) is essentially based on independence of particles' motions and a representation theorem for orthogonal martingale measure. In our case, the particles' motions are not independent. Moreover, we will see that actually in our stochastic partial differential equation,

there are two martingale measure terms which are orthogonal and our stochastic partial differential equation cannot be directly derived by a representation theorem of orthogonal martingale measure. In fact, we require a decomposition theorem of orthogonal martingale measure. Our approach is motivated by Walsh's work [61]. Here the notation coincides with that given in Section 1.2 of Chapter 1 if not specified. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of C^∞ -functions on $\mathbb{R}$ which are rapidly decreasing at infinity, $\mathcal{S}'(\mathbb{R})$ be the topological dual; namely, the space of Schwartz distributions. We have the following main result.

Theorem 7.1.1 *Assume $\rho(\cdot) \not\equiv 0$ is a smooth function and $c \neq 0$. Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu_t)$ be the interacting measure-valued branching diffusions governed by $\mathcal{L}$ and $\ell_t(x)$ be their density processes constructed in Chapter 6. Then there exist two orthogonal $\mathcal{S}'(\mathbb{R})$ -valued cylindrical Brownian motions V_t and W_t defined on an extended probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}_\mu)$ of the original space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ such that for every $\phi \in \mathcal{S}(\mathbb{R})$,*

$$(1.2) \quad \begin{aligned} \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle &= \frac{1}{2} \rho_c \int_0^t \langle \mu_s, \Delta \phi \rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \sqrt{\gamma(m_2 - 1)\ell_s(x)} \phi(x) V(dx, ds) \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle g(y - x) \nabla \phi(x), \mu_s(dx) \rangle W(dy, ds) \end{aligned}$$

holds for every $t \geq 0$, $\bar{\mathbb{P}}_\mu$ -almost surely.

7.2 SPDE for Empirical Measure Processes

The proof of Theorem 7.1.1 will be carried out by a series of lemmas. First, we derive stochastic partial differential equations for the empirical measure processes which correspond to the branching particle systems in a Brownian medium formulated in Section 1.2 of Chapter 1. Then we will discuss the tightness, L^1 and L^2 convergence

of several constructed sequences and the orthogonality of two martingale measures. Finally, we get two orthogonal cylindrical Brownian motions and the desired SPDE is derived from the SPDE for the empirical measure processes by taking limits. Now let us begin with our model. Recall $\rho_\epsilon = \rho(0) + \epsilon^2$ and our particle system has form

$$(2.3) \quad dx_s^\alpha = \int_{\mathbb{R}} g(y - x_s^\alpha) W(dy, ds) + \epsilon dB_s^\alpha,$$

where B_s^α are independent standard Brownian motions and $W(dy, ds)$ is a cylindrical Brownian motion which is independent of the B^α 's. By Itô's formula, for any $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$(2.4) \quad \begin{aligned} \phi(x_t^\alpha) - \phi(x_0^\alpha) &= \int_0^t \int_{\mathbb{R}} \nabla \phi(x_s^\alpha) g(y - x_s^\alpha) W(dy, ds) \\ &\quad + \epsilon \int_0^t \nabla \phi(x_s^\alpha) dB_s^\alpha + \frac{\epsilon^2}{2} \int_0^t \Delta \phi(x_s^\alpha) ds. \end{aligned}$$

Recall that our assumption on the particle systems is as follows: for any positive integer n , there is a system of particles initially with m_0^n particles, each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$. The initial positions of the particles are formulated in Section 2 of Chapter 4. Also we assume that the offspring distribution satisfies

$$\sum_{k=0}^{\infty} k p_k = 1, \quad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty,$$

and $m_0^n/\theta^n \leq \xi$, where $\xi > 0$ is a fixed constant.

Let $\mathfrak{R}$ be the set of all multi-indices, i.e. of strings of the form $\alpha = n_1 n_2 \cdots n_k$ where n_i are non-negative integers. Let $|\alpha|$ be the length of α . We provide $\mathfrak{R}$ with the arboreal ordering: $m_1 m_2 \cdots m_p \prec n_1 n_2 \cdots n_q$ iff $p \leq q$ and $m_1 = n_1, \dots, m_p = n_p$. if $|\alpha| = p$, then α has exactly $p - 1$ predecessors, which we shall denote respectively by $\alpha - 1, \alpha - 2, \dots, \alpha - |\alpha| + 1$. That is, if $\alpha = 6879$, then $\alpha - 1 = 687, \alpha - 2 =$

68, $\alpha - 3 = 6$. Define three families

$$\{B_t^\alpha, t \geq 0, \alpha \in \mathfrak{R}\}, \{S_n^\alpha(t), t \geq 0, \alpha \in \mathfrak{R}\}, \text{ and } \{N^\alpha, \alpha \in \mathfrak{R}\},$$

where the B_t^α are independent standard Brownian motions in $\mathbb{R}$ with $B_0^\alpha = 0$ for all α ; the S_n^α are i.i.d. exponential random variables with parameter $\gamma\theta^n$, which serve as lifetimes; and the N^α are i.i.d. random variable with

$$\mathbb{P}(N^\alpha = k) = p_k, \text{ for } k = 0, 1, 2, \dots \text{ and } p_1 = 0.$$

The families, $\{B^\alpha\}, \{S_n^\alpha\}, \{N^\alpha\}$, are independent.

The birth time $\beta(\alpha)$ of x^α is

$$\beta(\alpha) = \begin{cases} \sum_{i=1}^{|\alpha|-1} S_n^{\alpha-j} & \text{if } N^{\alpha-j} \geq 1, j = 1, \dots, |\alpha| - 1 \\ \infty & \text{otherwise.} \end{cases}$$

The death time $\zeta(\alpha)$ of x^α is

$$\zeta(\alpha) = \beta(\alpha) + S_n^\alpha.$$

Define $h^\alpha(t) = 1_{\{\beta(\alpha) \leq t < \zeta(\alpha)\}}$, which is the indicator function of the lifespan of x^α .

Let ∂ -the cemetery- be a point disjoint from $\mathbb{R}$, and put

$$x_t^\alpha = \partial \quad \text{if } t < \beta(\alpha) \text{ or } t \geq \zeta(\alpha).$$

We make the convention that any function f on $\mathbb{R}$ is extended to $\mathbb{R} \cup \{\partial\}$ by $f(\partial) = 0$.

Definition 7.2.1 Assume $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$. Let W and $\{B^\alpha\}$ be a cylindrical Brownian motion and a collection of standard one-dimensional Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

respectively. Suppose W and $\{B^i\}$ are independent of each other. Consider the following system of stochastic integral equations

$$(2.5) \quad y_i(t) - y_i(0) = \int_0^t \int_{\mathbb{R}} g(y - y_i(s)) W(dy, ds) + \epsilon B_t^i \text{ for } i = 1, \dots, n.$$

A continuous process $(y_1(t), \dots, y_n(t))$ which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a strong solution of (2.5) if (2.5) holds almost surely with respect to probability $\mathbb{P}$.

Lemma 7.2.1 Assume $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\mathbb{R}$. Let W and $\{B^i\}$, be a cylindrical Brownian motion and a collection of standard one-dimensional Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respectively. Suppose W and $\{B^i\}$, are independent of each other. Then the following system of stochastic integral equations

$$(2.6) \quad y_i(t) - y_i(0) = \int_0^t \int_{\mathbb{R}} g(y - y_i(s)) W(dy, ds) + \epsilon B_t^i \text{ for } i = 1, \dots, n$$

has a unique strong solution.

Proof: We prove the existence and uniqueness by using the method of successive approximation as follows.

We define a sequence $y^k(t) = \{(y_1^k(t), \dots, y_n^k(t))\}$ of n -dimensional continuous processes by

$$(2.7) \quad \begin{cases} y^0(t) = y(0) \\ y^k(t) = y(0) + (\int_0^t \int_{\mathbb{R}} g(y - y^{k-1}(s)) W(dy, ds), \dots, \\ \quad \dots, \int_0^t \int_{\mathbb{R}} g(y - y_n^{k-1}(s)) W(dy, ds)) + \epsilon (B_t^1, \dots, B_t^n), \end{cases}$$

where $y(0) = (y_1(0), \dots, y_n(0)) \in \mathbb{R}^n$ is fixed. Since

$$(2.8) \quad \begin{aligned} & \mathbb{E}\{\sup_{0 \leq t \leq T} |y^{k+1}(t) - y^k(t)|^2\} \\ &= \mathbb{E}\{\sup_{0 \leq t \leq T} \sum_{i=1}^n [\int_0^t \int_{\mathbb{R}} g(y - y_i^k(s)) W(dy, ds)]^2\} \end{aligned}$$

$$\begin{aligned}
& - \int_0^t \int_{\mathbb{R}} g(y - y_i^{k-1}(s)) W(dy, ds)]^2 \} \\
& \leq \mathbb{E} \sum_{i=1}^n \left[\int_0^T \int_{\mathbb{R}} (g(y - y_i^k(s)) - g(y - y_i^{k-1}(s))) W(dy, ds) \right]^2 \\
& = \mathbb{E} \sum_{i=1}^n \int_0^T \int_{\mathbb{R}} (g(y - y_i^k(s)) - g(y - y_i^{k-1}(s)))^2 dy ds \\
& = \sum_{i=1}^n \mathbb{E} \int_0^T 2[\rho(0) - \rho(y_i^k(s) - y_i^{k-1}(s))] ds \\
& \leq \sum_{i=1}^n \rho''(0) \mathbb{E} \int_0^T [y_i^k(s) - y_i^{k-1}(s)]^2 ds, \\
& = \rho''(0) \mathbb{E} \int_0^T |y^k(s) - y^{k-1}(s)|^2 ds,
\end{aligned}$$

we have

$$\begin{aligned}
(2.9) \quad & \mathbb{E}[\sup_{0 \leq s \leq T} |y^{k+1}(s) - y^k(s)|^2] \\
& \leq (\rho''(0))^k \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \mathbb{E}(|y^1(t_k) - y^0(t_k)|^2) \\
& \leq (\rho''(0))^k \frac{T^k}{k!} n \rho(0) T.
\end{aligned}$$

Consequently,

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq T} |y^{k+1}(s) - y^k(s)| > \frac{1}{2^k} \right\} \leq \text{const} (2\rho''(0))^k \frac{T^k}{k!}.$$

By the Borel-Cantelli lemma, we see that $y^k(s)$ converges uniformly on $[0, T]$ almost surely with respect to $\mathbb{P}$. Since T is arbitrary, $\lim_{k \rightarrow \infty} y^k(s) = y(s)$ determines a n -dimensional continuous process which is clearly a strong solution of (2.6). The uniqueness is obvious from the above argument. ■

Since the strong solution of (2.6) only depends on the initial state $y(0)$, B^* , and a

common W , we denote it by $x(y_i(0), B^i, t)$. Suppose μ_0 is a nonzero finite measure on $\mathbb{R}$ with compact support and $\mu_n = \frac{1}{\theta^n} \sum_{i=1}^{m_0^n} \delta_{x_i(0)} \Rightarrow \mu_0$ as $n \rightarrow \infty$. Let $N_1^n := \{1, 2, \dots, m_0^n\}$ and any $\alpha \in N_1^n \cap \mathfrak{R}$, define

$$(2.10) \quad x^\alpha(t) = \begin{cases} x(x_\alpha(0), B^\alpha, t) & 0 \leq t < S^\alpha \\ \partial & S^\alpha \leq t \end{cases}$$

and

$$x^\alpha(t) \equiv \partial \quad \text{for any } \alpha \in (\mathbb{N} - N_1^n) \cap \mathfrak{R} \text{ and } t \geq 0.$$

Let

$$\zeta_1 = \min\{S^\alpha : \alpha \in N_1^n \cap \mathfrak{R}\}.$$

By Ikeda-Nagasawa-Watanabe [35], given $\omega \in \Omega$ there exists $\alpha_0 \in N_1^n \cap \mathfrak{R}$ such that $\zeta_1(\omega) = S^{\alpha_0}(\omega)$. If $N^{\alpha_0}(\omega) = k \geq 2$, define

$$(2.11) \quad x^\alpha(t) = \begin{cases} x(x^{\alpha_0}(\zeta(\alpha_0)-), B^\alpha, t) & \beta(\alpha) \leq t < \zeta(\alpha) \\ \partial & \text{otherwise} \end{cases}$$

for $\alpha \in \{\alpha_0 i; i = 1, 2, \dots, k\}$. If $N^{\alpha_0}(\omega) = 0$, define

$$x^\alpha(t) = \partial \quad \text{for } 0 \leq t < \infty \quad \text{and} \quad \alpha \in \{\alpha_0 i : i \geq 1\}.$$

Suppose there have been m splits. Let $N_m^n \subset \mathfrak{R}$ be the all indices of the living particles and let

$$\zeta_{m+1} = \min\{S^\alpha : \alpha \in N_m^n\}.$$

Then for above given ω , as before there exists $\beta_0 \in N_m^n$ such that $\bar{\zeta}_{m+1}(\omega) = S^{\beta_0}(\omega)$.

If $N^{\beta_0}(\omega) = k \geq 2$, define

$$(2.12) \quad x^\alpha(t) = \begin{cases} x(x^{\beta_0}(\zeta(\beta_0)-), B^\alpha, t) & \beta(\alpha) \leq t < \zeta(\alpha) \\ \partial & \text{otherwise} \end{cases}$$

for $\alpha \in \{\beta_0 i; i = 1, 2, \dots, k\}$. If $N^{\beta_0}(\omega) = 0$, define

$$x^\alpha(t) = \partial \quad \text{for } 0 \leq t < \infty \quad \text{and} \quad \alpha \in \{\beta_0 i : i \geq 1\}.$$

Continuing in this way, we get a branching tree of particles for any given ω and initial state being one of the $\{x_1(0), \dots, x_{m_0}(0)\}$. Clearly this particle system is exchangeable. Define

$$(2.13) \quad \mu_t^n := \frac{1}{\theta^n} \sum_{\alpha \in \mathfrak{X}} \delta_{x_t^\alpha}.$$

For any $A \in \mathcal{B}(\mathbb{R})$, define

$$(2.14) \quad Z^n(A \times (0, t]) := \sum_{\alpha \in \mathfrak{X}} \frac{(N^\alpha - 1)}{\theta^n} 1_{\{x^\alpha(\zeta(\alpha)-) \in A, \zeta(\alpha) \leq t\}}.$$

By Lemma 4.2.1, μ_t^n is a right continuous strong Markov process and satisfies

$$(2.15) \quad \begin{aligned} \mu_t^n(\phi) - \mu_0^n(\phi) &= \int_0^t \int_{\mathbb{R}} \langle g(y - \cdot) \nabla \phi(\cdot), \mu_s^n \rangle W(dy, ds) \\ &\quad + \frac{\rho_\epsilon}{2} \int_0^t \langle \Delta \phi, \mu_s^n \rangle ds + \frac{\epsilon}{\theta^n} \sum_{\alpha \in \mathfrak{X}} \int_0^t h^\alpha(s) \nabla \phi(x_s^\alpha) dB_s^\alpha \\ &\quad + \frac{1}{\theta^n} \left\{ - \sum_{|\alpha|=1, \alpha \in \mathfrak{X}} \phi(x_{\zeta(\alpha)-}^\alpha) 1_{\{\zeta(\alpha) \leq t\}} \right. \\ &\quad \left. + \sum_{|\alpha| \geq 2, \alpha \in \mathfrak{X}} [\phi(x_{\beta(\alpha)}^\alpha) 1_{\{\beta(\alpha) \leq t\}} - \phi(x_{\zeta(\alpha)-}^\alpha) 1_{\{\zeta(\alpha) \leq t\}}] \right\} \\ &= \int_0^t \int_{\mathbb{R}} \langle g(y - \cdot) \nabla \phi(\cdot), \mu_s^n \rangle W(dy, ds) \\ &\quad + \frac{\rho_\epsilon}{2} \int_0^t \langle \Delta \phi, \mu_s^n \rangle ds + \frac{\epsilon}{\theta^n} \sum_{\alpha \in \mathfrak{X}} \int_0^t h^\alpha(s) \nabla \phi(x_s^\alpha) dB_s^\alpha \\ &\quad + \int_0^t \int_{\mathbb{R}} \phi(x) Z^n(dx, ds). \end{aligned}$$

So we have proved the following theorem.

Theorem 7.2.2 *$\forall n \in \mathbb{N}$, μ_t^n defined by (2.13) is a right continuous strong Markov process which is a unique strong solution of (2.15) in the sense that it is a unique*

solution of (2.15) for fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and given $W, \{B^\alpha\}, \{S_n^\alpha\}, \{N^n\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, all the $\{\mu_t^n; t \geq 0\}$ are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

7.3 Weak Convergence and Decomposition of Martingale Measure

For any $\phi \in \mathcal{S}(\mathbb{R})$, let

$$\begin{aligned} Z_t^n(\phi) &:= \int_0^t \int_{\mathbb{R}} \phi(x) Z^n(dx, ds), \\ X_t^n(\phi) &:= \int_0^t \int_{\mathbb{R}} \langle g(y - \cdot) \nabla \phi(\cdot), \mu_s^n \rangle W(dy, ds), \\ U_t^n(\phi) &:= \frac{\epsilon}{\sqrt{\theta^n}} \sum_{\alpha \in \mathbb{R}} \int_0^t h^\alpha(s) \nabla \phi(x_s^\alpha) dB_s^\alpha, \\ Y_t^n(\phi) &:= \frac{\rho \epsilon}{2} \int_0^t \langle \Delta \phi, \mu_s^n \rangle ds. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{R}_t^n &:= \sigma(\mu_s^n(\phi), Z_s^n(\phi), U_s^n(\phi), X_s^n(\phi), Y_s^n(\phi); 0 \leq s \leq t, \phi \in \mathcal{S}(\mathbb{R})), \\ \mathcal{F}_t^n &:= \mathcal{R}_t^n \vee \sigma\left(\int_0^u h(\mu_s^n(\phi)) ds, \int_0^u h(Z_s^n(\phi)) ds, \int_0^u h(U_s^n(\phi)) ds, \right. \\ &\quad \left. \int_0^u h(X_s^n(\phi)) ds, \int_0^u h(Y_s^n(\phi)) ds; 0 \leq u \leq t, \phi \in \mathcal{S}(\mathbb{R}), h \in B(\mathbb{R})\right). \end{aligned}$$

Lemma 7.3.1 *According to above notations, we have*

(i)

$$\mathbb{E} \left\{ \int_0^t \int_{\mathbb{R}} \phi(x) Z^n(dx, ds) \right\}^2 = \gamma(m_2 - 1) \mathbb{E} \int_0^t \langle \phi^2, \mu_u^n \rangle du;$$

(ii) *For any $t \geq 0$ and $n \geq 1$, $\langle \mu_t^n, 1 \rangle \geq 0$ and for any $T \geq 0$ we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \langle \mu_t^n, 1 \rangle \leq \langle \mu_0^n, 1 \rangle + (\gamma(m_2 - 1)T \langle \mu_0^n, 1 \rangle)^{1/2},$$

$$\mathbb{E} \sup_{0 \leq t \leq T} \langle \mu_t^n, 1 \rangle^2 \leq 2 \langle \mu_0^n, 1 \rangle^2 + 2\gamma(m_2 + 1)T \langle \mu_0^n, 1 \rangle.$$

(iii) $\{\mu_t^n; t \geq 0\}$ defined by (2.13) is tight as a family of processes with sample paths in $D([0, \infty), M_F(\mathbb{R}))$.

Proof: (i). First note that S_n^α are i.i.d. exponential random variables with parameter $\gamma\theta^n$ and $h^\alpha(t) = 1_{\{\beta(\alpha) \leq t < \zeta(\alpha)\}}$, for any $A \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned} (3.16) \quad & \mathbb{E}\{1_{(s,t]}(\zeta(\alpha))1_{(x^\alpha(\zeta(\alpha)-) \in A)}\} \\ &= \int_0^t \mathbb{E}[1_{(s,t]}(\beta(\alpha) + S_n^\alpha)1_{(x^\alpha((\beta(\alpha)+S_n^\alpha)-) \in A)} | S_n^\alpha = u] \mathbb{P}(S_n^\alpha \in du) \\ &= \mathbb{E}\left\{\int_0^\infty 1_{(s,t]}(\beta(\alpha) + u)1_{(x^\alpha((\beta(\alpha)+u)-) \in A)} \gamma\theta^n e^{-\gamma\theta^n u} du\right\} \\ &= \mathbb{E}\left\{\int_0^\infty 1_{(s,t]}(\beta(\alpha) + u)1_{(x^\alpha((\beta(\alpha)+u)-) \in A)} \gamma\theta^n \mathbb{P}(S_n^\alpha > u) du\right\} \\ &= \mathbb{E}\left\{\int_0^\infty 1_{(s,t]}(\beta(\alpha) + u)1_{(x^\alpha((\beta(\alpha)+u)-) \in A)} \gamma\theta^n \mathbb{P}(\beta(\alpha) + u < \zeta(\alpha)) du\right\} \\ &= \mathbb{E}\left\{\int_{\beta(\alpha)}^t 1_{(s,t]}(v)1_{(x^\alpha(v-) \in A)} \gamma\theta^n \mathbb{P}(\zeta(\alpha) > v) dv\right\} \\ &= \mathbb{E}\left\{\int_{\beta(\alpha)}^t 1_{(s,t]}(v)1_{(x^\alpha(v-) \in A)} \gamma\theta^n 1_{(\zeta(\alpha) > v)} dv\right\} \\ &= \mathbb{E}\left\{\int_s^t 1_{[\beta(\alpha), \zeta(\alpha))}(v)1_{(x^\alpha(v) \in A)} \gamma\theta^n dv\right\} \\ &= \mathbb{E}\left\{\int_s^t h^\alpha(u)1_{(x^\alpha(u) \in A)} \gamma\theta^n du\right\}. \end{aligned}$$

So, by (3.16), we have

$$\begin{aligned} (3.17) \quad & \mathbb{E}\left\{\int_0^t \int_{\mathbb{R}} \phi(x) Z^n(dx, ds)\right\}^2 \\ &= \mathbb{E} \sum_{\alpha \in \mathfrak{R}} \frac{(N^\alpha - 1)^2}{\theta^{2n}} \phi^2(x_{\zeta(\alpha)-}^\alpha) 1_{(\zeta(\alpha) \leq t)} \end{aligned}$$

$$\begin{aligned}
&= (m_2 - 1) \frac{1}{\theta^{2n}} \mathbb{E} \sum_{\alpha \in \mathbb{R}} \phi^2(x_{\zeta(\alpha)-}^\alpha) 1_{\{\zeta(\alpha) \leq t\}} \\
&= (m_2 - 1) \frac{1}{\theta^{2n}} \mathbb{E} \sum_{\alpha \in \mathbb{R}} \int_0^t h^\alpha(u) \phi^2(x_u^\alpha) \gamma \theta^n du \\
&= \gamma(m_2 - 1) \frac{1}{\theta^n} \mathbb{E} \sum_{\alpha \in \mathbb{R}} \int_0^t h^\alpha(u) \phi^2(x_u^\alpha) du \\
&= \gamma(m_2 - 1) \mathbb{E} \int_0^t \langle \phi^2, \mu_u^n \rangle du.
\end{aligned}$$

(ii). Consider the spatial motion-free branching process $\langle \mu_t^n, 1 \rangle$. Since $\int_0^t \int_{\mathbb{R}} Z^n(dx, ds)$ and

$$\langle \mu_t^n, 1 \rangle = \langle \mu_0^n, 1 \rangle + \int_0^t \int_{\mathbb{R}} Z^n(dx, ds)$$

are martingales, by Doob's submartingale inequality we have

$$\begin{aligned}
(3.18) \quad & \mathbb{E} \sup_{0 \leq t \leq T} \langle \mu_t^n, 1 \rangle \\
& \leq \langle \mu_0^n, 1 \rangle + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{R}} Z^n(dx, ds) \right| \\
& \leq \langle \mu_0^n, 1 \rangle + \mathbb{E} \left| \int_0^T \int_{\mathbb{R}} Z^n(dx, ds) \right| \\
& \leq \langle \mu_0^n, 1 \rangle + \sqrt{\mathbb{E} \gamma(m_2 - 1) \int_0^T \langle \mu_u^n, 1 \rangle du} \\
& \leq \langle \mu_0^n, 1 \rangle + \sqrt{\gamma(m_2 - 1) T \langle \mu_0^n, 1 \rangle}.
\end{aligned}$$

Again by Doob's submartingale inequality, we have

$$\begin{aligned}
(3.19) \quad & \mathbb{E} \sup_{0 \leq t \leq T} \langle \mu_t^n, 1 \rangle^2 \\
& \leq 2 \langle \mu_0^n, 1 \rangle^2 + 2 \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{R}} Z^n(dx, ds) \right|^2 \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \langle \mu_0^n, 1 \rangle^2 + 2\mathbb{E} \left| \int_0^T \int_{\mathbb{R}} Z^n(dx, ds) \right|^2 \\
&\leq 2 \langle \mu_0^n, 1 \rangle^2 + 2\gamma(m_2 - 1)\mathbb{E} \int_0^T \langle \mu_u^n, 1 \rangle du \\
&\leq 2 \langle \mu_0^n, 1 \rangle^2 + 2\gamma(m_2 - 1)T \langle \mu_0^n, 1 \rangle.
\end{aligned}$$

(iii). By Theorem 4.6.1 and Theorem 4.5.4 of Dawson [11], we only need to prove that for any given $\epsilon > 0$, $T > 0$, $\phi \in \mathcal{S}(\mathbb{R})$, and given stopping time τ_n bounded by T , then $\forall \eta > 0$, $\exists \delta, n_0$ such that

$$(3.20) \quad \sup_{n \geq n_0} \sup_{t \in [0, \delta]} \mathbb{P}\{|\mu_{\tau_n+t}^n(\phi) - \mu_{\tau_n}^n(\phi)| > \epsilon\} \leq \eta.$$

Note that $\mu_t^n(\phi)$ is a strong Markov process. From (ii), we obtain

$$\begin{aligned}
(3.21) \quad &\mathbb{P}(|\mu_{\tau_n+t}^n(\phi) - \mu_{\tau_n}^n(\phi)| > \epsilon) \\
&= \mathbb{P}(|\mu_t^n(\phi) - \mu_0^n(\phi)| > \epsilon) \\
&\leq \frac{C}{\epsilon^2} \{\mathbb{E}X_t^n(\phi)^2 + \mathbb{E}Y_t^n(\phi)^2 + \frac{1}{\theta^n} \mathbb{E}U_t^n(\phi)^2 + \mathbb{E}Z_t^n(\phi)^2\} \\
&\leq \frac{C_1}{\epsilon^2} \{\mathbb{E} \int_0^t \langle \rho(x-y) \nabla \phi(x) \nabla \phi(y), \mu_s^n(dx) \mu_s^n(dy) \rangle ds \\
&\quad + \frac{\rho_\epsilon^2}{4} t \mathbb{E} \int_0^t \langle \Delta \phi, \mu_s^n \rangle^2 ds + \frac{1}{\theta^n} \mathbb{E} \int_0^t \langle (\nabla \phi)^2, \mu_s^n \rangle ds \\
&\quad + \gamma(m_2 - 1) \mathbb{E} \int_0^t \langle \phi^2, \mu_u^n \rangle du\} \rightarrow 0 \quad \text{as } t \rightarrow 0.
\end{aligned}$$

(3.21) imply (3.20). The proof is complete. $\blacksquare$

Lemma 7.3.2 $\{Z^n\}$, $\{U^n\}$, and $\{Y^n\}$ are tight as families of processes with sample paths in $D([0, \infty), \mathcal{S}'(\mathbb{R}))$, respectively.

Proof: Since the same idea works for each sequence, here we only give the proof of $\{Z^n\}$. By Mitoma's theorem, we only need to prove that for a given $\phi \in \mathcal{S}(\mathbb{R})$,

$Z_t^n(\phi)$ is tight in $D([0, \infty), \mathbb{R})$. Let $C = \sup_x \phi(x)^2$, by Lemma 7.3.1, we have

$$\begin{aligned}
 (3.22) \quad \mathbb{P}\{Z_t^n(\phi) > n\} &\leq \frac{1}{n^2} \mathbb{E}\{Z_t^n(\phi)\}^2 \\
 &\leq \frac{1}{n^2} \gamma(m_2 - 1) \mathbb{E} \int_0^t \langle \phi^2, \mu_u^n \rangle du \\
 &\leq \frac{1}{n^2} \gamma(m_2 - 1) C \int_0^t \mathbb{E} \langle \mu_u^n, 1 \rangle du \\
 &\leq \frac{1}{n^2} \gamma(m_2 - 1) C \langle \mu_0^n, 1 \rangle t,
 \end{aligned}$$

which yields the compact containment condition holds. Now we use Kurtz's tightness criterion (cf. Ethier-Kurtz [21] p137. Theorem 8.6) to prove the tightness of $\{Z^n(\phi)\}$.

Let $\gamma_n^T(\delta) := \delta \gamma(m_2 - 1) C \sup_{0 \leq u \leq T} \langle \mu_u^n, 1 \rangle$, then for any $0 \leq t + \delta \leq T$,

$$\begin{aligned}
 (3.23) \quad &\mathbb{E}\{|Z_{t+\delta}^n(\phi) - Z_t^n(\phi)|^2 | \mathcal{F}_t^n\} \\
 &= \mathbb{E}\{\gamma(m_2 - 1) \int_t^{t+\delta} \langle \phi^2, \mu_u^n \rangle du | \mathcal{F}_t^n\} \\
 &\leq \mathbb{E}\{\gamma_n^T(\delta) | \mathcal{F}_t^n\}.
 \end{aligned}$$

By Lemma 7.3.1,

$$\lim_{\delta \rightarrow 0} \sup_n \mathbb{E}\{\gamma_n^T(\delta)\} = 0,$$

so $\{Z^n(\phi)\}$ is tight. ■

Lemma 7.3.3 (i) (μ^n, Z^n, U^n, Y^n) is tight on $D([0, \infty), (\mathcal{S}'(\mathbb{R}))^4)$.

(ii) Suppose the joint distribution

$$\mathcal{L}(\mu^{n_k}, Z^{n_k}, U^{n_k}, Y^{n_k}, W) \Longrightarrow \mathcal{L}(\mu^0, Z^0, U^0, Y^0, W)$$

(weakly converges), then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $D([0, \infty), \mathcal{S}'(\mathbb{R}))$ -valued sequences $\{\tilde{\mu}^{n_k}\}$, $\{\tilde{Z}^{n_k}\}$, $\{\tilde{U}^{n_k}\}$, $\{\tilde{Y}^{n_k}\}$, and $\tilde{W}^{n_k}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such

that $\mathbb{P} \circ (\mu^{n_k})^{-1} = \tilde{\mathbb{P}} \circ (\tilde{\mu}^{n_k})^{-1}$, $\mathbb{P} \circ (Z^{n_k})^{-1} = \tilde{\mathbb{P}} \circ (\tilde{Z}^{n_k})^{-1}$, $\mathbb{P} \circ (U^{n_k})^{-1} = \tilde{\mathbb{P}} \circ (\tilde{U}^{n_k})^{-1}$, $\mathbb{P} \circ (Y^{n_k})^{-1} = \tilde{\mathbb{P}} \circ (\tilde{Y}^{n_k})^{-1}$, $\mathbb{P} \circ (W)^{-1} = \tilde{\mathbb{P}} \circ (\tilde{W}^{n_k})^{-1}$, $\mathbb{P} \circ (Z^{n_k}, \mu^{n_k}, W)^{-1} = \tilde{\mathbb{P}} \circ (\tilde{Z}^{n_k}, \tilde{\mu}^{n_k}, \tilde{W}^{n_k})^{-1}$, and $\tilde{\mu}^{n_k} \rightarrow \tilde{\mu}^0$, $\tilde{Z}^{n_k} \rightarrow \tilde{Z}^0$, $\tilde{U}^{n_k} \rightarrow \tilde{U}^0$, $\tilde{Y}^{n_k} \rightarrow \tilde{Y}^0$, $(\tilde{Z}^{n_k}, \tilde{\mu}^{n_k}, \tilde{W}) \rightarrow (\tilde{Z}^0, \tilde{\mu}^0, \tilde{W}^{n_k})$ almost surely with respect to $\tilde{\mathbb{P}}$ on $D([0, \infty), \mathcal{S}'(\mathbb{R}))$ and $D([0, \infty), (\mathcal{S}'(\mathbb{R}))^3)$, respectively.

(iii) Given any $\phi \in \mathcal{S}(\mathbb{R})$, then $\tilde{\mu}^{n_k}(\phi) \rightarrow \tilde{\mu}^0(\phi)$, $\tilde{Z}^{n_k}(\phi) \rightarrow \tilde{Z}^0(\phi)$, $\tilde{U}^{n_k}(\phi) \rightarrow \tilde{U}^0(\phi)$, $\tilde{Y}^{n_k}(\phi) \rightarrow \tilde{Y}^0(\phi)$, $(\tilde{Z}^{n_k}(\phi), \tilde{\mu}^{n_k}(\phi), \tilde{W}^{n_k}(\phi)) \rightarrow (\tilde{Z}^0(\phi), \tilde{\mu}^0(\phi), \tilde{W}(\phi))$ in $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as $\mathbb{R}$ -valued and $\mathbb{R}^3$ -valued processes.

(iv) $\tilde{W}(dy, ds)$ and $\tilde{W}^{n_k}(dy, ds)$ are cylindrical Brownian motions and the stochastic integrals

$$\begin{aligned} \tilde{X}_t^{n_k}(\phi) &:= \int_0^t \int_{\mathbb{R}} \langle g(y - \cdot) \nabla \phi(\cdot), \tilde{\mu}_s^{n_k} \rangle \tilde{W}^{n_k}(dy, ds) \\ &\rightarrow \tilde{X}_t^0(\phi) := \int_0^t \int_{\mathbb{R}} \langle g(y - \cdot) \nabla \phi(\cdot), \tilde{\mu}_s^0 \rangle \tilde{W}(dy, ds) \quad \text{in } L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}). \end{aligned}$$

(v) $(\tilde{\mu}^0, \tilde{Z}^0, \tilde{X}^0)$ is unique in distribution and satisfies following equation

$$\begin{aligned} (3.24) \quad &\tilde{\mu}_t^0(\phi) - \tilde{\mu}_0^0(\phi) \\ &= \tilde{X}_t^0(\phi) + \frac{\rho_t}{2} \int_0^t \langle \Delta \phi, \tilde{\mu}_s^0 \rangle ds + \int_0^t \int_{\mathbb{R}} \phi(x) \tilde{Z}^0(dx, ds). \end{aligned}$$

(vi) $\tilde{Z}^{n_k}$ is orthogonal to $\tilde{X}^{n_k}$ and $\tilde{Z}^0$ is orthogonal to $\tilde{X}^0$.

Proof: (i) By Mitoma's theorem, it is enough to prove that $(\mu^n(\phi), Z^n(\phi), U^n(\phi), Y^n(\phi))$ is tight in $D([0, \infty), \mathbb{R}^4)$. This is equivalent to proving that each component and the sum of each pair of components are individually tight in $D([0, \infty), \mathbb{R})$ (cf. Ethier-Kurtz [21]p153 Ex.22). The latter can be proved in the same way as the proof of Lemma 7.3.2.

(ii) In the following, we will use a Skorohod's Representation Theorem for $\mathcal{S}'$ -valued

processes; namely, Theorem 3.1 of Jakubowski [38] (see appendix), to reach our conclusions. To this end, we must check following condition (i.e. assumption (13) of Jakubowski [38], see appendix) .

(SC): *There exist a countable family $\{f_i : D([0, \infty), S'(\mathbb{R})) \rightarrow [-\pi, \pi]\}_{i \in \mathbb{N}}$ of continuous functions (with respect Skorohod topology on $D([0, \infty), S'(\mathbb{R}))$), which separate points of $D([0, \infty), S'(\mathbb{R}))$.*

We have following facts:

(F.1) $S(\mathbb{R})$ is separable.

(F.2) $S'(\mathbb{R})$ is a completely regular topological space. (see Mitoma [46] p.998).

Suppose $\{g_i\}_{i \in \mathbb{N}}$ is the dense subset of $S(\mathbb{R})$ and $\{t_j\}_{j \in \mathbb{N}}$ is the set of all rational numbers in $\mathbb{R}$. Define

$$f_{ij} : x \in D([0, \infty), S'(\mathbb{R})) \rightarrow [-\pi, \pi]$$

by $f_{ij}(x) = \arctan \langle g_i, x(t_j) \rangle$, where

$$\langle \cdot, \cdot \rangle : S(\mathbb{R}) \times S'(\mathbb{R}) \rightarrow \mathbb{R}.$$

Then Theorem 1.7 of Jakubowski [36] implies condition (SC). The conclusions follow from Theorem 3.1 of Jakubowski [38].

(iii) From Lemma 7.3.1, given any $\phi \in S(\mathbb{R})$, we obtain the uniform integrability of $\tilde{\mu}^{n_k}(\phi)$, $\tilde{Z}^{n_k}(\phi)$, $\tilde{U}^{n_k}(\phi)$, $\tilde{Y}^{n_k}(\phi)$, and $(\tilde{Z}^{n_k}(\phi), \tilde{\mu}^{n_k}(\phi), \tilde{W}(\phi))$. So the conclusion follows from (ii) and Theorem 6.2.2.

(iv) Since $W, \tilde{W}$ and $\tilde{W}^{n_k}$ have same distribution, $\tilde{W}$ and $\tilde{W}^{n_k}$ are cylindrical Brownian motions.

In order to use Jakubowski's result on continuity of the Itô stochastic integral in Hilbert spaces, we have to introduce embedding techniques. Let $S(\mathbb{R}^d)$ be the

Schwartz space of test functions. We define the Fourier transform $\hat{u}$ of u by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi(x,\xi)} u(x) dx,$$

where (x, ξ) is the inner product on $\mathbb{R}^d$. If u is a tempered distribution, i.e. if $u \in \mathcal{S}'(\mathbb{R}^d)$, the topological dual of $\mathcal{S}(\mathbb{R}^d)$, we define $\hat{u}$ as a distribution by $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in \mathcal{S}(\mathbb{R}^d)$. Define a norm on $\mathcal{S}(\mathbb{R}^d)$ by

$$\|u\|_t = \int_{\mathbb{R}^d} (1 + |\xi|^2)^t |\hat{u}(\xi)|^2 d\xi \quad \text{for } t \in \mathbb{R}.$$

Let H_t be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the norm $\|\cdot\|_t$, then $M_F(\mathbb{R})$ can be identified as a subset of H_{-d-2} by the Sobolev embedding theorem (see Walsh [61] p.337 Example 4 or Dawson-Vaillancourt [15] p.20 Prop.5.1). The conclusion follows from Jakubowski ([37] p.3, Theorem 4, see appendix) and Lemma 7.3.1 of Chapter 7.

(v) Since

$$\begin{aligned} & \frac{\epsilon}{\theta^n} \sum_{\alpha \in \mathbb{R}} \int_0^t h^\alpha(s) \nabla \phi(x_s^\alpha) dB_s^\alpha \\ &= \frac{1}{\sqrt{\theta^n}} \tilde{U}_t^n(\phi) \rightarrow 0 \quad \text{a.s. } (\tilde{\mathbb{P}}), \end{aligned}$$

we have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \tilde{\mathbb{E}} |\tilde{\mu}_t^{n_k}(\phi) - \tilde{\mu}_0^{n_k}(\phi) - \tilde{X}_t^{n_k}(\phi) - \tilde{Y}_t^{n_k}(\phi) - \frac{1}{\sqrt{\theta^n}} \tilde{U}_t^{n_k}(\phi) - \tilde{Z}_t^{n_k}(\phi)| \\ &= \tilde{\mathbb{E}} |\tilde{\mu}_t^0(\phi) - \tilde{\mu}_0^0(\phi) - \tilde{X}_t^0(\phi) - \tilde{Y}_t^0(\phi) - \tilde{Z}_t^0(\phi)| \quad \phi \in C_c^\infty(\mathbb{R}), \forall t \geq 0. \end{aligned}$$

So (3.24) holds. By Itô's formula, we see that $\{\tilde{\mu}_t^0; t \geq 0\}$ is a solution to the $(\mathcal{A} + \mathcal{B}, \delta_{\mu_0})$ -MP. The uniqueness of $\{\tilde{\mu}_t^0; t \geq 0\}$ follows from the theorem 4.4.1 of Chapter 4. This also implies that $\tilde{X}^0 + \tilde{Z}^0$ is unique in distribution. From (iv), the uniqueness of $\tilde{X}^0$ is obvious. Combining these facts, we reach the conclusion.

(vi) Since $\tilde{Z}^{n_k}$ is a purely discontinuous martingale while $\tilde{X}^{n_k}$ is a continuous martingale, they are orthogonal. (cf. Theorem 43 Dellacherie-Meyer [16] p.353) From

Corollary 5.4.3, we have

$$\begin{aligned} < \tilde{X}^0(\phi) + \tilde{Z}^0(\phi) >_t = \gamma(m_2 - 1) \int_0^t < \phi^2, \tilde{\mu}_u^0 > du \\ + \int_0^t \int_{\mathbb{R}} < g(y - \cdot) \nabla \phi(\cdot), \tilde{\mu}_s^0 >^2 dy ds. \end{aligned}$$

So

$$< \tilde{X}^0(\phi), \tilde{Z}^0(\phi) >_t = 0 \quad \text{a.s.}(\mathbb{P}) \quad \forall t \geq 0.$$

This implies the orthogonality of $\tilde{Z}^0$ and $\tilde{X}^0$. ■

7.4 Proof of the Main Results

From Lemma 7.3.1 and Lemma 7.3.2, we have

$$< \int_0^t \int_{\mathbb{R}} \phi(x) \tilde{Z}^0(dx, ds) > = \gamma(m_2 - 1) \int_0^t < \phi^2, \tilde{\mu}_u^0 > du.$$

Now let V_t be an $\mathcal{S}'(\mathbb{R})$ -valued cylindrical Brownian motion which is independent of $\mu_s(dx)$ and $\tilde{X}_s^0$ (If necessary, we construct V_t on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_\mu)$ of the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_\mu)$).

Let $\ell_s(x)$ be the density process of $\tilde{\mu}_s^0$ constructed in Chapter 6. Set

$$\begin{aligned} (4.25) \quad V_t(\phi) &= \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{\gamma(m_2 - 1)\ell_s(x)}} 1_{(\ell_s(x) \neq 0)} \phi(x) \tilde{Z}^0(dx, ds) \\ &+ \int_0^t \int_{\mathbb{R}} 1_{(\ell_s(x) = 0)} \phi(x) V(dx, ds). \end{aligned}$$

Then it is easy to verify that V_t is an $\mathcal{S}'(\mathbb{R})$ -valued cylindrical Brownian motion.

Furthermore,

$$(4.26) \quad \tilde{Z}_t^0(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{\gamma(m_2 - 1)\ell_s(x)} \phi(x) V(dx, ds).$$

Note that all the terms in (2.15) converge in $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_\mu)$. Taking limits in (2.15) and using Lemma 7.3.3, then we get the desired stochastic partial differential equations (1.2) for $\tilde{\mu}_t^0$ and Theorem 7.1.1 is proved. ■

Appendix A

New Results of Jakubowski

In Chapter 7, we have applied Dr. Jakubowski's two theorems. Since these are unpublished results, for reader's convenience, we cite them in the following.

A.1 S' -valued Skorohod's Representation Theorem

Consider the following condition (SC).

(SC): *There exist a countable family $\{f_i : D([0, \infty), S'(\mathbb{R})) \rightarrow [-\pi, \pi]\}_{i \in \mathbb{N}}$ of continuous functions (with respect Skorohod topology on $D([0, \infty), S'(\mathbb{R}))$), which separate points of $D([0, \infty), S'(\mathbb{R}))$.*

Theorem A.1.1 (Theorem 1.3 of Jakubowski [38]) *Let (χ, τ) be a topological space satisfying condition (SC) and let $\{\mu_n\}_{n \in \mathbb{N}}$ be a uniformly tight sequence of laws on χ . Then there exists a subsequence $n_1 < n_2 < \dots$ and χ -valued random elements $Y_0, Y_1, Y_3, \dots$ defined on $([-\pi, \pi], \mathcal{B}_{[-\pi, \pi]}, \ell)$ such that*

$$(1.1) \quad \mu_{n_k} \sim Y_k \quad \text{for } k = 1, 2, \dots$$

$$(1.2) \quad Y_k(\omega) \xrightarrow{\tau} Y_0(\omega) \quad \text{as } k \rightarrow \infty, \omega \in [-\pi, \pi]$$

A.2 Limit Theorem of Hilbert Space-valued Itô Integrals

Given a partition $\sigma^n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$.

$$\int K_-^n \cdot X^n(t) \equiv \int K_-^{\sigma^n} \cdot X^{\sigma^n}(t) = \sum_{\{i: t_{i+1} \leq t\}} K(t_i)(X(t_{i+1}) - X(t_i))$$

is called the elementary stochastic integral of integrand K with respect to integrator X .

Consider the following condition (UT).

(UT): The family of elementary stochastic integrals $\{\int K_-^n \cdot X^n(T)\}$ with integrands bounded by 1 is uniformly tight.

Let H , K , and G be the real separable Hilbert spaces.

Theorem A.2.1 (Theorem 4 of Jakubowski [37]) *For each $n \in \mathbb{N}$, let X^n be an H -valued semimartingale with respect to the stochastic basis $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t \in [0, T]}, \mathbb{P}^n)$ and let K^n be adapted to $\{\mathcal{F}_t^n\}_{t \in [0, T]}$ and with trajectories in $D([0, T], K)$, where $K = L_2(H, G)$. Let $X^n(0) = 0$, for $n = 1, 2, \dots$. Suppose condition (UT) holds for $\{X^n\}$ and*

$$(K^n, X^n) \xrightarrow{\mathcal{D}} (K^0, X^0)$$

on the space $D([0, T], K \times H)$. Then X^0 is a semimartingale with respect to the natural filtration generated by (K^0, X^0) and

$$(K^n, X^n, \int K_-^n \cdot (dX^n)) \xrightarrow{\mathcal{D}} (K^0, X^0, \int K_-^0 \cdot (dX^0))$$

on $D([0, T], K \times H \times G)$.

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