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ON SUBDIRECT PRODUCT OF RINGS WITHOUT
SYMMETRIC DIVISORS OF ZERO

by

TAO-CHENG YIT

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ABSTRACT

Let R be a ring with involution. The ring R is called $*$ -compressible if $ax^n b = 0$ with $x = tt^*$ (norm) or $t+t^*$ (trace) and n a power of 2 imply $axb = 0$.

Prime examples of $*$ -compressible rings are the rings without nilpotent elements and the semi-prime rings with involution all of whose norms and traces do not annihilate themselves. We have shown that the latter rings are in fact the building blocks of any semi-prime $*$ -compressible ring. We proved the following theorem which generalizes Andrunakievic - Rjabuhin's theorem.

Theorem 1 A semi-prime ring R is $*$ -compressible iff R is subdirect product of skew-domains or orders in 2×2 matrices over a field with symplectic involution.

As an application of theorem 1 we have the following theorem.

Theorem 2 Let R be a semi-prime $*$ -compressible ring. Assume that for each symmetric prime ideal P of R , the factor ring R/P has all its traces and norms von Neumann regular. Then all traces and norms of R are von Neumann regular.

With respect to the rings without nilpotent elements which are direct product of division rings (studied by A.Abian in the commutative case and by M.Chacron in the general case) we have shown the following.

Theorem 3 Let R be a semi-prime and $*$ -compressible ring. Equip R with the binary relation " $\leq$ " defined by $a \leq b$ iff $ab = a^2$. Then R has a subdirect representation χ into rings R_i with unities whose non-zero traces and norms are invertible, such that χ implements a one-to-one correspondence between the traces(norms) of R and the traces(norms) of $\prod_i R_i$ if and only if R is hyperatomic for all its norms and is $*$ -orthogonally complete (i.e. R is orthogonally complete for both norms and traces).

One of the lemmas needed is that any $*$ -compressible ring has all its symmetric idempotents central (which is not generally true for rings without symmetric nilpotent elements).

Looking at other classical results about the structure of a ring without nilpotent elements we proved:

Theorem 4 The nilpotent elements of any $*$ -compressible ring are all skew-symmetric square-zero elements.

Theorem 5 Let R be a semi-prime and $*$ -compressible ring. Then

- 1) Each minimal $*$ -prime ideal P of R is $*$ -completely prime (that is , if $xy \in P$ with $s = t+t^*$ or tt^* , then $s \in P$ or $x \in P$).
- 2) For any norm or trace u in a minimal $*$ -prime ideal P of R , there exists a norm v outside P such that $uv = 0$.
- 3) For any minimal $*$ -prime ideal P of R and any $*$ -ideal I which is generated by a finite number of norms or traces,

we have $P \supseteq I$ or $P \supseteq \text{ann}(I)$ [$= \underline{r}(I) = \underline{\ell}(I)$], but not both.

- 4) The space X of all minimal $*$ -prime ideals of R under the topology admitting as open sets all sets $\sigma(A) = \{ P \in X \mid P \not\supseteq A \}$, where A is any $*$ -ideal of R , is a Hausdorff space.

Theorem 6 If R is semi-prime and $*$ -compressible, then for any ideal $I = I^*$ of R , $\text{ann}(I) = \bigcap \{ P \mid P \text{ } $*$ \text{-prime ideal of } R, P \not\supseteq I \}$.

Theorem 7 If R is (two-sided) noetherian and is as in the above theorem, then for every $*$ -ideal I , $\text{ann}(I)$ can be written in a unique way as an intersection of a finite number of minimal $*$ -prime ideals. Conversely, every minimal $*$ -prime ideal of R is the annihilator of a symmetric ideal. Thus the symmetric annihilator ideals form a finite boolean lattice coinciding with the lattice generated by the minimal $*$ -prime ideals.

Theorem 8 If R is a 2-torsion free or a semi-prime ring which is $*$ -compressible, then for any idempotent e of R , $ee^* = e^*e$. Consequently, all idempotents of R centralize all traces of R (in fact, all symmetric elements of R in the case of 2-torsion free).

Finally, we have given a condition which forces a $*$ -compressible ring to be without nilpotents. We proved

Theorem 9 Let R be a $*$ -compressible ring. Then any one of the following conditions implies that R is without nilpotents.

- (i) R does not have skew-symmetric nilpotents.
- (ii) R is semi-prime and is generated (as a ring) by its traces.

INTRODUCTION

Let R be a ring. An element a of R is called nilpotent if $a^n = 0$ for some integer $n \geq 1$. Of course, any skew-domain (that is, any ring without divisors of zero) admits zero for a unique nilpotent element (for brevity, we shall refer to such a ring as to ring without nilpotents). Conversely V. A. Andrunakievic and Ju. M. Rjabuhin have shown that a ring without nilpotents is a subdirect product of skew-domains, which is a generalization of the results of E. Cartan [6] , J. Wedderburn [36] and W. Krull [24] .

Recall that a ring R with involution is a ring together with a map $*$: $R \longrightarrow R$ such that for any a, b in R we have

$$1) \quad (a + b)^* = a^* + b^* , \quad 2) \quad (ab)^* = b^*a^* , \quad 3) \quad a^{**} = a.$$

A prototype of problem about rings with involution is how a condition on the symmetric elements ($x = x^*$) of a ring R will affect the global structure of the ring R ?

In this direction, C. Lanski has characterized a semi-prime ring R such that all its non-zero symmetric elements do not annihilate themselves in the case of 2-torsion free, namely ,

that R has one of the following types : i) a skew-domain ;
 ii) a subring of the direct sum of a skew-domain and its
 opposite with interchanging co-ordinate involution ; or iii)
 an order in 2×2 matrices over a field [26 , Theorem 6] .

S. Montgomery and I. N. Herstein extended Lanski's characterization
 to the case where R is any semi-prime ring with involution such
 all its non-zero traces $(t + t^*)$ do not
 annihilate themselves. Clearly a ring R as in the above has all
 its non-zero traces and norms not nilpotent. As for rings
 without nilpotents, one would like to show, at least in the case
 of 2-torsion free, that any ring without symmetric nilpotents
 is a subdirect product of rings considered by Lanski. Now this
 is not generally true. For if R is the 2×2 matrices over the
 integers modulo 3 equipped with the transpose of matrices then R
 is a simple ring without symmetric nilpotents but with non-
 trivial symmetric idempotents (e. g. $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$).

On the other hand, G. Thierrin has shown that R is without
 nilpotents iff R is compressible (that is , $ax^n b = 0$ implies

$axb = 0$, where n is a power of 2), [35]. One would like then to show the subdirect representation above at least in the case where R is *-compressible (i.e. $ax^n b = 0$ with $x = t + t^*$ or tt^* , n is a power of 2, implies $axb = 0$).

Although the compressible rings have zero nil radical, this property is no longer true for *-compressible ring. For Example,

if
$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

with the symplectic involution (that is, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$),

where F is any field, then R is *-compressible and has nil radical $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

We have shown that the nil radical of a *-compressible ring coincides with its prime radical and is an intersection of ideals P which are semi-prime and *-completely prime (i. e. $sx \in P$ with $s = t + t^*$ or tt^* implies $s \in P$ or $x \in P$). Clearly an ideal $I = I^*$ is *-completely prime iff the factor ring R/I equipped with the induced involution $x + I \longrightarrow x^* + I$ has all its non-zero traces and norms regular (for brevity, we call

the latter ring $\ast$ -completely prime). We have the following characterization of $\ast$ -compressible rings.

Theorem 1.4.1   Let R be a semi-prime ring. Then R is $\ast$ -compressible if and only if R is a subdirect product of $\ast$ -completely prime rings.

Using Lanski's characterization one derives that a ring as in the above theorem must be a subdirect product of skew-domains or 2×2 matrices over a field. In the process of the proof of this Theorem , we show two key propositions :

Proposition 1.4.1   Let $I = I^\ast$ be a $\ast$ -prime ideal. Then I is $\ast$ -completely prime if and only if I is $\ast$ -compressible.

Proposition 1.4.2   Let $I = I^\ast$ be a $\ast$ -compressible ideal. Let M be an m -system excluding I . Then I can be enlarged by a $\ast$ -compressible and $\ast$ -prime ideal J excluding M .

Here are two applications of the Theorem 1.4.1 :

Theorem 1.5.1 Let R be a *-compressible and semi-prime ring.

(i) If $s_1 s_2 \dots s_n = 0$ where s_i are traces or norms of R , then the product of the s_i 's is zero in any order.

(ii) If $sxdyt = 0$ where s, d, t are traces or norms of R , then

$sdtxy = 0$.

Call a ring R *-von Neumann regular (for abbreviation, *-v.N. regular) if for each norm or trace s there is x in R such that $s = sxs$.

Theorem 1.5.2 Let R be a *-compressible and semi-prime ring.

Assume that R/P is *-v. N regular for every *-prime ideal P .

Then R is *-v. N. regular.

A. Abian has shown that a commutative ring R without nilpotents equipped with the binary relation " $\leq$ " defined by $a \leq b$ iff $ab = a^2$ is isomorphic to a direct product of fields if and only if R is hyperatomic and orthogonally complete [1 , Theorem 1]. Abian's result has been extended to the non-commutative case by M. Chacron [8, Theorem 1]. His result reads as follows:

" Any ring R without nilpotents equipped with the binary relation " $\leq$ " defined by $a \leq b$ iff $ab = a^2$ is isomorphic to a direct product of division rings if and only if R is hyperatomic and orthogonally complete ".

Parallely to this, can we characterize the case where R has a subdirect representation γ into rings R_i with unities whose non-zero traces and norms are invertible such that γ implements a one-to-one correspondence between the traces(norms) of R and the traces(norms) of $\prod_i R_i$? Indeed, we have shown the following :

Theorem 2.2.1   Let R be a semi-prime and $*$ -compressible ring.

Equip R with the binary relation " $\leq$ " defined as in the above.

Then the following conditions are equivalent:

- i) R has a subdirect representation with the properties stated in the above question ;
- ii) R is hyperatomic for its norms and $*$ -orthogonally complete (that is, R is orthogonally complete for both norms and traces).

The proof of this Theorem follows the pattern of Chacron's proof on " Direct Product of Division Rings and a Paper of Abian " [8]. One of the lemmas needed is that any $*$ -compressible ring has all its symmetric idempotents central (which is not generally

true for rings without symmetric nilpotents).

In his notes [23], I. Kaplansky has proved that a ring R without nilpotents has the following properties:

- 1) Every minimal prime ideal is completely prime.
- 2) For any element u in a minimal prime ideal P there exist v outside P such that $uv = 0$.
- 3) Let X be the set of all minimal prime ideals of R . Then X is of a Hausdorff type (i.e. X is a Hausdorff space under the topology admitting as open sets all sets
$$\overline{\cap}(A) = \{ P \mid P \text{ minimal prime ideal} \not\subseteq A \}$$
 where A is any ideal of R).

The analogue for rings with involution of the above results is the following:

Proposition 3.2.1 Let R be a ring with involution which is semi-prime and *-compressible. Then every minimal *-prime ideal of R is *-completely prime.

Proposition 3.2.2   Let R be as in the above proposition.

Then for any norm or trace u in a minimal *-prime ideal P of R ,
there exists a norm v outside P such that uv = 0.

Proposition 3.2.3   Let R be as in the Proposition 3.2.1.

Let P be a minimal *-prime ideal of R. Then for any *-ideal I
which is generated by a finite number of norms or traces, we
have $P \supseteq I$ or $P \supseteq \text{ann}(I)$ [$= \underline{r}(I) = \underline{\ell}(I)$], but not both.

Theorem 3.2.1   Let R be as in the Proposition 3.2.1.

If X is the set of all minimal *-prime ideals of R then X is
a Hausdorff space under the topology admitting as open sets
all sets

$$\overline{\bigcap}^J(A) = \{ P \mid P \text{ minimal } *-prime \text{ ideals } \not\supseteq A \}$$

where A is any *-ideal of R.

As is well known, for commutative rings R without nil right
 ideals, any prime ideal of R contains a minimal prime ideal of R.

If in addition R is semi-prime , then for each ideal I of R ,

$\text{ann}(I) = \bigcap \{P \mid P \text{ prime ideals of } R, P \not\supseteq I\}$. Also in the noetherian case, $\text{ann}(I)$ can be written in a unique way as an intersection of a finite number of minimal prime ideals. It is quite natural to seek analogous results for rings with involution. We have shown the following results :

Theorem 3.3.1 Every *-prime ideal of a ring R with involution contains a minimal *-prime ideal.

Theorem 3.3.2 If R is semi-prime and *-compressible, then for any ideals $I = I^*$ of R , $\text{ann}(I) = \bigcap \{P \mid P \text{ *-prime ideal of R, } P \not\supseteq I\}$.

Theorem 3.3.3 If R is (two-sided) noetherian and is as in the above theorem, then for every *-ideal I, $\text{ann}(I)$ can be written in a unique way as an intersection of a finite number of minimal *-prime ideals. Conversely every minimal *-prime ideal of R is the annihilator of a symmetric ideal. Thus the symmetric annihilator ideals form a finite boolean lattice coinciding with the lattice generated by the minimal *-prime ideals.

Finally, we have given a condition which forces a
*-compressible ring to be compressible. We proved

Theorem 3.4.5   If R is *-compressible and is without
skew-symmetric nilpotents, then R is compressible (and
conversely).

Looking at the idempotents (not necessarily symmetric)
of any *-compressible ring R which is 2-torsion free or semi-
prime, we proved that they centralize all traces of R (in
fact, all symmetric elements of R in the case of 2-torsion free)
which gives a nice generalization of the classical case.
A consequence of this result is that any *-compressible semi-prime
ring which is generated by its traces is again a compressible
ring.

CHAPTER I

SUBDIRECT REPRESENTATION OF A *-COMPRESSIBLE RING

1.1 Introduction

In this chapter we first introduce the basic concepts of *-compressible rings, *-completely prime rings, and *-prime rings. In 1.3 we illustrate these by examples, and in 1.4 we establish the key theorem for this work. Finally, 1.5 provides several applications of this theorem.

1.2 Conventions and Definitions

R will always denote an associative ring with involution $*$.

We use the following notations :

For $x \in R$, $|x)$, $(x|$ and (x) denote the right, left, and two-sided ideals generated by x ;

$S = \{ r \in R \mid r^* = r \}$ (the set of symmetric elements of R);

$K = \{ r \in R \mid r^* = -r \}$ (the set of skew-symmetric elements of R);

$T = \{ x \in R \mid x = t + t^* \}$ (the set of all traces of R);

$N = \{ x \in R \mid x = tt^* \}$ (the set of all norms of R);

$S_0 = T \cup N \subseteq S$;

$S_0 \setminus I$ denotes the set of all traces and norms which are not in I .

$\mathcal{R}_e(I)$, where I is an ideal of R , denotes the sum of all ideals A of R such that $A^n \subseteq I$ for some $n \geq 1$.

Finally, if $I = I^*$ is a symmetric ideal of R , the factor ring R/I will always be equipped by the canonical involution $x + I \longrightarrow x^* + I$.

Definition 1.2.1 R is said to be *-compressible if for any s in S_0 , x and y in R , and any integer n equal to a power of 2, $xs^n y = 0$ implies $xsy = 0$ (we also allow the case where one of the elements x or y is a formal unity, although we do not require the existence of unity).

Definition 1.2.2 R is said to be *-completely prime if for any s in S_0 and x in R , $sx = 0$ implies $s = 0$ or $x = 0$.

Definition 1.2.3 R is said to be *-prime if for any symmetric ideals of R , $A = A^*$, $B = B^*$, $AB = 0$ implies $A = 0$ or $B = 0$.

Definition 1.2.4 Let $I = I^*$ be a symmetric ideal of R .

I is called respectively :

*-compressible if R/I is a *-compressible ring;

*-completely prime if R/I is a *-completely prime ring;

*-prime if R/I is a *-prime ring.

Definition 1.2.5 Let I be any ideal of R (not necessarily symmetric). (i) I is called *-compressible if $s \in S_0$ and $xs^n y \in I$ imply $xsy \in I$, where n is a power of two. (ii) I is called *-completely prime if xs or $sx \in I$ with $s \in S_0 \setminus I$ imply $x \in I$.

Remark 1.2.1 It is known that $I = I^*$ is *-prime if and only if I is semi-prime and $sRd \subseteq I$ with s, d in S implies $s \in I$ or $d \in I$

Remark 1.2.2 If I is *-compressible (*-completely prime) then $I \cap I^*$ is *-compressible (*-completely prime).

1.3 Examples

We begin with the

Remark 1.3.1 Let $R = F_2$ be the ring of matrices over a field F . Assume that for a given involution $*$, the symmetric divisors of zero are zero alone, then the involution $*$ must be the symplectic involution (that is ,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

In fact , if $*$ were not the symplectic involution then by a result of Jacobson [19, p. 311, Case A], the involution $*$ must be of the following type :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} r_1^{-1} \bar{a} r_1 & r_1^{-1} \bar{c} r_2 \\ r_2^{-1} \bar{b} r_1 & r_2^{-1} \bar{d} r_2 \end{pmatrix}$$

where F has an involution $\lambda \longrightarrow \bar{\lambda}$ and $\bar{r}_i = r_i$ ($i = 1, 2$) are

fixed invertible symmetric elements of F . Now take

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ then } x^2 = 0.$$

Then $xx^* = \begin{pmatrix} 0 & 0 \\ 0 & r_1^{-1}r_2 \end{pmatrix} \neq (0)$ and $x(xx^*) = 0$.

This means xx^* is a symmetric divisor of zero, a contradiction.

This shows that the involution $*$ must be the symplectic one.

Example 1.3.1 For brevity, call a ring R "Lanski" ring

if R is a semi-prime ring such that all its non-zero symmetric elements do not annihilate themselves. By Lanski's characterization, such a ring R is either (i) a skew-domain ; or (ii) a subring of $D \oplus D^{\text{op}}$ with interchanging involution, where D is a skew-domain and D^{op} is its opposite; or (iii) an order in F_2 .

In case (i), R is of course $*$ -compressible. In case (ii), R is a subring of the direct sum of $*$ -compressible rings and consequently R is again $*$ -compressible. In case (iii) by the preceding remark 1.3.1 , $S_0 \subseteq Z(F_2)$, the center of F_2 , So that F_2 is $*$ -compressible. This shows that any Lanski ring is $*$ -compressible.

Example 1.3.2 Let R be the 2×2 matrices over the integers modulo 6 equipped with the symplectic involution. Then R is

*-compressible but neither compressible nor *-completely prime.

Clearly R is *-compressible but not compressible, however it is not

*-completely prime because

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = 0 .$$

Example 1.3.3 The ring R of 2×2 matrices over the integers modulo 3 (with the transpose of matrices) is a ring without symmetric nilpotents but is not *-compressible because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 ,$$

while $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0 .$

Example 1.3.4 The ring R of 2×2 matrices over a commutative ring without nilpotents equipped with the symplectic involution is a *-compressible ring containing nilpotents $\neq 0$.

1.4 The Key Theorem

We begin with the following proposition

Proposition 1.4.0 Let R be a $*$ -compressible ring .

- (1) If $s \in \mathcal{S}_0$ with $s^n = 0$, then $s = 0$.
- (2) If $xx^* = 0$ then $x^*x = 0$.

Proof. (1) follows immediately from definition 1.2.1.

As for (2), note that $(x^*x)^2 = 0$ and apply (1).

Proposition 1.4.1 Let $I = I^*$ be a $*$ -prime ideal of R .

Then the following conditions are equivalent :

- (1) I is $*$ -completely prime ;
- (2) I is $*$ -compressible .

Proof. By definition, I is a $*$ -completely prime

($*$ -compressible) ideal of R iff the factor ring R/I is

a $*$ -completely prime ($*$ -compressible) ring. It suffices

therefore to show the proposition in the case $I = (0)$.

(1) $\Rightarrow$ (2). This is an immediate consequence of Example 1.3.1.

(2) $\Rightarrow$ (1). First we show that $sd = 0$ with s, d in S_0 implies

$s = 0$ or $d = 0$. Let x be in R . We have

$$\begin{aligned} s(sxd + dx*s)^2d &= s(sxdsxd + dx*s^2xd + sxd^2x*s + dx*sdx*s)d \\ &= 0 \quad (\text{because } sd = ds = 0). \end{aligned}$$

By the hypothesis, $s(sxd + dx*s)d = 0$. This means that

$s^2xd^2 = 0$ for all $x \in R$. Since R is $*$ -prime, $s^2 = 0$ or $d^2 = 0$.

Since R is $*$ -compressible, $s = 0$ or $d = 0$. By a result of

Lanski and its generalization to the case of characteristic 2

[26, Theorem 6 ; 17, Theorem 4 & its proof], R , which is

obviously semi-prime, must be $*$ -completely prime. For convenience

of the reader, we shall complete the proof. Let $st = 0$ with

$0 \neq s \in S_0$ and t a given element of R . Then $stt^* = 0$. By the

above, $tt^* = 0$, so $t^*t = 0$ (Proposition 1.4.0). Thus we have

$$(st^* + ts)^2 = ts^2t^*. \quad \text{Since } (ts^2t^*)^2 = 0, \quad (st^* + ts)^4 = 0.$$

Consequently $st^* + ts = 0$ and $s(st^* + ts) = 0$. Then $s^2t^* = 0$.

$$s^2(t + t^*) = s^2t + s^2t^* = 0. \quad \text{Therefore } t + t^* = 0. \quad \text{Hence}$$

t is in K . Now consider any r in R , we have $str = 0$. So $tr \in K$.

Then $tr = r^*t$. This means $(t | = | t) = (t)$. Since $t^2 = 0$,

$(t)^2 = 0$. By the semi-primeness of R , $(t) = 0$ and so $t = 0$.

Recall that a subset M_0 of a ring R is an m -system if $a, b \in M_0$ imply $axb \in M_0$ for some $x \in R$.

Proposition 1.4.2 Let $I = I^*$ be a $*$ -compressible ideal of R . Let M_0 be an m -system excluding I . Then I can be enlarged by a $*$ -compressible and $*$ -prime ideal P excluding M_0 .

Proof. Let M be an m -system containing M_0 and maximal with respect to the exclusion of the ideal I . As is well known, if J is the complement of M , $J = C(M)$, then J is a prime ideal of R containing I and excluding M_0 . Assuming for the moment that J is $*$ -compressible, if $P = J \cap J^*$ then $P \supseteq I$ is $*$ -prime and $*$ -compressible (Remark 1.2.2). It remains to show that J is $*$ -compressible.

Define $M_1 = M \cup \{ xs^n y \mid xsy \in M ; n = 1, 2, 2^2, \dots, 2^h, \dots \},$

$M_2 = M_1 \cup \{ x_1 s_1^n y_1 \mid x_1 s_1 y_1 \in M_1 ; n = 1, 2, \dots, 2^h, \dots \},$

.....

$$M_{k+1} = M_k \cup \{x_k s_k^n y_k \mid x_k s_k y_k \in M_k ; n = 1, 2, \dots, 2^h, \dots\},$$

Let $M' = \bigcup_k M_k$. Of course, $M' \supseteq M$. Now M' excludes I for if M_k

excludes I , then since I is $*$ -compressible, M_{k+1} will exclude I

and by construction, M excludes I . By induction on k , all M_k

exclude I and consequently $M' = \bigcup_k M_k$ excludes I . Next we

show that M' is an m -system. In fact, it is enough to show that

for any pair $a, b \in M_k$ there is $c \in R$ such that $acb \in M'$ (for

$M \subseteq M_1 \subseteq M_2 \subseteq \dots$). The property holds for $k = 1$. Now suppose

it is true for k , let us show this for $k + 1$. Let $a, b \in M_{k+1}$.

To find a $c \in R$ such that $acb \in M'$. There are three cases:

Case 1. Both a, b are in M_k . Here, c follows from the induction

step.

Case 2. Both a, b are in the set $\{x s_k^n y \mid x s_k y \in M_k\}$. Here

$a = x s_k^n y$, $b = z d^m t$ for some x, y, t and z such that $x s_k y$ and $z d t \in M_k$.

By the induction hypothesis, there exists c in R such that

$(x s_k y) c (z d t) \in M_{k'}$, for some k' . Then $x s_k^n (y c z d t) \in M_{k'+1}$ and

$(x s_k^n y c z) d^m t \in M_{k'+2}$. Hence $acb \in M'$.

Case 3. If $a \in M_k$ and $b \in \{xs^n y \mid xsy \in M_k\}$ then we have $a \in M_k$ and $b = xs^n y$ where $xsy \in M_k$. Thus there exists c in R such that $ac(xsy) \in M_{k'}$, for some k' . So $(acx)s^n y \in M_{k'+1}$. Hence $acb \in M'$.

This shows that M' is an m -system containing M . By the above, M' excludes I . By maximality, $M' = M$, and so, $J = C(M)$ is as required $*$ -compressible. The proof is now complete.

Proposition 1.4.3 Let $I = I^*$ and $J = J^*$ be ideals of R .
Suppose that $I \subseteq J$ and $I \cap S_0 = J \cap S_0$. Then $J \subseteq \mathcal{R}_e(I)$.

Proof. Let $\phi : R \longrightarrow R/I = R_1$ be the canonical map.
Let J_1 be the image of J in R_1 . For any $x_1 \in J_1$, we have $x_1 + x_1^* = (x + I) + (x^* + I)$, where $x \in J$ is a pre-image of x_1 .
Then $x_1 + x_1^* = (x + x^*) + I$. Since $x^* \in J^* = J$, $x_1 + x_1^* \in J_1$.
Now $x + x^* \in J \cap S_0 = I \cap S_0$, $x + x^* \in I$. Hence $x_1 + x_1^* = 0$.
Thus $x_1 \in K(R_1)$. Similarly we have $x_1 x_1^* = 0$ for all $x_1 \in J_1$.
Now let $x_1 \in J_1$. Since J_1 is an ideal of R_1 , for any r_1 in R_1 , we have $(x_1 r_1)^* = -x_1 r_1 = r_1^* x_1^* = -r_1^* x_1$. This means $x_1 R_1 = R_1 x_1$.

Thus $|x_1| = (x_1| = (x_1)$. Since $x_1^2 = 0$, $(x_1)^2 = 0$. Hence $J_1 \subseteq \mathcal{R}_e(0)$ in R_1 , and so $J \subseteq \mathcal{R}_e(I)$.

We are now in a position to prove the following key theorem.

Theorem 1.4.1   Let $I = I^*$ be a $*$ -compressible ideal of R .

Then $\mathcal{R}_e(I)$ is the intersection of all prime ideals of R containing I or equally the intersection of all $*$ -prime and $*$ -completely prime ideals containing I .

Proof.   Let $\mathcal{R}_1(I) = \bigcap \{ P \mid P \text{ prime ideal containing } I \}$,
 $\mathcal{R}_2(I) = \bigcap \{ P' \mid P' \text{ $*$ -prime and $*$ -completely prime ideal } \supseteq I \}$.

To prove that $\mathcal{R}_e(I) = \mathcal{R}_1(I) = \mathcal{R}_2(I)$. There are two cases :

Case 1. $S_0 \subseteq I$.

Since $S_0 \cap R = S_0 \cap I$, by proposition 1.4.3 , $\mathcal{R}_e(I) = R$.

Suppose that there exists a prime ideal $P \supset I$. Then

$S_0 \subseteq I \subseteq P \cap P^*$. Consider $\bar{R} = R/P \cap P^*$. For any $\bar{x} \in \bar{R}$, using

the argument in the proof of proposition 1.4.3 , we have

$\bar{x} + \bar{x}^* = 0$ and $\bar{x} \bar{x}^* = 0$ in $\bar{R}$. Then $|\bar{x}| = (\bar{x} | = (\bar{x})$ and $(\bar{x})^2 = 0$.

This would contradict the primeness of $\bar{R}$. Thus there are no

prime ideals $P \supset I$. By convention, $\mathcal{R}_1(I) = R$. Hence

$$\mathcal{R}_e(I) = \mathcal{R}_1(I) .$$

Similarly one proves $\mathcal{R}_e(I) = \mathcal{R}_2(I) .$

Case 2. $S_0 \not\subseteq I$.

Firstly we show $\mathcal{R}_1(I) \supseteq \mathcal{R}_2(I)$. In fact, if $P \neq R$

is a prime ideal containing I , then $M = C(P)$ is an m -system

excluding I . By Proposition 1.4.2 , the ideal I can be enlarged

to a $*$ -ideal P' which is $*$ -prime, $*$ -compressible and excludes M .

From this we obtain, $P' \subseteq C(M) = P$. By Proposition 1.4.1 , P' is

$*$ -completely prime. From this it follows that $\mathcal{R}_1(I) \supseteq \mathcal{R}_2(I)$.

Because both $\mathcal{R}_1(I)$ and $\mathcal{R}_2(I)$ are semi-prime ideals and that

$\mathcal{R}_1(I)$ is the least semi-prime ideal of R , $\mathcal{R}_1(I) = \mathcal{R}_2(I)$.

On the other hand, for each $s \in S_0 \setminus I$, $\bar{M}_s = \{s^n | n = 1, 2, \dots, 2^h \dots\}$

is an m -system excluding I for I is $*$ -compressible. Again by

Proposition 1.4.2 , I can be enlarged by a member P_s in $\mathcal{R}_2(I)$

excluding s . Let $\mathcal{R}_3(I) = \bigcap P_s$ for all $s \in S_0 \setminus I$. Then

$I \subseteq \mathcal{R}_2(I) \subseteq \mathcal{R}_3(I)$. Because $I \cap S_0 = \mathcal{R}_3(I) \cap S_0$, by Proposition 1.4.3 , $\mathcal{R}_3(I) \subseteq \mathcal{R}_e(I)$. Since both $\mathcal{R}_2(I)$ and $\mathcal{R}_3(I)$ are semi-prime, $\mathcal{R}_2(I) = \mathcal{R}_3(I) = \mathcal{R}_e(I)$. Therefore $\mathcal{R}_1(I) = \mathcal{R}_2(I) = \mathcal{R}_3(I) = \mathcal{R}_e(I)$.

Corollary 1.4.1.1 If I is a *-compressible symmetric ideal of R, then

$$\mathcal{R}_e(I) \supseteq I : S_0 = \{x \mid xS_0 \subseteq I\}.$$

Proof. We have $\mathcal{R}_e(I) = \bigcap P_s$, for all $s \in S_0 \setminus I$, where P_s *-prime and *-completely prime ideal excluding s . Let $x \in I : S_0$. Then $xS_0 \subseteq I$. For each $s \in S_0 \setminus I$, there is a P_s as required (Theorem 1.4.1 Case 2). Since $I \subseteq P_s$ and $xs \in I$, $xs \in P_s$. From this , $x \in P_s$ for P_s is *-completely prime. Since this is true for all $s \in S_0 \setminus I$, $x \in \mathcal{R}_e(I)$.

We specialize the Theorem 1.4.1 to the case $I = 0$ and we get

Theorem 1.4.2 Let R be a *-compressible ring. Let radR

denote the prime radical of R. Then we have :

- (i) $(\text{rad}R) \subseteq K$;
- (ii) $2(\text{rad}R)^3 = (0)$;
- (iii) $\text{rad}R$ coincides with the nil radical of R and is an intersection of *-prime and *-completely prime ideals ;
- (iv) $\text{rad}R$ consists precisely of all square zero quasi-central elements x of R (that is, the elements $x \in R$ such that $\underline{\text{and } |x} = (x| = (x)$).

Proof. (i) Let $x \in \text{rad}R$. Since $\text{rad}R$ is a symmetric ideal, $x^* \in \text{rad}R$, and consequently, $x + x^* \in (\text{rad}R) \cap S_0$. But $\text{rad}R$ is nil. Therefore $x + x^* = 0$, that is, $x \in K$.

(ii) Let $x, y \in \text{rad}R$. By the above, $(xy)^* = -xy = (-y)(-x)$,

and so, $xy = -yx$. Now for any z in $\text{rad}R$, we have

$$2xyz = xyz + xyz = xyz - xzy = xyz + zxy = xyz - xyz = 0.$$

(iii) This follows immediately from theorem 1.4.1.

(iv) Since $(\text{rad}R) \subseteq K$, $xR = Rx$ for any $x \in \text{rad}R$. Since

$$(\text{rad}R) \cap S_0 = 0, \quad xx^* = x^*x = x + x^* = 0. \quad \text{Consequently } x^2 = 0.$$

Conversely any square zero quasi-central element is in $\mathcal{R}_e(0) = \text{rad}R$.

We can now derive our main theorem.

Theorem 1.4.3   Let R be a semi-prime ring with involution.
Then R is *-compressible if and only if R is a subdirect product
of *-completely prime rings.

Proof. " Only if " part. By Theorem 1.4.2 , we have

$$(0) = \bigcap \{ P \mid P \text{ *-prime and *-completely prime ideal} \}.$$

Therefore R is a subdirect product of the *-completely prime rings
 R/P .

" If " part. Let R be a subdirect product of *-completely
 prime rings R_i . Let x, y be in R and s in S_0 with $xs^n y = 0$.
 We prove that $xsy = 0$. In fact, in each factor R_i , we have
 $x_i s_i^n y_i = 0$. Since R_i is semi-prime and *-completely prime,
 R_i is a Lanski ring . By Example 1.3.1 , R_i is *-compressible.
 Thus $x_i s_i y_i = 0$ and this is for all indices i . Therefore $xsy = 0$.

Corollary 1.4.3.1   Let R be a semi-prime and *-compressible
ring. Then R is a subdirect product of skew-domains or 2×2
matrices over a field.

Proof. It suffices to prove this for a Lanski ring (Theorem 1.4.3). But this is Lanski's characterization.

Corollary 1.4.3.2   A semi-prime ring R is *-compressible if and only if $xs^n y = 0$ implies $xsy = 0$ for any $s \in S_0$ and any integer n .

Corollary 1.4.3.3   If R is a *-compressible ring then R modulo its prime radical is a subdirect product of semi-prime rings which are *-completely prime. Conversely, if R is a subdirect product of semi-prime rings which are *-completely prime then R is *-compressible.

1.5   Applications

From Theorem 1.4.3 we derive several results extending the classical case [see 23 , 11].

Theorem 1.5.1   Let R be a *-compressible and semi-prime ring. Then

(1) If $s_1 s_2 \dots s_n = 0$ with $s_i \in S_0$, then for any permutation
 $i_1, i_2, \dots, i_n$ in the i 's, $s_{i_1} s_{i_2} \dots s_{i_n} = 0$

(that is, the product of the s_i 's is zero in any order).

(2) If $s x d y t = 0$ with $s, d, t \in S_0$ and $x, y \in R$ then $s d t x y = 0$.

Proof. (1). Let $(s_i^{(\lambda)})_{\lambda \in I}$ be the image of s_i under a
subdirect representation as in Theorem 1.4.3. If

$s_1 s_2 \dots s_n = 0$, then $s_1^{(\lambda)} s_2^{(\lambda)} \dots s_n^{(\lambda)} = 0$ for all $\lambda \in I$.

Using the regularity condition in each factor $R^{(\lambda)}$ of R ,

there must be $s_i^{(\lambda)} = 0$. Thus $s_{i_1}^{(\lambda)} s_{i_2}^{(\lambda)} \dots s_{i_n}^{(\lambda)} = 0$,

and this is for all $\lambda \in I$. Consequently $s_{i_1} s_{i_2} \dots s_{i_n} = 0$.

(2) It suffices to show the property in case where R is a
Lanski ring. If R is a skewdomain or a subring of direct product
of skew-domains, then R is without nilpotents and the property
follows at once. Otherwise, R must be an order in F_2 , in which
case S_0 is contained in the center of R and the property is then
obvious.

Theorem 1.5.2   Let R be as in Theorem 1.5.1. Assume that
 R/P is $\ast$ -v. N. regular for every $\ast$ -prime ideal P . Then R is
 $\ast$ -v. N. regular.

Proof and Explanation. R is $\ast$ -von Neumann regular

($\ast$ -v. N. regular) if for any $a \in S_0$ there is $x \in R$ such that
 $a = axa$. Let $a \in S_0$ and let E be the set of all elements of
the form

$$(a - ax_1a)(a - ax_2a) \dots (a - ax_na)$$

$x_1, x_2, \dots, x_n$ running over R . Clearly E is closed under

multiplication. Thus E is an m -system. We claim that $0 \in E$.

For if 0 were not in E , then by Proposition 1.4.2, the

$\ast$ -compressible ideal (0) can be enlarged by a $\ast$ -prime and

$\ast$ -compressible ideal P excluding E . By the hypothesis,

R/P is a $\ast$ -v. N. regular ring and consequently $a - aya \in P$

for a suitable y . But $a - aya \in E$, a contradiction. We

must conclude that $0 \in E$, that is, for some $x_i \in R$,

$$a(1 - x_1a)a(1 - x_2a) \dots a(1 - x_na) = 0.$$

By Theorem 1.5.1, we have

$$a^n(1 - x_1 a) \dots (1 - x_n a) = 0.$$

$$\text{Thus } a(1 - x_1 a) \dots (1 - x_n a) = 0.$$

Since the product $(1 - x_1 a) \dots (1 - x_n a)$ has the form

$$1 - za, \quad a(1 - za) = 0, \text{ that is, } a = aza.$$

Remark 1.5.1 A long standing conjecture of I. Kaplansky [22]

was that a ring R is v. N. regular iff R is semi-prime ring such that each prime image of R is v. N. regular. This conjecture was settled in the affirmative by I. N. Herstein in the case of rings without nilpotent elements [see, also J.W. Fisher and R. L. Snider, 11]. Theorem 1.5.2, whose proof follows the pattern of Herstein-Kaplansky's proof [see 23], is a generalization of Herstein's result. As for Fisher-Snider's result, it is likely that it can be generalized to rings with involution; any semi-prime ring R with involution in which Fisher - Snider's condition on chain of semi-prime ideals holds such that each *-prime image of R is *-v. N. regular should be *-v. N. regular.

CHAPTER II

SPLITTING OF A CERTAIN *-COMPRESSIBLE RING

2.1 Introduction

In this chapter we study subdirect representations γ of a ring R into rings R_i with unities whose non-zero traces and norms are invertible such that the representation γ implements a one-to-one correspondence between the traces(norms) of R and the traces(norms) of $\prod_i R_i$.

2.2 Conventions and Definitions

Unless otherwise stated, R will denote a semi-prime and *-compressible ring equipped with the binary relation $a \leq b$ iff $ab = a^2$.

Definition 2.2.1 A non-zero element $q \in R$ is called N-hyperatom of R if

(i) $x \leq q$ with $x \in N$ implies $x = 0$ or $x = q$

and

(ii) $rq \neq 0$ with r in N implies $srq = q$ for some s in S_0 .

Definition 2.2.2 The ring R is called hyperatomic for its norms if for each $0 \neq x \in N$ there exists $0 \neq q \in N$ such that $q \leq x$ and q is a N -hyperatom.

Definition 2.2.3 We say that an element s of R is a supremum of a subset $\{s^{(1)}, s^{(2)}, \dots, s^{(\alpha)}, \dots\}_{\alpha \in A}$ of R if $s^{(\alpha)} \leq s$ for all $\alpha \in A$ and $s^{(\alpha)} \leq u$ with $u \in R$ for all $\alpha \in A$ implies $s \leq u$.

Definition 2.2.4 We say that R is T -orthogonally complete if every orthogonal subset of T has a supremum s in T .

Definition 2.2.4' We say that R is N -orthogonally complete if every orthogonal subset of N has a supremum s in N .

Definition 2.2.5 We say that R is $*$ -orthogonally complete if it is both T -orthogonally complete and N -orthogonally complete.

2.3 Preliminary Lemmas and the Main Theorem.

Lemma 2.3.1   If $x \leq y$ with $x \in S_0$ and $y \in R$ then $xz \leq yz$ and $zx \leq zy$ for any $z \in R$.

Proof.   By Theorem 1.4.3 , R has a subdirect representation into $*$ -completely rings R_i . From $x \leq y$ (i.e. $xy = x^2$) follows $x_i y_i = x_i^2$ for all indices i . By $*$ -completely primeness of R_i , we have $x_i = 0$ or $y_i = x_i$. From this it follows that $x_i z_i y_i z_i = x_i z_i x_i z_i$ and therefore $(xz)(yz) = (xz)^2$. This shows that $xz \leq yz$. Similarly we have $zx \leq zy$.

Lemma 2.3.2   Let R be a $*$ -compressible ring (not necessarily semi-prime). Then all symmetric idempotents of R are central.

Proof.   Let $x^2 = 0$. Then $x^*x(x + x^*)^2xx^* = 0$.
Since R is $*$ -compressible, we have $x^*x(x + x^*)xx^* = 0$.
Thus $x^*xx^*xx^* = 0$ and so, $(xx^*)^3 = 0$. Hence $xx^* = 0$, and then

$x^*x = 0$ (Proposition 1.4.0). Consequently $(x + x^*)^2 = 0$

and so, $x + x^* = 0$.

Now let $x = ey - eye$ where $e = e^* = e^2$ and $y \in R$. Then $x^2 = 0$. Using the result above we have $x + x^* = 0$ and so $ey - eye + y^*e - ey^*e = 0$. Multiplying by e , we get $ey - eye + ey^*e - ey^*e = 0$, that is, $ey - eye = 0$. Similarly, we get $ye - eye = 0$. Thus $ey = eye = ye$ for all $y \in R$. This shows the lemma.

Lemma 2.3.3 For any $x \in S_0$ and $a \in R$, we have

- (i) $x^2a = x^2$ iff $xa = x$;
- (ii) $x^2a = x$ iff $xax = x$; $ax^2 = x$ iff $xax = x$;
- (iii) $x^2a = ax^2$ iff $xa = ax$;
- (iv) $x^2a = 0$ iff $xa = 0$;
- (v) $xa = 0$ iff $xax = 0$.

Proof. (i). Using the subdirect representation of R in Theorem 1.4.3, we get that $x^2a = x^2$ implies $x_i^2a_i = x_i^2$ for all indices i . Since each factor R_i is $*$ -completely prime,

$x_i = 0$ or $x_i = x_i a_i$ and so, $x_i a_i = x_i$ for all i . Consequently $xa = x$. The converse is obvious.

(ii). If $x^2 a = x$ then $x^2 ax = x^2$. Using (i), we get $xax = x$.

Conversely, if $xax = x$ then $x^2 ax = x^2$. So $(x^2 a)x = xx$.

Using the subdirect representation of Theorem 1.4.3 , we get

$x_i^2 a_i x_i = x_i x_i$ for all i . Since R_i is $*$ -completely prime ,
 $x_i^2 a_i = x_i$ or $x_i = 0$. From this follows $x_i^2 a_i = x_i$ and so, $x^2 a = x$.

This shows $x^2 a = x$ iff $xax = x^2$. Similarly , we have

$xax = x^2$ iff $ax^2 = x$.

The proof of (iii) , (iv), (v) are similar.

Lemma 2.3.4   If $0 \neq q \in N$ is a N -hyperatom, then there exists
a non-zero N -hyperatomic symmetric idempotent e such that $y \neq 0$
in S_0 with $q \leq y$, implies $ey \neq 0$.

Proof.   Since R is $*$ -compressible and $0 \neq q \in N$ is a
 N -hyperatom, $q^2 \neq 0$ and $sq^2 = q$ for some $s \in S_0$. By Lemma
 2.3.3-(ii), $q^2 s = q$. So $sq^2 = q^2 s$. By Lemma 2.3.3-(iii), we
 get $sq = qs$. Take $e = qs = sq$. Then $0 \neq e = e^2$ is a symmetric

idempotent element. It remains to show that e is an N -hyperatom.

In fact, if $x \leq e$ with $0 \neq x \in N$, then by Lemma 2.3.1, $xq \leq qsq$.

Since $qsq = q^2s = q$, we get $xq \leq q$. By definition, we have

$$(1) \quad (xq)_q = (xq)(xq).$$

Since $x \neq 0$, $xq \neq 0$ (for if $xq = 0$ then $x^2 = xe = xqs = 0$

and so, $x = 0$). Thus $nxq = q$ for some $n \in S_0$ (for q is a

N -hyperatom). Using (1) and the equality $nxq = q$, we get

$q^2 = qxq$ and $q^2s = qxs$. From this it follows

$$(2) \quad q = qxe = qex = q^2sx = qx.$$

Since $x \leq qs$, (3) $xqs = x^2$ and so, $xqsq = x^2q$.

Since $qsq = q$ we get (4) $xq = x^2q$. By (3) and (4), we have

$x^2 = xqs = x^2qs$. By Lemma 2.3.3-(i), $x = xqs$. Applying (2),

we derive that $x = xqs = xe = ex = sqx = sq = e$. This shows

that $x \leq e$ with $x \in N$ implies $x = 0$ or $x = e$. Now to check

Definition 2.2.1-(ii) for e . In fact, let $ye \neq 0$ with $y \in N$.

Then $yqs \neq 0$. Since $yq \neq 0$ and q is an N -hyperatom, we have $uyq = q$

for some $u \in S_0$. From this we have $uyqs = qs$ and $uye = e$.

Therefore we have shown that e is an N -hyperatom.

Finally , if $0 \neq y \in S_0$ and $q \leq y$ then $qy = q^2$ and consequently $ey = ye = qsy = sqy = sq^2 = q \neq 0$. The proof of this lemma is now complete.

Lemma 2.3.5 Let $E = \{e_i\}_{i \in I}$ be the set of all non-zero N-hyperatomic symmetric idempotents of R. Then E is an orthogonal set ; for any $i \in I$, $R_i = e_i R$ is a two-sided symmetric ideal of R; and $R_i \cap R_j = (0)$ for all i, j with $i \neq j$.

Proof. Since symmetric idempotents of R are central, we have $e_i e_j \leq e_j$. From the definition of N-hyperatom, it follows that $e_i e_j = e_j$ or $e_i e_j = 0$. Similarly , from $e_j e_i \leq e_i$ we get $e_j e_i = e_i$ or $e_j e_i = 0$. Consequently we get $e_i e_j = 0$ or $e_i = e_j e_i = e_i e_j = e_j$. This shows that E is an orthogonal set. Consequently, $R_i \cap R_j = e_i R \cap e_j R = (0)$ for any i, j with $i \neq j$. Finally, since every symmetric idempotent is central, for each $e_i \in E$, $e_i R = R e_i$ is a symmetric ideal of R.

Lemma 2.3.6 If R is hyperatomic for its norms and if e is a N-hyperatomic symmetric idempotent of R, then all non-zero traces and norms of eR are invertible.

Proof. Obviously, e is the identity of eR . Let $0 \neq s$ in $N(eR) = eN(R)$. Then $s = es = se \neq 0$. Since e is a N -hyperatom, we have $nse = e$ for some $n \in S_0$ and so, $(ne)(se) = e$. Hence all non-zero norms of eR are invertible.

Let $0 \neq x$ in $T(eR) = eT(R)$. Then $x = ex = xe \neq 0$ and so, $x^2e \neq 0$. Since x^2 is in N and e is a N -hyperatom, we have $mx^2e = e$ for some $m \in S_0$ and so, $(mx)(xe) = e$. This shows that all non-zero traces of eR are also invertible.

Lemma 2.3.7 If R is hyperatomic for its norms, then R has a subdirect representation γ into rings R_i with unities whose non-zero traces and norms are invertible.

Proof. Let E be as in lemma 2.3.5. By this lemma, $E \neq \emptyset$. Define the map $\gamma: R \longrightarrow \prod_{i \in I} R_i (= e_i R)$ by $\gamma(x) = (e_i x)_{i \in I}$. This is a ring homomorphism. By Lemma 2.3.6, all non-zero traces and norms of R_i are invertible. It remains to show that γ is one-to-one.

We first prove that $\gamma|_T$, the restriction of γ on T , and $\gamma|_N$, the restriction of γ on N , are one-to-one. In fact,

for any $0 \neq x \in T$, we have $0 \neq x^2 \in N$. Since R is hyperatomic for its norms, there exists $q \neq 0$ in N such that $q \leq x^2$ and q is an N -hyperatom. By Lemma 2.3.4, there is a non-zero symmetric idempotent N -hyperatom e_i such that $e_i x^2 \neq 0$ and so, $e_i x \neq 0$. This shows that $\gamma|_T$ is one-to-one. Similarly, $\gamma|_N$ is one-to-one.

We now prove that γ is one-to-one. In fact, let $x \in \text{Ker } \gamma$. Then $e_i x = 0$ and so, $e_i x^* = 0$, $e_i x x^* = 0$ for all indices i . Since $\gamma|_N$ is one-to-one, we get $x x^* = 0$. Since $e_i(x + x^*) = 0$ and $\gamma|_T$ is one-to-one, $x + x^* = 0$. Thus $\text{Ker } \gamma$ consists entirely of square-zero elements. If then $\text{Ker } \gamma \neq 0$, by Levitzki's Theorem [13, Lemma 1.1], R contains a nilpotent ideal, contradicting the semi-primeness of R . We must conclude that γ is one-to-one.

Lemma 2.3.8   The mapping γ in Lemma 2.3.7 has the following property:

If $\{s^{(1)}, s^{(2)}, \dots, s^{(\alpha)}, \dots\}_{\alpha \in A}$ is a family of traces(norms) of R having a supremum s in T (in N) then for any fixed $i \in I$, $s_i^{(\alpha)} = 0$ for all $\alpha \in A$ implies $s_i = 0$.

Proof. By Lemma 2.3.7, we have $s_i = e_i s$. Set $s' = s - e_i s$.

Since $\{e_i\}$ is orthogonal, we have

$$s'_\mu = e_\mu (s - e_i s) = \begin{cases} e_\mu s & \text{if } \mu \neq i \\ 0 & \text{if } \mu = i \end{cases}$$

Then $s^{(\alpha)}_\mu \leq s'$ for all $\mu \in I$ and so $s^{(\alpha)} \leq s'$ for all $\alpha \in A$.

Then $s \leq s'$ (for s is a supremum). By definition, $ss' = s^2$

and so, $s(s - e_i s) = s^2$. Thus $e_i s^2 = 0$. By Lemma 2.3.3-(v),

we have $e_i s = 0$ and $s_i = 0$. The proof of the norm case is similar.

Lemma 2.3.9 Let $\{s^{(1)}, s^{(2)}, \dots, s^{(\alpha)}, \dots\}_{\alpha \in A}$

be a family of traces(norms) of R such that its supremum

exists and is equal to a trace(norm) s . Let g be any symmetric

idempotent of R . Then the family

$$\{gs^{(1)}, gs^{(2)}, \dots, gs^{(\alpha)}, \dots\}_{\alpha \in A}$$

of traces(norms) of R admits a supremum equal to a trace(norm) gs .

Proof. Let γ be the subdirect representation in Lemma

2.3.7. Let $(s^{(\alpha)}) = (s^{(\alpha)}_i)_{i \in I}$ and $\gamma(g) = (g_i)_{i \in I}$.

Then since s is the supremum of $\{s^{(1)}, s^{(2)}, \dots, s^{(\alpha)}, \dots\}_{\alpha \in A}$

$s^{(\alpha)} \leq s$. By Lemma 2.3.1, $gs^{(\alpha)} \leq gs$ for all $\alpha \in A$. It remains to show that if $gs^{(\alpha)} \leq u$ for all $\alpha \in A$ and $u \in R$ then $gs \leq u$.

Suppose $gs \leq u$ were not true. Then $gsu \neq gsgs$. This means that

$g_i s_i u_i \neq g_i s_i g_i s_i$ for some $i \in I$. Thus $s_i \neq 0$. By Lemma 2.3.8,

$s_i^{(\alpha_0)} \neq 0$ for some $\alpha_0 \in A$. Since $s_i^{(\alpha_0)} \leq s_i$ and every non-zero

trace of R_i is invertible, we must have $s_i^{(\alpha_0)} = s_i$.

Since $gs^{(\alpha_0)} \leq u$, $gs^{(\alpha_0)}u = gs^{(\alpha_0)}gs^{(\alpha_0)}$ and so,

$g_i s_i^{(\alpha_0)} u_i = g_i s_i^{(\alpha_0)} g_i s_i^{(\alpha_0)}$. Thus $g_i s_i u_i = g_i s_i g_i s_i$, a contradiction.

So we must have $gsu = (gs)^2$, that is, $gs \leq u$. The proof of the

norm case is similar.

Lemma 2.3.10 Let R be hyperatomic for its norms and

*-orthogonally complete. Then γ in Lemma 2.3.8 implements a

one-to-one correspondence between the traces(norms) of R and

the traces(norms) of $\prod_i R_i$.

Proof. Since $T(R_i) = e_i T(R)$ and $N(R_i) = e_i N(R)$,

a trace(norm) of $\prod_i R_i$ has the form $(e_i x_i)_{i \in I}$ where x_i is

a trace(norm) of R . By lemmas 2.3.2 and 2.3.5, the set

$\{e_i x_i\}_{i \in I}$ is an orthogonal set of traces(norms) of R . Since R is $*$ -orthogonally complete, $\{e_i x_i\}_{i \in I}$ has a supremum (supremum) b in $T(\text{in } N)$. Then by Lemma 2.3.9, we have

$$e_j b = e_j \sup_i \{e_i x_i\} = \sup_i \{e_j e_i x_i\} = e_j x_j \text{ and so,}$$

$$\gamma(b) = (e_j b)_{j \in I} = (e_j x_j)_{j \in I}.$$

This shows that γ has the desired property.

We are now in a position to show the main theorem for this chapter.

Theorem 2.3.1   The ring R has a subdirect representation γ into rings R_i with unities whose non-zero traces and norms are invertible, such that γ implements a one-to-one correspondence between the traces(norms) of R and the traces(norms) of $\prod_i R_i$ if and only if R is hyperatomic for its norms and is $*$ -orthogonally complete.

Proof. " If " part. It is the immediate consequence of the above lemmas.

" Only if " part. Let $0 \neq x \in N$. By assumption, we have $x = (x_i)_{i \in I} \neq (0)$ where $x_i \in N(R_i)$ and any non-zero x_i are

invertible. Thus $x_j \neq 0$ for some $j \in I$. Take

$$q = (0, 0, \dots, 0, x_j, 0, \dots, 0).$$

Certainly $q \neq 0$ and q is in N . Since

$$\begin{aligned} qx &= (0, 0, \dots, 0, x_j, 0, \dots, 0) (x_i)_{i \in I} \\ &= (0, 0, \dots, 0, x_j x_j, 0, \dots, 0) \\ &= q^2, \end{aligned}$$

we have $q \leq x$.

We now show that q is a N -hyperatom. In fact, for any

$y \in N$ with $y \leq q$, we have $yq = y^2$ and

$$(0, 0, \dots, y_j x_j, 0, \dots, 0) = (y_1^2, y_2^2, \dots, y_j^2, \dots).$$

Consequently $y_i = 0$ for $i \neq j$ and $y_j = 0$ or x_j . Hence

$y = 0$ or $y = q$. If $rq \neq 0$ with $r \in N$ then $r_j x_j \neq 0$.

Since $r_j \neq 0$, r_j is invertible with its inverse in N (for if

$$r_j = tt^* \text{ then } r_j^{-1} = (t^{-1})^* t^{-1}). \text{ Take } u = (0, \dots, r_j^{-1}, \dots, 0).$$

Then u is in N and $urq = q$. This shows that R is hyperatomic

for its norms.

We now show that R is T -orthogonally complete. In fact,

let $B = \{s^{(1)}, s^{(2)}, \dots, s^{(\alpha)}, \dots\}_{\alpha \in A}$ be any

family of traces of R which is orthogonal. Since B is orthogonal,

$$s^{(\alpha)} s^{(\beta)} = 0 \text{ for } \alpha \neq \beta. \text{ From this it follows that } s_i^{(\alpha)} s_i^{(\beta)} = 0$$

for all $i \in I$ and so, $s_i^{(\alpha)} = 0$ or $s_i^{(\emptyset)} = 0$. Hence for fixed i , there is at most one α such that $s_i^{(\alpha)} \neq 0$. Take

$$s = (s_i)_{i \in I} \text{ where } s_i = \begin{cases} s_i^{(\alpha)} & \text{if for some } \alpha, s_i^{(\alpha)} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It remains to show that $\sup_{\alpha \in A} B = s$. If $s_i^{(\alpha_0)} \neq 0$ for $\alpha_0 \in A$, then $s_i^{(\alpha_0)} s_i = s_i^{(\alpha_0)} s_i^{(\alpha_0)}$ and $s_i^{(\alpha)} s_i = 0 = s_i^{(\alpha)} s_i^{(\alpha)}$ for $\alpha \neq \alpha_0$. Thus $s_i^{(\alpha)} s_i = s_i^{(\alpha)} s_i^{(\alpha)}$ for all $\alpha \in A$ and for all $i \in I$. This shows that $s^{(\alpha)} \leq s$ for all $\alpha \in A$.

Finally, let $s^{(\alpha)} \leq u$ with $u \in R$. Then (1) $s_i^{(\alpha)} u_i = s_i^{(\alpha)} s_i^{(\alpha)}$.

We must show that $s \leq u$, that is, $su = s^2$. Suppose that

$su \neq s^2$. Then there exists $i_0 \in I$ such that $s_{i_0} u_{i_0} \neq s_{i_0} s_{i_0}$.

From this we have $s_{i_0} \neq 0$. Thus (2) $s_{i_0} = s_{i_0}^{(\alpha_0)}$ for some $\alpha_0 \in A$.

From (1) and (2), we get $s_{i_0} u_{i_0} = s_{i_0} s_{i_0}$, a contradiction.

Hence we have $su = s^2$ and so, $s \leq u$. This shows that $\sup_{\alpha \in A} B = s$.

Similarly, one shows that R is N -orthogonally complete. The proof of the theorem is now complete.

CHAPTER III

ON THE STRUCTURE OF A *-COMPRESSIBLE RING

3.1 Introduction

In this chapter we study the minimal *-prime ideals of a *-compressible ring R , we describe the annihilators of the symmetric ideals keeping an eye to the classical cases [21, 23, 25], and we close this work by two conditions forcing a *-compressible ring to be a compressible ring.

3.2 The Space of Minimal *-prime Ideals

Proposition 3.2.1   Let R be a ring with involution which is semi-prime and *-compressible. Then every minimal *-prime ideal P of R is *-completely prime.

Proof. Since P is a *-prime ideal, by a result of Martindale [27, Theorem 6], we have $P = P_1 \cap P_1^*$ for some prime ideal P_1 of R . If $M = C(P_1)$ is the complement of P_1 , then M is an m -system excluding 0. By Proposition 1.4.2, there is a *-prime and *-completely prime $J = J^*$ such that $M \cap J = \emptyset$. Thus $J \subseteq P_1$ and

so, $J^* \subseteq P_1^*$. Consequently, $J \subseteq P_1 \cap P_1^* = P$. Since P is minimal, $J = P$. Therefore P is $*$ -completely prime.

Proposition 3.2.2 Let R and P be as in the above proposition.

Then for any norm or trace u in P , there exists a norm v outside P such that $uv = 0$

Proof. Let $M = \{s_1 s_2 \dots s_n \mid s_i \notin P, s_i \in S_0, i = 1, \dots, n\}$. Since P is $*$ -completely prime (Proposition 3.2.1), M is closed under multiplication. We claim that $0 \notin M$. For if 0 were in M , then $s_1 s_2 \dots s_n = 0 \in P$ for some s_i 's in $S_0 \setminus P$. By $*$ -completely primeness, $s_i \in P$ for some i , a contradiction. Let M_u be the multiplicative subset generated by $M \cup \{u\}$. Then M_u is a m -system. We now show that $0 \in M_u$. In fact, if 0 were not in M_u , then by Proposition 1.4.2, there is a $*$ -prime and $*$ -compressible ideal J of R such that $J \cap M_u = \emptyset$. Then $J \cap M = \emptyset$. Let $\bar{J}$ be the image of J in R/P . Then $\bar{J}$ is a $*$ -ideal of R/P . For any element $\bar{x} \in \bar{J}$, $\bar{x} = x + P$ for some $x \in J$, and so $x^* \in J$. Then $x + x^* \in J$ and $xx^* \in J$. Since $J \cap M = \emptyset$, $x + x^*$ and $xx^* \in P$. Hence $\bar{x} \bar{x}^* = 0$, $\bar{x} + \bar{x}^* = 0$ and $\bar{x}^2 = 0$. Therefore $\bar{J}$ consists entirely of square zero elements.

By Levitzki's Theorem [13 , Lemma 1.1], there exists a nilpotent ideal $\neq 0$, which contradicts the semi-primeness of the ideal P . Thus $\bar{J} = 0$ in R/P , that is , $J \subseteq P$. By the minimality of P , $J = P$. By hypothesis, $u \in P = J$, consequently $u \in J \cap M_u$, a contradiction . This shows that the m -system M_u contains 0. From this, we have $us_1s_2\ldots s_n = 0$ for some s_i 's not in P . Then $us_1s_2 \ldots s_n s_n \ldots s_2s_1 = 0$. Since $s_i \notin P$, $v = s_1s_2 \ldots s_n s_n \ldots s_2s_1 \notin P$ and $v \neq 0$. Hence $uv = 0$ with v a norm of R .

Given an ideal I of a ring R , let $\underline{\ell}(I) = \{x \in R \mid xI = 0\}$ be the left annihilator of I , $\underline{r}(I) = \{x \in R \mid Ix = 0\}$ the right annihilator of I , and $\text{ann}(I) = \underline{\ell}(I) \cap \underline{r}(I)$ the annihilator of I .

Remark 3.2.1 Let R be a semi-prime ring. For any ideal I of R , $\underline{\ell}(I) = \underline{r}(I) = \text{ann}(I)$ is a semi-prime ideal.

Proof. Let $x \in R$ such that $Ix = 0$. We have $(xI)^2 = xIxI = 0$, so $xI = 0$. By symmetry $\underline{\ell}(I) = \underline{r}(I) = \text{ann}(I)$ is an ideal of R . Suppose that $Y^2 \subseteq \text{ann}(I)$ for an ideal Y of R . From $Y^2I = 0$, we get $Y(YI) = 0$, and consequently $YIY = 0$, hence $(YI)^2 = YIYI = 0$ and so, $YI = 0$, that is , $Y \subseteq \text{ann}(I)$. This shows that $\text{ann}(I)$ is a

semi-prime ideal.

Remark 3.2.2 If R is a semi-prime ring with involution,
for each symmetric ideal $I = I^*$, $\text{ann}(I)$ is a symmetric semi-
prime ideal.

Proof. By Remark 3.2.1 .

Proposition 3.2.3 Let R be as in the Proposition 3.2.1 .
Let P be a minimal *-prime ideal of R . Then for any *-ideal I
which is generated (as a two-sided ideal) by a finite number of
norms or traces, we have $P \supseteq I$ or $P \supseteq \text{ann}(I)$, but not both.

Proof. We have $I \cdot \text{ann}(I) = 0 \subseteq P$ and consequently $I \subseteq P$ or
 $\text{ann}(I) \subseteq P$. Assume that $I \subseteq P$. Let $E = \{x_1, x_2, \dots, x_n\}$ be
a generating set for I as in the hypothesis. By Proposition
3.2.2 , there are norms $a_i \notin P$ such that $x_i a_i = 0$, $i = 1, \dots, n$.
Let $a = a_1 a_2 \dots a_n$. Then $a \notin P$ (by *-completely primeness of P)
and $x_i a = 0$, $i = 1, \dots, n$ (Theorem 1.5.1). Hence $(x_i) a = 0$.
Consequently $I a = 0$ with $a \notin P$. Thus $\text{ann}(I) \not\subseteq P$.

Theorem 3.2.1 Let R be as in Proposition 3.2.1 .

If X is the set of all minimal $*$ -prime ideals of R , then X is a Hausdorff space under the topology admitting as open sets all sets

$$\bigcap(A) = \{ P \mid P \text{ minimal } * \text{-prime ideal } \not\supset A \}$$

where A is any $*$ -ideal of R .

Proof. For any family of $*$ -ideals A_i , we have

$$\begin{aligned} \bigcup_i \bigcap(A_i) &= \{ P \in X \mid A_i \not\subset P \text{ for some } i \} \\ &= \{ P \in X \mid \sum_i A_i \not\subset P \} \\ &= \bigcap(\sum_i A_i) . \end{aligned}$$

For any $*$ -ideals A, B of R , we have

$$\begin{aligned} \bigcap(A) \cap \bigcap(B) &= \{ P \in X \mid A \not\subset P \text{ and } B \not\subset P \} \\ &= \{ P \in X \mid AB \not\subset P \text{ and } BA \not\subset P \} \\ &= \bigcap(AB + BA) . \end{aligned}$$

$$\text{Finally } \bigcap(R) = \{ P \in X \mid R \not\subset P \} .$$

This shows that X is a topological space.

Let $P_1 \neq P_2$ in X . Since P_2 is a minimal $*$ -prime ideal , $P_1 \not\subset P_2$. Then there must be $u \in R$ such that $u \in P_1$ and $u \notin P_2$.

Since P_1 and P_2 are semi-prime and $*$ -completely prime , we may

further assume that $u \in S_0$. For let $\bar{P}_1$ be the image of P_1 in R/P_2 . Then $\bar{P}_1$ is a $*$ -ideal of the factor ring R/P_2 . For any $\bar{x} \in \bar{P}_1$, we have $\bar{x} = x + P_2$ for some $x \in P_1$, $x^* \in P_1$ and hence $x + x^* \in P_1$ and $xx^* \in P_1$. Then there must be $x + x^*$ or xx^* not in P_2 . For if not, then $x + x^* \in P_2$ and $xx^* \in P_2$. Thus $\bar{x} + \bar{x}^* = 0$, $\bar{x} \bar{x}^* = 0$ and so, $\bar{x}^2 = 0$. By semi-primeness of R/P_2 and by Levitzki's Theorem, $\bar{P}_1 = 0$. Then $P_1 + P_2 \subseteq P_2$ and $P_1 \subseteq P_2$, a contradiction. This shows that $u = x + x^*$ or xx^* is in P_1 but not in P_2 . From this, $(u) \not\subseteq P_2$. By Proposition 3.2.3, we have $\text{ann}((u)) \subseteq P_2$. Also by the same proposition, $\text{ann}((u)) \not\subseteq P_1$. Then there are two disjoint open sets $\bigcap (\text{ann}((u)))$, $\bigcap ((u))$ containing P_1 , P_2 respectively. Hence X is a Hausdorff space. The proof of the theorem is now complete.

3.3 Annihilators of Symmetric Ideals

We turn to the study of annihilator of symmetric ideals.

To this end we prove

Theorem 3.3.1 Every $*$ -prime ideal of a ring R with involution contains a minimal $*$ -prime ideal.

Proof. Call a subset M of R a *-system if $s, d \in S_0 \cap M$ imply $sxd \in M$ for some $x \in R$. Clearly a semi-prime ideal $P = P^*$ is *-prime iff $C(P)$, the complement of P , is a *-system. We now show that the union of a chain of *-systems is a *-system. In fact, if $s, d \in \bigcup_i M_i$ and s, d are in S_0 , then $s \in M_i$ and $d \in M_j$; since $M_i \subseteq M_j$ or $M_j \subseteq M_i$, both $s, d \in M_i$ or both $s, d \in M_j$ and then $sxd \in M_i$ or $sxd \in M_j$ for some x , that is, $\bigcup_i M_i$ is a *-system. By Zorn's Lemma (reversed), any *-prime ideal must therefore contain a minimal *-prime ideal.

Corollary 3.3.1.1 In a semi-prime ring with involution, the following conditions are equivalent:

- (1) R is *-compressible ;
- (2) each minimal *-prime ideal is *-completely prime.

Proof. (1) $\Rightarrow$ (2). It has been proved in Theorem 3.2.1 . (2) $\Rightarrow$ (1). It suffices to show that (0) is an intersection of *-completely prime ideals . In fact, since R is semi-prime, $(0) = \bigcap \{ P \mid P \text{ *-prime ideals of } R \}$. However each P contains some minimal *-prime ideal P_1 (Theorem 3.3.1), which is *-completely prime (by hypothesis). A fortiori, $(0) = \bigcap P_i$.

Remark 3.3.1 If $P = P^*$ is a *-completely prime ideal of a ring R such that $Pu = 0$ for some u in S_0 but not in P , then $P = \underline{\ell}(u)$.

Proof. Since $Pu = 0$, $P \subseteq \underline{\ell}(u)$. Conversely, let $x \in \underline{\ell}(u)$. We have $xu = 0 \in P$ with $u \notin P$. By *-completely primeness of P , $x \in P$ and $P = \underline{\ell}(u)$.

Theorem 3.3.2 If R is semi-prime and *-compressible, then for any ideal $I = I^*$ of R , $\text{ann}(I) = \bigcap \{P \mid P \text{ *-prime ideal of } R, P \not\supseteq I\}$.

Proof. If $I = 0$ or $I = R$, then the result is obvious. Now suppose $(0) \subsetneq I \subsetneq R$. Certainly, R has a *-prime ideal not containing I (for if all *-prime ideals P were containing I , then $0 = \bigcap P \supseteq I$, a contradiction). Since $I \cdot \text{ann}(I) = 0$, $I \cdot \text{ann}(I) \subseteq P$ and consequently $I \subseteq P$ or $\text{ann}(I) \subseteq P$ for all *-prime ideals P of R . Thus $\text{ann}(I) \subseteq \bigcap \{P \mid P \text{ *-prime of } R, P \not\supseteq I\}$. Conversely, let $x \notin \text{ann}(I)$. Then $xI \neq 0$. By forthcoming Theorem 3.4.3, the right ideal xI can not be nil and hence, there exists a non-nilpotent element $z = xy \neq 0$ for some $y \in I$. Let P be a prime ideal not intersecting $M = \{z^n \mid n = 1, 2, \dots\}$.

Then $P_1 = P \cap P^*$ is a $*$ -prime ideal not containing z . It follows that $I \not\subseteq P_1$ and $x \notin P_1$. This shows that $x \notin \bigcap \{P \mid P \text{ } * \text{-prime ideal, } P \not\subseteq I\}$. Therefore the equality is proved.

Remark 3.3.2 Let R be a semi-prime and $*$ -compressible ring, I and J two symmetric ideals of R . Then $I \cdot J = 0$ implies
 $\mathcal{R}_e(I) \cdot \mathcal{R}_e(J) = 0$.

Proof. Suppose $X^n \subseteq I$ and $Y^m \subseteq J$. We have $X^n Y^m \subseteq IJ = 0$. From this $X^n Y^m = 0$. By Remark 3.2.1, $XY = 0$. This shows that
 $\mathcal{R}_e(I) \cdot \mathcal{R}_e(J) = 0$.

Theorem 3.3.3 If R is (two-sided) noetherian and is as
in the above remark, then for every $*$ -ideal I , $\text{ann}(I)$ can be
written in a unique way as an intersection of a finite number of
minimal $*$ -prime ideals. Conversely every minimal $*$ -prime ideal
of R is the annihilator of a symmetric ideal. Thus the symmetric
annihilator ideals form a finite boolean lattice coinciding with
the lattice generated by the minimal $*$ -prime ideals.

Proof. Let $\mathcal{F} = \{I_0 \mid I_0 \text{ semi-prime } *-ideal \text{ and can not be written as a finite intersection of } *-prime \text{ ideals} \}$.

If $\mathcal{F} \neq \emptyset$, then there is a maximal member I (by the noetherian condition). Certainly I is not $*-prime$, and hence for a pair of ideals we have $AB \subseteq I$ but $A \not\subseteq I$ and $B \not\subseteq I$. Thus $A + I \notin \mathcal{F}$ and $B + I \notin \mathcal{F}$ (by maximality of I in $\mathcal{F}$). Consequently $\mathcal{R}_e(A + I) \notin \mathcal{F}$ (for $\mathcal{R}_e(A + I)$ contains $A + I$). Similarly, $\mathcal{R}_e(B + I) \notin \mathcal{F}$.

Since $\mathcal{R}_e(A + I)$ and $\mathcal{R}_e(B + I)$ are semi-prime, they must be finite intersections of $*-prime$ ideals. From this $\mathcal{R}_e(A + I) \cap \mathcal{R}_e(B + I)$ is a finite intersection of $*-prime$ ideals and consequently

$\mathcal{R}_e(A + I) \cap \mathcal{R}_e(B + I) \notin \mathcal{F}$. But in the semi-prime ring R/I , we have $(A+I)(B+I) = AB + IB + AI + I^2 \subseteq I$. By Remark 3.3.2,

$\mathcal{R}_e(A+I/I) \cdot \mathcal{R}_e(B+I/I) = 0$ in R/I . Since R/I is semi-prime,

$\mathcal{R}_e(A+I) \cap \mathcal{R}_e(B+I) = I$, a contradiction. This shows that

$\mathcal{F} = \emptyset$. That is, every semi-prime $*-ideal$ I is a finite

intersection of $*-prime$ ideals. In particular, $(0) = \bigcap_{i=1}^n P_i$

for some $*-prime$ ideals P_i . Since each P_i contains a minimal

$*-prime$ ideal P'_i , we get $(0) = \bigcap_{i=1}^n P'_i$.

Changing eventually our notations we may assume that the P_i 's

are distinct. We claim that the P_i 's exhaust all the minimal

*-prime ideals of R . For if $(0) = \bigcap_{i=1}^m P_i'$ with $m > n$, then

we would get $P_1' \cdot P_2' \cdot \dots \cdot P_n' = 0 \subseteq P_{n+1}'$ and so, $P_i' \subseteq P_{n+1}'$

for some $i < n + 1$, contradicting the minimality of P_{n+1}' .

Let then $J = J^*$ be a semi-prime *-compressible ideal of R .

Going to the factor ring R/J , we see that J has a unique

representation as a finite intersection of *-prime ideal P

minimal over J (i.e. for any *-prime ideal P' such that

$P \supseteq P' \supseteq J$, $P = P'$). If, further, $J = \bigcap \{P \mid P \text{ minimal$

*-prime ideals}\}, then as the P 's are certainly minimal over

J , we get that they must be of a finite number. In particular

for $I = I^*$, we have

$$J = \text{ann}(I) = \bigcap \{ P \text{ minimal *-prime ideals } \not\supseteq I \}$$

$$= \bigcap_{i=1}^m \{ P_i \text{ minimal *-prime ideals} \}.$$

Let P be any minimal *-prime ideal. By the preceding, there

are finitely many minimal *-prime ideals P_i distinct from P

such that $0 = P \cap P_1 \cap P_2 \cap \dots \cap P_n$ and so,

$$0 = P_1 P_2 \dots P_n P,$$

that is, $0 \not\subseteq P_1 P_2 \dots P_n \subseteq \text{ann}(P)$. Thus $\text{ann}(P) \neq 0$.

Now , any non-zero symmetric ideal of the semi-prime ring R must intersect S_0 (this follows from the contrapositive of Proposition 1.4.3 , applied to $I = 0 = \mathcal{R}_e(0)$ and to $\text{ann}(P)$). Thus , there is $s \in S_0$ such that $Ps = 0$ with $s \neq 0$. By Remarks 3.2.1 and 3.3.1 , $P = \underline{\ell}(s) = \underline{r}(s) = \text{ann}(s)$.

Finally, by the above argument any intersection J' of minimal members P_i , $J' = \bigcap_{i \in I \subseteq \{1,2,\dots,n\}} P_i$,

being a semi-prime $*$ -compressible ideal , J' has therefore this unique representation. Thus the map which takes

$I \subseteq \{1,2,\dots,n\}$ to $\bigcap_{i \in I} P_i$

implements a lattice isomorphism from $\mathcal{B}(\{1,2,\dots,n\})$ to the lattice generated by the P_i 's .

Corollary 3.3.3.1 Let R be as in the Theorem 3.3.3 . Then
 R has a finite subdirect representation into Lanski rings.

Proof. For in R there are only a finite number of $*$ -completely prime ideals.

3.4 The Idempotents of *-compressible Rings.

In this last section we show that the nilpotent elements of any *-compressible ring R are all square-zero elements. Of course, this is a corollary of the Theorem 1.4.3 in the case of semi-prime rings. We derive certain informations on the idempotents of R generalizing the classical case.

Theorem 3.4.1 Let R be a *-compressible ring. Then for any square-zero element $x \in R$, $xx^* = 0$. Moreover, $x \in K$.

Proof. From $x^2 = 0$, we see that

$$x^*x(x + x^*)^2xx^* = x^*x(xx^* + x^*x)xx^* = x^*x^2x^*xx^* + x^*xx^*x^2x^* = 0.$$

Since R is *-compressible, we get $x^*x(x + x^*)xx^* = 0$, that is ,

$$x^*xx^*xx^* = 0 \text{ and so, } (xx^*)^3 = 0. \text{ Thus } (xx^*)^4 = 0 \text{ and then } xx^* = 0.$$

By Proposition 1.4.0, we have $x^*x = 0$. Since $(x + x^*)^2 = xx^* + x^*x = 0$, $x + x^* = 0$, and so $x \in K$.

Theorem 3.4.2 Let R be a *-compressible ring. Then all nilpotent elements of R are square-zero elements.

Proof. We first prove that $a^3 = 0$ implies $a^2 = 0$. In fact, let $b = a^2$. Then $b^2 = 0$ and so, $b \in K$ (Theorem 3.4.1).

Since $bTb \subseteq T$ and $(bTb)^2 = 0$, $bTb = 0$. Similarly, since $-bNb \subseteq N$ and $(bNb)^2 = 0$, $bNb = 0$. Consequently, for any $s \in S_0$, we have $(bs + (bs)^*)^2 = (bs - sb)^2 = bsbs + sbsb - bs^2b - sb^2s = 0$, thus $bs + (bs)^* = 0$, that is, $bs = sb$ for all $s \in S_0$. This shows that b centralizes all elements in S_0 . Now let $s \in S_0$, we have $(asa)^2 = asa^2sa = asbsa = abs^2a = a^3s^2a = 0 \cdot s^2a = 0$, and so, $asa \in K$ (Theorem 3.4.1). Consequently, we have $(asa)^* = -asa = a^*sa^*$. Setting $s = a + a^*$, we get $-a(a+a^*)a = a^*(a+a^*)a^*$ and so, $-aa^*a = a^*aa^*$. Multiplying on the right handside by a , we get $-aa^*a^2 = (a^*a)^2$, that is, $-aa^*b = (a^*a)^2$. Since b commutes with $aa^* \in N$, $(a^*a)^2 = -aa^*b = -baa^* = -a^3a^* = 0$. Thus $a^*a = 0$ and then $aa^* = 0$ (Proposition 1.4.0). Hence $(a + a^*)^3 = a^3 + a^{*3} = 0$ and so, $a + a^* = 0$, that is, $a \in K$ (Theorem 3.4.1). Thus $-b = -a^2 = aa^* \in N$. From $(-b)^2 = 0$, we get $b = 0$ and so, $a^2 = b = 0$.

Secondly we prove that $a^4 = 0$ implies $a^2 = 0$. In fact, let $b = a^2$. Then $b^2 = 0$, so b centralizes S_0 . Then for any

$s \in S_0$, $(asa)^3 = (asa^2sa)(asa) = asbsbsa = as^2b^2sa = 0$ and consequently, $(asa)^2 = 0$. Thus $asa \in K$ and then $a^*sa^* = -asa$.

Setting $s = a + a^*$, we get $a^*(a+a^*)a^* = -a(a+a^*)a$.

Multiplying this equality on the left handside by a , we get

$$(aa^*)^2 + aa^*^3 = -a^2a^*a = -ba^*a = -a^*ab = -a^*a^3 , \text{ that is ,}$$

$$aa^*^3 + a^*a^3 = -(aa^*)^2. \text{ Squaring this equality, we have}$$

$$(aa^*^3 + a^*a^3)^2 = (aa^*)^4 , \text{ that is , } aa^*^3aa^*^3 + a^*a^3a^*a^3 = (aa^*)^4,$$

Since $a^{*2} = -b$ and b commutes with $s \in S_0$, we get

$$aa^*(-b)aa^*(-b) + a^*aba^*ab = (aa^*)^4 \text{ and so ,}$$

$$aa^*aa^*(-b)^2 + a^*aa^*ab^2 = (aa^*)^4, \text{ that is , } (aa^*)^4 = 0.$$

Hence $aa^* = 0$ and then $a^*a = 0$. Thus $(a + a^*)^4 = (a^2 + a^{*2})^2$

$= a^4 + a^{*4} = 0$ and so, $a + a^* = 0$. Then $a \in K$ and $b \in N$. From

$b^2 = 0$, we get $b = 0$ and $a^2 = b = 0$.

Next assume $a^5 = 0$. It follows that $a^6 = 0$. By the above property, we have that $(a^2)^3 = 0$ implies $(a^2)^2 = 0$. By the above we get $a^2 = 0$.

Assume that $a^{3n} = 0$ ($n \geq 2$) implies $a^2 = 0$. Now let $a^{3(n+1)} = 0$. Then $(a^{n+1})^3 = 0$ and so, $(a^{n+1})^2 = 0$. Thus $a^{2n+2} = 0$ and hence $a^{3n} = 0$. By the induction assumption, we have $a^2 = 0$. The proof is now complete.

Theorem 3.4.3 Any semi-prime *-compressible ring is without nil right ideals ($\neq 0$).

Proof. For let I be a nil right ideal. By Theorem 3.4.2, each $x \in I$ is a square-zero element (no reference here to the subdirect representation of R). If I were not zero, then by Levitzki's Theorem, R would contain a non-zero nilpotent ideal, a contradiction. We must conclude that $I = 0$.

Theorem 3.4.4 If R is a 2-torsion free or a semi-prime ring which is *-compressible, then for any idempotent e of R , $ee^* = e^*e$. Consequently, all idempotents of R centralize all traces of R (in fact, all symmetric elements of R in the case of 2-torsion free).

Proof. Case 1. R is 2-torsion free.

Since $(ee^* - ee^*e)^2 = 0$, by Theorem 3.4.1, $ee^* - ee^*e \in K$.

Thus $-(ee^* - ee^*e) = ee^* - e^*ee^*$. Multiplying on the left handside by e , we get $ee^*e = e^*ee^*e$. Multiplying on the right handside

by e^* , we get $e^*ee^* = e^*ee^*e$. Hence $e^*ee^* = ee^*e$. Consequently,

$(e^*e - ee^*)^2 = 0$ and so, $e^*e - ee^* \in K \cap S$. Since R is 2-torsion free, $K \cap S = 0$ and then $e^*e = ee^*$.

We now prove that any idempotent e of R centralize all symmetric elements of R . Since $e^*e = ee^*$, ee^* is a symmetric idempotent element of R and so it is central. Thus for any $s \in S$, $(ee^*s - ese^*)^* = see^* - ese^* = ee^*s - ese^*$. That is, $ee^*s - ese^* \in S$. Since $(ee^*s - ese^*)^2 = 0$, $ee^*s - ese^* \in K$ and then $ee^*s - ese^* \in K \cap S$. Since R is 2-torsion free, $K \cap S = 0$. Thus $ee^*s - ese^* = 0$ for all $s \in S$. Since $e + e^* - ee^*$ is a symmetric idempotent of R , we have $(e + e^* - ee^*)s = s(e + e^* - ee^*)$ for all $s \in S$. Then $es + e^*s = se + se^*$. Multiplying on the left handside by e , we get $es + ee^*s = ese + ese^*$. Since $ee^*s = ese^*$, we get $es = ese$ for all $s \in S$. Similarly we get $se = ese$ for all $s \in S$. This shows that e centralizes S .

Case 2. R is semi-prime..

Let $(e_i)_{i \in I}$ be the image of e under a subdirect representation

as in Theorem 1.4.3. If $e_i e_i^* = 0$ then $e_i^* e_i = 0$ (Proposition 1.4.0).

If $e_i e_i^* \neq 0$ then $e_i^* e_i \neq 0$ and so for any $x_i \in R_i$, we have

$e_i e_i^* (e_i^* x_i - x_i) = 0$ and $(x_i - x_i e_i^*) e_i^* e_i = 0$. Thus $e_i^* x_i = x_i$ and

$x_i = x_i e_i^*$. Hence $e_i^* = 1$ in R_i . Similarly, $e_i = 1$ in R_i . Thus

$e_i e_i^* = e_i^* e_i$ for all $i \in I$. This shows that $ee^* = e^*e$.

We now prove that all idempotents e of R centralize all traces of R . Since $ee^* = e^*e$, ee^* is a symmetric idempotent and hence is central. Consequently, by setting $s = x + x^*$, we have

$$\begin{aligned} ee^*s - ese^* &= ee^*(x + x^*) - e(x + x^*)e^* \\ &= (ee^*x - ex^*e^*) + (ee^*x - ex^*e^*)^* . \end{aligned}$$

That is, $(ee^*s - ese^*) \in T$ for all $s \in T$. Since $(ee^*s - ese^*)^2 = 0$

and R is *-compressible, $ee^*s - ese^* = 0$ and then, $ee^*s = ese^*$

for all $s \in T$. Now $e + e^* - ee^*$ is symmetric idempotent and so is

central. Thus for any $s \in T$, we have

$$(e + e^* - ee^*)s = s(e + e^* - ee^*) \text{ and then } es + e^*s = se + se^*.$$

Multiplying by e , we get $es + ee^*s = ese + ese^*$. Since $ee^*s = ese^*$,

we get $es = ese$ for $s \in T$. Similarly, we $se = ese$. This shows

that e centralizes all $s \in T$.

Theorem 3.4.5   Let R be a $\ast$ -compressible ring. Then any one of the following conditions implies that R is compressible.

- (1) R does not have skew-symmetric nilpotents.
- (2) R is semi-prime and is generated (as a ring) by its traces.

Proof.   Assuming (1). This is a combination of theorems 3.4.1 and 3.4.2 .

Assuming (2). Clearly the condition is inherited by homomorphic image of R . Now R has a subdirect representation into Lanski rings R_i . Since all idempotents of R_i centralize all traces of R_i (Theorem 3.4.4) and since R_i is generated by its traces, all idempotents of R_i are central. Hence R_i can not be an order in the 2×2 matrices over a field . By Lanski's characterization , R_i must be either a skewdomain or a subring of a direct sum of skewdomains and hence R_i is compressible. Consequently , R is compressible.

We conclude this work by few questions.

Concerning the structure theorem 2.3.1 , one would like to investigate the $*$ -compressible rings R which split into a direct product of $*$ -completely prime rings and to investigate rings which have $*$ -orthogonal completion $\bar{R}$ in the direction of a recent study by W. Burgess and R. M. Raphael [5]. This , however , would require some developments on rings of quotients and injective envelopes of rings with involution, which are beyond the scope of this research. Other more elementary questions are the following :

Question 1. Is any $*$ -compressible ring strongly $*$ -compressible in the sense that $as^n b = 0$ implies $asb = 0$ for any integer n and $s \in S_0$?

Question 2. Do the idempotent elements of any $*$ -compressible ring commute with their involutes ?

Question 3. Can we exchange the role of the symmetric elements and the skew-symmetric elements in the forgoing study ?

We hope to come to all these problems in forthcoming studies.

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