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A Higher Order Panel Method for Linearized Unsteady Subsonic Aerodynamics

by

James Henry Sykes, B.A.Sc., B.Sc., Dip. Eng.

**A thesis submitted to the Faculty
of Graduate Studies and Research
in partial fulfilment of the
requirements for the degree of**

Master of Engineering

**Department of
Mechanical and Aerospace Engineering
Ottawa-Carleton Institute for Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario
Canada
February 1994**

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**A Higher Order Panel Method for Linearized
Unsteady Subsonic Aerodynamics**

submitted by

J. H. Sykes B.A.Sc., B.Sc., Dip. Eng.
in partial fulfilment of the requirements
for the degree of Master of Engineering

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Abstract

An aeroservoelastic system can be used to reduce or eliminate aeroelastic instabilities that may exist in an aircraft's flight envelope. The main difficulty in modeling an aeroelastic system for control design lies in the representation of the unsteady aerodynamics. The potential flow solution methods represent relatively inexpensive and efficient tools for predicting the unsteady aerodynamics for preliminary design purposes.

An approximate solution for the linear integral equation of unsteady small disturbance subsonic potential flow is presented. The formulation is based on that used by Morino to develop the SOUSSA unsteady aerodynamic panel method. SOUSSA uses a zeroth order method while the present work involves the formulation and implementation of a new higher order method that uses quadratic and cubic representation of the potential and surface geometry.

Several steady and unsteady test cases are analyzed to provide validation of the higher order method. The results are compared with SOUSSA results from the literature and analytic solutions where possible. Preliminary convergence studies indicate that the higher order method converges twice as fast as the zeroth order method.

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Nomenclature

a	Speed of sound
a_∞	Speed of sound in the free stream
$\hat{a}$	Difference between the local speed of sound and the free stream value
$\bar{a}_{1,2}$	Basis vectors (see Equation (2.114). (2.92))
$A_i^{(n_1)}$	Weights in the ξ_1 direction for Gaussian integration
$A_j^{(n_2)}$	Weights in the ξ_2 direction for Gaussian integration
$A_{i,j}$	Entries from Pascal's triangle. i.e. coefficients resulting from the chain rule expansion of derivatives
AR	Aspect ratio
b	Wing span
$\tilde{b}_k$	See Equation (2.80)
B_h	See Equation (2.43)
$B_{i,h}$	See Equation (2.47)
$\bar{c}$	Chord length
$\tilde{c}_k$	See Equation (2.80)
C_P	Pressure coefficient
C_h	Cardinal spline shape functions on the body
$\bar{C}_h$	See Equation (2.43)
$C_{i,h}$	See Equation (2.47)
C_n	Cardinal spline shape functions on the wake
$\tilde{C}_{\Phi_k}$	Velocity potential B-spline amplitudes
$\tilde{C}_{\Psi_k}$	Normalwash B-spline amplitudes
$\tilde{C}_{C_{p,k}}$	Pressure coefficient B-spline amplitudes

$\ddot{d}_k$	See Equation (2.80)
$\frac{d}{dt}$	Linear substantial derivative
$\frac{D}{Dt}$	Substantial derivative
D_h	See Equation (2.43)
D_{ih}	See Equation (2.47)
e_h	Generalized aerodynamic force
e_p	See Equation (3.3)
e_{L2}	$L2$ norm error
E	Domain function (see Equation (2.33))
$[E^1_{hm}]$	Matrix that relates the generalized coordinates to the normalwash at the control point on the body surface (see Equation (2.104))
$[E^{10}]$	Frequency independent part of $[E^1]$ matrix
$[E^{11}]$	Frequency dependent part of $[E^1]$ matrix
$[E^2_{ik}]$	Aerodynamic influence coefficient matrix that relates the velocity potential B-spline amplitudes to the normalwash B-spline amplitudes on the body surface (see Equation (2.86))
$[E^3_{hk}]$	Matrix that relates the velocity potential B-spline amplitudes to the pressure coefficient at the control points on the body surface (see Equation (2.126))
$[E^4_{nh}]$	Matrix that relates the pressure coefficient B-spline amplitudes to the generalized aerodynamic forces at the control points on the body surface (see Equation (2.134))
$\ddot{f}_l$	See Equation (2.80)
F_n	See Equation (2.43)
F_{in}	See Equation (2.47)
$\dot{g}_l$	See Equation (2.80)
G	A constant $G = 4$ (see Equation (3.3))
G	Green's function ($G \equiv 0$ at infinity)
G_n	See Equation (2.43)
G_{in}	See Equation (2.47)

h_o	Initial integration integral spacing for Romberg integration
h_k	Interval size in Romberg integration
H	Number of control points on the body
H_p	See Equation (3.2)
$\hat{i}, \hat{j}, \hat{k}$	Unit basis vectors in the physical space
i	Square root of -1 (i.e. $i = \sqrt{-1}$)
$\hat{I}, \hat{J}, \hat{K}$	Unit basis vectors in the Prandtl-Glauert space
$[I]$	Identity matrix
J	Jacobian
k	Reduced frequency ($k = \omega l / U_\infty$)
K	Kernel function
l	A reference length
$m(n)$	Identifies the trailing edge node which is on the same streamline as the n^{th} wake node
M	Free stream Mach number
$\tilde{M}_h$	Non-dimensional generalized force mode
$\tilde{M}_m$	Prescribed non-dimensional mode shapes used to define the motion of the body
$\hat{n}$	Unit normal vector in the physical $\hat{i}, \hat{j}, \hat{k}$ space
n_p	Number of integration points in the p direction
n_x	Component of unit normal $\hat{n}$ in x direction
N	Unit normal vector in the $\hat{I}, \hat{J}, \hat{K}$ space
N	Number of elements in one direction in an $N \times N$ grid
N_c	Number of elements in chordwise direction
N_X	Component of unit normal N in X direction
N	Number of control points on the wake
N	Unit normal in Prandtl-Glauert space
N	B-spline shape function

N_X	Component of unit normal N in X direction
N_k	B-spline shape functions on the body
N_l	B-spline shape functions on the wake
$N_{j,k}$	$N_k(\vec{P}_j)$ (see Equation (2.69))
N_{kj}	Transpose of N_{jk}
N_s	Number of elements in spanwise direction
p	Static pressure
p	Non-dimensional Laplace variable. i.e. complex frequency $p = (\gamma + i)k$
$\vec{p}$	Point having coordinates x, y, z in $\hat{i}, \hat{j}, \hat{k}$ space
p_∞	Free stream static pressure
$\vec{p}_{te}$	A point on the trailing edge of the body in the $\hat{i}, \hat{j}, \hat{k}$ space
$\vec{p}_w$	Point on the wake in the $\hat{i}, \hat{j}, \hat{k}$ space
$\vec{P}^*$	Field point in the $\hat{I}, \hat{J}, \hat{K}$ space
$\vec{P}_{TE}$	A point on the trailing edge of the body in the $\hat{I}, \hat{J}, \hat{K}$ space
$\vec{P}_W$	Point on the wake in the $\hat{I}, \hat{J}, \hat{K}$ space
q_m	Non-dimensional generalized coordinates
R	Relative distance from sending point to the field point
s	Laplace variable $s = U_\infty p / l$
S	Scalar equation defining body surface
S_B	Scalar equation that defines the steady state unperturbed geometry of the body
S_U	Scalar equation that defines the unsteady perturbation of the steady state unperturbed geometry of the body
S_{nh}	See Equation (2.41)
t	Time
T	Non-dimensional time in the Prandtl-Glauert space (see Equation (2.28))

T^*	A specific non-dimensional time in the Prandtl-Glauert space
T_m^k	Integral estimate at stage m and level k of Romberg integration
$\bar{u}$	Perturbation of the steady reference geometry
$\bar{u}_{1,2}$	See Equation (2.93)
U_∞	Velocity of the free stream in the $\hat{i}$ direction
$[(UL)_{nk}]$	See Equation (2.83)
$\vec{V}$	Fluid velocity with respect to the $\hat{i}, \hat{j}, \hat{k}$ frame of reference
x_a, x_b	Limits of integration
$\bar{x}$	Defines the body surface in terms of ξ_1, ξ_2
x, y, z	Coordinates in the physical space
x_*, y_*, z_*	Field point coordinates in the $\hat{i}, \hat{j}, \hat{k}$ space. i.e. coordinates of a specific point in the fluid volume
X, Y, Z	Non-dimensional coordinates in the Prandtl-Glauert space (see Equation (2.28))
X^*, Y^*, Z^*	Field point coordinates in the $\hat{I}, \hat{J}, \hat{K}$ space. i.e. coordinates of a specific point in the fluid volume
y_{ik}	See Equation (2.85)
Y_{ih}	See Equation (2.54)
Z_{ih}	See Equation (2.55)
z_{ik}	See Equation (2.85)
α	Angle of attack
$[\alpha_{hk}]$	Matrix that transforms B-splines to cardinal splines on the body
$[\alpha_{nl}]$	Matrix that transforms B-splines to cardinal splines on the wake
β	Subsonic compressibility correction factor $\beta = \sqrt{1 - M^2}$
γ	$Re\{p\}/Im\{p\}$ (p is the non-dimensional Laplace parameter)
δ	Dirac-delta function
δ_{ih}	Kronecker-Delta function (see Equation (2.56))
δ_p, δ	See Equation (3.13) and Figure 3.9

$\Delta\varphi$	Difference in disturbance velocity potential between upper and lower sides of the wake surface (see Equation (2.22))
$\Delta\vec{n}$	See Equation (2.91)
ϵ	Absolute error
η_1, η_2	Coordinates obtained from a linear transformation of ξ_1, ξ_2 used for evaluation of the singular integrals
θ	$\Theta l/U_\infty$
Θ	Time for a disturbance to propagate from the sending point, $\vec{P}$, to the field point, $\vec{P}^*$
Θ	Polar coordinate for a point on the surface of a circle
$\hat{\Theta}$	See Equation (2.31)
Θ_{th}	See Equation (2.47)
Θ_{in}	See Equation (2.47)
Θ_{th}	See Equation (2.47)
Θ_{in}	See Equation (2.47)
ξ_1, ξ_2	Convected curvilinear coordinates
$\xi_i^{(n_1)}$	Nodes in the ξ_1 direction for Gaussian integration
$\xi_j^{(n_2)}$	Nodes in the ξ_2 direction for Gaussian integration
Π	Time required for a fluid particle to travel from $\vec{P}_{TE}$ to $\vec{P}_W$ (see Equation (2.35))
ρ	Fluid density
ρ_∞	Free stream fluid density
$\hat{\rho}$	Difference between the local fluid density and the free stream value
σ_B	Surface of the body in the $\hat{i}, \hat{j}, \hat{k}$ space
Σ_B	Surface of the body in the $\hat{I}, \hat{J}, \hat{K}$ space
Σ_W	Surface of the wake in the $\hat{I}, \hat{J}, \hat{K}$ space
Σ_n	Quadrilateral surface panel on the wake
Σ_h	Quadrilateral surface panel on the body

τ	Time required for a fluid particle to travel from $\bar{p}_{te}$ to $\bar{p}_w$ (see Equation (2.24))
τ	Thickness to chord ratio
ϕ	Total velocity potential
φ	Disturbance velocity potential
Φ	Non-dimensional velocity potential in the Prandtl-Glauert space (see Equation (2.28))
Φ_S	Steady part of Φ
Φ_U	Unsteady part of Φ
ψ	Normalwash
Ψ	Non-dimensional normalwash (see Equation (2.28))
Ψ_S	Steady part of Ψ
Ψ_U	Unsteady part of Ψ
ω	Angular frequency

Subscripts and Superscripts

l	Lower side of wake surface
te, TE	Trailing edge
u	Upper side of wake surface
w, W	Wake

Special Notations

∇	Del operator
$[]^\Theta$	Evaluation at time $T^* - \Theta$
$\Delta()$	Discontinuity of $()$ across wake. i.e. $()_u - ()_l$
$(\dot{})$	Derivative of $()$ with respect to time
$(\sim)$	Laplace transform of $()$

$O()$	Order of ()
$Re\{ \}$	Real part of { }
$Im\{ \}$	Imaginary part of { }

Acronyms, Designations, and Abbreviations

AICM	Aerodynamic Influence Coefficient Matrix
ATFM	Aerodynamic Transfer Function Matrix
CFD	Computational Fluid Dynamics
DLM	Doublet Lattice Method
NASA	National Aeronautics and Space Administration
SOUSSA	Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics
SSOUP	Subsonic Steady, Oscillatory, and Unsteady Panel method
SSOUP-Z	Subsonic Steady, Oscillatory, and Unsteady Panel method – Zeroth Order Method
SSOUP-H	Subsonic Steady, Oscillatory, and Unsteady Panel method – Higher Order Method

Chapter 1

Introduction

1.1 Motivation and Statement of the Problem

Aeroelasticity involves the interaction of structural, aerodynamic, and inertial forces. This interaction may lead to static or dynamic instabilities in an aircraft's flight envelope. Aeroservoelasticity employs an active control system to attempt to reduce or eliminate the aeroelastic instabilities. The active control system includes sensors that detect the onset of the instabilities. The signals from the sensors are fed back through an appropriate control law to the control surface actuators, thus closing the feedback loop. The control surfaces deflect and affect the stability characteristics of the aircraft. Aeroservoelastic systems have applications including flutter suppression and gust alleviation. The use of an aeroservoelastic system may extend an aircraft's flight envelope or allow for the use of alternative structural designs that may result in a lighter, more efficient aircraft.

Control analysis techniques for aeroservoelastic systems^{[1]–[4]} require that the equations of motion be cast in a linear time-independent form. The frequency dependent unsteady aerodynamic forces are of primary importance and must be included in the analysis. The control system analysis techniques usually involve the equations of motion in the complex Laplace domain and employ finite state modeling to represent the unsteady aerodynamics (Karpel^[2]). Thus, an aerodynamic transfer function matrix (**ATFM**) is required at discrete complex frequencies in the

Laplace domain (s -plane). For general application, the ATFM must be available for complex configurations with arbitrary motions.

The classical approach has been to compute the ATFM at discrete frequencies for simple harmonic motions only (i.e. points along the imaginary axis in the s -plane). However, modern control analyses require the ATFM at arbitrary frequencies in the entire Laplace domain. To this end, several authors^{[5]–[9]} have developed techniques that use the ATFM for oscillatory motions at discrete frequencies to generate rational approximate solutions for arbitrary motions. All of these methods involve approximating the aerodynamic influence coefficients with rational functions of the Laplace variable, s . References [5]–[9] all use approximations based on the Padé approximants in which the ATFM is approximated by ratios of polynomials in s . The STARS aeroservoelastic analysis program^[1] uses both Padé and least squares approximations to curve fit the oscillatory aerodynamics. Once the oscillatory data has been curve fit, the approximating functions are assumed to be valid in the rest of the complex plane. However, the approximation is not expected to be very accurate for complex frequencies far from the imaginary axis. The main difficulty in modeling an aeroelastic system for control design lies in the representation of the unsteady aerodynamic loads (Karpel^[2]).

1.2 Review of Methods for Calculating Unsteady Aerodynamics

The area of computational fluid dynamics (CFD) involves computational techniques that use numerical approximations to solve the governing partial differential equations. The numerical approximations include discretization schemes and may employ finite element or finite difference methods to obtain a solution. Many aeroelastic phenomena involve viscosity dominated flow effects such as separation and vortex shedding. The Navier-Stokes equations are necessary to accurately model the flow field in these situations. However, the Euler equations can be used when viscous effects are negligible. Navier-Stokes calculations for three

dimensional flows (e.g. Chaderjian *et al*^[10]) have been primarily involved in computations on rigid configurations (Guruswamy^[11]). However, it is necessary to account for the flexibility of the aircraft structure for accurate prediction of the unsteady aerodynamics. The Euler equations were coupled with the modal structural equations of motion in a computer program called ENSAERO^[12]. The capabilities of ENSAERO were extended by including coupling between the Navier-Stokes and modal structural equations. The ability of the program to simulate unsteady laminar and turbulent viscous flows is demonstrated in references [11] and [13].

Although Euler or Navier-Stokes solutions are required for detailed design, the computational requirements and associated costs make them prohibitive for most preliminary design tasks. The Euler and Navier-Stokes solution methods must model the entire flow field under consideration and thus require a large number of elements. In the early stages of the design process, a large number of iterations are required before the optimum design configuration can be achieved. Thus, fast and efficient computation methods are required. Also, the increasing use of lightweight flexible structures in modern aircraft designs requires the consideration of aeroelastic effects at earlier design stages. The potential flow solution methods represent a relatively inexpensive and efficient alternative for preliminary design purposes.

The use of the potential flow equation rests on the assumption that the flow is isentropic and irrotational. For situations where non-linear behavior must be considered, such as transonic flows, techniques^{[14]-[16]} have been developed to include the non-linear terms from the full velocity potential equation. These methods require modeling of the flow field and often employ finite differencing schemes for numerical solution. However, for many purely subsonic flow situations, the non-linear terms can be neglected. Thus, most potential solutions for subsonic flow are based upon a linearized small perturbation version of the full unsteady potential equation. In linearized theory, the aerodynamic forces and moments depend upon the motion in a linear manner and the analysis is greatly simplified. Historical reviews of linear and non-linear unsteady potential flow methods are given in references [17]-[19].

1.3 Two-Dimensional Potential Flow Solution Methods

The most notable classical unsteady potential flow solution is the unsteady loading of a rigid thin airfoil oscillating in an incompressible flow. The solution was found almost simultaneously by several authors in the 1930's (Yates^[19]). The most well-known solution was found by Theodorsen^[20] in 1935. Theodorsen solved Laplace's equation for an airfoil harmonically oscillating in pitch and plunge. The resulting expressions gave the time-varying lift and moment about the elastic axis in terms of the pitch and plunge velocities and accelerations, and also in terms of a circulation function that depends on the reduced frequency. The original formulation also provided for the inclusion of an oscillating trailing edge control surface.

In an attempt to model three-dimensional flow, Theodorsen's solution was used in a simple strip theory^[19]. The wing was divided into spanwise strips on which the aerodynamic forces were assumed to be those associated with a two-dimensional airfoil undergoing the same motion as that particular section of the three-dimensional wing. This approach ignored spanwise interactions between strips. In an attempt to model the real situation more accurately, the strip theory was modified by allowing for spanwise variation in lift curve slope and location of the aerodynamic centre. The method was further extended using two dimensional compressible flow theory to account for effects of compressibility on the magnitude and phase of the aerodynamic forces.

1.4 Integral Representation of the Potential Flow Solution

The solution of the potential flow problem may be written in an integral form^[21] by application of Green's theorem^[22]. The use of Green's theorem results in a representation of the solution for the potential in terms of an integral of singularity

distributions over the surface of the body and wake. The steady flow panel methods are developed along similar lines. A review of the development of the steady panel methods is given in reference [23].

1.4.1 Subsonic Lifting Surface Methods

The integral equation formulation for a three-dimensional wing oscillating in subsonic compressible flow was given by Küssner in 1940^[19]. Küssner used the lifting surface approximation in which the wing is represented by a reference surface with zero thickness. His formulation led to an integral equation relating known normal velocity (downwash) on the reference surface to the unknown potential on the surface by means of a kernel function. Several authors^{[24][25]} were able to reduce Küssner's kernel function to a form suitable for solution^[19]. The resulting equation is usually referred to as the downwash-pressure integral equation and numerical approaches for its solution include both discrete and continuous methods.

The global pressure mode method^[24], sometimes referred to as the kernel-function method, is a continuous method that uses the acceleration potential to avoid having to integrate over the wake. If the disturbance associated with the motion is small, then the acceleration potential differs from the local pressure only by a constant. The boundary condition on the wake requires that the pressure discontinuity go to zero. Thus, discontinuities in acceleration potential (pressure) only exist on the body and integration need not be carried out over the wake. The distribution of pressure over a wing may be represented by a distribution of acceleration potential or pressure doublets over the planform (reference surface). The solution for the pressure distribution may be obtained by assuming no flow through the reference surface.

This solution process is implemented by representing the pressure at any point on the wing by a linear combination of pressure mode shapes. The pressure mode shapes are defined throughout the entire surface and are thus referred to as global. The global pressure modes are chosen such that the Kutta condition of zero pressure

difference at the trailing edge is automatically satisfied. In such a representation, each pressure mode is multiplied by a complex weight. The method selects the weights such that the boundary condition is satisfied exactly at a set of collocation or downwash control points. Thus, solving for the unknown pressure distribution requires solving for the unknown weights. To simplify the procedure, the integration of the kernel functions is broken up into several parts^[19].

The global pressure mode method can be applied to virtually any lifting surface planform for oscillatory motion as long as the first derivatives of the vibration modes used to represent the motion are continuous. With little modification, the method may be applied to configurations with multiple lifting surfaces. However, devising suitable global pressure modes for some configurations can be difficult. The inclusion of control surfaces presents a particular problem because of the sudden changes in geometry that arise due to control surface defections. Although an explicit wake model is not required in acceleration potential methods, a wake is implicitly defined downstream of each point on the lifting surface. Thus, applications to problems in which the implicit wake impinges on another body causes problems. In general, the method may be applied to lifting surfaces only and thus body or interference effects cannot be accounted for without special treatment.

The downwash-pressure integral equation may be solved alternatively by representing the distribution of pressure by discrete surface singularities. The lifting surface is divided into quadrilateral elements or panels. The pressure loading within each panel is assumed to be concentrated along a lifting line of pressure doublets (acceleration potential doublets) located at the quarter chord of each panel. To solve for the doublet line strength, a collocation procedure is used where the downwash control point is located at the three quarter chord point of each panel (see Figure 1.1). This discrete method where the wing is divided into a lattice of concentrated pressure line doublets is known as the doublet lattice method (DLM)^[26]. The DLM is a finite element lifting surface method that reduces to the vortex lattice method for steady flow (i.e. zero frequency). Although the approximation of the pressure distribution over the wing is somewhat crude, the method has the distinct

advantage that the user does not have to devise suitable global pressure modes that are defined over the entire lifting surface. The DLM is generally more applicable to multiple lifting surfaces with arbitrary configurations than the global pressure mode method. The strength of the concentrated pressure line doublets can vary discontinuously from panel to panel. Thus, lifting surfaces with sharp corners or control surfaces can be more easily modeled.

Although interference effects between lifting surfaces and bodies cannot be accounted for directly, most implementations of the DLM include rough approximations of body effects on wing loading. Kálmán *et al*^[27] used lifting surface elements to form an annular or ring wing that can be used to represent cylindrical bodies such as the fuselage or wing stores. Giesing *et al*^[28] uses an image system^[29] that eliminates the need for lifting surface elements on the body surface. These methods of accounting for body influence work adequately for conventional subsonic transport aircraft configurations where the fuselage is cylindrical over most of its length. However, for more complex configurations such as modern military fighter aircraft, such representation is difficult or impractical. Also, effects of wing thickness are not included in the DLM since the wing is approximated as a lifting surface with zero thickness.

1.4.2 Subsonic Surface Panel Methods

Although the lifting surface methods perform well for many problems, there is a need for a method that can account for wing-body interference and thickness effects and that can also be applied to arbitrary configurations. Morino^[30] has noted that the lifting surface/interference methods are usually quite cumbersome to use and always require human intervention. He further states that the use of these methods is to be considered an art, and therefore they cannot be easily automated. Several authors have developed methods that couple acceleration potential lifting surface theory with a surface panel method to improve the modeling of the wing-body interference. A method developed by Roos *et al*^[31] uses the DLM in combination with an unsteady source surface panel method to model the wing-

body interactions more effectively. The body surfaces are easily modeled using the surface panel method and thus the combination lifting surface/surface panel method is also relatively easy to use. A more recent method by Chen *et al*^[32] uses a similar approach and is able to account for the real body wake that may form due to flow separation. However, these methods are valid for harmonic oscillatory motion only.

A formulation by Morino^{[33][34]} applies the Green's function method to the linearized unsteady velocity potential equation to obtain a representation of the potential at any point in the flow field in terms of the values of the potential and its normal derivative on the actual body surface. An integral equation is obtained by imposing the condition that, as the flow field evaluation point approaches the surface, the value of the potential at the evaluation point approaches the value of the potential on the surface. The resulting integral equation is referred to as an integro-differential delay equation for the potential on the body. The integral equation can be interpreted as a distribution of velocity potential sources and doublets over the body and wake surfaces. This formulation accounts for the actual body shape, thickness, and orientation. Also, the formulation is valid for completely arbitrary motion. Thus, analyses involving oscillatory as well as diverging and converging motions are possible. The surface integrals may be evaluated computationally by spatial discretization with surface panels. The integrations may be carried out in the time domain or the Laplace transform may be used to obtain complex frequency domain results.

Morino's formulation for linearized subsonic flow was implemented in a computer program called SOUSSA (Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics) under contract by the NASA Langley research centre in Hampton, Virginia^{[35][36]}. The body surface is discretized by quadrilateral panels with hyperboloidal surfaces. The velocity potential within each panel is assumed to be spatially constant. The SOUSSA program is thus termed zeroth order. The Laplace transform is used to obtain a solution that gives complex frequency domain aerodynamics in the form of pressure coefficients and generalized forces. The only

inputs are the locations of the corners of the quadrilateral panels, Mach number, and reduced frequency. The wake is generated automatically, is assumed to have zero thickness, and extends downstream in the direction of the free stream. The wake geometry remains frozen during the calculation. The program is general and flexible and is very simple to use (Tseng *et al*^[37]). SOUSSA is applicable to general shapes such as complete aircraft or other bodies having arbitrary shapes, linear motions, and linear deformations. Numerous validations and critiques of the SOUSSA code are available in the literature^{[18][30][37]–[43]}. Generally, the SOUSSA results show good agreement with theory and experiment. The most general configurations to which the SOUSSA program has been applied are the space shuttle^[18] and the F-5 fighter wing model, with and without wing-mounted stores^[40].

The literature contains numerous references to future enhancements and improvements of the SOUSSA code including formulation and implementation of higher order elements to represent the geometry and velocity potential. In the zeroth order representation, there is a discontinuity in the velocity potential between panels. Also, the use of hyperboloidal surface geometry does not maintain continuity of slope between adjacent panels. Thus, the resulting surface is somewhat rough. It has been noted by Tinoco *et al*^[44] that lower order steady panel methods exhibit sensitivity to paneling density and distributions and that the success of these methods is highly dependent on the skill of the individual user. These problems are largely overcome by higher order methods in which the geometry and potential vary in a more continuous manner. It is anticipated that the implementation of higher order unsteady panel methods will also result in similar performance improvements. It is stated in the literature that the SOUSSA enhancements were to be implemented in a replacement code called UTSA, and that its development was continuing at a low level of effort (Yates^[18]). However, there do not seem to be any publications on this work in the recent literature.

1.5 Scope of the Present Work

The present work involves the implementation of an approximate solution for the linear integral equation of unsteady small disturbance subsonic potential flow in a computer program called SSOUP (Subsonic, Steady, Oscillatory, and Unsteady Panel method). The SSOUP program includes a zeroth order implementation (SSOUP-Z) similar to that of the SOUSSA code, as well as a so-called higher order implementation (SSOUP-H) involving bi-cubic variation of the body and wake surface geometry and bi-quadratic variation of the velocity potential.

The linearization of the fully unsteady velocity potential equation using the small perturbation assumption is presented. The numerical formulation of the solution for the integral equation using the finite element method is fully developed and presented. The formulation and discretization is based upon the method used by the SOUSSA program as outlined in reference [35].

The implementations of the numerical formulation for both the zeroth and higher order methods in the SSOUP program are presented. The similarities and differences between the zeroth order implementation of SSOUP and the SOUSSA program are also outlined.

Several steady and unsteady test cases that were analysed using SSOUP are presented and compared with theory and SOUSSA results from the literature. These test cases serve to provide validation of the SSOUP code. Limited comparisons are also made between the performance of the zeroth and higher order versions of the SSOUP program. Finally, improvements and recommendations for future work are suggested and discussed.

Chapter 2

Theoretical Formulation

2.1 Formulation of the Problem

2.1.1 Equation for Velocity Potential

Consider the irrotational, isentropic, inviscid flow of a perfect gas. Define a frame of reference $\hat{i}, \hat{j}, \hat{k}$ such that the fluid has a velocity $\vec{V}$. The velocity potential, ϕ , can be defined such that the fluid velocity is given by

$$\vec{V} = \nabla \phi \quad (2.1)$$

where : ∇ is the del operator

A differential equation for the velocity potential can be derived from the continuity equation and the Euler equation. Dowell^[45] derives the full equation for the unsteady velocity potential as

$$\nabla^2 \phi - \frac{1}{a^2} \left[\frac{\partial}{\partial t} (\nabla \phi \cdot \nabla \phi) + \frac{\partial^2 \phi}{\partial t^2} + \nabla \phi \cdot \nabla \left(\frac{\nabla \phi \cdot \nabla \phi}{2} \right) \right] = 0 \quad (2.2)$$

*where : ϕ is the velocity potential
a is the speed of sound*

Equation (2.2) is nonlinear and is very difficult to solve. Thus, we seek a linear approximation.

To linearize equation (2.2), consider a body immersed in a uniform flow. The flow velocity is U_∞ in the $\hat{i}$ direction. Assume

$$\phi = U_\infty x + \varphi \quad (2.3)$$

where : U_∞ is the undisturbed or free stream velocity
 φ is the disturbance velocity potential arising
 from the presence of the body

If equation (2.3) is substituted into (2.2), the result is

$$\begin{aligned} & \nabla^2 \varphi - \frac{1}{a^2} \left[2(U_\infty \hat{i} + \nabla \varphi) \cdot \frac{\partial}{\partial t} (U_\infty \hat{i} + \nabla \varphi) + \frac{\partial^2 \varphi}{\partial t^2} \right. \\ & \left. + (U_\infty \hat{i} + \nabla \varphi) \cdot \nabla \left(\frac{U_\infty^2}{2} + U_\infty \hat{i} \cdot \nabla \varphi + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \right) \right] \\ & = \nabla^2 \varphi - \frac{1}{a^2} \left[\frac{\partial^2 \varphi}{\partial t^2} + 2U_\infty \frac{\partial^2 \varphi}{\partial x \partial t} + U_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} + O(\varphi^2) \right] = 0 \end{aligned} \quad (2.4)$$

Further, if φ is small (small perturbation assumption), then we can neglect the nonlinear terms of $O(\varphi^2)$. Using (2.3), we can write the substantial derivative as

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} + \nabla \phi \cdot \nabla \\ &= \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} + \nabla \varphi \cdot \nabla \end{aligned} \quad (2.5)$$

Thus, neglecting nonlinear terms, we define the linear substantial derivative as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\infty \hat{i} \cdot \nabla \quad (2.6)$$

Now, assuming

$$a = a_\infty + \hat{a} \quad (2.7)$$

where : a_∞ is the speed of sound in the free stream
 $\hat{a}$ is the difference between the local speed of
 sound and the free stream value

and using the small perturbation assumption to say that $\hat{a}$ is small, and neglecting higher order terms, we can write (2.4) as

$$\frac{\partial^2 \varphi}{\partial x^2} (1 - M^2) + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{a_\infty^2} \left[\frac{\partial^2 \varphi}{\partial t^2} + 2U_\infty \frac{\partial^2 \varphi}{\partial x \partial t} \right] = 0$$

or

$$\nabla^2 \varphi - \frac{1}{a_\infty^2} \frac{d^2 \varphi}{dt^2} = 0 \quad (2.8)$$

Equation (2.8) is the small perturbation, linearized equation of the unsteady velocity potential.

2.1.2 Pressure Coefficient

Once the linearized velocity potential equation has been solved, the pressure can be evaluated from φ using the Bernoulli theorem for barotropic fluids. A barotropic fluid is a fluid for which the pressure is only a function of density^[46]. The exact nonlinear Bernoulli theorem for barotropic fluids is

$$\begin{aligned} \int_{p_\infty}^p \frac{1}{\rho} dp &= - \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \right) \\ &= - \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} + O(\varphi^2) \right) \end{aligned} \quad (2.9)$$

If we perform the integration assuming

$$\rho = \rho_\infty + \hat{\rho} \quad (2.10)$$

where : ρ_∞ is the density of the fluid in the free stream
 $\hat{\rho}$ is the difference between the local fluid
density and the free stream value

and we use the small perturbation assumption to assume that $\hat{\rho}$ is small, and we neglect the nonlinear terms, we get

$$\frac{p - p_\infty}{\rho_\infty} = - \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right) \quad (2.11)$$

Thus, using the definition for the pressure coefficient, we have

$$C_P = \frac{p - p_\infty}{\frac{1}{2} \rho_\infty U_\infty^2} = -2 \frac{1}{U_\infty^2} \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right) \quad (2.12)$$

or

$$C_P = -\frac{2}{U_\infty^2} \frac{d\varphi}{dt} \quad (2.13)$$

2.1.3 Boundary Condition

The boundary condition on the body is that the normal fluid velocity equals the normal body velocity. If the surface of the body is represented by

$$S(x, y, z, t) = 0 \quad (2.14)$$

then the boundary condition can be written as^[35]

$$\frac{\partial S}{\partial t} + \nabla \phi \cdot \nabla S = 0 \quad (2.15).$$

Using (2.3), we get

$$\begin{aligned} \frac{\partial S}{\partial t} + (U_\infty \hat{i} + \nabla \varphi) \cdot \nabla S &= 0 \\ \frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial x} + \nabla \varphi \cdot \nabla S &= 0 \end{aligned} \quad (2.16)$$

or

$$\nabla \varphi \cdot \nabla S = -\left(\frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial x}\right) \quad (2.17)$$

Multiplying both sides of (2.17) by $1/|\nabla S|$ gives

$$\nabla \varphi \cdot \frac{\nabla S}{|\nabla S|} = -\frac{dS}{dt} \frac{1}{|\nabla S|} \quad (2.18)$$

and defining the normalwash ψ as

$$\psi = \nabla \varphi \cdot \hat{n} = \nabla \varphi \cdot \frac{\nabla S}{|\nabla S|} \quad (2.19)$$

we get the boundary condition as

$$\psi = -\frac{dS}{dt} \frac{1}{|\nabla S|} \quad (2.20)$$

The boundary condition on the wake states that the pressure difference across the wake must be zero. Using the exact nonlinear Bernoulli theorem, we can write

$$\left[\frac{\partial}{\partial t} + U_{\infty} \frac{\partial}{\partial x} + \frac{1}{2} (\nabla \varphi_u + \nabla \varphi_l) \cdot \nabla \right] (\varphi_u - \varphi_l) = 0 \quad (2.21)$$

or, written in terms of the substantial derivative,

$$\frac{D}{Dt} \Delta \varphi = 0 \quad \text{where : } \Delta \varphi = \varphi_u - \varphi_l \quad (2.22)$$

In (2.22), the subscripts u and l are used to indicate the upper and lower sides of the wake.

Equation (2.22) can be integrated to give

$$\Delta \varphi(\vec{p}_w, t) = \Delta \varphi(\vec{p}_{te}, t - \tau) \quad (2.23)$$

where τ is the time required for a fluid particle to travel from the point $\vec{p}_{te}$ on the trailing edge of the body to the point $\vec{p}_w$ on the wake. Using the small perturbation assumption, the fluid particle velocity on the wake can be approximated by the free stream velocity, $U_{\infty} \hat{i}$. Thus,

$$\tau = \frac{1}{U_{\infty}} (x_w - x_{te}) \quad (2.24)$$

2.1.4 Generalized Aerodynamic Force

The generalized aerodynamic force, e_h , acting on the non-dimensional mode, $\vec{M}_h$, is given by

$$e_h = \frac{-1}{l^2} \iint_{\sigma_B} C_P \hat{n} \cdot \vec{M}_h(\sigma_B) d\sigma_B \quad (2.25)$$

where : $\hat{n}$ is the local surface unit normal vector
 l is a reference length
 σ_B is the surface of the body

The use of the non-dimensional mode shapes, $\vec{M}_h$, allows the total aerodynamic force to be separated into components that coincide with structural modes that

may be of particular interest. The mode shapes may be defined such that the total resultant aerodynamic force is separated into the standard lift and drag forces.

2.2 Subsonic Integral Equation

Equation (2.8) may be solved by developing an integral equation that relates the velocity potential on the body surface to the normalwash on the body surface. The development is based on the Green's function method or Green's theorem^[47]. Green's theorem obtains a representation of the velocity potential in terms of its value and its derivatives, on the surface of the body and wake.

In the development of the integral equation the unsteady motion of the aircraft is assumed to consist of small infinitesimal perturbations to the steady state motion. Further, the motion of the aircraft before time $t = 0$ is assumed to be steady state. Mathematically, the surface of the body is represented by

$$S(x, y, z, t) = S_B(x, y, z) + \epsilon S_U(x, y, z, t) = 0 \quad (2.26)$$

where : S_B is the steady state unperturbed geometry of the body

S_U is the unsteady perturbation

(note that for small perturbation $\epsilon \ll 1$)

The Green's function for the small perturbation, linearized unsteady velocity potential equation is defined by

$$\nabla^2 G - \frac{1}{a_\infty^2} \left(\frac{d^2 G}{dt^2} \right) = \delta(x - x_*, y - y_*, z - z_*, t - t_*) \quad (2.27)$$

where : G is the Green's function ($G \equiv 0$ at infinity)

δ is the Dirac delta function

x_*, y_*, z_* are the coordinates of a specific point in the fluid volume (x_*, y_*, z_* are constants)

For subsonic flow, the Green's function is an impulsive source^[35] and this allows for completely arbitrary time variation of the motion.

Morino^[35] derives the subsonic integral equation from the Green's theorem using the following nondimensionalization and Prandtl-Glauert transformation

$$\begin{aligned}\Phi &= \varphi/U_\infty l \\ X &= x/\beta l \\ Y &= y/l \\ Z &= z/l \\ T &= U_\infty t/l\end{aligned}\tag{2.28}$$

where : x, y, z are the surface coordinates in the physical $\hat{i}, \hat{j}, \hat{k}$ space
 l is a reference length

and β is the subsonic compressibility correction factor

$$\beta = \sqrt{(1 - M^2)}\tag{2.29}$$

The integral equation is then

$$\begin{aligned}4\pi E(\bar{P}^*)\Phi(\bar{P}^*, T^*) &= - \iint_{\Sigma_B} [\Psi]^\Theta \frac{1}{R} d\Sigma_B \\ &+ \iint_{\Sigma_B} \left\{ [\Phi]^\Theta \frac{\partial}{\partial N} \left(\frac{1}{R} \right) - \left[\frac{\partial \Phi}{\partial T} \right]^\Theta \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} \right\} d\Sigma_B \\ &+ \iint_{\Sigma_w} \left\{ [\Delta \Phi]^\Theta \frac{\partial}{\partial N_w} \left(\frac{1}{R} \right) - \left[\frac{\partial \Delta \Phi}{\partial T} \right]^\Theta \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N_w} \right\} d\Sigma_w\end{aligned}\tag{2.30}$$

In (2.30) $\bar{P}^* = (X^*, Y^*, Z^*)$ is the particular evaluation point in the fluid volume, as opposed to $\bar{P} = (X, Y, Z)$, which refers to a general point in the domain of the integration on the surface of the body. (Note that $\bar{P}$ and $\bar{P}^*$ are in the transformed $(\hat{I}, \hat{J}, \hat{K})$ space, not in the physical $(\hat{i}, \hat{j}, \hat{k})$ space.) Also,

$$\begin{aligned}R &= [(X - X^*)^2 + (Y - Y^*)^2 + (Z - Z^*)^2]^{\frac{1}{2}} \\ \Theta &= U_\infty \theta/l = [M(X - X^*) + R]M/\beta \\ \hat{\Theta} &= [M(X^* - X) + R]M/\beta\end{aligned}\tag{2.31}$$

where : M is the free stream Mach number

and the symbol $[]^\Theta$ indicates evaluation at a time $T^* - \Theta$. That is,

$$[]^\Theta = [] \Big|_{T=T^*-\Theta} \quad (2.32)$$

E is the domain function, defined as

$$\begin{aligned} E &= 1 \quad \text{outside } \Sigma_B \\ &= \frac{1}{2} \quad \text{on } \Sigma_B \text{ (regular point)} \\ &= 0 \quad \text{inside } \Sigma_B \end{aligned} \quad (2.33)$$

The relationship between $\Delta\Phi$ on the wake and $\Delta\Phi$ on the body can also be written in non-dimensional form from (2.23) and (2.28) as

$$\Delta\Phi(\vec{P}_W, T) = \Delta\Phi(\vec{P}_{TE}, T - \Pi) \quad (2.34)$$

where Π is found from (2.24) as

$$\Pi = \beta(X_W - X_{TE}) \quad (2.35)$$

2.3 Numerical Solution of the Subsonic Integral Equation

The subsonic integral equation is given by (2.30). A numerical approximation is required to solve (2.30) for general complex configurations. A finite-element representation is used to approximate the variation of the velocity potential, normalwash, and the surface geometry.

2.3.1 Spatial Discretization of Φ and Ψ using Cardinal Splines

The variation of the velocity potential, Φ , and the normalwash, Ψ , can be represented using a finite element representation of the form

$$\begin{aligned} \Phi(\vec{P}, T - \Theta) &= \sum_{h=1}^H \Phi_h(T - \Theta_h) C_h(\vec{P}) \\ \Psi(\vec{P}, T - \Theta) &= \sum_{h=1}^H \Psi_h(T - \Theta_h) C_h(\vec{P}) \end{aligned} \quad (2.36)$$

where:

H is the number of control points on the body.

$\Phi(\bar{P}, T - \Theta)$ and $\Psi(\bar{P}, T - \Theta)$ are the velocity potential and normalwash at an arbitrary point, $\bar{P}$, on the body surface at an arbitrary time $(T - \Theta)$. [Note that Θ is the delay time from the sending point, $\bar{P}$, to the receiving point, $\bar{P}^*$, and is therefore, a function of $\bar{P}$ and $\bar{P}^*$. i.e. $\Theta = \Theta(\bar{P}, \bar{P}^*)$].

$\Phi_h(T - \Theta_h)$ and $\Psi_h(T - \Theta_h)$ are the velocity potential and normalwash at control point, $\bar{P}_h$, on the body surface. Θ_h is the delay time from sending control point $\bar{P}_h$ to the receiving point, $\bar{P}^*$. The control point locations, $\bar{P}_h$, are constants. Thus, the delay time, Θ_h , is constant for a specific receiving point, $\bar{P}^*$, and only varies with $\bar{P}^*$. [i.e. $\Theta_h = \Theta_h(\bar{P}^*)$]

$C_h(\bar{P})$ are the global shape functions with unit magnitude at control point h and zero magnitude at all other control points. Their magnitudes are non-zero between control points. The cardinal spline has its global shape functions defined in this way.

A similar equation can be written for the variation of $\Delta\Phi$ over the wake

$$\Delta\Phi(\bar{P}_W, T - \Theta) = \sum_{n=1}^N \Delta\Phi_n(T - \Theta_n) C_n(\bar{P}_W) \quad (2.37)$$

where:

N is the number of control points on the wake.

$\Delta\Phi(\bar{P}_W, T - \Theta)$ is the value of $\Delta\Phi$ at an arbitrary point, $\bar{P}_W$, on the wake at an arbitrary time $(T - \Theta)$. [Note that Θ is the delay time from the sending point, $\bar{P}_W$, on the wake to the receiving point, $\bar{P}^*$, and is therefore a function of both $\bar{P}_W$ and $\bar{P}^*$. i.e. $\Theta = \Theta(\bar{P}_W, \bar{P}^*)$].

$\Delta\Phi_n(T - \Theta_n)$ is the velocity potential jump at control point $\bar{P}_n$ on the wake surface. Θ_n is the delay time from sending control point, $\bar{P}_n$, to the receiving point, $\bar{P}^*$. The control point locations, $\bar{P}_n$, are constants. Thus, the delay time, Θ_n , is constant for a specific receiving point, $\bar{P}^*$, and only varies with $\bar{P}^*$. [i.e. $\Theta_n = \Theta_n(\bar{P}^*)$]

$C_n(\bar{P})$ are the global shape functions with unit magnitude at wake control point n and zero magnitude at all other wake control points.

The wake is divided into strips. If the streamline that passes through control point $\bar{P}_n^W$ on the wake is followed back to the trailing edge of the body, the trailing edge point $\bar{P}_{m(n)}^{(TE)}$ can be located (Figure 2.1). Using (2.34), we can write $\Delta\Phi_n$ on the wake in terms of $\Delta\Phi_{m(n)}^{TE}$ (i.e. $\Delta\Phi$ at the point $\bar{P}_{m(n)}^{(TE)}$ on the trailing edge) as

$$\Delta\Phi_n(T) = \Delta\Phi_{m(n)}^{(TE)}(T - \Pi_n) \quad (2.38)$$

where : $m(n)$ identifies the trailing edge node which is on the same streamline as the n^{th} wake node $\bar{P}_n^W$

Π_n is the convection time from the trailing edge point, $\bar{P}_{m(n)}^{(TE)}$, to the wake control point, $\bar{P}_n^W$

Also,

$$\Delta\Phi_{m(n)}^{(TE)}(T) = \Phi_{h_u}(T) - \Phi_{h_l}(T) \quad (2.39)$$

where : Φ_u, Φ_l are the velocity potentials on the upper and lower surfaces at trailing edge point m

From (2.39), we can write

$$\Delta\Phi_{m(n)}^{(TE)}(T) = \sum_{h=1}^H S_{nh} \Phi_h(T) \quad (2.40)$$

where

$$\begin{aligned} S_{nh} &= +1 \quad \text{if } \bar{P}_h = \bar{P}_{m(n)}^{TE} \text{ is on the upper side of } \Sigma_B \\ &= -1 \quad \text{if } \bar{P}_h = \bar{P}_{m(n)}^{TE} \text{ is on the lower side of } \Sigma_B \\ &= 0 \quad \text{for all other points } \bar{P}_h \text{ on } \Sigma_B \end{aligned} \quad (2.41)$$

Now substituting (2.36) and (2.37) into (2.30), we have

$$\begin{aligned} 2E(\bar{P}^*)\Phi(\bar{P}^*, T^*) &= \sum_{h=1}^H B_h \Psi_h(T^* - \Theta_h) \\ &+ \sum_{h=1}^H C_h \Phi_h(T^* - \Theta_h) + \sum_{h=1}^H D_h \dot{\Phi}_h(T^* - \Theta_h) \\ &+ \sum_{n=1}^N F_n \Delta\Phi_n(T^* - \Theta_n) + \sum_{n=1}^N G_n \Delta\dot{\Phi}_n(T^* - \Theta_n) \end{aligned} \quad (2.42)$$

where

$$\begin{aligned}
 B_h &= -\frac{1}{2\pi} \iint_{\Sigma_B} C_h(\bar{P}) \frac{1}{R} d\Sigma_B \\
 C_h &= \frac{1}{2\pi} \iint_{\Sigma_B} C_h(\bar{P}) \frac{\partial}{\partial N} \left(\frac{1}{R} \right) d\Sigma_B \\
 D_h &= -\frac{1}{2\pi} \iint_{\Sigma_B} C_h(\bar{P}) \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} d\Sigma_B \\
 F_n &= \frac{1}{2\pi} \iint_{\Sigma_W} C_n(\bar{P}) \frac{\partial}{\partial N_u} \left(\frac{1}{R} \right) d\Sigma_W \\
 G_n &= -\frac{1}{2\pi} \iint_{\Sigma_W} C_n(\bar{P}) \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N_u} d\Sigma_W
 \end{aligned} \tag{2.43}$$

Now define a regular point to be a point on the body surface at which there exists a unique tangent plane to the surface (i.e. not a corner or sharp edge). If we set $\bar{P}^*$ to be a regular point on the body surface, then from (2.33) we have $E(\bar{P}^*) = 1/2$. If we further define the location of $\bar{P}^*$, and specify that it is located at control point i on Σ_B . (i.e. $\bar{P}^* = \bar{P}_i$), then (2.42) becomes

$$\begin{aligned}
 \Phi(\bar{P}_i, T^*) &= \sum_{h=1}^H B_h \Psi_h(T^* - \Theta_h) + \sum_{h=1}^H C_h \Phi_h(T^* - \Theta_h) \\
 &+ \sum_{h=1}^H D_h \dot{\Phi}_h(T^* - \Theta_h) + \sum_{n=1}^N F_n \Delta \Phi_n(T^* - \Theta_n) \\
 &+ \sum_{n=1}^N G_n \Delta \dot{\Phi}_n(T^* - \Theta_n)
 \end{aligned} \tag{2.44}$$

Using (2.39) and (2.40), we can write

$$\begin{aligned}
 \Delta \Phi_n(T^* - \Theta_n) &= \Delta \Phi_{m(n)}^{(TE)}(T^* - \Theta_n - \Pi_n) \\
 &= \sum_{h=1}^H S_{nh} \Phi_h(T^* - \Theta_n - \Pi_n)
 \end{aligned} \tag{2.45}$$

and now writing $\Phi(\bar{P}_i, T^*)$ as $\Phi_i(T^*)$ and substituting (2.45) into (2.44), we get

$$\begin{aligned}
\Phi_i(T^*) = & \sum_{h=1}^H B_{ih} \Psi_h(T^* - \Theta_{ih}) + \sum_{h=1}^H C_{ih} \Phi_h(T^* - \Theta_{ih}) \\
& + \sum_{h=1}^H D_{ih} \dot{\Phi}_h(T^* - \Theta_{ih}) + \sum_{n=1}^N \sum_{h=1}^H F_{in} S_{nh} \Phi_h(T^* - \Theta_{in} - \Pi_n) \\
& + \sum_{n=1}^N \sum_{h=1}^H G_{in} S_{nh} \dot{\Phi}_h(T^* - \Theta_{in} - \Pi_n)
\end{aligned} \tag{2.46}$$

where

$$\begin{aligned}
B_{ih} &= B_h \big|_{\vec{p}^* = \vec{p}_i} \\
C_{ih} &= C_h \big|_{\vec{p}^* = \vec{p}_i} \\
D_{ih} &= D_h \big|_{\vec{p}^* = \vec{p}_i} \\
\Theta_{ih} &= \Theta_h \big|_{\vec{p}^* = \vec{p}_i} \\
F_{in} &= F_n \big|_{\vec{p}^* = \vec{p}_i} \\
G_{in} &= G_n \big|_{\vec{p}^* = \vec{p}_i} \\
\Theta_{in} &= \Theta_n \big|_{\vec{p}^* = \vec{p}_i}
\end{aligned} \tag{2.47}$$

2.3.2 Laplace-Domain Analysis

Equation (2.46) can be solved for any arbitrary time. However, in chapter 1, the need for the aerodynamic transfer function matrix (ATFM) as a function of the complex frequency was outlined. Thus, we require a representation of (2.46) in the Laplace domain.

It was stated earlier that the motion of the aircraft was assumed to be steady state before time $T = 0$. This implies that if the potential and normalwash can be separated into steady and unsteady parts as

$$\begin{aligned}
\Phi &= \Phi_S + \Phi_U \\
\Psi &= \Psi_S + \Psi_U
\end{aligned} \tag{2.48}$$

then the unsteady part is equal to zero for $T < 0$.

$$\Phi_U = 0, \quad \Psi_U = 0 \quad (T < 0) \tag{2.49}$$

Also, if we are only interested in the solution for a time $T < T_{max}$, then the wake length need only extend as far as a fluid particle traveling at velocity U_∞ can travel in time T_{max} . Thus, the wake need not extend to infinity and can be truncated at some finite distance behind the body.

We can now take the Laplace transform of (2.46) and obtain

$$\begin{aligned}\tilde{\Phi}_i(p) = & \sum_{h=1}^H B_{ih} \tilde{\Psi}_h(p) e^{-p\Theta_{ih}} + \sum_{h=1}^H C_{ih} \tilde{\Phi}_h(p) e^{-p\Theta_{ih}} \\ & + \sum_{h=1}^H D_{ih} p \tilde{\Phi}_h(p) e^{-p\Theta_{ih}} + \sum_{n=1}^N \sum_{h=1}^H F_{in} S_{nh} \tilde{\Phi}_h(p) e^{-p(\Theta_{in} + \Pi_n)} \\ & + \sum_{n=1}^N \sum_{h=1}^H G_{in} S_{nh} p \tilde{\Phi}_h(p) e^{-p(\Theta_{in} + \Pi_n)}\end{aligned}\quad (2.50)$$

where : $(\sim)$ indicates the Laplace transform
 p is the nondimensional Laplace parameter

and

$$p = (\gamma + i)k \quad (2.51)$$

$$\text{where } k = \frac{\omega l}{U_\infty} \quad (2.52)$$

Note that $p = 0$ corresponds to steady state motion and $p = ik$ corresponds to harmonic motion. ($\gamma = \text{Re}\{p\}/\text{Im}\{p\}$ and $i = \sqrt{-1}$)

If we collect the $\tilde{\Phi}$ and $\tilde{\Psi}$ terms in equation (2.50). we get in matrix notation

$$\begin{bmatrix} \tilde{Y}_{ih} \end{bmatrix} \begin{Bmatrix} \tilde{\Phi}_h \end{Bmatrix} = \begin{bmatrix} \tilde{Z}_{ih} \end{bmatrix} \begin{Bmatrix} \tilde{\Psi}_h \end{Bmatrix} \quad (2.53)$$

where

$$\begin{aligned}\tilde{Y}_{ih} = & \delta_{ih} - (C_{ih} + pD_{ih})e^{-p\Theta_{ih}} \\ & - \sum_{n=1}^N (F_{in} + pG_{in})e^{-p(\Theta_{in} + \Pi_n)} S_{nh}\end{aligned}\quad (2.54)$$

$$\tilde{Z}_{ih} = B_{ih} e^{-p\Theta_{ih}} \quad (2.55)$$

and δ_{ih} is the Kronecker-Delta function, defined as

$$\begin{aligned}\delta_{ih} &= 1 & i=h \\ &= 0 & i \neq h\end{aligned}\tag{2.56}$$

The coefficients B_{ih} , C_{ih} , D_{ih} , F_{in} , and G_{in} in (2.54) and (2.55) involve integrals that include the Global shape functions or cardinal splines. The cardinal splines are nonzero over the entire surface. Consequently, integration involving the cardinal spline shape functions is somewhat complicated. It is possible to transform (2.54) and (2.55) so that the integrals involve the more usual B-splines. When using the B-splines, each shape function is nonzero over only a few elements. Therefore, integration is greatly simplified.

2.3.3 Spatial Discretization of Φ and Ψ using B-Splines

The cardinal spline shape functions, $C_h(\vec{P})$, can be written in terms of the B-spline shape functions, $N_k(\vec{P})$, via the transformation

$$C_h(\vec{P}) = \sum_{k=1}^H \alpha_{hk} N_k(\vec{P})\tag{2.57}$$

or in matrix form

$$\begin{bmatrix} \alpha_{hk} \end{bmatrix} \left\{ N_k(\vec{P}) \right\} = \left\{ C_h(\vec{P}) \right\}\tag{2.58}$$

To solve for the α_{hk} 's we use the definition of the cardinal splines. The cardinal spline shape functions are defined such that they have unit magnitude at control point h , and are zero at all other control points. i.e.

$$\begin{aligned}C_{hj} &= 1 & h=j \\ &= 0 & h \neq j\end{aligned}\tag{2.59}$$

where

$$C_{hj} = C_h(\vec{P}_j)\tag{2.60}$$

Thus, $[C_{hj}]$ is the identity matrix $[I]$. (2.58) becomes

$$\begin{bmatrix} \alpha_{hk} \end{bmatrix} \begin{bmatrix} N_k(\vec{P}_j) \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}\tag{2.61}$$

and the solution is

$$\begin{bmatrix} \alpha_{hk} \end{bmatrix} = \begin{bmatrix} N_k(\bar{P}_j) \end{bmatrix}^{-1} \quad (2.62)$$

(Note that the $N_k(\bar{P})$ are continuous functions, while the $N_k(\bar{P}_j)^{-1}$ entries are constants.) We now have a relationship between the B-splines and the cardinal splines.

Returning to the finite element representation, we can now write the variation of Φ and Ψ in terms of the B-splines.

$$\begin{aligned} \Phi(\bar{P}, T) &= \sum_{h=1}^H \Phi_h(T) \sum_{k=1}^H \alpha_{hk} N_k(\bar{P}) \\ \Psi(\bar{P}, T) &= \sum_{h=1}^H \Psi_h(T) \sum_{k=1}^H \alpha_{hk} N_k(\bar{P}) \end{aligned} \quad (2.63)$$

or

$$\begin{aligned} \Phi(\bar{P}, T) &= \sum_{h=1}^H \sum_{k=1}^H \Phi_h(T) \alpha_{hk} N_k(\bar{P}) \\ \Psi(\bar{P}, T) &= \sum_{h=1}^H \sum_{k=1}^H \Psi_h(T) \alpha_{hk} N_k(\bar{P}) \end{aligned} \quad (2.64)$$

In matrix form, we have

$$\begin{aligned} \Phi(\bar{P}, T) &= \left\{ \Phi_h(T) \right\}^T \begin{bmatrix} \alpha_{hk} \end{bmatrix} \left\{ N_k(\bar{P}) \right\} \\ \Psi(\bar{P}, T) &= \left\{ \Psi_h(T) \right\}^T \begin{bmatrix} \alpha_{hk} \end{bmatrix} \left\{ N_k(\bar{P}) \right\} \end{aligned} \quad (2.65)$$

Following the same procedure, the variation of $\Delta\Phi$ can also be represented using the B-splines, $N_l(\bar{P})$, on the wake as

$$\Delta\Phi(\bar{P}, T) = \sum_{n=1}^N \sum_{l=1}^N \Delta\Phi_n(T) \alpha_{nl} N_l(\bar{P}) \quad (2.66)$$

where

$$\begin{bmatrix} \alpha_{nl} \end{bmatrix} = \begin{bmatrix} N_l(\bar{P}_i) \end{bmatrix}^{-1} \quad (2.67)$$

and $\bar{P}_i$ are the control points on the wake.

Now for H evaluation points, $\vec{P}_j$, on the body, we have

$$\begin{aligned}\Phi(\vec{P}_j, T) &= \sum_{h=1}^H \sum_{k=1}^H \Phi_h(T) \alpha_{hk} N_k(\vec{P}_j) \\ \Psi(\vec{P}_j, T) &= \sum_{h=1}^H \sum_{k=1}^H \Psi_h(T) \alpha_{hk} N_k(\vec{P}_j)\end{aligned}\quad (2.68)$$

or, in matrix form

$$\begin{aligned}\left\{ \Phi(\vec{P}_j, T) \right\}^T &= \left\{ \Phi_h(T) \right\}^T \begin{bmatrix} \alpha_{hk} \end{bmatrix} \begin{bmatrix} N_k(\vec{P}_j) \end{bmatrix} \\ \left\{ \Psi(\vec{P}_j, T) \right\}^T &= \left\{ \Psi_h(T) \right\}^T \begin{bmatrix} \alpha_{hk} \end{bmatrix} \begin{bmatrix} N_k(\vec{P}_j) \end{bmatrix}\end{aligned}\quad (2.69)$$

Rearranging, we get

$$\begin{aligned}\left\{ \Phi(\vec{P}_j, T) \right\} &= \begin{bmatrix} N_{jk} \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \left\{ \Phi_h(T) \right\} \\ \left\{ \Psi(\vec{P}_j, T) \right\} &= \begin{bmatrix} N_{jk} \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \left\{ \Psi_h(T) \right\}\end{aligned}\quad (2.70)$$

where

$$\begin{aligned}\begin{bmatrix} N_{jk} \end{bmatrix} &= \begin{bmatrix} N_{kj} \end{bmatrix}^T \\ \begin{bmatrix} \alpha_{kh} \end{bmatrix} &= \begin{bmatrix} \alpha_{hk} \end{bmatrix}^T\end{aligned}\quad (2.71)$$

or, in indicial notation

$$\begin{aligned}\Phi_j(T) &= \sum_{h=1}^H \sum_{k=1}^H N_{jk} \alpha_{kh} \Phi_h(T) \\ \Psi_j(T) &= \sum_{h=1}^H \sum_{k=1}^H N_{jk} \alpha_{kh} \Psi_h(T)\end{aligned}\quad (2.72)$$

Now taking the Laplace transform of (2.72) gives

$$\begin{aligned}\tilde{\Phi}_j(p) &= \sum_{h=1}^H \sum_{k=1}^H N_{jk} \alpha_{kh} \tilde{\Phi}_h(p) \\ \tilde{\Psi}_j(p) &= \sum_{h=1}^H \sum_{k=1}^H N_{jk} \alpha_{kh} \tilde{\Psi}_h(p)\end{aligned}\quad (2.73)$$

and in matrix, form (2.73) becomes

$$\begin{aligned} \begin{bmatrix} N_{jk} \end{bmatrix} \begin{Bmatrix} \tilde{C}_{\Phi_k}(p) \end{Bmatrix} &= \begin{Bmatrix} \tilde{\Phi}_j(p) \end{Bmatrix} \\ \begin{bmatrix} N_{jk} \end{bmatrix} \begin{Bmatrix} \tilde{C}_{\Psi_k}(p) \end{Bmatrix} &= \begin{Bmatrix} \tilde{\Psi}_j(p) \end{Bmatrix} \end{aligned} \quad (2.74)$$

where

$$\begin{aligned} \begin{Bmatrix} \tilde{C}_{\Phi_k}(p) \end{Bmatrix} &= \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{Bmatrix} \tilde{\Phi}_h(p) \end{Bmatrix} \\ \begin{Bmatrix} \tilde{C}_{\Psi_k}(p) \end{Bmatrix} &= \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{Bmatrix} \tilde{\Psi}_h(p) \end{Bmatrix} \end{aligned} \quad (2.75)$$

or

$$\begin{aligned} \tilde{\Phi}_j(p) &= \sum_{k=1}^H N_{jk} \tilde{C}_{\Phi_k}(p) \\ \tilde{\Psi}_j(p) &= \sum_{k=1}^H N_{jk} \tilde{C}_{\Psi_k}(p) \end{aligned} \quad (2.76)$$

If we now say that the H evaluation points, P_j , are the control points, P_h , we get

$$\begin{aligned} \begin{Bmatrix} \tilde{\Phi}_h(p) \end{Bmatrix} &= \begin{bmatrix} N_{hk} \end{bmatrix} \begin{Bmatrix} \tilde{C}_{\Phi_k}(p) \end{Bmatrix} \\ \begin{Bmatrix} \tilde{\Psi}_h(p) \end{Bmatrix} &= \begin{bmatrix} N_{hk} \end{bmatrix} \begin{Bmatrix} \tilde{C}_{\Psi_k}(p) \end{Bmatrix} \end{aligned} \quad (2.77)$$

If we now substitute (2.77) into (2.50), we get

$$\begin{aligned} \sum_{k=1}^H N_{ik} \tilde{C}_{\Phi_k}(p) &= \sum_{h=1}^H \sum_{k=1}^H R_{ih} N_{hk} \tilde{C}_{\Psi_k}(p) e^{-p\Theta_{ih}} \\ &+ \sum_{h=1}^H \sum_{k=1}^H C_{ih} N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p\Theta_{ih}} \\ &+ \sum_{h=1}^H \sum_{k=1}^H D_{ih} p N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p\Theta_{ih}} \\ &+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H F_{in} S_{nh} N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p(\Theta_{in} + \Pi_n)} \\ &+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H G_{in} S_{nh} N_{hk} p \tilde{C}_{\Phi_k}(p) e^{-p(\Theta_{in} + \Pi_n)} \end{aligned} \quad (2.78)$$

Further, if we define B_h , C_h , D_h , F_n , and G_n in terms of the B -splines as

$$\begin{aligned}
 B_h &= \sum_{k=1}^H \tilde{b}_k \alpha_{kh} \\
 C_h &= \sum_{k=1}^H \tilde{c}_k \alpha_{kh} \\
 D_h &= \sum_{k=1}^H \tilde{d}_k \alpha_{kh} \\
 F_n &= \sum_{l=1}^N \tilde{f}_l \alpha_{ln} \\
 G_n &= \sum_{l=1}^N \tilde{g}_l \alpha_{ln}
 \end{aligned} \tag{2.79}$$

where

$$\begin{aligned}
 \tilde{b}_k &= -\frac{1}{2\pi} \iint_{\Sigma_B} N_k(\vec{P}) \frac{1}{R} d\Sigma_B \\
 \tilde{c}_k &= \frac{1}{2\pi} \iint_{\Sigma_B} N_k(\vec{P}) \frac{\partial}{\partial N} \left(\frac{1}{R} \right) d\Sigma_B \\
 \tilde{d}_k &= -\frac{1}{2\pi} \iint_{\Sigma_B} N_k(\vec{P}) \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} d\Sigma_B \\
 \tilde{f}_l &= \frac{1}{2\pi} \iint_{\Sigma_W} N_l(\vec{P}) \frac{\partial}{\partial N_u} \left(\frac{1}{R} \right) d\Sigma_W \\
 \tilde{g}_l &= -\frac{1}{2\pi} \iint_{\Sigma_W} N_l(\vec{P}) \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N_u} d\Sigma_W
 \end{aligned} \tag{2.80}$$

then (2.78) becomes

$$\begin{aligned}
 \sum_{k=1}^H N_{ik} \tilde{C}_{\Phi_k}(p) &= \sum_{h=1}^H \sum_{k=1}^H \tilde{b}_{ik} \alpha_{kh} N_{hk} \tilde{C}_{\Psi_k}(p) e^{-p\Theta_{ih}} \\
 &+ \sum_{h=1}^H \sum_{k=1}^H \tilde{c}_{ik} \alpha_{kh} N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p\Theta_{ih}} \\
 &+ \sum_{h=1}^H \sum_{k=1}^H \tilde{d}_{ik} p \alpha_{kh} N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p\Theta_{ih}} \\
 &+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H \sum_{l=1}^N \tilde{f}_{il} \alpha_{ln} S_{nh} N_{hk} \tilde{C}_{\Phi_k}(p) e^{-p(\Theta_{in} + \Pi_n)} \\
 &+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H \sum_{l=1}^N \tilde{g}_{il} \alpha_{ln} S_{nh} N_{hk} p \tilde{C}_{\Phi_k}(p) e^{-p(\Theta_{in} + \Pi_n)}
 \end{aligned} \tag{2.81}$$

or,

$$\begin{aligned}
\sum_{k=1}^H N_{ik} \tilde{C}_{\Phi_k}(p) &= \sum_{h=1}^H \sum_{k=1}^H \tilde{b}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \tilde{C}_{\Psi_k}(p) \\
&+ \sum_{h=1}^H \sum_{k=1}^H \tilde{c}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \tilde{C}_{\Phi_k}(p) \\
&+ \sum_{h=1}^H \sum_{k=1}^H p \tilde{d}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \tilde{C}_{\Phi_k}(p) \\
&+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H \sum_{l=1}^N \tilde{f}_{il} \alpha_{ln} e^{-p(\Theta_{in} + \Pi_n)} S_{nh} N_{hk} \tilde{C}_{\Phi_k}(p) \\
&+ \sum_{n=1}^N \sum_{h=1}^H \sum_{k=1}^H \sum_{l=1}^N p \tilde{g}_{il} \alpha_{ln} e^{-p(\Theta_{in} + \Pi_n)} S_{nh} N_{hk} \tilde{C}_{\Phi_k}(p) \quad (2.82)
\end{aligned}$$

Finally, defining

$$\left[(UL)_{nk} \right] = \left[S_{nh} \right] \left[N_{hk} \right] \quad (2.83)$$

and isolating the $\tilde{C}_{\Phi_k}$ and $\tilde{C}_{\Psi_k}$ terms, we get

$$\left[\tilde{y}_{ik} \right] \left\{ \tilde{C}_{\Phi_k} \right\} = \left[\tilde{z}_{ik} \right] \left\{ \tilde{C}_{\Psi_k} \right\} \quad (2.84)$$

where

$$\begin{aligned}
\tilde{y}_{ik} &= N_{ik} - \sum_{h=1}^H \tilde{c}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \\
&- \sum_{h=1}^H p \tilde{d}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \\
&- \sum_{h=1}^H \sum_{n=1}^N \sum_{l=1}^N \tilde{f}_{il} \alpha_{ln} e^{-p(\Theta_{in} + \Pi_n)} (UL)_{nk} \\
&- \sum_{h=1}^H \sum_{n=1}^N \sum_{l=1}^N p \tilde{g}_{il} \alpha_{ln} e^{-p(\Theta_{in} + \Pi_n)} (UL)_{nk} \\
\tilde{z}_{ik} &= \sum_{h=1}^H \tilde{b}_{ik} \alpha_{kh} e^{-p\Theta_{ih}} N_{hk} \quad (2.85)
\end{aligned}$$

or solving for $\{\tilde{C}_{\Phi_k}\}$ we have

$$\left\{ \tilde{C}_{\Phi_k} \right\} = \left[E^2 \right] \left\{ \tilde{C}_{\Psi_k} \right\} \quad (2.86)$$

where $[E^2]$ is the aerodynamic influence coefficient matrix (AICM) that relates the normalwash spline amplitudes to the velocity potential spline amplitudes and is given by

$$\begin{bmatrix} E^2 \end{bmatrix} = \begin{bmatrix} \tilde{y}_{ik} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{z}_{ik} \end{bmatrix} \quad (2.87)$$

Either of (2.84) or (2.53) can be solved numerically. However, equation (2.84) was the actual form that was implemented.

2.4 Formulation of the Normalwash in Terms of Generalized Coordinates

As stated earlier, the motion of the aircraft is assumed to consist of a very small perturbation of the steady state configuration. In developing the integral equation, (2.30), it was assumed that the surface of the body was time independent, or steady state. The unsteady contribution to the solution is retained in the boundary conditions.

The physical boundary condition in terms of a scalar equation defining the body surface, (2.26), was described in section 2.1.3 and is given by (2.20).

$$S(x, y, z, t) = S_B(x, y, z) + \epsilon S_U(x, y, z, t) = 0 \quad (2.26)$$

(for small perturbation $\epsilon \ll 1$)

$$\psi = -\frac{dS}{dt} \frac{1}{|\nabla S|} \quad (2.20)$$

It is more convenient to express the boundary condition in terms of the displacement $\vec{u}$ and surface coordinates ξ_1 and ξ_2 . Assume the surface of the body is given by

$$\vec{r} = \vec{r}(t) = \vec{p}(\xi_1, \xi_2) + \vec{u}(\xi_1, \xi_2, t) \quad (2.88)$$

where $\vec{p}$ is the steady state reference position rigidly connected to the $\hat{i}, \hat{j}, \hat{k}$ space and $\vec{u}$ is the steady or unsteady perturbation to the reference position. The convected curvilinear coordinates (ξ_1, ξ_2) move along with the surface of the body.

Using (2.88) as the definition for the surface of the body and assuming that the perturbation to the reference position is small, Morino^[35] develops the physical boundary condition as

$$\psi = \left(-\hat{i} + \frac{1}{U_\infty} \frac{\partial \vec{u}}{\partial t} \right) \cdot (\hat{n} + \Delta \vec{n}) \quad (2.89)$$

where

$$\hat{n} = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} \quad (2.90)$$

$$\Delta \vec{n} = \frac{(\vec{u}_1 \times \vec{a}_2 + \vec{a}_1 \times \vec{u}_2)}{|\vec{a}_1 \times \vec{a}_2|} \quad (2.91)$$

and

$$\vec{u}_1 = \frac{\partial \vec{p}}{\partial \xi_1}, \quad \vec{a}_2 = \frac{\partial \vec{p}}{\partial \xi_2} \quad (2.92)$$

$$\vec{u}_1 = \frac{\partial \vec{u}}{\partial \xi_1}, \quad \vec{u}_2 = \frac{\partial \vec{u}}{\partial \xi_2} \quad (2.93)$$

Equation (2.89) can be separated into steady, ψ_S , and unsteady, ψ_U , components

$$\begin{aligned} \psi_S &= \left[-\hat{i} \cdot \hat{n} \right] \\ \psi_U &= \left[\frac{1}{U_\infty} \frac{\partial \vec{u}}{\partial t} \cdot (\hat{n} + \Delta \vec{n}) - \hat{i} \cdot \Delta \vec{n} \right] \end{aligned} \quad (2.94)$$

Thus, the steady and unsteady problems can be solved separately. The steady and unsteady solutions may then be combined to give the total solution.

2.4.1 Unsteady Boundary Condition in Terms of Generalized Coordinates

The displacement vector $\vec{u}$ can be written in terms of generalized coordinates as

$$\vec{u}(\xi_1, \xi_2, t) = l \sum_{m=1}^M q_m(t) \vec{M}_m(\xi_1, \xi_2) \quad (2.95)$$

where : q_m are the nondimensional generalized coordinates

$\vec{M}_m$ are prescribed nondimensional mode shapes

l is a reference length

(Note that the non-dimensional modes $\vec{M}_m$ can include the six rigid body modes.)

Because the perturbation, $\vec{u}$, is small, the generalized coordinates, q_m , are also small. Using (2.95) we can write $\Delta\vec{n}$ as

$$\Delta\vec{n} = \sum_{m=1}^M q_m(t) \Delta\vec{n}_m \quad (2.96)$$

where

$$\Delta\vec{n}_m = l \frac{\left(\frac{\partial \vec{M}_m}{\partial \xi_1} \times \vec{a}_2 + \vec{a}_1 \times \frac{\partial \vec{M}_m}{\partial \xi_2} \right)}{|\vec{a}_1 \times \vec{a}_2|} \quad (2.97)$$

Also,

$$\frac{\partial \vec{u}}{\partial t} = l \sum_{m=1}^M \frac{\partial q_m}{\partial t}(t) \vec{M}_m(\xi_1, \xi_2) \quad (2.98)$$

Now substituting (2.96) and (2.98) into (2.94), we get

$$\begin{aligned} \Psi_U &= \frac{l}{U_\infty} \sum_{m=1}^M \frac{\partial q_m}{\partial t} \vec{M}_m \cdot \left(\hat{n} + \sum_{m=1}^M q_m \Delta\vec{n}_m \right) - \hat{i} \cdot \sum_{m=1}^M q_m \Delta\vec{n}_m \\ &= \sum_{m=1}^M \frac{l}{U_\infty} \frac{\partial q_m}{\partial t} \vec{M}_m \cdot \hat{n} + \frac{l}{U_\infty} \frac{\partial q_m}{\partial t} q_m \vec{M}_m \cdot \Delta\vec{n}_m - \hat{i} \cdot \Delta\vec{n}_m q_m \end{aligned} \quad (2.99)$$

The $\vec{M}_m \cdot \Delta\vec{n}_m$ term is nonlinear and can be neglected. Thus, (2.99) becomes

$$\Psi_U = \sum_{m=1}^M \frac{l}{U_\infty} \frac{\partial q_m}{\partial t} \vec{M}_m \cdot \hat{n} - \hat{i} \cdot \Delta\vec{n}_m q_m \quad (2.100)$$

If we now take the Laplace transform of (2.100), we get

$$\begin{aligned} \tilde{\Psi}_U &= \sum_{m=1}^M \frac{l}{U_\infty} s \tilde{q}_m (\vec{M}_m \cdot \hat{n}) - (\hat{i} \cdot \Delta\vec{n}_m) \tilde{q}_m \\ &= \sum_{m=1}^M \left[\frac{l}{U_\infty} s (\vec{M}_m \cdot \hat{n}) - (\hat{i} \cdot \Delta\vec{n}_m) \right] \tilde{q}_m \end{aligned} \quad (2.101)$$

where : s is the dimensional Laplace variable

and now setting

$$s = \frac{U_\infty p}{l} \quad (2.102)$$

we get

$$\tilde{\Psi}_U = \sum_{m=1}^M \left[p \vec{M}_m \cdot \hat{n} - \hat{i} \cdot \Delta\vec{n}_m \right] \tilde{q}_m \quad (2.103)$$

Setting $\tilde{\Psi}_h$ equal to the normalwash at the body surface control points $\vec{p}_h$ and writing (2.103) in matrix form, we have

$$\left[E_{hm}^1 \right] \left\{ \tilde{q}_m \right\} = \left\{ \tilde{\Psi}_h \right\} \quad (2.104)$$

with

$$\left[E_{hm}^1 \right] = \left[p \Lambda \vec{I}_m \cdot \hat{n} - \dot{i} \cdot \Delta \vec{n}_m \right] \Big|_{\vec{p}=\vec{p}_h} \quad (2.105)$$

or using (2.75) to write $\tilde{\Psi}_h$ in terms of the $\tilde{C}_{\Psi_k}$'s, we get

$$\left[\alpha_{kh} \right] \left[E_{hm}^1 \right] \left\{ \tilde{q}_m \right\} = \left\{ \tilde{C}_{\Psi_k} \right\} \quad (2.106)$$

We can substitute (2.106) into (2.86) to get

$$\left\{ \tilde{C}_{\Phi_k} \right\} = \left[E^2 \right] \left[\alpha_{kh} \right] \left[E_{hm}^1 \right] \left\{ \tilde{q}_m \right\} \quad (2.107)$$

2.5 Numerical Evaluation of Pressure Coefficient

The pressure coefficient is evaluated from the linearized small disturbance Bernoulli theorem and is given by equation (2.13) as

$$C_P = -\frac{2}{U_\infty^2} \frac{d\varphi}{dt} \quad (2.13)$$

In terms of the non-dimensional velocity potential, Φ , the pressure coefficient is

$$C_P = -2 \frac{l}{U_\infty} \left(\frac{\partial \Phi}{\partial t} + U_\infty \frac{\partial \Phi}{\partial x} \right) \quad (2.108)$$

The $\frac{\partial \Phi}{\partial x}$ term is evaluated by

$$\frac{\partial \Phi}{\partial x} = \nabla \Phi \cdot \hat{i} \quad (2.109)$$

and $\nabla \Phi$ can be evaluated as

$$\nabla \Phi = \frac{\partial \Phi}{\partial \xi_1} \vec{a}^1 + \frac{\partial \Phi}{\partial \xi_2} \vec{a}^2 + \frac{\partial \Phi}{\partial n} \hat{n} \quad (2.110)$$

where

$$\bar{a}^\alpha = a^{\alpha 1} \bar{a}_1 + a^{\alpha 2} \bar{a}_2 \quad (2.111)$$

with

$$\begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad (2.112)$$

The covariant metric tensor components, $a_{\alpha\beta}$, are

$$a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta \quad (2.113)$$

and the base vectors are given by

$$\bar{a}_\alpha = \frac{\partial \vec{p}}{\partial \xi_\alpha} \quad (2.114)$$

Thus, using (2.109) we can write the pressure coefficient as

$$C_P = -2l \left(\frac{1}{U_\infty} \frac{\partial \Phi}{\partial t} + \hat{i} \cdot \bar{a}^1 \frac{\partial \Phi}{\partial \xi_1} + \hat{i} \cdot \bar{a}^2 \frac{\partial \Phi}{\partial \xi_2} + \hat{i} \cdot \hat{n} \frac{\partial \Phi}{\partial n} \right) \quad (2.115)$$

Now from (2.72) and (2.88), we have

$$\Phi(\vec{p}(\xi_\beta), t) = \sum_{h=1}^H \sum_{k=1}^H N_k(\vec{p}(\xi_\beta)) \alpha_{kh} \Phi_h(t) \quad (2.116)$$

or

$$\Phi(\vec{p}(\xi_\beta), t) = \sum_{k=1}^H N_k(\vec{p}(\xi_\beta)) C_{\Phi_k}(t) \quad (2.117)$$

where

$$C_{\Phi_k}(t) = \sum_{h=1}^H \alpha_{kh} \Phi_h(t) \quad (2.118)$$

Now writing (2.117) for H points $\vec{p}_h$ gives

$$\Phi(\vec{p}_h(\xi_\beta), t) = \sum_{k=1}^H N_{hk} C_{\Phi_k}(t) \quad (2.119)$$

Thus,

$$\frac{\partial \Phi_h}{\partial \xi_\beta}(t) = \sum_{k=1}^H \frac{\partial N_{hk}}{\partial \xi_\beta} C_{\Phi_k}(t) \quad (2.120)$$

and

$$\begin{aligned}\frac{\partial \Phi_h}{\partial t}(t) &= \sum_{k=1}^H N_{hk} \frac{\partial C_{\Phi_k}(t)}{\partial t} \\ &= \sum_{k=1}^H N_{hk} \dot{C}_{\Phi_k}(t)\end{aligned}\quad (2.121)$$

Now writing (2.115) for a point $\bar{p}_h$ and using (2.119), (2.120), and (2.121) we get

$$C_P(\bar{p}_h, t) = \frac{-2l}{U_\infty} \sum_{k=1}^H \left[\dot{C}_{\Phi_k} N_{hk} \right]_{\bar{p}=\bar{p}_h} - 2l \sum_{k=1}^H \hat{i} \cdot \left[\bar{a}^1 C_{\Phi_k} \frac{\partial N_{hk}}{\partial \xi_1} + \bar{a}^2 C_{\Phi_k} \frac{\partial N_{hk}}{\partial \xi_2} \right]_{\bar{p}=\bar{p}_h} \quad (2.122)$$

where the term

$$\hat{i} \cdot \hat{n} \frac{\partial \Phi}{\partial n} = n_z \frac{\partial \Phi}{\partial n} \quad (2.123)$$

has been neglected due to the small perturbation assumption ($n_z \approx 0$). Now taking the Laplace transform of (2.122), we get

$$\tilde{C}_p(\bar{p}_h, s) = \sum_{k=1}^H \left[-2 \frac{sl}{U_\infty} N_{hk} - 2l \hat{i} \cdot \left(\bar{a}^1 \frac{\partial N_{hk}}{\partial \xi_1} + \bar{a}^2 \frac{\partial N_{hk}}{\partial \xi_2} \right) \right]_{\bar{p}=\bar{p}_h} \tilde{C}_{\Phi_k}(s) \quad (2.124)$$

and noting that

$$p = \frac{sl}{U_\infty}$$

we have

$$\tilde{C}_p(\bar{p}_h, p) = \sum_{k=1}^H \left[-2p N_{hk} - 2l \hat{i} \cdot \left(\bar{a}^1 \frac{\partial N_{hk}}{\partial \xi_1} + \bar{a}^2 \frac{\partial N_{hk}}{\partial \xi_2} \right) \right]_{\bar{p}=\bar{p}_h} \tilde{C}_{\Phi_k}(p) \quad (2.125)$$

or, in matrix form,

$$\left[E_{hk}^3 \right] \left\{ \tilde{C}_{\Phi_k} \right\} = \left\{ \tilde{C}_{p,h} \right\} \quad (2.126)$$

where

$$E_{hk}^3 = \sum_{k=1}^H \left[-2p N_{hk} - 2l \hat{i} \cdot \left(\bar{a}^1 \frac{\partial N_{hk}}{\partial \xi_1} + \bar{a}^2 \frac{\partial N_{hk}}{\partial \xi_2} \right) \right]_{\vec{p}=\vec{p}_h} \quad (2.127)$$

If we now express C_p using a similar expression as (2.119), we get

$$\tilde{C}_{P,h} = \sum_{k=1}^H N_{hk}(\xi_\beta) \tilde{C}_{C_{P,k}} \quad (2.128)$$

where

$$\tilde{C}_{C_{P,k}} = \sum_{h=1}^H \alpha_{kh} \tilde{C}_{P,h} \quad (2.129)$$

and we multiply both sides of (2.126) by α_{kh} , then we get

$$\left[\alpha_{kh} \right] \left[E_{hk}^3 \right] \left\{ \tilde{C}_{\Phi_k} \right\} = \left[\alpha_{kh} \right] \left\{ \tilde{C}_{P,h} \right\} \quad (2.130)$$

or, using (2.129), we get

$$\left[\alpha_{kh} \right] \left[E_{hk}^3 \right] \left\{ \tilde{C}_{\xi_k} \right\} = \left\{ \tilde{C}_{C_{P,k}} \right\} \quad (2.131)$$

and now substituting (2.107) into (2.131), we get

$$\left[\alpha_{kh} \right] \left[E_{hk}^3 \right] \left[E^2 \right] \left[\alpha_{kh} \right] \left[E_{hm}^1 \right] \left\{ \tilde{q}_m \right\} = \left\{ \tilde{C}_{C_{P,k}} \right\} \quad (2.132)$$

2.6 Numerical Evaluation of Generalized Aerodynamic Force

The generalized aerodynamic forces acting on the generalized force modes are given by (2.25) as

$$e_h = \frac{-1}{l^2} \iint_{\sigma_B} C_P \hat{n} \cdot \vec{M}_n(\sigma_B) d\sigma_B \quad (2.25)$$

Using (2.128), we can write (2.25) as

$$e_n = \frac{-1}{l^2} \iint_{\sigma_B} \left(\sum_{k=1}^H N_k(\xi_1, \xi_2) \tilde{C}_{CP,k} \right) \hat{n} \cdot \vec{M}_n(\sigma_B) d\sigma_B \quad (2.133)$$

or, in matrix form,

$$\begin{bmatrix} E_{nh}^4 \end{bmatrix} \left\{ \tilde{C}_{CP,k} \right\} = \left\{ e_n \right\} \quad (2.134)$$

where

$$E_{nh}^4 = \frac{-1}{l^2} \iint_{\sigma_B} N_k(\xi_1, \xi_2) \hat{n} \cdot \vec{M}_n(\sigma_B) d\sigma_B \quad (2.135)$$

Finally, substituting (2.132) into (2.134), we get

$$\begin{bmatrix} E_{nh}^4 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hk}^3 \end{bmatrix} \begin{bmatrix} E^2 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hm}^1 \end{bmatrix} \left\{ \tilde{q}_m \right\} = \left\{ e_n \right\} \quad (2.136)$$

2.7 Summary of Matrix Equations

The velocity potential spline amplitudes, $\tilde{C}_{\Phi_k}$, are given in terms of the normalwash spline amplitudes, $\tilde{C}_{\Psi_k}$, by equation (2.86).

$$\left\{ \tilde{C}_{\Phi_k} \right\} = \begin{bmatrix} E^2 \end{bmatrix} \left\{ \tilde{C}_{\Psi_k} \right\} \quad (2.86)$$

Equation (2.107) gives the velocity potential spline amplitudes in terms of the generalized coordinates, $\tilde{q}_m$.

$$\left\{ \tilde{C}_{\Phi_k} \right\} = \begin{bmatrix} E^2 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hm}^1 \end{bmatrix} \left\{ \tilde{q}_m \right\} \quad (2.107)$$

The pressure coefficient spline amplitudes, $\tilde{C}_{CP,k}$, are given in terms of the generalized coordinates by equation (2.132).

$$\begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hk}^3 \end{bmatrix} \begin{bmatrix} E^2 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hm}^1 \end{bmatrix} \left\{ \tilde{q}_m \right\} = \left\{ \tilde{C}_{CP,k} \right\} \quad (2.132)$$

Finally, the generalized aerodynamic forces are given in terms of the generalized coordinates by equation (2.136).

$$\begin{bmatrix} E_{nh}^4 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hk}^3 \end{bmatrix} \begin{bmatrix} E^2 \end{bmatrix} \begin{bmatrix} \alpha_{kh} \end{bmatrix} \begin{bmatrix} E_{hm}^1 \end{bmatrix} \left\{ \tilde{q}_m \right\} = \left\{ e_n \right\} \quad (2.136)$$

or,

$$\begin{bmatrix} ATFM \end{bmatrix} \left\{ \tilde{q}_m \right\} = \left\{ e_n \right\} \quad (2.137)$$

In equation (2.137), ATFM is the Aerodynamic Transfer Function Matrix that gives the Laplace transform of the generalized aerodynamic forces in terms of the Laplace transform of the generalized coordinates.

Chapter 3

Implementation

The formulation presented in chapter 2 was implemented in a computer program called SSOUP (Subsonic Steady, Oscillatory, and Unsteady Panel method). The program was designed so that analysis could be conducted on complex configurations with arbitrary shapes and motions. Although some emphasis was placed on speed and memory efficiency, the main objective was to implement the method as quickly and easily as possible. Many improvements can be made in regards to computational efficiency and are discussed in chapter 5.

3.1 Representation of Geometry, Velocity Potential, and Normalwash

The B-spline formulation presented in chapter 2 is implemented in SSOUP. The use of B-splines instead of cardinal splines greatly simplifies the computational procedure. Except for the lowest order splines, the cardinal spline shape functions are nonzero over most of the splined region. The B-spline shape functions are nonzero over only a small part of the splined region. Thus, even for high order B-splines, there are relatively few nonzero shape functions for any given point in the domain.

The SSOUP program uses either bi-linear or bi-cubic B-splines to represent

the geometry. For the bi-linear representation, four shape functions are nonzero at any point in the domain, while there are sixteen nonzero shape functions at any point for the bi-cubic representation. The velocity potential and normalwash variation are represented in SSOUP by either constant or bi-quadratic B-splines. There is one nonzero shape function at any point for the constant representation and nine nonzero shape functions at any point for the bi-quadratic representation.

3.1.1 Representation on the Body Surface

The surface of the aircraft is divided into regions referred to here as patches. The patches are further divided into smaller quadrilateral elements or panels, Σ_k . Figure 3.1 shows the arrangement of patches and panels used to represent a typical body geometry. The linear and cubic B-spline shape functions are shown in Figures 3.2 and 3.3. The use of bi-linear shape functions essentially means that the elements are hyperboloidal, and that the surface is continuous between all adjacent elements. The use of bi-cubic shape functions implies that the surface is continuous between all adjacent elements and that there is continuity of slope and curvature between adjacent elements on the same patch. The bi-cubic representation results in a much smoother surface than the bi-linear representation.

In the case of the bi-cubic representation, special attention must be given to the elements at the patch boundaries. The exact form of the cubic shape functions is determined by matching the displacement, slope, and curvature of adjacent elements located on the same patch. At the edge of a patch there are no adjacent elements, and some other condition must be used to define the shape functions. There are several options^[46] available to set the slope and curvature of the shape functions at the patch boundary. The well-known not-a-knot condition selects the shape function at the patch boundary by requiring that the shape functions on the first and second subinterval be identical. This is equivalent to eliminating the second knot but still requiring that the shape function pass through it. The cubic B-spline shape functions that result from the application of the not-a-knot condition at the patch boundaries are shown in Figure 3.4.

In SSOUP the variation of the velocity potential and normalwash over the quadrilateral elements defining the geometry are either constant (i.e. zeroth order) or bi-quadratic. The zeroth order shape functions are defined to be equal to unity over the element under consideration, and zero elsewhere. Thus, there is no continuity of displacement between adjacent elements. The use of zeroth order shape functions assumes that the potential and normalwash are constant within each element and equal to their value at the centre of the element. Thus, the evaluation or control point for the zeroth order shape functions is at the element centre.

The quadratic shape functions are defined for interior elements as shown in Figure 3.5. For interior elements, the displacement and slope are continuous between adjacent elements on the same patch. Again, the not-a-knot condition is used to define special shape functions for elements on the patch boundaries. The special boundary shape functions for the quadratic B-splines are shown in Figure 3.6.

The coordinates of the corner points of the elements on each patch are the only input requirements when using the system of patches and quadrilateral elements. The body surface geometry and variation of the potential and normalwash over the body surface are then completely represented.

3.1.2 Representations on the Wake Surface

In SSOUP, the wake is automatically generated from the geometric model of the body by extending the trailing edge nodes (element corners) in the direction of the free stream (positive x axis).

The element nodes that lie along the trailing edge of the body are assumed to form patch boundaries. Thus, the wake patches are defined by extending body patch boundaries. Consequently, for every body patch that has a boundary on the trailing edge, there is one wake patch. The geometric patch and element representation of a typical wake produced from the automatic wake generation process is shown in Figure 3.7.

Once the wake patches have been generated, they are divided up into quadri-

lateral elements, Σ_n , similar to the body elements, Σ_h . The representation of the geometry, velocity potential, and normalwash is the same as that used on the body. Note that the wake resulting from the automatic wake generation procedure is flat, and can be exactly represented by either bi-linear or bi-cubic B-splines. For steady state motion, the velocity potential is constant over the wake. Thus, the velocity potential on the wake can be exactly represented by zeroth or bi-quadratic B-splines for steady motion. Generally, the potential variation over the wake is not constant for unsteady motion and the quadratic variation is expected to have superior performance.

The input requirements for automatic wake generation are the patch boundaries that lie along the trailing edge, the wake length, and the element spacing in the free stream direction. The wake element spacing in the spanwise direction is determined by the spanwise spacing on the corresponding body patch from which the wake patch was generated.

3.1.3 Zeroth and Higher Order Methods

The use of bi-linear splines to represent the geometry, in combination with zeroth order splines to represent the potential and normalwash, will be referred to as the **Zeroth Order Method**. Note that the zeroth order method was only included so that results from the SOUSSA code could be reproduced.

The use of bi-cubic splines to represent the geometry, in combination with bi-quadratic splines to represent the potential and normalwash, will be referred to as the **Higher Order Method**.

However, it should be noted that in SSOUP it is possible to use any combination of bi-linear or bi-cubic splines to represent the geometry, and zeroth or bi-quadratic splines to represent the potential and normalwash.

3.2 Integration of Kernel-Shape Function Products

It is possible, for a zeroth order representation, to integrate the kernel-shape function products analytically. However, the analytic evaluation of the integrals is very complex and is generally impossible for higher order representations.

One approach to simplify the evaluation of the integrals is to approximate the integrands with functions that can be more easily integrated. These methods include series expansion and exponential approximation. Desmarais^[49] presents a summary of these methods with particular emphasis on exponential approximation.

Although computationally expensive, direct numerical integration, with special treatment of the singular integrals, is the simplest method of evaluating the kernel-shape function products. Therefore, direct numerical integration is used in the SSOUP program.

3.2.1 Evaluation of Singular Integrals

If the sending element/receiving point pair is such that the receiving point lies on the sending element, then the kernel functions have a singularity at the receiving point. The numerical integrations must be specially formulated to deal with the singularity. The formulation for the singular integrals used in the SSOUP program is based on the method given by Watson^[50].

In SSOUP the control points are at the element centres. Thus, the singularities occur at the element centres. The singularity is dealt with by dividing the quadrilateral element into four subelements as shown in Figure 3.8(a) and (b). The four subelements are triangular but are actually degenerate quadrilateral elements. One of the sides of the quadrilateral subelement converges to a point that coincides with the singularity as shown in Figure 3.8(c) and (d). Integration over each of the subelements is carried out with respect to a system of coordinates, (η_1, η_2) , that is obtained from a linear transformation of the parametric coordinates, (ξ_1, ξ_2) , of

the original element. If the transformed, (η_1, η_2) , coordinate system is used, then the integrand tends to a finite limit as the singularity is approached on each of the subelements^[50].

Gaussian quadrature was used to evaluate the integrals over the four subelements. Although schemes exist for varying the order of the numerical integration method according to the variation of the singular integrand, for simplicity, a constant number of integration points was used to evaluate the integrals over each of the subelements.

3.2.2 Evaluation of Non-Singular Integrals

If a constant number of integration points is used to evaluate the non-singular integrals, an unnecessarily accurate estimate of the integral would be achieved where the relative distance between sending element and receiving point was large. The integrals in (2.80) are nonsingular for the majority of the sending element/receiving point combinations. Therefore, it is desirable to minimize the computational effort (time) required for evaluation of the non-singular integrals. The order of the numerical integration method should be varied according to the rapidity of variation of the kernel-shape function products, so as to obtain roughly uniform precision for all sending element/receiving point combinations.

The selection of the order of the numerical integration is based on successive approximation and an exact numerical integration error formula. The SSOUP program provides two options for integration of the non-singular kernel-shape function products: (1.) Gaussian Quadrature and (2.) Successive Approximation using Romberg Integration.

3.2.3 Gaussian Quadrature

An n point Gaussian integration formula will exactly integrate a polynomial of degree $2n - 1$. Gaussian integration represents the best integration method in that there exists no other method for the same number of points that can as accurately

integrate a given polynomial^[48]. Thus, if the integrand behaves like a polynomial, then Gaussian integration is expected to perform well.

The scheme used in SSOUP to select the order of the Gaussian integration is based upon the method developed by Lachat and Watson^[51]. The method uses the Gaussian integration error formula originally given by Stroud and Secrest^[52]. Lachat and Watson give the error formula as

$$\left| \int_{-1}^1 \int_{-1}^1 f(\xi_1, \xi_2) d\xi_1 d\xi_2 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} A_i^{(n_1)} A_j^{(n_2)} f(\xi_i^{(n_1)}, \xi_j^{(n_2)}) \right| \leq 2 \sum_{p=1}^2 e_p H_p \quad (3.1)$$

where : $A_i^{(n_1)}, \xi_i^{(n_1)}$ are the weights and nodes in the ξ_1 direction
 $A_j^{(n_2)}, \xi_j^{(n_2)}$ are the weights and nodes in the ξ_2 direction

$$\left| \frac{\partial^{2n_p} f}{\partial \xi_p^{2n_p}} \right| \leq H_p \quad (3.2)$$

and

$$e_p \simeq \frac{G}{2^{2n_p} (2n_p)!} \quad (3.3)$$

where $G \approx 4$ ^[50]. Equation (3.1) places an upper bound on the absolute error in terms of the derivatives of the integrand. Equation (3.1) does not provide an exact error, but rather sets an upper error limit. Thus, if we can define the maximum value of the $2n_p$ th derivative in (3.2), then we have an error bound for the integral. In SSOUP several assumptions are used to simplify the error bound calculation. Thus, only an estimate of the error bound is used to select the order of the integration.

The integrands of (2.80) involve kernel-shape function products and, when written in terms of the parametric coordinates, take the form

$$f(\xi_1, \xi_2) = N(\xi_1, \xi_2) K(R(\xi_1, \xi_2)) J(\xi_1, \xi_2) \quad (3.4)$$

where : N is the shape function
 K is the kernel function
 J is the Jacobian
 R is the relative distance from the sending
element to the receiving point

Thus, to compute an upper bound for the error in the integrals we require the derivatives

$$\left| \frac{\partial^{2n_p} f}{\partial \xi_p^{2n_p}} \right| = \left| \frac{\partial^{2n_p}}{\partial \xi_p^{2n_p}} (N K J) \right| \quad (3.5)$$

These derivatives, (3.5), can become quite complex for large n_p . To simplify (3.5), the Jacobian is assumed to be constant over the element and equal to its value at the centre (control point) of the element. A constant Jacobian implies that the element curvature is constant. This approximation is reasonable providing that the element mesh is fine enough. Assuming a constant Jacobian, (3.5) becomes

$$\left| \frac{\partial^{2n_p} f}{\partial \xi_p^{2n_p}} \right| = J \left| \frac{\partial^{2n_p}}{\partial \xi_p^{2n_p}} (N K) \right| \quad (3.6)$$

The derivatives in (3.6) involve products of derivatives of the kernel and shape functions, and can be written as

$$\frac{\partial^{2n_p}}{\partial \xi_p^{2n_p}} (N K) = \sum_{i=0}^{2n_p} \sum_{j=0}^{2n_p} A_{ij} \frac{\partial^{2n_p-i}}{\partial \xi_p^{2n_p-i}} (K) \frac{\partial^j N}{\partial \xi_p^j} \quad (3.7)$$

where: A_{ij} are constants from Pascal's triangle (i.e. coefficients resulting from the chain rule expansion of the derivatives)

If we restrict our analysis to zeroth and bi-quadratic B-spline shape functions, then

$$\frac{\partial^j N}{\partial \xi_p^j} = 0 \quad \text{for } j > 2 \quad (3.8)$$

Thus, (3.7) becomes

$$\frac{\partial^{2n_p}}{\partial \xi_p^{2n_p}} (N K) = \sum_{i=0}^{2n_p} \sum_{j=0}^2 A_{ij} \frac{\partial^{2n_p-i}}{\partial \xi_p^{2n_p-i}} (K) \frac{\partial^j N}{\partial \xi_p^j} \quad (3.9)$$

The three fundamental kernel functions in (2.80) can be approximated as

$$\begin{aligned} K_b &= \frac{1}{R} = O\left(\frac{1}{R}\right) \\ K_{t,f} &= \frac{\partial}{\partial N} \frac{1}{R} = \frac{\hat{N} \cdot \vec{R}}{R^3} = O\left(\frac{1}{R^2}\right) \\ K_{d,f} &= \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} = \frac{M}{\beta} \left[\frac{\hat{N} \cdot \vec{R}}{R^2} - \frac{M N_X}{R} \right] = O\left(\frac{1}{R}\right) \end{aligned} \quad (3.11)$$

Assuming the Jacobian to be constant over the element, the derivatives of the approximate kernel functions, (3.11), can be written as

$$\frac{\partial^m K}{\partial \xi_p^m} = \frac{\partial^m K}{\partial R^m} \left(\frac{\partial R}{\partial \xi_p} \right)^m \quad (3.12)$$

and setting

$$\delta_p = \frac{\partial R}{\partial \xi_p} \quad (3.13)$$

we have

$$\frac{\partial^m K}{\partial \xi_p^m} = \frac{\partial^m K}{\partial R^m} \delta_p^m \quad (3.14)$$

Also,

$$\frac{\partial^m K_b}{\partial R^m} = \frac{\partial^m}{\partial R^m} \left(\frac{1}{R} \right) = \frac{m!}{R^{m+1}} \quad (3.15)$$

$$\frac{\partial^m K_{t,f}}{\partial R^m} = \frac{\partial^m}{\partial R^m} \left(\frac{1}{R^2} \right) = \frac{(m+1)!}{R^{m+2}} \quad (3.16)$$

$$\frac{\partial^m K_{d,g}}{\partial R^m} = \frac{\partial^m}{\partial R^m} \left(\frac{1}{R} \right) = \frac{m!}{R^{m+1}} \quad (3.17)$$

Now using (3.14) we can write (3.15), (3.16), and (3.17) as

$$\frac{\partial^m K_b}{\partial \xi_p^m} = (m)! \left(\frac{\delta_p}{R} \right)^m \frac{1}{R} \quad (3.18)$$

$$\frac{\partial^m K_{t,f}}{\partial \xi_p^m} = (m+1)! \left(\frac{\delta_p}{R} \right)^m \frac{1}{R^2} \quad (3.19)$$

$$\frac{\partial^m K_{d,g}}{\partial \xi_p^m} = (m)! \left(\frac{\delta_p}{R} \right)^m \frac{1}{R} \quad (3.20)$$

From (3.18), (3.19), and (3.20) we can write

$$\left[\frac{\partial^m K_b}{\partial \xi_p^m} \right] : \left[\frac{\partial^m K_{t,f}}{\partial \xi_p^m} \right] : \left[\frac{\partial^m K_{d,g}}{\partial \xi_p^m} \right]$$

as

$$\left[(m)! \left(\frac{\delta_p}{R} \right)^m \frac{1}{R} \right] : \left[m!(m+1) \left(\frac{\delta_p}{R} \right)^m \frac{1}{R^2} \right] : \left[(m)! \left(\frac{\delta_p}{R} \right)^m \frac{1}{R} \right] \quad (3.21)$$

giving

$$1 : \frac{(m+1)}{R} : 1 \quad (3.22)$$

For large R , (3.22) indicates that the derivative of K_b or K_d will be the largest. Thus, if the number of integration points is selected based on the error in K_b , then all of the integrals should be within the error limit. Therefore, the order of the integration used to compute the non-singular integrals was based upon the variation of the integrand of b_k in (2.80).

To get an estimate for the maximum error, we require an estimate of the maximum value of the derivative of the kernel with respect to the parametric coordinates. The maximum derivative of K_b with respect to ξ_p is found from

$$\left(\frac{\partial^m K_b}{\partial \xi_p^m} \right)_{\max} = (m)! \left(\frac{\delta_p}{R_{\min}} \right)^m \frac{1}{R_{\min}} \quad (3.23)$$

where $R_{\min}$ is the minimum distance between the receiving point and sending element surface. Thus, an estimate for the maximum value of H_p is found from

$$(H_p)_{\max} = J \sum_{i=0}^{2n_p} \sum_{j=0}^2 A_{ij} (2n_p - i)! \frac{\delta_p^{2n_p-i}}{R_{\min}^{2n_p-i+1}} \left(\frac{\partial^i N}{\partial \xi_p^i} \right)_{\max} \quad (3.24)$$

where it can be shown that for zeroth order splines

$$N_{\max} = 1, \quad \left(\frac{\partial N}{\partial \xi_p} \right)_{\max} = 0, \quad \left(\frac{\partial^2 N}{\partial \xi_p^2} \right)_{\max} = 0 \quad (3.25)$$

and for quadratic B-splines

$$N_{\max} = 1, \quad \left(\frac{\partial N}{\partial \xi_p} \right)_{\max} = \frac{2}{3}, \quad \left(\frac{\partial^2 N}{\partial \xi_p^2} \right)_{\max} = \frac{2}{3} \quad (3.26)$$

Equation (3.1) can be rewritten as

$$\epsilon \leq (2e_1 \tilde{H}_1 + 2e_2 H_2) \quad (3.27)$$

where: ϵ is the absolute error

The $2e_p H_p$ terms in (3.27) are the absolute errors in each of the parametric directions. In SSOP it is required that the integration be equally precise in each direction, so we set

$$e_1 H_1 = e_2 H_2 \quad (3.28)$$

Then we can write, from (3.27) and (3.28),

$$\frac{\epsilon}{4} = e_1 H_1 = e_2 H_2 \quad (3.29)$$

Also, although it is possible to compute the weights and nodes for any number of integration points, SSOUP places an upper limit on the maximum number of integration points. i.e.

$$1 \leq n_p \leq n_{pmax} \quad (3.30)$$

If more points are needed to satisfy the specified error limit, then the element is automatically subdivided into smaller quadrilateral elements.

The procedure used in SSOUP to determine the required number of integration points for a specified precision, ϵ , is as follows:

1. Specify a maximum absolute error limit ϵ .
2. Calculate an estimate of R_{min} for the current sending element/receiving point pair.
3. Determine the value of the Jacobian, J , and its components, δ_p , at the centre of the sending element.
4. Select $n_1 = 1$ and $n_2 = 1$.
5. Compute e_p .
6. Using R_{min} , n_p , and δ_p , calculate $(H_p)_{max}$ from (3.24).
7. If $e_p H_p > \epsilon/4$ and $n_p < n_{pmax}$, then increase n_p and repeat 5, 6, and 7.
8. If $e_p H_p < \epsilon/4$, then use n_p integration points in p direction.
9. If $n_p = n_{pmax}$ subdivide element in p direction and repeat 2, 3, 4, 5, 6, and 7.

This variable order numerical integration scheme was evaluated by computing the actual value for the integrand and comparing with the numerical approximation. The observed error was then calculated and compared with the specified error limit. Figure 3.9 shows a typical plot of the observed error versus the element size, δ , divided by the relative distance between sending element and receiving point.

3.2.4 Successive Approximation Using Romberg Quadrature

In a successive approximation scheme, an estimate for the integral is obtained at successive levels of a variable order integration scheme. At each level of the inte-

gration scheme, more integration points are added. The difference between integral approximations at two successive levels is compared with a preset tolerance. If the difference is not within the specified tolerance, the integral approximation at the next level is computed. The process continues adding more integration points at each level, until the successive difference is within the set tolerance. Successive approximation can be used with virtually any numerical integration scheme. However, successive approximation is much more efficient for some schemes than others.

Any numerical integration rule involves the product of the integrand evaluated at specified nodes and a corresponding weight. In some integration schemes the nodes for $n_p + 1$ integration points include the nodes for n_p integration points. Thus, if an estimate for the integral using n_p integration points has already been computed and we wish to calculate an estimate for $n_p + 1$ points, then we do not need to reevaluate the integrand at all of the nodes. We can use the results from the previous level and only evaluate the integrand at the nodes that are not included in the n_p integration point calculation.

If Gaussian integration represents the best integration rule, then we might expect it to work well for successive approximation. However, the weights and nodes for an n_p point Gaussian integration rule are unique. That is, none of the nodes for an $n_p + 1$ point rule are the same as for an n_p point rule. (There is one exception; all odd n_p have a node at the centre of the integration interval.) Thus, if using successive approximation with Gaussian integration, the integrand would have to be evaluated at all of the nodes at a given level to compute the approximation. Successive approximation with Gaussian integration results in a very computationally inefficient method.

The trapezoidal rule is an integration method in which the nodes for one level include the nodes from the previous level. In the trapezoidal rule, the integral is approximated by sums of trapezoids. Romberg integration is based on the trapezoidal rule and includes the principle of repeated iterated linear interpolation to increase the convergence rate and produce a very computationally efficient numerical integration method.

Consider the integral

$$I = \int_{x_a}^{x_b} f(x) dx \quad (3.31)$$

The method of successive approximations using the trapezoidal rule involves successive halving of the intervals (i.e. $h_k = h_0/2^k$) at each level, k , to obtain a more and more precise estimate for the integral, and is the first stage ($m = 0$) of Romberg integration. The trapezoidal approximation, T_0^k , for the integral (3.31) in terms of the integration interval spacing $h_0 = x_b - x_a$, is

$$T_0^k = \frac{h_k}{2} \left[f(x_a) + 2 \sum_{j=1}^{2^k-1} f(x_a + j h_k) + f(x_a + 2^k h_k) \right] \quad (3.32)$$

where $h_k = h_0/2^k$ is the spacing of the 2^k trapezoids.

In the second stage of Romberg integration ($m = 1$), linear interpolation is used between the integral estimates T_0^k and T_0^{k+1} (using the square of the interval size, h_k^2 , as the abscissa) to obtain a new integral estimate, T_1^k , with a theoretical interval size of zero. This procedure of linearly interpolating between successive levels is continued from stage to stage. The integral estimate at stage m and level k can be expressed as^[53]

$$T_m^k = \frac{4^m T_{m-1}^{k+1} - T_{m-1}^k h_{k+m}^2}{h_k^2 - h_{k+m}^2} \quad (3.32)$$

where

$$h_{k+m} = \frac{h_k}{2^m} \quad (3.32)$$

The procedure of linearly interpolating between successive levels of Romberg stages can be carried on indefinitely. McCalla^[53] has shown that the Romberg method converges as the number of interpolations approaches infinity.

In SSOUP, successive approximation using Romberg integration can be used to compute the non-singular integrals. At each level, approximation using iterated linear interpolation is used to try and satisfy the specified tolerance before proceeding to a new level.

3.3 Structure of the SSOUP Program

In the present version of SSOUP, the program must run from start to finish for a single problem. No restart capability has been included and the error handling is limited. Improvements are discussed in chapter 5. The organization of the main subroutines including brief descriptions follows:

START

1. **GInput** - Inputs geometric and related parameters.
2. **PGtrans** - Transforms body surface physical coordinates using the Prandtl-Glauert transformation and nondimensionalization (2.28).
3. **SplineGeom** - Splines the nodal coordinates that define the body geometry in the Prandtl-Glauert space in terms of local parametric coordinates.
4. **BodyInt** - Integrates kernel-shape function products (2.80) over the body elements to form the global b, c, and d matrices.
5. **Autowake** - Generates wake panels using the automatic wake generating algorithm.
6. **PGtrans** - Transforms wake surface physical coordinates using the Prandtl-Glauert transformation and nondimensionalization (2.28).
7. **SplineGeom** - Splines the nodal coordinates that define the wake geometry in the Prandtl-Glauert space in terms of local parametric coordinates.
8. **MakePi** - Computes the trailing edge wake convection times (2.30) and forms global Pi matrix (2.34).
9. **WakeInt** - Integrates kernel-shape function products over the wake elements to form the global f, and g matrices (2.80).
10. **AssembleAICM** - Assembles the N_{ik} , $\bar{b}_{ik}$, $\bar{c}_{ik}$, $\bar{d}_{ik}$, $\bar{f}_{il}$, $\bar{g}_{il}$, Θ_{ih} , Θ_{in} , Π_n , $(UL)_{nk}$, α_{kh} , and α_{in} matrices to form the $\bar{y}_{ik}$ and $\bar{z}_{ik}$ matrices that make up the AICM (2.87).
11. **Solve** - Solves the linear system and outputs the velocity potential at the

body element midpoints.

12. **MakeCp** – Calculates the pressure coefficients at the element centres using the velocity potential.
13. **GenForce** – Computes the generalized forces from the generalized force modes and the pressure coefficients.

END

Chapter 4

Verification of SSOUP Program

4.1 The SSOUP and SOUSSA Programs

This chapter presents results for several test problems. The primary purpose is to provide validation of the SSOUP code. Results obtained from the computer program SOUSSA^{[35][36]} are compared with experiment and theory in references [18], [30], and [37]–[43]. Generally, the SOUSSA results were found to be in good agreement with theory and experimental data. Thus, in the present work, the new higher order method implemented in SSOUP is only compared with the SOUSSA results.

The zeroth order and higher order implementations of SSOUP will be referred to as SSOUP-Z and SSOUP-H, respectively. The SOUSSA code uses a zeroth order method and thus a zeroth order formulation was implemented in SSOUP-Z so that results identical to those of SOUSSA could be reproduced and compared. However, the implementation in SOUSSA is somewhat different from the zeroth order implementation in SSOUP-Z.

4.1.1 Integration of Kernel-Shape Function Products

Both SOUSSA and SSOUP-Z use bi-linear shape functions to represent the geometry, and zeroth order shape functions to represent the potential. However,

the integration of the kernel-shape function products is different.

In SOUSSA, the kernel-shape function products are integrated analytically. This is possible because of the zeroth order formulation. Also, SOUSSA uses an approximation for the evaluation of the $[D_{ih}]$ and $[G_{in}]$ matrices. The approximation uses the entries of the $[C_{ih}]$ and $[F_{in}]$ matrices that are computed analytically. The approximation used for $[D_{ih}]$ is

$$\begin{aligned}
 D_{ih} &= -\frac{1}{2\pi} \iint_{\Sigma_h} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} d\Sigma_h & (2.43) \\
 &= -\frac{M}{2\pi\beta} \iint_{\Sigma_h} \frac{1}{R} \left(\frac{\partial R}{\partial N} - MN_X \right) d\Sigma_h \\
 &\approx -\frac{M}{2\pi\beta} R_{ih} \iint_{\Sigma_h} \frac{1}{R^2} \frac{\partial R}{\partial N} d\Sigma_h + \frac{1}{2\pi\beta} M^2 N_{X_h} \iint_{\Sigma_h} \frac{1}{R} d\Sigma_h \\
 &= \frac{M}{\beta} \left(R_{ih} C_{ih} - MN_{X_h} B_{ih} \right) & (4.1)
 \end{aligned}$$

and setting $N_{X_h} \approx 0$ (small perturbation assumption), we get

$$D_{ih} \approx \frac{M}{\beta} \left(R_{ih} C_{ih} \right) \quad (4.2)$$

Similarly, assuming that the wake has a small inclination with respect to the X -axis, we can write $[G_{in}]$ as

$$G_{in} \approx \frac{M}{\beta} \left(R_{in} F_{in} \right) \quad (4.3)$$

SSOUP-Z uses the variable order numerical integration scheme that was presented in chapter 3. For the higher order implementation, it is impossible to use analytic integration. Thus, numerical integration was used in SSOUP-Z so that the performance of the zeroth and higher order versions could be easily compared. The approximations given above for $[D_{ih}]$ and $[G_{in}]$ are not necessary, but were included in SSOUP so that results could be compared with SOUSSA.

4.1.2 Calculation of Pressure Coefficient

To calculate the pressure coefficient, the derivative of the velocity potential in each parametric direction is required at the element centres (see Equation 2.115).

For zeroth order implementation, the velocity potential is constant over the element. Therefore, the derivatives necessary for the evaluation of the pressure coefficient cannot be found directly from the zeroth order representation.

In SOUSSA, a weighted averaging scheme is used to first find the velocity potentials at the element nodes from the velocity potentials at the element centres. A 2D-linear spline is then used to interpolate the nodal values over the element. The derivatives at any point on the element can then be determined from the derivatives of the linear shape functions. Further, a special function is used to represent the pressure variation at the trailing edge. The pressure difference between the upper and lower surfaces is forced to go to zero with the square root of the distance from the trailing edge.

In SSOUP-Z, the velocity potentials at the element centres are spline fit using a 2D-quadratic spline. The required derivatives of the velocity potentials are then evaluated from the quadratic spline shape functions. There is no special treatment of the square root behavior of the pressure difference at the trailing edge. The pressure simply behaves like the derivatives of the special quadratic spline boundary shape functions determined from the not-a-knot condition.

4.2 Non-Lifting Steady State ($p = 0$) Verification Problems

As an initial step in evaluating the performance of the higher order method, several steady state test cases were analyzed using the SSOUP program. The numerical results are compared with analytical solutions where possible. In the following discussion, statements of SSOUP patch grid sizes refer to the number of elements along the spanwise direction first, followed by the number of elements along the chordwise direction. Thus, a 10×8 grid indicates 10 elements in the spanwise direction and 8 elements in the chordwise direction.

4.2.1 Sphere

Steady state potential distributions were calculated for a sphere in incompressible flow ($M=0$). A single patch was used to model the sphere surface. The element distribution for a 16×16 grid is shown in Figure 4.1. The element distribution is such that the elevation and azimuthal angular spacing is constant for each element. Thus, for the grid shown in Figure 4.1, the angular spacing for each element is $\frac{260^\circ}{16}$. Note that the three sided elements at the ends of the sphere are degenerate quadrilateral elements. These elements are actually four sided but one of the sides has collapsed to a point. The radius of the sphere was chosen arbitrarily to be 5.

The numerical results from the SSOUP program show good agreement with the analytical solution. The velocity potential for a 5×5 grid is shown in Figure 4.2. The results using the higher order method match the analytic solution almost exactly, as is evident by the overlapping of the SSOUP-H and analytic solution lines. However, for this grid size, the zeroth order method does not agree to the same accuracy. Further analyses were conducted using more elements, and the zeroth order solution eventually converged.

A convergence study was performed to study the effect of grid size on the numerical results. The results of the convergence study are shown in Figure 4.3. The L2 norm was used to evaluate the error in velocity potential over the sphere surface. For the calculation of the L2 norm error, the error was assumed to be constant over each element and equal to its value at the centre of the element, regardless of the order of the method. This allowed for comparison of the SSOUP-H and SSOUP-Z L2 norm error results.

It is clear from Figure 4.3 that the higher order method converges much faster than the zeroth order method. Thus, the higher order method requires fewer elements to achieve a specified accuracy.

4.2.2 Ellipsoids

The velocity potential distributions for various sized ellipsoids were calculated using the SSOUP program. The ellipsoids studied were based on perturbations of a sphere, and are shown in Figures 4.4-4.8. They are identified by their semi-major and semi-minor axes. Thus, an ellipsoid designated as 5 10 5 has a semi x-axis equal to 5, a semi y-axis equal to 10, and a semi z-axis equal to 5. The resulting ellipsoid has a span of 20 and a circular cross section with a radius of 5 at the semi span ($y=0$). Note that the 5 5 5 ellipsoid is a sphere of radius 5.

In each case, a single patch was used to model the ellipsoid surface. The grid was such that the element angular spacing along the chordwise and spanwise perimeter was uniform. The number of elements in the grid was determined by setting the number of elements in the chordwise direction and then calculating the number of elements required in the spanwise direction to keep the element dimensions roughly equal. The ellipsoids use degenerate elements at the ends, similar to the sphere test case in the previous section. However, in the previous section, the degenerate elements are at the leading edge of the sphere (i.e. along the x-axis), while the ellipsoids in this section, including the 5 5 5 sphere, have the degenerate elements at the tips (along the y axis).

Numerical results were obtained for several grid sizes. Again, the results were in good agreement with the analytic solution. The potential distribution at the semi span ($y = 0$) for the 5 20 5 ellipsoid with a 24×8 grid is shown in Figure 4.9. As before, the higher order results are in very close agreement with the analytic solution, while the zeroth order results require a finer mesh for convergence.

A convergence study was conducted that involved refinement of the patch grid at each stage. As the number of chordwise elements was increased, the number of spanwise elements was increased proportionally, so that the dimensions of the semi-span elements remained approximately equal. The L2 norm error is plotted versus the number of chordwise elements, N_c , in Figure 4.10. Again, the superior performance of the higher order method is clearly demonstrated by its larger rate

of convergence.

4.2.3 Circular Cylinder

An analytic solution for the velocity potential exists for 2D incompressible flow around a circle. The equivalent 3D problem is incompressible flow around an infinite span circular cylinder. The SSOUP program is set up for 3D analysis only. As the span is increased, the flow conditions at the semi-span of a circular cylinder approach the 2D flow conditions around a circle. If a circular cylinder with a large enough span is analyzed, the solution at the semi-span can be compared with the 2D analytic solution for flow around a circle. After preliminary investigation, it was determined that the span for a cylinder of radius 5 must be at least 150 to ensure that the flow conditions at the semi-span can be represented by 2D flow.

The surface of the cylinder was modeled by a single patch. A quarter span section for a 50×20 element grid is shown in Figure 4.11. The ends were left open, as investigations for cylinders with and without closed ends showed that the open ends had little influence on the solution at the semi-span. The grid was formed such that the element distribution was uniform in both the spanwise and chordwise directions. As the number of elements in the chordwise direction was increased, the number of elements in the spanwise directions was increased proportionally, so that the dimensions of the elements remained approximately equal.

The potential distribution at the semi-span of a radius 5, 150 span circular cylinder for incompressible flow was computed using SSOUP. The SSOUP results for a 50×8 element grid are compared with the analytic solution for a circle in Figure 4.12. The higher order solution shows good agreement with the analytic solution except at the trailing edge ($\Theta = 0$). For this grid, there are not enough points in the chordwise direction at the trailing edge to capture the comparatively rapid potential variation. Analysis with more elements showed that the variation at the trailing edge is eventually captured. As before, the zeroth order results are consistently less accurate than the higher order method for the same size grid.

4.2.4 Rectangular Planform Wing with a Kármán-Treftz Airfoil Section

The geometry of the Kármán-Treftz airfoil is generated from a circle by application of the well known Kármán-Treftz transformation^[54] given by

$$\frac{z - kl}{z + kl} = \frac{(\zeta - l)^k}{(\zeta + l)^k} \quad (4.4)$$

where : ζ -plane is the physical plane where the airfoil exists

z -plane is the plane where the circle exists

l is a real constant

$$k \leq 2$$

The Kármán-Treftz airfoil has a sharp trailing edge of finite angle and a rounded leading edge. The analytic solution for the velocity potential can be generated by application of (4.4) to the analytic solution for a circle. For analysis with SSOUP, the airfoil was modeled by considering the transformed geometry of the circular cylinder from the previous section. The solution at the semi-span is then compared with the 2D analytic solution for flow around the airfoil.

The circular cylinder from the previous section was transformed to generate a rectangular planform wing with a Kármán-Treftz airfoil. The cylinder was centered at $z = -0.5$ and the transformation parameters were $l=4.5$, and $k=1.7$. The resulting airfoil had a thickness to chord ratio of 0.33 and a trailing edge angle of 54° . The element distribution on the cylinder was the same as in the previous section. A typical element distribution resulting from application of the Kármán-Treftz transformation is shown for a quarter span section of the wing in Figure 4.13. The effect of the transformation on the element distribution is that the elements get smaller towards the leading and trailing edges. Thus, there are more elements concentrated at the leading and trailing edges. This is beneficial, as more elements are needed in these locations to capture the more rapid variation in velocity potential.

The problem was analyzed using SSOUP for several grid sizes. The analytic solution was compared with the numerical results at the semi-span of the Kármán-Treftz wing. The results for a 41×10 element grid are shown in Figure 4.14. The

numerical results show good agreement with the analytic solution. The higher order method seems to have slightly better performance than the zeroth order method.

4.2.5 Rectangular Planform Wing with a Circular Arc Airfoil Section

The steady state velocity potential and pressure distribution for an aspect ratio 3 rectangular planform wing, with a 5 percent thick circular arc airfoil, has been computed using SOUSSA by Morino *et al* [38] and Yates *et al* [39]. In the original papers, Morino and Yates compared the SOUSSA results to experimental pressure data obtained by Lessing *et al* [55] from a wind tunnel test model. The wind tunnel model dimensions are shown in Figure 4.15. For the present work, the SSOUP results will only be compared with the SOUSSA results.

The steady state ($p = 0$) velocity potential distribution was computed by Morino *et al* [38] using SOUSSA for an angle of attack (α) of zero, and $M = 0.24$. Figure 4.16 shows the potential distribution along the chord calculated by SOUSSA compared with results from SSOUP at the wing semi-span. The wing was modeled using two patches: one for the upper surface and one for the lower surface. Each patch had a uniformly distributed 20×27 element grid. The element distribution is shown in Figure 4.17, where the z -axis dimension has been exaggerated for clarity. The potential distribution computed by the higher and zeroth order versions of SSOUP are in agreement with the SOUSSA results.

Next, two patches with 18×10 element grids were used to calculate the steady state ($p = 0$) pressure distribution for $\alpha = 0$ and $M = 0.70$. The element distributions are shown in Figure 4.18, where again, the z -axis dimension has been exaggerated. The SOUSSA pressure distribution computed by Yates *et al* [39] is compared with the SSOUP results in Figure 4.19 at a spanwise station $|2y/b| = 0.5$.

The pressure distributions from SSOUP are in agreement with the SOUSSA results, except on the last elements at the leading and trailing edges. At the trailing edge, SOUSSA forces the pressure difference to go to zero with the square root of the distance from the trailing edge. In SSOUP, there is no such forcing of the

pressure difference at the trailing edge. Although it is not stated explicitly in any of the literature, it is also likely that the pressure difference has been forced to vary with the inverse of the square root of the distance from the leading edge. This would explain the discrepancy between the SSOUP and SOUSSA results at the leading and trailing edges.

4.3 Lifting Steady State Verification Problems

The following test problems involve asymmetric geometric flows and thus include a geometric wake model. Consistent with the body patch grids, statements of wake patch grid sizes refer to the number of elements along the spanwise direction first, followed by the number of elements along the chordwise direction.

4.3.1 Circular Cylinder

The analytic solution for a lifting cylinder is obtained by the addition of a vortex to the non-lifting analytic solution. The strength of the vortex and thus the lift force is determined from the Kutta condition. The point at which the Kutta condition is enforced determines the angle of attack. To model this situation in SSOUP, a wake must be added to the body geometry.

A lifting circular cylinder was analyzed using SSOUP. The cylinder body was modeled as in 4.2.3. The free stream was set at an angle of attack $\alpha = 0.6^\circ$ relative to the chord line of the cylinder and the flow was incompressible. The element distributions were again uniform in both directions. The lifting case requires the addition of a wake model. The wake was generated by SSOUP using the automatic wake generation procedure. The wake model consisted of a single patch with a 41×10 element grid. The wake length was 10 times the chord length (i.e. $10\bar{c}$). For steady state motion ($p = 0$) the potential difference, $\Delta\Phi$, is constant on the wake, and therefore the solution should be insensitive to the wake element size. The element midpoint (control point) was specified as the Kutta point. (The Kutta point is the point where the Kutta condition, Equation (2.23), is to be applied.)

The velocity potential distribution at the semi-span of the cylinder for a body patch with a 41×10 element grid is shown in Figure 4.20. Both the higher order and lower order solutions show good agreement with the analytic solution. For this problem, the higher order method seems to perform only slightly better than the zeroth order method. However, memory restrictions placed an upper limit on the patch grid sizes. As the number of elements in the chordwise direction was increased, the number of elements in the spanwise direction had to be correspondingly decreased to keep within the available memory limit. As a result, a high level of convergence could not be achieved for either method. The higher order method is expected to give better performance, but further analysis is required to draw specific conclusions regarding the relative performance of the methods. However, the results do provide validation of the SSOUP code.

4.3.2 Rectangular Planform Wing with a Kármán-Trefftz Airfoil Section

The Kármán-Trefftz airfoil, rectangular planform wing from section 4.2.4 was analyzed for an angle of attack of $\alpha = 0.6^\circ$ and $M = 0$. The body was modeled as described previously. The wake length was $5.55\bar{c}$ and was modeled by a single patch with a uniform element distribution. The wake patch consisted of a 41×10 element grid.

The problem was analysed using SSOUP for several grid sizes. The results for a body patch with a 41×10 element grid is presented in Figure 4.21. The zeroth and higher order methods show good agreement with the analytical solution. The zeroth order method seems to perform about as well as the higher order method. However, as discussed previously in section 4.3.1, memory restrictions made it impossible to achieve a high level of convergence. Thus, this test case simply serves to validate the SSOUP code.

4.3.3 Rectangular Planform Wing with a Circular Arc Airfoil Section

The pressure distribution for the same wing considered in 4.2.5 with $M = 0.70$, $\alpha = 5^\circ$, at $|2y/b| = 0.5$, is presented in Figure 4.22. The upper and lower surfaces were each modeled by separate patches with element grid sizes of 18×10 . The wake was modeled by a single patch with an 18×10 element grid. The wake was $5.55\bar{c}$ long and had a uniform element distribution.

The agreement between the SSOUP and SOUSSA results was good except at the leading and trailing edges. The discrepancy at the trailing edge can be attributed to the forced behaviour of the pressure at the trailing edge by the SOUSSA program. However, the SSOUP program failed to capture any of the leading edge peak. Thus, another analysis was performed.

In the second analysis, the upper and lower surfaces were split in two, forming separate leading edge and trailing edge patches. Thus, the wing body was composed of a total of four patches. The trailing edge patches consisted of 18×10 element grids that were uniformly distributed from the trailing edge to the $0.2\bar{c}$ chord position. The leading edge patches consisted of 18×10 element grids that were uniformly distributed from the $0.2\bar{c}$ chord position to the leading edge. The body geometry is shown in Figure 4.23.

The results of the second analysis are presented in Figure 4.24. The leading edge peak is now evident in the SSOUP results, but it is somewhat under predicted. A further refinement of the leading edge patch grid would be required to fully capture the leading edge peak. It would seem that the relatively coarse mesh used to produce the SOUSSA results should not be able to capture the leading edge peak. This seems to provide further evidence that in SOUSSA, the pressure at the leading edge must have been purposely forced to reflect the inverse square root behaviour in such a way that even a relatively coarse mesh can capture the leading edge peak. Also, the SSOUP-Z result has a large discontinuity in pressure coefficient between patches. This is due to the constant potential representation within the elements.

The higher order method is much better at representing the potential variation between the patch boundaries.

4.4 Oscillatory Test Case ($p = ik$)

4.4.1 Rectangular Planform Wing with a Circular Arc Airfoil Section

The unsteady pressure distributions for the wing of section 4.2.5 oscillating in its first bending mode has been calculated using SOUSSA by Yates *et al* [39]. The first bending mode shape was originally determined from a wind tunnel model by Lessing *et al* [55]. Morino *et al* [38] gives the normalized mode shape as

$$z = 0.18043|2y/b| + 1.70255|2y/b|^2 - 1.13688|2y/b|^3 + 0.25387|2y/b|^4 \quad (4.5)$$

where : z is the vertical displacement
 y is the distance along the span measured
 from the centre of the wing
 b is the span

The problem was analyzed using SSOUP for $M = 0.70$, $\alpha = 0^\circ$, and $k = \omega c/U_\infty = 0.94$. The body was modeled as before by two patches with 18×10 element grids. The wake model consisted of a single patch with an 18×35 element grid and had a length of $3.5\bar{c}$. The real and imaginary components of the complex pressure distribution along the chord for the $|2y/b| = 0.5$ spanwise station are shown in Figures 4.25 and 4.26.

The real component of the pressure distribution from SSOUP for both of the methods is in excellent agreement with the SOUSSA results except at the leading and trailing edges where the forced pressure variation in SOUSSA most likely explains the discrepancy. The imaginary component distribution is also in excellent agreement with the SOUSSA result.

To evaluate the performance of the zeroth and higher order methods, the element spacing in the chordwise direction was varied on the wake patch. The

results are shown in Figures 4.27 and 4.28. The higher order method seems to converge about twice as fast as the zeroth order method.

Chapter 5

Program Improvements and Recommendations for Future Work

5.1 Improvements

The SSOUP program is a preliminary code in that the main goal of its development was simply to implement the higher order formulation and to demonstrate the feasibility of the method. The formulation presented in chapter 2 has several important features that were not implemented in the SSOUP program. If these features are not included, the computer resources required for analysis of complex configurations such as complete aircraft may be prohibitive. Therefore, the improvements discussed below are recommended for future work and should result in a faster, more efficient, and user friendly program.

5.1.1 Computational Efficiency

All aerodynamic panel methods, steady and unsteady, involve computations with full matrices. Consequently, efficient memory storage techniques that have been developed for sparse matrices cannot be used. The memory requirements for the present method increase with the square of the number of panels used. Therefore, the memory requirements of the SSOUP program are inherently large. However, as previously mentioned, the SSOUP program is a preliminary code and

as such, memory efficiency was not of primary concern when it was first developed. Therefore, significant decreases in the memory requirements are still possible.

Currently, the SSOUP program is most efficient if the same number of panels are used on all patches. However, this is not always convenient. The formulation on which the SSOUP program is based is valid for geometric models that use patches containing different numbers of panels. In the present code, the memory storage allocation is static in that the maximum number of panels per patch and the maximum number of patches are fixed before the geometric parameters are input. Thus, an equal amount of memory is allocated for each patch even though some patches may contain more panels than others. It is possible to greatly reduce the memory requirements by utilizing a dynamic memory allocation scheme. A single vector that is properly partitioned can be used for memory storage of all variables. The overall size of the vector is fixed before input of the geometric model but the allocation of space within the vector is determined by the geometric parameters specific to the particular problem under analysis. Using this scheme, the memory allocation changes from problem to problem according to the individual requirements of each, and thus the memory allocation is dynamic.

The integration schemes used in the SSOUP program are direct in that they perform the numerical integration using the exact kernel-shape function expressions. Although they are computationally efficient, the integration of the kernel-shape function products still accounts for most of the execution time associated with the analysis. Methods have been developed^[49] that approximate the exact kernel-shape function expressions with expressions that can be more efficiently integrated. If such schemes were utilized, significant reductions in execution time could be realized.

Also, many problems contain geometric symmetries. The symmetries can be used to reduce the number of unknowns and therefore the amount of memory required. The execution time would also be reduced. The SSOUP code currently has no capacity to take advantage of geometric symmetry.

In summary, the use of a dynamic memory allocation scheme, a more computationally efficient approximate numerical integration technique, and the utilization

of geometric symmetries has the potential to greatly reduce the memory requirements and execution time of the program.

5.1.2 User-Defined Wake Geometry

The present version of the SSOUP program uses the automatic wake generation procedure described in chapter 3. The resulting wake geometry is adequate when no other information is available. However, the formulation is capable of handling any wake geometry as long as its inclination with respect to the x-axis is small (see equation (4.3)). Thus, if information regarding the wake geometry is known in advance, then the capability of specifying an alternate wake geometry that is input by the user may be included.

Also, the wake model resulting from the automatic wake generation procedure often involves more panels than the body model. Thus, the wake can account for a significant portion of the overall memory and execution time. If the wake is extended to infinity and divided into streamwise strips, then the streamwise integrations over the wake are the same incomplete modified Struve functions that occur as part of the kernel of the potential flow downwash integral equation and can be evaluated by methods developed for that purpose (Yates *et al*^[39]). The methods referred to involve exponential approximation of the integrands. The resulting integrals can be evaluated analytically. Thus, a reduction of the memory requirements and the overall execution time of the program would result if one of these methods was used.

5.1.3 Multiple Frequency Analysis and Restart Capability

As outlined in chapter 1, many aeroelastic calculations require the ATFM at a number of discrete complex frequencies. Thus, the capability of performing analyses involving multiple frequencies is desirable. Fortunately, the formulation on which the SSOUP program is based is particularly efficient for multiple frequency analyses. The AICM, given by equation (2.87), is formed from the $[\check{b}]$, $[\check{c}]$, $[\check{d}]$, $[\check{f}]$ and $[\check{g}]$

matrices that define the singularity influences at the control points. (See equation (2.80).) The computation of these matrices involves evaluation of surface integrals and can account for up to 90% of the CPU time for a single frequency run. However, these matrices are frequency independent and thus do not require reevaluation for multiple frequency analyses. Further, the $[E^1]$ boundary condition matrix that relates the vector of generalized coordinates to the normalwash vector (equation (2.104)) can be separated into frequency dependent and frequency independent parts as

$$\left[E_{hm}^1 \right] = p \left[\vec{M}_m \cdot \vec{n} \right] \Big|_{\vec{p}=\vec{p}_h} + \left[-\vec{i} \cdot \Delta \vec{n}_m \right] \Big|_{\vec{p}=\vec{p}_h} \quad (5.1)$$

or

$$\left[E_{hm}^1 \right] = p \left[E^{11} \right] + \left[E^{10} \right] \quad (5.2)$$

where

$$\left[E^{10} \right] = \left[-\vec{i} \cdot \Delta \vec{n}_m \right] \Big|_{\vec{p}=\vec{p}_h} \quad (5.3)$$

$$\left[E^{11} \right] = \left[\vec{M}_m \cdot \vec{n} \right] \Big|_{\vec{p}=\vec{p}_h} \quad (5.4)$$

Thus, the formulation is potentially very computationally efficient for multiple frequency runs. A restart capability is also desirable to enable restarting the program using the frequency independent matrices previously saved on disk. Thus, the development of multiple frequency analyses and restart capabilities would be very beneficial.

5.1.4 Interface with a Pre/Post-Processor

Currently, there is no facility for the input of a geometric model into the SSOUP program. The geometric models for the test cases presented in chapter 4 were hard coded and are available upon execution of the program. To allow for analysis of other problems, an interface subroutine would be required. This interface subroutine would read from an input file of predetermined format. The input file would contain the patch definitions and the coordinates of the nodes defining the corners of the quadrilateral elements. The generation of the nodes for

complicated geometries is not a trivial matter and can be quite time consuming. Therefore, the input file format could be designed so that it is compatible with a pre-processor or geometric modeling program such as PATRAN, MSC/XL, or IDEAS. Also, the amount of pressure data generated by the program may be quite large. Interpretation of the results may require some post-processing to provide graphical or visual presentation. Thus, a post-processor could be coupled with the output to allow fast verification and interpretation of results.

5.1.5 Calculation of Pressure at Leading and Trailing Edges

As discussed earlier, the SOUSSA program uses special functions to force the pressure difference to go to zero at the trailing edge as the square root of the distance from the trailing edge, and to force the pressure difference at the leading edge to vary with the inverse of the square root of the distance from the leading edge. This feature results in a pressure variation at the leading and trailing edges that better models the real situation. Thus, the aerodynamic forces found from integrating the pressure distribution over the surface are more accurate.

A similar procedure should be implemented in the SSOUP program. Note that the calculation of the pressure distribution can occur after the velocity potential has been found by solving the matrix equation (2.84) involving the AICM. Thus, the pressure calculation can be included in a post-processing step and could be considered if interfacing with a post-processor.

5.2 Recommendations for Future Work

The test cases presented in chapter 4 were primarily concerned with validation of the SSOUP program. Thus, after the implementation of the improvements discussed previously, the following recommendations for future work with the SSOUP program should be considered.

5.2.1 Extensive Convergence Studies

Although some convergence studies were presented, they were very preliminary, and more in depth analyses are required to fully assess the performance of the higher order method. Thus, future work should include detailed convergence studies with comparisons between the higher and zeroth order methods.

5.2.2 Wing-Body Interference

One of the advantages of the formulation is its ability to accommodate arbitrary configurations. Thus, analyses involving wing-body interference should be conducted and compared with the existing methods including the lifting surface/surface panel methods discussed in chapter 1.

5.2.3 Complex Frequency Analyses

Another feature of the present formulation is its ability to include arbitrary motions. The validation problems from chapter 4 give results for steady and harmonic motions only. Thus, some analyses involving converging and diverging motions are required to evaluate this capability.

5.2.4 Comparisons with Approximation Methods that Use Harmonic Motions Only

The motivation presented in chapter 1 outlined the need for a method that could compute the ATFM at any complex frequency. Current methods use approximations to obtain the complex frequency results by curve fitting the ATFM's for purely imaginary frequencies only. The current method is capable of computing the ATFM at any arbitrary complex frequency. Thus, a comparison between the approximation methods that use only the imaginary frequency ATFM's and the SSOUP program results is recommended.

Further, the approximation methods can be improved by including some complex frequency data in the curve fit. An investigation into the number of data points required to accurately represent the aerodynamics throughout the complex s -plane is also recommended.

Chapter 6

Conclusions

The implementation of the approximate solution of the linear integral equation of unsteady small disturbance subsonic potential flow has been presented. The SSOUP code belongs to the category of methods known as surface panel methods and is capable of analyzing complex configurations with steady, oscillatory, or fully unsteady motions. The formulation and discretization was based upon the method used by Morino in the computer program SOUSSA. The SOUSSA code uses a zeroth order method where the velocity potential is constant within each panel. The SSOUP code uses a new higher order method that employs a quadratic variation of the velocity potential over the panels. The SSOUP code also included a zeroth order method so that SOUSSA results could be duplicated.

Several steady and oscillatory test cases were analyzed to provide validation of the SSOUP code. The SSOUP results were compared with SOUSSA results taken from the literature and analytical solutions where possible. Generally, the SSOUP results were found to be in agreement with the SOUSSA results. Also, the results from the zeroth and higher order methods were compared and a few brief convergence studies were presented. The convergence studies involved a sphere and a series of ellipsoids in steady incompressible flow, and also a rectangular plan-form wing oscillating in its first bending mode in subsonic flow. The convergence studies indicated that the convergence rate of the higher order method was twice

that of the zeroth order method. In all of the test cases studied, the higher order method provided superior or equal performance compared to the zeroth order method. However, more extensive and complete analyses are required to fully assess the capabilities and performance of the higher order method.

Finally, some improvements to the SSOUP program were presented and discussed. These improvements are a necessity for analysis on more complex configurations such as complete aircraft to ensure that the computer resource requirements are not prohibitive. Recommendations for future work including more detailed convergence studies, wing-body interference, and complex frequency analysis were also presented.

2 OF 2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
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Figure 1.1: Idealization of lifting surface using quadrilateral elements for the Doublet Lattice Method(DLM)

Figure 2.1: Locating the trailing edge point, $\bar{P}_{m(n)}^{(TE)}$, from the wake element control point, $\bar{P}_n^W$.

Figure 3.1: Typical body geometry patch and panel definitions

Figure 3.2: Linear B-spline shape functions as a function of the local parametric coordinate ξ .

Figure 3.3: Cubic B-spline patch interior panel shape functions as a function of the local parametric coordinate ξ .

Figure 3.4: Cubic B-spline patch boundary panel shape functions as a function of the local parametric coordinate ξ .

Figure 3.5: Quadratic B-spline patch interior panel shape functions as a function of the local parametric coordinate ξ .

Figure 3.6: Quadratic B-spline patch boundary panel shape functions as a function of the local parametric coordinate ξ .

Figure 3.7: Typical wake geometry patch and panel definitions

Figure 3.8: (a) Quadrilateral element with singularity at element centre $(0,0)$. (b) Subdivision of quadrilateral element into four triangular subelements. (c) Degenerate subelement with singularity at vertex and transformed coordinates (η_1, η_2) . (d) Quadrilateral subelement showing the side $(\eta_2 = -1)$ that collapses to the singularity point in (c).

Figure 3.9: Absolute error as a function of δ/R for a moderately curved element with the error limit set to $1.0E-04$. ($s/\delta = 0.25$, $M = 0.5$)

Figure 4.1: Sphere: 16×16 grid.

Figure 4.2: Velocity potential distribution as a function of x for a sphere with a 5×5 grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$)

Figure 4.3: L_2 norm error in velocity potential as a function of N for a radius 5 sphere with an $N \times N$ grid in steady incompressible flow. ($p = 0$, $M = 0$)

Figure 4.4: 555 Ellipsoid: 16×16 grid.

Figure 4.5: 565 Ellipsoid: 18×16 grid.

Figure 4.6: 5 105 Ellipsoid: 26×16 grid.

Figure 4.7: 5 205 Ellipsoid: 47×16 grid.

Figure 4.8: 5405 Ellipsoid (half span only): 58×16 grid.

Figure 4.9: Velocity potential distribution as a function of x for a 5205 ellipsoid with a 24×8 grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$)

Figure 4.10: L_2 norm error in velocity potential as a function of N_c for ellipsoids with $N_s \times N_c$ grids in steady incompressible flow. ($p = 0$, $M = 0$) [N_c - number of chordwise elements, N_s - number of spanwise elements]

Figure 4.11: Circular Cylinder (quarter span only): Span=150, 50×20 grid

Figure 4.12: Velocity potential distribution as a function of Θ for a radius 5, span 150, circular cylinder with a 50×8 element grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$)

Figure 4.13: Rectangular wing with a Kármán-Treftz airfoil section (quarter span only): Span= 150, 41×20 grid, $k = 1.7$, $l = 4.5$. [The centre of the circle is at $x = -0.5$.]

Figure 4.14: Velocity potential distribution as a function of Θ for a rectangular wing with a Kármán-Treftz airfoil section, span of 150, transformation parameters $l = 4.5$ and $k = 1.7$, and a 41×10 grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$, $\alpha = 0^\circ$) [The centre of the circle is at $x = -0.5$.]

Figure 4.15: Wind tunnel model of the rectangular wing with a circular arc airfoil section reproduced from reference[55]. ($AR = 3$, $\tau = 0.05$)

Figure 4.16: Velocity potential distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 20×27 element grids in steady subsonic flow. ($AR = 3$, $\tau = 0.05$, $p = 0$, $y = 0$, $M = 0.24$, $\alpha = 0^\circ$)

Figure 4.17: Rectangular wing with a circular arc airfoil section and two 20×27 element grids. ($AR = 3$, $\tau = 0.05$) [Note: z-axis dimension is exaggerated.]

Figure 4.18: Rectangular wing with a circular arc airfoil section and two 18×10 element grids. ($AR = 3$, $\tau = 0.05$) [Note: z-axis dimension is exaggerated.]

Figure 4.19: Pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in steady subsonic flow. ($AR = 3$, $\tau = 0.05$, $p = 0$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 0^\circ$)

Figure 4.20: Velocity potential distribution as a function of Θ for a radius 5, span 150, circular cylinder with a 41×10 element grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$, $\alpha = 0.6^\circ$)

Figure 4.21: Velocity potential distribution as a function of Θ for a rectangular wing with a Kármán-Treftz airfoil section, span of 150, transformation parameters $l = 4.5$ and $k = 1.7$, and a 41×10 grid in steady incompressible flow. ($p = 0$, $y = 0$, $M = 0$, $\alpha = 0.6^\circ$)

Figure 4.22: Pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in steady subsonic flow. The wake patch contains an 18×10 element grid and has a length of $5.55\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = 0$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 5^\circ$)

Figure 4.23: Rectangular wing with a circular arc airfoil section and four 18×10 element grids. ($AR = 3$, $\tau = 0.05$) [Note: z-axis dimension is exaggerated.]

Figure 4.24: Pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and four 18×10 element grids in steady subsonic flow. The wake patch contains an 18×10 element grid and has a length of $5.55\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = 0$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 5^\circ$)

Figure 4.25: Real component of pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in subsonic flow. The wake patch contains an 18×35 element grid and has a length of $3.5\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = ik$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 0^\circ$, $k = \omega c/U_\infty = 0.94$)

Figure 4.26: Imaginary component of pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in subsonic flow. The wake patch contains an 18×35 element grid and has a length of $3.5\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = ik$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 0^\circ$, $k = \omega c/U_\infty = 0.94$)

Figure 4.27: Real component of pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in subsonic flow. The wake was modeled by a single patch with grid sizes indicated by the legend. For all cases the wake length was $3.5\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = ik$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 0^\circ$, $k = \omega c/U_\infty = 0.94$)

Figure 4.28: Imaginary component of pressure distribution as a function of x/c for a rectangular wing with a circular arc airfoil section and two 18×10 element grids in subsonic flow. The wake was modeled by a single patch with grid sizes indicated by the legend. For all cases the wake length was $3.5\bar{c}$. ($AR = 3$, $\tau = 0.05$, $p = ik$, $|2y/b| = 0.5$, $M = 0.70$, $\alpha = 0^\circ$, $k = \omega c/U_\infty = 0.94$)

5.2 6.3 18.7 Sykesoid: 13 x 18 grid.

END

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