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***T*-Stress Solutions for Cracks at Notches and in Cylinders**

By

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a thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Applied Science (Engineering)

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

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Abstract

The elastic T -stress is a parameter used to define the level of constraint at a crack-tip. It is important to provide T -stress solutions for practical geometries in order to apply the constraint-based fracture mechanics methodology. In the present work T -stress solutions are provided for cracks at a U-shaped notch, and circumferential through-wall cracks in cylinders. The U-shaped notch with length a and a notch-tip radius ρ in a finite width plate is analyzed using the finite element method. A crack with length l is located at the notch-tip. Three notch geometries were considered; defined by the notch length (a) to specimen width (W) ratios: $a/W = 0.3, 0.4$, and 0.5 . The ratio ρ/W was fixed at 0.025 for all geometries. The T -stress was obtained at nine cracks lengths for remote tension and remote bending. A cylinder with a circumferential through-wall crack was analyzed using the finite element method and the weight function method. Three cylinder geometries were considered; defined by the pipe radius (R) to wall thickness (t) ratios: $R/t = 5, 10$, and 20 . The T -stress was obtained at eight crack lengths for remote tension and remote bending loads. The T -stress was approximated using empirical formulas obtained from least-squares curve fitting. Weight functions for T -stress were then generated for general loading conditions. First the weight function method developed in two-dimensions was extended to a cylindrical surface. Then weight functions were obtained for the three geometries defined by $R/t = 5, 10$, and 20 . The weight functions were validated by comparing the weight function based T -stress to finite element results for remote tension and bending, and good agreement was achieved. The weight functions are valid for complex loading conditions and for cracks depths in the range $0 \leq \theta/\pi \leq 0.5$.

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Nomenclature

a	Notch length
$[k]$	Element stiffness matrix
l	Crack length at U-notch
q	Smooth function in domain integral
t	Cylinder / shell thickness & U-notch thickness
$t(x_1, a)$	Weight function for T -stress
$t(\beta, \theta)$	Weight function for T -stress defined for cylinder surface
u_j	Cartesian components for displacement
u, v, w	Displacement in x, y , and z directions
x_i	x, y, z coordinate directions
A	Crack area
$[B]$	Strain-displacement matrix
$C_1, \dots, C_6$	Constants
$D_1, D_2, \dots, D_n$	Constants for approximate weight function
$[D]$	Stress-strain matrix
E	Young's modulus
F	Boundary correction factor
G	Energy release rate
I	Interaction integral
J	J -integral
$[J]$	Jacobian
K	Stress intensity factor
K_T, K_t	Stress concentration factors
M	Moment
N_i	Shape function at node i
P	Force
Q	Constant equal to 0.4740
R	Cylinder / shell radius

T	T -stress
U	Potential energy
V	Normalized T -stress
V_{1i}, V_{2i}, V_{3i}	Local coordinate system on shell surface
W	Stress work or U-notch width
α, β	Rotations for shell element
β	Angular coordinate used for weight functions
δ_{ij}	Kronecker delta
ϵ_{ij}	Cartesian components for strain
ν	Poisson's ratio
θ	Crack half-angle for cylinder
ρ	Notch-tip radius
σ_{ij}	Cartesian components for stress
σ_n, σ_0	Nominal Stress
$\sigma(x_1)$	Crack face pressure
$\sigma(\beta)$	Crack face pressure defined for cylinder surface
ξ, η, ζ	Local coordinates for isoparametric elements
ψ	Kinetic energy density
Γ	Contour integral path

Chapter 1

Introduction

Fracture occurs when a crack reaches a critical length. The critical length of a crack is established using the materials fracture toughness, which is a material property. Traditional fracture theory uses the stress intensity factor to describe the stress at a crack. The stress intensity factor depends on the geometry and loading. Based on traditional fracture theory, fracture occurs when the stress intensity factor equals the materials fracture toughness. This is called one-parameter fracture theory since it assumes the stress intensity factor completely describes the stress at a crack. Here the materials fracture toughness is always taken as the lower bound value so it may be assumed constant for a given material. Strict testing procedures have been established to determine the materials fracture toughness. The two most commonly used test specimens are shown in Figure 1.1. These specimens are designed to ensure that the fracture toughness is always a lower bound value. In practice the fracture toughness may be higher for other types of geometries.

Two-parameter fracture theory does not assume the materials fracture toughness is fixed, and uses an additional parameter to determine the fracture toughness. The most commonly used is the elastic T -stress, or simply the T -stress. Both the stress intensity factor and the T -stress are defined using a mathematical representation of the crack-tip stress field. From this an infinite series expansion is produced, where the stress intensity factor is defined in the first term, and the T -stress is the second term. The T -stress has a unique value that depends on the geometry and loading. Most importantly is the sign and magnitude of the T -stress has been shown to correlate with the level of crack-tip constraint. Negative values of T -stress correspond with low crack-tip constraint, where positive values of T -stress correspond with high crack-tip constraint. The level of constraint impacts the fracture behaviour of materials. As constraint decreases in the negative region the measured fracture toughness increases. This is shown in Figure 1.2.

The conclusion is the T -stress provides a good index for the relative level of crack tip constraint for various geometries.

To use the T -stress for fracture assessment it is important to determine this parameter for practical geometries; in particular for geometries that experience low crack-tip constraint. Several methods may be used to determine the T -stress including the finite element method and the weight function method. In the present work T -stress solutions will be obtained using the finite element method and the weight function method. Two types of geometries will be considered; they are (1) cracks emanating from a U-shaped notch, and (2) circumferential through-wall cracks in cylinders.

The U-shaped notch is a geometry commonly encountered in engineering design. Notches in a structure cause a local stress concentration with the maximum stress located at the root of the notch. The elevated stress in this region leads to the development of fatigue cracks. The presence of cracks must be considered in the design of notched components. Analysis of cracks at notches is required for predicting the critical flaw size and the fatigue life. If cracks at notches experience low constraint then a higher value for fracture toughness could be used. This would allow for a larger critical flaw size and result in an increase in the predicted life of the component.

In the first part of this thesis the finite element method will be used to obtain the T -stress for cracks at a U-shaped notch in a finite width plate. The geometry of the U-shaped notch considered here is shown in Figure 1.3. Three notch lengths will be considered given by: $a/W = 0.3, 0.4, \text{ and } 0.5$. Each will include nine crack lengths given by: $l/\rho = 0.05, 0.10, 0.20, 0.30, 0.50, 1.00, 1.50, 2.00, \text{ and } 3.00$. The U-shaped notched plate will be subjected to remote tension and bending. The results for T -stress will be tabulated and compared to an equation derived by Wang et al. (2006).

Another important geometry is circumferential through-wall cracks in cylinders. Circumferential cracks are encountered in a variety of applications, such as pipelines, steam generator tubes, and tubular structures. Analysis of cracks in pipelines is important

when transporting pressurized fluids. Steam generator tubes are a specific application of pipelines, and are used in power plant heat exchangers. For nuclear power plants the assessment of fatigue cracks is vital. Analysis considers through-wall cracks in Leak-Before-Break (LBB) assessments. One important factor in LBB assessments is the critical flaw size. Low constraint and increased fracture toughness would allow for a larger critical flaw size. Another application for tubes is for offshore structures, where large tubular members are welded together. Welds are a common location for fatigue cracking due to imperfections introduced during the welding process. Offshore structures are also subjected to extreme conditions at sea, where failure could have severe consequences. A circumferential through-wall crack is a severe defect. Knowledge of the constraint level for these applications would lead to better fatigue and fracture assessment. Recently work has been done by Jayadevan et al. (2005) and Skallerud et al. (2005) to obtain the T -stress for surface cracks on pipes using the line-spring method, but no solutions for T -stress is available for through-wall cracks.

In the second part of this thesis the finite element method and the weight function method will be used to obtain T -stress solutions for circumferential through-wall cracks in cylinders. A circumferential through-wall crack in a cylinder is shown in Figure 1.4. Three geometries will be considered, given by: $R/t = 5, 10$, and 20 . Each will include eight crack lengths, given by: $\theta/\pi = 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375$, and 0.5000 . The T -stress will be obtained for remote tension and bending loads. The weight functions will be obtained for $R/t = 5, 10$, and 20 .

1.1. Outline for Thesis

The objective of this thesis is to provide T -stress solutions for cracks at a U-shaped notch, and for circumferential cracks in a cylinder. The finite element method and weight function method will be used for the analysis. The developed solutions will be useful for fatigue and fracture analysis of notch geometries and cylinder geometries.

Chapter 2 will review the background theory, and the methods used to calculate the T -stress. The T -stress solutions for the U-shaped notch are presented in Chapter 3, and the T -stress solutions for circumferential through-wall cracks are presented in Chapter 4. A discussion of the results is provided in Chapter 5.

Figure 1.1: Standard test specimens used to determine lower bound value of fracture toughness.

Figure 1.2: Relationship between T -stress and fracture toughness.

Figure 1.3: Through-thickness crack at the root of a U-shaped notch.

Figure 1.4: Circumferential through-wall crack in a cylinder.

Chapter 2

The Elastic T -stress in Fracture Mechanics

In this chapter the significance of the T -stress will be reviewed. Two methods used to determine the T -stress will be discussed, which are (1) the finite element method, and (2) the weight function method.

2.1. Stress Intensity Factor and the T -stress

The stress intensity factor and the T -stress are used to describe the stress at a crack, and they are defined using a theoretical solution for the crack-tip stress field.

2.1.1 Theoretical Solution for Crack Tip Stress Field

Crack Tip Stress Field

A crack may experience three possible deformation modes as shown in Figure 2.1. They can occur individually, or as a combination of modes. The stresses in the vicinity of a crack tip have been solved using a stress function attributed to Williams (1957). This approach yields an infinite power series for the crack tip stress field. For a two-dimensional crack, the first two terms of the Williams series expansion of the stresses for mode I loading are given by

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + T \quad (2.1)$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \quad (2.2)$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \quad (2.3)$$

where K_I is the mode I stress intensity factor, and r and θ are polar coordinates with the axes placed at the crack tip, as shown in Figure 2.2. The first term possesses a $1/\sqrt{r}$ singularity, and the second term is constant with r . The second term is non-singular, and acts in the plane of the crack (x_1 -direction). This term is defined as the T -stress. The remaining terms in the series expansion approach zero nearing the crack tip. Both the stress intensity factor and the T -stress depend on the geometry, crack depth, and loading. There also exists a critical value for the stress intensity factor that will result in crack extension, and is defined as the fracture toughness (K_C). For one-parameter fracture K is assumed to completely define the stress field at a crack-tip.

Two-Parameter Fracture Mechanics

From the Williams series expansion, the T -stress is a non-singular stress acting in the x_1 -direction. It has been shown by Larsson and Carlsson (1973), and Rice (1974) that the value of the T -stress has a significant impact on the size and shape of the plane strain plastic zone. The sign and magnitude of the T -stress will affect the level of crack tip triaxiality and thereby influence crack tip constraint. Positive values of T -stress increase crack tip triaxiality, resulting in high crack tip constraint. Negative values of T -stress decrease crack tip triaxiality, resulting in low crack tip constraint. The significance of this relationship has been explained by Du and Hancock (1991), O'Dowd and Shih (1991), and Betegon and Hancock (1991). The conclusion is the T -stress provides a good index for the relative crack tip constraint for various geometries. Geometries having a negative T -stress will exhibit a higher value of fracture toughness. Current testing to determine the fracture toughness enforces the high constraint condition, thus ensuring one parameter fracture. When $T \geq 0$ single parameter fracture theory based on K is rigorously correct. For negative T -stress values, single parameter fracture breaks down and the fracture toughness then depends on the T -stress. This dependence is defined as two-parameter fracture.

Evaluation of K and T

The methods used to calculate the stress intensity factor and T -stress are derived from the J -integral, which is equal to the energy release rate for linear elastic material behaviour.

2.1.2 The Energy Release Rate and the J -integral

Griffith (1920) defined the energy release rate ($\mathcal{G}$) as the rate of change in potential energy with crack area

$$\mathcal{G} = -\frac{dU}{dA} \quad (2.4)$$

where U is the potential energy, and A is the crack area. The energy release rate is valid for linear elastic materials. Rice (1968) proposed a line integral that is valid for non-linear material behaviour, and is given by

$$J = \int_{\Gamma} \left(W dx_2 - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x_1} ds \right) \quad (2.5)$$

where W is the strain energy density, $\mathbf{T}$ is the traction vector acting on Γ , $\mathbf{u}$ is the displacement vector at Γ , and ds is a length increment along Γ . An arbitrary contour used for integration is shown in Figure 2.3. Rice showed that this line integral is path independent. For linear elastic material behaviour $J = \mathcal{G}$. It was shown by Irwin (1956) that the stress intensity factor is related to the energy release rate

$$\mathcal{G} = \frac{K_I^2}{E'} \quad (2.6)$$

where,

$$\left. \begin{aligned} E' &= E && \text{for plane stress,} \\ E' &= \frac{E}{1-\nu^2} && \text{for plane strain,} \end{aligned} \right\} \quad (2.7)$$

and E is Young's modulus, and ν is Poisson's ratio. The J -integral allows the calculation the stress intensity factor for linear elastic material behaviour. Numerical methods are the most powerful tool available for evaluating the J -integral.

2.2. The Finite Element Method

Numerical methods have proven to be very useful in modelling fracture problems. The Finite Element Method (FEM) is a numerical method used extensively in fracture mechanics. The FEM allows evaluation of K and T for arbitrary geometries. This is done using special techniques designed to solve fracture problems with the FEM. They include modifications to the finite element mesh and the domain integral approach. In this section these techniques will be reviewed, as well as two isoparametric elements used in subsequent analysis.

2.2.1 Isoparametric Elements

The isoparametric element formulation greatly simplifies the development of the element equations. Equations are derived using a natural coordinate system that is identical for all elements. A relationship is used, called transformation mapping, to relate the natural coordinate system to the global coordinate system. The element shape functions are used to map the element geometry, as well as the nodal displacements. Two types of isoparametric elements were used in the present work: (1) the 8-node isoparametric plane strain element, and (2) the 8-node isoparametric general shell element.

Isoparametric Plane Strain Element

The isoparametric formulation of the 8-node quadrilateral is well documented. The element is shown in the natural coordinates and mapped into global coordinates in Figure 2.4. The geometry of the element is mapped into the global coordinate system using the following transformations

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{i=1}^8 N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \quad (2.8)$$

where x_i and y_i are the nodal coordinates in the global coordinate system, and N_i are the shape functions for node i . The shape functions are given by

$$\left. \begin{aligned} N_i(\xi, \eta) &= \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1) & \text{for } i = 1, 2, 3, 4 \\ N_i(\xi, \eta) &= \frac{1}{2}(1 - \xi^2)(1 + \eta\eta_i) & \text{for } i = 5, 7 \\ N_i(\xi, \eta) &= \frac{1}{2}(1 - \eta^2)(1 + \xi\xi_i) & \text{for } i = 6, 8 \end{aligned} \right\} \quad (2.9)$$

where (ξ_i, η_i) are the nodal coordinates in the natural coordinate system. The corner nodes have values of $(\xi, \eta = \pm 1)$, and the middle nodes lie on either $\xi = 0$ and $\eta = \pm 1$, or $\eta = 0$ and $\xi = \pm 1$. The displacements are approximated using

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_{i=1}^8 N_i(\xi, \eta) \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \quad (2.10)$$

The strains are related to the nodal displacements using the approximation given by Equations (2.10) as follows

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \quad (2.11)$$

where $[B] = [[B]_1, [B]_2, \dots, [B]_8]$, and

$$[B]_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}. \quad (2.12)$$

Since the shape functions N_i are in terms of the natural coordinates (ξ, η) direct evaluation of Equation (2.12) is not possible. The natural and global coordinate systems can be related using the Jacobian $[J]$ that takes the following form in two-dimensions

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (2.13)$$

where,

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}. \quad (2.14)$$

Equation (2.14) allows the evaluation of $[B]$. The stress-strain relationship is given by

$$\{\sigma\} = [D]\{\varepsilon\} \quad (2.15)$$

where $[D]$ is the constitutive matrix for plane strain. The element stiffness matrix is given by

$$[k] = \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] \det[J] d\xi d\eta \quad (2.16)$$

Isoparametric General Shell Elements

A shell forms a curved surface in space. The general shell is a Mindlin-type element, so it accounts for transverse shear deformation. Incorporating the transverse shear is done for analyzing thick shells. Although it is referred to as a thick shell, the element still models the field variables on the shell middle surface. The field variables for the general shell are three displacements and two rotations per node. The 8-node element shown in Figure 2.5 is used for the analysis of shells in the present work. The summary given here follows the presentation given by Cook (1989).

To define the shell geometry an orthogonal vector set $(\mathbf{V}_{1i}, \mathbf{V}_{2i}, \mathbf{V}_{3i})$ is defined at each node. At a general node i the thickness direction vector $\mathbf{V}_{3i}$ is given by

$$\mathbf{V}_{3i} = t_i \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix} \quad (2.17)$$

where l_{3i} , m_{3i} , and n_{3i} are directions cosines for the vector $\mathbf{V}_{3i}$, and t_i is the shell thickness at node i . The vectors $\mathbf{V}_{1i}$ and $\mathbf{V}_{2i}$ are orthogonal to $\mathbf{V}_{3i}$, and tangent to the shell middle surface. Details on defining these vectors are provided by Cook (1989). The positions of an arbitrary point in the element in the global coordinate system are

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^8 N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \sum_{i=1}^8 N_i(\xi, \eta) \zeta \frac{t_i}{2} \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix}. \quad (2.18)$$

The shape functions are functions of ξ and η however they are independent of ζ . The shape functions for the 8-node quadrilateral are given by equations (2.9). The Jacobian for three dimensions is given by

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (2.19)$$

The Jacobian is evaluated using positions defined by Equation (2.18). The displacements of an arbitrary point in the element are

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^8 N_i(\xi, \eta) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \sum_{i=1}^8 N_i(\xi, \eta) \zeta \frac{t_i}{2} \begin{bmatrix} -l_{2i} & l_{1i} \\ -m_{2i} & m_{1i} \\ -n_{2i} & n_{1i} \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \quad (2.20)$$

where α_i and β_i are the rotations at node i , and u_i , v_i and w_i are the displacements at node i . The stress is related to the displacements using

$$\begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^T = [\mathbf{B}] \begin{bmatrix} u_i & v_i & w_i & \alpha_i & \beta_i \end{bmatrix}^T \quad (2.21)$$

where $[\mathbf{B}] = [[\mathbf{B}]_1, [\mathbf{B}]_2, \dots, [\mathbf{B}]_8]$. The first row of a sub-matrix $[\mathbf{B}]_i$ is given by

$$\varepsilon_x = \frac{\partial u}{\partial x} = \left[\frac{\partial N_i}{\partial x} \quad 0 \quad 0 \quad -\zeta \frac{t_i}{2} \frac{\partial N_i}{\partial x} l_{2i} \quad \zeta \frac{t_i}{2} \frac{\partial N_i}{\partial x} l_{1i} \right] \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \alpha_i \\ \beta_i \end{Bmatrix}. \quad (2.22)$$

The remaining rows are determined in a similar manner using the displacements given by Equation (2.20). Note that all six strains are included in Equation (2.21) since the shell middle surface has an arbitrary orientation in the global coordinate system. The condition

that $\sigma_3 = 0$ is satisfied using the stress-strain relation defined for the local coordinate system. The stress-strain relations are given by

$$\{\sigma\} = [D]\{\varepsilon\} \quad \text{and} \quad \{\sigma'\} = [D']\{\varepsilon'\} \quad (2.23)$$

where $\{\sigma\}$ contains stresses in the global coordinate system xyz and $\{\sigma'\}$ contains stresses defined in the local directions normal and tangent to the shell mid-surface. $[D]$ is obtained from $[D']$ using the coordinate transformation $[D] = [T]^T [D'] [T]$. The stiffness matrix is

$$[\mathbf{k}] = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] \det[\mathbf{J}] d\xi d\eta d\zeta \quad (2.24)$$

$5N \times 5N \quad \quad \quad \begin{matrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{matrix} \quad \begin{matrix} 5N \times 6 & 6 \times 6 & 6 \times 5N \end{matrix}$

where N is the number of nodes per element. Equation (2.24) is valid for any element type. For the 8-node element shown in Figure 2.5 the thickness direction ζ is linear, and may be integrated explicitly. The $[B]$ matrix is divided into two parts, one that is independent of ζ and another that is linear in ζ . So that $[B] = [B_0] + \zeta[B_1]$. The linear terms integrated from minus one to one equal to zero, and Equation (2.24) becomes

$$[\mathbf{k}] = \int_{-1}^1 \int_{-1}^1 (2[B_0]^T [D] [B_0] + \frac{2}{3}[B_1]^T [D] [B_1]) \det[\mathbf{J}] d\xi d\eta \quad (2.25)$$

and $[\mathbf{J}]$ remains a 3×3 matrix, and is evaluated on the shell middle surface $\zeta = 0$.

2.2.2 Modelling the Crack Tip Stress Field

The leading term in the Williams series expansion predicts that the stress is proportional to $1/\sqrt{r}$ in a linear elastic material. The 8-node elements use quadratic interpolation polynomials, so are unable to capture the stress singularity at the crack tip. Barsoum

(1976) modified an 8-node quadrilateral by collapsing one side of the quadrilateral onto one point, thereby forming a triangle, as well as moving the mid-side nodes to the quarter points. This not only creates the singularity at the element corner, but also creates a stress proportional to $1/\sqrt{r}$ throughout the entire element. This modification is shown in Figure 2.6. These modifications are accomplished by mapping the isoparametric elements into a distorted configuration in the global coordinate system. The corner possessing the singularity is mapped to the crack tip.

For some elements it may not be possible to collapse one side. In this situation an artificial key hole is introduced at the crack tip. In addition, if the mid-side nodes are not moved then the crack tip elements do not possess the $1/\sqrt{r}$ singularity. This feature is usually desirable, but not necessary to obtain reasonable results as long as the mesh is properly refined around the crack-tip.

2.2.3 The Domain Integral Approach

The stress intensity factor is extracted from the finite element results using the J -integral, and the T -stress is extracted using an interaction integral based on Eshelby's theorem. These integrals can be converted to area integrals for two dimensional problems, and volume integrals for three dimensional problems using the domain integral approach. The domain integral approach is much easier to implement numerically, and provides greater accuracy.

The J-integral

Shih et al. (1986) proposed a form of the J -integral that includes inertia effects for crack growth in the x_1 direction. For dynamic conditions the J -integral is given by

$$J = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left[(W + \psi) \delta_{1j} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_j ds \quad (2.26)$$

where W and ψ are the stress work and kinetic energy densities respectively, σ_{ij} and u_i are the Cartesian components of stress and displacement, n_j is the unit vector normal to Γ and ds is the arc length along the contour. For quasi-static conditions $\psi = 0$ and Equation (2.26) takes the form

$$J = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left[W \delta_{1j} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_j ds. \quad (2.27)$$

Following the approach used by Li et al. (1985) a closed curve is defined that bounds an area A^* as shown in Figure 2.7. The closed curve can be written as $\Gamma = \Gamma_1 + \Gamma_+ - \Gamma_0 + \Gamma_-$. On curve Γ_0 the unit vector is defined as $m_i = -n_i$, and on the curve Γ_1 the unit vector is defined as $m_i = n_i$. Rewriting Equation (2.27) for the closed contour assuming there are no tractions on the crack faces

$$J = \int_{\Gamma} \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - W \delta_{1j} \right] m_j q_1 ds \quad (2.28)$$

where q_1 is a smooth function in A^* that is unity on Γ_0 and vanishes on Γ_1 . Applying the divergence theorem to Equation (2.28)

$$J = \int_A \frac{\partial}{\partial x_j} \left[\left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) q_1 \right] dA. \quad (2.29)$$

Using equilibrium and assuming there are no body forces results with the following simplification for Equation (2.29)

$$J = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right] \frac{\partial q_1}{\partial x_j} dA \quad (2.30)$$

Equation (2.30) is the domain integral expression for the J -integral, and is valid for a linear elastic material under quasi-static conditions in the absence of body forces and crack face tractions.

The Interaction Integral

A direct method for obtaining the T -stress uses the evaluation of an interaction integral based on Eshelby's theorem and is described by Kfouri (1986). The integral is defined using two stress-displacement fields. The primary field is due to a remote tension F shown in Figure 2.8, and the other is a superposition of the remote tension and an auxiliary field. The auxiliary field is produced by a point force f acting at the crack-tip in the x_1 direction. The point force f is balanced at the outer boundaries by the traction t . The analytical solution for the stress and displacement for the auxiliary field, denoted using the superscript ' L ', is given by:

$$\left. \begin{aligned} \sigma_{11}^L &= -\frac{f}{\pi r} \cos^3 \theta \\ \sigma_{22}^L &= -\frac{f}{\pi r} \cos \theta \sin^2 \theta \\ \sigma_{12}^L &= -\frac{f}{\pi r} \cos^2 \theta \sin \theta \end{aligned} \right\} \quad (2.31)$$

and,

$$\left. \begin{aligned} u_1^L &= -\frac{1-\nu^2}{E} \cdot \frac{f}{\pi} \left\{ \ln\left(\frac{r}{d}\right) + \frac{\sin^2 \theta}{2(1-\nu)} \right\} \\ u_2^L &= -\frac{1+\nu}{2E} \cdot \frac{f}{\pi} \{(1-2\nu)\theta - \cos \theta \sin \theta\}. \end{aligned} \right\} \quad (2.32)$$

Define $J(F)$ as the J -integral obtained from a remote contour for the stress-displacement field produced by remote tension F , and denote the stress, strain, and displacement fields as σ_{ij} , ε_{ij} and u_i respectively. Define $J(F, f, t)$ as the J -integral obtained from a remote contour for the stress-displacement field produced by the combined loading due to

the remote tension F and the auxiliary field. The first form of Eshelby's theorem is given by

$$J(F, f, t) = J(F) + Tf / E' . \quad (2.33)$$

Rearranging to solve for T , and substituting Equation (2.7) for plane strain gives

$$T = \frac{E}{1 - \nu^2} \cdot \frac{I}{f} \quad (2.34)$$

where I is defined as the interaction integral given by

$$I = J(F, f, t) - J(F) . \quad (2.35)$$

Using the J -integral given by Equation (2.27) the interaction integral is written as

$$I = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left[\sigma_{ij} \varepsilon_{ij} \delta_{1j} - \sigma_{ij} \frac{\partial u_i^L}{\partial x_1} - \sigma_{ij}^L \frac{\partial u_i}{\partial x_1} \right] n_j ds . \quad (2.36)$$

Nakamura and Parks (1992) provided the domain integral formulation for the interaction integral, and it is obtained in the same manner described for the J -integral. The two-dimensional domain integral formulation for the interaction integral is given by

$$I = \int_A \left[\left(\sigma_{ij} \frac{\partial u_i^L}{\partial x_1} + \sigma_{ij}^L \frac{\partial u_i}{\partial x_1} \right) \frac{\partial q_1}{\partial x_j} - \sigma_{ij} \varepsilon_{ij} \frac{\partial q_1}{\partial x_1} \right] dA . \quad (2.37)$$

The domain integral formulation for three dimensions was provided by Nakamura and Parks (1992). The integral has the same form as Equation (2.37); however, for three dimensions it is a volume integral.

Finite Element Implementation

In two-dimensions the domain integral approach defines an area over which to perform the integration. The inner contour Γ_0 , as shown in Figure 2.7 is taken at the crack tip. Consequently the area A^* is defined as the area inside the contour Γ_1 . The function q must be unity at the crack tip and vanish at the outer boundary defined by Γ_1 . Aside from these requirements the function q is arbitrary. One example of an acceptable function for q is the plateau function shown in Figure 2.9 (a), and is equal to one in all regions except the outer boundary. A pyramid function shown in Figure 2.9 (b) is also common, where q is equal to one at the crack tip and decreases linearly in all directions to zero at the outer boundaries. It was shown by Shih et al. (1986) that the J -integral is insensitive to the shape of the q function. The value of q must be specified at all nodes within the area of integration in accordance with the assumed function. Consistent with the isoparametric formulation the value of q within each element is approximated using the shape functions. The value for q is approximated within the element using

$$q = \sum_{i=1}^8 N_i(\xi, \eta) q_i \quad (2.38)$$

where N_i are the shape functions given by Equations (2.9) and q_i is the specified value for q at node i . The derivatives of Equation (2.38) with respect to the global coordinates for two-dimensions is given by

$$\begin{Bmatrix} \frac{\partial q}{\partial x_1} \\ \frac{\partial q}{\partial x_2} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \frac{\partial q}{\partial \xi} \\ \frac{\partial q}{\partial \eta} \end{Bmatrix} \quad (2.39)$$

where $[\mathbf{J}]^{-1}$ is the inverse of the Jacobian. Gaussian integration is used to evaluate the domain integral expressions for the J -integral and the interaction integral. The domain integral expression for the J -integral is numerically integrated using

$$J = \sum_{A^*} \sum_{p=1}^m \left\{ \left[\left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} \right] \det(\mathbf{J}) \right\}_p w_p \quad (2.40)$$

where m is the number of Gaussian points per element, and w_p are the weighting factors. The quantities contained within $\{ \}_p$ are evaluated at Gaussian points. Equation (2.40) does not account for crack face tractions. The domain integral expression for the interaction integral is numerically integrated using

$$I = \sum_{A^*} \sum_{p=1}^m \left\{ \left[\left(\sigma_{ij} \frac{\partial u_i^L}{\partial x_1} + \sigma_{ij}^L \frac{\partial u_i}{\partial x_1} \right) \frac{\partial q}{\partial x_j} - \sigma_{ij} \varepsilon_{ij} \frac{\partial q}{\partial x_1} \right] \det(\mathbf{J}) \right\}_p w_p. \quad (2.41)$$

Equation (2.40) and (2.41) are area integrals for two-dimensions.

Application for Shell Elements

For the shell element Equation (2.40) and (2.41) are evaluated as volume integrals for three dimensions. The 8-node shell element approximates the field variables at the shell middle surface. Using the same approach described for the stiffness matrix, the thickness direction is linear and may be integrated explicitly. This eliminates terms dependent on the thickness direction. Afterwards the Jacobian is evaluated at the middle surface, and integration is done over the shell surface area.

2.3. Superposition and Weight Function Methods

A numerical analysis to obtain the T -stress for a cracked body will yield a solution valid for one set of boundary conditions. For a specific geometry, different loadings will result in different values for the T -stress. The superposition method can be used to obtain the T -stress for simple loading conditions. For more complex situations it is also possible to incorporate the use of a weight function. These two methods are summarized here.

2.3.1 Superposition Method for Elastic T -stress

A cracked body is subjected to a load Q as shown in Figure 2.10 (a). It was shown by Chen (2000) that the T -stress could be obtained using the superposition of two stress fields. The first is the field obtained from the uncracked body subjected to load Q shown in Figure 2.10 (b). The load Q induces a stress distribution $\sigma(x_1)$ in the uncracked body. The stress distribution $\sigma(x_1)$ is equal to the normal stress acting in the x_2 direction along the x_1 -axis. The second field is obtained by loading the crack face with the stress distribution $\sigma(x_1)$ as shown in Figure 2.10 (c). Therefore the elastic T -stress is obtained as the summation of the T -stresses from these two stress fields

$$T = T_{\text{crack pressure}} + T_{\text{uncracked}} \quad (2.42)$$

The T -stress for the uncracked body shown in Figure 2.10 (b) subjected to load Q is obtained using

$$T_{\text{uncracked}} = (\sigma_{11} - \sigma_{22}) \Big|_{x_1=a} \quad (2.43)$$

where the T -stress is evaluated at $x_1 = a$ and $x_2 = 0$ in the uncracked body. Substituting Equation (2.43) into Equation (2.42) yields

$$T = T_{\text{crack pressure}} + (\sigma_{11} - \sigma_{22}) \Big|_{x_1=a} . \quad (2.44)$$

Equation (2.44) is the superposition method for T -stress.

2.3.2 Weight Function Method

The T -stress for the cracked body shown in Figure 2.10 (c) subjected to crack face pressure $\sigma(x_1)$ can be calculated using (Wang, 2002)

$$T_{\text{crack pressure}} = \int_0^a \sigma(x_1) \cdot t(x_1, a) dx_1 \quad (2.45)$$

where $\sigma(x_1)$ is the crack face pressure, a is the crack length, and $t(x_1, a)$ is the weight function for T -stress. The weight function is the T -stress obtained from a pair of unit loads acting on the crack faces. The weight function is independent of the loading, and depends only on the geometry. Substituting Equation (2.45) into Equation (2.44) yields

$$T = \int_0^a \sigma(x_1) \cdot t(x_1, a) dx_1 + (\sigma_{11} - \sigma_{22})|_{x_1=a} \cdot \quad (2.46)$$

To accommodate loading conditions the uncracked body is analyzed to determine $\sigma(x_1)$ and the T -stress at $x_1 = a$. Then using the weight function it is possible to compute the T -stress for the cracked body using Equation (2.46).

2.3.3 Approximate Weight Function

Fett (1997) proposed a general form for a weight function, which can be re-written as follows

$$t(x_1, a) = \frac{2}{\pi a} \left[D_1 \left(1 - \frac{x_1}{a} \right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a} \right)^{\frac{3}{2}} + D_3 \left(1 - \frac{x_1}{a} \right)^{\frac{5}{2}} + \dots + D_n \left(1 - \frac{x_1}{a} \right)^{n-\frac{1}{2}} \right] \quad (2.47)$$

where x_1 is defined using the coordinate system shown in Figure 2.10, a is the crack length, and $D_1, D_2, \dots, D_n$ are constants for a specific crack geometry. Considering only the first two terms from Equation (2.47),

$$t(x_1, a) = \frac{2}{\pi a} \left[D_1 \left(1 - \frac{x_1}{a} \right)^{\frac{1}{2}} + D_2 \left(1 - \frac{x_1}{a} \right)^{\frac{3}{2}} \right] \quad (2.48)$$

this weight function was used by Wang (2002) for single and double edge cracks as well as internal cracks, and excellent accuracy was achieved.

Reference Loadings

To solve the constants D_1 and D_2 two reference T -stress solutions are required. Following the approach used by Wang (2002) two crack-face pressures are used: (1) uniform crack pressure, and (2) linearly decreasing crack pressure. The crack face pressure for the first reference loading is given by

$$\sigma(x_1) = \sigma_0. \quad (2.49)$$

The corresponding T -stress is given by,

$$T_{1r} = \sigma_0 V_0 \quad (2.50)$$

where V_0 is the normalized T -stress for uniform crack face pressure. The crack face pressure for the second reference loading is given by

$$\sigma(x_1) = \sigma_0 \left(1 - \frac{x_1}{a} \right). \quad (2.51)$$

The corresponding T -stress is given by,

$$T_{2r} = \sigma_0 V_1 \quad (2.52)$$

where V_1 is the normalized T -stress for linearly decreasing crack face pressure. The T -stresses obtained from these two reference loads provide sufficient information to solve for the constants D_1 and D_2 .

Weight Function Solution

The general expressions for D_1 and D_2 are obtained by inserting Equations (2.49) to (2.52) into Equation (2.46) giving

$$\left. \begin{aligned} \sigma_0 V_0 &= \int_0^a \sigma_0 t(x_1, a) dx_1 \\ \sigma_0 V_1 &= \int_0^a \sigma_0 \left(1 - \frac{x_1}{a}\right) t(x_1, a) dx_1. \end{aligned} \right\} \quad (2.53)$$

Noting that $T_{\text{uncracked}} = 0$. Then substituting the two-term approximation for the weight function (Equation (2.48)) and integrating yields

$$\left. \begin{aligned} V_0 &= \frac{4}{\pi} \left[\frac{D_1}{3} + \frac{D_2}{5} \right] \\ V_1 &= \frac{4}{\pi} \left[\frac{D_1}{5} + \frac{D_2}{7} \right]. \end{aligned} \right\} \quad (2.54)$$

Solving Equations (2.54) for the constants D_1 and D_2 yields

$$D_1 = \frac{15}{16} \pi (5V_0 - 7V_1) \quad (2.55)$$

$$D_2 = \frac{5}{16} \pi (35V_0 - 21V_1). \quad (2.56)$$

With the normalized T -stress for both reference loadings (V_0, V_1) the coefficients of the weight function are obtained using Equations (2.55) and (2.56). With the constants D_1 and D_2 the approximate weight function is solved using Equation (2.48).

2.4. T -stress for Small Cracks at Notches

Wang et al. (2006) derived an equation to estimate the T -stress for small cracks emanating from notches. The equation is obtained using the superposition method for T -stress and the similarities between the elastic notch-tip stress field. The elastic notch-tip stress field has been investigated by Creager and Paris (1967), and Glinka (1985). The expressions for the notch-tip stress field are described by two parameters: (1) the stress concentration factor K_t , and (2) the notch-tip radius ρ . The stress components in the x_1 and x_2 direction at the root of an uncracked notch given by Glinka are

$$\sigma_{11} = \frac{K_t \sigma_n}{2} \left[\left(1 + 2 \frac{x_1}{\rho} \right)^{-1/2} - \left(1 + 2 \frac{x_1}{\rho} \right)^{-3/2} \right] \quad (2.57)$$

$$\sigma_{22} = \frac{K_t \sigma_n}{2} \left[\left(1 + 2 \frac{x_1}{\rho} \right)^{-1/2} + \left(1 + 2 \frac{x_1}{\rho} \right)^{-3/2} \right] \quad (2.58)$$

where σ_n is the nominal stress, and x_1 is measured from the root of the notch. These equations were verified for a wide variety of notch geometries using numerical and experimental methods and are valid for the range $x_1 < \rho$. Wang et al. (2006) used the superposition method for T -stress and the notch-tip stress field to solve for the T -stress. The equation presented by Wang et al. (2006)

$$\frac{T}{\sigma_n} = K_t \left[\frac{Q}{2} \left(1 + 2 \frac{l}{\rho} \right)^{-1/2} + \left(\frac{Q}{2} - 1 \right) \left(1 + 2 \frac{l}{\rho} \right)^{-3/2} \right] \quad (2.59)$$

where Q is a constant of 0.4740, and l is the crack length. This equation is valid for small cracks emanating from notches. Wang et al. (2006) predicts the T -stress to converge to a long crack solution as the crack extends past the notch-tip stress field. The T -stress solution for the long crack will be for an overall length equal to the notch length plus the crack length. Designating this effective crack length by c , and the specimen width by W , the T -stress for a long crack is given by

$$\frac{T_L}{\sigma_n} = V \left(\frac{c}{W} \right) \quad (2.60)$$

where $V(c/W)$ is the normalized T -stress, and is a function of the crack length and specimen width.

2.5. Summary

The background theory for the elastic T -stress was presented. Two methods used to calculate the T -stress were also reviewed, and are (1) the finite element method and, and (2) the weight function method. These methods will be used to determine the T -stress in subsequent chapters.

Figure 2.1: Three basic modes of crack surface displacement.

Figure 2.2: Coordinate system and stresses at crack tip for two dimensions.

Figure 2.3: An arbitrary counterclockwise contour around a crack-tip used for the evaluation of the J -integral.

Figure 2.4: 8-node isoparametric quadrilateral element.

Figure 2.5: 8-node isoparametric general shell element.

Figure 2.6: Collapsed side and movement of mid-side nodes to the quarter-point.

Figure 2.7: Closed contour used for two-dimensional domain integral approach.

Figure 2.8: Superposition of stress fields for evaluation of interaction integral in two-dimensions.

(a) Plateau function.

(b) Pyramid function.

Figure 2.9: Examples q functions in two dimensions.

Figure 2.10: Superposition method for T -stress

Chapter 3

Through Thickness Cracks at a U-shaped Notch

The U-shaped notch is a common shape encountered in engineering design. Fatigue cracks typically initiate at notches due to the local stress concentration. Knowledge of the constraint level for cracks at notches could lead to better assessment of fracture toughness and fatigue life. The objective of this chapter is to obtain T -stress solutions for cracks at a U-shaped notch in a finite width plate using the finite element method. Subsequently the results will be compared to an equation derived by Wang et al. (2006).

3.1. Finite Element Analysis

Before analyzing the U-shaped notch a series of test models were made. The test case is two through cracks extending symmetrically from a circular hole in an infinite plate, and is shown in Figure 3.1. The boundary correction factor and normalized T -stress obtained for these test models were compared to published data. Afterwards models for the U-shaped notch were made by modifying the test model.

ABAQUS version 6.4-1 was used for the finite element analysis. The finite element models were made using 8-node plane strain elements, and are defined as CPE8 in the ABAQUS element library. The material properties were defined as Young's Modulus $E = 30E6 \text{ lb/in.}^2$, and Poisson's ratio $\nu = 0.30$.

3.1.1 Crack Region

Both the test models and the models for the U-shaped notch have a similar crack region. A web mesh was used at the crack tip. Node positions within the contour integral area were automatically generated by ABAQUS. The ABAQUS node generation command creates the square root singularity at the crack-tip. This is done by collapsing the crack-tip elements into triangles, and moving the mid-side nodes to the quarter-points. Seven

contours of integration were used. The outer four contours were averaged to obtain the stress intensity factor and T -stress.

3.1.2 Test Models

A finite element model for the test problem is shown in Figure 3.2. Since there are two planes of symmetry only one-quarter of the plate was modelled. The analysis was performed for five crack lengths: $l/\rho = 0.05, 0.10, 0.20, 0.30$, and 0.50 . Here ρ is the radius of the hole. The width (W) and height (H) were made long enough to simulate a circular hole in an infinite plate.

The nodes along the horizontal plane of symmetry were constrained from motion in the vertical direction, but excluding the crack face. The nodes along the vertical plane of symmetry were constrained from motion in the horizontal direction. A uniform tensile load σ_n was applied to the remote end as shown in Figure 3.1.

Comparison with Published Data

In order to verify the finite element model the boundary correction factor and normalized T -stress were compared to published data. Published data was obtained from Broberg (2005), which contains boundary correction factor and normalized T -stress data for cracks at a circular hole using dislocation arrays. The boundary correction factor and normalized T -stress are defined as

$$F = \frac{K_I}{\sigma_n \sqrt{\pi(\rho + l)}} \quad (3.1)$$

$$V = \frac{T}{\sigma_n}. \quad (3.2)$$

The comparisons are presented in Tables 3.1 and 3.2 for boundary correction factor and normalized T -stress respectively. The results are graphed in Figures 3.3 and 3.4. The finite element results for the boundary correction factor differ from the published data by roughly 2% - 6%, and the finite element results for normalized T -stress differ from the published data by roughly 2% - 3%. These results are reasonable, thereby confirming that the finite element models are acceptable.

3.1.3 The U-shaped Notch

The dimensions of the U-shaped notch shown in Figure 3.5 are defined as follows:

$\rho \equiv$ Notch-tip radius = 3.0 in.

$W \equiv$ Specimen width = 120 in.

$t \equiv$ Specimen thickness = 1.0 in.

$a \equiv$ Notch length

Three notch lengths were considered: with a/W ratios equal to 0.3, 0.4, and 0.5. The ratio ρ/W is fixed at 0.025 for all notch geometries. The analysis was performed for nine crack lengths: $l/\rho = 0.05, 0.10, 0.20, 0.30, 0.50, 1.00, 1.50, 2.00$, and 3.00. Only a slight modification was required to adapt the test model to the U-shaped notch. This was done by removing the vertical plane of symmetry and extending elements from left side of the test model (Figure 3.2) to give the U-shaped notch shown in Figure 3.6. A model for the shortest crack length $l/\rho = 0.05$, and with $a/W = 0.4$ is depicted in Figure 3.6. The model shown in Figure 3.6 was used to model the short crack lengths in the range $0.05 \leq l/\rho \leq 0.50$. For the longer cracks in the range $1.0 \leq l/\rho \leq 3.0$ a model with less mesh refinement was used. A typical long crack model is shown in Figure 3.7 for $l/\rho = 1.0$ and $a/W = 0.3$.

Since there is one plane of symmetry only half of the U-shaped notch was modelled. The nodes along the horizontal plane of symmetry were constrained from motion in the

vertical direction, but excluding the crack face. Two types of loading were considered: (1) remote tension, and (2) remote bending. The tensile stress was applied as a uniform pressure as shown in Figure 3.8 (a), and the moment was applied to the model as a linearly decreasing pressure with σ_n defined as the extreme values as shown in Figure 3.8 (b).

T-stress Results for the U-shaped Notch

The T -stress was obtained for three geometries and nine crack lengths for both tension and bending loads. The results for T -stress were normalized for remote tension and bending by dividing the T -stress by the nominal stress σ_n defined in Figure 3.8. The results for normalized T -stress for geometries with a/W equal to 0.3, 0.4, and 0.5 are summarized in Tables 3.3, 3.4, and 3.5 respectively.

3.2. Comparison with Empirical Equation

In this section the equation derived by Wang et al. (2006) is compared with the finite element results. This equation was presented earlier as Equation (2.59). In order to evaluate the T -stress using Equation (2.59) the stress concentration factors for the U-shaped notch are required.

3.2.1 Stress Concentration at a U-shaped Notch

Equation (2.59) depends on two parameters: (1) the notch-tip radius ρ , and (2) the stress concentration factor K_t . The stress concentration factor is defined using the geometry of an uncracked notch. The geometry considered here is the U-shaped notch shown in Figure 3.5. The stress concentration factor for a U-shaped notch subjected to axial tension or bending is approximated by Pilkey (1997) using

$$K_T = C_1 + C_2 \left(\frac{a}{W} \right) + C_3 \left(\frac{a}{W} \right)^2 + C_4 \left(\frac{a}{W} \right)^3 \quad (3.3)$$

where $C_1, \dots, C_4$ are constants, and are determined using the functions of (a/ρ) given in Appendix 3-1. It is important to note that the stress concentration factor K_t in Equation (2.59) is defined differently than the stress concentration factor K_T in Equation (3.3). Both are defined using different nominal stresses. Equation (3.3) gives a stress concentration factor using a nominal stress defined at the minimum depth at the root of the notch $(W - a)$, as shown in Figure 3.9. In Equation (2.59) the stress concentration factor is defined using the nominal stress at the maximum depth away from the notch (W) . Using the different nominal stresses and the superposition principle it is possible to establish a relationship between the stress concentration factors. Noting that the remote tension will cause bending, as shown in Figure 3.10, the relationship for tension is given by

$$(K_t)_{tension} = \left(\frac{W}{W - a} \right) (K_T)_{tension} + 3a \left(\frac{\sqrt{W}}{W - a} \right)^2 (K_T)_{bending} . \quad (3.4)$$

For remote bending

$$(K_t)_{bending} = \left(\frac{W}{W - a} \right)^2 (K_T)_{bending} . \quad (3.5)$$

Appendix 3-2 contains details of how Equation (3.4) and (3.5) were obtained. The stress concentration factors K_T obtained using Equation (3.3) were inserted into either Equation (3.4) or (3.5) to give the stress concentration factor K_t required for Equation (2.59). A summary of the calculated stress concentrations factors K_t is provided in Table 3.6.

3.2.2 Calculation of the T -stress for cracks at a U-shaped Notch

The stress concentration factors K_t were inserted into Equation (2.59) to obtain the

normalized T -stress for short cracks for each model geometry and loading. Equation (2.59) is repeated next

$$\frac{T}{\sigma_n} = K_I \left[\frac{Q}{2} \left(1 + 2 \frac{l}{\rho} \right)^{-1/2} + \left(\frac{Q}{2} - 1 \right) \left(1 + 2 \frac{l}{\rho} \right)^{-3/2} \right] \quad (2.59)$$

where Q is a constant of 0.4740, l is the crack length, ρ is the notch-tip radius. This gives the approximation for the T -stress for short cracks. The T -stress for long cracks is for an edge crack with an effective length $c = a + l$. This was given by Equation (2.60)

$$\frac{T_L}{\sigma_n} = V \left(\frac{c}{W} \right) \quad (2.60)$$

The T -stress for edge cracks is well documented. Wang (2002) approximates the T -stress for edge cracks using the weight function method. The normalized T -stress for uniform crack face pressure, and linearly decreasing crack face pressure for an edge crack is approximated by Wang (2002) using

$$V_0 = 0.4012 + 38.5340 \left(\frac{c}{W} \right)^{8.0252} \quad (3.6)$$

$$V_1 = 0.2813 + 26.1749 \left(\frac{c}{W} \right)^{8.0877} . \quad (3.7)$$

These are used to obtain an approximate weight function for T -stress valid for edge cracks. The T -stress is predicted to converge from the short crack solution to the long crack solution as the crack extends past the notch-tip stress field.

3.2.3 Comparison

The normalized T -stress for short and long cracks is plotted in Figures 3.11 to 3.16 along with the corresponding finite element results. The finite element results differ from Equation (2.59) between 1-6% for small cracks in the range $l/\rho \leq 0.30$. For bending Equation (2.59) converges to the long crack solution. For this loading Equation (2.59) provides a good estimation for the transition to the long crack solution. In conclusion Equation (2.59) is consistent with the finite element results, and therefore can be used to estimate the T -stress for small cracks at notches in the range $l/\rho \leq 0.30$ for both tension and bending loads.

3.3. Summary

The finite element method was used to analyze the T -stress for cracks emanating from a U-shaped notch in a finite width plate. Three notch lengths were considered, given by: $a/W = 0.3, 0.4, \text{ and } 0.5$. The ratio ρ/W was fixed at 0.025. The T -stress was obtained for nine crack lengths, given by: $l/\rho = 0.05, 0.10, 0.20, 0.30, 0.50, 1.00, 1.50, 2.00, \text{ and } 3.00$. Each was analyzed for remote tension and remote bending loads.

The results show that small cracks at a U-shaped notch experience low crack-tip constraint. For small cracks in the range $0 \leq l/\rho \leq 0.5$, as the notch length is increased (i.e. $a/W = 0.3 \rightarrow a/W = 0.5$) the T -stress decreased. A longer notch length produces a greater stress concentration at the notch. This indicates that the increased stress around the notch lowers the constraint at the crack-tip. As the crack length increased the constraint increased. The applied load also influenced the constraint. Bending results in higher constraint relative to tension for a given geometry and crack length.

The finite element results were compared to Equation (2.59). The finite element results differ from Equation (2.59) by 1-6% for small cracks in the range $l/\rho \leq 0.30$. The equation derived by Wang et al. (2006) is valid for estimating the T -stress for small

cracks at notches for both tension and bending loads. In all cases the T -stress converged to the long crack solution given by Equation (2.60).

The general conclusion drawn from Equation (2.59) and the finite element results is that small cracks at any notch will experience low crack-tip constraint. The low constraint observed for small crack lengths will result in a higher fracture toughness and impact fatigue life assessment.

Appendix 3-1

Stress concentration factors K_T for the U-shaped notch given by Pilkey (1997). Valid for $2.0 \leq a/\rho \leq 20.0$.

$$K_T = C_1 + C_2 \left(\frac{a}{W} \right) + C_3 \left(\frac{a}{W} \right)^2 + C_4 \left(\frac{a}{W} \right)^3$$

For axial tension

$$C_1 = 0.953 + 2.136\sqrt{a/\rho} - 0.005a/\rho$$

$$C_2 = -3.255 - 6.281\sqrt{a/\rho} + 0.068a/\rho$$

$$C_3 = 8.203 + 6.893\sqrt{a/\rho} + 0.064a/\rho$$

$$C_4 = -4.851 - 2.793\sqrt{a/\rho} - 0.128a/\rho$$

For bending moment

$$C_1 = 2.966 + 0.502a/\rho - 0.009(a/\rho)^2$$

$$C_2 = -6.475 - 1.126a/\rho + 0.019(a/\rho)^2$$

$$C_3 = 8.023 + 1.253a/\rho - 0.020(a/\rho)^2$$

$$C_4 = -3.572 - 0.634a/\rho + 0.010(a/\rho)^2$$

Appendix 3-2

Conversion of the stress concentration factor K_T to the stress concentration factor K_t .

The stress concentration factor K_T is defined using the reduced width at the root of the notch, or $(W-a)$ in Figure 3.5. The stress concentration factor K_T is given by

$$K_T = \frac{\sigma_{\max}}{\sigma_N} \quad (\text{A3.1})$$

where $\sigma_{\max}$ is the maximum stress at the root of the notch. The nominal stresses using the reduced width are given by

$$\sigma_N = \frac{P}{t(W-a)} \quad \text{for axial tension, and} \quad (\text{A3.2})$$

$$\sigma_N = \frac{6M}{t(W-a)^2} \quad \text{for bending moment.} \quad (\text{A3.3})$$

These nominal stresses σ_N are shown in Figure 3.9 (a) and (b). Here the force P is applied at the centre of the cross section with minimum depth, as shown in Figure 3.9 (a). The stress concentration factor K_t is defined using the maximum width away from the notch, or W in Figure 3.5. The stress concentration factor K_t is given by

$$K_t = \frac{\sigma_{\max}}{\sigma_n} \quad (\text{A3.4})$$

where $\sigma_{\max}$ is the maximum stress at the root of the notch. The nominal stresses using the maximum width are given by

$$\sigma_n = \frac{P}{tW} \quad \text{for axial tension, and} \quad (\text{A3.5})$$

$$\sigma_n = \frac{6M}{tW^2} \quad \text{for bending moment.} \quad (\text{A3.6})$$

These nominal stresses are shown in Figure 3.8 (a) and (b). Here the force P is applied at the centre of the remote end of the plate, as shown in Figure 3.10. The remote tension shown in Figure 3.10 will induce a bending moment equal to $M = Pa/2$. Using the superposition principle the maximum stress at the root of the notch will be the sum of the stress due to tension and the stress due to bending. The maximum stress for remote tension is given by

$$\sigma_{\max} = (K_T \sigma_N)_{\text{tension}} + (K_T \sigma_N)_{\text{bending}} \quad (\text{A3.7})$$

The stress concentration factor K_t for remote tension is obtained by dividing Equation (A3.7) by the nominal stress for remote tension. Dividing Equation (A3.7) by Equation (A3.5) and substituting Equations (A3.2) and (A3.3)

$$(K_t)_{\text{tension}} = \frac{\frac{P}{t(W-a)}(K_T)_{\text{tension}} + \frac{6M}{t(W-a)^2}(K_T)_{\text{bending}}}{\frac{P}{tW}} \quad (\text{A3.8})$$

Rearranging and substituting $M = Pa/2$ yields

$$(K_t)_{\text{tension}} = \left(\frac{W}{W-a} \right) (K_T)_{\text{tension}} + 3a \left(\frac{\sqrt{W}}{W-a} \right)^2 (K_T)_{\text{bending}} \quad (\text{A3.9})$$

where $(K_T)_{tension}$ and $(K_T)_{bending}$ are obtained from Equation (3.3). For remote bending the maximum stress is given by

$$\sigma_{\max} = (K_T \sigma_N)_{bending} \quad (A3.10)$$

The stress concentration factor for remote bending is obtained by dividing Equation (A3.10) by the nominal stress for remote bending. Dividing Equation (A3.10) by (A3.6) and substituting Equation (A3.3) yields

$$(K_t)_{bending} = \frac{\frac{6M}{t(W-a)^2} (K_T)_{bending}}{\frac{6M}{tW^2}}. \quad (A3.11)$$

Rearranging yields

$$K_t = \left(\frac{W}{W-a} \right)^2 (K_T)_{bending} \quad (A3.12)$$

where $(K_T)_{bending}$ is obtained from Equation (3.3).

Crack Length l/ρ	Boundary Correction Factor, F		
	F Present	F Broberg	Difference (%)
0.05	0.669	0.656	1.98 %
0.10	0.852	0.824	3.40 %
0.20	1.002	0.955	4.92 %
0.30	1.056	1.000	5.60 %
0.50	1.083	1.019	6.28 %

Table 3.1: Boundary correction factor - two through cracks extending symmetrically from a circular hole.

Crack Length l/ρ	Normalized T -stress, V		
	V Present	V Broberg	Difference (%)
0.05	-1.303	-1.274	2.28 %
0.10	-1.057	-1.058	0.09 %
0.20	-0.774	-0.793	2.40 %
0.30	-0.635	-0.649	2.16 %
0.50	-0.546	-0.562	2.85 %

Table 3.2: Normalized T -stress - two through cracks extending symmetrically from a circular hole.

Crack Length l/ρ	Normalized T -stress, V : Finite Element Method U-shaped Notch, $a/W = 0.3$	
	Tension	Bending
0.05	-5.7628	-3.8133
0.10	-4.7238	-3.1300
0.20	-3.3638	-2.2194
0.30	-2.5128	-1.6339
0.50	-1.5632	-0.9514
1.00	-0.7391	-0.2800
1.50	-0.5194	-0.0551
2.00	-0.4381	0.0562
3.00	-0.3654	0.0186

Table 3.3: Results for normalized T -stress. U-shaped Notch, $a/W = 0.3$

Crack Length l/ρ	Normalized T -stress, V : Finite Element Method U-shaped Notch, $a/W = 0.4$	
	Tension	Bending
0.05	-8.2226	-4.8451
0.10	-6.7482	-3.9844
0.20	-4.8042	-2.8281
0.30	-3.5734	-2.0743
0.50	-2.1726	-1.1772
1.00	-0.9023	-0.2417
1.50	-0.4989	0.0941
2.00	-0.3289	0.2653
3.00	-0.1628	0.4615

Table 3.4: Results for normalized T -stress. U-shaped Notch, $a/W = 0.4$

Crack Length l/ρ	Normalized T -stress, V : Finite Element Method U-shaped Notch, $a/W = 0.5$	
	Tension	Bending
0.05	-12.3800	-6.3946
0.10	-10.1660	-5.2637
0.20	-7.2334	-3.7368
0.30	-5.3576	-2.7317
0.50	-3.1870	-1.5177
1.00	-1.0923	-0.2173
1.50	-0.4284	0.2594
2.00	-0.1274	0.5074
3.00	0.1865	0.7925

Table 3.5: Results for normalized T -stress. U-shaped Notch, $a/W = 0.5$

Notch Lengths a / W	Stress Concentration Factor, K_t Tension	Stress Concentration Factor, K_t Bending
0.3	12.56	8.310
0.4	18.06	10.56
0.5	26.69	13.79

Table 3.6: Stress concentration factors K_t for the U-shaped notch.

Figure 3.1: Test problem - Two through-thickness cracks extending symmetrically from a circular hole in an infinite plate.

Figure 3.2: Finite element model for two through-thickness cracks extending symmetrically from a circular hole in an infinite plate.

Figure 3.3: Comparison of finite element results to published data for boundary correction factor. Two cracks extending symmetrically from a circular hole in an infinite plate.

Figure 3.4: Comparison of finite element results to published data for Normalized T -stress. Two cracks extending symmetrically from a circular hole in an infinite plate.

Figure 3.5: Dimensions of the U-shaped notch.

Figure 3.6: Finite element model of U-shaped notch with $a/W = 0.4$ and $l/\rho = 0.05$.
 Example of a typical short crack model for the crack lengths $0.05 \leq l/\rho \leq 0.50$.

Figure 3.7: Finite element model of the U-shaped notch for $l/\rho = 1.0$ and $a/W = 0.3$.
 Example of a typical long crack model for crack lengths $1.0 \leq l/\rho \leq 3.0$.

Figure 3.8: Loading applied to the U-shaped notch.

Figure 3.9: Nominal stresses defined for stress concentration factor K_T .

Figure 3.10: Movement of remote tension load.

Figure 3.11: Comparison of T -stress results for tension with $a/W = 0.3$

Figure 3.12: Comparison of T -stress solutions for bending with $a/W = 0.3$

Figure 3.13: Comparison of T -stress solutions for tension with $a/W = 0.4$

Figure 3.14: Comparison of T -stress solutions for bending with $a/W = 0.4$

Figure 3.15: Comparison of T -stress solutions for tension with $a/W = 0.5$

Figure 3.16: Comparison of T -stress solutions for bending with $a/W = 0.5$

Chapter 4

Circumferential Through-Wall Flaws in Cylinders

Circumferential through-wall cracks are encountered in a variety of applications, including pipelines and tubular structures. A circumferential through-wall crack is a severe case. A fatigue and fracture analysis for pipelines and tubular structures must consider severe situations. Knowledge of the constraint levels for these applications would allow for more accurate assessment of fracture toughness and fatigue life.

In this chapter the finite element method will be used to model circumferential through-wall cracks in cylinders. The T -stress will be obtained for remote tension and bending, and subsequently the results will be approximated using least squares curve-fitting. The weight function method developed in two-dimensions will be extended to a cylindrical surface. Then weight functions will be obtained and validated using the finite element results.

4.1. Finite Element Analysis

A circumferential through-wall crack in a cylinder is shown in Figure 4.1. The key dimensions are defined as follows:

$R \equiv$ Mean radius of cylinder

$t \equiv$ Wall thickness of cylinder

$\theta \equiv$ Crack half-angle

Three geometries were considered, defined by the ratios: $R/t = 5, 10$, and 20 . The analysis was performed for eight crack lengths: with $\theta/\pi = 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375$, and 0.5000 . For all cases the cylinder was chosen long enough to simulate the conditions of an infinite cylinder.

ABAQUS version 6.4-1 was used for the finite element analysis. The finite element models were created using 8-node thick shell elements. This element type is defined as S8R in the ABAQUS element library, and includes transverse shear strain. For the shell geometry considered here ($R/t \geq 5$) thick shell elements are appropriate; however, for thicker shells ($R/t < 5$) a full three-dimensional analysis is required. The material properties were defined as Young's Modulus $E = 30E6 \text{ lb/in.}^2$, and Poisson's ratio $\nu = 0.30$. Since there are two planes of symmetry only $1/4$ of the cylinder was modelled. A model is shown in Figure 4.2 for a crack length of $\theta/\pi = 0.2500$. The models were subjected to remote tension and remote bending as shown in Figure 4.1. The tension load was applied as a uniform pressure, and the bending was applied as two equal but opposite point forces.

4.1.1 Crack Region

The S8R element cannot be degenerated into a triangle. Instead a very small key-hole was introduced at the crack tip with a radius $(1/10000) \times R$. This technique allowed the creation of a web-mesh design focused at the crack-tip, as shown in Figure 4.2. The nodes in the contour integral area were defined using a MATLAB program. This allowed easy manipulation of the node positions within the crack region. Twelve contours of integration were used. The outputs from the outer six contours were averaged to give the values for stress intensity factor and T -stress. The mesh was refined until contour independence was achieved.

4.1.2 Validation of Finite Element Models

To validate the finite element models the boundary correction factors were compared to published values. The stress intensity factor for a circumferential through-wall crack in a cylinder under axial tension and bending moment is given by

$$K_I = (F_t \sigma_t + F_b \sigma_b) \sqrt{\pi R \theta} \quad (4.1)$$

where F_t and F_b are the boundary correction factors for tension and bending respectively. The stresses are calculated using

$$\sigma_t = \frac{P}{2\pi R t} \quad (4.2)$$

$$\sigma_b = \frac{M}{\pi R^2 t} \quad (4.3)$$

for tension and bending respectively. The boundary correction factor for through-wall cracks in cylinders has been thoroughly investigated. Here the boundary correction factor is compared to early results from Zahoor (1989) and more recent results from Takahashi (2002). Zahoor used solutions provided by Sanders (1983) to produce empirical formulas. Sanders used a numerical approach based on complete shell theory including transverse shear strain to obtain solutions for tension and bending. Takahashi fitted results obtained by Lacire et al. (1999). Lacire et al. used shell elements for thin cylinders ($R/t > 10$), and used solid elements for thicker cylinders. Takahashi fitted empirical formulas to the results of Lacire et al. to within a few percent. The empirical formulas provided by Takahashi are given in Appendix 4-1, and those provided by Zahoor are given in Appendix 4-2.

The stress intensity factors obtained from the finite element analysis were converted to boundary correction factors using Equations (4.1) - (4.3). The boundary correction factors obtained using the finite element method was then compared to the empirical formulas given in Appendix 4-1 and 4-2. The values obtained from these formulas for each crack length are given in Tables 4.1 to 4.6 along with the corresponding finite element results. Figures 4.3 and 4.4 provide a graphical comparison for tension and bending respectively. Excellent agreement for the boundary correction factors was achieved. The percent difference between solutions provided by Takahashi is better than 3%, and the percent difference with those provided by Zahoor is between 1-7% for most results. The solution for the boundary correction factor confirms that the finite element models are acceptable.

4.1.3 Finite Element Results for T -stress

The finite element results for T -stress were obtained for $\theta/\pi = 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375, \text{ and } 0.5000$. These crack lengths were analyzed for cylinders with $R/t = 5, 10, \text{ and } 20$; with each subjected to remote tension and bending loads. The finite element results for T -stress were normalized using

$$V = \frac{T}{\sigma_t} \quad (4.4)$$

$$V = \frac{T}{\sigma_b} \quad (4.5)$$

for tension and bending respectively. The stresses σ_t and σ_b are given by Equations (4.2) and (4.3). The finite element results for normalized T -stress are provided in the first columns of Tables 4.7 - 4.12, and are plotted in Figures 4.5 and 4.6. All the T -stress solutions start close to minus one. This result is expected since a small circumferential crack in a cylinder should be equivalent to a centre crack in an infinite plate. After $\theta/\pi = 0.2$ the T -stress solutions diverge, with thicker cylinders predicting higher constraint levels.

4.2. Empirical Formulas for T -stress

The finite element results for normalized T -stress were approximated using least squares fitting. The T -stress for a centre crack in an infinite plate is minus one. Small cracks in the cylindrical tube should compare to a centre crack in an infinite plate. For this reason the T -stress approximation should have a zero slope and be equal to minus one at $\theta/\pi = 0$. In order to force the zero slope condition at $\theta/\pi = 0$ only even number exponents are used in the approximation polynomial (i.e. 2, 4,...). A tenth order polynomial was used to approximate the normalized T -stress for all three geometries and

both loadings conditions. Subsequently the coefficients for tension and bending were approximated using second order polynomials. This provided an approximation for any geometry in the range $5 \geq R/t \geq 20$. The approximation for normalized T -stress is given by

$$V = C_1 + C_2 \left(\frac{\theta}{\pi} \right)^2 + C_3 \left(\frac{\theta}{\pi} \right)^4 + C_4 \left(\frac{\theta}{\pi} \right)^6 + C_5 \left(\frac{\theta}{\pi} \right)^8 + C_6 \left(\frac{\theta}{\pi} \right)^{10} \quad (4.6)$$

where $C_1, \dots, C_6$ are coefficients determined using second order polynomials applicable for either tension or bending. The coefficients for remote tension are:

$$\begin{aligned} C_1 &= -0.9613 - 0.0062(R/t) + 0.0003(R/t)^2 \\ C_2 &= 1.0457 + 0.8247(R/t) - 0.0278(R/t)^2 \\ C_3 &= 33.671 - 22.126(R/t) + 0.5365(R/t)^2 \\ C_4 &= -210.6 + 120.23(R/t) - 1.5892(R/t)^2 \\ C_5 &= 852.45 - 322.72(R/t) + 0.2417(R/t)^2 \\ C_6 &= -1272 + 351.17(R/t) + 3.9989(R/t)^2 \end{aligned}$$

where the T -stress is calculated using Equation (4.4). The coefficients for remote bending are:

$$\begin{aligned} C_1 &= -0.934 - 0.01(R/t) + 0.0005(R/t)^2 \\ C_2 &= 4.3575 + 1.2671(R/t) - 0.0451(R/t)^2 \\ C_3 &= 74.273 - 35.312(R/t) + 0.9889(R/t)^2 \\ C_4 &= -680.39 + 266.98(R/t) - 6.3389(R/t)^2 \\ C_5 &= 2688.6 - 957.82(R/t) + 20.135(R/t)^2 \\ C_6 &= -3852.8 + 1325(R/t) - 25.744(R/t)^2 \end{aligned}$$

where the T -stress is calculated using Equation (4.5). The percent difference between the least-squares curve-fitting and the finite element results are better than 1%, except for bending with $R/t = 20$ where the percent difference is better than 3%. Equation (4.6) is graphed along with the finite element results in Figures 4.5 and 4.6 for tension and bending respectively. Equation (4.6) provides a fast and accurate means for estimating the T -stress for geometries in the range $5 \geq R/t \geq 20$, and for cracks in the range $0 \leq \theta/\pi \leq 0.5$. This approximation may only be used to predict the T -stress for remote tension and bending loads. A weight function is required to obtain general expressions valid for any loading condition.

4.3. Weight Functions for T -stress

In order use weight functions for the cylinder they must be redefined for the cylindrical geometry. This was accomplished by defining a new coordinate system. Approximate weight functions were obtained for all three pipe geometries using the new coordinate system. The weight functions were used to calculate the T -stress for remote tension and bending loads. These results were then compared to the finite element results already obtained for remote tension and bending.

4.3.1 Coordinate System

The coordinate system for two-dimensions is shown in Figure 4.7. This coordinate system is modified by replacing the coordinate axis along the crack length with a curved axis that lies along the pipe surface. This is shown in Figure 4.8, where Figure 4.7 is the original two-dimensional coordinate system, and Figure 4.8 shows the new curved coordinate system. The measurement along the angular β coordinate axis is defined in radians and θ is the crack-half angle as defined previously. In the present work weight functions are obtained for the cylindrical pipe using the coordinate system shown in Figure 4.8.

4.3.2 Approximate Weight Functions

A two-term expression to approximate the weight functions for T -stress was presented earlier as Equation (2.48). This equation was used by Wang (2002) for single and double edge cracks as well as internal cracks. Wang (2002) demonstrated the versatility of this equation for approximating weight functions. Here Equation (2.48) was used to approximate the weight functions for the cylindrical pipe. Equation (2.48) is re-written in terms of the $\beta - z$ coordinate system as follows

$$t(\beta, \theta) = \frac{2}{\pi \theta} \left[D_1 \left(1 - \frac{\beta}{\theta} \right)^{\frac{1}{2}} + D_2 \left(1 - \frac{\beta}{\theta} \right)^{\frac{3}{2}} \right] \quad (4.7)$$

where the constants D_1 and D_2 are determined using two reference T -stress solutions. Wang (2002) developed equations for the constants D_1 and D_2 that are in terms of the normalized T -stress for the two reference loading conditions. They were presented earlier as Equations (2.55) and (2.56), and are repeated next

$$D_1 = \frac{15}{16} \pi (5V_0 - 7V_1) \quad (2.55)$$

$$D_2 = \frac{5}{16} \pi (35V_0 - 21V_1). \quad (2.56)$$

Reference Loadings

The normalized T -stress solutions V_0 and V_1 are derived from two reference loads. They are: (1) uniform crack pressure, and (2) linearly decreasing crack pressure. The crack face pressure for the first reference load is given by

$$\sigma(\beta) = \sigma_0 \quad (4.8)$$

and the corresponding normalized T -stress is V_0 . The crack face pressure for the second reference load is given by

$$\sigma(\beta) = \sigma_0 \left(1 - \frac{\beta}{\theta}\right). \quad (4.9)$$

and the corresponding normalized T -stress is V_1 . These two crack face pressures are shown in Figures 4.9 and 4.10 for uniform and linearly decreasing crack face pressures respectively. These two reference loads were applied to all three model geometries and for all crack lengths. The results for normalized T -stresses V_0 and V_1 are provided in Tables 4.13 and 4.14 respectively. This data was approximated using least squares fitting. The approximations for $R/t = 5$ are

$$V_0 = 0.0121 + 4.4589 \left(\frac{\theta}{\pi}\right)^2 - 63.468 \left(\frac{\theta}{\pi}\right)^4 + 351.83 \left(\frac{\theta}{\pi}\right)^6 - 764.84 \left(\frac{\theta}{\pi}\right)^8 + 604.86 \left(\frac{\theta}{\pi}\right)^{10} \quad (4.10)$$

$$V_1 = 0.0069 + 1.951 \left(\frac{\theta}{\pi}\right)^2 - 19.158 \left(\frac{\theta}{\pi}\right)^4 + 39.745 \left(\frac{\theta}{\pi}\right)^6 + 127.02 \left(\frac{\theta}{\pi}\right)^8 - 304.43 \left(\frac{\theta}{\pi}\right)^{10}. \quad (4.11)$$

The approximations for $R/t = 10$ are

$$V_0 = 0.0095 + 6.494 \left(\frac{\theta}{\pi}\right)^2 - 133.32 \left(\frac{\theta}{\pi}\right)^4 + 827.97 \left(\frac{\theta}{\pi}\right)^6 - 2335.7 \left(\frac{\theta}{\pi}\right)^8 + 2624.1 \left(\frac{\theta}{\pi}\right)^{10} \quad (4.12)$$

$$V_1 = 0.0055 + 3.9115 \left(\frac{\theta}{\pi}\right)^2 - 84.303 \left(\frac{\theta}{\pi}\right)^4 + 521.66 \left(\frac{\theta}{\pi}\right)^6 - 1467.1 \left(\frac{\theta}{\pi}\right)^8 + 1643.8 \left(\frac{\theta}{\pi}\right)^{10}. \quad (4.13)$$

The approximations for $R/t = 20$ are

$$V_0 = 0.0376 + 6.7142\left(\frac{\theta}{\pi}\right)^2 - 203.47\left(\frac{\theta}{\pi}\right)^4 + 1673.6\left(\frac{\theta}{\pi}\right)^6 - 6092.1\left(\frac{\theta}{\pi}\right)^8 + 8373.7\left(\frac{\theta}{\pi}\right)^{10} \quad (4.14)$$

$$V_1 = 0.0233 + 3.8695\left(\frac{\theta}{\pi}\right)^2 - 128.58\left(\frac{\theta}{\pi}\right)^4 + 1043.7\left(\frac{\theta}{\pi}\right)^6 - 3690.9\left(\frac{\theta}{\pi}\right)^8 + 4913.6\left(\frac{\theta}{\pi}\right)^{10}. \quad (4.15)$$

The approximations (4.10) – (4.15) are graphed in Figures 4.11, 4.12 and 4.13 for $R/t = 5, 10$, and 20 . The percent difference between the least-squares approximations and the finite element data is better than 5% for most crack lengths. Larger percent difference is encountered for T -stress values close to zero; however, in this case the magnitude of the difference remains small. The normalized T -stress V_0 and V_1 for a specific geometry are inserted into Equations (2.55) and (2.56) to give the coefficients D_1 and D_2 . These coefficients are inserted into Equation (4.7) to give the approximate weight function.

4.3.3 Validation of Weight Functions

The weight functions were validated by comparing the weight function based T -stress to finite element results. This was done for remote tension and bending.

Remote Tension

The application of the superposition method to calculate the T -stress for remote tension is shown in Figure 4.14. The stress that acts along β and in the z -direction in the uncracked body will equal the remote tension

$$\sigma(\beta) = \sigma_0. \quad (4.16)$$

The T -stress in the uncracked body is

$$T_{\text{uncracked}} = (\sigma_{\beta} - \sigma_z) \Big|_{\beta=\theta} = -\sigma_0 \quad (4.17)$$

The T -stress in the cracked body subjected to crack face pressure $\sigma(\beta)$ is

$$T_{\text{crack pressure}} = \int_0^{\theta} \sigma(\beta) t(\beta, \theta) d\beta = \sigma_0 \int_0^{\theta} t(\beta, \theta) d\beta \quad (4.18)$$

where $t(\beta, \theta)$ is the approximate weight function. The T -stress for remote tension is given by

$$T = \sigma_0 \int_0^{\theta} t(\beta, \theta) d\beta - \sigma_0. \quad (4.19)$$

Inserting Equation (4.7) and dividing by σ_0 gives

$$\frac{T}{\sigma_0} = \frac{2D_1}{\pi\theta} \int_0^{\theta} \left(1 - \frac{\beta}{\theta}\right)^{\frac{1}{2}} d\beta + \frac{2D_2}{\pi\theta} \int_0^{\theta} \left(1 - \frac{\beta}{\theta}\right)^{\frac{3}{2}} d\beta - 1. \quad (4.20)$$

Equation (4.20) was used to obtain the normalized T -stress for remote tension for all three geometries. The weight function based T -stress for remote tension is plotted in Figure 4.17 along with the corresponding finite element results. The value of the T -stress is compared for each crack length in Tables 4.7, 4.9 and 4.11 for $R/t = 5, 10$, and 20 respectively. Excellent agreement was achieved between the weight function based T -stress and the finite element results. The percent differences between the normalized T -stress for axial tension are less than 2%.

Remote Bending

The crack face pressure $\sigma(\beta)$ is defined using the stress in the uncracked body. In particular it is the stress acting along β and in z -direction in the uncracked body. For bending the stress distribution will be linear, as shown in Figure 4.15. Defining the linear stress distribution in terms of the s coordinate system shown in Figure 4.16, the stress can be written as

$$\sigma(s) = \sigma_0 \left(1 - \frac{s}{R}\right). \quad (4.21)$$

To determine the crack face pressure $\sigma(\beta)$ we must rewrite the linear stress distribution $\sigma(s)$ so it is defined in terms of the angular coordinate β along the crack face. As shown in Figure 4.16, the s coordinate is related to the angle β using

$$s = R(1 - \cos \beta) \quad (4.22)$$

Inserting Equation (4.22) into (4.21) gives

$$\sigma(\beta) = \sigma_0 \cos \beta. \quad (4.23)$$

Equation (4.23) is the stress that acts along β and in the z -direction in the uncracked body. The T -stress in the uncracked body is

$$T_{\text{uncracked}} = (\sigma_\beta - \sigma_z) \Big|_{\beta=\theta} = -\sigma_0 \cos \theta. \quad (4.24)$$

The T -stress in the cracked body subjected to crack face pressure $\sigma(\beta)$ is

$$T_{\text{crack pressure}} = \int_0^\theta \sigma(\beta) t(\beta, \theta) d\beta = \sigma_0 \int_0^\theta \cos(\beta) t(\beta, \theta) d\beta \quad (4.25)$$

where $t(\beta, \theta)$ is the approximate weight function. The T -stress for remote bending is given by

$$T = \sigma_0 \int_0^\theta \cos(\beta) t(\beta, \theta) d\beta - \sigma_0 \cos \theta. \quad (4.26)$$

Inserting Equation (4.7) and dividing by σ_0 gives

$$\frac{T}{\sigma_0} = \frac{2D_1}{\pi\theta} \int_0^\theta \cos \beta \left(1 - \frac{\beta}{\theta}\right)^{\frac{1}{2}} d\beta + \frac{2D_2}{\pi\theta} \int_0^\theta \cos \beta \left(1 - \frac{\beta}{\theta}\right)^{\frac{3}{2}} d\beta - \cos \theta. \quad (4.27)$$

Equation (4.27) was used to obtain the normalized T -stress for remote bending for all three geometries. The weight function based T -stress for remote bending is plotted in Figure 4.18 along with the corresponding finite element results. The value of the T -stress is compared for each crack length in Tables 4.8, 4.10 and 4.12 for $R/t = 5, 10$, and 20 respectively. Good agreement was achieved between the weight function based T -stress and the finite element results. The differences between the normalized T -stress are less than 4%. For T -stress values near zero the percent difference was higher, however, the magnitude of the difference remains small.

In this section approximate weight functions were developed for circumferential through-wall flaws in cylinders by using a curved coordinate system. The weight functions were used to accurately predict the T -stress for remote tension and remote bending loads. This confirms that the weight functions are acceptable, and also validates the approach used to obtain the weight function for the cylindrical surface.

4.4. Summary

Finite element models were made for circumferential through-wall flaws in cylinders. Three geometries were considered, defined by $R/t = 5, 10$, and 20 . Each model was analyzed for eight crack lengths given by $\theta/\pi = 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375$, and 0.5000 . The models were validated by comparing the boundary correction factor to solutions available in the literature. Good agreement was obtained for the boundary correction factor. This confirmed the finite element models are acceptable.

Subsequently the T -stress was approximated using least-square curve fitting. The percent difference between the curve-fitting and finite element data is about 1-3%. The approximation given by Equation (4.6) may be used for remote tension and bending loads for $5 \leq R/t \leq 20$, and for crack lengths in the range $0 \leq \theta/\pi \leq 0.5$.

By defining a curved coordinate system, the two-dimensional weight function method was extended to the curved surface of the cylinder. Then approximate weight functions were obtained for $R/t = 5, 10$, and 20 . The weight functions were validated by comparing the weight function based T -stress to finite element results for remote tension and bending. The percent difference between the weight function prediction and the finite element data is better than 4%, thereby confirming that the weight functions are acceptable. The weight functions for T -stress are valid for complex loading conditions and for crack lengths in the range $0 \leq \theta/\pi \leq 0.5$.

Appendix 4-1

Boundary correction factors for stress intensity factor for a circumferential through-wall flaw in a cylinder, Takahashi (2002).

Applicable range: $1.5 < R/t < 80.5$, and $0 < \theta/\pi < 0.611$.

Boundary correction factor for axial tension

$$F_t = \left[A_t + B_t \left(\frac{\theta}{\pi} \right) + C_t \left(\frac{\theta}{\pi} \right)^2 + D_t \left(\frac{\theta}{\pi} \right)^3 + E_t \left(\frac{\theta}{\pi} \right)^4 \right] \quad (\text{A4.1})$$

$$A_t = 1$$

$$B_t = -1.040 - 3.1831\lambda - 4.83\lambda^2 - 2.369\lambda^3$$

$$C_t = 16.71 + 23.10\lambda + 50.82\lambda^2 + 18.02\lambda^3$$

$$D_t = -25.85 - 12.05\lambda - 87.24\lambda^2 - 30.39\lambda^3$$

$$E_t = 24.70 - 54.18\lambda + 18.09\lambda^2 + 6.745\lambda^3$$

$$\lambda = \log\left(\frac{t}{R}\right)$$

Boundary correction factor for bending moment

$$F_b = \left(1 + \frac{t}{2R} \right) \left[A_b + B_b \left(\frac{\theta}{\pi} \right) + C_b \left(\frac{\theta}{\pi} \right)^2 + D_b \left(\frac{\theta}{\pi} \right)^3 + E_b \left(\frac{\theta}{\pi} \right)^4 \right] \quad (\text{A4.2})$$

$$A_b = 0.65133 - 0.5774\lambda - 0.3427\lambda^2 - 0.0681\lambda^3$$

$$B_b = 1.879 + 4.795\lambda + 2.343\lambda^2 - 0.6197\lambda^3$$

$$C_b = -9.779 - 38.14\lambda - 6.611\lambda^2 + 3.972\lambda^3$$

$$D_b = 34.56 + 129.9\lambda + 50.55\lambda^2 + 3.374\lambda^3$$

$$E_b = -30.82 - 147.6\lambda - 78.38\lambda^2 - 15.54\lambda^3$$

$$\lambda = \log\left(\frac{t}{R}\right)$$

Appendix 4-2

Boundary correction factors for stress intensity factor for a circumferential through-wall flaw in a cylinder, Zahoor (1989).

Applicable range: $5 \leq R/t \leq 20$, and $0 < \theta/\pi < 0.5$.

Boundary correction factor for axial tension

$$F_t = 1 + A \left[5.3303 \left(\frac{\theta}{\pi} \right)^{1.5} + 18.773 \left(\frac{\theta}{\pi} \right)^{4.24} \right] \quad (\text{A4.3})$$

Boundary correction factor for bending moment

$$F_b = 1 + A \left[4.5967 \left(\frac{\theta}{\pi} \right)^{1.5} + 2.6422 \left(\frac{\theta}{\pi} \right)^{4.24} \right] \quad (\text{A4.4})$$

For both F_t and F_b

$$A = \left(0.125 \frac{R}{t} - 0.25 \right)^{0.25} \quad \text{for } 5 \leq \frac{R}{t} \leq 10$$

$$A = \left(0.4 \frac{R}{t} - 3.0 \right)^{0.25} \quad \text{for } 10 \leq \frac{R}{t} \leq 20$$

Tension - Boundary Correction Factors (B.C.F.): $R/t = 5$					
Crack Length θ / π	B.C.F., F_t Present	B.C.F., F_t Takahashi	Difference (%)	B.C.F., F_t Zahoor	Difference (%)
0.0625	1.0499	1.0412	0.83%	1.0653	1.45%
0.1250	1.1770	1.1747	0.19%	1.1865	0.80%
0.1875	1.3627	1.3657	0.22%	1.3508	0.88%
0.2500	1.6064	1.6048	0.10%	1.5625	2.80%
0.3125	1.9215	1.9076	0.73%	1.8347	4.74%
0.3750	2.3364	2.3154	0.91%	2.1874	6.81%
0.4375	2.8980	2.8946	0.12%	2.6484	9.42%
0.5000	3.6879	3.7369	1.31%	3.2522	13.40%

Table 4.1: Comparison of boundary correction factors for tension load on circumferential through-wall flaws in cylinders - $R/t = 5$.

Bending - Boundary Correction Factors (B.C.F.): $R/t = 5$					
Crack Length θ / π	B.C.F., F_b Present	B.C.F., F_b Takahashi	Difference (%)	B.C.F., F_b Zahoor	Difference (%)
0.0625	1.0391	1.0385	0.05%	1.0562	1.62%
0.1250	1.1352	1.1375	0.20%	1.1593	2.08%
0.1875	1.2627	1.2697	0.56%	1.2938	2.40%
0.2500	1.4138	1.4223	0.60%	1.4554	2.86%
0.3125	1.5927	1.5979	0.33%	1.6433	3.08%
0.3750	1.8125	1.8154	0.16%	1.8584	2.47%
0.4375	2.0971	2.1091	0.57%	2.1030	0.28%
0.5000	2.4868	2.5296	1.69%	2.3812	4.44%

Table 4.2: Comparison of boundary correction factors for bending load on circumferential through-wall flaws in cylinders - $R/t = 5$.

Tension - Boundary Correction Factors (B.C.F.): $R/t = 10$					
Crack Length θ / π	B.C.F., F_t Present	B.C.F., F_t Takahashi	Difference (%)	B.C.F., F_t Zahoor	Difference (%)
0.0625	1.0774	1.0674	0.93%	1.0653	0.56%
0.1250	1.2606	1.2570	0.29%	1.1865	1.80%
0.1875	1.5049	1.5147	0.65%	1.3508	3.91%
0.2500	1.8039	1.8197	0.87%	1.5625	4.94%
0.3125	2.1732	2.1841	0.50%	1.8347	5.16%
0.3750	2.6461	2.6532	0.27%	2.1874	5.11%
0.4375	3.2760	3.3052	0.88%	2.6484	5.46%
0.5000	4.1523	4.2514	2.33%	3.2522	7.07%

Table 4.3: Comparison of boundary correction factors for tension load on circumferential through-wall flaws in cylinders - $R/t = 10$.

Bending - Boundary Correction Factors (B.C.F.): $R/t = 10$					
Crack Length θ / π	B.C.F., F_b Present	B.C.F., F_b Takahashi	Difference (%)	B.C.F., F_b Zahoor	Difference (%)
0.0625	1.0665	1.0663	0.02%	1.0718	0.49%
0.1250	1.2168	1.2147	0.17%	1.2035	1.10%
0.1875	1.3987	1.4039	0.38%	1.3754	1.69%
0.2500	1.5977	1.6118	0.88%	1.5820	0.99%
0.3125	1.8185	1.8369	1.00%	1.8221	0.20%
0.3750	2.0777	2.0984	0.99%	2.0969	0.91%
0.4375	2.4028	2.4363	1.37%	2.4096	0.28%
0.5000	2.8408	2.9114	2.42%	2.7650	2.74%

Table 4.4: Comparison of boundary correction factors for bending load on circumferential through-wall flaws in cylinders - $R/t = 10$.

Tension - Boundary Correction Factors (B.C.F.): $R/t = 20$					
Crack Length θ / π	B.C.F., F_t Present	B.C.F., F_t Takahashi	Difference (%)	B.C.F., F_t Zahoor	Difference (%)
0.0625	1.1302	1.1173	1.15%	1.1248	0.48%
0.1250	1.4027	1.3829	1.43%	1.3564	3.41%
0.1875	1.7324	1.7245	0.45%	1.6704	3.71%
0.2500	2.1138	2.1107	0.15%	2.0750	1.87%
0.3125	2.5729	2.5503	0.89%	2.5949	0.85%
0.3750	3.1473	3.0931	1.75%	3.2691	3.73%
0.4375	3.8951	3.8294	1.72%	4.1499	6.14%
0.5000	4.9099	4.8901	0.41%	5.3037	7.42%

Table 4.5: Comparison of boundary correction factors for tension load on circumferential through-wall flaws in cylinders - $R/t = 20$.

Bending - Boundary Correction Factors (B.C.F.): $R/t = 20$					
Crack Length θ / π	B.C.F., F_b Present	B.C.F., F_b Takahashi	Difference (%)	B.C.F., F_b Zahoor	Difference (%)
0.0625	1.1012	1.1256	2.17%	1.1074	0.57%
0.1250	1.3385	1.3429	0.33%	1.3044	2.62%
0.1875	1.6019	1.5997	0.14%	1.5613	2.60%
0.2500	1.8761	1.8702	0.31%	1.8703	0.31%
0.3125	2.1708	2.1524	0.85%	2.2293	2.63%
0.3750	2.5047	2.4677	1.50%	2.6402	5.13%
0.4375	2.9043	2.8611	1.51%	3.1078	6.55%
0.5000	3.4181	3.4012	0.50%	3.6393	6.08%

Table 4.6: Comparison of boundary correction factors for bending load on circumferential through-wall flaws in cylinders - $R/t = 20$.

Tension: $R/t = 5$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9686	-0.9714	0.29%
0.1250	-0.9274	-0.9324	0.54%
0.1875	-0.8924	-0.8954	0.34%
0.2500	-0.8786	-0.8823	0.43%
0.3125	-0.8895	-0.8943	0.54%
0.3750	-0.8998	-0.9034	0.40%
0.4375	-0.8592	-0.8637	0.52%
0.5000	-0.7350	-0.7396	0.62%

Table 4.7: Normalized T -stress for remote tension – $R/t = 5$.

Bending: $R/t = 5$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9355	-0.9524	1.81%
0.1250	-0.8417	-0.8583	1.97%
0.1875	-0.7097	-0.7324	3.19%
0.2500	-0.5814	-0.5988	2.99%
0.3125	-0.4405	-0.4622	4.93%
0.3750	-0.2897	-0.3073	6.08%
0.4375	-0.1013	-0.1103	8.93%
0.5000	0.1621	0.1570	3.17%

Table 4.8: Normalized T -stress for remote bending – $R/t = 5$.

Tension: $R/t = 10$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9730	-0.9671	0.60%
0.1250	-0.9126	-0.9186	0.65%
0.1875	-0.9003	-0.8944	0.65%
0.2500	-0.9379	-0.9364	0.16%
0.3125	-1.0512	-1.0458	0.52%
0.3750	-1.1763	-1.1803	0.34%
0.4375	-1.2982	-1.2867	0.89%
0.5000	-1.3275	-1.3237	0.29%

Table 4.9: Normalized T -stress for remote tension – $R/t = 10$.

Bending: $R/t = 10$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9433	-0.9481	0.51%
0.1250	-0.8248	-0.8440	2.33%
0.1875	-0.7236	-0.7307	0.98%
0.2500	-0.6418	-0.6506	1.38%
0.3125	-0.6020	-0.6053	0.55%
0.3750	-0.5479	-0.5563	1.52%
0.4375	-0.4706	-0.4628	1.65%
0.5000	-0.3095	-0.3036	1.91%

Table 4.10: Normalized T -stress for remote bending – $R/t = 10$.

Tension: $R/t = 20$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9492	-0.9392	1.06%
0.1250	-0.8872	-0.9011	1.57%
0.1875	-0.9253	-0.9140	1.22%
0.2500	-1.0188	-1.0139	0.48%
0.3125	-1.1710	-1.1682	0.24%
0.3750	-1.3082	-1.3097	0.11%
0.4375	-1.4342	-1.4213	0.90%
0.5000	-1.4755	-1.4705	0.34%

Table 4.11: Normalized T -stress for remote tension - $R/t = 20$.

Bending: $R/t = 20$			
Crack Length θ / π	Normalized T -stress, V Finite Element Method	Normalized T -stress, V Weight Function Method	Difference (%)
0.0625	-0.9113	-0.9204	1.00%
0.1250	-0.8022	-0.8271	3.11%
0.1875	-0.7601	-0.7505	1.27%
0.2500	-0.7392	-0.7267	1.70%
0.3125	-0.7384	-0.7236	2.00%
0.3750	-0.6915	-0.6798	1.69%
0.4375	-0.6055	-0.5823	3.84%
0.5000	-0.4334	-0.4211	2.84%

Table 4.12: Normalized T -stress for remote bending - $R/t = 20$.

Crack Length θ / π	Normalized T -stress, V_0 : <i>Uniform Crack Face Pressure</i>		
	$R/t = 5$	$R/t = 10$	$R/t = 20$
0.0625	0.0279	0.0290	0.0529
0.1250	0.0690	0.0891	0.1148
0.1875	0.1037	0.1015	0.0774
0.2500	0.1174	0.0643	-0.0154
0.3125	0.1065	-0.0489	-0.1674
0.3750	0.0962	-0.1737	-0.3045
0.4375	0.1365	-0.2959	-0.4306
0.5000	0.2605	-0.3253	-0.4717

Table 4.13: The finite element results for normalized T -stress. Circumferential through-wall flaws in cylinders subjected uniform crack face pressure.

Crack Length θ / π	Normalized T -stress, V_1 : <i>Linearly Decreasing Crack Face Pressure</i>		
	$R/t = 5$	$R/t = 10$	$R/t = 20$
0.0625	0.0136	0.0171	0.0312
0.1250	0.0331	0.0524	0.0664
0.1875	0.0560	0.0573	0.0361
0.2500	0.0608	0.0269	-0.0364
0.3125	0.0645	-0.0509	-0.1443
0.3750	0.0440	-0.1404	-0.2419
0.4375	0.0499	-0.2287	-0.3274
0.5000	0.1171	-0.2610	-0.3583

Table 4.14: The finite element results for normalized T -stress. Circumferential through-wall flaws in cylinders subjected to linearly decreasing crack face pressure.

Figure 4.1: Circumferential through-wall flaw in a cylinder subjected to remote tension and bending moment.

Figure 4.2: Finite Element Model with crack length $\theta/\pi = 0.2500$.

Figure 4.3: Comparison of boundary correction factor for remote tension.

Empirical formulas by Takahashi (2002) given in Appendix 4-1.

Empirical formulas by Zahoor (1989) given in Appendix 4-2.

Figure 4.4: Comparison of boundary correction factor for bending moment.

Empirical formulas by Takahashi (2002) given in Appendix 4-1.

Empirical formulas by Zahoor (1989) given in Appendix 4-2.

Figure 4.5: Comparison of T -stress obtained from empirical formulas to finite element results for remote tension.

Figure 4.6: Comparison of T -stress obtained from empirical formulas to finite element results for bending moment.

Figure 4.7: Coordinate system defined for two-dimensions.

Figure 4.8: Coordinate system defined for cylindrical surface. The $\beta - z$ coordinate axes are placed with the origin at the centre of the crack.

Figure 4.9: Uniform crack face pressure.

Figure 4.10: Linearly decreasing crack face pressure.

Figure 4.11: Least-squares approximation of normalized T -stress for constant and linearly decreasing crack face pressure for $R/t = 5$.

Figure 4.12: Least-squares approximation of normalized T -stress for constant and linearly decreasing crack face pressure for $R/t = 10$.

Figure 4.13: Least-squares approximation of normalized T -stress for constant and linearly decreasing crack face pressure for $R/t = 20$.

Figure 4.14: Superposition method for calculation of T -stress for remote tension.

Figure 4.15: Remote bending load and corresponding linear stress distribution.

Figure 4.16: Relationship between s and β .

Figure 4.17: Comparison of weight function based T -stress to finite element results for remote tension.

Figure 4.18: Comparison of weight function based T -stress to finite element results for bending moment.

Chapter 5

Conclusions and Recommendations

5.1. Conclusions

The T -stress was analyzed for two types of geometries: (1) cracks emanating from a U-shaped notch in a finite width plate, and (2) for circumferential through-wall cracks in cylinders. The finite element method and the weight function method were used for the analysis.

Cracks at a U-shaped Notch in a Finite Width Plate

The U-shaped notch is a common geometry encountered in design. Fatigue cracks typically initiate at notches due to the local stress concentration. Analysis of cracks at notches is essential. Knowledge of the crack-tip constraint, or T -stress for this geometry would allow for more accurate assessment of fracture toughness and fatigue life. The finite element method was used to analyze the T -stress for cracks at a U-shaped notch. Three notch lengths were considered, given by: $a/W = 0.3, 0.4, \text{ and } 0.5$. The T -stress was obtained for nine crack lengths, given by: $l/\rho = 0.05, 0.10, 0.20, 0.30, 0.50, 1.00, 1.50, 2.00, \text{ and } 3.00$. Each was analyzed for remote tension and remote bending loads.

Next the finite element results for the U-shaped notch were compared to an equation derived by Wang et al. (2006). This equation approximates the T -stress for small cracks at notches. The T -stress obtained using the finite element method matched the equation between 1-6% for cracks in the range $l/\rho \leq 0.30$. When subjected to bending the equation converged to the long crack solution. In this case the equation also provided a good approximation of the transition from the short crack solution to the long crack solution. Based on the results obtained here it is concluded that the equation derived by Wang et al. provides a good estimation of the T -stress for small cracks at notches.

The results show that small cracks at a U-shaped notch experience low crack-tip constraint. As the notch length was increased from $a/W = 0.3 \rightarrow a/W = 0.5$, the T -stress for small cracks decreased. A longer notch length produces a larger stress concentration at the notch-tip. This shows that the increased stress around the notch lowers the constraint at the crack-tip. A general conclusion is that any crack in the vicinity of a notch will experience low crack-tip constraint. The application of constraint based fracture assessment for small cracks at notches would lead to more accurate fatigue and fracture assessment.

Circumferential Through-Wall Cracks in a Cylinder

The hollow cylinder is used for a variety of applications, including pressurized pipelines and tubular structures. A circumferential through-wall cracks is a severe defect and must be considered in a fracture/fatigue analysis. Knowledge of the constraint level would allow for more accurate fracture/fatigue assessment. The T -stress for circumferential through-wall cracks in cylinders was analyzed using the finite element method and the weight function method. Finite element models were made for three geometries, defined by the radius to thickness ratios: $R/t = 5, 10$, and 20 . The T -stress was obtained for eight crack lengths, given by: $\theta/\pi = 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375$, and 0.5000 . Each was subjected to remote tension and remote bending loads.

In order to approximate the T -stress empirical formulas were obtained using least-squares curve fitting. The formulas are valid for tension and bending loads for any geometry in the range $5 \geq R/t \geq 20$ and for crack lengths in the range $0 \leq \theta/\pi \leq 0.5$. Weight functions for T -stress were generated for general loading conditions. Before doing so the weight function method for two dimensions was extended to a cylindrical surface. Then approximate weight functions were defined for $R/t = 5, 10$, and 20 . They were validated by comparing the weight function based T -stress for tension and bending to the finite element results. The weight functions for T -stress are valid for general loading conditions and for crack lengths in the range $0 \leq \theta/\pi \leq 0.5$.

For all geometries and loading conditions the normalized T -stress at $\theta/\pi = 0.0625$ is close to minus one. This shows that small cracks in a cylinder are equivalent to a centre crack in an infinite plate. For crack lengths in the range $0 \leq \theta/\pi \leq 0.2$, the T -stress solutions for $R/t = 5, 10$, and 20 are equivalent for a specific loading condition. After about $\theta/\pi = 0.2$ the T -stress solutions diverge, with thicker cylinders predicting higher constraint levels. When subjected to remote bending the constraint increases as the crack length increases. When subjected to remote tension, the constraint for $R/t = 5$ increases slightly as the crack length increases. For $R/t = 10$ and 20 the constraint increases up to $\theta/\pi = 0.2$ then decreases in the range $0.2 \leq \theta/\pi \leq 0.5$.

The circumferential through-wall cracks experience low constraint for both tension and bending loads. Constraint based fracture assessment may be used for this geometry. Using a higher value for fracture toughness will increase the critical flaw size, and result in increased fatigue life predictions. This will impact any analysis that considers through-wall cracks in cylinders.

In conclusion, the geometries examined here experience low crack-tip constraint, and as a result constraint based fracture can be applied. Application of constraint based fracture will lead to better assessment of fracture and fatigue life.

5.2. Recommendations

Cracks emanating from other notch geometries (such as elliptical notches) could be analyzed using the finite element method and compared with the equation derived by Wang et al. (2006).

The present analysis of the T -stress for circumferential through-wall cracks in cylinders covers the range $5 \geq R/t \geq 20$. This could be expanded to cover more R/t ratios outside of this range. For thick cylinders with $R/t < 5$ the use of solid elements is recommended. Here the weight functions are only valid for $R/t = 5, 10$, and 20 . Further work could include developing weight functions for more R/t ratios.

The present analysis assumed the ends of the cylinder were free to rotate. In most cases the cylinder is supported, and the movement of the ends is restrained. Wang and Reinhardt (2003) provide stress intensity factor and limit load solutions taking into account end constraints. A similar analysis could be done for the T -stress.

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