INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700 800/521-0600

**Three Studies on Strategic Behaviour:
Allocation Mechanism, Vertical Contract and Debt Buyback**

by

Xiguang Liu

B.S.(Honours), Fudan University; M.E., Tsinghua University

A thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfilment of the
requirements for the degree of
Doctor of Philosophy
in the
Department of Economics

CARLETON UNIVERSITY

Ottawa, Ontario

November, 1997

Copyright©1997, Xiguang Liu

All rights reserved. This work may not be reproduced
in whole or in part, by photocopying or other means without
permission of the author.

National Library
of Canada

Acquisitions and
Bibliographic Services

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque nationale
du Canada

Acquisitions et
services bibliographiques

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file *Votre référence*

Our file *Notre référence*

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-32341-2

Canada

The undersigned hereby recommend to
the Faculty of Graduate Studies and Research
acceptance of the thesis,

**"Three Studies on Strategic Behaviour: Allocation Mechanism,
Vertical Contract and Debt Buyback"**

submitted by

Xiguang LIU, B.S., M.S.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Chair, Department of Economics

Thesis Supervisor

*for Professor Chen
as arranged.*

External Examiner

Carleton University
November 1997

To my father and mother

Abstract

This thesis comprises three chapters, each dealing with a topic related to strategic behaviours. Chapter 1 studies a topic of the theory of implementation. A planner wants to give an indivisible good to one of several individuals who values it most, without any monetary transfer. However, the planner does not know who is most deserving. The contribution of this chapter is that it presents several efficient mechanisms which the planner can use to achieve his objective in the situations where each individual's valuation of the good may or may not be common knowledge among the individuals.

The objective of Chapter 2 is to study the welfare implications of the policy prohibiting price floors in contracts between film distributors and exhibitors. A two-stage model is constructed to capture the basic features of the industry contracts: an exhibitor pays its distributor a guarantee as well as a percentage of the admission revenue; however, it retains all the revenue from selling snack food in theatres. It is shown that when movie-goers' reservation prices for movies and for snack foods are not negatively correlated, prohibiting price floor on admissions is welfare-improving. Furthermore, if a price floor is permitted, an attempt to bar integration between a distributor and an exhibitor will not be welfare-improving. These policy implications, however, crucially depend on the assumption of nonnegative correlation between movie-goers' reservation prices for movies and for snack foods.

In Chapter 3 the buyback option for settling LDC debts is evaluated through a theoretical model. Contrary to the majority of studies on the option, this analysis demonstrates that buybacks are welfare-improving to both a LDC and its creditors even at the ex post secondary market prices. It also shows that the so-called free-rider problem can't explain why creditors are unwilling to accept market-based discount buybacks. I propose and prove an alternative explanation that the unwillingness of creditors is caused by their market power. The predictions from this model are shown to be consistent with many historical facts and the current debt crises. An empirical study is also conducted to test the basic premise of the model.

Acknowledgments

I am grateful to many people from whom I benefitted during my Ph.D. studies at Carleton University and accomplishing this thesis. I would like to thank my supervisors of this thesis, Professor Keith Acheson, Professor Zhiqi Chen and Professor Chantale LaCasse. Professor LaCasse mainly supervises Chapter 1, Professor Acheson Chapter 2 and Professor Chen Chapter 3 respectively, while they also make contributions to other chapters. Without their advice, guidance and understanding, the completion of this thesis would not be possible. I would like especially to thank Professor Brian Erard, Director of the Joint Ph.D. Program with the University of Ottawa and Supervisor of Ph.D. Studies during the key stages of this thesis. His generous helps to me come from various aspects. I also owe a great deal to Professor Stephen Ferris not only for his excellent teaching but also for his nice personality. I gain much from these people in becoming an economist and most importantly an analytical person. Their knowledge, work, candidacy and enthusiasm have set a standard for a young economist.

Many thanks are due to Professor Michel Demers, who gave me many helps when he was Director of the Joint Ph.D. Program and Supervisor of Ph.D. studies. I would like to express my appreciation to all other members of the Joint Ph.D. program from whom I benefitted. These include Professors R.A. Brecher, R. Carson, E.U. Choudhri, D.G. McFetridge, S.B. Park, Stylianos Perrakis, L.L. Schembri, K. Day and André Plourde. I appreciate many fellow Ph.D. students in the Joint Ph.D. Program for their helps and friendship. I also wish to extend my thanks to Judy Poole, Ginette Harte, Shirley Beaton, Marilyn Kehrig and all other departmental staff for their providing an excellent supporting environment.

I wish to express my appreciation to my wife, Xiaohua Wang, for her love and understanding, especially during our transition to parenthood. She, as well as my son, Jiaying, did give up so much during my working on this thesis. I recognize that the moral

support from my parents are invaluable so that I dedicate this thesis to them. Many thanks are also due to my many friends in China, North America and other places for their warm friendship. Finally, I greatly appreciate the scholarships and financial assistance received from both the Faculty of Graduate Studies and Research of Carleton University and that of the University of Ottawa.

Table of Contents

Title Page	i
Acceptance Sheet	ii
Dedication	iii
Abstract	iv
Acknowledgments	v
Table of Contents	vii
List of Tables	ix
List of Figures	x
List of Appendices	xii
Chapter 1 Efficient Mechanisms for a Kind of Arrangement Problems	1
1. Introduction	1
2. Mechanisms with Complete Information	3
3. Mechanisms under Incomplete Information	19
4. Mechanisms for Multiple Individuals	26
5. Concluding Remarks	29
References	37
Chapter 2 Vertical Contract for Motion Pictures	38
1. Introduction	38
2. The Model	43
3. The Outcome of the Model	52
4. The Vertical Constraint vs. Policy Implications	65
5. The Policy Action vs. Merging	71
6. Negative Relationship between the Reservation Prices	74
7. Concluding Remarks	77

References	117
Chapter 3 The Buyback Option Revisited	119
1. Introduction	119
2. The Model	123
3. Buyback at Market Discount	127
4. Negotiated Buybacks	129
5. Creditors' Logic	137
6. Welfare Effects	140
7. An Empirical Study of Debt Overhang	141
8. Concluding Remarks	150
References	154

List of Tables

Tables of Chapter 2

Table 1 Summarization of Result 1	57
Table 2 Summarization of Result 2	58
Table 1-1 Result 1 if $c=0$	59
Table 2-1 Result 2 if $c=0$	60
Table 1-2 Result 1 if $d=0$	60
Table 2-2 Result 2 if $d=0$	61
Table 1-3 Result 1 if $c=a/2$	61
Table 2-3 Result 2 if $c=a/2$	62
Table 3 Summarization of Result 3	71
Table 4 Summarization of Results 4, 5 and 6	75
Table D-1	105
Table D-2	106
Table D-3	108
Table D-4	109
Table F-1	115

Tables of Chapter 3

Table 1 The Source Data of Mexico	144
Table 2 (Net) Foreign Direct Investment vs. Debt Overhang	146
Table 3 Foreign Long-term Loans vs. Debt Overhang	146
Table 4 Domestic Investment Supply vs. Debt Overhang	147

List of Figures

Figures of Chapter 1

Figure 1	6
Figure 2	11
Figure 3	17
Figure 4	21
Figure 5	24
Figure 6	26

Figures of Chapter 2

Figure 1	46
Figure 2	48
Figure 3	49
Figure 4	53
Figure 5	72
Figure 6	76
Figure B.1	84
Figure C.1	93
Figure C.2	94
Figure C.3	95
Figure C.4	97
Figure C.5	98
Figure C.6	100
Figure D.1	103
Figure E.1	111
Figure F.1	113

Figures of Chapter 3

Figure 1	120
Figure 2	131

List of Appendices

Appendix to Chapter 1

Appendix Proofs of Results	32
----------------------------------	----

Appendices to Chapter 2

Appendix A The Derivation of Demand Functions	80
Appendix B Proofs of Propositions	84
Appendix C Proof of Result 1	93
Appendix D Proof of Result 2	103
Appendix E Proof of Result 4	111
Appendix F Proof of Result 5	113

Appendices to Chapter 3

Appendix A Proof of the Lemma	152
Appendix B Investment Behavior under Debt Overhang	153

Chapter 1 Efficient Mechanisms for a Kind of Arrangement Problems

1. Introduction

The Old Testament tells of the dilemma faced by King Solomon. Two women came to King Solomon with a child, each claiming to be the child's true mother. King Solomon must identify the true mother and give her the child, at no charge either to her or to the imposter, despite the fact that he does not know which of the two women is in fact the true mother.

King Solomon's dilemma is the prototype of a category of arrangement problems, which exists in a number of economic and social contexts. The crucial elements of this arrangement problem are the following:

- (1) A planner or decision maker wants to allocate an indivisible good, which we call a "ticket",¹ to one and only one among several individuals.
- (2) The individuals are utility-maximizers. The utility attached to the ticket may be positive, zero or negative. It is assumed that there exists some amount of money such that an individual is indifferent between gaining (giving up) the amount of money and accepting the ticket of positive (negative) utility. Thus it is sufficient to consider the valuations of the individuals in terms of money.
- (3) The individual who is allocated the ticket must be the one who values it most.
- (4) The planner wants to give the ticket (rather than selling it).
- (5) The planner does not know which individual is most deserving (i.e., the individual who values the ticket most).
- (6) The individuals themselves know to whom the ticket should be allocated.

As an example, consider the assignment of tasks made by the Chair of an academic department. The Chair is the planner who wishes to allocate a duty, say teaching a course or

:

It can be a consumption good, a service, a task, a job offer, a license, etc..

taking charge of a committee, to one of a group of professors. These professors may be glad or averse to taking the duty. Some may consider the duty as bringing them more happiness than any other choice they face and so would like to take it (positive utility). Others would know the duty has a greater opportunity cost and would dislike taking it (negative utility). The Chair wants to allocate the duty to the professor to whom it will bring the most happiness. The Chair, however, does not know which professor this is. See Glazer and Ma (1989) for more examples.

Note that element (6) is an informational assumption. Different specifications can assure the assumption to be satisfied. In this chapter, when we say the problem is one of complete information, it means that each individual's true valuation for the ticket is common knowledge among all the individuals (except for the planner). Otherwise, the problem is one of incomplete information.

Economists have only recently started to pay attention to such arrangement problems, and the literature is correspondingly sparse. The study of such problems is an application of the theory of implementation, which focuses in part on the existence of mechanisms for solving such arrangement problems. Another branch of the literature focuses instead on elaborating simple mechanisms in specific contexts. A recent contribution to this vein is made by Glazer and Ma (1989). They provide a simple and efficient mechanism for the problem termed "the allocation of a prize". By "prize" they mean in fact that all individuals value the ticket positively. Moore (1991) elaborates a variant of Glazer and Ma's mechanism. But, both the mechanisms of Glazer and Ma's and Moore's do not apply to the situation where at least one individual values the ticket negatively. Neither applies to incomplete information contexts.²

In this chapter we provide several efficient mechanisms which apply to the general situation

²

We will give more details of Glazer and Ma's mechanism in section 2.

where the individuals may value the ticket for positive, zero or negative utility. Some of these mechanisms also apply to incomplete information contexts. In line with Glazer and Ma, these mechanisms are based on finite-multi-stage perfect information games and the concept of subgame perfect equilibrium. This is evidently natural and reasonable. A mechanism for such a problem is constructed as each individual will be asked some questions in sequence so as to reveal true information. The interactive process is a sequential game by nature. Such a sequential game must be of finite stages; otherwise, it is not practical. A finite-multi-stage perfect information game is guaranteed to have at least one subgame perfect equilibrium in pure strategies. The concept of subgame perfect equilibrium is equivalent to "credible deterrence", or time consistency. By "efficient", it means that the outcome of any subgame perfect equilibrium of the mechanism will give the ticket to the agent who values it most. That is, the mechanism implements the planner's goal.

The rest of this chapter is organized as follows. We illustrate the mechanisms of complete information for the situation where there are two individuals in section 2 and those of incomplete information in section 3. Section 4 extends the mechanisms of sections 2 and 3 to the situation where there are more than two individuals. Section 5 contains some concluding remarks. Some of the proofs are collected in an appendix.

2. Mechanisms with Complete Information

Formally, the problem can be specified as follows.

- (1) There are two individuals, A and B. A's and B's true valuations for the planner's ticket are denoted as u^* and v^* respectively.
- (2) Each individual knows her own valuation for the ticket.³
- (3) A and B also know each other's u^* , v^* .
- (4) The planner does not know u^* and v^* , and even does not know which of the two is larger.

³

For simplicity, we suppose the planner is male and all the individuals are female in this chapter.

(5) There exists a large enough positive number M such that $M > 4\max(|u^*|, |v^*|)$, and the planner knows M .

Recall that from element (2) in the introduction above, M , u^* and v^* can all be interpreted in terms of money. If the planner knows that u^* must fall in some interval $[a, b]$ and v^* in $[c, d]$, (5) is satisfied for any number M such that $M > 4\max(|a|, |b|, |c|, |d|)$. Thus (5) is a weaker requirement on what the planner needs to know. The planner can apply different M to different individuals. For simplicity, we illustrate our mechanisms with a uniform M .

The planner wants to give the ticket to the agent who values it most without any payment or transfers. Therefore:

(6) Planner's goal: let A accept the ticket if $u^* > v^*$, or B if $v^* > u^*$, or either A or B if $u^* = v^*$, without any payment transferred between A, B and himself.

We now turn to the consideration of a first mechanism by which the ticket can be allocated when individuals may value it positively, zero or negatively.

Mechanism 1:

Stage 0: Planner announces M .

Stage A1: Planner asks A whether the ticket should be given to B and A answers "yes" or "no".⁴ If "no", proceed to stage A2. If "yes", proceed to stage B1.

Stage A2: Planner asks B whether the ticket should be given to A or to herself and B answers "mine" or "hers". If "hers", A gets the ticket. If "mine", proceed to stage A3.

⁴

At this point we can assume that the planner decides whether to call A or B first at random by tossing a fair coin. We return to this point below.

- Stage A3: Planner asks A whether to agree that the ticket should be given to B and A answers "agree" or "not agree". If "agree", A pays $M/2$ and B gets the ticket with prize M . If "not agree", each pays M and then proceed to stage A4. At this stage, the agents enter a sequential bidding game, call it Γ_1 , in which both A and B are required to claim a bid from $[-M, +\infty)$.
- Stage A4: B bids v from $[-M, +\infty)$, then proceed to stage A5.
- Stage A5: A bids u from $[-M, +\infty)$. The individual with the higher bid (the winner) gets the ticket and pays her bid. If A loses, A pays $(u+M)$; if B loses, B pays $(v+M)$; if there is a tie, A is the winner and B the loser.
- Stage B1: Planner asks B whether the ticket should be given to A and B answers "yes" or "no". If "yes", each pays $M/3$. If "no", proceed to stage B2.
- Stage B2: Planner asks A whether the ticket should be given to B or to herself and A answers "mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage B3.
- Stage B3: Planner asks B whether to agree that the ticket should be given to A and B answers "agree" or "not agree". If "agree", A gets the ticket with prize M and B pays $M/2$. If "not agree", each pays M and then proceed to stage B4. At this stage, the agents enter a sequential bidding game, call it Γ_2 , in which A and B are required to claim a bid from $[-M, +\infty)$.
- Stage B4: A bids u from $[-M, +\infty)$ and then proceed to stage B5.
- Stage B5: B bids v from $[-M, +\infty)$. The individual with the higher bid (the winner) gets the ticket and pays her bid. If A loses, A pays $(u+M)$; if B loses, B pays $(v+M)$; if there is a tie, B is the winner and A the loser.

Mechanism 1's induced game form is shown in Figure 1.

In the bidding game Γ_1 , the payoffs (U_1, V_1) (where U_1 is the payoff to A) when B bids v and A bids u are:

$$U_l = U_l(u, v) = \begin{cases} u^* - u & \text{if } u \geq v \\ -u - M & \text{if } u < v \end{cases} \quad (1)$$

$$V_l = V_l(u, v) = \begin{cases} v^* - v & \text{if } v > u \\ -v - M & \text{if } v \leq u \end{cases} \quad (2)$$

In the bidding game Γ_2 , the payoffs (U_r, V_r) (where U_r is the payoff to A) when A bids u and B bids v are:

$$U_r = U_r(u, v) = \begin{cases} u^* - u & \text{if } u > v \\ -u - M & \text{if } u \leq v \end{cases} \quad (3)$$

$$V_r = V_r(u, v) = \begin{cases} v^* - v & \text{if } v \geq u \\ -v - M & \text{if } v < u \end{cases} \quad (4)$$

Figure 1

Result 1: The payoffs of any subgame perfect equilibrium of $\Gamma 1$ satisfy:

if $u^* > v^*$,

$$\begin{cases} U_l(\sigma_A^*, \sigma_B^*) = u^* + M > 0 \\ V_l(\sigma_A^*, \sigma_B^*) = 0 \end{cases} \quad (5)$$

if $u^* < v^*$,

$$\begin{cases} U_l(\sigma_A^*, \sigma_B^*) = 0 \\ V_l(\sigma_A^*, \sigma_B^*) = v^* - u^* > 0 \end{cases} \quad (6)$$

if $u^* = v^*$,

at some subgame perfect equilibria,

$$\begin{cases} U_l(\sigma_A^*, \sigma_B^*) = \theta(u^* + M) \quad \theta \in [0, 1] \\ V_l(\sigma_A^*, \sigma_B^*) = 0 \end{cases} \quad (7)$$

at all other subgame perfect equilibria,

$$\begin{cases} U_l(\sigma_A^*, \sigma_B^*) = u^* + M \\ V_l(\sigma_A^*, \sigma_B^*) = 0 \end{cases} \quad (8)$$

where in (7), θ is the probability that A bids $-M$.

Result 2: The payoffs of any subgame perfect equilibrium of $\Gamma 2$ satisfy :

if $u^* > v^*$,

$$\begin{cases} U_r(\sigma_A^*, \sigma_B^*) = u^* - v^* > 0 \\ V_r(\sigma_A^*, \sigma_B^*) = 0 \end{cases} \quad (9)$$

if $u^* < v^*$,

$$\begin{cases} U_r(\sigma_A^*, \sigma_B^*) = 0 \\ V_r(\sigma_A^*, \sigma_B^*) = v^* + M > 0 \end{cases} \quad (10)$$

if $u^* = v^*$,

at some subgame perfect equilibria.

$$\begin{cases} U_r(\sigma_A^*, \sigma_B^*) = 0 \\ V_r(\sigma_A^*, \sigma_B^*) = \theta(u^* + M), \quad \theta \in [0, 1] \end{cases} \quad (11)$$

at all other subgame perfect equilibria.

$$\begin{cases} U_r(\sigma_A^*, \sigma_B^*) = 0 \\ V_r(\sigma_A^*, \sigma_B^*) = v^* + M > 0 \end{cases} \quad (12)$$

where in (11), θ is the probability that B bids $-M$.

With results 1 and 2, theorem 1 follows.

Theorem 1: For mechanism 1, the outcome of any subgame perfect equilibrium is: (1) if $u^* > v^*$, A gets the ticket; (2) if $u^* < v^*$, B gets the ticket; (3) if $u^* = v^*$, either A or B gets the ticket; (4) no payment takes place between A, B and the planner.

Proof: If $u^* > v^*$, in the bidding game $\Gamma 1$, A gets $(u^* + M)$, while B gets zero. At stage A3, A will say "not agree": by saying so, A can get u^* while otherwise A loses $M/2$ (note that $u^* > -M/4$). Knowing A will say "not agree" at stage A3, B says "hers" at stage A2; otherwise, she would lose M . In $\Gamma 2$, because $u^* > v^*$, B will get nothing. Thus B says "agree" at stage B3, in which case she loses $M/2$ (otherwise she would lose M). Knowing that B must say "agree" at stage B3, A will say "mine" at stage B2. By doing so, A gets $(u^* + M) (> 0)$. Knowing A must say "mine" at stage B2, B will say "yes" at stage B1. By doing so, B only loses $M/3$ while otherwise she would lose $M/2$. Knowing that B must say "yes" at stage B1, A will say "no" at stage A1 to avoid a loss of $M/3$ due to B's saying "yes" at stage B1. Therefore, A gets the ticket. An analogous analysis shows that if $u^* < v^*$, B must get the

ticket while if $u^*=v^*$, either A or B may get the ticket. It is evident that no transfers takes place along the equilibrium path.

Q.E.D.

One can see that the auctions, Γ^1 and Γ^2 , and the payments for entrance to these auctions are constructed to have an incentive for the agent of higher valuation but to have a penalty for the agent of lower valuation to enter. Once the agent of higher valuation demonstrates to accept the ticket, the only way for the agent of lower valuation to avoid these auctions is to acknowledge that the ticket should be given to her opponent (the agent of higher valuation). When the agent of higher valuation has positive utility attached to the ticket, she of course likes to accept the ticket. When she has negative utility attached to the ticket, she must also show to accept the ticket. Otherwise, the agent of lower valuation would choose the action, i.e., the action "yes" in stage B1, which would lead both of them to face penalties, and the penalty enforced to the agent of higher valuation would leave her worse off than accepting the ticket. That is, the agent with a higher valuation is deterred from refusing the ticket, and the agent with a lower valuation is deterred from asking for it. Therefore, the equilibrium path will involve the agent of higher valuation accepting the ticket either in stage A1 or stage B1 and consequently, the other agent acknowledging that the ticket should be given to her in stage A2 or stage B2; the other stages after stage A2 and stage B2 will never take place and the action "yes" in stage B1 will never be taken.

Although we use " $-M/3$ " and " $-M/2$ " in mechanism 1, it is not necessary for M to be divisible. We can use " $-M_3$ " and " $-M_2$ " to substitute for " $-M/3$ " and " $-M/2$ " respectively, if besides M , the planner can figure out numbers M_3 and M_2 such that $-M < -M_2 < -M_3 < \max\{|u^*|, |v^*|\}$. Furthermore, with such M_3 and M_2 , M is not required to satisfy " $M > 4\max\{|u^*|, |v^*|\}$ ".

In mechanism 1, either A or B can be called upon to move first, without affecting the fulfilment of the planner's goal. It is in this sense that we can say that A and B are

symmetrically positioned. The first mover can be chosen at random; the distribution of probability does not matter to the fulfilment of the planner's goal. In the situation where $u^*=v^*$, however, the individual called upon first has an advantage. She can get the ticket if her valuation for it is positive or to refuse it if negative. This is because, in the situation where $u^*=v^*$, the individual called upon second will concur with the announcement of the one called upon first. Therefore, if the planner chooses the individual to be called upon first at random, the greater the probability of A (B) being chosen, the more likely it is for A (B) to make her wanted choice. If A and B have an equal probability of being chosen, then any such advantage vanishes in the sense that each faces an equal probability to make her preferred choice.

If the planner knows that someone's true valuation for the ticket, say B's, is positive, i.e., $v^*>0$, and that A's true valuation is strictly below-bounded by some number Q_a (and B's by Q_b ; evidently, Q_b can be zero), then mechanism 2 below also implements the planner's goal. One will note that mechanism 2 is simpler than mechanism 1. That is, the more information known to the planner, the simpler a mechanism will be available for the planner to implement his goal. When using mechanism 2, the planner does not need to figure out a large enough number M ; however, he must call upon B first, whose true valuation is positive. As stated in the paragraph above, either A or B can be called upon first in mechanism 1 without affecting the fulfilment of the planner's goal, while no such symmetry exists in mechanism 2.

Mechanism 2:

Stage 0: Planner announces any positive number "e", then proceed to stage 1.

Stage 1: Planner asks B whether the ticket should be given to A. If B answers "no", proceed to stage 2. If "yes", A gets the ticket.

Stage 2: Planner requires B to choose a number k where $0 < k < 1$. B announces her number k , then proceed to stage 3.

Stage 3: Planner asks A whether the ticket should be given to B or to herself and A answers

"mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage 4.

Stage 4: Planner asks B whether to agree that the ticket should be given to A and B answers "agree" or "not agree". If "agree", A gets the ticket with payment Q_a and B pays "ke". If "not agree", both pay the planner "e" and then proceed to stage 5. At this stage, the players enter a sequential bidding game, call it Γ , in which A is required to bid from $[Q_a, +\infty)$ and B from $[Q_b, +\infty)$.

Stage 5: A bids u from $[Q_a, +\infty)$. Then proceed to stage 6.

Stage 6: B bids v from $[Q_b, +\infty)$. The individual with the higher bid (the winner) gets the ticket and pays her bid. Further, if A loses, A pays " $u - Q_a$ "; if B loses, B pays " $v - Q_b$ "; if there is a tie, B is the winner and A the loser.

Mechanism 2's induced game form is shown in Figure 2.

Figure 2

The payoffs (U, V) for the bidding game Γ (U is the payoff to A) are:

$$U = U(u, v) = \begin{cases} u^* - u & \text{if } u > v \\ Qa - u & \text{if } u \leq v \end{cases} \quad (13)$$

$$V = V(u, v) = \begin{cases} v^* - v & \text{if } v \geq u \\ Qb - v & \text{if } v < u \end{cases} \quad (14)$$

Result 3: The payoffs of any subgame perfect equilibrium of Γ satisfy:

if $u^* > v^*$,

$$\begin{cases} U(\sigma_A^*, \sigma_B^*) = u^* - \max\{v^*, Qa\} > 0 \\ V(\sigma_A^*, \sigma_B^*) = 0 \end{cases} \quad (15)$$

if $u^* < v^*$,

$$\begin{cases} U(\sigma_A^*, \sigma_B^*) = 0 \\ V(\sigma_A^*, \sigma_B^*) = v^* - \max\{Qa, Qb\} > 0 \end{cases} \quad (16)$$

if $u^* = v^*$,

at some subgame perfect equilibria,

$$\begin{cases} U(\sigma_A^*, \sigma_B^*) = 0 \\ V(\sigma_A^*, \sigma_B^*) = \theta[v^* - \max\{Qa, Qb\}], \quad \theta \in [0, 1] \end{cases} \quad (17)$$

at all other subgame perfect equilibria,

$$\begin{cases} U(\sigma_A^*, \sigma_B^*) = 0 \\ V(\sigma_A^*, \sigma_B^*) = [v^* - \max\{Qa, Qb\}] \end{cases} \quad (18)$$

where in (17), θ is the probability that A bids $u = Qa$.

Using result 3, theorem 2 follows.

Theorem 2: The outcome of any subgame perfect equilibrium of the game induced by mechanism 2 is: (1) If $u^* > v^*$, A gets the ticket; (2) If $u^* < v^*$, and $v^* > 0$, B gets the ticket; (3) If $u^* = v^* \geq 0$, A or B may get the ticket; (4) In the three situations above, no payment takes place between A, B and the planner; (5) In the situations where $u^* < v^* \leq 0$ and where $u^* = v^* < 0$, mechanism 2 is not efficient, i.e., at some subgame perfect equilibria, it fails to meet the planner's goal.

Proof: (1) If $u^* > v^*$, B says "agree" at stage 4, since then she only loses ke ($< e$) instead of e . Knowing that B must say "agree" at stage 4, A will certainly say "mine" at stage 3 since $(u^* - Q_a) > 0$. Knowing that A must say "mine" at stage 3, B says "yes" at stage 1 to avoid loss in stage 4. Thus in the outcome of the subgame perfect equilibrium, A gets the ticket. (2) If $u^* < v^*$ and $v^* > 0$, B knows that she is able to get $(v^* - \max(Q_a, Q_b))$ (> 0) in the bidding game Γ . Furthermore, she knows that if she says "no" at stage 1 and chooses a positive number "k" such that $0 < k < 1$ and $-ke < (v^* - \max(Q_a, Q_b) - e)$, i.e., $k > \max(0, 1 - ((v^* - \max(Q_a, Q_b))/e))$, she makes A know that she will say "not agree" at stage 4, which means that A would incur a loss of "e" if A says "mine" at stage 3. Thus A says "hers" at stage 3 and consequently B gets the ticket. (3) If $u^* = v^*$ and $v^* \geq 0$, at stage 5, if B believes that A will bid $u = u^*$, then she chooses "agree" at stage 4 and A of course says "mine" at stage 3, and thus B must say "yes" at stage 1 to avoid a loss of "ke" at stage 4. Thus, A gets the ticket. If B believes that A will bid $u = Q_a$ with some positive probability θ , then she will say "no" at stage 1 and choose "k" such that $k > \max(0, 1 - ((v^* - \max(Q_a, Q_b))/e))$. Seeing this, A can infer that B must say "not agree" at stage 4. As a result, A says "hers" at stage 3 and thus B gets the ticket. (4) It is evident that no payment occurs along the equilibrium path. (5) If $u^* < v^* < 0$, B always says "yes" at stage 1; otherwise she loses at least $\max\{v^*, ke\}$. Then A gets the ticket, when she does not value it most. If $u^* < v^* = 0$, B can say "yes" or "no" at stage 1. If B says "yes", A gets the ticket. Then the ticket goes to the wrong individual in the view of the planner. If B says "no", she also chooses "k" such that $k > \max(0, 1 - ((v^* - \max(Q_a, Q_b))/e))$, and thus B herself gets the ticket. If $u^* = v^* < 0$, if the "e" announced by the planner is such that $-e > u^*$,

and if B believes that A will bid $u=Q_a$ with some positive probability not less than θ^* ,⁵ then B says "no" at stage 1 and chooses "k" such that $k > \max(0, 1 - (\theta^* (v^* - \max(Q_a, Q_b))/e))$. Having observed B's choices, A says "mine" at stage 3. Then B says "not agree" at stage 4. The outcome will be that A loses e and B gets the ticket with utility $-e + \theta[v^* \max\{Q_a, Q_b\}]$, which is at least as large as $-e$ for $\forall \theta \in [0, 1]$ and is thus greater than v^* . Then payments between A, B and the planner occur.

Q.E.D.

As compared to mechanism 1, mechanism 2 doesn't need to have an action to deter the agent of higher valuation from refusing the ticket. Instead of that the planner is required to figure out "M" in mechanism 1, the agent called upon first, B, should perhaps figure out "k" in mechanism 2. Only when B holds higher valuation than A does B need to figure out a pertinent "k" in order to signal to A not to challenge the fact that she should be given the ticket.

This is an appropriate place for us to review more details of Glazer and Ma (1989)'s mechanism.⁶ Their mechanism is based on the assumption that each agent's valuation comes from an interval, say $[0, d]$. In comparison with our mechanism 2, their mechanism has no stage 0 and begins with B's claiming a number "e" from $[0, d]$; it has no similar stage in which a number "k" is specified and thus lacks a threat of penalty "-ke" (i.e., -ke is replaced

⁵

Here θ^* denotes some fixed number larger than zero but less than 1.

⁶

Glazer and Ma (1989) construct two mechanisms. Their mechanism 1 is for the situation where each individual's valuation comes from a discrete set and their mechanism 2 for the situation where each individual's valuation comes from an interval. Their mechanism 1 is more restricted than their mechanism 2. For instance, their mechanism 1 can't apply to the situation where individuals equally value the ticket. Their mechanism 2 also applies to the situation where their mechanism 1 does. We are to say their mechanism 2 by "Glazer and Ma's mechanism".

by zero). That is, their mechanism starts with stage 1, where if the number claimed by B, e , is equal to zero, A gets the ticket; otherwise, directly proceed to stage 3. As we have done in the proof of our theorem 2, one can easily confirm that their mechanism can't meet the planner's goal either when $u^* \leq v^* \leq 0$ or when $v^* \leq u^* \leq 0$.

If the planner knows that at least one individual's true valuation for the ticket is positive, i.e., $\max\{u^*, v^*\} > 0$, and that the true valuation of A is strictly below-bounded by some number Q_a and B's by Q_b , the planner can also use mechanism 3 below to implement his goal. It is evident that the assumptions of mechanism 3 are weaker than those of mechanism 2, but stronger than those of mechanism 1. Consequently, mechanism 3 retains the symmetry of mechanism 1 but is a bit more complex than mechanism 2. Similar to mechanism 2, mechanism 3 does not require the planner to figure out a large enough number "M" but it requires the agent of higher valuation to figure out a positive number "k".

Mechanism 3:

Stage 0: Planner announces any positive number " e ", then proceed to stage B1.

Stage B1: Planner asks B whether the ticket should be given to A and B answers "yes" or "no".⁷ If "no", proceed to stage B2. If "yes", proceed to stage A1.

Stage B2: Planner requires B to choose any number k_1 where $0 < k_1 < 1$ and B announces her number k_1 , then proceed to stage B3.

Stage B3: Planner asks A whether the ticket should be given to B or to herself and A answers "mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage B4.

Stage B4: Planner asks B whether to agree that the ticket should be given to A and B says "agree" or "not agree". If "agree", A gets the ticket with payment Q_a and B pays " $k_1 e$ ". If "not agree", both pay the planner e and then proceed to stage B5. At this

⁷

Same as in mechanism 1, either A or B may be called upon first.

stage, the players enter a sequential bidding game, call it Γ_l , in which A is required to bid from $[Q_a, +\infty)$ and B from $[Q_b, +\infty)$.

Stage B5: A bids u from $[Q_a, +\infty)$. Then proceed to stage B6.

Stage B6: B bids v from $[Q_b, +\infty)$. The individual with the higher bid (the winner) gets the ticket and pays her bid. If A loses, A pays " $u - Q_a$ "; if B loses, B pays " $v - Q_b$ "; if there is a tie, B is the winner and A the loser.

Stage A1: Planner requires A to choose any number k_2 where $0 < k_2 < 1$ and A announces her number k_2 , then proceed to stage A2.

Stage A2: Planner asks B whether the ticket should be given to A or to herself and B says "mine" or "hers". If "hers", A gets the ticket. If "mine", proceed to stage A3.

Stage A3: Planner asks A whether to agree that the ticket should be given to B and A says "agree" or "not agree". If "agree", B gets the ticket with payment Q_b and A pays " $k_2 e$ ". If "not agree", both pay the planner e and then proceed to stage A4. At this stage, the players enter a sequential bidding game, call it Γ_r , in which B is required to bid from $[Q_b, +\infty)$ and A from $[Q_a, +\infty)$.

Stage A4: B bids v from $[Q_b, +\infty)$. Then proceed to stage A5.

Stage A5: A bids u from $[Q_a, +\infty)$. The individual with the higher bid (the winner) gets the ticket and pays her bid. If B loses, B pays " $v - Q_b$ "; if A loses, A pays " $u - Q_a$ "; if there is a tie, A is the winner and B the loser.

Mechanism 3's induced game form is shown in Figure 3.

Figure 3

In the bidding game Γ_l , the payoffs (U_l, V_l) (where U_l is the payoff to A) when B bids v and A bids u are:

$$U_l = U_l(u, v) = \begin{cases} u^* - u & \text{if } u > v \\ Qa - u & \text{if } u \leq v \end{cases} \quad (19)$$

$$V_l = V_l(u, v) = \begin{cases} v^* - v & \text{if } v \geq u \\ Qb - v & \text{if } v < u \end{cases} \quad (20)$$

In the bidding game Γ_r , the payoffs (U_r, V_r) (where U_r is the payoff to A) when A bids u and B bids v are:

$$U_r = U_r(u, v) = \begin{cases} u^* - u & \text{if } u \geq v \\ Qa - u & \text{if } u < v \end{cases} \quad (21)$$

$$V_r = V_r(u, v) = \begin{cases} v^* - v & \text{if } v > u \\ Qb - v & \text{if } v \leq u \end{cases} \quad (22)$$

The following theorem holds with mechanism 3.

Theorem 3: If $\max\{u^*, v^*\} > 0$, the outcome of any subgame perfect equilibrium of the game induced by mechanism 3 is: (1) If $u^* > v^*$, A gets the ticket; (2) If $u^* < v^*$, B gets the ticket; (3) If $u^* = v^*$, either A or B may get the ticket; (4) In the three situations above, no payment takes place between A, B and the planner.

Proof: If $u^* > v^*$, it must be true that $u^* > 0$ since $\max\{u^*, v^*\} > 0$. If A and B enter the sequential bidding game Γ_l , B would lose e . So B says “agree” at stage B4: by saying so, she would only lose “ $k_1 e$ ”. Knowing that B will say “agree” at stage B4, A would say “mine” at stage B3: by saying so, she would get $u^* - Qa > 0$. Thus B will not say “no” at stage B1, because if she says “yes”, she loses nothing. Then with theorem 2, A gets the ticket with no payment to happen between A, B and the planner. Similarly, if $u^* < v^*$, B gets the ticket without payments transferred; if $u^* = v^*$, either A or B may get the ticket without payments transferred.

Q.E.D.

The sequential bidding games are the key components of these mechanisms. However, if the true valuations of A and B are distinct, the sequential bidding games Γ_l and Γ_r of mechanism 3 can be replaced by simultaneous bidding games Γ_l' and Γ_r' respectively, which are defined as follows: both in Γ_l' and in Γ_r' , agents are required to submit a bid from $(-\infty, +\infty)$; the agent with the higher bid (the winner) gets the ticket and pays her bid; if there is a tie in Γ_r' , A is the winner; if there is a tie in Γ_l' , B is the winner. If $u^* \geq v^*$, in all the Nash equilibria of Γ_r' , which involve A to bid v^* and B to bid any from $(-\infty, v^*]$, the payoff to A is $(u^* - v^*)$ while that to B is zero. If $v^* \geq u^*$, in all the Nash equilibria of Γ_l' , which involve B to bid u^* and A to bid any from $(-\infty, u^*]$, the payoff to B is $(v^* - u^*)$ while that to A is zero. By these facts, one can easily show that mechanism 3 with such replacements also

implements the planner's goal if $u^* \neq v^*$. Such replacements can't be used when $u^* = v^*$, because then there is no "k" in $(0,1)$ for the individual (either A or B) who wants the ticket to choose for deterring her opponent. More specifically, suppose that Γ_l' and Γ_r' are substituted for Γ_l and Γ_r respectively and that $u^* = v^*$, the equilibrium path will involve either B saying "agree" at stage B4 or A saying "agree" at stage A3, with transfers to happen. Similarly, the auction Γ of mechanism 2 can be replaced by either Γ_r' or Γ_l' without affecting the efficiency of mechanism 2 if A and B value the ticket differently. The determination of whether there exist simultaneous bidding games for mechanism 1 awaits further research.

3. Mechanisms under Incomplete Information

If at least one individual's true valuation for the ticket is not common knowledge among the group of individuals, the arrangement problem is one of incomplete information. However, different specifications about incomplete information are possible, and different specifications will require different mechanisms in order for the planner to implement his goal. In this section, we specify three incomplete information contexts and find a corresponding mechanism for each.⁸

Context 1:(I)A does not know B's true valuation v^* , or B does not know A's true valuation u^* , or they do not know each other's true valuations, but both A and B know that their valuations are distinct and whose valuation is higher.

(II)(Behavioural Assumption)If an individual is indifferent between two actions, she will choose the action that is most preferred by her opponent.

⁸

Perry and Reny (1994) construct a mechanism, which is essentially different from ours, for an incomplete information context where (I)the individuals know that their valuations are distinct and whose is higher but neither rules out the valuation of the other and (II)the individual of lower valuation is able to place a finite upper bound on her opponent's valuation. Their mechanism does not work if one or some individuals negatively value the ticket.

The assumptions of context 1 are to be substituted for assumption (3) of mechanism 1. With all other assumptions from mechanism 1 in place, the following mechanism implements the planner's goal.

Mechanism 4:

- Stage 0: Planner announces his large enough M ($M > 2\max(|u^*|, |v^*|)$), then proceed to stage A1.
- Stage A1: Planner asks A whether the ticket should be given to B and A answers "yes" or "no". If "no", proceed to stage A2. If "yes", proceed to stage B1.
- Stage A2: Planner asks B whether the ticket should be given to A or to herself and B answers "mine" or "hers". If "hers", A gets the ticket. If "mine", proceed to stage A3.
- Stage A3: Planner asks A whether to agree that the ticket should be given to B and A answers "agree" or "not agree". If "agree", A pays M and B gets the ticket with prize M . If "not agree", both pay M and then enter a second-price sealed bidding game where each player can submit a bid from $(-\infty, +\infty)$.
- Stage B1: Planner asks B whether the ticket should be given to A and B answers "yes" or "no". If "yes", each pays $M/2$. If "no", proceed to stage B2.
- Stage B2: Planner asks A whether the ticket should be given to B or to herself and A answers "mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage B3.
- Stage B3: Planner asks B whether to agree that the ticket should be given to A and B answers "agree" or "not agree". If "agree", A gets the ticket with prize M and B pays M . If "not agree", both pay M and then enter a second-price sealed bidding game where each player can submit a bid from $(-\infty, +\infty)$.

The induced game form of this mechanism is shown in Figure 4.

Theorem 4: The outcome of any subgame perfect equilibrium of the induced game form of mechanism 4 is: (1)if $u^* > v^*$, A gets the ticket; (2)if $u^* < v^*$, B gets the ticket; (3)no payment takes place between A, B and the planner.

Proof: If $u^* > v^*$, at stage A3, A will say "not agree": by doing so, her payoff is $(-M - u^* - v^*)$, while otherwise her payoff is $-M$. Knowing that A will say "not agree" at stage A3, B says "hers" at stage A2; otherwise she will lose M . At stage B3, B is indifferent between choosing "agree" or "not agree". Therefore, she chooses "agree" due to the behaviour assumption (II). At stage B2, knowing that B will say "agree" at stage B3, A says "mine". By doing so she will get $(u^* - M) > 0$. Knowing that A will say "mine" at stage B2, B says "yes" at stage B1; by doing so, B only loses $M/2$ while otherwise she would lose M . Knowing that B will say "yes" at stage B1, A says "no" at stage A1, since accepting the ticket is better than losing $M/2$. As a result, A gets the ticket. An analogous argument shows that when $u^* < v^*$, B must get the ticket. It is evident that no payment happens along the equilibrium path.

Q.E.D.

In the following context of incomplete information, where the behavioural assumption is replaced, we have another mechanism for the planner to use.

Context 2: (I)The same as (I) of context 1.

(II)The planner, A and B have common knowledge of a lower bound " w " on the difference between A's and B's true valuations, i.e., $|u^* - v^*| > w$.

With all the assumptions from mechanism 1 except that the assumptions of context 2 are substituted for assumption (3) of mechanism 1, the following mechanism implements the planner's goal and satisfies theorem 4 above.

Mechanism 5:

- Stage 0: Planner announces his large enough M ($M > 2\max(|u^*|, |v^*|)$), then proceed to stage A1.
- Stage A1: Planner asks A whether the ticket should be given to B and A answers "yes" or "no". If "no", proceed to stage A2. If "yes", proceed to stage B1.
- Stage A2: Planner asks B whether the ticket should be given to A or to herself and B answers "mine" or "hers". If "hers", A gets the ticket. If "mine", proceed to stage A3.
- Stage A3: Planner asks A whether to agree that the ticket should be given to B and A answers "agree" or "not agree". If "agree", A pays $(M-w)$ and B gets the ticket with prize M . If "not agree", both pay M and then enter a second-price sealed bid auction where each player can submit a bid from $(-\infty, +\infty)$.
- Stage B1: Planner asks B whether the ticket should be given to A and B answers "yes" or "no". If "yes", each pays $M/2$. If "no", proceed to stage B2.
- Stage B2: Planner asks A whether the ticket should be given to B or to herself and A answers "mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage B3.
- Stage B3: Planner asks B whether to agree that the ticket should be given to A and B answers "agree" or "not agree". If "agree", A gets the ticket with prize M and B pays $(M-w)$. If "not agree", both pay M and then enter a second-price sealed bid auction where each player can submit a bid from $(-\infty, +\infty)$.

Mechanism 5's induced game form is shown in Figure 5.

Figure 5

Note that “ w ” can be any small positive number that satisfies the condition specified by (II) and $w < M/2$. Note also that $w < M/2$ implies “ $w - M < -M/2$ ”, which assures that B would answer “yes” at stage B1 if she were to answer “agree” at stage B3. The function of “ w ” is to make A or B to choose the action at stage A3 or B3 as she would be driven to do by the behavioural assumption.

Mechanisms 4 and 5 can be considered as the counterparts of mechanism 1 for incomplete information. The mechanism for the following context of incomplete information can be considered as the counterpart of mechanism 2.

Context 3: (I) The same as (I) of context 1.

(II) Except the planner, A and B have common knowledge of a lower bound “ w ”

on the difference between A's and B's true valuations, i.e., $|u^* - v^*| > w$. The planner, however, knows that B's true valuation is positive and that A's true valuation is strictly below-bounded by some number Q_a (and B's by Q_b ; evidently, Q_b can be zero).

Besides the assumptions of context 3, if all the assumptions except assumption (3) from mechanism 1 are in place, mechanism 6 below implements the planner's goal and satisfies theorem 4 above.

Mechanism 6:

Stage 0: Planner announces any positive number "e", then proceed to stage 1.

Stage 1: Planner asks B whether the ticket should be given to A and B says "yes" or "no".
If "no", proceed to stage 2. If "yes", A gets the ticket.

Stage 2: Planner requires B to choose a number k where $0 < k < 1$ and B announces her number k , then proceed to stage 3.

Stage 3: Planner asks A whether the ticket should be given to B or to herself and A answers "mine" or "hers". If "hers", B gets the ticket. If "mine", proceed to stage 4.

Stage 4: Planner asks B whether to agree that the ticket should be given to A and B answers "agree" or "not agree". If "agree", A gets the ticket with payment Q_a and B pays ke . If "not agree", both pay the planner e and then enter a second-price sealed bid auction, call it Γ , in which both A and B can submit bids from $(-\infty, +\infty)$.

Mechanism 6's induced game form is shown in Figure 6. In this mechanism, if $v^* > u^*$, B can signal to A that she should say "hers" at stage 3 by choosing k such that $0 < k < 1$ and $-ke < -e + w$ (note that $w < |v^* - u^*|$). Similarly, one can easily have the counterpart of mechanism 3 for context 3 of incomplete information.

Figure 6

4. Mechanisms for Multiple Individuals

The mechanisms of sections 2 and 3 can all be easily extended for the situation where there are N ($N > 2$) individuals. The principle is simple. For any pair of individuals, the planner can ask the series of questions as in those mechanisms. If the answers of the two individuals contradict each other, at least one of them will face a penalty. Therefore, one of the two must claim that she should be given the ticket (relative to the other) while the other must concur with her. That is, one individual can be excluded from the group possibly to be given the ticket and the other is still in the group. The planner can then pick another pair from the group and asks the pair the series of questions. When only one individual remains in the group, the planner gives her the ticket. For example, let the N individuals be numbered as $C_1, C_2, C_3, \dots, C_N$, then the counterpart of mechanism 1 is the following mechanism.

Mechanism 7:

Stage 0: (Start with $I=1$ and $L=1$) Planner announces M ,⁹ then proceed to stage CI.1.

Stage CI.1: Planner asks CI whether the ticket should be given to $C(I+L)$ and CI answers "yes" or "no". If "yes", proceed to stage $C(I+L).1$. If "no", then proceed to stage CI.2.

Stage CI.2: Planner asks $C(I+L)$ whether the ticket should be given to CI or to herself and $C(I+L)$ answers "mine" or "hers". If "hers" and $I+L < N$, set $L=L+1$, then proceed to stage CI.1. If "hers" and $I+L=N$, CI gets the ticket. If "mine", proceed to stage CI.3.

Stage CI.3: Planner asks CI whether to agree that the ticket should be given to $C(I+L)$ and CI answers "agree" or "not agree". If "agree", CI pays $M/2$, and $C(I+L)$ gets the ticket with prize M . If "not agree", CI and $C(I+L)$ both pay M and then proceed to stage CI.4. At this stage, CI and $C(I+L)$ enter a sequential bidding game, call it Γ_1 , where each player can bid from $[-M, +\infty)$.

Stage CI.4: $C(I+L)$ bids u_{i-L} from $[-M, +\infty)$, then proceed to stage CI.5.

Stage CI.5: CI bids u_i from $[-M, +\infty)$. If $u_i > u_{i-L}$, CI (the winner) gets the ticket and pays her bid, and $C(I+L)$ (the loser) pays $(M+u_{i-L})$. If $u_{i-L} > u_i$, $C(I+L)$ gets the ticket and pays her bid, and CI pays $(M+u_i)$. If there is a tie, CI is the winner and $C(I+L)$ the loser.

Stage $C(I+L).1$: Planner asks $C(I+L)$ whether the ticket should be given to CI and $C(I+L)$ answers "yes" or "no". If "yes", both CI and $C(I+L)$ pay $M/3$. If "no", proceed to stage $C(I+L).2$.

Stage $C(I+L).2$: Planner asks CI whether the ticket should be given to $C(I+L)$ or to herself and CI answers "mine" or "hers". If "hers" and $I+L=N$, $C(I+L)$ gets the ticket. If "hers" and $I+L < N$, set $I=I+L$ and $L=1$, then proceed to stage

⁹

In this situation of multiple individuals, M should be large enough that $M > 4\max(|u_1^*|, |u_2^*|, \dots, |u_N^*|)$, where u_i^* ($i=1, 2, \dots, N$) denotes the true valuation of CI.

CI.1. If "mine", proceed to stage C(I+L).3.

Stage C(I+L).3: Planner asks C(I+L) whether to agree that the ticket should be given to CI and C(I+L) answers "agree" or "not agree". If "agree", and $I+L=N$, CI gets the ticket with prize M and C(I+L) pays $M/2$. If "not agree", each pays M and then proceed to stage C(I+L).4. At this stage, CI and C(I+L) enter a sequential bidding game, call it Γ_2 , where each player can bid from $[-M, +\infty)$.

Stage C(I+L).4: CI bids u_i from $[-M, +\infty)$ and then proceed to stage C(I+L).5.

Stage C(I+L).5: C(I+L) bids u_{i-L} from $[-M, +\infty)$. If $u_i > u_{i-L}$, CI (the winner) gets the ticket and pays her bid, and C(I+L) (the loser) pays $(M+u_{i-L})$. If $u_{i-L} > u_i$, C(I+L) gets the ticket and pays her bid, and CI pays $(M+u_i)$. If there is a tie, C(I+L) is the winner and CI the loser.

The multi-agent counterpart of mechanism 2 is the following mechanism, where it is assumed that the planner knows CI positively values the ticket and that each CI's true valuation is strictly below-bounded by Q_i ($i=1, 2, \dots, N$).

Mechanism 8:

Stage 0: (Start with $I=1$ and $L=1$) Planner announces any positive number " e ", then proceed to stage CI.1.

Stage CI.1: Planner asks CI whether the ticket should be given to C(I+L) and CI answers "yes" or "no". If "no", proceed to stage CI.2. If "yes" and $I+L=N$, C(I+L) gets the ticket. If "yes" and $I+L < N$, set $I=I+L$ and $L=1$, then proceed to stage CI.1.

Stage CI.2: Planner requires CI to choose a number k_i where $0 < k_i < 1$ and CI announces her number k_i , then proceed to stage CI.3.

Stage CI.3: Planner asks C(I+L) whether the ticket should be given to CI or to herself and C(I+L) answers "mine" or "hers". If "hers" and $I+L=N$, CI gets the ticket. If "hers" and $I+L < N$, set $L=L+1$, then proceed to CI.1. If "mine", proceed to stage CI.4.

Stage CI.4: Planner asks CI whether to agree that the ticket should be given to C(I+L) and CI answers "agree" or "not agree". If "agree", C(I+L) gets the ticket with payment Q_{I-L} and CI pay " $k_i e$ ". If "not agree", both pay planner "e" and then proceed to stage CI.5. At this stage, CI and C(I+L) enter a sequential bid auction, call it Γ , where they can submit a bid from $[Q_I, +\infty)$ and from $[Q_{I-L}, +\infty)$ respectively.

Stage CI.5: C(I+L) bids u_{I-L} from $[Q_{I-L}, +\infty)$. Then proceed to stage CI.6.

Stage CI.6: CI bids u_I from $[Q_I, +\infty)$. The one with the higher bid gets the ticket and pays her bid. If C(I+L) loses, C(I+L) pays " $u_{I-L} - Q_{I-L}$ "; if CI loses, CI pays " $u_I - Q_I$ "; if there is a tie, CI is the winner and C(I+L) the loser.

One can also easily have the multi-agent counterparts of mechanisms 3, 4, 5 and 6, etc..

5. Concluding Remarks

The mechanisms of this chapter depend on the assumption that the planner knows some number M (the Q_a , Q_b , Q_I in the associated mechanisms can all be substituted by $-M$). In realistic application contexts, this assumption is easily satisfied. In King Solomon's dilemma, King Solomon's M can be each woman's life plus detaining the child for some days (note that the associated M_2 and M_3 can be obtained through adjusting the number of days of detaining the child). In the example introduced in section 1, the Chair's M can be specified as a cut-off of each professor's salary. In the context of an infinite repeated game, i.e., the planner has numerous tickets to allocate, M can be the eligibility of each candidate's applying for the unallocated tickets (note that the associated M_2 and M_3 can be specified as the eligibilities for the unallocated tickets excluding some particular tickets). It is also evident that this assumption is much weaker than assuming all individuals positively value the ticket (Glazer and Ma (1989), Moore (1991) and Perry and Reny (1994)). If it is assumed that the individuals positively value the ticket, mechanism 2 can be reduced to omitting stage 2 (set $k=0$) and setting $Q_a=Q_b=0$.

The main difference between the mechanisms with complete information and those with incomplete information lies in the bidding game. The mechanisms with complete information use sequential bidding games to deter the individual attaching lower valuation to the ticket from asking for the ticket while those with incomplete information use simultaneous bidding games instead. Except for mechanism 1 (7), the sequential bidding games of mechanisms 2 (8) and 3 can be replaced by the simultaneous ones where the winner pays either her bid or the second-priced bid if agents value the ticket differently. But the simultaneous bidding games of the mechanisms with incomplete information can not be replaced by the sequential ones of those with complete information. It may be possible to construct other simultaneous bidding games for the mechanisms with complete information. However, we consider it is hardly possible to construct sequential games for the mechanisms with incomplete information. Unsatisfactorily, the mechanisms with incomplete information do not work if some individuals equally value the ticket. The mechanism by Perry and Reny (1994) suffers from the same problem.

All mechanisms of this chapter embody a basic feature whereby the individual to be given the ticket is consistently revealed and acknowledged by her counterpart(s). That is, if the ticket should be given to some individual, then not only does the individual have to claim that, but also all other individuals must acknowledge it. This is appealing for most application contexts. We'd like to call this feature "objective consistency". Glazer and Ma's mechanism has this feature as well, but Perry and Reny's does not. How to formalize the objective consistency of a mechanism for such an arrangement problem is worthy of further research.

Mechanisms 1, 3, 4, 5 and 7 treat all individuals equally in the sense that the planner can call on any individual first. This symmetry reflects the fact that the planner knows no more information about one individual than another. Generally, if the information known to the planner is the same for all individuals, then a mechanism that implements the planner's goal should have this symmetry; otherwise, one can suspect that it can be refined to be simpler.

Mechanism 2 (6, 8), based on the planner knowing more information about one particular individual, is simpler than mechanism 1 (4 (5), 7), but it loses the symmetry. Thus asymmetric information will result in an asymmetric mechanism if the mechanism can not be further simplified. How formally to define asymmetric or symmetric information, asymmetric or symmetric mechanism, or to measure the simplicity of mechanism still calls for further research.

Further, designing more mechanisms for various incomplete information contexts is of interest, e.g., for the context where some individuals do not know their own valuations for the ticket but know those of others. One may like to do case studies of such arrangement problems to get convincing explanations of how they are solved in practice. Another avenue for further research relating to these mechanisms is to improve them so that information about individuals' valuations for the ticket can be obtained or estimated. For instance, if B gets the ticket in mechanism 2, her valuation for the ticket must be greater than $((1-k)e + \max(Q_a, Q_b))$. In certain contexts, the planner may be interested to know precisely the agents' valuations rather than just a lower bound.

Appendix Proofs of Results

This appendix provides proofs for results 1, 2 and 3. Since the proof of result 2 is same as that of result 1, it is omitted. The proof of result 3 is also similar to that of result 1. Thus, we just point out all the subgame perfect equilibria of Γ . We start by finding all the subgame perfect equilibria of Γ_1 .

Proposition 1:

If $v^* > u^*$, Γ_1 has a unique subgame perfect equilibrium $(u^{**}(v), v^{**})$, where

$$v^{**} = u^* \quad (A1)$$

$$u^{**}(v) = \begin{cases} -M & \text{if } v \geq u^* \\ v & \text{if } v < u^* \end{cases} \quad (A2)$$

If $v^* < u^*$, the mixed strategy subgame perfect equilibria of Γ_1 are characterized by:

$$\sigma_B^* = \Pr(v^{**}) = 1, v^{**} = -M, \quad (A3)$$

$$\sigma_A^* = (\Pr(u_1^{**}(v)), \Pr(u_2^{**}(v))) = (1-p, p), \forall p \in [0, 1] \quad (A4)$$

$$\text{where } u_1^{**}(v) = \begin{cases} -M & \text{if } v \geq u^* \\ v & \text{if } v < u^* \end{cases}, \quad (A5)$$

$$\text{and } u_2^{**}(v) = \begin{cases} -M & \text{if } v > u^* \\ v & \text{if } v \leq u^* \end{cases}. \quad (A6)$$

If $v^* = u^*$, the mixed strategy subgame perfect equilibria of Γ_1 are characterized by:

$$\sigma_B^* = (\Pr(v_1^{**}), \Pr(v_2^{**})) = (\theta, 1-\theta), \forall \theta \in [0, 1], \text{ where } v_1^{**} = -M, v_2^{**} = v^*; \quad (A7)$$

$$\sigma_A^* = \Pr(u_1^{**}(v)) = 1, \text{ where } u_1^{**}(v) \text{ is given by (A5)}. \quad (A8)$$

$$\text{Or: } \sigma_B^* = \Pr(v^{**}) = 1, \text{ where } v^{**} = -M; \quad (A9)$$

$$\sigma_A^* = (\Pr(u_1^{**}(v)), \Pr(u_2^{**}(v))) = (1-p, p), \forall p \in (0, 1], \quad (A10)$$

where $u_1^{**}(v)$ and $u_2^{**}(v)$ are given by (A5) and (A6) respectively.

Proof: To begin with, consider A's best reply at stage A5 in Figure 1.

If $v > u^*$, then according to (1),

when $u \geq v$, $U_i(u, v) = u^* - u \leq u^* - v < u^* - u^* = 0$;

when $u < v$,

$$U_i(u, v) = -M - u \begin{cases} < 0 & \text{if } u \neq -M \\ = 0 & \text{if } u = -M \end{cases}.$$

Thus, for $v > u^*$, A's best reply is uniquely

$$u^{**}(v) = -M. \quad (\text{A11})$$

If $v < u^*$, then for $\forall u > v$, $U_i(u, v) = u^* - u < u^* - v$,

for $u = v$, $U_i(u, v) = u^* - u = u^* - v > 0$, and

for $\forall u < v$ (note $u \geq -M$), $U_i(u, v) = -M - u \leq 0 < u^* - v$.

Thus for $v < u^*$, A's best reply is uniquely

$$u^{**}(v) = v. \quad (\text{A12})$$

If $v = u^*$, then for $\forall u > v$, $U_i(u, v) = u^* - u < u^* - v = u^* - u^* = 0$,

for $-M < \forall u < v$, $U_i(u, v) = -M - u < 0$,

for $u = -M$ or $u = v (= u^*)$, $U_i(u, v) = 0$.

Thus for $v = u^*$, A's best reply can be

$$u^{**}(v) = -M \text{ or } u^{**}(v) = u^*. \quad (\text{A13})$$

Combining (A11), (A12) and (A13), we get that A's best reply at stage A5 is (A5) and (A6).

Now consider B's best choice at stage A4.

Consider A to play mixed strategy (A4) at stage A5 and B to bid v at stage A4, then

$$V_i(\sigma_A^*, v) = (1-p)V_i(u_1^{**}(v), v) + pV_i(u_2^{**}(v), v).$$

If $v > u^*$, $u_1^{**}(v) = u_2^{**}(v) = -M$,

then $V_i(\sigma_A^*, v) = (1-p)V_i(u_1^{**}(v), v) + pV_i(u_2^{**}(v), v)$

$$= (1-p)(v^* - v) + p(v^* - v)$$

$$= v^* - v < v^* - u^*.$$

(A14)

If $v < u^*$, then $u_1^{**}(v) = u_2^{**}(v) = v < u^*$,

then $V_i(\sigma_A^*, v) = (1-p)V_i(u_1^{**}(v), v) + pV_i(u_2^{**}(v), v)$

$$= -M - v \leq 0 \quad (A15)$$

If $v = u^*$, then $u_1^{**}(v) = -M$ and $u_2^{**}(v) = v = u^*$,

$$\begin{aligned} \text{then } V_1(\sigma_A^*, v) &= (1-p)V_1(u_1^{**}(v), v) + pV_1(u_2^{**}(v), v) \\ &= (1-p)(v^* - u^*) + p(-M - u^*) . \end{aligned} \quad (A16)$$

If $v^* > u^*$, then given (A14), $\lim_{v \rightarrow u^*} V_1(\sigma_A^*, v) = v^* - u^*$.

If $p \neq 0$, then given (A16) and the assumption that $u^* > -M$,

$$V_1(\sigma_A^*, v^*) = (1-p)(v^* - u^*) + p(-M - u^*) < v^* - u^*,$$

which means there can be no such subgame perfect equilibrium with $p \neq 0$.

If $p = 0$, then given (A14), (A15) and (A16), B's unique best choice is $v^{**} = u^*$. This establishes (A1) and (A2).

If $v^* < u^*$, in (A16), $V_1(\sigma_A^*, v^*) < 0$, for $\forall p \in [0, 1]$;

in (A15), only $v = -M$ makes $V_1(\sigma_A^*, v) = 0$;

in (A14), $V_1(\sigma_A^*, v) = v^* - v < v^* - u^* < 0$.

Thus, (A3), (A4), (A5) and (A6) follow from them.

If $v^* = u^*$ and $p = 0$, that is, A plays $u_1^{**}(v)$,

in (A16), $V_1(\sigma_A^*, v^*) = 0$;

in (A15), only $v = -M$ makes $V_1(\sigma_A^*, v) = 0$, and otherwise $V_1(\sigma_A^*, v) < 0$;

in (A14), $V_1(\sigma_A^*, v) = v^* - v < v^* - u^* = 0$.

Thus, B has two best pure strategies, $v_1^{**} = -M$ and $v_2^{**} = u^*$. B may then play a mixed strategy:

$$\sigma_B^* = (\Pr(v_1^{**}), \Pr(v_2^{**})) = (\theta, 1 - \theta), \forall \theta \in [0, 1]. \quad (A17)$$

Owing to the fact that $V_1(\sigma_A^*, v_1^{**}) = V_1(\sigma_A^*, v_2^{**}) = 0$, (A17) are B's best mixed strategies.

Thus (A7) and (A8) follow.

If $v^* = u^*$ and $p \neq 0$,

in (A16), $V_1(\sigma_A^*, v^*) = p(-M - u^*) < 0$;

in (A15), only $v=-M$ makes $V_i(\sigma_A^*, v)=0$, and otherwise $V_i(\sigma_A^*, v)<0$;

in (A14), $V_i(\sigma_A^*, v)=v^*-v<v^*-u^*=0$.

Therefore B has a unique best choice $v^{**}=-M$, which establishes (A9) and (A10).

Q.E.D.

Proof of Result 1:

Suppose $v^*>u^*$. Substituting (A1) and (A2) into (1) and (2) yields:

$$U_i(\sigma_A^*, \sigma_B^*)=U_i(u^{**}(v^{**}), v^{**})=U_i(u^{**}(u^*), u^*)=U_i(-M, u^*)=0.$$

Similarly, $V_i(\sigma_A^*, \sigma_B^*)=V_i(-M, u^*)=v^*-u^*$. Thus (6) holds.

Suppose $v^*<u^*$. Substituting (A3) and (A4) into (1) and (2) yields:

$$\begin{aligned} U_i(\sigma_A^*, \sigma_B^*) &= (1-p)U_i(u_1^{**}(-M), -M) + pU_i(u_2^{**}(-M), -M) \\ &= (1-p)U_i(-M, -M) + pU_i(-M, -M) \\ &= u^* + M; \end{aligned}$$

$$V_i(\sigma_A^*, \sigma_B^*) = (1-p)V_i(-M, -M) + pV_i(-M, -M) = 0.$$

Thus (5) holds.

Suppose $v^*=u^*$. Substituting (A7) and (A8) into (1) and (2) yields:

$$\begin{aligned} U_i(\sigma_A^*, \sigma_B^*) &= \theta U_i(u_1^{**}(v_1^{**}), v_1^{**}) + (1-\theta)U_i(u_1^{**}(v_2^{**}), v_2^{**}) \\ &= \theta U_i(u_1^{**}(-M), -M) + (1-\theta)U_i(u_1^{**}(v^*), v^*) \\ &= \theta U_i(-M, -M) + (1-\theta)U_i(-M, v^*) \\ &= \theta(u^* + M); \end{aligned}$$

$$V_i(\sigma_A^*, \sigma_B^*) = \theta V_i(-M, -M) + (1-\theta)V_i(-M, v^*) = 0.$$

Thus (7) holds.

Substituting (A9) and (A10) into (1) and (2) yields:

$$\begin{aligned} U_i(\sigma_A^*, \sigma_B^*) &= (1-p)U_i(u_1^{**}(-M), -M) + pU_i(u_2^{**}(-M), -M) \\ &= (1-p)U_i(-M, -M) + pU_i(-M, -M) \\ &= u^* + M; \end{aligned}$$

$$V_i(\sigma_A^*, \sigma_B^*) = (1-p)V_i(-M, -M) + pV_i(-M, -M) = 0.$$

Thus (8) holds.

Q.E.D.

The subgame perfect equilibria of the bidding game Γ are characterized by proposition 2.

Proposition 2:

If $u^* > v^*$, Γ has a unique subgame perfect equilibrium:

$$u^{**} = \max(v^*, Qa), \quad (A18)$$

$$v_1^{**}(u) = \begin{cases} Qb & \text{if } u \geq v^* \\ \max(u, Qb) & \text{if } u < v^* \end{cases} \quad (A19)$$

If $u^* < v^*$, the mixed strategy subgame perfect equilibria of Γ are:

$$\sigma_A^* = \Pr(u^{**}) = 1, u^{**} = Qa; \quad (A20)$$

$$\sigma_B^* = (\Pr(v_1^{**}(u)), \Pr(v_2^{**}(u))) = (1-p, p), \forall p \in [0, 1], \quad (A21)$$

$$\text{where } v_1^{**}(u) = \begin{cases} Qb & \text{if } u \geq v^* \\ \max(u, Qb) & \text{if } u < v^* \end{cases}, \quad (A22)$$

$$\text{and } v_2^{**}(u) = \begin{cases} Qb & \text{if } u > v^* \\ \max(u, Qb) & \text{if } u \leq v^* \end{cases}. \quad (A23)$$

If $u^* = v^*$, the mixed strategy subgame perfect equilibria of Γ are:

$$\sigma_A^* = (\Pr(u_1^{**}), \Pr(u_2^{**})) = (\theta, 1-\theta), \forall \theta \in [0, 1], \text{ where } u_1^{**} = Qa, u_2^{**} = u^*;$$

$$\sigma_B^* = \Pr(v_1^{**}(u)) = 1, \text{ where } v_1^{**}(u) \text{ is given by (A22).}$$

Or: $\sigma_A^* = \Pr(u^{**}) = 1, \text{ where } u^{**} = Qa;$

$\sigma_B^* = (\Pr(v_1^{**}(u)), \Pr(v_2^{**}(u))) = (1-p, p), \forall p \in (0, 1], \text{ where } v_1^{**}(u) \text{ and } v_2^{**}(u) \text{ are given by (A22) and (A23) respectively.}$

References

- Abreu, D. and A. Sen (1990), Subgame Perfect Implementation: A Necessary and Almost Sufficient Condition, *Journal of Economic Theory*, 50, 285-199.
- Abreu, D. and A. Sen (1991), Virtual Implementation in Nash Equilibrium, *Econometrica*, 59, 997-1021.
- Glazer, J. and C-T. A. Ma (1989), Efficient Allocation of a "prize"---- King Solomon's Dilemma, *Game and Economic Behaviour*, V1, 222-233.
- Moore, J. (1991), Implementation in Environments with Complete Information. STICERD working paper TE/91/235, London School of Economics and Political Science.
- Perry Motty and Philip J. Reny (1994), A General Solution to King Solomon's Dilemma, paper for the conference "5th Annual Summer School in Economic Theory", Department of Economics and The Centre for Rationality and Interactive Decision Theory, Hebrew University of Jerusalem, 91904, Jerusalem, Israel.

Chapter 2 Vertical Contract for Motion Pictures

1. Introduction

This chapter will provide a theoretical study of vertical contract for motion pictures. Once a motion picture is produced, it is distributed to cinemas to “exhibit” it to an audience. The following are the important characteristics of the relationship between a distributor and a cinema: each motion picture is unique; producers generally sell or license exclusive rights to their productions by territory; and/or a motion picture is largely financed by a distributor; the distributor chooses a release pattern---the number and location of cinemas in which the motion picture is “booked” or licensed to play.

There is much variation in contracting. A typical contract is described by De Vany and Walls (1996) as having the following features. The cinema is usually required to exhibit it for a minimum number of weeks. For instance, 4 weeks is a common minimum, while 6 or 8 weeks minimum may be applied to some motion pictures. The cinema pays to the distributor a weekly rental for the privilege of exhibiting the motion picture. The rental usually is some percentage of the revenue from the admission tickets. The typical arrangement may call for the cinema to pay 90% of the revenue in excess of house expenses (the “house nut”) negotiated as a part of the contract. The cinema must pay as a minimum a stipulated percentage of total ticket revenue (without the deduction of the house nut). This minimum rental rate generally declines over the length of the run. The lower the rental rate the closer to the end of the run. For example, it is typical in North America to set minimum percentages of 70, 70, 60 and 60 in the first, second, third and fourth week, respectively of a four week run. A contract usually contains a “hold-over” clause that requires the cinema to continue exhibiting the motion picture some more weeks if the ticket revenue in the previous weeks exceeds a stipulated amount. Most contracts for a top-of-the-line picture also involve a guarantee, which represents a total amount that must be paid to the distributor for booking the film. The guarantee must be paid before the distributor ships the motion picture to the

cinema. The distributor will not receive additional payment from the cinema until the earnings on the percentage basis exceed the guarantee. Rental in excess of the guarantee is referred to as "overage" and is payable on a weekly basis. In some circumstances, the distributor calls for cinemas to bid for a contract, with their bids to be their offers of a guarantee, percentages and a minimum number of weeks, etc.. In Canada, distributors depend more on a process of negotiation with cinemas with which they have done business in the past. In 1984, an agreement was reached between the industry and the Director of Investigation and Research to make the lettering of contracts more competitive, i.e., more like an auction.¹ The cinema bidding the most preferred conditions by the distributor will obtain the contract. In its request for bids, a distributor usually specifies a minimum number of weeks for the run and a minimum percentage for each week, and perhaps a minimum guarantee.² In the contract, the distributor protects the cinema from contemporaneous and later exhibiting by "clearing" an area and time period in which it agrees not to license the same motion picture to a rival cinema. Therefore, a motion picture is usually played at one or more cinemas in a town in sequence. In addition, the distributor may set a minimum ticket price on the contract and leaves it possible for the cinema to upward adjust ticket pricing. Besides exhibiting the movie, the cinema may provide other services, e.g., selling snack foods. The availability of food and the film are related each other in attracting movie-goers so that each affects the revenues from admissions and from the other service.³ The contract, however, generally includes no clause on the cinema's other service. See De Vany and Eckert (1991), De Vany and Walls (1996), Fellman (1983) and May (1983) for more detailed

¹
See Annual Report, Director of Investigation and Research, Combines Investigation Act, Ottawa: Consumer and Corporate Affairs Canada, 1984.

²
Some examples of solicitation of bids are described in Fellman (1983).

³
Conant (1981), p564: "...refreshment profits are a major part of net earnings and theatres would close without such sales."

description about contracts.

In the 1970s, as the consequences of some cinemas' complaints, a number of Canadian competition policy actions were taken against distributors including a minimum ticket price restriction on their rental contracts with cinemas.⁴ Similar policy measures have been historically introduced in the United States.⁵ However, except for De Vany and Eckert (1991) who give some arguments for a distributor to place vertical constraint on ticket pricing, few theoretical assessments about such policy actions have been made.

In this chapter, we provide a formal analysis of the vertical constraint on ticket pricing. The analysis will be built up in a two-stage game setting participated in by one distributor and multiple cinemas and on the assumption of linear demands. For simplicity, the contract

4

In 1977, Columbia Pictures Industries, Inc. was considered guilty in not allowing a theatre owner to reduce entrance prices for a certain film and imposed a fine of \$1,250, which was raised to \$5,000 in 1978, and granted an Order prohibiting it and its directors, officers, servants and agents from doing any act directed towards the repetition of the offence. In 1979, United Artists Corporation was charged with unlawfully attempting to discourage the reduction of the price of theatre admissions with respect to three films, *One Flew Over the Cuckoo's Nest*, *Missouri Breaks*, and *A Bridge Too Far* and granted an Order prohibiting it from attempting to influence upward or discourage the reduction of the price at which motion picture exhibitors charge admission to their theatres by the Federal Court of Canada. In the same year, Bellevue Film Distributors Limited was charged with attempting to influence upward the price of admission for children to three drive-ins operated by Canadian Odeon Theatres Limited in Welland and St. Catharines, Ontario and granted by the Federal Court of Canada an Order prohibiting it from attempting to influence upward, or discourage the reduction of , the price of admission for children to films provided by it. For more information, see Annual Report, Director of Investigation and Research, Combines Investigation Act, Ottawa: Consumer and Corporate Affairs Canada, 1980 or later.

5

See, for example, Conant (1969, 1981).

modeled is assumed to involve a length of the run fixed by the distributor, a same percentage for the entire run and no “hand-over” clause; in addition, it is assumed that there exists no cost either to the distributor or to the cinema, and thus no “house nut” on the contract.

In the setting, the first stage is a bidding game participated in by the cinemas for the distributor to select one of them to contract a movie. That only one cinema will contract the movie embodies the implication of the “clearing” clause observed on realistic contracts. It is assumed that each cinema’s bid is its offer of a guarantee but does not involve an offer of a percentage for sharing the revenue from admissions. We will see later that it makes no sense in our model for the distributor to ask each cinema to bid an offer of a percentage in addition to an offer of a guarantee. The cinema bidding the highest guarantee will be selected by the distributor to have the contract. The second stage comprises sequential games, each of which embodies the vertical contract with the distributor as the upstream entity and the selected cinema as the downstream entity. Besides exhibiting the movie, the cinema will sell snack foods on its premises. The distributor sets a percentage for sharing the revenue from admissions and if permitted by law, perhaps a (minimum) ticket price.⁶ The cinema sets the price of its snack foods and, in the circumstance that the distributor does not limit ticket pricing, the price of tickets. Because of the clearance, only one cinema will exhibit the movie and a monopoly price of tickets will be set either by the distributor or by the cinema. It is assumed that the cinema forbids movie-goers from taking snack foods into it. This assumption is consistent with industry practice. Thus the price of the cinema’s snack foods is also set to maximize profit subject to the demand of those attending the movie. In accordance with industry practice, the revenue from admission tickets is shared by the distributor and the cinema according to the percentage set by the distributor. The revenue from snack foods accrues solely to the cinema. This is a significant source of profit, as Conant (1981) notes. Regarding the information structure, it is assumed that the cinemas

6

We will see later that a restriction of minimum ticket price set by the distributor is equivalent to fixing ticket pricing.

have full information in both stages; and the distributor lacks information as to whether all the cinemas would like to take its contract in the first stage. Once one of the cinemas is selected to take the contract in the first stage, the distributor has the same complete information as the cinema in the second stage. This structure of information makes it possible for the setting to be handled as one of complete information. It also explains why the distributor wants to select the cinema to take the contract through bidding.

With regard to demand, it is assumed that a fixed population of potential movie-goers exist for the whole run, each of which needs a ticket of admission to see the movie. It is assumed that no one would watch the movie twice or more. We also assume that each of those entering the cinema to view the movie may also demand one unit of snack foods from the cinema. The population are assumed to have “linear” demands for admission and for snack foods in the sense that there exist linear orders for describing their reservation prices for the movie and for describing those for the cinema’s snack foods.

With the linear demands, this setting allows us to analyze the vertical constraint on ticket pricing through comparing the subgame perfect equilibria of the game with and without the constraint. Under the “nonnegative assumption” that the potential movie-goers have a nonnegative relationship between their reservation prices for admission and those for a unit of snack foods, we demonstrate that only under certain competitive environments will the equilibrium contract derived from the model generate a conflict about ticket pricing between the distributor and the cinema. This is consistent with the empirical observation that only some of the industry contracts resulted in lawsuits. In the situation where such a conflict occurs, the policy action forbidding the distributor to limit ticket pricing will improve the total surplus of movie-goers and social welfare. If movie-goers’ reservation prices for snack foods are not very small relative to those for admission, the policy action will also improve the total profit of the industry. The model also explains why the distributor may want to control ticket pricing but only to set a price floor and why the cinema opposes that restriction. Furthermore, the model concludes that, if placing the vertical constraint on ticket

pricing is legal, a policy of preventing any integration between distributors and cinemas will generally worsen consumer surplus and social welfare.⁷ That is, forbidding distributors to limit ticket pricing is necessary for such a policy to be welfare-improving. The model may not support these conclusions if the “nonnegative assumption” is not satisfied, i.e., with a negative correlation, it is possible, but not necessarily true, that the policy action reduces social welfare.

Following this section, we specify our model in section 2 and investigate its outcome in section 3. The implications of the policy action are analyzed in section 4. We demonstrate the significance of the policy action on the regulations that bar integrations between distributors and cinemas in section 5. In section 6, an example of the model without the “nonnegative assumption” is examined. Some concluding remarks are given in section 7. There are six appendices to collect mathematical details.

2. The Model

Consider a setting where there are one distributor and n cinemas, which are named cinema 1, cinema 2, ..., cinema n respectively. The distributor wants to select one of them to contract a movie.

Let θ be a taste parameter for each individual of a fixed population of potential movie-goers. θ is assumed to be uniformly distributed on the interval $[0, 1]$ and is independent of the cinema exhibiting the movie. The parameter θ is used to identify each individual. Each individual needs one ticket for admission to the movie. No one wants to watch the movie more than once. The person θ gains utility measured in money of $a_i\theta$ from watching the movie within cinema i , where a_i is a positive constant, i.e., $a_i > 0$, $i = 1, 2, \dots, n$. That is, the people are distributed on the interval $[0, 1]$ according to the ascending order of their

⁷

Such regulations have been historically introduced in the United States. See De Vany and Eckert (1991).

reservation prices for the movie.

Besides exhibiting movies, the cinemas sell snack foods to those who attend a movie. Movie-goers are prohibited from taking snack foods into the cinemas, but they may buy snack foods within the cinemas. This means each of the cinemas can charge a monopoly price for snack foods to movie-goers. Each movie-goer has a demand price (a reservation price) for one unit of snack foods. The person θ gains utility measured in money of $c_i\theta + d_i$ if he/she takes snack foods within cinema i when cinema i is exhibiting the movie, where c_i and d_i are constants and d_i is nonnegative, i.e. $d_i \geq 0$, and $i=1, 2, \dots$ and n . Without loss of generality, we also assume $c_i^2 + d_i^2 \neq 0$.

Thus, the vector (a_i, c_i, d_i) characterizes cinema i when it takes the contract. If (a_i, c_i, d_i) is the same over all i , it means that each of the cinemas can give identical satisfactions to the population with the movie and its snack foods, i.e., the competition for the contract between the cinemas is very strong; it even approximates the situation of complete competition if n is very large. If (a_i, c_i, d_i) varies over i , it implies incomplete competition for the contract between the cinemas.

Denote the price of a ticket for admission as F and the price of per unit of snack foods as P . $F \geq 0$ and $P \geq 0$. Note that we need not distinguish the prices of admission and of snack foods between the cinemas, for only one cinema will exhibit the movie and any movie-goer is not allowed to bring snack foods into the cinema.

As usual, assume all of the people are utility-maximizers. Each of them will consider both the value of watching the movie and the option to consume snack foods within the cinema in deciding whether to buy a ticket or not. It is assumed that they have complete information related to their decisions, i.e., the full information of F and P . A person must first buy a ticket and then perhaps buy snack foods. This is because P , being a monopoly price charged by the cinema, will be generally higher and never lower than that outside the cinema, which is

assumed to be a competitive price, so that no one will just buy snack foods within the cinema but not watch the movie. Therefore, suppose that cinema i has contracted for the movie, for a particular F and P , the person θ such that $\theta \in [0, 1]$ and $a_i \theta \geq F$ will buy a ticket and will buy snack foods within the cinema if and only if $(c_i \theta + d_i - P) \geq 0$; the person θ such that $\theta \in [0, 1]$ and $a_i \theta < F$ but $(a_i \theta + c_i \theta + d_i - F - P) \geq 0$ will buy both a ticket and a unit of snack foods. It is assumed that a person will buy a ticket and/or a unit of snack foods if he or she is indifferent. These two assumptions make the demand functions for tickets and for snack foods be lower semi-continuous on P when $c_i = 0$,⁸ and do not affect the conclusions this chapter will make.

The parts of the population who buy a ticket and who buy snack foods form the total demands for tickets and for snack foods respectively. Suppose cinema i is assigned the contract. If $c_i > 0$, i.e., the population shows a nonnegative correlation between their reservation prices for admissions and those for snack foods, the total demand for tickets (D_u) is:

$$D_u = \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F + P \leq d_i; & (R_1) \\ 1 - \frac{F}{a_i} & \text{if } 0 \leq F \leq a_i, P \geq \frac{c_i}{a_i} F - d_i, P \leq c_i - d_i, P \geq 0; & (R_2) \\ 1 - \frac{F}{a_i} & \text{if } 0 \leq F \leq a_i, P \geq c_i + d_i, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a_i, F + P \geq a_i + c_i + d_i, P \geq 0; & (R_4) \\ 1 - \frac{F + P - d_i}{a_i + c_i} & \text{if } F \geq 0, P \geq 0, d_i \leq F + P \leq a_i + c_i - d_i, P \leq \frac{c_i}{a_i} F + d_i; & (R_5) \end{cases} \quad (1)$$

and the total demand for snack foods (D_{si}) is:

⁸

For instance, if $c_i = 0$, suppose $F = 0$, all the population go to the cinema and buy snack foods for $\forall P \leq d_i$ while no one will buy snack foods if $P > d_i$. That is, the demand function for snack foods is lower semi-continuous but not upper semi-continuous on P at $P = d_i$ if $c_i = 0$.

$$D_{st} = \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F+P \leq d_i; & (R_1) \\ 1 - \frac{P-d_i}{c_i} & \text{if } 0 \leq F \leq a_i, P \geq \frac{c_i}{a_i}F + d_i, P \leq c_i + d_i, P \geq 0; & (R_2) \\ 0 & \text{if } 0 \leq F \leq a_i, P \geq c_i + d_i, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a_i, F+P \geq a_i + c_i + d_i, P \geq 0; & (R_4) \\ 1 - \frac{F+P-d_i}{a_i+c_i} & \text{if } F \geq 0, P \geq 0, d_i \leq F-P \leq a_i + c_i - d_i, P \leq \frac{c_i}{a_i}F - d_i; & (R_5) \end{cases} \quad (2)$$

Figure 1 illustrates the regions R_1, R_2, R_3, R_4 and R_5 .⁹ If F and P are in region R_1 , all the people will go to the cinema and buy snack foods within the cinema. If F and P are in region R_2 , some of the people will go to the cinema and some of these movie-goers will buy snack foods. If F and P are in region R_3 , some of the people will go to the cinema but none of these movie-goers will buy snack foods. If F and P are in region R_4 , no one will go to the cinema. If F and P are in region R_5 , some of the people will go to the cinema and all of these movie-goers will buy snack foods.

Figure 1

If $c_i=0$, the total demands for tickets and for snack foods can be directly obtained from the above. With $c_i=0$, region R_2 vanishes. Exclude the second expressions in (1) and (2), plug $c_i=0$ into other expressions and modify the term " $P \geq d_i$ " (the term " $P \geq c_i + d_i$ " before plugging $c_i=0$) of the third expressions of (1) and (2) into " $P > d_i$ ". Leave (1) and (2) to be the total

9

Although Figure 1 is drawn with $a_i > d_i$, (1) and (2), as well as any result based on (1) and (2), apply to the situation where $a_i \leq d_i$.

demands respectively for tickets and for snack foods when $c_i=0$. This property allows the conclusions held for the situation where $c_i>0$ to apply to the situation where $c_i=0$.¹⁰ Alternatively, the total demands for tickets and for snack foods can be obtained through the method described in appendix A.¹¹

When $c_i<0$, which means the population shows a negative correlation between their reservation prices for admissions and those for snack foods, the total demands for tickets and for snack foods can also be easily obtained through the method described in appendix A. But, with $c_i<0$, to solve the model is a much more complicated job than that with $c_i\geq 0$.¹² Therefore, this chapter will not deal with the general case of $c_i<0$, but, in order to shed some light on the policy action with $c_i<0$, this chapter will analyze the special situation in which $c_i=-a_i$ and $a_i/4<d_i\leq a_i$.¹³ An implication of " $a_i\geq d_i$ " is that, among the people who attended the movie, the highest utility someone obtained from the movie is not lower than the highest utility some other individual obtained from snack foods.

If $c_i=-a_i$ and $a_i\geq d_i$, the total demand for tickets (D_{ti}) and that for snack foods (D_{si}) are:

¹⁰

See appendices C and D for details.

¹¹

The subscript i is suppressed in all the appendices.

¹²

The method used to solve the model with $c_i\geq 0$ does apply to the one with $c_i<0$. But with $c_i<0$, there are too many situations to be identified in order to obtain explicit solutions to the model.

¹³

That $a_i/4<d_i$ is in order to simplify solving the model. The model with $c_i=-a_i$ and $a_i\leq d_i$ can be similarly handled as that with $c_i=-a_i$ and $a_i\geq d_i$.

$$D_n = \begin{cases} (1 - \frac{F}{a_i}) & \text{if } 0 \leq F \leq a_i, F+P > d_i; P \geq 0 \quad (Q_1) \\ 1 & \text{if } 0 \leq F \leq a_i, F+P \leq d_i, P \geq 0; \quad (Q_2) \\ 0 & \text{if } F \geq a_i, P \geq 0; \quad (Q_3) \end{cases} \quad (3)$$

$$D_n = \begin{cases} 0 & \text{if } 0 \leq F \leq a_i, F+P > d_i; P \geq 0 \quad (Q_1) \\ \frac{d_i - P}{a_i} & \text{if } 0 \leq F \leq a_i, F+P \leq d_i, P \geq 0; \quad (Q_2) \\ 0 & \text{if } F \geq a_i, P \geq 0; \quad (Q_3) \end{cases} \quad (4)$$

The regions Q_1 , Q_2 and Q_3 are illustrated by Figure 2. If F and P are in region Q_2 , all of the people will go to the cinema but only some of the people buy snack foods within the cinema. If F and P are in region Q_1 , some of the people will go to the cinema but none will buy snack foods within the cinema. If F and P are in region Q_3 , no one will go to the cinema.

Figure 2

A typical contract specifies a guarantee, a minimum number of weeks for the run, a percentage for each week, a “house nut” clause and a “hand-over” clause, etc.. For simplicity, the contract modeled in this chapter is assumed to involve a length of the run fixed by the distributor, a same percentage for the entire run and no “hand-over” clause; besides, it is assumed that there exists no cost either to the distributor or to the cinema, and thus no “house nut” on the contract. Further, we assume that the distributor will select the cinema bidding the highest guarantee to have the contract, regardless of its offer of a percentage for sharing the revenue from admissions; such a percentage will be chosen by the distributor. Therefore, the model of this chapter has two stages, illustrated by panel (a) of Figure 3. In stage 1, the distributor calls for the cinemas to participate in a simultaneous

bidding game, in which the cinemas are requested to offer their guarantees. In stage 2, the distributor engages in a sequential game (vertical contract) with the cinema it just selected, where the percentage and prices of admission and of per unit of snack foods are decided and which will be detailed in the next section.

The bidding game is described by panel (b) of Figure 3 and is assumed to satisfy two assumptions. One is that the distributor will choose the cinema which bids the highest guarantee to have the contract, as stated above. The other is that, if there are more cinemas than one which bid the highest guarantee, the distributor will choose the cinema which can work out the largest joint profit or any of such cinemas if there are more than one of them capable of working out the largest joint profit. The key point here is that there will be only one cinema assigned the contract.

Figure 3

Regarding the structure of information for the model, we assume that the distributor has full information about the states of (a_i, c_i, d_i) , where $i=1, 2, \dots, n$, but has no idea whether a particular cinema will participate in the bidding game.¹⁴ This assumption makes each of the games in stage 2 one of complete information while the distributor does not have complete information in stage 1. It implies that it makes no sense for the distributor to request the cinemas to bid a percentage for sharing the revenue of admissions in addition to a guarantee: the distributor is able to set an optimal percentage for itself because it has complete information in stage 2. It also explains why the distributor wants to use the bidding game to choose the cinema that is to have the contract. In fact, if the distributor has complete information in stage 1, it can directly pick the cinema which can work out the largest joint profit to have the contract. We assume that the cinemas have full information about the states of (a_i, c_i, d_i) , where $i=1, 2, \dots, n$, and know that all the cinemas will participate in the bidding game. Thus, the bidding game is one of complete information. In particular, the cinemas know each other's bidding ranges. That is, cinema i will submit a bid from $[0, B_i]$, as presented by panel (b) of Figure 3, where $i=1, 2, \dots, n$ and B_i is the gross (joint) profit from admissions and snack foods cinema i can work out if it obtains the contract. B_i will be spelt out in the next section.

An important thing is that B_i , as well as the bidding behavior of cinema i , where $i=1, 2, \dots, n$, depends on whether the distributor wants to limit ticket pricing. We assume that the distributor will make it clear whether it wants to limit ticket pricing in its request for bids.¹⁵ It is also assumed to be common knowledge that the distributor is so "inflexible" that once it specified to limit ticket pricing in its request for bids it is committed to that policy. This

¹⁴

A cinema will not participate in the bidding game if it has a better alternative.

¹⁵

Note that the distributor is unable to specify a price floor for tickets in this model, because it does not know which of the cinemas will take the contract when it issues its request for bids.

“inflexible” assumption makes it simpler to characterize the equilibrium of the bidding game (see the theorem of the next section).

This “inflexible” assumption seems reasonable if (a_i, c_i, d_i) , where $i=1, 2, \dots, n$, are explained as common expectations about the potential movie-goers among the distributor and the cinemas. The distributor always wants to stipulate a percentage for sharing the revenue from admissions, because the earnings based on this percentage may be higher than any guarantee when the potential movie-goers deviate from these expectations, even though the percentage is derived from these expectations.

Suppose cinema i to take the contract, and we let N_i denote the guarantee offered by cinema i and K_i denote the percentage of the ticket revenue that accrues to the distributor, the profit of the distributor is $\pi^D(N)$ and that of cinema i is $\pi^{Ci}(N)$:

$$\pi^D(N) = \max \{K_i F D_{ti}, N_i\}, \quad (5)$$

$$\pi^{Ci}(N) = F D_{ti} + P D_{si} - \max \{K_i F D_{ti}, N_i\}. \quad (6)$$

Naturally, the payoffs of the other cinemas are assumed to be zero. Note that (5) embodies that the distributor will procure some “overage” only when $K_i F D_{ti} > N_i$.

As usual, it is assumed that the distributor and the cinemas want to maximize their own profits. Besides, the cinemas are assumed to be “friendly” to the distributor, i.e., when a cinema is indifferent it will choose the best alternative for the distributor among the indifference set. More specifically, the cinema taking the contract will satisfy demand for snack foods even if $P=0$; if the cinema is indifferent between two admission prices, it chooses the price which gives the distributor more profit. As one can see from appendices D and F, this “friendship” assumption makes it simpler to characterize the equilibrium contracts in some cases. On the other hand, it implies that the cinema will choose the highest admission price from its indifference set. So this “friendship” assumption will highlight the

effect of the policy action on making the price of admission generally lower without than with the vertical constraint on price. From a market perspective, this assumption reflects the competition between the cinemas for contracts. Similarly, if the cinema taking the contract wants to set $F=0$, which may imply no difference for the distributor to have such a contract or not, the distributor is assumed still to consent to the contract with the cinema.¹⁶

This chapter examines the impact on profits, consumer surplus and social welfare of the policy action. In the next section we investigate which cinema will obtain the contract and what is the equilibrium contract.

3. The Outcome of the Model

In panel (a) of Figure 3, each game of stage 2 embodies the vertical contract between the distributor and the cinema. In the context of such a vertical contract, the cinema, as the downstream entity, makes its optimal choices of its strategy variables while taking the upstream entity's strategies as given; the distributor, as the upstream entity, will oversee the optimal strategies of the cinema and then make its optimal strategies based on the cinema's optimal strategies. More concretely, suppose that cinema i takes the contract, if limiting ticket pricing is legal, the distributor has K_i and F as strategy variables¹⁷ while cinema i has strategy variable P ; if limiting ticket pricing is illegal, the distributor has K_i as strategy variable while cinema i has strategy variables P and F . Panels (a) and (c) of Figure 4 below illustrate the settings with and without the vertical constraints respectively. We will also investigate the situation where the distributor merges with a cinema, so Figure 4 includes panel (b).

¹⁶

Such a contract as $P=0$, or $F=0$, or there existing multiple P and F for the setter to choose without difference in the setter's profit implies a bargaining problem. This chapter does not address any bargaining issue on contracts.

¹⁷

The explanation why the distributor only sets a lower-bound for ticket pricing will be given later.

Figure 4

A contract is called an equilibrium contract, if all the strategy variables are optimally chosen by each entity. With the model of this chapter, a (vertical) equilibrium contract is a subgame perfect equilibrium of a game of stage 2.¹⁸

The equilibrium contracts governed by the objective functions of the distributor and cinema i described by (7) and (8) assume that cinema i takes the contracts. These contracts have particular importance to the purpose of this chapter. They are equivalent to those that do not involve a guarantee. Results 1 and 2 below describe them respectively corresponding to whether the distributor is permitted to vertically limit ticket pricing or not.

$$\pi^D = K_i F D_{ti}, \quad (7)$$

$$\pi^{Ci} = (1 - K_i) F D_{ti} + P D_{si}. \quad (8)$$

¹⁸

This chapter does not consider any mixed strategy equilibrium.

Result 1: With (7) and (8) as the objective functions of the distributor and cinema i , if $a_i > 0$, $c_i \geq 0$ and $d_i \geq 0$, and if the distributor is permitted to limit ticket pricing, the equilibrium contract between the distributor and cinema i as well as the associated surplus of movie-goers (S) are characterized by:¹⁹

$$(1) \text{ If } a_i + c_i \geq d_i, \quad d_i \leq \frac{a_i}{2} \quad \text{and} \quad \frac{a_i}{4} \geq \frac{(a_i + c_i - d_i)^2}{8(a_i + c_i)},$$

$$K_i^* = 1, \quad F^* = \frac{a_i}{2}, \quad P^* = \frac{c_i}{2} + d_i, \\ \pi^{C_i}_1 \equiv \pi^{C_i} = \frac{c_i}{4} + \frac{d_i}{2}, \quad \pi^{D_i}_1 \equiv \pi^D = \frac{a_i}{4}, \quad S = \frac{a_i + c_i}{8};$$

$$(2) \text{ If } a_i + c_i \geq d_i, \quad d_i \leq \frac{a_i}{2} \quad \text{and} \quad \frac{a_i}{4} \leq \frac{(a_i + c_i + d_i)^2}{8(a_i + c_i)},$$

$$\forall K_i^* \in [\frac{2}{3}, \min(1, K_{is})], \quad F^* = \frac{a_i + c_i + d_i}{2K_i^*}, \quad P^* = \frac{a_i + c_i + d_i}{2} [\frac{3}{2} - \frac{1}{K_i^*}], \\ \pi^{C_i}_1 \equiv \pi^{C_i} = \frac{(a_i + c_i + d_i)^2}{16(a_i + c_i)}, \quad \pi^{D_i}_1 \equiv \pi^D = \frac{(a_i + c_i + d_i)^2}{8(a_i + c_i)}, \quad S = \frac{(a_i + c_i + d_i)^2}{32(a_i + c_i)};$$

$$(3) \text{ If } a_i - c_i \geq d_i \quad \text{and} \quad d_i \geq \frac{a_i}{2},$$

$$\forall K_i^* \in [\frac{2}{3}, \min(1, K_{is})], \quad F^* = \frac{a_i + c_i + d_i}{2K_i^*}, \quad P^* = \frac{a_i + c_i + d_i}{2} [\frac{3}{2} - \frac{1}{K_i^*}], \\ \pi^{C_i}_1 \equiv \pi^{C_i} = \frac{(a_i + c_i + d_i)^2}{16(a_i + c_i)}, \quad \pi^{D_i}_1 \equiv \pi^D = \frac{(a_i + c_i + d_i)^2}{8(a_i + c_i)}, \quad S = \frac{(a_i + c_i + d_i)^2}{32(a_i + c_i)};$$

$$(4) \text{ If } a_i + c_i \leq d_i \quad \text{and} \quad d_i \leq 3(a_i + c_i),$$

¹⁹

See appendix C for proof.

$$\forall K_i^* \in [\frac{2}{3}, \min(1, K_{is})], \quad F^* = \frac{a_i + c_i + d_i}{2K_i^*}, \quad P^* = \frac{a_i + c_i + d_i}{2} [\frac{3}{2} - \frac{1}{K_i^*}],$$

$$\pi^{C_i}_1 \equiv \pi^{C_i} = \frac{(a_i + c_i + d_i)^2}{16(a_i + c_i)}, \quad \pi^{D_i}_1 \equiv \pi^D = \frac{(a_i + c_i + d_i)^2}{8(a_i + c_i)}, \quad S = \frac{(a_i + c_i + d_i)^2}{32(a_i + c_i)};$$

(5) If $d_i \geq 3(a_i + c_i)$,

$$\forall K_i^* \in [\frac{d_i - a_i - c_i}{d_i}, 1], \quad F^* = \frac{d_i - a_i - c_i}{K_i^*}, \quad P^* = d_i - F^*,$$

$$\pi^{C_i}_1 \equiv \pi^{C_i} = a_i + c_i, \quad \pi^{D_i}_1 \equiv \pi^D = d_i - a_i - c_i, \quad S = \frac{a_i + c_i}{2};$$

In the above, $K_{is} = \frac{2(a_i + c_i)(a_i + c_i + d_i)}{a_i[3(a_i + c_i) - d_i]}.$

Note that result 1 may give multiple equilibria for a set of values of a_i , c_i and d_i . In particular,

if $a_i + c_i \geq d_i$, $d_i \leq \frac{a_i}{2}$ and $\frac{a_i}{4} = \frac{(a_i + c_i + d_i)^2}{8(a_i + c_i)}$, the multiple equilibria may give different total

surplus of movie-goers and different profits of the cinema. This is because the distributor chooses the price for tickets only based on maximizing its own profit. It may choose a higher or a lower price for tickets but obtain the same profits due to a higher price resulting in fewer admissions, and a lower price in more, and/or the share percentage of ticket revenue being larger when the price is higher and smaller when it is lower. As a result of these interactions, it is not surprising that multiple equilibrium contracts may arise for a given set of values a_i , c_i and d_i .

Result 2: With (7) and (8) as the objective functions of the distributor and cinema i , if $a_i > 0$, $c_i \geq 0$ and $d_i \geq 0$, and if the distributor is not permitted to limit ticket pricing, the equilibrium contract between the distributor and cinema i as well as the associated surplus of movie-

goers (S) are characterized by:²⁰

(1) If $c_i \geq d_i$ and $c_i^3 - c_i d_i^2 - 2a_i d_i^2 \geq 0$;

$$K_i^* = 1, \quad F^* = \frac{a_i(c_i - d_i)}{2c_i}, \quad P^* = \frac{c_i + d_i}{2},$$

$$\pi^{C_i}_2 \equiv \pi^{C_i} = \frac{(c_i - d_i)^2}{4c_i}, \quad \pi^{D_i}_2 \equiv \pi^D = \frac{a_i(c_i^2 - d_i^2)}{4c_i^2}, \quad S = \frac{(a_i + c_i)(c_i + d_i)^2}{8c_i^2};$$

(2) If $c_i \geq d_i$ and $c_i^3 - c_i d_i^2 - 2a_i d_i^2 \leq 0$;

$$K_i^* \text{ is the unique root to } Z(K) = 0 \text{ with } K \in [0, 1],$$

$$F^* = \frac{[(1 - K_i^*)a_i + c_i - d_i]a_i}{2[(1 - K_i^*)a_i + c_i]}, \quad P^* = \frac{[(1 - K_i^*)a_i + c_i - d_i]c_i}{2[(1 - K_i^*)a_i + c_i]} + d_i,$$

$$\pi^{C_i}_2 \equiv \pi^{C_i} = \frac{[(1 - K_i^*)a_i + c_i + d_i]^2}{4[(1 - K_i^*)a_i + c_i]}, \quad \pi^{D_i}_2 \equiv \pi^D = K_i^* a_i \frac{[(1 - K_i^*)a_i + c_i]^2 - d_i^2}{4[(1 - K_i^*)a_i + c_i]^2},$$

$$S = \frac{(a_i + c_i)[(1 - K_i^*)a_i + c_i + d_i]^2}{8[(1 - K_i^*)a_i + c_i]^2};$$

(3) If $c_i \leq d_i$ and $a_i + c_i \geq d_i$;

$$K_i^* \text{ is the unique root to } Z(K) = 0 \text{ with } K \in [0, 1],$$

$$F^* = \frac{[(1 - K_i^*)a_i + c_i - d_i]a_i}{2[(1 - K_i^*)a_i + c_i]}, \quad P^* = \frac{[(1 - K_i^*)a_i + c_i - d_i]c_i}{2[(1 - K_i^*)a_i + c_i]} + d_i,$$

$$\pi^{C_i}_2 \equiv \pi^{C_i} = \frac{[(1 - K_i^*)a_i + c_i + d_i]^2}{4[(1 - K_i^*)a_i + c_i]}, \quad \pi^{D_i}_2 \equiv \pi^D = K_i^* a_i \frac{[(1 - K_i^*)a_i + c_i]^2 - d_i^2}{4[(1 - K_i^*)a_i + c_i]^2},$$

$$S = \frac{(a_i + c_i)[(1 - K_i^*)a_i + c_i + d_i]^2}{8[(1 - K_i^*)a_i + c_i]^2};$$

(4) If $c_i \leq d_i$ and $a_i + c_i \leq d_i$;

²⁰

See appendix D for proof.

$$\forall K_i^* \in [0, 1], \quad F^* = 0, \quad P^* = d_i, \quad \pi^{C_i}_2 \equiv \pi^{C_i} = d_i, \quad \pi^{D_i}_2 \equiv \pi^D = 0, \quad S = \frac{a_i + c_i}{2};$$

In the above, $Z(K) \equiv [(1-K)a_i + c_i]^3 - [(1-K)a_i + c_i]d_i^2 - 2a_id_i^2K$.

Results 1 and 2 are summarized in Table 1 and Table 2 respectively, with the subscript i being suppressed.

Table 1 Summarization of Result 1

$a > 0, c \geq 0, d \geq 0$	
$a + c \geq d$	
$d \leq a/2$	
$\frac{a}{4} \geq \frac{(a+c+d)^2}{8(a+c)}$	$\frac{a}{4} \leq \frac{(a+c+d)^2}{8(a+c)}$
$K^* = 1,$ $F^* = \frac{a}{2},$ $P^* = \frac{c}{2} + d,$ $\pi^C = \frac{c}{4} + \frac{d}{2},$ $\pi^D = \frac{a}{4},$ $S = \frac{a+c}{8};$	$\forall K^* \in [\frac{2}{3}, \min(1, K_i)],$ $F^* = \frac{a+c+d}{2K^*},$ $P^* = \frac{a+c+d}{2} [\frac{3}{2} - \frac{1}{K^*}],$ $\pi^C = \frac{(a+c+d)^2}{16(a+c)},$ $\pi^D = \frac{(a+c+d)^2}{8(a+c)},$ $S = \frac{(a+c+d)^2}{32(a+c)};$

Table 1 Summarization of Result 1 (continued)

$a > 0, c \geq 0, d \geq 0$		
$a + c \geq d$	$a - c \leq d$	
$d \geq a/2$	$d \leq 3(a+c)$	$d \geq 3(a+c)$
$\forall K^* \in [\frac{2}{3}, \min(1, K_r)],$ $F^* = \frac{a-c+d}{2K^*},$ $P^* = \frac{a-c+d}{2} [\frac{3}{2} - \frac{1}{K^*}],$ $\pi^C = \frac{(a+c+d)^2}{16(a+c)},$ $\pi^D = \frac{(a+c+d)^2}{8(a+c)},$ $S = \frac{(a+c+d)^2}{32(a+c)};$	$\forall K^* \in [\frac{2}{3}, \min(1, K_r)],$ $F^* = \frac{a+c+d}{2K^*},$ $P^* = \frac{a-c+d}{2} [\frac{3}{2} - \frac{1}{K^*}],$ $\pi^C = \frac{(a+c+d)^2}{16(a+c)},$ $\pi^D = \frac{(a+c+d)^2}{8(a+c)},$ $S = \frac{(a+c+d)^2}{32(a+c)};$	$\forall K^* \in [\frac{d-a-c}{d}, 1],$ $F^* = \frac{d-a-c}{K^*},$ $P^* = d - F^*,$ $\pi^C = a+c,$ $\pi^D = d-a-c,$ $S = \frac{a+c}{2};$

Table 2 Summarization of Result 2

$a > 0, c \geq 0, d \geq 0$			
$c \geq d$		$c \leq d$	
$c^3 - cd^2 - 2ad^2 \geq 0$	$c^3 - cd^2 - 2ad^2 \leq 0$	$d \leq a+c$	$d \geq a+c$
$K^* = 1,$ $F^* = \frac{a(c-d)}{2c},$ $P^* = \frac{c-d}{2},$ $\pi^C = \frac{(c-d)^2}{4c},$ $\pi^D = \frac{a(c^2-d^2)}{4c^2},$ $S = \frac{(a+c)(c+d)^2}{8c^2};$	K^* is the unique root to $Z(K) = 0$ with $K \in [0, 1],$ $F^* = \frac{[(1-K^*)a+c-d]a}{2[(1-K^*)a+c]},$ $P^* = \frac{[(1-K^*)a+c-d]c}{2[(1-K^*)a+c]} - d,$ $\pi^C = \frac{[(1-K^*)a+c-d]^2}{4[(1-K^*)a+c]},$ $\pi^D = K^* a \frac{[(1-K^*)a+c]^2 - d^2}{4[(1-K^*)a+c]^2},$ $S = \frac{(a+c)[(1-K^*)a+c-d]^2}{8[(1-K^*)a+c]^2};$	K^* is the unique root to $Z(K) = 0$ with $K \in [0, 1],$ $F^* = \frac{[(1-K^*)a+c-d]a}{2[(1-K^*)a+c]},$ $P^* = \frac{[(1-K^*)a+c-d]c}{2[(1-K^*)a+c]} - d,$ $\pi^C = \frac{[(1-K^*)a+c-d]^2}{4[(1-K^*)a+c]},$ $\pi^D = K^* a \frac{[(1-K^*)a+c]^2 - d^2}{4[(1-K^*)a+c]^2},$ $S = \frac{(a+c)[(1-K^*)a+c-d]^2}{8[(1-K^*)a+c]^2};$	$\forall K^* \in [0, 1],$ $F^* = 0,$ $P^* = d,$ $\pi^C = d,$ $\pi^D = 0,$ $S = \frac{a-c}{2};$

One may be interested in seeing some simple examples. So we calculate the following three special cases with Table 1 and Table 2.

(1) With $c=0$, Table 1 and Table 2 are reduced to Table 1-1 and Table 2-1 respectively. This case describes the situation where each member of the population has the same reservation price for snack foods.

Table 1-1 Result 1 if $c=0$

$a>0, c=0, d\geq 0$		
$d\leq(\sqrt{2}-1)a$	$(\sqrt{2}-1)a\leq d\leq 3a$	$d\geq 3a$
$K^*=1,$ $F^*=\frac{a}{2},$ $P^*=d,$ $\pi^C=\frac{d}{2},$ $\pi^D=\frac{a}{4},$ $S=\frac{a}{8};$	$\forall K^*\in[\frac{2}{3}, \min(1, K_s)],$ $F^*=\frac{a+d}{2K^*},$ $P^*=\frac{a+d}{2}[\frac{3}{2}-\frac{1}{K^*}],$ $\pi^C=\frac{(a+d)^2}{16a},$ $\pi^D=\frac{(a+d)^2}{8a},$ $S=\frac{(a+d)^2}{32a};$	$\forall K^*\in[\frac{d-a}{d}, 1],$ $F^*=\frac{d-a}{K^*},$ $P^*=d-F^*,$ $\pi^C=a,$ $\pi^D=d-a,$ $S=\frac{a}{2};$

In Table 1-1, $K_s \equiv \frac{2(a+d)}{3a-d}$.

Table 2-1 Result 2 if $c=0$

$a>0, c=0, d\geq 0$	
$d\leq a$	$d\geq a$
K^* is the unique root to $Z(K)=0$ with $K\in[0, 1]$, $F^* = \frac{[(1-K^*)a-d]a}{2[(1-K^*)a]},$ $P^*=d,$ $\pi^C = \frac{[(1-K^*)a-d]^2}{4[(1-K^*)a]},$ $\pi^D = K^*a \frac{[(1-K^*)a]^2 - d^2}{4[(1-K^*)a]^2},$ $S = \frac{a[(1-K^*)a+d]^2}{8[(1-K^*)a]^2};$	$\forall K^*\in[0, 1],$ $F^*=0,$ $P^*=d,$ $\pi^C=d,$ $\pi^D=0,$ $S=\frac{a}{2};$

In Table 2-1, $Z(K) \equiv [(1-K)a]^3 - [(1-K)a]d^2 - 2ad^2K$.

(2) With $d=0$, Table 1 and Table 2 are respectively reduced to Table 1-2 and Table 2-2.

Table 1-2 Result 1 if $d=0$

$a>0, c\geq 0, d=0$	
$a\geq c$	$a\leq c$
$K^*=1,$ $F^*=\frac{a}{2},$ $P^*=\frac{c}{2},$ $\pi^C=\frac{c}{4},$ $\pi^D=\frac{a}{4},$ $S=\frac{a+c}{8};$	$\forall K^*\in[\frac{2}{3}, \min(1, K_c)],$ $F^*=\frac{a+c}{2K^*},$ $P^*=\frac{a+c}{2}[\frac{3}{2}-\frac{1}{K^*}],$ $\pi^C=\frac{a-c}{16},$ $\pi^D=\frac{a-c}{8},$ $S=\frac{a-c}{32};$

In Table 1-2, $K_s \equiv \frac{2(a+c)}{3a}$.

Table 2-2 Result 2 if $d=0$

$a>0, c \geq 0, d=0$	
$K^* = 1,$	
$F^* = \frac{a}{2},$	
$P^* = \frac{c}{2},$	
$\pi^C = \frac{c}{4},$	
$\pi^D = \frac{a}{4},$	
$S = \frac{a+c}{8};$	

(3) With $c=a/2$, Table 1 and Table 2 are respectively reduced to Table 1-3 and Table 2-3.

Table 1-3 Result 1 if $c=a/2$

$a>0, c=a/2, d \geq 0$		
$d \leq (\sqrt{3} - \frac{3}{2})a$	$(\sqrt{3} - \frac{3}{2})a \leq d \leq \frac{9}{2}a$	$d \geq \frac{9}{2}a$
$K^* = 1,$ $F^* = \frac{a}{2},$ $P^* = \frac{a}{4} + d,$ $\pi^C = \frac{a}{8} + \frac{d}{2},$ $\pi^D = \frac{a}{4},$ $S = \frac{3a}{16};$	$\forall K^* \in [\frac{2}{3}, 1],$ $F^* = \frac{3a+2d}{4K^*},$ $P^* = \frac{3a+2d}{4} [\frac{3}{2} - \frac{1}{K^*}],$ $\pi^C = \frac{(3a+2d)^2}{96a},$ $\pi^D = \frac{(3a+2d)^2}{48a},$ $S = \frac{(3a+2d)^2}{192a};$	$\forall K^* \in [\frac{2d-3a}{2d}, 1],$ $F^* = \frac{2d-3a}{2K^*},$ $P^* = d - F^*,$ $\pi^C = \frac{3}{2}a,$ $\pi^D = d - \frac{3}{2}a,$ $S = \frac{3a}{4};$

Table 2-3 Result 2 if $c=a/2$

$a>0, c=a/2, d\geq 0$		
$d \leq \frac{\sqrt{5}}{10}a$	$\frac{\sqrt{5}}{10}a \leq d \leq \frac{3}{2}a$	$d \geq \frac{3}{2}a$
$K^* = 1,$ $F^* = \frac{a-2d}{2},$ $P^* = \frac{a-2d}{4},$ $\pi^C = \frac{(a-2d)^2}{8a},$ $\pi^D = \frac{a^2-4d^2}{4a},$ $S = \frac{3(a+2d)^2}{16a};$	K^* is the unique root to $Z(K)=0$ with $K \in [0, 1]$, $F^* = \frac{(3-2K^*)a-2d}{6-4K^*},$ $P^* = \frac{(3-2K^*)a-2d}{12-8K^*} + d,$ $\pi^C = \frac{[(3-2K^*)a+2d]^2}{(24-16K^*)a},$ $\pi^D = K^* \cdot \frac{[(3-2K^*)a]^2-4d^2}{(6-4K^*)^2a},$ $S = \frac{3a[(3-2K^*)a-2d]^2}{16[(3-2K^*)a]^2};$	$\forall K^* \in [0, 1],$ $F^* = 0,$ $P^* = d,$ $\pi^C = d,$ $\pi^D = 0,$ $S = \frac{3a}{4};$

In Table 2-3, $Z(K) \equiv (\frac{3}{2} - K)^3 a^2 - (\frac{3}{2} + K) d^2$.

The cinemas compete for the contract through bidding. Cinema i will submit its bid from $[0, B_{ih}]$, where $B_{ih} \equiv \pi_{ih}^{C_i} + \pi_{ih}^{D_i}$, and $h=1$ if the distributor specified to limit ticket pricing in its request for bids or $h=2$, if otherwise, and $\pi_{ih}^{C_i}$ and $\pi_{ih}^{D_i}$ are given by result 1 if $h=1$ or result 2 if $h=2$. Cinema i won't bid a guarantee larger than B_{ih} , because it would be worse off obtaining the contract with those terms.

Define $Y \equiv \{1, 2, 3, \dots, n\}$ and $Z_h \equiv \{i \mid i \in Y \text{ and } B_{ih} = \max\{B_{1h}, B_{2h}, \dots, B_{n-1h}, B_{nh}\}\}$. We assume that $\{(a_i, c_i, d_i) \mid i \in Y\}$ vary sufficiently that Z_h includes only one element, say $Z_h = \{zh\}$. In this case, the following theorem characterizes the pure strategy Nash equilibria of the bidding game.

Theorem: If $Z_h = \{zh\}$, where $h=1$ if the distributor specifies to limit ticket pricing in its request for bids or $h=2$ if otherwise, all the pure strategy Nash equilibria of the bidding game are characterized by

$$\{ \forall N_{zh}^* \in [N_{hm}^*, \max \{ \pi^{Dzh}_h, N_{hm}^* \}], \forall N_{jh}^* \in [0, \min \{ B_{jh}, N_{hm}^* \}], \forall j \in Y \setminus \{zh\} \}, \quad (9)$$

where $N_{hm}^* = \max \{ B_{ih} \mid i \in Y \setminus \{zh\} \}$, and π^{Dzh}_h is given by result 1 with $i=z1$ if $h=1$ or result 2 with $i=z2$ if $h=2$.²¹

Proof: If cinema zh submits a bid less than N_{hm}^* , some other cinema obtains the contract through bidding N_{hm}^* , leaving cinema zh with no contract. Suppose $\pi^{Dzh}_h > N_{hm}^*$, if cinema zh bids from $[N_{hm}^*, \pi^{Dzh}_h]$, it will win the contract with its profit being π^{Czh}_h ; if cinema zh were to submit a bid larger than π^{Dzh}_h , it would still obtain the contract, but its profit would be less than π^{Czh}_h . That is, cinema zh benefits most to bid from $[N_{hm}^*, \max \{ \pi^{Dzh}_h, N_{hm}^* \}]$. Cinema j , $\forall j \in Y \setminus \{zh\}$, should not submit a bid larger than $\min \{ B_{jh}, N_{hm}^* \}$, because it would be worse off obtaining the contract under those terms; it can bid in the interval $[0, \min \{ B_{jh}, N_{hm}^* \}]$, but will not win by doing so. Therefore, all the pure strategy Nash equilibria of the bidding game are characterized by (9).

Q.E.D.

These equilibria lead to the unique outcome of the model of this chapter that cinema zh obtains the contract with the guarantee N_{zh}^* paid to the distributor. With this outcome, the profits of the distributor and cinema zh are:

$$\pi^D(N) = \max \{ \pi^{Dzh}_h, N_{hm}^* \}, \quad (10)$$

$$\pi^{Czh}(N) = \pi^{Dzh}_h + \pi^{Czh}_h - \max \{ \pi^{Dzh}_h, N_{hm}^* \}, \quad (11)$$

²¹

Recall the "inflexible" assumption about the distributor. Without this assumption, a cinema may bid a bit larger guarantee in order for the distributor not to ask a percentage for sharing the profit from admissions (and not to limit ticket pricing if $h=1$), so that it can expand its profit like a merger if it obtains the contract. Then to characterize the equilibrium of the bidding game is correspondingly complicated.

The distributor in fact achieves three targets by means of bidding. The first, a cinema submitting a bid automatically reveals itself to the distributor to have an interest in taking the contract. The second, the distributor allocates its contract to the cinema which can work out the largest joint profit. The third, the guarantee may be enhanced by the competition from other cinemas for the contract. This is reflected by the facts that N_{hm}^* is independent of (a_{zh}, c_{zh}, d_{zh}) and the guarantee is at least N_{hm}^* . If the competition is so strong that $N_{hm}^* > \pi^{Dzh}_h$, the distributor procures more profit than just through asking a percentage of the profit from admissions (and limiting ticket pricing if $h=1$). In particular, if Z_h includes more than one element, e.g., in the situation of complete competition, the guarantee is the joint profit the cinema can produce, leaving the cinema itself with zero profit.

Note that $z1$, N_{1m}^* , $z2$ and N_{2m}^* are revealed by the cinemas; the distributor does not know them before it closes accepting bids. Therefore, the model of this chapter doesn't involve the distributor deciding whether to specify to limit ticket pricing or not in its request for bids according to $z1$, N_{1m}^* , $z2$ and N_{2m}^* .

For the purpose of this chapter, however, it doesn't matter whether the distributor specifies to limit ticket pricing in its request for bids or not. In either case, we can see why contracts tendered in some circumstances will introduce a conflict about ticket pricing.

Concretely, suppose that the distributor specified to limit ticket pricing in its request for bids, then cinema $z1$ will take the contract. If

$$\pi^{Dz1}_1 + \pi^{Cz1}_1 - \max \{ \pi^{Dz1}_1, N_{1m}^* \} < \pi^{Dz1}_2 + \pi^{Cz1}_2 - \max \{ \pi^{Dz1}_2, N_{1m}^* \}, \quad (12)$$

cinema $z1$ would prefer the distributor not to limit ticket pricing; if

$$\max \{ \pi^{Dz1}_2, N_{1m}^* \} < \max \{ \pi^{Dz1}_1, N_{1m}^* \}, \quad (13)$$

the distributor will persist in limiting ticket pricing. Similarly, suppose that the distributor did not specify that in its request for bids, then cinema $z2$ will take the contract. If

$$\max \{ \pi^{Dz2}_2, N_{2m}^* \} < \max \{ \pi^{Dz2}_1, N_{2m}^* \}, \quad (14)$$

the distributor will want to limit ticket pricing; if

$$\pi^{Dz2}_1 + \pi^{Cz2}_1 - \max \{ \pi^{Dz2}_1, N_{2m}^* \} < \pi^{Dz2}_2 + \pi^{Cz2}_2 - \max \{ \pi^{Dz2}_2, N_{2m}^* \}, \quad (15)$$

cinema z2 would prefer the distributor not to do that. Because N_{1m}^* is independent of (a_{z1}, c_{z1}, d_{z1}) and N_{2m}^* independent of (a_{z2}, c_{z2}, d_{z2}) , and we have propositions 1 and 3 of the next section, (12) and (13), or (14) and (15), can simultaneously hold for some values of (a_{z1}, c_{z1}, d_{z1}) and N_{1m}^* , or for some values of (a_{z2}, c_{z2}, d_{z2}) and N_{2m}^* . That is, under the competition environments that (12) and (13), or (14) and (15), simultaneously hold, the contract between the distributor and cinema z1, or cinema z2, will introduce a conflict about ticket pricing. Under other competition environments, a conflict about ticket pricing may not happen.

We now proceed to the next section for assessing the policy action.

4. The Vertical Constraint vs. Policy Implications

In this section, we concentrate on analyzing the implications of the policy action through a comparative approach. In particular, we identify whether the contract with the vertical constraint or that without the constraint will produce larger consumer surplus and social welfare. For this purpose, we need not distinguish which cinema receives the contract. Thus hereafter, the subscript describing a cinema is suppressed and the guarantee on the contract is denoted by N^* .

According to the “inflexible” assumption, any guarantee has no effect on the distributor’s intention whether to limit ticket pricing. That is, a guarantee will only affect the profit distribution between the distributor and the cinema, but not affect the prices of admission and of a unit of snack foods, the joint profit and the total surplus in an equilibrium contract. Therefore, except propositions 1 and 2 below, the conclusions hereafter, including those of sections 5 and 6, are independent of any guarantee, and this will not be mentioned if not necessary.

For simplicity, denote the joint profit, the total surplus of movie-goers and the social welfare (the sum of the joint profit and the total surplus of movie-goers) in the equilibrium contract

of result 1 (Table 1) as J_1 , S_1 and W_1 respectively and those in the equilibrium contract of result 2 (Table 2) as J_2 , S_2 and W_2 respectively. Based on results 1 and 2, the following propositions hold.

Proposition 1:

(i)**Ticket Price:** In the equilibrium contract of result 1, $F^* \geq a/2$; in the equilibrium contract of result 2, $F^* \leq a/2$, and if $d > 0$, $F^* < a/2$.

(ii)**Distributor's Profit:** $\pi^D_1 \geq a/4$; $\pi^D_2 \leq a/4$, and if $d > 0$, $\pi^D_2 < a/4$.

(iii)**Cinema's Profit:** $\pi^C_1 \leq \pi^C_2$. More specifically, if $d=0$ and the distributor chooses $F^*=a/2$ in the equilibrium contract of result 1, the cinema's profit will be the same in the equilibrium contract of result 1 and in that of result 2; otherwise, the cinema's profit will be lower in the equilibrium contract of result 1 than in that of result 2.

Proof: See appendix B.

As pointed out in the last section, this proposition helps to explain why the distributor may want to control ticket pricing and why the cinema may oppose that. The distributor may achieve a larger profit through controlling than not controlling ticket pricing. So may the cinema. In fact, with the guarantee of N^* , if the distributor controls ticket pricing, it can obtain a profit $\max\{\pi^D_1, N^*\}$, which is not less than $\max\{a/4, N^*\}$, while the cinema's profit is $\pi^D_1 + \pi^C_1 - \max\{N^*, \pi^D_1\}$; if the distributor does not control ticket pricing, it can only obtain a profit $\max\{N^*, \pi^D_2\}$, which is not larger than $\max\{N^*, a/4\}$, while the cinema's profit is $\pi^D_2 + \pi^C_2 - \max\{N^*, \pi^D_2\}$. As an example, if $N^* \leq \pi^D_2$, the distributor may get more profit through limiting ticket pricing than not limiting that, while accordingly, the cinema may get less profit.

If it is legal, a distributor generally sets a minimum price for tickets. From this proposition,

we see that if the guarantee is small enough, i.e., $N^* \leq \pi^D_2$,²² the cinema always prefers a lower price of tickets to the price that is optimal to the distributor. Thus the distributor can set its optimal ticket price as a minimum price and this will be equivalent to specifying the price of a ticket.

Proposition 2(Profit Distribution):

- (i) If $d \neq 0$, or $c \neq a$, or the distributor does not choose $F^* = a/2$, $\pi^D_1 > \pi^C_1 > 0$; if $d=0$ and $c=a$ and the distributor chooses $F^* = a/2$, $\pi^D_1 = \pi^C_1$.
- (ii) If $d \geq a+c$, $\pi^C_2 > 0$ but $\pi^D_2 = 0$; which of the two profits is larger is uncertain if $d < a+c$.

Proof: See appendix B.

Taking account of a guarantee N^* , (i) implies $\pi^D(N) \equiv \max\{\pi^D_1, N^*\} \geq \pi^D_1 + \pi^C_1 - \max\{\pi^D_1, N^*\} \equiv \pi^C(N)$. One can also easily make some pertinent changes on (ii) with N^* specified.

By this proposition, we see that the policy action worsens the profitability of the distributor and enhances that of the cinema. With the distributor controlling ticket pricing, if it decides to raise the price of admission, the cinema will respond by reducing the price of snack foods, so that the distributor is able to divert some revenue from snack foods to that from admissions. Therefore, the cinema's profit is limited by $(a+c)$, no matter how large "d" may be (see result 1). On the other hand, the distributor's profit will always be larger than the cinema's profit, no matter how small "a" and how large "d" (and "c") may be (result 1 and (i) of proposition 2). If the vertical constraint is illegal, the cinema may divert some admission revenue to that of snack foods by taking account the impact of reducing the price of admission on both sources of revenue.

²²

With proposition 3 below, this assumption can be relaxed a bit.

Proposition 3(Joint Profit):

(i) If $0 < d < \frac{\sqrt{3}}{6}(a-c)$ and $\frac{(a+c+d)^2}{8(a+c)} < \frac{a}{4}$, $J_1 > J_2$.

(ii) If $\frac{\sqrt{3}}{6}(a+c) < d < 3(a-c)$, or if $\frac{(a+c+d)^2}{8(a-c)} > \frac{a}{4}$ and $d < 3(a+c)$, or if either $d=0$ or

$$\frac{(a+c+d)^2}{8(a-c)} = \frac{a}{4} \quad \text{and } d < 3(a+c) \text{ but the distributor does not choose } F^* = \frac{a}{2} \text{ in the}$$

equilibrium contract of result 1, $J_1 < J_2$.

(iii) If $d \geq 3(a+c)$, or if either $d=0$ or $d = \frac{\sqrt{3}}{6}(a+c)$ and $\frac{(a-c+d)^2}{8(a+c)} < \frac{a}{4}$, or if either $d=0$

or $d = \frac{\sqrt{3}}{6}(a+c)$ and $\frac{(a+c-d)^2}{8(a+c)} = \frac{a}{4}$ and the distributor chooses $F^* = \frac{a}{2}$ in the equilibrium contract of result 1, $J_1 = J_2$.

Proof: See appendix B.

This proposition tells us that only if both c and d are small enough relative to a , may the joint profit with no vertical constraint allowed be less than under permitting it. That c and d are small means that the potential of the monopoly profit from snack foods is small. Making the vertical constraint illegal shifts the monopoly power of ticket pricing from the distributor to the cinema. With both the monopoly power of ticket pricing and that of pricing for snack foods, the cinema wants to lower the ticket price (relative to the optimal ticket price to the distributor) so that not only may a larger audience follow but also it can probably raise the price of snack foods. Although the decrease in ticket price may result in a decrease in the monopoly profit from admissions, the increases in movie-goers and in the price of snack foods will generate a more than offsetting rise in profit from snack foods. Because the cinema receives the total increase in the monopoly profit from snack foods but only bears part of the decrease in the monopoly profit from admissions, it always has an incentive to lower the ticket price to some extent. This is confirmed by proposition 1. With c and d being

small enough, the increase in the monopoly profit from snack foods is less than the decrease in that from admissions, so the joint profit decreases. With c and d being large enough, the increase in the monopoly profit from snack foods will not be less than the decrease in that from admissions, or perhaps there will not be a decrease in the profit of admissions, so the joint profit will not decrease.

Proposition 4(Movie-goers' Surplus and Social Welfare):

(i) $S_1 \leq S_2$; and if $0 < d < 3(a+c)$, $S_1 < S_2$.

(ii) If $0 < d < \frac{\sqrt{3}}{6}(a+c)$, or if $d=0$ but the distributor does not choose $F^* = \frac{a}{2}$ in the equilibrium contract of result 1, $W_1 < W_2$.

(iii) If $d \geq 3(a+c)$, or if $d=0$ and the distributor chooses $F^* = \frac{a}{2}$ in the equilibrium contract of result 1, $W_1 = W_2$.

Proof: See appendix B.

From proposition 3, the policy action may introduce a decrease in the joint profit. This proposition demonstrates that the increase in the total surplus of movie-goers introduced by the policy action will be enough to compensate for any possible decrease in the joint profit. That is, the policy action is justified in terms of generating more total consumer surplus and social welfare. The total surplus of movie-goers and the social welfare can increase because more people will go to the cinema for a reduced price of admission, though the movie-goers may face a raised price of snack foods. Now we investigate the effect of the policy action on the price of snack foods.

Proposition 5 (Price of Snack Foods):

If $d \geq a+c$, the price of snack foods in the equilibrium contract of result 1 will be lower than that in the equilibrium contract of result 2.

Proof: Self-evident with results 1 and 2.

Proposition 5 may not hold if $d < a + c$. For instance, if $0 < d < a/10$ and $c = a/2$, $P^* = a/4 + d$ in Table 1-3 while $P^* = a/4 + d/2$ in Table 2-3. With the special situations where $c = 0$ or $d = 0$, the effect of the policy action on the price of snack foods is more predictable.

Proposition 6:

(i) If $c = 0$, the price of snack foods will not be higher in the equilibrium contract of result 1 than in that of result 2; and if $c = 0$ and $d > (\sqrt{2} - 1)a$, the price of snack foods is lower in the equilibrium contract of result 1 than in that of result 2.

(ii) If $d = 0$, the price of snack foods will not be higher in the equilibrium contract of result 1 than in that of result 2.

Proof:

(i) With Table 1-1 and Table 2-1, we only need to show

$$P^* = \frac{a+d}{2} \left[\frac{3}{2} - \frac{1}{K^*} \right] < d \quad \text{when} \quad d > (\sqrt{2} - 1)a.$$

$$\text{This is done by } P^* = \frac{a+d}{2} \left[\frac{3}{2} - \frac{1}{K^*} \right] \leq \frac{a+d}{2} \left[\frac{3}{2} - 1 \right] < \frac{1}{4} \left[d + \frac{d}{\sqrt{2} - 1} \right] < d.$$

(ii) Self-evident with Table 1-2 and Table 2-2.

Q.E.D.

Although the price of snack foods may be raised or lowered by the policy action, the combined price, the sum of the price of a ticket and that of a unit of snack foods, will definitely not be raised. That is, if the policy action introduce an increase in the price of a unit of snack foods, they will introduce a decrease in the price of a ticket of greater or equal magnitude, so that the combined price will not be raised. We will further examine this and draw another important conclusion in the next section.

5. The Policy Action vs. Merging

The policy action has an important implication on the regulation that bars integration between a distributor and a cinema. Such regulations have been historically introduced in the United States. We start with investigating a merger's behavior in order to reveal this implication.

Result 3: If the distributor merges with a cinema, and if $a > 0$, $c \geq 0$ and $d \geq 0$, this merger will take a ticket for admission and a unit of snack foods as a combined good. If $d \leq a+c$, the merger will set the price for this combined good at $(a+c+d)/2$, i.e., $F^* + P^* = (a+c+d)/2$, subject

to $0 \leq P^* \leq \frac{(a+c-d)c}{2(a+c)} + d$, with the merger's profit $\pi^M = \frac{(a+c+d)^2}{4(a+c)}$ and the total surplus of

movie-goers $S = \frac{(a+c+d)^2}{8(a+c)}$; if $d \geq a+c$, the merger will set the price for this combined good

at d , i.e., $F^* + P^* = d$, subject to $0 \leq P^* \leq d$, with the merger's profit $\pi^M = d$ and the total surplus

of movie-goers $S = \frac{a+c}{2}$.

The following table summarizes this result.

Table 3 Summarization of Result 3

$d \leq a+c$	$d \geq a+c$
$F^* + P^* = \frac{a+c+d}{2}$ $(0 \leq P^* \leq \frac{(a+c-d)c}{2(a+c)} + d)$ $\pi^M = \frac{(a+c+d)^2}{4(a+c)}$ $S = \frac{(a+c+d)^2}{8(a+c)}$	$F^* + P^* = d$ $(0 \leq P^* \leq d)$ $\pi^M = d$ $S = \frac{a+c}{2}$

This result can be easily obtained from appendix D, where if $K=0$ and the distributor is not permitted to limit ticket pricing, the behavior of the cinema's pricing for tickets and snack foods is equivalent to that of the merger of the distributor and the cinema.

One important thing is that, the combined price for a ticket and a unit of snack foods, (F^*+P^*) , set by the merger will generally be lower than that in the equilibrium contract of result 1 but higher than that in the equilibrium contract of result 2. This is summarized in the following proposition.

Proposition 7: With $a>0$, $c\geq 0$ and $d\geq 0$,

$$F^*+P^* \text{ (separated without vertical constraint)} \leq F^*+P^* \text{ (merged)} \leq F^*+P^* \text{ (separated with vertical constraint)}$$

Proof: See appendix B.

Figure 5

Figure 5 illustrates proposition 7 when $d \leq a+c$ and $\frac{a}{4} < \frac{(a+c+d)^2}{8(a+c)}$. One can have similar

drawings for other ranges of the parameters. In Figure 5, panel (a) corresponds to $K_s > 1$ and panel (b) to $K_s < 1$, where K_s is defined in section 3. If $K_s = 1$, points G and N in panel (b) of Figure 5 are at a same position. The combined price for a ticket of admission and a unit of snack foods can be any point on EG in the equilibrium contract of result 1 while it is uniquely at point H in the equilibrium contract of result 2. With the distributor and the cinema being merged, the combined price can be any point on CD.

With results 1, 2 and 3, it is also evident that the merger produces a profit that is at least as great as the joint profits from results 1 and 2, and the largest total surplus of movie-goers is obtained by the equilibrium contract of result 2. Another important fact, corresponding to proposition 7 above, is that the equilibrium contract of result 2 will generally give rise to the largest social welfare. That is, if S and W denote the total surplus of movie-goers and social welfare respectively, we have proposition 8 below.

Proposition 8: With $a > 0$, $c \geq 0$ and $d \geq 0$,

$$(i) \quad S|_{\text{separated with vertical constraint}} \leq S|_{\text{merged}} \leq S|_{\text{separated without vertical constraint}} ;$$

$$(ii) \quad W|_{\text{separated with vertical constraint}} \leq W|_{\text{merged}} \leq W|_{\text{separated without vertical constraint}} .$$

Proof: See appendix B.

Since the 1950s, integration between a distributor and a cinema has been restricted in the United States. Proposition 8 states that such restrictions generally raise consumer surplus and social welfare, as long as distributors are not permitted to limit ticket pricing on their contracts with cinemas. In other words, forbidding the vertical constraint in contracts is necessary to achieve an increase in welfare by barring integration between a distributor and a cinema.

6. Negative Relationship between the Reservation Prices

The conclusions of sections 3, 4 and 5 are built up on the assumption that $c \geq 0$ ($c_i > 0$), i.e., the population shows a nonnegative relationship between their reservation prices for tickets and snack foods. This assumption appears necessary for the conclusions though. The example below with $c = -a$ and $a/4 < d \leq a$ will show this.

Result 4: If $a > 0$, $a \geq d > a/4$, $c = -a$, and if the distributor is permitted to limit ticket pricing, the equilibrium contract without a guarantee between the distributor and the cinema, the associated surplus of movie-goers (S) and social welfare (W) are characterized by:

$$K^* = 1, \quad F^* = d, \quad P^* = 0, \quad \pi^C = 0, \quad \pi^D = d, \quad S = \frac{(a-d)^2}{2a}, \quad W = \frac{a^2 + d^2}{2a}.$$

Result 5: If $a > 0$, $a \geq d > a/4$, $c = -a$, and if the distributor is not permitted to limit ticket pricing, the equilibrium contract without a guarantee between the distributor and the cinema, the associated surplus of movie-goers (S) and social welfare (W) are characterized by:

(1) If $d \leq \frac{a}{3}$,

$$K^* = 1 - \frac{d}{a}, \quad F^* = d, \quad P^* = 0, \quad \pi^C = \frac{d^2}{a}, \quad \pi^D = \frac{(a-d)d}{a}, \quad S = \frac{(a-d)^2}{2a}, \quad W = \frac{a^2 + d^2}{2a};$$

(2) If $\frac{a}{3} \leq d$,

$$K^* = \frac{a+d}{2a}, \quad F^* = \frac{a+d}{4}, \quad P^* = \frac{3d-a}{4}, \quad \pi^C = \frac{(a+d)^2}{16a}, \quad \pi^D = \frac{(a+d)^2}{8a},$$

$$S = \frac{(3a-d)^2}{32a}, \quad W = \frac{6(a+d)^2 - (3a-d)^2}{32a}.$$

Result 6: If $a > 0$, $a \geq d > a/4$, $c = -a$, and if the distributor and the cinema merge, this merger will set $F^* = d$ and $P^* = 0$, with the profit of the merged entity $\pi^M = d$. The total surplus of

movie-goers is $S = \frac{(a-d)^2}{2a}$ and the social welfare is $W = \frac{a^2+d^2}{2a}$.

The proofs of results 4 and 5 are given in appendices E and F respectively. Result 6 is in fact implied by appendix F. Interestingly, results 4 and 6 give same prices of admission and of a unit of snack foods, same profit of the industry, same consumer surplus and same social welfare.

These results are summarized in Table 4.

Table 4 Summarization of Results 4, 5 and 6

Result 4	Result 5		Result 6
$K^* = 1$ $F^* = d$ $P^* = 0$ $\pi^C = 0$ $\pi^D = d$ $S = \frac{(a-d)^2}{2a}$ $W = \frac{a^2+d^2}{2a}$	$d \leq \frac{a}{3}$	$d \geq \frac{a}{3}$	$F^* = d$ $P^* = 0$ $\pi^M = d$ $S = \frac{(a-d)^2}{2a}$ $W = \frac{a^2+d^2}{2a}$
	$K^* = 1 - \frac{d}{a}$ $F^* = d$ $P^* = 0$ $\pi^C = \frac{d^2}{a}$ $\pi^D = \frac{(a-d)d}{a}$ $S = \frac{(a-d)^2}{2a}$ $W = \frac{a^2+d^2}{2a}$	$K^* = \frac{a+d}{2a}$ $F^* = \frac{a+d}{4}$ $P^* = \frac{3d-a}{4}$ $\pi^C = \frac{(a+d)^2}{16a}$ $\pi^D = \frac{(a+d)^2}{8a}$ $S = \frac{(3a-d)^2}{32a}$ $W = \frac{6(a+d)^2 + (3a-d)^2}{32a}$	

Figure 6 below illustrates the comparisons between results 4 and 5. One can easily verify Figure 6. We see that forbidding the distributor to limit ticket pricing can increase the total surplus of movie-goers. But it may decrease social welfare. This is due to the profit of the industry being reduced. We also see that the price of a ticket and the profit of the distributor may be reduced, while the price of a unit of snack foods and the profit of the cinema may be raised by forbidding any price floor clause. The three results give the same combined price of a ticket and a unit of snack foods. Therefore, proposition 7 of section 5 still holds.

Figure 6

It is not surprising that under the stipulated circumstances, the social welfare may decrease when the cinema sets both the price of admission and that of snack foods. When the distributor sets the price of admission, the cinema chooses the price of snack foods to be zero and earns no profit. When the cinema controls both prices, it raises the price of snack foods while reducing the price of admission by the same amount. These changes in prices do not attract

more movie-goers. For this constant audience, the profit potential from snack foods is less than that from admissions.

Thus our model won't support the policy action with this example in terms of social welfare. Note that except panels (C) and (D), other panels of Figure 6 hold for the contract with a guarantee under the "inflexible" assumption.

7. Concluding Remarks

This chapter has focused on analyzing a policy action that prevents distributors from placing vertical constraint on ticket pricing in their contracts with cinemas. The analysis is built up in a two-stage game setting between one distributor and multiple cinemas. The model captures some of the features of the contracts observed in the industry. With the assumptions of linear demands and the "nonnegative assumption", we show that there will be conflict about ticket pricing between the distributor and the exhibitor only in some contracting environments. When conflict occurs, a policy forbidding the distributor from setting a minimum ticket price improves the total surplus of movie-goers and social welfare. In these circumstances, it is shown that (i) the distributor can procure a larger profit through controlling rather than not controlling ticket pricing; (ii) the cinema makes more profit in the absence of a ticket price minimum; and (iii) the cinema always prefers a lower price of admission to that which would be chosen by the distributor. If the reservation prices of snack foods are large enough relative to those of admission, the policy action will also raise the profit of the industry. These results derive from the price effect of the policy action of forbidding the vertical price constraint, which is a reduction of the combined price of per admission and per unit of snack foods. Furthermore, it is demonstrated that forbidding distributors vertically to limit ticket pricing is necessary for a policy of restricting integration between a distributor and a cinema to be welfare-improving. Therefore, for the assumed model and demand conditions, this chapter not only explains why a distributor and a cinema may have a conflict about ticket pricing, but also justifies the policy action.

Although the conclusions of this chapter are based on the assumptions of linear demands, we expect similar conclusions would result even without them if both the movie and snack foods are normal goods to the population. A basic principle is that, given the demand for a normal good, any policy action that weakens the supply-side monopoly power of the service should improve consumer surplus and social welfare. In the model of this chapter, forbidding the distributor vertically to limit ticket pricing is equivalent to shifting the monopoly power of pricing for tickets from the distributor to the cinema. But this power is scaled down after shifting, because the distributor weakens the incentive for the cinema to execute the power through a percentage for sharing the revenue of tickets. Thus even without the assumptions of linear demands, and/or even though the distribution of the population is not uniform, it is reasonable to expect the model still to support the policy action. Assuming the population to distribute within $[0, 1]$ is not a significant restriction. If we assume the population to be distributed within any interval, say $[s, t]$, the model's predictions will not be affected, but the calculations will involve some more constant terms and thus be more complicated.

The “nonnegative assumption” appears serious, as is illustrated by the example. One may expect the model to support the policy action even without the “nonnegative assumption” if the policy action can induce more movie-goers. But to investigate the general properties of the model without the “nonnegative assumption” is not an easy task.

The “inflexible” assumption plays a role in the model of this chapter. One may disagree with this assumption and suggest a “flexible” assumption that once the guarantee is large enough, the distributor will forgo asking for a percentage of the revenue from admissions and controlling ticket pricing, so that the cinema winning the contract, could expand its profit like a merger. With this “flexible” assumption, cinema i will bid from $[0, B_i]$ with B_i to be the profit it could produce like a merger. Then the equilibrium of the bidding game is uniquely with the winner, say cinema 1, to bid $\max \{\pi^{D1}_1, N^*\}$ if the distributor specified to limit ticket pricing in its request for bids or $\max \{\pi^{D1}_2, N^*\}$ if otherwise, where $N^* \equiv \max \{B_2, B_3, \dots, B_n\}$, and in either case, the distributor will finally give up to ask for a percentage

of the revenue from admissions and to control ticket pricing. Because the distributor knows that $\max \{\pi^{Di_1}, N\} \geq \max \{\pi^{Di_2}, N\}$ for $\forall i$ and $\forall N$, it will always specify that it will limit ticket pricing in its request for bids. However, with this “flexible” assumption, the model suggests no conflict about ticket pricing as the distributor would be willing to allow the cinema to adjust ticket pricing freely.

The contract studied by the model is a simplification of industry contracts. Among the simplifications are the no cost, no “house nut” clause, no “hold-over” clause, a fixed length of the run and a same percentage for the whole run assumptions. To relax any of these assumptions from the model may be desired. The model considers a single contract. To extend the model to handle two or more contracts signed between a distributor and a cinema in sequence, or between a distributor and different cinemas in sequence or simultaneously, would be an interesting extension. To study such a model may provide insights about the effect on the supply of motion pictures of the policy action. Mathematically, however, all of these possible extensions are not easy tasks.

Appendix A The Derivation of Demand Functions

Given F and P , if $a\theta \geq F$, i.e., $\theta \geq F/a$, the person θ must buy a ticket; if $a\theta < F$, but $a\theta + c\theta + d - F - P \geq 0$, the person θ will also buy a ticket, as well as a unit of snack foods. Thus we define the following two sets:

$$X_1(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta \geq F/a, F \geq 0, P \geq 0\},$$

$$X_2(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta < F/a, a\theta + c\theta + d - F - P \geq 0, F \geq 0, P \geq 0\}.$$

Obviously, $X_1 \cap X_2$ is empty and the total demand for tickets is the sum of the linear measures of these two sets. Since a finite number of points do not affect the linear measure of a set, $X_2(F, P)$ can also be defined as

$$X_2(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta \leq F/a, a\theta + c\theta + d - F - P \geq 0, F \geq 0, P \geq 0\}.$$

This makes $X_2(F, P)$ always closed.

Let $M(\cdot)$ denote the linear measure operator for a set. Then

$$M(X_1(F, P)) = \begin{cases} 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq 0; \\ 0 & \text{if } F \geq a, P \geq 0; \end{cases}$$

In the situation where $a+c > 0$,

$$M(X_2(F, P)) = \begin{cases} 0 & \text{if } F - P \geq a - c - d, F \geq 0, P \geq 0; \\ \frac{F}{a} & \text{if } 0 \leq F \leq a, F - P \leq d, P \geq 0; \\ 1 & \text{if } F \geq a, F - P \leq d, P \geq 0; \\ \frac{F}{a} - \frac{F - P - d}{a + c} & \text{if } F - P \geq d, P \leq \frac{c}{a}F - d, F \leq a, F \geq 0, P \geq 0; \\ 0 & \text{if } P \geq \frac{c}{a}F - d, F \geq 0, P \geq 0; \\ 1 - \frac{F + P - d}{a + c} & \text{if } F \geq a, d \leq F + P \leq a + c, P \geq 0; \end{cases}$$

The total demand for tickets is the sum of $M(X_1(F,P))$ and $M(X_2(F,P))$. A standard approach to calculate this sum comprises three steps. Step 1: add each expression of the right-hand of $M(X_2(F,P))$ to each expression of the right-hand of $M(X_1(F,P))$. Note that the associated constraints are also added. This will give us totally 12 expressions. Step 2: check its feasible set for each of the 12 expressions. If its feasible set of an expression is empty, exclude this expression. Step 3: exclude the redundant expressions of the remainder expressions. For example, the sum of the first expression of $M(X_1(F,P))$ and the third expression of $M(X_2(F,P))$ is redundant because it is contained by the sum of the first expression of $M(X_1(F,P))$ and the second expression of $M(X_2(F,P))$. Thus it should be excluded. With this approach, and with $a+c>0$, the total demand for tickets (D_t) can be expressed as

$$D_t = \begin{cases} 1 & \text{if } F-P \leq d, F \geq 0, P \geq 0; \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F-d, P \geq 0; \\ 0 & \text{if } F \geq a, F-P \geq a-c-d, P \geq 0; \\ 1 - \frac{F-P-d}{a-c} & \text{if } d \leq F-P \leq a-c-d, P \leq \frac{c}{a}F-d, F \geq 0, P \geq 0; \end{cases}$$

Now consider the total demand for snack foods. Some persons in X_1 may buy snack foods. They comprise the following set X_3 :

$$X_3(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta \geq F/a, F \geq 0, P \geq 0, c\theta + d - P \geq 0\}.$$

In the situation where $c>0$,

$$M(X_3(F,P)) = \begin{cases} 0 & \text{if } P \geq c-d, F \geq 0; \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, 0 \leq P \leq \frac{c}{a}F-d; \\ 1 - \frac{P-d}{c} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F-d, P \leq c-d; \\ 0 & \text{if } F \geq a, P \geq 0; \end{cases}$$

The total demand for snack foods for the situation where $a+c>0$ and $c>0$ is the sum of $M(X_2(F,P))$ and $M(X_3(F,P))$. With the same approach as stated above, the total demand for snack foods (D_s) can be expressed as

$$D_s = \begin{cases} 1 & \text{if } F+P \leq d, F \geq 0, P \geq 0; \\ 1 - \frac{P-d}{c} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F-d, P \leq c-d, P \geq 0; \\ 0 & \text{if } 0 \leq F \leq a, P \geq c-d, P \geq 0; \\ 0 & \text{if } F \geq a, F-P \geq a-c-d, P \geq 0; \\ 1 - \frac{F-P-d}{a-c} & \text{if } d \leq F-P \leq a-c-d, P \leq \frac{c}{a}F-d, F \geq 0, P \geq 0; \end{cases}$$

Now (1) and (2) of section 2 can be obtained by rewriting D_i above according to the feasible regions of D_s .

In the situation where $c=0$, $X_3(F, P)$ is reduced to $X_{31}(F, P)$:

$$X_{31}(F, P) = \{\theta \mid \theta \in [0, 1], \theta \geq F/a, F \geq 0, P \geq 0, d-P \geq 0\}.$$

$$M(X_{31}(F, P)) = \begin{cases} 0 & \text{if } P > d, F \geq 0; \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, 0 \leq P \leq d; \\ 0 & \text{if } F \geq a, P \geq 0; \end{cases}$$

Then it's easy to get the total demand functions for the situation where $c=0$. They are:

$$D_i = \begin{cases} 1 & \text{if } F+P \leq d, F \geq 0, P \geq 0; \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P > d; \\ 0 & \text{if } F \geq a, F-P \geq a-d, P \geq 0; \\ 1 - \frac{F+P-d}{a} & \text{if } d \leq F-P \leq a-d, P \leq d, F \geq 0, P \geq 0; \end{cases}$$

$$D_s = \begin{cases} 1 & \text{if } F+P \leq d, F \geq 0, P \geq 0; \\ 0 & \text{if } 0 \leq F \leq a, P > d; \\ 0 & \text{if } F \geq a, F+P \geq a+d, P \geq 0; \\ 1 - \frac{F+P-d}{a} & \text{if } d \leq F-P \leq a-d, P \leq d, F \geq 0, P \geq 0; \end{cases}$$

A subtle point is that “ $P > d$ ” in the first expression of $M(X_{31}(F, P))$ can not be written as “ $P \geq d$ ”. This implies the total demand function for snack foods is not continuous on P at $P=d$.

Now we see the derivation of (3) and (4) of section 2.

With $c=-a$, $X_2(F, P)$ becomes $X_{21}(F, P)$:

$$X_{21}(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta \leq F/a, d-F-P \geq 0, F \geq 0, P \geq 0\};$$

$X_3(F, P)$ becomes $X_{32}(F, P)$:

$$X_{32}(F, P) \equiv \{\theta \mid \theta \in [0, 1], \theta \geq F/a, F \geq 0, P \geq 0, -a\theta - d - P \geq 0\}.$$

$$M(X_{21}(F, P)) = \begin{cases} 0 & \text{if } F-P > d, F \geq 0, P \geq 0; \\ \frac{F}{a} & \text{if } 0 \leq F \leq a, F-P \leq d, P \geq 0; \\ 1 & \text{if } F \geq a, F-P \leq d, P \geq 0; \end{cases}$$

$$M(X_{32}(F, P)) = \begin{cases} 0 & \text{if } F \geq a, P \geq 0; \\ 0 & \text{if } P \geq d, F \geq 0; \\ 0 & \text{if } F-P \geq d, P \geq 0, F \geq 0; \\ \frac{P-d}{-a} - \frac{F}{a} & \text{if } F \geq 0, F-P \leq d, P \geq d-a, P \geq 0; \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, F-P \geq d, P \leq d-a, P \geq 0; \end{cases}$$

If $a \geq d$, $M(X_{32}(F, P))$ is reduced to

$$M(X_{32}(F, P)) = \begin{cases} 0 & \text{if } F-P \geq d, P \geq 0, F \geq 0; \\ \frac{P-d}{-a} - \frac{F}{a} & \text{if } F \geq 0, F-P \leq d, P \geq 0; \end{cases}$$

The sum of $M(X_1(F, P))$ and $M(X_{21}(F, P))$ is (3), the total demand for tickets. That of $M(X_{21}(F, P))$ and $M(X_{32}(F, P))$ is (4), the total demand for snack foods.

Appendix B Proofs of Propositions

In Figure B.1, some points of line VQ can be represented by a parameter K , with

$$F(K) = \frac{[(1-K)a+c-d]a}{2[(1-K)a-c]} \quad \text{and} \quad P(K) = \frac{[(1-K)a+c-d]c}{2[(1-K)a-c]} - d. \quad \text{Accordingly, at these points of line}$$

VQ, the joint profit is $J(K) = \frac{[(1-K)a+c-d]^2}{4[(1-K)a-c]} - Ka \frac{[(1-K)a-c]^2 - d^2}{4[(1-K)a-c]}$ and the total surplus of

the movie-goers is $S(K) = \frac{(a-c)[(1-K)a+c-d]^2}{8[(1-K)a-c]^2}$. The social welfare $W(K) \equiv J(K) + S(K)$.

In region ABVQ, the joint profit is $J = -\frac{1}{a+c} \left[F - P - \frac{a-c-d}{2} \right]^2 - \frac{[a-c-d]^2}{4[a-c]}$. Especially, at the

points $(F(K), P(K))$ of line VQ, $J(K) = -\frac{1}{a+c} \left[F(K) - P(K) - \frac{a-c-d}{2} \right]^2 - \frac{[a-c-d]^2}{4[a-c]}$.

Recall that $Z(K) \equiv [(1-K)a-c]^3 - [(1-K)a-c]d^2 - 2ad^2K$, as defined in result 2 of section 3. In

the equilibrium contract of result 2, with K^* as the unique root to $Z(K)=0$, if $0 \leq K^* \leq 1$, $J(K^*)$ and $S(K^*)$ are respectively the joint profit and the total surplus of movie-goers; if $K^* > 1$, the joint profit and the total surplus of movie-goers are respectively $J(1)$ and $S(1)$; if $K^* < 0$, the joint profit and the total surplus of movie-goers are respectively $J=d$ and $S=(a+c)/2$.

Figure B.1

Through direct calculations, we have

$$\frac{\partial J(K)}{\partial K} = -\frac{a^2 d^2 K}{2[(1-K)a-c]^3}, \quad (\text{B1})$$

$$\frac{\partial S(K)}{\partial K} = \frac{ad(a-c)[(1-K)a+c+d]}{4[(1-K)a+c]^3}, \quad (B2)$$

$$\frac{\partial W(K)}{\partial K} = \frac{ad\{(a-c)[(1-K)a+c+d]-2adK\}}{4[(1-K)a+c]^3}, \quad (B3)$$

$$Z'(K) = -3[(1-K)a+c]^2 a - ad^2 \leq 0 \text{ (equality holds only when } [(1-K)a+c]^2 + d^2 = 0) \quad (B4)$$

In the following, as in section 4, let J_1 and W_1 denote respectively the joint profit and the social welfare in the equilibrium contract of result 1 while J_2 and W_2 denote those in the equilibrium contract of result 2.

Proof of Proposition 3:

(i) In the equilibrium contract of result 1, if $\frac{(a+c-d)^2}{8(a-c)} < \frac{a}{4}$, which implies $d \leq a+c$ and $c < a$,

$F^* = \frac{a}{2}$, $P^* = \frac{c}{2} - d$, and thus $F^* + P^* = \frac{a+c-d}{2} - \frac{d}{2}$, the joint profit is

$J_1 = -\frac{1}{a+c} \left(\frac{d}{2}\right)^2 - \frac{(a+c-d)^2}{4(a-c)}$. In the equilibrium contract of result 2, with K^* as the unique

root to $Z(K) = 0$, $F^* + P^* = \frac{a+c+d}{2} - \frac{adK^*}{2[(1-K^*)a+c]}$ if $K^* \leq 1$ or $F^* + P^* = \frac{a(c-d)}{2} - \frac{c+d}{2}$ if

$K^* > 1$. Accordingly, the joint profit

$J_2 = J(K^*) = -\frac{1}{a-c} \left(\frac{adK^*}{2[(1-K^*)a+c]}\right)^2 + \frac{(a+c+d)^2}{4(a-c)}$ or $J_2 = J(1) = -\frac{1}{a-c} \left(\frac{ad}{2c}\right)^2 - \frac{(a+c-d)^2}{4(a-c)}$. Therefore,

if $K^* \leq 1$, $J_1 < J_2 - \frac{d}{2} > \frac{adK^*}{2[(1-K^*)a+c]}$.

Let K_0 denote the unique root to $\frac{d}{2} = \frac{adK}{2[(1-K)a+c]}$, then $K_0 = \frac{a+c}{2a}$, with $0 < K_0 < 1$ for $c < a$ and $(1-K_0)a+c = (a+c)/2 > 0$.

$$Z(K_0) = \frac{a+c}{2} \left[\frac{a-c}{2} - \sqrt{3}d \right] \left[\frac{a-c}{2} + \sqrt{3}d \right] . \quad (B5)$$

By (B4), $K_0 < K^* \Rightarrow Z(K_0) > Z(K^*) = \frac{a-c}{2} > \sqrt{3}d$.

By (B1) (note that $d \neq 0$), if $K_0 < K^*$, then $J_1 = J(K_0) > J(K^*)$. If $K^* > 1$, we directly have $J_1 = J(K_0) > J(1)$ for $K_0 < 1$.

(ii) If $\frac{\sqrt{3}}{6}(a+c) < d < 3(a+c)$. in the equilibrium contract of result 1,

$$F^* - P^* = \frac{a+c-d}{2} - \frac{d}{2} \text{ or } F^* - P^* = \frac{a+c+d}{2} - \frac{a-c-d}{4} . \text{ If } F^* - P^* = \frac{a+c-d}{2} - \frac{d}{2} , \text{ as above,}$$

$Z(K_0) < 0$. Therefore, $K_0 > K^*$ for such K^* that $Z(K^*) = 0$. If $K^* \geq 0$, by (B1), $J_1 < J_2$.

If $K^* < 0$, which implies $a+c < d$, then $J_1 = \frac{a-c}{4} + \frac{d}{2} < d = J_2$.

$$\text{If } F^* - P^* = \frac{a+c-d}{2} - \frac{a-c-d}{4} . \quad J_1 = -\frac{1}{a-c} \left[\frac{a-c-d}{4} \right]^2 + \frac{(a-c+d)^2}{4(a-c)}$$

With $d < 3(a+c)$, $J_1 < \frac{3}{16} \frac{[(4/3)d]^2}{d/3} = d$. Therefore, if $a+c \leq d < 3(a+c)$, $J_1 < J_2 = d$. If $d < a+c$, based

on result 2, with K^* as the unique root to $Z(K) = 0$,

$$J_2 = J(K^*) = -\frac{1}{a-c} \left(\frac{adK^*}{2[(1-K^*)a-c]} \right)^2 - \frac{(a-c+d)^2}{4(a-c)} \text{ if } K^* \leq 1 \text{ or } J_2 = J(1) \text{ if } K^* > 1 .$$

Let K_1 denote the unique root to $\frac{a+c+d}{4} = \frac{adK}{2[(1-K)a-c]}$, then $K_1 = \frac{(a+c)(a-c+d)}{a(a+c-3d)}$ (> 0),

with $(1-K_1)a-c = \frac{2d(a-c)}{a-c-3d} \geq 0$. Because

$$Z(K_1) = -\frac{2d^2(a-c)(a-c+d)}{(a-c-d)^3} [(a-c)^2 + 3d(a+c) - 18d^2] < 0 ,$$

we have $K_1 > K^*$, and by (B1), $J_1 < J_2$. (B6)

If $\frac{(a-c-d)^2}{8(a-c)} > \frac{a}{4}$ and $d < 3(a+c)$, or if either $d=0$ or $\frac{(a+c-d)^2}{8(a-c)} = \frac{a}{4}$ and $d < 3(a+c)$ but the

distributor will not choose $F^* = \frac{a}{2}$ in the equilibrium contract of result 1.

then $F^* - P^* = \frac{a-c-d}{2} - \frac{a-c-d}{4}$. Same as just shown. $J_1 < J_2$.

(iii) If $d \geq 3(a+c)$, $J_1 = d = J_2$. If $d = \frac{\sqrt{3}}{6}(a+c)$ and $\frac{(a-c-d)^2}{8(a+c)} < \frac{a}{4}$, or if $d = \frac{\sqrt{3}}{6}(a+c)$ and

$\frac{(a-c-d)^2}{8(a+c)} = \frac{a}{4}$ and the distributor will choose $F^* = \frac{a}{2}$ in the equilibrium contract of result

1, as shown in (i), $J_1 = J(K_0) = J(K^*) = J_2$ for $K_0 = K^*$. If $d=0$ and $\frac{(a+c+d)^2}{8(a+c)} \leq \frac{a}{4}$ and the

distributor will choose $F^* = \frac{a}{2}$ in the equilibrium contract of result 1, $J_1 = \frac{a+c}{4} = J_2$.

Q.E.D.

Proof of Proposition 4:

(i) is self-evident with results 1 and 2.

(ii) If $a-c \leq d < 3(a+c)$, in the equilibrium contract of result 1,

$$W_1 = \frac{7}{32} \frac{(a-c+d)^2}{(a+c)} \leq \frac{7}{32} \frac{(d+d/3)^2}{d/3} = \frac{7}{6} d ;$$

in the equilibrium contract of result 2, $W_2 = d + \frac{a+c}{2} > \frac{7}{6} d$.

If $\frac{\sqrt{3}}{6}(a+c) < d < a+c$, according to proposition 2 and (iii) of proposition 1, $W_1 < W_2$.

If $(0 \leq) d \leq \frac{\sqrt{3}}{6}(a+c)$ and $F^* - P^* = \frac{3(a-c-d)}{4}$ in the equilibrium contract of result 1,

according to (B6) and (iii) of proposition 1, $W_1 < W_2$.

If $0 < d \leq \frac{\sqrt{3}}{6}(a+c)$ and $F^* - P^* = \frac{a+c}{2} - d$ in the equilibrium contract of result 1, as known

in the proof of proposition 3, $W_1 = J(K_0) - \frac{a+c}{8} < W(K_0)$, where $K_0 = \frac{a+c}{2a}$. Note that, in result

1, the equilibrium contract with $F^* - P^* = \frac{a+c}{2} - d$ implies $c \leq a$ and especially, $c < a$ if $d > 0$.

Thus here $K_0 < 1$. In the equilibrium contract of result 2, $W_2 = W(K^*)$ if $0 \leq K^* \leq 1$

or $W_2 = W(1)$ if $K^* > 1$, where K^* is the unique root to $Z(K) = 0$. Here note that K^* must be nonnegative for $d < a+c$.

Let K_2 denote the unique root to $\frac{\partial W(K)}{\partial K} = 0$. By (B3), $K_2 = \frac{(a+c)(a-c-d)}{a(a-c-2d)} > 0$.

Because $(1-K_2)a-c = \frac{(a+c)d}{a-c+2d} \geq 0$, we have

$$\frac{\partial W(K)}{\partial K} > 0 \quad \text{for } \forall K \text{ such that } K < K_2. \quad (\text{B7})$$

$$Z(K_2) = \frac{d^2(a+c)^2}{a-c-2d} \left[\frac{d(a+c)}{(a-c-2d)^2} - 2 \right] - 3 \frac{d^3(a+c)}{a-c+2d} \leq 0 \quad \rightarrow \quad K^* \leq K_2. \quad (\text{B8})$$

By (B5), $K_0 \leq K^* \leq K_2$. Thus $W_1 < W(K_0) \leq W(K^*) = W_2$.

(iii) If $d \geq 3(a+c)$, $W_1 = d - \frac{a+c}{2} = W_2$.

If $d=0$, and $F^* = \frac{a}{2}$ in the equilibrium contract of result 1, $W_1 = \frac{3(a+c)}{8} = W_2$.

Q.E.D.

Proof of Proposition 2:

(i) Based on result 1, we only need to show $(\pi^D) = \frac{a}{4} > \frac{c}{4} - \frac{d}{2} (= \pi^C)$ when $0 < d \leq \frac{a}{2}$ and

$$\frac{a}{4} \geq \frac{(a+c+d)^2}{8(a+c)} . \text{ Suppose } \frac{a}{4} = \frac{c}{4} + \frac{d}{2} , \text{ i.e., } c=a-2d, \text{ then } \frac{(a-c+d)^2}{8(a+c)} = \frac{(2a-d)^2}{16(a-d)} \equiv q(d) .$$

$$q'(d) = \frac{(2a-d)d}{16(a-d)^2} > 0 \quad \text{for } 0 < d \leq a/2, \Rightarrow q(d) > q(0) = a/4. \text{ Therefore, the set}$$

$$\Omega \equiv \{(c, d) \mid 0 < d \leq \frac{a}{2} \text{ and } \frac{a}{4} \geq \frac{(a+c+d)^2}{8(a+c)}\} \text{ will be on one side of the plane } \frac{a}{4} = \frac{c}{4} + \frac{d}{2}$$

(taking c and d as variables). In particular, at $c=0$ and $d=a/4$, $\frac{a}{4} > \frac{c}{4} + \frac{d}{2}$ and $(0, a/4) \in \Omega$.

Thus $\frac{a}{4} > \frac{c}{4} + \frac{d}{2}$ for $\forall (c, d) \in \Omega$.

(ii) Evident from result 2 and Table 2-2.

Q.E.D.

Proof of Proposition 1:

(i) and (ii) are self-evident from Table 1 and Table 2.

(iii) If $d=0$ and $c \leq a$, $\pi^C = \frac{c}{4}$ in both Table 1-2 and Table 2-2.

If $d=0$ and $c > a$, $\pi^C = \frac{(a+c)}{16} < \frac{c}{8}$ in Table 1-2 while $\pi^C = \frac{c}{4}$ in Table 2-2.

If $a-c < d$, $\pi^C \leq a+c$ in Table 1 while $\pi^C = d$ in Table 2.

If $0 < d \leq a-c$, $\pi^C = \frac{(a-c+d)^2}{16(a-c)}$ or $\pi^C = \frac{c}{4} + \frac{d}{2}$ in Table 1. If $\pi^C = \frac{(a+c+d)^2}{16(a+c)}$ of result 1,

let K_3 denote the unique root to $\frac{(a+c+d)^2}{16(a+c)} = \frac{[(1-K^*)a+c-d]^2}{4[(1-K^*)a+c]}$ ($\equiv \pi^C(K^*)$).

That is, $K_3 = -\frac{d(a+c-d)}{a(a+c)} < 0$. In Table 2, $\pi^C = \frac{[(1-K^*)a+c+d]^2}{4[(1-K^*)a+c]} = \pi^C(K^*)$, where

$0 \leq K^* \leq 1$ and $(1-K^*)a+c \geq d$ (see appendix D for reasons).

Because $\frac{\partial \pi^C(K)}{\partial K} = a \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a+c]^2}$,

we have $\frac{\partial \pi^C(K)}{\partial K} \geq 0$ for $\forall K$ such that $(1-K)a+c \geq d$. Therefore, $\pi^C(K_3) < \pi^C(K^*)$.

If $\pi^C = \frac{c}{4} - \frac{d}{2}$ of result 1, $\frac{c}{4} - \frac{d}{2} < \frac{(a+c-d)^2}{4(a+c)} = \pi^C(0) \leq \pi^C(K^*)$.

Q.E.D.

Proof of Proposition 7:

If $d \geq a+c$, based on result 2, $F^* + P^* \text{ separated without vertical constraint} = d$;

based on result 3, $F^* - P^* \text{ merged} = d$;

based on result 1, $F^* - P^* \text{ separated with vertical constraint} = \begin{cases} \frac{3(a+c-d)}{4} & \text{if } d \leq 3(a+c) \\ d & \text{if } d \geq 3(a+c) \end{cases}$.

If $d \leq a+c$, based on result 2, $F^* + P^* \text{ separated without vertical constraint} = \frac{a+c+d}{2} - \frac{adK^*}{2[(1-K^*)a+c]}$,

where $0 \leq K^* \leq 1$ and $(1-K^*)a+c > 0$;

based on result 3, $F^* + P^* \text{ merged} = \frac{a+c+d}{2}$;

based on result 1, $F^* + P^* \text{ separated with vertical constraint} = \frac{3(a+c-d)}{4}$.

or $F^* - P^* \text{ separated with vertical constraint} = \frac{a+c-d}{2} - \frac{d}{2}$.

Q.E.D.

Proof of Proposition 8:

(i) If $d \geq a+c$, based on result 2, $S^* \text{ separated without vertical constraint} = \frac{a+c}{2}$;

based on result 3. $S|_{merged} = \frac{a+c}{2}$;

based on result 1. $S|_{separated\ with\ vertical\ constraint} = \begin{cases} \frac{(a-c+d)^2}{32(a-c)} \leq \frac{a-c}{2} & \text{if } d \leq 3(a-c) \\ \frac{a-c}{2} & \text{if } d \geq 3(a-c) \end{cases}$.

If $d \leq a+c$, based on result 2, result 3, (B2) and $0 \leq K^* \leq 1$,

$$S|_{separated\ without\ vertical\ constraint} = \frac{(a+c)[(1-K^*)a+c-d]^2}{8[(1-K^*)a+c]^2} \geq \frac{(a-c-d)^2}{8(a+c)} = S|_{merged} \geq \frac{a-c}{8} ;$$

based on result 1. $S|_{separated\ with\ vertical\ constraint} \leq \frac{a-c}{8}$.

(ii) If $d \geq a+c$, based on result 2, $W|_{separated\ without\ vertical\ constraint} = d - \frac{a+c}{2}$;

based on result 3, $W|_{merged} = d + \frac{a+c}{2}$;

based on result 1. $W|_{separated\ with\ vertical\ constraint} = \begin{cases} \frac{7(a-c+d)^2}{32(a-c)} \leq d + \frac{a+c}{2} & \text{if } d \leq 3(a-c) \\ d - \frac{a+c}{2} & \text{if } d \geq 3(a-c) \end{cases}$.

If $d \leq a+c$, based on result 3, $W|_{merged} = \frac{3(a+c+d)^2}{8(a-c)} = W(0)$;

based on result 2, with K^* as the unique root to $Z(K) = 0$.

$$W|_{separated\ without\ vertical\ constraint} = W(\min(K^*, 1)) .$$

By (B7) and (B8), $W|_{merged} \leq W|_{separated\ without\ vertical\ constraint}$.

Based on result 1, either $W|_{separated\ with\ vertical\ constraint} = \frac{7(a-c+d)^2}{32(a-c)} < W(0)$

$$\text{or } W \Big|_{\text{separated with vertical constraint}} = \frac{3(a+c)}{8} - \frac{d}{2} \leq W(0) \text{ .}$$

Q.E.D.

Appendix C Proof of Result 1

According to (1), (2), (7) and (8),

$$\pi^D = KF \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F - P \leq d; & (R_1) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F - d, P \leq c - d, P \geq 0; & (R_2) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq c - d, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a, F - P \geq a - c - d, P \geq 0; & (R_4) \\ 1 - \frac{F - P - d}{a - c} & \text{if } F \geq 0, P \geq 0, d \leq F - P \leq a - c - d, P \leq \frac{c}{a}F - d; & (R_5) \end{cases}$$

$$\pi^C = \begin{cases} P \\ P(1 - \frac{P - d}{c}) \\ 0 \\ 0 \\ P(1 - \frac{F - P - d}{a - c}) \end{cases} - (1 - K)F \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F - P \leq d; & (R_1) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F - d, P \leq c - d, P \geq 0; & (R_2) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq c - d, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a, F - P \geq a - c - d, P \geq 0; & (R_4) \\ 1 - \frac{F - P - d}{a - c} & \text{if } F \geq 0, P \geq 0, d \leq F - P \leq a - c - d, P \leq \frac{c}{a}F - d; & (R_5) \end{cases}$$

The vertical contract is: the cinema chooses P first, then the distributor chooses F and K .

First of all, the distributor will never choose $K=0$. Thus, we take $K \neq 0$ in the following.

Step 1: Cinema's best choice of P in each region.

Figure C.1

In region R_1 ,

$$P^* = d - F \text{ for } \forall F \in [0, d].$$

In region R_2 ,

$$\pi^C = P[1 - (P - d)/c] - (1 - K)F[1 - (F/a)].$$

$$\partial \pi^C / \partial P = [(c + d)/c] - [2P/c] = 0$$

$$\Rightarrow P = (c + d)/2.$$

Therefore.

Case A: if $c \geq d$,

$$P^* = \begin{cases} \frac{1}{2}(c - d) & \text{if } 0 \leq F \leq \frac{a}{2c}[c - d]; \\ \frac{c}{a}F - d & \text{if } \frac{a}{2c}[c - d] \leq F \leq a; \end{cases}$$

Figure C.2

Case B: if $c \leq d$, $P^* = [c/a]F - d$. See Figure C.1 for these two cases.

In region R_3 ,

$$\pi^C = [P - (1 - K)F](1 - \frac{F + P - d}{a - c}),$$

$$\frac{\partial \pi^C}{\partial P} = 0 \Rightarrow P = \frac{a - c + d - (2 - K)F}{2}. \quad (C1)$$

The line (C1) may take one of three possible positions, as shown by Figure C.2. Case 1 or case 2 happens if $a + c \leq d$. Case 3 happens if $a + c \geq d$.

$$\text{Case 1: if } a + c \leq (1 - K)d \text{ and } a + c \leq d, \text{ then } P^* = \begin{cases} 0 & \text{if } d \leq F \leq a - c + d; \\ d - F & \text{if } 0 \leq F \leq d. \end{cases}$$

$$\text{Case 2: if } a + c \geq (1 - K)d \text{ and } a + c \leq d, \text{ then point N has } F = \frac{d - a - c}{K},$$

therefore.

$$P^* = \begin{cases} d-F & \text{if } 0 \leq F \leq \frac{d-a-c}{K}; \\ \frac{1}{2}[a-c-d-(2-K)F] & \text{if } \frac{d-a-c}{K} \leq F \leq \frac{a+c-d}{2-K}; \\ 0 & \text{if } \frac{a+c-d}{2-K} \leq F \leq a-c-d. \end{cases}$$

Case 3: if $a+c \geq d$, point G has $F = \frac{a(a-c-d)}{(2-K)a-2c}$,

therefore,

$$P^* = \begin{cases} \frac{c}{a}F-d & \text{if } 0 \leq F \leq \frac{a(a-c-d)}{(2-K)a-2c}; \\ \frac{1}{2}[a-c-d-(2-K)F] & \text{if } \frac{a(a-c-d)}{(2-K)a-2c} \leq F \leq \frac{a-c+d}{2-K}; \\ 0 & \text{if } \frac{a+c-d}{2-K} \leq F \leq a-c-d. \end{cases}$$

It is evident that the cinema will never make a choice in regions R_3 and R_4 .²³

Step 2: Cinema's globally best choice of P.

If $c \leq d$, it is evident that the cinema's globally best choice is in region R_5 .

That is, case 1, case 2 and case 3 above are all possible.

Figure C.3

²³

Because the definitions of π^c and π^d on regions R_1 , R_2 and R_5 apply to the situation where $c=0$, this property implies result 1 also applies to the situation where $c=0$.

If $c \geq d$, then of course $a+c \geq d$, case 3 must happen. It is easy to show point G is upper to point E in Figure C.2 (Note the conditions $a > 0$, $c \geq 0$ and $d \geq 0$). Thus, the cinema's globally best choice of P is

$$P^* = \begin{cases} \frac{1}{2}(c-d) & \text{if } 0 \leq F \leq \frac{a(c-d)}{2c}; \\ \frac{c}{a}F - d & \text{if } \frac{a(c-d)}{2c} \leq F \leq \frac{a(a-c-d)}{(2-K)a-2c}; \\ \frac{1}{2}[a-c-d-(2-K)F] & \text{if } \frac{a(a-c-d)}{(2-K)a-2c} \leq F \leq \frac{a-c-d}{2-K}; \\ 0 & \text{if } \frac{a+c-d}{2-K} \leq F \leq a-c-d. \end{cases}$$

This P^* is illustrated by Figure C.3.

Step 3: Distributor's locally best choice if $c \geq d$.

(1) Corresponding to HE (refer to Figure C.3), the distributor's problem is

$$\left\{ \max_{K,F} \pi^D = KF(1 - \frac{F}{a}) \quad \text{s.t.} \quad 0 \leq K \leq 1 \text{ and } 0 \leq F \leq \frac{a(c-d)}{2c} \right\}.$$

The feasible set of this problem is illustrated by OEGD in Figure C.4. $\partial \pi^D / \partial F = 0 \Rightarrow F = a/2$. Because $[a(c-d)]/[2c] \leq a/2$, the solution to this problem must be at the boundaries of the feasible region, which are OD: $\{K=0, 0 \leq F \leq [a(c-d)]/[2c]\}$, EG: $\{K=1, 0 \leq F \leq [a(c-d)]/[2c]\}$, OE: $\{F=0, 0 \leq K \leq 1\}$ and GD: $\{F=[a(c-d)]/[2c], 0 \leq K \leq 1\}$. Then it is evident that $K^*=1$

$$\text{and } F^* = \frac{a(c-d)}{2c} \text{ with } \pi^D = \frac{a(c^2-d^2)}{4c^2}.$$

(2) Corresponding to EG (refer to Figure C.3), the distributor's problem is

$$\left\{ \max_{K,F} \pi^D = KF(1 - \frac{F}{a}) \quad \text{s.t.} \quad 0 \leq K \leq 1 \text{ and } \frac{a(c-d)}{2c} \leq F \leq \frac{a(a-c-d)}{(2-K)a-2c} \right\}.$$

The feasible set of this problem is illustrated by DGHC in Figure C.4. The first-order

conditions will give rise to a unique point $\{K=1, F=a/2\}$, which is not an internal point of the feasible set. In other words, the optimal point must be at the boundaries of the feasible set.

Consider at boundary GH: $\{K=1, [a(c-d)]/[2c] \leq F \leq [a(a+c-d)]/[(2-K)a+2c]\}$,

$\pi^D = F[1 - (F/a)]$, $\partial \pi^D / \partial F = 1 - [2F/a] = 0 \Rightarrow F = a/2$. Therefore,

if $\frac{a(a-c-d)}{a-2c} \geq \frac{a}{2}$, i.e., $d \leq \frac{a}{2}$,

$F^* = \frac{a}{2}$, $K^* = 1$, with $\pi^D = \frac{a}{4}$;

if $d \geq \frac{a}{2}$, $F^* = \frac{a(a-c-d)}{a-2c}$, $K^* = 1$,

with $\pi^D = \frac{a(a-c-d)(c-d)}{(a-2c)^2}$.

Figure C.4

Consider at boundary HC:

$\{0 \leq K \leq 1, KF = \frac{2(a-c)F}{a} - (a-c-d)\}$,

$\pi^D = KF(1 - \frac{F}{a})$.

$\frac{\partial \pi^D}{\partial F} = 0 \Rightarrow F = \frac{a[3(a-c)-d]}{4(a-c)}$. Therefore,

if $\frac{a[3(a-c)-d]}{4(a-c)} \geq \frac{a(a-c-d)}{a-2c}$, $K^* = 1$, $F^* = \frac{a(a-c-d)}{a-2c}$, with $\pi^D = \frac{a(a-c-d)(c+d)}{(a-2c)^2}$;

if $\frac{a[3(a-c)-d]}{4(a-c)} \leq \frac{a(a-c-d)}{a-2c}$, $K^* = \frac{2(a+c)(a-c-d)}{a[3(a-c)-d]}$, $F^* = \frac{a[3(a-c)-d]}{4(a-c)}$, with $\pi^D = \frac{(a+c+d)^2}{8(a-c)}$.

(3) Corresponding to GB (refer to Figure C.3), the distributor's problem is

$\{ \max_{K,F} \pi^D = \frac{KF(a+c-d-KF)}{2(a-c)} \text{ s.t. } 0 \leq K \leq 1 \text{ and } \frac{a(a-c-d)}{(2-K)a-2c} \leq F \leq \frac{a-c-d}{2-K} \}$.

The feasible set of this problem is illustrated by HNBC in Figure C.4.

$$\frac{\partial \pi^D}{\partial (KF)} = 0 \quad \rightarrow \quad KF = \frac{a-c-d}{2}.$$

Point S (see Figure C.4) is decided by
$$\begin{cases} F = \frac{a(a+c-d)}{(2-K)a-2c} \\ F = \frac{a-c-d}{2K} \end{cases}, \text{ which derives } K_s = \frac{2(a-c)(a-c-d)}{a[3(a-c)-d]}.$$

Point V (see Figure C.4) is decided by:
$$\begin{cases} F = \frac{a-c-d}{2-K} \\ F = \frac{a-c-d}{2K} \end{cases} \quad \rightarrow \quad K_v = \frac{2}{3}.$$

Therefore, if $K_s = \frac{2(a-c)(a-c-d)}{a[3(a-c)-d]} \leq 1$, $\forall K^* \in [\frac{2}{3}, K_s]$, $F^* = \frac{a-c-d}{2K^*}$, with $\pi^D = \frac{(a-c-d)^2}{8(a-c)}$;

if $K_s = \frac{2(a-c)(a-c-d)}{a[3(a-c)-d]} \geq 1$, $\forall K^* \in [\frac{2}{3}, 1]$, $F^* = \frac{a-c-d}{2K^*}$, with $\pi^D = \frac{(a-c-d)^2}{8(a-c)}$.

(4) Corresponding to BA (refer to Figure C.3), the distributor's problem is

$$\left\{ \max_{K, F} \pi^D = \frac{KF(a+c-d-F)}{a-c} \text{ s.t. } 0 \leq K \leq 1 \text{ and } \frac{a-c-d}{2-K} \leq F \leq a-c+d \right\}.$$

The feasible set of this problem is illustrated by NAB in Figure C.4. The first-order conditions of this problem will give rise to a unique point $\{K=1, F=(a+c+d)/2\}$, which is not an internal point of the feasible set. Thus, the optimal point must be at boundary BN: $\{0 \leq K \leq 1, F(2-K)=a+c+d\}$. On this boundary, $\pi^D = [2F-(a+c+d)][a+c-d-F]$. Then it is easy to check that the optimal point is point V.

Figure C.5

Step 4: Distributor's globally best choice if $c > d$.

Figure C.5 illustrates the situation where $[a(a+c-d)/(a+2c)] \geq a/2$. That is, point Q is below to point H. Under this situation, the distributor's globally best choice as well as the associated equilibrium contract are characterized by:

$$\text{if } d \leq \frac{a}{2}, \text{ and } \frac{(a-c-d)^2}{8(a-c)} \leq \frac{a}{4}, \quad K^* = 1, F^* = \frac{a}{2}, P^* = \frac{c}{2} - d, \pi^D = \frac{a}{4}, \pi^C = \frac{c}{4} - \frac{d}{2};$$

$$\text{if } d \leq \frac{a}{2}, \text{ and } \frac{(a-c-d)^2}{8(a-c)} \geq \frac{a}{4}, \quad \forall K^* \in [\frac{2}{3}, \min(1, K_1)], F^* = \frac{a-c-d}{2K^*},$$

$$P^* = \frac{a-c-d}{2} \left[\frac{3}{2} - \frac{1}{K^*} \right], \pi^D = \frac{(a-c-d)^2}{8(a-c)}, \pi^C = \frac{(a-c-d)^2}{16(a-c)};$$

$$\text{if } d \geq \frac{a}{2}, \quad \forall K^* \in [\frac{2}{3}, \min(1, K_1)], F^* = \frac{a-c-d}{2K^*},$$

$$P^* = \frac{a-c-d}{2} \left[\frac{3}{2} - \frac{1}{K^*} \right], \pi^D = \frac{(a-c-d)^2}{8(a-c)}, \pi^C = \frac{(a-c-d)^2}{16(a-c)}.$$

Step 5: Distributor's globally best choice if $c \leq d$.

To begin with, note that, with $c \leq d$, the cinema's globally best choice is those it may choose in region R_3 . Thus, case 1, case 2 and case 3 as shown in Figure C.2 should be analyzed separately.

$$\text{Under case 1, i.e., } a+c \leq (1-K)d \text{ and } a+c \leq d, \quad \pi^D = \begin{cases} KF(1 - \frac{F-d}{a+c}) & \text{if } d \leq F \leq a-c-d; \\ KF & \text{if } 0 \leq F \leq d; \end{cases} \text{ so the}$$

distributor's best choice is $K^* = (d-a-c)/d$, $F^* = d$, with $\pi^D = d-a-c$.

Under case 2, i.e., $a+c \geq (1-K)d$ and $a+c \leq d$,

$$\pi^D = \begin{cases} KF & \text{if } 0 \leq F \leq \frac{d-a-c}{K}; \\ KF \left[\frac{a-c-d-KF}{2(a-c)} \right] & \text{if } \frac{d-a-c}{K} \leq F \leq \frac{a-c-d}{2-K}; \\ KF \left[\frac{a-c-d-F}{a+c} \right] & \text{if } \frac{a+c+d}{2-K} \leq F \leq a-c-d; \end{cases}.$$

$$\frac{\partial \pi^D}{\partial F} = 0 \Rightarrow F = \frac{a-c-d}{2} \left(\leq \frac{a-c-d}{2-K} \right).$$

Therefore, the solution to this problem must be at the boundaries of its feasible set. On boundary HB (see Figure C.6), point B is evidently the optimal point, with $F^*=d$ and $\pi^D=d-a-c$. On boundary BN, $\pi^D=[2F-a-c-d](a+c+d-F)/(a-c)$, $\partial \pi^D/\partial F=0 \Rightarrow F=3[a+c+d]/4$ and $K=2/3$, i.e., point E.

To summarize, if $a+c>0$, $c \leq d$ and $a+c \leq d$, the distributor's globally best choice as well as the associated equilibrium contract are characterized by (C2)-(C5).

Under case 3, i.e., $a-c \geq d$ and $c \leq d$,

$$\pi^D = \begin{cases} KF(1 - \frac{F}{a}) & \text{if } 0 \leq F \leq \frac{a(a-c-d)}{(2-K)a-2c}; \\ KF[\frac{a+c-d-KF}{2(a+c)}] & \text{if } \frac{a(a-c-d)}{(2-K)a-2c} \leq F \leq \frac{a+c-d}{2-K}; \\ KF[\frac{a+c-d-F}{a-c}] & \text{if } \frac{a+c-d}{2-K} \leq F \leq a-c-d. \end{cases}$$

Obviously, the equilibria under case 3 are same as when $c \geq d$.

Step 6: Movie-goers' surplus.

From the above, we know that all the people who go to the cinema will take snack foods within the cinema in all the equilibrium contracts. Therefore, if $d \leq \frac{a}{2}$, and $\frac{(a-c-d)^2}{8(a-c)} \leq \frac{a}{4}$,

$$F^*-P^* = \frac{a-c}{2} - d, \quad (a-c)\theta - d - F^* - P^* = 0 \Rightarrow \theta_m = \frac{1}{2}.$$

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{a-c}{8};$$

if $d \geq 3(a-c)$, $F^*-P^*=d$, $(a-c)\theta - d - F^* - P^* = 0 \Rightarrow \theta_m = 0$,

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{a-c}{2} ;$$

Under other situations. $F^* - P^* = \frac{3(a-c-d)}{4}$. $(a-c)\theta - d - F^* - P^* = 0 \Rightarrow \theta_m = \frac{3(a-c)-d}{4(a-c)}$.

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{(a-c-d)^2}{32} .$$

Q.E.D.

Appendix D Proof of Result 2

According to (1), (2), (7) and (8),

$$\pi^D = KF \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F - P \leq d; & (R_1) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F - d, P \leq c - d, P \geq 0; & (R_2) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq c - d, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a, F - P \geq a - c - d, P \geq 0; & (R_4) \\ 1 - \frac{F - P - d}{a - c} & \text{if } F \geq 0, P \geq 0, d \leq F - P \leq a - c - d, P \leq \frac{c}{a}F - d; & (R_5) \end{cases}$$

$$\pi^C = \begin{cases} P \\ P(1 - \frac{P - d}{c}) \\ 0 \\ 0 \\ P(1 - \frac{F - P - d}{a - c}) \end{cases} - (1 - K)F \begin{cases} 1 & \text{if } F \geq 0, P \geq 0, F - P \leq d; & (R_1) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq \frac{c}{a}F - d, P \leq c - d, P \geq 0; & (R_2) \\ 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, P \geq c - d, P \geq 0; & (R_3) \\ 0 & \text{if } F \geq a, F - P \geq a - c - d, P \geq 0; & (R_4) \\ 1 - \frac{F - P - d}{a - c} & \text{if } F \geq 0, P \geq 0, d \leq F - P \leq a - c - d, P \leq \frac{c}{a}F - d; & (R_5) \end{cases}$$

No vertical pricing constraint in the contract: the cinema chooses F and P first, then the distributor chooses K .

Step 1: Cinema's locally best choice of F and P .

In region R_1 , $P^* = d$, $F^* = 0$ (note that the distributor will not choose $K = 0$), with $\pi^C = d$ and $\pi^D = 0$.

In region R_2 , the cinema's problem is $\{\max_{F,P} \pi^C = P[c + d - P]/c + (1 - K)F[a - F]/a$
S.T.: $0 \leq F \leq a$ and $(c/a)F + d \leq P \leq c + d\}$.

Figure D.1

$$\partial\pi^C/\partial P=0 \Rightarrow P=(c+d)/2 .$$

$$\partial\pi^C/\partial F=[1-K][a-2F]/a=0 \Rightarrow F=a/2 \text{ if } K \neq 1, \text{ or } \forall F \in [0,a] \text{ if } K=1.$$

If $K \neq 1$, plug $F=a/2$ into $P=(c/a)F+d$, $\Rightarrow P=(c/2)+d \geq (c+d)/2$ (note that $d \geq 0$). Therefore, the point $\{F=a/2, P=(c+d)/2\}$ is not an internal point of region R_2 , so that the solution to this problem must be on the boundaries of region R_2 . If $c \leq d$, $(c-d)/2 \leq d$, even though $K=1$, the solution to this problem is still on the boundaries. If $c > d$ and $K=1$, the point on QN (see Figure D.1) are all the solutions to this problem.

In region R_5 , $\pi^C=[P+(1-K)F][a+c+d-F-P]/[a+c]$. Denote the determinant of the Hessian matrix of π^C by Π . $\Pi=[4(1-K)-(2-K)^2]/(a+c)^2 \leq 0$ ($=0$ only if $K=0$). Therefore the solution to this problem must be on the boundaries.

Obviously, the cinema will never choose a point in regions R_3 and R_4 except boundary DE.²⁴

Step 2: Cinema's globally best choice of F and P .

It has been shown in step 1 that, (1)if $K=1$ and $c \geq d$, the cinema's globally best choice must be on AB, BC, and/or QN; (2)if $K < 1$ or $c \leq d$, the cinema's globally best choice must be on AB, BC, CD, CE and/or ED.

On AB, $\pi^C=(1-K)F[a+c-d-F]/[a+c]$, $\partial\pi^C/\partial F=0 \Rightarrow F=(a+c+d)/2$. Therefore, $F^*=(a+c+d)/2$ if $a+c \geq d$ or $F^*=d$ if $a+c \leq d$, and accordingly $\pi^C=(1-K)\frac{(a+c+d)^2}{4(a+c)}$ or $\pi^C=(1-K)d$.

On BC, note that the distributor will never set $K=0$, so the optimal point of the cinema is $\{F^*=0, P^*=d\}$, which is also a point of CD.

²⁴

Because the definitions of π^C and π^D on regions R_1 , R_2 , R_4 and R_5 applies to the situation where $c=0$, result 2 also applies to the situation where $c=0$.

On CD, $\pi^C = \left[\frac{(1-K)a-c}{a} F - d \right] \left[1 - \frac{F}{a} \right]$, $\partial \pi^C / \partial F = 0 \Rightarrow F^* = \frac{a[(1-K)a-c-d]}{2[(1-K)a+c]}$ if $d \leq (1-K)a+c$ or

$F^*=0$ if $d \geq (1-K)a+c$. Accordingly, $\pi^C = \frac{[(1-K)a-c-d]^2}{4[(1-K)a+c]}$ or $\pi^C = d$.

On CE, $\pi^C = P \left(1 - \frac{P-d}{c} \right)$, $F^*=0$, $P^*=(c+d)/2$ if $c \geq d$ or $P^*=d$ if $c \leq d$, accordingly $\pi^C = \frac{(c-d)^2}{4c}$ or $\pi^C = d$.

On ED, $\pi^C = (1-K)F \left(1 - \frac{F}{a} \right)$, $P^*=c+d$, $F^*=a/2$ if $K \neq 1$ or $\forall F^* \in [0, a]$ if $K=1$, with $\pi^C = (1-K)a/4$.

The optimal choices of the cinema on these boundaries as well as the associated profits are summarized in Table D-1 and Table D-2.

Table D-1

AB		CD	
$a+c \geq d$	$a-c \leq d$	$d \leq (1-K)a+c$	$d \geq (1-K)a+c$
$F^*=(a+c+d)/2$ $P^*=0$ $\pi^C = (1-K) \frac{(a-c-d)^2}{4(a-c)}$	$F^*=d$ $P^*=0$ $\pi^C = (1-K)d$	$F^* = \frac{a[(1-K)a-c-d]}{2[(1-K)a+c]}$ $P^* = \frac{c[(1-K)a-c-d]}{2[(1-K)a+c]} - d$ $\pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a+c]}$	$F^*=0$ $P^*=d$ $\pi^C = d$

Table D-2

CE		QN (if $K=1$ and $c \geq d$)	ED
$c \geq d$	$c \leq d$		
$F^*=0$ $P^*=(c+d)/2$ $\pi^C = \frac{(c-d)^2}{4c}$	$F^*=0$ $P^*=d$ $\pi^C=d$	$\forall F^* \in [0, \frac{a(c-d)}{2c}]$ $P^* = \frac{c-d}{2}$ $\pi^C = \frac{(c-d)^2}{4c}$	$F^*=a/2$ if $K \neq 1$ or $\forall F^* \in [0, a]$ if $K=1$ $P^*=c+d$ $\pi^C=(1-K)a/4$

Given $K \neq 1$ and $c^2-d^2 \neq 0$, $(1-K)(a+c+d)^2/[4(a+c)] > (1-K)a/4$;

if $a+c \leq d$, $(1-K)d \geq (1-K)(a-c) > (1-K)a/4$. Therefore, the maximum value of π^C on ED is strictly dominated by that on AB. That is, the cinema's globally best choice of F and P will not be on ED. If $d \leq (1-K)a+c$, then of course $d < a+c$.

Because $\partial \{[(1-K)a+c+(1-K)d]^2/[4((1-K)a+c)]\}/\partial c \geq 0$,

$$\frac{[(1-K)a-c-d]^2}{4[(1-K)a+c]} \geq \frac{[(1-K)a+c-(1-K)d]^2}{4[(1-K)a+c]} \geq \frac{[(1-K)(a-c+d)]^2}{4(1-K)(a-c)} = (1-K) \frac{[a-c-d]^2}{4[a-c]},$$

of which at least one strict inequality holds. If $d \geq (1-K)a+c$ and $a+c \geq d$, it can be shown that $(1-K)(a+c+d)^2/[4(a+c)] < d$.

In fact, $(1-K)(a+c+d)^2/[4(a+c)]$

$$= [(1-K)a + (1-K)c + (1-K)d](a+c+d)/[4(a+c)]$$

$$= [((1-K)a - c + d - K(c+d))](a+c+d)/[4(a+c)]$$

$$\leq [2d - K(c+d)](a+c+d)/[4(a+c)]$$

$$\leq [2d - K(c+d)][2(a+c)]/[4(a+c)]$$

$$< d.$$

If $a+c \leq d$, it is evident that the maximum value of π^C on AB is strictly dominated by that on CD. Therefore, given $K \neq 1$ and $c^2-d^2 \neq 0$, the cinema's optimal point will not be on AB. In other words, the cinema's globally best choice will be on CD or CE.

If $c \geq d$, then of course $(1-K)a+c \geq d$, it is easy to show $\frac{[(1-K)a+c-d]^2}{4[(1-K)a+c]} \geq \frac{[c-d]^2}{4c}$ (equality

holds only if $K=1$). Therefore, if $c \geq d$ and $K \neq 1$, the cinema's globally best choice is

$$\left\{ F^* = \frac{a[(1-K)a+c-d]}{2[(1-K)a+c]}, \quad P^* = \frac{c[(1-K)a+c-d]}{2[(1-K)a+c]} - d \right\}, \quad \text{with} \quad \pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a+c]} \quad \text{and}$$

$$\pi^D = Ka \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a+c]^2}.$$

If $c \leq d$, and if $(1-K)a+c \leq d$, $P^*=d$ and $F^*=0$, with $\pi^C=d$ and $\pi^D=0$.

If $c \leq d$, and if $(1-K)a+c \geq d$, the cinema's globally best choice is same as when $c \geq d$.

If $K=1$ and $c^2+d^2 \neq 0$, from Table D-1 and Table D-2, we know that the cinema's globally best choice is still not on AB and ED.

To conclude, Table D-3 below summarizes the cinema's globally best choice and the associated profits of the cinema and the distributor.

Table D-3

K ≠ 1		K = 1	
c ≥ d	c ≤ d		c ≤ d
$F^* = \frac{a[(1-K)a+c-d]}{2[(1-K)a-c]}$ $P^* = \frac{c[(1-K)a+c-d]}{2[(1-K)a-c]} + d$ $\pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a-c]}$ $\pi^D = Ka \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a-c]^2}$	$d \leq (1-K)a+c$	$d \geq (1-K)a-c$	$F^* \in [0, \frac{a(c-d)}{2c}]$ $P^* = \frac{c-d}{2}$ $\pi^C = \frac{(c-d)^2}{4c}$ $\pi^D = F^* (1 - \frac{F^*}{a})$
	$F^* = \frac{a[(1-K)a+c-d]}{2[(1-K)a-c]}$ $P^* = \frac{c[(1-K)a+c-d]}{2[(1-K)a-c]} + d$ $\pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a-c]}$ $\pi^D = Ka \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a-c]^2}$	$F^* = 0$ $P^* = d$ $\pi^C = d$ $\pi^D = 0$	

Step 3: Distributor's best choices of K and the associated equilibrium contracts.

With the "friendship assumption" (see section 2), if K=1 and c ≥ d, the cinema will choose $F^* = [a(c-d)]/[2c]$ (and $P^* = (c-d)/2$). Thus, Table D-3 is reduced to Table D-4.

Table D-4

$c \geq d$	$c \leq d$	
$F^* = \frac{a[(1-K)a+c-d]}{2[(1-K)a+c]}$ $P^* = \frac{c[(1-K)a+c-d]}{2[(1-K)a+c]} - d$ $\pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a+c]}$ $\pi^D = Ka \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a+c]^2}$	$d \leq (1-K)a+c$	$d \geq (1-K)a+c$
	$F^* = \frac{a[(1-K)a+c-d]}{2[(1-K)a+c]}$ $P^* = \frac{c[(1-K)a+c-d]}{2[(1-K)a+c]} + d$ $\pi^C = \frac{[(1-K)a+c-d]^2}{4[(1-K)a+c]}$ $\pi^D = Ka \frac{[(1-K)a+c]^2 - d^2}{4[(1-K)a+c]^2}$	$F^*=0$ $P^*=d$ $\pi^C=d$ $\pi^D=0$

Define $f(K) \equiv Ka \{ [(1-K)a+c]^2 - d^2 \} / [2(1-K)a+2c]^2$.

$f(0)=0$, and $f(1)=a[c^2-d^2]/[4c^2]$. Thus, $f(1) \geq 0$ if $c \geq d$ or $f(1) \leq 0$ if $c \leq d$.

$f'(K) = [a/4] \{ [(1-K)a+c]^3 - [(1-K)a+c]d^2 - 2ad^2K \} / [(1-K)a+c]^3$.

Therefore, $f'(K)=0 \Leftrightarrow [(1-K)a+c]^3 - [(1-K)a+c]d^2 - 2ad^2K=0$.

Define $Z(K) \equiv [(1-K)a+c]^3 - [(1-K)a+c]d^2 - 2ad^2K$.

$Z'(K) = -3[(1-K)a+c]^2a - ad^2 < 0$.

Therefore, there is at most one root to $f'(K)=0$ for $K \in [0, 1]$.

$Z(0) = (a+c)(a+c-d)$ (≥ 0 if $a+c \geq d$).

$Z(1) = c^3 - cd^2 - 2ad^2$.

$Z((a+c-d)/a) = -2d^2(a+c-d)$.

Then the distributor's globally best choice of K as well as the associated equilibrium contract are characterized by:

if $c \geq d$ and $Z(1) = c^3 - cd^2 - 2ad^2 \geq 0$, $K^*=1$, $F^*=(c-d)a/[2c]$, $P^*=(c+d)/2$, $\pi^C=[c+d]^2/[4c]$ and $\pi^D=a[c^2-d^2]/[4c^2]$;

if $c \geq d$ and $Z(1) = c^3 - cd^2 - 2ad^2 \leq 0$, K^* is the unique root to $Z(K) = 0$ for $K \in [0, 1]$,

$$F^* = \frac{a[(1-K^*)a-c-d]}{2[(1-K^*)a-c]}, \quad P^* = \frac{c[(1-K^*)a-c-d]}{2[(1-K^*)a-c]} - d, \quad \text{with} \quad \pi^C = \frac{[(1-K^*)a-c-d]^2}{4[(1-K^*)a-c]} \quad \text{and}$$

$$\pi^D = Ka \frac{[(1-K^*)a-c]^2 - d^2}{4[(1-K^*)a-c]^2};$$

if $c \leq d$ and $a+c \geq d$, the equilibrium contract is the same as if $c \geq d$ and $Z(1) = c^3 - cd^2 - 2ad^2 \leq 0$;

if $c \leq d$ and $a+c \leq d$, $F^* = 0$, $P^* = d$, $\pi^C = d$ and $\pi^D = 0$.

Step 4: Movie-goers' surplus.

From the above, we see that all the people who come to the cinema will take snack foods within the cinema in all the equilibrium contracts. Therefore, for the equilibrium contract

$$\text{that } F^* = \frac{a(c-d)}{2c} \text{ and } P^* = \frac{c-d}{2}, \quad (a-c)\theta - d - F^* - P^* = 0 \quad \Rightarrow \quad \theta_m = \frac{c-d}{2c},$$

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{(a-c)(c-d)^2}{8c^2};$$

$$\text{for the equilibrium contract that } F^* = \frac{[(1-K^*)a-c-d]a}{2[(1-K^*)a-c]} \text{ and } P^* = \frac{[(1-K^*)a-c-d]c}{2[(1-K^*)a-c]} + d,$$

$$(a-c)\theta - d - F^* - P^* = 0 \quad \Rightarrow \quad \theta_m = \frac{[(1-K^*)a-c-d]}{2[(1-K^*)a-c]},$$

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{(a-c)[(1-K^*)a-c+d]^2}{8[(1-K^*)a-c]^2};$$

for the equilibrium contract that $F^* = 0$ and $P^* = d$,

$$(a-c)\theta - d - F^* - P^* = 0 \quad \Rightarrow \quad \theta_m = 0.$$

$$S(\text{surplus}) = \int_{\theta_m}^1 [(a-c)\theta - d - F^* - P^*] d\theta = \frac{a-c}{2}.$$

Q.E.D.

Appendix E Proof of Result 4

According to (3), (4), (7) and (8),

$$\pi^D = KF \begin{cases} 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, F - P > d; & (Q_1) \\ 1 & \text{if } 0 \leq F \leq a, F - P \leq d, P \geq 0; & (Q_2) \\ 0 & \text{if } F \geq a, P \geq 0; & (Q_3) \end{cases}$$

$$\pi^C = \begin{cases} 0 \\ P(\frac{d-P}{a}) \\ 0 \end{cases} - (1-K)F \begin{cases} 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, F - P > d, P \geq 0; & (Q_1) \\ 1 & \text{if } F - P \leq d, F \geq 0, P \geq 0; & (Q_2) \\ 0 & \text{if } F \geq a, P \geq 0; & (Q_3) \end{cases}$$

The vertical contract is: the cinema to choose P first, then the distributor to choose F and K .

Step 1: Cinema's best choice of P .

In region Q_2 , the cinema's best choice is $P^* = d/2$ if $0 \leq F \leq d/2$ or $P^* = d - F$ if $d/2 \leq F \leq d$. In regions Q_1 and Q_3 , $P^* = 0$ if $F \geq d$ or $P^* = d - F$ if $0 \leq F \leq d$ ("friendship

Figure E.1

assumption"). Therefore, the cinema's globally best choice of P is shown as in Figure E.1.²⁵

²⁵

Mathematically, Q_2 can be extended to include the points on WG with the value of π^c (π^p) at a point on WG being defined as the limitation of a series of values of π^c (π^p) at a series of points within Q_2 which converges to this point. Such extension allows Q_2 to be closed and π^c (π^p) to be continuous on Q_2 . But two different values of π^c (π^p) exist at each point on WG , one defined from Q_1 and the other defined from Q_2 . However, the value of π^c (as well as π^p , except at point G) defined from Q_2 is larger than that defined from Q_1 . Thus this extension has no impact on analyzing the cinema's (the

Step 2: Distributor's best choice of F and K and the associated equilibrium contract.

In region Q_2 , $K^*=1$ and $F^*=d$, with $\pi^D=d$. In region Q_1 , the distributor's problem is

$$\{ \max_{K, F} \pi^D = KF(1-F/a) \text{ S.T. } 0 \leq K \leq 1, F \geq 0 \text{ and } F+P^* \geq d \}.$$

$$\partial[F(1-F/a)]/\partial F = 0 \Rightarrow F = a/2.$$

Owing to $a \geq d$, of course $a/2 \geq d/2$ (see Figure E.1). With $K^*=1$ and $F^*=a/2$ (then $P^*=\max(0, (2d-a)/2)$), $\pi^D=a/4$. Therefore, if $d \geq a/4$, the distributor should set $K^*=1$ and $F^*=d$ (then $P^*=0$, i.e., point W); if $d \leq a/4$, the distributor should set $K^*=1$ and $F^*=a/2$ (then $P^*=\max(0, (2d-a)/2)$, i.e., point V). It should be mentioned that, if $d \leq a/4$ and the distributor sets $F^*=a/2$, even though $P^*=0$, a half of the population will go to the cinema but none of these movie-goers will take snack foods within the cinema; if $d \geq a/4$ and the distributor sets $F^*=d$, all the people will go to the cinema but only some of them will take snack foods. Under the

equilibrium contract when $d \leq a/4$, the total surplus of movie-goers is $S_1 = \int_{1/2}^1 (a\theta - \frac{a}{2}) d\theta = \frac{a}{8}$.

Under the equilibrium contract when $d \geq a/4$, the total surplus of movie-goers from auditing

the movie $S_m = \int_0^1 (a\theta - d) d\theta = \frac{a-2d}{2}$; only the people such that $\theta \in [0, d/a]$ take snack foods.

with the total surplus from snack foods of them being $S_f = \int_0^{d/a} (-a\theta - d) d\theta = \frac{d^2}{2a}$. That is, if

$d \geq a/4$, the equilibrium contract will result in the total surplus of movie-goers to be

$$S_2 = S_m - S_f = \frac{(a-d)^2}{2a}.$$

Q.E.D.

distributor's) globally best choice.

Appendix F Proof of Result 5

According to (3), (4), (7) and (8),

$$\pi^D = KF \begin{cases} 1 - \frac{F}{a} & \text{if } 0 \leq F \leq a, F - P > d; & (Q_1) \\ 1 & \text{if } 0 \leq F \leq a, F - P \leq d, P \geq 0; & (Q_2) \\ 0 & \text{if } F \geq a, P \geq 0; & (Q_3) \end{cases}$$

$$\pi^C = \begin{cases} 0 & \text{if } 0 \leq F \leq a, F - P > d, P \geq 0; & (Q_1) \\ P(\frac{d-P}{a}) - (1-K)F & \text{if } 0 \leq F \leq a, F - P \leq d, P \geq 0; & (Q_2) \\ 0 & \text{if } F \geq a, P \geq 0; & (Q_3) \end{cases}$$

No vertical pricing constraint in the contract: the cinema chooses F and P first, then the distributor chooses K .

Step 1: Cinema's best choice of F and P in each region.²⁶

Figure F.1

In region Q_2 , the cinema's problem is

$$\{ \max_{F,P} \pi^C = (1-K)F + P[(d-P)/a], \text{ S.T. } F \geq 0, P \geq 0 \text{ and } F+P \leq d \}.$$

The determinant of Hessian matrix of π^C is zero. Therefore, the optimal point of the cinema must be among the points of the boundaries of the feasible set or among those satisfying the first-order conditions. The first-order conditions with the "friendship assumption", give rise

²⁶

The footnote of appendix E applies here.

to $F=d/2$ and $P=d/2$,²⁷ i.e., point N on WC (see Figure F.1). On boundary WO, the cinema will choose $F=d$ (and $P=0$) if $K<1$ or $\forall F \in [0, d]$ (and $P=0$) if $K=1$; on boundary OC, $F=0$ and $P=d/2$; on boundary WC, $F=[d+(1-K)a]/2$ and $P=[d-(1-K)a]/2$ if $(1-K)a \leq d$ or $F=d$ and $P=0$ if $(1-K)a \geq d$. With little effort, one can conclude that the cinema's best choice in region Q_2 is the point with $F=[d+(1-K)a]/2$ and $P=[d-(1-K)a]/2$ if $(1-K)a \leq d$ or $F=d$ and $P=0$ if $(1-K)a \geq d$, which respectively lead to $\pi^C=[d+(1-K)a]^2/(4a)$ or $\pi^C=(1-K)d$.

In region Q_1 , $\pi^C=(1-K)F(1-F/a)$. The maximum profit the cinema may obtain is $(1-K)a/4$ with F being chosen at $F=a/2$ (use the "friendship assumption").

The cinema will never pick up a point in region Q_3 .

Step 2: Cinema's globally best choice of F and P .

With the assumption that $d > a/4$,

$$\text{if } K=1 \ (\geq 1-(d/a)), \quad \pi^C = \frac{[d-(1-K)a]^2}{4a} > (1-K)\frac{a}{4} = 0 \quad ;$$

$$\text{if } 1 - \frac{d}{a} \leq K < 1, \quad \pi^C = \frac{[d-(1-K)a]^2}{4a} \geq (1-K)d > (1-K)\frac{a}{4} \quad ;$$

$$\text{if } K \leq 1 - \frac{d}{a}, \quad \pi^C = (1-K)d > (1-K)\frac{a}{4} \quad .$$

Therefore, the cinema's globally best choice is the point with $F = \frac{[d-(1-K)a]}{2}$

and $P = \frac{[d-(1-K)a]}{2}$ if $(1-K)a \leq d$, leading to $\pi^C = \frac{[d-(1-K)a]^2}{4a}$, or the point with $F=d$ and

$P=0$ if $(1-K)a \geq d$, leading to $\pi^C = (1-K)d$.

The cinema's globally best choice and the associated profits of itself and the distributor are

²⁷

Without the "friendship assumption", $P=d/2$ and $\forall F \in [0, d/2]$ satisfy the first-order conditions when $K=1$.

summarized in Table F-1.

Table F-1

$K \leq 1 - \frac{d}{a}$	$K \geq 1 - \frac{d}{a}$
$F^* = d$ $P^* = 0$ $\pi^C = (1-K)d$ $\pi^D = Kd$	$F^* = \frac{d-(1-K)a}{2}$ $P^* = \frac{d-(1-K)a}{2}$ $\pi^C = \frac{[d-(1-K)a]^2}{4a}$ $\pi^D = K \frac{d-(1-K)a}{2}$

Step 3: Distributor's best choice of K and the associated equilibrium contract.

With Table F-1, if $K \geq 1 - \frac{d}{a}$, $\pi^D = K \frac{d-(1-K)a}{2}$.

$$\frac{\partial}{\partial K} \left\{ K \frac{d-(1-K)a}{2} \right\} = 0 \quad \Rightarrow \quad K = \frac{a+d}{2a} \leq 1 : \quad \frac{a-d}{2a} \geq 1 - \frac{d}{a} \quad \Rightarrow \quad d \geq \frac{a}{3}.$$

With $K = \frac{a+d}{2a}$, $\pi^D = K \frac{d-(1-K)a}{2} = \frac{(a+d)^2}{8a} > (1 - \frac{d}{a})d$, where $(1 - \frac{d}{a})d$ is the maximum profit

the distributor can obtain when K is limited by $K \leq 1 - \frac{d}{a}$.

Therefore, if $d \geq \frac{a}{3}$, the distributor will choose $K^* = \frac{a+d}{2a}$, resulting in the equilibrium

contract to be $\{ F^* = \frac{d-a}{4}, P^* = \frac{3d-a}{4}, \pi^D = \frac{(a-d)^2}{8a} \text{ and } \pi^C = \frac{(a-d)^2}{16a} \}$;

if $d \leq \frac{a}{3}$, the distributor will choose $K^* = 1 - \frac{d}{a}$, resulting in the equilibrium contract to be

$\{ F^* = d, P^* = 0, \pi^D = \frac{(a-d)d}{a} \text{ and } \pi^C = \frac{d^2}{a} \}$.

Step 4: Movie-goers' Surplus.

All the population will go to the cinema. If $d \geq \frac{a}{3}$, the total surplus from viewing the movie

of the people is $S_{m1} = \int_0^1 (a\theta - \frac{a-d}{4})d\theta = \frac{a-d}{4}$. If $d \leq \frac{a}{3}$, the total surplus from viewing the

movie of the people is $S_{m2} = \int_0^1 (a\theta - d)d\theta = \frac{a-2d}{2}$. To calculate the total surplus from snack

foods is a bit complex. If $d \geq \frac{a}{3}$, we know that $F^* = \frac{d-a}{4}$ and $P^* = \frac{3d-a}{4}$. The solution

to $a\theta - \frac{a-d}{4} = 0$ is $\theta = \frac{d-a}{4a}$, which is also the solution to $-a\theta - d - \frac{3d-a}{4} = 0$. That is, such

persons as $\frac{d-a}{4a} < \theta \leq 1$ can get positive surplus from viewing the movie but will not take

snack foods with the cinema. All other persons will take snack foods within the cinema. The

total surplus from snack foods of them is $S_{f1} = \int_0^{(a-d)/(4a)} (-a\theta - d - \frac{3d-a}{4})d\theta = \frac{(a-d)^2}{32a}$. If $d \leq \frac{a}{3}$,

we know that $F^* = d$ and $P^* = 0$. The solution to $a\theta - F^* = 0$ is $\theta = \frac{d}{a}$, which is also the

solution to $-a\theta - d - P^* = 0$. Thus with $d \leq a/3$, such persons as $0 \leq \theta \leq \frac{d}{a}$ will take snack

foods within the cinema. The total surplus from snack foods of them is

$S_{f2} = \int_0^{d/a} (-a\theta - d)d\theta = \frac{d^2}{2a}$. Therefore, the total surplus of movie-goers under the equilibrium

contract is $S = S_{m1} - S_{f1} = \frac{(3a-d)^2}{32a}$ if $d \geq \frac{a}{3}$, or $S = S_{m2} - S_{f2} = \frac{(a-d)^2}{2a}$ if $d \leq \frac{a}{3}$.

Q.E.D.

References

- Cassady, Ralph. Jr. (1959), Impact of the Paramount Decision on Motion Picture Distribution and Price Making, *Southern California Law Review*, Vol.31, pp325-390.
- Conant, Michael (1960), *Antitrust in the Motion Picture Industry: Economic and Legal Analysis*, University of California Press, Berkeley and Los Angeles.
- (1981). The Paramount Decrees Reconsidered, *Law and Contemporary Problems*, Vol.144, pp79-107.
- De Vany, Arthur, and Ross D. Eckert (1991), Motion Picture Antitrust: The Paramount Cases Revisited, in Richard O. Zerbe, Jr. and Victor P. Goldberg, ed., *Research in Law and Economics*, Vol.14, pp51-112.
- , and W. David Walls (1996), Bose-Einstein Dynamics and Adaptive Contracting in the Motion Picture Industry, *The Economic Journal*, 106(November), pp14933-1514.
- Donahue, Suzanne Mary (1987), *American Film Distribution: The Changing Marketplace*, UMI Research Press, Ann Arbor.
- Fellman, Nat D. (1983), The Exhibitor, in Jason E. Squire, ed., *The Movie Business Book*, 3rd ed., pp323-326, Simon and Schuster, New York.
- Hellmuth, William F., Jr. (1962), The Motion Picture Industry, in Walter Adams, ed., *The Structure of American Industry: Some Case Studies*, 3rd ed., pp393-429, Macmillan, New York.
- Kenney, Roy W. (1983), and Benjamin Klein, The Economics of Block Booking, *The Journal of Law & Economics*, Vol.26 (October), pp497-540.
- Kindem, Gorham (1982), ed., *The American Movie Industry*, Southern Illinois University Press, Carbondale.
- Klein, Benjamin, Robert G. Crawford, and Armen A. Alchian (1978), Vertical Integration, Appropriable Rents, and the Competitive Contracting Press, *The Journal of Law & Economics*, Vol.21, pp297-326.
- May, Richard (1983), Sample Exhibition Contract, in Jason E. Squire, ed., *The Movie Business Book*, 3rd ed., pp323-326, Simon and Schuster, New York.
- Murphy, A.D. (1983), Distribution and Exhibition: an Overview, in Jason E. Squire, ed., *The*

Movie Business Book, 3rd ed., pp244-262, Simon and Schuster, New York.

Tirole, Jean (1988). The Theory of Industrial Organization, The MIT Press, Cambridge, Massachusetts.

Chapter 3 The Buyback Option Revisited

1. Introduction

Debt buyback refers to the practice that a debt-troubled country repurchases some or all of its debt obligations from its creditors at the secondary market price or a negotiated price. Since the Baker plan and the Brady plan introduced the buyback option, the majority of the studies about the option have concluded that a buyback may not be beneficial to a debtor country. The prevailing views on this issue are: (i) from a debtor's perspective, the buyback option is not an efficient approach to the reduction of debt obligation if a buyback has to be performed at the ex post secondary market price; (ii) a debtor country gains through a buyback only when either the creditors offer large enough concession in transaction price or there is a large enough third partner's fund (e.g., donation from IMF or the World Bank) to enhance the buyback.

These views on debt buyback are developed and formally analyzed by Dooley (1988), Claessens and Diwan (1989), Bulow and Rogoff (1988, 1991) and Bulow and Rogoff and Zhu (1994), etc.. Dooley (1988) argues that, if a LDC announces a repurchase of a significant amount of its debt on the secondary market, it will immediately raise the price at which the transaction must be undertaken to the level that corresponds to the ex post value of its debt, and consequently the buyback will produce no benefit for the LDC. Here the price of a LDC's debt at the secondary market is considered to be the value of the expected revenue of the debt from the country divided by the amount of its debt. Claessens and Diwan (1989) maintain that if investment in a LDC is not wasteful, a buyback will inevitably lead to a reduction in investment and will necessarily result in inefficient use of the country's resource. Bulow and Rogoff (1988) went further than Dooley, Claessens and Diwan, pointing out that even if a LDC buybacks a small amount (or "rides-up" the supply curve by repurchasing its debt one dollar after the other), the price at which the transaction is performed on the secondary market corresponds to the average value of the debt which is

(perhaps well) above the marginal price. This view is formally demonstrated in their 1991 paper. Building on these analyses, Bulow, Rogoff and Zhu (1994) estimate the “efficiency gain” of the buyback case of Mexico.

Figure 1 can be used to illustrate the key points of Dooley (1988) and Bulow and Rogoff (1988). Any transaction of a LDC’s debt on the secondary market not participated in by the LDC will not affect the price of the LDC’s debt on the secondary market (panel (a)). But if the country announces its plan to repurchase amount X of its debt D , the price of its debt on the secondary market will immediately be raised to $P(D-X)$ (panel (b)). Thus Dooley (1988) concludes that the country will overpay ABC for its repurchase X . Bulow and Rogoff (1988) argue that the country will bear a loss $AQNT$ even though the buyback is done along with the debt supply curve $P(D-X)$, for the country will face a raised price for its residual debt.

Figure 1

According to these studies, a highly indebted country should not buyback its debt at market discount. The facts, however, are that buybacks have been prevalent in the past as well as in the present. In a comprehensive study of the debt crisis of the 1930s, Eichengreen and Portes (1989) find that repurchases of debt obligation by sovereign debtors occurred extensively. In the period from 1988 to 1993, the annual debt stock reductions all over the developing

countries are 15,868; 16,022; 25,179; 4,999; 16,483 and 7,323 million U.S. dollars respectively, of which debt buybacks account for 1.64%, 15.2%, 17.3%, 20.6%, 46.6% and 12.4% respectively.¹ Furthermore, a debt-troubled country is generally more willing than its creditors to accept the buyback option in dealing with its debt troubles (Sachs (1989a)). These facts are obviously not consistent with the predictions of the studies cited above.

There have been some arguments supporting the option. Dornbusch (1988) points out that there should be efficiency gains from reducing its debt burden through any approach, including the buyback option, so that the LDC will attract more foreign investment after a debt reduction. Krugman (1989) shows that the market-based buyback scheme will ordinarily benefit a LDC if the country is on the wrong side of the "debt relief Laffer curve", that is, where a reduction in nominal claims actually increases expected payment.² Cohen and Verdier (1995) show that a "secret" buyback may benefit both a LDC and the creditors. Cabral (1996) studies the buyback option with a model in which buyback is taken as investment for assets. He demonstrates that it is reasonable for a LDC to undertake some buybacks if the parameters of the model satisfy a particular condition.³

A common feature of all existing theoretical models is that a buyback is assumed to be a zero-sum game among a LDC, its creditors and (perhaps) the third partners who may donate funds to enhance this buyback. In other words, no efficiency gain exists within a LDC through reducing its debt burden. As a result, either the LDC or the third partners or the both

1

Based on the data of The World Bank, World Debt Tables 1994-95, Volume 1, p195.

2

The empirical work by Claessens (1990), however, concludes that virtually no debtor country is on the wrong side of its debt Laffer curve.

3

Cabral (1996) does not give detailed discussion on whether the condition may be satisfied by the parameters.

must be the loser of buybacks. This feature is not consistent with the recognition that efficiency gain exists in a LDC through reducing its debt burden and that is the reason that the developed countries and international organisations should help LDCs to alleviate their debt burden. Efficiency gain arises from the fact that severe debt burden distorts the economic structure of a LDC and retards investment. Reducing its debt burden will help the country to adjust its economic structure and to attract investment so as to stimulate its economic growth (e.g., see Sachs (1989b)). Thus both theoretical perspectives and the observations of reality call for a more reasonable assessment of the buyback option.

In this chapter, we will study a model in which debt reduction generates efficiency gain within the debtor country itself. Here the efficiency gain is modelled as an improvement in productivity. The logic for this is that, debt reduction will enhance domestic investment, and increased investment level will enhance productivity. Behind this logic is the theory of endogenous economic growth (e.g., see Romer (1986, 1990, 1994) and Lucas (1988)). A number of interesting results emerge from our analysis. First, we show that a buyback is welfare-improving for both the debtor country and its creditors. This follows directly the efficiency gain of debt reduction. Second, we demonstrate that if the creditors are large and act as a Stackelberg leader, they may be reluctant to accept a buyback deal at the ex post secondary market price. The reason is that, with the expectation that a debt buyback will enhance the level of investment and the returns on the investment in the debtor economy, the creditors, acting as a Stackelberg-leader, may demand a price that is higher than the ex post secondary market price and perhaps even higher than the par value. (In other words, they may find the ex post secondary market price too low). Third, we show that the ex post secondary market price can be used as a benchmark dividing the efficiency gain of a buyback deal between the debtor country and its creditors. Finally, the predictions of this model are shown to be consistent with many facts in the historical and current debt crises.

The rest of this chapter is organized as follows. Section 2 presents the model. Section 3 shows why a buyback at the ex post secondary market price is welfare-improving for a LDC.

Section 4 analyses negotiated buybacks so as to explain why a LDC is active on the buyback option and why the so-called “free-rider” problem cannot explain the unwillingness on the part of creditors to agree to buybacks. We propose our own explanation to this unwillingness in section 5. Section 6 illustrates that a buyback is welfare-improving to both the LDC and its creditors. Section 7 presents an empirical study, which confirms the existence of the adverse effect of debt overhang on productivity. Some concluding remarks are given in section 8.

2. The Model

Consider a two-period framework. A debt-troubled country has borne a debt which will be due with a face value D in period 2. In period 1, the country has constant resource W , which can be used to consume (C_1), invest (I) and repurchase part or full amount (X) of its debt. That is, the country has the following budget constraint in period 1:

$$W = C_1 + I - XP, \quad (1)$$

where P is the transaction price of the buyback deal X .

In period 2, the country has constant resource W^* and random investment proceeds. That is, the total resource available to the country is

$$W^* - sf(I)H(D-X) \equiv W^* + sf(I)h(X). \quad (2)$$

In (2), $H(D-X)$ ($\equiv h(X)$, $h'(X) \geq 0$) specifies the impact of the “debt overhang” ($D-X$) on investment proceeds, or productivity. It captures the efficiency gain when the country buybacks some of its debt. The function $h(X)$ is normalized such that $h(0)=1$. $f(I)$ is investment effect function under current “debt overhang”. s represents the random factors that affect productivity but are independent of I and X . We motivate this specification as follows.

Debt-troubled countries share a fundamental characteristic that they suffered a noticeable fall in investment in the 1980s relative to the 1960s and 1970s. A popular explanation has to do

with the debt overhang. This hypothesis argues that when a LDC is unable to fully service its debt obligation, actual payments tend to depend on the country's economic performance. The existence of a heavy debt burden then depresses the return on investment so as to weaken the incentive to invest (Sachs (1989b), Helpman (1989a)).⁴ Formally, if the country's debt service is $s(D)$ ($s'(D) > 0$) and the country invests I that generates a proceed $f(I)$, the country's return on investment is only $(1-s(D))f(I)$, which is just a special case of (2). Husain (1993) uses a similar functional form to associate debt overhang with investment proceeds. Another explanation for the fall in investment is that the debt crisis is a liquidity issue, as opposed to a solvency issue (Krugman(1989)). According to this view, lucrative investment opportunities are available in a LDC, but some sort of market failure associated with the debt crisis has prevented creditors from lending any further. The Baker plan of the mid-1980s reflected this view and called for new lending by commercial banks. According to this explanation, the actual production of a debt-troubled country can be written as $A(I(D), D)f(I(D))$, where $I(D)$ is the country's investment plan under debt overhang D (and other resource constraints) with $I'(D) < 0$, and $A(I(D), D)$ represents gross productivity level. The key point is $(\partial A(I(D), D))/(\partial D) < 0$, because many investment projects will not be completed on schedule and many completed projects or firms will not be operated at full capacities due to liquidity constraint. Furthermore, the debt overhang may harm productivity via twisting investment structure. For instance, a country unable to fully service its debt obligation may divert its investment from its export-oriented industries to domestic-consumption ones, even though the former have higher productivities than the latter. Goldberg and Spiegel (1992) address this issue with a two-sector model and conclude that debt overhangs tend to distort the composition of investment in the debt nation and thus nominal debt reductions mitigate these distortions. Finally, debt overhang may damp resource mobility in the economy, which will also reduce the aggregate productivity (Fry(1991)).

4

We present a brief model to describe the investment behaviour under debt overhang in appendix B.

Indeed, using cross-country data, Cohen (1992) shows that, "the aggregate productivity growth experienced noticeable slowdown all over the world in the 1980s; the nondebtor countries experienced a significant productivity slowdown of about 2.3%; beyond this general slowdown, the moderately and severely indebted countries, as defined by the World Bank, experienced an additional fall of about 0.9%; the countries that rescheduled their debt more than once in the 1980s experienced an additional slowdown of 1.4%". "The debt-troubled countries experienced a specific slowdown of their growth rate, even after taking account of the fall of investment and the variations of their terms of trade, is confirmed by the econometric analysis." In an empirical study with the help of the time series of data of Mexico, we also find the existence of debt overhang effect on the supply of domestic investment funds (see section 7). These findings confirm the adverse effects of debt overhang on productivity progress.

Now return to (2). $f(I)$ and $h(X)$ are assumed to satisfy

$$f'(I) > 0 \text{ and } f''(I) < 0, \quad (3)$$

$$h'(X) \geq 0 \text{ and } h''(X) < 0, \quad (4)$$

$$h'(0) > \frac{1}{D} \cdot \frac{a}{1-a} . \quad (5)$$

The parameter a in (5) is the fraction of the debtor country's output that would be seized by its creditors if the country defaults part or full of its debt obligation. This assumption has been broadly adopted by the theoretical models of debt crisis (e.g., see Helpman (1989b), Froot (1989), Bulow and Rogoff (1988, 1991), Krugman (1989) and Cohen (1995)). a can be interpreted as the result of a bargaining game between the debtor country and its creditors, as in Bulow and Rogoff (1988). In such a game, the creditors threaten to impose a trade embargo or to eliminate trade credit — actions that will impose a cost on the debtor country. The debtor country then offers a fraction of its output to the creditors in order to avoid these penalties. Bulow and Rogoff (1988) argue on the basis of empirical evidence that a is likely to be small and close to zero. As they pointed out, the resources transferred by debtor

countries to (private) creditors in any given year have been at most 10 percent of GNP and at most 25 percent of exports. The 1988-89 edition of the World Debt Tables indicates that the highest ratio of debt service to GNP over the years 1980-87 for developing countries as a whole was 4.5 percent in 1987 and 5 percent for the highly indebted countries. The highest debt-service to export ratio for all countries was 20.2 percent in 1986; for highly indebted countries it was 29 percent in 1986. a may also be interpreted as a tax ceiling faced by the debtor country government, as in Helpman (1989b). Under Froot (1989)'s "gunboat technology" assumption, $a=1$. In general, however, $0 < a < 1$.

(4) and (5) imply that efficiency gain exists for at least a small amount of buyback X . (5) also means that the efficiency gain is not negligible at the initial debt overhang. With a being small, as pointed out in the last paragraph, and D being large, (5) is actually not a strong assumption. It ensures that it is always favourable for the country to buyback some of its debt. Note that (3) and (4) are not enough to guarantee the satisfaction of the second order conditions for the country's optimization problem (see below) with respect to I and X . Thus besides (3) and (4), we always assume the second order conditions will be satisfied.

The random term in (2), s , is assumed to have a distribution function $L(s)$ with support $(0, \infty)$. Without loss of generality, we also assume $E(s) \equiv \int_0^\infty s dL(s) = 1$ and $W^* = 0$.

In period 2, the country may or may not choose to default on its debt obligations, depending on the realization of its production level. If the country defaults on its debt obligations, the creditor-side will be able to force the country to pay $af(I)h(X)$. If $(D-X) > af(I)h(X)$, the country is better off defaulting on the residual debt obligations $(D-X)$. Thus the creditors can only expect payment of their loans in the amount of

$$R \equiv E(\min(D-X, af(I)h(X))) = af(I)h(X) \int_0^{s^*} s dL(s) + (D-X)(1-L(s^*)), \quad (6)$$

where $s^* \equiv s^*(I, X) \equiv (D-X)/(af(I)h(X))$. For convenience of notation, we define $\int_0^{s^*} s dL(s) \equiv S_s(I, X) \equiv S_s$.

We further assume that both the country and the creditors are risk neutral and have zero time discount rate. The creditors want to maximize its expected income from their loans, i.e., $\max G \equiv (XP + R)$ subject to the constraint that $0 \leq X \leq D$, while the country wants to maximize its expected utility level under some restrictions (see below). The utility function is assumed to take the form $U \equiv C1 + C2$, where $C1$ and $C2$ represent the consumption of the country in period 1 and period 2 respectively. The creditors and the country have the same information.

3. Buyback at Market Discount

Suppose the country can buy back part or full amount of its debt at the ex post secondary market price. Then the country's maximization problem is:

$$\max U \equiv C1 + EC2 \quad (7a)$$

$$\text{s.t.: } C1 = W - I - XP(X), \quad (7b)$$

$$EC2 = f(I)h(X) - R, \quad (7c)$$

$$C1 \geq 0, EC2 \geq 0, I \geq 0 \text{ and } 0 \leq X \leq D. \quad (7d)$$

In (7), E represents expectations operator.

We always use $P(X)$ to denote the ex post secondary market price if X units of debt is repurchased by the country. That is,

$$P(X) \equiv R/(D-X) = 1 - L(s^*) + af(I)h(X)S_t/(D-X). \quad (8)$$

Substituting (8) into (7), we obtain:

$$U \equiv W - I - D(1 - L(s^*)) + f(I)h(X) - (D/(D-X))af(I)h(X)S_t. \quad (9)$$

Differentiate (9) with respect to I and X :

$$\partial U / \partial I = -1 + f'(I)h(X) - (D/(D-X))af'(I)h(X)S_t, \quad (10)$$

$$\partial U / \partial X = f(I)h'(X) - (D/(D-X))af(I)h'(X)S_t - (D/(D-X)^2)af(I)h(X)S_t. \quad (11)$$

With (11) and assumption (5), it is easy to show that the country will always devote some resource into a buyback deal if the country makes non-zero investment (i.e., $f(I) \neq 0$). In fact,

for any amount of investment I , we have

$$h'(0) > \frac{1}{D} \cdot \frac{a}{1-a} \geq \frac{1}{D} \cdot \frac{aS_i(I,0)}{1-aS_i(I,0)} \quad (12)$$

(12) implies $\partial U / \partial X > 0$ at any $I > 0$ and $X = 0$. That is, $X = 0$ is not part of a solution to the maximization problem.⁶

From (10) and (11), one can see that, if no efficiency gain exists (i.e., $h(X) = h(0) = 1$ and $h'(X) = 0$ for any X), the country should never repurchase any of its debt for it is always $\partial U / \partial X < 0$, which is the conclusion in Bulow and Rogoff (1991). Therefore, we have the following proposition.

Proposition 1: If there is considerable efficiency gain through reducing the debt stock of a debt-troubled country (assumption (5) holds), it is always welfare-improving for the country to repurchase some of its debt, even if it must pay the ex post secondary market price for the buyback.

There is an intuitive explanation of (12). When the realization of productivity is so low that the country will have to default its debt obligation, the efficiency gain from repurchasing a marginal amount (ϵ) of its debt obligation can be expressed as $(1 - aS_i(I, 0))f(I)(h(\epsilon) - h(0))$ on one hand. On the other hand, the price of its debt in the secondary market is $(1/D)af(I)S_i(I, 0)$, and hence the cost for this marginal buyback can be represented by $(1/D)af(I)S_i(I, 0)\epsilon$. The efficiency gain is larger than the cost if $(1 - aS_i(I, 0))f(I)(h(\epsilon) - h(0)) > (1/D)af(I)S_i(I, 0)\epsilon$. Dividing both sides of this inequality by ϵ , letting ϵ approach 0 and re-

⁵

It always holds that $(aS_i / (1 - aS_i)) \leq (a / (1 - a))$ for $0 \leq S_i \leq 1$.

⁶

One of the first-order conditions for the maximization is $(\partial U / \partial X) \leq 0$ if $X = 0$.

arranging, we obtain (12). Hence, (12) implies that when the marginal efficiency gain exceeds the cost of the marginal buyback, the country will gain via a marginal buyback even though such a buyback may crowd out investment.

It is evident that we can even relax assumption (5) and reformulate proposition 1 as the following.

Proposition 2: For a resource allocation plan $(C1, I, 0)$ ($I > 0$) in period 1, if the marginal efficiency gain with this plan is large enough that $h'(0) > (1/D)(aS_t(I, 0)/(1 - aS_t(I, 0)))$, the country will gain if it uses some resource to repurchase some of its debt obligation, even though the repurchase deal must be subject to the ex post secondary market price and will crowd out the consumption and/or investment in period 1.⁷

With (5) and (11), we have an obvious comparative static result:

Proposition 3: The smaller the value of "a", the more favourable for the country to use some resource to repurchase its debt obligation on the secondary market, even though it has to pay the ex post market price.

The intuition behind this proposition is that, the smaller the value of "a", the smaller the part of the efficiency gain may go into the creditors' pockets and the larger the part of the efficiency gain will the country obtain.

4. Negotiated Buybacks

In practice, buybacks are often performed through negotiations between a debt-ridden country and its (some) creditors at market-based discount. It is also observed that a debt-

⁷

We will show that buybacks will in fact crowd in investment in the later.

ridden country usually shows greater interest in making a buyback deal than its creditors. In this section, we will explain why a debtor country is willing to have negotiated buybacks. We will also show that the so-called "free-rider" problem can not be the reason that creditors are often unwillingly to participate in negotiating buybacks.

Let us first clarify what we mean by "negotiated buybacks". For simplicity, we assume that the creditors of a LDC consolidate as a group in order to negotiate a buyback deal with the LDC. The group has the following objective function:

$$G = XP + af(I)h(X)S_t + (D-X)(1-L(s^*)). \quad (13)$$

The country's objective function is:

$$U = W - I - XP + (1-aS_t)f(I)h(X) - (D-X)(1-L(s^*)). \quad (14)$$

Given a transaction price P and the country's investment plan I , the group's best choice of the amount of the deal is derived from:⁸

$$0 = \partial G / \partial X = P + af(I)h'(X)S_t - (1-L(s^*)). \quad (15)$$

Given a transaction price P , the country's best choices of the amount of the deal and of its investment plan are derived from:⁹

8

There is a restriction that $0 \leq X \leq D$ for the creditor group's maximization problem. The P corresponding to $X=0$ as maximizing G should satisfy $(\partial G / \partial X)|_{X=0} \leq 0$, which implies that P also satisfies equation (19) below. The P corresponding to $X=D$ as maximizing G should satisfy $(\partial G / \partial X)|_{X=D} \geq 0$, which implies that P must be equal to or greater than 1, for $S_t=0$ and $L(s^*)=0$ at $X=D$. However, such P is not reasonable.

9

The country's maximization problem is also subject to constraints (7d). The P corresponding to $X=0$ as maximizing U should satisfy $(\partial U / \partial X)|_{X=0} \leq 0$. Such a P still satisfies the following (19) if (18) is satisfied. The P corresponding to $X=D$ as maximizing U should satisfy $(\partial U / \partial X)|_{X=D} \geq 0$, implying that P should be equal to or greater than 1, which is not reasonable. The constraint $EC2 \geq 0$ is always satisfied through the definition of R . If we add another assumption that

$$0 = \partial U / \partial I = -1 + (1 - aS_j)f'(I)h(X), \quad (16)$$

$$0 = \partial U / \partial X = -P + (1 - aS_j)f(I)h'(X) + 1 - L(s^*). \quad (17)$$

With (15) and (17), we can draw the following Figure 2, where $P_1(X)$ is derived from (15), $P_2(X)$ from (17) and $P(X)$, the ex post secondary market price, is expressed by (8). $P_1(X)$ and $P_2(X)$ intersect at point C, which corresponds to $P = 1 - L(s^*)$ and $h'(X) = 0$. That is, point C implies the amount of buyback which exhausts all possible efficiency gain.

Figure 2

With Figure 2, we define negotiated buybacks as follows.

$f'(0) > (1/(1-a))$, then the country will always make a positive investment. Corresponding to the case that the optimal resource allocation among consumption, investment and buybacks will make the constraint $C_1 \geq 0$ bound (i.e., $C_1 = 0$), (16) and (17) should be replaced by $(\partial U / \partial X) \leq -\lambda$ and $(\partial U / \partial X) \leq -\lambda P$ respectively, where λ is the Kuhn-Tucker multiplier for the constraint $C_1 \geq 0$ and $\lambda \geq 0$. The P corresponding to such a case still satisfies the following (19) if (18) is satisfied. These special cases will be ignored hereafter.

Definition: A buyback (X, P) is a negotiated buyback if (X, P) is within region ABC of Figure 2.

Such a definition is reasonable. Intuitively, a negotiated deal should bring benefits to both the country and its creditors, but it may not be an optimal deal from their individual perspectives (i.e., the deal may not satisfy their individual first-order conditions). Clearly, a negotiated buyback by this definition will have a price that the country is willing to offer and its creditors are willing to accept and have an amount that does not exceed the amount to exhaust all potential efficiency gain. We will see in section 6 that both the country and its creditors will gain from such a negotiated deal. Therefore, such definition is consistent with the intuition.

However, among the numerous deals (X, P) within region ABC, the country may prefer one to another. The same can be said about the creditors. Which of these negotiated buybacks will be the actual equilibrium outcome? To answer this question would require the construction and analysis of a formal bargaining model, which is beyond the scope of this thesis and is left to future research.

If a buyback has amount X of debt such that

$$h'(X) > \frac{h(X)}{D-X} \cdot \frac{aS_i(I, X)}{1 - aS_i(I, X)},^{10} \quad (18)$$

one can see that for such an amount X , it always holds that

$$P_1(X) < P(X) < P_2(X). \quad (19)$$

¹⁰

(18) is the necessary and sufficient condition for $P(X) < P_2(X)$ ((19) below). With few modifications, the intuitive explanation for (12) in the previous section can apply to (18).

Especially, for any possible buyback X with which it is still $\partial U/\partial X \geq 0$ in terms of (11), it must satisfy

$$h'(X) \geq \frac{h(X)}{D-X} \cdot \frac{aS_t \frac{D}{D-X}}{1 - aS_t \frac{D}{D-X}} > \frac{h(X)}{D-X} \cdot \frac{aS_t}{1 - S_t} \quad !! \quad (20)$$

because for any $D > X > 0$,

$$\frac{aS_t \frac{D}{D-X}}{1 - aS_t \frac{D}{D-X}} > \frac{aS_t}{1 - aS_t} \quad (21)$$

Therefore, with (18)-(21), we have proposition 4 below, which states that buybacks at market-based discount should be acceptable to both the country and its creditors.

Proposition 4: Let X^* be the optimal amount of debt the country would repurchase at the ex post secondary market price. For any $X < X^*$, $(X, P(X))$ is a negotiated deal; i.e., the ex post secondary market price should always be a price the country is willing to offer and its creditors are willing to accept.

Figure 2 also illustrates proposition 4. It shows that any possible buyback to the left of the LDC's optimal buyback plan (X^*) at the ex post secondary market price should satisfy (19). In addition, the optimal buyback plan (X^*) does not exceed the buyback amount which exhausts all possible efficiency gain.

Proposition 4 explains why a debt-troubled country may be quite willing to have a buyback

11

Note that $(\partial U/\partial X) \geq 0$, in terms of (11), and (4) imply that $(1 - (D/(D-X))aS_t) > 0$, which is necessary for (20).

deal with its creditors even if the deal is done at the ex post secondary market price. The country would have an even stronger incentive to work out a buyback deal if it expects the creditors to give it "concession" in buyback price.¹² (19) also has other significant implications. It is obvious that, the smaller the a , the closer $P_1(X)$ to $P(X)$. If we note the empirical result of Claessens and Sweder (1993) that the ex post secondary market price may be very close to the current (ex ante) secondary market price, we can reasonably predict that the current secondary market price largely reflects the creditors' evaluations ($P_1(X)$) for the actual value of their claims to debt-troubled countries. And the smaller the a , the more accurate will such reflection be. The difference between the ex post secondary market price (or as an empirical approximation, the ex ante secondary market price) and the price that the creditors are willing to accept (e.g., $P_1(X)$) gives an extent of possible "concession" to which the creditors may agree when negotiating a buyback deal with their debtor country. This also explains why the creditors may agree to some "concession". The larger the a , the larger the extent of possible concession and thus a debtor country may expect a larger concession from its creditors in buyback price. The prevailing (or ex post) secondary market price can be used as an objective benchmark for dividing the efficiency gain of a buyback deal between the debtor country and the creditors (see Figure 2). The deviation of the actual transaction price from the prevailing (or ex post) secondary market price, with the corresponding buyback deal X , is not enough to estimate the total efficiency gain, or the efficiency gain of either the creditors or the debtor country. However, it can be used to estimate the shift of the efficiency gain allocation due to the deviation of transaction price from the prevailing (or ex post) secondary market price. If a is small or even close to zero, and the remaining potential of efficiency gain after a buyback deal X is negligible, or more specifically, (18) continues to hold while $h'(X)=0$, then $P_1(X)$ and $P_2(X)$ are very close while (19) continues to hold. Under that circumstance, there is little manouvring room for the price of the buyback deal X , and

¹²

It is generally considered as a concession if the creditors accept a buyback price lower than the ex ante prevailing secondary market price.

the ex ante prevailing secondary market price will most likely be accepted by both the creditors and the debtor country as the transaction price of the buyback. Records of historical facts support the arguments here. In the debt crisis of the 1930s, most of the debt-troubled countries were cornered in a predicament of little international trade and international financial relations. This implied that the values of "a" were small or even close to zero for those countries and accordingly $P_1(X)$ was close to $P(X)$. If X is close or equal to D , $h'(X)$ must be close or equal to zero. That is, in the debt crisis of the 1930s, it were probably the case that $P_1(D)=P(D)=P_2(D)$. That is our explanation why the buyback option was extensively used to settle the debt crisis, with the transaction prices being largely the prevailing secondary market prices and the deals being mostly all of the debtor countries' debt obligations.

Since (19) predicts that the creditors should be glad to accept the ex post secondary market price for any repurchase deal, and the experiences in the debt crisis of the 1930s were largely consistent with this prediction, why is the situation of the debt crisis in the 1980s not well consistent with it? In the period from 1982 to 1987, no buyback agreement was reached, which is generally considered due to the creditors' unwillingness to accept buybacks. Let us call this inconsistency the "creditor-side puzzle". Some authors (e.g., Claessens and Diwan (1989)) have argued that the free-rider problem makes the creditors reluctant to participate in buyback deals with their debtor countries. In our model, can the free-rider problem explain the "creditor-side puzzle"? To answer this question, let us assume that a creditor of a debtor country has a loan with face value D_1 due in period 2. The country has total debt obligation of D due in period 2. The country seeks a buyback deal X ($0 \leq X \leq D_1 < D$) with this creditor. The creditor considers that the residual face value $(D_1 - X)$ of its loan will be equally treated as other creditors' loans if the country defaults in period 2, as is the so-called "equal-compensation" system. Thus the creditor in the deal will want to maximize

$$G = XP + (D_1 - X)(1 - L(s^*)) + ((D_1 - X)/(D - X))af(I)h(X)S_t. \quad (22)$$

The reaction function of this creditor regarding to transaction price P and the country's investment plan I is derived from

$$0 = \frac{\partial G}{\partial X} = P - (1 - L(s^*)) + \frac{D I - X}{D - X} a f(I) h'(X) S_t - \frac{D - D I}{(D - X)^2} a f(I) h(X) S_t. \quad (23)$$

With (23) and (8), we have

$$P - P(X) = -\frac{D I - X}{D - X} \left[h'(X) + \frac{h(X)}{D - X} \right] a f(I) S_t < 0. \quad (24)$$

(24) shows that, for any repurchase deal X , even though the deal happens between only one individual creditor and the country and the creditor knows that its residual loan will get no privilege treatment in period 2, the creditor would still accept the corresponding ex post secondary market price for the deal. So the "creditor-side puzzle" remains.

Proposition 5: In this model, the so-called free-rider problem can not explain the "creditor-side puzzle".

We will present an explanation to the "creditor-side puzzle" in the next section. However, to the end of this section, we would like to point out that the $P_1(X)$ derived from (15) and the P derived from (23) may be negative or zero for some buyback deal X . These special cases may explain why creditors may agree to write-off some of their claims or even to give more funds to their debtor country under some circumstances. The $P_2(X)$ derived from (17) will never be negative. It may be zero if $L(s^*)=1$ and $h'(X)=0$ for the associated investment and buyback deal. Since $L(s^*)=1$ implies that the debtor country will certainly default its debt obligation in period 2,¹³ any buyback by the country (with positive price) definitely signals its intention not to default its debt obligation. This implication is consistent with the conclusion of Acharya and Diwan (1993).¹⁴

¹³

$L(s^*)=1$ implies that the realization of random productivity in period 2 is such that $s \leq (D-X)/(af(I)h(X))$, i.e., $af(I)h(X) \leq (D-X)$.

¹⁴

Klog (1993) finds that Germany's buybacks at the secondary market price of its debt obligation in 1932-1939 were evidently driven by

5. Creditors' Logic

How do we explain the "creditor-side puzzle"? One possible reason is that many of the creditors are gigantic banks with market power over their debtor countries. As a result, the creditors may behave like a Stackelberg-leader when a debtor country tries to repurchase some debt obligation from them. That, as will be shown below, can explain the reluctance of creditors to allow debt buybacks.

With (13), and corresponding to the debtor's buyback deal X and investment I , the creditors (as Stackelberg leader) will ask a price according to

$$0 = \partial G / \partial P = X + af'(I)h(X)S_i(\partial I / \partial P) + f(I)h'(X)(\partial X / \partial P), \quad (25)$$

where I and X are the optimal choices of the debtor country as Stackelberg follower with P taken as given, i.e., the solution to (16) and (17).

(25), (16) and (17) together decide the Stackelberg-leader price. This means that, with the approved investment plan $I(X)$ and buyback deal X by the creditors, they want to charge the price $P_2(X)$ for the deal, which has been shown to be higher than the ex post secondary market price in the previous section ((19)). In some cases the Stackelberg-leader price may imply that no buyback deal can be reached by the country and the creditors. Let us consider the following special case in order to gain more insight into the "creditor-side puzzle".

Consider the situation where $f(I) = I^k$, $h(X) = X^r$ and there is no random productivity. The k and r satisfy $0 < k, r < 1$ and $k + r < 1$ for the second order condition to be satisfied. For simplicity, we also assume that both the initial wealth W received by the country and its debt stock D are large enough that "corner solution" will not happen for the country, i.e., if a solution to (25), (16) and (17) exists, it will satisfy (7d).¹⁵ The assumption of no random productivity

Germany's secret decision not to repudiate.

¹⁵

This assumption is not important for (33) below to hold.

implies $s=1$ and $L(s)=1$ if $s \geq 1$ or $L(s)=0$ if $s < 1$, and thus

$$S_t=1 \text{ if } s^* \geq 1 \text{ or } S_t=0 \text{ if } s^* < 1. \quad (26)$$

First, suppose that the debtor's investment plan I and buyback deal X are such that $s^* < 1$.

Then (25), (16) and (17) reduce to

$$X + rI^k X^{r-1} (\partial X / \partial P) = 0, \quad (25a)$$

$$-1 + kI^{k-1} X^r = 0, \quad (16a)$$

$$-P + rI^k X^{r-1} + 1 = 0. \quad (17a)$$

(25a), (16a) and (17a) will give us

$$\frac{k}{-k} \frac{1}{r} \frac{1-k}{n} \frac{1-k}{(P-1)} = 0, \quad (27)$$

where $n=1-k-r > 0$.

Owing to $1-k > 0$ and $n > 0$, (27) holds only if

$$P = +\infty. \quad (28)$$

Mathematically, $P = +\infty$ means no solution to (25a), (16a) and (17a). Intuitively, it means that the creditors will not agree to any buyback deal X if the deal X and the associated investment plan I are such that $s^* < 1$.

Second, suppose that the debtor's investment plan I and buyback deal X are such that $s^* \geq 1$.

Then (25), (16) and (17) reduce to

$$X - a k I^{k-1} X^r (\partial I / \partial P) + r I^k X^{r-1} (\partial X / \partial P) = 0, \quad (25b)$$

$$-1 + k I^{k-1} X^r - a k I^{k-1} X^r = 0, \quad (16b)$$

$$-P + r I^k X^{r-1} - a r I^k X^{r-1} = 0. \quad (17b)$$

(25b), (16b) and (17b) will give us

$$-k^{\frac{k}{n}} r^{\frac{1-k}{n}} \frac{1}{(1-a)^n} P^{-\frac{1-k}{n}} [(1-a)(r-1)+k+(1-k)P^{-\frac{2k}{n}}] = 0 \quad (29)$$

(29) implies

$$P = +\infty \text{ if } (1-a)(1-r)-k \leq 0, \text{ or} \quad (30a)$$

$$P = \left[\frac{(1-a)(1-r)-k}{1-k} \right]^{-\frac{n}{2k}} \text{ if } (1-a)(1-r)-k > 0. \quad (30b)$$

In (30), note that $(1-a)(1-r)-k < 1-k < 1$, and thus $P > 1$ in the situation where $(1-a)(1-r)-k > 0$.

The ex post secondary market price is given by

$$P(X) = 1 - L(s^*) - S_t/s^*, \quad (31)$$

i.e.,

$$P(X) = 1/s^* \leq 1 \text{ if } s^* \geq 1 \text{ or } P(X) = 1 \text{ if } s^* < 1. \quad (32)$$

With (28), (30) and (32), we always have

$$P > 1 \geq P(X). \quad (33)$$

(33) solves the "creditor-side puzzle". If the creditors act as Stackelberg leader, they will demand a price that is even higher than the ex post secondary market price. They may not agree to the deal (corresponding to $P = +\infty$), or ask extra gain over the par value for the deal (corresponding to $1 < P < +\infty$), neither of which will be acceptable to the debtor country.

Proposition 6: If the creditors of a debt-troubled country act as Stackelberg leader, they will demand a price that is higher than the ex post secondary market price or even higher than the par value for any buyback deal. As a result, they will not agree to any buyback deal at market-based discount.

6. Welfare Effect

Let $(X, P_n(X))$ denote the buyback deal and the associated transaction price. With the buyback deal X and the associated investment $I(X)$, one can calculate $P_1(X)$ from (15) and $P_2(X)$ from (17). Corresponding to the deal X and the associated investment $I(X)$, and the prices $P_i(X)$ ($i=n, 1$) for the creditors and the prices $P_j(X)$ ($j=n, 2$) for the debtor country, the welfare of the creditors and that of the debtor country are denoted by $G(I(X), X, P_i(X))$ and $U(I(X), X, P_j(X))$ respectively. We use $G(I(0), 0, P)$ and $U(I(0), 0, P)$ (P is any price)¹⁶ to denote the welfare of the creditors and that of the debtor country respectively if no buyback occurs.

We have the following lemma (see appendix A for proof).

Lemma: $\frac{\partial I(X)}{\partial X} \geq 0$, and for $\forall X \in [0, \text{Arg}(h'(X)=0))$, $\frac{\partial I(X)}{\partial X} > 0$.¹⁷

Relative to no buyback (i.e., $X=0$), a buyback deal $(X, P_n(X))$ has following welfare effects.

EGc ≡ The Efficiency Gain of the Creditors

$$\begin{aligned} &= G(I(X), X, P_n(X)) - G(I(0), 0, P) \\ &= G(I(X), X, P_n(X)) - G(I(X), X, P_1(X)) + G(I(X), X, P_1(X)) - G(I(0), 0, P) \\ &= X(P_n(X) - P_1(X)) + G(I(X), X, P_1(X)) - G(I(0), 0, P). \end{aligned} \quad (34)$$

EGd ≡ The Efficiency Gain of the debtor country

$$\begin{aligned} &= U(I(X), X, P_n(X)) - U(I(0), 0, P) \\ &= U(I(X), X, P_n(X)) - U(I(X), X, P_2(X)) + U(I(X), X, P_2(X)) - U(I(0), 0, P) \\ &= X(P_2(X) - P_n(X)) + U(I(X), X, P_2(X)) - U(I(0), 0, P). \end{aligned} \quad (35)$$

Propositions 7 and 8 follow from the above lemma and (34) and (35).

¹⁶

If $X=0$, P will not affect the welfare of the creditors and that of the debtor country.

¹⁷

$\text{Arg}(h'(X)=0)$ denotes the solution to $h'(X)=0$. One may recall that $h'(X)=0$ is the necessary and sufficient condition for $P_1(X)=P_2(X)$.

Proposition 7: Buybacks will not crowd out, but crowd in investment.

Proposition 8: For any buyback deal $(X, P_n(X)) (X > 0)$ such that $P_1(X) \leq P_n(X) \leq P_2(X)$, (i) the creditors will gain; (ii) the debtor country will gain if $P_2(X) \neq P_2(0)$ or $P_n(X) < P_2(X)$, and even if $P_2(X) = P_n(X) = P_2(0)$, the debtor country will not lose.

Proof: $\frac{\partial G(I, X, P)}{\partial I} = af'(I)h(X)S_I(I, X) > 0$, and by the lemma.

$$G(I(X), X, P_1(X)) = \max_Z G(I(X), Z, P_1(X)) \geq G(I(X), 0, P_1(X)) > G(I(0), 0, P_1(X)).$$

With (34), (i) must be true.

According to the derivations of $I(X)$ and $P_2(X)$,

$$U(I(X), X, P_2(X)) = \max_{I, Z} \{U(I, Z, P_2(X))\} \geq U(I(0), 0, P_2(X)).$$

If $P_2(X) \neq P_2(0)$, $Z=0$ and $I(0)$ will not satisfy (16) and (17), implying

$$U(I(X), X, P_2(X)) > U(I(0), 0, P_2(X)).$$

With (35), (ii) must be true.

Q.E.D.

It is evident that both the creditors and the debtor country may still gain with a buyback deal $(X, P_n(X))$ even for some $P_n(X)$ that is below $P_1(X)$ or above $P_2(X)$.

7. An Empirical Study of Debt Overhang

In this section, we conduct an empirical test of the key assumption in our theoretical model, that is, debt overhang has adverse effect on investment. There have been two strands of empirical studies related to the debt crisis. One is about the price movement of the debt-troubled economies' debt on the secondary market, e.g., Claessens and Sweder (1993) and Claessens (1990). The other is about the cross-section analysis of the relationship between debt overhang and growth, e.g., Cohen (1992). However, no econometric work based on time series of data about debt overhang and investment has been reported.

For this empirical study, we will use the time series of data of Mexico. Mexico is typical

among heavily indebted economies. Between 1950 and 1974 it enjoyed a remarkable period of high growth, low inflation and moderate external debt accumulation. Real growth averaged 6.4 percent, and inflation was in single digit throughout the period. However, the period of single digit inflation ended in 1973 as the real exchange rate started to appreciate and the accumulation of external debt accelerated above GNP growth. The subsequent period was characterized by rapidly expanding government revenues and vastly increased borrowing by its public sector. The government's share in total value added increased by almost one third, and in total investment from 33.5 percent in 1970-75 to substantially over 40 percent in later years. This expansion was largely fuelled from abroad: Mexico's external debt increased from \$18.23 billion (long-term debt, 15.61; short-term debt, 2.62) to \$86.08 billion (long-term debt, 59.87; short-term debt, 26.21) between 1975 and 1982. And the situation changed dramatically in 1982 when rising world interest rates and falling oil prices put an end to Mexico's increasingly expansionary policies. The refusal of external creditors to roll over Mexico's short-term debt left no option but fiscal retrenchment. Mexico, which had run non-interest current account deficits in each of the preceding 30 years, suddenly needed to run a surplus on that account in every following year through the 1980s. The ratio of external debt to GDP increased substantially under the influence of rising interest rates and falling growth rates. The gap between real interest rates on external debt and real GDP growth went from -6.3 percent in 1980-81 to 10.5 percent in 1983. This difference meant that, even without a non-interest current account deficit, the debt burden increased rapidly, simply through the compounding effect of interest on debt inherited from the past. In addition, there were substantial capital losses on external debt due to the necessary real exchange rate depreciation. Cornered in such a situation, Mexico chose to suspend debt-service payment in August 1982, which caused the world-wide debt crisis of the 1980s. An IMF-supported stabilization program in Mexico was launched in 1983. In the years that followed the announcement of the Baker plan, Mexico became the de facto paragon of the Baker plan. While Mexico's experience over the entire period of 1982-1988 demonstrated the eventual failure of the Baker plan, Mexico then initiated negotiations in early 1989 to restructure its external debt within the framework of the Brady plan, which came in March

1989. Thus Mexico appears as the most suitable subject for this empirical study.

In appendix B we present a simple model of investment behaviour under the debt overhang. Let I denote investment, Y the GDP level, TOT the terms of trade, R the interest rate or the cost of funds, and D a measure of debt (e.g., the total external debt stock outstanding or the ratio of the total external debt stock outstanding to the GDP (GNP) of the economy). As one can see from appendix B, investment behaviour in a debt-troubled economy can be specified as

$$I = f(Y, TOT, R, D). \quad (36)$$

Some other important factors, e.g., price level (inflation rate) and exchange rate, may also influence investment behaviour. In this brief study, however, the exchange rate and price level can be suppressed, because the terms of trade should reflect the changes in exchange rate and price level and because we will convert any nominal variable in the model from Mexican pesos into US dollars using the peso/dollar conversion factors estimated by the World Bank.

A linear version of (36) can be written as:

$$I = a + bY - cTOT + dR + eD + \varepsilon. \quad (37)$$

Several classical linear models based on (37) are estimated.¹⁸

In this empirical study all source data are drawn from publications by the World Bank and the International Monetary Funds (IMF). We chose not to use the original official statistics of Mexico for the following reasons. From 1973 on, the Mexican peso has fluctuated greatly and Mexico has experienced high inflation in most of the years. Mexican economic activities have been pushed to a large extent by its public sector with rapidly expanding government

¹⁸

We did not try nonlinear model formulas because the results from regressing the classic linear models appear satisfactory enough.

involvement in the economy. Under such severely unstable macroeconomic conditions and wide governmental involvement in the economy, market mechanisms in the Mexican economy can not be believed to be competitive. Thus the original official statistics of Mexico may not reflect the true reality of the Mexican economy. The data about Mexican economy published by the World Bank and IMF are based on the Mexican official statistics on one hand, but have been adjusted as necessary by the experts of the World Bank or IMF on the other hand. The data of the World Bank and IMF should therefore be more credible than the original Mexican official statistics for this study. The source data we used for this study are collected in Table 1.

Table 1: The Source Data of Mexico

Year	GDP	NDF	TOT	GDI	EDT	TBRUS	TBR	TAO	CFAA	FLL
1973	0.746	363.0	81.8	0.149	10.5	7.03		19.2	0.0125	1818.9
1974	0.966	550.0	117.7	0.21	13.95	7.87		19.7	0.0125	2668.9
1975	1.18	505.0	115.3	0.26	18.32	5.82		21.1	0.0125	3745.7
1976	1.47	541.0	117.7	0.31	23.97	5.05		27.5	0.0154	4504.0
1977	1.97	540.0	120.2	0.43	31.19	5.27		38.9	0.0226	4709.0
1978	2.49	830.0	104.1	0.56	35.71	7.22	12.75	35.5	0.0228	4536.9
1979	3.26	1327.0	113.2	0.80	42.77	10.04	17.89	32.6	0.0228	4081.0
1980	4.47	2186.0	146.5	1.21	57.38	11.62	27.73	31.9	0.0230	6820.6
1981	6.13	3076.0	163.4	1.68	78.22	14.08	33.28	34.0	0.0245	12498.7
1982	9.80	1900.0	159.1	2.24	86.08	10.72	57.44	55.5	0.0564	7968.7
1983	17.9	2192.0	148.1	3.7	92.97	8.62	59.07	69.8	0.120	2356.5
1984	29.5	1541.0	146.5	5.9	94.82	9.57	49.32	59.8	0.168	1623.4
1985	47.4	1984.0	144.8	10.0	96.87	7.49	63.20	57.5	0.257	-22.3
1986	79.2	2401.0	93.6	14.5	100.89	5.97	88.57	82.9	0.612	552.5
1987	193.3	2635.0	100.0	37.2	109.47	5.83	103.07	82.1	1.378	3778.9
1988	390.5	2880.0	88.4	79.7	99.21	6.67	69.15	60.3	2.273	-1487.7
1989	507.6	3176.0	96.4	108.7	93.84	8.11	45.01	47.3	2.461	-1459.4
1990	686.4	2633.0	111.4	150.3	106.03	7.51	34.76	44.9	2.813	8541.8
1991	865.2	4761.5	102.9	193.5	115.36	5.41	19.28	41.2	3.018	3887.1
1992	1019.2	4392.8	100.0	237.2	113.42	3.46	15.62	35.5	3.095	-347.5
1993	1070.3	4900.5	98.9	232.7	118.03	3.03	15.03	35.5	3.116	3055.7

NOTES:

1. NDF: net foreign direct investment, millions of US dollars.

TOT: terms of trade, index 1987=100.

GDI: gross domestic investment, billions of Mexican new pesos.

EDT: external debt (total), billions of US dollars.

BRUS: Treasury Bill rate of United States, percent.

TBR: Treasury Bill rate of Mexico, percent.

TAO: ratio of EDT to GNP, percent.

CFAA: annual averaged conversion factor, Mexican new pesos per US dollars.

FLL: net foreign long-term loan to Mexico, millions of US dollars.

2. Source:

NDF, TOT, GDI, EDT, CFAA and FLL: The World Bank: World Tables, 1995.

TBRUS and TBR, IMF: International Financial Statistics, January 1995, January 1990, January 1986, January 1982 and January 1977.

TAO: except the data for 1973-1978, The World Bank: World Debt Tables, 1994/95;
the data for 1973-1978, calculated through the GNP data from The World Bank,
Trend of Developing Economies, 1981 and 1978, and EDT above.

Tables 2, 3 and 4 below report some of the results we obtained from 8 different regressions. Statistically, gross domestic investment includes foreign direct investment and domestic investments financed by domestic funds and by foreign loans. This empirical study will reflect such classifications. In Table 2 the dependent variable is the foreign direct investment going to Mexico, denoted by NDFB. The dependent variable in Table 3 is the net foreign long-term loans going to Mexico, denoted by FLL. In Table 4, the dependent variable is GDIC, denoting the gross domestic investment of Mexico, or DIF, denoting the gross domestic investment net of the (net) foreign direct investment (i.e., $DIF \equiv GDIC - NDIB$), or ID, denoting the gross domestic investment net of the (net) foreign direct investment and the (net) foreign long-term loans (i.e., $ID \equiv GDIC - NDFB - FLL$). Evidently, DIF reflects the domestic investment financed by non-foreign direct investment funds and ID the domestic investment financed by domestic funds.

TBRUS, the Treasury Bill rate of the United States, is used to reflect the international capital cost while TBR, the annual-averaged Mexican Treasury Bill rates, is used to reflect the domestic capital cost. Table 2 contains the coefficients of the regression when either EDT or TAO is used to measure the debt burden of Mexico. The coefficients of the regression when EDT is replaced by TAO are not reported in Tables 3 and 4, because they did not yield any significantly different results from those reported here. The estimated equation for ID when TOT and TBRUS exist as explanatory variables is not reported either, because the absolute values of the t-statistic for the two variables are both less than 0.2 and the values of R^2 -ad are lower than those when TOT and TBRUS are not included. In these tables, the numbers in parentheses are t-statistics and R^2 _ad represents the adjusted error square. The

sample for Table 2 is the annual data from 1973 to 1993 inclusive. But the sample for Table 4 is the annual data from 1978 to 1993 inclusive, because the data for TBR from 1973 to 1977 are not available. The sample for each regression of Table 3 is specified in the table.

Table 2: (Net) Foreign Direct Investment vs. Debt Overhang

End. Var.	Exogenous Variables						D_W value	R ²	R ² _ad
	constant	GDPC	D		TOT	TBRUS			
			EDT	TAO					
NDFB	0.3967 (0.892)	0.0117 (7.102)	0.0129 (3.687)	Not used	-0.0124 (-2.57)	0.0321 (-0.719)	2.694	0.94	0.93
NDFB	-0.0244 (-0.053)	0.0161 (15.63)	Not used	0.0168 (3.878)	-0.0137 (-2.85)	0.0336 (0.770)	2.72	0.95	0.93

Table 3: Foreign Long-term Loans vs. Debt Overhang

End. Var.	Exogenous Variables					D_W value	R ²	R ² _ad	Sample
	constant	GDPC	EDT	TOT	TBRUS				
FLL	-4.8736 (-1.432)	0.0204 (1.630)	-0.0449 (-1.689)	0.0434 (1.160)	0.4077 (1.199)	1.956	0.46	0.32	73-93
FLL	-2.1427 (0.864)	0.0218 (1.734)	-0.0446 (-1.659)	Not used	0.4077 (2.871)	1.923	0.41	0.31	73-93
FLL	0.5808 (0.4655)	0.0579 (2.731)	0.0222 (0.656)	Not used	-0.4213 (-1.343)	1.972	0.91	0.86	73-82
FLL	-72.696 (-3.733)	-0.0070 (-0.556)	0.5972 (3.698)	Not used	2.1667 (3.300)	2.649	0.68	0.54	83-93

Table 4: Domestic Investment Supply vs. Debt Overhang

End. Var.	Exogenous Variables						D_W value	R ²	R ² _ad
	constant	GDPC	EDT	TOT	TBRUS	TBR			
GDIC	-9.7789 (-2.730)	0.3350 (14.69)	-0.2677 (-3.957)	0.0021 (0.007)	0.7115 (1.928)	0.1392 (2.932)	2.012	0.99	0.99
DIF	-9.7479 (-2.963)	0.3190 (15.24)	-0.2720 (-4.377)	0.0118 (0.402)	0.6729 (1.985)	0.1291 (2.653)	2.038	0.99	0.98
ID	-0.9632 (-0.978)	0.2830 (8.525)	0.2047 (2.426)	Not used	Not used	0.1264 (1.271)	2.409	0.97	0.96

From Table 2, we see that GDPC, EDT/TAO and TOT all have a significant effect on NDFB at 95% confidence level ($t_{0.95}(16)=2.120$).¹⁹ The positive relationship between NDFB and GDPC is easy to understand. The higher is the GDP, the higher is investment demand, and thus perhaps the higher is foreign direct investment. The negative relationship between NDFB and TOT, the terms of trade, is not surprising, either. This is because a country's exports should be negatively related to TOT and foreign direct investment projects are generally export-oriented.

However, the positive relationship between NDFB and EDT/TAO appear to contradict the implication of the debt overhang. There are several possible interpretations for this result. First, most foreign direct investment projects generally involve foreign loans. The contracts for such projects generally stipulate the proportions of foreign direct investment, foreign loans and domestic funds. Second, with higher EDT/TAO, the country will further prefer foreign direct investment to foreign borrowing once it starts to see the external debt burden. That is, the country will tend to adopt more preferential policies to attract foreign direct investment under higher debt overhang condition. Third, a project with foreign direct

¹⁹

All the critical values used in this study are read from Appendix B, Statistical Tables of J. Johnston (1984).

investment is generally long-term and of large and middle scale. This implies a large exit-cost for an investor to withdraw investment from such a project once it has started. Then the risky external debt level may only marginally deter further foreign direct investment if there is any deterrence effect. Fourth, a foreign direct investment project is generally export-oriented. Especially in the case of Mexico, a large part of the foreign direct investment it gets goes into its resource industry, i.e., oil industry. Thus from the view of the products market, foreign direct investments are not exposed to the adverse effect of the risky external debt level. Furthermore, preferential policies offered to foreign direct investments, e.g., low tax on or tax exemption of investment proceeds, priorities of foreign investors in profit distribution, etc, weaken or even eliminate any adverse effect of debt overhang on foreign direct investments. Therefore, the positive relationship between NDFB and EDT/TAO by itself is not sufficient to prove the absence of the debt overhang effect. It only shows that the debt overhang does not seem to have had a negative effect on a particular kind of investment in Mexico, namely, the foreign direct investment.

We also investigated whether any structural change in foreign direct investment behaviour occurred by dividing the sample period into two sub-periods, one from 1973 to 1982 and the other from 1983 to 1993. The regressions (not reported here) suggest no structural change.

Thus based on Table 2, we can say that the debt overhang does not seem to have had a negative effect on foreign direct investment in the Mexican economy.

The regressions reported in Table 3 were designed to determine whether the debt overhang adversely affected Mexico's ability to attract foreign long-term loans. The first two regressions show that the total external debt has an adverse effect on foreign long-term loans to Mexico at the confidence level of 90% ($t_{0.90}(16)=1.337$). The positive relationship between FLL and TBRUS reflects the capital-shortage feature of Mexican economy and/or wrong foreign capital usage policy: Mexico was willing to pay a higher cost in order to get more foreign loans. However, we should not put much confidence in these regressions

because their values of R^2 (R^2 -adj) are low.

The third and fourth regressions of Table 3 confirm that there was a structural change at the confidence level of 99% (with Chow-test, $F_{0.99}(4,11)=4.67$). We must interpret this result carefully. In the period from 1973 to 1982, GDPC and FLL have a significant positive relationship, while EDT seems to have no effect on FLL (note that FLL is the net foreign long-term loans going into Mexico). These results actually confirm that the expansion of the Mexican economy in that period was dependent on expanding foreign borrowing. The negative relationship between FLL and TBRUS reflects the demand-side feature of the international capital market in that period: there was enough international capital supply, but the capital cost discouraged Mexico from using more foreign capital. With the exposure to the debt crisis in 1982, the situation changed dramatically. In the period from 1983 to 1993, with the adoption of the structural reform program and the reversion of the feature of the international capital market, Mexico had to struggle to get some foreign loans in order to help its structural reform, to control inflation and to service its old debt obligation with new loans. The higher interest rate meant (to Mexico) a higher debt burden and thus Mexico needed more new loans to service its old due debt obligation. Also in this period, an increasingly large number of the long-term loans Mexico received were from international organizations, e.g., IMF and the World Bank, and the governments of some industrial countries, whose agreements to make loans to Mexico were mainly based on its large debt burden and other economic difficulties and on supporting its structural reform. Thus it is natural that both EDT and TBRUS were significantly (at the confidence level of 99%) and positively related to FLL, while FLL seemed not to be associated with GDPC.

The conclusion we can draw from Table 3, therefore, is that its debt overhang did not prevent Mexico from obtaining foreign long-term loans.

Table 4 contains the estimated coefficients related to GDIC, DIF and ID. It suggests the existence of the adverse effect of debt overhang on GDIC, DIF and ID at the confidence level

of 97.5% ($t_{0.975}(10)=2.228$). TOT appears to have no significant effect on GDIC and DIF. This can be explained by the fact that the conversion factors used in creating the data on GDIC and GDPC largely embody the information of TOT. The positive relationships between the investment (GDIC, DIF or ID) and the interest rates (TBRUS and TBR) reflect the fact that the Mexican economy was in a capital-shortage stage. The economy pushed up the interest rate on funds in order to attract investment supply.

To summarize, this empirical study confirms the existence of strong debt overhang effect on the supply of domestic investment funds. However, the debt overhang did not seem to affect foreign direct investment. Nor did it prevent Mexico from getting foreign long-term loans. In other words, the debt overhang severely deterred the supply of domestic investment funds, but not foreign funds.

8. Concluding Remarks

Debt overhang has adverse impact on investment and consequently on the productivities of debt-troubled countries. This is the source of efficiency gain from reducing those countries' debt overhangs. Built on this assumption, the theoretical analysis in this chapter shows that the debt buyback option will certainly benefit both the creditors and the debtor countries, even if a buyback deal is performed at the ex post secondary market price. Buybacks will crowd in, rather than crowd out, investment. These results hold even without external funds to enhance buybacks. Thus the efficiency gain of buybacks is from the inside of the debt-troubled economy, i.e., the improved productivity and enhanced investment, rather than from the outside of the economy, e.g., external funds. Furthermore, the analytical framework of this chapter is capable of explaining many facts in the historical and current debt crises.

A problem that may obstruct the use of the buyback option is that large creditors may act as Stackelberg leader towards their debtor countries. The resulting high price demanded by the large creditors makes it unattractive for the debtor countries to accept a buyback deal. This result is consistent with the observations that most small- and middle-sized creditors accept

buybacks at market discount but large banks generally do not like to do so.

Appendix A Proof of the Lemma

Recall that $s^*(I, X) = (D-X)/(af(I)h(X))$ and $S_t \equiv \int_0^t s dL(s)$.

$$\frac{\partial s^*(I, X)}{\partial I} = -\frac{D-X}{ah(X)} \cdot \frac{f'(I)}{f^2(I)} \leq 0.$$

$$\frac{\partial S_t(I, X)}{\partial I} = \frac{D-X}{af(I)h(X)} \cdot L'(s^*) \cdot \frac{\partial s^*(I, X)}{\partial I} \leq 0.$$

$$\frac{\partial U}{\partial I} = -1 + (1-aS_t)f'(I)h(X).$$

$$\frac{\partial^2 U}{\partial I^2} = (1-aS_t)f''(I)h(X) + (D-X)^2 \cdot (f'(I))^2 \cdot \frac{L'(s^*)}{ah(X)(f(I))^3}.$$

One of the second-order conditions for maximizing (14) with respect to I and X

is $\frac{\partial^2 U}{\partial I^2} \leq 0$. Differentiate (16) with respect to X:

$$(1-aS_t)f'(I)h'(X) + (1-aS_t)f''(I)h(X)\frac{\partial I}{\partial X} - af'(I)h(X)\frac{\partial S_t}{\partial X} = 0.$$

$$\begin{aligned} \text{i.e., } (1-aS_t)f'(I)h'(X) + \frac{D-X}{a(f(I))^3(h(X))^2} L'(s^*)f(I)h(X) \\ + (D-X)f(I)h'(X) + \frac{\partial^2 U}{\partial I^2} \cdot \frac{\partial I}{\partial X} = 0. \end{aligned}$$

Thus it must be $\frac{\partial I}{\partial X} \geq 0$. Especially, (16) implies $f'(I(X)) \neq 0$. Thus for $\forall X \in [0,$

$\text{Arg}(h'(X)=0))$, $\frac{\partial I(X)}{\partial X} > 0$.

Q.E.D.

Appendix B Investment Behaviour under Debt Overhang

Consider an investor living for two periods. In each period, the investor receives a wealth W_i , $i=1, 2$. In period 1, the investor chooses the levels of investment, borrowing and consumption for his life subject to his life-time budget constraint. He can borrow and lend in period 1 at interest rate r . In period 2, the investor will enjoy his investment revenue and service his debt obligation. Suppose that his utility function is $U(C1, EC2)$, where $C1$ and $C2$ are his consumption in periods 1 and 2 and E is the expectations operator. Then the decision problem of such an investor can be modelled as:

$$\text{Max } u=U(C1, EC2)$$

$$\text{s.t.: } C1=W1 + B - I, \quad (B1)$$

$$EC2=W2 + (EP(D,t))g(I) - (1+r)B, \quad (B2)$$

$$I \geq 0, C1 \geq 0, EC2 \geq 0, \quad (B3)$$

where B represents the investor's borrowing and I the investment; $g(I)$ the physical investment proceeds, with $g'(I) > 0$ and $g''(I) < 0$; $P(D,t)$ the real revenue factor when the economy in which he invested has debt stock outstanding D and faces the terms of trade t . The real revenue factor depends on D because the sovereign authority of the economy may levy taxes on economic activities in order to raise funds for servicing its sovereign debt obligation. It is reasonable to assume $[\partial P(D,t)/\partial D] < 0$. The real revenue factor depends on the terms of trade, because the economy, by assumption, is open.

We focus on the interior solution to the problem above. Such a solution is equivalent to the solution to the following problem:

$$\text{Max}_{B,I} u=U(W1 + B - I, W2 + (EP(D,t))g(I) - (1+r)B). \quad (B4)$$

The first-order conditions for (B4) yield:

$$(\partial U/\partial C1)=(1+r)(\partial U/\partial C2), \quad (B5)$$

$$(\partial U/\partial C1)=[E(P(D,t))]g'(I)(\partial U/\partial C2). \quad (B6)$$

(B5) and (B6) give rise to $(1+r)=[E(P(D,t))]g'(I)$, which implies the investment function $I=f(W1, W2, D, t, r)$. Furthermore, under the assumptions that $g''(I) < 0$ and $[\partial P(D,t)/\partial D] < 0$, $[\partial I/\partial D] < 0$.

References

- Acharya S. and I. Diwan (1993), Debt Buybacks Signal Sovereign Countries' Creditworthiness: Theory and Test, *International Economic Review*, 34 (November), pp795-817.
- Bulow, J. and K. Rogoff (1991), Sovereign Debt Repurchase: No Cure for Overhang, *Quarterly Journal of Economics*, (November) pp1219-1235.
- and ----- (1988), The Buyback Boondoggle. *Brookings Papers on Economic Activity* 2, pp675-698.
- , and ----- and Ning S. Zhu (1994), Estimating the Efficiency Gains of Debt Restructuring, The World Bank, July, WPS1317.
- Cabral, Célia C. (1996), Evaluating Debt Repurchases---What are the Alternatives to Investment?, *Journal of International Economics*, 40, pp477-494.
- Claessens S. (1990), Debt Laffer Curve: Some Estimates, *World Development* (December).
- and I. Diwan (1989), Market-based Debt Reductions, In M. Husain and I. Diwan (eds.), *Dealing with the Debt Crisis*, The World Bank, Washington D. C..
- and van-Wijinbergen, Sweder (1993), Secondary Market Prices and Mexico's Brady Deal. *Quarterly Journal of Economics*, 108(4), (November) pp967-982.
- Cohen, Daniel (1992), The Debt-Crisis: A Post-Mortem. *NBER Macroeconomics Annual*, MIT Press, pp65-114.
- (1991), *Private Lending to Sovereign States*, MIT Press.
- and Thierry Verdier (1995), "Secret" Buy-backs of LDC Debt. *Journal of International Economics*, 39, pp317-334.
- Dooley, M. (1988), Buy-Backs and the Market Valuation of External Debt, *IMF Staff Paper*, V35, N2, pp215-229.
- Dornbusch, R. (1988), Comments and Discussion, *Brookings Papers on Economic Activity* 2, pp699-704.
- Eichengreen, Barry and Richard Portes (1989), Dealing with Debt: The 1930s and 1980s, In M. Husain and I. Diwan (eds), *Dealing with the Debt Crisis*, The World Bank, Washington, D.C..

- Froot, K. A. (1989), Buybacks, Exit Bonds and the Optimality of Debt and Liquidity Relief, *International Economic Review*, 30, pp49-70.
- Fry, Maxwell J. (1991), Foreign Debt Reduction through Improved Domestic Resource Mobilization and Allocation, *Revue d'Economie Politique*, 101(4, July-August), pp623-637.
- Goldberg, Linda and Mark M. Spiegel (1992), Debt Write-Down and Debt-Equity Swaps in a Two-Sector Model, *Journal of International Economics*, 33(3-4, November), pp267-283.
- Helpman, Elhanan (1989a), Voluntary Debt Reduction: Incentives and Welfare, *International Monetary Fund Staff Papers*, 36(3, September), pp580-611.
- (1989b), The Simple Analytics of Debt-Equity Swaps, *American Economic Review*, 79(June), pp440-451.
- Husain, M. (1993), Debt Relief Schemes and Welfare, *Journal of International Money and Finance*, 12(6, December), pp632-643.
- Johnston, J. (1984), *Econometric Methods*, 3rd ed., McGraw-Hill Book Company.
- Klog, Adam (1993), *The German Buybacks 1932-1939: A Cure for Overhang?* Princeton Studies in International Finance, No.75, Princeton University, Department of Economics, International Financial Section.
- Krugman, Paul R. (1989), Market-Based Debt-Reduction Schemes, In Frankel, J. A., Dooley, M. P. and Wickham, P. (eds), *Analytical Issues in Debt*, Washington, D.C., International Monetary Fund, pp258-278.
- Lucas, Robert E. Jr. (1988), On the Mechanics of Economic Development, *Journal of Monetary Economics*, 22, pp3-42.
- Romer, Paul M. (1986), Increasing Returns and Long-run Growth, *Journal of Political Economy*, v94, pp1002-1037.
- (1990), Endogenous Technological Change, *Journal of Political Economy*, v98, ppS75-S102.
- (1994), The Origins of Endogenous Growth, *Journal of Economic Perspectives*, (Winter) pp3-22.

Sachs, Jeffery (1989a), Efficient Debt Reduction, In M. Husain and I. Diwan (eds), Dealing with the Debt Crisis, The World Bank, Washington, D. C..

----- (1989b), The Debt Overhang Problem of Developing Countries, In de Macedo and Findlay (eds), Debt, Growth and Stabilization: Essays in Memory of Carlos Diaz-Alejandro, Basil Blackwell, Oxford.

----- and Harry Huizinga (1987), U.S. Commercial Banks and the Developing-Country Debt Crisis, Brookings Papers on Economic Activity, v2, pp555-601.

IMAGE EVALUATION TEST TARGET (QA-3)

APPLIED IMAGE, Inc.
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989

© 1993, Applied Image, Inc., All Rights Reserved