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**SOME CONTRIBUTIONS TO ANALYSIS:
A NEW SPACE OF GENERALIZED FUNCTIONS
FOR INTEGRO-DIFFERENTIAL EQUATIONS
INVOLVING HILBERT TRANSFORMS,
AND A NOTE ON THE NON-NUCLEARITY OF A SPACE
OF TEST FUNCTIONS ON HILBERT SPACE**

by

Roderick Kerry Donnelly, B.Sc., M.Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

November, 1987

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Rod Donnelly

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Some Contributions to Analysis:

A New Space of Generalized Functions

For Integro-Differential Equations Involving Hilbert Transforms,

and a Note on the Non-Nuclearity of a Space

of Test Functions on Hilbert Space

submitted by Roderick Kerry Donnelly, B.Sc., M.Eng., in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In part one we investigate the equation $P(D, H)F = G$, where P is a constant coefficient polynomial in the differential operator D and Hilbert transform operator H , and F & G are elements of certain dual-spaces. We construct a space of test functions $\mathcal{G}$, whose dual $\mathcal{G}'$ is a natural setting for this equation, as it is closed with respect to the operations of differentiation and Hilbert transformation. We demonstrate the existence of solutions for various terms G .

In part two we consider the space $\mathcal{D}(\mathcal{H})$, constructed by Dudin, consisting of all the infinitely differentiable functions with bounded support on a real separable Hilbert space $\mathcal{H}$, all of whose derivatives have bounded range. We establish the non-nuclearity of $\mathcal{D}(\mathcal{H})$. The result follows by modifying a result of Meise that a certain space of infinitely differentiable functions (though not necessarily of bounded support) is non-nuclear.

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NOTATION FOR PART ONE

$\mathcal{D}, [\mathcal{D}']$	vector space of infinitely differentiable functions with compact support [and its dual].
$\mathcal{E}, [\mathcal{E}']$	vector space of infinitely differentiable functions [and its dual].
f, g	elements of $\mathcal{D}$.
F, G	elements of $\mathcal{G}$.
$\mathcal{F}$	Fourier transform operator.
$\mathcal{G}, [\mathcal{G}']$	vector space of functions which are sums of functions in $\mathcal{D}$ and $H(\mathcal{D})$ [and its dual].
H	Hilbert transform operator.
$H(\mathcal{D}), [H'(\mathcal{D})]$	vector space of functions which are Hilbert transforms of elements of $\mathcal{D}$ [and its dual].
	$\sqrt{-1}$.
$\mathcal{J}, [\mathcal{J}']$	vector space of functions which are Fourier transforms of elements of $H(\mathcal{D})$ [and its dual].
P, Q	constant coefficient polynomials.
$\mathcal{Q}, [\mathcal{Q}']$	vector space of functions which are Fourier transforms of elements of $\mathcal{G}$ [and its dual].
$\mathbb{R}$	the set of real numbers.
s	complex Fourier transform variable [possibly n -dimensional], given by $s = \sigma + i\tau$.
U	coordinate rotation operator [matrix] in $\mathbb{R}^n$.

x	spatial domain variable.
$\mathcal{Z}, [\mathcal{Z}']$	vector space of functions which are Fourier transforms of elements of $\mathcal{D}$ [and its dual].
α, β	elements of $\mathcal{Z}$.
δ	delta function [functional].
ϕ, ω	elements of $\mathcal{D}$.
θ	element of $\mathcal{G}$.
$\Theta, \bar{\Theta}$	elements of $\mathcal{Q}$.
χ	canonical isomorphism from $\mathcal{D}' \times H'(\mathcal{D})$ to $\mathcal{G}'$.
$\chi_{\mathcal{F}}$	canonical isomorphism from $\mathcal{Z}' \times \mathcal{J}'$ to $\mathcal{Q}'$.
Ξ, Λ, Υ	elements of $\mathcal{Q}'$.
ζ	element of $\mathcal{Z}$.
$\mathcal{H}, \tilde{\mathcal{H}}$	Hörmander's staircases, surfaces in $\tau_1 \sigma_1 \dots \sigma_n$ -space.

NOTATION FOR PART TWO

α	element of $C_{0,n}^\infty(\mathcal{H})$ which is 1 on Γ_N .
B_N	canonical embedding from $C_0^l([0, 2\pi]^N)$ to $L^2([0, 2\pi])$.
B_n	the closed ball $\{x : x \in \mathcal{H}, \ x\ \leq r\}$.
$C(K)$	vector space of continuous real valued functions on compact K , with the topology of uniform convergence.
$C(\mathcal{H}, E)$	vector space of continuous functions on $\mathcal{H}$ with bounded range

- in the real locally convex space E .
- C_N continuous linear mapping from H_N to $C_0^l([0, 2\pi]^N)$.
- $C^k(\mathcal{H}, \mathbb{R})$ Intersection of domains of definition of operators $D^l : C(\mathcal{H}, \mathbb{R}) \rightarrow C(\mathcal{H}, \mathcal{U}_k)$ where $l = 0, 1, \dots, k$.
- $C^l(\mathbb{R}^N)$ vector space of l -times continuously differentiable functions on $\mathbb{R}^N$.
- $C_0^l(\Delta_N)$ vector subspace of $C^l(\mathbb{R}^N)$ consisting of those functions with support contained in Δ_N .
- $C_0^l([0, 2\pi]^N)$ vector subspace of $C^l(\mathbb{R}^N)$ consisting of those functions with support contained in $[0, 2\pi]^N$.
- $C_0^l(\Gamma_N)$ vector space of real valued l -times continuously differentiable functions on E_N whose supports lie in Γ_N .
- $C_p^\infty(E)$ vector space of infinitely differentiable functions on a real locally convex space E .
- $C^\infty(\mathcal{H}, \mathbb{R})$ $\lim \text{proj} C^k(\mathcal{H}, \mathbb{R})$; the vector space of infinitely Fréchet differentiable functions, all of whose derivatives have bounded range.
- $C_{0,r}^\infty(\mathcal{H})$ vector subspace of $C^\infty(\mathcal{H}, \mathbb{R})$ consisting of those functions which vanish outside B_r .
- $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$ normed quotient vector space $C_{0,n}^\infty(\mathcal{H})/p_{\psi,l}^{-1}(0)$.
- $(\widehat{C_{0,n}^\infty(\mathcal{H})})_{p_{\psi,l}}$ the complete hull of $C_{0,n}^\infty(\mathcal{H})_{p_{\psi,l}}$.
- $C^\infty(E_N, \mathbb{R})$ vector space of infinitely differentiable real valued functions on E_N .
- $C_0^\infty(\Gamma_N)$ subset of $C^\infty(E_N, \mathbb{R})$ consisting of those functions with support in Γ_N .
- $C^0(\mathcal{H}, \mathbb{R})$ $C(\mathcal{H}, \mathbb{R})$.
- $D(\mathcal{H})$ $\lim \text{ind} C_{0,n}^\infty(\mathcal{H})$; vector space consisting of all infinitely Fréchet differentiable functions, all of whose derivatives have bounded range.
- D^l [weak] l -multiple differential operators.

D^0	identity operator.
D_N	continuous linear mapping from H_N to $L^2([0, 2\pi]^N)$, given by $B_N \circ C_N$.
Δ_N	subset of $\mathbb{R}^N$ given by $\{(t_1, \dots, t_n) : 0 \leq t_i \leq \frac{n}{2\sqrt{N}\ x_i\ }\}$.
E_N	linear hull of $\{x_1, \dots, x_N\}$.
(e_n)	orthogonal system in H_N .
(f_n)	orthogonal system in $L^2([0, 2\pi]^N)$.
Γ_N	"cube" in E_N (can also be considered as lying in H).
$\mathcal{H}^0$	$\mathbb{R}$.
$\mathcal{H}^k$	complete projective tensor product of k copies of $\mathcal{H}$.
H_N	certain Hilbert space of sequences.
i_N	continuous linear mapping from $C_0^l(\Delta_N)$ to $C_0^l(\Gamma_N)$.
j_N	continuous linear mapping from Δ_N to $\mathcal{H}$, taking values in $K \cap \Gamma_N$.
$\bar{j}_N$	continuous linear mapping from $C(K)$ to $L^2(\Gamma_N)$, given by $\bar{j}_N(f) = f \circ j_N$.
K	infinite dimensional compact absolutely convex subset of $\mathcal{H}$, contained in B_n .
λ	canonical mapping from $(\widehat{C_{0,n}^\infty(\mathcal{H})})_{p_\psi}$ to $C(K)$.
$L^2([0, 2\pi]^N)$	vector space of mod-squared integrable functions on $[0, 2\pi]^N$, with Lebesgue measure.
ℓ^p	vector space consisting of p -summing maps.
λ_N	composition map, given by $\bar{\lambda} \circ \sigma_N$.
N	arbitrary natural number.
$\mathbb{N}$	set of natural numbers.
N_0	$\mathbb{N} \cup \{0\}$.
$p_{\psi,0}$	seminorm on either $C^k(\mathcal{H}, \mathbb{R})$, $C^\infty(\mathcal{H}, \mathbb{R})$, or $C_{0,n}^\infty(\mathcal{H})$.
P_N	continuous linear map from $C_0^\infty(\Gamma_N)$ to $C_{0,n}^\infty(\mathcal{H})$.

Ψ	family of lower semicontinuous functions used in the description of Dudin's topology.
ψ	a member of Ψ .
π_N	continuous linear projection from $\mathcal{H}$ onto E_N .
q	canonical quotient mapping from $C_{0,n}^\infty(\mathcal{H})$ to $C(K)$.
$\mathbb{R}$	set of real numbers.
ρ_K	restriction map from $C_{0,n}^\infty(\mathcal{H})$ to $C(K)$.
$\tilde{\rho}_K$	map induced by ρ_K , from $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$ to $C(K)$.
σ_N	continuous linear composition map, given by $q \circ P_N$.
$\tilde{\sigma}_N$	continuous linear extension of σ_N , mapping $C_0^l(\Gamma_N)$ to $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$.
$\mathcal{U}_k$	vector space of all k -linear forms on $\mathcal{H} \times \dots \mathcal{H}$ (k -factors) that are continuous in all variables.
$\mathcal{U}_0$	$\mathbb{R}$.
$(x_j)_{j \in \mathbb{N}}$	sequence of linearly independent mutually orthogonal elements in K .
$[0, 2\pi]^N$	"cube" in $\mathbb{R}^N$.

CHAPTER 0. INTRODUCTION

Part one of this work is primarily concerned with establishing the existence of "generalized function" solutions of the equation $P(D, H)u = f$, or first the more basic $P(D, H)u = \delta$. In these last equations P is a constant [complex] coefficient polynomial in the differential operator $D \equiv id/dx$ and in the Hilbert transform operator H , where $H\phi(x) := P \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt$ for ϕ suitably well behaved ($\phi \in \mathcal{D}$, for example) (in this last equation P denotes the Cauchy principal value of the integral). The terms u , f , and δ represent continuous linear functionals on some space of test functions, as yet unspecified. In any reasonable definition of Hilbert transforms applied to elements of a dual space, we should expect to have both an identity like $H^2 u \equiv (-\pi^2)u$ and one like $(Hu)^{(k)} = Hu^{(k)}$, where the superscript (k) denotes k -fold differentiation (see Pandey [4, pp. 89] for these results when $u \in \mathcal{D}'$). These results allow us to write $P(D, H)u = \delta$ as $(P(D) + Q(D)H)u = \delta$, where now P and Q are new polynomials in $D \equiv id/dx$. The equation $(P(D) + Q(D)H)u = \delta$ clearly makes sense only in a [dual] space which is closed with respect to the operations of differentiation and Hilbert transformation. For example, the equation makes no sense in $\mathcal{D}'$, as $H : \mathcal{D}' \rightarrow H'(\mathcal{D})$, where $H'(\mathcal{D})$ denotes the dual of the space $H(\mathcal{D})$, consisting of Hilbert transforms of elements of $\mathcal{D}$ (see Pandey [1]). There are, however, at least three settings in which this equation makes sense. One involves the subspace $\mathcal{D}' \cap H'(\mathcal{D})$. Another involves the space $\mathcal{G}'$, which is the dual of the space $\mathcal{G}$ of test functions, consisting of sums of elements of $\mathcal{D}$ and $H(\mathcal{D})$. In yet another interpretation we can consider $P(D) + Q(D)H$ as an operator from $\mathcal{G}'$ to $\mathcal{D}'$. For the one dimensional case these interpretations are studied in chapter two, with the n -dimensional case being reserved for chapter three. We note that the space $(\mathcal{D}_{L^p})'$ is closed with respect to the operations of differentiation and Hilbert

transformation (see [5], for example), and so our equations would also make sense in this space. However, as we are ignorant of the existence of solutions of even differential equations in $(\mathcal{D}_{L^p})'$, we are hardly in a position to demand a solution of $(P(D) + Q(D)H)u = \delta$ in $(\mathcal{D}_{L^p})'$.

In chapter one we collect together and establish ourselves some results on the spaces $H(\mathcal{D})$ and $H(S)$, consisting of the Hilbert transforms of elements of the spaces $\mathcal{D}$ and S , respectively. The topologies on $H(\mathcal{D})$ and $H(S)$ are the ones transported, respectively, from $\mathcal{D}$ and S , via H . Both $H(\mathcal{D})$ and $H(S)$ are densely contained in $\mathcal{D}_{L^p}$, and convergence of a sequence in either of the first two spaces implies its convergence in $\mathcal{D}_{L^p}$ (see Pandey [4], for example, for the $H(\mathcal{D})$ case). Chaudhry & Pandey [2] have given an intrinsic description of convergence in $H(\mathcal{D})$ involving convergence in $\mathcal{D}_{L^p}$ (we state this result as Proposition 1.3). We isolate the essential part of the proof of this result (Lemma 1.10 & Remark 1.11) and, by counterexample (1.12), show that no intrinsic description of convergence in $H(S)$ in terms of convergence in $\mathcal{D}_{L^p}$ is possible.

We establish the important result (Proposition 1.14) that $\mathcal{D} \cap H(\mathcal{D}) = \{0\}$; this result is crucial for the construction of the space $\mathcal{G}$ given in chapter two. We show (example 1.19) that $S \cap H(S)$ is nonempty, and give two characterisations of this space (Proposition 1.15 & Corollary 1.16).

Chapter one is concluded with a study of the spaces $\mathcal{D}$, $H(\mathcal{D})$, $\mathcal{D}'$, & $H'(\mathcal{D})$ and the Fourier transforms of the elements of these spaces. The Fourier transform $\mathcal{F}$, with complex argument, establishes an isomorphism $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}$, where $\mathcal{Z}$ is a space of entire functions (in the transform variable s) satisfying certain inequalities (see our Remark 1.23 and Shilov [7, pp. 134 ff]). We are led to define the Fourier transformation $\mathcal{F}H\omega$, with complex argument, of an element $H\omega$ in $H(\mathcal{D})$. We show that $\mathcal{F} : H(\mathcal{D}) \rightarrow \mathcal{J}$ is an isomorphism.

where $\mathcal{J}$ is defined as the space of functions analytic to the left and right of the imaginary axis, and satisfying certain inequalities (see Definition 1.26 & Proposition 1.27). From these definitions we then define Fourier transforms $\mathcal{E} : \mathcal{D}' \rightarrow \mathcal{Z}'$ and $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$ in an obvious way. These definitions are consistent in that if $u \in \mathcal{D}' \cap H'(\mathcal{D})$ then its Fourier transform is the same, whether we consider it as an element of $\mathcal{D}'$ or $H'(\mathcal{D})$. We see that $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ and $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$ are isomorphisms.

In chapter two we define the vector space $\mathcal{Q}$, consisting of sums of functions in $\mathcal{D}$ and $H(\mathcal{D})$ (Definition 2.1). We show that there is a canonical isomorphism $\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$, the dual of $\mathcal{G}$ (Proposition 2.4). We define the Fourier transform $\mathcal{F}\theta$, with complex argument, of a function $\theta \in \mathcal{G}$ (Definition 2.6), and show that $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{Q}$ is an isomorphism. $\mathcal{Q}$ is a space of functions that are the sum of an entire function and a function which is analytic to the left and to the right of the imaginary axis, satisfying certain inequalities (Definition 2.8 & Proposition 2.9). This leads to defining a Fourier transform $\mathcal{F} : \mathcal{G}' \rightarrow \mathcal{Q}'$, with inverse $\mathcal{F}^{-1} : \mathcal{Q}' \rightarrow \mathcal{G}'$. The space $\mathcal{G}'$ is closed with respect to the operations of differentiation and Hilbert transformation. Applying a Fourier transform to the equation $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$ gives us the equation $(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \circ \mathcal{F}F = 1$ in $\mathcal{Q}'$ (Lemma 2.23 & Remark 2.24), where the complex variable $s = \sigma + i\tau$ has been used. Provided $\overline{P^2}(s) + \pi^2 \overline{Q^2}(s)$ is not zero at the origin (where the overbar denotes complex conjugation), we can construct a solution for $\mathcal{F}F$ in the equation $(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \mathcal{F}F = 1$ in $\mathcal{Q}'$ (Proposition 2.28). Under this restriction, we are thus ensured a solution of $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$.

In §2.3 we consider the equation $(P(D) + Q(D)H)F = \delta$ in $\mathcal{D}' \cap H'(\mathcal{D})$, where we can view $\mathcal{D}' \cap H'(\mathcal{D})$ as a subspace of $\mathcal{G}'$. We find a sufficient condition for $(P(D) + Q(D)H)u = \delta$ to have a solution in $\mathcal{D}' \cap H'(\mathcal{D})$ (Proposition 2.20). The solution of $(P(D) + Q(D)H)F$

$= \delta$ in $\mathcal{G}'$, constructed in the previous section, does not satisfy this condition (Remark 2.23) and we are led to conjecture that $(P(D) + Q(D)H)u = \delta$ has no solution in $\mathcal{D}' \cap H'(\mathcal{D})$ (Remark 2.31).

Finally, in §2.4, we consider the equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'$. We interpret $F \in \mathcal{G}'$, $\delta \in \mathcal{D}'$, and $\{P(D) + Q(D)H\} : \mathcal{G}' \rightarrow \mathcal{D}'$ (Definition 2.23). Interpreting $\{P(D) + Q(D)H\} : \mathcal{G}' \rightarrow \mathcal{D}'$ is equivalent to interpreting $\{P(D) + Q(D)H\}$ as a product mapping $P(D) \times (-Q(D)H) : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{D}'$ (Remark 2.34). Applying a Fourier transform to the equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'$ gives the equation $[[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]\mathcal{F}F = 1$ in $\mathcal{Z}'$ (Lemma 2.37), where $[[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]$ is interpreted as an operator from $\mathcal{Q}'$ to $\mathcal{Z}'$ (Definition 2.36). For arbitrary P & Q we can construct a solution for $\mathcal{F}F \in \mathcal{Z}'$ in the equation $[[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]\mathcal{F}F = 1$ (Proposition 2.39). This guarantees a solution for $F \in \mathcal{G}'$ in $\{P(D) + Q(D)H\}F = \delta$.

Chapter 3 is concerned with extending the ideas of chapter 2 to the n -dimensional case. In §3.1 we define the spaces $H(\mathcal{D})$, consisting of n -dimensional Hilbert transforms of elements in $\mathcal{D}(\mathbb{R}^n)$, and give it the topology transported from $\mathcal{D}(\mathbb{R}^n)$ via H . We thus define $H'(\mathcal{D})$ as the dual of $H(\mathcal{D})$. The space $\mathcal{J}$ is defined to consist of functions on $\mathbb{C}^n$ which can be written in the form $\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \alpha(s)$, ($s = (s_1, \dots, s_n) = (\sigma_1 + i\tau_1, \dots, \sigma_n + i\tau_n)$), where α is an entire function (in each of its arguments) which satisfies certain inequalities. The Fourier transform $\mathcal{F}H\phi$ is defined, for elements $H\phi$ in $H(\mathcal{D})$, and we establish the isomorphism $\mathcal{F} : H(\mathcal{D}) \rightarrow \mathcal{J}$. This allows us to establish an isomorphism $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$. We define the space $\mathcal{G}$ as the space of functions θ that can be written in the form $\theta = \phi + H\omega$ for $\phi, \omega \in \mathcal{D}$, and endow $\mathcal{G}$ with the obvious topology. There is a canonical isomorphism $\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$. We define a vector space $\mathcal{Q}$ of functions, and the Fourier transform of elements of $\mathcal{G}$, and show that $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{Q}$ is an isomorphism. The space $\mathcal{Q}'$ is the

isomorphic image of $\mathcal{G}'$ under $\mathcal{F}$. The space $\mathcal{G}'$ is closed with respect to the operations of Hilbert transformation and differentiation.

In §3.2 we consider the equation $P(D, H)F = \delta$ with an interpretation that $F, \delta \in \mathcal{G}'$. The equation reduces to $(P(D) + Q(D)H)F = \delta$. To construct solutions in $\mathcal{Q}'$ for the Fourier transformed equation, we use a modified form of the Hörmander's staircase method (Shilov [7, pp.176-178]). This method forces us to consider making a change of variables [coordinate rotation] $x \rightarrow Ux$ so that the rotated polynomials P_U & Q_U are such that $\overline{P_U}^2(s) - (-\pi^2)^n \overline{Q_U}^2(s)$ assumes a certain form. We show that such a rotation exists. We consider the effect of a rotation on an element F of $\mathcal{G}'$. The "rotated" form of F , F_U , is not an element of $\mathcal{G}'$, but rather of a space $(\mathcal{G}_U)'$. The rotated form of our equation becomes $P_U(D)F_U + Q_U(D)(HF)_U = \delta$ in $(\mathcal{G}_U)'$. Fourier transforming gives $(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}(U_j, \sigma)) Q_U(s)) \mathcal{F}F_U = 1$ in $(\mathcal{Q}_U)'$, where now $\mathcal{F} : (\mathcal{G}_U)' \rightarrow (\mathcal{Q}_U)'$. Provided P & Q satisfy a certain condition ($\spadesuit$, in remark 3.14, which is the analogue of $\overline{P}^2(s) + \pi^2 \overline{Q}^2(s) \neq 0$ at the origin in the finite dimensional case) we construct an element $\Lambda_U \in (\mathcal{Q}_U)'$ (definition 3.15) which is a solution for $\mathcal{F}F_U$. $(\Lambda_U, \tilde{\Theta}_U)$ is defined in terms of integrals of the two components of $\tilde{\Theta}_U$ over various surfaces in $\tau_1 \sigma_1 \dots \sigma_n$ -space. Under the restriction imposed by condition $\spadesuit$, we are thus ensured a solution of $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$.

In §3.3 we consider the equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'(\mathbb{R}^n)$, where we interpret $F \in \mathcal{G}'$, $\delta \in \mathcal{D}'$ and $\{P(D) + Q(D)H\} : \mathcal{G}' \rightarrow \mathcal{D}'$. We must again introduce a coordinate rotation so that the three polynomials associated with $\overline{P_U}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \overline{Q_U}(s)$ assume a certain form. We show that such a rotation exists. Applying a rotation to our equation gives $\{P_U(D) + Q_U(D)H\}F_U = \delta$ in $\mathcal{D}'$, where now F_U still denotes an element of $\mathcal{G}'$. Fourier transforming gives $[[P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q_U(s)]] \mathcal{F}F_U = 1$ in $\mathcal{Z}'$. We

construct a solution Λ for $\mathcal{FF}_U$ (definition 3.20 & proposition 3.22).

In §3.4 we consider the equation $\{P(D) + Q(D)H\}F = h$ in $\mathcal{D}'$, for arbitrary $h \in \mathcal{D}'$. We write $h = \sum_{k=0}^{\infty} h_k$, where the h_k are distributions with compact support. Our Fourier transformed equation becomes $[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] \mathcal{F}F = \mathcal{F}h$ in $\mathcal{Z}'$, where $\mathcal{F}h = \sum_{k=0}^{\infty} \mathcal{F}h_k$. We define $\Xi_k := \Lambda_k \mathcal{F}h_k$ in $\mathcal{Q}'$, where Λ_k is chosen so as to ensure both convergence of $\Xi := \sum_{k=0}^{\infty} \Xi_k$ in $\mathcal{Q}'$, and that $[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] \Xi_k = \mathcal{F}h_k$ in $\mathcal{Z}'$. This guarantees a solution Ξ for $\mathcal{F}F$.

In §3.5 we consider the equation $(P(D) + Q(D)H)F = G$ in $\mathcal{G}'$, for three cases of G . In Case (a) we interpret $G \in \mathcal{E}' \subset \mathcal{D}' \cap \mathcal{H}'(\mathcal{D})$, that is, $G = \chi(g, g)$ for some distribution g with compact support. We are able to construct a solution for $\mathcal{F}F$ in the transform domain. In Case (b) & Case (c) we consider, respectively, $G \in \mathcal{D}'$ (in the sense that $G = \chi(g, 0)$ for arbitrary $g \in \mathcal{D}'$) and arbitrary $G \in \mathcal{G}'$. Although we cannot construct transform domain solutions, we discuss various attempts and why they fail.

§3.6 contains some open problems regarding our work, and a suggestion for further research.

Our work relies heavily on the methods used to construct solutions in $\mathcal{D}'$ of constant coefficient linear partial differential equations. There are many excellent texts on this material, such as those by Shilov [7] or Trèves [9], for example. For no particular reason we choose Shilov [7] as a reference throughout this work.

In part two we consider a space $\mathcal{D}(\mathcal{H})$ that Dudin [1] has constructed on a real separable Hilbert space $\mathcal{H}$; this space consists of all the infinitely differentiable functions with bounded support, all of whose derivatives have bounded range. In Dudin's construction it is established that $\mathcal{D}'(\mathcal{H})$, the dual of $\mathcal{D}(\mathcal{H})$, is a certain space of measures. In this part we establish the non-nuclearity of $\mathcal{D}(\mathcal{H})$. This result follows by modifying result of Meise [4]

that a certain space C_p^∞ of infinitely differentiable functions on a real locally convex vector space E is non-nuclear, provided E contains an infinite dimensional compact absolutely convex set. Meise's proof can be adapted to deal with our case. On G_p^∞ Meise considers the topology of uniform convergence on compacts, topologised by the system of seminorms

$$p_{l,K,S}(f) := \sup_{0 \leq j \leq l} \sup_{x \in K} \sup_{y \in S} |D^j f(x)[y]|,$$

where $l \in \mathbb{N}_0$, compact $K \subset E$, and $S \in \wp$, where $\wp$ is a system of bounded subsets of E , directed by inclusion, which cover E and which contains all compact subsets of E . The topology that Dudin considers is significantly different and is described in part two.

CHAPTER 1. DEFINITIONS AND INITIAL RESULTS

The classical Hilbert transform of a function $f \in L^p$, for $p > 1$, is given by

$$Hf(x) = P \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt,$$

where the P denotes the Cauchy principal value of the integral. Titchmarsh [8] has shown that $H^2: L^p \rightarrow L^p$, with the inversion formula implied by the result $H^2 f = (-\pi^2)f$.

Definition 1.1 $\mathcal{D}_{L^p}$ is the vector space of complex valued C^∞ functions, defined on the real line, which, along with every derivative, belongs to L^p (for $p > 1$).

$\mathcal{D}_{L^p}$ is topologised by the family of seminorms γ_m for $m=0,1,2,\dots$, where, for $\phi \in \mathcal{D}_{L^p}$,

$$\gamma_m(\phi) := \left(\int_{-\infty}^{\infty} |\phi^{(m)}(x)|^p dx \right)^{1/p}.$$

Schwartz [6] has shown that $H: \mathcal{D}_{L^p} \rightarrow \mathcal{D}_{L^p}$ is a homeomorphism. Pandey & Chaudhry [6] have given an alternate proof of this result. Also, they give [2] a simple proof of the fact that if a sequence of functions $\{\phi_n\}$ tends to zero in $\mathcal{D}_{L^p}$, then for each $k=0,1,\dots$, $\phi_n^{(k)} \rightarrow 0$ uniformly for $x \in \mathbb{R}$, where the superscript (k) denotes the k^{th} derivative (this was first proved, using a different method, by Schwartz [6]). $\mathcal{D}$ is dense in $\mathcal{D}_{L^p}$ ($1 < p < \infty$), and convergence in $\mathcal{D}$ implies convergence in $\mathcal{D}_{L^p}$. The restriction of $f \in (\mathcal{D}_{L^p})'$ to $\mathcal{D}$ defines an element of $\mathcal{D}'$, that is $(\mathcal{D}_{L^p})' \subset \mathcal{D}'$ (see Schwartz [6, pp. 199]).

Following Pandey [4, pp. 87] we make the

Definition 1.2 $H(\mathcal{D})$ is the vector space of complex valued C^∞ functions ψ , defined on the real line, such that

$$\psi(x) = P \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt := H\phi(x), \text{ for } \phi \in \mathcal{D}.$$

The topology of $\mathcal{D}$ is transported to $H(\mathcal{D})$ via H . That is, a sequence $\{\phi_\nu\} \rightarrow 0$ in $H(\mathcal{D})$ iff the associated sequence $\{\phi_\nu\} \rightarrow 0$ in $\mathcal{D}$ (where $\psi_\nu = H\phi_\nu$). Chaudhry & Pandey [2] have demonstrated the following intrinsic definition of the topology on $H(\mathcal{D})$, (whose proof we shall discuss later)

Proposition 1.3 A sequence $\{\psi_\mu\} \rightarrow 0$ in $H(\mathcal{D})$ iff

- (i) the associated functions $\psi_\mu(z)$ are analytic outside a fixed closed interval $[a, b]$, or else $\psi_\mu(x)$ is analytic outside a fixed closed interval $[a, b]$, and
- (ii) $\{\psi_\mu\} \rightarrow 0$ in $\mathcal{D}_{L^p}$ as $\mu \rightarrow \infty$.

(In (i) the function $\psi_\mu(z)$ is defined in terms of $\phi_\mu \in \mathcal{D}$ via $(z = x + iy)$)

$$\psi_\mu(z) = \begin{cases} \int_{-\infty}^{\infty} \frac{\phi_\mu(t)}{t-z} dt, & \text{if } \text{Im}(z) \neq 0; \\ P \int_{-\infty}^{\infty} \frac{\phi_\mu(t)}{t-x} dt, & \text{if } \text{Im}(z) = 0. \end{cases}$$

We now consider Hilbert transforms of functions in the vector space S , consisting of all the C^∞ functions ("rapidly decreasing at infinity") for which $\lim_{|x| \rightarrow \infty} |x^\alpha \phi^{(k)}(x)| = 0$ for each $\alpha \in \mathbb{N}$ and each $k \in \mathbb{N}$. Convergence in S is defined by the family of seminorms $\{\gamma_{\alpha,k}\}$, for $\alpha, k \in \mathbb{N}$, where, for $\phi \in S$, $\gamma_{\alpha,k}(\phi) := \sup_{x \in \mathbb{R}} |x^\alpha \phi^{(k)}(x)|$.

Definition 1.4 $H(S)$ is the vector space of complex valued C^∞ functions ψ , defined on the real line, such that

$$\psi(x) = P \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt = H\phi, \text{ for } \phi \in S$$

Remark 1.5 Clearly $S \subset L^p$, for $p > 1$, and so the Hilbert transform of each element of S exists; $H : S \rightarrow H(S)$ is an injective map.

Again, the topology of S is transported to $H(S)$ via H . That is, a sequence $\{\psi_\nu\} \rightarrow 0$ in $H(S)$ iff the associated sequence $\{\phi_\nu\}$ converges to zero in S . The results of Pandey ([4], Theorems 1 & 2) are valid if we replace $\mathcal{D}$ by S , so we have:

Proposition 1.6 If $\psi = H\phi$, for $\phi \in S$, then $\psi^{(k)}(x) = H\phi^{(k)}(x)$.

Proposition 1.7 (i) $H(S) \subset \mathcal{D}_{L^p}$ and $H(S)$ is dense in $\mathcal{D}_{L^p}$. (ii) Convergence of a sequence in $H(S)$ implies its convergence in $\mathcal{D}_{L^p}$.

Remark 1.8 In the proof of 1.7(ii) we use the result that convergence in S implies convergence in $\mathcal{D}_{L^p}$.

Remark 1.9 The inversion formula $-H^2 f = \pi^2 f$, for $f \in L^p$, tells us that H is a linear homeomorphism from S onto $H(S)$.

We now consider the question of an intrinsic definition of the topology of $H(S)$, and contrast it with the already mentioned intrinsic definition of the topology of $H(\mathcal{D})$. To isolate the essential part of the proof of Proposition 1.3 we consider the

Lemma 1.10 Suppose that $\{\phi_\nu\}$ is a sequence in $\mathcal{D}$, all of whose members have supports contained in a common closed interval. Then $\phi_\nu \rightarrow 0$ in $\mathcal{D}$ iff $\phi_\nu \rightarrow 0$ in $\mathcal{D}_{L^p}$.

Proof: The necessity follows easily. Sufficiency comes from the previously mentioned result (proved simply in Chaudhry & Pandey [2] Lemma 1) that if $\phi_\nu \rightarrow 0$ in $\mathcal{D}_{L^p}$ then $\phi^{(k)} \rightarrow 0$ uniformly for all $x \in \mathbb{R}$, for each $k=0,1,2,\dots$.

Remark 1.11 This Lemma is used to prove Proposition 1.3, concerning the intrinsic description of convergence in $H(\mathcal{D})$, as follows. Suppose the sequence $\{\psi_\mu\} \rightarrow 0$ in $H(\mathcal{D})$, that is, the associated sequence $\{\phi_\mu\} \rightarrow 0$ in $\mathcal{D}$. Then Chaudhry & Pandey [2] show that condition (i) of Proposition 1.3 holds. Lemma 1.10 tells us that $\phi_\mu \rightarrow 0$ in $\mathcal{D}_{L^p}$. We have $\psi_\mu^{(k)} = H\phi_\mu^{(k)}$, by Proposition 1.6. Therefore, the following inequality holds (see Titchmarsh [8, pp. 132-133]): $\|\psi_\mu^{(k)}\|_p \leq C_p \|\phi_\mu^{(k)}\|_p$, where $\|\cdot\|_p$ denotes the L^p norm. Therefore, $\psi_\mu \rightarrow 0$ in $\mathcal{D}_{L^p}$. Now suppose that conditions (i) & (ii) of Proposition 1.3 hold. Then there exists a closed interval $[a,b]$ containing the supports of all the ϕ_μ . From the inversion formula, $\phi_\mu(x) = (-1/(\pi^2))H\psi_\mu(x)$, and so $\|\phi_\mu\|_p \leq \bar{C}_p \|\psi_\mu\|_p$. Thus, $\phi_\mu \rightarrow 0$ in $\mathcal{D}_{L^p}$, and so $\phi_\mu \rightarrow 0$ in $\mathcal{D}$, and so finally $\psi_\mu \rightarrow 0$ in $H(\mathcal{D})$.

It is clear that if a sequence of functions $\{\phi_\mu\} \rightarrow 0$ in $\mathcal{S}$ then $\{\phi_\mu\} \rightarrow 0$ in $\mathcal{D}_{L^p}$. The converse is false, as shown by our following

Counterexample 1.12 The following sequence $\{\phi_\nu\} \in \mathcal{S}$ converges to zero in $\mathcal{D}_{L^p}$ but not in $\mathcal{S}$. Let $\alpha \in \mathcal{D}$ be such that $0 \leq \alpha \leq 1$, $\text{supp } \alpha \subset [-1, 1]$, and $\alpha(0) = 1$. For example, consider $\alpha(x) := e \cdot \exp(1/(x^2 - 1))$ if $|x| < 1$, and $:= 0$ if $|x| \geq 1$. Then define $\phi_\nu(x) := \alpha(x - 2\nu)/\nu$, for $\nu = 1, 2, \dots$. Clearly $\phi_\nu \rightarrow 0$ in $\mathcal{D}_{L^p}$, since for each fixed $k = 0, 1, 2, \dots$ we have

$$\begin{aligned} \gamma_k(\phi) &:= \left\{ \int_{-\infty}^{\infty} \left| \frac{\alpha^{(k)}(x - 2\nu)}{\nu} \right|^p dx \right\}^{1/p} \\ &\leq \frac{1}{\nu} \left\{ \int_{-1}^1 |\alpha^{(k)}(x)|^p dx \right\}^{1/p} \leq \frac{c_k}{\nu} \rightarrow 0 \text{ as } \nu \rightarrow \infty. \end{aligned}$$

However, we see that

$$x^2 \phi_\nu(x) = \frac{x^2 \alpha(x - 2\nu)}{\nu} \geq \frac{(2\nu - 1)^2 \alpha(x - 2\nu)}{\nu} \geq \frac{\nu(\nu - 1) \alpha(x - 2\nu)}{\nu} = (\nu - 1) \alpha(x - 2\nu),$$

and this last sequence of functions clearly does not tend to zero uniformly $\forall x \in \mathbb{R}$ as $\nu \rightarrow \infty$.

Remark 1.13 Since a sequence converging to zero in $\mathcal{D}_{L^p}$ needn't converge to zero in $\mathcal{S}$, we have no intrinsic description of the topology of $\mathcal{H}(\mathcal{S})$, involving convergence in $\mathcal{D}_{L^p}$, which is analogous to that in Proposition 1.3.

We now consider how the spaces $\mathcal{H}(\mathcal{D})$ and $\mathcal{H}(\mathcal{S})$ relate to $\mathcal{D}$ & $\mathcal{S}$, respectively, as sets.

Proposition 1.14 $\mathcal{D} \cap \mathcal{H}(\mathcal{D}) = \{0\}$.

Proof: If $\phi \in \mathcal{D}$ then the Fourier transform of ϕ , $\mathcal{F}\phi$, must be an entire function of its complex argument s , where $\mathcal{F}\phi(s) := \int_{-\infty}^{\infty} \phi(x) e^{ixs} dx$. (See Shilov [7, pp. 136, where he uses $\mathcal{K} \equiv \mathcal{D}$]). In particular then, if we write $s = \sigma + i\tau$, then $\mathcal{F}\phi$ is an analytic function of σ . On the other hand, if $\phi = \mathcal{H}\psi$, where $\phi, \psi \in \mathcal{D}$, then it can be shown that

$\mathcal{F}H\psi(\sigma) = i\pi \operatorname{sgn}\sigma \mathcal{F}\psi(\sigma)$ a.e. ("almost everywhere") (see Butzer & Nessel [1, pp. 324]), where the function sgn is defined by

$$\operatorname{sgn}\sigma = \begin{cases} \sigma/|\sigma|, & \text{if } \sigma \neq 0; \\ 0 & \text{if } \sigma = 0. \end{cases}$$

If $\mathcal{F}H\psi$ is entire, then $\lim_{\sigma \rightarrow 0} \mathcal{F}H\psi(\sigma)$ exists independently of the manner in which σ approaches zero, therefore $\mathcal{F}\psi(0) = 0$. Also, $\lim_{\sigma \rightarrow 0} (\mathcal{F}H\psi)'(\sigma)$ exists (where a prime denotes differentiation) and so we must have $(\mathcal{F}\psi)'(0) = 0$. Continuing this process we see that $(\mathcal{F}\psi)^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots$ and so the entire function $\mathcal{F}\psi$ is zero in a neighbourhood of the origin. Thus $\mathcal{F}\psi \equiv 0$, $\psi \equiv 0$, and so $H\psi = \phi = 0$.

Proposition 1.15 $S \cap H(S) = \{\phi \in C^\infty : \phi = H\psi, \text{ for } \psi \in S, \text{ with } (\mathcal{F}\psi)^{(k)}(0) = 0 \forall k = 0, 1, \dots\}$.

Proof: Suppose that $\phi \in S \cap H(S)$. Then $\phi = H\psi$ for some $\psi \in S$. Therefore $\mathcal{F}\phi = \mathcal{F}H\psi$ as elements of S , and both functions equal $i\pi \operatorname{sgn}\sigma \mathcal{F}\psi(\sigma)$ a.e. (see Butzer & Nessel [1, pp. 324]). Following the proof of proposition 1.14 we conclude that $\mathcal{F}H\psi = i\pi \operatorname{sgn}\sigma \mathcal{F}\psi(\sigma) \forall \sigma$ and $(\mathcal{F}\psi)^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots$. Alternately, suppose that $\phi \in C^\infty$ is such that $\phi = H\psi$ for $\psi \in S$ with $(\mathcal{F}\psi)^{(k)}(0) = 0$. Clearly $\phi \in H(S)$. Then $\psi = (-1/\pi^2)H\phi$ in S and so $\mathcal{F}\psi = (-1/\pi^2)i\pi \operatorname{sgn}\sigma \mathcal{F}\phi(\sigma)$ a.e.. Again, since $\mathcal{F}\psi \in S \subset C^\infty$ we must have $\mathcal{F}H\phi = i\pi \operatorname{sgn}\sigma \mathcal{F}\phi(\sigma) \forall \sigma$ and $(\mathcal{F}\phi)^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots$. It follows that $i\pi \operatorname{sgn}\sigma \mathcal{F}\phi(\sigma) \in S$ and so $\mathcal{F}\phi \in S$ too. Therefore $\phi \in S$. Incidentally, we have shown that $\mathcal{F}H\psi = i\pi \operatorname{sgn}\sigma \mathcal{F}\psi(\sigma) \forall \sigma$ if $\psi \in S \cap H(S)$.

Corollary 1.16 $S \cap H(S) = \{\phi \in S : (\mathcal{F}\phi)^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots\}$.

Proof: If $\phi \in S \cap H(S)$ we saw in the proof of proposition 1.15 that $(\mathcal{F}\phi)^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots$. Alternately, if $\phi \in S$ is such that $(\mathcal{F}\phi)^{(k)}(0) = 0$ then, by proposition 1.15, if $\psi = H\phi$ we have $\psi \in S \cap H(S)$, and so $H\phi \in S \cap H(S)$.

Proposition 1.17 $S \cap H(S)$ is closed with respect to H .

Proof Suppose $\phi \in \mathcal{S} \cap H(\mathcal{S})$. Then $\phi = H\psi$ for $\psi \in \mathcal{S}$, and so $H\phi = -\pi^2\psi \in \mathcal{S}$. Therefore $H\phi \in \mathcal{S} \cap H(\mathcal{S})$.

Corollary 1.18 $H : \mathcal{S} \cap H(\mathcal{S}) \rightarrow \mathcal{S} \cap H(\mathcal{S})$ is an algebraic isomorphism.

Example 1.19 The characterisation of $\mathcal{S} \cap H(\mathcal{S})$ given in Corollary 1.16 allows us to construct the following function ϕ in $\mathcal{S} \cap H(\mathcal{S})$. Consider $\alpha \in \mathcal{D}$ defined by

$$\alpha(x) = \begin{cases} e^{1/(x^2-1)}, & \text{if } |x| < 1; \\ 0, & \text{otherwise,} \end{cases}$$

and then $\beta(x) := \alpha(x-1) + \alpha(x+1)$. Clearly $\beta \in \mathcal{S}$ and $\beta^{(k)}(0) = 0 \forall k = 0, 1, 2, \dots$

Therefore $\phi := \mathcal{F}^{-1}(\beta) \in \mathcal{S} \cap H(\mathcal{S})$.

Remark 1.20 We now return to a consideration of $H(\mathcal{D})$ and its dual, which we denote by $H'(\mathcal{D})$. We can define the generalized Hilbert transform Hf of $f \in \mathcal{D}'$ as that element of $H'(\mathcal{D})$ which satisfies $\langle Hf, \psi \rangle = \langle f, -H\psi \rangle$, for all $\psi \in H(\mathcal{D})$ (see for example, Pandey [4, pp. 89]). If $g \in H'(\mathcal{D})$ then its Hilbert transform Hg is likewise defined to be the distribution satisfying $\langle Hg, \phi \rangle = \langle g, -H\phi \rangle \forall \phi \in \mathcal{D}$. From these follow the inversion formula $H^2f = -\pi^2f \forall f \in \mathcal{D}'$, and so $H : \mathcal{D}' \rightarrow H'(\mathcal{D})$ is an isomorphism. We define the derivative g' of $g \in H'(\mathcal{D})$ to be that element of $H'(\mathcal{D})$ satisfying $\langle g', \psi \rangle = \langle g, -\psi' \rangle = \langle g, -H\phi' \rangle \forall \psi \in H(\mathcal{D})$, where $\psi = H\phi$, $\phi \in \mathcal{D}$.

Remark 1.21 $\mathcal{D}' \cap H'(\mathcal{D})$ is nonempty. For examples, both δ and $P.V.(1/x)$ lie in each space, and in fact $H\delta = -P.V.(1/x)$ (see remark 1.29 and Pandey [4]).

Proposition 1.22 $\mathcal{D}' \cap H'(\mathcal{D})$ is closed with respect to the operators $\frac{d}{dx}$ and H , and $\mathcal{D}' \cap H'(\mathcal{D})$ is a vector subspace of $\mathcal{D}'$ and of $H'(\mathcal{D})$.

Proof The first and third assertions are trivial. For the second, consider $f \in \mathcal{D}' \cap H'(\mathcal{D})$, so that certainly $Hf \in H'(\mathcal{D})$ (viewing $f \in \mathcal{D}'$) and $Hf \in \mathcal{D}'$ (viewing $f \in H'(\mathcal{D})$).

Remark 1.23 We now consider the Fourier transform of elements of $\mathcal{D}$ (see Shilov [7, pp. 134 ff.]). The Fourier transform $\Phi(\sigma)$ of $\phi \in \mathcal{D}$ is defined to be $\int_{-\infty}^{\infty} \phi(x)e^{i\sigma x} dx$. Since ϕ

has compact support we can extend the argument σ to complex values of $s = \sigma + i\tau$. The Fourier transform $\Phi(s)$ with complex argument is thus defined to be $\mathcal{F}\phi(s) := \Phi(s) := \int_{-\infty}^{\infty} \phi(x)e^{isx}dx$. It is known that $\mathcal{F}$, the Fourier transform with complex argument, establishes an isomorphism between $\mathcal{D}$ and the space $\mathcal{Z}$, consisting of all entire functions $\Phi(s)$ such that $|s|^q |\Phi(s)| \leq C_q e^{a|\tau|}$ for all $q = 0, 1, \dots$, where $s = \sigma + i\tau$ and the constants a and C_q depend on Φ . The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{Z} \rightarrow \mathcal{D}$ is defined by $\phi(x) := \mathcal{F}^{-1}\Phi(x) := (1/2\pi) \int_{-\infty}^{\infty} \Phi(\sigma)e^{-i\sigma x}d\sigma$. From Shilov [7, pp. 138]) we have the definition that a sequence $\Phi_\nu(s) \rightarrow 0$ in $\mathcal{Z}$ iff the associated sequence $\phi_\nu(x) \rightarrow 0$ in $\mathcal{D}$. Equivalently, $\Phi_\nu(s) \rightarrow 0$ in $\mathcal{Z}$ iff first the inequalities $|s|^q \Phi_\nu(s) \leq C_q e^{a|\tau|}$ hold (for each $q = 0, 1, 2, \dots$) for constant C_q and a not depending on ν , and if secondly $\Phi_\nu(s) \rightarrow 0$ uniformly on each bounded interval of the real line.

Remark 1.24 We consider Fourier transforms of elements of $H(\mathcal{D})$. For $\phi \in \mathcal{D}$ we have $\int_{-\infty}^{\infty} H\phi(x)e^{i\sigma x}dx = i\pi \operatorname{sgn}\sigma \mathcal{F}\phi(\sigma)$ a.e. (see Butzer & Nessel [1, pp. 324]), where, again, $\operatorname{sgn}\sigma := \sigma/|\sigma|$ if $\sigma \neq 0$, and $:= 0$ if $\sigma = 0$. Since $H\phi$ does not have compact support we cannot extend σ in the integrand to complex values of $s = \sigma + i\tau$. However, we can make the following initial

Definition 1.25 We define the Fourier transform, with complex argument, of a function $H\phi$ in $H(\mathcal{D})$ to be $\mathcal{F}H\phi(s) := i\pi \operatorname{sgn}\sigma \mathcal{F}\phi(s)$. This coincides a.e. with the conventional Fourier transform on the real axis.

Definition 1.26 We define $\mathcal{J}$ to be the vector space consisting of all functions that can be written in the form $i\pi \operatorname{sgn}\sigma \Phi(s)$, where $\Phi(s)$ is an entire function satisfying $|s|^q \Phi(s) \leq C_q e^{a|\tau|} \forall q = 0, 1, 2, \dots$, where the constants a and C_q may depend on Φ . $\mathcal{J}$ is endowed with the linear operations of addition and scalar multiplication in an obvious way.

Proposition 1.27 The Fourier transform with complex argument (given in Definition

1.25) $\mathcal{F} : H(\mathcal{D}) \rightarrow \mathcal{J}$ is an isomorphism.

Proof We see that $\mathcal{F} = i\pi \operatorname{sgn} \sigma \circ \mathcal{F} \circ (-H/\pi^2)$, where $(-H/\pi^2) : H(\mathcal{D}) \rightarrow \mathcal{D}$, given by $(-H/\pi^2)(H\phi) = \phi$, is an isomorphism. $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}$ is an isomorphism, and $i\pi \operatorname{sgn} \sigma : \mathcal{Z} \rightarrow \mathcal{J}$ given by $i\pi \operatorname{sgn} \sigma : \Phi(s) \mapsto i\pi \operatorname{sgn} \sigma \Phi(s)$ is an isomorphism. The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{J} \rightarrow H(\mathcal{D})$ is given by $\mathcal{F}^{-1}(i\pi \operatorname{sgn} \sigma \Phi(s))(x) := H\mathcal{F}^{-1}\Phi$.

Remark 1.28 We define convergence in $\mathcal{J}$ by saying that a sequence $\{i\pi \operatorname{sgn} \sigma \Phi_\nu\} \rightarrow 0$ in $\mathcal{J}$ iff the corresponding sequence $\{H\phi_\nu\} \rightarrow 0$ in $H(\mathcal{D})$ (and so iff $\{\phi_\nu\} \rightarrow 0$ in $\mathcal{D}$). Equivalently, $i\pi \operatorname{sgn} \sigma \Phi_\nu \rightarrow 0$ in $\mathcal{J}$ iff first the inequalities $|s^q \Phi_\nu(s)| \leq C_q e^{a|s|}$ hold ($\forall q = 0, 1, 2, \dots$) for constants C_q and a not depending on ν , and if secondly $\{\Phi_\nu\}(s) \rightarrow 0$ uniformly on each bounded interval of the real line.

Remark 1.29 In Remark 1.21 we mentioned that $\mathcal{D}' \cap H'(\mathcal{D})$ is nonempty. In a sense this statement is meaningless, since $\mathcal{D} \cap H(\mathcal{D}) = \{0\}$ and therefore, for example, an element of $\mathcal{D}'$ is defined only on $\mathcal{D}$ and not on $H(\mathcal{D})$. Nevertheless, we identify $\mathcal{D}' \cap H'(\mathcal{D})$ as those elements of $\mathcal{D}'$ which, when acting on $H(\mathcal{D})$ (that is, in the "prescription" for an element v of $\mathcal{D}'$ we act on elements of $H(\mathcal{D})$) define continuous linear forms. As examples we gave δ and $-P.V.(1/x)$. We know that $H(\mathcal{D}) \subset C^\infty(\mathbb{R})$ (proposition 1.6); if a sequence $\psi_\nu \rightarrow 0$ in $H(\mathcal{D})$ then $\psi_\nu \rightarrow 0$ in $\mathcal{D}_{L^p}$ (proposition 1.3), and so $\psi^{(k)} \rightarrow 0$ uniformly $\forall x \in \mathbb{R}$, for each $k = 0, 1, 2, \dots$ (see proof of proposition 1.10). Therefore $\psi_\nu \rightarrow 0$ in $\mathcal{E}$, where $\mathcal{E}$ denotes the vector space of smooth functions with the topology of uniform convergence of a function and its derivatives on compact sets. Therefore $H(\mathcal{D})$ may be embedded continuously in $\mathcal{E}$, and so we conclude that $\mathcal{E}' \subset H'(\mathcal{D})$. Thus $\mathcal{E}' \subset \mathcal{D}' \cap H'(\mathcal{D})$.

Remark 1.30 We have previously defined Fourier transform mappings $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{Z}$ (remark 1.23) and $\mathcal{F} : H(\mathcal{D}) \rightarrow \mathcal{J}$ (definition 1.25) and shown that they are isomorphisms. These definitions allow us to define Fourier transforms from $\mathcal{D}'$ to $\mathcal{Z}'$ (Shilov [7, pp.141])

and from $H'(\mathcal{D})$ to $\mathcal{J}'$, according to the formulas $\langle \mathcal{F}f, \zeta \rangle := 2\pi \langle f, \mathcal{F}^{-1}\zeta \rangle \forall \zeta \in \mathcal{Z}$, and $\langle \mathcal{F}g, \mathcal{F}H\phi \rangle := 2\pi \langle g, H\phi \rangle \forall \mathcal{F}H\phi \in \mathcal{J}$, where $f \in \mathcal{D}'$, and $g \in H'(\mathcal{D})$. This raises a dilemma. On the one hand, for example, if $f \in \mathcal{D}'$ then $\mathcal{F}f$ is only defined on $\mathcal{Z}$; that is, $\langle \mathcal{F}f, \alpha \rangle$ only makes sense if α is an entire function satisfying the given inequalities characterising $\mathcal{Z}$. On the other hand, if $f \in \mathcal{D}' \cap H'(\mathcal{D})$, we should want $\mathcal{F}f$ to lie in $\mathcal{J}'$ as well; that is, we should want to define $\langle \mathcal{F}f, i\pi \operatorname{sgn} \sigma \alpha(s) \rangle$, where α is an entire function as before. Put another way, we have $\mathcal{D}' \cap H'(\mathcal{D})$ nonempty, so we should like $\mathcal{F}(\mathcal{D}') \cap \mathcal{F}(H'(\mathcal{D}))$ nonempty. This prompts us to give the following (which we shall henceforth adopt)

Definitions 1.31 We define the Fourier transform of an element $f \in \mathcal{D}'$ to be an element of $\mathcal{Z}'$ defined by $\langle \mathcal{F}f, \alpha \rangle := 2\pi \langle f, \mathcal{F}^{-1}\alpha \rangle \forall \alpha \in \mathcal{Z}$. If, furthermore, $f \in \mathcal{D}' \cap H'(\mathcal{D})$, then the Fourier transform of f also defines an element of $\mathcal{J}'$ according to $\langle \mathcal{F}f, i\pi \operatorname{sgn} \sigma \alpha \rangle := 2\pi \langle f, \mathcal{F}^{-1}(i\pi \operatorname{sgn} \sigma \alpha(s)) \rangle \forall i\pi \operatorname{sgn} \sigma \alpha \in \mathcal{J}$. Similarly, we define the Fourier transform of an element g of $H'(\mathcal{D})$ to be an element of $\mathcal{J}'$ defined by

$$\langle \mathcal{F}g, i\pi \operatorname{sgn} \sigma \alpha \rangle := 2\pi \langle g, \mathcal{F}^{-1}(i\pi \operatorname{sgn} \sigma \alpha(s)) \rangle \forall i\pi \operatorname{sgn} \sigma \alpha \in \mathcal{J}.$$

If, further, $g \in \mathcal{D}' \cap H'(\mathcal{D})$, then the Fourier transform of g also defines an element of $\mathcal{Z}'$ according to $\langle \mathcal{F}g, \alpha \rangle := 2\pi \langle g, \mathcal{F}^{-1}\alpha \rangle \forall \alpha \in \mathcal{Z}$.

Remark 1.32 These definitions ensure that if $v \in \mathcal{D}' \cap H'(\mathcal{D})$, then the Fourier transform of v [treated as an element of $\mathcal{D}'$] agrees with the other Fourier transform of v [treated as an element of $H'(\mathcal{D})$]. As an example $1 \in \mathcal{Z}' \cap \mathcal{J}'$, and $\mathcal{F}\delta = 1$, where $\delta \in \mathcal{D}' \cap H'(\mathcal{D})$.

Definitions 1.33 We define the inverse Fourier transform of an element of $\mathcal{Z}'$ to be an element of $\mathcal{D}'$ defined by $\langle \mathcal{F}^{-1}f, \phi \rangle := (1/2\pi) \langle f, \mathcal{F}\phi \rangle \forall \phi \in \mathcal{D}$. If, furthermore, $f \in \mathcal{Z}' \cap \mathcal{J}'$, then the inverse Fourier transform of f also defines an element of $H'(\mathcal{D})$ according to $\langle \mathcal{F}^{-1}f, H\phi \rangle := (1/2\pi) \langle f, \mathcal{F}H\phi \rangle$. Similarly, the inverse Fourier transform of an element g of $\mathcal{J}'$ is defined to be an element of $H'(\mathcal{D})$ defined by $\langle \mathcal{F}^{-1}g, H\phi \rangle := (1/2\pi) \langle g, \mathcal{F}H\phi \rangle$. If,

furthermore, $g \in \mathcal{Z}' \cap \mathcal{J}'$, then the inverse Fourier transform of g also defines an element of $\mathcal{D}'$ according to $\langle \mathcal{F}^{-1}g, \phi \rangle := (1/2\pi) \langle g, \mathcal{F}\phi \rangle$.

Remark 1.34 These last definitions ensure that if $f \in \mathcal{Z}' \cap \mathcal{J}'$, then the various definitions of the inverse transforms of f agree.

Remark 1.35 From definitions 1.31 and 1.33 we conclude that $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ and $\mathcal{F} : \mathcal{H}'(\mathcal{D}) \rightarrow \mathcal{J}'$ are isomorphisms, and that $\mathcal{F} : \mathcal{D}' \cap \mathcal{H}'(\mathcal{D}) \rightarrow \mathcal{Z}' \cap \mathcal{J}'$ is an isomorphism.

CHAPTER 2. THE EQUATION $P(D, H)u = \delta$: ONE DIMENSIONAL CASE

§2.1 The Spaces $\mathcal{G}$, $\mathcal{Q}$, and their Duals

In light of Proposition 1.14; that $\mathcal{D} \cap H(\mathcal{D}) = \{0\}$, we make the following

Definition 2.1 We define the vector space $\mathcal{G} := \{\theta \in C^\infty(\mathbb{R}) : \theta = \phi + H\omega, \text{ where } \phi, \omega \in \mathcal{D}\}$ endowed with the linear operations of addition and scalar multiplication in an obvious way.

Remark 2.2 If $\theta = \phi + H\omega = 0$ then we must have $\phi = \omega = 0$ since $\mathcal{D} \cap H(\mathcal{D}) = \{0\}$. Thus, for $\theta \in \mathcal{G}$ the decomposition is unique.

Remark 2.3 We describe convergence in $\mathcal{G}$ by saying that a sequence $\{\theta_\nu\} \rightarrow 0$ in $\mathcal{G}$ iff both associated sequences $\{\phi_\nu\} \rightarrow 0$ in $\mathcal{D}$ and $\{H\omega_\nu\} \rightarrow 0$ in $H(\mathcal{D})$. There is a canonical algebraic isomorphism from $\mathcal{D} \times H(\mathcal{D}) \rightarrow \mathcal{G}$, where $(\phi, H\omega) \rightarrow \phi + H\omega$ and the topology on $\mathcal{G}$ introduced corresponds to the finest topology on $\mathcal{G}$, compatible with the vector space structure, making the isomorphism continuous.

Proposition 2.4 *There is a canonical [algebraic] isomorphism $\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$, the dual of $\mathcal{G}$.*

Proof: We define $\chi(f, Hg) := F \in \mathcal{G}'$, where $\langle F, \theta \rangle := \langle f, \phi \rangle + \langle Hg, H\omega \rangle$, and where $\theta = \phi + H\omega \in \mathcal{G}$. In this definition $f, g \in \mathcal{D}'$ ($H : \mathcal{D}' \rightarrow H'(\mathcal{D})$ is an isomorphism, see Pandey [4, pp. 89]). χ is clearly linear. To see that it is injective, suppose that $\chi(f, Hg) = 0$, so that $\langle f, \phi \rangle + \langle Hg, H\omega \rangle = 0 \forall \phi, \omega \in \mathcal{D}$. Fixing $\phi = 0$ and varying ω , and then fixing $\omega = 0$ and varying ϕ , we see that $f = g = 0$ in $\mathcal{D}'$. To see that χ is surjective, consider an arbitrary $F \in \mathcal{G}'$. Then F defines an element $f \in \mathcal{D}'$ through the relation $\langle f, \phi \rangle := \langle F, \theta \rangle$, where we fix $\omega = 0$ in $\theta = \phi + H\omega$. Similarly, F defines an element $Hg \in H'(\mathcal{D})$. Thus, we have $\langle F, \theta \rangle = \langle f, \phi \rangle + \langle Hg, H\omega \rangle$, where $\theta = \phi + H\omega$ is arbitrary, and

so $\chi(f, Hg) = F$.

Remark 2.5 We consider Fourier transforms of elements of $\mathcal{G}$. It can be shown (Butzer & Nessel [1, pp. 324]) that

$$\begin{aligned}\mathcal{F}\theta(\sigma) &= \mathcal{F}\phi(\sigma) + \mathcal{F}H\omega(\sigma) \\ &= \int_{-\infty}^{\infty} \phi(x)e^{i\sigma x}dx + \int_{-\infty}^{\infty} H\omega(x)e^{i\sigma x}dx \\ &= \mathcal{F}\phi(\sigma) + i\pi \operatorname{sgn}\sigma \mathcal{F}\omega(\sigma) \quad \text{a.e.}\end{aligned}$$

As $H\omega$ does not have compact support, we cannot extend σ in the second integral above to complex values of $s = \sigma + i\tau$. However, as in Definition 1.25 we make the following

Definition 2.6 We define the Fourier transform, with complex argument, $\mathcal{F}\theta(s)$ of an element $\theta \in \mathcal{G}$ by ($\theta = \phi + H\omega$)

$$\mathcal{F}\theta(s) := \Theta(s) := \mathcal{F}\phi(s) + i\pi \operatorname{sgn}\sigma \mathcal{F}\omega(s), \text{ where } s = \sigma + i\tau.$$

Remark 2.7 The functions $\mathcal{F}\phi(s)$ and $\mathcal{F}\omega(s)$ are entire and satisfy inequalities such as, for example, $|s^q \mathcal{F}\phi(s)| \leq C_q e^{a|\tau|} \forall q = 0, 1, 2, \dots$, where C_q and a are constants depending on $\mathcal{F}\phi$. $\mathcal{F}\omega$ satisfies a similar inequality.

Definition 2.8 We define $\mathcal{Q}$ to be the vector space consisting of all functions defined on the complex plane, that can be written in the form $\alpha(s) + i\pi \operatorname{sgn}\sigma \beta(s)$, where α and β are entire functions satisfying inequalities $|s^q \alpha(s)| \leq C_{q,\alpha} e^{a|\tau|}$, $|s^q \beta(s)| \leq C_{q,\beta} e^{a|\tau|} \forall q = 0, 1, 2, \dots$, where $C_{q,\alpha}$, $C_{q,\beta}$, and a are constants that may depend on either α or β , respectively. $\mathcal{Q}$ is endowed with the linear operations of addition and scalar multiplication in an obvious way.

Proposition 2.9 The Fourier transform $\mathcal{J}$, with complex argument, mapping $\mathcal{G}$ to $\mathcal{Q}$, is an isomorphism.

Proof By definition 2.6 we see that $\mathcal{F} = Y \circ \mathcal{F} \times \mathcal{F} \circ M$, where M is the canonical isomorphism from $\mathcal{D} \times H(\mathcal{D})$ to $\mathcal{G}$ (remark 2.3), $\mathcal{F} \times \mathcal{F} : \mathcal{D} \times H(\mathcal{D}) \rightarrow \mathcal{Z} \times \mathcal{J}$

is an isomorphism (remark 1.23. and Proposition 1.27), and $Y: \mathcal{Z} \times \mathcal{J} \rightarrow \mathcal{Q}$, given by $Y(\alpha(s), i\pi \operatorname{sgn} \sigma \beta(s)) := \alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)$, is an isomorphism. The inverse transform $\mathcal{F}^{-1}: \mathcal{Q} \rightarrow \mathcal{G}$ is given by $\mathcal{F}^{-1}(\alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)) := \mathcal{F}^{-1}\alpha + H\mathcal{F}^{-1}\beta$.

Definition 2.10 A sequence $\{\Theta_\nu\} \rightarrow 0$ in $\mathcal{Q}$ iff the associated sequence $\{\theta_\nu\} \rightarrow 0$ in $\mathcal{G}$.

Proposition 2.11 A sequence $\{\Theta_\nu\} \rightarrow 0$ in $\mathcal{Q}$ iff the following two conditions are both met ($\theta_\nu = \alpha_\nu + i\pi \operatorname{sgn} \sigma \beta_\nu$)

(i). For each $q = 0, 1, 2, \dots$ we have $|s^q \alpha_\nu(s)| \leq C_{q,\alpha} e^{a|s|} \forall \nu$, and $|s^q \beta_\nu(s)| \leq C_{q,\beta} e^{a|s|} \forall \nu$,

where constants $a, C_{q,\alpha}$, and $C_{q,\beta}$ are independent of ν , and

(ii) Both $\alpha_\nu(\sigma) \rightarrow 0$ and $\beta_\nu(\sigma) \rightarrow 0$ uniformly on each bounded interval of the real line.

Proof: Parallels that for the case of convergence in $\mathcal{Z}$ (remark 1.23), *mutatis mutandis* (see Shilov [7, pp 138]).

Remark 2.12 $\mathcal{F}: \mathcal{G} \rightarrow \mathcal{Q}$ is clearly continuous.

Definition 2.13 We define the Fourier transform mapping $\mathcal{F}: \mathcal{G}' \rightarrow \mathcal{Q}'$ by $\langle \mathcal{F}F, \Theta \rangle := (2\pi) \langle F, \mathcal{F}^{-1}\Theta \rangle \forall \Theta \in \mathcal{Q}$, and the inverse Fourier transform $\mathcal{F}^{-1}: \mathcal{Q}' \rightarrow \mathcal{G}'$ by $\langle \mathcal{F}^{-1}\Lambda, \theta \rangle := (2\pi)^{-1} \langle \Lambda, \mathcal{F}\theta \rangle \forall \theta \in \mathcal{G}$.

Remark 2.14 In light of Proposition 2.9 we see $\mathcal{F}: \mathcal{G}' \rightarrow \mathcal{Q}'$ is an isomorphism (indeed, this should be verified before we define the inverse Fourier transform). Also, in these last two, and in subsequent definitions we take licence to use the symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote all the so-called Fourier and inverse Fourier transforms between the various spaces. The particular "version" of $\mathcal{F}$ or $\mathcal{F}^{-1}$ to be interpreted will be clear from the argument of the operator, usually.

Remark 2.15 In Chapter 1 we introduced the space $\mathcal{J}$ (Definition 1.26), consisting of the Fourier transforms of elements in $H(\mathcal{D})$ (Proposition 1.27). We defined the Fourier transform mapping $\mathcal{F}: H'(\mathcal{D}) \rightarrow \mathcal{J}'$ (definitions 1.31), with inverse $\mathcal{F}^{-1}: \mathcal{J}' \rightarrow H'(\mathcal{D})$.

We mentioned the notion of the Fourier transform mapping $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ (Definitions 1.31) with its inverse $\mathcal{F}^{-1} : \mathcal{Z}' \rightarrow \mathcal{D}'$. We claim that there is a canonical linear isomorphism $\chi_{\mathcal{F}} : \mathcal{Z}' \times \mathcal{J}' \rightarrow \mathcal{Q}'$ defined by $\langle \chi_{\mathcal{F}}(\mathcal{F}f, \mathcal{F}Hg), \Theta \rangle := \langle \mathcal{F}f, \mathcal{F}\phi(s) \rangle + \langle \mathcal{F}Hg, \mathcal{F}H\omega(s) \rangle$, where $\Theta(s) = \mathcal{F}\phi(s) + \mathcal{F}H\omega(s)$. Clearly $\chi_{\mathcal{F}}(\mathcal{F}f, \mathcal{F}Hg)$ so defined is a continuous linear form on $\mathcal{Q}$, for any choice of $\mathcal{F}f \in \mathcal{Z}'$, $\mathcal{F}Hg \in \mathcal{J}'$. To see that $\chi_{\mathcal{F}}$ is surjective, consider any $\mathcal{F}F \in \mathcal{Q}'$. By fixing $\mathcal{F}H\omega(s) = 0$ in $\Theta(s)$, $\mathcal{F}F$ defines an element $\mathcal{F}f$ in $\mathcal{Z}'$ according to $\langle \mathcal{F}f, \mathcal{F}\phi(s) \rangle := \langle \mathcal{F}F, \Theta(s) \rangle$, where $\Theta(s) = \mathcal{F}\phi(s) + \mathcal{F}H\omega(s)$. Similarly, $\mathcal{F}F$ defines an element $\mathcal{F}Hg$ of $\mathcal{J}'$ and it follows that $\mathcal{F}F = \chi_{\mathcal{F}}(\mathcal{F}f, \mathcal{F}Hg)$. It is clear that $\chi_{\mathcal{F}}$ is injective.

§2.2 The Equation $P(D,H)F=\delta$ in $\mathcal{G}'$

Definition 2.16 The Hilbert transform HF of an element F in $\mathcal{G}'$ is defined to be that element $f \in \mathcal{G}'$ given by $\langle HF, \theta \rangle := \langle F, -H\theta \rangle$.

Lemma 2.17 For any $F \in \mathcal{G}'$ the identity $H^2F = (-\pi^2)F$ holds.

Proof $\langle H^2F, \theta \rangle := -\langle HF, H\theta \rangle := \langle F, H^2\theta \rangle = \langle F, -\pi^2\theta \rangle := \langle -\pi^2F, \theta \rangle$, where we've used the fact that $H^2\theta = (-\pi^2)\theta \forall \theta \in \mathcal{G}$.

Definition 2.18 Suppose $P(D)$ is a constant coefficient polynomial in the operator $D \equiv id/dx$. Then, for any $F \in \mathcal{G}'$, $P(D)F \in \mathcal{G}'$ is defined by $\langle P(D)F, \theta \rangle := \langle F, \bar{P}(-D)\theta \rangle$. (If $P(x) = \sum_k a_k x^k$ then $\bar{P}(x) = \sum_k \bar{a}_k x^k$, where the overbar denotes complex conjugation).

Since $(H\phi)^{(k)} = H\phi^{(k)} \forall \phi \in \mathcal{D}$ (see Proposition 1.6), clearly $(H\theta)^{(k)} = H\theta^{(k)} \forall \theta \in \mathcal{G}$ (the superscript (k) denotes k -fold differentiation). This prompts the following

Lemma 2.19 For any $F \in \mathcal{G}'$ we have $(HF)^{(k)} = HF^{(k)}$.

Proof According to definitions 2.18 and 2.16,

$$\langle (HF)^{(k)}, \theta \rangle := \langle HF, (-1)^{(k)}\theta^{(k)} \rangle = -\langle F, (-1)^{(k)}(H\theta)^{(k)} \rangle = -\langle F^{(k)}, H\theta \rangle = \langle HF^{(k)}, \theta \rangle.$$

Remark 2.20 We consider the equation $P(D,H)F=\delta$ in $\mathcal{G}'$, where P is a constant coefficient polynomial. This equation makes sense because $\mathcal{G}'$ is closed with respect to the operations of differentiation and Hilbert transformation. Moreover, since $\delta \in \mathcal{D}' \cap H'(\mathcal{D})$, we can consider $\delta \in \mathcal{G}'$ as $\langle \delta, \theta \rangle := \langle \delta, \phi \rangle + \langle \delta, H\omega \rangle$ makes sense (for $\theta = \phi + H\omega$) and defines a continuous linear form on $\mathcal{G}$. In the notation of Proposition 2.4 we can interpret $\delta \in \mathcal{G}'$ as $\chi(\delta, \delta)$. Indeed, for any $f \in \mathcal{D}' \cap H'(\mathcal{D})$ we can interpret $f \in \mathcal{G}'$ as $\chi(f, f)$. From Lemmas 2.17 and 2.19 we see that $P(D,H)F=\delta$ reduces to the equation $(P(D)+Q(D)H)F = \delta$, where [new] P and Q are constant coefficient polynomials in the term $D \equiv id/dx$. From Definitions 2.16 and 2.18 we see that, for all $\theta \in \mathcal{G}$, $\langle (P(D)+Q(D)H)F, \theta \rangle := \langle F, (\overline{P}(-D) - \overline{Q}(-D)H)\theta \rangle$.

Definition 2.21 We define the continuous operator $i\pi \operatorname{sgn} \sigma : \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$i\pi \operatorname{sgn} \sigma (\alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)) := (-\pi^2)\beta(s) + i\pi \operatorname{sgn} \sigma \alpha(s).$$

For the polynomials P & Q we define the "multiplication" operator $(\overline{P}(s) - i\pi \operatorname{sgn} \sigma \overline{Q}(s)) : \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$\begin{aligned} (\overline{P}(s) - i\pi \operatorname{sgn} \sigma \overline{Q}(s)) \Theta(s) &:= (\overline{P}(s) \mathcal{F}\phi(s) + \pi^2 \overline{Q}(s) \mathcal{F}\omega(s)) \\ &\quad + i\pi \operatorname{sgn} \sigma (\overline{P}(s) \mathcal{F}\omega(s) - \overline{Q}(s) \mathcal{F}\phi(s)), \end{aligned}$$

where $\Theta(s) = \mathcal{F}\phi(s) + i\pi \operatorname{sgn} \sigma \mathcal{F}\omega(s)$. The multiplication so defined is continuous.

Definition 2.24 For polynomials P & Q we define the operator $(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) : \mathcal{Q}' \rightarrow \mathcal{Q}'$ by $\langle (P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \Lambda, \Theta \rangle := \langle \Lambda, (\overline{P}(s) - i\pi \operatorname{sgn} \sigma \overline{Q}(s)) \Theta(s) \rangle$. This operator is continuous.

Lemma 2.23 For any $F \in \mathcal{G}$ we have $\mathcal{F} \circ (P(D) + Q(D)H) F = (P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \mathcal{F} F$ as elements of $\mathcal{Q}'$.

Proof: Appealing to the various definitions we have

$$\begin{aligned}
 \langle \mathcal{F} \circ (P(D) + Q(D)H) F, \Theta \rangle &= (2\pi) \langle (P(D) + Q(D)H) F, \mathcal{F}^{-1} \Theta \rangle \\
 &= (2\pi) \langle F, \bar{P}(-D) \mathcal{F}^{-1} \Theta - \bar{Q}(-D) H \mathcal{F}^{-1} \Theta \rangle \\
 &= (2\pi) \langle F, \mathcal{F}^{-1} (\bar{P}(s) \Theta(s)) - \mathcal{F}^{-1} (i\pi \operatorname{sgn} \sigma \bar{Q}(s) \Theta(s)) \rangle \\
 &= \langle \mathcal{F} F, (\bar{P}(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s)) \Theta(s) \rangle \\
 &= \langle (P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \mathcal{F} F, \Theta \rangle,
 \end{aligned}$$

and the result follows.

Remark 2.24 Applying a Fourier transform to the equation $(P(D) + Q(D)H) F = \delta$ in $\mathcal{G}'$ we get, using the last Lemma, $(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \mathcal{F} F = 1$. We note that, according to remark 2.15, we have $\mathcal{F} \delta = 1$ in $\mathcal{Q}'$, where by 1 we mean $\chi_{\mathcal{F}(1,1)}$ (see definition 2.13).

Remark 2.25 Considering definition 2.21, define $\tilde{\Theta}(s) := (\bar{P}(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s)) \Theta(s)$, where $\Theta \in \mathcal{Q}$ can be written $\Theta(s) = \mathcal{F} \phi(s) + i\pi \operatorname{sgn} \sigma \mathcal{F} \omega(s)$. We shall write $\tilde{\Theta} = \alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)$, so that from definition 2.21 we have

$$\alpha(s) = (\bar{P}(s) \mathcal{F} \phi(s) + \pi^2 \bar{Q}(s) \mathcal{F} \omega(s)), \quad \beta(s) = (\bar{P}(s) \mathcal{F} \omega(s) - \bar{Q}(s) \mathcal{F} \phi(s)).$$

Solving these last two equations formally we find that

$$\mathcal{F} \phi(s) = \frac{\pi^2}{\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)} (\alpha(s) \bar{P}(s) / \pi^2 - \beta(s) \bar{Q}(s))$$

and

$$\mathcal{F} \omega(s) = \frac{1}{\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)} (\alpha(s) \bar{Q}(s) + \beta(s) \bar{P}(s)).$$

These results hold for all s such that $\bar{P}^2(s) + \pi^2 \bar{Q}^2(s) \neq 0$.

Definition 2.26 Suppose that $\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)$ is not zero at the origin. We then define

the linear form Λ on $\mathcal{Q}$ as follows:

$$\begin{aligned} \langle \Lambda, \tilde{\Theta} \rangle := & \int_{\Gamma} \frac{\pi^2}{\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)} (\alpha(s) \bar{P}(s)/\pi^2 - \beta(s) \bar{Q}(s)) ds \\ & + i\pi \int_{\tilde{\Gamma}} \frac{1}{\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)} (\alpha(s) \bar{Q}(s) + \beta(s) \bar{P}(s)) ds \\ & + 2i\pi \int_0^{\tau_0} \frac{\alpha(s) \bar{Q}(s) + \beta(s) \bar{P}(s)}{\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)} ds, \end{aligned}$$

where we use $\tilde{\Theta} \in \mathcal{Q}$, and write $\tilde{\Theta}(s) = \alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)$. In this definition Γ is a horizontal contour (running from $-\infty + ia_0$ to $\infty + ia_0$, $a_0 \in \mathbb{R}$) in the complex s -plane, chosen so as to miss the zeros of $\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)$. The contour $\tilde{\Gamma}$ is also horizontal, but broken into two sections: one running from $i\tau_0$ to $-\infty + i\tau_0$, and the other from $i\tau_0$ to $\infty + i\tau_0$, where $\tau_0 > 0$ is chosen so that it is less than every positive imaginary component of any non-real roots of $\bar{P}^2(s) + \pi^2 \bar{Q}^2(s)$. The contours are shown in figure 1.

Remark 2.27 The linear form Λ just defined on $\mathcal{Q}$ is continuous. The magnitude of the denominator is bounded away from zero in each integrand. The continuity of Λ then follows from the [equivalent] definition of convergence in $\mathcal{Q}$ as given in Proposition 2.11.

Proposition 2.28 Suppose $P(s)$ and $Q(s)$ are such that $\bar{P}^2(0) + \pi^2 \bar{Q}^2(0) \neq 0$. Then Λ , in definition 2.26, is a solution for $\mathcal{F}\mathcal{F}$ in the equation $(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \mathcal{F}\mathcal{F} = 1$.

Proof Appealing to the various definitions we have

$$\begin{aligned} \langle (P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \Lambda, \Theta \rangle &= \langle \Lambda, (\bar{P}(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s)) \Theta(s) \rangle \\ &= \langle \Lambda, \tilde{\Theta} \rangle, \end{aligned}$$

where $\tilde{\Theta} := (\bar{P}(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s)) \Theta(s)$. Writing $\Theta(s) = \mathcal{F}\phi(s) + i\pi \operatorname{sgn} \sigma \mathcal{F}\omega(s)$ and $\tilde{\Theta}(s) = \alpha(s) + i\pi \operatorname{sgn} \sigma \beta(s)$, we can solve for $\mathcal{F}\phi(s)$ and $\mathcal{F}\omega(s)$ in terms of $\alpha(s)$ and $\beta(s)$; the results are given in remark 2.25. Thus, from definition 2.26 we have:

$$\langle \Lambda, \tilde{\Theta} \rangle = \int_{\Gamma} \mathcal{F}\phi(s) ds + i\pi \int_{\tilde{\Gamma}} \mathcal{F}\omega(s) ds + 2i\pi \int_0^{\tau_0} \mathcal{F}\omega(s) ds.$$

Figure 1. Contours Γ & $\bar{\Gamma}$, in the s_1 -plane, for defining A .

Both $\mathcal{F}\phi$ and $\mathcal{F}\omega$ are entire functions satisfying the inequalities given in remark 1.23.

Therefore, by Cauchy's theorem $\int_{\Gamma} \mathcal{F}\phi(s)ds = \int_{-\infty}^{\infty} \mathcal{F}\phi(\sigma)d\sigma$. Also,

$$\int_{\Gamma} \mathcal{F}\omega(s)ds = \int_0^{-\infty} \mathcal{F}\omega(\sigma)d\sigma + \int_0^{\infty} \mathcal{F}\omega(\sigma)d\sigma - 2 \int_0^{i\tau_0} \mathcal{F}\omega(s)ds,$$

that is, $\int_{\Gamma} \mathcal{F}\omega(s)ds = \int_{-\infty}^{\infty} \operatorname{sgn}\sigma \mathcal{F}\omega(\sigma)d\sigma - 2 \int_0^{i\tau_0} \mathcal{F}\omega(s)ds$. Therefore

$$\begin{aligned} \langle \Lambda, \tilde{\Theta} \rangle &= \int_{-\infty}^{\infty} \mathcal{F}\phi(\sigma)d\sigma + i\pi \int_{-\infty}^{\infty} \operatorname{sgn}\sigma \mathcal{F}\omega(\sigma)d\sigma \\ &= \langle 1, \Theta(s) \rangle \end{aligned}$$

and the proof is complete.

Corollary 2.29 *With the same conditions on P & Q as in 2.28, we have $\mathcal{F}^{-1}(\Lambda)$ is a solution in $\mathcal{G}'$ for the equation $(P(D) + Q(D)H)F = \delta$.*

Remark 2.30 It is unfortunate that Corollary 2.29 places restrictions on P & Q in order that a solution of $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$ be constructed by the method of Proposition 2.28. It would be desirable to have a theory of the division of elements of $\mathcal{Q}'$ by "polynomials" of the form $\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)$ (see Hörmander [3] for the case of division of elements of S by polynomials). Suppose that we could show the following: "if P and Q are polynomials that do not vanish identically, then the mapping $\Theta(s) \rightarrow (\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)) \Theta(s)$ has a continuous inverse". With this result we could show the following: if we are given Λ in $\mathcal{Q}'$ and polynomials P and Q not identically zero, then there exists $\Xi \in \mathcal{Q}'$ such that $\Lambda = (P(s) + i\pi \operatorname{sgn}\sigma Q(s)) \Xi$ (see Hörmander [3, theorems 1.3 & 5] for the equivalent statements in S'). To see this, consider the linear form $(\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)) \Theta \rightarrow \langle \Lambda, \Theta \rangle$ which is defined on the subset $(\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)) \mathcal{Q}$ of $\mathcal{Q}$. It is continuous, for if $(\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)) \Theta \rightarrow 0$ in $\mathcal{Q}$ then by supposition $\Theta \rightarrow 0$ in $\mathcal{Q}$. Therefore $\langle \Lambda, \Theta \rangle \rightarrow 0$. Using the Hahn-Banach theorem we can extend the linear form to a continuous linear form $\Theta \rightarrow \langle \Xi, \Theta \rangle$ on all of $\mathcal{Q}$. Then we shall have $\langle \Xi, (\bar{P}(s) - i\pi \operatorname{sgn}\sigma \bar{Q}(s)) \Theta \rangle = \langle \Lambda, \Theta \rangle$; that is

$(P(s) + i\pi \operatorname{sgn} s Q(s))\Xi = \Lambda$. Continuing, from this last result we get: if $P(D)$ and $Q(D)$ are constant coefficient polynomials in the operator $D \equiv id/dx$, and if $T \in \mathcal{G}'$, then there exists $S \in \mathcal{G}'$ such that $(P(D) + Q(D)H)S = T$. To see this, apply a Fourier transform to this equation to get $(P(s) + i\pi \operatorname{sgn} s Q(s))\Xi = \Lambda$, where $\Xi = \mathcal{F}S$ and $\Lambda = \mathcal{F}T$. By the previous result this transformed equation has a solution Ξ in $\mathcal{Q}'$, and so inverting Ξ we arrive at a solution S for $(P(D) + Q(D)H)S = T$.

§2.3 The Equation $P(D, H)u = \delta$ in $\mathcal{D}' \cap H'(\mathcal{D})$

We saw in Proposition 1.22 that the vector subspace $\mathcal{D}' \cap H'(\mathcal{D})$ is closed with respect to the operations of differentiation and Hilbert transformation. Thus, the equation $(P(D) + Q(D)H)u = \delta$ makes sense for $u \in \mathcal{D}' \cap H'(\mathcal{D})$ (see remark 2.20)

Proposition 2.30 *A sufficient condition for $(P(D) + Q(D)H)u = \delta$ to have a solution in $\mathcal{D}' \cap H'(\mathcal{D})$ is that $(P^2(D) + \pi^2 Q^2(D))v = \delta$ has a solution v in $\mathcal{D}' \cap H'(\mathcal{D})$.*

Proof We can check that if $v \in \mathcal{D}' \cap H'(\mathcal{D})$ satisfies $(P^2(D) + \pi^2 Q^2(D))v = \delta$ then $(P(D) - Q(D)H)v \in \mathcal{D}' \cap H'(\mathcal{D})$ satisfies $(P(D) + Q(D)H)u = \delta$.

Remark 2.31 We now consider conditions under which $(P^2(D) + \pi^2 Q^2(D))v = \delta$ has a solution in $\mathcal{D}' \cap H'(\mathcal{D})$. From remark 1.35 we recall that $\mathcal{F} : \mathcal{D}' \cap H'(\mathcal{D}) \rightarrow \mathcal{Z}' \cap \mathcal{J}'$ is an isomorphism, whether we consider $\mathcal{F}$ being $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ or $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$. For any solution in $\mathcal{D}' \cap H'(\mathcal{D})$ we must have then that both $\langle (P^2(D) + \pi^2 Q^2(D))v, \phi \rangle = \langle \delta, \phi \rangle \forall \phi \in \mathcal{D}$, and $\langle (P^2(D) + \pi^2 Q^2(D))v, H\phi \rangle = \langle \delta, H\phi \rangle \forall H\phi \in H(\mathcal{D})$. For any polynomial P we can define the multiplication operators $P^2(s) : \mathcal{Z}' \rightarrow \mathcal{Z}'$ and $P^2(s) : \mathcal{J}' \rightarrow \mathcal{J}'$ in an obvious way (see Shilov [7, pp. 140] for the definition in $\mathcal{Z}'$). First consider the equation in $\mathcal{D}'$. Applying the Fourier transform $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ yields the equation $(P^2(s) + \pi^2 Q^2(s))\mathcal{F}v = 1$ in $\mathcal{Z}'$. Alternately, considering the equation in $H'(\mathcal{D})$ and applying the Fourier transform

$\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$ we get $(P^2(s) + \pi^2 Q^2(s)) \mathcal{F}v = 1$ in $\mathcal{J}'$. For consistency, both of these equations must be satisfied for $\mathcal{F}v \in \mathcal{Z}' \cap \mathcal{J}'$ (see definition 1.33). That is, since $v \in \mathcal{D}' \cap H'(\mathcal{D})$, $\mathcal{F}v \in \mathcal{Z}' \cap \mathcal{J}'$ must satisfy both:

- (a) $\langle (P^2(s) + \pi^2 Q^2(s)) \mathcal{F}v, \mathcal{F}\phi \rangle = \langle 1, \mathcal{F}\phi(s) \rangle \forall \mathcal{F}\phi(s) \in \mathcal{Z}$, and
- (b) $\langle (P^2(s) + \pi^2 Q^2(s)) \mathcal{F}v, i\pi \operatorname{sgn} \sigma \mathcal{F}\phi(s) \rangle = \langle 1, i\pi \operatorname{sgn} \sigma \mathcal{F}\phi(s) \rangle \forall i\pi \operatorname{sgn} \sigma \mathcal{F}\phi(s) \in \mathcal{J}$

An $\mathcal{F}v$ satisfying (a) is given by $1/(\overline{P}^2(s) + \pi^2 \overline{Q}^2(s)) \in \mathcal{Z}'$, where we define

$$\langle 1/(\overline{P}^2(s) + \pi^2 \overline{Q}^2(s)), \mathcal{F}\phi(s) \rangle := \int_{\Gamma} (\mathcal{F}\phi(s) / (\overline{P}^2(s) + \pi^2 \overline{Q}^2(s))) ds,$$

where Γ is the same contour as given in definition 2.26. Similarly, if we make the further assumption that $\overline{P}^2(0) + \pi^2 \overline{Q}^2(0) \neq 0$ we define an $\mathcal{F}v \in \mathcal{J}'$ satisfying (b) as $1/(\overline{P}^2(s) + \pi^2 \overline{Q}^2(s))$, where

$$\begin{aligned} \langle 1/(\overline{P}^2(s) + \pi^2 \overline{Q}^2(s)), i\pi \operatorname{sgn} \sigma \mathcal{F}\phi(s) \rangle &:= i\pi \int_{\tilde{\Gamma}} (\mathcal{F}\phi(s) / (\overline{P}^2(s) + \pi^2 \overline{Q}^2(s))) ds \\ &\quad + 2i\pi \int_0^{\tau_0} (\mathcal{F}\phi(s) / (\overline{P}^2(s) + \pi^2 \overline{Q}^2(s))) ds, \end{aligned}$$

and where $\tilde{\Gamma}$ and τ_0 are as given in definition 2.26. However, although this last definition of $\mathcal{F}v$ also defines a continuous linear form on $\mathcal{Z}'$, we can check that it does not satisfy (a). We are therefore led to conjecture that $(P^2(D) + \pi^2 Q^2(D))v = \delta$ has no solution in $\mathcal{D}' \cap H'(\mathcal{D})$, and further that $(P(D) + Q(D)H)u = \delta$ has no solution in $\mathcal{D}' \cap H'(\mathcal{D})$.

Remark 2.32 We claim that the solution of $(P(D) + Q(D)H)F = \delta$ constructed in proposition 2.26 does not lie in $\mathcal{D}' \cap H'(\mathcal{D})$, where we view this as a subspace of $\mathcal{G}'$. In remark 2.20 we observe that we can interpret any $f \in \mathcal{D}' \cap H'(\mathcal{D})$ as $f \in \mathcal{G}'$, that is, as $\chi(f, f)$. Similarly, for any $g \in \mathcal{Z}' \cap \mathcal{J}'$ we can interpret $g \in \mathcal{Q}'$ as $\chi_{\mathcal{F}}(g, g)$, where the isomorphism $\chi_{\mathcal{F}} : \mathcal{Z}' \times \mathcal{J}' \rightarrow \mathcal{Q}'$ is defined in remark 2.15. Examining the proof of Proposition 2.28 we see that Λ so defined is not of the form $\chi_{\mathcal{F}}(g, g)$ for $g \in \mathcal{Z}' \cap \mathcal{J}'$, and so Λ cannot be considered as lying in $\mathcal{Z}' \cap \mathcal{J}'$. It follows then that $\mathcal{F}^{-1}\Lambda$ does not lie in $\mathcal{D}' \cap H'(\mathcal{D})$.

§2.4 The Equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'$

The equation $\{P(D) + Q(D)H\}F = \delta$ makes sense if we interpret $\delta \in \mathcal{D}'$, $F \in \mathcal{G}'$, and $\{P(D) + Q(D)H\}$ as an operator from $\mathcal{G}' \rightarrow \mathcal{D}'$.

Definition 2.33 We define the operator $\{P(D) + Q(D)H\} : \mathcal{G}' \rightarrow \mathcal{D}'$ by $\langle \{P(D) + Q(D)H\}F, \phi \rangle := \langle F, (\bar{P}(-D) - \bar{Q}(-D)H) \phi \rangle \forall \phi \in \mathcal{D}$.

Remark 2.34 There is another "direct" way of defining $\{P(D) + Q(D)H\}$ that is equivalent to 2.33. Namely, we can define $\{P(D) + Q(D)H\} : \mathcal{G}' \rightarrow \mathcal{D}'$ by $\{P(D) + Q(D)H\}F := P(D)f + \pi^2 Q(D)g$, where $F = \chi(f, Hg)$ and $f, g \in \mathcal{D}'$ (see proposition 2.4 for notation). To see that these two definitions are equivalent note that

$$\begin{aligned} \langle \{P(D) + Q(D)H\}F, \phi \rangle &:= \langle P(D)f + \pi^2 Q(D)g, \phi \rangle \\ &= \langle f, \bar{P}(-D)\phi \rangle + \pi^2 \langle g, \bar{Q}(-D)\phi \rangle \\ &= \langle f, \bar{P}(-D)\phi \rangle - \langle Hg, \bar{Q}(-D)H\phi \rangle \\ &= \langle F, (\bar{P}(-D) - \bar{Q}(-D)H) \phi \rangle \\ &= \langle \{P(D) + Q(D)H\}F, \phi \rangle \text{ by definition 2.33.} \end{aligned}$$

We note that since, by proposition 2.4, we can identify $\mathcal{D}' \times H'(\mathcal{D})$ with $\mathcal{G}'$ using the isomorphism χ , we are equivalently identifying $\{P(D) + Q(D)H\}$ with a product mapping $P(D) \times (-Q(D)H) : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{D}'$ given by $P(D) \times (-Q(D)H)(f, Hg) := P(D)f + \pi^2 Q(D)g$.

Definition 2.35 We define $(\bar{P}(s) - i\pi \text{sgn} \sigma \bar{Q}(s)) : \mathcal{Z} \rightarrow \mathcal{Q}$ according to

$$(\bar{P}(s) - i\pi \text{sgn} \sigma \bar{Q}(s)) \zeta(s) := \bar{P}(s)\zeta(s) - i\pi \text{sgn} \sigma \bar{Q}(s)\zeta(s).$$

This operation is continuous.

Definition 2.36 We define $[[P(s) + i\pi \text{sgn} \sigma Q(s)]] : \mathcal{Q}' \rightarrow \mathcal{Z}'$ by

$$\langle [[P(s) + i\pi \text{sgn} \sigma Q(s)]]\Xi, \zeta \rangle := \langle \Xi, (\bar{P}(s) - i\pi \text{sgn} \sigma \bar{Q}(s)) \zeta(s) \rangle \forall \zeta \in \mathcal{Z}.$$

These definitions lead to the following

Lemma 2.37 For any $F \in \mathcal{G}'$ we have $\mathcal{F} \circ \{P(D) + Q(D)H\}F = [[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]\mathcal{F}F$ as elements of $\mathcal{Z}'$.

Proof: Appealing to the various definitions we have

$$\begin{aligned}
 \langle \mathcal{F} \circ \{P(D) + Q(D)H\}F, \zeta \rangle &= (2\pi) \langle \{P(D) + Q(D)H\}F, \mathcal{F}^{-1}\zeta \rangle \\
 &= (2\pi) \langle F, \bar{P}(-D)\mathcal{F}^{-1}\zeta - \bar{Q}(-D)H\mathcal{F}^{-1}\zeta \rangle \\
 &= (2\pi) \langle F, \mathcal{F}^{-1}(\bar{P}(s)\zeta(s)) - \mathcal{F}^{-1}(i\pi \operatorname{sgn} \sigma \bar{Q}(s)\zeta(s)) \rangle \\
 &= \langle \mathcal{F}F, \bar{P}(s)\zeta(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s)\zeta(s) \rangle \\
 &= \langle [[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]\mathcal{F}F, \zeta \rangle
 \end{aligned}$$

and the proof follows.

Applying a Fourier transform to the equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'$ gives $\mathcal{F} \circ \{P(D) + Q(D)H\}F = 1$ in $\mathcal{Z}'$; that is, from the preceding Lemma, $[[P(s) + i\pi \operatorname{sgn} \sigma Q(s)]]\mathcal{F}F = 1$ in $\mathcal{Z}'$.

Definition 2.38 We define the linear form $1/(P(s) + i\pi \operatorname{sgn} \sigma Q(s))$ on $\mathcal{Q}$ according to

$$\left\langle \frac{1}{(P(s) + i\pi \operatorname{sgn} \sigma Q(s))}, \Theta \right\rangle := \int_{\Gamma} \frac{\Theta(s)}{(\bar{P}(s) - i\pi \operatorname{sgn} \sigma \bar{Q}(s))} ds,$$

where the horizontal contour Γ , running from left to right in the complex s -plane, is chosen so as to miss the zeros of $\bar{P}(s) - i\pi \bar{Q}(s)$ in the half-plane $\sigma < 0$, the zeros of $\bar{P}(s) + i\pi \bar{Q}(s)$ in the half-plane $\sigma > 0$, and the zeros of $\bar{P}(i\tau)$ on the imaginary axis. We can see that $1/(P(s) + i\pi \operatorname{sgn} \sigma Q(s)) \in \mathcal{Q}'$.

Proposition 2.39 $\mathcal{F}^{-1}(1/(P(s) + i\pi \operatorname{sgn} \sigma Q(s)))$ is a solution of the equation

$$\{P(D) + Q(D)H\}F = \delta$$

in $\mathcal{D}'$.

Proof Appealing to the various definitions we have

$$\begin{aligned}
 & \langle (P(D) + Q(D))H\mathcal{F}^{-1}(1/(P(s) + i\pi\operatorname{sgn}\sigma Q(s))), \phi \rangle \\
 &= \langle \mathcal{F}^{-1}(1/(P(s) + i\pi\operatorname{sgn}\sigma Q(s))), \bar{P}(-D)\phi - \bar{Q}(-D)H\phi \rangle \\
 &= (2\pi)^{-1} \langle 1/(P(s) + i\pi\operatorname{sgn}\sigma Q(s)), \mathcal{F}(\bar{P}(-D)\phi) - \mathcal{F}(\bar{Q}(-D)H\phi) \rangle \\
 &= (2\pi)^{-1} \langle 1/(P(s) + i\pi\operatorname{sgn}\sigma Q(s)), \bar{P}(s)\mathcal{F}\phi(s) - \bar{Q}(s)i\pi\operatorname{sgn}\sigma\mathcal{F}\phi(s) \rangle \\
 &= (2\pi)^{-1} \int_{\Gamma} \mathcal{F}\phi(s)ds = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathcal{F}\phi(\sigma)d\sigma \text{ (by Cauchy's theorem)} \\
 &= (2\pi)^{-1} \langle 1, \mathcal{F}\phi \rangle = \langle \mathcal{F}^{-1}1, \phi \rangle = \langle \delta, \phi \rangle,
 \end{aligned}$$

and the proof follows.

CHAPTER 3. THE MULTIDIMENSIONAL CASE

§3.1 Initial Definitions & Results

We shall now consider the equation $P(D, H)F = \delta$ in $\mathcal{G}'$, the dual of the space of test functions $\mathcal{G}$ defined on the real n -dimensional space $\mathbb{R}^n$. Most of the definitions in chapter two carry over with little difficulty. In the above equation, P is a constant coefficient polynomial in the terms $i\partial/\partial x_1, \dots, i\partial/\partial x_n$, and H denotes the n -dimensional Hilbert transform operator given by $H = H_1 \circ \dots \circ H_n$, where $H_i \phi := P \int_{-\infty}^{\infty} \frac{\phi(t_1, \dots, t_n)}{t_i - x_i} dt_i \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$. We denote by $H(\mathcal{D})$ the vector space consisting of [n -dimensional] Hilbert transforms of elements of $\mathcal{D}(\mathbb{R}^n)$. We have the identity $H^2 \phi = (-\pi^2)^n \phi \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$. Again, the result $\mathcal{D}(\mathbb{R}^n) \cap H(\mathcal{D}) = \{0\}$ holds. The dual of $H(\mathcal{D})$ (where $H(\mathcal{D})$ is topologised analogously to the one dimensional case) is denoted by $H'(\mathcal{D})$. We define the generalized Hilbert transform Hf of $f \in \mathcal{D}'(\mathbb{R}^n)$ as that element of $H'(\mathcal{D})$ satisfying $\langle Hf, H\phi \rangle = (-1)^n \langle f, H^2 \phi \rangle \quad \forall H\phi \in H(\mathcal{D})$. If $g \in H'(\mathcal{D})$ then its Hilbert transform Hg is defined to be the distribution satisfying $\langle Hg, \phi \rangle = -\langle g, H\phi \rangle \quad \forall \phi \in \mathcal{D}$. The identity $H^2 f = (-\pi^2)^n f$ holds $\forall f \in \mathcal{D}'(\mathbb{R}^n)$ and so $H : \mathcal{D}' \rightarrow H'(\mathcal{D})$ is an isomorphism. Again, $\mathcal{D}'(\mathbb{R}^n) \cap H'(\mathcal{D})$ is nonempty, and indeed $\mathcal{D}' \cap H'(\mathcal{D})$ is closed with respect to the operations of differentiation and Hilbert transformation (we henceforth drop explicit reference to $\mathbb{R}^n$).

The Fourier transform of an element $\phi \in \mathcal{D}$ is defined to be

$$\mathcal{F}\phi(s) := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi(x_1, \dots, x_n) e^{i(x, s)} dx,$$

where again we've extended the usual n -dimensional transform argument $\sigma = (\sigma_1, \dots, \sigma_n)$ to complex values $s = (s_1 + i\tau_1, \dots, s_n + i\tau_n)$, since ϕ has compact support (see Shilov

[7, pp. 148 ff.]). The term (x, s) stands for $x_1 s_1 + \dots + x_n s_n$. The Fourier transform with complex argument establishes an isomorphism between $\mathcal{D}$ and $\mathcal{Z}$, the vector space of all functions $\Phi(s)$ which are entire with respect to each of the arguments $s_1, \dots, s_n$, and which satisfy inequalities of the form $|s_1^{q_1} \dots s_n^{q_n} \Phi(s)| \leq C_q e^{a_1|q_1| + \dots + a_n|q_n|}$ for all $|q| = 0, 1, 2, \dots$, where constants $a_1, \dots, a_n$, & C_q depend on Φ ($|q| = \sum q_i$). The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{Z} \rightarrow \mathcal{D}$ is defined by

$$\begin{aligned} \phi(x_1, \dots, x_n) &:= \mathcal{F}^{-1} \Phi(x_1, \dots, x_n) \\ &:= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi(\sigma_1, \dots, \sigma_n) e^{-i(x, \sigma)} d\sigma_1 \dots d\sigma_n \end{aligned}$$

where (x, σ) denotes $x_1 \sigma_1 + \dots + x_n \sigma_n$. Convergence in $\mathcal{Z}$ is characterised by saying that a sequence $\{\Phi_\nu\} \rightarrow 0$ in $\mathcal{Z}$ iff the associated sequence $\{\phi_\nu\} \rightarrow 0$ in $\mathcal{D}$. Equivalently, $\{\Phi_\nu\} \rightarrow 0$ in $\mathcal{Z}$ iff

- (i) the inequalities $|s_1^{q_1} \dots s_n^{q_n} \Phi_\nu(s)| \leq C_q e^{a_1|q_1| + \dots + a_n|q_n|}$ hold for all multi-indices $q = (q_1, \dots, q_n)$ such that $|q| = \sum_{i=1}^n q_i = 0, 1, 2, \dots$, and
- (ii) $\{\Phi_\nu\} \rightarrow 0$ uniformly on each bounded subset of $\mathbb{R}^n$.

We define the Fourier transform $\mathcal{F}f$ of an element $f \in \mathcal{D}'$ as that element of $\mathcal{Z}'$ given by $\langle \mathcal{F}f, \Phi \rangle := (2\pi)^n \langle f, \phi \rangle$, where $\mathcal{F}\phi = \Phi$. We define the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{Z}' \rightarrow \mathcal{D}'$ according to $\langle \mathcal{F}^{-1}g, \phi \rangle := (1/(2\pi)^n) \langle g, \mathcal{F}\phi \rangle \forall \phi \in \mathcal{D}$. We have that $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ is an isomorphism.

As in the one dimensional case, we define the Fourier transform, with complex argument, of a function $H\phi$ in $H(\mathcal{D})$ to be $\mathcal{F}H\phi(s) := \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\phi(s)$. We define $\mathcal{J}$ to be the vector space consisting of all functions that can be written in the form $\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \Phi(s)$, where $\Phi(s)$ is an entire function in each of its arguments, satisfying $|s^q \Phi(s)| \leq C_q e^{a_1|q_1| + \dots + a_n|q_n|}$ for all $|q| = 0, 1, 2, \dots$, where the constants $a_1, \dots, a_n$, & C_q

depend on Φ (we use s^q to denote $s_1^{q_1} \dots s_n^{q_n}$). $\mathcal{J}$ is endowed with the linear operations of addition and scalar multiplication in an obvious way. As in Proposition 1.27, we can show that the Fourier transform with complex argument, $\mathcal{F} : H(\mathcal{D}) \rightarrow \mathcal{J}$, is an isomorphism. Convergence in $\mathcal{J}$ is defined by saying that a sequence $\{\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \Phi_\nu\} \rightarrow 0$ in $\mathcal{J}$ iff the associated sequence $\{\phi_\nu\} \rightarrow 0$ in $\mathcal{D}$, where $\mathcal{F}\phi_\nu = \Phi_\nu$. Equivalently, $\{\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \Phi_\nu\} \rightarrow 0$ in $\mathcal{J}$ iff

- (i) the inequalities $|s^q \Phi_\nu(s)| \leq C_q e^{a_1|r_1| + \dots + a_n|r_n|}$ hold for all $|q| = 0, 1, 2, \dots$, for constants $a_1, \dots, a_n$, & C_q not depending on ν , and
- (ii) $\{\Phi_\nu\} \rightarrow 0$ uniformly on each bounded set in $\mathbb{R}^n$.

We define the Fourier transform of an element g of $H'(\mathcal{D})$ to be an element of $\mathcal{J}'$ defined by $\langle \mathcal{F}g, \mathcal{F}H\phi \rangle := (2\pi)^n \langle g, H\phi \rangle \forall \mathcal{F}H\phi \in \mathcal{J}$. We define the inverse Fourier transform of an element g of $\mathcal{J}'$ to be the element of $H'(\mathcal{D})$ given by $\langle \mathcal{F}^{-1}g, H\phi \rangle := (1/(2\pi)^n) \langle g, \mathcal{F}\phi \rangle$. We have that $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$ is an isomorphism.

As in the one dimensional case we impose the same consistency conditions on $\mathcal{F} : \mathcal{D}' \rightarrow \mathcal{Z}'$ and $\mathcal{F} : H'(\mathcal{D}) \rightarrow \mathcal{J}'$, and on $\mathcal{F}^{-1} : \mathcal{Z}' \rightarrow \mathcal{D}'$ and $\mathcal{F}^{-1} : \mathcal{J}' \rightarrow H'(\mathcal{D})$ as given, respectively, in definitions 1.31 and 1.33. These consistency conditions ensure that if $f \in \mathcal{D}' \cap H'(\mathcal{D})$ then the two definitions of the Fourier transform of f agree. Similarly, if $g \in \mathcal{Z}' \cap \mathcal{J}'$, then the two definitions of the inverse transform of g will agree. The mapping $\mathcal{F} : \mathcal{D}' \cap H'(\mathcal{D}) \rightarrow \mathcal{Z}' \cap \mathcal{J}'$ will be an isomorphism.

As in the one dimensional case we introduce the vector space $\mathcal{G} := \{\theta \in C^\infty(\mathbb{R}^n) : \theta = \phi + H\omega, \text{ where } \phi, \omega \in \mathcal{D}\}$, endowed with the linear operations of addition and scalar multiplication in an obvious way. The topology of $\mathcal{G}$ is defined in an obvious way (cf remark 2.3). Again, there is a canonical isomorphism $\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$ (cf proposition 2.4). We define the Fourier transform, with complex argument, of an element $\theta \in \mathcal{G}$ (where

$\theta = \phi + H\omega$ by $\mathcal{F}\theta(s) := \Theta(s) := \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s)$ (cf definition 2.6). The functions $\mathcal{F}\phi(s)$ and $\mathcal{F}\omega(s)$ are entire in each of their arguments and satisfy inequalities such as, for example, $|s^q \mathcal{F}\phi(s)| \leq C_q e^{a_1|\tau_1| + \dots + a_n|\tau_n|}$ for all $|q| = 0, 1, 2, \dots$, where the constants $a_1, \dots, a_n$ & C_q depend on $\mathcal{F}\phi$. We define $\mathcal{Q}$ to be the vector space consisting of all functions defined on $\mathbb{C}^n$, that can be written in the form $\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s)$, where α and β are entire functions (in each of their arguments) satisfying inequalities such as the one just given for $\mathcal{F}\phi(s)$. We can show that $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{Q}$ is an isomorphism (cf proposition 2.9). The topology of $\mathcal{Q}$ is defined in a way analogously to the one dimensional case (definition 2.10) and we have an intrinsic description of convergence in $\mathcal{Q}$ that is evident (cf proposition 2.11).

The Hilbert transform of an element in $\mathcal{G}'$ is defined to be that element of $\mathcal{G}'$ given by $\langle HF, \theta \rangle := \langle F, -H\theta \rangle$. For any $F \in \mathcal{G}'$ the identity $H^2 F = (-\pi^2)^n F$ holds. For any $F \in \mathcal{G}'$ we have $(HF)^{(k)} = H F^{(k)}$, where the superscript (k) denotes multiple partial differentiation $(\partial/\partial x_1)^{k_1} \dots (\partial/\partial x_n)^{k_n}$, where $k := (k_1, \dots, k_n)$.

§3.2 The Equation $P(D, H)F = \delta$ in $\mathcal{G}'$

• In view of the facts from the preceding section it is clear that if P is a constant coefficient polynomial in the terms $i\partial/\partial x_1, \dots, i\partial/\partial x_n$, and in the n -dimensional Hilbert transform operator H , then the equation $P(D, H)F = \delta$ in $\mathcal{G}'$ reduces to $(P(D) + Q(D)H)F = \delta$, where [different] P & Q are polynomials in the various partial derivatives. To show that a solution exists, in $\mathcal{G}'$, to this equation, we shall use a modified form of the Hörmander's staircase method (see Shilov [7, pp. 176 ff.]). This method hinges on being able to introduce

a coordinate rotation so as to reduce both polynomials, P & Q to their "normal forms"

$$\begin{aligned} a_0 \left(i \frac{\partial}{\partial x_1} \right)^m + \sum_{k=0}^{m-1} P_k \left(i \frac{\partial}{\partial x_2}, \dots, i \frac{\partial}{\partial x_n} \right) \left(i \frac{\partial}{\partial x_1} \right)^k \quad a_0 \neq 0, \\ b_0 \left(i \frac{\partial}{\partial x_1} \right)^\mu + \sum_{l=0}^{\mu-1} Q_l \left(i \frac{\partial}{\partial x_2}, \dots, i \frac{\partial}{\partial x_n} \right) \left(i \frac{\partial}{\partial x_1} \right)^l \quad b_0 \neq 0, \end{aligned} \quad (3.1)$$

respectively, where we assume that P is an m^{th} order and Q a μ^{th} order polynomial. In order to do this we must introduce the concept of the rotation of an element of $\mathcal{G}'$.

Remark 3.1 If we make the change of variables $x = Uy$ in $\mathbb{R}^n$, where U is a rotation matrix, then a function $\phi(x) \in \mathcal{D}(\mathbb{R}^n)$ is carried into the function $\phi_U(y) \in \mathcal{D}(\mathbb{R}^n)$, where $\phi_U(y) := \phi(Uy)$. The function $H\omega(x) \in H(\mathcal{D})$ is carried into the function $H\omega(Uy)$ defined by

$$H\omega(Uy) := P \int_{-\infty}^{\infty} \frac{1}{t_1 - \langle U_1, y \rangle} \dots P \int_{-\infty}^{\infty} \frac{1}{t_n - \langle U_n, y \rangle} \omega(t_1, \dots, t_n) dt_1 \dots dt_n,$$

where U_j denotes the j^{th} row of matrix U . We see that $(H\omega)_U(y) := H\omega(Uy)$ is not an element of $H(\mathcal{D})$ for arbitrary U . For suppose $(H\omega)_U(y) = H\psi(y)$ for some $\psi \in \mathcal{D}$. Taking [n-dimensional] Fourier transforms of both sides gives $\prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathcal{F}\omega(\sigma) = \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\psi(\sigma)$. Since $\mathcal{F}\omega$ and $\mathcal{F}\psi$ are both analytic in σ , this last equation is not valid unless $U \equiv I$, the identity matrix.

Definition 3.2 $\mathcal{G}_U := \{ \theta_U \in C^\infty(\mathbb{R}^n) : \theta_U(x) := \theta(Ux) = \phi(Ux) + H\omega(Ux), \theta \in \mathcal{G} \}$.

Convergence in $\mathcal{G}_U$ is defined in an obvious way: $\{(\theta_U)_\mu\} \rightarrow 0$ in $\mathcal{G}_U$ iff $\{\theta_\mu\} \rightarrow 0$ in $\mathcal{G}$, where $(\theta_U)_\mu(x) = \theta_\mu(Ux)$.

Remark 3.3 Taking the [classical] Fourier transform of $\theta_U \in \mathcal{G}_U$ we get $\mathcal{F}\theta_U(\sigma) := \Theta_U(\sigma) = \mathcal{F}\phi(U\sigma) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathcal{F}\omega(U\sigma)$ a.e. This leads us to define the Fourier transform, with complex argument, of an element $\theta_U \in \mathcal{G}_U$ by

$$\mathcal{F}\theta_U(s) = \Theta_U(s) := \mathcal{F}\phi(Us) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathcal{F}\omega(Us).$$

By $\mathcal{F}\phi(U_s)$, for example, we mean $\mathcal{F}\phi(\langle U_1, \sigma \rangle + i\langle U_1, \tau \rangle, \dots, \langle U_n, \sigma \rangle + i\langle U_n, \tau \rangle)$. The functions $\mathcal{F}\phi(U_s)$ and $\mathcal{F}\omega(U_s)$ are entire in each of the arguments $s_1, \dots, s_n$. $\mathcal{F}\phi$ and $\mathcal{F}\omega$ satisfy inequalities such as, for example, $|s^q \mathcal{F}\phi(U_s)| \leq C_q e^{a_1 |\langle U_1, \tau \rangle| + \dots + a_n |\langle U_n, \tau \rangle|}$; that is, we have $|s^q \mathcal{F}\phi(U_s)| \leq C_q e^{b_1 |\tau_1| + \dots + b_n |\tau_n|}$ for all $|q| = 0, 1, 2, \dots$, where the constants $b_1, \dots, b_n$, & C_q depend on $\mathcal{F}\phi$ & U . We can define $\mathcal{Q}_U$ to be the vector space consisting of all functions, defined on $\mathbb{C}^n$, that can be written in the form $\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s)$, where α and β are entire functions (in each of their arguments) satisfying inequalities such as the one just given for $\mathcal{F}\phi(U_s)$. We can show that $\mathcal{F} : \mathcal{G}_U \rightarrow \mathcal{Q}_U$ is an isomorphism (cf proposition 2.9), with inverse $\mathcal{F}^{-1} : \mathcal{Q}_U \rightarrow \mathcal{G}_U$ given by

$$\mathcal{F}^{-1}\left(\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s)\right) := \mathcal{F}^{-1}\alpha + \Pi \mathcal{F}^{-1}\beta,$$

where $\Theta_U(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s)$. The topology of $\mathcal{Q}_U$ is defined in a way analogously to the one dimensional case (see definition 2.10) and we have an intrinsic description of convergence in $\mathcal{Q}_U$ (cf proposition 2.11). We can define the Fourier transform mapping $\mathcal{F} : (\mathcal{G}_U)' \rightarrow (\mathcal{Q}_U)'$ by $\langle \mathcal{F}F_U, \Theta_U \rangle := \langle F_U, \mathcal{F}^{-1}\Theta_U \rangle \forall \Theta_U \in \mathcal{Q}_U$, and the inverse Fourier transform $\mathcal{F}^{-1} : (\mathcal{Q}_U)' \rightarrow (\mathcal{G}_U)'$ by $\langle \mathcal{F}^{-1}\Lambda_U, \theta_U \rangle := \langle \Lambda_U, \mathcal{F}\theta_U \rangle \forall \theta_U \in \mathcal{G}_U$. Motivated by definition 2.21 we define the continuous operator $\prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) : \mathcal{Q}_U \rightarrow \mathcal{Q}_U$ by

$$\prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \left(\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s) \right) := (-\pi^2)^n \beta(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \alpha(s).$$

For polynomials P_U & Q_U we define the continuous operator

$$\left(\overline{P_U}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \overline{Q_U}(s) \right) : \mathcal{Q}_U \rightarrow \mathcal{Q}_U$$

by

$$\left(\overline{P_U}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \overline{Q_U}(s) \right) \left(\mathcal{F}\phi(U_s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \mathcal{F}\omega(U_s) \right) :=$$

$$\overline{P_U}(s)\mathcal{F}\phi(U_s) + (\pi^2)^n \overline{Q_U}(s)\mathcal{F}\omega(U_s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) (\overline{P_U}(s)\mathcal{F}\omega(U_s) + (-1)^n \overline{Q_U}(s)\mathcal{F}\phi(U_s)),$$

where $\Theta_U(s) := \mathcal{F}\phi(U_s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle)\mathcal{F}\omega(U_s)$. This in turn leads us to define the [weakly] continuous operator $(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle)Q_U(s)) : (\mathcal{Q}_U)' \rightarrow (\mathcal{Q}_U)'$ by

$$\begin{aligned} \left\langle \left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle)Q_U(s) \right) \Lambda_U, \Theta_U \right\rangle \\ := \left\langle \Lambda_U, \left(\overline{P_U}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \overline{Q_U}(s) \right) \Theta_U \right\rangle. \end{aligned}$$

Definition 3.4 If P_U is a polynomial in n variables we define $P_U(D) : (\mathcal{G}_U)' \rightarrow (\mathcal{G}_U)'$ according to $\langle P_U(D)F_U, \theta_U \rangle := \langle F_U, \overline{P_U}(-D)\theta_U \rangle \forall \theta_U \in \mathcal{G}_U$. This makes sense because $\overline{P_U}(-D)\theta_U \in \mathcal{G}_U \forall \theta_U \in \mathcal{G}_U$.

Definition 3.5 We define the rotation (coordinate change) mapping $U : (\mathcal{G}_U)' \rightarrow (\mathcal{G}_U)'$ according to $U : F \rightarrow F_U$, where $\langle F_U, \theta_U \rangle := \langle F, \theta \rangle$, where $\theta_U(x) = \theta(Ux)$, $\theta \in \mathcal{G}$.

Remark 3.6 If P is a polynomial in n variables then $P(D)\theta \in \mathcal{G} \forall \theta \in \mathcal{G}$. Therefore $(P(D)\theta)_U(x) := (P(D)\theta)(Ux) = P_U(D)\theta(Ux) = P_U(D)\theta_U(x)$, where now $P_U(D)$ denotes the polynomial differential operator obtained from $P(D)$ by making the change of variables $x \rightarrow Ux$. Definition 3.5 tells us that $\langle (P(D)F)_U, \theta_U \rangle := \langle P(D)F, \theta \rangle = \langle F, \overline{P}(-D)\theta \rangle = \langle F_U, (\overline{P}(-D)\theta)_U \rangle = \langle F_U, \overline{P_U}(-D)\theta_U \rangle = \langle P_U(D)F_U, \theta_U \rangle$, according to definition 3.4, where $\hat{F} \in \mathcal{G}'$, $\theta_U \in \mathcal{G}_U$. That is, we have $(P(D)F)_U = P_U(D)F_U$ in $(\mathcal{G}_U)'$.

Proposition 3.7 $\mathcal{F} \circ ((P(D) + Q(D)H)F)_U = \left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle)Q_U(s) \right) \mathcal{F}F_U$ as elements of $(\mathcal{Q}_U)'$.

Proof Appealing to the various definitions we have

$$\begin{aligned} \langle \mathcal{F} \circ ((P(D) + Q(D)H)F)_U, \Theta_U \rangle &= \langle \mathcal{F} \circ (P(D)F)_U, \Theta_U \rangle + \langle \mathcal{F} \circ (Q(D)HF)_U, \Theta_U \rangle \\ &= \langle \mathcal{F} \circ (P_U(D)F_U), \Theta_U \rangle + \langle \mathcal{F} \circ (Q_U(D)(HF)_U), \Theta_U \rangle \\ &= \langle P_U(s)\mathcal{F}F_U, \Theta_U \rangle + \langle Q_U(s)\mathcal{F}(HF)_U, \Theta_U \rangle. \end{aligned}$$

Now, we see that

$$\begin{aligned}
(2\pi)^{-1} \langle \mathcal{F}(\mathbf{H}\mathbf{F})_U, \Theta_U \rangle &= \langle (\mathbf{H}\mathbf{F})_U, \mathcal{F}^{-1}\Theta_U \rangle \\
&= \langle \mathbf{H}\mathbf{F}, \mathcal{F}^{-1}\Theta \rangle \\
&= (-1)^n \langle \mathbf{F}, \mathbf{H}\mathcal{F}^{-1}\Theta \rangle \\
&= (-1)^n \left\langle \mathbf{F}, \mathcal{F}^{-1} \left(\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \Theta(s) \right) \right\rangle \\
&= (-1)^n \left\langle \mathbf{F}_U, \left(\mathcal{F}^{-1} \left(\prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \Theta(s) \right) \right)_U \right\rangle \\
&= (-1)^n \left\langle \mathbf{F}_U, \mathcal{F}^{-1} \left(\prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \Theta(Us) \right) \right\rangle \\
&= (-1)^n (2\pi)^{-1} \left\langle \mathcal{F}\mathbf{F}_U, \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \Theta_U(s) \right\rangle \\
&= (2\pi)^{-1} \left\langle \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathcal{F}\mathbf{F}_U, \Theta_U \right\rangle.
\end{aligned}$$

Therefore

$$\langle \mathcal{F} \circ ((\mathbf{P}(\mathbf{D}) + \mathbf{Q}(\mathbf{D})\mathbf{H})\mathbf{F})_U, \Theta_U \rangle = \left\langle \left(\mathbf{P}_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathbf{Q}_U(s) \right) \mathcal{F}\mathbf{F}_U, \Theta_U \right\rangle$$

and the result follows.

Remark 3.8 We could also prove proposition 3.7 in a more general way. First we consider the “rotation” operator [isomorphism] $U : \mathcal{Q}' \rightarrow (\mathcal{Q}_U)'$ given by $U(\mathcal{F}\mathbf{F}) := (\mathcal{F}\mathbf{F})_U$, where $(\mathcal{F}\mathbf{F})_U$ is defined by $\langle (\mathcal{F}\mathbf{F})_U, \Theta_U \rangle := \langle \mathcal{F}\mathbf{F}, \Theta \rangle \forall \Theta_U \in \mathcal{Q}_U$. Therefore $\langle (\mathcal{F}\mathbf{F})_U, \Theta_U \rangle = (2\pi) \langle \mathbf{F}, \mathcal{F}^{-1}\Theta \rangle = (2\pi) \langle \mathbf{F}_U, (\mathcal{F}^{-1}\Theta)_U \rangle = (2\pi) \langle \mathbf{F}_U, \mathcal{F}^{-1}\Theta_U \rangle = \langle \mathcal{F}\mathbf{F}_U, \Theta_U \rangle$, and so $(\mathcal{F}\mathbf{F})_U = \mathcal{F}\mathbf{F}_U$ (note that in passing from the third to the fourth steps in this last chain of equalities we have made use of the results in remark 3.3: if $\Theta_U(s) = \mathcal{F}\phi(Us) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \mathcal{F}\omega(Us)$ then $\mathcal{F}^{-1}\Theta_U(x) = \phi(Ux) + \mathbf{H}\omega(Ux)$ and so clearly $\mathcal{F}^{-1}\Theta_U = (\mathcal{F}^{-1}\Theta)_U$). This result has been tacitly used also in the string of equalities in

the proof of proposition 3.7). Similarly,

$$\begin{aligned}
 & \left\langle \left((P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)) \mathcal{F}F \right)_U, \Theta_U \right\rangle = \left\langle (P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)) \mathcal{F}F, \Theta \right\rangle \\
 & = \left\langle \mathcal{F}F, \left(\overline{P}(s) + (-1)^n \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \overline{Q}(s) \right) \Theta \right\rangle \\
 & = \left\langle (\mathcal{F}F)_U, \left(\left(\overline{P}(s) + (-1)^n \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \overline{Q}(s) \right) \Theta \right)_U \right\rangle \\
 & = \left\langle (\mathcal{F}F)_U, \left(\overline{P}_U(s) + (-1)^n \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \overline{Q}_U(s) \right) \Theta_U(s) \right\rangle \\
 & = \left\langle \left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) Q_U(s) \right) (\mathcal{F}F)_U, \Theta_U \right\rangle.
 \end{aligned}$$

Now let us consider the Fourier transform of F_U , the rotation of F (from definition 3.5).

We have $\langle \mathcal{F}F_U, \Theta_U \rangle := \langle F_U, \mathcal{F}^{-1} \Theta_U \rangle = \langle F_U, (\mathcal{F}^{-1} \Theta)_U \rangle = \langle F, \mathcal{F}^{-1} \Theta \rangle = \langle \mathcal{F}F, \Theta \rangle = \langle (\mathcal{F}F)_U, \Theta \rangle$. That is, $\mathcal{F}F_U = (\mathcal{F}F)_U$ in $(Q_U)'$. We can show, analogously to the one dimensional case, that $\mathcal{F} \circ (P(D) + Q(D)H)F = (P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)) \mathcal{F}F$ (cf Lemma 2.23). Therefore

$$\begin{aligned}
 \mathcal{F} \circ \left((P(D) + Q(D)H)F \right)_U &= \left((P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)) \mathcal{F}F \right)_U \\
 &= (P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) Q_U(s)) \mathcal{F}F_U,
 \end{aligned}$$

as was to be shown.

Remark 3.9 Consider the equation $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$. Applying a rotation mapping gives $((P(D) + Q(D)H)F)_U = \delta_U$ in $(\mathcal{G}_U)'$, so that applying a Fourier transform gives $\left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) Q_U(s) \right) \mathcal{F}F_U = 1$ in $(Q_U)'$.

Remark 3.10 (cf remark 2.25) Considering remark 3.3, define

$$\tilde{\Theta}_U(s) := \left(\overline{P}_U(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \overline{Q}_U(s) \right) \Theta_U(s).$$

where $\Theta_U \in Q_U$ can be written $\Theta_U(s) = \mathcal{F}\phi(Us) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \mathcal{F}\omega(Us)$. We shall write $\bar{\Theta}_U(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s)$, so that from remark 3.3, defining the operator $(\bar{P}_U(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \bar{Q}_U(s)) : Q_U \rightarrow Q_U$, we have

$$\alpha(s) = \bar{P}_U(s) \mathcal{F}\phi(Us) + (\pi^2)^n \bar{Q}_U(s) \mathcal{F}\omega(Us)$$

$$\beta(s) = \bar{P}_U(s) \mathcal{F}\omega(Us) + (-1)^n \bar{Q}_U(s) \mathcal{F}\phi(Us).$$

Solving these last two equations formally we find that

$$\mathcal{F}\phi(Us) = \frac{(\alpha(s) \bar{P}_U(s) - (\pi^2)^n \beta(s) \bar{Q}_U(s))}{\bar{P}_U^2(s) - (-\pi^2)^n \bar{Q}_U^2(s)}$$

and

$$\mathcal{F}\omega(Us) = \frac{\beta(s) \bar{P}_U(s) - (-1)^n \alpha(s) \bar{Q}_U(s)}{\bar{P}_U^2(s) - (-\pi^2)^n \bar{Q}_U^2(s)}.$$

These results hold for all s such that $\bar{P}_U^2(s) - (-\pi^2)^n \bar{Q}_U^2(s) \neq 0$.

Remark 3.11 Consider the denominators in each of the equations for $\mathcal{F}\phi(Us)$ and $\mathcal{F}\omega(Us)$ in remark 3.10: $\bar{P}_U^2(s) - (-\pi^2)^n \bar{Q}_U^2(s)$. Later we shall construct solutions of the equation $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$ using a slightly modified form of the Hörmander's staircase method (see Shilov [7, pp. 174 ff]). For this method it will be necessary to introduce a rotation of coordinates $x \mapsto Ux$ so that $\bar{P}_U^2(s) - (-\pi^2)^n \bar{Q}_U^2(s)$ assumes the form

$$as_1^\nu + \sum_{j=1}^{\nu-1} T_j(s_2, \dots, s_n) s_1^j, \quad a \neq 0. \quad (3.1)$$

We suppose that P is an m^{th} -order and Q a μ^{th} -order polynomial. We distinguish two cases: (a) $m > \mu$, and (b) $m = \mu$ (if $\mu > m$ the reasoning will be as in (a), *mutatis mutandis*). The results we shall now obtain are summarised at the end of the remark.

Case (a): $m > \mu$ In this case $\nu = 2m$ in equation (3.1) and the coefficient a in equation (3.1) is determined solely by the highest $[m^{\text{th}}]$ degree terms in $\bar{P}_U(s)$. We saw, either in remark 3.3 or remark 3.8, that each rotation $x \mapsto Ux$ in $\mathbb{R}^n$ induces a corresponding

rotation $s \rightarrow Us$ in the Fourier transform domain. We see that the highest degree terms in $\overline{P_U}(s)$ will be determined by the highest degree terms in $\overline{P_U}(-D)$, and so in turn by the highest degree terms in $P_U(D)$. Since $\mu < m$, a will be nonzero in equation (3.1) iff the coefficient $\overline{a_0}$ of s_1^m in $\overline{P_U}(s)$ is nonzero, and so in turn iff the coefficient a_0 of s_1^m in $P_U(s)$ is nonzero. Therefore, we want to introduce the coordinate rotation $x \rightarrow Ux$ in $\mathbb{R}^n$ so that, in the transform domain, $P_U(\sigma)$ has the so-called "normal-form" (Shilov [7, pp. 174-175])

$$P_U(\sigma) = a_0 \sigma_1^m + \sum_{k=0}^{m-1} P_k(\sigma_2, \dots, \sigma_n) \sigma_1^k, \quad a_0 \neq 0. \quad (3.2)$$

We recall that $P_U(\xi) := P(U\xi)$, where $\xi \in \mathbb{R}^n$. Suppose that $P(\xi)$ is **not** initially of the form (3.2). We make the coordinate rotation $\xi = U\sigma$ in $\mathbb{R}^n$, so that $\xi_j = \sum_{k=1}^n u_{jk} \sigma_k$, where the rotation matrix U is as yet unspecified. Suppose that

$$P_0(\xi) = \sum_r a_r \xi^{q_{1r}} \dots \xi_n^{q_{nr}}, \quad q_{1r} + \dots + q_{nr} = m$$

is the set of highest degree terms of $P(\xi)$. Then the coefficient of σ_1^m in $P_U(\sigma)$ will be

$$P_0(U_{c_1}) = \sum_r a_r u_{11}^{q_{1r}} \dots u_{n1}^{q_{nr}}, \quad q_{1r} + \dots + q_{nr} = m,$$

where the notation $P_0(U_{c_1})$ tells us of the dependence of P_0 on the first column of matrix U . Therefore, if we want a nonzero coefficient of σ_1^m in $P_U(\sigma)$, then we must choose $u_{11}, \dots, u_{n1}$ to be any normalised set of numbers (i.e. $\sum u_{i1}^2 = 1$) such that $P_0(U_{c_1}) \neq 0$. That is, the vector U_{c_1} can lie anywhere on the unit sphere in $\mathbb{R}^n$ except on the cone $P_0(\xi) = 0$. The remaining columns of U can be chosen arbitrarily, provided they are mutually orthogonal and normalised.

Thus, a sufficient (and necessary) condition for writing $P_U(\sigma)$ as in (3.2) is that the σ_1 -axis of the $(\sigma_1, \dots, \sigma_n)$ coordinate system should not be directed along the so-called

"characteristic cone" $P_0(\xi) = 0$. This is evident when we consider that a point on the σ_1 -axis will have coordinates $(u_{11}, \dots, u_{n1})$ relative to the ξ coordinate system.

Case (b): $\mu = m$ This case is somewhat more delicate, in as much as the highest $[m^{\text{th}}]$ degree term of $\overline{P_U^2}(s) - (-\pi^2)^n \overline{Q_U^2}(s)$ may vanish. For example, if $P(\xi) = \xi_1 \xi_2 - \xi_1$, $Q(\xi) = (1/\pi^2) \xi_1 \xi_2 - \xi_2$, where $n = 2$, then $(P_0(U_{c_1}))^2 - (-\pi^2)^n (Q_0(U_{c_1}))^2 = (u_{11} u_{21})^2 - \pi^4 (u_{11} u_{21} / \pi^2)^2 = 0$ for any choice of rotation matrix U . Again, the highest degree $[2m^{\text{th}}]$ terms in $\overline{P_U^2}(s) - (-\pi^2)^n \overline{Q_U^2}(s)$ (i.e. in (3.1)) will be determined solely by the highest $[m^{\text{th}}]$ degree terms in $\overline{P_U}(s)$ and $\overline{Q_U}(s)$, and so in turn by the highest degree terms in $P_U(s)$ and $Q_U(s)$. As a step towards $\overline{P_U^2}(s) - (-\pi^2)^n \overline{Q_U^2}(s)$ assuming the form (3.1) we should like to introduce a coordinate rotation $x \rightarrow Ux$ in $\mathbb{R}^n$ so that $P_U(\sigma)$ and $Q_U(\sigma)$ both have the "normal forms"

$$\begin{aligned} P_U(\sigma) &= a_0 \sigma_1^m + \sum_{k=0}^{m-1} P_k(\sigma_2, \dots, \sigma_n) \sigma_1^k, \quad a_0 \neq 0 \\ Q_U(\sigma) &= b_0 \sigma_1^m + \sum_{k=0}^{m-1} Q_k(\sigma_2, \dots, \sigma_n) \sigma_1^k, \quad b_0 \neq 0. \end{aligned} \quad (3.30)$$

If this rotation can be made, which satisfies the further condition that $(\overline{a_0})^2 - (-\pi^2)^n (\overline{b_0})^2 \neq 0$, then $\overline{P_U^2}(s) - (-\pi^2)^n \overline{Q_U^2}(s)$ will assume the form (3.1), with $\nu_d = 2m$.

Suppose that one or the other (or possibly both) of $P(\xi)$ or $Q(\xi)$ is not initially of the form in (3.3). We make the coordinate rotation $\xi = U\sigma$ in $\mathbb{R}^n$, so that $\xi_j = \sum_{k=1}^n u_{jk} \sigma_k$, where the rotation matrix U is unspecified. P and Q become, respectively, the m^{th} degree polynomials P_U and Q_U in the terms $\sigma_1, \dots, \sigma_n$. The coefficients of σ_1^m in $P_U(\sigma)$ and $Q_U(\sigma)$ come from the highest degree terms in $P(\xi)$ and $Q(\xi)$, respectively. Specifically, suppose that

$$\begin{aligned} P_0(\xi) &= \sum_r a_r \xi_1^{q_{1r}} \dots \xi_n^{q_{nr}}, \quad q_{1r} + \dots + q_{nr} = m \\ Q_0(\xi) &= \sum_r b_r \xi_1^{p_{1r}} \dots \xi_n^{p_{nr}}, \quad p_{1r} + \dots + p_{nr} = m \end{aligned}$$

constitute the set of highest degree terms of $P(\xi)$ and $Q(\xi)$, respectively. Then the coefficients of σ_1^m in $P_U(\sigma)$ and $Q_U(\sigma)$ will be, respectively

$$P_0(U_{c_1}) = \sum_r a_r u_{11}^{q_{1r}} \dots u_{nr}^{q_{nr}}, \quad Q_0(U_{c_1}) = \sum_l b_l u_{11}^{p_{1l}} \dots u_{nl}^{p_{nl}},$$

where again the notation $P_0(U_{c_1})$, for example, indicates the dependence of P_0 on the first column of matrix U . Therefore we choose $u_{11}, \dots, u_{n1}$ to be any normalised set of numbers ($\sum u_{i1}^2 = 1$) satisfying the conditions either

- (i) $P_0(U_{c_1}) \neq 0$, or (ii) $Q_0(U_{c_1}) \neq 0$, together with
- (iii) $(\overline{P_0}(U_{c_1}))^2 - (-\pi^2)^n (\overline{Q_0}(U_{c_1}))^2 \neq 0$.

That is, the vector U_{c_1} can lie anywhere on the unit sphere in $\mathbb{R}^n$ except on both of the cones $P_0(\xi) = 0$, $Q_0(\xi) = 0$, or on the cone $(\overline{P_0}(\xi))^2 - (-\pi^2)^n (\overline{Q_0}(\xi))^2 = 0$. The remaining columns of U can be chosen arbitrarily, provided they are mutually orthogonal and normalised. Thus, a sufficient condition for writing $P_U(\sigma)$ and $Q_U(\sigma)$ as in (3.3) (and hence $\overline{P_U}(s) - (-\pi^2)^n \overline{Q_U}(s)$ as in (3.1)) is that the σ_1 axis of the $(\sigma_1, \dots, \sigma_n)$ coordinate system should not be directed along both of the characteristic cones $P_0(\xi) = 0$ or $Q_0(\xi) = 0$, nor along the cone $(\overline{P_0}(U_{c_1}))^2 - (-\pi^2)^n (\overline{Q_0}(U_{c_1}))^2 = 0$.

As noted above, condition (iii) is not always met. Indeed, if $\overline{P_U}(s) - (-\pi^2)^n \overline{Q_U}(s) \equiv 0$ is satisfied then (iii) is violated and so we exclude this pathological case from the analysis. If (iii) is violated otherwise then we consider the next highest [typically the $(m-1)^{\text{th}}$] degree terms, $P_1(\xi)$ and $Q_1(\xi)$, and demand that either

- (i) $P_1(U_{c_1}) \neq 0$, or (ii) $Q_1(U_{c_1}) \neq 0$, together with
- (iii) $(\overline{P_1}(U_{c_1}))^2 - (-\pi^2)^n (\overline{Q_1}(U_{c_1}))^2 \neq 0$.

(we allow one of (i) or (ii) to be violated, obviously, in the event that one of $P_1(\xi)$ or $Q_1(\xi)$ is identically zero, for example). If (iii) again cannot be met we consider successively lower degree terms of P & Q until we find an integer λ such that either

- (i)' $P_\lambda(U_{c_1}) \neq 0$, or (ii)' $Q_\lambda(U_{c_1}) \neq 0$, together with
 (iii)' $(\overline{P}_\lambda(U_{c_1}))^2 - (-\pi^2)^n (\overline{Q}_\lambda(U_{c_1}))^2 \neq 0$.

If no such λ can be found we conclude that $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s) \equiv 0$, and, as stated previously, we exclude this case.

To summarise, assuming $(\overline{P}(\xi))^2 - (-\pi^2)^n (\overline{Q}(\xi))^2 \neq 0$, a sufficient condition for ensuring that $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)$ can be written as in equation (3.1) is that we find a lowest value of $\lambda = 0, 1, 2, \dots$ such that either (i)' or (ii)' above holds, together with (iii)'. These conditions will be met if the first column vector, U_{c_1} , of the rotation matrix U lies anywhere on the unit sphere in $\mathbb{R}^n$ except on both of the cones $P_\lambda(\xi) = 0$ or $Q_\lambda(\xi) = 0$, or on the cone $(\overline{P}_\lambda(\xi))^2 - (-\pi^2)^n (\overline{Q}_\lambda(\xi))^2 = 0$ (the terms P_λ & Q_λ denote, for increasing $\lambda = 0, 1, 2, \dots$ the successively lower degree terms of P & Q , respectively).

Remark 3.12 We return to remark 3.9. Applying first a rotation U and then a Fourier transform to the equation $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$ gives us the new equation

$$\left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}(U_j, \sigma)) Q_U(s) \right) \mathcal{F}F_U = 1$$

in $(Q_U)'$. According to remark 3.11 we can find, provided $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s) \neq 0$, a rotation U such that $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)$ assumes the form given in equation (3.1) or remark 3.11. That is, we can write

$$\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s) = as_1^\nu + \sum_{j=1}^{\nu-1} T_j(s_2, \dots, s_n) s_1^j, \quad a \neq 0 \quad (3.4)$$

Definition 3.13 Following Shilov [7, pp. 176-179] we consider the real $(n+1)$ -dimensional Euclidean space whose coordinate axes correspond to the real parts $\sigma_1, \dots, \sigma_n$ of $s_1, \dots, s_n$, and the imaginary part τ_1 of s_1 . We decompose the $(n-1)$ -dimensional $\sigma_2 \dots \sigma_n$ -space into a locally finite family of subsets $\Delta_1, \dots, \Delta_r, \dots$ (that is, locally finite in each neighbourhood of $\sigma_2, \dots, \sigma_n$ -space) by $(n-2)$ -dimensional hyperplanes parallel to the coordinate

hyperplanes. With each Δ_j we associate a fixed value of $\tau_1 = \tau_1^{(j)}$. Hörmander's staircase, $\mathcal{H}$, is defined to be the set of all points $(\sigma_1, \dots, \sigma_n, \tau_1)$ such that $-\infty < \sigma_1 < \infty$ and $\tau_1 = \tau_1^{(j)}$ if $(\sigma_2, \dots, \sigma_n) \in \Delta_j$, for $j = 1, 2, \dots$. Figure 2, below, shows an example of Hörmander's staircase when $n = 2$.

Remark 3.14 In order to be able to construct solutions of the equation

$$\left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) Q_U(s) \right) \mathcal{F}F_U = 1$$

in $(Q_U)'$ using the modified Hörmander's staircase method we must place restrictions on the polynomials P & Q (cf conditions on P & Q in the one dimensional case in proposition 2.28). We state these in the form of

Remark ♠ Consider the n -dimensional hyperplanes in $\tau_1 \sigma_1 \sigma_2 \dots \sigma_n$ space formed by setting one $\sigma_j = 0$, for $j = 1, 2, \dots$. We require that, in each of these n -dimensional hyperplanes, there exist an n -dimensional "strip" of finite "width" about the $(n-1)$ -dimensional coordinate hyperplane $\sigma_1 \dots \sigma_{j-1} \sigma_{j+1} \dots \sigma_n$, within which

$$|\overline{P}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)|$$

is bounded away from zero. These strips are formed by $(n-1)$ -dimensional hyperplanes running parallel to the coordinate hyperplanes $\sigma_1 \dots \sigma_{j-1} \sigma_{j+1} \dots \sigma_n$. That is, there exist constants $\kappa > 0$ and $\eta_0 > 0$ such that, for each $j = 1, 2, \dots$, we have

$$|\overline{P}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| \geq \kappa$$

for all $\{\tau_1, \sigma_1, \sigma_2, \dots, \sigma_n\}$ such that $|\tau_1| \leq \eta_0$ and $\sigma_j = 0$.

We note that if P & Q satisfy condition ♠ then $|\overline{P}_U^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q}_U^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)|$ will be bounded away from zero in a strip in the n -dimensional hyperplanes in $\tau_1 \xi_1 \dots \xi_n$ -space formed by setting one $\xi_j = 0$, where $\xi = U\sigma$ in $\mathbb{R}^n$. A few moment's

Figure 2. Examples of Hörmander's staircase when $n = 2$

Figure 3. Examples of strips within which $|\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2) - (-\pi^2)^2 \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2)|$ is bounded away from zero

thought convinces us that we needn't impose condition $\spadesuit$ on some "skewed" strip in $s_1, \dots, s_n$ -space on P & Q in order that P_U & Q_U satisfy a similar inequality in a strip in the n -dimensional hyperplanes $\tau_1 \xi_1 \dots \xi_n$, where $\xi_j = 0$. For $n = 2$ we show, in figure 3, an example of the strips (centred on the rotated coordinate axes σ_1 & σ_2) within which $|\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2) - (-\pi)^2 \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2)|$ is bounded away from zero.

We now assume that if $\overline{P_U}^2(s) - (-\pi^2)^n \overline{Q_U}^2(s)$ is in the form given in equation (3.4), then it is possible to construct a Hörmander's staircase $\mathcal{H}$ on which $|\overline{P_U}^2(s) - (-\pi^2)^n \overline{Q_U}^2(s)| > c > 0$ for some constant c , and on which the $|\tau_1^{(j)}|$ are uniformly bounded by the constant η_0 in condition $\spadesuit$. We shall later show (Lemma 3.19) that such a construction is possible. Assuming that such a staircase $\mathcal{H}$ exists, we make the following

Definition 3.15 Suppose that P & Q satisfy condition $\spadesuit$ of remark 3.14, and that $\overline{P_U}^2(s) - (-\pi^2)^n \overline{Q_U}^2(s)$ can be written as in equation (3.4). We then define a linear form Λ_U on $\mathcal{Q}_U$ as follows

$$\begin{aligned}
 \langle \Lambda_U, \tilde{\Theta}_U \rangle := & \int_{\mathcal{H}} \left\{ \frac{\alpha(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} \right. \\
 & \left. - \frac{(\pi^2)^n \beta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{Q_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} \right\} ds_1 d\sigma_2 \dots d\sigma_n \\
 & + (i\pi)^n \int_{\mathcal{H}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) ds_1 d\sigma_2 \dots d\sigma_n \\
 & + 2(i\pi)^n \sum_{I_k \in \text{PLUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \in \Delta_j \cap I_k \\ \sigma_1 \in I_k}} \int_0^{\tau_1^{(j)}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) ds_1 d\sigma_2 \dots d\sigma_n, \\
 & - 2(i\pi)^n \sum_{I_k \in \text{MINUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \in \Delta_j \cap I_k \\ \sigma_1 \in I_k}} \int_0^{\tau_1^{(j)}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) ds_1 d\sigma_2 \dots d\sigma_n,
 \end{aligned} \tag{3.5}$$

where

$$W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) := \frac{\beta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} - \frac{(-1)^n \alpha(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{Q_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)},$$

and where we use $\tilde{\Theta}_U \in \mathcal{Q}_U$, writing $\tilde{\Theta}_U = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}(U_j, \sigma)) \beta(s)$. In the above expression we have already described the Hörmander's staircase $\mathcal{H}$ in definition 3.13. $\tilde{\mathcal{H}}$, in the second integral above, is a modified form of $\mathcal{H}$, which contains the same points as $\mathcal{H}$ but on which the sense of integration is different, as we shall explain below. The last two terms in the above definition also need clarification.

Consider the coordinate axes of $\xi_1, \dots, \xi_n$ lying in $\sigma_1 \dots \sigma_n$ -space, where we have the coordinate rotation $\xi = U\sigma$ in $\mathbb{R}^n$. $\sigma_1 \dots \sigma_n$ can be thought of as being decomposed (except for a set of measure zero) into a number of disjoint "quadrants", or sectors (actually, 2^n of them); each quadrant depends on a particular choice of the sequence $\xi_j \geq 0$ for $j = 1, 2, \dots, n$. With each such sector we can associate either a plus or minus sign, depending on whether $\prod_{j=1}^n (i\pi \operatorname{sgn}(U_j, \sigma))$ is greater than or less than zero, respectively. $\tilde{\mathcal{H}}$ contains the same points as $\mathcal{H}$, and we still consider $\sigma_1 \dots \sigma_n$ -space as being decomposed into a locally finite family $\Delta_1, \dots, \Delta_r, \dots$ of subsets. The sense of integration over $\tilde{\mathcal{H}}$ is the same as over $\mathcal{H}$, with the exception of the variable σ_1 . As we vary only the σ_1 coordinate of a point in $\sigma_1 \dots \sigma_n$ -space, the sense of integration, with respect to σ_1 , depends on the sign associated with the quadrant in which our point lies. That is to say, consider $\sigma_2, \dots, \sigma_n$ fixed in a particular Δ_j . Then over any particular quadrant the limits of integration for the σ_1 variable will be l_1 and l_2 (where $l_1 \leq l_2$, and one of them may be infinite). The σ_1 integration will run from l_1 to l_2 if the sign of the quadrant is positive, and from l_2 to l_1 if it is negative. The sense of integration for the σ_1 variable, for the case $n = 2$ and a particular coordinate rotation, is shown in figure 4.

Figure 4. Sense of integration for σ_1 variable on $\mathcal{H}$.

The various sectors (just described) in $\sigma_1 \dots \sigma_n$ -space are separated from each other by $(n - 1)$ -dimensional hyperplanes. Any such hyperplane will correspond to setting a particular $\langle U_j, \sigma \rangle = 0$, and then choosing a particular combination (sequence of $n - 1$ terms) from $\langle U_1, \sigma \rangle \geq 0, \dots, \langle U_{j-1}, \sigma \rangle \geq 0, \langle U_{j+1}, \sigma \rangle \geq 0, \dots, \langle U_n, \sigma \rangle \geq 0$. We can associate either a plus or minus sign to each of these separating hyperplanes, depending on whether, in the sense of integration with respect to σ_1 , we go away from or come towards the hyperplane, respectively. The signs associated to two of the four separating hyperplanes, for the case $n = 2$, are shown in figure 4 (either a + or - sign). We denote the collection of separating hyperplanes with an associated + sign by **PLUS**, and those with a - sign by **MINUS**. A typical element of PLUS or MINUS will be denoted I_k . We see that in the third term in equation (3.5), defining Λ_U , we are integrating over the "hedge" running from $\{\sigma_1 \dots \sigma_n\} \in I_k$ and $\tau_1 = 0$, up to $\{\sigma_1 \dots \sigma_n\} \in I_k$ and $\tau_1 = \tau_1^{(j)}$, depending on whether $\{\sigma_2, \dots, \sigma_n\} \in \Delta_j$. We then sum the contributions from each I_k in PLUS. Similarly, in the fourth term in equation (3.5), we sum contributions from each I_k in MINUS. For the case $n = 2$ we have shown (shaded) an example of one of the four "hedges" (in particular, for $I_k \in \text{PLUS}$), for a typical Hörmander's staircase, in figure 5.

Our reasons for the clumsy construction given in equation (3.5) will emerge in proposition 3.17.

Remark 3.16 The linear form Λ_U just defined on $\mathcal{Q}_U$ is continuous. We take a sequence $\{(\tilde{\Theta}_U)_\mu\} \rightarrow 0$ in $\mathcal{Q}_U$ and consider the various terms in the right hand side of equation (3.5) when we substitute an element from the sequence in the left hand side. Since P & Q satisfy condition $\spadesuit$, and $|\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)| > c > 0$ on $\mathcal{H}$ (see remark 3.14) we see that the denominators in the integrands in equation (3.5) are all bounded away from zero over the range of integration. We recall (remark 3.3), that a sequence $\{(\tilde{\Theta}_U)_\mu\} \rightarrow 0$ in

Figure 5. Example of a range of integration "hedge" for an $I_k \in PLUS$ in the third term of equation (3.5) ($n = 2$).

$\mathcal{Q}_U$ iff the corresponding sequence $\{(\bar{\theta}_U)_\mu\} \rightarrow 0$ in $\mathcal{G}_U$ (where $\bar{\theta}_U(x) = \phi_U(x) + (H\omega)_U(x)$), or equivalently iff $\{(\Theta_U)_\mu\}$ satisfies the following two conditions (cf proposition 2.11):

- (i) For each $|q| = 0, 1, 2, \dots$ we have $|s^q \alpha_\mu(s)| \leq C_{q,\alpha} e^{b_1|\tau_1| + \dots + b_n|\tau_n|}$ and $|s^q \beta_\mu(s)| \leq C_{q,\beta} e^{b_1|\tau_1| + \dots + b_n|\tau_n|}$, where constants b_i , $C_{q,\alpha}$, & $C_{q,\beta}$ are independent of μ , and
- (ii) both $\{\alpha_\mu(\sigma)\} \rightarrow 0$ and $\{\beta_\mu(\sigma)\} \rightarrow 0$ uniformly on each bounded subset of $\mathbb{R}^n$.

In (i) & (ii) we have used $(\bar{\Theta}_U)_\mu(s) := \alpha_\mu(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}(U_j, \sigma)) \beta_\mu(s)$. Conditions similar to (i) and (ii) must be satisfied by $\mathcal{F}\phi_\mu(s)$ and $\mathcal{F}\omega_\mu(s)$, where $\bar{\Theta}_\mu(s) := \mathcal{F}\phi_\mu(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega_\mu(s)$.

In order to show that $|\langle \Lambda_U, (\Theta_U)_\mu \rangle| \rightarrow 0$ we first consider the first integral on the right hand side of equation (3.5). We have

$$\begin{aligned} & \left| \int_{\mathcal{H}} \frac{\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \overline{Q_U}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} ds_1 d\sigma_2 \dots d\sigma_n \right| \\ & \leq \frac{1}{c} \int_{\mathcal{H}} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n. \end{aligned}$$

Now,

$$\begin{aligned} & \int_{\mathcal{H}} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n \\ & = \int_{\substack{\mathcal{H} \\ |\sigma_i| \leq N}} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n \\ & \quad + \int_{\substack{\mathcal{H} \\ |\sigma_i| > N}} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n. \end{aligned}$$

Using condition (i) above, for α_μ , we see that we can make the second integral on the right

of this last equation as small as we like by choosing N sufficiently large. Furthermore

$$\begin{aligned}
& \int_{|\sigma_i| \leq N} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n \\
& \leq \rho_0 \int_{|\sigma_i| \leq N} |\alpha_\mu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n \quad (\rho_0 \text{ constant}) \\
& = \frac{\rho_0}{(2\pi)^n} \int_{|\sigma_i| \leq N} \left| \int_{\mathbb{R}^n} (\phi_U)_\mu(x_1, \dots, x_n) e^{-i(x, (\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n))} dx_1 \dots dx_n \right| d\sigma_1 \dots d\sigma_n \\
& \leq \frac{\rho_0}{(2\pi)^n} \int_{|\sigma_i| \leq N} \left| (\phi_U)_\mu(x_1, \dots, x_n) \left\{ \int_{|\sigma_i| \leq N} e^{i(x, \sigma)} d\sigma_1 \dots d\sigma_n \right\} \right| dx_1 \dots dx_n \\
& \leq \frac{\rho_0}{(2\pi)^n} \int_{|\sigma_i| \leq N} \left| (\phi_U)_\mu(x_1, \dots, x_n) \left\{ \int_{|\sigma_i| \leq N} \kappa_0(b_1, \tau_0) d\sigma_1 \dots d\sigma_n \right\} \right| dx_1 \dots dx_n \\
& \leq \frac{\rho_0 \kappa_1}{(2\pi)^n} \int_{|\sigma_i| \leq N} \left| (\phi_U)_\mu(x_1, \dots, x_n) \right| dx_1 \dots dx_n \quad (\kappa_0, \kappa_1 \text{ constants}),
\end{aligned}$$

and this last term can be made arbitrarily small by a judicious choice of μ (in this last series of inequalities we have used $(\phi_U)_\mu = \mathcal{F}^{-1}(\alpha_\mu)$). We can apply a similar argument to the second component of the first integral on the right hand side of (3.5). Thus, the magnitude of this first integral can be made arbitrarily small. Similarly, the magnitude of the second right hand integral in (3.5) can be made arbitrarily small. Appealing to condition $\clubsuit$ of remark 3.14, we can also show that the magnitudes of the third and fourth right hand integrals of equation (3.5) can be made arbitrarily small. This implies that $\{|\langle \Lambda_U, (\Theta_U)_\mu \rangle|\} \rightarrow 0$, and the continuity of Λ_U follows.

Proposition 3.16 Suppose that P & Q satisfy condition $\clubsuit$ of remark 3.14, and that $\overline{P_U}^2(s) - (-\pi^2 \overline{Q_U}^2(s))$ can be written as in equation (3.4). Then Λ_U , as in definition 3.15, is a solution for $\mathcal{FF}_U$ in the equation $\left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) Q_U(s) \right) \mathcal{FF}_U = 1$.

Proof Appealing to the various definitions we have

$$\begin{aligned}
& \left\langle \left(P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) Q_U(s) \right) \Lambda_U, \Theta_U \right\rangle \\
& = \left\langle \Lambda_U, \left(P_U(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) Q_U(s) \right) \Theta_U(s) \right\rangle \\
& = \langle \tilde{\Lambda}_U, \tilde{\Theta}_U \rangle,
\end{aligned}$$

where $\tilde{\Theta}_U(s) := \left(P_U(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) Q_U(s) \right) \Theta_U$. Writing $\Theta_U(s) = \mathcal{F}\phi(Us) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \mathcal{F}\omega(Us)$ and $\tilde{\Theta}_U(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \beta(s)$, we can solve for $\mathcal{F}\phi(Us)$ and $\mathcal{F}\omega(Us)$ in terms of $\alpha(s)$ and $\beta(s)$. The result is as given in remark 3.10.

Thus, from definition 3.15 we have

$$\begin{aligned} \langle \Lambda_U, \tilde{\Theta}_U \rangle &= \int_{\mathcal{H}} \mathcal{F}\phi(Us) ds_1 d\sigma_2 \dots d\sigma_n + (i\pi)^n \int_{\mathcal{H}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n \\ &\quad + 2(i\pi)^n \sum_{I_k \in \text{PLUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, nI_k \\ \sigma_1 \in I_k}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n \\ &\quad - 2(i\pi)^n \sum_{I_k \in \text{MINUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, nI_k \\ \sigma_1 \in I_k}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n \end{aligned}$$

Both $\mathcal{F}\phi(Us)$ and $\mathcal{F}\omega(Us)$ are entire functions in each of their arguments, satisfying inequalities such as those in remark 3.3. Therefore, by Cauchy's theorem,

$$\int_{\mathcal{H}} \mathcal{F}\phi(Us) ds_1 d\sigma_2 \dots d\sigma_n = \int_{\mathbb{R}^n} \mathcal{F}\phi(U\sigma) d\sigma_1 \dots d\sigma_n.$$

Also, from Cauchy's theorem and the definition of integration over $\mathcal{H}$ (definition 3.15), we find

$$\begin{aligned} &(i\pi)^n \int_{\mathcal{H}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n \\ &= (i\pi)^n \int_{\mathbb{R}^n} \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \mathcal{F}\omega(U\sigma) d\sigma_1 \dots d\sigma_n \\ &\quad - 2(i\pi)^n \sum_{I_k \in \text{PLUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, nI_k \\ \sigma_1 \in I_k}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n \\ &\quad + 2(i\pi)^n \sum_{I_k \in \text{MINUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, nI_k \\ \sigma_1 \in I_k}} \mathcal{F}\omega(Us) ds_1 d\sigma_2 \dots d\sigma_n. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \Lambda_U, \tilde{\Theta}_U \rangle &= \int_{\mathbb{R}^n} \mathcal{F}\phi(U\sigma) d\sigma_1 \dots d\sigma_n + \int_{\mathbb{R}^n} \prod_{j=1}^n (i\pi \operatorname{sgn}\langle U_j, \sigma \rangle) \mathcal{F}\omega(U\sigma) d\sigma_1 \dots d\sigma_n \\ &= \langle 1, \Theta_U \rangle \end{aligned}$$

and the proof is complete.

Corollary 3.17 With the same conditions on P & Q as in proposition 3.16, we have

$\mathcal{F}^{-1}(\Lambda_U)$ is a solution in $(\mathcal{G}_U)'$ for the equation $P_U(D)F_U + Q_U(D)(HF)_U = \delta$.

Corollary 3.18 With the same conditions on P & Q as in proposition 3.16, we have

$\mathcal{F}^{-1}(\Lambda)$ is a solution in $\mathcal{G}'$ for the equation $(P(D) + Q(D)H)F = \delta$, where $U : \Lambda \rightarrow \Lambda_U$

(see remark 3.8).

Lemma 3.19 Assuming that $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)$ is of the form given in equation (3.4), then it is possible to construct a Hörmander's staircase $\mathcal{H}$ on which $|\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)| > c > 0$ for some constant c , and on which the $\tau_1^{(j)}$ are uniformly bounded by the constant $\eta_0 > 0$ (see remark 3.14).

Proof (see Shilov [7, pp.177-178]) Consider the polynomial

$$\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s) = as_1^\nu + \sum_{k=0}^{\nu-1} T_k(s_2, \dots, s_n) s_1^k, \quad a \neq 0,$$

for fixed arbitrary values of $s_2 = \sigma_2, \dots, s_n = \sigma_n$. It has at most ν roots $s_1^{(1)}, \dots, s_1^{(\nu)}$ in the s_1 -plane and so can be factored

$$\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s) = a(s - s_1^{(1)}) \dots (s - s_1^{(\nu)}). \quad (3.6)$$

In the strip $|\tau_1| < \eta_0$ it is possible to draw a line $\tau_1 = \text{constant}$ whose distance from each of the roots $s_1^{(1)}, \dots, s_1^{(\nu)}$ is greater than η_0/ν . From the factorisation (3.6) it is clear that $|\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)| > c$, some constant, on this line (incidentally, if condition $\spadesuit$ is also satisfied we see that at those n values of σ_1 satisfying $\langle U_1, \sigma \rangle = 0, \dots, \langle U_n, \sigma \rangle = 0$ (remember $\sigma_2, \dots, \sigma_n$ are fixed) we have $|\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)| \geq \kappa$ for all τ_1 such that $\tau_1 \leq \eta_0$).

The roots of a polynomial with a constant leading coefficient depend continuously on the remaining coefficients (Shilov [7, pp.177]). Thus we can find a sufficiently small neighbourhood of the point $(\sigma_2, \dots, \sigma_n)$ within which the corresponding roots $s_1^{(1)}, \dots, s_1^{(\nu)}$

of $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)$ lie arbitrarily close to their original positions. Therefore $|\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)| > c$ will still hold, on the line $\tau_1 = \text{constant}$, in some neighbourhood of the chosen point $(\sigma_2, \dots, \sigma_n)$.

We can associate such a neighbourhood to every point of $\sigma_2, \dots, \sigma_n$ -space. Without loss of generality we can consider these neighbourhoods as being bounded by $(n-1)$ -dimensional hyperplanes parallel to the coordinate hyperplanes. From this covering of $\sigma_2, \dots, \sigma_n$ -space we can select a locally finite covering $V_1, V_2, \dots, V_j, \dots$. We define $\Delta_1 = V_1$, and $\Delta_j = V_j - (\Delta_1 \cup \dots \cup \Delta_{j-1})$ for $j > 1$. The $\{\Delta_j\}$ are disjoint regions which, together with the values $\tau_1^{(j)}$, prescribe the Hörmander's staircase $\mathcal{H}$.

Remark 3.20 The construction of our continuous linear form Λ_U in definition 3.15 rests upon the assumption that P & Q satisfy condition $\spadesuit$ of remark 3.14. We consider two simple examples to illustrate this feature. First suppose that $\overline{P}^2(\sigma_1 + i\tau_1) - (-\pi^2)^2 \overline{Q}^2(\sigma_1 + i\tau_1) = (\sigma_1 + i\tau_1)^2 \sigma_2^2 - (-\pi^2)^2 (i/\pi^2)^2$ for $n = 2$. In this case, in the plane $\sigma_1 = 0$ the polynomial behaves like $-\tau_1^2 \sigma_2^2 + 1$. So, the polynomial has zeros on the hyperbolas $\tau_1 = \pm 1/|\sigma_2|$, and we see that $|\overline{P}^2(i\tau_1, \sigma_2) - (-\pi^2)^2 \overline{Q}^2(i\tau_1, \sigma_2)|$ is not bounded away from zero, in the $\tau_1 \sigma_2$ -plane, in a strip of finite width about the σ_2 -axis. Thus, condition $\spadesuit$ is violated. Now suppose that $\overline{P}(-D) = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ and $\overline{Q}(-D) = i/\pi^2$, so that $\overline{P}^2(\sigma_1 + i\tau_1, \sigma_2) - (-\pi^2)^2 \overline{Q}^2(\sigma_1 + i\tau_1, \sigma_2) = ((\sigma_1 + i\tau_1)^2 + \sigma_2^2)^2 + 1$. We see that $\overline{P}_U^2(s) - (-\pi^2)^n \overline{Q}_U^2(s)$ is already of the form of equation (3.4), so that no coordinate rotation is necessary. Thus, we need only consider the behaviour of the polynomial in the plane $\sigma_1 = 0$. We have

$$\overline{P}^2(i\tau_1, \sigma_2) - (-\pi^2)^2 \overline{Q}^2(i\tau_1, \sigma_2) = (-\tau_1^2 + \sigma_2^2)^2 + 1 = \tau_1^4 - 2\tau_1^2 \sigma_2^2 + \sigma_2^4 + 1.$$

We can check that the polynomial is zero whenever $\tau_1^2 = \sigma_2^2 \pm i$, so that there are no zeros in the $\tau_1 \sigma_2$ -plane. In fact, it is easy to show that the polynomial is greater than or equal

to one in the whole $\tau_1\sigma_2$ plane, and so condition $\spadesuit$ is clearly satisfied.

§3.3 The Equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'(\mathbb{R}^n)$

As in the one dimensional case (§ 2.4), the equation $\{P(D) + Q(D)H\}F = \delta$ makes sense if we interpret $\delta \in \mathcal{D}'$, $F \in \mathcal{G}'$, and $\{P(D) + Q(D)H\}F$ as an operator from $\mathcal{G}' \rightarrow \mathcal{D}'$.

We define (cf definition 2.33) this operator by

$$\langle \{P(D) + Q(D)H\}F, \phi \rangle := \langle F, (\bar{P}(-D) + (-1)^n \bar{Q}(-D)H) \phi \rangle \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

Equivalently, we can say $\{P(D) + Q(D)H\}F := P(D)f + (\pi^2)^n Q(D)g$, where $F = \chi(f, Hg)$ and $f, g \in \mathcal{D}'(\mathbb{R}^n)$ (cf remark 2.34) ($\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$ is a canonical isomorphism, cf proposition 2.4).

As in the preceding section, our ability to find a solution of $\{P(D) + Q(D)H\}F = \delta$ will rest on being able to introduce a coordinate rotation $x \rightarrow Ux$ that reduces $P(D)$ & $Q(D)$ to $P_U(D)$ & $Q_U(D)$ having normal forms (see equation (3.1) at the beginning of § 3.2). Interpreting the equation $\{P(D) + Q(D)H\}F = \delta$ in $\mathcal{D}'$ makes the introduction of coordinate rotations somewhat simpler than in the previous section, where we interpreted the equation in $\mathcal{G}'$. To see why, let us recall a few ideas about rotations of test functions and distributions. The coordinate rotation $x = Uy$ in $\mathbb{R}^n$ carries a test function $\phi(x) \in \mathcal{D}$ into a test function $\phi_U(y) \in \mathcal{D}$, where $\phi_U(y) := \phi(Uy)$ (remark 3.1). This allows us to define the rotation mapping $U : \mathcal{D}' \rightarrow \mathcal{D}'$ according to $U : f \rightarrow f_U$, where $\langle f_U, \phi \rangle := \langle f, \phi_{U^{-1}} \rangle$ (see Shilov [7, pp.150-151]). If we denote $\mathcal{F}\phi(\sigma) := \Phi(\sigma)$ then we should find that $\Phi_U(\sigma) := \mathcal{F}\phi_U(\sigma) = \Phi(U\sigma)$. Also, we can define the rotation mapping $U : \mathcal{Z}' \rightarrow \mathcal{Z}'$ according to $U : \mathcal{F}f \rightarrow (\mathcal{F}f)_U$, where $\langle (\mathcal{F}f)_U, \Phi \rangle := \langle \mathcal{F}f, \Phi_{U^{-1}} \rangle$. This leads eventually to the result $\mathcal{F}(f_U) = (\mathcal{F}f)_U$. These results should be contrasted with the facts that

$U : \mathcal{G}' \rightarrow (\mathcal{G}_U)'$ and $U : \mathcal{Q}' \rightarrow (\mathcal{Q}_U)'$, which follow from the unfortunate situation that the rotation of an element of $\mathcal{G}$ is not, in general, another element of $\mathcal{G}$. With these notions at hand we can apply a rotation mapping to both sides of $\{P(D) + Q(D)H\}F = \delta$ to get $(\{P(D) + Q(D)H\}F)_U = \delta_U$. Now, $\delta_U = \delta$ in $\mathcal{D}'$. Furthermore,

$$(\{P(D) + Q(D)H\}F)_U = (P(D)f + (\pi^2)^n Q(D)g)_U = (P(D)f)_U + (\pi^2)^n (Q(D)g)_U,$$

where $F = \chi(f, Hg)$. It is easy to show that $(P(D)f)_U = P_U(D)f_U$, for example, where $P_U(D)$ denotes the polynomial differential operator obtained from $P(D)$ by making the change of variables $x \rightarrow Ux$. Therefore, our rotated equation becomes

$$\{P_U(D) + Q_U(D)H\}F_U = \delta$$

in $\mathcal{D}'$, where $F_U = \chi(f_U, Hg_U)$ (not to be confused with $F_U \in (\mathcal{G}_U)'$ used in the preceding section).

The properties of the Fourier transforms between the various spaces carry over from the one dimensional case. For polynomials P & Q we have the continuous multiplication operator $(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s)) : \mathcal{Z} \rightarrow \mathcal{Q}$ given by

$$(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s))\zeta(s) := \bar{P}(s)\zeta(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s)\zeta(s).$$

We can thus define $[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] : \mathcal{Q}' \rightarrow \mathcal{Z}'$ according to

$$\langle [[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]]\Xi, \zeta \rangle := \langle \Xi, (\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s))\zeta(s) \rangle.$$

We can show (cf Lemma 2.37) that $\mathcal{F} \circ \{P(D) + Q(D)H\}F = [[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]]\mathcal{F}F$ as elements of $\mathcal{Z}'$. Therefore, applying a Fourier transform to the equation $\{P_U(D) + Q_U(D)H\}F_U = \delta$ gives us

$$[[P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q_U(s)]]\mathcal{F}F_U = \delta \text{ in } \mathcal{Z}'. \quad (3.7)$$

In order to construct a solution of this equation (3.7) we shall make direct use of the Hörmander's staircase method (Shilov [7, pp. 176-178]). We assume that P & Q are, respectively, m^{th} and μ^{th} order polynomials (without loss of generality we assume $m \geq \mu$). The Hörmander's staircase method will rely on us being able to find a coordinate rotation $x \rightarrow Ux$ that forces each of the three polynomials $\overline{P_U}(s) - (i\pi)^n \overline{Q_U}(s)$, $\overline{P_U}(s) + (i\pi)^n \overline{Q_U}(s)$, and $\overline{P_U}(s)$ into a form resembling equation (3.1) (see remark 3.11); that is, each of these three polynomials must resemble

$$as_1^m + \sum_{j=1}^{m-1} T_j(s_2, \dots, s_n) s_1^j, \quad a \neq 0. \quad (3.8)$$

Following the argument in remark 3.11, we realise that this condition will be met if we can find a coordinate rotation matrix U that simultaneously reduces $P \rightarrow P_U$ and $Q \rightarrow Q_U$, where

$$\begin{aligned} P_U(\sigma) &= a_0 \sigma_1^m + \sum_{k=0}^{m-1} P_k(\sigma_2, \dots, \sigma_n) \sigma_1^k, \quad a_0 \neq 0, \\ Q_U(\sigma) &= b_0 \sigma_1^\mu + \sum_{l=0}^{\mu-1} Q_l(\sigma_2, \dots, \sigma_n) \sigma_1^l, \quad b_0 \neq 0, \end{aligned} \quad (3.9)$$

and, in the event $\mu = m$, satisfies the constraint $a_0 \pm (i\pi)^n b_0 \neq 0$. We exclude from our analysis the pathological cases where either $\overline{P}(\sigma) + (i\pi)^n \overline{Q}(\sigma) \equiv 0$ or $\overline{P}(\sigma) - (i\pi)^n \overline{Q}(\sigma) \equiv 0$. From remark 3.11, we suppose that one of $P(\xi)$ or $Q(\xi)$ is not of the form given in equation (3.9), and let

$$\begin{aligned} P_0(\xi) &= \sum_r a_r \xi_1^{q_{1r}} \dots \xi_n^{q_{nr}} \quad q_{1r} + \dots + q_{nr} = m \\ Q_0(\xi) &= \sum_l b_l \xi_1^{p_{1l}} \dots \xi_n^{p_{nl}} \quad p_{1l} + \dots + p_{nl} = \mu \end{aligned}$$

constitute the sets of highest degree terms of $P(\xi)$ & $Q(\xi)$, respectively. We again conclude that each of the three polynomials $\overline{P_U}(s) - (i\pi)^n \overline{Q_U}(s)$, $\overline{P_U}(s) + (i\pi)^n \overline{Q_U}(s)$, & $\overline{P_U}(s)$ can be written in the form of (3.8) (with different coefficients) provided the following conditions are met

(i) $P_0(U_{c_1}) \neq 0$, (ii) $Q_0(U_{c_1}) \neq 0$,
 (iii) [if $\mu = m$] $P_0(U_{c_1}) \neq \pm(i\pi)^n Q_0(U_{c_1})$,
 (U_{c_1} denotes the vector in $\mathbb{R}^n$ whose components constitute the first column of rotation matrix U). That is, the vector U_{c_1} can lie anywhere on the unit sphere in $\mathbb{R}^n$ except on any of the cones $P_0(\xi) = 0$, $Q_0(\xi) = 0$, or [if $\mu = m$] on the cones $P_0(\xi) \pm (i\pi)^n Q_0(\xi) = 0$. Equivalently, the σ_1 -axis of the $\sigma_1 \dots \sigma_n$ coordinate system ($\xi = U\sigma$) should not be directed along any of these cones. The remaining columns of U can be chosen arbitrarily, provided they are mutually orthogonal and normalised.

It is possible that one condition in (iii) above is violated for any choice of U_{c_1} . For example, let us suppose that $P_0(U_{c_1}) + (i\pi)^n Q_0(U_{c_1}) = 0$ for any choice of U_{c_1} . In this event we consider successively lower degree terms, P_λ & Q_λ , of $P(\xi)$ and $Q(\xi)$ ($\lambda > 0$) and stop when we find a λ such that

$$(iii) \quad P_0(U_{c_1}) - (i\pi)^n Q_0(U_{c_1}) \neq 0 \text{ and } P_\lambda(U_{c_1}) + (i\pi)^n Q_\lambda(U_{c_1}) \neq 0.$$

Thus, we can find a coordinate rotation U such that the three polynomials associated with $\overline{P_U}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \overline{Q_U}(s)$ have the form given in equation (3.8). It is possible to construct a Hörmander's staircase $\mathcal{H}$ (described in definition 3.13) on which $|\overline{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \overline{Q}(s)| > c > 0$. We need only slightly modify the construction given in Lemma 3.19. Fixing the arbitrary values $s_2 = \sigma_2, \dots, s_n = \sigma_n$ we realise that each of the polynomials $\overline{P_U}(s) - (i\pi)^n \overline{Q_U}(s)$, $\overline{P_U}(s) + (i\pi)^n \overline{Q_U}(s)$, & $\overline{P_U}(s)$ has at most m roots in the s_1 plane (without loss of generality we assume $m \geq \mu$). As each polynomial can be written as in (3.8), we can factor them as

$$\overline{P_U}(s) + (i\pi)^n \overline{Q_U}(s) = a_1(s - s_{11}^{(1)}) \dots (s - s_{11}^{(m)}) \quad a_1 \neq 0,$$

$$\overline{P_U}(s) - (i\pi)^n \overline{Q_U}(s) = a_2(s - s_{12}^{(1)}) \dots (s - s_{12}^{(m)}) \quad a_2 \neq 0,$$

$$\overline{P_U}(s) = a_3(s - s_{13}^{(1)}) \dots (s - s_{13}^{(m)}) \quad a_3 \neq 0.$$

It is always possible to draw a line $\tau_1 = \text{constant}$, in the region $|\tau_1| \leq 3m+1$, whose distance from each root of the above polynomial is greater than one. From the above factorisations it is clear that $|\left(\overline{P}(s) + \prod_{j=1}^n (-i\pi \text{sgn} \sigma_j) \overline{Q}(s)\right)| > c > 0$ on any such line, where $c = \min(a_1, a_2, a_3)$. The construction of the Hörmander's staircase then follows as in Lemma 3.19. With such a staircase $\mathcal{H}$ at hand we introduce the

Definition 3.20 We define the linear functional Λ on $\mathcal{Q}$ by

$$\langle \Lambda, \Theta \rangle := \int_{\mathcal{H}} \frac{\Theta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) + \prod_{j=1}^n (-i\pi \text{sgn} \sigma_j) \overline{Q_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} d\sigma_1 \dots d\sigma_n.$$

The integral exists, as the magnitude of the denominator in the integrand exceeds c , and the numerator is integrable.

Remark 3.21 Λ is a continuous linear form on $\mathcal{Q}$. To see this, consider a sequence $\{\Theta_\nu\} \rightarrow 0$ in $\mathcal{Q}$, and write $\Theta_\nu(s) = \alpha_\nu(s) + \prod_{j=1}^n (i\pi \text{sgn} \sigma_j) \beta_\nu(s)$. We must show that $|\langle \Lambda, \Theta_\nu \rangle| \rightarrow 0$. Observe that

$$\begin{aligned} & \left| \int_{\mathcal{H}} \frac{\alpha_\nu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\overline{P_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) + \prod_{j=1}^n (-i\pi \text{sgn} \sigma_j) \overline{Q_U}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} d\sigma_1 \dots d\sigma_n \right| \\ & \leq \frac{1}{c} \int_{\mathcal{H}} |\alpha_\nu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| d\sigma_1 \dots d\sigma_n \\ & = \int_{|\sigma_j| \leq N} \frac{|\alpha_\nu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)|}{c} d\sigma_1 \dots d\sigma_n \\ & \quad + \int_{|\sigma_j| > N} \frac{|\alpha_\nu(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)|}{c} d\sigma_1 \dots d\sigma_n \end{aligned}$$

and both of these integrals can be made arbitrarily small by a suitable choice of ν and N (see remark 3.16). A similar result holds for the integral involving β , and the result follows.

Proposition 3.22 $\Lambda \in \mathcal{Q}'$, defined in 3.20, is a solution for $\mathcal{FF}_U$ in the equation $[(P_U(s) + \prod_{j=1}^n (i\pi \text{sgn} \sigma_j) Q_U(s))] \mathcal{FF}_U = 1$ in $\mathcal{Z}'$.

Proof For any $\zeta \in \mathcal{Z}$ we have

$$\begin{aligned} \langle [P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q_U(s)] \Lambda, \zeta(s) \rangle &= \langle \Lambda, (\overline{P_U(s)} + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \overline{Q_U(s)}) \zeta(s) \rangle \\ &= \int_{\mathcal{H}} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\sigma_1 \dots d\sigma_n \\ &= \sum_j \int_{(\sigma_2, \dots, \sigma_n) \in \Delta_j} \left\{ \int_{-\infty}^{\infty} \zeta(\sigma_1 + i\tau_1^{(j)}, \sigma_2, \dots, \sigma_n) d\sigma_1 \right\} d\sigma_2 \dots d\sigma_n. \end{aligned}$$

The inner integral, for fixed j , is taken over a line parallel to the real axis in the $\sigma_1 + i\tau_1$ plane, at a distance $|\tau_1^{(j)}|$ from the axis. Cauchy's theorem allows us to replace the integral by one along the σ_1 -axis. Thus, finally

$$\begin{aligned} \langle [P_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q_U(s)] \Lambda, \zeta(s) \rangle &= \sum_j \int_{(\sigma_2, \dots, \sigma_n) \in \Delta_j} \int_{-\infty}^{\infty} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\sigma_1 \dots d\sigma_n \\ &= \int_{\mathbb{R}^n} \zeta(\sigma) d\sigma = \langle 1, \zeta \rangle, \end{aligned}$$

and the result follows.

Corollary 3.23 $\mathcal{F}^{-1}\Lambda$ is a solution for F_U in the equation $\{P_U(D) + Q_U(D)H\}F_U = \delta$, and $(\mathcal{F}^{-1}\Lambda)_{U^{-1}}$ is a solution for F in the equation $\{P(D) + Q(D)H\}F = \delta$.

§3.4 The Equation $\{P(D) + Q(D)H\}F = h$ in $\mathcal{D}'$

In this section we shall demonstrate that the equation $\{P(D) + Q(D)H\}F = h$ in $\mathcal{D}'$ has a solution F in $\mathcal{G}'$ for any $h \in \mathcal{D}'$. We interpret $\{P(D) + Q(D)H\}F: \mathcal{G}' \rightarrow \mathcal{D}'$ as defined in the preceding section. The concept of "convolution" of elements of $\mathcal{G}'$ with elements of $\mathcal{D}'$ is not clear (even if those elements of $\mathcal{D}'$ have compact support). To show that solutions exist we adapt a technique used in the study of partial differential equations with non-trivial right hand sides (see Shilov [7, pp.180-183]). This approach allows us to bypass the concept of convolution.

Throughout this section we shall assume that P & Q are already such that $P(\sigma)$ and $Q(\sigma)$ have the "normal forms" given in equation (3.9) of §3.3. For, if not, we realise that a rotation of coordinates can be carried out to bring them into this form. We also assume that the coefficient of the highest order term (in s_1) of both $\bar{P}(s) + (i\pi)^n \bar{Q}(s)$ and $\bar{P}(s) - (i\pi)^n \bar{Q}(s)$ is nonzero. Again, if not, then a different coordinate rotation will change this (see §3.3). Therefore, we shall not use the subscript U in this section.

Fourier transforming our equation gives

$$[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] \mathcal{F}P = 1 \text{ in } \mathcal{Z}'. \quad (3.10)$$

We know that arbitrary $h \in \mathcal{D}'$ can be represented by the weakly convergent series

$$h = \sum_{k=0}^{\infty} h_k \quad (3.11)$$

of functionals with compact support (see Shilov [7, pp.80-81, eqn (10)]). Therefore, in $\mathcal{Z}'$ we have the weakly convergent representation $\mathcal{F}h = \sum_{k=0}^{\infty} \mathcal{F}h_k$, where, since the h_k have compact support, the $\mathcal{F}h_k(s)$ are entire functions (in each of $s_1, \dots, s_n$) having first order exponential growth in the imaginary domain, and power growth in the real domain (Shilov [7, pp.151-152, eqn. (17)]).

In $\mathcal{Q}'$ we define

$$\Xi_k := \Lambda_k \mathcal{F}h_k(s) \text{ for } k = 1, 2, \dots \quad (3.12)$$

where $\Lambda_k \in \mathcal{Q}'$ is defined by

$$\langle \Lambda_k, \Theta(s) \rangle := \int_{\mathcal{H}_k} \frac{\Theta(s)}{(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s))} ds, \quad (3.13)$$

for some Hörmander's staircase $\mathcal{H}_k$ (observe that the properties of $\mathcal{F}h_k$ make it a "multiplier" in $\mathcal{Q}'$: from §3.1 we recall that $\mathcal{Q}$ is characterised as the space of functions on

$\mathcal{Q}$ that can be written as $\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s)$, where α and β are entire functions satisfying inequalities such as, for example, $|s^q \alpha(s)| \leq C_q e^{a_1 |\tau_1| + \dots + a_n |\tau_n|}$ for all multi-indices q such that $|q| = 0, 1, 2, \dots$, for constants $a_1, \dots, a_n$, & C_q depending on α . Therefore, for any $\Theta \in \mathcal{Q}$, we have $(\mathcal{F}h_k)^* \Theta \in \mathcal{Q}$ also, where $(\mathcal{F}h_k)^*(s) := \overline{\mathcal{F}h_k(\bar{s})}$ (an overbar denotes complex conjugation). We thus define $\mathcal{F}h_k \Lambda$, for any $\Lambda \in \mathcal{Q}'$, according to $\langle \mathcal{F}h_k \Lambda, \Theta \rangle := \langle \Lambda, (\mathcal{F}h_k)^* \Theta \rangle$. We note then that

$$\begin{aligned}
 & \left\langle \left[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)] \Xi_k, \zeta(s) \right] \right\rangle \\
 &= \left\langle \Xi_k, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \zeta(s) \right\rangle \\
 &= \left\langle \Lambda_k \mathcal{F}h_k, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \zeta(s) \right\rangle \\
 &= \left\langle \Lambda_k, (\mathcal{F}h_k)^*(s) \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \zeta(s) \right\rangle \\
 &= \int_{\mathcal{H}_k} (\mathcal{F}h_k)^*(s) \zeta(s) ds \\
 &= \int_{\mathbb{R}^n} (\mathcal{F}h_k)^*(\sigma) \zeta(\sigma) d\sigma = \langle \mathcal{F}h_k, \zeta \rangle.
 \end{aligned} \tag{3.14}$$

Now, suppose it is possible to select the functionals Λ_k (through a suitable choice of the staircases H_k) so as to ensure [weak] convergence of the series

$$\Xi = \sum_{k=0}^{\infty} \Xi_k \tag{3.15}$$

in $\mathcal{Q}'$. In this event we see that Ξ represents a solution for $\mathcal{F}F$ in equation (3.10), since

$$\begin{aligned}
 \left[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)] \Xi \right] &= \sum_{k=0}^{\infty} \left[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)] \Xi_k \right] \\
 &= \sum_{k=0}^{\infty} \mathcal{F}h_k \quad (\text{from (3.14)}) \\
 &= \mathcal{F}h,
 \end{aligned}$$

by supposition (note that $[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] : Q' \rightarrow \mathcal{Z}'$ is weakly continuous, thus justifying the first step in this last string of equalities).

So, our problem becomes one of finding the correct expansion (3.11) of functionals Λ_k so as to ensure the weak convergence of the expansion in (3.15).

First, we explain how a Hörmander's staircase can be "warped" to affect certain properties of the solution (Shilov [7, pp.178-179]). In previously constructed staircases, if we fixed $\sigma_2, \dots, \sigma_n$ then the corresponding cross section of the staircase was a line in the s_1 -plane parallel to the real axis. We can choose the cross sections differently. For example, consider the line Γ in the s_1 -plane defined by

$$\tau_1 = (d_1 \log^+ |\sigma_1| - d_2)^+ \quad (3.16)$$

for constants d_1 & d_2 , where $d_1 > 0$, and we use $f^+(x) = \max\{f(x), 0\}$. Examples of these curves for various values of d_2 are shown in figure 6.

Consider the strip $|\tau_1 - d_1 \log^+ |\sigma_1|| \leq 3m+1$, as shown in figure 7. By suitable shifting we can find a curve Γ_j of the form (3.16) on which $|\left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s)\right)| > c > 0$ on the whole curve ($s_2, \dots, s_n$ fixed). In this case Hörmander's staircase $\mathcal{H}$ is defined to be the set of points $(\sigma_1, \dots, \sigma_n, \tau_n)$ such that $s_1 = \sigma_1 + i\tau_1 \in \Gamma_j$ if $(\sigma_2, \dots, \sigma_n) \in \Delta_j$, where the Δ_j are as described in, say, Lemma 3.19. We give the

Definition 3.24 We define the linear form Υ on Q according to

$$\langle \Upsilon, \Theta \rangle := \quad (3.17)$$

$$\int_{\mathcal{H}} \frac{\Theta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\bar{P}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \bar{Q}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} ds_1 d\sigma_2 \dots d\sigma_n.$$

Remark 3.25 The integral exists, and defines $\Upsilon \in Q'$. First of all, the denominator exceeds c in magnitude. The following inequality holds for s on $\mathcal{H}$, for any k and $|\sigma_1|$

Figure 6. Examples of the curves $\tau_1 = (d_1 \log^+ |\sigma_1| - d_2)^+$ for various d_2

Figure 7. Strip within which Hörmander's staircase cross-section Γ_j lies.

large enough

$$\begin{aligned}
 |s_1^k \sigma_2^2 \dots \sigma_n^2 \Theta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)| &\leq C_{k,2} e^{b_1 |\tau_1|} \\
 &\leq C_{k,2} e^{b_1 (d_1 \log^+ |\sigma_1| + (3m+1))} \\
 &\leq C'_{k,2} |\sigma_1|^{b_1 d_1}.
 \end{aligned} \tag{3.18}$$

So, the integral in (3.17) exists if we consider these inequalities with, say, $k > b_1 d_1 + 1$.

To show that Υ is continuous we then proceed as in remark 3.16.

Proposition 3.26 Υ , defined above, is a solution for $\mathcal{FF}$ in the equation

$$[[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] \mathcal{FF} = 1 \text{ in } \mathcal{Z}'$$

Proof We have

$$\begin{aligned}
 \left\langle [[P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)]] \Upsilon, \zeta \right\rangle &= \left\langle \Upsilon, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \zeta(s) \right\rangle \\
 &= \int_{\mathcal{H}} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) ds_1 d\sigma_2 \dots d\sigma_n.
 \end{aligned}$$

The proof follows if we can replace the inner integral (with respect to s_1) in this last term by a real axis integral. Thus, we must show that the integral along the vertical line segment,

$$\int_0^{d_1 \log^+ |\sigma_1| - d_2} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\tau_1$$

tends to zero as $|\sigma_1| \rightarrow \infty$. We realise that a string of inequalities such as in (3.18) holds for ζ as well as Θ . Therefore, for $|\sigma_1|$ large enough,

$$\begin{aligned}
 \left| \int_0^{d_1 \log^+ |\sigma_1| - d_2} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\tau_1 \right| &\leq (d_1 \log^+ |\sigma_1| + |d_2|) C'_{k,2} |\sigma_1|^{b_1 d_1 - k} \\
 &\leq C''_{k,2} |\sigma_1|^{b_1 d_1 + 1 - k},
 \end{aligned}$$

for any k . Choosing k large enough gives us what we want, so that

$$\int_{\mathcal{H}} \zeta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) ds_1 d\sigma_2 \dots d\sigma_n = \int_{\mathbb{R}^n} \zeta(\sigma_1, \dots, \sigma_n) d\sigma,$$

and the proof follows.

Remark 3.27 Shilov [7, pp.179] comments that it is possible, when defining Hörmander's staircase $\mathcal{H}$, to extend the staircase into the complex space of the other variables (not just s_1 as above). Warped staircases, such as the one above, only in several variables, will be useful in defining the Λ_k in (3.13), so as to ensure the weak convergence of (3.15). The method we shall use to choose a proper decomposition (3.11), and suitable warped staircases in (3.13), is the same as that used by Shilov [7, pp.181-183] in the case of the equation $P(D)u = f$ in $\mathcal{D}'$, *mutatis mutandis*. As it is somewhat technical we shall merely outline the steps.

Let $\{\gamma_k\}$ be a sequence of positive numbers tending to infinity. We decompose h , as in (3.11), so that the term h_0 has support in a ball centred at the origin, and for $k \geq 1$ the support of each functional h_k falls inside the volume $O_k := \{x : x_j \leq -\gamma_k, \text{ for } j = 1, 2, \dots, n\}$ if a suitable rotation is performed (that such a decomposition is possible is described by Shilov [7, pp.80-81]).

First we assume that the support of h_k naturally lies in O_k ($k > 0$). We then construct a staircase in the strip

$$|\tau_j - \rho_k \log^+ |\sigma_j|| \leq 3m + 1 \quad \text{for } j = 1, 2, \dots, n,$$

where the ρ_k are positive constants to be chosen later (the construction of such a staircase is essentially the same as in Lemma 3.19, with the difference being that initially we set $s_2, \dots, s_n$ as arbitrary constants in the above strips, instead of on the real axes as in Lemma 3.19. All we want to do is bound $|\left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s)\right)|$ away from zero on the staircase). We can obtain bounds on the magnitudes of the terms $(\mathcal{F}h_k)^*(s)$, $\Theta(s)$, and $\left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s)\right)$ on the warped staircase. These estimates, in turn,

allow us to estimate the magnitude of

$$\langle \Xi_k, \Theta \rangle := \int_{\mathcal{H}_k} \frac{(\mathcal{F}h_k)^*(s)\Theta(s)}{\left(\overline{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \overline{Q}(s)\right)} ds.$$

From this estimate we can show that if γ_k is chosen properly, then, for sufficiently large k , $|\langle \Xi_k, \Theta \rangle| \leq \text{constant}/k^2$, where the constant depends on Θ but not on k .

Now we suppose that the support of h_k lies within O_k only after a suitable coordinate rotation U_k is performed, so that

$$\operatorname{supp}(h_k(U_k y)) \subset O_k = \{y : y_j \leq \gamma_k, \text{ for } j = 1, 2, \dots, n\}.$$

Writing $(h_k)_U(x) = h_k(Ux)$, the previous argument shows that there exists Hörmander's staircase $\mathcal{H}_k$ on which

$$\left| \int_{\mathcal{H}_k} \frac{(\mathcal{F}(h_k)_U)^*(s)\Theta(U_k s)}{\overline{P}(U_k s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \langle U_{kj}, \sigma \rangle) \overline{Q}(U_k s)} ds \right| \leq \frac{\text{constant}}{k^2}$$

for any $\Theta \in \mathcal{Q}$, for sufficiently large k . From previous work (the beginning of §3.3) we realise that $\mathcal{F}(h_k)_U = (\mathcal{F}h_k)_U$, where $(\mathcal{F}h_k)_U(s) = (\mathcal{F}h_k)(U_s)$. Substituting $s' = U_k s$ in the integral over $\mathcal{H}_k$ we get an integral over $U_k^{-1}\mathcal{H}_k$. Thus, for sufficiently large k we have

$$\left| \int_{U_k^{-1}\mathcal{H}_k} \frac{(\mathcal{F}h_k)^*(s')\Theta(s')}{\overline{P}(s') + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j') \overline{Q}(s')} ds' \right| \leq \frac{\text{constant}}{k^2}.$$

Thus, for each h_k in our chosen expansion, (3.11) of h , where $k > 0$, we have found a warped staircase ($\mathcal{H}_k$ or $U_k^{-1}\mathcal{H}_k$, depending on whether a rotation U_k is needed to put the support of h_k into O_k , defined above) on which

$$|\langle \Xi_k, \Theta \rangle| \leq \frac{\text{constant}}{k^2}$$

for any $\Theta \in \mathcal{Q}$ (for sufficiently large k). So $\sum_{k=0}^{\infty} \langle \Xi_k, \Theta \rangle$ converges absolutely, $\sum_{k=0}^{\infty} \Xi_k$ converges weakly, and we are guaranteed a solution of (3.10) for any choice of $h \in \mathcal{D}'$.

§3.5 The Equation $P(D, H)F = G$ in $\mathcal{G}'$

Case (a) We first consider a rather simple case of the above equation, namely, when $G \in \mathcal{E}'$, the space of distributions with compact support (from remark 1.29 we realise that $\mathcal{E}' \subset \mathcal{D}' \cap H'(\mathcal{D})$, and so we can consider $\mathcal{E}'$ as a subspace of $\mathcal{G}'$). That is, we interpret $G \in \mathcal{G}'$ as $\chi(g, g)$, where $g \in \mathcal{E}'$ and χ is the canonical isomorphism from $\mathcal{D}' \times H'(\mathcal{D})$ to $\mathcal{G}'$.

From previous considerations, the equation reduces to $(P(D) + Q(D)H)F = G$. We shall assume that P & Q are such that $\bar{P}^2(s) - (-\pi^2)^n \bar{Q}^2(s)$ can be written as in equation (3.4). If not, we realise that a rotation of coordinate axes can bring the latter polynomial into the desired form. Fourier transforming our equation gives

$$(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)) \mathcal{F}F = \mathcal{F}G \text{ in } \mathcal{Q}'. \quad (3.19)$$

From remark 2.15, we can identify $\mathcal{F}G$ in $\mathcal{Q}'$ with $\chi_{\mathcal{F}}(\mathcal{F}g, \mathcal{F}g)$, where $\chi_{\mathcal{F}}$ is the canonical isomorphism from $\mathcal{Z}' \times \mathcal{J}'$ to $\mathcal{G}'$. As noted in §3.4, $\mathcal{F}g$ is an entire function (in each of $s_1, \dots, s_n$) having first order exponential growth in the imaginary domain and power growth in the real domain (Shilov [7, pp.151-152, eqn (17)]). Therefore $\mathcal{F}g$ is a multiplier in $\mathcal{Q}$, in the sense that $(\mathcal{F}g)^* \Theta \in \mathcal{Q}$ for all $\Theta \in \mathcal{Q}$, and $\mathcal{F}G$ is a multiplier in $\mathcal{Q}'$, in the sense that $\langle (\mathcal{F}G)\Lambda, \Theta \rangle := \langle \Lambda, (\mathcal{F}g)^* \Theta \rangle$ for $\Lambda \in \mathcal{Q}'$ (where $(\mathcal{F}g)^*(s) := \overline{\mathcal{F}g(\bar{s})}$, an overbar denoting complex conjugation). We note that $(\mathcal{F}g)^*$ can be considered as a multiplier in the sense introduced in §3.3, where we defined "multiplication" of elements of $\mathcal{Q}_U$ by terms such as $(\bar{P}_U(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \bar{Q}_U(s))$; that is, we can identify $(\mathcal{F}g)^*$ with $((\mathcal{F}g)^* + \prod_{j=1}^n (i\pi \operatorname{sgn} \langle U_j, \sigma \rangle) \cdot 0)$.

In $\mathcal{Q}'$ we define $\Xi := (\mathcal{F}G)\Lambda$, where $\Lambda \in \mathcal{Q}'$ is given in definition (3.15):

$$\langle \Lambda, \tilde{\Theta} \rangle :=$$

$$\begin{aligned}
& \int_{\mathcal{H}} \left\{ \frac{\alpha(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \bar{P}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\bar{P}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \bar{Q}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} \right. \\
& \quad \left. - (\pi^2)^n \frac{\beta(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) \bar{Q}(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)}{\bar{P}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) - (-\pi^2)^n \bar{Q}^2(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n)} \right\} d\sigma_1 d\sigma_2 \dots d\sigma_n \\
& + (i\pi)^n \int_{\mathcal{H}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\sigma_1 d\sigma_2 \dots d\sigma_n \\
& + 2(i\pi)^n \sum_{I_k \in \text{PLUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, \sigma_1 \in I_k}} \int_0^{\tau_1^{(j)}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\sigma_1 d\sigma_2 \dots d\sigma_n \\
& - 2(i\pi)^n \sum_{I_k \in \text{MINUS}} \sum_{\Delta_j} \int_{\substack{(\sigma_1, \dots, \sigma_n) \\ (\sigma_2, \dots, \sigma_n) \in \Delta_j, \sigma_1 \in I_k}} \int_0^{\tau_1^{(j)}} W(\sigma_1 + i\tau_1, \sigma_2, \dots, \sigma_n) d\sigma_1 d\sigma_2 \dots d\sigma_n,
\end{aligned}$$

where W is as defined immediately following equation (3.5), and where we use $\tilde{\Theta} \in \mathcal{Q}$ writing $\tilde{\Theta}(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s)$. The explanations of the various terms above are as in definition 3.15.

We claim that Ξ defined above is a solution for $\mathcal{FF}$ in equation (3.19). To see this consider

$$\begin{aligned}
& \left\langle \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Xi, \Theta \right\rangle \\
& = \left\langle (\mathcal{F}G)\Lambda, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta \right\rangle \\
& = \left\langle \Lambda, (\mathcal{F}g)^*(s) \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta \right\rangle \\
& = \langle \Lambda, (\mathcal{F}g)^*(s) \tilde{\Theta}(s) \rangle \\
& = \int_{\mathbb{R}^n} (\mathcal{F}g)^*(\sigma) \mathcal{F}\phi(\sigma) d\sigma \\
& \quad + \int_{\mathbb{R}^n} (\mathcal{F}g)^*(\sigma) \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(\sigma) d\sigma \\
& = \langle (\mathcal{F}G), \Theta \rangle,
\end{aligned}$$

where the transition from the third to the second last step, in this string of equalities, follows (with much previous effort) from the proof of proposition 3.16 (we have used $\Theta \in \mathcal{Q}$ given by $\Theta(s) = \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s)$, and $\tilde{\Theta}(s) = \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta(s)$).

Case (b) We can consider the equation $P(D,H)F=G$ when $G \in \mathcal{D}'$, if we interpret $G=\chi(g,0)$ for some $g \in \mathcal{D}'$ ($\chi : \mathcal{D}' \times H'(\mathcal{D}) \rightarrow \mathcal{G}'$ is the canonical isomorphism). From previous considerations the equation reduces to $(P(D) + Q(D)H)F = G$.

Fourier transforming our equation gives

$$\left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s)\right) \mathcal{F}F = \mathcal{F}G \text{ in } \mathcal{Q}', \quad (3.20)$$

where we identify $\mathcal{F}G = \chi_{\mathcal{F}}(\mathcal{F}g, 0)$. As in §3.4, we have the representation

$$g = \sum_{k=0}^{\infty} g_k \quad (3.21)$$

in $\mathcal{D}'$, where the g_k are distributions with compact support. It is straightforward to show that the [weak] topology on $\mathcal{Q}'$ is the finest one making $\chi_{\mathcal{F}}$ continuous (assuming $\mathcal{Z}' \times \mathcal{J}'$ to have the product topology of the weak topologies on $\mathcal{Z}'$ & $\mathcal{J}'$). Therefore we have the convergent expansion $\mathcal{F}G = \sum_{k=0}^{\infty} \mathcal{F}G_k$ in $\mathcal{Q}'$, where $\mathcal{F}G_k := \chi_{\mathcal{F}}(\mathcal{F}g_k, 0)$.

We now sidetrack for a brief discussion on multipliers of generalized functions. Multiplication of elements of $\mathcal{Z}'$ by a function $\varpi(s)$ is defined formally by

$$\langle \varpi(s) \mathcal{F}f, \zeta(s) \rangle := \langle \mathcal{F}f, \varpi^*(s) \zeta(s) \rangle,$$

and is valid for all functions ϖ such that $\varpi^* \zeta \in \mathcal{Z}$ for all $\zeta \in \mathcal{Z}$ (Shilov [7, pp.140]). In particular then, we see that the properties of $\mathcal{F}g_k$ make it a multiplier of $\mathcal{Z}'$.

Definition 3.28 We define $\mathcal{M}$ to be the space of all pairs of functions $(\varpi_{\alpha}, \varpi_{\beta})$ satisfying the condition that $(\varpi_{\alpha}^*, \varpi_{\beta}^*) \diamond \Theta \in \mathcal{Q}$ for all $\Theta \in \mathcal{Q}$, where we define the operation $\diamond$ according to

$$(\varpi_{\alpha}^*, \varpi_{\beta}^*) \diamond \Theta(s) := \varpi_{\alpha}^* \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \varpi_{\beta}^*(s) \beta(s)$$

for $\Theta(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s)$. For $(\varpi_{\alpha}, \varpi_{\beta}) \in \mathcal{M}$ we can thus define the mapping $(\varpi_{\alpha}, \varpi_{\beta}) \diamond \mathcal{Q}' \rightarrow \mathcal{Q}'$ according to

$$\langle (\varpi_{\alpha}, \varpi_{\beta}) \diamond \Lambda, \Theta \rangle := \langle \Lambda, (\varpi_{\alpha}^*, \varpi_{\beta}^*) \diamond \Theta \rangle$$

[we note then that $(\varpi_\alpha, \varpi_\beta) \diamond \Lambda \in \mathcal{Q}'$, for all $\Lambda \in \mathcal{Q}'$, whenever ϖ_α & ϖ_β are multipliers of $\mathcal{Z}'$]

With this definition at hand we define

$$\Xi_k := (\mathcal{F}g_k, 0) \diamond \Lambda_k, \quad (3.22)$$

where $\Lambda_k \in \mathcal{Z}'$ is unspecified for now. Following the line of reasoning in §3.4, we should like to choose the Λ_k in (3.22) so as to ensure the [weak] convergence of $\sum_{k=0}^{\infty} \Xi_k$ in $\mathcal{Q}'$, and to have

$$\left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Xi_k = \mathcal{F}G_k$$

in $\mathcal{Q}'$. We have

$$\begin{aligned} & \left\langle \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Xi_k, \Theta \right\rangle \\ &= \left\langle (\mathcal{F}g_k, 0) \diamond \Lambda_k, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta(s) \right\rangle \\ &= \left\langle (\mathcal{F}g_k, 0) \diamond \Lambda_k, \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s) \right\rangle \\ &= \left\langle \Lambda_k, ((\mathcal{F}g_k)^*, 0) \diamond \left(\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s) \right) \right\rangle \\ &= \left\langle \Lambda_k, ((\mathcal{F}g_k)^*, 0) \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \cdot 0 \right\rangle, \end{aligned}$$

where $\alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s) := \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta(s)$, and $\Theta(s) = \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s)$. From previous working (remark 3.10) we have

$$\alpha(s) = \bar{P}(s) \mathcal{F}\phi(s) + (\pi^2)^n \bar{Q}(s) \mathcal{F}\omega(s).$$

Our problem then is to find Λ_k in (3.22) so as to ensure convergence of $\sum_{k=0}^{\infty} \Xi_k$, and

to satisfy

$$\begin{aligned}\langle \Lambda_k, (\mathcal{F}g_k)^*(s)\alpha(s) \rangle &= \int_{\mathbb{R}^n} (\mathcal{F}g_k)^*(\sigma)\alpha(\sigma)d\sigma \\ &= \langle \mathcal{F}g_k, \mathcal{F}\phi \rangle + \langle 0, \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(\sigma) \rangle \\ &= \langle \mathcal{F}G_k, \Theta \rangle\end{aligned}$$

(recall, from remark 2.15, that this is how we define $\mathcal{F}G_k = \chi_{\mathcal{F}}(\mathcal{F}g_k, 0)$). Unfortunately, we are unable to find such a Λ_k .

There is another approach we might take. We recall some concepts from §3.2. For polynomials P & Q we defined the continuous operator

$$\left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) : \mathcal{Q} \rightarrow \mathcal{Q}$$

by

$$\left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \left(\mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s) \right) :=$$

$$\bar{P}(s)\mathcal{F}\phi(s) + (\pi^2)^n \bar{Q}(s)\mathcal{F}\omega(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \left(\bar{P}(s)\mathcal{F}\omega(s) + (-1)^n \bar{Q}(s)\mathcal{F}\phi(s) \right),$$

for $\Theta(s) := \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s)$, and hence the continuous operator

$$\left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) : \mathcal{Q}' \rightarrow \mathcal{Q}'$$

by

$$\begin{aligned}\left\langle \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Lambda, \Theta \right\rangle \\ := \left\langle \Lambda, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) \Theta \right\rangle.\end{aligned}$$

Clearly we can extend this definition to operators of the form

$$\left(A(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B(s) \right) : \mathcal{Q}' \rightarrow \mathcal{Q}'$$

where A and B are any two functions such that $A^*\zeta \in \mathcal{Z}$ and $B^*\zeta \in \mathcal{Z}$ for all $\zeta \in \mathcal{Z}$ (that is, for all A & B that are multipliers of $\mathcal{Z}$). Moreover, we can check that these operators commute:

$$\begin{aligned} \left(A(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B(s) \right) \left(C(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) D(s) \right) \Lambda \\ = \left(C(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) D(s) \right) \left(A(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B(s) \right) \Lambda \end{aligned}$$

for all $\Lambda \in \mathcal{Z}'$.

Now we define

$$\Xi_k = \left(A(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B(s) \right) \Lambda_k, \quad (3.23)$$

where A & B are unspecified for now, and where $\Lambda_k \in \mathcal{Q}'$ is given by

$$\langle \Lambda_k, \tilde{\Theta} \rangle := \int_{\mathcal{H}_k} \frac{\alpha(s)}{\overline{P}(s)} ds$$

for some Hörmander's staircase $\mathcal{H}_k$ (assuming P is in normal form), where $\tilde{\Theta}(s) = \alpha(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \beta(s)$. Consider

$$\begin{aligned} \left\langle \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Xi_k, \Theta \right\rangle \\ = \left\langle \left(A(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B(s) \right) \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Lambda_k, \Theta \right\rangle. \end{aligned}$$

Suppose that we could choose A^* & B^* so that

$$\left(A^*(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) B^*(s) \right) \Theta(s) = (\mathcal{F}g_k)^*(s) \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \cdot 0 \quad (3.24)$$

for all $\Theta \in \mathcal{Q}$, where $\Theta(s) = \mathcal{F}\phi(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}\omega(s)$. In this event we see that

$$\begin{aligned}
 & \left\langle \left(P(s) + \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) Q(s) \right) \Xi_k, \Theta \right\rangle \\
 &= \left\langle \Lambda_k, \left(\bar{P}(s) + \prod_{j=1}^n (-i\pi \operatorname{sgn} \sigma_j) \bar{Q}(s) \right) (\mathcal{F}g_k)^*(s) \mathcal{F}\phi(s) \right\rangle \\
 &= \int_{\mathcal{H}_k} (\mathcal{F}g_k)^*(s) \mathcal{F}\phi(s) ds \\
 &= \int_{\mathbb{R}^n} (\mathcal{F}g_k)^*(\sigma) \mathcal{F}\phi(\sigma) d\sigma \\
 &= \langle \mathcal{F}G_k, \Theta \rangle,
 \end{aligned}$$

which is what we want. However, we can check that it is impossible to fix A^* & B^* so that (3.24) is satisfied for all $\Theta \in \mathcal{Q}$.

Case (c) We consider the equation $P(D,H)F=G$ for arbitrary $G \in \mathcal{G}'$. The equation becomes $(P(D)+Q(D)H)F=G$. If we follow the line of reasoning in §3.4 it would be natural to write $G = \chi(g^{(\alpha)}, Hg^{(\beta)})$, where $g^{(\alpha)}, g^{(\beta)} \in \mathcal{D}'$, and decompose $g^{(\alpha)} = \sum_{k=0}^{\infty} g_k^{(\alpha)}$, $g^{(\beta)} = \sum_{k=0}^{\infty} g_k^{(\beta)}$, where $\{g_k^{(\alpha)}\}$ and $\{g_k^{(\beta)}\}$ are sequences of distributions with compact support. Therefore, $\mathcal{F}G = \sum_{k=0}^{\infty} \mathcal{F}G_k$, where $\mathcal{F}G_k = \chi_{\mathcal{F}}(\mathcal{F}g_k^{(\alpha)}, \mathcal{F}Hg_k^{(\beta)})$. We see immediately that

$$(\mathcal{F}g_k^{(\alpha)}, \mathcal{F}Hg_k^{(\beta)}) \circ \Lambda$$

makes no sense for $\Lambda \in \mathcal{Q}$, since $\mathcal{F}Hg_k^{(\beta)} (= \prod_{j=1}^n (i\pi \operatorname{sgn} \sigma_j) \mathcal{F}g_k^{(\beta)})$ is certainly not a multiplier of $\mathcal{J}'$.

It would seem that a different approach is necessary.

§3.6 Some Further Questions

In the one dimensional case, interpreting $(P(D) + Q(D)H)F = \delta$ in $\mathcal{G}'$, we should like to know if the condition $\bar{P}^2(0) + \pi^2 \bar{Q}^2(0) \neq 0$ is necessary for constructing a solution Λ in

$\mathcal{Q}'$ for the Fourier transformed equation (see definition 2.26 and proposition 2.28).

In the multidimensional case we should like to know how much (if any) condition (remark 3.14) can be relaxed in order that we can still construct solutions in $\mathcal{Q}'$ for the Fourier transformed version of $(P(D) + Q(D)H)F = \delta$. Again, what condition is necessary for construction of solutions? Regardless of [non-pathological] conditions on P & Q , how are we to construct transform domain solutions for $(P(D) + Q(D)H)F = G$ in $\mathcal{G}'$, when G is arbitrary (§3.5 Case (c)), or even when $G = \chi(g, 0)$ for $g \in \mathcal{D}'$ (§3.5 Case (c))?

Having settled these questions, it would be worthwhile considering boundary value problems. That is, we investigate equations of the form

$$\frac{\partial^m F(t, x)}{\partial t^m} = \sum_{k=0}^{m-1} P_k(D, H) \frac{\partial^k F(t, x)}{\partial t^k}, \quad x = (x_1, \dots, x_n),$$

where H is a Hilbert transform operator in the x variables, and where we require a solution to satisfy certain initial conditions imposed on $F(0, x), \partial F(0, x)/\partial t, \dots, \partial^{m-1} F(0, x)/\partial t^{m-1}$. This will require a modified form of the space $\mathcal{G}$.

§3.7 Appendix

In this appendix we shall give a proof of the result, stated in the proof of Lemma 3.19, that the roots of a polynomial with a constant leading coefficient depend continuously on the remaining coefficients. We are indebted to Professor John Dixon for pointing out this simple proof. To conform somewhat with the notation of Lemma 3.19 we shall consider the normal form polynomial $P(s_1, s_2, \dots, s_n) = as_1^\nu + \sum_{k=0}^{\nu-1} P_k(s_2, \dots, s_n)s_1^k$, where $a \neq 0$. For arbitrary fixed $s_2 = \sigma_{20}, \dots, s_n = \sigma_{n0}$ we consider the ν zeros $s_1^{(1)}, \dots, s_1^{(\nu)}$ of $P(s_1, \sigma_{20}, \dots, \sigma_{n0})$ (as a polynomial in s_1). Multiple roots are counted separately. We consider disjoint circles in the s_1 -plane, all of the same radius ϵ , centred on the

distinct roots of $P(s_1, \sigma_2, \dots, \sigma_n)$. Clearly, we can find some constant $M > 0$ such that $|P(s_1, \sigma_2, \dots, \sigma_n)| \geq M$ for s_1 on [but not necessarily within] each circle. We can choose $\delta > 0$ such that, for all $\sigma_2, \dots, \sigma_n$ satisfying $\{(\sigma_2 - \sigma_{2_0})^2 + \dots + (\sigma_n - \sigma_{n_0})^2\}^{1/2} < \delta$, we have

$$\begin{aligned} & |P(s_1, \sigma_2, \dots, \sigma_n) - P(s_1, \sigma_{2_0}, \dots, \sigma_{n_0})| \\ &= \left| \sum_{k=0}^{\nu-1} (P_k(\sigma_2, \dots, \sigma_n) - P_k(\sigma_{2_0}, \dots, \sigma_{n_0})) s_1^k \right| \\ &\leq (\nu - 1) \max_{0 \leq k \leq \nu-1} |P_k(\sigma_2, \dots, \sigma_n) - P_k(\sigma_{2_0}, \dots, \sigma_{n_0})| \cdot \text{constant} \\ &\leq M/2 \end{aligned}$$

for any s_1 lying on one of our circles. Therefore

$$|P(s_1, \sigma_2, \dots, \sigma_n) - P(s_1, \sigma_{2_0}, \dots, \sigma_{n_0})| < |P(s_1, \sigma_{2_0}, \dots, \sigma_{n_0})|$$

for s_1 lying on any circle. $P(s_1, \sigma_2, \dots, \sigma_n)$ and $P(s_1, \sigma_{2_0}, \dots, \sigma_{n_0})$ are both analytic in s_1 , so that Rouché's theorem tells us that both polynomials have the same number of zeros within each of our circles. The radius of the circles was arbitrary, so we have the result that the roots of $P(s_1, \sigma_2, \dots, \sigma_n)$, in the s_1 -plane, depend continuously on the point $(\sigma_2, \dots, \sigma_n)$ in $\mathbb{R}^{n-1}$, and our proof follows.

We note that this approach may be bypassed using a different method (see Trèves [10]).

CHAPTER 4. ON THE NON-NUCLEARITY OF A SPACE OF TEST FUNCTIONS ON HILBERT SPACE

§4.1 Description of Dudin's Space $\mathcal{D}(\mathcal{H})$

Consider the vector space $C(\mathcal{H}, E)$, consisting of all continuous functions on the real separable Hilbert space $\mathcal{H}$ with a bounded range in the real locally convex space E . Let $\Psi(\mathcal{H})$ be the family of all functions $\psi : \mathcal{H} \rightarrow (0, \infty]$ satisfying the condition that the set $\{x : x \in \mathcal{H}, \psi(x) \leq k\}$ is compact for any $k \in \mathbb{R}$. This last condition implies that every member of $\Psi(\mathcal{H})$ is lower semicontinuous and takes its own least value, so that $\inf_{x \in \mathcal{H}} \psi(x) > 0$ for each $\psi \in \Psi(\mathcal{H})$. We topologise $C(\mathcal{H}, E)$ by the family of seminorms

$$p_{\psi, q}(\phi) := \sup_{x \in \mathcal{H}} \frac{q(\phi(x))}{\psi(x)},$$

where $\psi \in \Psi(\mathcal{H})$ and q is any continuous seminorm on E . We set $a/\infty = 0$ for any $a \in \mathbb{R}$.

Let $\mathcal{H}^k$ denote the complete projective tensor product of k copies of $\mathcal{H}$. The dual of $\mathcal{H}^k$ is the space $\mathcal{U}_k$ of all k -linear forms on $\mathcal{H} \times \dots \times \mathcal{H}$ (k factors) that are continuous in all the variables. The function $\|F\|_{\mathcal{U}_k} := \sup_{\|x_i\| \leq 1} F(x_1, \dots, x_k)$, for $F \in \mathcal{U}_k$, is a norm on $\mathcal{U}_k$. Dudin's work is valid for $\mathcal{U}_k$ with any locally convex topology lying inclusively between the weak topology $\sigma(\mathcal{U}_k, \mathcal{H}^k)$ and the strong topology $\beta(\mathcal{U}_k, \mathcal{H}^k)$. We shall specifically choose the latter, which is to say that $\mathcal{U}_k$ is normed as just described. By a derivative, unless otherwise specified, we shall mean a weak derivative.

The k -multiple differentiation operator can be considered as an operator from $C(\mathcal{H}, \mathbb{R})$ to $C(\mathcal{H}, \mathcal{U}_k)$. By $C^k(\mathcal{H}, \mathbb{R})$ we denote the intersection of the domains of definition of the operators D^l , $l = 0, 1, \dots, k$, and we equip $C^k(\mathcal{H}, \mathbb{R})$ with the weakest topology in which all of these operators are continuous. That is, we topologise $C^k(\mathcal{H}, \mathbb{R})$ by the family of

seminorms

$$p_{\psi,k}(\phi) := \sup_{\substack{x \in \mathcal{H} \\ 0 \leq j \leq k}} \frac{\|D^j \phi(x)\|_{\mathcal{U}_k}}{\psi(x)}.$$

We denote $\mathcal{U}_0 \equiv \mathcal{H}^0 := \mathcal{R}$, $C^0(\mathcal{H}, \mathcal{R}) = C(\mathcal{H}, \mathcal{R})$, and let D^0 denote the identity operator. We set $C^\infty(\mathcal{H}, \mathcal{R}) := \lim \text{proj} C^k(\mathcal{H}, \mathcal{R})$, so that $C^\infty(\mathcal{H}, \mathcal{R})$ is topologised by the family of seminorms

$$p_{\psi,k}(\phi) := \sup_{\substack{x \in \mathcal{H} \\ 0 \leq j \leq k}} \frac{\|D^j \phi(x)\|_{\mathcal{U}_k}}{\psi(x)} \quad k = 0, 1, 2, \dots$$

We note that since $D^k : C(\mathcal{H}, \mathcal{R}) \rightarrow C(\mathcal{H}, \mathcal{U}_k)$, we have by assumption that $D^k \phi$ has bounded range in $\mathcal{U}_k$. From this it can be shown that $C^\infty(\mathcal{H}, \mathcal{R})$ consists, in fact, of the infinitely Fréchet differentiable functions, all of whose derivatives have bounded range.

We let B_r denote the closed ball $\{x : x \in \mathcal{H}, \|x\| \leq r\}$ and let $C_{0,r}^\infty(\mathcal{H})$ denote the subspace of $C^\infty(\mathcal{H}, \mathcal{R})$ consisting of those functions vanishing outside B_r . We set $\mathcal{D}(\mathcal{H}) := \lim \text{ind} C_{0,n}^\infty(\mathcal{H})$. It follows from the characterisation of $C^\infty(\mathcal{H}, \mathcal{R})$ that we can show that $\mathcal{D}(\mathcal{H})$ consists of all the infinitely Fréchet differentiable functions with bounded support, all of whose derivatives have bounded range.

§4.2 The Non-Nuclearity of $\mathcal{D}(\mathcal{H})$

To establish the non-nuclearity of $\mathcal{D}(\mathcal{H})$ we shall need some auxiliary results. For any locally convex space (l.c.s.) F , a linear map $u : \mathcal{D}(\mathcal{H}) \rightarrow F$ is continuous iff the restriction $u_n : C_{0,n}^\infty(\mathcal{H}) \rightarrow F$ is continuous for each n . We shall use the following without proof (see Schaefer [6, pp. 101])

Lemma 4.1 *If E is a l.c.s. then E is nuclear iff every continuous linear map of E into any Banach space is nuclear.*

Lemma 4.2 Let E, F, G , & H be l.c.s. and let u, v , & w be continuous linear maps from E to F , F to G , and G to H , respectively. If v is a nuclear map from F to G then $v \circ u$ and $w \circ v$ (hence $w \circ v \circ u$) are nuclear maps.

We use these results as follows. If we can show that $C_{0,n}^\infty(\mathcal{H})$ is not nuclear for all n , then Lemma 1 tells us there exists some Banach space F and some continuous linear map $u_n : C_{0,n}^\infty(\mathcal{H}) \rightarrow F$ which is not nuclear. If $i_n : C_{0,n}^\infty(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ denotes the canonical injection, then the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{D}(\mathcal{H}) & \xrightarrow{u} & F \\ i_n \swarrow & & \nearrow u_n \\ & C_{0,n}^\infty(\mathcal{H}) & \end{array}$$

along with Lemma 2 tells us that u cannot be nuclear, where u is any continuous linear extension of u_n to the whole of $\mathcal{D}(\mathcal{H})$. Thus, to show that $\mathcal{D}(\mathcal{H})$ is non-nuclear it suffices to show that $C_{0,n}^\infty(\mathcal{H})$ is non-nuclear, for arbitrary n .

Proposition 4.3 Dudin's space $C_{0,n}^\infty(\mathcal{H})$ is non-nuclear, for arbitrary n .

Proof (We use the notation of Meise [4]). The strategy of the proof is first to assume that $C_{0,n}^\infty(\mathcal{H})$ is nuclear. From this assumption it will eventually follow that the canonical embedding $B_N : C_0^1([0, 2\pi]^N) \rightarrow L^2([0, 2\pi]^N)$ from the space of 1-times continuously differentiable functions with support contained in the subset $[0, 2\pi]^N$ of $\mathbb{R}^N$, into $L^2([0, 2\pi]^N)$ is of type $\ell^{1/2}$ (see Pietsch [5, pp. 125, 8.2.1]). This will lead to a contradiction.

Suppose that K is an infinite dimensional compact absolutely convex subset of $\mathcal{H}$ that is contained in B_n . We suppose further that K is such that it contains a linearly independent sequence $(x_j)_{j \in \mathbb{N}}$ of mutually orthogonal elements. If we suppose $C_{0,n}^\infty(\mathcal{H})$ to

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be nuclear, then for the continuous seminorm $p_{\psi_K,0}$ on $C_{0,n}^\infty(\mathcal{H})$ there exists a continuous seminorm $p_{\psi,l} \geq p_{\psi_K,0}$ such that the canonical mapping

$$\chi : (C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}} \rightarrow (C_{0,n}^\infty(\mathcal{H}))_{p_{\psi_K,0}}$$

is of type $\ell^{1/2}$ (see Pietsch [5, Theorem 8.6.1, pp. 141]). The notation $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$, for example, refers to the quotient space $C_{0,n}^\infty(\mathcal{H})/p_{\psi,l}^{-1}(0)$ obtained by considering $C_{0,n}^\infty(\mathcal{H})$ as a vector space with a topology generated by the seminorm $p_{\psi,l}$. $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$, with the quotient topology, is then a normed space. The seminorm $p_{\psi_K,0}$ is defined by

$$p_{\psi_K,0}(\phi) := \sup_{x \in \tilde{B}_\infty} \frac{|\phi(x)|}{\psi_K(x)} = \sup_{x \in K} |\phi(x)|,$$

where $\psi \in \Psi(\mathcal{H})$ is defined by $\psi_K(x) = 1$ if $x \in K$, and $= \infty$ if $x \notin K$.

The restriction map

$$\rho_K : C_{0,n}^\infty(\mathcal{H}) \rightarrow C(K), \quad \rho_K(\phi) := f|_K$$

induces the continuous linear map $\tilde{\rho}_K : (C_{0,n}^\infty(\mathcal{H}))_{p_{\psi_K,0}} \rightarrow C(K)$. Maps of the type ℓ^p have the ideal property (see Pietsch [5, pp. 128, 8.2.8]), so that the map $\chi_K := \tilde{\rho}_K \circ \chi$ is also of type $\ell^{1/2}$. Since $C(K)$ is complete, χ_K can be extended to a map $\tilde{\chi}_K$ on the complete hull $(\widehat{C_{0,n}^\infty(\mathcal{H})})_{p_{\psi,l}}$ of $(C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$. The map $\tilde{\chi}_K$ is of type $\ell^{1/2}$ as well.

Figure 8. Mappings between the various spaces so far.

For each $N \in \mathbb{N}$ we denote the linear hull of $\{x_1, \dots, x_N\}$ by E_N . We define Γ_N as the largest closed "cube" such that $2\Gamma_N$ fits inside the closed n -sphere in E_N (so that the corners of Γ_N have coordinates $(\pm \frac{n}{2\|x_1\|\sqrt{N}}, \dots, \pm \frac{n}{2\|x_N\|\sqrt{N}})$ relative to the basis $\{x_1, \dots, x_N\}$ in E_N , and the volume of Γ_N tends to zero as $N \rightarrow \infty$). Consider Γ_N as a closed subset of $\mathcal{H}$, and the complement of the interior of B_n as a closed subset disjoint from Γ_N . Then, by Lemma 3 (see Appendix) there exists a function $\alpha \in C_{0,n}^\infty(\mathcal{H})$ such that $\alpha(x) = 1$ if $x \in \Gamma_N$ (and so $\alpha(x) = 0$ if $x \notin \text{int} B_n$). Let $C_0^\infty(\Gamma_N)$ denote the subspace of $C^\infty(E_N, \mathbb{R})$ consisting of all those functions whose support lies in Γ_N . We note that in this case the topology introduced by Dudin coincides with the usual one of uniform convergence of a function and its derivatives on Γ_N . We can extend a function $f \in C_0^\infty(\Gamma_N)$ to a function in $C_{0,n}^\infty(\mathcal{H})$ via a map P_N as follows:

$$P_N : C_0^\infty(\Gamma_N) \rightarrow C_{0,n}^\infty(\mathcal{H}), \quad P_N(f) := (f \circ \pi_N)(\alpha),$$

where π_N is a continuous linear projection from $\mathcal{H}$ onto E_N (that is, $P_N(f)(x) = (f \circ \pi_N(x))(\alpha(x))$). P_N is a continuous linear map (see Lemma 4 in the Appendix).

It should be noted that this last stage differs from Meise's proof. He is able to construct a canonical continuous linear map $\tilde{\pi}_N : C^\infty(E_N) \rightarrow C_\rho^\infty(\mathcal{H})$ given by $\tilde{\pi}_N(f) := f \circ \pi_N$. This approach is clearly not available here.

We compose P_N with the canonical map $q : C_{0,n}^\infty(\mathcal{H}) \rightarrow (C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$ to get the continuous linear map $\sigma_N := q \circ P_N$. The map σ_N is continuous with respect to the topology on $C_0^\infty(\Gamma_N)$ induced by $C_0^l(\Gamma_N)$; $C_0^\infty(\Gamma_N)$ is dense in $C_0^l(\Gamma_N)$. Thus, σ_N can be continuously extended to get the continuous linear map

$$\bar{\sigma}_N : C_0^l(\Gamma_N) \rightarrow (C_{0,n}^\infty(\mathcal{H}))_{p_{\psi,l}}$$

Therefore, the map $\lambda_N := \tilde{\chi}_K \circ \sigma_N$, where $\lambda_N : C_0^l(\Gamma_N) \rightarrow C(K)$, is of the type $\ell^{1/2}$.

Figure 9. Mappings between the various spaces introduced so far

Define $\Delta_N := \{(t_1, \dots, t_N) \in \mathbb{R}^N : 0 \leq t_i \leq \frac{n}{2\sqrt{N}\|x_i\|}\}$. Then $j_N : \Delta_N \rightarrow \mathcal{H}$, where $j_N(t_1, \dots, t_N) := \sum_{n=1}^N t_n x_n$, is a continuous linear map taking values in $K \cap \Gamma_N$, since K is absolutely convex (in fact, j_N maps Γ_N onto $K \cap \Gamma_N$). Thus, the map

$$\tilde{j}_N : C(K) \rightarrow L^2(\Delta_N), \quad \tilde{j}_N(f) := f \circ j_N$$

is continuous and linear.

The space $C_0^l(\Delta_N)$ of functions in $C^l(\mathbb{R}^N)$ with [compact] support contained in Δ_N is embedded continuously in $C_0^l(\Gamma_N)$ by the inclusion i_N , where

$$i_N(f)\left(\sum_{j=1}^N t_j x_j\right) := f(t_1, \dots, t_N), \quad \text{for } f \in C_0^l(\Gamma_N).$$

Thus, we have the following situation:

Figure 10. Mappings between the various spaces introduced so far.

Therefore the map

$$F_N := \tilde{j}_N \circ \lambda_N \circ i_N : C_0^l(\Delta_N) \rightarrow L^2(\Delta_N)$$

is of type $\ell^{1/2}$. F_N is just the inclusion mapping. Thus, by a stretching argument we see that the canonical embedding

$$B_N : C_0^l([0, 2\pi]^N) \rightarrow L^2([0, 2\pi]^N)$$

is of type $\ell^{1/2}$.

We shall now show that this result is false. Our proof is virtually the same as that of Meise [4], who uses a technique borrowed from Gramsch [2]. To this end consider the Hilbert space

$$H_N := \left\{ (a_N) \in \mathbb{R}^N \mid \sum_{n \in \mathbb{N}^N} \left(\prod_{j=1}^N n_j^2 \right) \left(\sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leq l}} n^{2\alpha} \right) |a_n|^2 < \infty \right\}.$$

Then the map $C_N : H_N \rightarrow C_0^l([0, 2\pi]^N)$ given by

$$C_N((a_n)_{n \in \mathbb{N}^N})(t_1, \dots, t_N) := \sum_{n \in \mathbb{N}^N} a_n \prod_{j=1}^N \sin n_j t_j$$

is continuous and linear. Therefore $D_N : H_N \rightarrow L^2([0, 2\pi]^N)$ is of type $\ell^{1/2}$, where $D_N := B_N \circ C_N$.

We know that mappings of the type ℓ^p are precompact (see Pietsch [4, pp. 127, 8.2.6]) and that precompact mappings from a normed space to a Banach space are in fact compact. Thus $D_N : H_N \rightarrow L^2([0, 2\pi]^N)$ is compact. According to Pietsch [4, pp. 129-30, 8.3.1], for two Hilbert spaces E & F and a compact mapping $T \in \mathcal{L}(E, F)$ there exist two orthonormal systems $(\hat{e}_i)$ and $(\hat{f}_i)$ in E & F , respectively, as well as a family of numbers (λ_i) with $\lambda_i > 0$ such that

$$Tx = \sum_i \lambda_i \langle x, \hat{e}_i \rangle \hat{f}_i \quad \forall x \in E \quad (1).$$

Furthermore, for each compact mapping $T \in \mathcal{L}(E, F)$ the identities

$$\alpha_r(T) = \lambda_r(T) \quad \forall r \in \mathbb{N}$$

are valid, where $\alpha_r(T)$ denotes the r^{th} approximation number of T (see Pietsch [4, pp. 132]).

We apply this knowledge to our system as follows. The functions $f_n : x \rightarrow \prod_{j=1}^N \sin n_j x_j$, for $n \in \mathbb{N}^N$, form an orthogonal system in $L^2([0, 2\pi]^N)$. In H_N we have the orthogonal system $e_n = (\delta_{n,m})_{m \in \mathbb{N}^N}$, where $\delta_{n,m}$ denotes the Kronecker delta. On H_N we have the inner product

$$\langle (a_n), (b_n) \rangle = \sum_{n \in \mathbb{N}^N} \left(\prod_{j=1}^N n_j^2 \right) \left(\sum_{\substack{m \in \mathbb{N}^N \\ |m| \leq l}} n^{2\alpha} |a_n b_n| \right).$$

The $\ell^{1/2}$ [compact] map D_N can be written as

$$\begin{aligned} D_N((a_n)_{n \in \mathbb{N}^N}) &= \sum_{n \in \mathbb{N}^N} a_n f_n \\ &= \sum_{n \in \mathbb{N}^N} \langle (a_n)_{m \in \mathbb{N}^N}, \frac{e_n}{\|e_n\|_{H_N}} \rangle \frac{f_n}{\|f_n\|_{L^2}} \end{aligned}$$

(Comparing this last form with that in (1) we see that

$$\lambda_n = \frac{1}{\|e_n\|_{H_N} \|f_n\|_{L^2}} = \frac{1}{\left\{ \prod_{j=1}^N n_j^2 \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq l}} n^{2\alpha} \right) \right\}^{1/2} \left\{ \int_{[0,2\pi]^N} \left| \prod_{j=1}^N \sin n_j x_j \right|^2 dx_1 \dots dx_j \right\}^{1/2}}.$$

That is

$$\lambda_n = \frac{1}{\pi^{N/2} \prod_{j=1}^N n_j \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq l}} n^{2\alpha} \right)^{1/2}} \quad \text{for } n \in \mathbb{N}^N.$$

We claim that $\sum_{n \in \mathbb{N}^N} \lambda_n^{1/2}$ does not converge for N large enough. To see this note that

$$\sum_{n \in \mathbb{N}^N} \lambda_n^{1/2} = \sum_{k=1}^{\infty} \sum_{|n|=k} \lambda_n^{1/2}.$$

There are $\binom{k+N-1}{k}$ terms in the series $\sum_{|n|=k} \lambda_n^{1/2}$; that is to say, at least $k^{N-1}/(N-1)!$ terms. For $|n| := \sum n_i = k$ we get

$$\begin{aligned} \lambda_n &= \frac{1}{\pi^{N/2} \prod_{j=1}^N n_j \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq l}} k^{2\alpha_1} \dots k^{2\alpha_N} \right)^{1/2}} \\ &= \frac{1}{\pi^{N/2} \prod_{j=1}^N n_j \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq l}} k^{2|\alpha|} \right)^{1/2}} \\ &\geq \frac{1}{\pi^{N/2} k^N \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq l}} k^{2|\alpha|} \right)^{1/2}} \geq \frac{1}{\pi^{N/2} k^N ((l+1)^N k^{2l})^{1/2}} \end{aligned}$$

Therefore

$$\sum_{n \in \mathbb{N}^N} \lambda_n^{1/2} \geq \sum_{k=1}^{\infty} \frac{k^{N-1}}{(N-1)!} \frac{1}{(\pi(l+1))^{N/4} k^{(N+l)/2}} = \text{constant} \cdot \sum_{k=1}^{\infty} \frac{k^{N-1}}{k^{(N+l)/2}},$$

and this last series diverges for $N \geq l$.

Thus D_N is not a mapping of type $\ell^{1/2}$ and so $C_{0,n}^{\infty}(\mathcal{H})$ cannot have been nuclear.

The proof is complete.

§4.3 Appendix

Lemma 4.4 (Notation same as in the text) *With $\Gamma_N \subset \text{int} B_n$, there exists a function $\alpha \in C_{0,n}^\infty(\mathcal{H})$ such that $\alpha(x) = 1$ if $x \in \Gamma_N$ (and so $\alpha(x) = 0$ if $x \notin \text{int} B_n$).*

Proof We recall some notation and definitions from Lang [3]. Let M be a metric space with distance function d . An open subset V of M is said to be *scalped* if there exist open balls $U, U_1, \dots, U_m$ in M such that $V = U \cap (\overline{U_1})^c \cap \dots \cap (\overline{U_m})^c$, where $(\overline{U_i})^c$ denotes the complement of the closure of the ball U_i . We shall need the following (for a proof see Lang [3], pp. 29).

Lemma A Let M be a metric space and $\{U_i\}$, $i = 1, 2, \dots$ be a countable covering of a subset W by open balls. Then there exists a locally finite open covering $\{V_i\}$, ($i = 1, 2, \dots$) of W such that $V_i \subset U_i$ for all i , and such that V_i is scalped for all i .

Next we need a modification of another result of Lang ([3], pp. 29):

Lemma B Let U be an open ball in a Hilbert space $\mathcal{H}$ and let $V = U \cap (\overline{U_1})^c \cap \dots \cap (\overline{U_m})^c$ be a scalped open subset. Then there exists a C^∞ -function $\omega : \mathcal{H} \rightarrow \mathbb{R}$, having, along with all its derivatives, a bounded range, such that $\omega(x) > 0$ if $x \in V$ and $\omega(x) = 0$ otherwise.

Proof of Lemma B The square of the norm $\|x\|^2 = \langle x, x \rangle$, in a Hilbert space, is everywhere differentiable. If we denote $f(x) := \|x\|^2$, then $f'(x)(h) = 2\langle x, h \rangle$, and $f''(x)(h)(k) = 2\langle h, k \rangle$, so that clearly $f(x)$ and all its derivatives will have bounded range if we restrict x to a bounded set. Consider the function $\eta : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$\eta(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \|x\| > 1; \\ 0, & \|x\| \leq 1. \end{cases}$$

and then define $\zeta(x) := 1 - \eta(x)$. ζ is a C^∞ -function having, along with all its derivatives, bounded range. By shifting and scaling ζ we can construct, for each U_i , a function ϕ_i

such that

$$0 \leq \phi_i(x) < 1, \quad \text{if } x \in (\overline{U_i})^c$$

$$\phi_i(x) = 1; \quad \text{if } x \in \overline{U_i}.$$

Now consider the function $\tau : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$\tau(x) = \begin{cases} e^{\frac{1}{\|x\|^2-1}}, & \|x\| < 1; \\ 0, & \|x\| \geq 1. \end{cases}$$

Again, τ is a C^∞ -function having, along with all its derivatives, bounded range. By shifting and scaling τ we can construct a function ϕ such that $\phi(x) > 1$ on U and $\phi(x) = 0$ outside U . Let $\omega(x) = \phi(x) \prod (1 - \phi_i(x))$ to obtain a desired function.

We now proceed with a proof of our Lemma 3, which is almost the same as a theorem of Lång ([3], pp. 29-30). By Lindelöf's theorem we can find a countable collection of open balls $\{U_i\}$ ($i = 1, 2, \dots$) covering Γ_N (considered as a closed subset of $\mathcal{H}$) and such that each $U_i \subset \text{int} B_n$. Let W be the union of the U_i , so that Lemma A gives us a locally finite refinement $\{V_i\}$. Using Lemma B we get a function ω_i which is greater than zero on V_i and equals zero outside V_i . Let $\omega = \sum \omega_i$ (the sum is finite at each point of W). Then $\omega(x) > 0$ if $x \in \Gamma_N$ and $\omega(x) = 0$ if $x \notin \text{int} B_n$.

Let U be the open neighbourhood of Γ_N on which ω is greater than zero. Then Γ_N and U^c are disjoint closed sets and so we can apply the above construction to obtain a function $\sigma : \mathcal{H} \rightarrow \mathbb{R}$ which is greater than zero on U^c and equals zero on Γ_N . We note that σ and its derivatives need not have bounded range on all of $\mathcal{H}$. We set $\alpha := \omega/(\sigma + \omega)$ to get the desired function.

Lemma 4.5 (Notation same as text) *Suppose that π_N is a continuous linear projection from $\mathcal{H}$ on E_N . Then the map $P_N : C_0^\infty(\Gamma_N) \rightarrow C_{0,n}^\infty(\mathcal{H})$ defined by $P_N(f) := (f \circ \pi_N)(\alpha)$ is continuous and linear.*

Proof The linearity of P_N is clear. To see continuity, consider the arbitrary zero

neighbourhood in $C_{0,n}^\infty(\mathcal{H})$ defined by

$$V := \left\{ \phi \in C_{0,n}^\infty(\mathcal{H}) \mid \sup_{\substack{0 \leq j \leq l \\ x \in B_n \\ \|v\| \leq 1}} \frac{|D^j \phi(x)(y_1, \dots, y_j)|}{\psi(x)} < \epsilon \right\}.$$

If there exists $\psi_N \in \Psi(E_N)$, $\psi_N : E_N \rightarrow (0, \infty]$ such that $\psi_N(\pi_N x) \leq \psi(x) \forall x \in \mathcal{H}$, then we see that if $f \in U$, the zero neighbourhood in $C_{0,n}^\infty(\mathcal{H})$ defined by

$$U := \left\{ g \in C_0^\infty(\Gamma_N) \mid \sup_{\substack{0 \leq j \leq l \\ \xi \in E_N \cap B_n \\ \|v\| \leq 1}} \frac{|D^j g(\xi)(\eta_1, \dots, \eta_j)|}{\kappa(l) \sup_{\substack{x \in \mathcal{H} \\ 0 \leq m \leq l}} \|D^m \alpha(x)\|_{\mathcal{U}_m}} \right\},$$

where $\kappa(l)$ is a constant to be determined later, then the following holds:

$$\begin{aligned} & \sup_{\substack{0 \leq j \leq l \\ x \in B_n \\ \|v\| \leq 1}} \frac{|D^j \{(f \circ \pi_N)(\alpha)\}(x)(y_1, \dots, y_j)|}{\psi(x)} \\ &= \sup_{\substack{0 \leq j \leq l \\ x \in B_n \\ \|v\| \leq 1}} \frac{\left| \sum_{i=0}^j \binom{j}{i} D^i(f \circ \pi_N)(x)(y_1, \dots, y_i) D^{j-i} \alpha(x)(y_{i+1}, \dots, y_j) \right|}{\psi(x)} \\ &\leq \sup_{\substack{0 \leq j \leq l \\ x \in B_n \\ \|v\| \leq 1}} \frac{a(j) \sup_{\substack{z \in \mathcal{H} \\ 0 \leq m \leq j}} \|D^m \alpha(z)\|_{\mathcal{U}_m} \sup_{\substack{x \in B_n \\ 0 \leq r \leq j \\ \|v\| \leq 1}} |D^r(f \circ \pi_N)(x)(y_1, \dots, y_r)|}{\psi(x)} \\ &\leq \sup_{\substack{0 \leq m \leq l \\ z \in \mathcal{H}}} b(l) \|D^m \alpha(z)\|_{\mathcal{U}_m} \sup_{\substack{0 \leq j \leq l \\ x \in B_n \\ \|v\| \leq 1}} \frac{|D^j f(\pi_N x)(\pi_N y_1, \dots, \pi_N y_j)|}{\psi(x)} \\ &\leq b(l) \sup_{\substack{0 \leq m \leq l \\ z \in \mathcal{H}}} \|D^m \alpha(z)\|_{\mathcal{U}_m} \sup_{\substack{0 \leq j \leq l \\ \xi \in E_N \cap B_n \\ \|\eta_i\| \leq 1}} \frac{|D^j f(\xi)(\eta_1, \dots, \eta_j)|}{\psi_N(x)} \\ &< \epsilon \end{aligned}$$

by assumption, where we choose $\kappa(l) = b(l)$ for the definition of the neighbourhood U .

This would imply that $P_N(U) \subset V$ and so P_N is continuous.

To show the existence of the required function $\psi_N : E_N \rightarrow (0, \infty]$ satisfying $\psi_N(\pi_N x) \leq \psi(x) \forall x \in \mathcal{H}$ consider the following. The set $S_\psi = \{x \in \mathcal{H} \mid \psi(x) < \infty\}$ is σ -compact, and $S_\psi = \bigcup_{n=1}^\infty K_n$, where compact $K_n := \{x \in \mathcal{H} \mid \psi(x) \leq n\}$. Without loss of generality

Assume that $0 < \inf_{x \in H} \psi(x) \leq 1$. Clearly $\pi_N(S)$ is σ -compact in E_N and $\pi_N(S_\psi) = \bigcup_{n=1}^{\infty} \pi_N K_n$, where the $\pi_N K_n$ are compact. On E_N define ψ_N by

$$\psi_N(\xi) = \begin{cases} \infty, & \text{if } \xi \notin \pi_N(S_\psi); \\ \inf_{x \in H} \psi(x), & \text{if } \xi \in \pi_N(K_1); \\ 1, & \text{if } \xi \in \pi_N(K_2) \setminus \pi_N(K_1); \\ \vdots & \\ n-1, & \text{if } \xi \in \pi_N(K_n) \setminus \bigcup_{j=1}^{n-1} \pi_N(K_j); \\ \vdots & \end{cases}$$

Then $\psi_N : E_N \rightarrow (0, \infty]$ and $\{\xi | \psi_N(\xi) \leq c\}$ is compact $\forall c \in \mathbb{R}$. Moreover $\psi_N(\pi_N x) \leq \psi(x) \forall x \in H$.

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