The Strong Topological Center and the Dual Factorization
Property

by

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Abstract

We develop the concept of the strong topological centre to measure the degree of Arens regularity of an Arens regular Banach algebra. First, we study the strong dual factorization property, introduced by Lau–Ülger, through a new approach. This leads us to give a new characterization of this property and to prove that the strong dual factorization property implies the equivalence between the norms of the algebra and of its left or right multiplier algebra. We then link the strong dual factorization property with the new concept of the strong topological centre by providing another new characterization of the strong dual factorization property, which characterizes at the same time the existence of a specific isomorphism between the strong topological centre and the left multiplier algebra. We use the strong topological centre to compare, in terms of Arens regularity, l^1 equipped with the pointwise product and infinite-dimensional commutative C^* -algebras, and prove that l^1 is more Arens regular. We also present a list of all behaviours of the strong topological centre, as well as, various application of this new concept to the study of different problems in Banach algebra theory and abstract harmonic analysis. Using the strong topological centre, we finish by producing a list of Banach algebras satisfying a conjecture of Lau–Ülger.

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À toi papa et à tous tous mes amours

*The art of doing mathematics consists
in finding that special case which
contains all the germs of generality*
David Hilbert

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Chapter 1

Introduction

A Banach algebra is said to be Arens regular if the two Arens products, introduced by R. Arens in [2] and [1], on the second dual are identical. The extreme opposite property is when the two products coincide only on the algebra itself, in which case we say that the algebra is strongly Arens irregular. Any operators algebra is Arens regular; on the other hand, group algebras over locally compact groups are strongly Arens irregular. So, determining if a Banach algebra is Arens regular, strongly Arens irregular or neither, is an interesting question and forms a big area of research. There are many unsolved questions, such as, for which locally compact groups is the Fourier algebra Arens regular? Or, for which Banach space X , is the algebra $B(X)$ of linear operators on X Arens regular? At this time, just determining the topological centre of a Banach algebra is a difficult question, but when we know that the algebra is Arens regular, then what else? It is time to pass to the second step! Can we compare Banach algebras in terms of Arens regularity? This thesis answers this question by providing a new tool: the strong topological centre. This new object will measure the degree of regularity of Arens regular Banach algebras.

Chapter 2 contains examples and definitions that will be used in subsequent chapters. Section 2.1 provides background and notation. Section 2.2 is about a specific class of Banach algebras, Segal algebras. The last section of this chapter gives information about Banach algebras of type (M).

The third chapter is about the dual factorization property. We generalize this notion, introduced in [17], and give examples. A new approach to studying this property is presented in Section 3.1. With the technique developed in this section, we give a new way of studying the algebra $(A^*A)^*$. In Section 3.2, we study the strong dual factorization property for Banach algebras without bounded approximate identity, which is a new investigation. We use abstract Segal algebras to deduce Theorem 3.2.1 which says that the left strong dual factorization property implies the equivalence of the algebra norm with the norm of its right multiplier algebra.

The main work of this thesis is contained in chapter 4. We introduce the notion of the strong topological centre. To illustrate this new concept, some examples are given in Section 4.1. In that section, there are general results about the strong topological centre that will help us deduce Theorem 4.1.1 which states that the right strong topological centre of $K(X)$ is $W(X)$. In Section 4.2, we study in greater details the strong topological centre for Banach algebras with a bounded approximate identity. The Theorem 4.2.1 is a new characterization of the strong dual factorization property using the strong topological centre. For a Banach algebra A , we have the following situation:

$$A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \subseteq A^{**}$$

All these inclusions can be strict, and we give a complete list of all the possible cases in Section 4.3. We then focus on the Neufang convolution algebra $N(L^p(G))$, where G is a discrete group. We determine that the right topological centre and the right

strong topological centre are equal for this algebra. In particular, this provides us an example of a Banach algebra which is not a right ideal in its bidual, not Arens regular and not the algebra itself, for which we still have the equality between the right strong topological centre and the right topological centre. In Section 1.4, we tackle the left multipliers of the left strong topological centre. For a large class of Banach algebras including those with a bounded approximate identity, we show that

$$LM(SZ_\ell(A^{**})) = SZ_\ell(LM(A^{**}))$$

Finally, the thesis uses the strong topological centre as a tool in chapter 5. We give a criterion for the bidual of a Banach algebra to be left or right faithful. We present a very short proof, relying on the concept of the strong topological centre, for the result of Lau showing that Arens regularity of the Fourier algebra $A(G)$ over an amenable locally compact group G implies that G is finite. We also study the following problem raised by Lau–Ulgert.

Conjecture. *A weakly sequentially complete Banach algebra with a BAI enjoying the left or right strong dual factorization property is unital.*

Ulgert established this conjecture for Arens regular Banach algebras, and Lau–Ulgert proved it for Banach algebras with a sequential bounded approximate identity. Using the strong topological centre, we prove this conjecture for many other classes of Banach algebras, in particular for strongly Arens irregular ones and Banach algebras which are ideals in their biduals.

We finish this introduction by mentioning that all results not attributed to others are new results obtained in this thesis. Also, the proofs following the attributed results are new.

Chapter 2

Preliminaries

In this chapter we provide the background for this thesis. Definitions and examples that are included will be considered as acquired for the rest of this thesis. In Subsection [2.1](#), we introduce the notation for the Arens products and topological centres for the second dual of a Banach algebra. We then discuss about two special embeddings of the left multiplier algebra of a Banach algebra. They will be very useful tools for the rest of the thesis. In Section [2.2](#) we consider a specific class of subalgebras of the group algebra, Segal algebras. The main interest of Segal algebras is Theorem [2.2.1](#). After the concept of Segal algebra is generalized with the goal to obtain an analog to Theorem [2.2.1](#) for abstract Segal algebras. We finish this chapter with presenting a new class of Banach algebras introduced in [[3](#)], Banach algebras of type (M).

2.1 Banach algebras

Definition 2.1.1. A linear algebra A over the scalar field $\mathbb{C}$ is called a Banach algebra if there is a norm $\|\cdot\|$ turning A into a Banach space and such that

$$\|ab\| \leq \|a\| \|b\|$$

for all $a, b \in A$.

We denote by X^* the dual of a Banach space X . For $x \in X$ and $f \in X^*$, indifferently we denote by $\langle x, f \rangle$ or $\langle f, x \rangle$ the natural duality between X and X^* . We present here a few examples that will be used later.

Example 2.1.2

- (i) Let G be a locally compact group. Let λ be a left Haar measure. We denote by $L^1(G)$ the space of equivalence classes of functions absolutely integrable over G with respect to λ and with the norm

$$\|f\|_1 = \int_G |f| \, d\lambda$$

The Banach space $(L^1(G), \|\cdot\|_1)$ equipped with the convolution multiplication defined by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s) \, d\lambda(t)$$

for $f, g \in L^1(G)$ and $s \in G$, is a Banach algebra. This algebra has an identity if and only if G is discrete. This Banach algebra is called the *group algebra*

- (ii) A continuous function $\omega : G \rightarrow \mathbb{C}$ is a weight if $\omega(e_G) = 1$, where e_G is the identity of G , and $\omega(st) \leq \omega(s)\omega(t)$ for all $s, t \in G$. Let ω be a weight on G . The Beurling

algebra $L^1(G, \omega)$ is defined as

$$L^1(G, \omega) = \{\phi : G \rightarrow \mathbb{C} \mid \phi\omega \in L^1(G)\}$$

The dual of $L^1(G, \omega)$ is $L^\infty(G, \omega^{-1}) = \{f : G \rightarrow \mathbb{C} \mid f\omega^{-1} \in L^\infty(G)\}$

- (iii) Let G be a locally compact group. Let λ be a left Haar measure. We denote for $1 \leq p < \infty$ by $L^p(G)$ the space of equivalence classes of absolutely p -integrable functions, i.e. such that

$$\|f\|_p = \left(\int_G |f|^p d\lambda \right)^{\frac{1}{p}} < \infty$$

When G is compact, $L^1(G)$ with $\|\cdot\|_1$ is a Banach algebra for all $1 \leq p < \infty$

- (iv) Let G be a locally compact group. Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The space $A_p(G)$ consists of all functions $f \in C_0(G)$ such that there are sequences $\{g_n\}_{n \in \mathbb{N}} \in L^p(G)$ and $\{h_n\}_{n \in \mathbb{N}} \in L^q(G)$ with

$$\sum_{n=1}^{\infty} \|g_n\|_p \|h_n\|_q < \infty$$

and $f = \sum_{n=1}^{\infty} g_n * h_n$ where $h(\cdot) = h(\cdot^{-1})$ for $a \in G$. For $f \in A_p(G)$ we define

$$\|f\|_A = \inf \left\{ \sum_{n=1}^{\infty} \|g_n\|_p \|h_n\|_q \mid f = \sum_{n=1}^{\infty} g_n * h_n \text{ for } g_n \in L^p(G), h_n \in L^q(G) \right\}$$

The set $A_p(G)$ with $\|\cdot\|_A$ and the pointwise product forms a Banach algebra which is called the *Figa-Talamanca Herz algebra*. When $p = 2$ the algebra $A_2(G)$ is called the Fourier algebra denoted by $A(G)$.

- (v) Let Ω be a locally compact Hausdorff space. A function $f : \Omega \rightarrow \mathbb{C}$ is said to vanish at infinity if for each $\epsilon > 0$ there is a compact $K \subseteq \Omega$ such that $|f(r)| < \epsilon$ for all $r \in \Omega \setminus K$. The set of all continuous functions vanishing at infinity is denoted by $C_0(\Omega)$. This set equipped with the uniform norm $\|\cdot\|_\infty$ and the pointwise product is a Banach algebra. This Banach algebra has an identity if and only if Ω is compact.

- (vi) Let X be a Banach space. The space of all bounded, respectively compact, linear operator on X , denoted by $B(X)$, respectively $K(X)$, is a Banach algebra when it is equipped with the operator norm and composition as the multiplication.
- (vii) Let X be a Banach space and $T \in B(X)$. Then, T is called *nuclear* if there exist $\{y_n\}_{n \in \mathbb{N}} \subseteq X$ and $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq X^*$ with

$$\sum_{n=1}^{\infty} \|y_n\| \|\lambda_n\| < \infty \quad \text{and} \quad T(x) = \sum_{n=1}^{\infty} \langle \lambda_n, x \rangle y_n$$

for all $x \in X$. Thus, we have $T = \sum_{n=1}^{\infty} y_n \otimes \lambda_n$ in $B(X)$. The set of nuclear operators on X is denoted by $N(X)$. For $T \in N(X)$, we define $\|T\|_N$ by

$$\|T\|_N = \inf \left\{ \sum_{n=1}^{\infty} \|y_n\| \|\lambda_n\| \mid T = \sum_{n=1}^{\infty} y_n \otimes \lambda_n \right\}$$

for $y_n \in X$ and $\lambda_n \in X^*$. We have that $\|T\| \leq \|T\|_N$ and $(N(X), \|\cdot\|_N)$ is a Banach algebra.

Definition 2.1.3. A C^* -algebra is a Banach algebra A on which there is an involution $a \mapsto a^*$ such that

$$\|a^*a\| = \|a\|^2 \quad (2.1)$$

for all $a \in A$.

Remark 2.1.4. Equation (2.1) is called the C^* -condition.

In fact, the C^* -condition imposes a very rigid structure. For example, any commutative C^* -algebra is isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω .

The next forms a very important class of C^* -algebras.

Definition 2.1.5. A von Neumann algebra is a C^* -algebra A such that there is a Banach space X with $X^* = A$.

As we have seen among the examples, not all Banach algebras have an identity. However, many Banach algebras have an approximate identity.

Definition 2.1.6. Let A be a Banach algebra. A left approximate identity is a net $\{e_\alpha\}_{\alpha \in I}$ in A such that

$$\lim_{\alpha} e_\alpha a = a$$

for all $a \in A$. A right approximate identity is defined similarly. An approximate identity is a left and right approximate identity.

A bounded left approximate identity (BLAI) bounded by M is a left approximate identity $\{e_\alpha\}_{\alpha \in I}$ such that

$$\sup_{\alpha \in I} \|e_\alpha\| \leq M < \infty$$

A bounded right approximate identity (BRAI) bounded by M is defined similarly. A bounded approximate identity (BAI) is a bounded left and right approximate identity.

As we said before the definition, many Banach algebras have a BAI, such as the group algebra and the Fourier algebra defined over an amenable group. Also, a C^* -algebra A has a BAI.

Definition 2.1.7. Let A be a Banach algebra. A left Banach A -module is a Banach space X with a bilinear map

$$A \times E \rightarrow E$$

$$(a, x) \mapsto ax$$

such that

$$a(bx) = (ab)x$$

$$\|ax\| \leq \|a\| \|x\|$$

for all $a, b \in A$ and all $x \in X$. A right Banach A -module is a Banach space X with a bilinear map

$$\begin{aligned} \cdot : A \times E &\rightarrow E \\ (a, x) &\mapsto a \cdot x \end{aligned}$$

such that

$$\begin{aligned} (x \cdot a) \cdot b &= x \cdot (ab) \\ \|x \cdot a\| &\leq \|x\| \|a\| \end{aligned}$$

for all $a, b \in A$ and all $x \in X$. A Banach A -bimodule is a Banach space with respect to which it is a left Banach A -module and a right Banach A -module and which are such that

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

for all $a, b \in A$ and all $x \in X$.

For this thesis, the most important A -module is the dual A^* . The next example describes the module actions.

Example 2.1.8. The dual A^* of a Banach algebra A is a Banach A -bimodule. The module actions are given by

$$\begin{aligned} \langle f \cdot a, b \rangle &= \langle f, ab \rangle, \\ \langle a \cdot f, b \rangle &= \langle f, ba \rangle. \end{aligned}$$

for $f \in A^*$ and $a, b \in A$.

We denote by $\langle A^*A \rangle$ and $\langle AA^* \rangle$, respectively, the linear span of the sets A^*A and AA^* ; their closures will be denoted by $\overline{\langle A^*A \rangle}$ and $\overline{\langle AA^* \rangle}$, respectively. We denote by A^*A , respectively AA^* , by the set of all $f \cdot a$, respectively $a \cdot f$, where $f \in A^*$ and $a \in A$.

2.1.1 The second dual

For a Banach algebra A , we know that A embedded isometrically in A^{**} . In 1952 Arens [1] showed that there are two natural ways to extend the product of A to A^{**} . The resulting products on A^{**} are called the Arens products.

Definition 2.1.9. Let A be a Banach algebra. Let $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$. The left Arens product is defined by

$$\begin{aligned}\langle m \sqcap n, f \rangle &= \langle m, n \rceil f \rangle \\ \langle n \sqcap f, a \rangle &= \langle n, f \sqcap a \rangle \\ \langle f \sqcap a, b \rangle &= \langle f, ab \rangle\end{aligned}$$

The right Arens product is defined by

$$\begin{aligned}\langle m \triangle n, f \rangle &= \langle n, f \triangle m \rangle \\ \langle f \triangle m, a \rangle &= \langle m, a \triangle f \rangle \\ \langle a \triangle f, b \rangle &= \langle f, ba \rangle\end{aligned}$$

Observe that $\sqcap$ and $\triangle$ make A^* a right and left Banach A -module respectively. Both module actions coincide with those presented in Example 2.1.8. Another way to define the Arens product is the following which uses the weak* topology on A^{**} .

Definition 2.1.10. Let A be a Banach algebra. Let $m, n \in A^{**}$. Let $\{m_\alpha\}_{\alpha \in I}$ and $\{n_\beta\}_{\beta \in J}$ be nets in A such that

$$m = w^*\text{-}\lim_{\alpha} m_\alpha \quad \text{and} \quad n = w^*\text{-}\lim_{\beta} n_\beta$$

The left Arens product, denoted by $\square$ is given by

$$m \square n = w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} m_\alpha n_\beta$$

The right Arens product denoted by $\triangle$ is given by

$$m \triangle n = w^* - \lim_{\beta} u^* - \lim_{\alpha} m_{\alpha} n,$$

Starting with Definition 2.1.10 one can easily check that the expression for the Arens products in Definition 2.1.10 are well-defined. As we can see there is no reason for $\square$ and $\triangle$ to agree on A^{**} .

Since the two Arens products are usually different it is interesting to know for which elements they coincide. We will define the two topological centres which characterize this property.

Definition 2.1.11. Let A be a Banach algebra. The left topological centre is defined by

$$\begin{aligned} Z_\ell(A^{**}) &= \{m \in A^{**} \mid m \square n = m \triangle n \ \forall n \in A^{**}\} \\ &= \{m \in A^{**} \mid \lambda_m : A^{**} \ni n \mapsto m \square n \text{ is weak}^* \text{-weak}^* \text{-continuous}\} \end{aligned}$$

The right topological centre is defined by

$$\begin{aligned} Z_r(A^{**}) &= \{m \in A^{**} \mid n \square m = n \triangle m \ \forall n \in A^{**}\} \\ &= \{m \in A^{**} \mid \rho^m : A^{**} \ni n \mapsto n \triangle m \text{ is weak}^* \text{-weak}^* \text{-continuous}\} \end{aligned}$$

Then $A \subseteq Z_\ell(A^{**}) \cap Z_r(A^{**})$ and both topological centres are closed subalgebras of A^{**} and are weak*-dense in A^{**} . We mentioned that of course not all Banach algebras have an identity but many have a bounded approximate identity. This has repercussions in the second dual.

Definition 2.1.12. Let A be a Banach algebra. A mixed identity in A^{**} is an element $E \in A^{**}$ such that $m = m \square E = E \triangle m$ for all $m \in A^{**}$.

It can be proved that A^{**} has a mixed identity if and only if A has a bounded approximate identity. The next theorem is Theorem 5.1 of [17] which characterizes the elements of $Z_\ell(A^{**})$ for a Banach algebra with a BAI using the concept of mixed identity.

Theorem 2.1.13. (Lau – Ülger) *Let A be a Banach algebra with a BAI. For $m \in A^{**}$ the following are equivalent.*

- (i) $m \in Z_\ell(A^{**})$;
- (ii) (a) $m \square A \subseteq Z_\ell(A^{**})$,
 (b) $m \triangle E = m$ for each mixed unit E ,
 (c) $f \triangle m \in A^* A$ for all $f \in A^*$.

Definition 2.1.14. *A Banach algebra A is called*

- (i) Arens regular if $Z_\ell(A^{**}) = Z_r(A^{**}) = A^{**}$.
- (ii) Arens irregular if A is not Arens regular.
- (iii) left, respectively right, strongly Arens irregular if $Z_\ell(A^{**}) = A$, respectively $Z_r(A^{**}) = A$.
- (iv) strongly Arens irregular if $Z_\ell(A^{**}) = Z_r(A^{**}) = A$.

A first observation is that $Z_\ell(A^{**}) = A^{**}$ if and only if $Z_r(A^{**}) = A^{**}$. Therefore it is enough that $Z_r(A^{**}) = A^{**}$ for A to be Arens regular.

Example 2.1.15.

- (i) Any closed subalgebra and quotient algebra obtained by the quotient of a closed ideal of an Arens regular Banach algebra are Arens regular [18, Corollary 2.6.18].

- (ii) Any C^* -algebra is Arens regular [6, Corollary 3.2.37].
- (iii) The group algebra $L^1(G)$ is strongly Arens irregular [17, Theorem 1].
- (iv) Let X be a Banach space. If the Banach algebra $K(X)$ is Arens regular, then X is reflexive [6, Theorem 2.6.23].

It is unknown if there exist an infinite group for which the Fourier algebra is Arens regular. We know for instance the following result by Forrest, which is Theorem 3.2 of [10].

Theorem 2.1.16. (*Forrest*) *Let G be a locally compact group. If $A(G)$ is Arens regular, then G is a discrete group.*

Lau in [11] characterizes, with Theorem 3.7 of [11], for which groups $A(G)$ is an ideal in its second dual. This is the next result.

Theorem 2.1.17. (*Lau*) *Let G be a locally compact group. The Fourier algebra $A(G)$ is an ideal in its second dual if and only if G is discrete.*

This is dual to Watanabe's result stating that $L^1(G)$ is an ideal in its second dual if and only if G is compact.

If A is a Banach algebra with a BAI, the set A^*A is a closed subspace of A^* , by Cohen's factorization theorem, and a A -submodule of A^* . Moreover, it is left introverted in A^* ; that is, $m \square x \in A^*A$ for all $x \in A^*A$ and $m \in (A^*A)^*$, where $m \square x = \tilde{m} \square r$ for any extension $\tilde{m} \in A^{**}$ of m . This is an equivalent formulation to A^*A being a left $(A^{**}, \square)$ -submodule of A^* [7, Proposition 5.2]. We make $(A^*A)^*$ a Banach algebra by

defining the following multiplication

$$\langle m \sqcup n, r \rangle = \langle m, n \sqcup r \rangle$$

for $m, n \in (\Gamma \setminus \Gamma)^*$ and $r \in \Gamma^*A$. The multiplication $\sqcup$ on $(\Gamma \setminus \Gamma)^*$ is induced by the left Arens product on A^{**} . This means that if $m, n \in (\Gamma \setminus \Gamma)^*$ and $\tilde{m}, \tilde{n} \in A^{**}$ are extensions of m, n to A^* respectively, then $\tilde{m} \sqcup \tilde{n}$ is an extension of $m \sqcup n$ to A^* .

2.1.2 Multipliers

Definition 2.1.18. A Banach algebra A is called left faithful if

$$\text{Ann}_l(A) = \{a \in A : aA = 0\} = \{0\}$$

A right faithful Banach algebra is defined similarly. A faithful Banach algebra is a left and right faithful Banach algebra.

Definition 2.1.19. Let A be a Banach algebra and $a \in A$. Let L_a and R_a be the linear maps in $B(A)$ defined by

$$L_a(b) = ab \text{ and } R_a(b) = ba$$

for $b \in A$.

The left regular representation of A is the map

$$\begin{aligned} L : A &\rightarrow B(A) \\ a &\mapsto L_a \end{aligned}$$

The right regular representation of A is the map

$$\begin{aligned} R : A &\rightarrow B(A) \\ a &\mapsto R_a \end{aligned}$$

The injectivity of L is equivalent to the left faithfulness of A , and the same holds for the right

Definition 2.1.20. Let A be a Banach algebra.

(i) A left multiplier on A is a bounded linear operator $T : A \rightarrow A$ such that

$$T(ab) = T(a)b$$

for $a, b \in A$. A right multiplier is defined similarly. The set of all left (resp. right) multipliers is denoted by $LM(A)$ (resp. $RM(A)$)

(ii) A multiplier on A is a bounded linear operator $T : A \rightarrow A$ such that

$$T(a)b = aT(b)$$

for $a, b \in A$. The set of all multipliers on A is denoted by $M(A)$

(iii) A pair of linear operators (T, S) from A to A is called a double multiplier if

$$xT(y) = S(x)y$$

for all $x, y \in A$. The set of all double multipliers is denoted by $DM(A)$.

Remark 2.1.21. It is known that for a double multiplier (T, S) , T and S are continuous, $T \in LM(A)$ and $S \in RM(A)$. In the literature, double multipliers are also called double centralizers.

Example 2.1.22. Let X be a Banach space. The left multiplier algebra of $A = K(X)$ is $B(X)$ [6, Theorem 2.5.13].

The left multiplier algebra, the right multiplier algebra and the multiplier algebra are closed subalgebras of $B(A)$. In fact, for a left faithful Banach algebra A , the image

of the left regular representation is a left ideal in $LM(A)$. A similar result holds for $RM(A)$.

Now, we will focus on the algebra $LM(A)$. First, assume that A has a BLAI. Consider the map $T \mapsto T^{**}(E)$ from $LM(A)$ to (A^{**}, Δ) , where E is a fixed w^* -cluster point of a BLAI of A . This map is an isometric embedding when the BLAI is bounded by 1. McKilligan [21, Theorem 1.1] proves the following theorem.

Theorem 2.1.23. *(McKilligan) For a Banach algebra A , the following are equivalent*

- (i) *A has a left approximate identity bounded by 1;*
- (ii) *$LM(A)$ is isometrically isomorphic to a subalgebra of (A^{**}, Δ) via $T \mapsto \phi_T$, where T and ϕ_T satisfy $\phi_T \wedge a = T(a)$ in A^{**} for each $a \in A$.*

In view of this theorem, when A has a BAI, we will always use the isomorphism as above and we will consider, from now on, $LM(A)$ as a subalgebra of (A^{**}, Δ) . Of course, the right version of this result is also true.

The left multiplier algebra can also be viewed as a subalgebra of another Banach algebra. Consider the map $T \mapsto T^*$ from $LM(A)$ into $Hom_{A_l}(A^*, A^*)$, the set of all bounded linear left A -module homomorphisms from A^* to A^* . This is an isometric anti-isomorphism. We remark here that A does not need to have a BAI. We can describe $LM(A)$ in $Hom_{A_l}(A^*, A^*)$ as follows:

$$LM(A) = \{T \in Hom_{A_l}(A^*, A^*) \mid T \text{ is weak}^*\text{-weak}^*\text{-continuous} \}$$

The right version of this result holds, too. We will also consider from now on, $LM(A)$ as a subalgebra of $Hom_{A_l}(A^*, A^*)$, with the opposite product. When the situation requires it, we will specify the algebra in which we see $LM(A)$ as subalgebra.

2.2 Segal algebras

In this section, we present a specific class of Banach algebras, Segal algebras. We will generalize those algebras in chapter 3. So this section is here to provide a general historical background of Segal algebras and examples. We finish it with the general definition.

In 1932, Wiener presented in [33] following algebra.

Example 2.2.1. (Wiener) Let $K = [0, 1]^n$. For $f \in C(\mathbb{R}^n)$ we define

$$M_\infty(f) = \sum_{m \in \mathbb{Z}^n} \max_{x \in K} |f(x - m)|.$$

The set of all functions $f \in C(\mathbb{R}^n)$ for which $M_\infty(f) < \infty$ forms a Banach algebra with M_∞ as a norm and convolution as product. This Banach algebra is contained in $L^1(\mathbb{R}^n)$. The norm given by M_∞ is not invariant under translation. The following

$$\|f\| = \sup_{y \in \mathbb{R}^n} M_\infty(L_y(f)),$$

where $L_y(f)(x) = f(yx)$ for $x, y \in \mathbb{R}^n$, is an equivalent norm to M_∞ and is translation invariant.

Segal in [29] extracted from this example an underlying abstract structure. However, he did not give the axioms to define Segal algebras. It was Reiter who gave the axiomatization in [27].

Definition 2.2.2. Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is a Segal algebra if it satisfies the following conditions:

(i) $S^1(G)$ is a dense ideal in $L^1(G)$.

(iv) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ satisfying

$$\|f\|_1 \leq \|f\|_S$$

for $f \in S^1(G)$.

(v) $S^1(G)$ is left invariant.

$$f \in S^1(G) \Rightarrow L_y f \in S^1(G)$$

for $y \in G$, and for each $f \in S^1(G)$, the mapping $y \mapsto L_y f$ of G into $S^1(G)$ is continuous.

(vi) the norm $\|\cdot\|_S$ is left invariant:

$$\|L_y f\|_S = \|f\|_S$$

for $f \in S^1(G)$.

Many Banach algebras are Segal algebras. The following examples are quite well known in abstract harmonic analysis.

Example 2.2.3.

- (i) The Lebesgue–Fourier algebra, $(L^1(G) \cap A(G), \|\cdot\|)$ where $\|f\| = \|f\|_1 + \|f\|_{A(G)}$ with G a locally compact infinite group.
- (ii) The Figa-Talamanca–Herz algebra $A_p(G)$, defined in Example 2.2.1.
- (iii) The Figa-Talamanca–Herz–Lebesgue algebra $A_p^r(G) = A_p(G) \cap L^r(G)$ for a non-compact locally compact group G , $p \in (1, \infty)$ and $r \in [1, \infty)$. The norm on $A_p^r(G)$ is given by $\|f\|_{A_p^r} = \|f\|_{A_p} + \|f\|_r$, where $\|f\|_r$ is the norm on $L^r(G)$ [11].

With all those examples we see that Segal algebras form a very rich class of Banach algebras. Now we present one of the most important theorems on Segal algebras. See [18, Theorem 9.1].

Theorem 2.2.4. *(Reiter) Let G be a locally compact group. Let $S^1(G)$ be a Segal algebra and let I_S be a closed ideal of $S^1(G)$. Then there exists a unique closed ideal I of $L^1(G)$ such that $I_S = I \cap S^1(G)$ and I is the closure of I_S in $L^1(G)$.*

We can deduce from that theorem and a little work that there is an isomorphism between the maximal ideal space of a Segal algebra and the maximal ideal space of $L^1(G)$. This property concerning the ideal structure is important and the next step was to know if Theorem 2.2.4 can be true for a bigger class of Banach algebras.

Definition 2.2.5. *Let $(A, \|\cdot\|_A)$ be a Banach algebra. A Banach algebra $(B, \|\cdot\|_B)$ is a left (right) abstract Segal algebra in A if the following conditions are satisfied*

- (i) *The algebra B is a dense left (right) ideal of A .*
- (ii) *There is a constant $C > 0$ such that $\|b\|_A \leq C\|b\|_B$ for each $b \in B$.*
- (iii) *There is a constant $M > 0$ such that*

$$\|ab\|_B \leq M\|a\|_A\|b\|_B \quad (2.2)$$

for all $a \in A$ and for all $b \in B$.

A Banach algebra $(B, \|\cdot\|_B)$ is a symmetric abstract Segal algebra in A , if it is a left and right abstract Segal algebra in A .

Of course any Segal algebra is a left abstract Segal algebra. The following are other examples.

Example 2.2.6

- (i) Let H be a Hilbert space. Let $p \in [0, \infty)$. Define for $T \in K(H)$

$$\|T\|_{S_p} = (\text{Tr}(|T|^p))^{\frac{1}{p}}$$

where Tr is the trace. The p^{th} Schatten class $S_p(H)$ is the set of all $T \in K(H)$, such that $\|T\|_{S_p} < \infty$: for example when $p = 1$, $S_p(H)$ is the algebra of trace class operators, if $p = 2$, $S_p(H)$ is the algebra of Hilbert-Schmidt operators. For all $1 < p < \infty$, $S_p(H)$ is an abstract symmetric Segal algebra in $K(H)$.

- (ii) The Lebesgue Fourier algebra is an abstract symmetric Segal algebra in $A(G)$.
- (iii) The Figa-Talamanca Herz Lebesgue algebra $A_p^*(G)$ is an abstract symmetric Segal algebra in $A_p(G)$.

Burnham proved in [5] the same theorem as Theorem 2.2.6, but for abstract Segal algebras with an approximate identity. It is Leinert who gave the general theorem that follows. See [18, Theorem 2.3].

Theorem 2.2.7. (Leinert) *Let A be a Banach algebra and B be a left abstract Segal algebra in A . If J is a closed left ideal in B , then there is a unique closed left ideal I of A such that $J = I \cap B$ and I is the closure of J in A .*

With the help of this theorem, we can prove in particular that there is a 1-1 correspondence between the maximal modular left ideals of B and those of A . See [18, Theorem 3.7] for a proof.

2.3 Banach algebras of type (M)

We collect, in this section, definitions and theorems concerning a type of Banach algebras introduced by Hu, Neufang and Ruan in [13]. The motivation to introduce this new class is Theorem 3 of [13] which follows. Note that we only give the left version (this theorem, but the right version is also true).

Theorem 2.3.1. *(Hu-Neufang-Ruan) Let A be a Banach algebra. Let $T \in LM(A)$. Consider the following ;*

- (i) $T \in A$;
- (ii) $T^{**} = \lambda_m$ for some $m \in A^{**}$;
- (iii) $T^*(A^*) \subseteq \overline{\langle A^*A \rangle}$.

Then the following conditions hold.

- (a) (i) $\rightarrow$ (ii) $\rightarrow$ (iii).
- (b) (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), if A has a BAI.
- (c) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), if A is weakly sequentially complete with a sequential BAI.

As remarked in [13], if A is weakly sequentially complete with a sequential approximate identity, the implication (iii) $\Rightarrow$ (i) holds. The point in this, is that the cardinality of the approximate identity is dominated by the cardinal level of the weak completeness of A . To avoid the problem of this cardinal level and still have the same implication.

Hu–Neufang–Ruan defines the class of Banach algebras of type (M) using Theorems 4 and 5 of [3] which follow

Theorem 2.3.2 (Hu–Neufang–Ruan) *Let A be a Banach algebra with a BAI. Assume that there exists a family $\{A_i\}_{i \in I}$ of closed left and right ideals in A respectively with the following properties*

- (I) *Each A_i is weakly sequentially complete with a sequential approximate identity*
- (II) *For each $i \in I$ there is a right and left A_i module projection p_i from A to A_i respectively*
- (III) *For any $T \in LM(A)$ and $T \in RM(A)$ if $T|_{A_i} \in A_i$ for all $i \in I$ then $T \in A$*

Then the assertion (i)–(ii) and (iii) of Theorem 2.3.1 are equivalent for all $T \in LM(A)$ and $T \in RM(A)$ respectively

Theorem 2.3.3. (Hu–Neufang–Ruan) *Let A be a Banach algebra with a BAI. Assume that for any $T \in LM(A)$ and $T \in RM(A)$ respectively A has a closed subalgebra B of the type as in Theorem 2.3.2 such that*

- (a) *for each $f \in A^*A$ and $f \in AA^*$ we have that $f|_B \in B^*B$ and $f|_B \in BB^*$ respectively*
- (b) *$I \cap B \subseteq B$ and $I \in A$ if $I|_B \in B$*

Then, the assertions (i)–(ii) and (iii) of Theorem 2.3.1 are equivalent for all $T \in LM(A)$ and $T \in RM(A)$ respectively

Now, we introduce the main concept of this section. See [3], Definition 6

Definition 2.3.4. (Hu-Neufang-Ruan) Let A be a Banach algebra as in Theorem 2.3.3. Then A is said to be of type (LM) and of type (RM), respectively. If A is both of type (LM) and of type (RM), then A is said to be of type (M).

Of course, a unital weakly sequentially complete Banach algebra is of type (M). Many algebras arising abstract harmonic analysis are of type (LM), (RM) and (M). The following is contained in Proposition 8, Theorem 10 and Theorem 11 of [3].

Theorem 2.3.5. (Hu-Neufang-Ruan)

- (i) Let G be a locally compact group. Then, $L^1(G)$ is of type (M).
- (ii) Let G be a locally compact group and ω a weight such that $\omega \geq 1$. Then, $(L^1(G), \omega)$ is of type (M).
- (iii) Let G be a locally compact amenable group. Then, $A(G)$ is of type (M).

Chapter 3

The Dual Factorization Property

In this chapter, we will study the following property $A^* = A^*1$ for a Banach algebra A . The first authors to introduce a terminology to describe this property are Lau and Ülger in [1]. They say that a Banach algebra A with a BAI factors on the left (resp. right) if $1 = 4^*A$ (resp. $1^* = A1^*$). In the first section we generalize the notion of Lau and Ülger and give examples. Some of the characterizations of Lau and Ülger are also part of this section. We finish the section with a simple application of the weak dual factorization property by giving a generalization of a result by Ülger [3], Lemma 3.1]. The second section consists of another approach to study the dual factorization property for a Banach algebra A with a BAI. Through this approach, we give a new characterization for the factorization $1^* = 1^*A$ involving the left multiplier algebra of A . The last section of this chapter will treat the dual factorization property for Banach algebras without BAI. We will be interested in a specific class of Banach algebras abstract Segal algebras. We will prove that there is no proper abstract Segal algebra B with $B^* = B^*B$.

3.1 Definition and characterization

As we mentioned before, Lau and Ülger in [17] were the first to study in depth, the property $A^* = A^*A$, for a Banach algebra A with a BAI. We will use a different terminology than the one introduced in [17]. The reason is, if A has a BAI, then by Cohen's factorization theorem, A^*A is a closed subspace of A^* . However, nothing insures that A^*A is a subspace of A^* in general.

Definition 3.1.1. *Let A be a Banach algebra. We say that*

- (i) *A has the left strong dual factorization property if $A^* = A^*A$,*
- (ii) *A has the left dual factorization property if $A^* = \overline{A^*A}$,*
- (iii) *A has the left weak dual factorization property if $A^* = \overline{\langle A^*A \rangle}$.*

The definitions of the right and two sided are defined analogously.

Of course, when A has a BAI, then (i), (ii) and (iii) are the same. Also, (i) implies (ii) which implies (iii) in the previous definition.

Now, we present some examples to illustrate the above properties .

Example 3.1.2.

- (i) Any faithful reflexive Banach algebra A has the left and right weak dual factorization property. We only prove the left version since the right one is similar.

Suppose that $A^* \neq \overline{\langle A^*A \rangle}$. Then, there exists a non-zero $m \in A^{**} \setminus A$ such that $m|_{\overline{\langle A^*A \rangle}} = 0$. In particular,

$$0 = \langle m, f \cdot a \rangle = \langle a \sqsubset m, f \rangle$$

for any $f \in A^*$ and $a \in A$. This implies that $a \sqsubset m = 0$ for all $a \in A$ which is a contradiction, since A is faithful.

- (ii) The Banach algebra $l^2(I)$, where I is an index set, equipped with the pointwise multiplication has the dual factorization property. We have that $\overline{l^2(I) \cdot l^2(I)} = \overline{l^1(I)} = l^2(I)$. Here, the closure is taken in the norm of $l^2(I)$.
- (iii) The Banach algebra $K(c_0)$ has the left strong dual factorization property but not the right one [17, Example 2.5].
- (iv) Any Arens regular Banach algebra with a BAI has the strong dual factorization property [30, Theorem 3.1]

The list of examples above does not cover all possible cases. Here, we did not present a Banach algebra without BAI with the strong dual factorization property. We will deal with this case in Section 3.4. Another missing example is a Banach algebra A with the weak dual factorization property, but without the dual factorization property, such example is not known.

Now, we present two characterizations, due to Lau and Ülger, of the strong dual factorization property for a Banach algebra with a BAI. We refer the reader to [17] for proofs of these results. The main reason to present them is that they constitute the starting point to our study of the dual factorization property. They will be used in chapter 4, as well.

Lemma 3.1.3 (Lau-Ulger [1, Lemma 2.1]) *Let A be a Banach algebra with a BAI $\{e_\alpha\}_{\alpha \in I}$. Then, A has the left (resp. right) strong dual factorization property if and only if for each f in A^* the net $\{f|_{e_\alpha}\}_{\alpha \in I}$ (resp. $\{e_\alpha|_f\}_{\alpha \in I}$) converges weakly to f .*

Proposition 3.1.4. (Lau-Ulger [1, Proposition 2.2]) *Let A be a Banach algebra with a BAI. Then the following assertions hold:*

(i) *A has the left strong dual factorization property if and only if the algebra $(A^{**}, \square)$ is unital.*

(ii) *A has the right strong dual factorization property if and only if the algebra $(A^{**}, \triangle)$ is unital.*

We present here an application of the notion of weak dual factorization property. This application can be viewed as a generalization of Lemma 3.1 of [1] which follows

Lemma 3.1.5. (Ulger) *Any weakly sequentially complete Arens regular Banach algebra with a sequential bounded approximate identity is unital.*

Lemma 3.1.6. *Let A be a weakly sequentially complete Banach algebra with a sequential approximate identity. If A has the left weak dual factorization property then A is unital.*

Proof. For any $\mu \in LM(1)$, $\mu^*(A^*) \subseteq A^* = \overline{\langle A^*1 \rangle}$. By the remark after Theorem 3.1 we have that $\mu \in A$. In particular, we can deduce that the identity operator is in A , thus A is unital. $\square$

As we can see, using Lemma 3.1.6 and Example 3.1 (iv), one deduces Lemma 3.1.7.

3.2 Banach algebras with the dual factorization property with a BAI

In this part of chapter 3, our goal is to give a new characterization of the strong dual factorization property for a Banach algebra with a BAI. We also use a new approach to study this property. Our method can also be employed to understand the Banach algebra $(A^*A)^*$. We obtain as corollaries various results about this algebra which are contained in [17].

The work of Lau and Ülger on the strong dual factorization property depends to a large extent on Lemma 3.1. They use it to prove Proposition 3.2 which is their main tool to deduce the left or right strong dual factorization property for a Banach algebra. We prove Proposition 3.2 independently of Lemma 3.1 and from this deduce a new characterization of the strong dual factorization property. The inspiration for our approach is Theorem 1.1 of [1] which follows.

Theorem 3.2.1. *(Baker Lau Pym) Let A be a Banach algebra with a BRAI. Also, $(A^*A)^\perp$ is the left ideal of the right annihilators in A^{**} . $(A^*A)^*$ has a natural Banach algebra structure under which it is isomorphic (as a Banach algebra, but not necessarily isometric) to the Banach algebra $\text{Hom}_{A,r}(A^*, A^*)$ of bounded linear of the right A -module homomorphism on A^* . In particular, $(A^*A)^*$ has a two-sided identity*

Remark 3.2.2. If the BRAI is bounded by 1, then $(A^*A)^*$ is isometric isomorphic to $\text{Hom}_{A,r}(A^*, A^*)$.

The link between $\text{Hom}_{A,r}(A^*, A^*)$ and $(A^*A)^*$ is used also by Neufang in [21] to prove that $RV(A)$ can be identify with the topological centre of $(A^*A)^*$, which can be defined analogously to the left topological centre of A^{**} , for Banach algebras with the

F_κ property and the Mazur property of level κ , where $\kappa > \aleph_0$ is a cardinal. Here, we present Proposition 4.1 of [25] which we will use later and also to establish our notation through this section.

Proposition 3.2.3. *Let A be a Banach algebra with a BRAI $\{e_\alpha\}_{\alpha \in I}$. Define $\rho: (A^*A)^* \rightarrow \text{Hom}_{A, \tau}(A^*, A^*)$ by*

$$\langle \rho(m)(f), a \rangle = \langle m, f \cdot a \rangle$$

for $m \in (A^*A)^*$, $f \in A^*$ and $a \in A$. For a fixed mixed unit E of a BRAI in A^{**} , define $\tau_E: \text{Hom}_{A, \tau}(A^*, A^*) \rightarrow (A^*A)^*$ by

$$\langle \tau_E(\omega), g \rangle = \langle E, \omega(g) \rangle$$

for $\omega \in \text{Hom}_{A, \tau}(A^*, A^*)$ and $g \in A^*A$. Then ρ and τ_E are algebra homomorphisms which are inverse to each other.

The following diagram illustrate the aboves, where E is a fixed mixed unit in A^{**}

$$\begin{array}{ccc} (A^{**}, \square) & \xrightarrow{\pi} & (A^*A)^* \\ & \uparrow \begin{array}{c} \tau_E \quad \rho \\ \downarrow \end{array} & \\ & \text{Hom}_{A, \tau}(A^*, A^*) & \end{array} \quad (3.1)$$

In this diagram, π is the projection map. We define a homomorphism from $(A^{**}, \square)$ to $\text{Hom}_{A, \tau}(A^*, A^*)$ that will make the diagram commuting. This construction is known, but has not been used for general Banach algebras.

Define $\Phi : (A^{**}, \square) \rightarrow \text{Hom}_{A,r}(A^*, A^*)$ by $\Phi(m) = {}_mT$ where ${}_mT(f) = m \square f$ for $m \in A^{**}$ and $f \in A^*$. We now add the map Φ to the previous diagram where E is a fixed mixed unit in A^{**} . Now, let us prove that this diagram commutes.

$$\begin{array}{ccc}
 A^{**} & \xrightarrow{\pi} & (A^*A)^* \\
 & \searrow \Phi & \uparrow \tau_E \\
 & & \text{Hom}_{A,r}(A^*, A^*) \\
 & & \downarrow \rho
 \end{array} \tag{3.2}$$

Lemma 3.2.4. *Let A be a Banach algebra with a BRAI. Then $\Phi = \rho \circ \pi$.*

Proof. Let $m \in A^{**}$ and ${}_mT : A^* \rightarrow A^*$ be defined by ${}_mT(f) = m \square f$. Let ρ and τ_E as in Proposition 3.2.3. We have

$$\begin{aligned}
 \langle \tau_E(\Phi(m)), f \cdot a \rangle &= \langle E, (\Phi(m))(f \cdot a) \rangle \\
 &= \langle E, {}_mT(f \cdot a) \rangle \\
 &= \langle E, m \square (f \cdot a) \rangle \\
 &= \langle E, (m \square f) \triangle a \rangle \\
 &= \langle a \square E, m \square f \rangle \\
 &= \langle a \triangle m, f \rangle \\
 &= \langle m, f \cdot a \rangle \\
 &= \langle \pi(m), f \cdot a \rangle
 \end{aligned}$$

for $f \in A^*$ and $a \in A$. □

The definition of Φ holds for general Banach algebras, since we only used the module action of A^{**} on A^* . In this generality, the next two propositions address the injectivity

and surjectivity of Φ . Those two properties of Φ characterize natural properties of the Banach algebra A .

Proposition 3.2.5. *Let A be a Banach algebra. Then, $\ker(\Phi) = \overline{\langle A^*A \rangle}^\perp$. So, Φ is injective if and only if $A^* = \overline{\langle A^*A \rangle}$.*

Proof. We shall show that $\ker(\Phi) = (\overline{\langle A^*A \rangle})^\perp$. Note that $(A^*A)^\perp$ is the right annihilator of $(A^{**}, \square)$, denoted by $\text{Ann}_r(A^{**}, \square)$. Let us prove this claim. Let $m \in \text{Ann}_r(A^{**}, \square)$. Then,

$$0 = \langle a \square m, f \rangle = \langle m, f \cdot a \rangle$$

for each $a \in A$ and $f \in A^*$. We can reverse this argument and using the weak*-weak*-continuity of the map $A^{**} \ni n \rightarrow n \square m$ to obtain the other inclusion. The last step is to prove that $\text{Ann}_r(A^{**}, \square) = \ker(\Phi)$. Let $m \in \ker(\Phi)$. Let $a \in A$ and $f \in A^*$. We have that

$$\begin{aligned} 0 = \langle \Phi(m)(f), a \rangle &= \langle {}_m T(f), a \rangle \\ &= \langle m \square f, a \rangle \\ &= \langle a \square m, f \rangle \end{aligned}$$

Thus, $m \in \text{Ann}_r(A^{**}, \square)$. The reverse argument shows the other inclusion as well. We conclude that $\ker(\Phi) = A^*A^\perp$. $\square$

Proposition 3.2.6. *Let A be a Banach algebra. Then, Φ is surjective if and only if A has a BRAI.*

Proof. Suppose that Φ is surjective. In particular, there is $E \in A^{**}$ such that $\Phi(E) = \text{id}$. Thus, $f = {}_E T(f) = E \square f$ for any $f \in A^*$. For all $n \in A^{**}$,

$$\langle n, f \rangle = \langle n, E \square f \rangle = \langle n \square E, f \rangle.$$

So, E is a right unit for $(A^{**}, \square)$, therefore A has a bounded right approximate identity.

Now suppose that A has a BRAI. Theorem 3.2.6 and Lemma 3.2.5 give the conclusion. $\square$

Corollary 3.2.7. *Let A be a Banach algebra with a BAI. If id is in the image of Φ , then Φ is surjective.*

Proof. By assumption id is the image of Φ , there exists $E \in A^{**}$ such that $\Phi(E) = id$. Let $m \in A^{**}$ and $f \in A^*$. Observe that

$$\begin{aligned} \langle m, f \rangle &= \langle m, id(f) \rangle \\ &= \langle m, \Phi(E)(f) \rangle \\ &= \langle m, E \square f \rangle \\ &= \langle m \square E, f \rangle. \end{aligned}$$

Therefore, the element E is a right identity for the left Arens product. Then A has a BRAI. Using Proposition 3.2.6, we have the result. $\square$

Proposition 3.2.6 gives us a lot of information about $Hom_{A^*}(A^*, A^*)$ when A has a BRAI. If we are interested in a particular properties of $(A^*A)^*$ or in the projection map π , it is useful to look at $Hom_{A^*,r}(A^*, A^*)$. The next result is an example of Theorem 1.3(1) of [1] which affirms that if A has a BAI, π is injective on A regarded as a subalgebra of A^{**} . We will generalize this and show that, in fact, π is injective on two generally bigger sets than A , both of which contain A as a subalgebra.

Proposition 3.2.8. *Let A be a Banach algebra with a BAI. Then, $\pi : A^{**} \rightarrow (A^*A)^*$ is injective on $Z_r(A^{**})$ and on $RM(A)^{op}$, regarded as a subalgebras of $(A^{**}, \square)$.*

Proof. By Lemma 3.1, we only need to show that Φ is injective on $Z_r(A^{**})$ and $RM(A)$. First, consider the case of $Z_r(A^{**})$. Let $m_1, m_2 \in Z_r(A^{**})$ be such that $\Phi(m_1) = \Phi(m_2)$. Let $n \in A^{**}$. We have that ${}_n T(f) = {}_{m_2} T(f)$ for all $f \in A^*$. Thus,

$$\begin{aligned}
 \langle n \Delta m_1, f \rangle &= \langle n \sqcap m_1, f \rangle \\
 &= \langle n - m_1 \sqcap f \rangle \\
 &= \langle n, {}_{m_1} T(f) \rangle \\
 &= \langle n, {}_{m_2} T(f) \rangle \\
 &= \langle n, m_2 \sqcap f \rangle \\
 &= \langle n \sqcap m_2, f \rangle \\
 &= \langle n \Delta m_2, f \rangle.
 \end{aligned}$$

Since this is true for an arbitrary $f \in A^*$, we conclude that $n \Delta m_1 = n \Delta m_2$. In particular for $n = E$, where E is a mixed unit, and so we have that $m_1 = m_2$.

Now, we move on to the case of $RM(A)$. Let $L_1, L_2 \in RM(A)$ such that $\Phi(L_1^{**}(E)) = \Phi(L_2^{**}(E))$, where E is a fixed weak*-cluster point of a BAI. For all $f \in A^*$ and $a \in A$, we have then

$$\begin{aligned}
 \langle L_1(a), f \rangle &= \langle E, L_1^*(f \cdot a) \rangle \\
 &= \langle L_1^{**}(E), f \cdot a \rangle \\
 &= \langle L_1^{**}(E) \sqcap f, a \rangle \\
 &= \langle L_2^{**}(E) \sqcap f, a \rangle \\
 &= \langle L_2(a), f \rangle
 \end{aligned}$$

□

Define

$$\begin{aligned} M_1 &= \{m \in A^{**} \mid A \square m \subseteq A\}, \\ \widetilde{M}_1 &= \{m \in (A^*A)^* \mid A \square m \subseteq A\}. \end{aligned}$$

In [17], the authors show that if A is a Banach algebra with a BAI bounded by 1, then $\widetilde{M}_1$ is isometrically isomorphic to $RM(A)$. They also use the set M_1 to characterize the left strong dual factorization property in Proposition 2.1 of [17], which follows.

Theorem 3.2.9. (*Lau–Ülger*) *Let A be a Banach algebra with a BAI. Then A has the left strong dual factorization property if and only if $M_1 \subseteq Z_\ell(A^{**})$.*

Remark 3.2.10. The proof of Lau and Ülger shows that if a Banach algebra A , with or without a BAI, has the left weak dual factorization property, then $M_1 \subseteq Z_\ell$. If A has no BAI, then $M_1 \subseteq Z_\ell$ does not imply any level of dual factorization property. A counter-example is $(l^1, \cdot)$.

The sets M_1 and $\widetilde{M}_1$ are very important. In fact, in the context of the commutative diagram (), we prove in the next lemma that there is a strong link between $RM(1)$ and M_1 . After that, we give a new shorter proof of Theorem 4.4 of [17], which is Theorem 3.2.11. We recall that we identified $RM(A)$ in $Hom_{A^r}(A^*, A^*)$ with the set of elements of $Hom_{A^r}(A^*, A^*)$ which are weak*-weak*-continuous. This identification is not only an isomorphism, but also an isometry.

Lemma 3.2.11. *Let A be a Banach algebra and $m \in A^{**}$. If $\Phi(m) \in RM(1) \subseteq Hom_{A^r}(A^*, A^*)$, then $m \in M_1$. In particular, when A has a BRAI, the preimage of $RM(A)$ is exactly M_1 .*

Proof. Let $m \in A^{**}$ such that $\Phi(m) \in RM(A) \subseteq Hom_{A^r}(A^*, A^*)$. Then, ${}_m f$ is weak*-

weak*-continuous. For a net $\{f_\alpha\}_{\alpha \in I}$ in A^* converging weak* to 0, we have that

$$\begin{aligned} \langle a \sqcap m, f_\alpha \rangle &= \langle m \sqcap f_\alpha, z \rangle \\ &= \langle {}_m T(f_\alpha, a) \rangle \rightarrow 0 \end{aligned}$$

for each $a \in A$. Thus, $a \sqcap m$ is weak*-continuous for each $a \in A$ which implies that $m \in M_1$.

Now, suppose that A has a BRAI. Then, by Proposition 3.2.10, Φ is surjective. The remaining thing to show is that $\Phi(M_1) \subseteq RM(A)$. Let $m \in M_1$. We have to show that ${}_m T$ is weak*-weak*-continuous, but

$$\begin{aligned} \langle {}_m T(f_\alpha, a) \rangle &= \langle m \sqcap f, a \rangle \\ &= \langle a \sqcap m, f_\alpha \rangle \rightarrow 0 \end{aligned}$$

for all $a \in A$.

□

Theorem 3.2.12. (Lau-Ülger [11], Theorem 4.4]) Let A be a Banach algebra with a BRAI bounded by 1. Then $RM(A)$ is isometrically isomorphic to $\widetilde{M}_1$.

Proof. First, it is easy to see that $\pi(M_1) \subseteq \widetilde{M}_1$. By Lemma 3.2.11, $\Phi(M_1) = RM(A)$. So by using Lemma 3.2.12, we have $\tau_E(RM(A)) \subseteq \widetilde{M}_1$, where E is a fixed mixed unit. To prove the reverse inclusion, we proceed by contradiction. Suppose there exists $m_0 \in \widetilde{M}_1$ such that $\rho(m_0) \notin RM(A)$. By Lemma 3.2.11, there exists $m \in A^{**}$ such that $\pi(m) = m_0$, and so $\Phi(m) \notin RM(A)$. Hence, there exists a net $\{f_\alpha\}$ converging weak* to 0 in A^* such that $\{{}_m T(f_\alpha)\}$ does not converge weak* to 0. Let $a \in A$. By Cohen's

factorization theorem, there is $b, c \in A$ such that $a = bc$. Then,

$$\begin{aligned}
 \langle mT(f_\alpha), a_i \rangle &= \langle m \square f_\alpha, a \rangle \\
 &= \langle \epsilon \square m, f_\alpha \cdot b \rangle \\
 &= \langle \pi(\epsilon \square m), f_\alpha \cdot b \rangle \\
 &= \langle b \square (\pi(\epsilon) \square \pi(m)), f_\alpha \rangle \rightarrow 0
 \end{aligned}$$

which is a contradiction. So $\rho(\widetilde{M}_1) \subseteq RM(A)$. Since ρ and τ_E are isomorphism, by Proposition 3.2.12, we conclude that $RM(A)$ is isometrically isomorphic to $\widehat{M}_1$. $\square$

Remark 3.2.13. Here, the isometry is just due to the contractness of the BAI. Without the bound of 1, we still have an isomorphism.

Now that we have studied various properties of Φ , we can return to the strong dual factorization property. As we mentioned, Lau and Ülger used Proposition 3.2.12 as their main tool. The proof of this proposition depends on Lemma 3.2.11. Here, we will present a different approach.

Theorem 3.2.14. (*Lau-Ülger*) *Let A be a Banach algebra with a BAI. Then,*

(i) $A^* = A^*A$ if and only if $(A^{**}, \square)$ is unital;

(ii) $A^* = AA^*$ if and only if $(A^{**}, \triangle)$ is unital.

Proof. We prove only the first assertion. The second one can be proved similarly. Let Φ be as defined before. This is a homomorphism and from Proposition 3.2.12, $\ker(\Phi) = (A^*A)^\perp = \text{Ann}_r(A^{**}, \square)$. Also, for any mixed unit E of A^{**} , $\Phi(E) = id$.

Suppose that $A^* = A^*A$. Then $(A^*A)^* = \{0\}$, and so Φ is injective. Let E be a mixed unit of A^{**} and $m \in A^{**}$. Then,

$$\begin{aligned}\Phi(E \sqcup m) &= \Phi(E) \circ \Phi(m) \\ &= \Phi(m).\end{aligned}$$

So, by the injectivity of Φ , we conclude that $E \sqcup m = m$. Thus, $(A^{**}, \sqcup)$ is unital.

Suppose that $(A^{**}, \sqcup)$ is unital with the unit E . Let $m \in \ker(\Phi) = \text{Ann}_r(A^{**}, \sqcup)$. Thus, in particular, $m = E \sqcup m = 0$. Hence, Φ is injective, i.e., $A^* = A^*A$ by Proposition . $\square$

We have that $RM(A)$ is embedded into different algebras like A itself. The following diagram, which is easily checked to commute, shows the complete picture of the situation.

$$\begin{array}{ccc} A^{**} & \xrightarrow{\pi} & (A^*A)^* \\ & \searrow \Phi & \uparrow \tau_E \quad \rho \\ & & \text{Hom}_{A,r}(A^*, A^*) \\ & \nearrow & \uparrow \\ & RM(A) & \\ \uparrow & \nearrow & \\ A & & \end{array}$$

Lau Ülger mention, in [17], that the characterization of the Theorem may be not easy to use, since it is hard to determine the set $\{m \in A^{**} \mid m \cdot A \subseteq A\}$. One checks directly that $RM(A) \subseteq M_1$. Therefore, a natural question to ask is: When does M_1 coincide with $RM(A)$? In fact, the answer is a new characterization of the left strong dual factorization property.

Theorem 3.2.15. *Let A be a Banach algebra with a BAI. Then $A^*A = A^*$ if and only if $M_1 = RM(A)$ in $(A^{**}, \square)$.*

Proof. We only need to show that $M_1 \subseteq RM(A)$. Since Φ is injective, by Proposition 3.2.14 and the diagram commutes, it is enough to verify that $\Phi(M_1) \subseteq RM(A)$ in $Hom_{A,r}(A^*, A^*)$. But, this holds by Lemma 3.2.13.

Now, suppose that $M_1 = RM(A)$. We have $A^{**} = (A^*A)^* \oplus (A^*A)^\perp$ on the Banach space level, and $(A^*A)^\perp = Ann_r(A^{**}, \square)$. Observe that $Ann_r(A^{**}, \square) \subseteq M_1 = RM(A)$. Now, since Φ is injective on $RM(A)$, by Lemma 3.2.13 and Proposition 3.2.14, $Ann_r(A^{**}, \square) = \ker(\Phi) = \{0\}$, due to Proposition 3.2.14. Therefore, we conclude that $A^*A = A^*$. $\square$

3.3 Banach algebras with the strong dual factorization property without a BAI

The strong dual factorization property is, indeed, very a strong property. For example, for the Fourier algebra $A(G)$, the strong dual factorization property implies that $A(G)$ is unital. The strong dual factorization property has mostly been studied for Banach algebras with a BAI (the Fourier algebra is an exception). In this chapter, we will pay attention to a specific class of Banach algebras: abstract Segal algebras. Those algebras do not have a BAI. This is a consequence of Cohen's factorization theorem. The axioms needed to define abstract Segal algebras are very simple, and they are satisfied by a huge number of Banach algebras. For a faithful Banach algebra A , the image of the left regular representation of A , denoted by L_A , is a left ideal in $LM(A)$. Then A is an

left abstract Segal algebra in $\overline{L_A}$. So any faithful Banach algebra is an abstract Segal algebra. In this section we prove two useful lemmas on the set B^*B for a Segal algebra B . After that, we show that there is no proper right abstract Segal algebra with the left strong dual factorization property.

The next lemma is a generalization of a part of the proof of Lemma 1 of [23]. Actually the proof in [23] holds for right abstract Segal algebras. We present it here for the convenience of the reader.

Lemma 3.3.1. *Let $(A, \|\cdot\|_A)$ be a Banach algebra and $(B, \|\cdot\|_B)$ a right abstract Segal algebra in A . Then, each element of B^*B extends uniquely to an element of A^* . Thus, B^*B can be identified with a subset of A^* .*

Proof. Let $\phi \in B^*$ and $b \in B$. Observe that

$$|\langle \phi, b \cdot c \rangle| = |\langle \phi, bc \rangle| \leq \|\phi\|_{B^*} \|bc\|_B \leq M \|\phi\|_B \|b\|_B \|c\|_A$$

where M is a constant. This inequality shows that $\phi \cdot b$ is bounded on B with respect to the norm of $\|\cdot\|_A$. Since B is dense in A , $\phi \cdot b$ can be extended in a unique way to A^* . $\square$

A natural question is: When can B^*B identify with a subset of A^* ? Again the proof of Lemma 1 in [23] gives a partial answer, and the argument still holds for abstract symmetric Segal algebras.

Corollary 3.3.2. *Let $(A, \|\cdot\|_A)$ be a Banach algebra with a BAI and $(B, \|\cdot\|_B)$ an abstract symmetric Segal algebra in A . Then, B^*B can be identified with a subset of A^*A .*

Proof. By Cohen's factorization theorem $B = BA$. So $B^*B = B^*(BA) = (B^*B)A$.

Let $h \in B^*B$. Then $h = f \cdot a$, for some $f \in B^*B$ and $a \in A$. By Lemma 3.3.2, there is an extension $\tilde{f} \in A^*$ such that $\tilde{f}|_B = f$. One checks that $\tilde{f} \cdot a \in A^*A$ extends h and the extension is unique by Lemma 3.3.2. $\square$

With Lemma 3.3.2, we have all we need to prove the main theorem of this section. This result is a new fact on the strong dual factorization property.

Theorem 3.3.3. *Let $(A, \|\cdot\|_A)$ be a Banach algebra. Then, there is no proper right abstract Segal algebra $(B, \|\cdot\|_B)$ in A such that $B^* = B^*B$.*

Proof. Suppose there is a proper right abstract Segal algebra $(B, \|\cdot\|_B)$ in A such that $B^* = B^*B$. Let $\iota : (B, \|\cdot\|_B) \rightarrow (A, \|\cdot\|_A)$ be the natural inclusion map which is continuous by the definition of a right abstract Segal algebra. Let $P : A^* \rightarrow B^*$ the adjoint of ι . So, $P(f) = f|_B$ for $f \in A^*$. By the density of B , P is injective. In view of Lemma 3.3.2 and the hypothesis that $B^* = B^*B$, P is surjective and thus, a homeomorphism. The operator $P^* : B^{**} \rightarrow A^{**}$ is also a homeomorphism. Hence, there are constants D_1 and D_2 such that

$$D_1 \|m\|_{B^{**}} \leq \|P^*(m)\|_{A^{**}} \leq D_2 \|m\|_{B^{**}}$$

for all $m \in B^{**}$. But, $P^*(b) = \iota^{**}(b) = \iota(b)$ for all $b \in B$. $\square$

Using the previous result we show that the following Banach algebras do not have the left or the right strong dual factorization property:

- (i) any proper Segal algebra of $L^1(G)$ where G an infinite non-discrete locally compact group;

- (ii) any p^{th} -Schatten class, in particular $N(H)$, the algebra of nuclear operators and the algebra of Hilbert Schmidt operators, $HS(H)$, over an infinite dimensional Hilbert space H ;
- (iii) the Lebesgue Fourier algebra $L^1(G) \cap A(G)$, where G an infinite locally compact group;
- (iv) the Figa-Talamanca Herz Lebesgue algebra $A_p^r(G)$, where G an infinite locally compact group.

Even if this is an long list, we can say more. The most important consequence is the following result. It is a corollary of Theorem 3.3.3, but since it is an important result in itself, we present it as a theorem.

Theorem 3.3.4. *Let A be a faithful Banach algebra. If A has the left, respectively right, strong dual factorization property, then the norm of A is equivalent to the norm of $RM(A)$, respectively $LM(A)$.*

Proof. Suppose that the norm of A is not equivalent to the norm of its right multiplier algebra. Therefore, A is a proper right abstract Segal algebra in $\overline{RA}$, the closure of its right regular representation, and so, we obtain a contradiction due to Theorem 3.3.3. □

The reverse question makes sense. For example, there are no proper Segal algebras in $L^1(G)$ when G is discrete, which is exactly when $L^1(G)$ has the strong dual factorization property. We could ask if it is possible to get a proper right, respectively left, abstract Segal algebra in a Banach algebra with the left, respectively right, strong dual factorization property. Yes it is, even if it is not the case of Segal algebras in $L^1(G)$, for

an infinite locally compact group G . For example, being a C^* -algebra $K(H)$ is Arens regular, has a BAI, and so has the strong dual factorization property [30, Theorem 3.1]. But, the algebra of Hilbert-Schmidt operators is a proper abstract Segal algebra in $K(H)$, when H is an infinite dimensional Hilbert space.

Chapter 4

The Strong Topological Centre

This chapter is devoted to solving the following question: Between two classes of Banach algebras, like $(l^1, \cdot)$ and C^* -algebras, which one is more Arens regular? Of course, if we want to compare Banach algebras in a specific aspect, such as Arens regularity, we need a tool to measure this property. We introduce the notion of strong topological centre. This will be our tool to quantify Arens regularity. The definition of the strong topological centre and examples compose essentially the first section of this chapter. We show the strong topological centre is contained in the topological centre, but also in another specific set of the second dual of a Banach algebra, which is sometimes isomorphic to its left or right multiplier algebra. The second part of this chapter studies this new concept for Banach algebras with a BAI. We characterize Banach algebras A such that the left strong topological centre of A is isomorphic to $LM(A)$. Since the left strong topological centre of a Banach algebra A lies between A and $Z_\ell(A^{**})$, many possible cases can arise. With this characterization, we are able to produce a list that covers all those cases. In Section 4.3, we determine the right strong topological centre of convolution algebra $N(l^p(G))$ of the nuclear operators on $l^p(G)$, where G is a discrete

group. We finish this chapter with the study of the multiplier algebra of the strong topological centre.

4.1 Definition and examples

Before introducing the definition of the strong topological centre, we make some observations. Recall that an operator T between dual Banach spaces is weak*-weak*-continuous if and only if T has a pre-adjoint. Also, the left topological centre of a Banach algebra A can be defined as

$$Z_\ell(A^{**}) = \{m \in A^{**} \mid \lambda_m \text{ is weak*-weak*-continuous}\}.$$

So, an element $m \in A^{**}$ belongs to $Z_\ell(A^{**})$ if and only if there is an operator $S \in B(A^*)$ such that $\lambda_m = S^*$. But it is natural to go one level further, the idea of the strong topological centre comes from this observation: instead of asking that λ_m be the adjoint of an operator $S \in B(A^*)$, we will ask that λ_m be the second adjoint of an operator $T \in B(A)$.

Definition 4.1.1. *Let A be a Banach algebra. The left strong topological centre and the right strong topological centre of A are defined, respectively, by*

$$\begin{aligned} SZ_\ell(A^{**}) &= \{m \in A^{**} \mid \lambda_m = T^{**} \text{ for some } T \in B(A)\}; \\ SZ_r(A^{**}) &= \{m \in A^{**} \mid \rho^m = T^{**} \text{ for some } T \in B(A)\}. \end{aligned}$$

Remark 4.1.2. The condition to be a second adjoint for the left multiplication by m , for $m \in A^{**}$, already appears in [13]. There, the authors ask that the left multiplication be a second adjoint of a left multiplier of A . In fact, this is equivalent. The reason why the authors studied this condition was to characterize a Banach algebra A in $LM(A)$ and in A^{**} . It was not link it to Arens regularity.

Of course, we have that $A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**})$. We also note that $SZ_\ell(A^{**})$ is a closed subalgebra of $(A^{**}, \sqcup)$. Obviously, the condition to be in the strong topological centre is more restrictive than the one for the topological centre. There is no reason why λ_m should be a second adjoint of a $T \in B(A)$ for $m \in Z_\ell(A^{**})$.

Hence, if two Arens regular Banach algebras A and B are such that $SZ_\ell(A^{**}) = A^{**}$ and $SZ_\ell(B^{**}) \neq B^{**}$, we can affirm that A is more Arens regular than B . For a Banach algebra A , the condition $SZ_\ell(A^{**}) = A^{**}$ seems to be very strong. At first glance, reflexive Banach algebras have this property. A very different example is the following.

Example 4.1.3. Let $A = (l^1, \cdot)$. This is a commutative Banach algebra and is also Arens regular. Since $(l^1)^{**} = c_0^{***}$, we can decompose as follows: $(l^1)^{**} = l^1 \oplus c_0^-$. Also, $l^1 \cdot l^\infty \subseteq c_0$. Let $a \in l^1$, $\lambda \in l^\infty$, and $m \in c_0^-$. Then, $\langle m \sqcup 1, \lambda \rangle = \langle m, a \cdot \lambda \rangle = 0$. Thus $m \sqcup n = 0$ for all $n \in (l^1)^{**}$. Therefore, for $m \in (l^1)^{**}$, $\lambda_m = (L_{m_1})^{**}$ for some $m_1 \in l^1$. An analogous argument with ρ^m can be done. So, we conclude that $SZ_\ell(A^{**}) = SZ_r(A^{**}) = A^{**}$.

In Definition 4.1.1, we only ask that the left multiplication by an element $m \in A^{**}$ is a second adjoint of $T \in B(A)$. As we mentioned in Remark 4.1.2, the operator $T \in B(A)$ that we are looking for is in fact in $LM(A)$. This observation leads us to present another characterization of the strong topological centre, which is contained in Theorem 17 of [13].

Lemma 4.1.4. (*Hu Neaufang Ruan*) *Let A be a Banach algebra. Then,*

$$SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \sqcap A \subseteq A\}.$$

$$SZ_r(A^{**}) = Z_r(A^{**}) \cap \{m \in A^{**} \mid A \triangle m \subseteq A\}.$$

Proof. We will prove the left version of the result. The proof for the right one is

similar. The inclusion $SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \square A \subseteq A\}$ is clear by the comment before Lemma 4.1.4. We only need to prove the reverse inclusion. Let $n \in Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \square A \subseteq A\}$. Since, $n \in Z_\ell(A^{**})$, then the map λ_n is weak*-weak*-continuous. Let $T \in B(A^*)$ such that $T^* = \lambda_n$. It remains to show that T is weak*-weak*-continuous, but since n belongs to the set $\{m \in A^{**} \mid m \square A \subseteq A\}$, we have indeed that $T^*(A) \subseteq A$. $\square$

This result allows us to determine the strong topological centre in some extreme cases. Of course, it is obvious that if A is left strongly Arens irregular, then $SZ_\ell(A^{**}) = A$. Another extreme case is when A is Arens regular. With Lemma 4.1.4, another case is covered: the one where A is a left ideal in A^{**} .

Corollary 4.1.5. *Let A be a Banach algebra.*

(i) *If A is Arens regular, then $SZ_\ell(A^{**}) = \{m \in A^{**} \mid m \square A \subseteq A\}$.*

(ii) *If A is a left ideal in A^{**} , then $SZ_\ell(A^{**}) = Z_\ell(A^{**})$.*

Corollary 4.1.6. *Let A be a C^* -algebra. Then, $SZ_\ell(A^{**}) = \{m \in A^{**} \mid m \square A \subseteq A\}$.*

With Corollary 4.1.5 and Example 4.1.3, we can now say, comparing l^1 with a commutative C^* -algebra, which one is more Arens regular. The answer is l^1 unless the commutative C^* -algebra is an ideal in its second dual. Note that an infinite dimensional unital C^* -algebra A cannot be a left or right ideal in A^{**} . Here, we specify a commutative C^* -algebra since l^1 is also commutative.

For the Arens regularity, there is no difference between the left and the right since $Z_\ell(A^{**}) = A^{**}$ if and only if $Z_r(A^{**}) = A^{**}$. However, for the strong topological centre,

this situation may not occur. Again, this is a consequence of Lemma 4.1.6. The following example shows how big the difference can be between strong topological centres.

Example 4.1.7. Let S be an infinite discrete right zero semigroup with more than two elements, i.e., S is a set with the product defined by $st = t$ for $s, t \in S$. Consider $l^1(S)$ with the convolution product. We may identify $l^1(S)^{**}$ with $M(\beta S)$ where βS is the Stone-Ćech compactification of S . For $u, v \in M(\beta S)$, we have

$$u \square v = u \triangle v = \langle u, 1 \rangle v \quad (4.1)$$

Thus, $l^1(S)$ is Arens regular. See [4, Example 7.30]. From equation (4.1), we see that for $u \in M(\beta S)$ and $v \in l^1(S)$, $u \square v \in l^1(S)$ and so, $l^1(S)$ is a left ideal in $M(\beta S)$. Since $l^1(S)$ is Arens regular and is a left ideal in its second dual, we have $SZ_l(M(\beta S)) = M(\beta S)$, by Lemma 4.1.6. Again, from equation (4.1), we see that $\{v \in M(\beta S) \mid l^1(S) \triangle v \subseteq l^1(S)\} = l^1(S)$, thus $SZ_r(M(\beta S)) = l^1(S)$.

This example is very revealing. Even if a Banach algebra is Arens regular, the right strong topological centre can be minimal while the left one is maximal. This kind of phenomena does not occur for Arens regularity. Also, we will see in the next section (Corollary 4.1.8) that there exists non-reflexive Arens regular Banach algebras such that their strong topological centres are minimal.

Another consequence of Lemma 4.1.6 is the identification of the right topological centre of $K(X)$ where X is a non-reflexive Banach space such that X^* has the bounded approximation property and the Radon-Nikodym property. Recall that for a such X , we have $K(X)^{**} = B(X^{**})$ [7, Page 60]. Before we state the theorem, we establish our notation, which comes from [17]. Let X be a Banach space. For a set $Y \in B(X)$ we define Y^a as follows $Y^a := \{T^* \mid T \in Y\}$. We denote by $W(X)$ the set of weakly compact operators on X .

Theorem 4.1.8. *Let X be a non reflexive Banach space such that X^* has the bounded approximation property and the Radon Nikodym property. Let $A = K(Y)$. Then $SZ_r(A^{**}) = W(X)^{**}$.*

Proof. Dales and Lau [m] obtain the following

$$\begin{aligned} Z_\ell(A^{**}) &= B(X^*)^* \\ Z_r(A^{**}) &= \{T \in B(X^{**}) \mid U(Y^{**}) \subseteq Y\} \\ Z_\ell(A^{**}) \cap Z_r(A^{**}) &= W(X)^{**} \end{aligned}$$

Now observe that the canonical embedding of $K(X)$ in $B(X^{**})$ is the map $T \mapsto T^{**}$. By Lemma 4.1.7 we have

$$SZ_r(A^{**}) = Z_r(A^{**}) \cap \{m \in A^{**} \mid 1 \triangle m \subseteq 1\}$$

Let us show first this inclusion

$$SZ_r(A^{**}) \supseteq Z_\ell(A^{**}) \cap Z_r(A^{**})$$

Let $U \in Z_\ell(A^{**}) \cap Z_r(A^{**})$. To show that for $T \in K(X)$ $T^{**} \triangle U = T^{**} \circ U^* \in K(X)$ we need to find $S \in K(X)$ such that $S^{**} = T^{**} \circ U^*$. Since $U \in Z_\ell(A^{**})$ we have that $T^{**} \circ U|_X$ maps Y to X . Let $S = T^{**} \circ U|_X \in K(X)$. Now we have $S^{**}|_Y = T^{**} \circ U|_X$. Since $U \in Z_\ell(A^{**})$ U is weak*-weak*-continuous operator so $T^{**} \circ U$ is too. Then by the weak*-density of X in X^{**} we conclude that $S^{**} = T^{**} \circ U^*$.

Now we show the reverse inclusion. We only need to show that $SZ_r(A^{**}) \subseteq Z_\ell(A^{**})$.

Let $U \in SZ_r(A^{**})$. Since $SZ_r(A^{**}) \subseteq Z_R(A^{**})$ $V \triangle U = V \square U = V \circ U$ for any $V \in B(X^{**})$. By Lemma 4.1.7 for each $V \in K(Y)$ there is $S \in K(X)$ such that $I^{**} \triangle U = I^{**} \circ U^* = S^*$. Let $R = I \circ U|_X$. We will show that $R = S$. In fact this S

is $T^{**} \circ U|_Y$. Now, observe that

$$\begin{aligned} T^{**} \circ U &= S^{**} \\ &= (T \circ U|_Y)^{**} \\ &= T^{**} \circ (U|_Y)^{**} \end{aligned}$$

for each $T \in K(X)$. Let $R_\alpha \in K(X)$ be a net converging weak* to $id \in B(Y^{**})$. Thus

$$\begin{aligned} U &= id \circ U \\ &= id \square U \\ &= w^*\text{-}\lim_{\alpha} R_\alpha^{**} \square U \\ &= w^*\text{-}\lim_{\alpha} R_\alpha^{**} \circ U \\ &= w^*\text{-}\lim_{\alpha} R_\alpha^{**} \circ (U|_X)^{**} \\ &= id \circ (U|_X)^{**} \\ &= (U|_X)^{**} \end{aligned}$$

Therefore, $U \in Z_\ell(A^{**})$. We conclude that $SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap Z_r(A^{**}) = W(X)^{aa}$

□

Corollary 4.1.9.

$$(i) \quad SZ_r(K(c_0)^{**}) = K(c_0)$$

$$(ii) \quad \text{Let } J \text{ be the James space with the pointwise product. Then, } SZ_r(K(J)^{**}) = W(J)^{aa} \neq K(J)$$

Proof. Let us show (i) first. It is known that $l^1 = c_0^*$ has the bounded approximation property and the Radon-Nikodým property. Then by Theorem 4.1.8, $SZ_r(K(c_0)^{**}) = W(K(c_0))^{aa}$. Next, we prove that $K(c_0) = W(c_0)$. Let $T \in W(c_0)$. Then $T^* \in W(l^1)$,

Let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence in l^1 , so $\{T(x_n)\}_{n \in \mathbb{N}}$ is also a bounded sequence in l^1 , and hence has a weakly convergent subsequence. However, l^1 has the Schur property: each weakly convergent sequence in l^1 is norm-convergent. We deduce then that T^* is a compact operator, and so T is compact.

To prove (ii), observe that J^{**}/J has dimension 1 from which we deduce that J^{**}/J is separable. Recall that if X is a Banach space such that X^{**}/X is separable, then X^* has the Radon-Nikodým property [1, page 219]. The James space has the bounded approximation property [1, page 63]. So, we can apply Theorem 2.2 and we get that $SZ_r(K(J)^{**}) = W(J)^{1,1}$. To see that $K(J) \neq W(J)$, we argue as follow. The closed ideal $W(J)$ is of codimension 1 in $B(J)$, however, $K(J)$ has infinite codimension in $B(J)$, thus $K(J) \neq W(J)$ [1, page 63]. $\square$

As we saw in Example 4.1.9, there is a difference between the left and the right for the strong topological centre. Here, we found the right strong topological centre for $K(X)^{**}$ for certain Banach space X . We will determine the left one in the next section and see that they are different in general.

Next, we interpret Theorem 2.2 of [1], which is Theorem 4.1.10, in terms of the strong topological centre. Again, here, the only tool we need is Lemma 4.1.9.

Theorem 4.1.10. (Baker-Lau-Pym) *Let A be a weakly sequentially complete Banach algebra with a BAI $\{\epsilon_\alpha\}_{\alpha \in I}$. Let $m \in Z_\ell(A^{**})$ such that $m \square A \subseteq A$. Let S be a closed linear subspace of A containing each ϵ_α . Suppose that $A \square \overline{S}^{u*} \subseteq A$. Then, $m \in A$.*

With Lemma 4.1.9, we see that the elements of the topological centre considered above are those of the left strong topological centre. So, in terms of the strong topo-

logical centre we obtain the following

Theorem 4.1.11. *Let A be a weakly sequentially complete Banach algebra with a BAI $\{e_\alpha\}_{\alpha \in I}$. Let S be a closed linear subspace of A containing each e_α . Suppose that $A \square S^{w*} \subseteq A$. Then, $SZ_t(A^{**}) = A$.*

Now, the most natural set S to consider is A . The condition that $A \square \overline{S}^t \subseteq A$ implies that A is a right ideal in A^{**} .

Corollary 4.1.12. *Let A be a weakly sequentially complete Banach algebra with a BAI. Suppose that A is a right ideal in A^{**} . Then, $SZ_t(A^{**}) = A$.*

We finish with another interpretation (but of Theorem 18 of [1]). This result presented immediately below determines which elements in the topological centre of a Banach algebra A of type (LM) or (M) are in A .

Theorem 4.1.13. *(Hu-Neufang-Ruan) Let A be a Banach algebra and $m \in A^{**}$. If A is of type (LM) the following are equivalent*

$$(i) \quad m \in A,$$

$$(ii) \quad m \square A \subseteq A \text{ and } m \in Z_t(A^{**})$$

$$(iii) \quad m \square A \subseteq A, \quad m - 1^* \in \overline{\langle A^*A \rangle} \text{ and } m \triangle E = m \text{ for some mixed unit } E$$

Corollary 4.1.14. *Let A be a Banach algebra of type (LM) or (M). Then,*

$$SZ_t(A^{**}) = A$$

Proof. This is a direct consequence of Lemma □

4.2 Banach algebras with a BAI

In this section, we restrict our study of the strong topological centre to Banach algebras with a BAI. In fact, this is not really a big restriction since many Banach algebras enjoy this property, such as the Fourier algebra over amenable groups, C^* -algebras, group algebras and many others. With a BAI, we can use the powerful Cohen's factorization theorem. We also have that the left multiplier algebra of a Banach algebra A can be embedded in A^{**} . In this section, we begin with a characterization of $LM(A)$ in A^{**} . Therefore, we prove the equivalence between the left strong dual factorization property and the identification $SZ_\ell(A^{**}) = LM(A)$. Then, we link the Arens semi-regularity with the strong dual factorization property using the strong topological centre.

As McKilligan proved in [31], the existence of a BAI for a Banach algebra A is equivalent to the existence of a certain isomorphism from $LM(A)$ into (A^{**}, Δ) . We want to know how to recognize the elements of $LM(A)$ in A^{**} .

Lemma 4.2.1. *Let A be a Banach algebra with a BAI. Fixcd weak*-cluster point F of a BAI (which then yields an embedding of $LM(A)$ into (A^{**}, Δ)). Then,*

$$LM(A) = \{m \in A^{**} \mid m \square A \subseteq A \text{ and } m \Delta F = m\}$$

Proof. First, for $T \in LM(A)$, we have that $T^{**}(n) = T^{**}(E) \Delta n$ and $T^{**}(f) = f \Delta T^{**}(E)$ for all $n \in A^{**}$ and $f \in A'$ [13, Theorem 3]. Via the embedding $T \mapsto T^{**}(E)$, we have $T^{**}(E) \square a = T^{**}(E) \Delta a = T(a) \in A$ for all $a \in A$. Also, $T^{**}(E) \Delta E = T^{**}(E)$. Thus, we have that

$$LM(A) \subseteq \{m \in A^{**} \mid m \square A \subseteq A \text{ and } m \Delta E = m\}.$$

Now, take m in the set of the right hand side. Let $T = \lambda_m|_A$. Then, $T \in LM(A)$. We need to show that $T^{**}(E) = m$. Let $\{\epsilon_\alpha\}$ be a BAI such that E is a weak*-cluster

point. Then

$$\begin{aligned}
 T^{**}(E) &= \lim_{\alpha} T^{**}(e_{\alpha}) \\
 &= \lim_{\alpha} m \Delta e_{\alpha} \\
 &= m \Delta E \\
 &= m.
 \end{aligned}$$

□

Remark 4.2.2. Of course, the result

$$RM(A) = \{m \in A^{**} \mid A \Delta m \subseteq A \text{ and } E \sqsubset m \text{ for some mixed unit } E\},$$

can be proven in a similar way. Also, with this lemma, we see that

$$LM(A) \subseteq \{m \in A^{**} \mid m \sqsubset A \subseteq A\}$$

for any mixed unit E . Note that in general, if $m \Delta E = m$ for some mixed unit E , then the same equality is not true for all mixed units. Corollary 4.2.1 will show for which $m \in \{m \in A^{**} \mid m \sqsubset A \subseteq A\}$ this holds.

Take $m \in SZ_{\ell}(A^{**})$. Then, there is $T \in LM(A)$ such that $T^{**} = \lambda_m$. This restriction to $LM(A)$ and the fact that $LM(A) \subseteq \{m \in A^{**} \mid m \sqsubset A \subseteq A\}$ suggest that the set $\{m \in A^{**} \mid m \sqsubset A \subseteq A\}$ can be replaced by $LM(A)$ to determine the strong topological centre as in Lemma 4.2.1.

Theorem 4.2.3. *Let A be a Banach algebra with a BAI. For any mixed unit E (which yields an embedding of $LM(A)$, respectively $RM(A)$, in A^{**}), we have*

$$SZ_{\ell}(A^{**}) = Z_{\ell}(A^{**}) \cap LM(A).$$

$$SZ_r(A^{**}) = Z_r(A^{**}) \cap RM(A).$$

Proof. We will prove only the left version, the other being similar. Let $m \in SZ_\ell(A^{**})$. Then, by Lemma 4.2.3, $n \in Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \sqsubseteq A \subseteq A\}$. Therefore, by Lemma 4.2.2, we need to prove that $n \triangle E = n$ for some mixed unit E . But, $n \in Z_\ell(A^{**})$ and so, $n \triangle E = n \sqcap E = n$. Hence, $SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \cap LM(A)$. The reverse inclusion is clear by Lemma 4.2.1 and Remark 4.2.3. $\square$

Remark 4.2.4. We can also use Theorem 4.2.2 with Lemma 4.2.3 to prove Theorem 4.2.2.

We see in the proof of Theorem 4.2.2 that if $m \in SZ_\ell(A^{**})$, then $m \triangle E = m$ for any mixed unit E since the choice of E does not affect the proof. This is a consequence of Theorem 4.2.2. The next lemma, which is Lemma 21 of [13], characterizes a left multipliers T satisfying $T^{**}(E_1) = T^{**}(E_2)$ for all mixed units E_1 and E_2 of A^{**} .

Lemma 4.2.5. (*Hu-Neufang-Ruan*) *Let A be a Banach algebra with a BAI. Then, for $T \in LM(A)$, we have $T^*(A^*) \subseteq A^*A$ if and only if $T^{**}(E_1) = T^{**}(E_2)$ for all mixed units E_1 and E_2 of A^{**} .*

Corollary 4.2.6. *Let A be a Banach algebra with a BAI. Then, $m \in SZ_\ell(A^{**})$ if and only if $m \sqsubseteq A \subseteq A$ and $m \triangle E = m$ for all mixed units E of A^{**} .*

Proof. Suppose first that $m \in Z_\ell(A^{**})$. By Lemma 4.2.3, we have that $m \sqsubseteq A \subseteq A$. Since $m \in Z_\ell(A^{**})$, we deduce that $m \triangle E = m$ for all mixed unit E of A^{**} by Theorem 4.2.2. Now, let us prove the other implication. Suppose that $m \sqsubseteq A \subseteq A$ and $m \triangle E = m$ for all mixed units E . Define $T_m \in LM(A)$ by $T_m(a) = m \sqcap a$ for all $a \in A$. Observe that

$$\begin{aligned} \langle T_m^*(f), a \rangle &= \langle f, m \sqcap a \rangle \\ &= \langle f \triangle m, a \rangle \end{aligned}$$

for all $f \in A^*$ and $a \in A$, so $T^*(f) = f \Delta m$. Hence, for any mixed unit E , we have

$$\begin{aligned} \langle m, f \rangle &= \langle m \Delta E, f \rangle \\ &= \langle E, f \wedge m \rangle \\ &= \langle E, T_m^*(f) \rangle \\ &= \langle L_m^{**}(E), f \rangle. \end{aligned}$$

for all $f \in A^*$. In particular, we deduce that $L_m^{**}(E_1) = L_m^{**}(E_2)$ for all mixed units E_1 and E_2 . Then, for all $f \in A^*$, we have

$$f \Delta m = T_m^*(f) \in A^* A.$$

by Lemma 4.1. Since $m \sqcap A \subseteq A \subseteq Z_\ell(A^{**})$, $m \Delta E = m$ for all mixed units E and $f \Delta m \in A^* A$ for each $f \in A^*$, we obtain that $m \in Z_\ell(A^{**})$, by Theorem 4.2. It remains to prove, by Theorem 4.3, that $m \in LM(A)$. But, $m \sqcup A \subseteq A$ and $m \wedge E = m$ for all mixed units E , and so, by Lemma 4.1, $m \in LM(A)$. $\square$

The sets $\{m \in A^{**} \mid m \sqcap A \subseteq A\}$ and $\{m \in A^{**} \mid A \Delta m \subseteq A\}$ already appeared in [14]. The authors mention that the condition $m \sqcap A \subseteq A$ is very difficult to verify. In that context, it is natural to suppose that A be a left ideal in A^{**} . Theorem 4.4 replaces $\{m \in A^{**} \mid m \sqcap A \subseteq A\}$ by $LM(A)$ in Lemma 4.1. If we look at Lemma 4.1, we still have the condition $m \sqcup A \subseteq A$, and another one. However, this way of looking at Theorem 4.4 is not the correct one. In fact, Theorem 4.5 avoids the condition $m \sqcap A \subseteq A$. We start with a Banach algebra A with a BAI. It is easier and more natural to find $LM(A)$ first. Then, using the embedding of $LM(A)$ in A^{**} , we are able to find the strong topological centre when we know usually topological centres.

Now we give some interesting corollaries to Theorem 4.5.

Corollary 4.2.7. *Let A be a Banach algebra with a BAI. If A is Arens regular then $SZ_\ell(A^{**}) = LM(A)$. In particular, the left strong topological centre of a C^* -algebra A is $LM(A)$.*

Corollary 4.2.8. *Let A be a unital Banach algebra. Then $SZ_\ell(A^{**}) = A$. In particular, the left strong topological centre of any unital C^* -algebra A is A .*

Before we state the next corollary, recall that $K(X)$ has a BAI if and only if X^* has the bounded approximation property. See [1, page 219].

Corollary 4.2.9. *Let X be a non reflexive Banach space such that X^* has the bounded approximation property and the Radon-Nikodým property. Then*

$$SZ_\ell(K(X)^{**}) = B(X)$$

Proof. As we mentioned in the proof of Theorem 4.2.6, $SZ_\ell(K(X)^{**}) = B(X)^a$. Moreover, we have $LM(K(X)) = B(X)$ [6, Theorem 2.5.13]. The embedding of $LM(K(X))$ into $K(X)^* = B(X^{**})$ presented in Section 4.2.3 is given by the map $B(\Gamma \ni T \mapsto T^{**})$. By Theorem 4.2.6, $SZ_\ell(K(X)^{**}) = B(X)$ since $B(X)^a \subseteq B(X)$. $\square$

If we recall the initial comparison between (l^1) and a unital C^* -algebra, we see now that the unital C^* -algebra has the smallest level of Arens regularity, and l^1 has the highest, which is a surprising fact.

If we take two Banach algebras which have the same second dual, there is no link between the strong topological centre of each algebra, as the following example shows.

Example 4.2.10. Consider c and c_0 , the set of all convergent sequences and all sequences converging to zero, respectively, equipped with the supremum norm and pointwise multiplication. Both have l^∞ as second dual. We have that $SZ_\ell(c^{**}) = c$ by Corollary

and $SZ_\ell((c_0)^{**}) = l^\infty$ by Corollary 4.2.10. Thus, the second dual does not determine the strong topological centre.

We have seen with Lemma 4.2.9 that any weakly sequentially complete Banach algebra with a BAI is unital if $A^* = A^*A$. From Corollary 4.2.10, we deduce that $SZ_\ell(A^{**}) = A$. If one makes the condition of A being unital to having a sequential BAI, where A^* is different of A^*A in general, but the same conclusion still holds provided that A is weakly sequentially complete.

Corollary 4.2.11. *Let A be a weakly sequentially complete Banach algebra with a sequential BAI. Then, $SZ_\ell(A^{**}) = A$.*

Proof. Let $m \in SZ_\ell(A^{**})$. By Theorem 4.2.10, $SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap LM(A)$. So, there is $T \in LM(A)$ such that $T^{**} = \lambda_m$. By Theorem 4.2.9, we see that $T \in A$, and hence $m \in A$. $\square$

From the last corollary and Corollary 4.2.10, one may conjecture that the strong topological centre of any weakly sequentially complete Banach algebra A with a BAI is A . Of course, this is certainly not true if A has no BAI like Example 4.2.8 (v) will show.

Lau and Ülger in [17] use $\{m \in A^{**} \mid A\Delta m \subseteq A\}$ to characterize the left strong dual factorization property, see Theorem 4.2.12. They mention that it is unexpected to characterize a left property, i.e., $A^* = A^*A$, with a right one, i.e., $\{m \in A^{**} \mid A\Delta m \subseteq A\}$. Their approach is based on the topological centre. The next theorem shows that the strong topological centre is the good one to obtain all the left properties together.

Theorem 4.2.12. *Let A be a Banach algebra with a BAI. Then, A has the left strong dual factorization property, $A^* = A^*A$, if and only if $SZ_\ell(A^{**}) = LM(A)$.*

Proof. Fixed a mixed unit E . Suppose that $A^* = A^*A$. Then, by Theorem 4.2.12 (b) for each $T \in LM(A)$, there is $m \in A^{**}$ such that $\lambda_m = T^{**}$. So, $LM(A) \subseteq SZ_\ell(A^{**})$. Hence, by Theorem 4.2.13, we have that $SZ_\ell(A^{**}) = LM(A)$.

Suppose that $SZ_\ell(A^{**}) = LM(A)$. We have $E = id^{**}(E) \in LM(A) = SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**})$. We conclude that $(A^{**}, \square)$ is unital and by Theorem 4.2.13 $A^* = A^*A$. $\square$

In [11], Lau–Úlger prove that $K(c_0)$ has the left strong dual factorization property. In fact, their proof works for all non-reflexive Banach spaces X such that X^* has the bounded approximation property and the Radon-Nikodym property. Here, we can obtain this result as a corollary of Theorem 4.2.13, as follows.

Corollary 4.2.13. *(Lau–Úlger) Let X be a non-reflexive Banach space such that X^* has the bounded approximation property and the Radon-Nikodym property. Then, $K(X)$ has the left strong dual factorization property*

Proof. By Corollary 4.2.13, $SZ_\ell(K(X)^{**}) = B(X) = LM(K(X))$. We conclude, using Theorem 4.2.13, that $K(X)$ has the left strong dual factorization property. $\square$

Remark 4.2.14. It seems unknown, in general, for which Banach spaces X , the algebra $K(X)$ has the left or right strong dual factorization property.

We study the case where the left strong topological centre is equal to the second dual using Theorem 4.2.13.

Corollary 4.2.15. *Let A be a Banach algebra with a BAI. Then, $SZ_\ell(A^{**}) = A^{**}$ if and only if $LM(A) = A^{**}$.*

Proof If $SZ_\ell(A^{**}) = A^{**}$ then by Theorem 4.2.15 $LM(A) = A^{**}$. Suppose that $LM(A) \neq A^{**}$. Then $(A^{**} \wedge)$ is unital and so there is only one mixed unit in A^{**} say E . We represent an element of A^{**} as $T^{**}(I)$ for $T \in LM(A)$. We have that $T^{**}(E) \triangle E = T^{**}(F)$ and $T^{**}(F) \sqcap A \subseteq A$. So by Corollary 4.2.16 we obtain that $SZ_\ell(A^{**}) = A^{**}$. $\square$

Remark 4.2.16 The situation mentioned in Corollary 4.2.15 occurs naturally. For instance one can take $A = c_0$ (Example 4.2.10). In general for a commutative C^* -algebra A the equality $SZ_\ell(A^{**}) = A^{**}$ means that the idealizer of the representation in $B(H)$ for some Hilbert space H and the von Neumann envelope of the representation of A in $B(H)$ are the same. One can also show that the James space equipped with the pointwise multiplication satisfies the equivalent conditions of Corollary 4.2.15.

Corollary 4.2.17. *Let A be a Banach algebra with a BAI. If $LM(A) = A^{**}$ then A has the left and right strong dual factorization property.*

Proof By Corollary 4.2.15 $LM(A) = A^{**} = SZ_\ell(A^{**})$ so A is Arens regular and by Theorem 4.2.14 we obtain the assertion. $\square$

We finish this section with an interesting link between the strong dual factorization property and a concept introduced by Grosser in [12], Arens semi-regularity.

Definition 4.2.18. (Grosser [12], page 45) *Let A be a Banach algebra with a BAI. Then A is called Arens semi regular if $S^{**}(E) = T^{**}(E)$ for all double centralizers (S, T) of A and all mixed units E .*

Grosser, in [12] gives an interesting characterization of Arens semi-regularity. We present this result since we will need it later. We first need to introduce some notations.

from [13]. Let A be a Banach algebra with a BAI. Let E be a fixed mixed unit in A^{**} . Define

$$\pi_1 : DM(A) \rightarrow LM(A),$$

$$(T, S) \mapsto T,$$

$$\pi_2 : DM(A) \rightarrow RM(A),$$

$$(T, S) \mapsto S$$

$$\epsilon : LM(A) \rightarrow A^{**},$$

$$T \mapsto T^{**}(E),$$

$$\nu : RM(A) \rightarrow A^{**},$$

$$S \mapsto S^{**}(E).$$

We also introduce the the sets $D_\epsilon = \epsilon\pi_1(DM(A))$ and $D_\nu = \nu\pi_2(DM(A))$.

Theorem 4.2.19. (Grosser [14, Theorem 3]) *Let A be a Banach algebra with a BAI. For a mixed unit E , let D be one of the following sets, D_ϵ , D_ν or $D_\epsilon \cup D_\nu$. Then A is Arens semi-regular if and only if the two Arens products coincide on D for each mixed unit E .*

Theorem 4.2.20. *Let A be a Banach algebra with a BAI. If A has the left or right strong dual factorization property, then A is Arens semi-regular.*

Proof. By Theorem 4.2.19, $LM(A) = SZ_\ell(A^{**}) \subset Z_\ell(A^{**})$. The proof of Theorem 4.2.20 does not depend on the choice of the mixed unit E . Thus, by Theorem 4.2.19, A is Arens semi-regular. $\square$

Remark 4.2.21. In fact, this is a generalization of Theorem 1 of [12], which states that an Arens regular Banach algebra with a BAI is Arens semi-regular. The proof of Theorem 1 of [12] uses the fact that for a Banach algebra with a BAI, Arens regularity implies

the left and the right strong dual factorization properties. However, Theorem 4.1.1 only needs left or right strong dual factorization, which is an improvement.

The converse of Theorem 4.1.1 is false in general, for example, for $A = L^1(G)$ with G abelian non-discrete locally compact group [12, Examples].

Theorem 4.1.1 is interesting because it is the starting point for studying the question mentioned in Remark 4.1.2, for which Banach space X , the algebra $K(X)$ has the left strong dual factorization property? It is true that we need $K(X)$ with a BAI to use Theorem 4.1.1, and this is equivalent to X^* having the bounded approximation property.

4.3 Examples

Obviously, these inclusions are clear: $A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \subseteq A^{**}$. There are many possible cases where the different inclusions can be strict or equal. In this section, we present examples for all these cases. Most of the examples in this section have already been seen in this thesis. With this section, we want to show that all possible behaviours of the strong topological centre. We make the comparison with the strong topological centre with both topological centre and the left multiplier algebra, in view of Theorem 4.1.1. The left and right topological centre do not have to be equal in general even when A is left strongly Arens irregular, as it was first shown by Neufang in [23]. Other examples are given in [7]. The same is true for the strong topological centre.

Example 4.3.1. Examples of possible cases involving SZ_ℓ and Z_ℓ .

$$(i) \quad SZ_\ell(A^{**}) = Z_\ell(A^{**}) = A$$

Any strongly Arens irregular Banach algebras such as the group algebra $L^1(G)$ over a locally compact group G .

$$(ii) \quad A = SZ_\ell(A^{**}) \subsetneq Z_\ell(A^{**}) \subsetneq A^{**}$$

The Banach algebra $A = A(SU(3))$ is an example of this type. This is a unital Banach algebra, so, by Corollary 4.3.1, $SZ_\ell(A^{**}) = A$. But Losert showed that $A(SU(3))$ is not strongly Arens irregular in his presentation: *On the centre of the bidual of Fourier algebra (the compact case)*, at the 2004 Istanbul International Conference on Abstract Harmonic Analysis. Finally, the algebra $A(SU(3))$ is not Arens regular since, otherwise, $SU(3)$ being an amenable group would have to be finite by Corollary 4.3.2.

$$(iii) \quad A = SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) = A^{**}$$

By Theorem 4.3.3, any unital Arens regular Banach algebra is of this type. For example, any unital C^* -algebra, in particular, any von Neumann algebra.

$$(iv) \quad A \subsetneq SZ_\ell(A^{**}) \subsetneq Z_\ell(A^{**}) \subsetneq A^{**}$$

By Corollary 4.3.1, the Banach algebra $A = K(X)$ of compact operators on a non-reflexive Banach space X , such that X^* has the bounded approximation property and the Radon-Nikodym property, has the left strong dual factorization property. By Corollary 4.3.1, $SZ_\ell(K(X)^{**}) = B(X) \subsetneq B(X^*)^{aa} = Z_\ell(K(X)^{**})$. Since X is not reflexive, $K(X)$ is not Arens regular.

$$(v) \quad A \subsetneq SZ_\ell(A^{**}) = Z_\ell(A^{**}) \subsetneq A^{**}$$

One can take $A = A(G)$, the Fourier algebra over any discrete group G containing the free group over 2 generator, $\mathbb{F}_2$, as a subgroup. A consequence of a result of

Losert, Theorem 19 of [14], is that $A(G)$ is not (left) strongly Arens irregular. Since G is discrete, $A(G)$ is an ideal in $A(G)^{**}$, by Theorem 4.3.1, so by Corollary 4.3.2, $SZ_\ell(A(G)^{**}) = Z_\ell(A(G)^{**})$. Finally, note that $A(\mathbb{Z})$ is a subalgebra of $A(G)$. By Pontryagin duality, we have that $A(\mathbb{Z}) = L^1(\mathbb{T})$, and the group algebra is strongly Arens irregular, thus, $A(G)$ is not Arens regular.

$$(vi) \quad SZ_\ell(A^{**}) = Z_\ell(A^{**}) = A^{**}$$

By Corollary 4.3.2, any Arens regular Banach algebra which is a left ideal in A^{**} is an example. The algebra $(l^1, \cdot)$ is one of them, as seen in Example 4.3.1, also c_0 and $K(H)$, where H is a Hilbert space.

Remark 4.3.2. For the case (v) in the previous example, any Banach algebra which is a left ideal in its second dual and not Arens regular nor left strongly Arens irregular, is an example. The really interesting case is when A is not a left ideal in A^{**} . The next section presents such an example.

In view of Theorem 4.3.1, we can produce a similar list with $LM(A)$ instead of $SZ_\ell(A^{**})$. The reason why the next list is shorter is the following: if A is a Banach algebra with a BAI and $LM(A) = A^{**}$, then by Corollary 4.3.2, $SZ_\ell(A^{**}) = A^{**}$ and automatically we have $SZ_\ell(A^{**}) = LM(A) = A^{**}$. Of course, we assume that all Banach algebras, we consider, have a BAI since we want to see $LM(A)$ as a subalgebra of A^{**} .

Example 4.3.3. Examples of possible cases involving SZ_ℓ and $LM(A)$

$$(i) \quad A = SZ_\ell(A^{**}) \subsetneq LM(A) \subsetneq A^{**}$$

Any non-unital Banach algebra of type (M) is an example. By Corollary 4.3.2, $A = SZ_\ell(A^{**})$. Since A is not unital, then $A \subsetneq LM(A)$. The last strict inclusion follow from the remark preceding Example 4.3.1. A concrete example is $A =$

$A(G)$, where G is a locally compact non-compact group. By Theorem 4.1.1, $A(G)$ is a Banach algebra of type (M).

$$(ii) \ A = SZ_\ell(A^{**}) = LM(A) = A^{**}$$

With Corollary 4.1.2, one can see that this condition is equivalent to A being reflexive and unital.

$$(iii) \ A \subsetneq SZ_\ell(A^{**}) = LM(A) \subsetneq A^{**}$$

By Corollary 4.1.2, the Banach algebra $A = K(X)$ of compact operators on an infinite dimensional non-reflexive Banach space X , such that X^* has the bounded approximation property and the Radon-Nikodym property, has the left strong dual factorization property. By Corollary 4.1.2 and the fact that X is infinite dimensional, $K(X) \subsetneq SZ_\ell(K(X)^{**}) = B(X) \subsetneq B(X^*)^{aa} \subseteq B(X^{**})$.

$$(iv) \ A \subsetneq SZ_\ell(A^{**}) \subsetneq LM(A) \subsetneq A^{**}$$

Clearly, if a Banach algebra A with a BAI satisfies this condition, then A is not left strongly Arens irregular and by Theorem 4.1.3, neither Arens regular. Also, by Theorem 4.1.3, $A^* \neq A^*A$. A Banach algebra satisfying the requested conditions is the algebra $A = K(J)$, equipped with the opposite product, of compact operators on the James space. By Corollary 4.1.2, we have $A = K(J)^{op} \subsetneq SZ_\ell(A^{**}) = W(X)^{aa}$. Since, $LM(A) = B(J)$, and J not being reflexive, $SZ_\ell(A^{**}) \subsetneq LM(A) = B(J)$. The last inclusion is strict since, otherwise, by Corollary 4.1.3, $K(J)^{op}$ would have to be Arens regular, which is not the case since J is not reflexive.

$$(v) \ A \subsetneq SZ_\ell(A^{**}) = LM(A) = A^{**}$$

This condition describes Banach algebra A , with a BAI, which are non-unital, Arens regular and whose second duals are isomorphic to $LM(A)$. The natural examples are c_0 and $K(H)$, where H is an infinite dimensional Hilbert space. The James space J equipped with the pointwise multiplication is another example.

This algebra is called the James algebra and has a BAI bounded by one (cf. Example 4.1.15 (vi)). Also, J^{**} is isometrically isomorphic to $LM(J)$ [6, Example 4.1.15 (xii)]. But J is not reflexive.

As it is possible for the left and the right topological centre, the left and right strong topological centre can be different. In fact, Example 4.3.3 shows that even if the left strong topological centre is all the second dual, the right strong topological centre can be equal to the algebra itself. Such extreme difference cannot occur for the topological centre. If the left, respectively the right, topological centre equals all the second dual, the right, respectively the left, does too. Hence, we see that the strong topological centre not only measure the degree of Arens regularity, but also makes a finer difference than the topological centre between the left and the right.

The following is another example where one strong topological centre is strictly included into the other.

Example 4.3.4. By Corollary 4.3.2, we have $SZ_l(K(c_0)^{**}) = B(c_0)$, whereas, by Corollary 4.3.3, $SZ_r(K(c_0)^{**}) = K(c_0)$.

4.4 The convolution algebra of nuclear operators

We now study, for $q < p < \infty$, the algebra of nuclear operators on $l^p(G)$ for a discrete group G with a different product than composition. Neufang, in [24], introduces a new product on $N(l^p(G))$ for any locally compact group G . This algebra is the first example which is left but not right strongly Arens irregular. This wonderful object will also be a good example for the strong topological centre. In fact, the right strong and

the right topological centre coincide for any discrete group G and are different from the algebra itself and its second dual. We will use the description of $Z_r(\Lambda(l^p(G))^{**})$ as lemmas which come from [3].

We first introduce the product defined by Neufang in [34] for any locally compact group. Therefore, we make the assumption that G is discrete and obtain the result announced above.

Let G be a locally compact group. For $T \in B(l^p(G))$ we denote by $M_{L_t I_t I_t^{-1} \rho}$ the multiplication operator of the function

$$t \longmapsto \langle L_t T I_t^{-1} \rho \rangle$$

where, $L_t(f)(s) = f(ts)$ for $f \in l^p(G)$ and $s, t \in G$. For $f \in l^p(G)$ and $r \in G$ we have

$$(M_{L_t I_t I_t^{-1} \rho} f)(x) = \langle L_x T I_t^{-1} \rho \rangle f(x)$$

We define a module action of $N(l^p(G))$ on $B(l^p(G)) = \Lambda(l^p(G))^*$ by

$$T \curvearrowright \rho = M_{L_t T I_t^{-1} \rho}$$

for $T \in B(l^p(G))$ and $\rho \in \Lambda(l^p(G))$. We can now define the *convolution* product on $\Lambda(l^p(G))$. For $\rho, \tau \in \Lambda(l^p(G))$ and $T \in B(l^p(G))$, set

$$\langle \rho * \tau \rangle = \langle \tau, T \curvearrowright \rho \rangle$$

Proposition 4.4.1. (Neufang [34, Definition und Satz 5.2.1]) *Let G be a locally compact group. Then $(\Lambda(l^p(G)), *)$ is a Banach algebra.*

In this section we always consider $\Lambda(l^p(G))$ as equipped with the convolution product.

Remark 4.4.2 The reason why we call $\ast$ a convolution on $\mathcal{N}(L^p(G))$ is given in Proposition 4.4.3. Let

$$\tau : L^\infty \rightarrow B(L^p(G))$$

$$f \mapsto M_f$$

where $M_f(g) = \Delta(g)^{-1}f(g^{-1})$ and Δ is the modular function of the group. Since τ is a weak * -weak * -continuous isometry, its pre adjoint is a quotient on $L^1(G)$.

Proposition 4.4.3. (*Nakajima [Nak53, 1]*) *Let G be a locally compact group. The map*

$$\pi : (\mathcal{N}(L^p(G)), \ast) \rightarrow (L^1(G), \ast)$$

which is the pre adjoint of τ defined above, is an algebra homomorphism. If G is discrete then the product in $\mathcal{N}(L^p(G))$ extends the convolution product in $L^1(G)$.

For a locally compact group G we denote by $M_{L^\infty(G)}$ the space of multiplication operators of $L^\infty(G)$ functions in $B(L^p(G))$. For $X \subseteq B(L^p(G))$ we write $X^\perp$ for its pre-annihilator in $\mathcal{N}(L^p(G))$ and $X^\perp$ for its annihilator in $B(L^p(G))$.

We shall be interested in infinite discrete group G . Then $\mathcal{N}(L^p(G))^\perp$ is not an ideal in its second dual. Also we have the following

Lemma 4.4.4 *Let G be a locally compact discrete group. Then*

$$\mathcal{N}(L^p(G))^\perp = l^1(G)^\perp = (M_{\ell^\infty(G)})^\perp$$

as Banach spaces

In [3] the next two lemmas which follow directly from the definition of the operator convolution product have been used to determine the right topological centre of $\mathcal{N}(L^p(G))$.

For us they are the keys to prove that the right strong and the right topological centre are equals

Lemma 4.4.5. *Let G be a locally compact group. Then*

$$\mathcal{N}(l^p(G)) \cap (M_{l^\infty(G)})_\perp = 0$$

Lemma 4.4.6. *Let G be a locally compact group. Then*

$$(M_{l^\infty(G)})_\perp \cap \mathcal{N}(l^p(G)) \subseteq (M_{l^\infty(G)})_\perp$$

Let G be a discrete group. With the two previous lemmas, we see that $(M_{l^\infty(G)})_\perp$ is an ideal in $\mathcal{N}(l^p(G))$. Also, by Proposition 4.4.3, $l^1(G)$ is a subalgebra of $\mathcal{N}(l^p(G))$ so we have a module action of $l^1(G)$ on $(M_{l^\infty(G)})_\perp$. We can define a product $\circ$ on $l^1(G) \cap (M_{l^\infty(G)})_\perp$ by

$$(a, f) \circ (b, g) = (a * b, a \times g + f * b)$$

This product turns $l^1(G) \cap (M_{l^\infty(G)})_\perp$ into a Banach algebra. This definition makes sense for a general Banach algebra A and a Banach A -module X where we replace the action of l^1 on $(M_{l^\infty(G)})_\perp$ which is the product $*$ on $\mathcal{N}(l^p(G))$, by the module action of A on X .

Proposition 4.4.7. *(Lugger-Neufang [3, Proposition 3.3.3]) Let G be a discrete group. Then*

$$l^1(G) \cap (M_{l^\infty(G)})_\perp \otimes = (\mathcal{N}(l^p(G)) \cap *)$$

This proposition allows us to get our hands easily on $\mathcal{N}(l^p(G))^{**}$. We have that $l^1(G) \cap (M_{l^\infty(G)})_\perp = \mathcal{N}(l^p(G))$, hence

$$(\mathcal{N}(l^p(G)))^{**} = l^1(G)^{**} \cap (M_{l^\infty(G)})_\perp^{**} = l^1(G)^{**} \cap M_{l^\infty(G)}$$

For $(m, \phi), (n, \psi) \in l^1(G)^{**} \ominus (M_{l^\infty(G)})^\perp$, we obtain

$$(m, \phi) \Delta (n, \psi) = (m \Delta n, m \Delta \psi + \phi \Delta n)$$

Theorem 4.4.8. (Auger–Neufang [1, Theorem 3.3.2]) Let G be a discrete group. Then

$$Z_l(N(l^p(G))^{**}) = N(l^p(G)),$$

$$Z_r(N(l^p(G))^{**}) = l^1(G) \ominus (M_{l^\infty(G)})^\perp$$

Now, we have collected all the results we need to determine the right strong topological centre of $V(l^p(G))$

Theorem 4.4.9. Let G be a discrete group. Then,

$$SZ_l(N(l^p(G))) = V(l^p(G)),$$

$$SZ_r(N(l^p(G))^{**}) = Z_r(N(l^p(G))^{**}) = l^1(G) \ominus (M_{l^\infty(G)})^\perp.$$

Proof. The first assertion is clear. For the second, by Lemma 4.4.8, we only need to show that $N(l^p(G)) \Delta Z_r(V(l^p(G))^{**}) \subseteq N(l^p(G))$. We start by observing following. Let $\Phi \in l^1(G)$ and $\lambda \in (M_{l^\infty(G)})^\perp$. By Lemma 4.4.8, we have that $\Phi \Delta \lambda = 0$. Then, for $\Psi \in M_{l^\infty(G)}^\perp$ we see that $\Phi \Delta \Psi = 0$.

Let $m \in Z_r(N(l^p(G))^{**})$. By Theorem 4.4.9, there exist $b \in l^1(G)$ and $\psi \in M_{l^\infty(G)}^\perp$ such that $m = (b, \psi)$. Let $n \in N(l^p(G))$. By Lemma 4.4.8, there is $a \in l^1$ and $\phi \in (M_{l^\infty(G)})^\perp$ such that $n = (a, \phi)$. In view of Proposition 4.4.7, we need to conclude that $(a, \phi) \Delta (b, \psi) \in N(l^p(G))$. Note that

$$(a, \phi) \Delta (b, \psi) = (a * b, a \Delta \psi + \phi * b) \quad (4.2)$$

Here, $a * b \in l^1(G)$. By the observation at the beginning of this proof, we have

that $a\Delta t = 0$ and by Lemma 4.3.10 $a + b \in (M_{l^\infty(G)})_1$. So we conclude that $\mathcal{N}(l^p(G)) \cap Z_r(\mathcal{N}(l^p(G))^{**}) \subseteq \mathcal{N}(l^p(G))$. $\square$

4.5 The multiplier algebra of the strong topological centre

The conjecture of Ghahramani-Lau which was that $M(G)$ is strongly Arens irregular for any locally compact group G .

This has been recently solved by Losert–Neufang–Pachl–Stepaniuk affirmatively in [21]. A possible alternative approach suggested by Fouladizadeh is to prove the following equality

$$LM(Z_\ell(A^{**})) = Z_\ell(LM(A)^{**}) \quad (4.3)$$

for $A = L^1(G)$ where G is a locally compact group. When A is Arens regular and unital in particular for any unital C^* -algebra this relation holds. If we replace the unit by a BAI this is no longer true in general. For a counter example take c_0 . Since c_0 is Arens regular, we have $LM(Z_\ell(c_0^{**})) = LM(l^\infty) = l^\infty$. But $Z_\ell(LM(c_0)^{**}) = Z_\ell((l^\infty)^{**}) = (l^\infty)^{**}$.

We are interested in equation (4.3) but for the strong topological centre. The new relation is

$$LM(SZ_\ell(1)) \supseteq Z_\ell(LM(A)^{**}) \quad (4.4)$$

For $A = L^1(G)$ this equation is true for any locally compact group but being equal to

$M(G)$ since $L^1(G)$ is left strongly Arens irregular

$$LM(SZ_\ell(L^1(G))) = LM(L^1(G)) = M(G)$$

since $M(G)$ is unital we have $SZ_\ell(M(G)^{**}) = M(G)$ by Corollary 4.5.2

The first observation we can make regarding equation (4.4) for general Banach algebras is that $LM(A)$ being unital and then, by Corollary 4.5.2 $SZ_\ell(LM(A)^{**}) = LM(A)$ for any Banach algebra A . Thus equation (4.4) becomes

$$LM(SZ_\ell(A^{**})) = LM(A) \quad (4.5)$$

This observation allows us to give sufficient conditions on A obtain equation (4.5). Obviously any Banach algebra A with $SZ_\ell(A^{**}) = A$ or $LM(A)$ satisfies equation (4.5). If A has a BAI and A enjoys the left strong dual factorization property then $SZ_\ell(A^{**}) = LM(A)$ by Theorem 4.5.3. Therefore

$$LM(SZ_\ell(A^{**})) = LM(LM(A)) = LM(A)$$

Thus Arens regular Banach algebras with a BAI satisfy equation (4.5) in particular the algebra c_0 for which equation (4.4) is false. There is also another class of Banach algebras that satisfies equation (4.5) Banach algebras of type (M), due to Corollary 4.5.2.

The group algebra on any locally compact group is in this class as the Fourier algebra over an amenable group. All those examples have something in common, they have a BAI.

Theorem 4.5.1. *Let A be a Banach algebra such that $A = \overline{\langle A^2 \rangle}$. If $(A^* \square A)$ is left faithful, then $LM(SZ_\ell(A^{**})) = LM(A)$.*

Proof. Let $\iota : LM(SZ_\ell(A^{**})) \rightarrow LM(A)$ be defined by $\iota(T) = T|_A$ for $T \in LM(SZ_\ell(A^{**}))$.

To see that indeed $\iota(T) \subseteq A$ note that by hypothesis, for each $a \in A$ there are

$b_i, c_i \in A$ such that $a = \sum_{i=1}^{\infty} b_i c_i$, so we see

$$T(a) = T\left(\sum_{i=1}^{\infty} b_i c_i\right) = \sum_{i=1}^{\infty} T(b_i) \cdot c_i \in A,$$

since, by Lemma 4.1.1, A is an ideal in $SZ_{\ell}(A^{**})$. Thus, the map r is well-defined. The next step is to show that r is injective. Let $T \in \ker(r)$. Suppose that T is not the zero multiplier. Then, there is $m_0 \in SZ_{\ell}(A^{**})$ such that $T(m_0) \neq 0$. Let $n \in A^{**}$. There is a net $\{n_{\alpha}\} \subseteq A$ converging weak* to n . Since $SZ_{\ell}(A^{**}) \subseteq Z_{\ell}(A^{**})$, we have

$$\begin{aligned} T(m_0) \square n &= w^*\text{-}\lim_{\alpha} T(m_0) \square n_{\alpha} \\ &= w^*\text{-}\lim_{\alpha} T(m_0 \square n_{\alpha}) \\ &= w^*\text{-}\lim_{\alpha} 0 \\ &= 0. \end{aligned}$$

This is a contradiction since $(A^{**}, \square)$ is left faithful.

Let $U \in LM(A)$. Then, $U^{**} : (A^{**}, \Delta) \rightarrow (A^{**}, \Delta)$ is a left multiplier. Let $m \in SZ_{\ell}(A^{**})$. We have, for $n \in A^{**}$ and $f \in A^*$,

$$\begin{aligned} \langle U^{**}(m) \square n, f \rangle &= \langle U^{**}(m), n \square f \rangle \\ &= \langle m, U^*(n \square f) \rangle \\ &= \langle m, n \square U^*(f) \rangle \\ &= \langle m \square n, U^*(f) \rangle \\ &= \langle m \Delta n, U^*(f) \rangle \\ &= \langle U^{**}(m \Delta n), f \rangle \\ &= \langle U^{**}(m) \Delta n, f \rangle. \end{aligned}$$

Thus, $U^{**}(m) \in Z_{\ell}(A^{**})$ for each $U \in LM(A)$ and $m \in SZ_{\ell}(A^{**})$. Note that $U^{**}(m) \Delta A \subset A$, using that U^{**} is a left multiplier of (A^{**}, Δ) as well as Lemma 4.1.1. Again, by

Lemma 4.5.1 we conclude that $U^{**}(m) \in SZ_\ell(A^{**})$

Define $\phi: LM(A) \rightarrow LM(SZ_\ell(A^{**}))$ by $\phi(U) = U^{**}|_{SZ_\ell(A^{**})}$. The map ϕ is an isometry. To see this note that $\|U^{**}|_{SZ_\ell(A^{**})}\| \leq \|U\|$ and $\|U^{**}|_{SZ_\ell(A^{**})}\| \geq \|U^{**}|_A\| = \|U\|$. Combining that τ is injective, ϕ is an isometry and $\tau \circ \phi = id$ we have that ϕ is an isometric isomorphism. $\square$

Corollary 4.5.2. *Let A be a Banach algebra with a BAI. Then $LM(SZ_\ell(A^{**})) = LM(A)$.*

As we can see, the condition $A = \overline{\langle A^2 \rangle}$ insure that the map τ is well define. The hypothesis that $(A^{**})^\perp$ is left faithful guaranties that τ is injective.

Chapter 5

Applications

As we have seen, the notion of strong topological centre is a source of intriguing problems. But as the topological centre, the strong topological centre may be used as a tool. An obvious application of this is that a Banach algebra is not strongly Arens irregular if we know that the strong topological centre is not minimal. This chapter shows many different ways to use the strong topological centre as a versatile and powerful tool. Among other things, we employ it to check the faithfulness of the second dual of Banach algebra. We also use it to study the group algebra and the Fourier algebra, in particular the problem of Arens regularity of the latter. In the last section, we present new proofs of some results, mainly concerning weakly sequentially complete Banach algebra, of Lau and Ülger.

5.1 Faithfulness of the second dual

We have seen in Example 5.1.1 that the Banach algebra $(l^1(S), \cdot)$ for a right zero semi-group is Arens regular, but $SZ_\ell((l^1(S))^{**}) = l^1(S)$. There are also other types of Banach algebras which are Arens regular and such that the left, the right or both strong topological centres are minimal: for instance, any unital C^* -algebra. The next theorem links this phenomenon with the weak dual factorization property.

Theorem 5.1.1. *Let A be a left faithful, Arens regular Banach algebra. If $SZ_\ell(A^{**}) = A$, then $A^* = \overline{AA^*}$.*

Proof. First, observe that $A^* = \overline{AA^*}$ is equivalent to $\text{Ann}_l(A^{**}, \Delta) = \{0\}$. Since A is Arens regular, this is equivalent to $\text{Ann}_l(A^{**}, \square) = \{0\}$.

We will proceed by contradiction. Suppose that $\text{Ann}_l(A^{**}, \Delta) = \{0\}$. Then, there is a non-zero $m \in A^{**}$ such that $m \square n = 0$ for all $n \in A^{**}$. Notice that $m \square A = 0$. Thus, $m \in SZ_\ell(A^{**})$ since A is Arens regular by Corollary 5.1.1 (i). By hypothesis, $SZ_\ell(A^{**}) = A$, so $m \in A$. This is a contradiction, since A is left faithful. $\square$

This result is similar to Theorem 5.1.2. Both link the strong topological centre with one of the three levels of the dual factorization property. Of course, if we add the hypothesis that A has a BAI, then the above theorem is known, since, in this case, the Arens regularity implies the strong dual factorization property.

The next theorem shows that, with some additional assumptions on A in Theorem 5.1.1, we obtain an equivalence.

Corollary 5.1.2. *Let A be a left faithful, weakly sequentially complete, Arens regular Banach algebra with a sequential approximate identity. Then, $SZ_\ell(A^{**}) = A$ if and only if $A^* = \overline{\langle \bar{1}A^* \rangle}$, if and only if A is unital.*

Proof. If $A^* = \overline{\langle AA^* \rangle}$, then A is unital by Lemma 5.1.1. Conversely, if A is unital, we, of course, have that $A^* = \overline{\langle AA^* \rangle}$. Therefore, if $A^* = \overline{\langle AA^* \rangle}$, then A being unital satisfies $SZ_\ell(A^{**}) = A$, by Corollary 5.1.1. The reverse follows from Theorem 5.1.2. $\square$

If we look more carefully at the proof of Theorem 5.1.1, we see that the left faithfulness of $(A^{**}, \square)$ is linked with the left strong topological centre as follows

Theorem 5.1.3. *Let A be a left faithful Banach algebra. If $SZ_\ell(A^{**}) = A$, then $(A^{**}, \square)$ is left faithful.*

Proof. Let $m \in \text{Ann}_\ell(A^{**}, \square)$. So, the left multiplication by m is identically 0. Thus it is weak*-weak*-continuous. Hence, $m \in Z_\ell(A^{**})$. Moreover, $m \square A = \{0\} \subset A$, so $m \in SZ_\ell(A^{**}) = A$, by Lemma 5.1.1. We conclude that $m = 0$ since A is left faithful. $\square$

Remark 5.1.4. Note that the condition $SZ_\ell(A^{**}) = A$ is much more weaker than the assumption $Z_\ell(A^{**}) = A$, there are examples such that A is Arens regular with the strong topological being minimal.

Corollary 5.1.5. *Let A be a left faithful Banach algebra. If $(A^{**}, \square)$ is not left faithful, then A is not left strongly Arens irregular.*

5.2 The group algebra and the Fourier algebra

In this section, we shall present shorter proofs of, and generalize, some results of [13]. We first give a new proof of Proposition 8 of [13] which is Proposition 5.2.1. We generalize this result to obtain some corollaries. Thereafter, we use the strong topological centre to study the Arens regularity of the Fourier algebra.

Proposition 5.2.1. (*Hu-Neufang-Ruan [13], Proposition 8*) *Let G be a locally compact group. For $\mu \in M(G)$, the following are equivalent:*

- (i) $\mu \in L^1(G)$;
- (ii) there exists $m \in L^\infty(G)^*$ such that $\mu \square n = m \square n$ for all $n \in L^\infty(G)^*$;
- (iii) $L^\infty(G) \triangle \mu \subseteq LUC(G)$, where $LUC(G)$ is the space of bounded left uniformly continuous functions on G .

Proof. Suppose (i). This obviously implies (ii).

Suppose (ii). Then, $m \in SZ_\ell(L^\infty(G)^*)$, and to get (iii), we just apply Theorem 5.1.1.

Suppose (iii). By Lemma 5.1.2, Theorem 5.1.1 and Lemma 5.1.3, we have that $\mu \in SZ_\ell(A^{**})$. Since $L^1(G)$ is strongly Arens irregular, we get (i). $\square$

The authors in [13] use the fact that $L^1(G)$ is of type (M), but the proof given here shows that there is no need of it. The crucial point is that $SZ_\ell(L^\infty(G)^*) = L^1(G)$. Indeed, we generalize the previous proposition.

Theorem 5.2.2. *Let A be a Banach algebra with a BAI such that $SZ_\ell(A^{**}) = A$. For $T \in LM(A)$, the following are equivalent:*

- (i) $F \in A$,
- (ii) there exists $m \in A^{**}$ such that $T^{**}(n) = m \square n$ for all $n \in A^{**}$;
- (iii) $T^*(A^*) = A^* \Delta T^{**}(F) \subset A^*A$ for some mixed unit F .

Proof. We argue similarly as in the proof of Proposition 5.2.2. □

From this result, we obtain, as a corollary, Theorem 10 and 11 of [13], which are presented both in the next result

Corollary 5.2.3. (*Hu Neufang Ruan*) *Let G be a locally compact group*

(i) *Let ω be a weight on G with $\omega \geq 1$. For $\mu \in M(G, \omega)$, the following are equivalent:*

- (a) $\mu \in L^1(G, \omega)$;
- (b) *there exists $m \in L^\infty(G, \omega^{-1})^*$ such that $\mu \square n = m \square n$ for all $n \in L^\infty(G, \omega^{-1})^*$,*
- (c) $L^\infty(G, \omega^{-1}) \Delta \mu \subseteq LUC(G, \omega^{-1})$.

(ii) *Suppose that G is amenable. For $\phi \in B(G)$, the following are equivalent:*

- (a) $\phi \in A(G)$;
- (b) *there exists $m \in A(G)^{**}$ such that $\phi \square n = m \square n$ for all $n \in A(G)^{**}$,*
- (c) $VN(G) \Delta \phi \subseteq UCB(G)$, where $UCB(\hat{G}) = VN(G) \cdot A(G)$.

Proof. By Theorem 5.2.2, $L^1(G, \omega)$ and $A(G)$, in the latter case for G being amenable, are Banach algebras of type (M). By Corollary 5.2.2, $SZ_\ell(A^{**}) = A$ for any Banach algebra A of type (M). We obtain the equivalences in both cases by applying Theorem

□

Now, we use the notion of strong topological centre to work on the question of Arens regularity of the Fourier algebra. We only treat the case of an amenable group.

Theorem 5.2.4. (*Lau–Losert [16, Theorem 6.5]*) *Let G be a locally compact amenable group. If G is discrete, then $A(G)$ is strongly Arens irregular.*

Proof. Since G is amenable, $A(G)$ has a BAI and $A(G)$ is of type (M), by Theorem 5.2.3. In view of Corollary 5.2.2, $SZ_t(A(G)^{**}) = SZ_r(A(G)^{**}) = A(G)$. Now, since G is discrete, $A(G)$ is an ideal in its second dual, by Theorem 5.2.1, and by Corollary 5.2.2 (ii), we obtain that $A(G)$ is strongly Arens irregular, due to Corollary 5.2.2 (ii). $\square$

Here, the fact that, for all locally compact amenable groups,

$$SZ_t(A(G)^{**}) = SZ_r(A(G)^{**}) = A(G),$$

is very interesting. We can ask if this is true all locally compact groups in general. However, this is not true. If G is a discrete group, then by Corollary 5.2.2 (ii), the strong topological centre and the topological centre coincide. If you restrict to discrete group, we need to get a non-amenable group G such that $A(G)$ is not strongly Arens irregular. An example of such a group is the free group on two generators, $\mathbb{F}_2$, as we mentioned in Example 5.2.1 (v). The next step is to characterize all locally compact groups such that $SZ_t(A(G)^{**}) = SZ_r(A(G)^{**}) = A(G)$. We do not have an answer for this, since it would solve the question of the Arens regularity of $A(G)$, which is a long-standing problem.

With Theorem 5.2.4, we can easily deduce that G is finite if G is amenable and $A(G)$ is Arens regular.

Corollary 5.2.5. (*Lau [14, Proposition 3.3]*) *Let G be a locally compact amenable group. If $A(G)$ is Arens regular, then G is finite.*

Proof. By Theorem 5.1.1, G is discrete. Due to Theorem 5.1.2, $A(G)$ is strongly Arens irregular, but, by assumption, $A(G)$ is Arens regular, too, which implies that G is finite. $\square$

5.3 Weakly sequentially complete Banach algebras

The results in this section are due to Lau and Ülger. They are contained in [14] and [15]. The proofs presented here are totally different from the original ones. We really want to say that even the *philosophy* of the approach is completely different. The proofs in [14, 7] and [32] can be qualified as *local proofs* in the sense that they study the behaviour of one element and after to prove the result. Our proofs can be called *global*, in the sense that they rather use sets as tools.

Before restricting our attention to the class of weakly sequentially complete Banach algebras, we present a new proof of Proposition 2.10 of [17].

Proposition 5.3.1. (*Lau–Ülger*) *Let A be a Banach algebra with a BAI. If A has the left but not the right strong dual factorization property, then $Z_\ell(A^{**}) \neq Z_r(A^{**})$.*

Proof. By Theorem 5.1.1, we have $RM(A) = \{m \in A^{**} \mid 4\Delta m \subset 4\}$. Due to Theorem 5.1.2, $RM(A) \subseteq Z_\ell(A^{**})$. If $Z_\ell(A^{**}) = Z_r(A^{**})$, then, $RM(A) = SZ_r(A^{**})$, by Theorem 5.1.3. We deduce, by the right version of Theorem 5.1.2, that A has the right strong dual factorization property, which is a contradiction. $\square$

Here we really see the difference between the approaches. At any time, we use sets instead of an element of A^{**} .

Now, we move to the case our Banach algebras are weakly sequentially complete.

Proposition 5.3.2. *Let A be a weakly sequentially complete Banach algebra with a sequential BAI. If A is a left ideal in A^{**} , then A is left strongly Arens irregular.*

Proof. By Corollary 5.3.1, we have $SZ_\ell(A^{**}) = A$. Corollary 5.3.1 (iii) entails that A is left strongly Arens irregular. $\square$

We can also give another proof of Lemma 3.1 in [31], which is the next lemma.

Lemma 5.3.3. *(Ülger) Any weakly sequentially complete, Arens regular Banach algebra A with a sequential BAI is unital.*

Proof. By Corollary 5.3.1, $SZ_\ell(A^{**}) = A$. Since, A is Arens regular, $SZ_\ell(A^{**}) = IN(A)$, by Corollary 5.3.1. We conclude that A is unital. $\square$

Remark 5.3.4. This lemma can be also deduced from Lemma 5.3.3, since an Arens regular Banach algebra with a BAI has the left strong dual factorization property [40, Theorem 3.1].

Lau and Ülger make the following conjecture in a comment after Theorem 2.6 of [57].

Conjecture 5.3.5. *(Lau–Ülger) A weakly sequentially complete Banach algebra with a BAI enjoying the left or right strong dual factorization property is unital.*

This conjecture is proven in [17], assuming the BAI being sequential. We have generalized their result in Lemma 5.3.3. By using the strong topological centre, we now present a list of Banach algebras satisfying the Conjecture 5.3.5.

Theorem 5.3.6. *The Conjecture is true for Banach algebras A in the following classes of Banach algebras.*

- (i) *Banach algebras with a sequential BAI*
- (ii) *Arens regular Banach algebras.*
- (iii) *Banach algebras with $SZ_c(A^{**}) = A$, in particular,*
 - a) *left strongly Arens irregular Banach algebras;*
 - b) *Banach algebra of type (LM) or (M);*
 - c) *Banach algebras which are an ideal in their second duals.*

Proof.

- (i) Use Lemma ;
- (ii) [31, Theorem 3.3]:
- (iii) Since, A has the left strong dual factorization property, $SZ_c(A^{**}) = LM(4)$, by Theorem . Thus, the condition $SZ_c(A^{**}) = A$ implies that A is unital. The assertion (a) is obvious. (b) follows from Corollary and (c) follows from Corollary .

□

Chapter 6

Conclusion

As it is natural in research, we cannot study a specific subject without finding links with other topics. This thesis is not an exception. We found links, for instance, between the strong topological centre and the dual factorization property.

We first introduced and studied different levels of the dual factorization property. A possible conjecture is that the strong dual factorization property implies the existence of a bounded approximate identity, as it is the case for the Fourier algebra, in which case, one even obtain a unit. It is clear that the set A^*A , is linked to many properties of a Banach algebra A . For example, in the proof of Theorem 2.2 in [32], Ulger proves that $\overline{\langle A^*A \rangle} = \overline{\langle \Sigma \rangle}$ for a semi-simple commutative Banach algebra which is an ideal in its second dual, where Σ is the spectrum of A . The annihilator of A^*A and the radical of $(1^{**}, \square)$ are also related as Mustafayev shows in [22]. Theorem 3.1 shows that the strong dual factorization property implies the Arens semi-regularity. The converse is false in general. For which Banach algebras is it true? Another question is, for which Banach space X the algebra of compact operators $K(X)$ has the strong dual

factorization property?

The main purpose of this thesis was to measure the degree of Arens regularity of Arens regular Banach algebras and to compare them. The strong topological centre achieved this goal. Among the proven results, we saw that the algebra $(l^1, \cdot)$ is much more Arens regular than most of the C^* -algebras or any infinite-dimensional von Neumann algebras. We provided a complete list of examples that covers all the possible cases of the behaviour of the strong topological centre. For Banach algebras with BAI and without the left strong dual factorization property, we only know that the left strong topological centre is a subalgebra of the left multiplier algebra containing the algebra itself as a left ideal. In Section 4.4, we showed that the right strong topological centre and the right topological centre coincide for the convolution algebra $(\mathcal{N}(l^p(G)), *)$ when G is a discrete group. This provided an example of a Banach algebra which is not an ideal in its second dual, and neither Arens regular nor right strongly Arens irregular, but where the equality holds. We saw that the strong topological centre can be used as a tool for studying the Arens regularity of the Fourier algebra $A(G)$. The proof of Corollary 4.3.1 proposed another approach to this problem. This question can be solved by determining for which discrete group, the strong topological centre of the Fourier algebra is $A(G)$. Example 4.3.1(v) shows, by Losert's result, that this can be true only for discrete groups that do not contain a copy of $\mathbb{F}_2$.

Dales and Lau introduced in [7], the notion of sets, in the second dual of a Banach algebra, which are determining for the topological centre, abbreviated by *DTC* sets. A similar notion can also be developed, in the second dual of a Banach algebra, for the strong topological centre. Since the strong topological centre is defined with the idea that we consider a level lower than for the topological centre, we can maybe define a corresponding notion in the dual of a Banach algebra. This question

on its own is clearly intriguing.

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