Circuit-Reduction Techniques and Application to High-Speed Systems and Interconnects

by

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in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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Abstract

Rapid developments in VLSI technology and an ever-increasing quest for high-speed applications has highlighted previously negligible interconnect effects. These effects if not predicted at early design stages can severely degrade system performance. Generally, interconnect networks result in large systems of equations, simulation of which is prohibitively CPU expensive. Model-reduction is the key to fast simulation of interconnect networks. Conventional model-reduction techniques using Krylov-subspace methods, handle the distributed transmission lines by discretizing them into a set of ordinary differential equations. Discretization generally needs large number of sections and consequently leads to large circuit matrices.

In this thesis, a new technique is presented which enables the inclusion of *distributed* transmission line networks described by Telegrapher's equations directly using a congruent transformation based analysis. Next, an *efficient sensitivity analysis algorithm* of transmission line networks is presented that handles distributed transmission line circuit stamps including frequency dependent parameters. While this algorithm performs small-scale sensitivity analysis on large interconnect circuits, a new method to perform *large-scale sensitivity* is suggested that reduces a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit. Next, a novel algorithm was developed for forming reduced-circuits for large interconnect networks *without the need for partitioning* the network into linear and nonlinear parts. This algorithm projects the full *nonlinear* circuit equations into a subspace of lower dimension. The reduced system can be solved using any of the conventional numerical integration techniques, resulting in significant reduction in the CPU time.

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Dedicated to my Guru

Sri Yajnavalkya Munimanikyam

for showering unconditional love,
boundless patience and supreme guidance

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List of Acronyms

VLSI	Very Large Scale Integration
CAD	Computer-Aided Design
MMT	Moment Matching Techniques
EM	Electromagnetic
PVL	Padé via Lanczos
MNA	Modified Nodal Analysis
TEM	Transverse Electromagnetic
AWE	Asymptotic Waveform Evaluation
CFH	Complex Frequency Hopping
RLC	Resistor-Inductor-Capacitor (networks)
CPU	Central Processing Unit
NCR	Nonlinear Circuit Reduction
p.u.l	per-unit-length

List of Symbols

$\mathbf{A}$	Matrix A
$\mathbf{A}^T$	Matrix A transposed
$\mathbf{A}^{(r)}$	r^{th} derivative of A
$\mathbf{0}$	Matrix of zeros
$\mathbf{I}$	Identity matrix
$\mathbf{I}_q$	Identity matrix of size $q \times q$
$\mathbb{R}^N$	The set of all real vectors of size N
$\mathbb{R}^{N \times N}$	The set of all real matrices of size $N \times N$
$(d_{i,j})$	Matrix composed of elements $d_{i,j}$ in the i^{th} row and j^{th} column
$\hat{a}$	Approximation of a
s	Laplace operator ($= j\omega$)
$\tilde{\mathbf{R}}, \tilde{\mathbf{L}}, \tilde{\mathbf{G}}, \tilde{\mathbf{C}}$	matrix p.u.l parameters of the transmission line
$\mathbf{Z}_L, \mathbf{Y}_L$	Matrix p.u.l impedance and admittance parameters of the transmission line
$\mathbf{C}$	MNA matrix representing the lumped memory components of a network

$\mathbf{G}$	MNA matrix representing the lumped memoryless components of a network
$\mathbf{i}_p, \mathbf{v}_p$	Port currents and port voltages of a network
$\mathbf{D}_k$	Selector matrix that maps the currents of an interconnect subnetwork k into the node-space of a lumped network
$\mathbf{A}_k$	Y-parameters of the interconnect subnetwork k
$\mathbf{F}(s)$	MNA matrix containing contributions due to all interconnects in the network
$\mathbf{b}$	Vector of contributions due to independent voltage/current sources
$\kappa(\mathbf{A}, \mathbf{R}, n)$	Krylov subspace
$\mathbf{Q}$	Orthogonal matrix
$\mathbf{M}_n^i$	n^{th} vector for an excitation at port i
λ	Parameter w.r.t which sensitivity is calculated
$\mathbf{F}(x)$	Vector containing the contributions due to nonlinear elements in the network
$\mathbf{M}^{(i,j)}$	i^{th} moment w.r.t frequency and j^{th} moment w.r.t parameter λ
$\mathbf{a}_k$	Normalized time-domain derivatives of a vector-function
$\mathbf{J}(x)$	Jacobian matrix of the nonlinear circuit

Chapter 1

Introduction

1.1 Background and Motivation

Rapid developments in VLSI technology, design and computer-aided design (CAD) techniques at both the chip and package level have pushed the central processor cycle times to the vicinity of 1ns and communication switches are being designed to transmit data that have bit rates faster than 1Gb/s. The ever-increasing quest for high-speed applications is placing higher demands on interconnect performance and has highlighted the previously negligible effects of interconnects such as ringing, signal delay, distortion, reflections and crosstalk [1]–[15] (see Fig. 1.1). These effects if not predicted at early design stages can severely degrade system performance. Generally, interconnect networks result in large systems of equations, simulation of which is prohibitively CPU expensive.

With higher signal speeds, the electrical length of interconnects becomes a significant portion of the wavelength. This gives rise to signal distorting effects that do not

Figure 1.1: Interconnect problem

exist at lower frequencies. At high signal speeds the interconnect exhibits distributed capacitive and inductive effects which makes the receiver no longer a direct load of the driver [1]–[10]. This leads to various transmission line effects such as reflections, overshoot, undershoot, crosstalk and modeling of these needs the blending of EM and circuit theory. Generally, high-speed interconnect effects are neither handled efficiently nor accurately by conventional circuit simulators such as SPICE [16].

High-speed effects if neglected at early design stages can cause logic glitches that render a fabricated digital circuit inoperable or lead to a distortion in an analog signal which could have been crucial for the given specifications. Extra iterations in the design cycle are costly and involve a substantial amount of man-power. Delay in time-to-market, which is a crucial factor, when affected severely, has the potential to damage a healthy industry. Due to these factors, accurate prediction of high-speed

interconnect effects is a necessity in high-speed designs. It is imperative that designers simulate their designs along with interconnect effects as accurately as possible without a substantial loss in computational efficiency. One of the main problems encountered in trying to predict interconnect effects is that in general, interconnect networks result in large systems of equations, simulation of which is prohibitively CPU expensive. Forming a reduced linear system is the key to fast simulation of interconnect networks. This process of forming a reduced-model of a large linear system is termed as *model-reduction* [17]–[28]. Model-reduction is a paradigm where a large system of equations is reduced to a small set of equations such that the response obtained from the smaller system resembles that of the original system.

There have been numerous publications using moment-matching techniques (MMT) and Padé approximation for reducing the order of the linear system [17]–[20]. However, Padé approximation has an ill-conditioning problem associated with it. In order to overcome the ill-conditioning problem, several techniques based on Krylov-subspace methods and congruent transformations were developed [21]–[28]. Padé via Lanczos (PVL) was first suggested to overcome the difficulties due to ill-conditioning [21]. The passivity of the reduced-order model generated by PVL is ensured for only two-element kind networks but not for general RLC networks. A co-ordinate transformed Arnoldi algorithm was suggested to generate stable reduced-order models for RLC circuits [22], however passivity is not ensured. [24] extends the block Arnoldi technique to generate passive reduced-order models for RLC networks. Krylov-subspace techniques were further advanced in [25], [26] where passivity is ensured for multipoint multiport network reductions.

Approaches like the PVL and Arnoldi are based on Krylov-subspace methods and

oblique/congruent transformations. These techniques overcome the ill-conditioning associated with the direct Padé approximation and are very efficient for simulating large *lumped linear networks*. However, these techniques handle the distributed transmission lines governed by Telegrapher's partial differential equations by *discretizing* them into a set of ordinary differential equations. However, there are several disadvantages with the discretization based representation of distributed networks.

1. Large number of sections are required in order to ensure reasonable accuracy, which leads to large circuit matrices. This problem deteriorates with higher frequencies or sharper rise/fall times.
2. The method cannot directly handle measured subnetworks or transmission lines with frequency-dependent parameters.
3. Discretization of transmission lines leads to Gibbs' phenomenon, which becomes more pronounced at higher frequencies [10].

In this thesis, a new technique is presented in chapter 3 to overcome the above difficulties, which enables the inclusion of *distributed* transmission line networks described by Telegrapher's equations *directly* using a congruent transformation based analysis. Chapter 4 presents a new method to perform sensitivity analysis of transmission line networks based on congruent transformation techniques that handles distributed transmission line circuit stamps including frequency dependent parameters without the need for discretization of Telegrapher's equations. The algorithm is based on projecting the adjoint network equations on a reduced order subspace that preserves the circuit moments.

All the above-mentioned techniques present in the literature perform model-reduction with respect to a single parameter (frequency). However, it is also necessary to include variations, due to environmental effects and changes in interconnect parameters such as width and height. These effects induce a change in the MNA (Modified Nodal Analysis) matrices of the circuit. It is often not viable to perform simulation of large circuits repeatedly for changes due to these effects. Hence, a reduction technique is required, that performs analysis of large circuits simultaneously with respect to frequency and other design parameters of the circuit. Chapter 5 presents an algorithm to reduce the order of a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit. These parameters could also include physical parameters of the transmission lines.

In general, model-reduction algorithms commonly found in the literature, address the interconnect problem by partitioning the network into linear and nonlinear parts [24], [27]–[30]. The linear network is reduced using moment matching techniques to form an n-port macromodel. This macromodel is then coupled with the nonlinear part of the network to obtain the circuit response. However, the efficiency of these methods drop significantly when the number of ports of the linear subnetwork increase as a result of having more nonlinear terminations. To overcome this disadvantage, a new method is presented in chapter 6 which directly reduces the whole network containing both the interconnects and nonlinear terminations without the need to partition the network into linear and nonlinear parts. Consequently, the efficiency of the new algorithm is completely independent of the number of ports in the interconnect network.

1.2 Contributions

The main subject of the thesis is the simulation of high-speed interconnect networks. The work presented here will provide an easy means to handle high-speed interconnect models in circuit simulators. In view of the problems mentioned in the previous section, the main contributions of this thesis are as follows:

1. A new technique (chapter 3) which enables the inclusion of *distributed* transmission line networks described by Telegrapher's equations *directly* using a congruent transformation based analysis. Previous methods that relied on congruent transformation techniques, handled the transmission lines by discretization of the Telegrapher's equations. The new method takes advantage of the congruent transformation techniques while including the interconnects as *distributed* transmission lines. Also presented is the proof of preservation of moments and passivity of the reduced-order system for networks containing distributed elements. In addition, the proposed method can efficiently handle distributed networks with frequency-dependent parameters [31]. A speed-up of an order of magnitude was observed when compared to conventional simulation techniques.
2. An efficient algorithm for sensitivity analysis of transmission line networks (see chapter 4) based on congruent transformation techniques that handles distributed transmission line circuit stamps including frequency dependent parameters without the need for discretization of the Telegrapher's equations. The algorithm is based on projecting the adjoint network equations on a reduced order subspace that preserves the circuit moments. The sensitivity can then be evaluated over any number of frequency points for the cost of one QR

factorization. A speed-up of two orders of magnitude was observed.

3. A new algorithm is presented to reduce the order of a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit (see chapter 5). These parameters could also include physical parameters of the transmission lines. The reduction algorithm is based on congruent transformation techniques applied to multiple dimensions. The proposed algorithm provides efficient means to estimate the response of large circuits simultaneously as a function of several design parameters. This algorithm performs *large-scale sensitivity* analysis on interconnect networks as opposed to small-scale sensitivity analysis performed by the algorithm mentioned in the previous item. This means that this algorithm can be used to predict variations in circuit responses for large variations in the external parameters effecting it. A speed-up of two orders of magnitude was observed when compared to conventional techniques.
4. A novel algorithm is developed (chapter 6) for forming reduced-circuits for large interconnect networks without the need for partitioning the network into linear and nonlinear parts. This is accomplished by projecting the full *nonlinear* circuit equations into a subspace of lower dimension. As a result, the original set of nonlinear time-domain equations is replaced by a smaller set of nonlinear equations such that the first 'q' derivatives of the time-response of the original system and the response obtained using the reduced system are identical. The reduced system can then be solved using any of the conventional numerical integration techniques, resulting in significant reduction in the CPU time due to the

fact that at each time point, a much smaller set of nonlinear algebraic equations need to be solved. This algorithm reduces the whole network containing both the interconnects and nonlinear terminations without the need to partition the network into linear and nonlinear parts. Conventional model-reduction techniques need to partition interconnect networks into linear and nonlinear parts and hence suffer from inefficiencies as the number of ports increase. The efficiency of the new algorithm is completely independent of the number of ports in the interconnect network. A speed-up of an order of magnitude was observed using this algorithm when compared to conventional techniques.

It is demonstrated that these techniques provide a marked improvement in the simulation of various interconnect networks.

1.3 Organization of the Thesis

This thesis is organized as follows: chapter 2 presents a brief overview of the interconnect problem and the existing model-reduction techniques. This chapter also provides some background on interconnect simulation and highlights the difficulties encountered by conventional circuit simulators while handling large interconnect networks. In chapter 3, a new algorithm is proposed that enables the inclusion of distributed transmission line networks described by Telegrapher's equations directly using a congruent transformation based analysis. Chapter 4 presents an efficient sensitivity analysis algorithm for transmission line networks based on congruent transformation techniques. This algorithm handles distributed transmission line circuit stamps including frequency dependent parameters without the need for discretization of the Teleg-

rapher's equations. Chapter 5 outlines a new technique to reduce the order of a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit. In short, this algorithm performs *large-scale sensitivity* analysis on interconnect networks. In chapter 6, a novel algorithm is outlined that simulates large nonlinear interconnect networks efficiently by projecting the full *nonlinear* circuit equations into a subspace of lower dimension. Finally, conclusions are presented in chapter 7.

Chapter 2

Review of Model Order Reduction Techniques

As the rise times of digital signals drop into sub-nanosecond range, the effects of off-chip circuitry become increasingly important. Advances in fabrication methods have made interconnect analysis imperative for diverse technologies such as packages, multi-chip modules, printed circuit boards and backplanes. Shrinking device sizes coupled with increasing operating speeds are highlighting the effects, such as ringing, signal delay, distortion, and reflections on single interconnect lines as well as crosstalk between adjacent lines [1]–[15]. Interconnections can be from various levels of design hierarchy, such as on-chip, packaging, multi-chip modules and printed circuit boards. At relatively higher frequencies lumped models become inaccurate and distributed quasi-TEM models based on Telegrapher’s equations become necessary.

In general we find that high-speed effects are not always handled appropriately by conventional circuit simulators such as SPICE [16]. Interconnect effects if not con-

sidered at early design stages can cause logic glitches that render a fabricated digital circuit inoperable or distort an analog signal enough to fail the desired specifications. Since extra iterations in the design cycle are costly, accurate prediction of these effects is a necessity in high-speed designs. Hence, it becomes extremely important for designers to simulate their design along with interconnect effects as accurately as possible without a loss in computational efficiency. One of the main problems encountered in trying to predict interconnect effects is that in general interconnect networks result in large systems of equations, simulation of which is prohibitively CPU expensive. Forming a reduced linear system is the key to fast simulation of interconnect networks.

Presented in this chapter is a brief overview of the interconnect problem and the existing model-reduction techniques. This chapter also provides some background on interconnect simulation and highlights the difficulties encountered by conventional circuit simulators while handling large interconnect networks. The organization of this chapter is as follows: section 2.1 defines the term “high-speed interconnect” and talks about the conditions when the interconnect effects become important. Section 2.2 describes broadly the various interconnect models used in practice and section 2.3 talks about the Telegrapher’s equations that describe a distributed transmission line set. Section 2.4 illustrates the set-up of the modified nodal analysis (MNA) matrices of the system and section 2.5 discusses the issues involved in simulating networks with distributed interconnects. Next, section 2.6 introduces the concepts behind the application of the paradigm of model-reduction in simulating interconnect networks, followed by sections 2.7 and 2.8 which briefly describe *direct* moment matching techniques (MMTs) such as AWE and CFH. Section 2.9 introduces the preliminaries

of *indirect* MMTs and is followed by a description of the Arnoldi algorithm in section 2.10. A discussion on passivity is ensued in section 2.11 and modifications to the Arnoldi algorithm are presented to generate passive macromodels.

2.1 “High-Speed” Interconnects

In a broad sense a “high-speed interconnect” is an interconnect where the time taken by the propagating signal to travel between its end points cannot be neglected. An important but obvious factor that influences this definition is the physical extent of the interconnect; the longer the interconnect, the more time the signal takes to travel between its end points. The smoothness of the signal is also effected as it propagates through the line. The interconnect exhibits distributed capacitive and inductive effects which makes the receiver no longer a direct load of the driver [1]–[10]. This leads to various transmission line effects such as reflections, overshoot, undershoot, crosstalk and modeling of these needs the blending of EM and circuit theory.

The term “high-speed” can also be understood from a different view point; in terms of the frequency content of the signal. At low frequencies an interconnect behaves effectively as a short circuit. However, this is not the case at higher frequencies. The same interconnect exhibits inductive/capacitive effects at higher frequencies. Faster clock speeds and sharper slew rates tend to add a considerable amount of high-frequency content.

An important criterion used for classifying interconnects is the electrical length of an interconnect. An interconnect is “electrically short” if, at the highest operating frequency of interest, the interconnect length is physically shorter than approximately

one-tenth of the wavelength. Otherwise, the interconnect is referred to as “electrically long” [1], [10]. Electrically short interconnects can be represented by lumped models whereas electrically long interconnects need quasi-TEM distributed or full-wave models. For the categories shown in Table 2.1 [8], the wavelength is in the order of $1 - 10cm$. The propagated signal rise times are in the range $100 - 1000ps$. Hence the line lengths are either comparable or much longer than the signal wavelengths. Depending on the operating frequency, signal rise times and nature of the structure, these interconnects can be modeled as lumped (RC or RLC), distributed (frequency independent/dependent RLCG parameters, lossy, coupled) or full-wave models.

Interconnection type	Line width (μm)	Line thickness (μm)	Line resistance (Ω/cm)	Maximum length(cm)
On-chip	0.5 – 2	0.7 – 2	100 – 1000	0.3 – 1.5
Thin film	10 – 25	5 – 8	1.25 – 4	20 – 45
Ceramic	75 – 100	16 – 25	0.4 – 0.7	20 – 50
PCBs	60 – 100	30 – 50	0.06 – 0.08	40 – 70
Cables	100 – 450	35 – 450	0.0013 – 0.033	150 – 500

Table 2.1: Interconnect technologies

In most digital applications the desired highest operating frequency of interest is governed by the rise/fall time of the propagating signal. A practically useful relationship between the maximum frequency of interest (f_{max}) and the rise/fall time of the signal (t_r) can be expressed as [2], [32]–[34]

$$f_{max} \approx 0.35/t_r. \quad (2.1)$$

In view of the various problems encountered through interconnect effects it becomes

imperative to form interconnect models that capture high-frequency responses efficiently and accurately. In the next section, a brief description of the various interconnect models used in practice is presented.

2.2 High-Speed Interconnect Models

Depending on the operating frequency, signal rise times and nature of the structure, the interconnects can be modeled as lumped, distributed or distributed models with frequency dependent parameters.

2.2.1 Lumped Models

At lower frequencies, the interconnect circuits could be modeled using lumped RC or RLC circuit models. RC circuits responses are monotonic in nature. However, in order to account for ringing in signal waveforms, RLC circuit models may be required. Usually, lumped interconnect circuits extracted from layouts contain a large number of nodes that make the simulation highly CPU intensive.

2.2.2 Distributed Transmission Line Models

At relatively higher signal-speeds, electrical length of interconnects becomes a significant fraction of the operating wavelength, giving rise to signal distorting effects that do not exist at lower frequencies. Consequently, the conventional lumped impedance interconnect models become inadequate and transmission line models based on quasi-transverse electromagnetic mode (TEM) assumptions are needed. The TEM approximation represents the ideal case, where both the E and H fields are perpendicular to

the direction of propagation and is valid under the condition that the line cross section is much smaller than the wavelength. However, the inhomogeneities in practical wiring configurations give rise to E or H fields in the direction of propagation. If the line cross section or the extent of these non-uniformities remain a small fraction of the wavelength in the frequency range of interest, the solution to Maxwell's equations are given by the so-called quasi-TEM modes and are characterized by distributed R , L , G , C per unit length (p.u.l.) parameters [10]. In practical situations, owing to complex interconnect geometries and varying cross-sectional areas, the interconnects might have to be modeled as nonuniform lines. In this case, the p.u.l. parameters are functions of the distance, along the length of the transmission line [35]–[37].

2.2.3 Distributed Models with Frequency-Dependent Parameters

At low frequencies, the current in a conductor is distributed uniformly throughout its cross section. However, as the operating frequency increases, the current distribution gets uneven and gets concentrated near the surface or edges of the conductor. This phenomenon can be categorized as follows: skin, edge, and proximity effects [10], [31], [38], [39]. The skin effect causes the current to concentrate in a thin layer near the conductor surface and this reduces the effective cross section available for signal propagation. This leads to an increase in the resistance to signal propagation and other related effects [8]. The edge effect causes the current to concentrate near the sharp edges of the conductor. The proximity effect causes the current to concentrate in the sections of ground plane that are close to the signal conductor. To account for these effects, modeling based on frequency-dependent p.u.l. parameters

may be necessary.

In the next section we shall take a look at the equations that govern the distributed interconnect models.

2.3 Distributed Transmission Line Equations

Transmission line characteristics are in general described by Telegrapher's equations. Consider the transmission line system shown in Fig. 2.1(a). Telegrapher's equations for such a structure can be derived by discretizing the line into infinitesimal sections (see Fig. 2.1(b)) of length Δz , assuming uniform p.u.l. parameters of resistance (R), inductance (L), conductance (G), and capacitance (C). Each section then includes a resistance $R\Delta z$, inductance $L\Delta z$, conductance $G\Delta z$, and capacitance $C\Delta z$. Using Kirchhoff's current and voltage laws for the infinitesimal section described above, and taking the limit $\Delta z \rightarrow 0$ one can derive the Telegrapher's equations to be

$$\frac{\partial}{\partial z}v(z, t) = -Ri(x, t) - L\frac{\partial}{\partial t}i(x, t) \quad (2.2a)$$

$$\frac{\partial}{\partial z}i(z, t) = -Gv(x, t) - C\frac{\partial}{\partial t}v(x, t) \quad (2.2b)$$

Taking the Laplace transform of (2.2), we have,

$$\frac{\partial}{\partial z}V(z, s) = -(R + sL)I(x, s) = -ZI(x, s) \quad (2.3a)$$

$$\frac{\partial}{\partial z}I(z, s) = -(G + sC)V(x, s) = -YV(x, s) \quad (2.3b)$$

where Z and Y represent the p.u.l. impedance and admittance of the transmission

(a) A transmission line system

(b) An infinitesimal section of the transmission line

Figure 2.1: Single transmission line system

line, given by

$$Z = R + sL \quad (2.4a)$$

$$Y = G + sC \quad (2.4b)$$

In the case of a multi-conductor transmission line (MTL) system with N coupled conductors (see Fig. 2.2), we can derive multi-conductor transmission line equations. Per-unit-length parameters ($\tilde{\mathbf{R}}, \tilde{\mathbf{L}}, \tilde{\mathbf{G}}, \tilde{\mathbf{C}}$) become matrices and voltage-current variables become vectors represented by $\mathbf{v}$ and $\mathbf{i}$, respectively. Noting these changes, we can rewrite (2.2) as

$$\frac{\partial}{\partial z} \mathbf{v}(z, t) = -\tilde{\mathbf{R}} \mathbf{i}(x, t) - \tilde{\mathbf{L}} \frac{\partial}{\partial t} \mathbf{i}(x, t) \quad (2.5a)$$

$$\frac{\partial}{\partial z} \mathbf{i}(z, t) = -\tilde{\mathbf{G}} \mathbf{v}(x, t) - \tilde{\mathbf{C}} \frac{\partial}{\partial t} \mathbf{v}(x, t) \quad (2.5b)$$

Figure 2.2: A multi-conductor transmission line system

Taking the Laplace transform of (2.5), we have,

$$\frac{\partial}{\partial z} V(z, s) = -(\tilde{\mathbf{R}} + s\tilde{\mathbf{L}})\mathbf{I}(x, s) = -\mathbf{Z}\mathbf{I}(x, s) \quad (2.6a)$$

$$\frac{\partial}{\partial z} \mathbf{I}(z, s) = -(\tilde{\mathbf{G}} + s\tilde{\mathbf{C}})\mathbf{V}(x, s) = -\mathbf{Y}\mathbf{V}(x, s) \quad (2.6b)$$

where $\mathbf{Z}$ and $\mathbf{Y}$ represent the p.u.l. impedance and admittance of the transmission line, given by

$$\mathbf{Z} = \tilde{\mathbf{R}} + s\tilde{\mathbf{L}} \quad (2.7a)$$

$$\mathbf{Y} = \tilde{\mathbf{G}} + s\tilde{\mathbf{C}} \quad (2.7b)$$

2.4 Formulation of Network Equations

Prior to introducing interconnect simulation algorithms, it would be useful to review a generic formulation of network equations. For both frequency or time-domain analysis, the first step is to set up the modified nodal analysis (MNA) matrices [30], [40]. A general multiport linear subnetwork consisting of lumped RLC and distributed components can be described through its MNA equations as

$$\mathbf{C} \frac{d}{dt} \mathbf{x} + \mathbf{G} \mathbf{x} + \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{i}_k(t) = \mathbf{B} \mathbf{v}_p \quad (2.8a)$$

$$\mathbf{i}_p = \mathbf{B}^T \mathbf{x} \quad (2.8b)$$

where

$\mathbf{x} \in \mathbb{R}^n$ is the vector of node voltage waveforms appended by independent voltage

source currents, linear inductor currents and port currents,

$\mathbf{C} \in \mathbb{R}^{n \times n}$, $\mathbf{G} \in \mathbb{R}^{n \times n}$ are constant matrices describing the lumped memory and memoryless elements of the network, respectively,

$\mathbf{i}_p$, $\mathbf{v}_p$ denote the port currents and port voltages respectively, p being the number of ports,

$\mathbf{B} = (b_{i,j})$ with elements $b_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, p\}$, is a selector matrix that maps the port voltages into the node space $\mathbb{R}^n$ of the network and n is the total number of variables in the MNA formulation,

$\mathbf{D}_k = (d_{i,j})$ with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, 2n_k\}$ with a maximum of one non-zero in each row or column, is a selector matrix that maps $\mathbf{i}_k(t) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^n$ of the network,

N_t is the total number of distributed coupled transmission line sets and n_k is the number of coupled conductors in the linear subnetwork k .

The first and second terms in (2.8a) cover the network's lumped components and the third term describes currents at the interconnect subnetwork terminals and then maps them into rest of the network through the matrix $\mathbf{D}_k$.

The frequency-domain equations of interconnect subnetwork 'k' containing a distributed transmission line model can be written in the form

$$\mathbf{I}_k(s) = \mathbf{A}_k(s)\mathbf{V}_k(s) = \mathbf{A}_k(s)\mathbf{D}_k^T\mathbf{X}(s) \quad (2.9)$$

where $\mathbf{V}_k(s)$, $\mathbf{I}_k(s)$ and $\mathbf{A}_k(s)$ represent the Laplace-domain terminal voltages and currents and the admittance parameters for the subnetwork k respectively. $\mathbf{X}(s)$ is the Laplace transform of $\mathbf{x}(t)$.

The admittance parameters for the subnetwork k can be computed as follows. Consider the multi-conductor transmission line (MTL) system with N conductors in Fig. 2.2. The system can be represented in the frequency-domain using Telegrapher's equations given in (2.6).

The solution to (2.6) can be expressed as

$$\begin{pmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{pmatrix} = e^{\mathbf{E}} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{pmatrix} \quad (2.10)$$

where

$$\mathbf{E} = d \begin{pmatrix} \mathbf{0} & -\mathbf{Z} \\ -\mathbf{Y} & \mathbf{0} \end{pmatrix} \quad (2.11)$$

and $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$ and $\mathbf{I}_2$ are the terminal voltage/current vectors of the transmission line network (see Fig. 2.2) and d is the length of the transmission line.

Equation (2.10) can be rearranged to compute the admittance parameters of the transmission line subnetwork as

$$\begin{pmatrix} \mathbf{I}_1 \\ -\mathbf{I}_2 \end{pmatrix} = \mathbf{A}_k(s) \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} \quad (2.12)$$

Using (2.9) and the Laplace transform of (2.8), assuming zero initial conditions, we have

$$\mathbf{H}(s)\mathbf{X}(s) = \mathbf{B}\mathbf{v}_p \quad (2.13a)$$

$$\mathbf{i}_p = \mathbf{B}^T \mathbf{X}(s) \quad (2.13b)$$

where

$$\mathbf{H}(s) = \mathbf{G} + s\mathbf{C} + \mathbf{F}(s) \quad \text{and} \quad (2.14a)$$

$$\mathbf{F}(s) = \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{A}_k(s) \mathbf{D}_k^T. \quad (2.14b)$$

An example is given to illustrate the formulation of the MNA (Modified Nodal Analysis) equations used in this paper. Consider the network shown in Fig. 2.3. The network can be described using (2.8) as

$$\mathbf{G} = \begin{pmatrix} G_1 & -G_1 & 0 & 0 & 0 & -1 \\ -G_1 & G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & G_2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad (2.15a)$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.15b)$$

and $\mathbf{X}(s) = (v_1 \ v_2 \ v_3 \ v_4 \ I_L \ I_E)^T$

Figure 2.3: An example circuit for formulation of network equations

The hybrid parameters of the transmission line are

$$\begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = e^{\mathbf{E}d} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}; \quad \mathbf{E} = \begin{pmatrix} 0 & -r \\ -g & 0 \end{pmatrix} + s \begin{pmatrix} 0 & -l \\ -c & 0 \end{pmatrix} \quad (2.16)$$

V_2 , I_2 , V_3 and I_3 can be rearranged to compute the admittance parameters of the transmission line subnetwork as

$$\mathbf{A}_1(s) = \frac{1}{Z_0} \begin{pmatrix} \frac{1+e^{-2\gamma d}}{1-e^{-2\gamma d}} & \frac{-2e^{-\gamma d}}{1-e^{-2\gamma d}} \\ \frac{-2e^{-\gamma d}}{1-e^{-2\gamma d}} & \frac{1+e^{-2\gamma d}}{1-e^{-2\gamma d}} \end{pmatrix}; \quad \begin{aligned} Z_0 &= \sqrt{(r+sl)/(g+sc)} \\ \gamma &= \sqrt{(r+sl)(g+sc)} \end{aligned} \quad (2.17)$$

These admittance parameters are mapped into the network equations through $D_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$ to form

$$\mathbf{F}(s) = \mathbf{D}_1 \mathbf{Y}_1(s) \mathbf{D}_1^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{Z_0} \left(\frac{1+e^{-2\gamma d}}{1-e^{-2\gamma d}} \right) & \frac{1}{Z_0} \left(\frac{-2e^{-\gamma d}}{1-e^{-2\gamma d}} \right) & 0 & 0 & 0 \\ 0 & \frac{1}{Z_0} \left(\frac{-2e^{-\gamma d}}{1-e^{-2\gamma d}} \right) & \frac{1}{Z_0} \left(\frac{1+e^{-2\gamma d}}{1-e^{-2\gamma d}} \right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.18)$$

2.4.1 Formulation of Circuit Equations with Nonlinear Terminations

The subnetwork described in (2.8) is usually terminated with nonlinear components in general circuit configurations. The effect of the nonlinear components is included in the MNA formulation. Thus, the modified equations governing the entire network after each port is terminated with a voltage/current source or a linear/nonlinear termination, can be described as

$$\mathbf{C}_\phi \frac{d}{dt} \mathbf{x} + \mathbf{G}_\phi \mathbf{x} + \sum_{k=1}^{N_t} \mathbf{D}_k i_k(t) + \mathbf{F}_\phi(\mathbf{x}) = \mathbf{b}_\phi(t) \quad (2.19)$$

where,

$\mathbf{C}_\phi, \mathbf{G}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ are the modified $\mathbf{C}$ and $\mathbf{G}$ matrices to include the linear terminations,

$\mathbf{x}_\phi \in \mathbb{R}^{N_\phi}$ is the augmented vector of unknowns,

$\mathbf{b}(t) \in \mathbb{R}^{N_\phi}$ is a vector with entries determined by the independent voltage/current sources and

$\mathbf{F}_\phi(\mathbf{x})$ is a function describing the nonlinear terminations of the circuit.

As we shall see, the presence of nonlinear components along with a distributed representation of the interconnects is associated with a mixed frequency/time problem which shall be discussed in the next section.

2.5 Interconnect Simulation Issues

Simulation of large interconnect networks is generally associated with two major bottlenecks: mixed frequency/time problem and CPU expense.

2.5.1 Mixed Frequency/Time Problem

The major difficulty in simulating high-frequency models such as distributed transmission lines is due to the fact that, while described in terms of partial differential equations, they are best represented in the frequency domain. As seen in the previous sections, they do not have a direct representation in the time-domain. On the other hand, nonlinear devices can only be described in the time-domain. These simultaneous formulations are difficult to handle using a traditional ordinary differential equation solver such as SPICE [16], [40]–[45].

2.5.2 CPU Expense

Frequency-domain simulation of large linear networks is conventionally done by solving (2.8) at each frequency point using LU decomposition and forward-backward substitution. For time-domain simulation, due to the association of the mixed frequency/time problem discussed in section 2.5.1, the interconnect subnetworks in (2.19) are discretized using lumped segmentation (see section 2.2.1) to form a large set of equations. Integration techniques are then used to convert this large set of time-domain differential equations into a set of difference equations [45]. To solve the resulting large set of nonlinear difference equations at each time point, Newton iterations are required, which may need several LU decompositions. This causes the CPU cost of a time-domain analysis to be expensive.

The objectives of interconnect simulation algorithms are to address both the mixed frequency/time problem as well as to handle large linear circuits without considerable CPU expense. There have been several attempts in the literature to address both the above-mentioned problems by applying model-order reduction on the linear subsystem containing lumped as well as distributed subnetworks to form miniature models. These reduced-models can then be coupled with nonlinear terminations to compute the time-domain response. The details of these algorithms shall be discussed in the following sections.

2.6 Model-Reduction Based Interconnect Simulation Algorithms

Several attempts have been made in the literature to address the interconnect problem using model-order reduction techniques [17]–[28]. Model-reduction techniques are based heavily on the fact that the time-domain behavior of a large linear system is governed mostly by a small set of so-called “dominant poles” of the system. “Dominant poles” are poles that are close to the imaginary axis and significantly influence the time as well as the frequency characteristics of the system. The moment-matching techniques (MMTs) [17]–[28] capitalize on the fact that irrespective of the presence of a large number of poles in a system, only the dominant poles are sufficient to accurately characterize a given system.

2.6.1 Concepts of Moment Matching Techniques

A brief mathematical description of the underlying concepts of MMTs is given below. Consider a single input/single output system and let $H(s)$ be the transfer function. $H(s)$ can be represented in a rational form as

$$H(s) = \frac{P(s)}{Q(s)} \quad (2.20)$$

where $P(s)$ and $Q(s)$ are polynomials in s . Equivalently, (2.20) can be written as

$$H(s) = c + \sum_{i=1}^{N_p} \frac{k_i}{s - p_i} \quad (2.21)$$

where p_i and k_i are the i^{th} pole residue pair, N_p is the total number of system poles, and c is the direct coupling constant. The time-domain impulse response can be computed in a closed form using inverse Laplace transform as

$$h(t) = c\delta(t) + \sum_{i=1}^{N_p} k_i e^{p_i t} \quad (2.22)$$

In case of large networks, N_p , the total number of poles can be of the order of thousands. Computing all N_p poles will be highly CPU intensive even for a small network and for large networks it is completely impractical. Model-reduction techniques address the above issue by deriving a reduced-order approximation in terms of dominant poles, instead of trying to compute all the poles of a system. Assuming that only dominant poles were extracted, (2.20) can be rewritten to obtain approximate frequency/time responses, as

$$H(s) \approx \hat{H}(s) = \frac{\hat{P}(s)}{\hat{Q}(s)} = \hat{c} + \sum_{i=1}^L \frac{\hat{k}_i}{s - \hat{p}_i} \quad (2.23)$$

$$h(t) \approx \hat{h}(t) = \hat{c}\delta(t) + \sum_{i=1}^L \hat{k}_i e^{\hat{p}_i t}. \quad (2.24)$$

Note that $L \ll N_p$.

2.6.2 Moments of the Response

Consider the Taylor series expansion of the transfer function $H(s)$ at point $s = 0$

$$H(s) \approx \tilde{H}(s) = H(0) + s \frac{(H(s))^{(1)}}{1!} + s^2 \frac{(H(s))^{(2)}}{2!} + \dots + s^n \frac{(H(s))^{(n)}}{n!} \quad (2.25)$$

where the superscript (n) denotes the n^{th} derivative. Using a simpler notation, (2.25) can be rewritten as

$$H(s) \approx \tilde{H}(s) = m_0 + m_1s + m_2s^2 + \cdots + m_ns^n = \sum_{i=0}^n s^i m_i, \quad m_i = \frac{H(0)^{(i)}}{i!} \quad (2.26)$$

It has been shown that the moments provide an estimation of delay and rise times [46], [47]. Elmore delay [46], which approximates the midpoint of the monotonic step response waveform by the mean of the impulse response, essentially matches the first moment of the response. This can be considered as one of the basic forms of approximation. However, in order to get an accurate prediction of interconnect effects, it is essential that the reduced-order model must match (or preserve) as many moments as possible.

A brief discussion of the MMT techniques already available in the literature shall follow in the subsequent sections.

2.7 Asymptotic Waveform Evaluation (AWE)

Moments matching techniques such as AWE use the moments described in (2.26) to form a Padé approximation to extract the dominant poles and residues of a given system.

2.7.1 Padé Approximation

Consider a system transfer function $H(s)$ that is approximated by a rational function $\bar{H}(s)$ as

$$H(s) \approx \bar{H}(s) = \frac{a_0 + a_1s + \dots + a_Ls^L}{1 + b_1s + \dots + b_Ms^M} = \frac{P_L(s)}{Q_M(s)} \quad (2.27)$$

where $a_0, \dots, a_L, b_1, \dots, b_M$ are the unknowns (total of $L + M + 1$ variables). Considering also the Taylor series expansion of $H(s)$ at $s = 0$ we have,

$$\frac{a_0 + a_1s + \dots + a_Ls^L}{1 + b_1s + \dots + b_Ms^M} = m_0 + m_1s + m_2s^2 + \dots + m_{L+M}s^{L+M}. \quad (2.28)$$

Cross multiplying and equating the coefficients of similar powers of s starting from s^{L+1} to s^{L+M} on both sides of (2.28), the denominator polynomial coefficients can be evaluated as

$$\begin{pmatrix} m_{L-M+1} & m_{L-M+2} & \dots & m_L \\ m_{L-M+2} & \ddots & \vdots & m_{L+1} \\ \vdots & \dots & \ddots & \vdots \\ m_L & m_{L+1} & \dots & m_{L+M+1} \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{L+M} \end{pmatrix} \quad (2.29)$$

The numerator coefficients can be found by equating the remaining powers of s

(from s^0 to s^L) as

$$\begin{aligned}
a_0 &= m_0 \\
a_1 &= m_1 + b_1 m_0 \\
&\vdots \\
a_L &= m_L + \sum_{i=1}^{\min(L,M)} b_i m_{L-i}
\end{aligned} \tag{2.30}$$

Equations (2.29) and (2.30) yield an approximate transfer function in terms of rational polynomials. The computation of the moments $m_0, m_1 s, m_2 s^2, \dots, m_{L+M}$ is discussed in the following section.

2.7.2 Computation of Moments

MMTs such as AWE approximate the transfer function of a system using Padé approximation. However, in order to perform Padé approximation, we need to evaluate the moments of the system, which are required by (2.29) and (2.30). A recursive relation for the evaluation of moments of $\mathbf{X}(s)$ in (2.13) can be obtained in the form [18]

$$[\mathbf{H}(0)]\mathbf{M}_n = - \sum_{r=1}^n \frac{\frac{\partial^r}{\partial s^r} \mathbf{H}(s) \big|_{s=0}}{r!} \mathbf{M}_{n-r} \tag{2.31a}$$

with

$$[\mathbf{H}(0)]\mathbf{M}_0 = \mathbf{b} \tag{2.31b}$$

where $\mathbf{M}_i$ is the i^{th} moment of $\mathbf{X}(s)$ in (2.13) and $\mathbf{b} \in \mathcal{C}^N$ is a vector containing the contribution of the Laplace transforms of the current and voltage sources which were used as terminations at the ports. The moments of a particular node voltage (say the

j^{th} element in $\mathbf{X}(s)$) $m_0, m_1, \dots, m_{L+M}$ can be extracted as the j^{th} elements of the corresponding moment vector $\mathbf{M}_i$.

To evaluate the moments using (2.31), the derivatives of $\mathbf{H}(s)$ (denoted by $\mathbf{H}^{(r)}$) are needed, the computation of which further requires the derivatives of the Y-parameters of the transmission line subnetworks $\mathbf{A}_k^{(r)}$. $\mathbf{A}_k^{(r)}$ can be obtained using any of the previously published methods [18]–[20]. In this thesis, the matrix exponential method was used as a method of choice to find the derivatives of the admittance parameters of each interconnect subnetwork. These derivatives are obtained as follows.

The hybrid parameter matrix of the transmission line subnetwork k computed in (2.10) is split as

$$\begin{pmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{pmatrix} = e^{\mathbf{E}} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{pmatrix} \quad (2.32)$$

Matrix exponential method is used on (2.32) to find the derivatives of the hybrid parameters as

$$\begin{pmatrix} \mathbf{V}_2^{(r)} \\ \mathbf{I}_2^{(r)} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{11}^{(r)} & \mathbf{T}_{12}^{(r)} \\ \mathbf{T}_{21}^{(r)} & \mathbf{T}_{22}^{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^{(r)} \\ \mathbf{I}_1^{(r)} \end{pmatrix} \quad (2.33)$$

Rearranging $\mathbf{V}_1^{(r)}, \mathbf{I}_1^{(r)}, \mathbf{V}_2^{(r)}$ and $\mathbf{I}_2^{(r)}$ in (2.33), the derivatives of the admittance parameters of the subnetwork k can be computed to be

$$\mathbf{Y}_k^{(r)} = \begin{pmatrix} \mathbf{T}_{22}^{(r)}(\mathbf{T}_{12}^{(r)})^{-1} & \mathbf{T}_{21}^{(r)} - \mathbf{T}_{22}^{(r)}(\mathbf{T}_{12}^{(r)})^{-1}\mathbf{T}_{11}^{(r)} \\ -(\mathbf{T}_{12}^{(r)})^{-1} & (\mathbf{T}_{12}^{(r)})^{-1}\mathbf{T}_{11}^{(r)} \end{pmatrix} \quad (2.34)$$

In the case of subnetworks containing frequency-dependent parameters, full-wave

elements or measured networks, accurate moments can be obtained using the techniques described in [31], [48], [49].

The moments obtained using the recursive relationship described in (2.31) are then used in equations (2.29) and (2.30) to form a Padé approximant which approximates the transfer function of the system with respect to frequency.

2.7.3 Limitations of Single Expansion MMT Algorithms

Due to the inherent limitations of Padé approximants, MMTs based on single expansion often give inaccurate results. The following is a list of those properties that have the most impact on MMTs.

- The matrix in (2.29) (known as Toeplitz matrix) becomes increasingly ill-conditioned as its size increases. This implies that only six to eight accurate poles can be detected from a single expansion.
- Padé often produces unstable poles on the right-hand side of the complex plane.
- Padé accuracy deteriorates as we move away from the expansion point.
- Padé provides no estimates for error bounds.

In addition, there is no guarantee that the reduced-model obtained as above is passive. Passivity implies that a network cannot generate more energy than it absorbs, and no passive termination of the network will cause the system to go unstable [22]–[24], [28]. The loss of passivity can be a serious problem because transient simulations of reduced networks may encounter artificial oscillations.

In systems containing distributed elements the number of dominant poles will be significantly higher, and it is very difficult to capture all of them with a single Padé expansion. This lead to the development of multipoint expansion techniques such as complex frequency hopping (CFH), which is summarized in the next section.

2.8 Complex Frequency Hopping

CFH extends the process of moment matching to multiple expansion points (hops) in the complex plane near or on the imaginary axis using a binary search algorithm [18]. With a minimized number of frequency point expansions, enough information is obtained to enable the generation of an approximate transfer function that matches the original function up to a predefined highest frequency of interest. Using the information from all the expansion points, CFH extracts a dominant pole-set. In addition, CFH provides an error criterion for the selection of accurate poles and transfer functions.

Selection and Minimization of Hops in CFH: A Padé approximation is accurate only near the point of expansion and its accuracy decreases as we move away from the point of expansion (hop). In order to validate the accuracy of such an approximation, at least two expansion points are necessary. Accuracies of these two expansions can be verified by matching the poles generated at these two hops (referred as pole-matching-based approach) [18]. Alternatively, the two hops can be verified for their accuracy by comparing the value of the transfer functions due to both these hops at a point intermediate to them (referred as transfer-function-based approach) [50]. CFH relies on a binary search algorithm to determine the expansion points and to minimize the

number of expansions.

In the next section, we shall look at *indirect* moment matching techniques as opposed to AWE and CFH, which are direct moment matching techniques.

2.9 Preliminaries of Indirect MMTs

Direct MMTs such as AWE, discussed in Section 2.7, have some disadvantages associated with them. First amongst them is the ill-conditioning associated with the moment-matrix. Due to the ill-conditioning problem, the number of dominant poles that could be extracted from any expansion point is generally fewer than ten poles. The second major difficulty is that they do not guarantee the passivity of reduced models. In order to address these difficulties, a parallel class of algorithms, which can be classified as *indirect moment-matching techniques* were developed [21]–[28].

These algorithms are based on what is known as Krylov-subspace techniques and congruent transformation. One of the main features of these algorithms is that they construct the reduced-model based on the extraction of leading eigenvalues (those with the largest magnitude) of a given system (on the contrary, the reduced models from the CFH technique is based on extracting the dominant poles of a given system). In the rest of this section, we will describe preliminaries required to understand indirect MMTs.

Recall from Section 2.4, the time domain MNA equations for a lumped network

can be represented as

$$\mathbf{C} \frac{d}{dt} \mathbf{x}(t) + \mathbf{G} \mathbf{x}(t) = \mathbf{B} \mathbf{u}(t) \quad (2.35a)$$

$$\mathbf{w}(t) = \mathbf{B}^T \mathbf{x}(t) \quad (2.35b)$$

where $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are the input and output vectors respectively.

Premultiplying (2.35) by $\mathbf{G}^{-1}$ we have

$$\mathbf{A} \frac{d}{dt} \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{R} \mathbf{u}(t) \quad (2.36a)$$

$$\mathbf{w}(t) = \mathbf{B}^T \mathbf{x}(t) \quad (2.36b)$$

where,

$$\mathbf{A} = -\mathbf{G}^{-1} \mathbf{C}, \quad \mathbf{R} = \mathbf{G}^{-1} \mathbf{B} \quad (2.36c)$$

Taking the Laplace transform of (2.36), we can write

$$s \mathbf{A} \mathbf{X}(s) = \mathbf{X}(s) - \mathbf{R} \mathbf{U}(s) \quad (2.37a)$$

$$\mathbf{W}(s) = \mathbf{B}^T \mathbf{X}(s) \quad (2.37b)$$

Rearranging (2.37) the transfer function $\mathbf{Y}(s)$ of the system can be written as

$$\mathbf{Y}(s) = \mathbf{B}^T (\mathbf{I} - s \mathbf{A})^{-1} \mathbf{R} \quad (2.38)$$

where $\mathbf{I}$ is the identity matrix.

2.9.1 Ill-conditioning of Padé

Expanding (2.38) in terms of Taylor series,

$$\begin{aligned}
 Y(s) &= \mathbf{B}^T (\mathbf{I} + s\mathbf{A} + \dots + s^q \mathbf{A}^q + \dots) \mathbf{R} \\
 &= \sum_{k=0}^q s^k (\mathbf{B}^T \mathbf{A}^k \mathbf{R}) \\
 &= \sum_{k=0}^q s^k m_k \quad \text{where } m_k = \mathbf{B}^T \mathbf{A}^k \mathbf{R}.
 \end{aligned} \tag{2.39}$$

As can be seen from (2.39), when successive moments are explicitly calculated, they are obtained as powers of $\mathbf{A}$. With the increasing values of k , this process quickly converges to an eigenvector corresponding to an eigenvalue of the largest magnitude [51]. As a result, for relatively large values of k , the explicitly calculated moments $m_k, m_{k+1}, \dots$ will not add any extra information to the moments, as all of them contain information only about the largest eigenvalue. In other words, the “moments beyond k ” are almost identical (or parallel to each other), giving rise to the ill-conditioning problem associated with Padé approximation. We shall see later in this chapter, how the indirect MMTs overcome this problem.

2.9.2 Eigenvalues and Poles

Another thing to note is that the inverses of the eigenvalues of $\mathbf{A}$ are the poles of the system (2.35). This can be easily seen by the following procedure.

Performing an eigenvalue decomposition on $\mathbf{A}$ we have, $\mathbf{A} = \mathbf{F} \lambda \mathbf{F}^{-1}$, (where $\lambda = \text{diag}[\lambda_1, \dots, \lambda_n]$ and $\mathbf{F}$ contains the eigenvectors of $\mathbf{A}$), and substituting in (2.38) we

have,

$$\begin{aligned}
\mathbf{Y}(s) &= \mathbf{B}^T (\mathbf{I} - s\mathbf{F}\lambda\mathbf{F}^{-1})^{-1} \mathbf{R} \\
&= \mathbf{B}^T \mathbf{F} (\mathbf{I} - s\lambda) \mathbf{F}^{-1} \mathbf{R} \\
&= \mathbf{B}^T \mathbf{F} \begin{pmatrix} \frac{1}{1-s\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-s\lambda_n} \end{pmatrix} \mathbf{F}^{-1} \mathbf{R}
\end{aligned} \tag{2.40}$$

which can be simplified (say when $\mathbf{Y}(s)$ is a scalar, for the sake of simplicity) as

$$\mathbf{Y}(s) = \sum_i \frac{\eta_i}{1 - s\lambda_i} = \sum_i \frac{-(\eta_i/\lambda_i)}{s - (1/\lambda_i)} = \sum_i \frac{k_i}{s - p_i} \tag{2.41}$$

where η_i is a function of the eigenvectors of the matrix $\mathbf{A}$ and k_i and p_i represent the i^{th} pole-residue pair. From (2.41) it can be clearly seen that the inverses of the eigenvalues of $\mathbf{A}$ are the poles of the system (2.35). Another thing to note is that the transfer function $\mathbf{Y}(s)$ can be easily obtained in terms of poles and residues once the eigenvectors and eigenvalues of $\mathbf{A}$ are known. However, for large interconnect circuits, it would be impractical to compute all the eigenvalues and eigenvectors of $\mathbf{A}$. Hence, in the following sections, we will review some of the efficient techniques to extract the leading eigenvalues, i.e. the dominant poles of the system by virtue of being the inverses of the poles of the system. Once the leading eigenvalues (i.e. dominant poles) are captured, a reduced-order model can be formed. This is explained in the next section which describes the Arnoldi algorithm.

2.10 The Arnoldi Algorithm

The Arnoldi algorithm was suggested in the literature to overcome the ill-conditioning problem associated with the Padé approximation. This algorithm [51] first starts by computing a subspace which spans the same space as the moments of the system of equations (2.37). This subspace is called the Krylov-subspace and is denoted by $\kappa(A, R, n)$. In other words, any vector that is a linear combination of the moments of the system, can be expressed as a linear combination of the columns of $\kappa(A, R, n)$. This can be mathematically expressed as

$$\kappa(A, R, n) = \text{colsp}(\mathbf{K}) \quad (2.42)$$

where,

$$\mathbf{K} = \begin{pmatrix} \mathbf{R} & \mathbf{A}\mathbf{R} & \dots & \mathbf{A}^{n-1}\mathbf{R} \end{pmatrix}. \quad (2.43)$$

The matrix $\mathbf{K}$ by itself is likely to be ill-conditioned since the columns of $\mathbf{K}$ are formed based on the sequence $\mathbf{A}^i\mathbf{R}$, which quickly converges to the eigenvector of the largest eigenvalue [51]. To overcome this problem (which is the ill-conditioning problem), the Arnoldi algorithm computes another orthogonal matrix $\mathbf{Q}$ such that

$$\text{colsp}(\mathbf{Q}) = \text{colsp}(\mathbf{K}). \quad (2.44)$$

In contrast to the matrix $\mathbf{K}$, the matrix $\mathbf{Q}$ is well conditioned and as many leading columns of $\mathbf{Q}$ can be constructed to get an accurate solution. The matrix $\mathbf{Q}$ can

be computed using techniques such as the *modified Gram-Schmidt algorithm* [51]. A simple implementation of the Arnoldi algorithm which uses modified Gram-Schmidt is given below.

```

function Arnoldi
     $q_1 = \mathbf{R} / \|\mathbf{R}\|_2$  // Get the direction of  $q_1$ 
    for  $j = 1$  to  $k$  //  $k$  is the desired number of columns of the  $\mathbf{Q}$  matrix
         $z = \mathbf{A}q_j$ ;
        for  $i = 1$  to  $j$ 
             $h_{ij} = q_i^T z$ 
             $z = z - h_{ij}q_i$ ;
        end for
         $h_{j+1,j} = \|z\|$ ; if  $h_{j+1,j} = 0$ , quit
         $q_j = z / h_{j+1,j}$ ;
    end for
end function

```

The columns q_j computed by the Arnoldi algorithm are called Arnoldi vectors. The loop over i updating z in the pseudo-code, corresponds to the modified Gram Schmidt algorithm, which subtracts the components of $\mathbf{A}q_j$ in the directions q_1 to q_j away from z , leaving them orthogonal to z . This is demonstrated in Fig. 2.4. Computing a total of “ k ” Arnoldi vectors costs k matrix vector multiplications involving $\mathbf{A}$, plus $\mathcal{O}(k^2n)$ related cost.

There are several alternative methods available in the literature for finding the basis for Krylov-subspace. For example, one can use multiple passes of orthogonalization to increase the robustness of the modified Gram Schmidt orthogonalization process. It can be shown that the eigenvalues of $\mathbf{H} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}$ are a good approximation of the leading eigenvalues of $\mathbf{A}$ [51].

To recap, we started with the circuit equations $\mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{G}\mathbf{x}(t) = \mathbf{B}u(t)$ and $w(t) =$

Figure 2.4: Computation of column q_2 through Arnoldi

$\mathbf{B}^T \mathbf{x}(t)$. We formed the product $\mathbf{A} = -\mathbf{G}^{-1}\mathbf{C}$. Using orthogonal transformation, we were able to determine the leading eigenvalues of $\mathbf{A}$ that corresponds to the dominant poles of the transfer-function. In the following section, we will show how to use this information to perform model-reduction on the original circuit equations.

2.10.1 Model-Reduction through Arnoldi

Finding the reduced-order circuit equations can be explained by a change of variables in (2.35) by mapping the vector $\mathbf{x}$ of dimension n into a smaller vector $\hat{\mathbf{x}}$ of dimension q ($q \ll n$) using an orthogonal matrix $\mathbf{Q}$. This mapping is performed as

$$\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}} \quad (2.45)$$

Using (2.45), we can rewrite Laplace-domain circuit equations in (2.37) as

$$s\mathbf{A}\mathbf{Q}\hat{\mathbf{X}}(s) = \mathbf{Q}\hat{\mathbf{X}}(s) - \mathbf{R}\mathbf{U}(s) \quad (2.46a)$$

$$\mathbf{W}(s) = \mathbf{B}^T \mathbf{Q}\hat{\mathbf{X}}(s) \quad (2.46b)$$

Premultiplying both sides of (2.46a) by $\mathbf{Q}^T$ and using the relation $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, we have

$$\begin{aligned} s\mathbf{Q}^T\mathbf{A}\mathbf{Q}\hat{\mathbf{X}}(s) &= \hat{\mathbf{X}}(s) - \mathbf{Q}^T\mathbf{R}\mathbf{U}(s) \\ \Rightarrow \hat{\mathbf{X}}(s) &= (\mathbf{I} - s\mathbf{Q}^T\mathbf{A}\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{R}\mathbf{U}(s) \end{aligned} \quad (2.47)$$

$$\begin{aligned} \mathbf{W}(s) &= \mathbf{B}^T\mathbf{Q}\hat{\mathbf{X}}(s) \\ \Rightarrow \mathbf{W}(s) &= \mathbf{B}^T\mathbf{Q}(\mathbf{I} - s\mathbf{Q}^T\mathbf{A}\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{R}\mathbf{U}(s) \end{aligned} \quad (2.48)$$

Hence, the transfer-function of the reduced system can be written as

$$\hat{\mathbf{Y}}(s) = \mathbf{B}^T\mathbf{Q}(\mathbf{I} - s\mathbf{Q}^T\mathbf{A}\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{R} \quad (2.49)$$

Comparing the original transfer-function $\mathbf{Y}(s)$ represented by (2.38) with the transfer-function of the reduced system represented by (2.49), we can draw the following conclusions. The eigenvalues of $\hat{\mathbf{Y}}(s)$ are given by the eigenvalues of $\mathbf{H}_q = \mathbf{Q}^T\mathbf{A}\mathbf{Q}$. However, since the eigenvalues of $\mathbf{H}_q$ are a good approximation of the leading eigenvalues of $\mathbf{A}$, we can conclude that the eigenvalues of the transfer function of the reduced system are a good approximation to the eigenvalues of the original transfer function.

An important criterion during the above reduction is the accuracy of the response of the reduced system given by (2.46). The frequency response of the reduced system (2.49) is also a good approximation of the frequency-response of the original transfer function (2.38). An indicator for the accuracy of the response of the reduced system is the total number of moments it can preserve (match), for a given order of reduction (q). It can be proved [24] that the reduced system (2.46) of order q preserves the first

q moments of the original network. In essence, we are able to implicitly match the moments and obtain a reduced-model without the need to directly use the moments as in the AWE algorithm. Hence, Arnoldi does not suffer from the same numerical ill-conditioning that is associated with direct moment-matching algorithms. The accuracy of the Arnoldi approximation gradually increases as the order is increased since more moments of the original transfer function will be matched [24]. A comparison of the poles captured from AWE, CFH and Arnoldi is illustrated in Fig. 2.5.

Figure 2.5: Illustration of poles captured from AWE, CFH and Arnoldi

In an alternative approach, the reduction based on Lanczos algorithm can preserve double the moments when compared to Arnoldi for the same order q . The difference between Arnoldi and Lanczos algorithms is that in Lanczos algorithm the matrix $\mathbf{A}$ is transformed to a tridiagonal matrix $\mathbf{T}_q$. In addition, Lanczos algorithm uses two bi-orthogonal Krylov-subspaces, to recursively compute $\mathbf{T}_q$ [21]. However, macromodels formed using the above Lanczos-based reduction scheme are not guaranteed to be passive. Passivity is a very important property of a system, which shall be discussed

in the next section.

2.11 Passivity Preservation

Passivity implies that a network cannot generate more energy than it absorbs, and no passive termination of the network will make the system unstable. Passivity is an important property, because stable but not passive macromodels can lead to unstable systems when connected to other passive systems. The loss of passivity can be a serious problem because transient simulations of non-passive networks may encounter artificial oscillations. This is illustrated in Fig. 2.6, which represents the transient response of a reduced-order macromodel of a large linear circuit, when connected to an external load of 50 k. In this section, important algorithms that are available in the literature, for preservation of passivity during the reduction of linear networks will be reviewed [22]–[28], [52].

2.11.1 Review of Passivity Properties

A network is passive iff its admittance (or impedance) parameter matrix is positive real. The necessary and sufficient conditions for the network admittance matrix $\mathbf{Y}(s)$ to be passive are the following.

1. $\mathbf{Y}(s^*) = \mathbf{Y}^*(s)$ for all complex s , where $*$ is the complex conjugate operator,
2. $\mathbf{Y}(s)$ is positive real, that is, $z^{*T}(\mathbf{Y}(s) + \mathbf{Y}^T(s^*))z \geq 0$ for all complex values of s satisfying $\text{Re}(s) \geq 0$ and for any complex vector z .

Figure 2.6: Transient response of a nonpassive macromodel with passive terminations

2.11.2 Passive Reduction

Recall from Section 2.10.1, where model-reduction was achieved by applying Arnoldi on the set of equations represented by (2.37) (we refer this approach as classical Arnoldi). However, this approach does not preserve the passivity of the reduced system. Instead, if the Arnoldi is applied on the original set of circuit equations represented by (2.35), reducing the $\mathbf{C}$ and $\mathbf{G}$ matrices directly, passivity can be preserved (with certain conditions, which will be discussed later in this section). A brief description of this process is given below. Using (2.45), (2.35) can be rewritten in the Laplace-domain as

$$s\mathbf{C}\mathbf{Q}\hat{\mathbf{X}}(s) + \mathbf{G}\mathbf{Q}\hat{\mathbf{X}}(s) = \mathbf{B}\mathbf{U}(s) \quad (2.50a)$$

$$\mathbf{W}(s) = \mathbf{B}^T\mathbf{Q}\hat{\mathbf{X}}(s) \quad (2.50b)$$

Premultiplying both sides of (2.50a) by $\mathbf{Q}^T$, we have

$$s\mathbf{Q}^T\mathbf{C}\mathbf{Q}\hat{\mathbf{X}}(s) + \mathbf{Q}^T\mathbf{G}\mathbf{Q}\hat{\mathbf{X}}(s) = \mathbf{Q}^T\mathbf{B}\mathbf{U}(s) \quad (2.51)$$

$$\Rightarrow \hat{\mathbf{X}}(s) = (\hat{\mathbf{G}} + s\hat{\mathbf{C}})^{-1}\hat{\mathbf{B}}\mathbf{U}(s)$$

$$\mathbf{W}(s) = \hat{\mathbf{B}}^T \hat{\mathbf{X}}(s) \quad (2.52)$$

$$\Rightarrow \mathbf{W}(s) = \hat{\mathbf{B}}^T (\hat{\mathbf{G}} + s\hat{\mathbf{C}})^{-1} \hat{\mathbf{B}} \mathbf{U}(s)$$

where,

$$\hat{\mathbf{G}} = \mathbf{Q}^T\mathbf{G}\mathbf{Q}; \quad \hat{\mathbf{C}} = \mathbf{Q}^T\mathbf{C}\mathbf{Q}; \quad \hat{\mathbf{B}} = \mathbf{Q}^T\mathbf{B} \quad (2.53)$$

Hence, the transfer-function of the reduced system can be written as

$$\hat{\mathbf{Y}}(s) = \hat{\mathbf{B}}^T (\hat{\mathbf{G}} + s\hat{\mathbf{C}})^{-1} \hat{\mathbf{B}} \quad (2.54)$$

It can be proved that the reduced model, given by (2.54) is passive [24]. One of the conditions required to prove the passivity of the reduced model is that the original $\mathbf{C}$ matrix must be a symmetric and nonnegative definite matrix. This condition can be satisfied with a slight change in the MNA formulation. The $\mathbf{G}$ and $\mathbf{C}$ matrices are formulated using the modified MNA formulation suggested in [24]. This formulation ensures that $\mathbf{C}$ is nonnegative definite and $\mathbf{G}$ is of the form $\begin{pmatrix} \mathbf{N} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{pmatrix}$ where $\mathbf{N}$ is nonnegative definite. Following the suggested modification, the $\mathbf{C}$ and $\mathbf{G}$ matrices for the example presented in section 2.4 should be modified to be

$$\mathbf{G} = \begin{pmatrix} G_1 & -G_1 & 0 & 0 & 0 & -1 \\ -G_1 & G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & G_2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad (2.55a)$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.55b)$$

This guarantees that the reduced model obtained in (2.51) is provably passive. However, the reduced models formed by PVL are not passive due to the fact that the Lanczos method uses oblique projections as opposed to the orthogonal projections used by Arnoldi.

An important point to note is that, with respect to transmission lines, Arnoldi based methods present in the literature, base their treatment on the discretization of Telegrapher's equations. As we have seen in the previous chapter, discretization has many disadvantages associated with it. In this thesis, efficient techniques are presented that overcome the above problem and further improve interconnect simulation techniques.

Chapter 3

Model-Reduction of Distributed Interconnects Using Congruent Transformation Techniques

3.1 Introduction

As we have seen from the previous chapter, recently, there have been several attempts to address the interconnect problem using moment-matching techniques (MMT) and Padé approximation by reducing the order of the linear system [17]–[20]. In order to overcome the ill-conditioning associated with the direct Padé approximation, several techniques based on Krylov-subspace methods were developed [21]–[28]. The Padé via Lanczos (PVL) was first suggested to overcome the difficulties due to ill-conditioning [21]. However, PVL does not ensure the passivity of the reduced-model. The passivity of the reduced-order model generated by PVL is ensured for only two-element kind

networks but not for general RLC networks. Next, a co-ordinate transformed Arnoldi algorithm was suggested to generate stable but not passive reduced-order models for RLC circuits [22]. [24] extends the block Arnoldi technique to generate passive reduced-order models for RLC networks. The Krylov-space techniques were further advanced in [25], [26] where passivity is ensured for multipoint multiport network reductions.

However, with respect to distributed transmission line networks, the above techniques base their treatment on discretization of Telegrapher's equations. The discretization can be either straight-forward lumped RLC component representation or based on the numerical preprocessing to approximate the frequency-characteristics of Telegrapher's equations in terms of a set of ordinary differential equations [27], [28]. However, there are several disadvantages with the discretization based representation of distributed networks. (a) Large number of sections are required in order to ensure reasonable accuracy, which leads to large circuit matrices. This problem deteriorates with higher frequencies or sharper rise/fall times. (b) The method cannot directly handle measured subnetworks or transmission lines with frequency-dependent parameters. (c) Discretization of transmission lines leads to Gibbs' phenomenon, which becomes more pronounced at higher frequencies [10].

In this chapter a new technique [30], [52]–[58] is presented to overcome the above difficulties. This algorithm enables the inclusion of distributed transmission line networks described by Telegrapher's equations directly using a congruent transformation based analysis. Also presented is the proof of preservation of moments and passivity of the reduced-order system for networks containing distributed elements for both the multipoint and multiport cases. In addition, the proposed method can efficiently

handle distributed networks with frequency-dependent parameters [31].

This chapter is organized as follows. Section 3.2 formulates the network equations and section 3.3 presents the new algorithm, which enables the inclusion of distributed transmission line networks described by Telegrapher's equations directly using a congruent transformation based analysis. Section 3.4 presents the proof of preservation of moments, followed by section 3.5, which presents the proof that the reduced-models generated from this algorithm are passive. Section 3.6 extends this technique, for multiple point expansions and section 3.7 presents the numerical examples.

3.2 Formulation of Network Equations

The MNA formulation presented in this section is very similar to that presented in section 2.4, however it is repeated again for the sake of clarity and completeness. A multiport linear subnetwork consisting of lumped RLC and distributed components can be expressed as

$$\mathbf{C}\dot{\mathbf{x}} + \mathbf{G}\mathbf{x} + \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{i}_k(t) = \mathbf{B}\mathbf{v}_p \quad (3.1a)$$

$$\mathbf{i}_p = \mathbf{B}^T \mathbf{x} \quad (3.1b)$$

where

$\mathbf{x} \in \mathbb{R}^n$ is the vector of node voltage waveforms appended by independent voltage source currents, linear inductor currents and port currents,

$\mathbf{C} \in \mathbb{R}^{n \times n}$, $\mathbf{G} \in \mathbb{R}^{n \times n}$ are constant matrices describing the lumped memory and memoryless elements of the network, respectively. $\mathbf{G}$ and $\mathbf{C}$ are formulated using the modified MNA formulation suggested in [24]. This formulation ensures that $\mathbf{C}$

is nonnegative definite and $\mathbf{G}$ is of the form $\begin{pmatrix} \mathbf{N} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{pmatrix}$ where $\mathbf{N}$ is nonnegative definite,

$\mathbf{i}_p$, $\mathbf{v}_p$ and denote the port currents and port voltages respectively, p being the number of ports,

$\mathbf{B} = (b_{i,j})$ with elements $b_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, p\}$, is a selector matrix that maps the port voltages into the node space $\mathbb{R}^n$ of the network and n is the total number of variables in the MNA formulation,

$\mathbf{D}_k = (d_{i,j})$ with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, 2n_k\}$ with a maximum of one non-zero in each row or column, is a selector matrix that maps $\mathbf{i}_k(t) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^n$ of the network,

N_t is the total number of distributed coupled transmission line sets and n_k is the number of coupled conductors in the linear subnetwork k .

The first and second terms in (3.1a) cover network's lumped components and the third term describes currents at interconnect subnetwork terminals and then maps them into rest of the network through the matrix $\mathbf{D}_k$.

The frequency-domain equations of interconnect subnetwork 'k' containing distributed transmission line models can be written in the form

$$\mathbf{I}_k(s) = \mathbf{A}_k(s)\mathbf{V}_k(s) = \mathbf{A}_k(s)\mathbf{D}_k^T\mathbf{X}(s) \quad (3.2)$$

where $\mathbf{V}_k(s)$, $\mathbf{I}_k(s)$ and $\mathbf{A}_k(s)$ represent the Laplace-domain terminal voltages and currents and the admittance parameters for the subnetwork k respectively. $\mathbf{X}(s)$ is

the Laplace transform of $x(t)$.

The admittance parameters for the subnetwork k can be computed as follows. Consider the multi-conductor transmission line (MTL) system with N conductors. The system can be represented in the frequency-domain using Telegrapher's equations as

$$\frac{\partial}{\partial z} \mathbf{V}(z, s) = -\mathbf{Z}_L \mathbf{I}(z, s) \quad (3.3a)$$

$$\frac{\partial}{\partial z} \mathbf{I}(z, s) = -\mathbf{Y}_L \mathbf{V}(z, s) \quad (3.3b)$$

where $\mathbf{Z}_L$ and $\mathbf{Y}_L$ represent the per-unit-length impedance and admittance matrices, given by

$$\mathbf{Z}_L = \tilde{\mathbf{R}} + s\tilde{\mathbf{L}} \quad (3.4a)$$

$$\mathbf{Y}_L = \tilde{\mathbf{G}} + s\tilde{\mathbf{C}} \quad (3.4b)$$

where $\tilde{\mathbf{R}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{C}}$ are per-unit-length resistance, inductance, conductance and capacitance matrices and $\mathbf{V}$, $\mathbf{I}$ are the terminal voltage/current vectors of the transmission line network. It is to be noted that $\tilde{\mathbf{R}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{C}}$ are nonnegative definite matrices [10].

Solution of (3.3) can be expressed as

$$\begin{pmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{pmatrix} = e^{\mathbf{E}} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{pmatrix} \quad (3.5)$$

where

$$\mathbf{E} = d \begin{pmatrix} \mathbf{0} & -\mathbf{Z}_L \\ -\mathbf{Y}_L & \mathbf{0} \end{pmatrix} \quad (3.6)$$

and V_1 , V_2 , I_1 and I_2 are the terminal voltage/current vectors of the transmission line network (see Fig. 3.1) and d is the length of the transmission line.

Figure 3.1: A multi-conductor transmission line subnetwork

Equation (3.5) can be rearranged to compute the admittance parameters of the transmission line subnetwork as

$$\begin{pmatrix} I_1 \\ -I_2 \end{pmatrix} = \mathbf{A}_k(s) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (3.7)$$

Using (3.2) and the Laplace transform of (3.1), assuming zero initial conditions, we have

$$\mathbf{H}\mathbf{X}(s) = \mathbf{B}v_p \quad (3.8a)$$

$$i_p = \mathbf{B}^T \mathbf{X}(s) \quad (3.8b)$$

where

$$\mathbf{H}(s) = \mathbf{G} + s\mathbf{C} + \mathbf{F}(s) \quad \text{and} \quad (3.9a)$$

$$\mathbf{F}(s) = \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{A}_k(s) \mathbf{D}_k^T. \quad (3.9b)$$

This general formulation of an interconnect network will be used in the next section to develop an algorithm that performs model-reduction using congruent transformation techniques by including transmission lines *directly* during its analysis.

3.3 Reduction Algorithm for Networks with Distributed Components

In this section, an algorithm based on congruent transformation to reduce networks containing distributed elements is presented. The proposed method can handle distributed models directly, without the need for discretization into lumped components.

Let us consider the system represented by (3.1) when a unit excitation is applied at port i

$$\mathbf{H}(s)\mathbf{X}(s) = \mathbf{b}_i \quad (3.10)$$

where $\mathbf{b}_i$ is a binary vector with non-zero entry at location i . Let $\mathbf{M}_j^i$ be the corresponding j^{th} moment vector of $\mathbf{X}(s)$. The new modified subspace (denoted by $\mathbf{K}$) for a single expansion point at $s = 0$ is formed to be

$$\mathbf{K} = \begin{pmatrix} \mathbf{M}_0^1 & \dots & \mathbf{M}_{n_1}^1 \end{pmatrix} \quad (3.11)$$

where $(n_1 + 1)$ represents the number of moments corresponding to the excitation at port i . Next, the moments at a frequency point $s = \alpha$ are calculated by expanding $\mathbf{X}(s)$ in (3.10) using Taylor series as

$$\mathbf{X}(s) = \sum_{n=0}^{\infty} \mathbf{M}_n^i (s - \alpha)^n \quad (3.12)$$

3.3.1 Evaluation of Moments

The moments M_j^i in (3.12) can be evaluated by using a recursive relation in the form [18]

$$[H(\alpha)]M_n^i = - \sum_{r=1}^n \frac{\frac{\partial^r}{\partial s^r} H(s)|_{s=\alpha} M_{n-r}^i}{r!} \quad (3.13a)$$

with

$$[H(\alpha)]M_0^i = b_i \quad (3.13b)$$

To evaluate the moments using (3.13), the derivatives of $H(s)$ (denoted by $H^{(r)}$) are needed, the computation of which further requires the derivatives of $F(s)$ (denoted by $F^{(r)}$). They can be obtained using (3.9) as

$$F^{(r)} = \sum_{k=1}^{N_t} D_k A_k^{(r)} D_k^T. \quad (3.14)$$

where $A_k^{(r)}$ can be obtained using any of the previously published methods [18]–[20]. In this chapter, the matrix exponential method was used to find the derivatives of the admittance parameters of each interconnect subnetwork k (see section 2.4).

3.3.2 Model-Reduction through Congruent Transformation

Since the matrix $\mathbf{K}$ obtained from (3.11) is generally ill-conditioned, it is transformed into a new matrix $\mathbf{Q}$ through orthonormalization [22]–[24], [51]. The $\mathbf{Q}$ thus obtained is used in the congruent transformation

$$\mathbf{X}(s) = \mathbf{Q} \hat{\mathbf{X}}(s) \quad (3.15)$$

of the linear distributed system described by (3.8). This reduces the system to

$$(\hat{\mathbf{G}} + s\hat{\mathbf{C}} + \hat{\mathbf{F}}(s))\hat{\mathbf{X}}(s) = \hat{\mathbf{b}}_i \quad (3.16)$$

where

$$\hat{\mathbf{G}} = \mathbf{Q}^T \mathbf{G} \mathbf{Q}; \quad \hat{\mathbf{C}} = \mathbf{Q}^T \mathbf{C} \mathbf{Q}; \quad (3.17a)$$

$$\hat{\mathbf{F}}(s) = \mathbf{Q}^T \mathbf{F}(s) \mathbf{Q}; \quad \hat{\mathbf{b}}_i = \mathbf{Q}^T \mathbf{b}_i \quad (3.17b)$$

It is to be noted that the size of the reduced system (3.16) $q \ll N$. Hence the reduced system can be simulated efficiently. The proof that this transformation preserves the first $n_i + 1$ moments of $\mathbf{X}(s)$ at port i is given in the next section.

3.4 Proof of Preservation of Moments

Consider the matrix $\mathbf{K}$ represented by (3.11). Performing an orthonormal decomposition on $\mathbf{K}$ we have

$$\mathbf{K} = \mathbf{Q} \mathbf{R} \quad (3.18)$$

where $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}_q$. Next, using the congruence transformation in (3.15) we have

$$\mathbf{Q}^T \mathbf{H}(s) \mathbf{Q} \hat{\mathbf{X}}(s) = \mathbf{Q}^T \mathbf{b}_i \quad (3.19)$$

Since $\mathbf{H}(s)$ is positive definite, $\mathbf{Q}^T \mathbf{H}(s) \mathbf{Q}$ is also positive definite and thus nonsin-

gular. Substituting (3.18) in (3.19) we get

$$\mathbf{R}^{-T} \mathbf{K}^T \mathbf{H}(s) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{X}}(s) = \mathbf{R}^{-T} \mathbf{K}^T \mathbf{b}_i \quad (3.20)$$

or

$$\mathbf{K}^T \mathbf{H}(s) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{X}}(s) = \mathbf{K}^T \mathbf{b}_i \quad (3.21)$$

Proof that the reduced system (3.19) conserves the first $(n_i + 1)$ moments of the original system (3.10) corresponding to a unit excitation at port i is based on mathematical induction. First we will prove that the 0^{th} moment obtained from the reduced order model is equivalent to that obtained from the original system represented by (3.10). Next, we will show that the first m moments are preserved if the previous $(m - 1)$ moments are preserved.

Consider the system at $s = \alpha_0 = 0$. Using (3.13) and (3.19) we have

$$\mathbf{Q}^T \mathbf{H}(\alpha_0) \mathbf{Q} \hat{\mathbf{M}}_0^i = \mathbf{Q}^T \mathbf{b}_i \quad (3.22)$$

Using (3.21), (3.22) can be expressed as

$$\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{K}^T \mathbf{b}_i \quad (3.23)$$

Using (3.13), (3.23) can be written as

$$\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{M}_0^i \quad (3.24)$$

Since $\mathbf{Q}^T \mathbf{H}(\alpha_0) \mathbf{Q}$ is nonsingular, (3.24) has a unique solution. Hence $\mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{e}_r$,

where $\mathbf{e}_r$ is the r^{th} column of the identity matrix $\mathbf{U}_q \in \mathbb{R}^{q \times q}$ and $r = \sum_{d=1}^{i-1} (n_d + 1) + 1$, represents the unique solution for (3.24). Next, from (3.15) the 0^{th} moment of $\mathbf{X}(s)$ is obtained as

$$\mathbf{Q}\hat{\mathbf{M}}_0^i = \mathbf{K}\mathbf{R}^{-1}\hat{\mathbf{M}}_0^i = \begin{pmatrix} \mathbf{M}_0^1 & \dots & \mathbf{M}_{n_p}^p \end{pmatrix} \mathbf{e}_r = \mathbf{M}_0^i \quad (3.25)$$

Here, as seen from the RHS of (3.25), 0^{th} moment of the system is preserved.

We now proceed to prove that if the hypothesis holds good for all l from $l = 0$ to $l = m - 1$, i.e. $\mathbf{Q}\hat{\mathbf{M}}_l^i = \mathbf{M}_l^i$ for $0 \leq l \leq m - 1$, then it also holds good for $l = m$, i.e. $\mathbf{Q}\hat{\mathbf{M}}_m^i = \mathbf{M}_m^i$. Using (3.13) to find the moments of $\mathbf{X}(s)$ in (3.22), we have

$$\mathbf{Q}^T \mathbf{H}(\alpha_0) \mathbf{Q} \hat{\mathbf{M}}_m^i = - \sum_{r=0}^{m-1} \frac{\mathbf{Q}^T \mathbf{H}^{m-r}(\alpha_0) \mathbf{Q} \hat{\mathbf{M}}_r^i}{(m-r)!} \quad (3.26)$$

Using (3.21), (3.26) can be expressed as

$$\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = - \sum_{r=0}^{m-1} \mathbf{K}^T \frac{\mathbf{H}^{m-r}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_r^i}{(m-r)!} \quad (3.27)$$

Since the hypothesis holds good for all l from $l = 0$ to $l = m - 1$, $\mathbf{R}^{-1} \hat{\mathbf{M}}_l^i = \mathbf{e}_r$ where $r = \sum_{d=1}^{i-1} (n_d + 1) + (l + 1)$. This reduces (3.27) to

$$\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = - \mathbf{K}^T \sum_{r=0}^{m-1} \frac{\mathbf{H}^{m-r}(\alpha_0) \mathbf{M}_r^i}{(m-r)!} \quad (3.28)$$

Using (3.13), (3.28) can be re-written as

$$\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = -\mathbf{K}^T \mathbf{H}(\alpha_0) \mathbf{M}_m^i \quad (3.29)$$

Again, since $\mathbf{Q}^T \mathbf{H}(\alpha_0) \mathbf{Q}$ is nonsingular, (3.29) has a unique solution. Hence $\mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = \mathbf{e}_r$, where $r = \sum_{d=1}^{i-1} (n_d + 1) + (m + 1)$, represents the unique solution for (3.29).

Next, from (3.15) the m^{th} moment of $\mathbf{X}(s)$ is obtained as

$$\mathbf{Q} \hat{\mathbf{M}}_m^i = \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = \begin{pmatrix} \mathbf{M}_0^1 & \dots & \mathbf{M}_{n_p}^p \end{pmatrix} \mathbf{e}_r = \mathbf{M}_m^i \quad (3.30)$$

Hence, the m^{th} moment of the system corresponding to an excitation at port i is conserved if all the moments corresponding to the excitation at the same port are conserved from $n = 0$ to $n = m - 1$. Thus by *Mathematical Induction* we can conclude that the first $(n_i + 1)$ moments are conserved for the system. In other words the first $(n_i + 1)$ moments of $\mathbf{Q} \hat{\mathbf{X}}(s) = \mathbf{Q} \hat{\mathbf{H}}^{-1}(s) \hat{\mathbf{b}}_i$ are the same as the first $(n_i + 1)$ moments of $\mathbf{X}(s) = \mathbf{H}^{-1} \mathbf{b}_i$.

For the special case $q = N$, where N is the size of $\mathbf{H}(s)$ (i.e. no model reduction), it can be proved that infinite moments of $\mathbf{X}(s)$ are conserved.

Consider the reduced order system (3.19) with an excitation at port i . Since $q = N$, $\mathbf{Q}^T$ is a square matrix and is nonsingular. Hence,

$$\hat{\mathbf{X}}(s) = \mathbf{Q}^{-1} \mathbf{H}(s) \mathbf{b}_i \quad (3.31)$$

Using (3.15), $\mathbf{X}(s)$ can be found from the reduced order model to be,

$$\mathbf{Q}\hat{\mathbf{X}}(s) = \mathbf{H}(s)\mathbf{b}_i \equiv \mathbf{X}(s) \quad (3.32)$$

From (3.32) we can say that infinite moments of $\mathbf{X}(s)$ are conserved when $q = N$.

3.5 Preservation of Passivity

Section 2.11 highlighted the importance of passivity. Passivity implies that a network cannot generate more energy than it absorbs, and no passive termination of the network will make the system unstable. Passivity is an important property, because stable but not passive macromodels can lead to unstable systems when connected to other passive systems. A network is passive iff its admittance (or impedance) parameter matrix is positive real. The necessary and sufficient conditions for the network admittance matrix $\mathbf{Y}(s) = \hat{\mathbf{B}}^T \mathbf{H}^{-1}(s) \hat{\mathbf{B}}$ to be passive are the following.

1. $\mathbf{Y}(s^*) = \mathbf{Y}^*(s)$ for all complex s , where $*$ is the complex conjugate operator,
2. $\mathbf{Y}(s)$ is positive real, that is, $z^{*T}(\mathbf{Y}(s) + \mathbf{Y}^T(s^*))z \geq 0$ for all complex values of s satisfying $\text{Re}(s) \geq 0$ and for any complex vector z .

Since

$$\hat{\mathbf{F}}(s) = \mathbf{Q}^T \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{A}_k(s) \mathbf{D}_k^T \mathbf{Q} \quad (3.33)$$

$\hat{\mathbf{F}}(s)$ is positive real if each $\mathbf{A}_k(s)$ is positive real. Proof that $\mathbf{A}_k(s)$ is positive real is given below.

Consider the multi-conductor transmission line (MTL) system with N conductors. The exponential term in (3.5) can be expanded as

$$e^{\mathbf{E}} \equiv \lim_{n \rightarrow \infty} \left(\mathbf{I} + \frac{\mathbf{E}}{n} + \frac{\mathbf{E}^2}{2n^2} \right)^n \quad (3.34)$$

This expansion can be visualized as an infinite series of networks with each of their hybrid parameters being $(1 + \mathbf{E}/n + \mathbf{E}^2/2n^2)$. Let us consider one of such networks

$$\left(\mathbf{I} + \frac{\mathbf{E}}{n} + \frac{\mathbf{E}^2}{2n^2} \right) = \begin{pmatrix} \mathbf{I} + \frac{\mathbf{Z}_L \mathbf{Y}_L d^2}{2n^2} & -\frac{\mathbf{Z}_L d}{n} \\ -\frac{\mathbf{Y}_L d}{n} & \mathbf{I} + \frac{\mathbf{Y}_L \mathbf{Z}_L d^2}{2n^2} \end{pmatrix} \quad (3.35)$$

i.e.,

$$\begin{pmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{pmatrix} = \prod_{i=1}^n \begin{pmatrix} \mathbf{I} + \frac{\mathbf{Z}_L \mathbf{Y}_L d^2}{2n^2} & -\frac{\mathbf{Z}_L d}{n} \\ -\frac{\mathbf{Y}_L d}{n} & \mathbf{I} + \frac{\mathbf{Y}_L \mathbf{Z}_L d^2}{2n^2} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ -\mathbf{I}_1 \end{pmatrix} \quad (3.36)$$

The admittance parameters $\mathbf{y}_{par}$ of the infinitesimal section of the transmission line subnetwork are derived from (3.36) to be

$$\mathbf{y}_{par} = \begin{pmatrix} n \frac{\mathbf{Z}_L^{-1}}{d} + \frac{\mathbf{Y}_L d}{2n} & -n \frac{\mathbf{Z}_L^{-1}}{d} \\ -n \frac{\mathbf{Z}_L^{-1}}{d} - \frac{\mathbf{Y}_L \mathbf{Z}_L \mathbf{Y}_L}{4n^3} & n \frac{\mathbf{Z}_L^{-1}}{d} + \frac{\mathbf{Y}_L d}{2n} \end{pmatrix} \cong \begin{pmatrix} n \frac{\mathbf{Z}_L^{-1}}{d} + \frac{\mathbf{Y}_L d}{2n} & -n \frac{\mathbf{Z}_L^{-1}}{d} \\ -n \frac{\mathbf{Z}_L^{-1}}{d} - \frac{\mathbf{Y}_L \mathbf{Z}_L \mathbf{Y}_L}{4n^3} & n \frac{\mathbf{Z}_L^{-1}}{d} + \frac{\mathbf{Y}_L d}{2n} \end{pmatrix} \quad (3.37)$$

Expressing (3.37) as a sum of two positive real matrices, we have

$$\begin{aligned}
\mathbf{y}_{par} &= \begin{pmatrix} n \frac{\mathbf{Z}_L^{-1}}{d} & -n \frac{\mathbf{Z}_L^{-1}}{d} \\ -n \frac{\mathbf{Z}_L^{-1}}{d} & n \frac{\mathbf{Z}_L^{-1}}{d} \end{pmatrix} + \begin{pmatrix} \frac{\mathbf{Y}_L d}{2n} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{Y}_L d}{2n} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} n \frac{\mathbf{Z}_L^{-1}}{d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} + \begin{pmatrix} \frac{\mathbf{Y}_L d}{2n} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{Y}_L d}{2n} \end{pmatrix}
\end{aligned} \tag{3.38}$$

This implies that $\mathbf{y}_{par}$ is positive real. Hence the matrix representing the admittance parameters of an infinitesimal section of the transmission line subnetwork is proved to be passive. The entire transmission line subnetwork is formed by cascading an infinite number of such passive sections in series with each other. Since a combination of passive networks results in a passive network, the admittance parameter matrix $\mathbf{A}_k(s)$ of the transmission line network as $n \rightarrow \infty$ represents a passive network and is consequently a positive real matrix. Since each $\mathbf{A}_k(s)$ is positive real, it follows from (3.33) that $\hat{\mathbf{F}}(s)$ is also positive real.

Verifying condition 1 for the admittance matrix obtained from the reduced order system, we have

$$\mathbf{Y}(s^*) = \hat{\mathbf{B}}^T (\hat{\mathbf{G}} + s^* \hat{\mathbf{C}} + \hat{\mathbf{F}}(s^*))^{-1} \hat{\mathbf{B}} = \hat{\mathbf{B}}^T ((\hat{\mathbf{G}} + s \hat{\mathbf{C}} + \hat{\mathbf{F}}(s))^{-1})^* \hat{\mathbf{B}} = \mathbf{Y}^*(s) \tag{3.39}$$

since $\hat{\mathbf{F}}(s)$ is positive real. Hence condition 1 is satisfied. Now, we have

$$\begin{aligned}
z^{*T}(\mathbf{Y}(s) + \mathbf{Y}^T(s^*))z &= z^{*T}(\hat{\mathbf{B}}^T((\hat{\mathbf{G}} + s\hat{\mathbf{C}} + \hat{\mathbf{F}}(s))^{-1})^*\hat{\mathbf{B}} \\
&\quad + \hat{\mathbf{B}}^T(\hat{\mathbf{G}} + s^*\hat{\mathbf{C}} + \hat{\mathbf{F}}(s^*))^{-T}\hat{\mathbf{B}})z \\
&= z^{*T}\hat{\mathbf{B}}^T(\hat{\mathbf{G}} + s\hat{\mathbf{C}} + \hat{\mathbf{F}}(s))^{-1} \\
&\quad \cdot \left[\hat{\mathbf{G}} + s\hat{\mathbf{C}} + \hat{\mathbf{F}}(s) + (\hat{\mathbf{G}} + s^*\hat{\mathbf{C}} + \hat{\mathbf{F}}(s^*))^T \right] \\
&\quad \cdot (\hat{\mathbf{G}} + s^*\hat{\mathbf{C}} + \hat{\mathbf{F}}(s^*))^{-T}\hat{\mathbf{B}}z
\end{aligned} \tag{3.40}$$

Letting $w = (\hat{\mathbf{G}} + s^*\hat{\mathbf{C}} + \hat{\mathbf{F}}(s^*))^{-T}\hat{\mathbf{B}}z$ (3.40) reduces to

$$\begin{aligned}
z^{*T}(\mathbf{Y}(s) + \mathbf{Y}^T(s^*))z &= w^{*T} \left[(\hat{\mathbf{G}} + s\hat{\mathbf{C}}) + (\hat{\mathbf{G}} + s^*\hat{\mathbf{C}})^T + \hat{\mathbf{F}}(s) + \hat{\mathbf{F}}(s^*)^T \right] w \\
&= w^{*T} \left[(\hat{\mathbf{G}} + s\hat{\mathbf{C}}) + (\hat{\mathbf{G}} + s^*\hat{\mathbf{C}})^T \right] w \\
&\quad + w^{*T} \left[\hat{\mathbf{F}}(s) + \hat{\mathbf{F}}(s^*)^T \right] w
\end{aligned} \tag{3.41}$$

The first term in (3.41) has been proved to be positive in [24]. The second term is also positive since $\hat{\mathbf{F}}(s)$ is positive real. Hence we can say that

$$z^{*T}(\mathbf{Y}(s) + \mathbf{Y}^T(s^*))z \geq 0 \tag{3.42}$$

Thus, the system admittance matrix derived from the reduced order model satisfies condition 2. Consequently we can conclude that the system is passive. It is to be noted that in this proof, no assumption on the transformation matrix $\mathbf{Q}$ has been made other than the fact that it is a real matrix.

3.6 Extension to the Multi-Point Case

The technique presented in section 3.3 can be very easily extended to the case of multi-point expansions. This is done by generating the moments at various frequency points and using them in the subspace $\mathbf{K}$. Since the moments computed at frequency points other than zero are complex, they are split into real and imaginary parts before being appended to the subspace. Let $\Re_j^i(\alpha_k)$ and $\Im_j^i(\alpha_k)$ represent the real and imaginary parts of the j^{th} moment vector corresponding to an excitation at port i computed at frequency point $s = \alpha_k$. The subspace $\mathbf{K}$ now becomes

$$\mathbf{K} = \begin{pmatrix} M_0^1(\alpha_0) & \dots & M_{n_p}^p(\alpha_0) & \Re_0^1(\alpha_1) & \Im_0^1(\alpha_1) & \dots & \Re_{n_p}^p(\alpha_1) & \Im_{n_p}^p(\alpha_1) \\ & & & \dots & \Re_0^1(\alpha_s) & \Im_0^1(\alpha_s) & \dots & \Re_{n_p}^p(\alpha_s) & \Im_{n_p}^p(\alpha_s) \end{pmatrix} \quad (3.43)$$

Proceeding along similar lines as in section 3.4, we will prove that the reduced system (3.19) (with the modified (3.43)) evaluated at a frequency point $s = \alpha_k \neq 0$, conserves the first $(n_i + 1)$ moments of the original system. First we will prove that the 0^{th} moment obtained from the reduced-order model is equivalent to that obtained from the original system represented by (18). Next, we will show that the first m moments are preserved if the previous $(m - 1)$ moments are preserved.

Using (3.13) and (3.19) at $s = \alpha_k$, we have

$$\mathbf{Q}^T \mathbf{H}(\alpha_k) \mathbf{Q} \hat{\mathbf{M}}_0^i = \mathbf{Q}^T \mathbf{b}_i \quad (3.44)$$

Using (3.18), (3.44) can be expressed as

$$\mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{K}^T \mathbf{b}_i \quad (3.45)$$

Using (3.13), (3.45) can be written as

$$\mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{M}_0^i \quad (3.46)$$

Since $\mathbf{Q}^T \mathbf{H}(\alpha_k) \mathbf{Q}$ is nonsingular, (3.46) has a unique solution. Hence $\mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \mathbf{e}_r + j\mathbf{e}_{r+1}$ where $r = (2k-1) \sum_{d=1}^p (n_d+1) + \sum_{d=1}^{i-1} 2(n_d+1) + 1$ and $j = \sqrt{-1}$, represents the unique solution for (3.46). Next, from (3.15) the 0^{th} moment of $\mathbf{X}(s)$ is obtained as

$$\mathbf{Q} \hat{\mathbf{M}}_0^i = \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_0^i = \begin{pmatrix} \mathbf{M}_0^1(\alpha_0) & \dots & \mathfrak{S}_{n_p}^p(\alpha_s) \end{pmatrix} (\mathbf{e}_r + j\mathbf{e}_{r+1}) = \mathbf{M}_0^i \quad (3.47)$$

Here, as seen from the RHS of (3.47), 0^{th} moment of the system is preserved.

We now proceed to prove that if the hypothesis holds good for all l from $l=0$ to $l=m-1$, i.e. $\mathbf{Q} \hat{\mathbf{M}}_l^i = \mathbf{M}_l^i$ for $0 \leq l \leq m-1$, then it also holds good for $l=m$, i.e. $\mathbf{Q} \hat{\mathbf{M}}_m^i(\alpha_k) = \mathbf{M}_m^i(\alpha_k)$. Using (3.13) to find the moments of $\mathbf{X}(s)$ in (3.44), we have

$$\mathbf{Q}^T \mathbf{H}(\alpha_k) \mathbf{Q} \hat{\mathbf{M}}_m^i(\alpha_k) = - \sum_{r=0}^{m-1} \frac{\mathbf{Q}^T \mathbf{H}^{m-r}(\alpha_k) \mathbf{Q} \hat{\mathbf{M}}_r^i(\alpha_k)}{(m-r)!} \quad (3.48)$$

Using (3.21), (3.48) can be expressed as

$$\mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i(\alpha_k) = - \sum_{r=0}^{m-1} \mathbf{K}^T \frac{\mathbf{H}^{m-r}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_r^i(\alpha_k)}{(m-r)!} \quad (3.49)$$

Since the hypothesis holds good for all l from $l = 0$ to $l = m - 1$, $\mathbf{R}^{-1} \hat{\mathbf{M}}_l^i = \mathbf{e}_r + j\mathbf{e}_{r+1}$ where $r = (2k - 1) \sum_{d=1}^p (n_d + 1) + \sum_{d=1}^{i-1} 2(n_d + 1) + (l + 1)$. This reduces (3.49) to

$$\mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i(\alpha_k) = - \mathbf{K}^T \sum_{r=0}^{m-1} \frac{\mathbf{H}^{m-r}(\alpha_k) \mathbf{M}_r^i(\alpha_k)}{(m-r)!} \quad (3.50)$$

Using (3.13), (3.50) can be re-written as

$$\mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i(\alpha_k) = - \mathbf{K}^T \mathbf{H}(\alpha_k) \mathbf{M}_m^i(\alpha_k) \quad (3.51)$$

Since $\mathbf{Q}^T \mathbf{H}(\alpha_k) \mathbf{Q}$ is nonsingular, (3.51) has a unique solution. Hence $\mathbf{R}^{-1} \hat{\mathbf{M}}_l^i = \mathbf{e}_r + j\mathbf{e}_{r+1}$, where $r = (2k - 1) \sum_{d=1}^p (n_d + 1) + \sum_{d=1}^{i-1} 2(n_d + 1) + (m + 1)$, represents the unique solution for (3.51). Next, from (3.15) the m^{th} moment of $\mathbf{X}(s)$ is obtained as

$$\mathbf{Q} \hat{\mathbf{M}}_m^i = \mathbf{K} \mathbf{R}^{-1} \hat{\mathbf{M}}_m^i = \left(\mathbf{M}_0^1(\alpha_0) \quad \dots \quad \mathfrak{S}_{n_p}^p(\alpha_s) \right) (\mathbf{e}_r + j\mathbf{e}_{r+1}) = \mathbf{M}_m^i \quad (3.52)$$

Hence the m^{th} moment computed at a frequency point $s = \alpha_k$ is conserved if all the moments at the same frequency point for a unit excitation at the same port are conserved from $l = 0$ to $l = m - 1$. Thus by *Mathematical Induction* we can conclude that the first $(n_i + 1)$ moments are conserved at the frequency point $s = \alpha_k$. In other words the first $(n_i + 1)$ moments of $\mathbf{Q} \hat{\mathbf{X}}(s) = \mathbf{Q} \hat{\mathbf{H}}^{-1}(s) \hat{\mathbf{b}}_i$ are the same as the first

$(n_i + 1)$ moments of $\mathbf{X}(s) = \mathbf{H}^{-1}\mathbf{b}_i$ at an expansion $s = \alpha_k$.

Pseudo-code for the proposed algorithm is given below.

```

[G, C, F(s), B] = FormMNA();
K = null;
ReducedModel = SearchExpansionPtsAndFormModel(G, C, F(s), B, 0, highest_freq, K);
quit;

function SearchExpansionPtsAndFormModel(G, C, F(s), B, lowest_freq, highest_freq, K)
    reduced_model_1 = FormModel(G, C, F(s), B, lowest_freq, K);
    reduced_model_2 = FormModel(G, C, F(s), B, highest_freq, K);
    middle_freq = (lowest_freq + highest_freq)/2;
    if (reduced_model_1_at_middle_freq - reduced_model_2_at_middle_freq)
                                                > error_criteria
        SearchExpansionPtsAndFormModel(G, C, F(s), B, lowest_freq, middle_freq, K);
        SearchExpansionPtsAndFormModel(G, C, F(s), B, middle_freq, highest_freq, K);
    end if
    return(resultant_reduced_order_model);
end function

function FormModel(G, C, F(s), B, fpoint, K)
    for j = 1 to num_ports
        K = Append(K, getMoments(G, C, F(s), Bj, fpoint));
    end for
    Q = orthonormalize(K);
    G_hat = Q' G Q;
    C_hat = Q' C Q;
    F_hat(s) = Q' F(s) Q;
    B_hat = Q' B;
    return(G_hat, C_hat, F_hat(s), B_hat)
end function

function orthonormalize(A)
    [m, n] = size(A)
    for i = 1 to min(m-1, n)
        u_i = A(i:m, i) + sign(A(i, i))e_1 // e_1 is the first column of the

```

```

         $u_i = u_i / \text{norm}(u_i)$  // identity matrix
         $A(i:m, i:n) = A(i:m, i:n) - 2u_i(u_i^T A(i:m, i:n))$ 
    end for
    return(A);
end function

function getMoments(G, C, F(s), Bi, fpoint)
    w = 2* $\pi$ *j*fpoint;
    H = G + w*C + F(w);
    [L,U] = lu(H);
    N1 = forward_backward_subst(L,U,Bi);
    if fpoint != 0
        M1 = real(N1);
        M2 = imag(N1);
    end if
    sum = 0;
    for i = 1 to num_moments_needed
        for k = 1 to i-1
            sum -= kth_derivative_of_H(fpoint)*N(n-k)/factorial(i);
            /* The kth derivative
               of H(s) is computed as described in Section 3.3.1*/
        end for
        Ni = forward_backward_subst(L,U,sum);
        if fpoint != 0
            M2i+1 = real(Ni);
            M2i+2 = imag(Ni);
        end if
    end for
    if fpoint = 0
        return N;
    else
        return M;
    end if
end function

```

3.7 Numerical Results

This section presents numerical examples to demonstrate the efficiency and accuracy of the algorithm presented in the chapter.

3.7.1 Example 1

A network consisting of seven transmission lines and several lumped components, shown in Fig. 3.2, was considered for this example. The proposed algorithm was used to find the reduced-order system. Six moments were considered at frequency points zero and 1.75 GHz to form the subspace $\mathbf{K}$. This was then used to form the reduced-order model, which was used to find the Y-parameters of the entire linear distributed network. Fig. 3.3 shows the Y-parameters of the network calculated from both the proposed method and SPICE. As seen they match accurately.

In the case where the Telegrapher's equations are discretized (e.g., each interconnect discretized into 50 sections each), the size of the matrix obtained from MNA (Modified Nodal Analysis) equations would be 1100×1100 . But, in the case of the proposed algorithm, the size of the matrix was 36×36 .

Fig. 3.4 shows the transient response at the output of an inverter connected between port 2 and a capacitive load of 0.1pF. The input to the circuit was connected through a driver at port 1, with rise/fall times of 0.1ns and pulse width of 5ns. As seen, the response from the proposed method matches with the SPICE simulations accurately. A speed up of 48 is obtained using the proposed algorithm for this example.

Figure 3.2: Seven transmission line network

Figure 3.3: Y-parameters for example 1

Figure 3.4: Transient response for example 1

3.7.2 Example 2

This example illustrates the capability of the proposed algorithm to directly handle frequency-dependent elements in a congruent transformation based approach. The network consists of several lumped components and one transmission line whose parameters are frequency-dependent. The frequency dependent parameters of the transmission lines $\mathbf{R}(s)$, $\mathbf{L}(s)$, $\mathbf{C}(s)$ and $\mathbf{G}(s)$ are approximated by rational functions which are positive real. This ensures the positive real nature of $\mathbf{F}(s)$. In turn, this guarantees the passivity of the macromodel according to the proof given in this paper. The subspace $\mathbf{K}$ is then formed using moments found through the method described in [16] for frequency-dependent networks. Three moments at zero, one moment at 1 GHz and three moments at 2 GHz were used to form the subspace. The subspace thus obtained is used in the reduction of the system to find the reduced-order model. The responses from the proposed algorithm and direct methods are compared in Fig. 3.5

and are found to match accurately.

Figure 3.5: Frequency response of example 2

Chapter 4

Sensitivity Analysis of Transmission-Line Networks using Model-Reduction Techniques

4.1 Introduction

Advances in fabrication methods and decreasing device sizes have significantly increased the relative contribution of high-speed interconnects to overall signal degradation [6]. Transmission line analysis has therefore become imperative at all levels of the design hierarchy from backplanes to printed circuit boards, packages, and integrated circuits [8]. Generally, interconnect networks result in large systems of equations, simulation of which is prohibitively CPU expensive. Recently, model-reduction techniques were presented as key to fast simulation of interconnect networks. Model-reduction techniques present a fast way of evaluating the response of large circuits

without performing expensive L/U decompositions on the large system of equations at all frequency points of interest. This is extremely efficient in order to evaluate the response of a given circuit. However, sensitivity analysis of such large circuits is also a necessity. Generally, techniques to perform sensitivity analysis, such as the adjoint sensitivity method depend heavily on the fact that the L/U decomposition of the original system of equations is known at all frequency points of interest. But, when model-reduction techniques are used to evaluate the frequency response of a circuit, the L/U decomposition of the MNA matrices is not known at all frequency points. Therefore, in order to perform sensitivity analysis on these circuits using conventional sensitivity techniques, expensive L/U decompositions have to be performed on the original circuit matrices. Performing expensive L/U decompositions on the MNA matrices of the original system is not an option. Hence, we need an efficient technique to perform sensitivity analysis on large circuits using model-reduction techniques.

In literature, several techniques can be found that use model reduction based on Padé approximation to perform sensitivity analysis of linear lumped networks [59], [60]. Besides the fact that these techniques cannot handle distributed networks, their analysis requires finding the derivatives of the poles and zeros of the reduced model. This can be both cumbersome and inefficient. In this chapter, an efficient algorithm [61], [62] is proposed for sensitivity analysis of transmission line networks based on congruent transformation techniques. The proposed technique handles distributed transmission line circuit stamps including frequency dependent parameters without the need for discretization of the Telegrapher's equations. The algorithm is based on projecting the adjoint network equations on a reduced order subspace that preserves the circuit moments. The sensitivity can then be evaluated over any

number of frequency points for the cost of one QR factorization.

This chapter is organized as follows. Section 4.2 presents the formulation of the circuit equations. This is followed in section 4.3 by a brief description of adjoint sensitivity analysis. The proposed model reduction algorithm is then discussed in section 4.4. Finally, the results are presented in section 4.5.

4.2 Formulation of Network Equations

A general linear distributed subnetwork with its elements as a function of parameter λ can be written as

$$\mathbf{Y}(s, \lambda) \mathbf{X}(s, \lambda) = \mathbf{B} \quad (4.1a)$$

$$\mathbf{Y}(s, \lambda) = \mathbf{G}(\lambda) + s\mathbf{C}(\lambda) + \mathbf{F}(s, \lambda) \quad (4.1b)$$

where,

$\mathbf{X}(s, \lambda) \in \mathcal{C}^N$ is the vector of the complex variables describing the subnetwork which includes node voltage waveforms appended by independent and dependent voltage source currents and inductor currents,

$\mathbf{G}(\lambda), \mathbf{C}(\lambda) \in \mathbb{R}^{N \times N}$ are matrices describing the lumped memory and memoryless elements of network. $\mathbf{G}, \mathbf{C}$ are formulated as suggested in [24]

$\mathbf{B} \in \mathcal{C}^N$ is a vector with entries determined by the independent voltage and current sources.

$\mathbf{F}(s, \lambda) = \sum_{k=1}^N \mathbf{D}_k \mathbf{A}_k(s, \lambda) \mathbf{D}_k^T$ is a matrix containing the frequency dependent y-parameters of all the distributed transmission line sets. $\mathbf{D}_k = [d_{i,j}]$ with ele-

ments $d_{i,j} \in \{0,1\}$ where $i \in \{1,2,\dots,N\}$, $j \in \{1,2,\dots,2n_k\}$ with a maximum of one non-zero in each row or column, is a selector matrix that maps $\mathbf{I}_k(t) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^N$ of the network. The y-parameters of transmission line set k are given by [7]

$$\mathbf{A}_k \mathbf{V}_k = \mathbf{I}_k \quad (4.2)$$

where $\mathbf{V}_k \in \mathbb{R}^N$ is the vector of terminal voltages of the interconnect subnetwork k and

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} \\ \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} \end{pmatrix} \quad (4.3)$$

and $\mathbf{E}_1$ and $\mathbf{E}_2$ are the diagonal matrices,

$$\mathbf{E}_1 = \text{diag} \left\{ \frac{1 + e^{-2\gamma_i d}}{1 - e^{-2\gamma_i d}}, i = 1 \dots n_k \right\} \quad (4.4a)$$

$$\mathbf{E}_2 = \text{diag} \left\{ \frac{2}{e^{-\gamma_i d} - e^{\gamma_i d}}, i = 1 \dots n_k \right\} \quad (4.4b)$$

where d is the length of the transmission line set, and γ_i^2 are the eigenvalues of the matrix $\mathbf{Z}_L \mathbf{Y}_L$ with associated eigenvectors $\mathbf{x}_i$ where,

$$\mathbf{Z}_L = \tilde{\mathbf{R}} + s\tilde{\mathbf{L}} \quad (4.5a)$$

$$\mathbf{Y}_L = \tilde{\mathbf{G}} + s\tilde{\mathbf{C}} \quad (4.5b)$$

and $\tilde{\mathbf{R}}, \tilde{\mathbf{L}}, \tilde{\mathbf{G}}$ and $\tilde{\mathbf{C}} \in \mathbb{R}^{n_k \times n_k}$ are the per-unit-length resistance, inductance, conductance and capacitance matrices of the transmission line set k . $\mathbf{S}_v$ is a matrix with the eigenvectors $\mathbf{x}_i$ in the columns, and $\mathbf{S}_i = \mathbf{Z}_L^{-1} \mathbf{S}_v \Gamma$ where

$\Gamma = \text{diag}\{\gamma_i, i = 1 \dots n_k\}$. It is to be noted that the per unit length $\tilde{\mathbf{R}}, \tilde{\mathbf{L}}, \tilde{\mathbf{G}}, \tilde{\mathbf{C}}$ parameters of the transmission line can also be a function of frequency and the parameter λ .

4.3 Sensitivity Analysis

Several techniques can be found in the literature to determine the sensitivity of the output w.r.t any parameter λ in the network. Amongst these, the adjoint technique has been proved to be the most efficient method [45]. The present section briefly describes this technique. This forms a basis to propose the new algorithm discussed in section 4.4.

Let V be the output variable of interest. We have

$$V = \mathbf{d}^T \mathbf{X} \quad (4.6)$$

where $\mathbf{d}$ is a selector vector with non-zero entries corresponding to the output variable. The sensitivity of V w.r.t λ can be written as

$$\frac{\partial V}{\partial \lambda} = -\mathbf{X}_a^T \frac{\partial \mathbf{Y}}{\partial \lambda} \mathbf{X}_a \quad (4.7)$$

where $\mathbf{X}$ is the solution of the network equations, and $\mathbf{X}_a$ is the solution of the adjoint equations

$$\mathbf{Y}^T(s, \lambda) \mathbf{X}_a = \mathbf{d} \quad (4.8)$$

Equations (4.1), (4.7) and (4.8) can be solved to obtain the sensitivity of V w.r.t

λ . The relative sensitivity is defined as

$$S_\lambda = \lambda \frac{\partial V}{\partial \lambda} \quad (4.9)$$

The definition in (4.9) is directly related to the partial derivatives, however it is more meaningful when comparing sensitivity w.r.t. components of different magnitudes.

The computational cost of this method is one L/U factorization of $\mathbf{Y}(s, \lambda)$ at each frequency point. It is to be noted that for large circuits this method can be computationally intensive. To address this problem, in the next section we propose an algorithm to perform sensitivity analysis on large networks using model reduction techniques.

4.4 Sensitivity Analysis using Model Reduction

The proposed algorithm to find the sensitivity of voltage and current waveforms of the transmission line network described in (4.1a) is presented in the following subsections. First an algorithm is presented for the reduction of the equations representing the original system (4.1a) and its sensitivity (4.8). This is followed by a description of the computation of the sensitivity of the MNA matrix, required in (4.7).

The original system in (4.1a) can be reduced by a congruent transformation into a smaller system of size q [52], [54]. The congruent transformation matrix $\mathbf{Q}$ is an orthonormal basis, constructed such that

$$\text{colsp}[\mathbf{Q}] = \text{colsp}[\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_q] \quad (4.10)$$

where $\mathbf{Q} \in \mathbb{R}^{N \times q}$, and $\mathbf{M}_i$ is the i^{th} moment of the circuit response. The moments $\mathbf{M}_i$ are evaluated using the recursive relation given by [18]

$$[\mathbf{Y}]_{s=0} \mathbf{M}_i = \sum_{r=1}^n \frac{\left. \frac{\partial^r \mathbf{Y}}{\partial s^r} \right|_{s=0}}{r!} \mathbf{M}_{i-r} \quad (4.11)$$

with

$$[\mathbf{Y}]_{s=0} \mathbf{M}_0 = \mathbf{B} \quad (4.12)$$

The reduced system is formed by performing the congruent transformation

$$\mathbf{X}(s) = \mathbf{Q} \hat{\mathbf{X}}(s) \quad (4.13)$$

on the original system (4.1). Using (4.13) and (4.1a) the reduced system can be written as

$$\hat{\mathbf{Y}}(s) \hat{\mathbf{X}}(s) = \hat{\mathbf{B}} \quad (4.14)$$

where

$$\hat{\mathbf{Y}} = \mathbf{Q}^T \mathbf{Y} \mathbf{Q}; \quad \hat{\mathbf{B}} = \mathbf{Q}^T \mathbf{B} \quad (4.15)$$

The solution of the original network is obtained by mapping the solution of the reduced circuit back to the original space using (4.13). Such a congruent transformation was shown to produce a passive system while preserving the first q derivatives of the original system [52], [54].

From (4.13) it can be seen that

$$\frac{\partial \mathbf{X}}{\partial \lambda} = \frac{\partial \mathbf{Q}}{\partial \lambda} \hat{\mathbf{X}} + \mathbf{Q} \frac{\partial \hat{\mathbf{X}}}{\partial \lambda} \quad (4.16)$$

Equation (4.16) demonstrates that finding the sensitivity of $\mathbf{Q}$ w.r.t λ involves finding the derivative of the congruent transformation w.r.t λ which is difficult and cumbersome to evaluate. In order to avoid the evaluation of $\frac{\partial \mathbf{Q}}{\partial \lambda}$, we perform model reduction directly on the adjoint system of equations (4.8) through a congruent transformation $\mathbf{Q}_a$. As in the case of model reduction of the original system (4.1), the moments of $\mathbf{X}_a$ in (4.8) are needed for the construction of the congruent transformation $\mathbf{Q}_a$. These moments are evaluated using the recursive relationship given by [18]

$$[\mathbf{Y}^t]_{s=0} \mathbf{M}_i^a = \sum_{r=1}^i \frac{\left. \frac{\partial^r \mathbf{Y}^t}{\partial s^r} \right|_{s=0}}{r!} \mathbf{M}_{i-r}^a \quad (4.17)$$

with

$$[\mathbf{Y}^t]_{s=0} \mathbf{M}_0^a = \mathbf{d} \quad (4.18)$$

where $\mathbf{M}_0^a$ is the i^{th} moment of the adjoint circuit response. It should be noted that the factorization of $\mathbf{Y}^t$ can be easily deduced from that of $\mathbf{Y}$ which was evaluated in (4.12). Reducing the adjoint circuit equations (4.8), we have

$$\hat{\mathbf{Y}}_a \hat{\mathbf{X}}_a(s) = \hat{\mathbf{d}}_a \quad (4.19)$$

where

$$\hat{\mathbf{Y}}_a = \mathbf{Q}_a^T \mathbf{Y}^T \mathbf{Q}_a; \quad \hat{\mathbf{d}}_a = \mathbf{Q}_a^T \mathbf{d} \quad (4.20)$$

The solution of the adjoint network can therefore be found by mapping $\hat{\mathbf{X}}_a$ back to the original space through the congruent transformation

$$\mathbf{X}_a(s) = \mathbf{Q}_a \hat{\mathbf{X}}_a(s) \quad (4.21)$$

The sensitivity can then be found at any frequency by simply solving the reduced systems in (4.14) and (4.19). Such congruent transformation applied to a linear system, is shown to preserve the first q derivatives with respect to frequency, thus preserving the accuracy of the response [52], [54]. In addition, the matrices involved in (4.14) and (4.19) are very small when compared to the original equations (4.1) and (4.8). Hence the solution of the reduced system of equations is much less computationally expensive when compared to that of the original system. It is to be noted that the system moments and the adjoint moments are computed using the same L/U decomposition in (4.11) and (4.17). However, there is no obvious relationship between $\mathbf{Q}$ and $\mathbf{Q}_a$. Therefore two separate QR decompositions are required to find the two transformation matrices. Furthermore, the algorithm does not require $\mathbf{Q}$ and $\mathbf{Q}_a$ to be of the same order, although using the same order for both transformations is recommended.

4.4.1 Calculation of $\frac{\partial \mathbf{Y}}{\partial \lambda}$

From (4.7) it can be seen that the derivative of the MNA equations is required to calculate the output sensitivities. The method of calculating this derivative however depends on the type of the parameter λ . The following subsections briefly describe the computation of $\frac{\partial \mathbf{Y}}{\partial \lambda}$ when λ represents lumped and distributed elements.

1. *Sensitivity with Respect to Parameters of Lumped Components*

When λ represents a parameter of a lumped element it can be seen from (4.1b) and (4.7) that

$$\frac{\partial \mathbf{Y}}{\partial \lambda} = \frac{\partial \mathbf{G}}{\partial \lambda} + s \frac{\partial \mathbf{C}}{\partial \lambda} \quad (4.22)$$

which can be easily evaluated [45].

2. Sensitivity with Respect to Electrical Parameters of Transmission Lines

If λ represents an electrical parameter of a distributed element, it can be seen from (4.1b), (4.2) and (4.7) that

$$\frac{\partial \mathbf{Y}}{\partial \lambda} = \frac{\partial \mathbf{F}(s)}{\partial \lambda} = \frac{\partial}{\partial \lambda} \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{A}_k(s) \mathbf{D}_k^T \quad (4.23)$$

In order to compute $\frac{\partial \mathbf{Y}}{\partial \lambda}$ using (4.23) we need to find the derivatives of $\mathbf{A}_k(s)$ w.r.t. to λ . It can be shown that $\mathbf{A}_k(s)$ satisfies the relation [60]

$$\begin{aligned} \frac{\partial \mathbf{A}_k}{\partial \lambda} \begin{pmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{pmatrix} &= \begin{pmatrix} \frac{\partial \mathbf{S}_i}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_i}{\partial \lambda} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{S}_i & 0 \\ 0 & \mathbf{S}_i \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda} & \frac{\partial \mathbf{E}_2}{\partial \lambda} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda} & \frac{\partial \mathbf{E}_1}{\partial \lambda} \end{pmatrix} - \mathbf{A}_k \begin{pmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda} \end{pmatrix} \end{aligned} \quad (4.24)$$

From (4.24) it can be seen that the sensitivity of $\mathbf{A}_k$ depends on the partial derivatives of the eigenvalues γ_i^2 and eigenvectors $\mathbf{x}_i$ of the matrix $\mathbf{Z}_L \mathbf{Y}_L$ which can be obtained using

$$\begin{pmatrix} \gamma_i \mathbf{I} - \mathbf{Z}_L \mathbf{Y}_L & \mathbf{x}_i \\ \mathbf{x}_i^T & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{x}_i}{\partial \lambda} \\ \frac{\partial \gamma_i^2}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial}{\partial \lambda} (\mathbf{Z}_L \mathbf{Y}_L) \right) \mathbf{x}_i \\ 0 \end{pmatrix} \quad (4.25)$$

3. Sensitivity with Respect to Physical Parameters of Transmission Lines

While sensitivity with respect to electrical parameters of transmission lines is

a useful measure, it is often an intermediate step in the calculation of sensitivity w.r.t. to physical parameters such as width because ultimately physical parameters are the design parameters. Suppose λ is a physical parameter of a transmission line system with N conductors. A simple expression for the output sensitivity can be written as

$$\frac{\partial V}{\partial \lambda} = \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial V}{\partial R_{i,j}} \frac{\partial R_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial L_{i,j}} \frac{\partial L_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial G_{i,j}} \frac{\partial G_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial C_{i,j}} \frac{\partial C_{i,j}}{\partial \lambda} \right) \quad (4.26)$$

The above formulation requires the computation of the sensitivity of the output with respect to all $4 \times N^2$ electrical parameters of the transmission lines. The derivatives of the electrical parameters with respect to the physical parameters can be found using numerical methods [13], [63], [64], but for many cases simple analytical expressions for the electrical parameters (such as those in [65]) can be used.

Pseudo code of the algorithm is given below.

```

[G,C,F(s),B,d] = FormMna();
Y(s) = G + sC + F(s);

// Form reduced order model
[K,Ka]= getMoments(Y,B,d);
Q = orthonormalize(K);
Qa = orthonormalize(Ka);
Y(s) = Q'Y(s)Q;
Ya(s) = Qa'Y'(s)Qa;
B = Q'B;
da = Qa'd;

// Perform sensitivity analysis on reduced system

```

```

for each frequency point
    solve  $\hat{Y}(s)\hat{x} = \hat{B}$ ;
    solve  $\hat{Y}_a(s)\hat{x}_a = \hat{d}_a$ ;
     $x = Q\hat{x}$ ;
     $x_a = Q\hat{x}_a$ ;
    evaluate  $\partial Y/\partial \lambda$  as described in section IV.B
     $\partial V/\partial \lambda = x'_a(\partial Y/\partial \lambda)x$ ;
end for
quit;

function orthonormalize(Y)
    [m,n] = size(Y)
    for i = 1 to min(m-1,n)
         $u_i = Y(i:m,i) + \text{sign}(Y(i,i))e_1$  //  $e_1$  is the first column of the
         $u_i = u_i/\text{norm}(u_i)$  // identity matrix
         $Y(i:m,i:n) = Y(i:m,i:n) - 2u_i(u_i^T Y(i:m,i:n))$ 
    end for
    return(Y);
end function

function getMoments(Y,B,d)
    [L,U] = lu();
    sum1 = 0; sum2 = 0;
    for i = 1 to num_moments_needed
        for k = 1 to i-1
            sum1 -= kth_derivative_of_Y*N(n-k)/factorial(i);
            sum2 -= transpose(kth_derivative_of_Y)*M(n-k)/factorial(i);
            /* The kth derivative of is computed as described
               in Section 4.4.*/
        end for
         $N_i = \text{forward\_backward\_subst}(L,U,\text{sum1})$ ;
         $M_i = \text{forward\_backward\_subst}(U',L',\text{sum2})$ ;
    end for
    return [N,M]
end function

```

Figure 4.2: Inductance per unit length of transmission line B

Figure 4.3: Resistance per unit length of transmission line B

Figure 4.4: Relative sensitivity of V_{out} w.r.t. capacitance per unit length of transmission line A

4.5.2 Example 2

For the second example, a relatively large interconnect circuit consisting of 458 resistors, inductors and capacitors and 12 transmission lines is considered. The input pulse and the output response at the node of interest are shown in Fig. 4.6. The relative sensitivity with respect to the parameters of one of the transmission lines was computed using the proposed algorithm. The time domain sensitivity was obtained using FFT and is shown in Figs 4.7 to 4.10. The response obtained from the proposed reduced order adjoint technique matched those of the conventional adjoint technique with no noticeable difference. Two expansions were used to match the sensitivity up to 1GHz. Only two L/U decompositions of the original system and one QR factorization of the subspace were required to obtain the response at 100 frequency points. Note that computation cost of the QR factorization needed to find the congruent transformation is comparable to that of one L/U decomposition. The adjoint technique required 100 L/U decompositions. Table 4.1 compares the number of L/U decompositions required for sensitivity analysis with respect to 10 different parameters at 100 frequency points.

Figure 4.5: A microstrip line from the numerical example

Figure 4.6: Time domain response

Figure 4.7: Relative sensitivity w.r.t. resistance per unit length

Figure 4.8: Relative sensitivity w.r.t. inductance per unit length

Figure 4.9: Relative sensitivity w.r.t. capacitance per unit length

Figure 4.10: Relative sensitivity w.r.t. transmission line width

	Equivalent L/Us	Speed-up
Perturbation	1001	1
Conventional adjoint technique	100	10
Reduced-order adjoint technique	3	334

Table 4.1: Comparison of the total expense for different methods

Chapter 5

Analysis of Transmission Line Circuits using Multi-Dimensional Model Reduction Techniques

5.1 Introduction

As we have seen in chapters 1 to 3, numerous algorithms can be found in the literature that efficiently simulate circuits by reducing the order of the linear system [17], [18], [24], [52]. These techniques perform model reduction with respect to a single parameter (frequency). However, at times, it is also necessary to include large-scale variations, due to environmental effects and changes in interconnect parameters such as width and height. These effects induce a change in the MNA matrices of the circuit. It is often not viable to perform simulation of large circuits repeatedly for changes due to these effects. Hence, a reduction technique is required, that performs

analysis of large circuits simultaneously with respect to frequency and other design parameters of the circuit.

In this chapter a new technique [66]–[68] is presented to reduce the order of a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit. These parameters could also include physical parameters of the transmission lines. The reduction algorithm is based on congruent transformation techniques extended to multiple dimensions. This technique provides efficient means to estimate the response of large circuits simultaneously as a function of several design parameters.

This chapter is organized as follows. Section 5.2 presents an outline of the proposed algorithm followed by section 5.3, which discusses the proof of preservation of moments. Section 5.4 presents the numerical examples.

5.2 Algorithm for Reduction of Multi-Dimensional Systems

This section describes the proposed algorithm which performs multi-dimensional model reduction of linear distributed networks using congruent transformation techniques. The formulation of the network equations and the terminology used to present this algorithm is similar to the formulation and terminology used in section 4.2 of chapter 4. The algorithm first starts by computing the multi-dimensional moments of the system and uses them to form a subspace. This subspace is then used to form a reduced model. The process is explained in the following sections.

5.2.1 Computation of Moments

The moments of $\mathbf{X}(s, \lambda)$ are computed recursively with respect to s and λ along with their cross-terms using the recursive relationship

$$\mathbf{Y}(s_i, \lambda_j) \mathbf{M}^{(m,n)} = - \sum_{r=1}^n \frac{1}{r!} \sum_{q=0}^m \frac{\mathbf{Y}^{(q,r)} \mathbf{M}^{(m-q,n-r)}}{q!} - \sum_{q=1}^m \frac{\mathbf{Y}^{(q,0)} \mathbf{M}^{(m-q,n)}}{q!} \quad (5.1a)$$

$$\mathbf{Y}(s_i, \lambda_j) \mathbf{M}^{(0,0)} = b \quad (5.1b)$$

where $\mathbf{M}^{(m,n)}$ is the m^{th} and n^{th} moment with respect to s and λ respectively, s_i and λ_j are the i^{th} and j^{th} expansion points of s and λ and $\mathbf{Y}^{(q,r)}$ is the q^{th} and r^{th} derivative with respect to s and λ , evaluated at $s = s_i$ and $\lambda = \lambda_j$.

It is to be noted that in order to compute the moments using (5.1a) and (5.1b), evaluation of the term $\mathbf{Y}^{(q,r)}$ is necessary. Computation of $\mathbf{Y}^{(q,r)}$ is trivial when λ represents a lumped element [45]. Following is a brief description for the case when λ represents a parameter of a transmission line.

It can be seen that computing the derivative of $\mathbf{Y}(s, \lambda)$ w.r.t λ requires the calculation of the derivatives of the $\mathbf{Y}$ parameters $\mathbf{A}_k$ defined in (4.2). It can be shown that $\mathbf{A}_k$ satisfies the relation [69]

$$\begin{aligned} \frac{\partial \mathbf{A}_k}{\partial \lambda} \begin{pmatrix} \mathbf{S}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_v \end{pmatrix} &= \begin{pmatrix} \frac{\partial \mathbf{S}_i}{\partial \lambda} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{S}_i}{\partial \lambda} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{S}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_i \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda} & \frac{\partial \mathbf{E}_2}{\partial \lambda} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda} & \frac{\partial \mathbf{E}_1}{\partial \lambda} \end{pmatrix} - \mathbf{A}_k \begin{pmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{S}_v}{\partial \lambda} \end{pmatrix}. \end{aligned} \quad (5.2)$$

From (5.2) it is seen that the derivative of $\mathbf{A}_k$ depends on the partial derivatives

of the eigenvalues γ_i^2 and eigenvectors $\mathbf{x}_i$ of the matrix $\mathbf{Z}_L \mathbf{Y}_L$ which can be obtained using

$$\begin{pmatrix} \gamma_i \mathbf{I} - \mathbf{Z}_L \mathbf{Y}_L & \mathbf{x}_i \\ \mathbf{x}_i^T & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{x}_i}{\partial \lambda} \\ \frac{\partial \gamma_i^2}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{Z}_L \mathbf{Y}_L}{\partial \lambda} \mathbf{x}_i \\ 0 \end{pmatrix}. \quad (5.3)$$

In the case when λ represents a physical parameter of the transmission-line system with N conductors, a simple expression for the first derivative can be written as

$$\frac{\partial \mathbf{Y}}{\partial \lambda} = \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial \mathbf{Y}}{\partial R_{i,j}} \frac{\partial R_{i,j}}{\partial \lambda} + \frac{\partial \mathbf{Y}}{\partial L_{i,j}} \frac{\partial L_{i,j}}{\partial \lambda} + \frac{\partial \mathbf{Y}}{\partial G_{i,j}} \frac{\partial G_{i,j}}{\partial \lambda} + \frac{\partial \mathbf{Y}}{\partial C_{i,j}} \frac{\partial C_{i,j}}{\partial \lambda} \right). \quad (5.4)$$

Higher derivatives w.r.t λ can be found by further differentiation of (5.2) and (5.4).

5.2.2 Model-Reduction through Congruent Transformation

Once the moments of the network are computed the multi-dimensional modified Krylov subspace (denoted by $\mathbf{K}$) is computed as follows

$$\mathbf{K}_{N \times p} = [\mathbf{M}^{(0,0)} \dots \mathbf{M}^{(m,n)}] \quad (5.5)$$

It is to be noted that $\mathbf{K}$ also contains the cross-derivatives w.r.t s and λ ; p represents the total number of multi-dimensional moments contained in $\mathbf{K}$. Since the matrix $\mathbf{K}$ is generally ill-conditioned, it is transformed into a new matrix $\mathbf{Q}$ through orthonormalization [24], [51]. Using the congruent transformation matrix $\mathbf{Q}$, the original large set of unknowns represented by $\mathbf{X}(s, \lambda)$ is reduced to a small set of unknowns, denoted by $\tilde{\mathbf{X}}(s, \lambda)$ as follows. Define

$$\mathbf{X}(s, \lambda) = \mathbf{Q} \tilde{\mathbf{X}}(s, \lambda). \quad (5.6)$$

Using (4.1) and (5.6),

$$\tilde{\mathbf{Y}}(s, \lambda) \tilde{\mathbf{X}}(s, \lambda) = \tilde{\mathbf{b}} \quad (5.7)$$

where

$$\tilde{\mathbf{Y}}(s, \lambda) = \mathbf{Q}^T \mathbf{Y}(s, \lambda) \mathbf{Q}; \quad \tilde{\mathbf{b}} = \mathbf{Q}^T \mathbf{b} \quad (5.8)$$

It is to be noted that the dimension of the reduced model is much smaller when compared to the size of the original circuit equations (4.1). Next, the response of the reduced model is obtained from (5.7) and is transformed to the original system through (5.6). In the next section a proof is given that the congruent transformation (5.6) preserves p moments with respect to s and λ .

5.3 Proof of Conservation of Moments

Consider the matrix $\mathbf{K}$ represented by (5.5). Performing an orthogonal decomposition on $\mathbf{K}$ we have

$$\mathbf{K} = \mathbf{Q}\mathbf{R} \quad (5.9)$$

where. $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}_p$. Next, using the congruence transformation in (5.6) we have

$$\mathbf{Q}^T \mathbf{Y}(s, \lambda) \mathbf{Q} \tilde{\mathbf{X}}(s, \lambda) = \mathbf{Q}^T \mathbf{b}. \quad (5.10)$$

Substituting (5.9) in (5.10) and simplifying we get

$$\mathbf{K}^T \mathbf{Y}(s, \lambda) \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{X}}(s, \lambda) = \mathbf{K}^T \mathbf{b} \quad (5.11)$$

Proof that the reduced system (5.10) conserves p moments with respect to s and

λ is based on mathematical induction. First we prove that the $(0,0)^{th}$ moment obtained from the reduced model is equivalent to that obtained from the original system represented by (4.1). Next, we show that the $(m,n)^{th}$ moment is conserved if all the previous moments, are conserved (i.e. holds good for $(l,u)^{th}$ moment where $\{0 \leq l \leq m-1, 0 \leq u \leq n-1\}$, $\{0 \leq l \leq m, 0 \leq u \leq n-1\}$ and $\{0 \leq l \leq m-1, 0 \leq u \leq n\}$).

Using (5.1b), (5.10) and (5.9) we have

$$\mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(0,0)} = \mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{M}^{(0,0)} \quad (5.12)$$

Since $\mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K}$ is nonsingular, (5.12) has a unique solution. Hence $\mathbf{R}^{-1} \tilde{\mathbf{M}}^{(0,0)} = \mathbf{e}_r$ where $\mathbf{e}_r$ is the r^{th} column of the identity matrix $\mathbf{I}_p$ and $r=1$ represents a unique solution of (5.12). Next from (5.6) the $(0,0)^{th}$ moment of $\mathbf{X}(s, \lambda)$ is obtained as

$$\mathbf{Q} \tilde{\mathbf{M}}^{(0,0)} = \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(0,0)} = \mathbf{K} \mathbf{e}_r = \mathbf{M}^{(0,0)} \quad (5.13)$$

Here, as seen from the RHS of (5.13), the $(0,0)^{th}$ moment of the system is conserved.

We now proceed to prove that if the hypothesis holds good for all $\tilde{\mathbf{M}}^{(l,u)}$ where $\{0 \leq l \leq m-1, 0 \leq u \leq n-1\}$, $\{0 \leq l \leq m, 0 \leq u \leq n-1\}$ and $\{0 \leq l \leq m-1, 0 \leq u \leq n\}$, then it also holds good for $l=m, u=n$ i.e. $\mathbf{Q} \tilde{\mathbf{M}}^{(m,n)} = \mathbf{M}^{(m,n)}$.

Using (5.1a) to find the moments of $\mathbf{X}(s, \lambda)$ and simplifying using (5.11), we have

$$\begin{aligned} \mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m,n)} &= - \sum_{r=1}^n \frac{1}{r!} \sum_{q=0}^m \frac{\mathbf{K}^T \mathbf{Y}^{(q,r)} \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m-q,n-r)}}{q!} \\ &\quad - \sum_{q=1}^m \frac{\mathbf{K}^T \mathbf{Y}^{(q,0)} \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m-q,n)}}{q!} \end{aligned} \quad (5.14)$$

Since the hypothesis holds good for all $\tilde{\mathbf{M}}^{(l,u)}$ such that $\{0 \leq l \leq m-1, 0 \leq u \leq n-1\}$, $\{0 \leq l \leq m, 0 \leq u \leq n-1\}$ and $\{0 \leq l \leq m-1, 0 \leq u \leq n\}$, $\mathbf{R}^{-1}\tilde{\mathbf{M}}^{(l,u)} = e_r$ where r corresponds to the location of $\mathbf{M}^{(l,u)}$ in $\mathbf{K}$. This reduces (5.14) to

$$\begin{aligned} \mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m,n)} = & - \sum_{r=1}^n \frac{1}{r!} \sum_{q=0}^m \frac{\mathbf{K}^T \mathbf{Y}^{(q,r)} \mathbf{M}^{(m-q,n-r)}}{q!} \\ & - \sum_{q=1}^m \frac{\mathbf{K}^T \mathbf{Y}^{(q,0)} \mathbf{M}^{(m-q,n)}}{q!} \end{aligned} \quad (5.15)$$

Using (5.1a), (5.15) can be re-written as

$$\mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m,n)} = -\mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{M}^{(m,n)} \quad (5.16)$$

Again, since $\mathbf{K}^T \mathbf{Y}(s_i, \lambda_j) \mathbf{K}$ is nonsingular, (5.16) has a unique solution. Hence $\mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m,n)} = e_r$ where r corresponds to the location of $\mathbf{M}^{(m,n)}$ in $\mathbf{K}$, represents the unique solution for (5.16). Next from (5.6) the $(m, n)^{th}$ moment of $\mathbf{X}(s, \lambda)$ is obtained as

$$\mathbf{Q} \tilde{\mathbf{M}}^{(m,n)} = \mathbf{K} \mathbf{R}^{-1} \tilde{\mathbf{M}}^{(m,n)} = \mathbf{K} e_r = \mathbf{M}^{(m,n)} \quad (5.17)$$

Hence the $(m, n)^{th}$ moment of the system with respect to s and λ is conserved if all the previous moments $\tilde{\mathbf{M}}^{(l,u)}$ where $\{0 \leq l \leq m-1, 0 \leq u \leq n-1\}$, $\{0 \leq l \leq m, 0 \leq u \leq n-1\}$ and $\{0 \leq l \leq m-1, 0 \leq u \leq n\}$ are conserved. Thus by *Mathematical Induction* we can conclude that p moments of $\mathbf{Q} \tilde{\mathbf{X}}(s, \lambda) = \mathbf{Q} \tilde{\mathbf{Y}}^{-1}(s, \lambda) \tilde{\mathbf{b}}$ are the same as p moments of $\mathbf{X}(s, \lambda) = \mathbf{Y}^{-1}(s, \lambda) \mathbf{b}$.

5.4 Results

5.4.1 Example 1

An interconnect circuit consisting of 460 resistors, inductors and capacitors and 12 transmission lines is considered for this example. The physical parameters of the transmission lines are shown in Table 5.1. The size of the MNA matrices of this circuit was 494×494 . The proposed algorithm was used to reduce this circuit as a function of frequency and the width of a particular transmission line. The reduced-order model formed, was of size 20×20 . Fig. 5.1 and Fig. 5.2 show a comparison of the responses obtained from the original and the reduced-order model as the width of the transmission line is varied from 2.6mm to 3.8mm. As can be seen from the plots, the responses match accurately. The error in the response of the reduced-order model was less than 1%. Only one L/U decomposition was required to obtain the 3-dimensional response using the reduced model. On the other hand, 650 L/U decompositions were required to obtain the same response from the original system. This led to a speedup of two orders of magnitude when compared to the original system of equations.

Physical Parameter	Value
Width of interconnect (w)	3.2mm
Thickness of interconnect	0.03mm
Substrate thickness (h)	1.2mm
ϵ_r of substrate	2.7

Table 5.1: Physical parameters of transmission lines

Figure 5.1: Frequency response as the width of the interconnect is varied from 2.6mm to 3.8mm

Figure 5.2: Output response as a function of frequency and width of transmission line

5.4.2 Example 2

A 10 transmission-line circuit consisting of 4380 resistors, inductors and capacitors is considered for the second example. The dimensions of the MNA matrices for this circuit were 2208×2208 . The proposed algorithm was used to reduce this circuit as a function of frequency and the width of a particular transmission line. Frequency domain moments, moments w.r.t the width of the transmission line and cross derivatives w.r.t frequency and transmission line width were used to form the reduced-order model. The reduced-order model so formed, was of size 35×35 . The reduced model was then used to find the response of the circuit simultaneously with variation along frequency and the width of the transmission line. A comparison of the frequency response at two nodes, as the width of the transmission line is varied from 2.6mm to 3.8mm is shown in Fig. 5.3 and Fig. 5.4. Fig. 5.5 and Fig. 5.6 show the 3-dimensional comparison of the responses obtained from the original and the reduced-order model for both nodes. Fig. 5.7 and Fig. 5.8 show the comparison of the contour plots obtained from the original system and the reduced model for both nodes. As seen, the responses obtained from the original system and the reduced-order model match accurately. The error in the response of the reduced-order model was less than 1%. Only one L/U decomposition was required to obtain the 3-dimensional response using the reduced model. On the other hand, 650 L/U decompositions were required to obtain the same response from the original system. This led to a speedup of two orders of magnitude when compared to the original system of equations.

Figure 5.3: Frequency response as the width of the interconnect is varied from 2.6mm to 3.8mm for node 4

Figure 5.4: Frequency response as the width of the interconnect is varied from 2.6mm to 3.8mm for node 5

Figure 5.5: Output response of the original system and the reduced model as a function of frequency and the width of transmission line for node 4

Figure 5.6: Output response of the original system and the reduced model as a function of frequency and the width of transmission line for node 5

Figure 5.7: Contour plot for the output response of the original system and the reduced model as a function of frequency and width of transmission line for node 4

Figure 5.8: Contour plot for the output response of the original system and the reduced model as a function of frequency and width of transmission line for node 5

5.4.3 Example 3

Another application of the theory presented in this paper is demonstrated using the following example. A network consisting of 60 small-signal models of transistors with an MNA size of 404×404 was considered for this example. The proposed algorithm was used to reduce this network as a function of frequency and the h_{fe} of the transistor. 14 frequency moments and 5 moments with respect to h_{fe} were considered to form the reduced model. A plot of the transfer function of the original network and the reduced model is shown with respect to frequency and h_{fe} in Fig. 5.9. Thus, the characteristics of the transfer function of a linearized network can be studied w.r.t a circuit parameter using the proposed algorithm.

Figure 5.9: Linearized transistor network

Chapter 6

Nonlinear Circuit-Reduction of High-Speed Interconnect Networks

6.1 Introduction

Several attempts can be found in the literature to solve the interconnect problem by partitioning the network into linear and nonlinear parts [24], [27]–[30]. The linear network is reduced using moment matching techniques to form an n -port macromodel. This macromodel is then coupled with the nonlinear part of the network to obtain the circuit response. Most of these reduction techniques are based on projecting the linear system into a subspace of lower dimension. However, the efficiency of these methods drop significantly when the number of ports of the linear subnetwork increase as a result of having more nonlinear terminations. To overcome this disadvantage, a new method is proposed which directly reduces the whole network containing both the interconnects and nonlinear terminations.

In this chapter, an algorithm [70]–[73] is described for projecting the full *nonlinear* circuit equations into a Krylov subspace of lower dimension. As a result, the original set of nonlinear time-domain equations is replaced by a smaller set of nonlinear equations such that the first ‘q’ derivatives of the time-response of the original system and the response obtained using the reduced system are identical. The reduced system can be solved using any of the conventional numerical integration techniques, resulting in significant reduction in the CPU time due to the fact that at each time point, a much smaller set of nonlinear algebraic equations need to be solved.

This chapter is organized as follows. Section 6.2 formulates the problem and section 6.3 presents a systematic approach to form the Krylov subspace and derive the reduced-order nonlinear circuit. Section 6.4 presents a proof that the proposed circuit-reduction algorithm preserves the first ‘q’ derivatives of the time-domain response. Section 6.5 explains the refreshing strategy of the algorithm and section 6.6 discusses computational issues involved in implementing the proposed algorithm. Finally numerical results are presented in section 6.7.

6.2 Problem Formulation

6.2.1 Formulation of Network Equations

The MNA formulation which will be presented in this section is very similar to that presented in section 2.4.1, however it is repeated again for the sake of clarity and completeness. Consider a general network containing an arbitrary number of lumped components and interconnects, terminated with nonlinear components. The MNA

formulation of such a network can be written as [30], [40]

$$\mathbf{C}_\phi \dot{\mathbf{x}}_\phi + \mathbf{G}_\phi \mathbf{x}_\phi + \mathbf{F}_\phi(\mathbf{x}_\phi) + \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{i}_k(t) = \mathbf{b}_\phi(t) \quad (6.1)$$

where,

- $\mathbf{x}_\phi \in \mathbb{R}^{N_\phi}$ is the vector of node voltage waveforms appended by independent voltage source currents, linear inductor currents, nonlinear capacitor charges, nonlinear inductor flux waveforms and currents and voltages due to nonlinear components,
- $\mathbf{C}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ and $\mathbf{G}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ are constant matrices describing the lumped memory and memoryless elements of the network, respectively,
- $\mathbf{b}_\phi(t) \in \mathbb{R}^{N_\phi}$ is a vector with entries determined by the independent voltage/current sources,
- $\mathbf{F}_\phi(\mathbf{x}_\phi)$ is a function describing the nonlinear elements of the circuit,
- N_ϕ is the total number of variables in the MNA formulation,
- $\mathbf{D}_k = [d_{i,j}]$ with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, n\}$ with a maximum of one non-zero in each row or column, is a selector matrix that maps $\mathbf{i}_k(t) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^n$ of the network,
- N_t is the total number of distributed coupled interconnect sets and n_k is the number of coupled conductors in the linear subnetwork k .

The first and second terms in (6.1) cover the network's lumped components, the third term represents the nonlinear components and the fourth term describes currents at each interconnect subnetwork terminal and then maps them into the rest of the network through matrix $\mathbf{D}_k$.

The frequency-domain equations of interconnect subnetwork ' k ' containing distributed interconnects can be written in the form

$$\mathbf{I}_k(s) = \mathbf{Y}_k(s)\mathbf{V}_k(s) = \mathbf{Y}_k(s)\mathbf{D}_k^T\mathbf{X}_\phi(s) \quad (6.2)$$

where $\mathbf{V}_k(s)$, $\mathbf{I}_k(s)$ and $\mathbf{Y}_k(s)$ represent the Laplace-domain terminal voltages and currents and the admittance parameters for the subnetwork k respectively. $\mathbf{X}_\phi(s)$ is the Laplace transform of $\mathbf{x}_\phi(t)$.

The distributed interconnect subnetwork described in (6.2) can be discretized using an appropriate discretization method [32], [74], [75] (see section 2.2.1). The vector of unknowns $\mathbf{x}_\phi(t)$ is expanded to $\mathbf{x}(t)$ to accommodate the new variables introduced due to discretization. Similarly, the matrices and vectors, $\mathbf{C}_\phi$, $\mathbf{G}_\phi$, $\mathbf{F}_\phi$ and $\mathbf{b}_\phi$ are expanded to form $\mathbf{C}$, $\mathbf{G}$, $\mathbf{F}$ and $\mathbf{b}$. The MNA equations of the network can now be written as,

$$\mathbf{C}\dot{\mathbf{x}} + \mathbf{G}\mathbf{x} + \mathbf{F}(\mathbf{x}) = \mathbf{b}(t) \quad (6.3)$$

where, $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^N$, $\mathbf{b}(t) \in \mathbb{R}^N$, $\mathbf{C} \in \mathbb{R}^{N \times N}$, $\mathbf{G} \in \mathbb{R}^{N \times N}$ and $N > N_\phi$.

6.2.2 Transient Analysis using Conventional Techniques

The time-domain solution to the network can be obtained by solving the MNA equations in (6.3) using backward difference formulae and newton iterations. The back-

ward difference formulae include methods such as Backward Euler, Gear's Method etc. Using these techniques, (6.3) is transformed to a set of nonlinear algebraic equations in the form

$$\mathbf{A}(h, \mathbf{G}, \mathbf{C})\mathbf{x}_{k+1} + \mathbf{F}(\mathbf{x}_{k+1}) - \mathbf{K}(\mathbf{x}_k) = 0 \quad (6.4)$$

where $\mathbf{A}(h, \mathbf{G}, \mathbf{C})$ and $\mathbf{K}(\mathbf{x}_k)$ are functions of $h, \mathbf{G}, \mathbf{C}, \mathbf{x}_k$ and are dependent on the chosen backward difference formula, h is the step size used in the simulation and $\mathbf{x}_i$ is the vector of unknowns at the i^{th} time point. The solution is obtained by performing Newton iterations on (6.3). These Newton iterations in general require the computation and subsequent L/U decomposition of the Jacobian matrix $\mathbf{M} = \mathbf{A}(h, \mathbf{G}, \mathbf{C}) + \partial\mathbf{F}/\partial\mathbf{x}$. It is to be noted that the size of the matrix $\mathbf{M}$ would be extremely large due to the presence of discretized interconnect subnetworks. Performing L/U decompositions on $\mathbf{M}$ repeatedly at each time point to obtain the solution can prove to be computationally unviable. In the next section, we propose an algorithm to overcome this problem. The proposed method is based on projecting the original system of nonlinear equations into a subspace of lower dimension. The solution of the reduced-circuit would be relatively inexpensive to compute when compared to the original circuit owing to smaller dimensions of circuit matrices. The solution obtained in the reduced-order space is then mapped to that of the original system.

6.3 Circuit-Reduction of Interconnect Circuits

The main idea of the new algorithm presented in this chapter is centered around the replacement of the original nonlinear circuit equations by a smaller set of nonlinear equations such that the first 'q' derivatives of the time-response of the original system

and the response obtained from the reduced system are identical.

The main steps involved in the new algorithm are as follows:

1. Formulation of the modified nodal analysis (MNA) circuit equations including the nonlinear components,
2. Computation of the time derivatives of the system,
3. Formulation of Krylov subspace using the derivatives computed from step 2 and obtaining the modified congruent transformation matrix ($\mathbf{Q}$) using orthonormalization [24], [51] and
4. Model-reduction on the entire network using congruent transformation. Details of the algorithm are discussed briefly in the following sections.

6.3.1 Computation of Time Domain Derivatives of the Original System

$\mathbf{x}(t)$ in (6.3) is expanded in a Taylor series as

$$\mathbf{x}(t) = \sum_{k=0} \mathbf{a}_k (t - t_0)^k \quad (6.5)$$

where $\mathbf{a}_k = \mathbf{x}^{(k)}(t_0)/k!$, $k = 0, 1, 2, \dots$ are the normalized time domain derivatives of $\mathbf{x}$ and are computed using

$$(k+1)\mathbf{C}\mathbf{a}_{k+1} + \mathbf{G}\mathbf{a}_k + \mathbf{f}_k = \mathbf{b}_k \quad (6.6)$$

where $\mathbf{b}_i$ and $\mathbf{f}_i$ denote the i^{th} derivatives of $\mathbf{b}(t)$ and $\mathbf{F}(\mathbf{x})$ evaluated at $t = t_0$ respectively and $\mathbf{a}_0$ is assigned the initial conditions $\mathbf{x}_0$. Assuming for the moment that the matrix $\mathbf{C}$ is invertible, the coefficients $\mathbf{a}_k, k = 1, 2, \dots$ can be computed recursively using (6.6). The first derivative of $\mathbf{F}(\mathbf{x})$ is computed as

$$\frac{d}{dt}\mathbf{F}(\mathbf{x}) = \mathbf{J} \left(\sum_{k=0} \mathbf{a}_k t^k \right) \sum_{k=1} k \mathbf{a}_k t^{k-1} = \zeta(t) \sum_{k=1} k \mathbf{a}_k t^{k-1} \quad (6.7)$$

where $\mathbf{J}(\mathbf{x}) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$.

Using (6.7) and for simplicity assigning $t_0 = 0$, the m^{th} derivative of $\mathbf{F}(\mathbf{x})$ at $t = 0$ can be computed as

$$\left. \frac{d^m}{dt^m} \mathbf{F}(\mathbf{x}) \right|_{t=0} = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-j-1)!} (j+1) [\zeta^{m-j-1}(0)] \mathbf{a}_{j+1} = \mathbf{f}_m \quad (6.8)$$

As an example of the calculation of derivatives, consider

$$F(v_d) = I_s (\exp(v_d/V_T) - 1) \quad (6.9)$$

where $v_d = \sum_{i=0} c_i t^i$ and $F(v_d) = \sum_{i=0} d_i t^i$. Expansion of the exponential gives

$$d_0 = I_s (\exp(c_0/V_T) - 1) \quad (6.10)$$

$$d_i = \frac{1}{iV_T} \sum_{j=0}^{i-1} d_j c_{i-j} (i-j) + \frac{I_s}{V_T} c_i \quad (6.11)$$

Similar expansions can be found for other functions as shown in Table 6.1 [76].

Equation	Terms $x = \sum c_i, y = \sum e_i, f = \sum d_i$
$f = x + y$	$d_i = c_i + e_i$
$f = x - y$	$d_i = c_i - e_i$
$f = x * y$	$d_i = \sum_{j=0}^i c_j e_{i-j}$
$f = x/y$	$d_i = 1/e_0 \left(c_i - \sum_{j=0}^{i-1} d_j e_{i-j} \right)$
$f = \exp(x)$	$d_0 = \exp(c_0)$ $d_i = 1/i \sum_{j=0}^{i-1} d_j c_{i-j} (i-j)$
$f = \log(x)$	$d_0 = \log(c_0)$ $d_i = \frac{1}{c_0} \left(c_i - \sum_{j=1}^{i-1} c_j d_{i-j} \frac{i-j}{i} \right)$
$f = x^p$	$d_0 = c_0^p$ $d_i = \frac{1}{c_0^i} \left(p d_0 c_i i + \sum_{j=1}^{i-1} d_j c_{i-j} (p i - j(p+1)) \right)$
$f = \sin(x)$ $y = \cos(x)$	$d_0 = \sin(c_0), e_0 = \cos(c_0)$ $d_i = \frac{1}{i} \sum_{j=0}^{i-1} e_{(i-1-j)} c_j(j)$ $e_i = \frac{-1}{i} \sum_{j=0}^{i-1} d_{(i-1-j)} c_j(j)$

Table 6.1: Recursive formulae for the derivatives of simple functions.

Using (6.8), (6.6) becomes

$$(k+1)\mathbf{C}\mathbf{a}_{k+1} + \mathbf{G}\mathbf{a}_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!} [\zeta^{k-j-1}(0)] \mathbf{a}_{j+1} = \mathbf{b}_k \quad (6.12)$$

In the case where $\mathbf{C}$ is not invertible, (6.6) may be separated into

$$(k+1) \begin{pmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{k+1} \\ \mathbf{w}_{k+1} \end{pmatrix} + \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{u}_k \\ \mathbf{w}_k \end{pmatrix} + \begin{pmatrix} \mathbf{F}(\mathbf{u}_k, \mathbf{w}_k) \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{u,k} \\ \mathbf{b}_{w,k} \end{pmatrix} \quad (6.13)$$

where $\mathbf{a}_k$ has been divided into the two disjoint differential and algebraic sets $\mathbf{u}_k$ and $\mathbf{w}_k$ such that there are no empty rows or columns in $\mathbf{C}_{11}$. The first row of (6.13) is an

explicit equation for $\mathbf{u}_{k+1}$ in terms of $\mathbf{u}_k$ and $\mathbf{w}_k$ requiring $\mathbf{C}_{11}$ to be invertible. The second row of (6.13) is an explicit equation for $\mathbf{w}_k$ in terms of $\mathbf{u}_k$ and $\mathbf{b}_k$ requiring $\mathbf{G}_{22}$ to be invertible. The required invertible conditions are usually met by the modified nodal analysis formulation used to develop circuit equations. Special formulation methods can be used to ensure that $\mathbf{C}_{11}$ and $\mathbf{G}_{22}$ are invertible without the need for state space formulation [45].

6.3.2 Reduced Circuit via Congruent Transformation

The original system (6.3) is reduced to a smaller set of unknowns through a congruent transformation obtained from the modified Krylov subspace $\mathbf{K}$. This subspace, formed by the derivatives computed in (6.12) is defined as [71]

$$\mathbf{K} = \begin{pmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \dots & \mathbf{a}_q \end{pmatrix} \quad (6.14)$$

where q is the order of reduction and $q \ll N_\phi$. Performing an orthogonal decomposition [24], [51] on $\mathbf{K}$ we have

$$\mathbf{K} = \mathbf{Q}\mathbf{R} \quad (6.15)$$

where $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_q$. Using the matrix $\mathbf{Q}$ obtained from (6.15) we perform a congruent transformation on the original system (6.3) given by

$$\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}} \quad (6.16)$$

where $\hat{\mathbf{x}} \in \mathbb{R}^q$. This change of variable reduces the original system (6.3) to a system with a smaller set of unknowns, given by

$$\hat{\mathbf{C}}\dot{\hat{\mathbf{x}}} + \hat{\mathbf{G}}\hat{\mathbf{x}} + \hat{\mathbf{F}}(\hat{\mathbf{x}}) = \hat{\mathbf{b}}(t) \quad (6.17)$$

where

$$\hat{\mathbf{G}} = \mathbf{Q}^T \mathbf{G} \mathbf{Q}; \quad \hat{\mathbf{C}} = \mathbf{Q}^T \mathbf{C} \mathbf{Q}; \quad \hat{\mathbf{F}}(\hat{\mathbf{x}}) = \mathbf{Q}^T \mathbf{F}(\mathbf{Q}\hat{\mathbf{x}}); \quad \hat{\mathbf{b}}(t) = \mathbf{Q}^T \mathbf{b}(t) \quad (6.18)$$

The reduced set of equations (6.17) can be solved using any of the conventional numerical integration techniques [45] to get $\hat{\mathbf{x}}$. The solution for the original system (6.3) is obtained using the transformation given in (6.16). It is to be noted that the computational cost involved in solving (6.17) is drastically reduced when compared to (6.3) as the order of the reduced model is significantly less than the order of the original system.

6.4 Proof of Preservation of Derivatives

Proof that the reduced-order system (6.17) preserves the first q derivatives of the original system (6.3) is given by Mathematical Induction. First it shall be proved that the first derivative obtained from the reduced-order model is equivalent to that obtained from the original system (6.3). Next, we will show that the $(k + 1)^{th}$ derivative is conserved if the previous k derivatives are conserved.

$\hat{\mathbf{x}}(t)$ in (6.17) is expanded in a Taylor series as

$$\hat{\mathbf{x}} = \sum_{k=0} \hat{\mathbf{a}}_k t^k \quad (6.19)$$

The coefficients $\hat{\mathbf{a}}_k$ are computed using the recursive relationship

$$(k+1)\hat{\mathbf{C}}\hat{\mathbf{a}}_{k+1} + \hat{\mathbf{G}}\hat{\mathbf{a}}_k + \hat{\mathbf{f}}_k = \hat{\mathbf{b}}_k \quad (6.20)$$

where $\hat{\mathbf{a}}_0$ represents the initial conditions of the reduced system and is chosen such that it satisfies

$$\mathbf{Q}\hat{\mathbf{a}}_0 = \mathbf{a}_0 \quad (6.21)$$

From (6.20), it is seen that $\hat{\mathbf{a}}_1$ is given by

$$\hat{\mathbf{C}}\hat{\mathbf{a}}_1 + \hat{\mathbf{G}}\hat{\mathbf{a}}_0 + \hat{\mathbf{F}}(\mathbf{Q}\hat{\mathbf{a}}_0) = \hat{\mathbf{b}}_0 \quad (6.22)$$

i.e.,

$$\mathbf{Q}^T \mathbf{C} \mathbf{Q} \hat{\mathbf{a}}_1 + \mathbf{Q}^T \mathbf{G} \mathbf{Q} \hat{\mathbf{a}}_0 + \mathbf{Q}^T \mathbf{F}(\mathbf{Q} \hat{\mathbf{a}}_0) = \mathbf{Q}^T \mathbf{b}_0 \quad (6.23)$$

Using (6.15) and (6.21), (6.23) can be written as

$$\mathbf{K}^T (\mathbf{C} [\mathbf{a}_0 \ \mathbf{a}_1 \ \dots \ \mathbf{a}_q] \mathbf{R}^{-1} \hat{\mathbf{a}}_1 + \mathbf{G} \mathbf{a}_0 + \mathbf{F}(\mathbf{a}_0) - \mathbf{b}_0) = 0 \quad (6.24)$$

Substituting $\mathbf{R}^{-1} \hat{\mathbf{a}}_1 = \mathbf{e}_2$, where $\mathbf{e}_r$ is the r^{th} column of the identity matrix $\mathbf{I}_q \in \mathbb{R}^{q \times q}$ and simplifying (6.24) using (6.6), we have

$$\mathbf{K}^T (\mathbf{C} \hat{\mathbf{a}}_1 + \mathbf{G} \mathbf{a}_0 + \mathbf{F}(\mathbf{a}_0) - \mathbf{b}_0) = 0 \quad (6.25)$$

Hence, $\mathbf{R}^{-1}\hat{\mathbf{a}}_1 = \mathbf{e}_2$ is a solution of (6.24). Therefore the first derivative of the original system obtained from the reduced system is

$$\mathbf{Q}\hat{\mathbf{a}}_1 = \mathbf{K}\mathbf{R}^{-1}\hat{\mathbf{a}}_1 = [\mathbf{a}_0 \ \mathbf{a}_1 \ \dots \ \mathbf{a}_q]\mathbf{e}_2 = \mathbf{a}_1 \quad (6.26)$$

From (6.26) it can be seen that the first derivative of the system is conserved.

We now proceed to prove that if the hypothesis holds good for all l from $l = 0$ to $l = k$, i.e. $\mathbf{Q}\hat{\mathbf{a}}_l = \mathbf{a}_l$ for $0 \leq l \leq k$, then it also holds good for $l = k + 1$, i.e. $\mathbf{Q}\hat{\mathbf{a}}_{k+1} = \mathbf{a}_{k+1}$.

The first derivative of $\hat{\mathbf{F}}(\hat{\mathbf{x}})$ is computed as

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{F}}(\hat{\mathbf{x}}) &= \frac{\partial}{\partial \mathbf{x}}(\mathbf{Q}^T \mathbf{F}(\mathbf{Q}\hat{\mathbf{x}})) \frac{\partial}{\partial t} \mathbf{Q}\hat{\mathbf{x}} = \mathbf{Q}^T \frac{\partial}{\partial \mathbf{x}}(\mathbf{F}(\mathbf{x})) \mathbf{Q} \frac{\partial}{\partial t} \hat{\mathbf{x}} \\ &= \mathbf{Q}^T \zeta(t) \mathbf{Q} \sum_{k=1} k \hat{\mathbf{a}}_k t^{k-1} \quad (6.27) \end{aligned}$$

Using (6.27), the m^{th} derivative of $\hat{\mathbf{F}}(\hat{\mathbf{x}})$ at $t = 0$ is computed to be

$$\left. \frac{d^m}{dt^m} \hat{\mathbf{F}}(\hat{\mathbf{x}}) \right|_{t=0} = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-j-1)!} (j+1) \mathbf{Q}^T [\zeta^{m-j-1}(0)] \mathbf{Q} \hat{\mathbf{a}}_{j+1} = \hat{\mathbf{f}}_m \quad (6.28)$$

Using (6.28), (6.20) becomes

$$(k-1) \mathbf{C} \hat{\mathbf{a}}_{k-1} + \mathbf{G} \hat{\mathbf{a}}_k + \sum_{j=0}^{k-1} \frac{(j+1)!}{(k-j-1)!k!} \mathbf{Q}^T [\zeta^{k-j-1}(0)] \mathbf{Q} \hat{\mathbf{a}}_{j+1} = \hat{\mathbf{b}}_k; \quad k \geq 1 \quad (6.29)$$

Using (6.15) and (6.18), (6.29) can be written as

$$\mathbf{Q}^T \left((k+1)\mathbf{C}[a_0 \ a_1 \ \dots \ a_q]\mathbf{R}^{-1}\hat{\mathbf{a}}_{k+1} + \mathbf{G}\mathbf{Q}\hat{\mathbf{a}}_k + \sum_{j=0}^{k-1} \frac{(j+1)!}{(k-j-1)!k!} [\zeta^{k-j-1}(0)]\mathbf{Q}\hat{\mathbf{a}}_{j+1} - \mathbf{b}_k \right) = 0 \quad (6.30)$$

Since $\mathbf{Q}\hat{\mathbf{a}}_l = \mathbf{a}_l$ for $0 \leq l \leq k$, (6.30) is reduced to

$$\mathbf{K}^T \left((k+1)\mathbf{C}[a_0 \ a_1 \ \dots \ a_q]\mathbf{R}^{-1}\hat{\mathbf{a}}_{k+1} + \mathbf{G}\mathbf{a}_k + \sum_{j=0}^{k-1} \frac{(j+1)!}{(k-j-1)!k!} [\zeta^{k-j-1}(0)]\mathbf{a}_{j+1} - \mathbf{b}_k \right) = 0 \quad (6.31)$$

Substituting $\mathbf{R}^{-1}\hat{\mathbf{a}}_{k+1} = \mathbf{e}_{k+2}$ in (6.31), and simplifying using (6.12) we have

$$\mathbf{K}^T \left((k+1)\mathbf{C}\mathbf{a}_{k+1} + \mathbf{G}\mathbf{a}_k + \sum_{j=0}^{k-1} \frac{(j+1)!}{(k-j-1)!k!} [\zeta^{k-j-1}(0)]\mathbf{a}_{j+1} - \mathbf{b}_k \right) = 0 \quad (6.32)$$

Hence $\mathbf{R}^{-1}\hat{\mathbf{a}}_{k+1} = \mathbf{e}_{k+2}$ is a solution of (6.30). Therefore the $(k+1)^{th}$ derivative of the original system obtained from the reduced system is

$$\mathbf{Q}\hat{\mathbf{a}}_{k+1} = \mathbf{K}\mathbf{R}^{-1}\hat{\mathbf{a}}_{k+1} = [a_0 \ a_1 \ \dots \ a_q]\mathbf{e}_{k+2} = \mathbf{a}_{k+1} \quad (6.33)$$

From (6.33) it can be seen that the $(k+1)^{th}$ derivative of the system is conserved if all its previous derivatives are conserved. Thus by *Mathematical Induction* we can conclude that the first q derivatives are conserved for the system.

6.5 Refreshing

The derivatives calculated at a time point will be accurate in the vicinity of the expansion point. This necessitates the refreshment of the algorithm at intermediate expansion points. An outline of the refreshing strategy followed in this algorithm is described below.

The original circuit is initially reduced at $t = 0s$ to form a reduced circuit. The response of the reduced circuit is computed using a suitable integration method on (6.17). The response ($\hat{x}$) obtained from the reduced circuit is used to compute the derivative of the response ($\dot{x}$) as

$$\dot{\hat{x}} = \hat{C}^{-1}(\hat{b}(t) - \hat{G}\hat{x} - \hat{F}(\hat{x})) \quad (6.34)$$

The response and derivative of the reduced circuit are then mapped to the original system space at each time point using the congruent transformation to obtain the response (x) and derivative ($\dot{x}$) of the original circuit. These values are then plugged into the original system equations (6.3) to verify the solution obtained from the reduced circuit. An error criterion (δ) at any time-point t_n can be obtained by measuring the variance of the solution in satisfying the original system equations as

$$\delta = ||C\dot{x}(t_n) + Gx(t_n) + F(x(t_n)) - b(t_n)|| \quad (6.35)$$

It is to be noted that the error criterion δ is actually the norm of the residual error in satisfying the KCL equations. If $\delta \geq \epsilon$, where ϵ is the required tolerance, then the reduced circuit is refreshed by performing another reduction at $t = t_n$. The

above process is repeated until the time-domain response is obtained up to the desired time-point of interest.

6.6 Computational Considerations

The main portion of computational expense to calculate the response of the original circuit arises from the L/U decompositions of the Jacobian matrix in (6.4). However, since the matrices of the reduced-order circuit in (6.17) are relatively small ($q \ll N$), the expense to compute the L/U decomposition of the corresponding Jacobian matrix is significantly less when compared to that of the original system. This difference in expense converts into a saving in the computational time required to find the response of the circuit through the reduced-order circuit.

A practical implementation of the proposed algorithm would require the computation of the nonlinear part of the Jacobian of the reduced-order circuit, $\hat{\mathbf{J}}(\hat{\mathbf{x}}) = \mathbf{Q}^T \mathbf{J} \mathbf{Q}$, n_i times at each time point, where n_i is the number of Newton iterations required for convergence at that particular time point. Note that $\mathbf{J}$ is the nonlinear part of the Jacobian of the original circuit. In the case when the matrix $\mathbf{J}$ is dense (i.e. the original system has a relatively large number of nonlinear components), the number of multiplications required for this operation would be $N^2 q + N q^2$ which makes it to be of $O(N^2)$ (where N is the size of the original system (6.3) and q is the size of the reduced-circuit). This computation can prove to be expensive and would consequently reduce the usefulness of the circuit-reduction algorithm. However, when the original system is a large interconnect network with nonlinear terminations, the matrix $\mathbf{J}$ would be extremely sparse. If the original network has n nonlinear equa-

tions, this operation would require a maximum of $2nq + nq^2$ multiplications (assuming that on the average, each nonlinear equation depends on voltages at two independent nodes of the circuit). It is to be noted that in this case, the number of multiplications is independent of the size N of the original system. Thus, being relatively inexpensive, the computational expense of the Jacobian for the reduced-order circuit becomes inconsequential.

These factors discussed above together greatly decrease the computational complexity involved in solving large interconnect networks with nonlinear terminations. The examples shown in the following section demonstrate the computational efficiency of the proposed algorithm.

6.7 Numerical Results

6.7.1 Example 1

A distributed interconnect network with nine coupled lines (Fig. 6.1) is considered for this example [72], [77]. The network was discretized using lumped segmentation. The size of the MNA matrices was 1800×1800 . The proposed algorithm was used to find the reduced-order circuit of the original system by considering the first 20 time-derivatives to form the modified Krylov subspace. The response of the original system was computed from the reduced-circuit by refreshing at suitable time points using a relative tolerance of 10^{-2} and an absolute tolerance of 10^{-5} . The responses computed from the original system and the reduced circuit are shown in Fig. 6.2–6.4. The CPU times required to compute these responses are shown in Table 6.2. As we see a speed up of 15.7 is achieved using the proposed algorithm.

(a) Interconnect network

$$L = \begin{pmatrix} 5.04 & 0.182 & 0.315 & 0.057 & 0.315 & 0.182 & 0.315 & 0.057 & 0.315 \\ 0.182 & 5.04 & 0.315 & 0.182 & 0 & 0 & 0 & 0 & 0.315 \\ 0.315 & 0.315 & 5.04 & 0.315 & 0.182 & 0 & 0.057 & 0 & 0.182 \\ 0.057 & 0.182 & 0.315 & 5.04 & 0 & 0 & 0 & 0 & 0 \\ 0.315 & 0 & 0.182 & 0 & 5.04 & 0.315 & 0.182 & 0 & 0 \\ 0.182 & 0 & 0 & 0 & 0.315 & 5.04 & 0.315 & 0.182 & 0 \\ 0.315 & 0 & 0.057 & 0 & 0.182 & 0.315 & 5.04 & 0.315 & 0.182 \\ 0.057 & 0 & 0 & 0 & 0 & 0.182 & 0.315 & 5.04 & 0 \\ 0.315 & 0.315 & 0.182 & 0 & 0 & 0 & 0.182 & 0 & 5.04 \end{pmatrix} nH/cm$$

$$C = \begin{pmatrix} 187.6 & -0.385 & -2.23 & -0.0046 & -2.23 & -0.385 & -2.23 & -0.0046 & -2.23 \\ -0.385 & 187.6 & -2.23 & -0.385 & 0 & -0.385 & -2.23 & -0.0046 & -2.23 \\ -2.23 & -2.23 & 187.6 & -2.23 & -0.385 & 0 & -0.0046 & 0 & -0.385 \\ -0.0046 & -0.385 & -2.23 & 187.6 & 0 & 0 & 0 & 0 & 0 \\ -2.23 & 0 & -0.385 & 0 & 187.6 & -2.23 & -0.385 & 0 & 0 \\ -0.385 & 0 & 0 & 0 & -2.23 & 187.6 & -2.23 & -0.385 & 0 \\ -2.23 & 0 & -0.0046 & 0 & -0.385 & -2.23 & 187.6 & -2.23 & -0.385 \\ -0.0046 & 0 & 0 & 0 & 0 & -0.385 & -2.23 & 187.6 & 0 \\ -2.23 & -2.23 & -0.385 & 0 & 0 & 0 & -0.385 & 0 & 187.6 \end{pmatrix} pF/m$$

(b) Per-unit-length parameters

Figure 6.1: Coupled interconnect system

	Size	Time (secs)	Speed-up
Original Circuit	1800 × 1800	336	—
Reduced Circuit	20 × 20	21.4	15.7

Table 6.2: CPU comparison for linear circuit

Figure 6.2: Comparison of the responses obtained from the original and reduced circuits at the far end of the signal line (Vout1)

Figure 6.3: Comparison of the responses obtained from the original and reduced circuits at the far end of the victim line #2 (Vout2)

Figure 6.4: Comparison of the responses obtained from the original and reduced circuits at the near end of the victim line #6 (Vout3)

6.7.2 Example 2

For this example the distributed transmission line set is terminated with nonlinear components as shown in Fig. 6.5. The network was again discretized using lumped segmentation. The size of the MNA matrices was 1802×1802 . The proposed algorithm was used to find the reduced-order circuit of the original system by considering the first 20 time-derivatives to form the modified Krylov subspace. The response of the original system was computed from the reduced-circuit. It is to be noted that the nonlinear part of the Jacobian of the original system is extremely sparse and hence the computational cost of obtaining a projection of this matrix in the subspace of the reduced-order circuit is trivial. The reduced circuit was simulated with a relative tolerance of 10^{-2} and an absolute tolerance of 10^{-5} . The responses computed from the original system and the reduced circuit are shown in Fig. 6.6 and Fig. 6.7. The

CPU times required to compute these responses are shown in Table 6.3. As we see again a speed up of 12.3 is achieved using the proposed algorithm.

(a) Nonlinear interconnect network

$$\mathbf{L} = \begin{pmatrix} 5.04 & 0.182 & 0.315 & 0.057 & 0.315 & 0.182 & 0.315 & 0.057 & 0.315 \\ 0.182 & 5.04 & 0.315 & 0.182 & 0 & 0 & 0 & 0 & 0.315 \\ 0.315 & 0.315 & 5.04 & 0.315 & 0.182 & 0 & 0.057 & 0 & 0.182 \\ 0.057 & 0.182 & 0.315 & 5.04 & 0 & 0 & 0 & 0 & 0 \\ 0.315 & 0 & 0.182 & 0 & 5.04 & 0.315 & 0.182 & 0 & 0 \\ 0.182 & 0 & 0 & 0 & 0.315 & 5.04 & 0.315 & 0.182 & 0 \\ 0.315 & 0 & 0.057 & 0 & 0.182 & 0.315 & 5.04 & 0.315 & 0.182 \\ 0.057 & 0 & 0 & 0 & 0 & 0.182 & 0.315 & 5.04 & 0 \\ 0.315 & 0.315 & 0.182 & 0 & 0 & 0 & 0.182 & 0 & 5.04 \end{pmatrix} nH/cm$$

$$\mathbf{C} = \begin{pmatrix} 187.6 & -0.385 & -2.23 & -0.0046 & -2.23 & -0.385 & -2.23 & -0.0046 & -2.23 \\ -0.385 & 187.6 & -2.23 & -0.385 & 0 & 0 & 0 & 0 & -2.23 \\ -2.23 & -2.23 & 187.6 & -2.23 & -0.385 & 0 & -0.0046 & 0 & -0.385 \\ -0.0046 & -0.385 & -2.23 & 187.6 & 0 & 0 & 0 & 0 & 0 \\ -2.23 & 0 & -0.385 & 0 & 187.6 & -2.23 & -0.385 & 0 & 0 \\ -0.385 & 0 & 0 & 0 & -2.23 & 187.6 & -2.23 & -0.385 & 0 \\ -2.23 & 0 & -0.0046 & 0 & -0.385 & -2.23 & 187.6 & -2.23 & -0.385 \\ -0.0046 & 0 & 0 & 0 & 0 & -0.385 & -2.23 & 187.6 & 0 \\ -2.23 & -2.23 & -0.385 & 0 & 0 & 0 & -0.385 & 0 & 187.6 \end{pmatrix} pF/m$$

(b) Per-unit-length parameters

Figure 6.5: Coupled interconnect system with nonlinear terminations

Figure 6.6: Comparison of the responses obtained from the original and reduced circuits at Vout1

Figure 6.7: Comparison of the responses obtained from the original and reduced circuits at the far end of the signal line (Vout2)

	Size	Time (secs)	Speed-up
Original Circuit	1802×1802	340	—
Reduced Circuit	20×20	27.6	12.3

Table 6.3: CPU comparison for nonlinear circuit

6.7.3 Example 3

A distributed interconnect network with three sets of nine coupled lines and six sets of two coupled lines is considered for this example. The network was discretized using the matrix-rational Padé approximation [75]. The size of the MNA matrices was 1288×1288 . A uniform lumped segmentation of the same network resulted in MNA matrices of size 9933×9933 . The network which was discretized using the matrix-rational Padé approximation was macromodelled using PRIMA and the pole selection technique presented in [73]. The Y11 and Y12 parameters of the interconnect network and the macromodel are shown in Figs. 6.8 and 6.9 respectively. A comparison of the CPU time required to perform time-domain simulation using the macromodel obtained using each method is shown in Table 6.4. Performance of the macromodelling techniques decreased with an increase in the number of ports. For instance, when the number of ports was increased to seven PRIMA needed 69s to simulate the time domain response. In addition, the overhead cost involved in forming the macromodel was found to be around 87s.

Next, both the discretized networks, were simulated using the time-domain circuit reduction technique presented in this chapter. A comparison of the CPU time taken for the above simulations is presented in Table 6.5. It is to be noted that the simulation time required for the time-domain circuit reduction technique is independent of

the number of ports in the network. A comparison of the transient response obtained from the original and the reduced circuits is shown in Fig. 6.10 and Fig. 6.11.

From the above example it can be seen that if we are interested in forming a *macromodel* of the interconnect network in order to use it *repeatedly* in different configurations, then using PRIMA coupled with the pole selection technique is efficient. Whereas, if we are only interested in the transient solution of the interconnect network, then using the Matrix Padé approximation to discretize the network and using the nonlinear time-domain reduction technique to find the transient response is more CPU efficient.

Table 6.4: Macromodel simulation (excluding overhead of around 70s)

	PRIMA	PRIMA + Pole selection
Size	420×420	235×235
CPU time (excluding overhead)	5.2s	1.7s
Simulation Speed-up	1	3

Table 6.5: CPU comparison for transient analysis

	Uniform Discretization (UD)	Matrix Pad� Approxima- tion (MPA)	UD + NCR	MPA + NCR
Size	9933×9933	1288×1288	15×15	15×15
CPU time	310.3s	61.4s	63.2s	11.6s
Speed-up	1	5.0	4.9	26.7

Figure 6.8: $|Y_{11}|$ of the network

Figure 6.9: $|Y_{12}|$ of the network

Figure 6.10: Transient response at the far end of the signal line

Figure 6.11: Transient response at the far end of the victim line

Chapter 7

Conclusions and Future Work

This chapter contains a summary of the work presented in this thesis. In addition, the direction of future work is also discussed.

7.1 Summary

This thesis presents solutions to problems related to simulation of high-speed interconnect networks. In this work, several methods were presented that enhance the accuracy and efficiency of simulation of large interconnect networks. First, a new technique was presented that enabled the inclusion of distributed transmission line networks described by Telegrapher's equations directly using a congruent transformation based analysis. This method takes advantage of congruent transformation techniques while including the interconnects as *distributed* transmission lines. Next, an efficient sensitivity analysis algorithm for transmission line networks was developed based on congruent transformation techniques. This method also includes the interconnects as distributed transmission lines described by Telegrapher's partial differen-

tial equations. The new sensitivity algorithm performs efficient small-scale sensitivity analysis by projecting the adjoint circuit equations on a reduced order subspace that preserves the circuit moments.

Most model-reduction techniques perform reduction with respect to a single parameter (frequency). However, it is also necessary to include variations, due to environmental effects and changes in interconnect parameters such as width and height. To address this issue a new method was suggested that performs reduction of a *distributed transmission line system* simultaneously with respect to frequency and any other parameter of the circuit. These parameters could also include physical parameters of the transmission lines. The reduction algorithm is based on congruent transformation techniques applied to multiple dimensions. The proposed algorithm also provides efficient means to estimate the response of large circuits simultaneously as a function of several design parameters.

Next, a novel algorithm was developed for forming reduced-circuits for large interconnect networks without the need for partitioning the network into linear and nonlinear parts. Conventional model-reduction techniques need to partition interconnect networks into linear and nonlinear parts and hence suffer from inefficiencies as the number of ports increase. This algorithm overcomes this difficulty by projecting the full *nonlinear* circuit equations into a subspace of lower dimension. As a result, the original set of nonlinear time-domain equations is replaced by a smaller set of nonlinear equations such that the first 'q' derivatives of the time-response of the original system and the response obtained using the reduced system are identical. The reduced system can be solved using any conventional numerical integration technique, resulting in significant reduction in the CPU time due to the fact that at each time

point, a much smaller set of nonlinear algebraic equations need to be solved. This algorithm reduces the whole network containing both the interconnects and nonlinear terminations without the need to partition the network into linear and nonlinear parts. The efficiency of the new algorithm is completely independent of the number of ports in the interconnect network. A speed-up of an order of magnitude was observed using this algorithm when compared to conventional techniques.

These techniques presented in this thesis provide a marked improvement in the simulation of various interconnect networks.

7.2 Future Work/Directions

The work presented in this thesis has the potential to be applied in various fields other than that of interconnect networks. This work also has the potential to form a basis for further development of new algorithms that facilitate not only the field of interconnects but also that of general circuit simulation. Given below is a brief list of the future work that can be undertaken in light of the work presented in this thesis.

1. Application of Arnoldi and the proposed algorithms to the field of linear Microelectromechanical systems (MEMS).
2. Extension of the nonlinear circuit reduction technique to nonlinear MEMS in the case of large-scale deformations.
3. The multi-dimensional reduction technique can be efficiently used for optimization of large systems with respect to circuit parameters.

4. The multi-dimensional reduction technique can be combined with space mapping techniques to produce interesting results.
5. The nonlinear circuit reduction technique can be extended to time variant systems as opposed to time invariant systems.
6. Combining the nonlinear circuit reduction technique with parametric nonlinear behavioral models for the driver and receiver circuits.
7. Application of the nonlinear circuit reduction technique to the shooting method in harmonic balance.
8. The multi-dimensional reduction technique can be used to perform efficient thermal analysis of silicon micro-structures.

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