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FINITE ELEMENT ANALYSIS
OF INTERACTION BETWEEN AN ELASTIC CIRCULAR PLATE
AND AN ISOTROPIC ELASTIC MEDIUM

)
by

Md. Musharrafuzzaman, B.Sc. Engg.

A THESIS SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
AND RESEARCH IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF ENGINEERING

FACULTY OF ENGINEERING
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The undersigned recommend to the Faculty of Graduate
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"FINITE ELEMENT ANALYSIS OF INTERACTION BETWEEN
AN ELASTIC CIRCULAR PLATE AND AN ISOTROPIC
ELASTIC SOLID MEDIUM"

submitted by Md. Musharrafuzzaman, B.Sc. Engg., in
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of
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ABSTRACT

The axially symmetric interaction of circular plates with a free edge, resting on an isotropic elastic solid medium, is examined by the finite element method. Displacement method is used in the finite element formulation. The plate may be of uniform thickness or it may have differential thickness. The supporting isotropic elastic medium is modelled as a homogeneous material or as a non-homogeneous material where elastic properties vary linearly with depth. In both cases, the Poisson's ratio of the elastic medium varies between 0.0 and 0.49, while that of the plate remains constant at 0.3. The contact between the two materials is assumed to be smooth. Flexural behaviour of the plate is represented by the classical Poisson-Kirchhoff thin plate theory.

To model the isotropic elastic medium, axisymmetric solid elements of triangular cross-section are used. Two generalized displacements (viz. radial and vertical) are considered at each node. Circular plate is modelled as an assemblage of annular plate elements, with two degrees of freedom (viz. vertical deflection and slope) per node. To improve accuracy of the solution, a special formulation is adopted for the central element of the circular plate.

On the basis of these models, a finite element program is developed in FORTRAN IV language. The program is machine independent and runs on the batch mode of operation. Also, two separate programs are written to speed up data input and to detect probable errors in the input data.

The flexural behaviour of an axisymmetrically loaded circular plate (the raft) resting on an isotropic elastic solid medium (the soil) is influenced by many factors. The work presented in this thesis involves investigation of some of these parameters. Giving due consideration to the numerous factors involved, attention is focused only on the following items:

- i) influence of a soil medium of finite depth;
- ii) influence of differential plate thickness;
- iii) influence of the external loading configurations; and
- iv) influence of non-homogeneous elastic half-space.

For each parameter, numerical results developed include influence coefficients for the plate displacement and flexural moments, the contact stress at the interface, together with the stress distribution within the medium.

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TABLE OF CONTENTS

	Page
ABSTRACT	(i)
ACKNOWLEDGEMENTS	(iii)
TABLE OF CONTENTS	(iv)
LIST OF FIGURES	(vi)
LIST OF TABLES	(xii)
LIST OF SYMBOLS	(xv)
CHAPTER 1. INTRODUCTION	
1.1 General.....	1
1.2 Survey of Available Literature	3
1.3 Scope and Objectives of the Work	7
1.4 Method of Analysis	8
1.4.1 The finite element method	8
1.5 Content of the Thesis	12
CHAPTER 2. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON HOMOGENEOUS ISOTROPIC ELASTIC HALF-SPACE	
2.1 Assumptions	15
2.2 Development of Axisymmetric Solid Program	17
2.2.1 Choice of element	18
2.2.2 Derivation of the element stiffness matrix..	19
2.2.3 Consistent load vector for uniformly distributed load	22
2.2.4 Numerical accuracy.....	24
2.3 Development of Axisymmetric Plate Bending Program ..	27
2.3.1 Derivation of the element stiffness matrix ..	28
2.3.2 Special circular plate bending element	31
2.3.3 Consistent load vector for uniformly distributed load	31
2.3.4 Flexural moments and shearing forces	32
2.3.5 Numerical accuracy	34
2.4 Interface Condition	35

2.5	A Finite Element Computer Program to Solve the Present Interaction Problem	36
2.5.1	Main features	37
2.5.2	Limitations	38
2.6	Circular Plate Resting on Homogeneous Elastic Half-Space	40
2.6.1	General	40
2.6.2	Comparison of finite element results with existing solutions	41
2.6.3	Comments on the overall accuracy	44
2.7	Extension of the Work	46
CHAPTER 3. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON AN ISOTROPIC ELASTIC SOLID MEDIUM		
3.1	Effect of Finite Depth of Soil Medium	82
3.2	Effect of Non-Uniform Plate Thickness	85
3.3	Effect of Loading Configurations	87
CHAPTER 4. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE		
4.1	General	139
4.2	Non-homogeneous Half-Space Selected for Study	139
4.3	Computed Results	140
CHAPTER 5. CONCLUSIONS AND RECOMMENDATIONS		
5.1	Summary	156
5.2	Conclusions	158
5.3	Recommendations for Future Work	161
REFERENCES		164
APPENDICES		
Appendix A		169
Appendix B		281

LIST OF FIGURES

Figure		Page
1.1	Elastic circular plate resting on an isotropic elastic solid medium.	13
1.2	Uses of circular plate.	14
2.1	Effect of interface conditions on central deflection of circular plates (after Hooper, 1974).	47
2.2	Cross-sections of axisymmetric ring elements.....	48
2.3	A typical axisymmetric triangular ring element.	49
2.4	Assumed deflection model due to uniform pressure.....	49
2.5	Thick cylinder under internal pressure: finite element subdivisions.....	50
2.6	Thick cylinder under internal pressure: comparison of finite element results.	51
2.7	Surface load on homogeneous elastic solid medium of finite depth ($H/a=5.0$): finite element subdivisions...	52
2.8	Surface load on homogeneous elastic solid medium of finite depth ($H/a=5.0$): comparison of $I_{\omega(r)}$	53
2.9	Annular plate element with associated degrees of freedom.	54
2.10	Solid circular plate element with associated degrees of freedom.	54
2.11	Simply supported circular plate under uniform pressure: comparison of finite element results.....	55
2.12	Partially loaded simply supported circular plate: comparison of finite element results.....	57
2.13	Clamped circular plate under uniform pressure: comparison of finite element results.....	59
2.14	Partially loaded clamped circular plate: comparison of finite element results.	60

Figure		Page
2.15	Finite frictional contact.....	62
2.16	Contact stress distribution between rigid circular punch and homogeneous elastic half-space (after Spence, 1973).	62
2.17	Outline flow diagram.....	63
2.18	Finite element subdivisions for elastic half-space region.....	67
2.19	Completely flexible plate on homogeneous elastic half-space: comparison of $I_{\omega(r)}$ for $\nu_s=0.0$	68
2.20	Completely flexible plate on homogeneous elastic half-space: comparison of $I_{\omega(r)}$ for $\nu_s=0.49$	68
2.21	Rigid plate ($K_r=100$) on homogeneous elastic half-space; comparison of $I_{\omega(r)}$ for $\nu_s=0.0$	69
2.22	Rigid plate ($K_r=100$) on homogeneous elastic half-space: comparison of $I_{\omega(r)}$ for $\nu_s=0.49$	69
2.23	Rigid plate ($K_r=100$) on homogeneous elastic half-space: contact stress distribution.....	70
2.24	Circular plates on homogeneous elastic half-space: comparison of differential deflections for $\nu_s=0.0$..	71
2.25	Plates on homogeneous elastic half-space: surface deformations for various relative rigidities (K_r)..	72
2.26	Plates on homogeneous elastic half-space: contact stress distribution for various relative rigidities (for $\nu_s=0.0$).....	73
2.27	Plates on homogeneous elastic half-space: distribution of bending moment for various K_r and $\nu_s=0.3$	74
2.28	Plates on homogeneous elastic half-space: distribution of normal stress (σ_{zz}) along the axis of symmetry.....	75

Figure		Page
3.1	Influence coefficients for surface displacement for various relative rigidities (for $H/a=1.0$, $\nu_s=0.0$).....	90
3.2	Influence coefficients for surface displacement for various relative rigidities (for $H/a=1.0$, $\nu_s=0.49$).....	90
3.3	Bending moment distributions for various relative rigidities (for $H/a=1.0$, $\nu_s=0.0$).....	91
3.4	Variation of bending moment distributions with Poisson's ratio ν_s for $H/a=1.0$	91
3.5	Contact stress distributions for various relative rigidities, for $\nu_s=0.0$ and $H/a=1.0$	92
3.6	Variation of normal stress (σ_{zz}) with depth along the centerline of the raft for $H/a=1.0$ and $\nu_s=0.0$	92
3.7	Influence coefficients for surface displacement for various relative rigidities and $H/a=2.5$ (with $\nu_s=0.0$).....	93
3.8	Influence coefficients for surface displacement for various relative rigidities and $H/a=2.5$ (with $\nu_s=0.49$).....	93
3.9	Bending moment distributions for various relative rigidities (for $H/a=2.5$, $\nu_s=0.0$).....	94
3.10	Variation of bending moment distributions with Poisson's ratio ν_s for $H/a=2.5$	95
3.11	Contact stress distributions for various relative rigidities, for $\nu_s=0.0$ and $H/a=2.5$	96
3.12	Variation of normal stress (σ_{zz}) with depth along the centerline of the raft for $H/a=2.5$ and $\nu_s=0.0$	96
3.13	Influence coefficients for surface displacement for various relative rigidities (for $H/a=5.0$ and $\nu_s=0.0$).....	97

Figure		Page
3.14	Influence coefficients for surface displacements for various relative rigidities (for $H/a=5.0$ and $\nu_s=0.49$).....	97
3.15	Bending moment distributions for various relative rigidities (for $H/a=5.0$ and $\nu_s=0.0$).	98
3.16	Variation of bending moment distribution with Poisson's ratio ν_s for $H/a=5.0$	99,
3.17	Contact stress distributions for various relative rigidities for $\nu_s=0.0$ and $H/a=5.0$	100
3.18	Variation of normal stress (σ_{zz}) with depth along the centerline of the raft for $H/a=5.0$ and $\nu_s=0.0$	101
3.19	Effect of layer depth (H) on central deflection of the raft (I_{ω}) for various relative rigidities (K_r).	102
3.20	Variation of maximum differential deflection with relative rigidity for different layer depths.....	103
3.21	Variation of maximum flexural moment with relative rigidity of the raft for various layer depths.....	104
3.22	Bending moment (M_x, M_θ) distributions for different layer depths and $K_r=0.1$ for $\nu_s=0.0$	105
3.23	A typical circular raft having non-uniform thickness.....	106
3.24	Effect of differential plate thickness on influence coefficients for surface displacement (for $\nu_s=0.0$).	107
3.25	Effect of differential plate thickness on vertical deflection of the plate for $\nu_s=0.0$	108
3.26	Effect of differential plate thickness on vertical deflection of the plate for $\nu_s=0.49$	109
3.27	Effect of differential plate thickness on bending moment distribution for $\nu_s=0.0$	110

Figure		Page
3.28	Effect of differential plate thickness on bending moment distribution for $\nu_s=0.49$	111
3.29	Effect of differential plate thickness on contact stress distribution for $\nu_s=0.0$	112
3.30	Effect of loading configuration on the influence coefficients for surface displacements ($I_{\omega(r)}$) for $K_r=1.0$ and $H/a=5.0$	113
3.31	Effect of loading configuration on bending moment distribution for $K_r=1.0$, $H/a=5.0$ and $\nu_s=0.0$	114
3.32	Effect of loading configuration on bending moment distribution for $K_r=1.0$, $H/a=5.0$ and $\nu_s=0.49$	115
3.33	Effect of loading configuration on contact stress distribution for $K_r=1.0$, $H/a=5.0$ and $\nu_s=0.0$	116
3.34	Effect of loading configuration on the influence coefficients for surface displacement ($I_{\omega(r)}$) for $K_r=0.1$, $H/a=5.0$ and $\nu_s=0.0$	117
3.35	Effect of loading configuration on the influence coefficients for surface displacement ($I_{\omega(r)}$) for $K_r=0.1$, $H/a=5.0$ and $\nu_s=0.49$	118
3.36	Effect of loading configuration on bending moment distribution for $K_r=0.1$, $H/a=5.0$ and $\nu_s=0.0$	119
3.37	Effect of loading configuration on bending moment distribution for $K_r=0.1$, $H/a=5.0$ and $\nu_s=0.49$	120
4.1	Non-homogeneous elastic half-space.....	143
4.2	Plates on non-homogeneous elastic half-space: influence coefficients for surface displacement for various relative rigidities and $\psi=40$	144
4.3	Plates on non-homogeneous elastic half-space: variation of maximum differential plate deflection with K_r for $\psi=40$	145

Figure		Page
4.4	Plates on non-homogeneous elastic half-space: bending moment distributions for various relative rigidities for $\nu_s=0.0$ and $\psi=40$	145
4.5	Plates on non-homogeneous elastic half-space: variation of bending moment distributions with Poisson's ratio ν_s for $\psi=40$	146
4.6	Plates on non-homogeneous elastic half-space: contact stress distribution for various relative rigidities for $\psi=40$ and $\nu_s=0.0$	147
4.7	Plates on non-homogeneous elastic half-space: variation of normal stress (σ_{zz}) with depth along the centerline of the plate for $\psi=40$ and $\nu_s=0.0$	148
4.8	Effect of non-homogeneity on the influence coefficient for surface displacement for $K_r=0.1$, $\psi=40$ and $H/a=20$	149
4.9	Effect of non-homogeneity on bending moment distributions for $K_r=10$, $\psi=40$	150
4.10	Effect of non-homogeneity on contact stress distribution for $K_r=10$, $\nu_s=0.0$ and $\psi=40$	151

LIST OF TABLES

Table		Page
2.1	Surface load on homogeneous elastic solid medium of finite depth ($H/a=5.0$): influence coefficients for surface displacement.....	76
2.2	Finite element representation of elastic half-space.....	77
2.3	Uniformly loaded circular plate on homogeneous elastic half-space: influence coefficient for surface displacements for $\nu_s=0.0$	78
2.4	Uniformly loaded circular plate on homogeneous elastic half-space: influence coefficients for surface displacement for $\nu_s=0.49$	79
2.5	Uniformly loaded circular plate on homogeneous elastic half-space: distribution of flexural moments.....	80
2.6	Uniformly loaded circular plate on homogeneous elastic half-space: distribution of contact pressure for $\nu_s=0.0$	81
2.7	Uniformly loaded circular plate on homogeneous elastic half-space: distribution of normal stress (σ_{zz}) along the centerline of the plate for $\nu_s=0.0$	81
3.1	Effect of layer depth on the influence coefficients for surface displacement for $K_r=0.01$	121
3.2	Effect of layer depth on the influence coefficients for surface displacement for $K_r=0.10$	122
3.3	Effect of layer depth on the influence coefficients for surface displacement for $K_r=1.0$	123
3.4	Effect of layer depth on the influence coefficients for surface displacement for $K_r=10.0$	124
3.5	Effect of layer depth on the influence coefficients for surface displacement for $K_r=100.0$	125

Table		Page
3.6	Effect of layer depth on bending moment distributions for $K_r=0.1$	126
3.7	Effect of layer depth on bending moment distributions for $K_r=1.0$	127
3.8	Effect of layer depth on bending moment distributions for $K_r=10.0$	128
3.9	Effect of layer depth on contact stress distributions.	129
3.10	Summary of the effect of layer depth.....	130
3.11	Effect of differential plate thickness on the influence coefficients for surface deformation for $\nu_s=0.0$ and $H/a=5.0$	131
3.12	Effect of differential plate thickness on the influence coefficients for surface deformation for $\nu_s=0.49$ and $H/a=5.0$	132
3.13	Effect of differential plate thickness on bending moment distributions for $\nu_s=0.0$ and $H/a=5.0$	133
3.14	Effect of differential plate thickness on bending moment distributions for $\nu_s=0.49$ and $H/a=5.0$	133
3.15	Effect of differential plate thickness on contact stress distributions for $\nu_s=0.0$ and $H/a=5.0$	134
3.16	Effect of loading configuration on $I_{\omega(r)}$ for $K_r=1.0$ and $H/a=5.0$	134
3.17	Effect of loading configuration on bending moment distribution for $K_r=1.0$ and $H/a=5.0$	135
3.18	Effect of loading configuration on contact stress distribution for $K_r=1.0$ and $H/a=5.0$	135
3.19	Effect of loading configuration and K_r on $I_{\omega(r)}$ for $\nu_s=0.0$ and $H/a=5.0$	136

Table		Page
3.20	Effect of loading configuration and K_r on $I_{\omega(r)}$ for $\nu_s=0.49$ and $H/a=5.0$	137
3.21	Effect of loading configuration and relative rigidity on bending moment distributions for $\nu_s=0.0$ and $H/a=5.0$	138
4.1	Influence coefficients for surface displacement of circular plate on non-homogeneous elastic half-space for $\nu_s=0.0$ and $\psi=40$	152
4.2	Influence coefficients for surface displacement of circular plate on non-homogeneous elastic half-space for $\nu_s=0.49$ and $\psi=40$	153
4.3	Plates on non-homogeneous elastic half-space: distribution of bending moments for $\nu_s=0.0$ and $\psi=40$	154
4.4	Plates on non-homogeneous elastic half-space: distribution of contact stress for $\nu_s=0.0$ and $\psi=40$	154
4.5	Plates on non-homogeneous elastic half-space: distribution of normal stress (σ_{zz}) along the centerline of the plate for $\nu_s=0.0$ and $\psi=40$	155

LIST OF SYMBOLS

Symbol	Definition
$[A]$	Matrix which relates the nodal displacement vector $\{\delta\}_e$ with the generalized coordinate vector $\{\alpha\}$.
a	Radius of plate (or raft). Also denotes outer radius of an annular plate (or raft).
$[B]$	Matrix which relates the strain vector $\{\epsilon\}$ with the displacement vector $\{\delta\}$, for an element.
b	Inner radius of an annular plate.
D	Rigidity of the plate (see Eqn. 2.25).
$[D]$	Constitutive relation matrix (i.e. it relates the stress vector $\{\sigma\}$ with the strain vector $\{\epsilon\}$, for an element.
dA	Differential area.
DOF	Degrees of freedom.
dV	Differential volume.
E_b	Young's modulus for plate (or raft).
E_s	Young's modulus for solid (or soil).
F_i, F_k	Nodal forces at nodes i and k respectively.
i, j, k	Node numbers. When used as subscript, indicates that the quantity is referred to a particular node (e.g. w_i denotes the vertical displacement at node i).
$I_p(r)$	Influence coefficient for normal contact stress (pressure) at a horizontal distance r from the axis of symmetry.
$I_w(r)$	Influence coefficients for surface displacement (vertical) at a point having radial coordinate r .
$I_w(0)$	Influence coefficient for surface displacement (vertical) at the centre of the plate (or the raft).

Symbol	Definition
$I_{\Delta\omega}$	Influence coefficient for maximum differential deflection (vertical) of the plate (or the raft).
I_1, I_2, I_3	Stress invariants.
K_r	Relative rigidity of the plate (or the raft).
$[K]_e$	Element stiffness matrix.
$[K]_\alpha$	Integral part of the element stiffness matrix.
$\ln$	Natural logarithm (i.e. to the base e).
M_j	Equivalent externally applied radial moment at node j.
M_r	Radial component of bending moment.
M_θ, M_t	Circumferential component of bending moment.
P	Externally applied total load.
$p(r)$	Normal stress developed at the interface between the plate (or the raft) and the solid (or the soil) medium, at a point having radial coordinate r.
p_0	Intensity of uniform internal pressure.
q	Intensity of externally applied uniform pressure (normal).
Q_r	Shearing force (normal) at a point having radial coordinate r.
r	Radial coordinate of a point.
r_i, r_j, r_k	Radial coordinates of nodes i, j, k respectively.
t	Thickness of the elastic circular plate (or raft).
U_b	Strain energy due to bending.
u	Component of the displacement along the radial direction.
u_i, u_j, u_k	Radial displacement of nodes i, j, k respectively.

SYMBOL	DEFINITION
$[V]$	Matrix used to represent the strain energy due to bending in matrix form (Eqn. 2.27b).
ω	Component of the displacement along z direction.
$\omega_i, \omega_j, \omega_k$	Vertical displacement at nodes i, j and k respectively.
$\omega(r)$	Vertical displacement at a point having radial coordinate r.
W_a	Work done by the actual system of forces.
W_e	Work done by the equivalent system of forces.
z	Z-coordinate of a point.
z_i, z_j, z_k	Z-coordinate of nodes i, j, k respectively.
$\alpha_i (i=1,6)$	Generalized coordinates.
αt	Thickness of the inner region of a plate (or raft) having non-uniform thickness.
βa	Radius of the uniformly loaded area, in the case of partial loads.
$\Delta \omega$	Maximum differential deflection (i.e. the deflection at the center of the plate - the deflection at the edge of the plate).
$\{\delta\}_e$	Nodal displacement vector for an element.
$\{\epsilon\}$	Strain vector for an element.
ϵ_{rr}	Radial component of strain.
ϵ_{zz}	Normal (vertical) component of strain.
$\epsilon_{\theta\theta}$	Circumferential component of strain.
ϵ_{rz}	Shear strain.
$\{\chi\}$	Matrix used to represent the strain energy due to bending in matrix form (Eqn. 2.27a).
$\{F\}_e$	Consistent load vector due to uniform pressure on element.
$\{\alpha\}$	Generalized coordinate vector.

SYMBOL	DEFINITION
ν_b	Poisson's ratio for plate (or raft).
ν_s	Poisson's ratio for solid (or soil).
$\{\sigma\}$	Stress vector for an element.
σ_{rr}	Radial component of stress.
σ_{zz}	Normal (vertical) component of stress.
$\sigma_{\theta\theta}$	Circumferential component of stress.
σ_{rz}	Shear stress.
$\sigma_1, \sigma_2, \sigma_3$	Components of principal stresses.
θ, Z, R	Components of spherical coordinate system.
$\theta\theta$	When used as subscript, refers to circumferential direction.
$\tau_{\max}$	Maximum shear stress at a point.

1. INTRODUCTION

1.1 GENERAL

The flexural behaviour of an axisymmetrically loaded finite circular plate with a free edge, resting on a deformable elastic medium, has several engineering applications. Solutions to such problems are of particular importance in the analysis and design of structural foundations resting on soil or rock media (Selvadurai; 1978, 1979a). The classical problem relating to the axisymmetric flexure of a completely flexible and an infinitely rigid circular plate resting on an isotropic elastic half-space was examined by Boussinesq (1885). In engineering applications, however, a circular plate is neither completely flexible nor infinitely rigid, rather it has some finite rigidity.

The rigidity (or flexibility) of a plate depends upon the elastic and geometric properties of the plate-elastic medium system, and is defined by

$$K_R = \frac{E_b}{E_s} (1 - \nu_s^2) \left(\frac{t}{a}\right)^3 \quad (1.1)$$

In this expression, E_s and ν_s are the elastic constants of the solid medium, and E_b , t and a denote the Young's modulus, thickness and radius of the plate respectively. The term K_R is usually referred to as the relative rigidity of the plate. $K_R=0$ represents a completely

flexible plate (i.e. the load is directly applied on the elastic solid medium), whereas a very stiff plate is represented by $K_r \gg 100$.

The definition of the relative rigidity (K_r) in Eqn. (1.1) is not unique. It can also be defined as

$$K_r = \frac{E_b}{E_s} \frac{(1-\nu_s^2)}{(1-\nu_b^2)} \left(\frac{t}{a}\right)^3 \quad (1.1b)$$

where ν_b is the Poisson's ratio of the plate. Both of these definitions of K_r are used by researchers (see e.g. Brown, 1969a,b; Boswell and Scott, 1975; Selvadurai, 1979a). Since in the present study, Poisson's ratio of the plate (ν_b) is assumed to remain constant at 0.3, the former definition of K_r is used.

The flexural behaviour of a circular plate of finite flexibility resting on an isotropic elastic solid medium is influenced by many factors such as the material and geometric properties of the plate and the supporting medium, interface conditions, external loading configurations, boundary conditions, etc. (see e.g. Brown, 1969a,b; Hooper, 1975; Carrier and Christian, 1973). To be able to predict the behaviour of such a plate under anticipated loads, these factors must be accounted for.

The axisymmetric soil-structure interaction problem, that has been examined in this thesis, is illustrated in Fig. 1.1[†]. An elastic

[†] Figures and Tables are presented at the end of each Chapter.

circular plate of radius a , thickness t , Young's modulus E_b and Poisson's ratio ν_b is resting on an isotropic elastic solid medium having thickness H , and the elastic properties E_s and ν_s . The plate is acted upon by a uniform pressure of intensity q , distributed over an area of radius βa . The plate may be of uniform thickness or its thickness may vary, as shown in Fig. 1.1. The isotropic elastic solid medium may behave as a homogeneous material or as a non-homogeneous material where properties may change linearly with depth. The contact between the plate and the supporting elastic medium is assumed to be smooth.

It should be noted that the interaction problem described above is essentially a three-dimensional problem. However, since the present study is restricted to only axisymmetric loads and plates, the problem under consideration is reduced to a two-dimensional one. Furthermore, this study is restricted to only elastic analysis.

1.2 SURVEY OF AVAILABLE LITERATURE

Based on the methods of analysis employed by previous investigators, the available literature relating to the present interaction problem may be discussed under the following headings. A complete exposition of these methods is given by Selvadurai (1978).

Power Series Method

The classical problem relating to the axisymmetric flexure of a circular plate resting on isotropic elastic half-space was examined

by Borowicka (1936) by using power series expansion technique. He represented the deflections of the plate and the displacements of the elastic half-space as power series in the radial coordinate r . By equating the coefficients of corresponding terms in the two power series, a set of linear equations was derived. These equations were solved to determine the contact stress distribution. Benito (1948) also employed this technique for the analysis of a circular plate stiffened at the center by an infinitely rigid concentric region. This analytical procedure was subsequently extended by Ishkova (1951), and Brown (1969a, b) who incorporated the effect of a singular term in the approximation of the contact stress distribution. Brown (1969b) also examined the effect of thickness of supporting soil medium and presented numerical results for the deflections, bending moments, contact stresses for a number of cases of practical interest. Zemochkin (1947) analyzed the axisymmetric problem of a circular plate by considering the compatibility of displacements between the elastic half-space and the plate along a series of concentric circular regions. Similar procedures were also employed by Holmberg (1946) who assumed that the contact stress was composed of three and four annular regions, respectively, of uniform stress. Barden (1965) used an analogous technique in his treatment of the circular plate problem, and extended the discussion to transversely isotropic media.

Finite Difference Method

Chattopadhyay and Ghosh (1969) employed the finite difference approach to analyze a circular plate resting on an elastic half-space, subjected to six concentrated loads placed symmetrically around the center of the plate. They also carried out some experiments using a 'Perspex' plate and rubber subgrade to represent the elastic solid medium. Deflections were measured at various locations of the plate. Satisfactory agreement was obtained between the computed results and the measured values. This analysis technique was extended by Chakravorty and Ghosh (1975) to incorporate variation of thickness of the plate and to allow for arbitrary loads. Shirokov (1970) used an analogous technique to examine the behaviour of a rigid circular plate, resting on a non-linearly deformable elastic half-space. Numerical results were presented for the deflections, variation of normal stress with depth, and for the contact stresses.

Finite Element Method

Cheung and Zienkiewicz (1965) investigated the problems of slabs and tanks resting either on an elastic half-space (both isotropic and orthotropic) or on a so-called Winkler's foundation by the finite element method. Smith (1967) used a similar technique to examine the axisymmetric flexure of a circular plate on idealized Winkler medium, or on homogeneous elastic half-space. He also performed a series of laboratory tests. A satisfactory agreement between the finite element

results and experimental values was obtained. Christian (1968) extended the analysis to incorporate the effect of a term which becomes singular as the Poisson's ratio of the elastic medium equals 0.5. Svec and Gladwell (1973) developed a refined plate bending element for the problem of a thin elastic plate on an elastic half-space. Carrier and Christian (1973), and Bowswell and Scott (1975) considered the elastic half-space to be heterogeneous and presented numerical results both for a rigid circular plate and a flexible circular plate. Hooper (1974,1975) examined the same problem by considering the contact between the plate and the elastic half-space (isotropic or transversely isotropic) to be adhesive. Of related interest are the works reported by Wood (1977), King and Chandrasekaran (1974), Svec (1976), etc.

Energy Method

Selvadurai (1979a) examined the application of an energy method for the analysis of axisymmetric flexure of a uniformly loaded circular plate in smooth contact with a homogeneous isotropic elastic half-space. In this analysis, the deflected shape of the circular plate was represented in the form of a power series in terms of the radial coordinate r . This deflected shape was specified to within a set of undetermined constant and satisfied the kinematic constraints of the axisymmetric deformation. The total potential energy functional for the plate-elastic half-space system was expressed in terms of the constants characterizing the deflected shape of the plate. Two of

these constants were eliminated by invoking the Kirchhoff boundary conditions applicable to the free edge of the plate, and the remaining constants were determined from a set of linear equations generated by the minimization of the total potential energy functional. Subsequently, Selvadurai (1979b) extended this analysis for the case of a thick plate, and also for plates with a restrained edge (Selvadurai, 1979c).

1.3 SCOPE AND OBJECTIVES OF THE WORK

In recent years, circular plates have found increased application in raft foundations used for structures such as nuclear reactors, storage tanks, silos, and in chimney and tower foundations (Fig. 1.2). It has been found that rafts of low relative rigidity (K_r) may suffer excessive differential deflection (Brown, 1969a). On the other hand, relatively stiff rafts may be subjected to excessive bending moments. Thus, in the design of raft foundations, it is desirable to be able to predict the differential deflections, flexural moments and contact stresses under anticipated loads in order that a raft of suitable stiffness may be selected.

The flexural behaviour of an axisymmetrically loaded circular plate (raft) resting on an isotropic elastic medium (soil) is influenced by many parameters. The works presented in this thesis involve investigation of some of these selected parameters. The finite element method is used as a tool for analysis. The primary objectives of this work are as mentioned below:

- i) To develop a finite element computer program which is capable of solving the present interaction problem.
- ii) To examine the behaviour of a uniformly loaded circular plate in smooth contact with a homogeneous isotropic elastic half-space.
- iii) To examine the effect of some selected parameters on the flexural behaviour of an axisymmetrically loaded circular plate, resting on an isotropic elastic solid medium. The parameters which were selected for study include:
 - a) the thickness of the supporting elastic medium,
 - b) the differential thickness of the plate,
 - c) the external loading configurations, and
 - d) the non-homogeneity of the elastic medium.

1.4 METHOD OF ANALYSIS

The finite element method is used for the analysis of the present interaction problem. Assumptions made for this analysis are outlined in Section 2.1. A brief general description of the method is presented below.

1.4.1 The Finite Element Method

The finite element method has recently become a useful tool for the analysis of practical problems encountered in Structural and Geotechnical Engineering (Zienkiewicz, 1971; Bowes and Russel, 1975;

Desai and Abel, 1972). The extensive use of the method has been made practical due to the availability of fast computers with large storage capacity, and due to the development of the method itself. One of the main advantages of the finite element method is that it does not deal with the partial differential equations of equilibrium or compatibility. Rather, it converts the problem directly into one requiring the solution of simultaneous linear equations. Although these equations contain a large number of unknowns, they can be arranged in such a way that the resulting matrices are sparse and/or banded. Efficient algorithms can then be developed for the solution of such equations.

The solution of a problem by the finite element method follows an orderly step-by-step procedure. A detailed description of these steps given in any standard text (see e.g. Bowes and Russel, 1975; Zienkiewicz, 1971; Desai and Abel, 1972; Gallagher, 1975) can be summarized as follows (Desai and Abel, 1972).

1. Discretization of the continuum

The continuum is the physical body, structure, or solid being analyzed. Discretization may be described as the process in which the given body is subdivided into an equivalent system of the finite element. For the present problem, axisymmetric solid elements of triangular cross-section are used to subdivide the elastic solid medium. Annular elements and solid circular elements are used for the discretization of the plate.

2. Selection of the displacement models

Deflection functions are assumed to represent approximately the actual or exact distribution of the displacements. A deflection function is commonly assumed in the form of a polynomial. The number of terms that can be retained in the polynomial depends on the nodal degrees of freedom chosen. In order to ensure that the finite element results approach the correct solution, the assumed deflection functions should satisfy the following:

- i) The deflection functions must be continuous within the element, and the deflections must be compatible between adjacent elements.
- ii) The deflection functions must include the rigid body displacements of the element.
- iii) The deflection functions must include the constant strain states of the element.

The assumed deflection functions for both the axisymmetric solid and axisymmetric plate bending elements satisfy the above requirements, except that compatibility is not fully maintained at the interface nodes.

3. Derivation of the element stiffness matrix

The stiffness matrix consists of the coefficients of the equilibrium equations derived from the material and geometric properties of an element. It can be derived by making use of the principle of

minimum potential energy. A description of derivation of the element stiffness matrices is given in Sections 2.2.2 and 2.3.1.

4. Assembly of the algebraic equations for the overall discretized continuum

This process includes the assembly of the overall stiffness matrix of the entire structure from the individual element stiffness matrices, and the overall force (the load vector) from the element nodal force vectors. In this work, a standard technique, known as the "Code Number Method" (Beaufait, 1970), is used for this purpose.

5. Solutions for the unknown displacements

The algebraic equations assembled in step 4 are solved for the unknown displacements. Gaussian elimination technique along with the method of Payne and Iron (1963) is used for this purpose.

6. Computations of element stresses, flexural moments, etc. from the nodal displacements

In general, the stresses, strains, flexural moments, etc. are the derivatives of the nodal displacements. Since the displacements at any point within an element are uniquely determined from the nodal displacements, their derivatives can be calculated easily, and from there, stresses, flexural moments, etc. can be determined.

1.5 CONTENT OF THE THESIS

An outline of the interaction problem that has been examined in this thesis is presented in Chapter 1. A brief discussion of the previous works related to the present problem is also made in the same chapter.

Chapter 2 is concerned with the development and calibration of a finite element computer program which is capable of handling the present interaction problem.

The finite element results, associated with the investigation of parameters which govern the behaviour of a circular plate resting on an isotropic elastic solid medium, are presented in Chapters 3 and 4.

Conclusions and recommendations are made in Chapter 5.

A listing of the FORTRAN statements, and a user's manual are given in Appendix A .

* Young's Modulus - E_b

Poisson's Ratio - ν_b

** Young's Modulus - E_s (or $E_s(z)$: see Fig. 4.1)

Poisson's Ratio - ν_s

FIGURE 1.1 AN ELASTIC CIRCULAR PLATE RESTING ON AN ISOTROPIC ELASTIC SOLID MEDIUM

a) A Nuclear Reactor (after Ho and Stoykovich, 1973)

b) A Tower Silo (after Karol, 1960)

FIGURE 1.2 USES OF CIRCULAR PLATE

2. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON HOMOGENEOUS ISOTROPIC ELASTIC HALF-SPACE

In the past few years, the finite element method has been extensively used to solve many soil-structure interaction problems. This has been possible due to the efficiency of the method itself and the relatively large storage capacity and very high speed of the modern digital computers (Smith, 1970).

The solution of a continuum problem by the finite element method always follows an orderly step-by-step process. These steps are described in any standard text (e.g. Zienkiewicz, 1967; Bowes and Russel, 1975) and have been summarized in the previous Chapter. Application of the finite element method to solve the problem of a normally loaded circular plate resting on an isotropic elastic solid medium necessarily involves the same steps. A description of the development of axisymmetric finite element technique for solving the present interaction problem will be presented in subsequent sections of this Chapter. However, before considering the interaction problem related to a circular plate resting on an isotropic elastic half-space, it is necessary to outline the assumptions involved in the finite element idealization.

2.1 ASSUMPTIONS

The following assumptions are made for the finite element analysis:

1. The flexural behaviour of the structural foundation is represented by the classical Poisson-Kirchhoff thin plate theory. This assumption is realized in a large variety of structural foundation problems encountered in Civil Engineering Practice (Bowles, 1968; Selvadurai, 1978). The fundamental assumptions of the Poisson-Kirchhoff's thin plate theory can be summarized as follows:
 - i) The thickness of the plate is small in comparison with its lateral dimensions.
 - ii) The deflection of the plate is small compared to its thickness.
 - iii) The strain energy is composed of only the flexural energy.
2. The solid medium (the soil) which supports the circular plate (the raft) is assumed to behave as an isotropic and elastic continuum.
3. The contact between the circular plate and solid medium is assumed to be smooth and capable of sustaining tensile tractions. The former part of this assumption signifies that only the vertical forces can be transmitted at the interface nodes but not the lateral forces. The latter part, on the other hand, ensures that no gap occurs between the circular plate and the supporting elastic medium, at any stage of loading.

In general, this is not a realistic assumption because it neglects the frictional forces that are developed at the interface. The

effects of this assumption on central deflection are illustrated in Fig. 2.1. For a completely rigid plate resting on a very compressible elastic half-space, the central deflection for frictionless contact is approximately 9.0 percent higher than the corresponding deflection for adhesive contact. For a completely flexible plate, the corresponding increase is of the order of 16.0 percent. However, as the Poisson's ratio of the elastic medium ν_s approaches 0.5, the effects of contact between the two materials become insignificant. Other effects of the assumed interface condition are discussed in Section 2.4.

2.2 DEVELOPMENT OF AXISYMMETRIC SOLID PROGRAM

The first step in the finite element analysis consists of subdividing the continuum region into elements by imaginary lines. The selection of element for a given problem is an exercise of engineering judgement and greatly influenced by the nature of the problem to be solved (Desai and Abel, 1972). In general, the number, shape, size and configuration of the elements should be such that the original body is simulated as closely as possible. The factors to be considered in this process of discretization are the geometry, loading, and material properties of the structure; bandwidth, element aspect ratio, finite representation of infinite bodies, etc.

To model the isotropic elastic solid medium, axisymmetric solid element of triangular cross-section is used. This type of element was

introduced by Clough and Rashid (1965). Smith (1967) also used the same type of element to model the elastic solid foundation. Boswell^P and Scott (1975) used six noded triangular elements, shown in Fig. 2.2, for analyzing a flexible circular plate on a heterogeneous elastic half-space. Carrier and Christian (1973) used quadrilateral elements composed of four constant strain triangles for the analysis of a rigid plate resting on homogeneous and non-homogeneous elastic half-spaces. These elements were first proposed by Wilson (1965). In this work, the central node was eliminated by static condensation. Hooper (1974,1975) used eight noded isoparametric quadrilateral elements to model the foundation.

2.2.1 Choice of Element

In general, the choice of an element (Desai and Abel, 1972) greatly depends on the nature of the problem to be solved and is influenced by the geometry of the structure and also by the number of independent spatial coordinates (e.g. R , θ , Z referring to the cylindrical polar coordinate system) necessary to describe the problem. Since the present problem possesses an axis of symmetry, and the solid elements are assumed to have no rotational stiffness, only the two generalized displacements per node can be specified. In this situation, an improved higher order polynomial approximation could be used by increasing the number of nodes per element. But, incorporation of secondary nodes will obviously increase the bandwidth and require more core space and computing cost. On the other hand, the selection of lower order poly-

nomial requires, in general, the use of relatively refined mesh for adequate accuracy. In all practical problems, there is a compromise between the two methods. However, the reasons for the author's option for three noded triangular elements are stated below:

1. A triangular element is the simplest of all two-dimensional elements, but still very efficient in subdividing a continuum region, especially one which has an irregular boundary. Although the present work is associated with regular boundaries, an extension of it may involve an irregular one.
2. A modified and efficient seven point Gaussian quadrature scheme (Cowper, 1972) was available for performing integrations related to triangular elements.
3. For the same number of elements, the bandwidth is considerably less.

2.2.2 Derivation of the Element Stiffness Matrix

Fig. 2.3 shows a typical axisymmetric triangular ring element with nodes i, j, k numbered in the counter-clockwise order. For the degrees of freedom shown, the simplest representation of deflection functions is given by two linear polynomials

$$u = \alpha_1 + \alpha_2 r + \alpha_3 z \quad (2.1)$$

$$w = \alpha_4 + \alpha_5 r + \alpha_6 z \quad (2.2)$$

where u and w are the components of the displacement vector in the R and Z directions respectively, and α_i ($i=1, \dots, 6$) are generalized coordinates. With the assumed displacement functions, standard steps of equilibrium method (see e.g. Desai and Abel, 1972) can be followed and the required element stiffness matrix $[K]_e$ can be represented as

$$[K]_e = [A^{-1}]^T \left(\int_{vol} [B]^T [D] [B] dV \right) [A^{-1}] \quad (2.3)$$

where

$$[A^{-1}] = \begin{bmatrix} \bar{A}^{-1} & 0 \\ 0 & \bar{A}^{-1} \end{bmatrix} \quad (2.4)$$

$$[\bar{A}] = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{bmatrix} \quad (2.5)$$

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{r} & 1 & \frac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad (2.6)$$

and

$$[D] = \frac{E_s(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \begin{bmatrix} 1 & \frac{\nu_s}{1-\nu_s} & \frac{\nu_s}{1-\nu_s} & 0 \\ & 1 & \frac{\nu_s}{1-\nu_s} & 0 \\ & & 1 & 0 \\ \text{symmetric} & & & \frac{1-2\nu_s}{2(1-\nu_s)} \end{bmatrix} \quad (2.7)$$

where E_s and ν_s are the Young's modulus and Poisson's ratio respectively of the elastic solid medium.

In Eqn. (2.5), subscripts of r and z denote node numbers. Also, $[D]$ is referred to as the constitutive relation matrix (i.e. it relates the stress vector $\{\sigma\}$ with the strain vector $\{\epsilon\}$).

$$\{\sigma\} = [D]\{\epsilon\} \quad (2.8)$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{Bmatrix}; \quad \{\epsilon\} = \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \epsilon_{rz} \end{Bmatrix} \quad (2.9)$$

and subscript $\theta\theta$ refers to direction perpendicular to rz plane.

Using Eqns. (2.6) and (2.7), Eqn. (2.3) can be simplified to the following form (see Desai and Abel, 1972)

$$[K]_e = [A^{-1}]^T [K_\alpha] [A^{-1}] \quad (2.10)$$

where

$$[K_\alpha] = \int_{vol} [B]^T [D] [B] dV = \int_{vol} \begin{bmatrix} \frac{D_{11}}{r^2} & \frac{D_{11}+D_{12}}{r} & \frac{D_{11}z}{r^2} & 0 & 0 & \frac{D_{12}}{r} \\ 2(D_{11}+D_{12}) & (D_{11}+D_{12})\frac{z}{r} & 0 & 0 & 2D_{12} & \\ D_{11}\frac{z^2}{r^2}+D_{22} & 0 & D_{22} & D_{12}\frac{z}{r} & 0 & \\ 0 & 0 & 0 & 0 & D_{22} & 0 \\ \text{Symmetric} & & & & D_{11} & \end{bmatrix} dV \quad (2.11)$$

and

$$D_{11} = \frac{E_s(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)}, \quad (2.12)$$

$$D_{12} = \frac{E_s \nu_s}{(1+\nu_s)(1-2\nu_s)}, \quad D_{22} = \frac{E_s}{2(1+\nu_s)} \quad (2.13)$$

$$dV = 2\pi r dr dz = 2\pi r dA. \quad (2.14)$$

It is to be noted that the integrals in $[K_\alpha]$ involve polynomials in r and z . It is possible to obtain closed form integrations for these polynomials. However, numerical integrations, using the Gaussian quadrature formulae (Cowper, 1972) with area coordinate system, give satisfactory results for all practical purposes.

Having established the element stiffness matrices, it is now necessary to assemble them to get the total stiffness matrix $[K]$ of the solid medium. A standard technique known as the 'code number method' (Beaufait, F.W. et al, 1970) is used for this purpose.

It should be mentioned here that the present finite element analysis cannot be employed for $\nu_s=0.5$ because of the occurrence of the term $(1-2\nu_s)$ in the denominator of all non-zero elements of the constitutive relation matrix $[D]$ (Eqn. (2.7)). There are different ways to eliminate this numerical difficulty and some of these will be discussed in Section 2.5.2.

2.2.3 Consistent Load Vector for Uniformly Distributed Loads

If the solid medium is directly acted upon by a line or ring load, it is converted to a concentrated load and treated as a known force

component. However, the treatment of a uniformly distributed load is different. While static equivalents at the end nodes i and k (Fig. 2.4) could be determined, a more accurate method would be to find the equivalent nodal forces which constitute the consistent load vector.

Referring to Fig. 2.4, the work done W_e by the equivalent nodal forces F_i and F_k is

$$W_e = F_i \omega_i + F_k \omega_k \quad (2.15)$$

In matrix form

$$W_e = \begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_i \\ 0 \\ F_k \end{Bmatrix} \quad (2.16)$$

On the other hand, the work done by the actual system of forces W_a can be written as (see Appendix B for derivation)

$$W_a = \int_{r_i}^{r_k} 2\pi q \begin{Bmatrix} u \\ \omega \end{Bmatrix}^T \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r \, dr \quad (2.17)$$

Using Eqns. (2.1) and (2.2) and relating $\alpha_{i(i=1, \dots, 6)}$ with the nodal displacements, the above equation can be simplified as follows:

$$W_a = \begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix} 2\pi q [A^{-1}]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{r_k^2 - r_i^2}{2} \\ \frac{r_k^3 - r_i^3}{3} \\ \frac{(r_k^2 - r_i^2)z}{2} \end{Bmatrix} \quad (2.18)$$

It may be noted that for a horizontal surface $z=z_i=z_k$. Making the two loading systems equivalent, the closed form expressions for P_i and F_k are obtained.

$$F_i = \frac{(r_k - r_i)}{12 \text{ Area}} \{z_i (-2r_i^2 + r_i r_k + r_k^2) + z_j (2r_i^2 - r_i r_k - r_k^2)\} 2\pi q \quad (2.19a)$$

$$F_k = \frac{(r_k - r_i)}{12 \text{ Area}} \{z_i (-r_i^2 - r_i r_k + 2r_k^2) + z_j (r_i^2 + r_i r_k - 2r_k^2)\} 2\pi q \quad (2.19b)$$

where

$$\text{Area} = \frac{1}{2} \{r_i (z_j - z_i) + r_k (z_i - z_j)\}. \quad (2.20)$$

2.2.4 Numerical Accuracy

The methods described in the preceding sections have been programmed to solve a wide range of axisymmetric problems. The output of this program includes nodal displacements and forces, and the stress

components σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz} . The principal stresses (σ_1 , σ_2 , σ_3 , $\tau_{\max}$) and stress invariants (I_1 , I_2) are also calculated at the user's request. To check the numerical accuracy of the program, the following problems were solved and the computed results were compared with the corresponding exact solutions.

i) Thick cylinder under internal pressure: The problem investigated by Bowes and Russel (1975) is illustrated in Fig. 2.5. Finite element subdivisions together with the nodal loads[†] necessary to represent an internal pressure of 1000 psi (6.895 MPa) are also shown on the same Figure. The exact expressions for the radial stress σ_{rr} and the circumferential stress $\sigma_{\theta\theta}$ (Timoshenko and Goodier, 1970) are

$$\sigma_{rr} = \frac{b^2 p_o}{(a^2 - b^2)} \left(1 - \frac{a^2}{r^2} \right) \quad (2.21)$$

$$\sigma_{\theta\theta} = \frac{b^2 p_o}{(a^2 - b^2)} \left(1 + \frac{a^2}{r^2} \right) \quad (2.22)$$

where

a = outer radius,

b = inner radius and

p_o = intensity of uniform internal pressure.

[†] Instead of specifying the nodal loads, as shown in Fig. 2.5, the problem could be solved by specifying the intensity of pressure, in which case the consistent loads would be calculated by the program. In fact, the problem was examined for both the loading conditions and consistent results were obtained.

The comparison for σ_{rr} and $\sigma_{\theta\theta}$ is shown graphically in Fig. 2.6. The results obtained from the finite element method compare very closely with the analytical results to within an accuracy of approximately 0.2 percent.

ii) Surface load on homogeneous elastic solid medium of finite

depth: The results for this problem are given in graphical form by Terzaghi (1943). The same problem is solved by the finite element method for a selected ratio of the layer depth H to the radius of the loaded area a (Fig. 2.8). The contact between the elastic solid medium and the rigid base is assumed to be smooth and a rigid frictionless vertical boundary is introduced at a distance $10a$ from the center of the load. The finite element subdivisions are shown in Fig. 2.7. A comparison of results derived by the finite element method and those presented by Terzaghi (1943) is shown in Fig. 2.8. For ease of comparison, the surface displacement $\omega(r)$ at a distance r is expressed as

$$\omega(r) = \frac{a q}{E_s} I_{\omega(r)} \quad (2.23)$$

where

$I_{\omega(r)}$ = influence coefficient for displacement,

a = radius of the loaded area,

q = intensity of the applied stress and

E_s = Young's modulus for the homogeneous solid medium.

It is observed from Fig. 2.8 that the computed results approximate closely the curves obtained by Terzaghi (1943). For $\nu_s=0.0$, the finite element results for $I_{\omega(0)}$ are accurate to within 2.6 percent. For $\nu_s=0.5$, the corresponding difference is of the order of 6.5 percent.

As mentioned in Section 2.2.2, the finite element analysis cannot be employed for $\nu_s=0.5$. From a set of trial computer runs, it was found that the finite element results for $\nu_s=0.49$ approximate the corresponding analytical values for $\nu_s=0.5$ to within 6.5 percent. The computed results for other values of ν_s are presented in Table 2.1.

2.3 DEVELOPMENT OF AXISYMMETRIC PLATE BENDING PROGRAM

Axisymmetrically loaded elastic circular plate is modelled as an assemblage of annular plate elements. This type of element was used by Smith (1970) for the analysis of circular plate resting on idealized (Winkler) and actual foundations. Wood (1977) used rectangular bending elements as described by Cheung and Zienkiewicz (1965) to subdivide a quarter of the circular raft. Hooper (1974,1975), on the other hand, used eight noded isoparametric quadrilateral elements to model the flexural behaviour of a circular raft foundation. However, from the symmetric geometry and the loading considerations, it was felt that the annular elements would perform better in the study of the axisymmetric problem related to the flexure of a thin circular plate resting on an elastic medium.

2.3.1 Derivation of the Element Stiffness Matrix

Fig. 2.9 shows an annular plate element with the associated degrees of freedom. An approach similar to that described in Section 2.2.2 could be used here to obtain the element stiffness matrix for the annular plate. A simpler technique employs a Raleigh-Ritz type formulation.

Considering the small deflection Poisson-Kirchhoff thin plate theory, the elastic strain energy of a circular plate subjected to the axisymmetric deflection $\omega(r)$ is composed of only the flexural energy U_b , given by (Timoshenko and Woinowsky-Krieger, 1959):

$$U_b = \frac{D}{2} \iint_{\text{Area}} \left[\{\nabla^2 \omega(r)\}^2 - \frac{2(1-\nu_b)}{r} \frac{d\omega(r)}{dr} \frac{d^2\omega(r)}{dr^2} \right] r dr d\theta \quad (2.24)$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} ; \quad D = \frac{E_b t^3}{12(1-\nu_b^2)} \quad (2.25)$$

and E_b and ν_b are respectively the Young's modulus and Poisson's ratio for the plate material. Eqn. (2.24) can be re-arranged into the matrix form (Kirkhope and Wilson, 1976) as

$$U_b = \frac{D}{2} \iint_{\text{Area}} \{\chi\}^T [V] \{\chi\} r dr d\theta \quad (2.26)$$

where

$$\{\chi\} = \begin{Bmatrix} -\frac{d^2\omega(r)}{dr^2} \\ -\frac{1}{r} \frac{d\omega(r)}{dr} \end{Bmatrix} \quad (2.27a)$$

and

$$[V] = \begin{bmatrix} 1 & v_b \\ v_b & 1 \end{bmatrix} \quad (2.27b)$$

The highest order polynomial that can be utilized as a deflection function is

$$\omega(r) = \alpha_1 + \alpha_2 r + \alpha_3 r^2 + \alpha_4 r^3 \quad (2.28)$$

The generalized coordinates $\alpha_i (i=1,2,3,4)$ can be now related to the nodal displacements by the following expression:

$$\begin{Bmatrix} \omega_i \\ \theta_i \\ \omega_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & r_i & r_i^2 & r_i^3 \\ 0 & 1 & 2r_i & 3r_i^2 \\ 1 & r_j & r_j^2 & r_j^3 \\ 0 & 1 & 2r_j & 3r_j^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (2.29)$$

Substituting the expression for $\omega(r)$ in Eqn. (2.27a) and using Eqns. (2.25) and (2.27b), U_b in Eqn. (2.26) can be written as (see e.g.

Kirkhope and Wilson, 1976):

$$U_b = \begin{Bmatrix} \omega_i \\ \theta_i \\ \omega_j \\ \theta_j \end{Bmatrix}^T [A^{-1}]^T [K_\alpha] [A^{-1}] \begin{Bmatrix} \omega_i \\ \theta_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (2.30)$$

where

$$[A] = \begin{bmatrix} 1 & r_i & r_i^2 & r_i^3 \\ 0 & 1 & 2r_i & 3r_i^2 \\ 1 & r_j & r_j^2 & r_j^3 \\ 0 & 1 & 2r_j & 3r_j^2 \end{bmatrix} \quad (2.31)$$

and

$$[K_\alpha] = \pi D \begin{bmatrix} 0 & 0 & 0 & 0 \\ \ln \frac{r_j}{r_i} & 2(1+\nu_b)(r_j-r_i) & \frac{3}{2}(1+2\nu_b)(r_j^2-r_i^2) & \\ & 4(1+\nu_b)(r_j^2-r_i^2) & 6(1+\nu_b)(r_j^3-r_i^3) & \\ \text{Symmetric} & & \frac{9}{4}(5+4\nu_b)(r_j^4-r_i^4) \end{bmatrix} \quad (2.32)$$

The minimization of U_b with respect to the nodal displacements will result in a set of simultaneous equations, from which the element stiffness matrix $[K]_e$ can be extracted.

$$[K]_e = 2[A^{-1}]^T [K_\alpha] [A^{-1}] \quad (2.33)$$

It is to be noted that the matrix $[K_\alpha]$ contains the term $\ln \frac{r_j}{r_i}$ which approaches infinity as $r_i \rightarrow 0$ (i.e. the annular plate element becomes a solid element). From a computational point of view, this difficulty can be eliminated by allowing a small circular hole to be located at the center of the plate. This approximation does not cause a significant error in the displacement, but the flexural moments and

stresses are in error. A more accurate approach, therefore, would be to use solid circular element with modified degrees of freedom.

2.3.2 Special Circular Plate Bending Element

A solid circular plate element of radius r_j along with the associated degrees of freedom is shown in Fig. 2.10. Due to the complete symmetry of plate deflection, rotation θ_1 at the center of the plate is zero, hence it need not be considered as an active degree of freedom. The appropriate form of the deflection function is

$$\omega(r) = \alpha_1 + \alpha_2 r^2 + \alpha_3 r^3 \quad (2.34)$$

Now, the element stiffness matrix $[K]_e$ can be derived following the same procedure outlined in the previous section. The closed form expressions for $[K]_e$ are

$$[K]_e = \frac{2\pi D}{r_j} \begin{bmatrix} 9 & -9 & \frac{9}{2} r_j \\ & 9 & -\frac{9}{2} r_j \\ \text{Symmetric} & & (\frac{13}{4} + \nu_b) r_j^2 \end{bmatrix} \quad (2.35)$$

2.3.3 Consistent Load Vector for Uniformly Distributed Load

Determination of the consistent load vector $\{F\}_e$ for uniformly distributed load on annular plate elements follows a method similar to that outlined in Section 2.2.3. Referring to Fig. 2.9, $\{F\}_e$ can be expressed as

$$\{F\}_e = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} = 2\pi q [A^{-1}]^T \begin{Bmatrix} \frac{r_j^2 - r_i^2}{2} \\ \frac{r_j^3 - r_i^3}{3} \\ \frac{r_j^4 - r_i^4}{4} \\ \frac{r_j^5 - r_i^5}{5} \end{Bmatrix} \quad (2.36)$$

where $[A]$ is obtained from Eqn. (2.31).

The corresponding expressions for the central plate element are

$$\{F\}_e = \begin{Bmatrix} F_i \\ F_j \\ M_j \end{Bmatrix} = 2\pi q [A^{-1}]^T \begin{Bmatrix} \frac{r_j^2}{2} \\ \frac{r_j^4}{4} \\ \frac{r_j^5}{5} \end{Bmatrix} \quad (2.37)$$

where

$$[A^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & r_j^2 & r_j^3 \\ 0 & 2r_j & 3r_j^2 \end{bmatrix}^{-1} \quad (2.38)$$

and q is the intensity of uniform pressure.

2.3.4 Flexural Moments and Shearing Forces

The radial component M_r and circumferential component M_θ of the flexural moments induced due to axisymmetric bending of the circular

plate can be calculated from the following relationships (Timoshenko and Woinowsky-Krieger, 1959).

$$M_r = -D \left(\frac{\partial^2 \omega(r)}{\partial r^2} + \frac{\nu_b}{r} \frac{\partial \omega(r)}{\partial r} \right) \quad (2.39)$$

$$M_\theta^* = -D \left(\nu_b \frac{\partial^2 \omega(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \omega(r)}{\partial r} \right) \quad (2.40)$$

Substituting the expression for $\omega(r)$ from Eqn. (2.28) and making use of Eqns. (2.29) and (2.31), the above equations can be written as follows: (see Appendix B)

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & \frac{\nu_b}{r} & 2(1+\nu_b) & 3(2+\nu_b)r \\ 0 & \frac{1}{r} & 2(1+\nu_b) & 3(2\nu_b+1)r \end{bmatrix} [A^{-1}] \{\delta\}_e \quad (2.41)$$

where

$$\{\delta\}_e = \begin{Bmatrix} \omega_i \\ \theta_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (2.42)$$

Thus, when $[A^{-1}]$ and $\{\delta\}_e$ for an annular element are known, the flexural moments can be determined at any point within the element. The corresponding expressions for the central element are

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & 2(1+\nu_b) & 3r(2+\nu_b) \\ 0 & 2(1+\nu_b) & 3r(1+2\nu_b) \end{bmatrix} [A^{-1}] \begin{Bmatrix} \omega_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (2.43)$$

*

Circumferential component of flexural moment is also denoted by M_t

Shearing force per unit length Q_r of the circular plate can similarly be determined from

$$Q_r = D \left[\frac{\partial^3 \omega(r)}{\partial r^3} + \frac{1}{r} \frac{\partial^2 \omega(r)}{\partial r^2} - \frac{1}{r^2} \frac{\partial \omega(r)}{\partial r} \right] \quad (2.44)$$

The final expressions for Q_r at any point within an element and at a distance r from the center of the plate are (see Appendix B)

$$Q_r = D \left\{ 0 - \frac{1}{r^2} \quad 0 \quad 9 \right\} [A^{-1}] \{\delta\}_e \quad (2.45)$$

when $r \neq 0$ (annular element)

and

$$Q_r = D \left\{ 0 \quad 0 \quad 9 \right\} [A^{-1}] \begin{Bmatrix} \omega_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (2.46)$$

when $r=0$ (solid circular element)

2.3.5 Numerical Accuracy

The finite element techniques outlined in the preceding sections (2.3.1 to 2.3.4) have been used to examine the axisymmetric deformations of a circular plate having arbitrary boundary conditions. It is, however, desirable that the numerical accuracy of the computed results be checked with known theoretical solutions. The following problems were solved for this purpose:

- a) A simply supported circular plate under the action of, uniform stress:
 - i) fully loaded; ii) partially loaded.

b) A clamped circular plate under the action of uniform stress:

i) fully loaded; ii) partially loaded.

Graphical comparisons between the finite element results and exact solutions (Timoshenko and Woinowsky-Krieger, 1959) for the above problems are presented in Figs. 2.11 through 2.14. Again, a very close agreement is obtained between the finite element computations and the corresponding exact solutions. The finite element results for $\omega(r)$ compare almost exactly with the theoretical curves. However, the computed values of flexural moments exhibit a maximum error which is of the order of 4.2 percent.

2.4 INTERFACE CONDITION

To develop a complete finite element model for the present interaction problem, it is essential that some assumptions must be made with regard to the interface conditions that can exist between the flexible plate and the elastic medium. For the present study, the contact between the two elastic materials is considered to be smooth. The interface could also have been assumed as either rough[†], or frictional^{††}.

† A rough contact assumes that interfacial slip may or may not occur. This condition is simulated in the finite element analysis by introducing interface elements.

†† A frictional contact may be considered as coulomb frictional, or finite frictional. Coulomb frictional contact assumes that the shear stress τ at the interface is proportional to the applied stress intensity q , that is

$$\tau = \mu q \quad (2.47)$$

where μ is the coefficient of friction between the two surfaces. Finite frictional contact, on the other hand, considers that τ assumes a constant value when the intensity of the applied stress exceeds certain limiting values (Fig. 2.15).

A smooth contact assumes that no frictional forces develop at the interface and that interfacial slip can occur. In the finite element analysis, this is accounted for by matching only the vertical component of displacement at the interface nodes. This assumption has a two-fold effect: Firstly, it increases the central and maximum differential deflections of a plate resting on an elastic medium. This point has been illustrated in Fig. 2.1. Secondly, this approximation reduces the normal stresses at the interface. For a rigid circular punch on homogeneous elastic half-space, the distribution of contact stress for various interface conditions is shown in Fig. 2.16 (Hooper, 1974). For $\nu_s = 0.0$, it may be noted that for the case of smooth contact the normal stress at the center of the plate is approximately 15.0 percent lower than the corresponding stress for adhesive contact. As ν_s approaches 0.5, the effect of interface conditions on contact stress distribution becomes insignificant (Spence, 1973). For a very flexible plate, the distribution of normal stresses at the interface remains unaffected due to the change of interface condition.

2.5 A FINITE ELEMENT COMPUTER PROGRAM TO SOLVE PRESENT INTERACTION PROBLEM

Axisymmetric solid and plate bending programs of sections 2.2 and 2.3 respectively when combined, together with the assumed frictionless interface condition, gives a complete finite element program which is capable of solving the present interaction problem. A brief and simplified flowchart of this program is shown in Fig. 2.17. A complete listing of the statements will be found in Appendix A.

2.5.1 Main Features

1. The program is written in FORTRAN IV language and needs approximately 40K words core space to solve a problem having 350 degrees of freedom. It is designed to run in the batch mode and was tested on a Honeywell Sigma-9 computer.
However, the program is machine independent and should run on any installation having a FORTRAN compiler and having similar attributes.
2. The program takes input in rigid format. Preparation of such an input is very much troublesome because the computer misinterprets information which is not correctly positioned on the cards. To overcome this difficulty, a separate program was written which runs in the time-sharing mode, takes input in free format[†] and writes them in the rigid format on a specified file. This file is later used as an input file for the main program.
3. Before using the main program for solving a problem, it is desirable that probable mistakes in the data input be detected and eliminated. A separate program was written for this purpose. This program also runs in the time-sharing mode, and with the aid of a package program CALPLOT (see ref. CALPLOT) draws finite element grids on a CRT (Cathode Ray Tube) screen. A hard copy of this grid is also provided by the computer at the request of the user.

[†] To achieve free format input and to make the program machine independent, a special subroutine, called REED, has been used in this program. The general purpose 'free format' subroutine has been developed by Dr. E.W. Wright of Public Works, Canada.

4. The program calculates nodal displacements and forces, and the average displacement of the plate. It also calculates the flexural moments (M_r , M_θ) and shearing forces at each node of the plate elements. At the user's request, the stress components (σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} , σ_{rz}), the principal stresses (σ_1 , σ_2 , σ_3) and the stress invariants (I_1 , I_2) at the centroid of solid elements are also evaluated.
5. The determination of the contact stress distribution from conventional forms of finite element analysis is not entirely straightforward especially in cases where the elastic properties of the solids in contact with each other are strongly dissimilar. For the case of a smooth rigid circular punch indenting a homogeneous isotropic elastic half-space, normal contact stresses evaluated at the interface node are found to fluctuate widely about the theoretical line (Hooper, 1974). However, closest agreement with the theoretical curve is obtained if stresses at the centroid of solid elements which are immediately beneath the plate region are used for this purpose. Such a procedure was used in all subsequent analyses.

2.5.2 Limitations

1. The constitutive matrix $[D]$ in Eqn. (2.7) contains the term $(1-2\nu_s)$ in the denominator which becomes zero for an incompressible solid ($\nu_s=0.5$) and takes a very small value as ν_s approaches 0.5. The corresponding stiffness terms become very large.

This indicates that the equilibrium method of finite element formulation can not be used for an incompressible elastic medium. To avoid this numerical difficulty, a different type of variational formulation, the so-called "hybrid finite element method", was used by Carrier and Christian (1973). This method was originally developed by Herrmann (1965) and a general description of it is given by Desai and Abel (1972). In the present work, however, attention is restricted to the more practical nearly incompressible solid medium rather than the truly incompressible elastic medium. From a series of trial runs, it was found that a value of $\nu_s = 0.49$ can approximate a nearly incompressible solid medium without a significant loss of accuracy.

2. For a rigid circular raft on homogeneous elastic half-space, Boussinesq (1885) found that the normal contact stress becomes singular at the edge of the raft. The finite element analysis, on the other hand, yields a finite value at the edge. Thus, for a rigid plate resting on an isotropic elastic half-space, the finite element method cannot truly represent the exact mathematical model. Therefore, for such a case, a larger error may be expected.

In a real soil-foundation interaction problem, always some local yielding takes place at regions of high stresses. This usually results in a re-distribution of the contact pressure. With the type of formulation used, the finite element method cannot take into account the effects of

such plastic deformations. For most practical purposes, the effect of local yielding on displacements of a plate on elastic half-space is negligible, but the flexural moments M_r and M_θ are in error (Brown, 1968).

2.6 CIRCULAR PLATE RESTING ON HOMOGENEOUS ELASTIC HALF-SPACE

2.6.1 General

The analysis performed in this section assumes that the soil medium can be idealized as a homogeneous, isotropic, elastic half-space. For finite element modelling of the half-space region, a vertical boundary is imposed at a distance $R=20a$ from the axis of symmetry and a smooth horizontal rigid boundary is introduced at a depth $H=20a$. At these distances, the boundaries have virtually no effect on the flexural interaction of the plate for small values of ν_s (less than 0.47), irrespective of whether the boundary is perfectly smooth or completely rough. In the present analysis, however, both the vertical and horizontal boundaries are considered to be perfectly smooth. Such a representation of the elastic half-space region is not entirely new; other investigators (see e.g. Carrier and Christian, 1973; Hooper, 1974, 1975; Boswell and Scott, 1975) have used similar models (Table 2.2) of the same half-space.

Several finite element grids were examined in an attempt to optimize the bandwidth and storage requirement of the computer. The grid that was ultimately selected is shown in Fig. 2.18. It consists of 270 elements connected at 160 nodes. This grid was used with a range

of relative rigidities ($10^{-2} \leq K_r \leq 10^2$) to examine the flexural interaction between the plate and the elastic half-space region. The finite element results were then compared with available solutions (e.g. Borowicka, 1936; Boussinesq, 1885; Brown, 1969a,b; Selvadurai, 1979a,etc.).

2.6.2 Comparison of Finite Element Results with Existing Solutions

Completely flexible plate: The limiting solution for the completely flexible plate can be obtained by considering the problem where the normal load of uniform intensity is applied directly to the surface of the half-space. For a uniform stress q applied normal to the surface over a circular area of radius a , Boussinesq's (1885) results for maximum surface displacement $\omega(o)$ and differential surface displacement ($\omega(o)-\omega(a)$) between the center and edge of the loaded area are given by

$$\omega(o) = \frac{2(1-\nu_s^2)}{E_s} qa \quad (2.49)$$

$$\Delta\omega = \omega(o) - \omega(a) = \left(2 - \frac{4}{\pi}\right) \left(\frac{1-\nu_s^2}{E_s}\right) qa \quad (2.50)$$

These equations may be re-arranged into non-dimensional form as

$$I_{\omega(o)} = \frac{\omega(o) E_s}{qa} = 2(1-\nu_s^2) \quad (2.51)$$

$$I_{\Delta\omega} = \frac{\Delta\omega E_s}{qa} = \left(2 - \frac{4}{\pi}\right) (1-\nu_s^2) \quad (2.52)$$

where $I_{\omega(o)}$ and $I_{\Delta\omega}$ are influence coefficients for the central and differential deflections respectively. Fig. 2.19 compares the finite element results for $I_{\omega(r)}$ with the theoretical values developed for $\nu_s=0$ (see e.g. Jumikis, 1969). For $I_{\omega(o)}$, the computed result is approximately 5.0 percent less than the theoretical value. On the other hand, the differential deflection $I_{\Delta\omega}$ is over-estimated by approximately 1.3 percent. The corresponding results for $\nu_s=0.49$ are shown in Fig. 2.20. For this case, however, the finite element result for $I_{\omega(o)}$ is underestimated by approximately 8.5 percent. The corresponding error on $I_{\Delta\omega}$ is relatively less, to the order of approximately 7.0 percent. For both $\nu_s=0.0$ and 0.49 , the plate is assumed to be completely flexible when K_r is approximately $0.0\bar{r}$.

Completely rigid plate: A second bound to the problem discussed in this Chapter can be obtained by considering the case of a rigid circular punch indenting the half-space. For smooth contact, Boussinesq's (1885) solution for the normal displacements are

$$\omega(r) = \frac{\pi}{2} \frac{(1-\nu_s^2)}{E_s} qa \quad (0 \leq r \leq a) \quad (2.53)$$

$$\omega(r) = \frac{qa(1-\nu_s^2)}{E_s} \sin^{-1}(r/a) \quad (a \leq r < \infty) \quad (2.54)$$

Again, these equations can be expressed in the non-dimensional form as

$$I_{\omega(r)} = \frac{\omega(r)E_s}{aq} = \frac{\pi}{2} (1-\nu_s^2) \quad (0 \leq r \leq a) \quad (2.55)$$

$$I_{\omega(r)} = \frac{\omega(r)E_s}{aq} = (1-\nu_s^2)\sin^{-1}(r/a) \quad (a \leq r < \infty) \quad (2.56)$$

It may be recalled that in the finite element analysis an infinitely rigid plate is represented by a plate having finite but high relative rigidity (K_r). From a set of trial computer runs, it was observed that $K_r=100$ represents an approximately rigid plate. Fig. 2.21 compares the finite element results for $I_{\omega(r)}$ with the theoretical values developed for $\nu_s=0$. For $I_{\omega(0)}$, the finite element method underestimates the exact value by approximately 11.0 percent. The corresponding comparison for nearly incompressible solid medium ($\nu_s=0.49$) is shown in Fig. 2.22. In this case, the computed results for $I_{\omega(r)}$ ($0 \leq r \leq a$) underpredict the exact solution by approximately 26.0 percent.

Under certain circumstances, the displacement formulation for the finite element method provides an upper bound to the true stiffness of the structure (Fraeijs de Veubeke, B., 1964). In other words, the stiffness coefficients for a given displacement model have magnitudes higher than those for the exact solution. Thus, under a given load, the simulated structure deforms less than the actual structure. For the present problem, the simulated structure becomes particularly stiff when the supporting soil medium is truly or nearly incompressible ($\nu_s=0.49$). For such a case, the soil medium can propagate almost any loading to the boundaries with little or no decay, giving rise to unacceptably small deflections of the plate. However, when the boundary conditions were

changed to an unrestricted vertical boundary for the side, and free to move laterally horizontal boundary for the bottom, approximately 9.0 percent improvement in the results was obtained.

Contact stress distribution: For frictionless contact, the Boussinesq's solution for contact stress $p(r)$ at the interface between a rigid circular plate and a homogeneous elastic half-space is

$$p(r) = \frac{q}{2\sqrt{1 - \frac{r^2}{a^2}}} \quad (2.57)$$

In non-dimensional form

$$I_{p(r)} = \frac{p(r)}{q} = \frac{1}{2\sqrt{1 - \frac{r^2}{a^2}}} \quad (2.58)$$

where $I_{p(r)}$ is the influence coefficient for contact stress at a distance r from the center of the plate.

The finite element results for $I_{p(r)}$ are shown graphically in Fig. 2.23. At the center of the raft (for $k_r=100$), the finite element result underestimates the corresponding exact value by approximately 9.5 percent.

2.6.3 Comments on the Overall Accuracy

From the comparison made in the preceding section, it can be observed that in the context of the present interaction problem the finite element method always underestimates the actual deflections. The

differential deflections, however, determined from the finite element method compare favourably with the equivalent exact solutions. It is further observed that the magnitude of error increases when the solid medium is nearly incompressible, and it becomes quite significant for the case of a rigid plate. Thus, for a rigid plate on an incompressible homogeneous isotropic elastic half-space, the computed results for plate deformations are approximately 17.0 percent lower than the corresponding exact values. With the exception of an infinitely rigid plate on an incompressible solid medium, the finite element results are within the range of acceptable accuracy, for geotechnical application of the results. Therefore, it may be concluded that the finite element analysis, described in this thesis, will yield acceptable results for a plate of finite rigidity resting on an elastic solid medium which is compressible or nearly incompressible.

Being satisfied with the overall performance of the finite element analysis in the limiting cases, further computations were carried out for values of relative plate rigidities in the range $10^{-2} \leq K_r \leq 10^2$. In many cases, the computed results were compared with values obtained from other numerical techniques (e.g. Brown, 1969a,b; Hooper, 1975; Selvadurai, 1979a) and satisfactory agreement was obtained. In Fig. 2.24, the computed results of differential deflections are compared with values obtained from the energy method (Selvadurai, 1979a). Other comparisons, however, are not shown. A summary of the results is presented in Figs. 2.25 to 2.28 and in Tables 2.3 to 2.7.

2.7 EXTENSION OF THE WORK

Having completed the analysis of a uniformly loaded circular plate resting on a homogeneous elastic half-space, it is of interest to investigate the effect of some further parameters which may influence the flexural behaviour of an elastic plate resting on an isotropic elastic solid medium. Giving due consideration to the numerous parameters involved, attention is focussed only on the following items:

- i) Influence of a soil medium of finite layer depth
- ii) Influence of differential plate thickness
- iii) Influence of the external loading configuration
- iv) Influence of non-homogeneous elastic half-space.

Throughout this work, the elastic properties E_b and ν_b of the plate are considered to be 3.0×10^6 psi and 0.3 respectively, and a rigid frictionless or unrestrained vertical boundary is imposed at a distance of $10a$ from the center of the plate. Furthermore, a completely flexible plate is denoted by $K_r = 0.01$, and an infinitely rigid plate is represented by $K_r = 100.0$. The effects of these parameters are discussed in the subsequent chapters.

Rigid Plate

Smooth Contact

$$w(o) = \frac{P(1-v_s^2)}{2 E_s a}$$

where E_s = Young's modulus of the soil medium
 v_s = Poisson's ratio of the soil medium.

For $v_s = 0.0$, $\{w(o)\}_{\text{smooth}} \times 0.91 = \{w(o)\}_{\text{rough}}$;

For $v_s = 0.5$, $\{w(o)\}_{\text{smooth}} = \{w(o)\}_{\text{rough}}$.

Rigid Plate

Rough or Bonded Contact

$$w(o) = \frac{P F_1 (1-v_s^2)}{2 E_s a}$$

where

$$F_1 = \frac{1-2v_s}{(1-v_s) \ln(3-4v_s)}$$

Flexible Plate or Surface Load

Smooth Contact

$$w(o) = \frac{2(1-v_s^2)qa}{E_s}$$

Flexible Plate or Surface Load

Rough or Bonded Contact

$$w(o) = \frac{2 F_2 (1-v_s^2)qa}{E_s}$$

where

$$F_2 = 1 - \frac{1}{2\pi} \left(\frac{1-2v_s}{1-v_s} \right)^2$$

For $v_s = 0.0$, $\{w(o)\}_{\text{smooth}} \times 0.84 = \{w(o)\}_{\text{rough}}$;

For $v_s = 0.5$, $\{w(o)\}_{\text{smooth}} = \{w(o)\}_{\text{rough}}$.

Fig. 2.1. Effect of interface conditions on central deflection of circular plates (after Hooper, 1974).

a) Three Noded Triangular Element

b) Six Noded Triangular Element

c) Quadrilateral Element Composed of Four Triangles

d) Eight Noded Quadrilateral Element

FIGURE 2.2 CROSS-SECTIONS OF AXISYMMETRIC RING ELEMENTS

FIGURE 2.3 A TYPICAL AXISYMMETRIC TRIANGULAR RING ELEMENT

FIGURE 2.4 ASSUMED DEFLECTION MODEL DUE TO UNIFORM PRESSURE

FIGURE 2.5 THICK CYLINDER UNDER INTERNAL PRESSURE; FINITE ELEMENT SUBDIVISIONS

FIGURE 2.6 THICK CYLINDER UNDER INTERNAL PRESSURE:
COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.7 SURFACE LOAD ON HOMOGENEOUS ELASTIC SOLID MEDIUM OF FINITE DEPTH
($H/a=5.0$): FINITE ELEMENT SUBDIVISIONS

FIGURE 2.8 SURFACE LOAD ON HOMOGENEOUS ELASTIC SOLID MEDIUM OF FINITE DEPTH ($H/a = 5.0$): COMPARISON OF $I_w(r)$

FIGURE 2.9 ANNULAR PLATE ELEMENT WITH ASSOCIATED DEGREES OF FREEDOM

FIGURE 2.10 SOLID CIRCULAR PLATE ELEMENT WITH ASSOCIATED DEGREES OF FREEDOM

FIGURE 2.11 SIMPLY SUPPORTED CIRCULAR PLATE
UNDER UNIFORM PRESSURE: COMPARISON OF
FINITE ELEMENT RESULTS

FIGURE 2.11 (Cont'd) SIMPLY SUPPORTED CIRCULAR PLATE UNDER UNIFORM PRESSURE: COMPARISON OF FINITE ELEMENT RESULTS

● FIGURE 2.12 PARTIALLY LOADED SIMPLY SUPPORTED CIRCULAR PLATE:
COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.12 PARTIALLY LOADED SIMPLY SUPPORTED CIRCULAR PLATE:
(Cont'd) COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.13 CLAMPED CIRCULAR PLATE UNDER UNIFORM PRESSURE:
COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.14 PARTIALLY LOADED CLAMPED CIRCULAR PLATE:
COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.14 (Cont'd) PARTIALLY LOADED CLAMPED CIRCULAR PLATE:
COMPARISON OF FINITE ELEMENT RESULTS

FIGURE 2.15 FINITE FRICTIONAL CONTACT

FIGURE 2.16 CONTACT STRESS DISTRIBUTION BETWEEN RIGID CIRCULAR PUNCH AND HOMOGENEOUS ELASTIC HALF-SPACE (after Spence, 1973)

* () Refers to the name of the SUBROUTINE.

FIGURE 2.17. OUTLINE FLOW DIAGRAM.

FIGURE 2.17 (cont'd). OUTLINE FLOW DIAGRAM.

FIGURE 2.17 (cont'd) OUTLINE FLOW DIAGRAM.

FIGURE 2.17 (cont'd) OUTLINE FLOW DIAGRAM.

FIGURE 2.18 FINITE ELEMENT SUBDIVISIONS FOR ELASTIC HALF-SPACE REGION

FIGURE 2.19 COMPLETELY FLEXIBLE PLATE ON HOMOGENEOUS ELASTIC. HALF-SPACE: COMPARISON OF $I_w(r)$ FOR $\nu_s = 0.0$

FIGURE 2.20 COMPLETELY FLEXIBLE PLATE ON HOMOGENEOUS ELASTIC HALF-SPACE: COMPARISON OF $I_w(r)$ FOR $\nu_s = 0.49$

FIGURE 2.21 RIGID PLATE ON HOMOGENEOUS ELASTIC HALF-SPACE:
COMPARISON OF $I_w(r)$ FOR $\nu_s = 0.0$

FIGURE 2.22 RIGID PLATE ($K_L = 100$) ON HOMOGENEOUS ELASTIC HALF-
SPACE: COMPARISON OF $I_w(r)$ FOR $\nu_s = 0.49$

FIGURE 2.24 CIRCULAR PLATE ON HOMOGENEOUS ELASTIC HALF-SPACE:
COMPARISON OF DIFFERENTIAL DEFLECTION FOR $\nu_s = 0.0$

FIGURE 2.25 PLATES ON HOMOGENEOUS ELASTIC HALF-SPACE:
SURFACE DEFORMATIONS FOR VARIOUS RELATIVE RIGIDITIES (K_r)

FIGURE 2.26 PLATES ON HOMOGENEOUS ELASTIC HALF-SPACE:
CONTACT STRESS DISTRIBUTION FOR VARIOUS
RELATIVE RIGIDITIES (K_r) FOR $\nu_s = 0.0$

FIGURE 2.27 PLATES ON HOMOGENEOUS ELASTIC HALF-SPACE:
DISTRIBUTION OF BENDING MOMENTS FOR VARIOUS
RELATIVE RIGIDITIES (K_r) AND $\nu_s = 0.0$

FIGURE 2.28 PLATES ON HOMOGENEOUS ELASTIC HALF-SPACE:
DISTRIBUTION OF NORMAL STRESS (σ_{zz}) ALONG THE
AXIS OF SYMMETRY

2

A

Table 2.1. Surface load on homogeneous elastic solid medium
of finite depth ($H/a=5.0$): Influence coefficients for
surface displacement

$$I_{\omega(r)} = \frac{\omega(r)E_s}{aq} \quad (2.23)$$

r/a	$I_{\omega(r)}$				
	$\nu_s=0.0$	$\nu_s=0.3$	$\nu_s=0.45$	$\nu_s=0.47$	$\nu_s=0.49$
0.00	1.7136	1.5718	1.3648	1.3058	1.2296
0.20	1.6740	1.5274	1.3084	1.2462	1.1712
0.40	1.5962	1.4552	1.2338	1.1650	1.0800
0.60	1.4706	1.3412	1.1248	1.0482	0.9454
0.80	1.2774	1.1670	0.9640	0.8828	0.7684
1.00	0.9664	0.8732	0.7100	0.6440	0.5506
1.20	0.6802	0.6052	0.4882	0.4450	0.3860
1.45	0.4924	0.4382	0.3518	0.3220	0.2832
1.80	0.3460	0.3074	0.2474	0.2302	0.2104
2.30	0.2244	0.1978	0.1598	0.1534	0.1488
3.00	0.1284	0.1083	0.0772	0.0708	0.0670
4.05	0.0573	0.0430	0.0178	0.0011	0.0061
5.25	0.0203	0.0092	-0.0130	-0.0201	-0.0270
6.55	0.0042	-0.0048	-0.0234	-0.0279	-0.0300
8.15	-0.0026	-0.0108	-0.0284	-0.0326	-0.0349
10.00	-0.0033	-0.0104	-0.0250	-0.0275	-0.0281

Table 2.2. Finite element representation of elastic half-space.

Investigator	Radial Distance R	Depth H	Boundary Condition at Side	Boundary Condition at Bottom
Carrier and Christian (1973)	8.5a	7.7a	R or UR	R
Hooper (1974)	6.5a	6.5a	R or UR	R
Boswell & Scott (1975)	8.0a	8.0a	R	R
Present Work	20a	20a	R or UR	R

UR = Unrestrained or free boundary
 R = Frictionless rigid boundary

Table 2.3. Uniformly loaded circular plate on homogeneous elastic half-space: Influence coefficients for surface displacement for $\nu_s = 0.0$

r/a	$I_{\omega(r)}$				
	$K_r=0.01$	$K_r=0.10$	$K_r=1.00$	$K_r=10.0$	$K_r=100.0$
0.000	1.8962	1.7498	1.4456	1.3922	1.3824
0.167	1.8570	1.7338	1.4450	1.3916	1.3824
0.333	1.8162	1.6856	1.4322	1.3902	1.3822
0.500	1.7276	1.6064	1.4116	1.3878	1.3820
0.667	1.5948	1.4990	1.3850	1.3846	1.3816
0.833	1.4058	1.3700	1.3544	1.3810	1.3812
1.000	1.1582	1.2302	1.3222	1.3774	1.3810
1.167	0.8966	0.9244	0.9650	0.9762	0.9776
1.333	0.7378	0.7530	0.7754	0.7816	0.7824
1.500	0.6286	0.6378	0.6516	0.6554	0.6558
1.700	0.5342	0.5398	0.5484	0.5508	0.5510
1.933	0.4554	0.4588	0.4642	0.4658	0.4660
2.267	0.3744	0.3764	0.3796	0.3806	0.3806
2.633	0.3100	0.3114	0.3132	0.3138	0.3138
3.100	0.2528	0.2536	0.2548	0.2550	0.2552
3.733	0.1996	0.2000	0.2006	0.2008	0.2008
4.667	0.1480	0.1481	0.1484	0.1485	0.1485
5.667	0.1114	0.1115	0.1117	0.1118	0.1118
7.000	0.0806	0.0806	0.0807	0.0808	0.0808

Table 2.4. Uniformly loaded circular plate on homogeneous elastic half-space: Influence coefficients for surface displacement for $\nu_s = 0.49$.

r/a	$I_{\omega(r)}$				
	$K_r = 0.01$	$K_r = 0.10$	$K_r = 1.00$	$K_r = 10.00$	$K_r = 100.00^*$
0.000	1.3716	1.2318	0.9664	0.8726	0.8608
0.167	1.3162	1.2182	0.9622	0.8720	0.8608
0.333	1.2754	1.1772	0.9496	0.8704	0.8606
0.500	1.2018	1.1100	0.9294	0.8680	0.8604
0.667	1.0914	1.0196	0.9034	0.8648	0.8600
0.833	0.9452	0.9118	0.8734	0.8612	0.8596
1.000	0.7802	0.7962	0.8420	0.8574	0.8594
1.167	0.6140	0.6336	0.6682	0.6798	0.6814
1.333	0.5222	0.5324	0.5522	0.5590	0.5598
1.500	0.4448	0.4542	0.4612	0.4782	0.4790
1.700	0.3648	0.3716	0.3842	0.3884	0.3890
1.933	0.2998	0.3042	0.3124	0.3152	0.3156
2.267	0.2384	0.2408	0.2454	0.2468	0.2470
2.633	0.1913	0.1926	0.1951	0.1960	0.1961
3.100	0.1501	0.1508	0.1523	0.1528	0.1529
3.733	0.1110	0.1114	0.1122	0.1124	0.1125
4.667	0.0725	0.0728	0.0732	0.0733	0.0733
5.667	0.0448	0.0449	0.0451	0.0452	0.0452
7.000	0.0242	0.0243	0.0244	0.0245	0.0245

* Corresponds to frictionless rigid boundary.

Table 2.5. Uniformly loaded circular plate on homogeneous elastic half-space: Distribution of flexural moments.

r/a	M_r/qa^2 for $\nu_s=0.0$				
	$K_r=0.01$	$K_r=0.10$	$K_r=1.00$	$K_r=10.00$	$K_r=100.0$
0.000	0.0026	0.0326	0.0896	0.1057	0.1066
0.167	0.0023	0.0321	0.0868	0.1023	0.1041
0.333	0.0033	0.0316	0.0797	0.0932	0.0947
0.500	0.0042	0.0293	0.0671	0.0776	0.0788
0.667	0.0052	0.0243	0.0494	0.0561	0.0569
0.833	0.0055	0.0156	0.0269	0.0300	0.0303
1.000	0.00	0.00	0.00	0.00	0.00

M_t/qa^2					
0.000	0.0026	0.0326	0.0896	0.1057	0.1066
0.167	0.0024	0.0323	0.0877	0.1033	0.1052
0.333	0.0032	0.0322	0.0838	0.0982	0.1000
0.500	0.0038	0.0310	0.0765	0.0893	0.0908
0.667	0.0044	0.0286	0.0664	0.0769	0.0781
0.833	0.0047	0.0241	0.0534	0.0615	0.0624
1.000	0.0036	0.0172	0.0384	0.0442	0.0449

Table 2.6. Uniformly loaded circular plate on homogeneous elastic half-space: Distribution of contact pressure for $\nu_s = 0.0$

	$P(r) = \sigma_{zz}/q$				
r/a	$K_r = 0.01$	$K_r = 0.10$	$K_r = 1.00$	$K_r = 10.0$	$K_r = 100.0$
0.111	1.1340	1.1130	0.7193	0.4986	0.4671
0.278	1.1020	1.0560	0.7003	0.5105	0.4836
0.444	1.0820	0.9875	0.6939	0.5515	0.5315
0.611	1.0290	0.8963	0.7114	0.6335	0.6227
0.780	0.8538	0.7745	0.7870	0.8046	0.8073
0.946	0.5252	0.6982	1.0680	1.2440	1.2690

Table 2.7. Uniformly loaded circular plate on homogeneous elastic half-space. Distribution of normal stress (σ_{zz}) along the centerline of the plate for $\nu_s = 0.0$.

	σ_{zz}/q				
z/a	$K_r = 0.01$	$K_r = 0.10$	$K_r = 1.00$	$K_r = 10.0$	$K_r = 100.0$
0.111	1.016	0.9953	0.6634	0.4749	0.4471
0.222	0.9718	0.9453	0.6466	0.4804	0.4567
0.400	0.9378	0.9025	0.6444	0.5048	0.4850
0.611	0.8156	0.7797	0.5935	0.4962	0.4823
0.867	0.6597	0.6310	0.5156	0.4564	0.4480
1.200	0.4932	0.4752	0.4139	0.3829	0.3785
1.622	0.3422	0.3334	0.3060	0.2923	0.2904
2.122	0.2322	0.2283	0.2166	0.2107	0.2099
2.733	0.1517	0.1501	0.1453	0.1429	0.1426
3.467	0.1005	0.0997	0.0976	0.0966	0.0964
4.333	0.0662	0.0659	0.0650	0.0646	0.0645
5.389	0.0439	0.0438	0.0434	0.0432	0.0431
6.722	0.0283	0.0283	0.0281	0.0280	0.0280

3. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON AN ISOTROPIC ELASTIC SOLID MEDIUM

In the previous chapter, the classical problem relating to the axisymmetric flexure of a circular plate resting on a homogeneous isotropic elastic half-space has been examined by the finite element method. Throughout the analysis, it was assumed that the plate is of uniform thickness and that a uniform pressure is applied over the entire region of the plate. Such assumptions are not realistic for many practical situations (Bowles, 1968). However, the application of the finite element method need not be restricted to these simplifying assumptions. The non-classical problems (e.g. thickness variations, non-homogeneity, etc.) can also be easily examined by the finite element method without encountering additional difficulty.

To investigate the effect of some selected parameters, several problems relating to the flexural behaviour of a circular plate having finite stiffness resting on a deep elastic solid medium ($H/a=5.0$) [see e.g. Fig. 3.23] were solved. The results are presented in subsequent sections of this chapter.

3.1 EFFECT OF FINITE DEPTH OF SOIL MEDIUM

In many natural deposits, the layer of soil which supports a raft is comparatively shallow, and below this there may be a stratum of stiffer material. For such situations, the depth of the supporting medium can have an influence on the flexural behaviour of the raft.

To investigate the effect of finite thickness of the supporting layer, three different values of H/a (1.0, 2.5 and 5.0) were selected. A layer having $H/a=1.0$ may be considered as shallow whereas $H/a=2.5$ and 5.0 may be regarded as moderately deep and deep respectively. Considering the soil medium to be homogeneous and isotropic, computations were carried out for each of these three cases for values of relative raft rigidities in the range $10^{-2} \leq K_r \leq 10^2$. Poisson's ratios of the soil medium were taken as 0.0 and 0.49.

The results relating to the case of a shallow layer ($H/a=1.0$) are shown in Figs. 3.1 through 3.6. The corresponding results for moderately deep ($H/a=2.5$) and deep ($H/a=5.0$) layers are presented in Figs. 3.7 through 3.12 and Figs. 3.13 through 3.18 respectively.

The influence coefficients for surface deformation ($I_{\omega(r)}$) for various layer depths are presented in Tables 3.1 through 3.5. A summary of these results is shown in Fig. 3.19. It is observed that an increase of layer depth increases deflections considerably. For $\nu_s=0.0$, an increase of H/a from 1.0 to 2.5 increases the central deflection of a flexible raft ($K_r=0.01$) by approximately 49%. When the layer depth is further increased from 2.5a to 5.0a, the corresponding central deflection increases by only 17%. Thus, as the foundation becomes relatively deep ($H/a \geq 5.0$), the variation of raft displacement with change of layer depth becomes insignificant. Moreover, it is further observed that the relative rigidity of the raft K_r and Poisson's ratio ν_s have comparatively little effect on the variation of deflections due to the variation of layer depth.

The variation of differential deflection with K_r and layer depth is shown in Fig. 3.20 for values of $\nu_s = 0.0$ and 0.49 . From this figure, it is evident that the differential deflection increases with the increase of layer depth. This increment is very significant when the foundation is relatively shallow ($H/a < 2.0$) and becomes less significant for a deeper foundation. For a particular layer depth and relative rigidity K_r , the differential deflection reduces with the increase of ν_s . This effect of ν_s on Δw becomes more significant for a deeper foundation.

The finite element results for flexural moments for various relative rigidities of the raft and layer depths are tabulated in Tables 3.6 to 3.8. A summarized form of these results for the central moments is shown in Fig. 3.21. It is clearly seen that a rapid increase in moment occurs due to increase in raft stiffness. The flexural moments at a point increase further with increasing layer depth and take a maximum value when the foundation materials become nearly incompressible. For a deep or moderately deep foundation ($H/a \geq 2.0$), the circumferential moment at a point (Fig. 3.22) is greater than the radial moment. On the other hand, for a shallow foundation ($H/a < 2.0$), the radial moment is greater than the circumferential moment for a portion of the raft ($K_r < 1.0$).

Table 3.9 shows the effect of layer depth on contact stress distributions for $K_r = 1.0$ and 100.0 . As would be expected, increasing layer depths reduces the normal stress at the interface. The reduction becomes negligible for the case of a deep layer.

Figs. 3.6, 3.12 and 3.18 illustrate the variation of normal stress σ_{zz} along the centerline of the raft. It is seen that σ_{zz} increases rapidly with decreasing relative rigidity K_r , and for a given relative rigidity, it reduces with increasing depth. For a relatively deep layer ($H/a \geq 5.0$), σ_{zz} becomes quite insignificant at a depth of approximately $2.5a$.

A summary of the effect of depth of supporting soil stratum on the flexural behaviour of a circular plate is presented in Table 3.10.

3.2 EFFECT OF NON-UNIFORM PLATE THICKNESS

From economic and other considerations, it is sometimes necessary to use a circular raft having non-uniform thickness. This type of foundation is generally known as a pedestal foundation and is usually used as a tower silo foundation or as a storage tank foundation. For the safe operation of such structures, it is important to be able to predict the behaviour of the foundations under the desired loads so that procedures can be adopted for their design.

A typical non-uniform raft is shown in Fig. 3.23. For the purpose of the present investigation, it is assumed that the thickness of the outer region, from $0.6a$ to a , remains constant at t , whereas the thickness of the inner region (0 to $0.6a$) increases gradually. It is also assumed that the raft is supported by a deep ($H/a=5.0$) homogeneous elastic soil medium, and that it is acted upon by a uniform pressure of intensity q . α is a measure of non-uniformity defined as the ratio of the thickness of the inner region to the thickness of the outer region.

Values of α are selected in such a manner that the flexural behaviour of the raft could still be approximated by the thin plate theory.

Fig. 3.24 shows the variation of $I_{\omega(r)}$ for selected limiting values of α . It is easily observed that variation of α causes a considerable change in displacements under the raft; however, surface deformations outside the raft do not change significantly. As would be expected, an increase in the thickness of the inner region leads to a decrease in the center displacements and an increase in the corresponding edge displacements. Also, Tables 3.11 and 3.12, and Figs. 3.25 and 3.26 show that when α assumes a relatively high value ($\alpha > 1.4$), the inner region tends to behave as a rigid raft, and a change in the raft displacements gradually becomes insignificant with increasing α .

The variation of bending moment distributions with α is shown in Tables 3.13 and 3.14 and in Figs. 3.27 and 3.28. With increasing α , the flexural moments at the center of the raft increase. It is also noticed that for any value of α the radial moment M_r equals the circumferential moment M_θ at the center of the raft. Moreover, as would be expected, with increasing α , the distributions of flexural moments become discontinuous.

The effect of non-uniform thickness on the contact distribution is shown in Table 3.15 and Fig. 3.29. A glance at this figure shows that the contact stress at the center of the raft decreases significantly with increasing α . Thus, changing α from 1.0 to 1.2 leads to a reduction

of contact stress at the center of the raft by approximately 10%. A change of α from 1.2 to 1.4, however, decreases the corresponding contact stress by only about 6%. The contact stresses at or near the edge of a moderately stiff raft ($K_r > 1.0$) become very high with increasing α , and the probability of local bearing failure increases.

3.3 EFFECT OF LOADING CONFIGURATIONS

In many practical situations (e.g. a silo foundation), the load is not applied uniformly over the entire foundation surface. In areas where the bearing capacity of the soil medium is very low, the diameter of the foundation may be substantially larger than the diameter of the silo. To be able to predict the flexural behaviour of such a foundation, it is necessary to investigate the effect of partial loading configurations.

For the purpose of investigation, it is assumed that a raft of uniform thickness and finite stiffness ($K_r = 1.0$) is resting on a deep ($H/a = 5.0$) homogeneous elastic soil medium. The radius of the loaded area is denoted by βa , a being the radius of the raft. It is further assumed that for different values of β , the total load applied remains the same.

For three selected values of β , the influence coefficients for displacement $I_{w(r)}$ are presented in Table 3.16 and shown graphically in Fig. 3.30. It is observed that, for $\nu_s = 0.0$, decreasing the radius of the loaded area from a to $0.8a$ increases the central deflection by approximately 11.2%. The corresponding increase for nearly incompressible soil medium ($\nu_s = 0.49$) is about 14%. A further reduction of the loaded area increases the central deflection of the raft even more.

As would be expected, when the load is applied over a portion of the raft, the differential deflection is higher. For $\nu_s=0.0$, decreasing β from 1.0 to 0.8 and 0.8 to 0.6 increases the maximum differential deflection by approximately 75% and 41%, respectively. The corresponding increments for $\nu_s=0.49$ are 55% and 35%, respectively. Thus, irrespective of the Poisson's ratio of the soil medium, reduction of the loaded area has a significant adverse effect on differential deflection.

The effect of partial loading on the flexural moment distribution is shown in Figs. 3.31 and 3.32, and in Table 3.17. It is noticed that for very compressible medium ($\nu_s=0.0$), the change in flexural moments is very susceptible to changes in β . For such a medium, reducing β from 1.0 to 0.8 approximately doubles the value of M_r at the center of the raft. Further reduction of β from 0.8 to 0.6 increases M_r by approximately 60%. At the edge of the raft, however, the corresponding increase of M_t is relatively less, of the order of 29%. As ν_s approaches 0.5, the change in flexural moments becomes less significant.

Distributions of contact stress in non-dimensional form for various values of β are shown in Table 3.18 and Fig. 3.33. As would be expected, decreasing β significantly increases contact stress near the central region of the raft. However, the contact stress at or near the edge of the raft decreases with reducing β , and the possibility of loss of contact between the raft and the supporting soil medium increases.

Effect of relative rigidity: To show the effect of relative rigidity, computations were carried out for a relatively flexible ($K_r=0.1$) raft, for the same values of β . The computed results are shown graphically in Figs. 3.34 to 3.37. Tables 3.19 and 3.20 summarize the effect of relative rigidity on the influence coefficient for displacement. It is noticed that with decreasing relative rigidity both the central and the maximum differential deflections suffer more change for a certain change in β . A similar effect on flexural moment is also observed from Table 3.21. For $\beta=0.6$ and $K_r=0.1$, a change in the sign of radial bending moment M_r is also noted.

FIGURE 3.1 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a = 1.0$ AND $\nu_s = 0.0$

FIGURE 3.2 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a = 1.0$ AND $\nu_s = 0.49$

FIGURE 3.3 BENDING MOMENT DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=1.0$ AND $\nu_s=0.0$

FIGURE 3.4 VARIATION OF BENDING MOMENT DISTRIBUTIONS WITH POISSON'S RATIO ν_s FOR $H/a=1.0$

FIGURE 3.5 CONTACT STRESS DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $\nu_s = 0.0$ AND $H/a = 1.0$

FIGURE 3.6 VARIATION OF NORMAL STRESS (σ_{zz}) WITH DEPTH ALONG THE AXIS OF SYMMETRY FOR $H/a = 1.0$ AND $\nu_s = 0.0$

FIGURE 3.7 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=2.5$ AND $\nu_s=0.0$

FIGURE 3.8 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=2.5$ AND $\nu_s=0.49$

FIGURE 3.9 BENDING MOMENT DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=2.5$ AND $\nu_a=0.0$

FIGURE 3.10 VARIATION OF BENDING MOMENT DISTRIBUTIONS .
WITH POISSON'S RATIO v_s FOR $H/a = 2.5$

FIGURE 3.11 CONTACT STRESS DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $\nu_s = 0.0$ AND $H/a = 2.5$

FIGURE 3.12 VARIATION OF NORMAL STRESS (σ_{zz}) WITH DEPTH ALONG THE CENTERLINE OF THE RAFT FOR $H/a = 2.5$ AND $\nu_s = 0.0$

FIGURE 3.13 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=5.0$ AND $\nu_s=0.0$

FIGURE 3.14 INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=5.0$ AND $\nu_s=0.49$

FIGURE 3.15 BENDING MOMENT DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $H/a=5.0$ AND $\nu_s=0.0$

FIGURE 3.16 VARIATION OF BENDING MOMENT DISTRIBUTION WITH POISSON'S RATIO v_s FOR $H/a = 5.0$

FIGURE 3.17 CONTACT STRESS DISTRIBUTIONS FOR VARIOUS RELATIVE RIGIDITIES FOR $\nu_s = 0.0$ AND $H/a = 5.0$

FIGURE 3.18 VARIATION OF NORMAL STRESS (σ_{zz}) WITH DEPTH ALONG THE AXIS OF SYMMETRY FOR $H/a=5.0$ AND $\nu_s=0.0$

FIGURE 3.19 EFFECT OF LAYER DEPTH (H) ON CENTRAL DEFLECTION OF THE RAFT ($I_w(0)$) FOR VARIOUS RELATIVE RIGIDITIES (K_r)

FIGURE 3.20 VARIATION OF MAXIMUM DIFFERENTIAL DEFLECTION WITH RELATIVE RIGIDITY FOR DIFFERENT LAYER DEPTHS

FIGURE 3.21 VARIATION OF MAXIMUM FLEXURAL MOMENT WITH RELATIVE RIGIDITY OF THE RAFT FOR VARIOUS LAYER DEPTHS

FIGURE 3.22 BENDING MOMENT (M_r , M_t) DISTRIBUTIONS FOR DIFFERENT LAYER DEPTHS AND $\nu_r = 0.1$, $\nu_s = 0.0$

FIGURE 3.23 A TYPICAL CIRCULAR RAFT FOUNDATION HAVING NON-UNIFORM THICKNESS

FIGURE 3.24 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT FOR $\nu = 0.0$

FIGURE 3.25 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON VERTICAL DEFLECTION OF THE PLATE FOR $\nu_s = 0.0$

FIGURE 3.26 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON VERTICAL DEFLECTION OF THE PLATE FOR $\nu_B = 0.49$

FIGURE 3.27 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON BENDING MOMENT DISTRIBUTION FOR $\nu_s = 0.0$

FIGURE 3.28 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON BENDING MOMENT DISTRIBUTION FOR $\nu_s = 0.49$

FIGURE 3.29 EFFECT OF DIFFERENTIAL PLATE THICKNESS ON CONTACT STRESS DISTRIBUTION FOR $\nu_s = 0.0$

FIGURE 3.30 EFFECT OF LOADING CONFIGURATION ON THE INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT ($I_w(r)$) FOR $K_r = 1.0$ AND $H/a = 5.0$

FIGURE 3.31 EFFECT OF LOADING CONFIGURATION ON BENDING MOMENT DISTRIBUTION FOR $K_r = 1.0$, $H/a = 5.0$ AND $\nu_s = 0.0$

FIGURE 3.32 EFFECT OF LOADING CONFIGURATION ON BENDING MOMENT DISTRIBUTION FOR $K_r = 1.0$, $H/a = 5.0$, AND $\nu_s = 0.49$

FIGURE 3.33 EFFECT OF LOADING CONFIGURATION ON CONTACT STRESS DISTRIBUTION FOR $K_r = 1.0$, $H/a = 5.0$ AND $\nu_s = 0.0$

FIGURE 3.34 EFFECT OF LOADING CONFIGURATION ON THE INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT ($I_w(r)$) FOR $K=0.1$, $H/a=5.0$ AND $\nu=0.0$

FIGURE 3.35 - EFFECT OF LOADING CONFIGURATION ON THE INFLUENCE COEFFICIENT FOR SURFACE DISPLACEMENT ($I_w(r)$) FOR $K=0.1$, $H/a=5.0$ AND $\nu_s=0.49$

FIGURE 3.36 EFFECT OF LOADING CONFIGURATIONS ON BENDING MOMENT DISTRIBUTION FOR $K_r = 0.10$, $H/a = 5.0$ AND $\nu_s = 0.0$

FIGURE 3.37 EFFECT OF LOADING CONFIGURATION ON BENDING MOMENT DISTRIBUTION FOR $K_r = 0.10$, $H/a = 5.0$ AND $\nu_s = 0.49$

Table 3.1. Effect of layer depth on the influence coefficients for surface displacements for $K_r=0.01$.

r/a	$I_{\omega}(r)$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a					
	1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.9716	1.4564	1.7040	0.6903	1.0718	1.2086
0.20	0.9630	1.4441	1.6740	0.6750	1.0523	1.1667
0.40	0.9300	1.3875	1.5960	0.6450	0.9831	1.0748
0.60	0.8445	1.2720	1.4646	0.5805	0.8600	0.9402
0.80	0.6975	1.0820	1.2578	0.4650	0.7002	0.7622
1.00	0.4275	0.7857	0.9738	0.2859	0.5052	0.5523
1.20	0.1875	0.5162	0.6826	0.1245	0.3552	0.3864
1.40	0.0855	0.3640	0.5375	0.0465	0.2612	0.3000
1.70	0.0270	0.2282	0.3875	0.0105	0.1580	0.2252
2.00	0.0028	0.1420	0.2900	-0.0019	0.0802	0.1775
2.50	-0.0040	0.0642	0.1875	-0.0038	0.0200	0.1125
3.00	-0.0024	0.0300	0.1284	-0.0040	0.0047	0.0667
3.50	-0.0011	0.0167	0.0875	-0.0043	-0.0044	0.0275
4.00	-0.0007	0.0032	0.0573	-0.0050	-0.0164	0.0062
5.00	-0.0000	-0.0030	0.0250	-0.0060	-0.0183	-0.0023
6.00		-0.0024	0.0075	-0.0068	-0.0186	-0.0248
8.00		-0.0005	-0.0026	-0.0071	-0.0163	-0.0330
10.00		-0.0002	-0.0033	-0.0072	-0.0150	-0.0250

Table 3.2. Effect of layer depth on the influence coefficients for surface displacement for $K_r=0.10$.

r/a	$I_{\omega(r)}$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.9681	1.4337	1.6500	0.6903	1.0281	1.1334
0.20	0.9450	1.4105	1.6215	0.6675	1.0058	1.1034
0.40	0.8852	1.3373	1.5375	0.6314	0.9372	1.0220
0.60	0.7800	1.2107	1.4022	0.5580	0.8249	0.8973
0.80	0.6454	1.0335	1.2188	0.4455	0.6795	0.7392
1.00	0.4812	0.8215	1.0079	0.3080	0.5166	0.5649
1.20	0.1920	0.5300	0.6959	0.1216	0.3630	0.3917
1.40	0.0933	0.4012	0.5375	0.0514	0.2692	0.3000
1.70	0.0315	0.2340	0.3976	0.0075	0.1600	0.2304
2.00	0.0075	0.1445	0.2975	-0.0002	0.0824	0.1805
2.50	-0.0038	0.0642	0.1952	-0.0038	0.0242	0.1200
3.00	-0.0027	0.0304	0.1290	-0.0039	0.0034	0.0680
3.50	-0.0016	0.0142	0.0854	-0.0043	-0.0087	0.0325
4.00	-0.0005	0.0034	0.0578	-0.0051	-0.0163	0.0066
5.00	-0.0000	-0.0029	0.0252	-0.0062	-0.0183	-0.0010
6.00		-0.0023	0.0075	-0.0068	-0.0186	-0.0028
8.00		-0.0005	-0.0026	-0.0071	-0.0163	-0.0033
10.00		-0.0002	-0.0033	-0.0071	-0.0150	-0.0025

Table 3.3. Effect of layer depth on the influence coefficients for surface displacement for $K_r=1.0$.

r/a	$I_{\omega(r)}$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.7659	1.1588	1.3505	0.5318	0.7872	0.8511
0.20	0.7535	1.1469	1.3377	0.5206	0.7761	0.8397
0.40	0.7302	1.1120	1.3011	0.4982	0.7436	0.8025
0.60	0.6840	1.0571	1.2444	0.4616	0.6921	0.7476
0.80	0.6334	0.9875	1.1736	0.4178	0.6276	0.6798
1.00	0.5750	0.9114	1.0970	0.3683	0.5577	0.6072
1.20	0.2375	0.5673	0.7322	0.1600	0.3908	0.4107
1.40	0.1202	0.3800	0.5575	0.0788	0.2870	0.3275
1.70	0.0383	0.2370	0.4088	0.0340	0.1752	0.2500
2.00	0.0064	0.1470	0.2300	0.0074	0.1002	0.1463
2.50	-0.0034	0.0730	0.1963	-0.0037	0.0292	0.1238
3.00	-0.0027	0.0310	0.1306	-0.0038	0.0084	0.0715
3.50	-0.0017	0.0072	0.0888	-0.0042	-0.0036	0.0338
4.00	-0.0005	0.0030	0.0585	-0.0052	-0.0156	0.0082
5.00	-0.0000	-0.0028	0.0282	-0.0060	-0.0181	-0.0196
6.00		-0.0024	0.0126	-0.0067	-0.0186	-0.0279
8.00		-0.0005	-0.0025	-0.0070	-0.0162	-0.0336
10.00		-0.0002	-0.0033	-0.0072	-0.0150	-0.0247

Table 3.4. Effect of layer depth on the influence coefficients for surface displacement for $K_r=10.0$.

r/a	$I_{w(r)}$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.6476	0.9932	1.1789	0.4263	0.6195	0.6696
0.20	0.6435	0.9914	1.1771	0.4260	0.6179	0.6677
0.40	0.6383	0.9864	1.1718	0.4215	0.6129	0.6624
0.60	0.6300	0.9786	1.1639	0.4155	0.6050	0.6542
0.80	0.6278	0.9689	1.1540	0.4065	0.5952	0.6441
1.00	0.6212	0.9584	1.1435	0.4025	0.5847	0.6335
1.20	0.2775	0.5871	0.7512	0.1755	0.4089	0.4226
1.40	0.1200	0.3975	0.5713	0.0842	0.3000	0.3350
1.70	0.0435	0.2460	0.4055	0.0165	0.1840	0.2505
2.00	0.0142	0.1565	0.3006	0.0044	0.1020	0.2000
2.50	-0.0033	0.0690	0.1954	-0.0036	0.0442	0.1305
3.00	-0.0027	0.0345	0.1314	-0.0038	0.0042	0.0737
3.50	-0.0016	0.0072	0.0913	-0.0041	-0.0024	0.0413
4.00	-0.0005	0.0044	0.0612	-0.0050	-0.0153	0.0152
5.00	-0.0000	-0.0028	0.0285	-0.0065	-0.0180	-0.0111
6.00		-0.0024	0.0127	-0.0067	-0.0186	-0.0277
8.00		-0.0005	-0.0024	-0.0071	-0.0165	-0.0330
10.00		-0.0002	-0.0032	-0.0073	-0.0150	-0.0247

Table 3.5. Effect of layer depth on the influence coefficients for surface displacement for $K_r=100.0$.

r/a	$I_{\omega(r)}$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.6305	0.9687	1.1538	0.4101	0.5927	0.6413
0.20	0.6304	0.9686	1.1537	0.4100	0.5925	0.6411
0.40	0.6301	0.9681	1.1531	0.4092	0.5921	0.6405
0.60	0.6298	0.9672	1.1523	0.4083	0.5912	0.6398
0.80	0.6281	0.9662	1.1513	0.4078	0.5901	0.6387
1.00	0.6278	0.9651	1.1501	0.4072	0.5891	0.6375
1.20	0.2556	0.5900	0.7541	0.1650	0.4119	0.4244
1.40	0.1245	0.3925	0.5625	0.0788	0.3005	0.3275
1.70	0.0435	0.2420	0.4006	0.0188	0.1840	0.2505
2.00	0.0105	0.1570	0.3025	-0.0010	0.1004	0.1950
2.50	-0.0032	0.1055	0.1963	-0.0039	0.0372	0.1188
3.00	-0.0027	0.0340	0.1315	-0.0040	0.0107	0.0741
3.50	-0.0017	0.0156	0.0925	-0.0043	-0.0023	0.0375
4.00	-0.0006	0.0044	0.0612	-0.0050	-0.0153	0.0093
5.00	-0.0000	-0.0027	0.0285	-0.0062	-0.0179	-0.0190
6.00		-0.0024	0.0111	-0.0068	-0.0185	-0.0276
8.00		-0.0005	-0.0024	-0.0070	-0.0163	-0.0332
10.00		-0.0002	-0.0032	-0.0072	-0.0150	-0.0247

Table 3.6. Effect of layer depth on bending moment distributions for $K_r=0.1$.

r/a	M_r/qa^2					
	$\nu_s=0.0$			$\nu_s=0.49$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.0102	0.0140	0.0163	0.0100	0.0178	0.0206
0.20	0.0105	0.0160	0.0163	0.0105	0.0194	0.0205
0.40	0.0120	0.0163	0.0159	0.0122	0.0182	0.0177
0.60	0.0178	0.0158	0.0145	0.0138	0.0144	0.0143
0.80	0.0097	0.0126	0.0097	0.0110	0.0093	0.0086
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
r/a	M_t/qa^2					
	1.0	2.5	5.0	1.0	2.5	5.0
	H/a					
0.00	0.0102	0.0140	0.0163	0.0100	0.0178	0.0201
0.20	0.0101	0.0144	0.0165	0.0104	0.0180	0.0214
0.40	0.0110	0.0150	0.0162	0.0120	0.0181	0.0194
0.60	0.0120	0.0152	0.0154	0.0128	0.0163	0.0172
0.80	0.0113	0.0140	0.0132	0.0119	0.0135	0.0140
1.00	0.0015	0.0099	0.0122	0.0080	0.0099	0.0116

Table 3.7. Effect of layer depth on bending moment distributions for $K_r=1.0$.

r/a	M_r/qa^2						
	$\nu_s=0.0$			$\nu_s=0.49$			
	H/a 1.0	2.5	5.0	1.0	2.5	5.0	
0.00	0.0551	0.0720	0.0727	0.0600	0.0879	0.0946	
0.20	0.0530	0.0706	0.0716	0.0592	0.0868	0.0920	
0.40	0.0478	0.0630	0.0624	0.0543	0.0774	0.0774	
0.60	0.0372	0.0490	0.0472	0.0441	0.0585	0.0572	
0.80	0.0218	0.0289	0.0255	0.0267	0.0327	0.0304	
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
	M_t/qa^2						
	0.00	0.0551	0.0720	0.0727	0.0600	0.0879	0.0946
	0.20	0.0541	0.0710	0.0727	0.0595	0.0865	0.0943
	0.40	0.0509	0.0663	0.0672	0.0570	0.0818	0.0854
	0.60	0.0453	0.0587	0.0584	0.0513	0.0715	0.0732
	0.80	0.0362	0.0473	0.0462	0.0421	0.0568	0.0572
	1.00	0.0240	0.0324	0.0401	0.0278	0.0390	0.0495

Table 3.8. Effect of layer depth on bending moment distributions
for $K_r = 10.0$.

r/a	M_r/qa^2						
	$\nu_s=0.0$			$\nu_s=0.49$			
	H/a 1.0	2.5	5.0	1.0	2.5	5.0	
0.00	0.0780	0.1045	0.1056	0.0910	0.1355	0.1382	
0.20	0.0750	0.1011	0.1016	0.0880	0.1322	0.1374	
0.40	0.0664	0.0880	0.0886	0.0785	0.1168	0.1171	
0.60	0.0498	0.0662	0.0663	0.0608	0.0874	0.0840	
0.80	0.0283	0.0369	0.0334	0.0342	0.0476	0.0438	
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
	M_t/qa^2						
	0.00	0.0780	0.1045	0.1056	0.0910	0.1355	0.1382
	0.20	0.0765	0.1016	0.1035	0.0890	0.1328	0.1332
	0.40	0.0712	0.0944	0.0947	0.0838	0.1245	0.1273
	0.60	0.0620	0.0820	0.0820	0.0740	0.1081	0.1084
	0.80	0.0488	0.0650	0.0633	0.0590	0.0853	0.0842
	1.00	0.0280	0.0444	0.0548	0.0400	0.0580	0.0727

Table 3.9. Effect of layer depth on contact stress distributions.

r/a	σ_{zz}/q ($\nu_s=0.0$)					
	$K_r=1.0$			$K_r=100.0$		
	H/a 1.0	2.5	5.0	1.0	2.5	5.0
0.00	0.7600	0.6840	0.6800	0.5820	0.4400	0.4200
0.10	0.7670	0.6780	0.6635	0.5830	0.4420	0.4250
0.20	0.7620	0.6695	0.6520	0.5900	0.4510	0.4370
0.30	0.7540	0.6610	0.6475	0.5940	0.4690	0.4580
0.40	0.7405	0.6570	0.6410	0.5980	0.4900	0.4820
0.50	0.7200	0.6550	0.6420	0.6100	0.5260	0.5270
0.60	0.7105	0.6600	0.6500	0.6360	0.5770	0.5800
0.70	0.7200	0.6805	0.6760	0.6383	0.6480	0.6500
0.80	0.7525	0.7330	0.7300	0.7788	0.7520	0.7650
0.90	0.8475	0.8415	0.8290	0.9620	0.9300	0.9400
0.95	0.9460	0.9450	0.9100	1.1730	1.1500	1.1400

$$I_{\omega}(0) = \frac{\omega(0)E_s}{aq} \quad M_r(0) = \frac{M_r}{aq^2} \text{ at } r=0;$$

$$I_{\Delta\omega} = \frac{\{\omega(0) - \omega(a)\}E_s}{aq(1-\nu_s^2)} \quad M_t(0) = \frac{M_t}{aq^2} \text{ at } r=0.$$

Table 3.10. Summary of the effect of layer depth.

$K_r = 0.01$

	$\nu_s = 0.00$	$\nu_s = 0.49$
$I_{\omega}(0)$	0.9716	0.6903
$I_{\Delta\omega}$	0.5441	0.4444
$M_r(0)$	0.0006	0.0003
$M_t(0)$	0.0006	0.0003

	$\nu_s = 0.00$	$\nu_s = 0.49$
$I_{\omega}(0)$	1.4564	1.0718
$I_{\Delta\omega}$	0.6707	0.6226
$M_r(0)$	0.0011	0.0020
$M_t(0)$	0.0011	0.0020

$K_r = 100.0$

$I_{\omega}(0)$	0.6305	0.4101
$I_{\Delta\omega}$	0.0027	0.0032
$M_r(0)$	0.0820	0.0957
$M_t(0)$	0.0820	0.0957

$I_{\omega}(0)$	0.9687	0.5927
$I_{\Delta\omega}$	0.0036	0.0039
$M_r(0)$	0.1092	0.1431
$M_t(0)$	0.1092	0.1431

	$\nu_s = 0.00$	$\nu_s = 0.49$
$I_{\omega}(0)$	1.7040	1.2086
$I_{\Delta\omega}$	0.7302	0.7212
$M_r(0)$	0.0014	0.0020
$M_t(0)$	0.0014	0.0020

$I_{\omega}(0)$	1.1538	0.6413
$I_{\Delta\omega}$	0.0037	0.0042
$M_r(0)$	0.1105	0.1450
$M_t(0)$	0.1105	0.1450

Table 3.11. Effect of differential plate thickness on the influence coefficients for surface deformation for $\nu_s = 0.0$ and $H/a = 5.0$.

r/a	$I_{\omega(r)}$			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.00	1.3505	1.2965	1.2668	1.2431
0.20	1.3377	1.2881	1.2605	1.2387
0.40	1.3011	1.2637	1.2425	1.2257
0.60	1.2444	1.2253	1.2141	1.2053
0.80	1.1736	1.1720	1.1706	1.1696
1.00	1.0970	1.1085	1.1153	1.1207
1.20	0.7322	0.7371	0.7398	0.7421
1.45	0.5160	0.5183	0.5196	0.5207
1.80	0.3567	0.3579	0.3585	0.3590
2.30	0.2292	0.2297	0.2300	0.2303
3.00	0.1306	0.1308	0.1309	0.1310
4.05	0.0583	0.0584	0.0584	0.0585
5.25	0.0207	0.0208	0.0208	0.0208
6.55	0.0045	0.0045	0.0045	0.0045
8.15	-0.0025	-0.0025	-0.0025	-0.0025
10.00	-0.0033	-0.0033	-0.0033	-0.0033

Table 3.12. Effect of differential plate thickness on the influence coefficients for surface deformation for $\nu_s=0.49$ and $H/a=5.0$.

r/a	$I_{\omega(r)}$			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.00	0.8900	0.8289	0.7934	0.7644
0.20	0.8754	0.8187	0.7857	0.7587
0.40	0.8336	0.7893	0.7635	0.7424
0.60	0.7691	0.7439	0.7293	0.7172
0.80	0.6885	0.6816	0.6774	0.6740
1.00	0.6011	0.6081	0.6122	0.6156
1.20	0.4080	0.4113	0.4133	0.4148
1.45	0.2990	0.3015	0.3030	0.3042
1.80	0.2216	0.2233	0.2246	0.2255
2.30	0.1541	0.1550	0.1555	0.1560
3.00	0.0710	0.0716	0.0720	0.0723
4.05	0.0079	0.0082	0.0083	0.0085
5.25	-0.0265	-0.0264	-0.0264	-0.0264
6.55	-0.0294	-0.0294	-0.0294	-0.0294
8.15	-0.0337	-0.0337	-0.0337	-0.0337
10.00	-0.0248	-0.0248	-0.0248	-0.0248

Table 3.13. Effect of differential plate thickness on bending moment distributions for $\nu_s=0.0$ and $H/a=5.0$.

r/a	M_r/qa^2			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.0	0.0732	0.0923	0.1022	0.1100
0.2	0.0716	0.0884	0.0994	0.1054
0.4	0.0620	0.0790	0.0874	0.0940
0.6	0.0470	0.0623	0.0692	0.0747
0.8	0.0255	0.0330	0.0352	0.0369
1.0	0.0000	0.0000	0.0000	0.0000
r/a	M_t/qa^2			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.0	0.0732	0.0923	0.1021	0.1100
0.2	0.0728	0.0896	0.0992	0.1068
0.4	0.0672	0.0844	0.0934	0.1004
0.6	0.0584	0.0749	0.0830	0.0893
0.8	0.0460	0.0414	0.0375	0.0344
1.0	0.0401	0.0274	0.0241	0.0216

Table 3.14. Effect of differential plate thickness on bending moment distributions for $\nu_s=0.49$ and $H/a=5.0$.

r/a	M_r/qa^2			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.0	0.0900	0.1152	0.1299	0.1419
0.2	0.0815	0.1056	0.1197	0.1312
0.4	0.0707	0.0917	0.1039	0.1140
0.6	0.0543	0.0711	0.0809	0.0890
0.8	0.0308	0.0364	0.0396	0.0423
1.0	0.0000	0.0000	0.0000	0.0000
r/a	M_t/qa^2			
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$
0.0	0.0900	0.1152	0.1299	0.1419
0.2	0.0841	0.1085	0.1228	0.1345
0.4	0.0774	0.1000	0.1132	0.1240
0.6	0.0675	0.0876	0.0993	0.1088
0.8	0.0538	0.0475	0.0438	0.0407
1.0	0.0370	0.0314	0.0282	0.0255

Table 3.15. Effect of differential plate thickness on contact stress distributions for $\nu_s=0.0$ and $H/a=5.0$.

r/a	σ_{zz}/q				ν_s
	$\alpha=1.0$	$\alpha=1.2$	$\alpha=1.4$	$\alpha=1.6$	
0.133	0.6596	0.5914	0.5543	0.5249	0.00
0.333	0.6540	0.6042	0.5760	0.5537	
0.535	0.6483	0.6283	0.6158	0.6060	
0.735	0.6843	0.6918	0.6949	0.6974	
0.935	0.8653	0.8987	0.9186	0.9343	
0.133	1.2410	1.1670	1.1240	1.0880	0.49
0.333	1.0410	0.9913	0.9622	0.9383	
0.535	0.8725	0.8668	0.8634	0.8606	
0.735	0.7438	0.7835	0.8065	0.8254	
0.935	0.6135	0.6831	0.7236	0.7568	

Table 3.16. Effect of loading configuration on $I_{\omega(r)}$ for $K_r=1.0$ and $H/a=5.0$.

r/a	$I_{\omega(r)} = \frac{\omega(r)E_s \pi a \beta^2}{P}$					
	$\nu_s=0.0$			$\nu_s=0.49$		
	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$
0.00	1.3505	0.9610	0.5951	0.8511	0.6206	0.3920
0.20	1.3377	0.9467	0.5827	0.8397	0.6081	0.3815
0.40	1.3011	0.9051	0.5459	0.8025	0.5721	0.3518
0.60	1.2444	0.8408	0.4920	0.7436	0.5168	0.3078
0.80	1.1736	0.7607	0.4303	0.6798	0.4490	0.2576
1.00	1.0970	0.6767	0.3688	0.6072	0.3779	0.2077
1.20	0.7322	0.4581	0.2528	0.4107	0.2580	0.1428
1.45	0.5160	0.3254	0.1807	0.3007	0.1890	0.1045
1.80	0.3567	0.2261	0.1261	0.2229	0.1400	0.0775
2.30	0.2292	0.1456	0.0814	0.1520	0.0977	0.0543
3.00	0.1306	0.0831	0.0465	0.0715	0.0448	0.0248
4.05	0.0583	0.0731	0.0208	0.0082	0.0048	0.0025
5.25	0.0207	0.0132	0.0073	-0.0026	-0.0170	-0.0096
6.55	0.0045	0.0028	0.0016	-0.0294	-0.0189	-0.0106
8.15	-0.0025	-0.0016	-0.0010	-0.0337	-0.0215	-0.0121
10.00	-0.0033	-0.0021	-0.0012	-0.0248	-0.0159	-0.0089

Table 3.17. Effect of loading configuration on bending moment distribution for $K_T=1.0$ and $H/a=5.0$.

r/a	$M_r \pi \beta^2 / P$						
	$\nu_s = 0.0$			$\nu_s = 0.49$			
	$\beta = 1.0$	$\beta = 0.8$	$\beta = 0.6$	$\beta = 1.0$	$\beta = 0.8$	$\beta = 0.6$	
0.0	0.0740	0.0865	0.0780	0.0946	0.0569	0.0850	
0.2	0.0716	0.0823	0.0722	0.0920	0.0524	0.0773	
0.4	0.0620	0.0719	0.0577	0.0774	0.0495	0.0614	
0.6	0.0470	0.0529	0.0316	0.0572	0.0325	0.0337	
0.8	0.0255	0.0240	0.0075	0.0304	0.0145	0.0080	
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
	$M_t \pi \beta^2 / P$						
	0.0	0.0740	0.0865	0.0780	0.0946	0.0569	0.0850
	0.2	0.0728	0.0836	0.0740	0.0943	0.0538	0.0796
	0.4	0.0672	0.0780	0.0657	0.0854	0.0493	0.0703
	0.6	0.0584	0.0670	0.0509	0.0732	0.0420	0.0544
	0.8	0.0460	0.0508	0.0347	0.0572	0.0317	0.0372
	1.0	0.0401	0.0343	0.0249	0.0495	0.0215	0.0267

Table 3.18. Effect of loading configuration on contact stress distribution for $K_T=1.0$ and $H/a=5.0$.

r/a	$\sigma_z \pi \beta^2 a^2 / P \quad (\nu_s = 0.0)$		
	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$
0.133	0.6596	0.5376	0.3715
0.333	0.6540	0.5110	0.3354
0.535	0.6483	0.4700	0.2848
0.735	0.6843	0.4388	0.2416
0.935	0.8653	0.4757	0.2329

Table 3.19. Effect of loading configuration and K_r on $I_{\omega(r)}$ for $\nu_s = 0.0$ and $H/a = 5.0$.

r/a	$I_{\omega(r)} = \frac{\omega(r) E_s \pi a \beta^2}{P}$					
	$K_r = 0.1$			$K_r = 1.0$		
	$\beta = 1.0$	$\beta = 0.8$	$\beta = 0.6$	$\beta = 1.0$	$\beta = 0.8$	$\beta = 0.6$
0.00	1.6500	1.3031	0.9057	1.3505	0.9610	0.5951
0.20	1.6215	1.2716	0.8700	1.3377	0.9467	0.5827
0.40	1.5375	1.1751	0.7659	1.3011	0.9051	0.5459
0.60	1.4022	1.0141	0.6102	1.2444	0.8408	0.4920
0.80	1.2188	0.8033	0.4457	1.1736	0.7607	0.4303
1.00	1.0078	0.5789	0.2982	1.0970	0.6767	0.3688
1.20	0.6959	0.4181	0.2234	0.7322	0.4581	0.2528
1.45	0.5444	0.3069	0.1670	0.5160	0.3254	0.1807
1.80	0.3491	0.2175	0.1196	0.3567	0.2261	0.1261
2.30	0.2258	0.1418	0.0785	0.2292	0.1456	0.0814
3.00	0.1290	0.0813	0.0452	0.1306	0.0831	0.0465
4.05	0.0576	0.0363	0.0202	0.0583	0.0731	0.0208
5.25	0.0204	0.0182	0.0071	0.0207	0.0132	0.0073
6.55	0.0043	0.0026	0.0014	0.0045	0.0028	0.0016
8.15	-0.0026	-0.0017	-0.0010	-0.0025	-0.0016	-0.0010
10.00	-0.0033	-0.0021	-0.0012	-0.0033	-0.0021	-0.0012

Table 3.20. Effect of loading configuration and K_r on $I_{\omega(r)}$ for $\nu_s=0.49$ and $H/a=5.0$.

r/a	$I_{\omega(r)} = \frac{\omega(r) E_s \pi a \beta^2}{P}$					
	$K_r=0.10$			$K_r=1.0$		
	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$
0.00	1.1334	0.9045	0.6341	0.8511	0.6206	0.3920
0.20	1.1034	0.8762	0.6060	0.8397	0.6081	0.3815
0.40	1.0220	0.7944	0.5259	0.8025	0.5721	0.3518
0.60	0.8973	0.6649	0.4077	0.7436	0.5168	0.3078
0.80	0.7392	0.5029	0.2858	0.6798	0.4490	0.2576
1.00	0.5649	0.3366	0.1800	0.6072	0.3779	0.2077
1.20	0.3917	0.2393	0.1297	0.4107	0.2580	0.1428
1.45	0.2871	0.1752	0.0944	0.3007	0.1890	0.1045
1.80	0.2130	0.1303	0.0699	0.2229	0.1400	0.0775
2.30	0.1500	0.0929	0.0506	0.1520	0.0977	0.0543
3.00	0.0680	0.0413	0.0223	0.0715	0.0448	0.0248
4.05	0.0065	0.0033	0.0015	0.0082	0.0048	0.0025
5.25	-0.0027	-0.0175	-0.0099	-0.0026	-0.0170	-0.0096
6.55	-0.0029	-0.0191	-0.0108	-0.0294	-0.0189	-0.0106
8.15	-0.0034	-0.0215	-0.0121	-0.0337	-0.0215	-0.0121
10.00	-0.0025	-0.0160	-0.0090	-0.0248	-0.0159	-0.0089

Table 3.21. Effect of loading configuration and relative rigidity K_r on bending moment distributions for $\nu_s=0.0$ and $H/a=5.0$.

r/a	$M_r \pi \beta^2 / P$					
	$K_r=0.10$			$K_r=1.0$		
	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$	$\beta=1.0$	$\beta=0.8$	$\beta=0.6$
0.0	0.0168	0.0121	0.0079	0.0740	0.0865	0.0780
0.2	0.0163	0.0116	0.0072	0.0716	0.0823	0.0722
0.4	0.0159	0.0115	0.0061	0.0620	0.0719	0.0577
0.6	0.0145	0.0105	0.0022	0.0470	0.0529	0.0316
0.8	0.0097	0.0047	-0.0010	0.0255	0.0240	0.0075
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
r/a	$M_t \pi \beta^2 / P$					
	0.0	0.0168	0.0121	0.0079	0.0740	0.0865
	0.2	0.0165	0.0118	0.0074	0.0728	0.0836
	0.4	0.0161	0.0122	0.0068	0.0672	0.0780
	0.6	0.0154	0.0116	0.0049	0.0584	0.0670
	0.8	0.0132	0.0089	0.0026	0.0460	0.0508
	1.0	0.0122	0.0058	0.0021	0.0401	0.0343

4. FINITE ELEMENT ANALYSIS OF CIRCULAR PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE

4.1 GENERAL

Although it is often convenient to estimate settlements of a structure on the basis of homogeneous elastic half-space, it frequently occurs in practice that the supporting strata are non-homogeneous. This is generally true of several extensive deposits of over-consolidated clay and it is natural to enquire to what extent the settlements are affected by the heterogeneous nature of the soil.

In the past years, less attention has been given to the investigation of non-homogeneous elastic half-space primarily because of the computational difficulties involved. The finite element method described in Chapter 2 is fairly general in this regard. It allows the investigation of any non-homogeneous anisotropic problem without encountering additional difficulties.

4.2 NON-HOMOGENEOUS HALF-SPACE SELECTED FOR STUDY

There is an infinite variety of elastic half-spaces to be studied, but it is generally accepted that many natural deposits of soils have stiffnesses which increase with depth. The particular non-homogeneous half-space that was selected for investigation, shown in Fig. 4.1, is defined by

$$E_s(z) = E_0 + \lambda z \quad (4.1)$$

and $\nu_s = \text{constant}$ (4.2)

where E_0 and $E_s(z)$ are the Young's moduli of elasticity of the soil medium at the surface and at a depth z respectively, λ is the slope of the straight line expressing the linear increase of E_s with depth.

This half-space has two important properties. Firstly, for $\lambda=0$, the half-space is the conventional homogeneous elastic half-space. Secondly, for $\lambda=0$ and $\nu_s=0.5$, Gibson (1967,1968,1969) has shown that the settlement is independent of the size and shape of the plate. Assumption of a finite surface modulus removes this displacement singularity.

The degree of non-homogeneity is expressed by a factor ψ defined by $\psi=E_0/a\lambda$, where a is the radius of the plate and λ is the slope of the line expressing the increase of Young's modulus of the elastic medium with depth. For $\psi=40$, a series of computations were carried out, using the finite element grid of Fig. 2.18, for values of relative rigidities in the range $10^{-2} \leq K_r \leq 10^2$. In the evaluation of K_r , E_0 was used instead of E_s . The computed results are presented in the following section.

4.3 COMPUTED RESULTS

The finite element results for $I_{\omega(r)}$ for various values of relative rigidity of the plate (K_r) are presented in Tables 4.1 and 4.2, and are shown graphically in Fig. 4.2. For the limiting case $K_r=0$, the finite element results for the central deflection of a flexible plate approximate closely (within 6%) the corresponding analytical values given by Brown and Gibson (1972).

The finite element results for the corresponding maximum differential plate displacement are plotted in Fig. 4.3. For the limiting case of $K_r=0.0$, a close agreement is observed between computed and theoretical values.

Distribution of flexural moments (M_r, M_θ) at various locations of a circular raft of finite flexibility are shown in Tables 4.3 and are plotted in Fig. 4.4. It may be noted that, with the exception of a very flexible raft ($K_r < 0.1$), the maximum bending moments occur at the center of the raft. For a very flexible raft, the maximum values are located at approximately $0.8a$ (where a is the radius of the raft). In this context, a similar observation was made by Hooper (1974). The effect of Poisson's ratio of the elastic medium (ν_s) on the distribution of flexural moments is shown in Fig. 4.5.

Values of normal stresses at the interface between a finite plate and a non-homogeneous elastic half-space are tabulated in Table 4.4 and are plotted in Fig. 4.6. The corresponding variation of normal stresses with depth, along the center line of the raft, is shown in Table 4.5 and Fig. 4.7.

The effect of increasing Young's modulus of the supporting medium with depth on the behaviour of a plate resting on it, is seen when the finite element results for a plate on homogeneous elastic half-space is compared with the corresponding results related to the non-homogeneous case. For a particular degree of non-homogeneity ($\psi=40$), such a comparison

is shown in Figs. 4.8, 4.9 and 4.10. As would be expected, for a given value of K_r , the deflections of the plate reduce with increasing non-homogeneity. A significant change of normal deflections beyond the plate region is also observed. From Figs. 4.9 and 4.10, it may be noticed that both the flexural moments and contact stresses of a plate resting on a non-homogeneous elastic half-space are less than the corresponding values related to homogeneous half-spaces.

FIGURE 4.1 NON-HOMOGENEOUS ELASTIC HALF-SPACE

FIGURE 4.2 PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE:
INFLUENCE COEFFICIENTS FOR SURFACE DISPLACEMENT
FOR VARIOUS RELATIVE RIGIDITIES FOR $\nu = 40.0$

FIGURE 4.4 PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE: BENDING MOMENT DISTRIBUTION FOR VARIOUS K_r FOR $\nu_s = 0.0$ AND $\nu = 40$

FIGURE 4.5 PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE: VARIATION OF BENDING MOMENT DISTRIBUTIONS WITH POISSON'S RATIO FOR $\psi=40.0$

FIGURE 4.6 PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE: CONTACT STRESS DISTRIBUTION FOR VARIOUS RELATIVE RIGIDITIES (K_r) FOR $\nu=0.0$ AND $\nu_B=0.0$

FIGURE 4.7 PLATES ON NON-HOMOGENEOUS ELASTIC HALF-SPACE:
VARIATION OF NORMAL STRESS (σ_{zz}) WITH DEPTH ALONG
THE CENTERLINE OF PLATE FOR $\nu = 40$ AND $\nu_s = 0.0$

FIGURE 4.8 EFFECT OF NON-HOMOGENEITY ON THE INFLUENCE COEFFICIENT FOR SURFACE DISPLACEMENT FOR $\kappa_r = 0.1$, $H/a = 20$ AND $\nu_r = 40$

FIGURE 4.9 EFFECT OF NON-HOMOGENEITY ON BENDING MOMENT DISTRIBUTIONS FOR $K_r=10$ AND $\psi=40$

FIGURE 4.10 EFFECT OF NON-HOMOGENEITY ON CONTACT STRESS DISTRIBUTION FOR $K_T=10$, $\Psi=40$ AND $\nu_s=0.0$

Table 4.1. Influence coefficients for surface displacement of circular plate on non-homogeneous elastic half-space, for $\nu_s=0.0$ and $\psi=40$.

r/a	$I_{\omega(r)} = \frac{\omega(r)E_0}{aq}$				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.000	1.8004	1.6850	1.3898	1.3036	1.2934
0.166	1.7886	1.6692	1.3854	1.3032	1.2934
0.333	1.7452	1.6218	1.3728	1.3016	1.2932
0.500	1.6622	1.5436	1.3524	1.2992	1.2930
0.666	1.5318	1.4378	1.3260	1.2962	1.2926
0.833	1.3454	1.3104	1.2956	1.2926	1.2922
1.000	1.1078	1.1724	1.2638	1.2890	1.2900
1.166	0.8428	0.8704	0.9108	0.9220	0.9234
1.333	0.6872	0.7022	0.7246	0.7308	0.7314
1.500	0.5806	0.5898	0.6036	0.6074	0.6078
1.700	0.4894	0.4948	0.5034	0.5056	0.5060
1.933	0.4132	0.4168	0.4220	0.4236	0.4238
2.266	0.3358	0.3378	0.3410	0.3418	0.3420
2.633	0.2748	0.2760	0.2780	0.2786	0.2786
3.100	0.2212	0.2218	0.2230	0.2232	0.2234
3.733	0.1717	0.1721	0.1727	0.1729	0.1729
4.666	0.1245	0.1246	0.1250	0.1251	0.1251
5.666	0.0918	0.0919	0.0920	0.0921	0.0921
7.000	0.0647	0.0647	0.0648	0.0648	0.0605
8.500	0.0452	0.0456	0.0453	0.0453	0.0453

Table 4.2. Influence coefficients for surface displacement of circular plate on non-homogeneous elastic half-space, for $\nu_s=0.49$ and $\psi=40$.

r/a	$I_{\omega(r)} = \frac{\omega(r)E_0}{aq}$				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.000	1.2690	1.1780	0.9154	0.8228	0.8112
0.166	1.2558	1.1644	0.9112	0.8222	0.8112
0.333	1.2160	1.1240	0.8986	0.8208	0.8110
0.500	1.1440	1.0574	0.8788	0.8184	0.8108
0.666	1.0358	0.9678	0.8530	0.8152	0.8104
0.833	0.8924	0.8608	0.8234	0.8116	0.8102
1.000	0.7206	0.7460	0.7922	0.8078	0.8098
1.166	0.5680	0.5868	0.6220	0.6336	0.6352
1.333	0.4790	0.4890	0.5090	0.5158	0.5166
1.500	0.4044	0.4134	0.4314	0.4376	0.4384
1.700	0.3276	0.3340	0.3466	0.3510	0.3514
1.933	0.2656	0.2698	0.2780	0.2808	0.2812
2.266	0.2078	0.2100	0.2146	0.2162	0.2164
2.633	0.1642	0.1658	0.1679	0.1687	0.1688
3.100	0.1266	0.1274	0.1288	0.1293	0.1294
3.733	0.0915	0.0919	0.0927	0.0929	0.0930
4.666	0.0578	0.0580	0.0584	0.0585	0.0586
5.666	0.0341	0.0342	0.0343	0.0344	0.0344
7.000	0.0173	0.0173	0.0174	0.0175	0.0175
8.500	0.0043	0.0044	0.0044	0.0045	0.0045

Table 4.3. Plates on non-homogeneous elastic half-space;
distribution of bending moments for $\nu_s=0.0$ and $\psi=40$.

r/a	M_r/qa^2				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.000	0.0025	0.0321	0.0887	0.1048	0.1067
0.166	0.0023	0.0316	0.0860	0.1014	0.1032
0.333	0.0032	0.0312	0.0789	0.0924	0.0940
0.500	0.0041	0.0289	0.0665	0.0770	0.0781
0.666	0.0051	0.0241	0.0490	0.0573	0.0565
0.833	0.0054	0.0153	0.0267	0.0298	0.0301
1.000	0.0000	0.0000	0.0000	0.0000	0.0000
r/a	M_t/qa^2				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.000	0.0025	0.0321	0.0887	0.1048	0.1067
0.166	0.0023	0.0317	0.0868	0.1024	0.1043
0.333	0.0031	0.0318	0.0829	0.0974	0.0991
0.500	0.0037	0.0306	0.0758	0.0885	0.0900
0.666	0.0043	0.0282	0.0658	0.0762	0.0775
0.833	0.0047	0.0238	0.0529	0.0610	0.0620
1.000	0.0036	0.0170	0.0380	0.0439	0.0446

Table 4.4. Plates on non-homogeneous elastic half-space;
distribution of contact stresses for $\nu_s=0.0$ and $\psi=40$.

r/a	σ_{zz}/q				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.111	1.0100	0.9125	0.5596	0.4542	0.4417
0.278	1.0180	0.8852	0.5648	0.4721	0.4611
0.444	1.0000	0.8367	0.5813	0.5108	0.5025
0.611	0.9343	0.7739	0.6220	0.5831	0.5785
0.780	0.7739	0.7143	0.7313	0.7389	0.7399
0.9466	0.5026	0.7462	1.0600	1.1450	1.1555

Table 4.5. Plates on non-homogeneous elastic half-space;
distribution of normal stress (σ_{zz}) along the center line
of the plate for $\nu_s=0.0$ and $\psi=40^\circ$.

z/a	σ_{zz}/q				
	$K_r=0.01$	$K_r=0.1$	$K_r=1.0$	$K_r=10.0$	$K_r=100.0$
0.111	1.0070	0.9245	0.5635	0.4544	0.4414
0.222	0.9302	0.8425	0.5407	0.4509	0.4402
0.400	0.9061	0.8101	0.5474	0.4704	0.4613
0.611	0.8018	0.7175	0.5272	0.4723	0.4658
0.866	0.6586	0.5978	0.4479	0.4437	0.4397
1.200	0.4970	0.4613	0.3969	0.3787	0.3766
1.622	0.3510	0.3332	0.3027	0.2941	0.2931
2.122	0.2406	0.2323	0.2185	0.2147	0.2142
2.733	0.1597	0.1561	0.1501	0.1485	0.1483
3.466	0.1064	0.1048	0.1022	0.1015	0.1014
4.333	0.0709	0.0702	0.0691	0.0687	0.0687
5.388	0.0475	0.0472	0.0466	0.0465	0.0465
6.722	0.0311	0.0309	0.0307	0.0306	0.0306
8.444	0.0203	0.0203	0.0202	0.0202	0.0202
10.500	0.0133	0.0135	0.0135	0.0135	0.0135
13.277	0.0091	0.0090	0.0090	0.0090	0.0090
17.000	0.0067	0.0067	0.0067	0.0067	0.0067

5. ~~CONCLUSIONS~~ AND RECOMMENDATIONS

5.1 SUMMARY

In recent years, circular plates have found increased application in raft foundations used for structures such as nuclear reactors, storage tanks, silos, and in chimney and tower foundations (Fig. 1.2). It has been observed that rafts of low relative rigidity may suffer excessive differential deflection (Brown, 1969a) which may be detrimental to its function. On the other hand, a relatively stiff raft may be subjected to excessive bending moments. Thus, in the design of raft foundations, it is desirable to be able to predict the differential deflections, flexural moments and contact stresses under anticipated loads in order that a raft of suitable stiffness may be selected.

In this work, the axially symmetric interaction of an elastic circular plate with a free edge, resting on an isotropic elastic solid medium, has been examined by the finite element method. Displacement method is used in the finite element formulation. The plate under consideration may be of uniform thickness or its thickness may vary, as shown in Fig. 1.1. The supporting isotropic elastic medium may behave as a homogeneous material or as a non-homogeneous material where elastic properties may change linearly with depth. In both cases, it is assumed that the Poisson's ratio of the elastic medium may vary between 0.0 and 0.49. The contact between the two materials is assumed to be smooth.

L

Flexural behaviour of the plate is represented by the classical Poisson-Kirchhoff thin plate theory.

To model the isotropic elastic solid medium, axisymmetric solid elements of triangular cross-section are used. Two generalized displacements at each node (viz. radial and vertical) are considered as degrees of freedom (DOF). The axisymmetrically loaded elastic circular plate is modelled as an assemblage of annular plate elements with two DOF per node (viz. vertical displacement and rotation). To improve accuracy of the solution, a special formulation is used for the central solid circular plate element.

On the basis of these models, a finite element program has been written in FORTRAN IV language. The program is machine independent, runs in the batch mode of operation, and needs approximately a 40K word core space to solve a problem having 350 DOF. This capability is sufficient for most practical purposes. Also, two other programs have been written to speed up data input and to detect probable errors in the input. Both these programs are designed to run either on the time-sharing or on the batch mode of operation.

The program calculates nodal displacements and forces, and the average displacement of the plate. It also calculates the flexural moments (M_r, M_θ) and the shearing forces (Q_r) at each node of the plate elements. At the user's request, the stress components ($\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}$), the principal stresses ($\sigma_1, \sigma_2, \sigma_3$) and the stress invariants

(I_1, I_2) at the centroid of solid elements are also evaluated.

The flexural behaviour of an axisymmetrically loaded circular plate (the raft) resting on an isotropic elastic solid medium (the soil) is influenced by many factors. The work presented in this thesis involves investigation of some of these parameters. Giving due consideration to the numerous factors involved, attention is focused only on the following items:

- i) influence of a soil medium of finite depth
- ii) influence of differential plate thickness
- iii) influence of the external loading configuration and
- iv) influence of non-homogeneous elastic half-space.

Numerical results developed include influence coefficients for the plate displacement and flexural moments, the contact stress at the interface, together with the stress distribution within the medium.

5.2 CONCLUSIONS

- 1) In the context of present interaction problems, the finite element method always underestimates the actual deflections. The error on the flexural moments and shearing forces is even more. The differential deflections, however, compare favourably with the equivalent exact solutions. The magnitude of error increases when the supporting solid medium becomes nearly incompressible ($\nu_s = 0.49$), and it becomes quite significant for the case of a rigid plate. For a rigid plate on an incompressible homogeneous elastic half-

space, finite element results for the vertical displacement at the center of the plate underestimate the corresponding exact solutions by approximately 17%. With the exception of an infinitely rigid plate on a truly incompressible solid medium ($\nu_s = 0.5$), however, the finite element analysis described in this thesis yields acceptable results for the analysis of initial displacements of circular foundations on soils which can be reasonably viewed as linearly elastic material.

- 2) The flexural behaviour of a circular plate is greatly influenced by the relative rigidity of the plate-elastic medium system (K_r). Depending on the elastic properties of the supporting elastic medium, the same plate may have different relative rigidities. With increasing K_r , both the central deflection and differential deflection decrease causing a significant increase in the flexural moments.
- 3) i) The central deflection, differential deflection and flexural moments of a raft largely depend on the thickness of the elastic solid medium. With increasing thickness of the elastic medium, all these quantities increase. These increments are very significant when the medium is relatively shallow ($H/a \leq 2.0$), and become less significant for a deeper medium ($H/a \geq 5.0$).

- ii) For deep layers ($H/a \geq 5.0$), an increase in the relative rigidity (K_r) causes a significant decrease in the central deflection, but Poisson's ratio has comparatively little effect. In this regard, a similar observation was made by Brown (1969a).
 - iii) For shallow layers ($H/a \leq 2.0$), a decrease in Poisson's ratio of the foundation material causes a significant increase in the influence coefficient for central deflection, but relative rigidity has comparatively little effect.
- 4) A variation of α (i.e. as the thickness of the inner region of the plate changes) causes a considerable change in displacements under the plate. Surface deformations outside the plate region, however, do not change significantly. As would be expected, an increase in thickness of the inner region of the plate leads to a decrease in deflection at the center and an increase in the corresponding edge deflection. Moreover, with increasing α , the distribution of flexural moments becomes discontinuous, but maximum values still occur at the center of the plate.
- 5) A decrease in the radius of the loaded area βa (i.e. as the same total load is applied over a smaller area) increases the central deflection and differential deflection of the plate significantly, but Poisson's ratio of the elastic medium has relatively little effect. The corresponding effect on flexural moments is even more.

A reduction of β from 1.0 to 0.8 approximately doubles the values of M_r and M_θ at the center of the plate, for a very compressible medium ($\nu_s=0.0$). For nearly incompressible mediums, the change in flexural moments becomes less significant.

- 6) The principal effect of soil non-homogeneity, relative to an equivalent homogeneous medium, is to reduce the central deflection and differential deflection for uniformly applied loads. With increasing non-homogeneity, a significant change of normal deflections beyond the plate region occurs. Also, the flexural moments at a point reduce with increasing non-homogeneity. The effect of non-homogeneity on contact stress distribution is relatively insignificant.

5.3 RECOMMENDATIONS FOR FUTURE WORK

This study has attempted to investigate a number of factors which are of importance to the present soil structure interaction problem. The study has indicated the need for further research in the following areas:

- 1) Displacement formulation of the finite element method is not applicable to a situation where the Poisson's ratio of the supporting elastic medium equals 0.5. Special formulation, known as 'hybrid formulation', is required for such a case. The finite element computer program, developed in this work, could be extended to handle such cases.

- 2) In most practical situations, the circular plate which is generally used as a raft foundation does not have a free edge; rather the edge is partially restrained. Selvadurai (1979c) has examined the effect of restraining an edge using a variational approach. It is recommended that the same problem be investigated by the finite element method.
- 3) The present study assumes that the plate is perfectly smooth and that interfacial slip between the plate and the elastic medium can occur. In actual practice, the interface is not perfectly smooth; rather it is seen to be frictional. It is therefore recommended that the analysis be extended to incorporate the effect of frictional contact. (Friction could be Coulomb friction or finite friction.)
- 4) Also, the present study assumes that in all ranges of loading both the plate and the supporting solid medium behave elastically. In a practical situation, however, always some local yielding near the edge of the plate takes place and a redistribution of stress occurs. It would be desirable to take into account such redistribution of stresses in the analysis.
- 5) The present work is concerned with the regular geometry of the elastic half-space region. There are situations where the supporting elastic medium has irregular geometry. It is therefore recommended that the effect of irregular

geometry of the elastic half-space region, on the flexural behaviour of a finite circular plate, be investigated.

- 6) The present study assumes that the circular plate under consideration is thin (i.e. its thickness is small compared to its other dimensions), and the strain energy due to shear deformation of the plate is negligible. This assumption is realistic for most cases. However, for some other cases, the circular plate may be moderately thick. The present analysis will not be adequate for such cases. Hence, it is felt that the present analysis should be extended for the case of a moderately thick plate.

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APPENDIX A

FINITE ELEMENT ANALYSIS OF AXISYMMETRIC
SOIL-STRUCTURE INTERACTION PROBLEMS

PROGRAM USER'S MANUAL

July 1979

DEPARTMENT OF CIVIL ENGINEERING
FACULTY OF ENGINEERING
CARLETON UNIVERSITY
OTTAWA, CANADA

CONTROL CARDS

The following control cards are used to execute the program for the Finite Element Analysis of Axisymmetric Soil-Structure Interaction Problems. It should be noted that with the passage of time, some changes may occur in the control cards.

```
!JOB Account No., Name, 9, ENG. (ENG. is optional, needed only
!LIMIT (CORE,40),(TIME,2),(LO,200) if output is wanted from
!ASSIGN F:2,(FILE,DATAR),(IN),(SAVE) C.J. Mackenzie Building RBT)
!ASSIGN F:3,(DEVICE,LP)
!ASSIGN F:4,(FILE,DUM1),(OUTIN),(REL)
!ASSIGN F:7,(FILE,DUM2),(OUTIN),(REL)
!ASSIGN F:9,(FILE,DUM3),(OUTIN),(REL)
!RUN (LMN,LFANCI)
```

NOTE: DATAR contains the Input Data and must be on the disk before the execution of the program. Also, LFANCI is the load module name and must be on the disk.

Execution of the Preparation of Input Data Program
(using a time-sharing terminal)

```
!SET F:5/DATAF ; IN; SAVE
!SET F:6/DATAR ; OUT; SAVE
!LFREE.
```

NOTE: DATAF is a file which contains the required input data in a free format. DATAR is the output file, LFREE being the load module name.

Execution of the Program to Draw Finite Element Grid

(The following commands are applicable to any user who has access to FORT4, i.e. on-line FORTRAN.)

```
!SET F:3/DATAR ; IN; SAVE
!SET F:5 ME
!LYNX OPLOT OVER LPLOT; CALPLOT
!LPLOT.
```

NOTE: OPLOT is the object module of the plotting program. DATAR is a file which contains the input data in the specified format. These two files must be on the disk prior to the execution of this program.

DISCLAIMER

The program described in this manual is developed to study the axisymmetric flexural behaviour of an elastic circular plate resting on an isotropic elastic solid medium. Many soil-structure interaction problems which possess an axis of symmetry can be solved using this program. While every effort has been made to ensure the accuracy of the information presented in this manual, and the reliability of the program described therein, use of this material can be made only with the understanding that neither the author nor the manual writer will assume any liability of any kind arising from such use.

CONTENTS OF APPENDIX A

	Page
A.1 INTRODUCTION.....	170
A.2 ASSUMPTIONS.....	174
A.3 METHOD OF ANALYSIS.....	175
A.4 MAIN FEATURES	177
A.5 LIMITATIONS	180
A.6 SIGN CONVENTIONS AND UNITS	181
A.7 A PROGRAM FOR THE FINITE ELEMENT ANALYSIS OF AXISYMMETRIC SOIL-STRUCTURE INTERACTION PROBLEMS	181
A.7.1 Input	181
A.7.1.1 Descretization of the continuum	182
A.7.1.2 Numbering of nodes	182
A.7.1.3 Numbering of degrees of freedom	185
A.7.1.4 Numbering of elements	186
A.7.1.5 Format of the input data	186
A.7.2 Output	191
A.7.3 Outline Flow Diagram	192
A.7.4 FORTRAN Symbols Used	196
A.7.5 Example	212
A.8 COMPUTER PROGRAM FOR THE PREPARATION OF INPUT DATA	225
A.9 PROGRAM FOR DRAWING FINITE ELEMENT GRID	226
A.9.1 General	226
A.9.2 Output	226
A.10 FORTRAN LISTINGS	226
A.10.1 Main Program	229
A.10.2 Program for the preparation of input data	265
A.10.3 Program for drawing finite element grid	277
REFERENCES	

3

A.1 INTRODUCTION

The finite element computer program described in this user's manual was developed by the author to examine an axisymmetric soil-structure interaction problem. The program is concerned with only the elastic analysis. The problem which was examined extensively, using this program, is illustrated in Fig. A.1. It is concerned with the analysis of a thin elastic circular plate (the raft) of radius a , elastic properties E_b and ν_b resting on an isotropic elastic solid medium (the soil) having depth H , Young's modulus E_s and Poisson's ratio ν_s . The plate may be of uniform thickness or its thickness may vary. The supporting isotropic elastic solid medium may behave as a homogeneous material or as a non-homogeneous material. Some other typical problems which could be investigated using this program are shown in Fig. A.2. It should be noted that all these problems are, in fact, three-dimensional problems. But, since attention is restricted to axisymmetric geometry and loads, they are reduced to two-dimensional problems.

This program, together with two other programs which were developed to speed up the input data and to correct the probable errors in the

* Young's Modulus - E_b

Poisson's Ratio - ν_b

** Young's Modulus - E_s (or $E_s(z)$: see Fig. 4.1)

Poisson's Ratio - ν_s

FIGURE A.1 AN ELASTIC CIRCULAR PLATE RESTING ON AN ISOTROPIC ELASTIC SOLID MEDIUM

a) Simply Supported Circular Plate

b) Clamped Circular Plate

c) Bore-hole Problem

d) Bonded Circular Plate

FIGURE A.2 SOME TYPICAL AXISYMMETRIC PROBLEMS

e) Embedded Anchor Problem

f) Thick Cylinder Under Internal Pressure

FIGURE A.2 (Cont'd) SOME TYPICAL AXISYMMETRIC PROBLEMS

input, constitutes a part of a thesis submitted by the author to the Carleton University, Ottawa in partial fulfillment of the requirements of an M.Eng. degree in Civil Engineering.

To make the manual self-contained, a brief description of the analysis method and program operations is included. For a more detailed information, the original thesis may be referred to. This manual also presents outline flow diagrams, a list of FORTRAN variables used, the FORTRAN source listings and output of the program for a typical problem.

A.2 ASSUMPTIONS

The displacement finite element method is used to analyze the soil-structure interaction problem. The following assumptions are made for the analysis:

1. The flexural behaviour of the structural foundation (the raft) is represented by the classical Poisson-Kirchhoff thin plate theory. This assumption is realized in a large variety of structural foundation problems encountered in Civil Engineering (see e.g. Bowles, 1968; Selvadurai, 1978). The fundamental assumptions of the Poisson-Kirchhoff's thin plate theory may be summarized as follows:
 - i) The thickness of the plate is small in comparison with its lateral dimensions.
 - ii) The deflection of the plate is small compared to its thickness.

- iii) The strain energy is composed of only the flexural energy.
- 2. The solid medium (the soil) which supports the circular plate (the raft) is assumed to behave as an isotropic and elastic continuum.
- 3. The contact between the circular plate and solid medium is assumed to be smooth and capable of sustaining tensile tractions. The first part of this assumption signifies that only vertical forces can be transmitted at the nodes which are common to both the plate and the supporting solid medium. The latter part of the assumption ensures that no gap occurs between the materials at any stage of loading.

A.3 METHOD OF ANALYSIS

The solution of a problem using the finite element method follows an orderly step-by-step process. A description of these steps can be found in any standard text on this method (see, e.g. Bowes and Russel, 1975; Desai and Abel, 1972). A very brief description, however, relating to the development of the finite element model for the interaction problem is given below.

The elastic circular plate is modelled as an assemblage of annular plate elements with two degrees of freedom (DOF) per node, as shown in Fig. A.4. To improve the accuracy of the solution, special formulation

FIGURE A.3 A TYPICAL AXISYMMETRIC TRIANGULAR RING ELEMENT

(with modified DOF, as shown in Fig. A.5) is used for the central solid circular plate element. The isotropic elastic solid-medium is modelled using axisymmetric solid element of triangular cross-section, as shown in Fig. A.3.

Using these models and following the standard steps of displacement method, the element stiffness matrices are evaluated first. These matrices are assembled together to obtain the total stiffness matrix for the plate. The element stiffness matrices for the axisymmetric solid elements are then evaluated and added with the plate stiffness matrix to get the global stiffness matrix $[K]$ for the total structure. The global force vector $\{F\}$ is then set up for the given loads. These matrices are related to the global displacement vector $\{U\}$ as follows:

$$[K]\{U\} = \{F\} \quad (A.1)$$

This equation represents a set of simultaneous linear equations, which when solved, yield the nodal displacements. Since the other quantities of interest (e.g. the flexural moments, shearing forces and stresses) are derivatives or functions of the nodal displacements, they are evaluated easily.

A.4 MAIN FEATURES

1. The program has been written in FORTRAN IV language and should run on any machine that accepts FORTRAN IV language. The limiting dimensions in the present version of the program are as follows:

FIGURE A2.4 ANNULAR PLATE ELEMENT WITH ASSOCIATED DEGREES OF FREEDOM

FIGURE A.5 SOLID CIRCULAR PLATE ELEMENT WITH ASSOCIATED DEGREES OF FREEDOM

No. of degrees of freedom	350
Bandwidth	30
No. of nodes	175
No. of solid elements	300
No. of plate elements	20

The program requires a core space of approximately 40K words to solve a problem having 350 degrees of freedom and a bandwidth of 30.

2. The program takes input data in a rigid format. Preparation of such an input is time-consuming and difficult, because the computer misinterprets an information which is not correctly positioned on the cards. To aid in the preparation of input data, a separate program has been written also in the FORTRAN IV language. This program accepts the input data in a free format and writes it in a rigid format on a specified file. This data file is used later by the analysis program.
3. Before executing the analysis program, it is desirable that probable errors in the input data be detected and eliminated. To aid in detecting errors in the input data, another program has been written. This program, with the aid of a graphic software package, known as CALPLOT, draws the finite element grid and displays it on a CRT (Cathode Ray Tube) screen. A hard copy of the display is provided at the user's request.

A.5 LIMITATIONS

1. The element stiffness matrices for the axisymmetric solid contain the term $(1-2\nu_s)$ in the denominator, ν_s being the Poisson's ratio of the solid medium. This term approaches a value of zero as ν_s approaches 0.5 and the stiffness terms become infinity when ν_s equals 0.5. Thus, the displacement formulation of finite element method can not be employed when the solid medium is incompressible. By a set of trial solutions, however, it has been found that the program can handle the case of $\nu_s=0.49$ without introducing much error in the solution.
2. Only axisymmetric loads and axisymmetric geometry of the plate-elastic medium system can be handled by the program. Furthermore, the analysis is restricted to the elastic range only.
3. Only the following loads could be handled by the program:
 - a) Uniform pressure on plate element(s) acting in a direction parallel to Z axis.
 - b) Uniform pressure on solid element(s) acting in a direction parallel to Z axis. The boundary on which the pressure acts must be parallel to R-axis (i.e. uniform pressure acting on an inclined boundary cannot be handled).
 - c) Ring load acting in a direction parallel to coordinate axes R and Z.
 - d) Axisymmetric ring moments (for plate elements only).

- e) Concentrated load parallel to the axis of symmetry and acting through a point on it.

In all these cases, a load is considered as positive when acting in the positive direction of a coordinate axis.

A.6 SIGN CONVENTIONS AND UNITS

Nodal displacements (including rotations) and moments are treated as positive in the positive directions of degrees of freedom as shown in Figs. A.3, A.4 and A.5. **Tensile** stresses are treated as positive.

Any set of consistent units may be used; the program does not carry out any transformation. A typical set of units is shown below.

<u>Parameters</u>	<u>Units</u>
Coordinates	inch
Plate radius	inch
Plate thickness	inch
Young's modulus	lb/in ²
Concentrated load	lb
Uniform pressure	lb/in ²
Radial moment in plate	lb-in/in
Circumferential moment in plate	lb-in/in
Stress	lb/in ²

A.7 A PROGRAM FOR THE FINITE ELEMENT ANALYSIS OF AXISYMMETRIC SOIL-STRUCTURE INTERACTION PROBLEMS

A.7.1 Input

The following subsections illustrate the preparation of input data for the finite element analysis of an interaction problem.

A.7.1.1 Descretization of the continuum

The first step in the preparation of input data for the finite element analysis of an interaction problem involves the subdivision of the continuum region into elements (usually known as grid) by the user. Circular plate is divided into a number of annular-ring elements and the elastic solid medium into axisymmetric solid elements having triangular cross-sections. Since attention is restricted to only axisymmetric cases, the annular plate elements could be thought of as bar elements and the axisymmetric solid elements as plane triangular elements.

The selection of an efficient finite element grid needs engineering judgement and is greatly influenced by the nature of the problem to be solved. The accuracy of the solution is influenced by the grid chosen. Some finite element grids are shown in Figs. A.6 and A.7.

A.7.1.2 Numbering of nodes

After a finite element grid has been prepared, the user should establish coordinate axes with the vertical Z axis along the axis of symmetry, and the R axis lying in a horizontal plane (as shown in Figs. A.6 and A.7). The user then assigns a number to each node. Any arbitrary scheme can be used for this purpose with one exception. For the nodes which are common to both the plate and the elastic solid medium, the number assigned to each node must increase with increasing distance from the axis of symmetry. The choice of node numbers for the elastic solid medium is arbitrary.

FIGURE A.6 CIRCULAR PLATE ON HOMOGENEOUS ELASTIC SOLID MEDIUM:
FINITE ELEMENT GRID

FIGURE A.6 (Contd.) CIRCULAR PLATE ON HOMOGENEOUS ELASTIC SOLID MEDIUM: FINITE ELEMENT GRID SHOWING NODE NUMBERS & COMPONENT NUMBERS

FIGURE A.7 CLAMPED CIRCULAR PLATE: FINITE ELEMENT MODEL

The numbering system has an influence on the bandwidth. In most cases, an arbitrary numbering scheme increases the bandwidth rapidly. By a skillful assignment of node numbers, the bandwidth can be minimized. A simple procedure, to minimize bandwidth, is to number the smaller dimension at an extremity on the center of the continuum, and then to number the adjacent rows or columns until the whole continuum has been covered.

A.7.1.3 Numbering of degrees of freedom

Having completed the numbering of nodes, the user should assign a number to each degree of freedom. These numbers are usually referred to as component numbers and are necessary to specify the boundary conditions (i.e. known displacements including rotations) of a problem. They are also required to input a concentrated load or a ring load. Unlike numbering of nodes, this numbering system is not arbitrary. For a particular node, the R component of displacement is numbered first, followed by the Z component of displacement and then the rotation, if any. Numbers related to degrees of freedom of a node n may be obtained as follows:

$$r \text{ component of displacement} = 2n + \ell - 1 = m \text{ (say);}$$

$$z \text{ component of displacement} = m + 1;$$

$$\text{rotation component, if any} = m + 2;$$

where ℓ is the number of plate nodes which have a node number less than n .

It should be mentioned here that once the node numbers are supplied by the user as specified in Section A.7.1.5, the program automatically calculates the component number corresponding to each degree of freedom. Therefore, it is not necessary for the user to assign component numbers to all degrees of freedom. The user needs to know only those component numbers which are required either to specify the boundary conditions of a problem or to specify the loads.

A.7.1.4 Numbering of elements

At this stage, the user must assign a number to each plate element and solid element. This numbering scheme is purely arbitrary except that the numbers must be integers and continuous. The plate elements and solid elements are to be numbered separately[†].

A.7.1.5 Format of the input data

The program accepts input data in a rigid format. The format in which the input must be supplied is shown below.

[†] For further illustration, the user may refer to example problem of Section A.7.5.

<u>CARD GROUP</u>	<u>NO. OF CARDS</u>	<u>FORMAT</u>	<u>DESCRIPTION AND REMARKS</u>
A	1	20A4	Case Title
B	NN=no. of nodes	I5,2F10.4	One card for each node containing; node number, R coordinate, Z coordinate.
C	1		Blank card to indicate end of node data.
D	1	F10.1,F10.4	Young's modulus and Poisson's ratio for the plate. NOTE: This card must be omitted if there is no plate.
E	NELP=no. of plate elements	3I5,F10.4	One card for each plate element containing; element number, left node number, right node number, thickness. NOTE: Input must start at the left-most element (i.e. element nearer to the axis of symmetry) and proceed away from the axis of symmetry. If thickness of an element is the same as that of the previous one, the relevant data field may be left blank. Cards in this group must be omitted if there is no plate.
F	1		Blank card to indicate end of plate element data.
G	1	I5	OPTION :JOPT JOPT=0 if Young's modulus for each solid element is supplied by the user;

CARD GROUP	NO. OF CARDS	FORMAT	DESCRIPTION AND REMARKS
G	1	I5	JOPT=1 for non-homogeneous solid of linearly varying Young's modulus; JOPT=-1 if there is no solid element.
H	NTE=no. of solid elements	4I5,F10.1,F10.4	One card for each solid element containing element number, connecting nodes (to be counted in a counter clockwise direction), Young's modulus and Poisson's ratio. NOTE: If Young's modulus and/or Poisson's ratio for an element are the same as that of the previous element, the relevant data fields may be left blank. Cards in this group must be omitted if JOPT=0.
I	1	F10.1,F10.4	Card containing Young's modulus at the surface of the solid medium and Poisson's ratio of the solid medium.
J	NTE=no. of solid elements	4I5,F10.4	One card for each solid element containing element number, connecting nodes (to be counted in a counter clockwise direction) and Poisson's ratio. NOTE: If Poisson's ratio of an element is the same as that of the previous element, the relevant field may be left blank. Cards in Groups I and J must be omitted if JOPT≠1.
K	1		Blank card to indicate end of solid element data. NOTE: This card must be omitted if JOPT=-1.

CARD GROUP	NO. OF CARDS	FORMAT	DESCRIPTION AND REMARKS
L	NUDLP=no. of plate elements having uniform pressures	2I5,F10.4	One card for each plate element having uniform pressure containing, left node number, right node number, and the intensity of pressure. NOTE: Cards in this group must be omitted if there is no uniform pressure on any plate element.
M	1		Blank card to indicate end of uniform pressure load data for plate elements.
N	NUDLS=no. of solid elements having uniform pressure	3I5,F10.4	One card for each solid element having uniform pressure containing, connecting nodes for the element (starting from the top left node and proceeding in a counter clockwise direction) and the intensity of the load. NOTE: Cards in this group must be omitted if there is no uniform pressure on any solid element.
O	1		Blank card to indicate the end of uniform pressure load data for solid elements.
P	NC=No. of concentrated loads, ring loads and ring moments	I5,F10.4	One card for each concentrated load (both horizontal loads and vertical loads), ring load and ring moment containing component number and the magnitude of load. NOTE: A ring load may be either horizontal or vertical. The magnitude of a ring load or ring moment must be calculated by the user. Component number refers to a number which corresponds to the degree of freedom along which the load (or moment) acts. Cards in this group must be omitted if there is no such load.

CARD GROUP	NO. OF CARDS	FORMAT	DESCRIPTION AND REMARKS
Q	1		Blank card to indicate end of concentrated loads and/or ring loads and/or ring moment data.
R	ND=No. of known displacements, including rotations, if any	I5,F10.4	One card for each known displacement and/or rotations containing component number and the magnitude of displacements or rotations. NOTE: Zero displacements (including rotation) must be specified.
S	1		Blank card to indicate the end of known displacement data.
T	1	I5	OPTION: NREQ NREQ=2 if the stress components, the stress invariants and principal stresses for the solid elements are required; NREQ>0 but ≠2 if only the stress components are required, but the stress invariants and principal stresses are not required; NREQ≤0 when stresses are not required.
U	1	I5	OPTION: NEXT NEXT=1 when the same problem is to be solved again for another set of loading and/or boundary conditions. For such a case, the rest of the input data must start from card group L. NEXT=2 when another problem is to be solved. For such a case, the input must start from card group A. NEXT≠1 or 2. The program run is terminated.

A.7.2 Output

The program prints out the following information:

1. A copy of the input data supplied by the user.
2. The force and displacement components at each node, and the average displacement of the plate, if any.
3. The flexural moments and shearing forces at each node of the plate.
4. The components of stress (normal, radial, circumferential and shear) at the centroid of each solid element, when requested by the user.
5. The stress invariants (I_1, I_2) at the centroid of each solid element, if requested by the user.
6. The principal stresses and their directions for each solid element, if requested by the user.

* () Refers to the name of the SUBROUTINE.

A.7.4 FORTRAN symbols used

Variables listed below have been used for the main program and the subprograms. Type A stands for an array and S for a scalar.

MAIN PROGRAM

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
ABAR	A	Symmetric non-zero part of matrix A which relates the nodal displacement vector $\{\delta\}_e$ with the generalized coordinate vector $\{\alpha\}$ for an element.
AIN	A	Inverse of the matrix A. Matrix A relates the nodal displacement vector $\{\delta\}_e$ with the generalized coordinate vector $\{\alpha\}$ for an element. $\{\delta\}_e = [A]\{\alpha\}$
AREA	S	Cross-sectional area of an axisymmetric solid element.
BB	A	Relates the strain vector $\{\epsilon\}$ with the displacement vector $\{\delta\}_e$ for an element. This matrix is usually referred to as $[B]$ in the finite element literature. $\{\epsilon\} = [B][A^{-1}]\{\delta\}_e$
BTDB	A	Denotes the integral part of an element stiffness matrix, i.e. $\int_{vol} [B^T][D][B]dv$ where $[B]$ is as defined above (see BB) and $[D]$ is the constitutive relation matrix.
D	A	Relates the stress vector $\{\sigma\}$ with the strain vector $\{\epsilon\}$ as $\{\sigma\} = [D]\{\epsilon\}$ This matrix is usually known as the constitutive relation matrix.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
ES	A	Element stiffness matrix for the solid element.
ESP	A	Element stiffness matrix for the plate element.
F	A	Global force vector.
I, IA, IC, II, IM, IP, IR, JA, JB, JJ	S	Counters or local variables.
INDEX	S	Array containing the component numbers of each known displacement. This array is used in solving a set of simultaneous linear equations by the Payne and Iron's method.
IRR, IRW	S	Reading unit and writing unit respectively.
JOPT	S	User's option. JOPT=0 if the modulus of elasticity for each solid element is supplied by the user; JOPT=1 if the elastic solid medium is non-homogeneous where the modulus of elasticity varies linearly with depth; JOPT=-1 if there is no solid element.
LBAND	S	Maximum bandwidth allowed.
NBAND	S	Calculated bandwidth for a given problem.
NBANDP	S	Calculated bandwidth for plate elements only.
NCON	A	Contains the corner node numbers of each solid element.
NDF	S	Total number of degrees of freedom for a problem.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
NDMAX	S	Maximum number of degrees of freedom allowed.
NELP	S	Number of plate elements.
NEMAX	S	Maximum number of solid elements allowed.
NEXT	S	User's option: NEXT=1 when a given problem is to be solved for another set of loads or boundary conditions; NEXT=2 when a new problem is to be solved; NEXT=1 or 2 when the program run is terminated.
NNL	S	Left node number of a plate element.
NNMAX	S	Maximum number of nodes allowed by the program.
NNR	S	Right node number of a plate element.
NON	S	Number of nodes for a given problem.
NPL	A	Contains the connecting node numbers for the plate elements.
NPMAX	S	Maximum number of plate elements allowed.
NRD	A	Contains the component numbers which correspond to radial degree of freedom for each node.
NREQ	S	User's option: NREQ=2 if the stress components, stress invariants and the principal stresses are required; NREQ>0 but ≠2 if only the stress components are required; NREQ<0 if the stresses are not required.
NTE	S	Total number of solid elements for a problem.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
PR	S	Poisson's ratio of the plate material.
R	A	Contains the radial coordinate of each node.
RMOM	A	Array containing the values of radial-moment at the nodes of each plate element.
S	A	Rectangular matrix consisting of the lower part of the banded global stiffness matrix.
THICK	A	Contains the thickness of each plate element.
TMOM	A	Contains the values of circumferential moment at the nodes of each plate element.
U	A	Global displacement vector.
XNU	A	Contains the Poisson's ratio of each solid element.
YM	A	Contains Young's modulus for each solid element.
YMOD	S	Young's modulus for the plate material.
Z	A	Contains the Z coordinate of each node.

SUBROUTINE TITLE[†]

IRR,IRW	S	Same as defined in the main program.
Name	A	Array containing the Case Title.

[†] Functions of each subroutine are explained in the program itself.

SUBROUTINE CORD

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
I	S	Counter used for printing coordinates of each node.
IRR,IRW	S	Same as defined in the main program.
NDF	S	Same as defined in the main program.
NF	S	Number of degrees of freedom at each node.
NN	S	Node number.
NNMAX	S	Same as defined in the main program.
NON	S	Number of nodes for a given problem.
R	A	Same as defined in the main program.
RR	S	Radial coordinate of a particular node.
Z	A	Same as defined in the main program.
ZZ	S	Z coordinate of a particular node.

SUBROUTINE ELMP

IRR,IRW	S	Same as defined in the main program.
NBANDP	S	Same as defined in the main program.
NDE,THIC,XT3	S	Local variables.
NEL	S	Element number (for plate elements).
NELP,NNMAX,NPL, NPMAX,NRD	A or S	Same as defined in the main program.
N1,N2	S	Left node and right node respectively of a plate element
PR,THICK,YMOD	A or S	Same as defined in the main program.

SUBROUTINE STIFFP

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
AA	A	Relates the nodal displacement vector $\{\delta\}_e$ with the generalized coordinate vector $\{\alpha\}$ for an element. $\{\delta\}_e = [AA]\{\alpha\}$ <p>This matrix is usually denoted by $[A]$ in the literature, related to the finite element method.</p>
DP	S	Rigidity of the plate defined as $DP = \frac{E_b t^3}{12(1-\nu_b^2)}$ <p>where E_b and ν_b are the elastic properties of the plate material, t being the thickness of the plate.</p>
ESP, NNMAX, NPMAX, PR, R, THICK	A or S	Same as defined in the main program.
I, IM, J, PR1, RLOG	S	Either counters or local variables.
R1, R2	S	Radial coordinates of left and right nodes respectively of a plate element.
XKD	A	Matrix which contains the integral part of the plate element stiffness matrix, i.e. $\int_{vol} [B^T][D][B]dv$
YOM	S	Same as YMOD defined in the main program.

SUBROUTINE AMAT

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
AA	A	Same as defined in the SUBROUTINE STIFFP.
NNMAX	S	Same as defined in the main program.
NN1, NN2	S	Left and right nodes respectively of a plate element.
R	A	Same as defined in the main program.
R1, R2	S	R coordinates of left and right nodes respectively for a plate element.

SUBROUTINE MATIN[†]SUBROUTINE ESTIFF

AIN, BTDB, ES	A	Same as defined in the main program.
I, I1, I2, J, JJ, J2, N, SUM	S	Either counters or local variables.

SUBROUTINE FILLSP

ESP, LBAND, NDMAX, NRD,>NNL, NNR	A or S	Same as defined in the main program.
II, JJ, NSUB, NXI	S	Counters or local variables.
NX	A	Same as INDEX defined in the main program.
SP	A	Same as S defined in the main program.

[†] This SUBROUTINE, which inverts an nxn matrix, was developed by Bowes and Russel (1975). Definition of FORTRAN symbols therefore is not included here.

SUBROUTINE CENTRE

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
CONST	S	Constant equal to 2π .
DP	S	Same as defined in the SUBROUTINE STIFFP.
ESP, NNMAX, NPMAX, PR, R, THICK	A or S	Same as defined in the main program.
I, J	S	Counters
NN1, NN2	S	Same as>NNL and NNR respectively as defined in the main program.
R2	S	Outer radius of a plate element.
YOM	S	Same as defined in the SUBROUTINE STIFFP.

SUBROUTINE FILCEN

ESP, LBAND, NDMAX, NNL, NNR, NPMAX, NRD, S	A or S	Same as defined in the main program.
II, JJ, NSUB, NXI	S	Either counter or local variables.
NXX	A	Local dummy array, used in storing component numbers.

SUBROUTINE ELMT

ES0	S	Young's modulus at the surface of an elastic solid medium (i.e. at $z=0$) of linearly varying elastic properties with depth.
I, J, NAD, NB, NC1, NC2, NC3, NC4, XT1, XT2, XNUS, YMM	S	Counters or local variables.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
IRR, IRW, JOPT, LBAND, NBAND, NBANDP, NCON, NEMAX, NNMAX, NPL, NPMAX, NRT, NTE, XNU, YM, Z	A or S	Same as defined in the main program.
NEL	S	Same as defined in the SUBROUTINE ELMP.
NMAX	S	Out of three horizontal degrees of freedom (DOF) at three nodes of a triangular element, NMAX corresponds to the highest DOF.
NMIN	S	Same as defined above except NMIN corresponds to the lowest DOF.
SLOPE	S	Rate of increase of Young's modulus with depth for the case of non-homogeneous solid medium.

SUBROUTINE AB

ABAR, AREA, NCON, NEMAX, NNMAX, R, Z	A or S	Same as defined in the main program.
NEL	S	Same as defined in the SUBROUTINE ELMP.
N1, N2, N3	S	Connecting node numbers (in the counter clockwise direction) for a solid element.

SUBROUTINE DMAT

D, NEMA, XNU, YM	A	Same as defined in the main program.
NEL	S	Same as defined in the SUBROUTINE ELMP.
PXNU	S	Poisson's ratio of a particular solid element.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
PYM	S	Young's modulus of a particular solid element.
TERM	S	Local variable.

SUBROUTINE AINVR5

ABIN	A	Inverse of matrix ABAR (i.e. $[\bar{A}]$) defined in the main program.
AIN	A	Same as defined in the main program.
I,J	S	Counters used in matrix operations.

SUBROUTINE EST1

AIN,BB,NCON,NEMAX, NNMAX,R,Z	A	Same as defined in the main program.
ARR	S	R coordinate at the centroid of a solid element.
ARZ	S	Z coordinate at the centroid of a solid element.
EE	A	Element stiffness matrix, but used here as a dummy matrix to aid in matrix multiplications.
IA,IB,IC,IKK,JA,JB, JC,SUM	S	Counters or local variables.
NEL	S	Same as defined in the SUBROUTINE ELMP.
N1,N2,N3	S	Same as defined in the SUBROUTINE AB.

SUBROUTINE INTBDB

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
AREA,BTDB,D,NCON, NEMAX,NNMAX,R,Z	A or S	Same as defined in the main program.
D11,D12,D13,D22,D23, D33,D44,I,II,J,JJ,W1, W2,XLC,XL1,XL2,XL3	S	Either counters or local variables.
NEL	S	Same as defined in the SUBROUTINE ELMP.
N1,N2,N3	S	Same as defined in the SUBROUTINE AB.
R1,R2,R3	S	R coordinates of connecting nodes of a solid element.
XL	A	Local dummy array used for numerical integration.
Z1,Z2,Z3	S	Z coordinates of connecting nodes of a solid element.

SUBROUTINE FILLS

ES,LBAND,NCON, NDMAX,NEMAX,NNMAX, NRD,S	A or S	Same as defined in the main program.
I,II,IR,J,JJ,NADR, NSIZE,NSUB,N1,N2	S	Either counters or local variables.
NEL	S	Same as defined in the SUBROUTINE ELMP.
NN	S	Number of nodes per element.
NX	A	Same as INDEX defined in the main program, but used here as a dummy array.

SUBROUTINE CLVECP

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
CONST	S	Constant equal to 2π times intensity of uniform pressure.
DLOAD	S	Intensity of uniform pressure.
F,IRR,IRW,NDMAX, NNMAX,NRD,R	A or S	Same as defined in the main program.
IA,IB,NR1,NR2, SUM,TEMP	S	Counters or local variables.
N1,N2	S	Same as>NNL and NNR respectively defined in the main program.
R1,R2	S	R coordinates at left and right nodes respectively of a plate element.
TERM	A	Local dummy array used in setting up global force vector {F}.

SUBROUTINE CSLOAD

ABAR,F,NDMAX,NNMAX, NRD,R,Z	A or S	Same as defined in the main program.
NR1,NR3,TERM1,TERM2, TERM3	S	Local variables.
TPI	S	Constant equal to 2π .
UDL	S	Intensity of uniform pressure on solid elements.

SUBROUTINE FVEC

F,IRR,IRW,NDMAX	A or S	Same as defined in the main program.
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<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
FORCE	S	Magnitude of concentrated load (acting in a direction parallel to coordinate axes) or ring load (including ring moments).
N	S	Component number along which the above force acts (i.e. the number which corresponds to the degree of freedom along which the above force acts).

SUBROUTINE DVEC

DISP	S	Magnitude of known displacement (or rotation) including sign.
I	S	Counter.
INDEX, IRR, IRW, NDF, NDMAX, U	A or S	Same as defined in the main program.
NNC	S	Component number which corresponds to the above known displacement (or rotation).

SUBROUTINE SOLVE

C, I, IB, IC, II, IJ, J, JA, JJ, NA, NR	S	Either counters or local variables.
CP	S	A large number (10^{30}) used in the solution of simultaneous equations using Payne and Iron's method.
F, INDEX, LBAND, NBAND, NDF, NDMAX, S, R, U	A or S	Same as defined in the main program.

SUBROUTINE ADISP

A, AD, B, C, I, IEN, IEVC, IMN, IMVC, ISN, ISVC, NOP, NR, VOL	S	Counters or local variables.
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<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
IRW,NDMAX,NELP, NNMAX,NPL,NPMAX, NRD,R	A or S	Same as defined in the main program.
NNLP	S	Number of plate elements plus one.
R1,R3	S	R coordinates of inner node and outer node respectively of a plate element.

SUBROUTINE NDFORC

F,IRW,LBAND,NBAND, NDF,NDMAX,NNMAX,NON, NPL,NPMAX,NRD,S,U	A or S	Same as defined in the main program.
I,IJ,J,JNB,K,KNB, NK,NR,NRJ	S	Either counters or local variables.

SUBROUTINE BMSF

ALP	A	Local dummy array used for printing bending moments and shear forces.
AMATP	A	A matrix for plate element. Matrix A has been defined in the main program under AIN.
CONST,CONST1	S	Local variables.
DER1	S	First derivative of vertical displacement ω with respect to r, i.e.
		$\frac{\partial \omega}{\partial r}$
DER2	S	Second derivative of vertical displacement ω with respect to r, i.e.

$$\frac{\partial^2 \omega}{\partial r^2}$$

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
DER3	S	Third derivative of vertical displacement with respect to r, i.e. $\frac{\partial^3 w}{\partial r^3}$
DISP	A	Same as U explained in the main program.
I,J,K,NHL,NHR,NL, NR,SUM,TEM	A or S	Either counters or local variables.
IRW,NDMAX,NELP, NNMAX,NPL,NPMAX, NRD,PR,R,THICK,YMOD	A or S	Same as explained in the main program.
QRL	S	Shear force Q_r at the left node of a plate element.
QRR	S	Shear force Q_r at the right node of a plate element.
R1,R2	S	Same as R1 and R3 respectively explained in the SUBROUTINE ADISP.
RMOM	A	Contains radial moment M_r for each plate node.
TMOM	A	Contains circumferential moment M_θ for each plate node.
XMRL,XMRR	S	Radial component of flexural moment (i.e. M_r) at the left node and right node respectively of a plate element.
XMTL,XMTR	S	Circumferential component of flexural moment (i.e. M_θ) at the left node and right node respectively of a plate element.

SUBROUTINE EST2

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
BB, F, IRW, NCON, NDMAX, NEMAX, NNMAX, NPL, NPMAX, NRD, NTE, R, U, Z	A or S	Same as defined in the main program.
IA, IB, IC, IG, II, IK, IM, INT, JJ, KP, NADR, SUM	S	Either counters or local variables.
IND	A	Same as INDEX defined in the main program.
N1, N2, N3	S	Same as defined in the SUBROUTINE AB.
RC	S	R coordinate at the centroid of a solid element.

SUBROUTINE SINVAR

I	S	Counter.
IRW, NCON, NEMAX, NNMAX, NTE, R, Z	A or S	Same as defined in the main program.
N1, N2, N3	S	Same as defined in the SUBROUTINE AB.
RC, ZC	S	R coordinate and Z coordinate respectively at the centroid of a solid element.
SR, SRZ, ST, SZ	S	Stress components (i.e. $\sigma_{rr}, \sigma_{rz}, \sigma_{\theta\theta}, \sigma_{zz}$).
XI1, XI2	S	Stress invariants I_1 and I_2 respectively.

SUBROUTINE PSTRESS

IA, IS, XL1, XL2, XL3, XL4	S	Either counters or local variables.
IRW, NDMAX, NTE	A or S	Same as defined in the main program.

<u>Symbol</u>	<u>Type</u>	<u>Definition</u>
PST1,PST2,PST3	S	Components of principal stress (i.e. σ_1 , σ_2 and σ_3).
SMAX	S	Maximum principal stress.
SMIN	S	Minimum principal stress.
SSMAX	S	Maximum shear stress.
STR	A	Global force vector, but used here as a dummy array in storing stress components.
THETA1,THETA2	S	Direction of principal stresses (i.e. angles related to principal planes).

A.7.5 Example

To show a typical output of the program, the problem illustrated in Fig. A.6 has been solved using $H=2a$ and $R=4a$. The output is shown below.

8	.9000E 00	.1800E 01
9	.1200E 01	.3100E 01
10	.1800E 01	.5000E 01
11	.1600E 01	.0000E 00
12	.1600E 01	.8000E 00
13	.1900E 01	.1800E 01
14	.2500E 01	.3100E 01
15	.3700E 01	.5000E 01
16	.2500E 01	.0000E 00
17	.2500E 01	.8000E 00
18	.2900E 01	.1800E 01
19	.3600E 01	.3200E 01
20	.3400E 01	.0000E 00
21	.3400E 01	.8000E 00
22	.4100E 01	.1900E 01
23	.4900E 01	.3300E 01
24	.5800E 01	.5000E 01
25	.4500E 01	.0000E 00
26	.4500E 01	.9000E 00
27	.5400E 01	.2300E 01
28	.6400E 01	.3500E 01
29	.7800E 01	.5000E 01
30	.6000E 01	.0000E 00
31	.6000E 01	.1100E 01
32	.7000E 01	.2500E 01
33	.8600E 01	.3000E 01
34	.1000E 02	.5000E 01
35	.7800E 01	.0000E 00
36	.7800E 01	.1300E 01
37	.1000E 02	.3300E 01
38	.1000E 02	.0000E 00
39	.1000E 02	.1700E 01

TOTAL NO OF NODES = 39

ELEMENTS WITH ROTATION DOF

YOUNGS MODULUS = .3000E 07
 POISSONS RATIO = .3000E 00

ELEM.NO. CONNECTING NODES THICKNESS

ELEM.NO.	CONNECTING NODES		THICKNESS
	LT	RT	
1	1	6	.5700E-01
2	6	11	.5700E-01
3	11	16	.5700E-01

TOTAL NO OF ELEMENTS HAVING
ROTATION DEGREES OF FREEDOM = 3

ELEMENTS WITHOUT ROTATION DOF

ELEM.NO.	CONNECTING NODE NUMBERS		YON.MOD.	P-RATIO
1	1	2	.3000E 02	.4700E 00
2	1	7	.3000E 02	.4700E 00
3	6	7	.3000E 02	.4700E 00
4	6	12	.3000E 02	.4700E 00
5	11	12	.3000E 02	.4700E 00
6	11	17	.3000E 02	.4700E 00
7	16	17	.3000E 02	.4700E 00
8	16	21	.3000E 02	.4700E 00
9	20	21	.3000E 02	.4700E 00
10	20	26	.3000E 02	.4700E 00
11	25	26	.3000E 02	.4700E 00
12	25	31	.3000E 02	.4700E 00
13	30	31	.3000E 02	.4700E 00
14	30	36	.3000E 02	.4700E 00
15	35	36	.3000E 02	.4700E 00
16	35	39	.3000E 02	.4700E 00
17	2	3	.3000E 02	.4700E 00
18	7	3	.3000E 02	.4700E 00
19	7	8	.3000E 02	.4700E 00
20	12	13	.3000E 02	.4700E 00
21	12	17	.3000E 02	.4700E 00
22	17	18	.3000E 02	.4700E 00
23	17	21	.3000E 02	.4700E 00
24	21	22	.3000E 02	.4700E 00
25	21	26	.3000E 02	.4700E 00

26	26	22	27	.3000E 02	.4700E 00
27	26	27	31	.3000E 02	.4700E 00
28	31	27	32	.3000E 02	.4700E 00
29	31	32	36	.3000E 02	.4700E 00
30	36	32	33	.3000E 02	.4700E 00
31	36	33	37	.3000E 02	.4700E 00
32	36	37	39	.3000E 02	.4700E 00
33	3	4	8	.3000E 02	.4700E 00
34	8	4	9	.3000E 02	.4700E 00
35	8	9	13	.3000E 02	.4700E 00
36	13	9	14	.3000E 02	.4700E 00
37	13	14	18	.3000E 02	.4700E 00
38	18	14	19	.3000E 02	.4700E 00
39	18	19	22	.3000E 02	.4700E 00
40	22	19	23	.3000E 02	.4700E 00
41	22	23	27	.3000E 02	.4700E 00
42	27	23	28	.3000E 02	.4700E 00
43	27	28	32	.3000E 02	.4700E 00
44	32	28	29	.3000E 02	.4700E 00
45	32	29	33	.3000E 02	.4700E 00
46	33	29	34	.3000E 02	.4700E 00
47	33	34	37	.3000E 02	.4700E 00
48	4	5	9	.3000E 02	.4700E 00
49	9	5	10	.3000E 02	.4700E 00
50	9	10	14	.3000E 02	.4700E 00
51	14	10	15	.3000E 02	.4700E 00
52	14	15	19	.3000E 02	.4700E 00
53	19	15	23	.3000E 02	.4700E 00
54	23	15	24	.3000E 02	.4700E 00
55	23	24	28	.3000E 02	.4700E 00
56	28	24	29	.3000E 02	.4700E 00

TOTAL NO. OF ELEMENTS WITHOUT ROTATION DOF = 56

MAXIMUM BAND WIDTH ALLOWED = 30
CALCULATED BAND WIDTH = 21

*** APPLIED LOADS ***

UDL ON ELEMENTS HAVING RDOF BETWEEN NODES		INTENSITY
1	6	.1000E 03
6	11	.1000E 03
11	16	.1000E 03

UDL ON ELEMENTS WITHOUT RDOF
BETWEEN NODES INTENSITY

KNOWN NON-ZERO CONC. LOAD COMPONENTS
COMPONENT NUMBER LOAD

KNOWN DISPLACEMENTS COMPONENT NUMBER	DISPLACEMENT
3	.0000E 00
4	.0000E 00
6	.0000E 00
8	.0000E 00
10	.0000E 00
11	.0000E 00
22	.0000E 00
33	.0000E 00
52	.0000E 00
62	.0000E 00
72	.0000E 00

AVERAGE DISPLACEMENT = .49464E 01

FORCE AND DISPLACEMENT COMPONENTS

NODE NO	FORCES			DISPLACEMENTS		
	R COMP	Z COMP	T COMP	R COMP	Z COMP	T COMP
1	-.5684E-13	.6032E 02	.0000E 00	-.1238E-01	.6950E 01	.0000E 00
2	-.7120E 01	-.4725E-12		.2696E-32	.5949E 01	
3	-.2906E 02	-.1847E-12		.7564E-32	.4697E 01	
4	-.9827E 01	-.2984E-12		.1854E-32	.2761E 01	
5	-.2120E 01	-.1491E 03		.5740E-33	.6302E-30	
6	.8527E-12	.4021E 03	.2145E 02	.2497E 00	.6742E 01	-.5143E 00
7	-.5684E-13	-.2782E-11		.4015E 00	.5859E 01	
8	-.3979E-12	-.2217E-11		.6007E 00	.4522E 01	
9	-.3688E-11	.3979E-12		.8084E 00	.2591E 01	
10	-.3979E-12	-.6272E-03		.1125E 01	.3754E-30	
11	.9095E-12	.8705E-03	.4023E 02	.5039E 00	.6144E 01	-.9443E 00
12	-.2274E-12	.3411E-12		.7776E 00	.5336E 01	
13	-.7284E-11	-.1933E-11		.1188E 01	.3855E 01	
14	-.6025E-11	.1037E-11		.1485E 01	.1752E 01	
15	-.3922E-11	-.8518E 03		.1741E 01	.2218E-30	
16	.5684E-13	.6305E 03		.9273E 00	.5176E 01	
17	.1990E-11	-.1108E-10	-.9076E 02	.1131E 01	.4294E 01	-.1131E 01
18	-.4889E-11	-.8015E-11		.1634E 01	.2519E 01	
19	-.1097E-10	-.5674E-11		.1596E 01	.1068E 01	
20	-.7958E-12	-.3183E-11		.7024E 00	.2486E 01	
21	-.5855E-11	-.1125E-10		.1443E 01	.2437E 01	
22	-.4029E-11	-.7503E-11		.1631E 01	.1363E 01	
23	-.5780E-11	-.1609E-11		.1667E 01	.5034E 00	
24	-.2558E-11	-.4820E 03		.1604E 01	.6969E-31	
25	-.7958E-12	-.4544E-11		.7064E 00	.1270E 01	
26	-.1990E-11	-.1762E-11		.1220E 01	.1173E 01	
27	-.6889E-11	-.1421E-11		.1511E 01	.5277E 00	
28	-.9891E-11	.1421E-11		.1492E 01	.1761E 00	
29	-.9095E-11	-.6752E 02		.1391E 01	.9682E-32	
30	-.1535E-11	-.3597E-12		.7455E 00	.5641E 00	
31	-.1933E-11	.1140E-11		.1115E 01	.4539E 00	
32	-.5855E-11	.2899E-11		.1318E 01	.1687E 00	
33	-.3897E-11	.3634E-11		.1187E 01	.9522E-01	
34	-.1727E-11	.2140E 03		.1198E 01	-.3934E-31	

35	-.3951E-11	-.2330E-11	.7235E 00	.2295E 00
36	-.4224E-11	.7514E-11	.1027E 01	.1613E 00
37	.2697E-11	-.1023E-11	.1061E 01	-.1873E-01
38	-.8491E-12	.5116E-12	.6657E 00	.1241E 00
39	.6253E-12	.1251E-11	.9463E 00	.4576E-01

BENDING MOMENTS AND SHEAR FORCES FOR PLATE ELEMENTS

EL NO	MR LT	MT LT	QR LT	MR RT	MT RT	QR RT
1	.4445E 02	.4445E 02	.1092E 00	.4104E 02	.4208E 02	.5555E 01
2	.4369E 02	.4287E 02	.1487E 02	.2982E 02	.3627E 02	.2210E 02
3	.3263E 02	.3711E 02	.3299E 02	.4442E 01	.2229E 02	.3918E 02

STRESS AT CENTROID OF ELEMENT

ELMT NO	R COORD AT CG	R COMP	T COMP	Z COMP	SHEAR
1	.267E 00	-.3192E 02	-.3224E 02	-.6771E 02	-.9933E 00
2	.533E 00	-.5357E 02	-.5211E 02	-.8278E 02	-.7276E 00
3	.107E 01	-.2029E 02	-.2078E 02	-.5241E 02	-.4731E 01
4	.133E 01	-.4308E 02	-.4175E 02	-.7019E 02	-.4125E 01
5	.190E 01	-.2303E 02	-.2242E 02	-.5168E 02	-.8324E 01
6	.220E 01	-.2934E 02	-.3102E 02	-.6144E 02	-.8381E 01
7	.280E 01	-.4718E 02	-.4574E 02	-.7674E 02	-.1845E 02
8	.310E 01	-.1984E 01	.9859E 01	.1871E 01	-.2106E 02
9	.377E 01	-.1378E 02	-.1852E 01	-.9176E 01	-.2222E 01
10	.413E 01	.1725E 02	.2151E 02	.1497E 02	-.5446E 01
11	.500E 01	-.1130E 02	-.4173E 01	-.1052E 02	.1084E 01
12	.550E 01	.1354E 02	.1619E 02	.1097E 02	-.1377E 01
13	.660E 01	-.8204E 01	-.3473E 01	-.8494E 01	.1881E 01
14	.720E 01	.7887E 01	.1049E 02	.7065E 01	.4870E 00
15	.853E 01	-.5830E 01	-.2063E 01	-.5284E 01	.1945E 01
16	.927E 01	.1328E 01	.3579E 01	.9238E 00	.1195E 01

17	.267E 00	-.2949E 02	-.6528E 02	-.1151E 01
18	.567E 00	.2343E 01	-.3658E 02	-.6335E 00
19	.110E 01	-.3231E 02	-.6786E 02	-.5113E 01
20	.147E 01	-.5674E 01	-.4381E 02	-.4415E 01
21	.200E 01	-.2790E 02	-.5906E 02	-.8830E 01
22	.243E 01	-.3135E 02	-.6577E 02	-.1032E 02
23	.293E 01	-.1289E 02	-.3933E 02	-.1734E 02
24	.347E 01	.7857E 01	.3963E 00	-.7648E 01
25	.400E 01	-.2607E 02	-.2666E 02	-.8239E 01
26	.467E 01	.5220E 01	.7971E 01	-.2973E 01
27	.530E 01	-.7365E 01	-.8584E 01	-.1841E 01
28	.613E 01	-.8759E 00	.1265E 01	.3623E 00
29	.693E 01	-.1302E 01	-.1807E 01	.4403E 00
30	.780E 01	-.2551E 01	-.3090E 00	.1181E 01
31	.880E 01	-.2415E 01	.7983E-01	.7211E 00
32	.927E 01	.2034E 01	.2227E 01	.2689E 00
33	.300E 00	-.1105E 02	-.5506E 02	-.1986E 01
34	.700E 00	-.3495E 01	-.4689E 02	-.1402E 01
35	.133E 01	-.3136E 01	-.4230E 02	-.6558E 01
36	.187E 01	-.1778E 02	-.5533E 02	-.6705E 01
37	.243E 01	.1467E 02	-.1486E 02	-.1340E 02
38	.300E 01	-.1816E 02	-.3589E 02	-.6466E 01
39	.353E 01	-.1925E 02	-.3105E 02	-.9614E 01
40	.420E 01	.1092E 02	.1988E 01	-.4189E 01
41	.480E 01	-.1638E 02	-.2001E 02	-.4640E 01
42	.557E 01	.1479E 01	.1557E 01	-.1119E 01
43	.627E 01	-.6118E 01	-.5825E 01	-.1168E 01
44	.707E 01	-.3480E 01	-.1097E 01	-.1594E-01
45	.780E 01	-.9216E 00	-.6326E-01	.3443E 00
46	.880E 01	-.5257E 00	.2888E 00	.6800E 00
47	.953E 01	.1627E 01	.4048E 01	-.3129E-01
48	.400E 00	-.3137E 01	-.4654E 02	-.1446E 01
49	.100E 01	-.2251E 01	-.4284E 02	-.3131E 00
50	.183E 01	.7753E 01	-.2654E 02	-.6559E 01
51	.267E 01	-.2074E 01	-.2750E 02	-.7138E 00
52	.327E 01	.5868E 01	-.7526E 01	-.5055E 01
53	.407E 01	-.1705E 02	-.2972E 02	-.3195E 01
54	.480E 01	-.3436E 01	-.8150E 01	-.2913E-01
55	.570E 01	-.8190E 01	-.9703E 01	-.1694E 01
56	.667E 01	-.2137E 01	-.2354E 01	.3232E 00

STRESS INVARIANTS AT CG OF SOIL EL.

EL NO	COORDINATES AT CG			INVARIANTS		
	R-COORD	Z-COORD	I1	I2		
1	.2667E 00	.5333E 00	-.1319E 03	.5373E 04		
2	.5333E 00	.2667E 00	-.1885E 03	.1154E 05		
3	.1067E 01	.5333E 00	-.9349E 02	.2552E 04		
4	.1333E 01	.2667E 00	-.1550E 03	.7736E 04		
5	.1900E 01	.5333E 00	-.9713E 02	.2796E 04		
6	.2200E 01	.2667E 00	-.1218E 03	.4549E 04		
7	.2800E 01	.5333E 00	-.1697E 03	.8948E 04		
8	.3100E 01	.2667E 00	.9747E 01	-.4482E 03		
9	.3767E 01	.5667E 00	-.2480E 02	.1640E 03		
10	.4133E 01	.3000E 00	.5373E 02	.9215E 03		
11	.5000E 01	.6667E 00	-.2599E 02	.2088E 03		
12	.5500E 01	.3667E 00	.4070E 02	.5435E 03		
13	.6600E 01	.8000E 00	-.2017E 02	.1241E 03		
14	.7200E 01	.4333E 00	.2545E 02	.2124E 03		
15	.8533E 01	.1000E 01	-.1318E 02	.4995E 02		
16	.9267E 01	.5667E 00	.5830E 01	.7855E 01		
17	.2667E 00	.1133E 01	-.1243E 03	.4718E 04		
18	.5667E 00	.1467E 01	-.3030E 02	-.2207E 03		
19	.1100E 01	.1133E 01	-.1311E 03	.5261E 04		
20	.1467E 01	.1467E 01	-.5524E 02	.5140E 03		
21	.2000E 01	.1133E 01	-.1123E 03	.3777E 04		
22	.2433E 01	.1467E 01	-.1265E 03	.4811E 04		
23	.2933E 01	.1133E 01	-.6243E 02	.7394E 03		
24	.3467E 01	.1500E 01	.2571E 02	.8872E 02		
25	.4000E 01	.1200E 01	-.6678E 02	.1368E 04		
26	.4667E 01	.1700E 01	.2874E 02	.2379E 03		
27	.5300E 01	.1433E 01	-.1619E 02	.6364E 02		
28	.6133E 01	.1967E 01	.7002E 01	.1337E 01		
29	.6933E 01	.1633E 01	.4200E 00	-.8812E 01		
30	.7800E 01	.2267E 01	.3237E 00	-.9712E 01		
31	.8800E 01	.2533E 01	.2895E 00	-.6841E 01		
32	.9267E 01	.2100E 01	.9537E 01	.2694E 02		
33	.3000E 00	.2233E 01	-.7715E 02	.1335E 04		

34	.7000E 00	.2667E 01	-.5393E 02	.3407E 03
35	.1333E 01	.2233E 01	-.4730E 02	.1747E 03
36	.1867E 01	.2667E 01	-.8882E 02	.2087E 04
37	.2433E 01	.2233E 01	.1741E 02	-.4010E 03
38	.3000E 01	.2700E 01	-.6375E 02	.1134E 04
39	.3533E 01	.2300E 01	-.6019E 02	.1003E 04
40	.4200E 01	.2800E 01	.3062E 02	.2328E 03
41	.4800E 01	.2500E 01	-.4348E 02	.5641E 03
42	.5567E 01	.3033E 01	.1285E 02	.3085E 02
43	.6267E 01	.2767E 01	-.1066E 02	.1895E 02
44	.7067E 01	.3667E 01	-.7835E 00	-.1354E 02
45	.7800E 01	.3500E 01	.3546E 01	-.4523E 01
46	.8800E 01	.4333E 01	.3942E 01	-.1604E 01
47	.9533E 01	.3767E 01	.1196E 02	.4224E 02
48	.4000E 00	.3733E 01	-.5281E 02	.2997E 03
49	.1000E 01	.4367E 01	-.4694E 02	.1799E 03
50	.1833E 01	.3733E 01	-.8964E 01	-.4332E 03
51	.2667E 01	.4367E 01	-.2716E 02	-.1488E 02
52	.3267E 01	.3767E 01	.1232E 02	-.9291E 02
53	.4067E 01	.3833E 01	-.5645E 02	.9493E 03
54	.4800E 01	.4433E 01	-.6589E 01	-.2989E 02
55	.5700E 01	.3933E 01	-.1795E 02	.7754E 02
56	.6667E 01	.4500E 01	.1284E 00	-.1582E 02

PRINCIPAL STRESSES AND MAX. SHEAR STRESS

EL NO	STR-1	STR-2	STR-3	MAX SHEAR STRESS	ANGLE MAX S2-S3	ANGLE MIN S2-S3
1	-.3224E 02	-.3190E 02	-.6774E 02	.1792E 02	-.571E 00	.894E 02
2	-.5211E 02	-.5355E 02	-.8280E 02	.1534E 02	-.306E 00	.897E 02
3	-.2078E 02	-.1961E 02	-.5310E 02	.1674E 02	-.371E 01	.863E 02
4	-.4175E 02	-.4247E 02	-.7080E 02	.1453E 02	-.208E 01	.879E 02
5	-.2242E 02	-.2079E 02	-.5392E 02	.1657E 02	-.628E 01	.837E 02
6	-.3102E 02	-.2729E 02	-.6350E 02	.1811E 02	-.523E 01	.848E 02
7	-.4574E 02	-.3832E 02	-.8560E 02	.2364E 02	-.829E 01	.817E 02
8	.9859E 01	.2109E 02	-.2120E 02	.2114E 02	-.449E 02	.451E 02
9	-.1852E 01	-.8278E 01	-.1467E 02	.6411E 01	-.548E 01	.845E 02
10	.2151E 02	.2167E 02	.1055E 02	.5564E 01	.934E 01	.993E 02

11	-.4173E 01	-.9759E 01	-.1206E 02	.3945E 01	.284E 01	.928E 02
12	.1619E 02	.1414E 02	.1037E 02	.2908E 01	.321E 01	.932E 02
13	-.3473E 01	-.6462E 01	-.1024E 02	.3381E 01	.635E 01	.963E 02
14	.1049E 02	.8113E 01	.6839E 01	.1828E 01	.186E 01	.881E 02
15	-.2063E 01	-.3593E 01	-.7521E 01	.2729E 01	.965E 01	.996E 02
16	.3579E 01	.2338E 01	-.8664E-01	.1833E 01	-.234E 02	.666E 02
17	-.2949E 02	-.2945E 02	-.6532E 02	.1793E 02	-.696E 00	.893E 02
18	.2343E 01	.3943E 01	-.3659E 02	.2027E 02	-.111E 01	.889E 02
19	-.3090E 02	-.3159E 02	-.6858E 02	.1884E 02	-.291E 01	.871E 02
20	-.5758E 01	-.5169E 01	-.4431E 02	.1957E 02	-.506E 01	.849E 02
21	-.2539E 02	-.2557E 02	-.6139E 02	.1800E 02	-.574E 01	.843E 02
22	-.2940E 02	-.2850E 02	-.6862E 02	.2006E 02	-.600E 01	.840E 02
23	-.1020E 02	-.4310E 01	-.4792E 02	.2180E 02	-.168E 02	.732E 02
24	.1746E 02	.1264E 02	-.4383E 01	.1092E 02	.308E 02	.121E 03
25	-.1405E 02	-.1812E 02	-.3461E 02	.1028E 02	-.868E 01	.813E 02
26	.1555E 02	.9872E 01	.3320E 01	.6114E 01	.121E 02	.102E 03
27	-.2389E 00	-.6035E 01	-.9914E 01	.4838E 01	-.650E 01	.835E 02
28	.6613E 01	.1325E 01	-.9355E 00	.3774E 01	-.309E 02	.591E 02
29	.3529E 01	-.1047E 01	-.2062E 01	.2795E 01	.791E 01	.979E 02
30	.3184E 01	.1980E 00	-.3058E 01	.3121E 01	.198E 02	.110E 03
31	.2624E 01	.2733E 00	-.2608E 01	.2616E 01	.159E 02	.106E 03
32	.5277E 01	.2416E 01	.1844E 01	.1716E 01	-.360E 01	.864E 02
33	-.1105E 02	-.1096E 02	-.5515E 02	.2210E 02	-.172E 01	.883E 02
34	-.3549E 01	-.3450E 01	-.4693E 02	.2174E 02	-.159E 01	.884E 02
35	-.1873E 01	-.2067E 01	-.4336E 02	.2075E 02	-.805E 01	.819E 02
36	-.1571E 02	-.1662E 02	-.5649E 02	.2039E 02	-.520E 01	.848E 02
37	.1761E 02	.1984E 02	-.2004E 02	.1994E 02	-.448E 02	.452E 02
38	-.9690E 01	-.1606E 02	-.3800E 02	.1416E 02	-.673E 01	.833E 02
39	-.9886E 01	-.1387E 02	-.3643E 02	.1327E 02	-.105E 02	.795E 02
40	.1772E 02	.1257E 02	.3300E 00	.8694E 01	.165E 02	.106E 03
41	-.7085E 01	-.1321E 02	-.2318E 02	.8047E 01	-.715E 01	.828E 02
42	.9818E 01	.2638E 01	.3978E 00	.4710E 01	.182E 02	.108E 03
43	.1282E 01	-.4794E 01	-.7149E 01	.4216E 01	-.554E 01	.845E 02
44	.3793E 01	-.1097E 01	-.3480E 01	.3637E 01	-.200E 00	.898E 02
45	.4531E 01	.5780E-01	-.1043E 01	.2787E 01	.175E 02	.107E 03
46	.4179E 01	.6742E 00	-.9111E 00	.2545E 01	.401E 02	.130E 03
47	.6282E 01	.4048E 01	.1627E 01	.2328E 01	.316E 00	.903E 02
48	-.3137E 01	-.3089E 01	-.4659E 02	.2175E 02	-.167E 01	.883E 02
49	-.1854E 01	-.2249E 01	-.4284E 02	.2049E 02	-.398E 00	.896E 02
50	.9818E 01	.8965E 01	-.2775E 02	.1878E 02	-.175E 02	.725E 02

51	.2414E 01	--.2054E 01	--.2752E 02	.1497E 02	--.138E 01	.886E 02
52	.1398E 02	.7562E 01	--.9220E 01	.1160E 02	--.403E 02	.497E 02
53	--.9678E 01	--.1629E 02	--.3048E 02	.1040E 02	--.389E 01	.861E 02
54	.4996E 01	--.3435E 01	--.8150E 01	.6573E 01	--.144E 00	.899E 02
55	--.5241E-01	--.7091E 01	--.1080E 02	.5375E 01	--.536E 01	.846E 02
56	.4619E 01	--.1904E 01	--.2587E 01	.3603E 01	.409E 01	.941E 02
STOP						

A.8 COMPUTER PROGRAM FOR THE PREPARATION OF INPUT DATA

This program has been written to aid in the preparation of input data for the main program for the finite element analysis of interaction between an elastic circular plate and an isotropic elastic solid medium. The program takes the input data in free format and writes it in a rigid format on a specified file. To achieve 'free format' input, a general purpose subroutine, called REED[†], has been used. The arrangement of input data is exactly the same as described in Section A.7.1 of this manual, except that no specified format need be used for the input. Any two consecutive data items on a single line of input must be separated by one or more blanks. The program is designed to run either in the time-sharing or the batch mode of operation. Since the FORTRAN symbols used for this program have been explained in the program itself, a separate definition of these symbols is not presented here. A FORTRAN listing of the program is given in Section A.10.

[†] Subroutine 'REED' was developed by Dr. E.W. Wright (1978) of Public Works, Canada.

A.9 COMPUTER PROGRAM FOR DRAWING FINITE ELEMENT GRID

A.9.1 General

This program has been written to draw the finite element grid using the input data for the main program for the finite element analysis of interaction between an elastic circular plate and an isotropic elastic solid medium. This program uses the graphic software available under the 'CALPLOT' subroutine package. The program is designed to run either on the time-sharing or the batch mode of operation. Using the input data of Section A.7.1 of this manual, the program draws the finite element grid on a CRT screen (for time-sharing operation). A hard copy of the grid is also provided by the computer at the user's request. A typical output of the program and FORTRAN listings are presented in subsequent sections.

A.9.2 Output

A hard copy of the finite element grid related to the example problem of Section A.7.5 is presented on page 227.

A.10 FORTRAN LISTINGS

A listing of FORTRAN statements is presented in the subsequent sections.

Finite Element Grid

REFERENCES

1. Bowles, J.E. (1968). Foundation Analysis and Design. International Student Edition, McGraw-Hill, Inc., Japan.
2. Bowes, W.H. and Russel, L.T. (1975). Stress Analysis by the Finite Element Method for Practicing Engineers. Lexington Books, Massachusetts.
3. CALPLOT User's Manual (1978). Carleton University, Ottawa, Canada.
4. Desai, C.S. and Abel, J.F. (1972). Introduction to the Finite Element Method, Van Nostrand Reinhold, New York.
5. Selvadurai, A.P.S. (1978). Elastic Analysis of Soil-Foundation Interaction. Development in Geotechnical Engineering, Vol. 19, Elsevier Scientific Publishing Company, New York.
6. Wright, E.W. (1978). SUBROUTINE REED, the CAD Project, Technological Research and Development Branch, Public Works, Canada.

A.10.1 Main Program

```

1.000 C
2.000 C
3.000 C
4.000 C
5.000 C
6.000 C
7.000 C
8.000 C
9.000 C
10.000 C
11.000 C
12.000 C
13.000 C
14.000 C
15.000 C
16.000 C
17.000 C
18.000 C
19.000 C
20.000 C
21.000 C
22.000 C
23.000 C
24.000 C
25.000 C
26.000 C
27.000 C
28.000 C
29.000 C
30.000 C
31.000 C
32.000
33.000
34.000
35.000
36.000
37.000 C
38.000 C
39.000
40.000

*****
=====
          A FINITE ELEMENT PROGRAM
    FOR THE ANALYSIS OF AXISYMMETRIC SOIL-
    STRUCTURE INTERACTION PROBLEMS
=====
          UNIFORMLY DISTRIBUTED AND LINE LOADS ONLY
=====
          PROGRAMMED BY:
            MD. MUSHARRAFUZZAMAN
            CIVIL ENGINEERING DEPARTMENT
            CARLETON UNIVERSITY, OTTAWA
            CANADA
            JUNE 1979

          SUPERVISORS: PROF. A.P.S. SELVADURAI
                        AND PROF. W.H. BOWES
          -----
          *****
          DIMENSION R(175),Z(175),NCON(3,300),YM(300),XNU(300),
          +D(4,4),ABAR(3,3),AIN(6,6),BTDB(6,6),ES(6,6),
          +S(30,350),F(350),U(350),INDEX(350),BB(4,6),
          + THICK(20),RMOM(20),TMOM(20)
          +,ESP(4,4),NPL(20),NRD(175)

          SET LIMITS OF DIMENSIONS FOR SUBROUTINES
          LBAND=30
          NNMAX=175

```

```

41.000 NEMAX=300
42.000 NDMAX=350
43.000 NPMAX=20
44.000 C
45.000 C ASSIGN READING AND WRITING UNITS
46.000 C
47.000 IRR=2
48.000 IRW=3
49.000 C
50.000 C READ AND PRINT CASE TITLE
51.000 C
52.000 4 CALL TITLE
53.000 C
54.000 C READ COORDINATES OF NODES
55.000 C
56.000 CALL CORD(NDF,2,R,Z,NON,NNMAX)
57.000 C
58.000 C READ ELEMENT PROPERTIES AND INCIDENCES FOR
59.000 C ELEMENTS HAVING ROTATION DEGREES OF FREEDOM (RDOF)
60.000 C INPUT MUST BE GIVEN FROM LEFT TO RIGHT IN ORDER
61.000 C
62.000 CALL ELMP(NELP, YMOD, PR, THICK, NPL, NRD, NBANDP, NPMAX, NNMAX)
63.000 IF (NELP.GT.0) NDF=NDF+(NELP+1)
64.000 C
65.000 C CHECK IF NO OF ELEMENTS WITH RDOF IS ZERO.
66.000 C IF SO, SKIP SOME OPERATIONS AND GO TO 35
67.000 C
68.000 IF (NELP.EQ.0) GO TO 35
69.000 C
70.000 C INITIALIZE S-MATRIX TO ZERO
71.000 C
72.000 DO 6 JA=1, LBAND
73.000 DO 6 JB=1, NDF
74.000 S(JA,JB)=0.
75.000 C
76.000 C
77.000 C CALCULATE STIFFNESS MATRIX FOR ALL ELEMENTS
78.000 C HAVING RDOF AND SUBSEQUENTLY FILL IN S-MATRIX
79.000 C
80.000 DO 20 IM=1, NELP

```

```

81.000 NNL=NPL(IM)
82.000 NNR=NPL(IM+1)
83.000 IF(IM.EQ.1.AND.R(NNL).EQ.0.)GO TO 16
84.000 CALL STIFFP(R,IM,NNL,NNR,THICK,YMOD,PR,ESP,NNMAX,NPMAX)
85.000 CALL FILLSP(S,ESP,NNL,NNR,INDEX,NRD,LBAND,NDMAX,NNMAX)
86.000 GO TO 20
87.000 16 CALL CENTRE(R,ESP,YMOD,PR,THICK,NNL,NNR,NPMAX,NNMAX)
88.000 CALL FILCEN(S,ESP,NRD,NNL,NNR,LBAND,NDMAX,NNMAX,NPMAX)
89.000 20 CONTINUE
90.000 C
91.000 C READ OPTION : JOPT
92.000 C JOPT = 0 FOR HOMOGENEOUS SOLID
93.000 C JOPT = 1 FOR NON-HOMOGENEOUS SOLID WITH LINEARLY
94.000 C VARYING MODULUS OF ELASTICITY, ES
95.000 C JOPT = -1 IF NO. OF AXISYMMETRIC SOLID ELEMENT
96.000 C IS ZERO
97.000 C
98.000 35 READ(IRR,301)JOPT
99.000 IF(JOPT.EQ.-1)GO TO 61
100.000 C
101.000 C READ PROPERTIES AND INCIDENCES FOR AXISYMMETRIC
102.000 C SOLID ELEMENTS
103.000 C
104.000 CALL ELMT(NTE,NCON,YM,XNU,NPL,NRD,NBAND,NBANDP,
105.000 +NELP,LBAND,NEMAX,NNMAX,NPMAX,JOPT,Z)
106.000 38 IF(NELP.GT.0)GO TO 42
107.000 DO 40 IP=1,NDF
108.000 DO 40 IR=1,NBAND
109.000 S(IR,IP)=0.
110.000 40 CONTINUE
111.000 42 DO 45 IP=1,NDF
112.000 F(IP)=0.
113.000 45 CONTINUE
114.000 C
115.000 C - UNIT 4 - FOR WRITING BB MATRIX, D*B*A -1
116.000 C - UNIT 7 - FOR WRITING STRESSES (FOR EST2)
117.000 C - UNIT 9 - FOR WRITING S-MATRIX BEFORE ELIMINATION
118.000 C
119.000 REWIND 4
120.000 REWIND 7

```

```

121.000
122.000 C
123.000 C
124.000 C
125.000 C
126.000
127.000
128.000
129.000
130.000
131.000
132.000
133.000
134.000
135.000 C
136.000 C
137.000 C
138.000 C
139.000
140.000
141.000
142.000
143.000
144.000
145.000
146.000
147.000 C
148.000 C
149.000 C
150.000
151.000 C
152.000 C
153.000 C
154.000
155.000 C
156.000 C
157.000 C
158.000
159.000 C
160.000 C

REWIND 9

CALCULATE STIFFNESS MATRIX FOR ELEMENTS WITHOUT
RDOF AND FILL IN S-MATRIX

DO 60 I=1,NTE
CALL AB(I,NCON,R,Z,AREA,ABAR,NEMAX,NNMAX)
IF (I.EQ.1) GO TO 48
IF (YM(I).EQ.YM(I-1).AND.XNU(I).EQ.XNU(I-1))
+GO TO 49
48 CALL DMAT(I,D,YM,XNU,NEMAX)
49 CALL MATIN(ABAR,3)
CALL AINVR(ABAR,AIN)
CALL EST1(I,NCON,R,Z,D,BB,ES,AIN,NEMAX,NNMAX)

WRITE BB (D*B*A(-1) ) ON UNIT 4 (TO BE USED FOR
CALCULATING STRESSES FOR SOIL ELEMENTS)

WRITE(4)((BB(I,J),JJ=1,6),II=1,4)
CALL INTBDB(I,D,NCON,BTDB,AREA,R,Z,NEMAX,NNMAX)
CALL ESTIFF(6,AIN,BTDB,ES)
CALL FILLS(2,3,6,ES,S,NCON,I,LBAND,NDMAX,NEMAX,INDEX,
+NRD,NNMAX)
60 CONTINUE
61 IF(JOPT.EQ.-1)NBAND=NBANDP
62 WRITE(IRW,300)

CALCULATE CONSISTENT LOADS FOR UDL ON PLATE, IF ANY

CALL CLVECP(R,F,NRD,ESP,NNMAX,NDMAX,NDF)

CALCULATES CONSISTENT LOADS FOR UDL ON SOIL, IF ANY

CALL CSLOAD(R,Z,F,ABAR,NRD,NNMAX,NDMAX)

READ KNOWN NON-ZERO CONC. LOAD COMPONENTS, IF ANY

CALL FVEC(F,NDMAX)

READ KNOWN DISPLACEMENTS , IF ANY

```



```

161.000 C
162.000
163.000 C
164.000 C
165.000 C
166.000
167.000
168.000
169.000 C
170.000 C
171.000 C
172.000 C
173.000
174.000 C
175.000 C
176.000 C
177.000
178.000
179.000 C
180.000 C
181.000 C
182.000
183.000
184.000
185.000
186.000 C
187.000 C
188.000 C
189.000
190.000
191.000
192.000 C
193.000 C
194.000 C
195.000 C
196.000
197.000
198.000
199.000 C
200.000 C

      CALL DVEC(U,INDEX,NDF,NDMAX)

      WRITE S ON UNIT-9 FOR LATER USE IN NODAL FORCE

      DO 63 IA=1,NBAND
      WRITE(9)(S(IA,IC),IC=1,NDF)
63 CONTINUE

      SOLVE FOR DISPLACEMENTS USING GAUSSIAN ELIMINATION
      AND PAYNE AND IRON'S METHOD

      CALL SOLVE(S,F,U,INDEX,NDF,NBAND,LBAND,NDMAX)

      CALCULATE AVERAGE DISPLACEMENTS

      IF(NELP.EQ.0)GO TO 64
      CALL ADISP(NELP,U,R,NNMAX,NDMAX,NRD,NPL,NPMAX)

      READ S-MATRIX FROM UNIT-9

64 REWIND 9
      DO 65 IA=1,NBAND
      READ(9)(S(IA,IC),IC=1,NDF)
65 CONTINUE

      CALCULATE NODAL FORCES F = K * U

      CALL NDFORC(S,U,F,NPL,NRD,NON,NDF,NBAND,LBAND
      +,NDMAX,NNMAX,NPMAX)
      IF(NELP.EQ.0)GO TO 70

      CALCULATE BENDING MOMENTS MR,MT AND SHEAR FORCE QR
      FOR PLATE ELEMENTS

      CALL BMSF(R,U,YMOD,PR,THICK,NELP,NNMAX,NDMAX,NPMAX
      +,RMOM,TMOM,NRD,NPL)
      IF(JOPT.EQ.-1)GO TO 75

      READ OPTION : NREQ

```

```

201.000 C      NREQ = 2 WHEN STRESS COMPONENTS, STRESS INVARIANTS
202.000 C      AND PRINCIPAL STRESSES ARE REQUIRED
203.000 C      NREQ = POSITIVE BUT NOT 2, WHEN ONLY STRESS
204.000 C      COMPONENTS ARE REQUIRED
205.000 C      NREQ = NEGATIVE, WHEN STRESSES ARE NOT REQUIRED
206.000 C
207.000 C      70 READ(IRR,301)NREQ
208.000 C      IF(NREQ.LE.0)GO TO 75
209.000 C
210.000 C      CALCULATE STRESSES SR,ST,SZ,SRZ , IF REQUIRED
211.000 C
212.000 C      CALL EST2(NCON,R,Z,U,BB,F,NTE,INDEX,NEMAX,NNMAX,
213.000 C      +NDMAX,NPL,NRD)
214.000 C      IF(NREQ.NE.2)GO TO 75
215.000 C
216.000 C      CALCULATE STRESS INVARIANTS I1 AND I2, IF REQD.
217.000 C
218.000 C      CALL SINVAR(NCON,R,Z,NTE,NNMAX,NEMAX)
219.000 C
220.000 C      CALCULATE PRINCIPAL STRESSES, IF REQUIRED
221.000 C
222.000 C      CALL PSTRES(NTE,F,NDMAX)
223.000 C      75 READ(IRR,301)NEXT
224.000 C
225.000 C      READ OPTION : NEXT
226.000 C      NEXT = 1, WHEN SAME PROBLEM IS TO BE SOLVED FOR
227.000 C      A DIFFERENT LOADING
228.000 C      NEXT = 2, WHEN COMPLETELY NEW PROBLEM IS TO BE
229.000 C      SOLVED
230.000 C      NEXT = OTHER NUMBERS, NO OTHER PROBLEM TO BE SOLVED
231.000 C
232.000 C      IF(NEXT.EQ.1)GO TO 62
233.000 C      IF(NEXT.EQ.2)GO TO 4
234.000 C      STOP
235.000 C      300 FORMAT(////' *** APPLIED LOADS ***')
236.000 C      301 FORMAT(I5)
237.000 C      END
238.000 C
239.000 C      * * * * *
240.000 C      * * * * *

```

```

241.000 SUBROUTINE TITLE
242.000 C
243.000 C READS AND PRINTS CASE TITLE
244.000 C
245.000 DIMENSION NAME(20)
246.000 IRR=2
247.000 IRW=3
248.000 READ(IRR,500)NAME
249.000 WRITE(IRW,502)
250.000 WRITE(IRW,504)NAME
251.000 RETURN
252.000 500 FORMAT(20A4)
253.000 502 FORMAT(1H1,///)
254.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
255.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
256.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
257.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
258.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
259.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
260.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
261.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
262.000 +3X,'* * * * * * * * * * * * * * * * * * * * * *',/
263.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
264.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
265.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
266.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
267.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
268.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
269.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
270.000 +2X,'* * * * * * * * * * * * * * * * * * * * * *',/
271.000 +10X,'* * * * * * * * * * * * * * * * * * * * * *',/
272.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
273.000 504 FORMAT(//)
274.000 +5X,'CASE TITLE ....',2X,20A4,///
275.000 +5X,'* * * * * * * * * * * * * * * * * * * * * *',/
276.000 END
277.000 C
278.000 C * * * * * * * * * * * * * * * * * * * * * *
279.000 C
280.000 SUBROUTINE CORD(NDF,NF,R,Z,NNON,NNMAX)

```

```

281.000 C
282.000 C
283.000 C
284.000 C
285.000 C
286.000 C
287.000
288.000
289.000
290.000
291.000
292.000
293.000
294.000
295.000
296.000
297.000
298.000
299.000
300.000
301.000
302.000
303.000
304.000
305.000
306.000
307.000
308.000 C
309.000 C
310.000 C
311.000
312.000
313.000 C
314.000 C
315.000 C
316.000 C
317.000 C
318.000 C
319.000 C
320.000 C

PROGRAM TO READ CO-ORDINATES OF NODES AND
CALCULATE TOTAL NUMBER OF DEGREES OF FREEDOM
NDF= TOTAL NUMBER OF DEGREES OF FREEDOM
NF= DEGREES OF FREEDOM AT EACH NODE

DIMENSION R(NNMAX),Z(NNMAX)
IRR=2
IRW=3
NON=0
NDF=0
10 READ(IRR,106)NN,RR,ZZ
IF(NN.LE.0)GO TO 12
NON=NON+1
R(NN)=RR
Z(NN)=ZZ
NDF=NDF+NF
GO TO 10
12 WRITE(IRW,100)
WRITE(IRW,104)(I,R(I),Z(I),I=1,NN)
WRITE(IRW,102)NON
RETURN
100 FORMAT(// ' NODE NO',2X,'R-COORD',3X,'Z CO-ORD')
102 FORMAT(// ' TOTAL NO OF NODES =',I3)
104 FORMAT(I5,2E12.4)
106 FORMAT(I5,2F10.4)
END

* * * * *
SUBROUTINE ELMP(NELP,YMOD,PR,THICK,NPL,NRD,NBANDP,
+NPMAX,NNMAX)
READS AND WRITES (ON UNIT 1) ELEMENT PROPERTIES
(E.G., CONNECTING NODES, THICKNESS, YOUNG'S
MODULOUS AND POISSON'S RATIO) FOR PLATE
ELEMENTS
NOTE: ANY DEFAULT VALUE OF 'THICK' IS
REPLACED BY ALREADY SUPPLIED NON ZERO VALUE

```

```

321.000
322.000
323.000
324.000
325.000
326.000
327.000
328.000
329.000
330.000
331.000
332.000
333.000
334.000
335.000
336.000
337.000
338.000
339.000
340.000
341.000
342.000
343.000
344.000
345.000
346.000
347.000
348.000
349.000
350.000
351.000
352.000
353.000
354.000
355.000
356.000
357.000
358.000
359.000
360.000

* DIMENSION THICK(NPMAX),NPL(NPMAX),NRD(NNMAX)
  IRR=2
  IRW=3
  NELP=0
  NBANDP=0
  READ(IRR,101)YMOD,PR
  IF(YMOD.LE.0.)GO TO 15
  WRITE(IRW,103)YMOD,PR
  READ(IRR,102)NEL,N1,N2,THICK(NEL)
  NELP=NELP+1
  NPL(NEL)=N1
  NBANDP=N2-N1
  NRD(N1)=N1*2-1+(NEL-1)
  WRITE(IRW,104)
  WRITE(IRW,106)NEL,N1,N2,THICK(NEL)
  XT3=THICK(NEL)
10 READ(IRR,102)NEL,N1,N2,THIC
  IF(NEL.LE.0)GO TO 14
  NELP=NELP+1
  NDE=N2
  THICK(NEL)=THIC
  NPL(NEL)=N1
  IF(N2-N1).GT.NBANDP)NBANDP=N2-N1
  NRD(N1)=N1*2-1+(NEL-1)
  IF(THICK(NEL).GT.0.)XT3=THICK(NEL)
  IF(THICK(NEL).EQ.0.)THICK(NEL)=XT3
  WRITE(IRW,106)NEL,N1,N2,THICK(NEL)
  GO TO 10
14 NRD(NDE)=NDE*2-1+NELP
  NPL(NELP+1)=NDE
  NBANDP=(NBANDP+1)*3
  WRITE(IRW,105)NELP
15 RETURN
101 FORMAT(F10.1,F10.4)
102 FORMAT(3I5,F10.4)
103 FORMAT('ELEMENTS WITH ROTATION DOF',//
+ ' YOUNGS MODULUS =',E12.4,//
+ ' POISSONS RATIO =',E12.4)
104 FORMAT('ELEM.NO.',2X,'CONNECTING NODES',3X,'THICKNESS',//

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361.000      +15X, 'LT TO RT')
362.000      105 FORMAT(// ' TOTAL NO OF ELEMENTS HAVING' /
363.000      + ' ROTATION DEGREES OF FREEDOM = ', I3)
364.000      106 FORMAT(I5, 5X, I5, 6X, I5, E12.4)
365.000      END
366.000 C
367.000 C
368.000 C
369.000      * * * * *
370.000      SUBROUTINE STIFFP(R, IM, NN1, NN2, THICK, YOM, PR, ESP
371.000      +, NMAX, NPMAX)
372.000 C
373.000 C
374.000 C
375.000      CALCULATES PLATE STIFFNESS MATRIX USING STRAIN ENERGY
376.000      BENDING PART OF STIFFNESS
377.000 C
378.000      DIMENSION ESP(4, 4), XKD(4, 4), AA(4, 4), R(NMAX), THICK(NPMAX)
379.000
380.000      R1=R(NN1)
381.000      IF(R1.EQ.0..AND.IM.EQ.1) R1=.0001
382.000      R2=R(NN2)
383.000      DP=YOM*(THICK(IM)**3)/(12.*(1.-PR*PR))
384.000      DP=DP*2.*3.14159264
385.000      PR1=1.+PR
386.000      DO 10 I=1, 4
387.000      DO 10 J=1, 4
388.000      XKD(I, J)=0.
389.000
390.000      10 CONTINUE
391.000      RLOG=ALOG(R2/R1)
392.000      XKD(2, 2)=RLOG
393.000      XKD(2, 3)=2.*PR1*(R2-R1)
394.000      XKD(2, 4)=1.5*(1.+2.*PR)*(R2*R2-R1*R1)
395.000      XKD(3, 3)=4.*PR1*(R2*R2-R1*R1)
396.000      XKD(3, 4)=6.*PR1*(R2**3-R1**3)
397.000      XKD(4, 4)=2.25*(5.+4.*PR)*(R2**4-R1**4)
398.000      XKD(3, 2)=XKD(2, 3)
399.000      XKD(4, 2)=XKD(2, 4)
400.000      XKD(4, 3)=XKD(3, 4)
      CALL AMAT(AA, R, NN1, NN2, NMAX)
      CALL MATIN(AA, 4)
      CALL ESTIFF(4, AA, XKD, ESP)
      DO 20 I=1, 4

```

```

401.000      DO 20 J=1,4
402.000      ESP(I,J)=DP*ESP(I,J)
403.000      20 CONTINUE
404.000      RETURN
405.000      END
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437.000 C
438.000 C
439.000
440.000 C

      *****

SUBROUTINE AMAT(AA,R,NN1,NN2,NNMAX)

CALCULATES A-MATRIX FOR BENDING PART OF PLATE STIFFNESS
MATRIX

DIMENSION AA(4,4),R(NNMAX)
R1=R(NN1)
R2=R(NN2)
AA(1,1)=1.0
AA(1,2)=R1
AA(1,3)=R1*R1
AA(1,4)=R1**3
AA(2,1)=0.
AA(2,2)=1.0
AA(2,3)=2.*R1
AA(2,4)=3.*R1*R1
AA(3,1)=1.0
AA(3,2)=R2
AA(3,3)=R2*R2
AA(3,4)=R2**3
AA(4,1)=0.
AA(4,2)=1.0
AA(4,3)=2.*R2
AA(4,4)=3.*R2*R2
RETURN
END

      *****

SUBROUTINE MATIN(A,N)

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THIS PROGRAM INVERTS AN N*N MATRIX
DIMENSION A(N,N),B(21,21)
FN=N
SUM=0.0
DO 4 I=1,N
DO 2 J=1,N
B(I,J)=0.0
2 CONTINUE
SUM=SUM+ABS(A(I,I))
B(I,I)=1.
4 CONTINUE
AVA=SUM/FN
RER=0.000001*AVA
DO 40 I=1,N
IF(ABS(A(I,I))-RER)6,6,20
6 IF(I-N)10,7,10
7 WRITE(3,100)
CALL EXIT
10 DO 12 II=I,N
IF(ABS(A(II,I))-RER)12,12,14
12 CONTINUE
GO TO 7
14 DO 16 JJ=1,N
A(I,JJ)=A(I,JJ)+A(II,JJ)
16 B(I,JJ)=B(I,JJ)+B(II,JJ)
20 R=1./A(I,I)
DO 22 J=1,N
A(I,J)=R*A(I,J)
B(I,J)=R*B(I,J)
22 CONTINUE
DO 37 II=1,N
IF(II-I)32,37,32
32 C=-A(II,I)
DO 35 J=1,N
A(II,J)=A(II,J)+C*A(I,J)
B(II,J)=B(II,J)+C*B(I,J)
35 CONTINUE
37 CONTINUE
40 CONTINUE

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520.000 C

DO 42 I=1,N
DO 42 J=1,N
A(I,J)=B(I,J)
42 CONTINUE
RETURN
100 FORMAT(// ' ***** MATRIX IS SINGULAR ***** ' )
END

* * * * *

SUBROUTINE ESTIFF(N,AIN,BTDB,ES)

THIS SUB-ROUTINE CALCULATES ELEMENT STIFFNESS MATRIX , ES
FROM A-INVERSE, BTDB (INTEGRATED) MATRICES

DIMENSION AIN(N,N),BTDB(N,N),ES(N,N)
DO 20 I=1,N
DO 20 J=1,N
SUM=0.0
DO 10 JJ=1,N
SUM=SUM+BTDB(I,JJ)*AIN(JJ,J)
10 CONTINUE
ES(I,J)=SUM
20 CONTINUE

DO 40 I1=1,N
DO 40 J1=1,N
SUM=0.0
DO 30 JJ=1,N
SUM=SUM+AIN(JJ,I1)*ES(JJ,J1)
30 CONTINUE
BTDB(I1,J1)=SUM
40 CONTINUE

DO 50 I2=1,N
DO 50 J2=1,N
ES(I2,J2)=BTDB(I2,J2)
50 CONTINUE
RETURN
END

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* * * * *
SUBROUTINE FILLSP(SP,ESP,NNL,NNR,NX,NRD,LBAND,NDMAX
+,NNMAX)

FILLS IN S-MATRIX FOR ANNULAR PLATE ELEMENTS ONLY
GLOBAL DOF ARE ADJUSTED IF SPECIAL ELEMENT WERE
USED FOR CENTRAL PLATE ELEMENT

DIMENSION SP(LBAND,NDMAX),ESP(4,4),NX(NDMAX),NRD(NNMAX),
NX(1)=NRD(NNL)+1
NX(2)=NX(1)+1
NX(3)=NRD(NNR)+1
NX(4)=NX(3)+1
DO 30 II=1,4
NXI=NX(II)
DO 30 JJ=II,4
NSUB=NX(JJ)-NX(II)+1
SP(NSUB,NXI)=SP(NSUB,NXI)+ESP(JJ,II)
30 CONTINUE
RETURN
END

* * * * *
SUBROUTINE CENTRE(R,ESP,YOM,PR,THICK,NN1,NN2,NPMAX,NNMAX)

CALCULATES SPECIAL ELEMENT STIFFNESS MATRIX FOR
CENTRAL PLATE ELEMENT. DOF ARE ONLY 3 E.G., VERTICAL
DISPLACEMENTS AND ROTATION AT OUTER END

DIMENSION ESP(4,4),THICK(NPMAX),R(NNMAX)
DO 10 I=1,4
DO 10 J=1,4
ESP(I,J)=0.
10 CONTINUE
R2=R(NN2)
DP=YOM*(THICK(1)**3)/(12.*(1.-PR*PR))
CONST=6.283185308*DP/(R2*R2)
ESP(1,1)=9.*CONST

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LEAF 243 OMITTED IN PAGE NUMBERING

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ESP(1,2)=-ESP(1,1)
ESP(1,3)=4.5*R2*CONST
ESP(2,2)=ESP(1,1)
ESP(2,3)=-ESP(1,3)
ESP(3,3)=CONST*(3.25+PR)*R2*R2
ESP(2,1)=ESP(1,2)
ESP(3,1)=ESP(1,3)
ESP(3,2)=ESP(2,3)
RETURN
END

* * * * *
SUBROUTINE FILCEN(S,ESP,NRD,NNL,NNR,LBAND,NDBAND,NMAX,NMAX
+,NPMAX)

* * * * *
FILLS IN S-MATRIX BY ELEMENTS OF CENTRAL PLATE
ELEMENT STIFFNESS MATRIX

DIMENSION S(LBAND,NDBAND),ESP(4,4),NRD(NMAX),NXX(3)
NXX(1)=NRD(NNL)+1
NXX(2)=NRD(NNR)+1
NXX(3)=NXX(2)+1
DO 15 II=1,3
  NXI=NXX(II)
  DO 15 JJ=II,3
    NSUB=NXX(JJ)-NXX(II)+1
    S(NSUB,NXI)=S(NSUB,NXI)+ESP(II,JJ)
  15 CONTINUE
  NXI=NXX(1)+1
  S(1,NXI)=10.**5
END

* * * * *
SUBROUTINE ELMT(NTE,NCON,YM,XNU,NPL,NRD,NBAND,NBANDP,
+NELP,LBAND,NEMAX,NMAX,NPMAX,JOPT,Z)

PROGRAM TO READ ELEMENT PROPERTIES AND INCIDENCES
I.E CORNER NUMBERS; YOUNGS MODULUS, POISSON'S RATIO

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```

601.000 C      YM = YOUNGS MODULUS OF ELEMENT
602.000 C      XNU= POISSON'S RATIO
603.000 C      NOTE: ANY DEFAULT VALUE OF YM OR XNU IS REPLACED
604.000 C      BY ALREADY SUPPLIED IMMEDIATE NON ZERO VALUE
605.000 C
606.000 C      DIMENSION NCON(3,NEMAX),YM(NEMAX),XNU(NEMAX),NRD(NNMAX)
607.000 C      +,NPL(NPMAX),Z(NNMAX)
608.000 C      IRR=2
609.000 C      IRW=3
610.000 C      NBAND=0
611.000 C      IF(NELP.EQ.0)NPL(1)=NNMAX
612.000 C      IF(JOPT.EQ.1)GO TO 8
613.000 C      READ(IRR,101)NEL,NC1,NC2,NC3,YMM,XNUS
614.000 C      GO TO 10
615.000 C      8 READ(IRR,99)ES0,SLOPE
616.000 C      WRITE(IRR,109)ES0,SLOPE
617.000 C      READ(IRR,101)NEL,NC1,NC2,NC3,XNUS
618.000 C      YMM=ES0+SLOPE*(Z(NC1)+Z(NC2)+Z(NC3))/3.
619.000 C      10 NTE=0
620.000 C      XT1=YMM
621.000 C      XT2=XNUS
622.000 C      12 NTE=NTE+1
623.000 C      NCON(1,NEL)=NC1
624.000 C      NCON(2,NEL)=NC2
625.000 C      NCON(3,NEL)=NC3
626.000 C      DO 20 I=1,3
627.000 C      NC4=NCON(I,NEL)
628.000 C      NAD=1
629.000 C      15 IF(NC4.LE.NPL(NAD))GO TO 17
630.000 C      NAD=NAD+1
631.000 C      IF(NAD.LE.(NELP+1))GO TO 15
632.000 C      NRD(NC4)=2*NC4+NELP
633.000 C      GO TO 20
634.000 C      17 NRD(NC4)=2*NC4-1+(NAD-1)
635.000 C      20 CONTINUE
636.000 C      YM(NEL)=YMM
637.000 C      XNU(NEL)=XNUS
638.000 C      IF(YM(NEL).GT.0.)XT1=YM(NEL)
639.000 C      IF(XNU(NEL).GT.0.)XT2=XNU(NEL)
640.000 C      IF(YM(NEL).EQ.0.)YM(NEL)=XT1

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641.000 IF(XNU(NEL).EQ.0.)XNU(NEL)=XT2
642.000 NMAX=MAX0(NRD(NC1),NRD(NC2),NRD(NC3))
643.000 NMIN=MIN0(NRD(NC1),NRD(NC2),NRD(NC3))
644.000 NB=NMAX-NMIN
645.000 IF(NB.GT.NBAND)NBAND=NB
646.000 IF(JOPT.EQ.1)GO TO 21
647.000 READ(IRR,101)NEL,NC1,NC2,NC3,YMM,XNUS
648.000 IF(NEL.GT.0)GO TO 12
649.000 GO TO 22
650.000 21 READ(IRR,101)NEL,NC1,NC2,NC3,XNUS
651.000 IF(NEL.LE.0)GO TO 22
652.000 YMM=ES0+SLOPE*(Z(NC1)+Z(NC2)+Z(NC3))/3.
653.000 GO TO 12
654.000 22 NBAND=NBAND+2
655.000 IF(JOPT.EQ.1)GO TO 24
656.000 WRITE(IRW,102)
657.000 24 WRITE(IRW,104)(I,(NCON(J,I),J=1,3),YM(I),XNU(I),I=1,NTE)
658.000 WRITE(IRW,105)NTE
659.000 IF(NBANDP.GT.NBAND)NBAND=NBANDP
660.000 IF(LBAND-NBAND)25,30,30
661.000 25 WRITE(IRW,106)LBAND,NBAND
662.000 CALL EXIT
663.000 30 WRITE(IRW,108)LBAND,NBAND
664.000 35 RETURN
665.000 99 FORMAT(F10.1,F10.4)
666.000 100 FORMAT(I5)
667.000 101 FORMAT(4I5,F10.1,F10.4)
668.000 102 FORMAT(// ' ELEMENTS WITHOUT ROTATION DOF',/,
669.000 + ' ELEM.NO.',2X,'CONNECTING NODE NUMBERS',/,
670.000 +3X,'YON.MOD.',4X,'P-RATIO')
671.000 104 FORMAT(2X,I3,3I8,4X,2E12.4)
672.000 105 FORMAT(// ' TOTAL NO. OF ELEMENTS WITHOUT ROTATION DOF =',I3)
673.000 106 FORMAT(//
674.000 + ' BAND WIDTH TOO LARGE'//
675.000 + ' MAXIMUM BAND WIDTH ALLOWED =',I3,/,
676.000 + ' CALCULATED BAND WIDTH',6X,'=',I3)
677.000 108 FORMAT(//
678.000 + ' MAXIMUM BAND WIDTH ALLOWED =',I3,/,
679.000 + ' CALCULATED BAND WIDTH',6X,'=',I3)
680.000 109 FORMAT(// ' ELEMENTS WITHOUT RDOF',/,/

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681.000 + ' YOUNG MODULUS AT SURFACE =' ,E12.4,/  

682.000 + ' SLOPE (RATE OF INC. OF ES)=' ,E12.4//  

683.000 + ' ELEM.NO.' ,2X,'CONNECTING NODE NUMBERS' ,  

684.000 +3X,'YON.MOD.' ,4X,'P-RATIO')  

685.000 END  

686.000 C  

687.000 C  

688.000 C  

689.000 SUBROUTINE AB(NEL,NCON,R,Z,AREA,ABAR,NEMAX,NNMAX)  

690.000 C  

691.000 C  

692.000 G  

693.000 C  

694.000 DIMENSION NCON(3,NEMAX),ABAR(3,3),R(NNMAX),Z(NNMAX)  

695.000 ABAR(1,1)=1.  

696.000 N1=NCON(1,NEL)  

697.000 N2=NCON(2,NEL)  

698.000 N3=NCON(3,NEL)  

699.000 ABAR(1,2)=R(N1)  

700.000 ABAR(1,3)=Z(N1)  

701.000 ABAR(2,1)=1.  

702.000 ABAR(2,2)=R(N2)  

703.000 ABAR(2,3)=Z(N2)  

704.000 ABAR(3,1)=1.  

705.000 ABAR(3,2)=R(N3)  

706.000 ABAR(3,3)=Z(N3)  

707.000 AREA=((ABAR(2,2)-ABAR(1,2))*(ABAR(3,3)-ABAR(1,3))-  

708.000 +(ABAR(2,3)-ABAR(1,3))*(ABAR(3,2)-ABAR(1,2)))/2.  

709.000 IF (AREA)15,25,30  

710.000 15 AREA=-AREA  

711.000 GO TO 30  

712.000 25 WRITE(3,122)NEL  

713.000 CALL EXIT  

714.000 30 RETURN  

715.000 122 FORMAT(//'WARNING: AREA OF ELEMENT',I3,' =0.00')  

716.000 END  

717.000 C  

718.000 C  

719.000 C  

720.000 SUBROUTINE DMAT(NEL,D,YM,XNU,NEMAX)

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THIS SUBROUTINE GENERATES D-MATRIX (ELASTICITY-
COEFFICIENT MATRIX)

DIMENSION D(4,4), YM(NEMAX), XNU(NEMAX)
PYM=YM(NEL)
PXNU=XNU(NEL)
TERM=PYM*(1.-PXNU)/((1.+PXNU)*(1.-2.*PXNU))
D(1,1)=TERM
D(1,2)=PXNU*TERM/(1.-PXNU)
D(1,3)=D(1,2)
D(1,4)=0.0
D(2,2)=D(1,1)
D(2,3)=D(1,2)
D(2,4)=0.0
D(3,3)=D(1,1)
D(3,4)=0.0
D(4,4)=0.5*(1.-2.*PXNU)*TERM/(1.-PXNU)
D(2,1)=D(1,2)
D(3,1)=D(1,3)
D(3,2)=D(2,3)
D(4,1)=0.0
D(4,2)=0.0
D(4,3)=0.0
RETURN
END

* * * * *
SUBROUTINE AINVRS(ABIN,AIN)
THIS PROGRAM GENERATES A-INVERSE FROM A-BAR-INVERSE

DIMENSION ABIN(3,3), AIN(6,6)
DO 10 I=1,3
DO 10 J=1,3
AIN(I,J)=ABIN(I,J)
AIN(I+3,J)=0.0
AIN(I,J+3)=0.0
AIN(I+3,J+3)=ABIN(I,J)

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10 CONTINUE
RETURN
END

* * * * *
SUBROUTINE EST1(NEL,NCON,R,Z,DD,BB,EE,AIN,NEMAX,NNMAX)
FINDS THE PRODUCT D*B*A-INVERSE, PRODUCT B*A-INVERSE
IS STORED IN 'EE' ON RETURN 'BB' ASSUMES THE VALUE
OF THE REQD. PRODUCT

DIMENSION NCON(3,NEMAX),R(NNMAX),Z(NNMAX),DD(4,4),BB(4,6)
+,AIN(6,6),EE(6,6)
N1=NCON(1,NEL)
N2=NCON(2,NEL)
N3=NCON(3,NEL)
ARR=(R(N1)+R(N2)+R(N3))/3.
ARZ=(Z(N1)+Z(N2)+Z(N3))/3.
DO 14 IK=1,4
DO 14 IKK=1,6
BB(IK,IKK)=0.0
14 CONTINUE
BB(1,2)=1.0
BB(2,1)=1.0/ARR
BB(2,2)=1.0
BB(2,3)=ARZ/ARR
BB(3,6)=1.0
BB(4,3)=1.0
BB(4,5)=1.0
DO 24 IA=1,4
DO 24 IB=1,6
SUM=0.0
DO 18 IC=1,4
SUM=SUM+DD(IA,IC)*BB(IC,IB)
18 CONTINUE
EE(IA,IB)=SUM
24 CONTINUE
DO 35 JA=1,4
DO 35 JB=1,6

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801.000 SUM=0.0
802.000 DO 32 JC=1,6
803.000 SUM=SUM+EE(JA,JC)*AIN(JC,JB)
804.000 32 CONTINUE
805.000 BB(JA,JB)=SUM
806.000 35 CONTINUE
807.000 RETURN
808.000 END
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814.000 C
815.000 C
816.000 C
817.000 C
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      SUM=0.0
      DO 32 JC=1,6
      SUM=SUM+EE(JA,JC)*AIN(JC,JB)
32 CONTINUE
      BB(JA,JB)=SUM
35 CONTINUE
      RETURN
      END

      * * * * *
      SUBROUTINE INTBDB(NEL,D,NCON,BTDB,AREA,R,Z,NEMAX,NNMAX)

      THIS SUB-ROUTINE GENERATES AND INTEGRATES B-T*D*B, 6*6
      MATRIX USING GAUSSIAN QUADRATURE FORMULA
      NOTE.. NUMBER OF SAMPLING POINTS = 7

      DIMENSION D(4,4),NCON(3,NEMAX),BTDB(6,6),R(NNMAX),
      +Z(NNMAX),XL(3)
      DO 10 I=1,6
      DO 10 J=1,6
      BTDB(I,J)=0.0
10 CONTINUE
      N1=NCON(1,NEL)
      N2=NCON(2,NEL)
      N3=NCON(3,NEL)
      R1=R(N1)
      R2=R(N2)
      R3=R(N3)
      Z1=Z(N1)
      Z2=Z(N2)
      Z3=Z(N3)
      TPI=6.283185308
      D11=D(1,1)
      D12=D(1,2)
      D22=D(2,2)
      D13=D(1,3)
      D23=D(2,3)
      D33=D(3,3)
      D44=D(4,4)

```

```

841.000 BTDB(1,2)=D12+D22
842.000 BTDB(1,6)=D23
843.000 W1=0.375
844.000 W2=0.1041666666666667
845.000 XLC=0.3333333333333333
846.000 XL(1)=0.736712498968435
847.000 XL(2)=0.237932366472434
848.000 XL(3)=0.025355134559132
849.000 ARR=(R1+R2+R3)*XLC
850.000 ARZ=(Z1+Z2+Z3)*XLC
851.000 BTDB(1,1)=BTDB(1,1)+W1*D22/ARR
852.000 BTDB(1,3)=BTDB(1,3)+W1*D22*ARZ/ARR
853.000 BTDB(2,2)=BTDB(2,2)+W1*(D11+2.*D12+D22)*ARR
854.000 BTDB(2,3)=BTDB(2,3)+W1*(D12+D22)*ARZ
855.000 BTDB(2,6)=BTDB(2,6)+W1*(D13+D23)*ARR
856.000 BTDB(3,3)=BTDB(3,3)+W1*(D22*ARZ*ARZ/ARR)+W1*D44*ARR
857.000 BTDB(3,5)=BTDB(3,5)+W1*D44*ARR
858.000 BTDB(3,6)=BTDB(3,6)+W1*D23*ARZ
859.000 BTDB(5,5)=BTDB(5,5)+W1*D44*ARR
860.000 BTDB(6,6)=BTDB(6,6)+W1*D33*ARR
861.000 DO 35 J=1,3
862.000 DO 30 JJ=1,2
863.000 XL1=XL(1)
864.000 XL2=XL(2)
865.000 XL3=XL(3)
866.000 IF (JJ.EQ.1) GO TO 22
867.000 XL2=XL(3)
868.000 XL3=XL(2)
869.000 22 ARR=R1*XL1+R2*XL2+R3*XL3
870.000 ARZ=Z1*XL1+Z2*XL2+Z3*XL3
871.000 BTDB(1,1)=BTDB(1,1)+W2*D22/ARR
872.000 BTDB(1,3)=BTDB(1,3)+W2*D22*ARZ/ARR
873.000 BTDB(2,2)=BTDB(2,2)+W2*(D11+2.*D12+D22)*ARR
874.000 BTDB(2,3)=BTDB(2,3)+W2*(D12+D22)*ARZ
875.000 BTDB(2,6)=BTDB(2,6)+W2*(D13+D23)*ARR
876.000 BTDB(3,3)=BTDB(3,3)+W2*(D22*ARZ*ARZ/ARR)+W2*D44*ARR
877.000 BTDB(3,5)=BTDB(3,5)+W2*D44*ARR
878.000 BTDB(3,6)=BTDB(3,6)+W2*D23*ARZ
879.000 BTDB(5,5)=BTDB(5,5)+W2*D44*ARR
880.000 BTDB(6,6)=BTDB(6,6)+W2*D33*ARR

```

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920.000

30 CONTINUE
  XL(1)=XL3
  XL(2)=XL2
  XL(3)=XL1
35 CONTINUE
  DO 40 II=1,6
  DO 40 JJ=II,6
    BTDB(II,JJ)=TPI*AREA*BTDB(II,JJ)
    IF(II.NE.JJ) BTDB(JJ,II)=BTDB(II,JJ)
40 CONTINUE
  RETURN
  END

* * * * *
SUBROUTINE FILLS(NF,NN,NSIZE,ES,S,NCON,NEL,LBAND
+,NDMAX,NEMAX,NX,NRD,NNMAX)

  GENERATES S- MATRIX FROM ELMT. STIFFNESS MAT., ES
  NF= DEGREES OF FREEDOM PER NODE
  NN= NO. OF NODE PER ELEMENT
  NSIZE= SIZE OF THE ELEMENT STIFFNESS MATRIX, ES
  NEL= ELEMENT NUMBER
  NBAND= BAND WIDTH
  NCON= CORNER NODES MATRIX
  NDF= TOTAL DEGREES OF FREEDOM
  NTE= TOTAL NO OF ELEMENTS FOR THIS PROBLEM

  DIMENSION ES(NSIZE,NSIZE),S(LBAND,NDMAX),NCON(3,NEMAX)
  +,NX(NDMAX),NRD(NNMAX)
  NADR=0
  DO 5 I=1,NF
  DO 5 J=1,NN
    NADR=NADR+1
  IR=NCON(J,NEL)
  NX(NADR)=NRD(IR)+I-1
5 CONTINUE
  NL=NX(JJ)

```

```

921.000 N2=NX(I)
922.000 IF (N1-N2) 25,30,30
923.000 25 N1=NX(I)
924.000 N2=NX(I)
925.000 10 NSUB=N1-N2+1
926.000 S(NSUB,N2)=S(NSUB,N2)+S(JJ,II)
927.000 40 CONTINUE
928.000 RETURN
929.000 END
930.000
931.000 *****
932.000 *****
933.000 *****
934.000 *****
935.000 *****
936.000 *****
937.000 *****
938.000 *****
939.000 *****
940.000 *****
941.000 *****
941.500 *****
941.700 *****
942.000 *****
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960.000 *****

SUBROUTINE CLVECP(R,F,NRO,AMATP,NNMAX,NNMAX,NDF)
DETERMINES CONSISTENT LOAD VECTOR FOR UNIFORMLY
DISTRIBUTED LOAD ON ELEMENTS WITH ROOF
DIMENSION R(NNMAX),F(NNMAX),NRO(NNMAX),AMATP(4,4)
+ ,TERM(4),TEMP(4)
IRR=2
IRW=3
DO 8 IA=1,NDF
8 F(IA)=0
WRITE(IRW,102)
10 READ(IR,100)N1,N2,DLOAD
IF(N1.LE.0)GO TO 40
WRITE(IRW,104)N1,N2,DLOAD
R1=R(N1)
R2=R(N2)
CALL AMAT(AMATP,R,N1,N2,NNMAX)
CALL MATIN(AMATP,4)
CONST=5.281185307*DLOAD
TERM(1)=CONST*(R2-R1)*R1/2.
TERM(2)=CONST*(R2**3-R1**3)/3.
TERM(3)=CONST*(R2**4-R1**4)/4.
TERM(4)=CONST*(R2**5-R1**5)/5.
DO 30 IA=1,4
30 R(IA)=R(IA)+TERM(IA)
SUM=0.
DO 20 IB=1,4
20 IB=IB+1
SUM=SUM+TERM(IB)*AMATP(IB,IA)
20 CONTINUE
TEMP(IA)=SUM

```

```

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962.000
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975.000 C
976.000 C
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979.000 C
980.000 C
981.000 C
982.000 C
983.000 C
984.000 C
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996.000
997.000
998.000
999.000
1000.000

30 CONTINUE
NR1=NRD(N1)
NR2=NRD(N2)
F(NR1+1)=F(NR1+1)+TEMP(1)
F(NR2+1)=F(NR2+1)+TEMP(3)
F(NR1+2)=F(NR1+2)+TEMP(2)
F(NR2+2)=F(NR2+2)+TEMP(4)
GO TO 10
40 RETURN
100 FORMAT(2I5,F10.4)
102 FORMAT(//,' UDL ON ELEMENTS HAVING RDOF',/
+' BETWEEN NODES',5X,'INTENSITY')
104 FORMAT(3X,I3,4X,I3,3X,E12.4)
END
* * * * *
SUBROUTINE CSLQAD(R,Z,F,ABAR,NRD,NNMAX,NDMAX)
GENERATES CONSISTANT LOAD VECTOR FOR UNIFORMLY
DISTRIBUTED LOAD OF INTENSITY 'UDL'
CONNECTING NODES AND INTENSITY OF LOAD ARE INPUT
FOR THIS SUBPROGRAM
DIMENSION R(NNMAX),Z(NNMAX),F(NDMAX),ABAR(3,3)
+,NRD(NNMAX)
TPI=6.283185307
WRITE(3,104)
5 READ(2,100)N1,N2,N3,UDL
IF(N1.LE.0) GO TO 15
ABAR(1,1)=1.
ABAR(2,1)=1.
ABAR(3,1)=1.
ABAR(1,2)=R(N1)
ABAR(2,2)=R(N2)
ABAR(3,2)=R(N3)
ABAR(1,3)=Z(N1)
ABAR(2,3)=Z(N2)
ABAR(3,3)=Z(N3)
CALL MATIN(ABAR,3)

```

```

1001.000      TERM1=0.5*TPI*UDL*(R(N3)**2-R(N1)**2)
1002.000      TERM2=TPI*UDL*(R(N3)**3-R(N1)**3)/3.
1003.000      TERM3=0.5*TPI*UDL*Z(N1)*(R(N3)**2-R(N1)**2)
1004.000      NR1=NRD(N1)
1005.000      NR3=NRD(N3)
1006.000      F(NR1+1)=F(NR1+1)+ABAR(1,1)*TERM1+ABAR(2,1)
1007.000      +*TERM2+ABAR(3,1)*TERM3
1008.000      F(NR3+1)=F(NR3+1)+ABAR(1,3)*TERM1+ABAR(2,3)
1009.000      +*TERM2+ABAR(3,3)*TERM3
1010.000      WRITE(3,102)N1,N3,UDL
1011.000      GO TO 5
1012.000      15 RETURN
1013.000      100 FORMAT(3I5,F10.4)
1014.000      102 FORMAT(1X,2I5,4X,E12.4)
1015.000      104 FORMAT(//,' UDL ON ELEMENTS WITHOUT RDOF',/
1016.000      +' BETWEEN NODES',5X,'INTENSITY')
1017.000      END
1018.000      C
1019.000      C
1020.000      C
1021.000      C
1022.000      C
1023.000      C
1024.000      C
1025.000      C
1026.000      C
1027.000      C
1028.000      C
1029.000      C
1030.000      C
1031.000      C
1032.000      C
1033.000      C
1034.000      C
1035.000      C
1036.000      C
1037.000      C
1038.000      C
1039.000      C
1040.000      C

      ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** 
      SUBROUTINE FVEC(F,NDMAX)
      THIS SUBROUTINE READS NON ZERO NODAL FORCE COMPONENTS

      DIMENSION F(NDMAX)
      IRR=2
      IRW=3
      WRITE(IRW,100)
      4 READ(IRR,110)N,FORCE
      IF(N)10,10,5
      5 F(N)=F(N)+FORCE
      WRITE(IRW,120)N,FORCE
      GO TO 4
      10 RETURN
      100 FORMAT(//,' KNOWN NON-ZERO CONC. LOAD COMPONENTS' /
      +' COMPONENT NUMBER',3X,'LOAD ')
      110 FORMAT(5I5,F10.4)
      120 FORMAT(7X,13,7X,E12.4)
      END

```

```

1041.000 C
1042.000 C
1043.000 C
1044.000 C
1045.000 C
1046.000 C
1047.000 C
1048.000 C
1049.000 C
1050.000
1051.000
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*****
SUBROUTINE DVEC(U, INDEX, NDF, NDMAX)

THIS SUBROUTINE READS KNOWN DISPLACEMENTS AND FILLS IN
DISPLACEMENT VECTOR, U ALSO SUBSEQUENTLY FILLS INDEX VECTOR
INDEX= 1, IF DISPLACEMENT IS KNOWN
INDEX= 0, IF DISPLACEMENT IS NOT KNOWN

DIMENSION U(NDMAX), INDEX(NDMAX)
IRR=2
IRW=3
DO 10 I=1,NDF
  INDEX(I)=0
  U(I)=0.0
10 CONTINUE
  WRITE(IRW,100)
12 READ(IRR,102)NNC,DISP
  IF(NNC)20,20,15
15 INDEX(NNC)=1
  U(NNC)=DISP
  WRITE(IRW,104)NNC,DISP
  GO TO 12
20 RETURN
100 FORMAT(//
  + ' KNOWN DISPLACEMENTS' /
  + ' COMPONENT NUMBER', 3X, 'DISPLACEMENT')
102 FORMAT(I5,F10.4)
104 FORMAT(7X,I3,7X,E12.4)
END

*****
SUBROUTINE SOLVE(S,F,U, INDEX, NDF, NBAND, LBAND, NDMAX)

SOLVES A SYSTEM OF SIMULTANEOUS LINEAR EQNS. BY
PAYNE AND IRON METHOD USING GAUSSIAN ELIMINATION
S- MATRIX IS USED IN ELIMINATION

DIMENSION S(LBAND, NDMAX), F(NDMAX), U(NDMAX), INDEX(NDMAX)

```



```

1081.000 CP=10.**30
1082.000 DO 10 JA=1,NDF
1083.000 IF (INDEX(JA).EQ.0) GO TO 10
1084.000 S(1,JA)=S(1,JA)*CP
1085.000 F(JA)=S(1,JA)*U(JA)
1086.000 10 CONTINUE
1087.000 IJ=1
1088.000 NA=NBAND
1089.000 NR=NDF-NBAND
1090.000 12 R=1./S(1,IJ)
1091.000 S(1,IJ)=1.0
1092.000 F(IJ)=F(IJ)*R
1093.000 IF (NA.EQ.1) GO TO 35
1094.000 DO 30 II=2,NA
1095.000 C=S(II,IJ)
1096.000 S(II,IJ)=C*R
1097.000 DO 20 JJ=2,II
1098.000 I=II-JJ+1
1099.000 J=JJ-1+IJ
1100.000 S(I,J)=S(I,J)-C*S(JJ,IJ)
1101.000 20 CONTINUE
1102.000 F(J)=F(J)-F(IJ)*C
1103.000 30 CONTINUE
1104.000 NR=NR-1
1105.000 IJ=IJ+1
1106.000 IF (NR.LT.0) NA=NA-1
1107.000 GO TO 12
1108.000 35 U(NDF)=F(NDF)
1109.000 C
1110.000 C
1111.000 C
1112.000 DO 50 IB=2,NDF
1113.000 NA=NDF-IB+1
1114.000 NR=IB-1
1115.000 IF (NR.GE.NBAND) NR=NBAND-1
1116.000 DO 40 IC=1,NR
1117.000 F(NA)=F(NA)-S(IC+1,NA)*U(NA+IC)
1118.000 40 CONTINUE
1119.000 U(NA)=F(NA)
1120.000 50 CONTINUE

```

BACK SUBSTITUTION

```

1121.000      RETURN
1122.000      END
1123.000      C
1124.000      C
1125.000      C
1126.000      C
1127.000      C
1128.000      C
1129.000      C
1130.000      C
1131.000      C
1132.000      C
1133.000      C
1134.000      C
1135.000      C
1136.000      C
1137.000      C
1138.000      C
1139.000      C
1140.000      C
1141.000      C
1142.000      C
1143.000      C
1144.000      C
1145.000      C
1146.000      C
1147.000      C
1148.000      C
1149.000      C
1150.000      C
1151.000      C
1152.000      C
1153.000      C
1154.000      C
1155.000      C
1156.000      C
1157.000      C
1158.000      C
1159.000      C
1160.000      C

      * * * * *
SUBROUTINE ADISP(NELP, DISP, R, NNMAX, NDMAX, NRD, NPL, NPMAX)
CALCULATES AND PRINTS AVERAGE DISPLACEMENT OF THE PLATE
DIMENSION R(NNMAX), DISP(NDMAX), NRD(NNMAX), NPL(NPMAX)
IRW=3
NNLP=NELP+1
NR=MOD(NNLP, 2)
NOP=NELP/2
VOL=0.0
DO 10 I=1, NOP
  ISN=NPL(2*I-1)
  IMN=NPL(2*I)
  IEN=NPL(2*I+1)
  ISVC=NRD(ISN)+1
  IMVC=NRD(IMN)+1
  IEVC=NRD(IEN)+1
  A=DISP(ISVC)
  B=(DISP(IMVC)-DISP(ISVC))/(R(IMN)-R(ISN))
  C=(DISP(IEVC)-DISP(ISVC)-B*(R(IEN)-R(ISN)))/
    +((R(IEN)-R(ISN))*(R(IEN)-R(IMN)))
  R3=R(IEN)
  R1=R(ISN)
  VOL=VOL+2.*(A*(R3*R3-R1*R1)/2.+B*(R3**3-R1**3)/3.-
    +B*(R3*R3-R1*R1)/2.+C*(R3**4-R1**4)/4.-C*(R1+R(IMN)
    +)*(R3**3-R1**3)/3.+C*R1*R(IMN)*(R3*R3-R1*R1)/2.)
10 CONTINUE
IF (NR.EQ.1) GO TO 15
IEN=NPL(NNLP)
ISN=NPL(NELP)
IEVC=NRD(NNLP)
ISVC=NRD(NELP)
R3=R(IEN)
R1=R(ISN)
A=DISP(ISVC+1)

```

```

1161.000      B=DISP(IEVC+1)
1162.000      VOL=VOL+2.*(A*(R3*R3-R1*R1)/2.-(A-
1163.000      +B)*((R3**3-R1**3)/3.-R1*(R3*R3-R1*R1)/2.)/
1164.000      +(R3-R1)
1165.000      15 IEN=NPL(NNLP)
1166.000      AD=VOL/R(IEN)**2
1167.000      WRITE(IRW,100)AD
1168.000      RETURN
1169.000      100 FORMAT(// ' AVERAGE DISPLACEMENT = ',E12.5/)
1170.000      END
1171.000      C
1172.000      C
1173.000      C
1174.000      C
1175.000      C
1176.000      C
1177.000      C
1178.000      C
1179.000      C
1180.000      C
1181.000      C
1182.000      C
1183.000      C
1184.000      C
1185.000      C
1186.000      C
1187.000      C
1188.000      C
1189.000      C
1190.000      C
1191.000      C
1192.000      C
1193.000      C
1194.000      C
1195.000      C
1196.000      C
1197.000      C
1198.000      C
1199.000      C
1200.000      C

      * * * * *
      SUBROUTINE NDFORC(S,U,F,NPL,NRD,NON,NDF,NBAND
      +,LBAND,NDMAX,NNMAX,NPMAX)

      THIS SUBROUTINE CALCULATES NODAL FORCES; F=S*U
      AND ALSO PRINTS DISPLACEMENTS AND FORCES

      DIMENSION S(LBAND,NDMAX),U(NDMAX),F(NDMAX)
      +,NPL(NPMAX),NRD(NNMAX)
      IRW=3
      NR=1
      DO 30 I=1,NDF
      F(I)=0.0
      IF(I-NBAND)10,10,8
      8 NR=I-NBAND+1
      10 DO 12 J=NR,I
      JNB=I-J+1
      F(I)=F(I)+S(JNB,J)*U(J)
      12 CONTINUE
      DO 20 K=2,NBAND
      KNB=I+K-1
      IF(KNB.GT.NDF)GO TO 20
      F(I)=F(I)+S(K,I)*U(KNB)
      20 CONTINUE
      30 CONTINUE
      NR=1
      WRITE(IRW,100)
      DO 40 IJ=1,NON

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1201.000 NRJ=NRD(IJ)
1202.000 IF(IJ.EQ.NPL(NK))GO TO 38
1203.000 WRITE(IRW,110)IJ,F(NRJ),F(NRJ+1),U(NRJ)
1204.000 +,U(NRJ+1)
1205.000 GO TO 40
1206.000 38 NK=NK+1
1207.000 WRITE(IRW,120)IJ,F(NRJ),F(NRJ+1),F(NRJ+2)
1208.000 +,U(NRJ),U(NRJ+1),U(NRJ+2)
1209.000 40 CONTINUE
1210.000 RETURN
1211.000 100 FORMAT(///
1212.000 + 'FORCE AND DISPLACEMENT COMPONENTS',/,/,/
1213.000 +
1214.000 +21X,'FORCES',28X,'DISPLACEMENTS',/,/
1215.000 + 'NODE NO',2X,'R COMP',6X,'Z COMP',6X,'T COMP',
1216.000 +6X,'R COMP',6X,'Z COMP',6X,'T COMP')
1217.000 110 FORMAT(15,2E12.4,12X,2E12.4)
1218.000 120 FORMAT(15,6E12.4)
1219.000 END
1220.000 C
1221.000 C
1222.000 C
1223.000 C
1224.000 C
1225.000 C
1226.000 C
1227.000 C
1228.000 C
1229.000 C
1230.000 C
1231.000 C
1232.000 C
1233.000 C
1234.000 C
1235.000 C
1236.000 C
1237.000 C
1238.000 C
1239.000 C
1240.000 C

* * * * *
SUBROUTINE BMSF(R,DISP,YMOD,PR,THICK,NELP,NNMAX,
+NDMAX,NPMAX,RMOM,TMOM,NRD,NPL)
CALCULATES AND PRINTS BENDING MOMENTS AND SHEAR
FORCES FOR PLATE ELEMENTS AT EACH NODE
DIMENSION R(NNMAX),DISP(NDMAX),THICK(NPMAX),RMOM(
+NPMAX),TMOM(NPMAX),NRD(NNMAX),NPL(NPMAX),TEM(4),ALP(4)
+,AMATP(4,4)
IRW=3
WRITE(IRW,100)
CONST1=YMOD/((1.-PR*PR)*12.)
DO 40 I=1,NELP
CONST=CONST1*THICK(I)**3
NL=NPL(I)
NR=NPL(I+1)
R1=R(NL)
R2=R(NR)

```

```

1241.000 CALL AMAT(AMATP,R,NL,NR,NNMAX)
1242.000 CALL MATIN(AMATP,4)
1243.000 NHL=NRD(NL)
1244.000 NHR=NRD(NR)
1245.000 TEM(1)=DISP(NHL+1)
1246.000 TEM(2)=DISP(NHL+2)
1247.000 TEM(3)=DISP(NHR+1)
1248.000 TEM(4)=DISP(NHR+2)
1249.000 DO 20 J=1,4
1250.000 SUM=0.
1251.000 DO 10 K=1,4
1252.000 SUM=SUM+AMATP(J,K)*TEM(K)
1253.000 10 CONTINUE
1254.000 ALP(J)=SUM
1255.000 20 CONTINUE
1256.000 DER2=2.*ALP(3)+6.*ALP(4)*R1
1257.000 DER3=6.*ALP(4)
1258.000 IF(R1.EQ.0.)GO TO 25
1259.000 DER1=ALP(2)+2.*ALP(3)*R1+3.*ALP(4)*R1*R1
1260.000 XMRL=-CONST*(DER2+PR*DER1/R1)
1261.000 XMTL=-CONST*(DER1/R1+PR*DER2)
1262.000 QRL=CONST*(DER3+DER2/R1-DER1/(R1*R1))
1263.000 GO TO 30
1264.000 25 XMRL=-CONST*(1.+PR)*DER2
1265.000 XMTL=XMRL
1266.000 QRL=1.5*DER3
1267.000 30 DER1=ALP(2)+2.*ALP(3)*R2+3.*ALP(4)*R2*R2
1268.000 DER2=2.*ALP(3)+6.*ALP(4)*R2
1269.000 XMRR=-CONST*(DER2+PR*DER1/R2)
1270.000 XMTR=-CONST*(DER1/R2+PR*DER2)
1271.000 QRR=CONST*(DER3+DER2/R2-DER1/(R2*R2))
1272.000 WRITE(IRW,104)I,XMRL,XMTL,QRL,XMRR,XMTR,QQR
1273.000 40 CONTINUE
1274.000 RETURN
1275.000 100 FORMAT(///
1276.000 + 'BENDING MOMENTS AND SHEAR FORCES',/
1277.000 + 'FOR PLATE ELEMENTS',/
1278.000 +
1279.000 + 'EL NO',3X,'MR LT',7X,'MT LT',7X,'QR LT',7X,'MR RT',
1280.000 +7X,'MT RT',7X,'QR RT')

```

```

1281.000
1282.000
1283.000 C
1284.000 C
1285.000 C
1286.000 C
1287.000
1288.000
1289.000 C
1290.000 C
1291.000 C
1292.000 C
1293.000 C
1294.000
1295.000
1296.000
1297.000
1298.000
1299.000
1300.000
1301.000
1302.000
1303.000
1304.000
1305.000
1306.000
1307.000
1308.000
1309.000
1310.000
1311.000
1312.000
1313.000
1314.000
1315.000
1316.000
1317.000
1318.000
1319.000
1320.000

104 FORMAT(I4,6E12.4)
END

* * * * *

SUBROUTINE EST2(NCON,R,Z,U,BB,F,NTE,IND,NEMAX,NNMAX,
+,NDMAX,NPL,NRD)

CALCULATES AND PRINTS STRESSES FOR EACH ELEMENT
SIGMA= (D*B*A-INVERSE)*U
(D*B*A-INVERSE) IS CALCULATED IN EST1

DIMENSION NCON(3,NEMAX),R(NNMAX),Z(NNMAX),U(NDMAX)
+,BB(4,6),F(NDMAX),IND(NDMAX),NPL(NPMAX),NRD(NNMAX)
REWIND 4
IRW=3
WRITE(IRW,112)
DO 30 IA=1,NTE
N1=NCON(1,IA)
N2=NCON(2,IA)
N3=NCON(3,IA)
RC=(R(N1)+R(N2)+R(N3))/3.
NADR=0
DO 12 IB=1,2
DO 12 IC=1,3
NADR=NADR+1
KP=NCON(IC,IA)
IND(NADR)=NRD(KP)+IB-1
12 CONTINUE
READ(4)((BB(II,JJ),JJ=1,6),II=1,4)
DO 18 IG=1,4
SUM=0.0
DO 16 IK=1,6
INT=IND(IK)
SUM=SUM+BB(IG,IK)*U(INT)
16 CONTINUE
F(IG)=SUM
18 CONTINUE
WRITE(7)(F(IM),IM=1,4)

```

```

1321.000 WRITE(IRW,114)IA,RC,(F(IM),IM=1,4)
1322.000 30 CONTINUE
1323.000 RETURN
1324.000 112 FORMAT(///' STRESS AT CENTROID OF ELEMENT',/
1325.000 +'-----',/
1326.000 +' ELMT NO',2X,'R COORD',4X,'R COMP',6X,'T COMP',6X,
1327.000 +'Z COMP',6X,'SHEAR',/
1328.000 +'11X,' AT CG')
1329.000 114 FORMAT(I5,2X,E10.3,4E12.4)
1330.000 END
1331.000 C
1332.000 C
1333.000 C
1334.000 C
1335.000 C
1336.000 C
1337.000 C
1338.000 C
1339.000 DIMENSION R(NNMAX),Z(NNMAX),NCON(3,NEMAX)
1340.000 IRW=3
1341.000 REWIND 7
1342.000 WRITE(IRW,200)
1343.000 DO 20 I=1,NTE
1344.000 N1=NCON(1,I)
1345.000 N2=NCON(2,I)
1346.000 N3=NCON(3,I)
1347.000 RC=(R(N1)+R(N2)+R(N3))/3.
1348.000 ZC=(Z(N1)+Z(N2)+Z(N3))/3.
1349.000 READ(7)SR,ST,SZ,SRZ
1350.000 XI1=SR+ST+SZ
1351.000 XI2=SR*ST+ST*SZ+SZ*SR-SRZ*SRZ
1352.000 WRITE(IRW,202)I,RC,ZC,XI1,XI2
1353.000 20 CONTINUE
1354.000 RETURN
1355.000 200 FORMAT(///
1356.000 +' STRESS INVARIANTS AT CG OF SOIL EL.',/
1357.000 +'-----',/
1358.000 +'2X,' EL NO',3X,' COORDINATES AT CG',10X,' INVARIANTS',/
1359.000 +'10X,'R-COORD',5X,'Z-COORD',7X,'I1',10X,'I2')
1360.000 202 FORMAT(1X,I5,4E12.4)
1361.000 END

```

```

1362.000 C
1363.000 C
1364.000 C
1365.000 C
1366.000 C
1367.000 C
1368.000 C
1369.000 C
1370.000 C
1371.000 C
1372.000
1373.000
1374.000
1375.000
1376.000
1377.000
1378.000
1379.000
1380.000
1381.000
1382.000
1383.000
1384.000
1385.000
1386.000
1387.000
1388.000
1389.000
1390.000
1391.000
1392.000
1393.000
1394.000
1395.000
1396.000
1397.000
1398.000
1399.000
1400.000
1401.000
1402.000

*****
SUBROUTINE PSTRES(NTE,STR,NDMAX)
CALCULATES AND PRINTS PRINCIPAL STRESSES,
ALSO CALCULATES MAXIMUM SHEAR STRESS AND THE PLANE
OF PRINCIPAL STRESSES

DIMENSION STR(NDMAX)
REWIND 7
IRW=3
WRITE(IRW,115)
XL3=90./3.141592654
DO 32 IS=1,NTE
  READ(7)(STR(IA),IA=1,4)
  PST1=STR(2)
  XL1=0.5*(STR(1)+STR(3))
  XL4=(0.5*(STR(1)-STR(3)))*2+STR(4)*STR(4)
  XL2=$QRT(XL4)
  PST2=XL1+XL2
  PST3=XL1-XL2
  SMAX=AMAX1(STR(1),STR(3))
  SMIN=AMIN1(STR(1),STR(3))
  THETA1=ATAN(-STR(4)/XL1)
  THETA1=THETA1*XL3
  THETA2=THETA1+90.
  SMAX=AMAX1(PST1,PST2,PST3)
  SMIN=AMIN1(PST1,PST2,PST3)
  SSMAX=0.5*ABS(SMAX-SMIN)
  WRITE(IRW,116)IS,PST1,PST2,PST3,SSMAX,THETA1,THETA2
32 CONTINUE
RETURN
115 FORMAT(' PRINCIPAL STRESSES AND MAX. SHEAR STRESS',/
+ ' EL NO',2X,'STR-1',7X,'STR-2',6X,'STR-3',6X,
+ 'MAX SHEAR',4X,'ANGLE',4X,'ANGLE',/
+ 43X,'STRESS',4X,'MAX S2-S3',2X,'MIN S2-S3')
116 FORMAT(1X,I3,4E12.4,2E10.3)
END

```


A.10.2 Program for the preparation of input data

```

=====

```

```

1.000 C
2.000 C
3.000 C
4.000 C
5.000 C
6.000 C
7.000 C
8.000 C
9.000 C
10.000 C
11.000 C
12.000 C
13.000 C
14.000 C
15.000 C
16.000 C
17.000 C
18.000 C
19.000 C
20.000 C
21.000 C
22.000 C
23.000 C
24.000 C
25.000 C
26.000 C
27.000 C
28.000 C
29.000 C
30.000 C
31.000 C
32.000 C
33.000 C
34.000 C
35.000 C
36.000 C
37.000 C
38.000 C
39.000 C
40.000 C

```

```

=====

```

```

PROGRAMMED BY: MD. MUSHARRAFUZZAMAN
                CIVIL ENGINEERING DEPARTMENT
                CARLETON UNIVERSITY, OTTAWA
                JUNE 1979

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SUPERVISORS:  PROF. A.P.S. SELVADURAI AND
                PROF. W.H. BOWES

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```

NOTE:  * * * TWO CONSECUTIVE DATA MUST BE SEPERATED
        BY ONE OR MORE BLANKS * * *

```

```

DIMENSION NAME(20)

```

```

COMMON/BLOCK1/IR,IW,IN,ITOT,JA(80),IA(48),IFI(6),AFL(6),
+IND(6),JBND(8,6),IBND(4,6),FBND(4,6)

```

```

IR=5

```

```

IW=6

```

```

IN=30

```

```

DO 4 I=1,4

```

```

DO 4 J=1,6

```

```

FBND(I,J)=0.

```

```

4 CONTINUE

```

```

DO 6 I=1,8

```

```

DO 6 J=1,6

```

```

JBND(I,J)=0

```

```

6 CONTINUE

```

```

ITOT=6

```

```

READ CASE TITLE * * * *

```

```

7 READ(IR,300)NAME

```

```

WRITE(IW,300)NAME

```

```

READ COORDINATES (R,Z) OF EACH NODE * * * *

```

```

41.000      8 CALL REED
42.000      NN=IFI(1)
43.000      RR=AFL(2)
44.000      ZZ=AFL(3)
45.000      WRITE(IW,302)NN,RR,ZZ
46.000      IF(NN.LE.0)GO TO 10
47.000      GO TO 8
48.000      READ YOUNG'S MODULUS AND POISSON'S RATIO
49.000      FOR THE PLATE * * * *
50.000      C
51.000      10 CALL REED
52.000      YMOD=AFL(1)
53.000      PR=AFL(2)
54.000      WRITE(IW,304)YMOD,PR
55.000      IF(YMOD.LE.0.)GO TO 20
56.000      READ ELEMENT NUMBER N1,CONNECTING NODE NOS.
57.000      N2,N3, AND THICKNESS$ FOR EACH PLATE EL.* *
58.000      C
59.000      12 CALL REED
60.000      N1=IFI(1)
61.000      N2=IFI(2)
62.000      N3=IFI(3)
63.000      THICK=AFL(4)
64.000      WRITE(IW,306)N1,N2,N3,THICK
65.000      IF(N1.LE.0)GO TO 20
66.000      GO TO 12
67.000      C
68.000      READ OPTION: JOPT * * * *
69.000      C
70.000      JOPT=0 IF ELASTIC PROPERTIES FOR ALL THE SOLID
71.000      C      ELEMENTS ARE GIVEN BY THE USER
72.000      C
73.000      JOPT=1 FOR NON-HOMOG. SOLID OF LINEARLY VARYING
74.000      C      MODULUS OF ELASTICITY WITH DEPTH,
75.000      C
76.000      JOPT=-1 WHEN THERE IS NO SOLID ELEMENT.
77.000      C
78.000      20 CALL REED
79.000      C
80.000      JOPT=IFI(1)
      WRITE(IW,308)JOPT
      IF(JOPT.EQ.1)GO TO 24
      IF(JOPT.EQ.-1)GO TO 28
      READ EL. NO. N1, CONNECTING NODES: N2,N3,N4;
      Y. MOD. AND P. RATIO FOR EACH SOLID EL.* *

```

4 4

OF / DE

```

81.000 C
82.000
83.000
84.000
85.000
86.000
87.000
88.000
89.000
90.000
91.000
92.000 C
93.000 C
94.000 C
95.000
96.000
97.000
98.000
99.000 C
100.000 C
101.000 C
102.000
103.000
104.000
105.000
106.000
107.000
108.000
109.000
110.000 C
111.000 C
112.000 C
113.000
114.000
115.000
116.000
117.000
118.000
119.000
120.000 C

22 CALL REED
N1=IFI(1)
N2=IFI(2)
N3=IFI(3)
N4=IFI(4)
YMOD=AFL(5)
PR=AFL(6)
WRITE(IW,310)N1,N2,N3,N4,YMOD,PR
IF(N1.LE.0)GO TO 28
GO TO 22

READ YOUNG'S MODULUS AT THE SURFACE OF THE
SOLID MEDIUM: ES, AND RATE OF INCREASE OF ES.

24 CALL REED
YMOD0=AFL(1)
SLOPE=AFL(2)
WRITE(IW,304)YMOD0,SLOPE
. . . JOPT=1: READ EL. NO. N1, CONNECTING NODES: N2,
N3,N4, AND P-RATIO FOR EACH SOLID EL. * * *

26 CALL REED
N1=IFI(1)
N2=IFI(2)
N3=IFI(3)
PR=AFL(4)
WRITE(IW,312)N1,N2,N3,PR
IF(N1.LE.0)GO TO 28
GO TO 26

READ APPLIED LOADS: UNIFORMLY DISTRIBUTED
LOADS ON PLATE ELEMENTS ARE TO BE READ FIRST.
(INPUT: LEFT NODE,RT. NODE, INTENSITY) * * *

28 CALL REED
N1=IFI(1)
N2=IFI(2)
UDL=AFL(3)
WRITE(IW,316)N1,N2,UDL
IF(N1.LE.0)GO TO 30
GO TO 28

READ UDL ON SOLID EL. (INPUT: CONNECTING NO-

```

```

121.000 C      DES( STARTING FROM TOP LEFT), AND INTENSITY) *
122.000 C
123.000
124.000
125.000
126.000
127.000
128.000
129.000
130.000
131.000 C
132.000 C
133.000 C
134.000
135.000
136.000
137.000
138.000
139.000
140.000 C
141.000 C
142.000 C
143.000
144.000
145.000
146.000
147.000
148.000
149.000 C
150.000 C
151.000 C
152.000 C
153.000 C
154.000 C
155.000
156.000
157.000
158.000 C
159.000 C
160.000 C

30 CALL REED
   N1=IFI(1)
   N2=IFI(2)
   N3=IFI(3)
   UDL=AFL(4)
   WRITE(IW,306)N1,N2,N3,UDL
   IF(N1.LE.0)GO TO 32
   GO TO 30
      READ CONC. LOADS: INPUT COMPONENT NUMBER,
      AND MAGNITUDE OF LOAD FOR EACH LOAD * * * *
32 CALL REED
   N1=IFI(1)
   CL=AFL(2)
   WRITE(IW,320)N1,CL
   IF(N1.LE.0)GO TO 34
   GO TO 32
      READ KNOWN DISPLACEMENTS. FOR EACH DISP. INPUT:
      COMPONENT NUMBER, MAGNITUDE OF DISPL. * * * *
34 CALL REED
   N1=IFI(1)
   U=AFL(2)
   WRITE(IW,320)N1,U
   IF(N1.LE.0)GO TO 36
   GO TO 34
      READ OPTION: NREQ * * * *
      NREQ=2 WHEN STRESS COMP., STRESS INVARIANTS,
      ARE REQUIRED; NREQ=POSITIVE BUT NOT 2:
      WHEN ONLY STRESS COMP. ARE REQUIRED;
      NREQ=NEG. OR ZERO STRESSES ARE NOT REQ.
36 CALL REED
   INOP=IFI(1)
   WRITE(IW,308)INOP
      READ OPTION: NEXT * * * *
      NEXT=1 WHEN SAME PROB. IS TO BE SOLVED
      FOR ANOTHER LOADING OR BOUNDARY COND.;

```

```

161.000 C
162.000 C
163.000 C
164.000 C
165.000
166.000
167.000
168.000
169.000
170.000
171.000
172.000
173.000
174.000
175.000
176.000
177.000
178.000
179.000
180.000
181.000 C
182.000 C
183.000 C
184.000
185.000 C
186.000 C
187.000 C
188.000 C
189.000 C
190.000
191.000
192.000
193.000 C
194.000
195.000
196.000
197.000
198.000
199.000 C
200.000 C

NEXT=2 WHEN COMPLETELY DIFFERENT PROB.
IS TO BE SOLVED; NEXT= OTHER NUMBER
WHEN NO OTHER PROB. IS TO BE SOLVED.

CALL REED
INOP=IFI(1)
WRITE(IW,308) INOP
IF(INOP.EQ.1) GO TO 28
IF(INOP.EQ.2) GO TO 7
STOP
300 FORMAT(20A4)
302 FORMAT(I5,2F10.4)
304 FORMAT(F10.1,F10.4)
306 FORMAT(3I5,F10.4)
308 FORMAT(I5)
310 FORMAT(4I5,F10.1,F10.4)
312 FORMAT(4I5,F10.4)
316 FORMAT(2I5,F10.4)
320 FORMAT(I5,F10.4)
END
* * * * *
SUBROUTINE REED
SUBROUTINE FOR FREE FORMAT READ
VERSION OF UTILITY SUBROUTINE REED: MAY 4, 1978
COMMON/BLOK1/IR,IW,IN,ITOT,JA(80),IA(48),IFI(6)
+,AFL(6),IND(6),JBND(8,6),LBND(4,6),FBND(4,6)
DIMENSION JBNTT(6)
DATA IO,I1,I2,I3,I4,I5,I6,I7,I8,I9/
+LH0,LH1,LH2,LH3,LH4,LH5,LH6,LH7,LH8,LH9/
DATA IBLANK,IMINUS,IDOT/LH-,LH-,LH-/
DATA II,IO,IQ,IT,IU,IY/LHI,LHO,LHQ,LHT,LHU,LHY/
DATA IF,INN/LHF,LHN/
INITIALIZE INPUT DATA ARRAYS TO BLANK OR ZERO

```

```

201.000 112 DO 100 I=1,48
202.000 100 IA(I)=IBLANK
203.000 DO 101 I=1,6
204.000 IFI(I)=0
205.000 IND(I)=0
206.000 JBNTT(I)=0
207.000 101 AFL(I)=0.
208.000 C
209.000 DO 603 J=1,6
210.000 DO 603 K=1,4
211.000 603 JBNTT(J)=JBNTT(J)+JBND(K,J)+JBND(K+4,J)
212.000 C
213.000 C READ UNPROCESSED ALPHANUMERIC INPUT
214.000 C M70$
215.000 110 READ(IR,1,END=108)(JA(I),I=1,IN)
216.000 C M70$
217.000 GO TO 109
218.000 C MAL$
219.000 C 110 READ(IR,1)(JA(I),I=1,IN)
220.000 C MAL$
221.000 C IF(ENDFILE IR)108,109
222.000 108 WRITE(IW,3)
223.000 GO TO 110
224.000 C
225.000 C SET COUNTS ETC TO INITIAL VALUES
226.000 109 KOUNT=1
227.000 IOUNT=1
228.000 INUMB=1
229.000 NUMB=0
230.000 ISIGN=1
231.000 IDES=0
232.000 IDET=1
233.000 IWORD=0
234.000 ISIZE=0
235.000 IBIG=0
236.000 IZERO=0
237.000 C
238.000 C PROCESS AS MUCH AS REQUESTED OF INPUT ARRAY UNLESS TERMINATED
239.000 C BY UPPER LIMITS ON NO. OF DATA UNITS TO BE GENERATED
240.000 DO 102 I=1,IN

```

```

241.000 C RECOGNIZE BLANK AND TRANSFER CONTROL FOR PROCESSING OF BLANK
242.000 IF(JA(I).NE.IBLANK)GOTO 103
243.000 IF(IWORD.EQ.0)GOTO 102
244.000 IF(INUMB.EQ.0)GOTO 107
245.000 GOTO 113
246.000 C
247.000 103 IWORD=1
248.000 IF(ISIZE.GT.8)GOTO 106
249.000 C LOAD A1 DATA INTO ALPHANUMERIC ARRAY IF DATA UNIT.LENGTH DOES NOT
250.000 C EXCEED EIGHT
251.000 IA(IOUNT)=JA(I)
252.000 IOUNT=IOUNT+1
253.000 ISIZE=ISIZE+1
254.000 106 IF(INUMB.EQ.0)GOTO 102
255.000 C
256.000 C CONVERT ALPHANUMERIC DIGIT TO NUMERICAL EQUIVALENT
257.000 IF(JA(I).EQ.I0.OR.JA(I).EQ.I0)GOTO 500
258.000 IF(JA(I).EQ.I1.OR.JA(I).EQ.I1)GOTO 501
259.000 IF(JA(I).EQ.I2)GOTO 502
260.000 IF(JA(I).EQ.I3)GOTO 503
261.000 IF(JA(I).EQ.I4)GOTO 504
262.000 IF(JA(I).EQ.I5)GOTO 505
263.000 IF(JA(I).EQ.I6)GOTO 506
264.000 IF(JA(I).EQ.I7)GOTO 507
265.000 IF(JA(I).EQ.I8)GOTO 508
266.000 IF(JA(I).EQ.I9)GOTO 509
267.000 IF(JA(I).EQ.IDOT)GOTO 150
268.000 IF(JA(I).EQ.IMINUS)GOTO 122
269.000 INUMB=0
270.000 ISIGN=1
271.000 IDES=0
272.000 IBIG=0
273.000 IF(IKOUNT)=0
274.000 AFL(KOUNT)=0.
275.000 IND(KOUNT)=0
276.000 GOTO 102
277.000 C
278.000 150 IF(IDES.EQ.1) INUMB=0
279.000 IDES=1
280.000 C IDES=1 MEANS THAT A DECIMAL POINT HAS BEEN ENCOUNTERED

```



```

281.000      GOTO 102
282.000 C
283.000      122 IF((ISIGN.EQ.-1).OR.(NUMB.NE.0).OR.(IZERO.NE.0)
284.000      +.OR.(IDES.NE.0)) INUMB=0
285.000      ISIGN=-1
286.000      GOTO 102
287.000 C
288.000      500 NUM=0
289.000      IF(NUMB.EQ.0) IZERO=1
290.000      GOTO 104
291.000      501 NUM=1
292.000      GOTO 104
293.000      502 NUM=2
294.000      GOTO 104
295.000      503 NUM=3
296.000      GOTO 104
297.000      504 NUM=4
298.000      GOTO 104
299.000      505 NUM=5
300.000      GOTO 104
301.000      506 NUM=6
302.000      GOTO 104
303.000      507 NUM=7
304.000      GOTO 104
305.000      508 NUM=8
306.000      GOTO 104
307.000      509 NUM=9
308.000      104 CONTINUE
309.000      IF(NUMB.GT.3275) GOTO 105
310.000      114 IF(IDET.EQ.10000) GOTO 119
311.000      NUMB=10*NUMB+NUM
312.000      IF(IDES.EQ.1) IDET=10*IDET
313.000      GOTO 102
314.000 C
315.000      119 ANUM=NUM
316.000      IF(1BIG.EQ.2) GOTO 117
317.000      ANUMB=NUMB
318.000      GOTO 115
319.000 C
320.000      105 ANUM=NUM

```

```

321.000 IF (IBIG.EQ.1)GOTO 116
322.000 IF (IBIG.EQ.2)GOTO 117
323.000 IF (NUMB.EQ.3276.AND.NUM.LE.7)GOTO 114
324.000 ANUMB=NUMB
325.000 116 IF (IDES.EQ.1)GOTO 115
326.000 ANUMB=10.*ANUMB+ANUM
327.000 IBIG=1
328.000 GOTO 102
329.000 C
330.000 115 ADET=IDET
331.000 IBIG=2
332.000 ANUMB=ANUMB/ADET
333.000 117 ADET=10.*ADET
334.000 ANUMB=ANUMB+ANUM/ADET
335.000 GOTO 102
336.000 C
337.000 118 IND(KOUNT)=1
338.000 IF (ISIGN.LT.1)ANUMB=-ANUMB
339.000 AFL(KOUNT)=ANUMB
340.000 GOTO 121
341.000 C
342.000 C CARRY OUT FINAL COMPUTATION AND STORAGE OF NUMERICAL DATA
343.000 113 IF (IBIG.GE.1)GOTO 118
344.000 NUMB=NUMB*ISIGN
345.000 C DO FINAL COMPUTATION AND STORAGE OF FIXED POINT NUMERICAL DATA
346.000 IFI(KOUNT)=NUMB/IDET
347.000 AIDET=IDET
348.000 ANUMB=NUMB
349.000 C DO FINAL COMPUTATION AND STORAGE OF FLOATING POINT NUMERICAL DATA
350.000 AFL(KOUNT)=ANUMB/AIDET
351.000 IND(KOUNT)=2
352.000 C
353.000 C PERFORM RANGE CHECKS IF REQUIRED
354.000 121 JBND=JBND(1,KOUNT)+JBND(2,KOUNT)+JBND(3,KOUNT)+JBND(4,KOUNT)
355.000 IF (JBND.EQ.0)GO TO 524
356.000 IF (JBND(1,KOUNT).EQ.0)GO TO 521
357.000 IF (IFI(KOUNT).LE.IBND(1,KOUNT))GO TO 521
358.000 WRITE(IW,5)IFI(KOUNT)
359.000 GO TO 112
360.000 521 IF (JBND(2,KOUNT).EQ.0)GO TO 522

```

```

361.000 IF (IFI(KOUNT).GE.IBND(2,KOUNT))GO TO 522
362.000 WRITE(IW,6)IFI(KOUNT)
363.000 GO TO 112
364.000 522 IF (JBND(3,KOUNT).EQ.0)GO TO 523
365.000 IF (IFI(KOUNT).LE.IBND(3,KOUNT))GO TO 523
366.000 WRITE(IW,7)IFI(KOUNT)
367.000 READ(IR,1)IYN
368.000 IF ((IYN.NE.IY).AND.(IYN.NE.IO).AND.(IYN.NE.IO))GOTO 112
369.000 523 IF (JBND(4,KOUNT).EQ.0)GO TO 524
370.000 IF (IFI(KOUNT).GE.IBND(4,KOUNT))GO TO 524
371.000 WRITE(IW,8)IFI(KOUNT)
372.000 READ(IR,1)IYN
373.000 IF ((IYN.NE.IY).AND.(IYN.NE.IO).AND.(IYN.NE.IO))GOTO 112
374.000 524 JBNT=JBND(5,KOUNT)+JBND(6,KOUNT)+JBND(7,KOUNT)+JBND(8,KOUNT)
375.000 IF (JBNT.EQ.0)GO TO 528
376.000 IF (JBND(5,KOUNT).EQ.0)GO TO 525
377.000 IF (AFL(KOUNT).LE.FBND(1,KOUNT))GO TO 525
378.000 WRITE(IW,9)AFL(KOUNT)
379.000 GO TO 112
380.000 525 IF (JBND(6,KOUNT).EQ.0)GO TO 526
381.000 IF (AFL(KOUNT).GE.FBND(2,KOUNT))GO TO 526
382.000 WRITE(IW,10)AFL(KOUNT)
383.000 GO TO 112
384.000 526 IF (JBND(7,KOUNT).EQ.0)GO TO 527
385.000 IF (AFL(KOUNT).LE.FBND(3,KOUNT))GO TO 527
386.000 WRITE(IW,11)AFL(KOUNT)
387.000 READ(IR,1)IYN
388.000 IF ((IYN.NE.IY).AND.(IYN.NE.IO).AND.(IYN.NE.IO))GOTO 112
389.000 527 IF (JBND(8,KOUNT).EQ.0)GO TO 528
390.000 IF (AFL(KOUNT).GE.FBND(4,KOUNT))GO TO 528
391.000 WRITE(IW,12)AFL(KOUNT)
392.000 READ(IR,1)IYN
393.000 IF ((IYN.NE.IY).AND.(IYN.NE.IO).AND.(IYN.NE.IO))GOTO 112
394.000 528 CONTINUE
395.000 C
396.000 NUMB=0
397.000 ISIGN=1
398.000 IDES=0
399.000 IDET=1
400.000 IBIG=0

```

```

401.000 C
402.000 C DATA ITEM COUNT FOR BOTH NUMERIC AND ALPHANUMERIC DATA
403.000 107 KOUNT=KOUNT+1
404.000 IF(KOUNT.GT.ITOT.OR.KOUNT.GT.6)GOTO 510
405.000 IOUNT=8*KOUNT-7
406.000 ISIZE=0
407.000 IWORD=0
408.000 INUMB=1
409.000 IDES=0
410.000 NUMB=0
411.000 ISIGN=1
412.000 IBIG=0
413.000 IDET=1
414.000 IZERO=0
415.000 102 CONTINUE
416.000 C
417.000 510 CONTINUE
418.000 IF(IA(1).EQ.IQ.AND.IA(2).EQ.IU.AND.IA(3).EQ.II.AND.IA(4).EQ.IT)
419.000 + GOTO 333
420.000 IF(IA(1).EQ.IF.AND.IA(2).EQ.II.AND.IA(3).EQ.INN) GOTO 333
421.000 GO TO 111
422.000 333 WRITE(IW,4)
423.000 STOP
424.000 C
425.000 601 WRITE(IW,13)KKKK
426.000 GOTO 112
427.000 C
428.000 111 CONTINUE
429.000 DO 602 J=1,ITOT
430.000 KKK=J
431.000 IF(IND(J).EQ.0.AND.JBNTT(J).GT.0)GOTO 601
432.000 602 CONTINUE
433.000 DO 120 I=1,8
434.000 DO 120 J=1,ITOT
435.000 120 JBND(L,J)=0
436.000 831 CONTINUE
437.000 RETURN
438.000 C
439.000 1 FORMAT(130A1)
440.000 3 FORMAT(' NO DATA SUPPLIED. TRY AGAIN')

```

```

441.000 4 FORMAT(' PROGRAM HAS BEEN INTERRUPTED BY USER')
442.000 5 FORMAT(' DATA VALUE JUST SUPPLIED (' ,I7,') IS TOO LARGE.' ,/,
443.000 + ' TRY AGAIN')
444.000 6 FORMAT(' DATA VALUE JUST SUPPLIED (' ,I7,') IS TOO SMALL.' ,7,
445.000 + ' TRY AGAIN')
446.000 7 FORMAT(' DATA VALUE JUST SUPPLIED (' ,I7,') IS UNUSUALLY LARGE.' ,7,
447.000 + ' DO YOU WISH TO USE IT? Y/N')
448.000 8 FORMAT(' DATA VALUE JUST SUPPLIED (' ,I7,') IS UNUSUALLY SMALL.' ,/
449.000 + ' DO YOU WISH TO USE IT? Y/N')
450.000 9 FORMAT(' DATA VALUE JUST SUPPLIED (' ,G12.4,') IS TOO LARGE.' ,/,
451.000 + ' TRY AGAIN')
452.000 10 FORMAT(' DATA VALUE JUST SUPPLIED (' ,G12.4,') IS TOO SMALL.' ,/,
453.000 + ' TRY AGAIN')
454.000 11 FORMAT(' DATA VALUE JUST SUPPLIED (' ,G12.4,') IS UNUSUALLY',
455.000 + ' LARGE.' ,/, ' DO YOU WISH TO USE IT? Y/N')
456.000 12 FORMAT(' DATA VALUE JUST SUPPLIED (' ,G12.4,') IS UNUSUALLY',
457.000 + ' SMALL.' ,/, ' DO YOU WISH TO USE IT? Y/N')
458.000 13 FORMAT(
459.000 + ' THERE IS AN ERROR IN TERM NO.' ,I3, ' OF INPUT DATA JUST SUPPLIED.
460.000 + ' ,/, ' PLEASE TRY AGAIN')
461.000 END

```

```

A.10.3 Program for drawing finite element grid
=====
1.000 C
2.000 C
3.000 C
4.000 C
5.000 C
6.000 C
7.000 C
8.000 C
9.000 C
10.000 C
11.000 C
12.000 C
13.000 C
14.000 C
15.000 C
16.000 C
17.000 C
18.000 C
19.000 C
20.000 C
21.000 C
22.000 C
23.000 C
24.000 C
25.000 C
26.000 C
27.000 C
28.000 C
29.000 C
30.000 C
31.000 C
32.000 C
33.000 C
34.000 C
35.000 C
36.000 C
37.000 C
38.000 C
39.000 C
40.000 C

A PROGRAM FOR DRAWING FINITE ELEMENT GRID USING
THE INPUT DATA FOR THE MAIN PROGRAM. ( FANCI-
FINITE ELEMENT ANALYSIS OF INTERACTION BETWEEN
AN ELASTIC CIRCULAR PLATE AND AN ISOTROPIC
SOLID MEDIUM ) * * * * *

-----
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JUNE 1979
SUPERVISORS: PROF. A.P.S. SELVADURAI AND
PROF. W.H. BOWES
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=====

NOTE: * * * THIS PROGRAM USES THE BASIC SOFTWARE
AVAILABLE UNDER THE 'CALPLOT' SUBROUTINE
PACKAGE * * *

DIMENSION R(175),Z(175),NCON(3,300),NPL(20),NAME(20)

ASSIGN READING UNITS * * *

IR=3
IRR=5

READ CASE TITLE (DUMMY READ) * * *
READ(IR,100)NAME
READ COORDINATES OF NODES (R,Z); INPUT: NN=
NODE NO., RR= R-COORD., ZZ=Z-COORD. * * *
10 READ(IR,102)NN,RR,ZZ
IF(NN.LE.0)GO TO 15
R(NN)=RR
Z(NN)=ZZ
GO TO 10

READ YOUNG'S MOD. AND POISSON'S RATIO FOR

```

```

41.000 C
42.000
43.000
44.000
45.000 C
46.000 C
47.000 C
48.000
49.000
50.000
51.000
52.000
53.000
54.000
55.000
56.000 C
57.000 C
58.000
59.000
60.000
61.000
62.000 C
63.000 C
64.000 C
65.000
66.000
67.000
68.000
69.000
70.000
71.000
72.000 C
73.000 C
74.000 C
75.000
76.000 C
77.000 C
78.000 C
79.000
80.000

      THE PLATE (DUMMY READ) * * *
15 READ(IR,104)YMOD,PR
    NPLP=0
    IF(YMOD.LE.0.)GO TO 25

      READ ELEMENT NO. N1, LEFT NODE NO. N2, RIGHT
      NODE NO. N3 FOR EACH PLATE ELEMENT * * *
17 READ(IR,102)N1,N2,N3
    IF(N1.LE.0)GO TO 20
    NPLP=NPLP+1
    N4=N1
    N5=N3
    NPL(N1)=N2
    GO TO 17
20 NPL(N4+1)=N5

      READ OPTION: JOPT (DUMMY READ) * * *
25 READ(IR,108)JOPT
    NSEL=0
    IF(JOPT.EQ.1)GO TO 30
    IF(JOPT.EQ.-1)GO TO 40

      READ ELEMENT NO. N1, CONNECTING NODES; N2,N3,N4
      FOR EACH SOLID ELEMENT * * *
27 READ(IR,110)N1,N2,N3,N4
    IF(N1.LE.0)GO TO 40
    NCON(1,N1)=N2
    NCON(2,N1)=N3
    NCON(3,N1)=N4
    NSEL=NSEL+1
    GO TO 27

    IF JOPT=1; READ YOUNG'S MODULUS AT THE SURFACE OF
    THE SOLID MEDIUM YMOD, AND SLOPE * * *
30 READ(IR,104)YMOD,SLOPE

      READ ELEMENT NO., CONNECTING NODES; N2,N3,N4 FOR
      EACH SOLID ELEMENT ( IF JOPT=1) * * *
35 READ(IR,110)N1,N2,N3,N4
    IF(N1.LE.0)GO TO 40

```

```

81.000 NCON(1,N1)=N2
82.000 NCON(2,N1)=N3
83.000 NCON(3,N1)=N4
84.000 NSEL=NSEL+1
85.000 GO TO 35
86.000 C
87.000 C
88.000 C
89.000 C
90.000 C
91.000 C
92.000 40 READ(IRR,108)FAC
93.000 IF(FAC.LE.0.)FAC=1.
94.000 C
95.000 C
96.000 CALL START AND SET PAGE SIZE * * * *
97.000 CALL START
98.000 CALL PAGE(20.0,26.0,.5,.5)
99.000 CALL FACTOR(FAC)
100.000 C
101.000 C
102.000 DRAW STRAIGHT LINES FOR PLATE ELEMENTS, IF ANY * * * *
103.000 IF(NELP.LE.0)GO TO 75
104.000 DO 70 I=1,NELP
105.000 N1=NPL(I)
106.000 N2=NPL(I+1)
107.000 R1=R(N1)
108.000 R2=R(N2)
109.000 Z1=Z(N1)
110.000 Z2=Z(N2)
111.000 CALL PLOT(R1,Z1,3)
112.000 CALL PLOT(R2,Z2,2)
113.000 C
114.000 C
115.000 70 CONTINUE
116.000 C
117.000 C
118.000 IF THE NO. OF SOLID ELEMENT = ZERO, STOP; OTHERWISE
119.000 DRAW TRIANGULAR SOLID ELEMENTS * * * *
120.000 75 IF(JOPT.EQ.-1)GO TO 85
DO 80 I=1,NSEL
N1=NCON(1,I)
N2=NCON(2,I)
N3=NCON(3,I)
R1=R(N1)

```



```
121.000      R2=R(N2)
122.000      R3=R(N3)
123.000      Z1=Z(N1)
124.000      Z2=Z(N2)
125.000      Z3=Z(N3)
126.000      CALL PLOT(R1,Z1,3)
127.000      CALL PLOT(R2,Z2,2)
128.000      CALL PLOT(R3,Z3,2)
129.000      CALL PLOT(R1,Z1,2)
130.000      80 CONTINUE
131.000      85 CALL PLOT(R1,Z1,999)
132.000      STOP
133.000      100 FORMAT(20A4)
134.000      102 FORMAT(3G)
135.000      104 FORMAT(2G)
136.000      108 FORMAT(1G)
137.000      110 FORMAT(4G)
138.000      END
```

APPENDIX B

B.1 CONSISTENT LOAD VECTOR DUE TO UNIFORM PRESSURE ON SOLID ELEMENT

Referring to Fig. 2.4, the work done W_e by the equivalent nodal forces F_i and F_k is

$$W_e = F_i \omega_i + F_k \omega_k \quad (B.1.1)$$

In matrix form

$$W_e = \begin{bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_i \\ 0 \\ F_k \end{bmatrix} \quad (B.1.2)$$

On the other hand, the work done by the actual system of forces (pressure) W_a can be written as

$$W_a = \int_{r_i}^{r_k} \int_0^{2\pi} q \omega \, r \, dr \, d\theta \quad (B.1.3)$$

or

$$W_a = 2\pi q \int_{r_i}^{r_k} \omega r \, dr \quad (B.1.4)$$

In matrix form,

$$W_a = 2\pi q \int_{r_i}^{r_k} \begin{Bmatrix} U \\ \omega \end{Bmatrix}^T \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r dr \quad (\text{B.1.5})$$

where

$$u = \alpha_1 + \alpha_2 r + \alpha_3 z$$

and

$$\omega = \alpha_4 + \alpha_5 r + \alpha_6 z \quad (\text{B.1.6})$$

Relating the generalized coordinates $\alpha_i (i=1,2,\dots,6)$ with the nodal displacements (i.e. $u_i, u_j, \dots, \omega_j, \omega_k$) one can obtain

$$\begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix} = [A] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad (\text{B.1.7})$$

where

$$[A] = \begin{bmatrix} 1 & r_i & z_i & 0 & 0 & 0 \\ 1 & r_j & z_j & 0 & 0 & 0 \\ 1 & r_k & z_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_i & z_i \\ 0 & 0 & 0 & 1 & r_j & z_j \\ 0 & 0 & 0 & 1 & r_k & z_k \end{bmatrix} \quad (\text{B.1.8})$$

Substituting Eqn. (B.1.6) in Eqn. (B.1.5)

$$W_a = 2\pi q \int_{r_i}^{r_k} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}^T \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix}^T \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r dr \quad (B.1.9)$$

Using Eqn. (B.1.7), the above equation can be written as

$$W_a = 2\pi q \begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix}^T [A^{-1}]^T \int_{r_i}^{r_k} \begin{bmatrix} r & 0 \\ r^2 & 0 \\ rz & 0 \\ 0 & r \\ 0 & r^2 \\ 0 & rz \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} dr \quad (B.1.10)$$

This expression can easily be simplified into the following form

$$W_a = 2\pi q \begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix}^T [A^{-1}]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (r_k^2 - r_i^2)/2 \\ (r_k^3 - r_i^3)/3 \\ (r_k^2 - r_i^2)z/2 \end{Bmatrix} \quad (B.1.11)$$

It may be noted that, for a horizontal surface $z=z_i=z_k$.

Since the two loading system is equivalent, the Eqn. (B.1.2) can be equated with Eqn. (B.1.11)

$$\begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_i \\ 0 \\ F_k \end{Bmatrix} = \begin{Bmatrix} u_i \\ u_j \\ u_k \\ \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix}^T 2\pi q [A^{-1}]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (r_k^2 - r_i^2)/2 \\ (r_k^3 - r_i^3)/3 \\ (r_k^2 - r_i^2)z/2 \end{Bmatrix} \quad (B.1.12)$$

or

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_i \\ 0 \\ F_k \end{Bmatrix} = 2\pi q [A^{-1}]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (r_k^2 - r_i^2)/2 \\ (r_k^3 - r_i^3)/3 \\ (r_k^2 - r_i^2)z/2 \end{Bmatrix} \quad (B.1.13)$$

Now (see Desai and Abel, 1972),

$$[A^{-1}] = \frac{1}{2 \cdot \text{Area}} \begin{bmatrix} r_j z_k - r_k z_j & r_k z_i - r_i z_k & r_i z_j - r_j z_i & 0 & 0 & 0 \\ z_j - z_i & z_k - z_i & z_i - z_j & 0 & 0 & 0 \\ r_k - r_j & r_i - r_k & r_j - r_i & 0 & 0 & 0 \\ 0 & 0 & 0 & r_j z_k - r_k z_j & r_k z_i - r_i z_k & r_i z_j - r_j z_i \\ 0 & 0 & 0 & z_j - z_i & z_k - z_i & z_i - z_j \\ 0 & 0 & 0 & r_k - r_j & r_i - r_k & r_j - r_i \end{bmatrix} \quad (B.1.14)$$

where

$$\text{Area} = \frac{1}{2} \{r_j(z_k - z_i) + r_i(z_j - z_k) + r_k(z_i - z_j)\} \quad (\text{B.1.15})$$

Substitution of Eqn. (B.1.14) into Eqn. (B.1.13) and then carrying out the required matrix multiplication will result in the following.

$$F_i = \frac{(r_k - r_i)}{12 \text{ Area}} \{z_i(-2r_i^2 + r_i r_k + r_k^2) + z_j(2r_i^2 - r_i r_k - r_k^2)\} 2\pi q \quad (\text{B.1.16})$$

$$F_k = \frac{(r_k - r_i)}{12 \text{ Area}} \{z_i(-r_i^2 - r_i r_k + 2r_k^2) + z_j(r_i^2 + r_i r_k - 2r_k^2)\} 2\pi q \quad (\text{B.1.17})$$

B.2 FLEXURAL MOMENTS AND SHEARING FORCES

The radial component M_r and the circumferential component M_θ of the flexural moments induced due to axisymmetric bending of circular plate can be calculated from the following relationships (Timoshenko and Woinowsky-Krieger, 1959).

$$M_r = -D \left(\frac{\partial^2 \omega(r)}{\partial r^2} + \frac{\nu_b}{r} \frac{\partial \omega(r)}{\partial r} \right) \quad (\text{B.2.1})$$

$$M_\theta = -D \left(\nu_b \frac{\partial^2 \omega(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \omega(r)}{\partial r} \right) \quad (\text{B.2.2})$$

Annular Plate Element: Flexural Moments

For an annular plate element (Fig. 2.9)

$$\omega(r) = \alpha_1 + \alpha_2 r + \alpha_3 r^2 + \alpha_4 r^3 \quad (\text{B.2.3})$$

Therefore,

$$\frac{\partial \omega(r)}{\partial r} = \alpha_2 + 2\alpha_3 r + 3\alpha_4 r^2 \quad (\text{B.2.4})$$

and

$$\frac{\partial^2 \omega(r)}{\partial r^2} = 2\alpha_3 + 6\alpha_4 r \quad (\text{B.2.5})$$

Substituting Eqns. (B.2.4) and (B.2.5) into Eqns. (B.2.1) and (B.2.2) one can get

$$M_r = +D\{(2\alpha_3 + 6\alpha_4 r) + \frac{v_b}{r}(\alpha_2 + 2\alpha_3 r + 3\alpha_4 r^2)\} \quad (\text{B.2.6})$$

and

$$M_\theta = -D\{v_b(2\alpha_3 + 6\alpha_4 r) + \frac{1}{r}(\alpha_2 + 2\alpha_3 r + 3\alpha_4 r^2)\} \quad (\text{B.2.7})$$

In matrix form,

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & v_b/r & 2(1+v_b) & 3(2+v_b)r \\ 0 & 1/r & 2(1+v_b) & 3(2v_b+1)r \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (\text{B.2.8})$$

The generalized coordinates $\alpha_i (i=1,2,3,4)$ can be related to the nodal displacement vector as follows (see e.g. Bowes and Russel, 1975)

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = [A]^{-1} \{\delta\}_e \quad (\text{B.2.9})$$

where

$$A = \begin{bmatrix} 1 & r_i & r_i^2 & r_i^3 \\ 0 & 1 & 2r_i & 3r_i^2 \\ 1 & r_j & r_j^2 & r_j^3 \\ 0 & 1 & 2r_j & 3r_j^2 \end{bmatrix} \quad (\text{B.2.10})$$

and

$$\{\delta\}_e = \begin{Bmatrix} \omega_i \\ \theta_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (\text{B.2.11})$$

Substitution of Eqn. (B.2.9) in Eqn. (B.2.8) yields the following

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & \nu_b/r & 2(1+\nu_b) & 3(2+\nu_b) \\ 0 & 1/r & 2(1+\nu_b) & 3(2\nu_b+1) \end{bmatrix} [A]^{-1} \{\delta\}_e \quad (\text{B.2.12})$$

Annular Plate Element: Shearing Forces

Referring to Fig. 2.9, the shearing force per unit length Q_r of the circular plate can be determined from (Timoshenko and Woinowsky-Krieger, 1959).

$$Q_r = D \frac{\partial^3 \omega(r)}{\partial r^3} + \frac{1}{r} \frac{\partial^2 \omega(r)}{\partial r^2} - \frac{1}{r^2} \frac{\partial \omega(r)}{\partial r} \quad (\text{B.2.13})$$

where

$$\omega(r) = \alpha_1 + \alpha_2 r + \alpha_3 r^2 + \alpha_4 r^3 \quad (\text{B.2.3})$$

Therefore

$$\frac{\partial^3 \omega(r)}{\partial r^3} = 6\alpha_4 \quad (\text{B.2.14})$$

Substitution of Eqns. (B.2.4), (B.2.5) and (B.2.14) into Eqn. (B.2.13) yields

$$Q_r = D[6\alpha_4 + \frac{1}{r}(2\alpha_3 + 6\alpha_4 r) - \frac{1}{r^2}(\alpha_2 + 2\alpha_3 r + 3\alpha_4 r^2)] \quad (\text{B.2.15})$$

or

$$Q_r = D \begin{bmatrix} 0 & -\frac{1}{r^2} & 0 & 9 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (\text{B.2.16})$$

Using Eqn. (B.2.9), the above expression can be written as follows

$$Q_r = D \begin{bmatrix} 0 & -\frac{1}{r^2} & 0 & 9 \end{bmatrix} [A]^{-1} \{\delta\}_e \quad (\text{B.2.17})$$

Solid Circular Plate Element: Flexural Moments.

Eqns. (B.2.1) and (B.2.2) can be used to calculate the flexural moments at any point of a solid circular plate element. Referring to Fig. 2.10, the deflection function can be written as follows.

$$\omega(r) = \alpha_1 + \alpha_2 r^2 + \alpha_3 r^3 \quad (\text{B.2.18})$$

Now

$$\frac{\partial \omega(r)}{\partial r} = 2\alpha_2 r + 3\alpha_3 r^2 \quad (\text{B.2.19})$$

$$\frac{\partial^2 \omega(r)}{\partial r^2} = 2\alpha_2 + 6\alpha_3 r \quad (\text{B.2.20})$$

$$\frac{\partial^3 \omega(r)}{\partial r^3} = 6\alpha_3 \quad (\text{B.2.21})$$

Substituting Eqns. (B.2.19) and (B.2.20) in Eqns. (B.2.1) and (B.2.2) one can obtain the following.

$$M_r = -D \left\{ 2\alpha_2 + 6\alpha_3 r + \frac{\nu_b}{r} (2\alpha_2 r + 3\alpha_3 r^2) \right\} \quad (\text{B.2.22})$$

$$M_\theta = -D \left\{ \nu_b (2\alpha_2 + 6\alpha_3 r) + \frac{1}{r} (2\alpha_2 r + 3\alpha_3 r^2) \right\} \quad (\text{B.2.23})$$

In matrix form

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & 2(1+\nu_b) & 3r(2+\nu_b) \\ 0 & 2(1+\nu_b) & 3r(2\nu_b+1) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad (\text{B.2.24})$$

The generalized coordinates $\alpha_i (i=1,2,3)$ can now be related to the nodal displacements as follows

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & r_j^2 & r_j^3 \\ 0 & 2r_j & 3r_j^2 \end{bmatrix} \begin{Bmatrix} \omega_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (\text{B.2.25})$$

Substitution of Eqn. (B.2.25) into Eqn. (B.2.24) yields

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & 2(1+\nu_b) & 3r(2+\nu_b) \\ 0 & 2(1+\nu_b) & 3r(2\nu_b+1) \end{bmatrix} [A^{-1}] \begin{Bmatrix} \omega_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (\text{B.2.26})$$

where

$[A]^{-1}$ is shown in Eqn. (B.2.25).

Solid Circular Plate Element: Shearing Force

Eqn. (B.2.13) can be used to obtain the shearing force per unit length Q_r of the solid circular plate element.

Substituting Eqns. (B.2.19), (B.2.20) and (B.2.21) into Eqn. (B.2.13) one can obtain the following.

$$Q_r = D \left[6\alpha_3 + \frac{1}{r} (2\alpha_2 + 6\alpha_3 r) - \frac{1}{r^2} (2\alpha_2 r + 3\alpha_3 r^2) \right] \quad (\text{B.2.27})$$

or

$$Q_r = D \begin{bmatrix} 0 & 0 & 9 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad (\text{B.2.28})$$

Using Eqn. (B.2.25), the above expression can be written as follows

$$Q_r = D \begin{bmatrix} 0 & 0 & 9 \end{bmatrix} [A]^{-1} \begin{Bmatrix} \omega_i \\ \omega_j \\ \theta_j \end{Bmatrix} \quad (\text{B.2.29})$$

END

0 2 H 0 9 H 8 2

FIN